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Johann Bouali

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The Hodge realization functor on the derived category of relative motives

Johann Bouali

January 24, 2022

Abstract

We give, for a complex algebraic variety S , a Hodge realization functor \mathcal{F}_S^{Hdg} from the (unbounded) derived category of constructible motives $DA_c(S)$ over S to the (undounded) derived category $D(MHM(S))$ of algebraic mixed Hodge modules over S . Moreover, for $f : T \rightarrow S$ a morphism of complex quasi-projective algebraic varieties, \mathcal{F}_-^{Hdg} commutes with the four operations $f^*, f_*, f!, f^!$ on $DA_c(-)$ and $D(MHM(-))$, making in particular the Hodge realization functor a morphism of 2-functor on the category of complex quasi-projective algebraic varieties which for a given S sends $DA_c(S)$ to $D(MHM(S))$, moreover \mathcal{F}_S^{Hdg} commutes with tensor product. We also give an algebraic and analytic Gauss-Manin realization functor from which we obtain a base change theorem for algebraic De Rham cohomology and for all smooth morphisms a relative version of the comparaison theorem of Grothendieck between the algebraic De Rahm cohomology and the analytic De Rahm cohomology.

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1 Introduction

Saito's theory of mixed Hodge modules associate to each complex algebraic variety S a category $MHM(S)$ which is a full subcategory of $\text{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S)$ which extend variations of mixed Hodge structure and admits a canonical monoidal structure given by tensor product, and associate to each morphism of complex algebraic varieties $f : X \rightarrow S$, four functor $Rf_{Hdg!}, Rf_{Hdg*}, f^*Hdg, f^*Hdg$. In the case of a smooth proper morphism $f : X \rightarrow S$ with S and X smooth, $H^n Rf_{Hdg*} \mathbb{Z}_X^{Hdg}$ is the variation of Hodge structure given by the Gauss-Manin connexion and the local system $H^n Rf_* \mathbb{Z}_X$. Moreover, these functors induce the six functor formalism of Grothendieck. We thus have, for a complex algebraic variety S a canonical functor

$$MH(/S) : \text{Var}(\mathbb{C})/S \rightarrow D(MHM(S)), (f : X \rightarrow S) \mapsto Rf_{!Hdg} \mathbb{Z}_X^{Hdg}$$

and

$$MH(/-) : \text{Var}(\mathbb{C}) \rightarrow \text{TriCat}, S \mapsto (MH(S) : \text{Var}(\mathbb{C})/S \rightarrow D(MHM(S))),$$

is a morphism of 2-functor. In this work, we extend $MH(/-)$ to motives by constructing, for each complex algebraic variety S , a canonical functor $\mathcal{F}_S^{Hdg} : \text{DA}(S) \rightarrow D(MHM(S))$ which is monoidal, that is commutes with tensor product, together with, for each morphism of complex algebraic varieties $g : T \rightarrow S$ a canonical transformation map $T(g, \mathcal{F}^{Hdg})$, which make

$$\mathcal{F}_-^{Hdg} : \text{Var}(\mathbb{C}) \rightarrow \text{TriCat}, S \mapsto (\mathcal{F}_S^{Hdg} : \text{DA}(S) \rightarrow D(MHM(S))),$$

is a morphism of 2-functor : this is the contain of theorem 43. A partial result in this direction has been obtained by Ivorra in [18] using a different approach. We already have a Betti realization functor

$$\text{Bti}_- : \text{Var}(\mathbb{C}) \rightarrow \text{TriCat}, S \mapsto (\text{Bti}_S^* : \text{DA}(S) \rightarrow D(S)),$$

which extend the Betti realization. The functor $\mathcal{F}_-^{FDR} := (\mathcal{F}_-^{FDR}, \text{Bti}_-)$ is obtained by constructing the De Rham part

$$\mathcal{F}_-^{FDR} : \text{Var}(\mathbb{C}) \rightarrow \text{TriCat}, S \mapsto (\mathcal{F}_S^{FDR} : \text{DA}(S) \rightarrow D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I))),$$

which takes values in the derived category of filtered algebraic D -modules $D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I)) := K_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I))([E_1]^{-1})$ obtained by inverting the classes filtered Zariski local equivalences $[E_1]$ modulo ∞ -filtered homotopy and then using the following key theorem (theorem 32)

Theorem 1. (i) *Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then the full embedding*

$$\iota_S : MHM(S) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

induces a full embedding

$$\iota_S : D(\text{MHM}(S)) \hookrightarrow D_{\mathcal{D}(1,0)\text{fil},rh}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{an})$$

whose image consists of $((M_I, F, W), u_{IJ}), (K, W), \alpha \in D_{\mathcal{D}(1,0)\text{fil},rh}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{an})$ such that

$$((H^n(M_I, F, W), H^n(u_{IJ})), H^n(K, W), H^n \alpha) \in \text{MHM}(S)$$

for all $n \in \mathbb{Z}$ and such that for all $p \in \mathbb{Z}$, the differential of $(\text{Gr}_W^p M_I, F)$ are strict for the filtration F (in particular, the differentials of (M_I, F, W) are strict for the filtration F).

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then the full embedding

$$\iota_S : \text{MHM}(S) \hookrightarrow C_{\mathcal{D}(1,0)\text{fil},rh}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{an})$$

induces a full embedding

$$\iota_S : D(\text{MHM}(S)) \hookrightarrow D_{\mathcal{D}(1,0)\text{fil},\infty,rh}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{an})$$

whose image consists of $((M_I, F, W), u_{IJ}), (K, W), \alpha \in D_{\mathcal{D}(1,0)\text{fil},\infty,rh}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{an})$ such that

$$((H^n(M_I, F, W), H^n(u_{IJ})), H^n(K, W), H^n \alpha) \in \text{MHM}(S)$$

for all $n \in \mathbb{Z}$ and such that there exist $r \in \mathbb{N}$ and an r -homotopy equivalence $((M_I, F, W), u_{IJ}) \rightarrow ((M'_I, F, W), u_{IJ})$ such that for all $p \in \mathbb{Z}$, the differential of $(\text{Gr}_W^p M'_I, F)$ are strict for the filtration F (in particular, the differentials of (M'_I, F, W) are strict for the filtration F).

Note that the category $D_{\mathcal{D}(1,0)\text{fil},\infty,rh}(S/(\tilde{S}_I))$ is NOT triangulated. More precisely the canonical triangles of $D_{\mathcal{D}(1,0)\text{fil},\infty,rh}(S/(\tilde{S}_I))$ does NOT satisfy the 2 of 3 axiom of a triangulated category. Moreover there exist canonical triangles of $D_{\mathcal{D}(1,0)\text{fil},\infty,rh}(S/(\tilde{S}_I))$ which are NOT the image of distinguish triangles of $\pi_S(D(\text{MHM}(S)))$. This method can be seen as a relative version of the construction of F.Lecomte and N.Wach in [21].

In section 6.1.1 and 6.2.1, we construct an algebraic and analytic Gauss-Manin realization functor, but this functor does NOT give a complex of filtered D -module, BUT a complex of filtered O -modules whose cohomology sheaves have a structure of filtered D_S modules. Hence, it does NOT get to the desired category. Moreover the Hodge filtration is NOT the right one : see proposition 118 and proposition 111 However this functor gives some interesting results. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$ smooth. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define the filtered algebraic Gauss-Manin realization functor defined as

$$\begin{aligned} \mathcal{F}_S^{GM} : C(\text{Var}(\mathbb{C})^{sm}/S) &\rightarrow C_{O\text{fil},\mathcal{D}}(S/(\tilde{S}_I))^\vee, M \mapsto \\ \mathcal{F}_S^{GM}(F) &:= ((e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet), F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F)), \end{aligned}$$

see definition 122. Note that the canonical triangles of $D_{O\text{fil},\mathcal{D},\infty}(S/(\tilde{S}_I))$ does NOT satisfy the 2 of 3 axiom of a triangulated category. The filtered algebraic Gauss-Manin realization functor induces by proposition 107

$$\begin{aligned} \mathcal{F}_S^{GM} : \text{DA}_c(S)^{op} &\rightarrow D_{O\text{fil},\mathcal{D},\infty}(S/(\tilde{S}_I)), M \mapsto \\ \mathcal{F}_S^{GM}(M) &:= ((e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$. We then prove (theorem 35):

Theorem 2. (i) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S}$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_S^{GM}(M) \rightarrow \mathcal{F}_T^{GM}(g^*M)$$

is an isomorphism in $D_{O_T \text{fil}, \mathcal{D}, \infty}(T/(Y \times \tilde{S}_I))$.

(ii) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T^O(g, \mathcal{F}^{GM})(M) : Lg^{*mod} \mathcal{F}_S^{GM}(M) \rightarrow \mathcal{F}_T^{GM}(g^*M)$$

is an isomorphism in $D_{O_T}(T)$.

(iii) A base change theorem for algebraic De Rham cohomology : Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $h : U \rightarrow S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Then the map (see definition 1)

$$T_w^O(g, h) : Lg^{*mod} Rh_*(\Omega_{U/S}^\bullet, F_b) \rightarrow Rh'_*(\Omega_{U_T/T}^\bullet, F_b)$$

is an isomorphism in $D_{O_T}(T)$.

Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define the filtered analytic Gauss-Manin realization functor defined as

$$\begin{aligned} \mathcal{F}_S^{GM} : \text{DA}_c(S)^{op} &\rightarrow D_{O_{\text{fil}, \mathcal{D}, \infty}}(S/(\tilde{S}_I))^\vee, M \mapsto \\ \mathcal{F}_S^{GM}(M) &:= ((e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{et}(\Omega_{\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 143. We then prove (theorem 37):

Theorem 3. (i) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S}$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, \mathcal{F}_{an}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_{S, an}^{GM}(M) \rightarrow \mathcal{F}_{T, an}^{GM}(g^*M)$$

is an isomorphism in $D_{O_T \text{fil}, \mathcal{D}, \infty}(T/(Y \times \tilde{S}_I))$.

(ii) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, \mathcal{F}_{an}^{GM})(M) : Lg^{*mod[-]} \mathcal{F}_{S, an}^{GM}(M) \rightarrow \mathcal{F}_{T, an}^{GM}(g^*M)$$

is an isomorphism in $D_{O_T}(T)$.

A consequence of the construction of the transformation map between the algebraic and analytic Gauss-Manin realization functor is the following (theorem 39)

Theorem 4. (i) Let $S \in \text{Var}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$\mathcal{J}_S(-) \circ H^n T(\text{An}, \mathcal{F}_{an}^{GM})(M) : \mathcal{J}_S(H^n(\mathcal{F}_S^{GM}(M))^{an}) \xrightarrow{\sim} H^n \mathcal{F}_{S, an}^{GM}(M)$$

is an isomorphism in $\text{PSh}_{\mathcal{D}}(S^{an}/(\tilde{S}_I^{an}))$.

(ii) A relative version of Grothendieck GAGA theorem for De Rham cohomology Let $h : U \rightarrow S$ a smooth morphism with $S, U \in \text{SmVar}(\mathbb{C})$. Then,

$$\mathcal{J}_S(-) \circ J_S T_\omega^O(an, h) : J_S((R^n h_* \Omega_{U/S}^\bullet)^{an}) \xrightarrow{\sim} R^n h_* \Omega_{U^{an}/S^{an}}^\bullet$$

is an isomorphism in $\text{PSh}_{\mathcal{D}}(S^{an})$.

In section 6.1.2, using results of sections 2, 4 and 5, we construct the algebraic filtered De Rham realization functor \mathcal{F}_S^{FDR} . We construct it via a larger category and use theorem 32: Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. we define in definition 133(ii) which use definition 129 and definition 41, the filtered algebraic De Rham realization functor defined as

$$\begin{aligned} \mathcal{F}_S^{FDR} : C(\text{Var}(\mathbb{C})^{sm}/S) &\rightarrow C_{\mathcal{D}fil}(S/(\tilde{S}_I)), F \mapsto \mathcal{F}_S^{FDR}(F) := \\ &(e'(\tilde{S}_I)_* \text{Hom}^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)). \end{aligned}$$

By proposition 114(ii), it induces

$$\begin{aligned} \mathcal{F}_S^{FDR} : \text{DA}_c(S) &\rightarrow D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)), M \mapsto \mathcal{F}_S^{FDR}(M) := \\ &(e'(\tilde{S}_I)_* \text{Hom}^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$. We compute this functor for an homological motive in proposition 116 and we get by corollary 4, for $S \in \text{Var}(\mathbb{C})$ and $M \in \text{DA}_c(S)$, $\mathcal{F}_S^{FDR}(M) \in \pi_S(D(MHM(S)))$, and the following (theorem 36):

Theorem 5. (i) Let $g : T \rightarrow S$ a morphism, with $S, T \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. Let $M \in \text{DA}_c(S)$. Then map in $\pi_T(D(MHM(T)))$

$$T(g, \mathcal{F}^{FDR})(M) : g_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_T^{FDR}(g^* M)$$

given in definition 138 is an isomorphism.

(ii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. Then, for $M \in \text{DA}_c(X)$, the map given in definition 139

$$T_l(f, \mathcal{F}^{FDR})(M) : Rf_l^{Hdg} \mathcal{F}_X^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_S^{FDR}(Rf_l M)$$

is an isomorphism in $\pi_S(D(MHM(S)))$.

(iii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. We have, for $M \in \text{DA}_c(X)$, the map given in definition 139

$$T_*(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_S^{FDR}(Rf_* M) \xrightarrow{\sim} Rf_*^{Hdg} \mathcal{F}_X^{FDR}(M)$$

is an isomorphism in $\pi_S(D(MHM(S)))$.

(iv) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. Then, for $M \in \text{DA}_c(S)$, the map given in definition 139

$$T^l(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_X^{FDR}(f^! M) \xrightarrow{\sim} f_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M)$$

is an isomorphism in $\pi_X(D(MHM(X)))$.

(v) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$, the map in $\pi_S(D(\text{MHM}(S)))$

$$T(\mathcal{F}_S^{FDR}, \otimes)(M, N) : \mathcal{F}_S^{FDR}(M) \otimes_{O_S}^{Hdg} \mathcal{F}_S^{FDR}(N) \xrightarrow{\sim} \mathcal{F}_S^{FDR}(M \otimes N)$$

given in definition 139 is an isomorphism.

We also have a canonical transformation map between the Gauss-Manin and the De Rham functor given in definition 135 which satisfy (see proposition 118) :

Proposition 1. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

(i) For $M \in \text{DA}_c(S)$ the map in $D_{O_S, \mathcal{D}}(S/(\tilde{S}_I)) = D_{O_S, \mathcal{D}}(S)$

$$o_{fil}T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : o_{fil}\mathcal{F}_S^{GM}(\mathbb{L}\mathbb{D}_S M) \xrightarrow{\sim} o_{fil}\mathcal{F}_S^{FDR}(M)$$

given in definition 135 is an isomorphism if we forgot the Hodge filtration F .

(ii) For $M \in \text{DA}_c(S)$ and all $n, p \in \mathbb{Z}$, the map in $\text{PSh}_{O_S, \mathcal{D}}(S/(\tilde{S}_I))$

$$F^p H^n T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : F^p H^n \mathcal{F}_S^{GM}(\mathbb{L}\mathbb{D}_S M) \hookrightarrow F^p H^n \mathcal{F}_S^{FDR}(M)$$

given in definition 135 is a monomorphism. Note that $F^p H^n T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M)$ is NOT an isomorphism in general : take for example $M(S^\circ/S)^\vee = D(\mathbb{A}^1, et)(j_* E_{et}(\mathbb{Z}(S^\circ/S)))$ for an open embedding $j : S^\circ \hookrightarrow S$, then

$$\mathcal{F}_S^{GM}(\mathbb{L}\mathbb{D}_S M(S^\circ/S)^\vee) = \mathcal{F}_S^{GM}(\mathbb{Z}(S^\circ/S)) = j_* E(O_{S^\circ}, F_b) \notin \pi_S(\text{MHM}(S))$$

and hence NOT isomorphic to $\mathcal{F}_S^{GM}(\mathbb{L}\mathbb{D}_S M(S^\circ/S)^\vee) \in \pi_S(\text{MHM}(S))$, see remark 9. It is an isomorphism in the very particular cases where $M = D(\mathbb{A}^1, et)(\mathbb{Z}(X/S))$ or $M = D(\mathbb{A}^1, et)(\mathbb{Z}(X^\circ/S))$ for $f : X \rightarrow S$ is a smooth proper morphism and $n : X^\circ \hookrightarrow X$ is an open subset such that $X \setminus X^\circ = \cup D_i$ is a normal crossing divisor and such that $f|_{D_i} = f \circ i_i : D_i \rightarrow X$ are SMOOTH morphism with $i_i : D_i \hookrightarrow X$ the closed embedding and considering $f|_{X^\circ} = f \circ n : X^\circ \rightarrow S$ (see proposition 111).

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_i S_i$ an open cover such that there exists closed embedding $i_i : S \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We define the Hodge realization functor as, using definition 133,

$$\mathcal{F}_S^{Hdg} := (\mathcal{F}_S^{FDR}, \text{Bti}_S^*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

which is on objects given by, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, taking $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F, W)$ gives the weight structure on $D(\mathbb{A}^1, et)(F)$,

$$\begin{aligned} \mathcal{F}_S^{Hdg}(F) &:= (\mathcal{F}_S^{FDR}(F), \text{Bti}_S^* F, \alpha(F)) := \\ &(e(S)_* \mathcal{H}om((\hat{R}_{\tilde{S}_I}^{CH}(\rho_{\tilde{S}_I}^* Li_{I*} j_I^*(F, W)), \hat{R}^{CH}(T^q(D_{IJ})(-))), (E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}), T_{IJ})), \\ &e(S)_* \text{sing}_{\mathbb{D}^*} \text{An}_S^* L(F, W), \alpha(F)) \in D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}), \end{aligned}$$

where $\alpha(F)$ is given in definition 169 and on morphism, for $m : F_1 \rightarrow F_2$ with $F_1, F_2 \in C(\text{Var}(\mathbb{C})^{sm}/S)$,

$$\mathcal{F}_S^{Hdg}(m) := (\mathcal{F}_S^{FDR}(m), \text{Bti}_S^*(m), \theta(m)),$$

where $\theta(m)$ is given in definition 169. This functor induces by proposition 114 the functor

$$\begin{aligned} \mathcal{F}_S^{Hdg} &:= (\mathcal{F}_S^{FDR}, \text{Bti}_S^*) : \text{DA}(S) \rightarrow D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}), \\ M = D(\mathbb{A}^1, et)(F) &\mapsto \mathcal{F}_S^{Hdg}(M) := \mathcal{F}_S^{Hdg}(F) = (\mathcal{F}_S^{FDR}(M), \text{Bti}_S^* M, \alpha(M)), \end{aligned}$$

with $\alpha(M) = \alpha(F)$.

The main result of this article is the following (theorem 43) :

Theorem 6. Let $k \subset \mathbb{C}$ be a subfield.

(i) For $S \in \text{Var}(\mathbb{C})$, we have $\mathcal{F}_S^{\text{Hdg}}(\text{DA}_c(S)) \subset D(\text{MHM}(S))$,

$$\iota_S : D(\text{MHM}(S)) \hookrightarrow D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{an})$$

being a full embedding by theorem 32.

(ii) The Hodge realization functor $\mathcal{F}_{\text{Hdg}}(-)$ define a morphism of 2-functor on $\text{Var}(\mathbb{C})$

$$\mathcal{F}_-^{\text{Hdg}} : \text{Var}(\mathbb{C}) \rightarrow (\text{DA}_c(-) \rightarrow D(\text{MHM}(-)))$$

whose restriction to $\text{QPVar}(\mathbb{C})$ is an homotopic 2-functor in sense of Ayoub. More precisely,

(ii0) for $g : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(S)$, the the maps of definition 138 and of definition 165 induce an isomorphism in $D(\text{MHM}(T))$

$$\begin{aligned} T(g, \mathcal{F}^{\text{Hdg}})(M) &:= (T(g, \mathcal{F}^{\text{FDR}})(M), T(g, \text{bti})(M), 0) : \\ g^{\hat{*}\text{Hdg}} \mathcal{F}_S^{\text{Hdg}}(M) &:= \iota_T^{-1}(g_{\text{Hdg}}^{\hat{*}\text{mod}} \mathcal{F}_S^{\text{FDR}}(M), g^{*w} \text{Bti}_S(M), g^*(\alpha(M))) \\ &\xrightarrow{\sim} \iota_T^{-1}(\mathcal{F}_T^{\text{FDR}}(g^*M), \text{Bti}_T^*(g^*M), \alpha(g^*M)) =: \mathcal{F}_T^{\text{Hdg}}(g^*M), \end{aligned}$$

(ii1) for $f : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(T)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(\text{MHM}(S))$

$$\begin{aligned} T_*(f, \mathcal{F}^{\text{Hdg}})(M) &:= (T_*(f, \mathcal{F}^{\text{FDR}})(M), T_*(f, \text{bti})(M), 0) : \\ Rf_{\text{Hdg}*} \mathcal{F}_T^{\text{Hdg}}(M) &:= \iota_S^{-1}(Rf_*^{\text{Hdg}} \mathcal{F}_T^{\text{FDR}}(M), Rf_{*w} \text{Bti}_T(M), f_*(\alpha(M))) \\ &\xrightarrow{\sim} \iota_S^{-1}(\mathcal{F}_S^{\text{FDR}}(Rf_*M), \text{Bti}_S^*(Rf_*M), \alpha(Rf_*M)) =: \mathcal{F}_S^{\text{Hdg}}(Rf_*M), \end{aligned}$$

(ii2) for $f : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(T)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(\text{MHM}(S))$

$$\begin{aligned} T_!(f, \mathcal{F}^{\text{Hdg}})(M) &:= (T_!(f, \mathcal{F}^{\text{FDR}})(M), T_!(f, \text{bti})(M), 0) : \\ Rf_{!\text{Hdg}} \mathcal{F}_T^{\text{Hdg}}(M) &:= \iota_S^{-1}(Rf_{!}^{\text{Hdg}} \mathcal{F}_T^{\text{FDR}}(M), Rf_{!w} \text{Bti}_T^*(M), f_!(\alpha(M))) \\ &\xrightarrow{\sim} \iota_S^{-1}(\mathcal{F}_S^{\text{FDR}}(Rf_!M), \text{Bti}_S^*(Rf_!M), \alpha(f_!M)) =: \mathcal{F}_S^{\text{Hdg}}(Rf_!M), \end{aligned}$$

(ii3) for $f : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(S)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(\text{MHM}(T))$

$$\begin{aligned} T^!(f, \mathcal{F}^{\text{Hdg}})(M) &:= (T^!(f, \mathcal{F}^{\text{FDR}})(M), T^!(f, \text{bti})(M), 0) : \\ f^{*\text{Hdg}} \mathcal{F}_S^{\text{Hdg}}(M) &:= \iota_T^{-1}(f_{\text{Hdg}}^{*\text{mod}} \mathcal{F}_S^{\text{FDR}}(M), f^{!w} \text{Bti}_S(M), f^!(\alpha(M))) \\ &\xrightarrow{\sim} \iota_T^{-1}(\mathcal{F}_T^{\text{FDR}}(f^!M), \text{Bti}_T^*(f^!M), \alpha(f^!M)) =: \mathcal{F}_T^{\text{Hdg}}(f^!M), \end{aligned}$$

(ii4) for $S \in \text{Var}(\mathbb{C})$, and $M, N \in \text{DA}_c(S)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(\text{MHM}(S))$

$$\begin{aligned} T(\otimes, \mathcal{F}^{\text{Hdg}})(M, N) &:= (T(\otimes, \mathcal{F}_S^{\text{FDR}})(M, N), T(\otimes, \text{bti})(M, N), 0) : \\ \iota_S^{-1}(\mathcal{F}_S^{\text{FDR}}(M) \otimes_{\mathcal{O}_S}^{\text{Hdg}} \mathcal{F}_S^{\text{FDR}}(N), \text{Bti}_S(M) \otimes \text{Bti}_S(N), \alpha(M) \otimes \alpha(N)) \\ &\xrightarrow{\sim} \mathcal{F}_S^{\text{Hdg}}(M \otimes N) := \iota_S^{-1}(\mathcal{F}_S^{\text{FDR}}(M \otimes N), \text{Bti}_S(M \otimes N), \alpha(M \otimes N)). \end{aligned}$$

(iii) For $S \in \text{Var}(\mathbb{C})$, the following diagram commutes :

$$\begin{array}{ccc} \text{Var}(\mathbb{C})/S & \xrightarrow{\text{MH}(/S)} & D(\text{MHM}(S)) \\ \downarrow M(/S) & & \downarrow \iota^S \\ \text{DA}(S) & \xrightarrow{\mathcal{F}_S^{\text{Hdg}}} & D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{an}) \end{array}$$

We obtain theorem 6 from theorem 5 and from the result on the Betti factor after checking the compatibility of these transformation maps with the isomorphisms $\alpha(M)$.

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2 Generalities and Notations

2.1 Notations

- After fixing a universe, we denote by
 - Set the category of sets,
 - Top the category of topological spaces,
 - Ring the category of rings and $\text{cRing} \subset \text{Ring}$ the full subcategory of commutative rings,
 - RTop the category of ringed spaces,
 - * whose set of objects is $\text{RTop} := \{(X, O_X), X \in \text{Top}, O_X \in \text{PSh}(X, \text{Ring})\}$
 - * whose set of morphism is $\text{Hom}((T, O_T), (S, O_S)) := \{(f : T \rightarrow S), (a_f : f^*O_S \rightarrow O_T)\}$
 - and by $ts : \text{RTop} \rightarrow \text{Top}$ the forgetfull functor.
 - Cat the category of small categories which comes with the forgetful functor $o : \text{Cat} \rightarrow \text{Fun}(\Delta^1, \text{Set})$, where $\text{Fun}(\Delta^1, \text{Set})$ is the category of simplicial sets,
 - RCat the category of ringed topos
 - * whose set of objects is $\text{RCat} := \{(\mathcal{X}, O_{\mathcal{X}}), \mathcal{X} \in \text{Cat}, O_{\mathcal{X}} \in \text{PSh}(\mathcal{X}, \text{Ring})\}$,
 - * whose set of morphism is $\text{Hom}((\mathcal{T}, O_{\mathcal{T}}), (\mathcal{S}, O_{\mathcal{S}})) := \{(f : \mathcal{T} \rightarrow \mathcal{S}), (a_f : f^*O_{\mathcal{S}} \rightarrow O_{\mathcal{T}})\}$
 - and by $tc : \text{RCat} \rightarrow \text{Cat}$ the forgetfull functor.
- Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be a functor with $\mathcal{C}, \mathcal{C}' \in \text{Cat}$. For $X \in \mathcal{C}$, we denote by $F(X) \in \mathcal{C}'$ the image of X , and for $X, Y \in \mathcal{C}$, we denote by $F^{X,Y} : \text{Hom}(X, Y) \rightarrow \text{Hom}(F(X), F(Y))$ the corresponding map.
- For $\mathcal{C} \in \text{Cat}$, we denote by $\mathcal{C}^{op} \in \text{Cat}$ the opposite category whose set of object is the one of $\mathcal{C} : (\mathcal{C}^{op})^0 = \mathcal{C}^0$, and whose morphisms are the morphisms of \mathcal{C} with reversed arrows.
- Let $\mathcal{C} \in \text{Cat}$. For $S \in \mathcal{C}$, we denote by \mathcal{C}/S the category
 - whose set of objects $(\mathcal{C}/S)^0 = \{X/S = (X, h)\}$ consist of the morphisms $h : X \rightarrow S$ with $X \in \mathcal{C}$,
 - whose set of morphism $\text{Hom}(X'/S, X/S)$ between $X'/S = (X', h'), X/S = (X, h) \in \mathcal{C}/S$ consits of the morphisms $(g : X' \rightarrow X) \in \text{Hom}(X', X)$ such that $h \circ g = h'$.

We have then, for $S \in \mathcal{C}$, the canonical forgetful functor

$$r(S) : \mathcal{C}/S \rightarrow \mathcal{C}, \quad X/S \mapsto r(S)(X/S) = X, \quad (g : X'/S \rightarrow X/S) \mapsto r(S)(g) = g$$

and we denote again $r(S) : \mathcal{C} \rightarrow \mathcal{C}/S$ the corresponding morphism of (pre)sites.

- Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be a functor with $\mathcal{C}, \mathcal{C}' \in \text{Cat}$. Then for $S \in \mathcal{C}$, we have the canonical functor

$$F_S : \mathcal{C}/S \rightarrow \mathcal{C}'/F(S), \quad X/S \mapsto F(X/S) = F(X)/F(S), \\ (g : X'/S \rightarrow X/S) \mapsto (F(g) : F(X')/F(S) \rightarrow F(X)/F(S))$$

- Let $\mathcal{S} \in \text{Cat}$. Then, for a morphism $f : X' \rightarrow X$ with $X, X' \in \mathcal{S}$ we have the functor

$$\begin{aligned} C(f) : \mathcal{S}/X' &\rightarrow \mathcal{S}/X, \quad Y/X' = (Y, f_1) \mapsto C(f)(Y/X') := (Y, f \circ f_1) \in \mathcal{S}/X, \\ (g : Y_1/X' &\rightarrow Y_2/X') \mapsto (C(f)(g) := g : Y_1/X \rightarrow Y_2/X) \end{aligned}$$

- Let $\mathcal{S} \in \text{Cat}$ a category which admits fiber products. Then, for a morphism $f : X' \rightarrow X$ with $X, X' \in \mathcal{S}$, we have the pullback functor

$$\begin{aligned} P(f) : \mathcal{S}/X &\rightarrow \mathcal{S}/X', \quad Y/X \mapsto P(f)(Y/X) := Y \times_X X'/X' \in \mathcal{S}/X', \\ (g : Y_1/X &\rightarrow Y_2/X) \mapsto (P(f)(g) := (g \times I) : Y_1 \times_X X' \rightarrow Y_2 \times_X X') \end{aligned}$$

which is right adjoint to $C(f) : \mathcal{S}/X' \rightarrow \mathcal{S}/X$, and we denote again $P(f) : \mathcal{S}/X' \rightarrow \mathcal{S}/X$ the corresponding morphism of (pre)sites.

- Let $\mathcal{C}, \mathcal{I} \in \text{Cat}$. Assume that \mathcal{C} admits fiber products. For $(S_\bullet) \in \text{Fun}(\mathcal{I}^{op}, \mathcal{C})$, we denote by $\mathcal{C}/(S_\bullet) \in \text{Fun}(\mathcal{I}, \text{Cat})$ the diagram of category given by

- for $I \in \mathcal{I}$, $\mathcal{C}/(S_\bullet)(I) := \mathcal{C}/S_I$,
- for $r_{IJ} : I \rightarrow J$, $\mathcal{C}/(S_\bullet)(r_{IJ}) := P(r_{IJ}) : \mathcal{C}/S_I \rightarrow \mathcal{C}/S_J$, where we denoted again $r_{IJ} : S_J \rightarrow S_I$ the associated morphism in \mathcal{C} .

- Let $(F, G) : \mathcal{C} \rightleftarrows \mathcal{C}'$ an adjunction between two categories.

- For $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$, we consider the adjunction isomorphisms
 - * $I(F, G)(X, Y) : \text{Hom}(F(X), Y) \rightarrow \text{Hom}(X, G(Y))$, $(u : F(X) \rightarrow Y) \mapsto (I(F, G)(X, Y)(u) : X \rightarrow G(Y))$
 - * $I(F, G)(X, Y) : \text{Hom}(X, G(Y)) \rightarrow \text{Hom}(F(X), Y)$, $(v : X \rightarrow G(Y)) \mapsto (I(F, G)(X, Y)(v) : F(X) \rightarrow Y)$.
- For $X \in \mathcal{C}$, we denote by $\text{ad}(F, G)(X) := I(F, G)(X, F(X))(I_{F(X)}) : X \rightarrow G \circ F(X)$.
- For $Y \in \mathcal{C}'$ we denote also by $\text{ad}(F, G)(Y) := I(F, G)(G(Y), Y)(I_{G(Y)}) : F \circ G(Y) \rightarrow Y$.

Hence,

- for $u : F(X) \rightarrow Y$ a morphism with $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$, we have $I(F, G)(X, Y)(u) = G(u) \circ \text{ad}(F, G)(X)$,
- for $v : X \rightarrow G(Y)$ a morphism with $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$, we have $I(F, G)(X, Y)(v) = \text{ad}(F, G)(Y) \circ F(v)$.

- Let \mathcal{C} a category.

- We denote by (\mathcal{C}, F) the category of filtered objects : $(X, F) \in (\mathcal{C}, F)$ is a sequence $(F^\bullet X)_{\bullet \in \mathbb{Z}}$ indexed by \mathbb{Z} with value in \mathcal{C} together with monomorphisms $a_p : F^p X \hookrightarrow F^{p-1} X \hookrightarrow X$.
- We denote by (\mathcal{C}, F, W) the category of bifiltered objects : $(X, F, W) \in (\mathcal{C}, F, W)$ is a sequence $(W^\bullet F^\bullet X)_{\bullet, \bullet \in \mathbb{Z}^2}$ indexed by \mathbb{Z}^2 with value in \mathcal{C} together with monomorphisms $W^q F^p X \hookrightarrow F^{p-1} X, W^q F^p X \hookrightarrow W^{q-1} F^p X$.

- For \mathcal{C} a category and $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ an endofunctor, we denote by (\mathcal{C}, Σ) the corresponding category of spectra, whose objects are sequence of objects of \mathcal{C} $(T_i)_{i \in \mathbb{Z}} \in \text{Fun}(\mathbb{Z}, \mathcal{C})$ together with morphisms $s_i : T_i \rightarrow \Sigma T_{i+1}$, and whose morphism from (T_i) to (T'_i) are sequence of morphisms $T_i \rightarrow T'_i$ which commutes with the s_i .

- Let \mathcal{A} an additive category.

- We denote by $C(\mathcal{A}) := \text{Fun}(\mathbb{Z}, \mathcal{A})$ the category of (unbounded) complexes with value in \mathcal{A} , where we have denoted \mathbb{Z} the category whose set of objects is \mathbb{Z} , and whose set of morphism between $m, n \in \mathbb{Z}$ consists of one element (identity) if $n = m$, of one element if $n = m + 1$ and is \emptyset in the other cases.
- We have the full subcategories $C^b(\mathcal{A}), C^-(\mathcal{A}), C^+(\mathcal{A})$ of $C(\mathcal{A})$ consisting of bounded, resp. bounded above, resp. bounded below complexes.
- We denote by $K(\mathcal{A}) := \text{Ho}(C(\mathcal{A}))$ the homotopy category of $C(\mathcal{A})$ whose morphisms are equivalent homotopic classes of morphism and by $Ho : C(\mathcal{A}) \rightarrow K(\mathcal{A})$ the full homotopy functor. The category $K(\mathcal{A})$ is in the standard way a triangulated category.

• Let \mathcal{A} an additive category.

- We denote by $C_{fil}(\mathcal{A}) \subset (C(\mathcal{A}), F) = C(\mathcal{A}, F)$ the full additive subcategory of filtered complexes of \mathcal{A} such that the filtration is biregular : for $(A^\bullet, F) \in (C(\mathcal{A}), F)$, we say that F is biregular if $F^\bullet A^r$ is finite for all $r \in \mathbb{Z}$.
- We denote by $C_{2fil}(\mathcal{A}) \subset (C(\mathcal{A}), F, W) = C(\mathcal{A}, F, W)$ the full subcategory of bifiltered complexes of \mathcal{A} such that the filtration is biregular.
- For $A^\bullet \in C(\mathcal{A})$, we denote by $(A^\bullet, F_b) \in (C(\mathcal{A}), F)$ the complex endowed with the trivial filtration (filtration bete) : $F^p A^n = 0$ if $p \geq n + 1$ and $F^p A^n = A^n$ if $p \leq n$. Obviously, a morphism $\phi : A^\bullet \rightarrow B^\bullet$, with $A^\bullet, B^\bullet \in C(\mathcal{A})$ induces a morphism $\phi : (A^\bullet, F_b) \rightarrow (B^\bullet, F_b)$.
- For $(A^\bullet, F) \in C(\mathcal{A}, F)$, we denote by $(A^\bullet, F(r)) \in C(\mathcal{A}, F)$ the filtered complex where the filtration is given by $F^p(A^\bullet, F(r)) := F^{p+r}(A^\bullet, F)$.
- Two morphisms $\phi_1, \phi_2 : (M, F) \rightarrow (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ are said to be r -filtered homotopic if there exist a morphism in $\text{Fun}(\mathbb{Z}, (\mathcal{A}, F))$

$$h : (M, F(r-1))[1] \rightarrow (N, F), h := (h^n : (M^{n+1}, F(r-1)) \rightarrow (N^n, F))_{n \in \mathbb{Z}},$$

where \mathbb{Z} have only trivial morphism (i.e. h is a graded morphism but not a morphism of complexes) such that $d'h + hd = \phi_1 - \phi_2$, where d is the differential of M and d' is the differential of N , and we have $h(F^p M^{n+1}) \subset F^{p-r+1} N^n$, note that by definition r does NOT depend on p and n ; we then say that

$$(h, \phi_1, \phi_2) : (M, F)[1] \rightarrow (N, F)$$

is an r -filtered homotopy. By definition, an r -filtered homotopy $(h, \phi_1, \phi_2) : (M, F)[1] \rightarrow (N, F)$ is an r' -filtered homotopy for all $r' \geq r$, and a 1-filtered homotopy is an homotopy of $C(\mathcal{A}, F)$. By definition, an r -filtered homotopy $(h, \phi_1, \phi_2) : (M, F)[1] \rightarrow (N, F)$ gives if we forgot filtration an homotopy $(h, \phi_1, \phi_2) : M[1] \rightarrow N$ in $C(\mathcal{A})$.

- Two morphisms $\phi_1, \phi_2 : (M, F) \rightarrow (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ are said to be ∞ -filtered homotopic if there exist $r \in \mathbb{N}$ such that $\phi_1, \phi_2 : (M, F) \rightarrow (N, F)$ are r -filtered homotopic. Hence, if $\phi_1, \phi_2 : (M, F) \rightarrow (N, F)$ are ∞ -filtered homotopic, then $\phi_1, \phi_2 : M \rightarrow N$ are homotopic ; of course the converse is NOT true since r does NOT depend on $p, n \in \mathbb{Z}$.
- We will use the fact that by definition if $\phi : M \rightarrow N$ with $M, N \in C(\mathcal{A})$ is an homotopy equivalence, then $\phi : (M, F_b) \rightarrow \phi(N, F_b)$ is a 2-filtered homotopy equivalence.
- A morphism $\phi : (M, F) \rightarrow (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ is said to be an r -filtered homotopy equivalence if there exist a morphism $\phi' : (N, F) \rightarrow (M, F)$ such that
 - * $\phi' \circ \phi : (M, F) \rightarrow (M, F)$ is r -filtered homotopic to I_M and
 - * $\phi \circ \phi' : (N, F) \rightarrow (N, F)$ is r -filtered homotopic to I_N .

If $\phi : (M, F) \rightarrow (N, F)$ is an r -filtered homotopy equivalence, then it is an s -filtered homotopy equivalence for $s \geq r$. If $\phi : (M, F) \rightarrow (N, F)$ is an r -filtered homotopy equivalence, $\phi : M \rightarrow N$ is an homotopy equivalence

- A morphism $\phi : (M, F) \rightarrow (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ is said to be an ∞ -filtered homotopy equivalence if there exist $r \in \mathbb{Z}$ such that $\phi : (M, F) \rightarrow (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ is an r -filtered homotopy equivalence. If $\phi : (M, F) \rightarrow (N, F)$ is an ∞ -filtered homotopy equivalence, $\phi : M \rightarrow N$ is an homotopy equivalence ; the converse is NOT true since r does NOT depend on $p, n \in \mathbb{Z}$.
- We denote by $K_r(\mathcal{A}, F) := \text{Ho}_r(C(\mathcal{A}, F))$ the homotopy category of $C(\mathcal{A}, F)$ whose objects are those of $C(\mathcal{A}, F)$ and whose morphisms are the morphisms of $C(\mathcal{A}, F)$ modulo r -filtered homotopies, and by $\text{Ho}_r : C(\mathcal{A}, F) \rightarrow K_r(\mathcal{A}, F)$ the full homotopy functor. However, for $r > 1$ the category $K_r(\mathcal{A}, F)$ with the canonical triangles the standard ones does NOT satisfy the 2 of 3 axiom of a triangulated category. Indeed, for $r > 1$, a commutative diagram in $K_r(\mathcal{A}, F)$

$$\begin{array}{ccccc} (A^\bullet, F) & \xrightarrow{m} & (B^\bullet, F) & \longrightarrow & \text{Cone}^{i_2}(m) = ((A^\bullet, F)[1] \oplus (B^\bullet, F), d, d^{p-1} m) & \longrightarrow & (A^\bullet, F)[1] \\ \phi \downarrow & & \psi \downarrow & & & & \downarrow \phi[1] \\ (A'^\bullet, F) & \xrightarrow{m'} & (B'^\bullet, F) & \longrightarrow & \text{Cone}^{i_2}(m') = ((A'^\bullet, F)[1] \oplus (B'^\bullet, F), d, d^{p-1} m') & \longrightarrow & (A'^\bullet, F)[1] \end{array}$$

i.e. which commutes modulo r -filtered homotopy can NOT be completed with a third vertical arrow. In particular, for $r > 1$ and commutative diagram in $C(\mathcal{A}, F)$

$$\begin{array}{ccccc} (A^\bullet, F) & \xrightarrow{m} & (B^\bullet, F) & \longrightarrow & \text{Cone}^{i_2}(m) = ((A^\bullet, F)[1] \oplus (B^\bullet, F), d, d^{p-1} m) & \longrightarrow & (A^\bullet, F)[1] \\ \phi \downarrow & & \psi \downarrow & & \downarrow (\phi[1], \psi) & & \downarrow \phi[1] \\ (A'^\bullet, F) & \xrightarrow{m'} & (B'^\bullet, F) & \longrightarrow & \text{Cone}^{i_2}(m') = ((A'^\bullet, F)[1] \oplus (B'^\bullet, F), d, d^{p-1} m') & \longrightarrow & (A'^\bullet, F)[1] \end{array}$$

if ψ and ϕ are r -filtered homotopy equivalence, then $(\phi[1], \psi)$ is NOT necessary an r -filtered homotopy equivalence.

- We denote by $K_{fil,r}(\mathcal{A}) := \text{Ho}_r(C_{fil}(\mathcal{A}))$ the homotopy category of $C_{fil}(\mathcal{A})$ whose objects are those of $C_{fil}(\mathcal{A},)$ and whose morphisms are the morphisms of $C_{fil}(\mathcal{A})$ modulo r -filtered homotopies, and by $\text{Ho}_r : C_{fil}(\mathcal{A}) \rightarrow K_{fil,r}(\mathcal{A})$ the full homotopy functor. However, for $r > 1$ the category $K_{fil,r}(\mathcal{A})$ with the canonical triangles the standard ones does NOT satisfy the 2 of 3 axiom of a triangulated category.
- We denote by $K_{fil,\infty}(\mathcal{A}) := \text{Ho}_\infty(C_{fil}(\mathcal{A}))$ the homotopy category of $C_{fil}(\mathcal{A})$ whose objects are those of $C_{fil}(\mathcal{A},)$ and whose morphisms are the morphisms of $C_{fil}(\mathcal{A})$ modulo ∞ -filtered homotopies, and by $\text{Ho}_\infty : C_{fil}(\mathcal{A}) \rightarrow K_{fil,\infty}(\mathcal{A})$ the full homotopy functor. However, the category $K_{fil,\infty}(\mathcal{A})$ with the canonical triangles the standard ones does NOT satisfy the 2 of 3 axiom of a triangulated category.
- We have the Deligne decalage functor

$$\begin{aligned} \text{Dec} : C(\mathcal{A}, F) &\rightarrow C(\mathcal{A}, F), (M, F) \mapsto \text{Dec}(M, F) := (M, \text{Dec } F), \\ \text{Dec } F^p M^n &:= F^{p+n} M^n \cap d^{-1}(F^{p+n+1} M^{n+1}) \end{aligned}$$

It is the right adjoint of the shift functor

$$S : C(\mathcal{A}, F) \rightarrow C(\mathcal{A}, F), (M, F) \mapsto S(M, F) := (M, SF), SF^p M^n := F^{p-n} M^n$$

The dual decalage functor

$$\begin{aligned} \text{Dec}^\vee : C(\mathcal{A}, F) &\rightarrow C(\mathcal{A}, F), (M, F) \mapsto \text{Dec}^\vee(M, F) := (M, \text{Dec}^\vee F), \\ \text{Dec}^\vee F^p M^n &:= F^{p+n} M^n + d(F^{p+n-1} M^{n+1}) \end{aligned}$$

is the left adjoint of the shift functor. Note that $\text{Dec}((M, F)[1]) \neq (\text{Dec}(M, F))[1]$, $\text{Dec}^\vee((M, F)[1]) \neq (\text{Dec}^\vee(M, F))[1]$ and $S((M, F)[1]) \neq (S(M, F))[1]$.

- Let \mathcal{A} be an abelian category. Then the additive category (\mathcal{A}, F) is an exact category which admits kernel and cokernel (but is NOT an abelian category). A morphism $\phi : (M, F) \rightarrow (N, F)$ with $(M, F) \in C(\mathcal{A}, F)$ is strict if the inclusion $\phi(F^n M) \subset F^n N \cap \text{Im}(\phi)$ is an equality, i.e. if $\phi(F^n M) = F^n N \cap \text{Im}(\phi)$.

- Let \mathcal{A} be an abelian category.

- For $(A^\bullet, F) \in C(\mathcal{A}, F)$, considering $a_p : F^p A^\bullet \hookrightarrow A^\bullet$ the structural monomorphism of the filtration, we denote by, for $n \in \mathbb{N}$,

$$H^n(A^\bullet, F) \in (\mathcal{A}, F), \quad F^p H^n(A^\bullet, F) := \text{Im}(H^n(a_p) : H^n(F^p A^\bullet) \rightarrow H^n(A^\bullet)) \subset H^n(A^\bullet)$$

the filtration induced on the cohomology objects of the complex. In the case $(A^\bullet, F) \in C_{\text{fil}}(\mathcal{A})$, the spectral sequence $E_r^{p,q}(A^\bullet, F)$ associated to (A^\bullet, F) converge to $\text{Gr}_F^p H^{p+q}(A^\bullet, F)$, that is for all $p, q \in \mathbb{Z}$, there exist $r_{p+q} \in \mathbb{N}$, such that $E_s^{p,q}(A^\bullet, F) = \text{Gr}_F^p H^{p+q}(A^\bullet, F)$ for all $s \leq r_{p+q}$.

- A morphism $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$ with $(A^\bullet, F), (B^\bullet, F) \in C(\mathcal{A}, F)$ is said to be a filtered quasi-isomorphism if for all $n, p \in \mathbb{Z}$,

$$H^n \text{Gr}_F^p(m) : H^n(\text{Gr}_F^p A^\bullet) \xrightarrow{\sim} H^n(\text{Gr}_F^p B^\bullet)$$

is an isomorphism in \mathcal{A} . Consider a commutative diagram in $C(\mathcal{A}, F)$

$$\begin{array}{ccccc} (A^\bullet, F) & \xrightarrow{m} & (B^\bullet, F) & \longrightarrow & \text{Cone}^{i_2}(m) = ((A^\bullet, F)[1] \oplus (B^\bullet, F), d, d' \stackrel{p_1}{=} m) & \longrightarrow & (A^\bullet, F)[1] \\ \phi \downarrow & & \psi \downarrow & & \downarrow (\phi[1], \psi) & & \downarrow \phi[1] \\ (A'^\bullet, F) & \xrightarrow{m'} & (B'^\bullet, F) & \longrightarrow & \text{Cone}^{i_2}(m') = ((A'^\bullet, F)[1] \oplus (B'^\bullet, F), d, d' \stackrel{p_1}{=} m') & \longrightarrow & (A'^\bullet, F)[1] \end{array}$$

If ϕ and ψ are filtered quasi-isomorphisms, then $(\phi[1], \psi)$ is an filtered quasi-isomorphism. That is, the filtered quasi-isomorphism satisfy the 2 of 3 property for canonical triangles.

- If two morphisms $\phi_1, \phi_2 : (M, F) \rightarrow (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ are r -filtered homotopic, then for all $p, q \in \mathbb{Z}$ and $s \geq r$.

$$E_s^{p,q}(\phi_1) = E_s^{p,q}(\phi_2) : E_s^{p,q}(M, F) \rightarrow E_s^{p,q}(N, F).$$

Hence if $\phi : (M, F) \rightarrow (N, F)$ with $(M, F), (N, F) \in C(\mathcal{A}, F)$ is an r -filtered homotopy equivalence then for all $p, q \in \mathbb{Z}$ and $s \geq r$.

$$E_r^{p,q}(\phi) : E_r^{p,q}(M, F) \xrightarrow{\sim} E_r^{p,q}(N, F)$$

is an isomorphism in \mathcal{A} .

- Let $r \in \mathbb{N}$. A morphism $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$ with $(A^\bullet, F), (B^\bullet, F) \in C(\mathcal{A}, F)$ is said to be an r -filtered quasi-isomorphism if it belongs to the submonoid of arrows generated by filtered quasi-isomorphism and r -filtered homotopy equivalence, that is if there exists $m_i : (C_i^\bullet, F) \rightarrow (C_{i+1}^\bullet, F)$, $0 \leq i \leq s$, with $(C_i^\bullet, F) \in C(\mathcal{A}, F)$ ($C_0^\bullet, F) = (A^\bullet, F)$ and $(C_s^\bullet, F) = (B^\bullet, F)$, such that

$$m = m_s \circ \cdots \circ m_i \circ \cdots \circ m_0 : (A^\bullet, F) \rightarrow (B^\bullet, F)$$

and $m_i : (C_i^\bullet, F) \rightarrow (C_{i+1}^\bullet, F)$ either a filtered quasi-isomorphism or an r -filtered homotopy equivalence. Note that our definition is stronger than the one given in [9] in order to get a multiplicative system. Indeed, if $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$ with $(A^\bullet, F), (B^\bullet, F) \in C(\mathcal{A}, F)$ is an r -filtered quasi-isomorphism then for all $p, q \in \mathbb{Z}$ and $s \geq r$,

$$E_r^{p,q}(m) : E_r^{p,q}(A^\bullet, F) \xrightarrow{\sim} E_r^{p,q}(B^\bullet, F)$$

is an isomorphism in \mathcal{A} , but the converse is NOT true. If a morphism $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$, with $(A^\bullet, F), (B^\bullet, F) \in C_{fil}(\mathcal{A})$ is an r -filtered quasi-isomorphism, then for all $n \in \mathbb{Z}$

$$H^n(m) : H^n(A^\bullet, F) \xrightarrow{\sim} H^n(B^\bullet, F)$$

is a filtered isomorphism, i.e. an isomorphism in (\mathcal{A}, F) . The converse is true if there exist $N_1, N_2 \in \mathbb{Z}$ such that $H^n(A^\bullet) = H^n(B^\bullet) = 0$ for $n \leq N_1$ or $n \geq N_2$. A filtered quasi-isomorphism is obviously a 1-filtered quasi-isomorphism. However for $r > 1$, the r -filtered quasi-isomorphism does NOT satisfy the 2 of 3 property for morphisms of canonical triangles.

- A morphism $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$ with $(A^\bullet, F), (B^\bullet, F) \in C(\mathcal{A}, F)$ is said to be an ∞ -filtered quasi-isomorphism if there exist $r \in \mathbb{N}$ such that $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$ an r -filtered quasi-isomorphism. If a morphism $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$, with $(A^\bullet, F), (B^\bullet, F) \in C_{fil}(\mathcal{A})$ is an ∞ -filtered quasi-isomorphism, then for all $n \in \mathbb{Z}$

$$H^n(m) : H^n(A^\bullet, F) \xrightarrow{\sim} H^n(B^\bullet, F)$$

is a filtered isomorphism. The converse is true if there exist $N_1, N_2 \in \mathbb{Z}$ such that $H^n(A^\bullet) = H^n(B^\bullet) = 0$ for $n \leq N_1$ or $n \geq N_2$. The ∞ -filtered quasi-isomorphism does NOT satisfy the 2 of 3 property for morphisms of canonical triangles. If $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$ is such that $m : A^\bullet \rightarrow B^\bullet$ is a quasi-isomorphism but $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$ is not an ∞ -filtered quasi-isomorphism, then it induces an isomorphisms $H^n(m) : H^n(A^\bullet) \xrightarrow{\sim} H^n(B^\bullet)$, hence injective maps

$$H^n(m) : F^p H^n(A^\bullet, F) \hookrightarrow F^p H^n(B^\bullet, F)$$

which are not isomorphism (the non strict case). If $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$ is such that $m : A^\bullet \rightarrow B^\bullet$ is a quasi-isomorphism but $m : (A^\bullet, F) \rightarrow (B^\bullet, F)$ is not an ∞ -filtered quasi-isomorphism (the non strict case), then $H^n \text{Cone}(m) = 0$ for all $n \in \mathbb{N}$, hence $\text{Cone}(m) \rightarrow 0$ is an ∞ -filtered quasi-isomorphism ; this shows that the ∞ -filtered quasi-isomorphism does NOT satisfy the 2 of 3 property for morphism of canonical triangles.

- Let \mathcal{A} be an abelian category.
 - We denote by $D(\mathcal{A})$ the localization of $K(\mathcal{A})$ with respect to the quasi-isomorphisms and by $D : K(\mathcal{A}) \rightarrow D(\mathcal{A})$ the localization functor. The category $D(\mathcal{A})$ is a triangulated category in the unique way such that D a triangulated functor.
 - We denote by $D_{fil}(\mathcal{A})$ the localization of $K_{fil}(\mathcal{A})$ with respect to the filtered quasi-isomorphisms and by $D : K_{fil}(\mathcal{A}) \rightarrow D_{fil}(\mathcal{A})$ the localization functor.
- Let \mathcal{A} be an abelian category. We denote by $\text{Inj}(\mathcal{A}) \subset \mathcal{A}$ the full subcategory of injective objects, and by $\text{Proj}(\mathcal{A}) \subset \mathcal{A}$ the full subcategory of projective objects.
- For $\mathcal{S} \in \text{Cat}$ a small category, we denote by
 - $\text{PSh}(\mathcal{S}) := \text{PSh}(\mathcal{S}, \text{Ab}) := \text{Fun}(\mathcal{S}, \text{Ab})$ the category of presheaves on \mathcal{S} , i.e. the category of presheaves of abelian groups on \mathcal{S} ,
 - $\text{PSh}(\mathcal{S}, \text{Ring}) := \text{Fun}(\mathcal{S}, \text{Ring})$ the category of presheaves of ring on \mathcal{S} , and $\text{PSh}(\mathcal{S}, \text{cRing}) \subset \text{PSh}(\mathcal{S}, \text{Ring})$ the full subcategory of presheaves of commutative ring.
 - for $F \in \text{PSh}(\mathcal{S})$ and $X \in \mathcal{S}$, $F(X) = \Gamma(X, F)$ the sections on X and for $h : X' \rightarrow X$ a morphism with $X, X' \in \mathcal{S}$, $F(h) := F^{X, Y}(h) : F(X) \rightarrow F(X')$ the morphism of abelian groups,
 - $C(\mathcal{S}) = \text{PSh}(\mathcal{S}, C(\mathbb{Z})) = C(\text{PSh}(\mathcal{S})) = \text{PSh}(\mathcal{S} \times \mathbb{Z})$ the big abelian category of complexes of presheaves on \mathcal{S} with value in abelian groups,
 - $K(\mathcal{S}) := K(\text{PSh}(\mathcal{S})) = \text{Ho}(C(\mathcal{S}))$ In particular, we have the full homotopy functor $Ho : C(\mathcal{S}) \rightarrow K(\mathcal{S})$,

- $C_{(2)fil}(\mathcal{S}) := C_{(2)fil}(\text{PSh}(\mathcal{S})) \subset C(\text{PSh}(\mathcal{S}), F, W)$ the big abelian category of (bi)filtered complexes of presheaves on \mathcal{S} with value in abelian groups such that the filtration is biregular, and $\text{PSh}_{(2)fil}(\mathcal{S}) = (\text{PSh}(\mathcal{S}), F, W)$,
- $K_{fil}(\mathcal{S}) := K_{fil}(\text{PSh}(\mathcal{S})) = \text{Ho}(C_{fil}(\mathcal{S}))$,
- $K_{fil,r}(\mathcal{S}) := K_{fil,r}(\text{PSh}(\mathcal{S})) = \text{Ho}_r(C_{fil}(\mathcal{S}))$, $K_{fil,\infty}(\mathcal{S}) := K_{fil,\infty}(\text{PSh}(\mathcal{S})) = \text{Ho}_\infty(C_{fil}(\mathcal{S}))$.

For $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism a presite with $\mathcal{T}, \mathcal{S} \in \text{Cat}$, given by the functor $P(f) : \mathcal{S} \rightarrow \mathcal{T}$, we will consider the adjunctions given by the direct and inverse image functors :

- $(f^*, f_*) = (f^{-1}, f_*) : \text{PSh}(\mathcal{S}) \rightleftarrows \text{PSh}(\mathcal{T})$, which induces $(f^*, f_*) : C(\mathcal{S}) \rightleftarrows C(\mathcal{T})$, we denote, for $F \in C(\mathcal{S})$ and $G \in C(\mathcal{T})$ by

$$\text{ad}(f^*, f_*)(F) : F \rightarrow f_* f^* F, \text{ ad}(f^*, f_*)(G) : f^* f_* G \rightarrow G$$

the adjunction maps,

- $(f_*, f^\perp) : \text{PSh}(\mathcal{T}) \rightleftarrows \text{PSh}(\mathcal{S})$, which induces $(f_*, f^\perp) : C(\mathcal{T}) \rightleftarrows C(\mathcal{S})$, we denote for $F \in C(\mathcal{S})$ and $G \in C(\mathcal{T})$ by

$$\text{ad}(f_*, f^\perp)(F) : G \rightarrow f^\perp f_* G, \text{ ad}(f_*, f^\perp)(G) : f_* f^\perp F \rightarrow F$$

the adjunction maps.

- For $(\mathcal{S}, O_S) \in \text{RCat}$ a ringed topos, we denote by

- $\text{PSh}_{O_S}(\mathcal{S})$ the category of presheaves of O_S modules on \mathcal{S} , whose objects are $\text{PSh}_{O_S}(\mathcal{S})^0 := \{(M, m), M \in \text{PSh}(\mathcal{S}), m : M \otimes O_S \rightarrow M\}$, together with the forgetful functor $o : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}_{O_S}(\mathcal{S})$,
- $C_{O_S}(\mathcal{S}) = C(\text{PSh}_{O_S}(\mathcal{S}))$ the big abelian category of complexes of presheaves of O_S modules on \mathcal{S} ,
- $K_{O_S}(\mathcal{S}) := K(\text{PSh}_{O_S}(\mathcal{S})) = \text{Ho}(C_{O_S}(\mathcal{S}))$, in particular, we have the full homotopy functor $Ho : C_{O_S}(\mathcal{S}) \rightarrow K_{O_S}(\mathcal{S})$,
- $C_{O_S(2)fil}(\mathcal{S}) := C_{(2)fil}(\text{PSh}_{O_S}(\mathcal{S})) \subset C(\text{PSh}_{O_S}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of O_S modules on \mathcal{S} such that the filtration is biregular and $\text{PSh}_{O_S(2)fil}(\mathcal{S}) = (\text{PSh}_{O_S}(\mathcal{S}), F, W)$,
- $K_{O_S fil}(\mathcal{S}) := K_{fil}(\text{PSh}_{O_S}(\mathcal{S})) = \text{Ho}(C_{O_S fil}(\mathcal{S}))$,
- $K_{O_S fil,r}(\mathcal{S}) := K_{fil,r}(\text{PSh}_{O_S}(\mathcal{S})) = \text{Ho}_r(C_{O_S fil}(\mathcal{S}))$, $K_{O_S fil,\infty}(\mathcal{S}) := K_{fil,\infty}(\text{PSh}_{O_S}(\mathcal{S})) = \text{Ho}_\infty(C_{O_S fil}(\mathcal{S}))$.

- For $\mathcal{S}_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat})$ a diagram of (pre)sites, with $\mathcal{I} \in \text{Cat}$ a small category, we denote by

- $\Gamma \mathcal{S}_\bullet \in \text{Cat}$ the associated diagram category
 - * whose objects are $\Gamma \mathcal{S}_\bullet^0 := \{(X_I, u_{IJ})_{I \in \mathcal{I}}\}$, with $X_I \in \mathcal{S}_I$, and for $r_{IJ} : I \rightarrow J$ with $I, J \in \mathcal{I}$, $u_{IJ} : X_J \rightarrow r_{IJ}(X_I)$ are morphism in \mathcal{S}_J noting again $r_{IJ} : \mathcal{S}_I \rightarrow \mathcal{S}_J$ the associated functor,
 - * whose morphism are $m = (m_I) : (X_I, u_{IJ}) \rightarrow (X'_I, v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ}(m_J) \circ u_{IJ}$ in \mathcal{S}_J ,
- $\text{PSh}(\mathcal{S}_\bullet) := \text{PSh}(\Gamma \mathcal{S}_\bullet, \text{Ab})$ the category of presheaves on \mathcal{S}_\bullet ,
 - * whose objects are $\text{PSh}(\mathcal{S}_\bullet)^0 := \{(F_I, u_{IJ})_{I \in \mathcal{I}}\}$, with $F_I \in \text{PSh}(\mathcal{S}_I)$, and for $r_{IJ} : I \rightarrow J$ with $I, J \in \mathcal{I}$, $u_{IJ} : F_J \rightarrow r_{IJ*} F_I$ are morphism in $\text{PSh}(\mathcal{S}_I)$, noting again $r_{IJ} : \mathcal{S}_J \rightarrow \mathcal{S}_I$ the associated morphism of presite,
 - * whose morphism are $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ*} m_J \circ u_{IJ}$ in $\text{PSh}(\mathcal{S}_I)$,

- $\text{PSh}(\mathcal{S}_\bullet, \text{Ring}) := \text{PSh}(\Gamma\mathcal{S}_\bullet, \text{Ring})$ the category of presheaves of ring on \mathcal{S}_\bullet given in the same way, and $\text{PSh}(\mathcal{S}_\bullet, \text{cRing}) \subset \text{PSh}(\mathcal{S}_\bullet, \text{Ring})$ the full subcategory of presheaves of commutative ring.
- $C(\mathcal{S}_\bullet) := C(\text{PSh}(\mathcal{S}_\bullet))$ the big abelian category of complexes of presheaves on \mathcal{S}_\bullet with value in abelian groups,
- $K(\mathcal{S}_\bullet) := K(\text{PSh}(\mathcal{S}_\bullet)) = \text{Ho}(C(\mathcal{S}_\bullet))$, in particular, we have the full homotopy functor $Ho : C(\mathcal{S}_\bullet) \rightarrow K(\mathcal{S}_\bullet)$,
- $C_{(2)fil}(\mathcal{S}_\bullet) := C_{(2)fil}(\text{PSh}(\mathcal{S}_\bullet)) \subset C(\text{PSh}(\mathcal{S}_\bullet), F, W)$ the big abelian category of (bi)filtered complexes of presheaves on \mathcal{S}_\bullet with value in abelian groups such that the filtration is biregular, and $\text{PSh}_{(2)fil}(\mathcal{S}_\bullet) = (\text{PSh}(\mathcal{S}_\bullet), F, W)$, by definition $C_{(2)fil}(\mathcal{S}_\bullet)$ is the category
 - * whose objects are $C_{(2)fil}(\mathcal{S}_\bullet)^0 := \{((F_I, F, W), u_{IJ})_{I \in \mathcal{I}}\}$, with $(F_I, F, W) \in C_{(2)fil}(\mathcal{S}_I)$, and for $r_{IJ} : I \rightarrow J$ with $I, J \in \mathcal{I}$, $u_{IJ} : (F_I, F, W) \rightarrow r_{IJ*}(F_J, F, W)$ are morphism in $C_{(2)fil}(\mathcal{S}_I)$, noting again $r_{IJ} : \mathcal{S}_J \rightarrow \mathcal{S}_I$ the associated morphism of presite,
 - * whose morphism are $m = (m_I) : ((F_I, F, W), u_{IJ}) \rightarrow ((G_I, F, W), v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ*}m_J \circ u_{IJ}$ in $C_{(2)fil}(\mathcal{S}_I)$,
- $K_{fil}(\mathcal{S}_\bullet) := K_{fil}(\text{PSh}(\mathcal{S}_\bullet)) = \text{Ho}(C_{fil}(\mathcal{S}_\bullet))$,
- $K_{fil,r}(\mathcal{S}_\bullet) := K_{fil,r}(\text{PSh}(\mathcal{S}_\bullet)) = \text{Ho}_r(C_{fil}(\mathcal{S}_\bullet))$, $K_{fil,\infty}(\mathcal{S}_\bullet) := K_{fil,\infty}(\text{PSh}(\mathcal{S}_\bullet)) = \text{Ho}_\infty(C_{fil}(\mathcal{S}_\bullet))$.

Let $\mathcal{I}, \mathcal{I}' \in \text{Cat}$ be small categories. Let $(f_\bullet, s) : \mathcal{T}_\bullet \rightarrow \mathcal{S}_\bullet$ a morphism a diagrams of (pre)site with $\mathcal{T}_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat})$, $\mathcal{S}_\bullet \in \text{Fun}(\mathcal{I}', \text{Cat})$, which is by definition given by a functor $s : \mathcal{I} \rightarrow \mathcal{I}'$ and morphism of functor $P(f_\bullet) : \mathcal{S}_{s(\bullet)} := \mathcal{S}_\bullet \circ s \rightarrow \mathcal{T}_\bullet$. Here, we denote for short, $\mathcal{S}_{s(\bullet)} := \mathcal{S}_\bullet \circ s \in \text{Fun}(\mathcal{I}, \text{Cat})$. We have then, for $r_{IJ} : I \rightarrow J$ a morphism, with $I, J \in \mathcal{I}$, a commutative diagram in Cat

$$D_{fIJ} := \begin{array}{ccc} \mathcal{S}_{s(J)} & \xrightarrow{r_{IJ}^s} & \mathcal{S}_{s(I)} \\ f_J \uparrow & & \uparrow f_I \\ \mathcal{T}_J & \xrightarrow{r_{IJ}^t} & \mathcal{T}_I \end{array}$$

We will consider the the adjunction given by the direct and inverse image functors :

$$\begin{aligned} ((f_\bullet, s)^*, (f_\bullet, s)_*) &= ((f_\bullet, s)^{-1}, (f_\bullet, s)_*) : \text{PSh}(\mathcal{S}_{s(\bullet)}) \rightleftarrows \text{PSh}(\mathcal{T}_\bullet), \\ F &= (F_I, u_{IJ}) \mapsto (f_\bullet, s)^* F := (f_I^* F_I, T(D_{fIJ})(F_J) \circ f_I^* u_{IJ}), \\ G &= (G_I, v_{IJ}) \mapsto (f_\bullet, s)_* G := (f_{I*} G_I, f_{I*} v_{IJ}). \end{aligned}$$

It induces the adjunction $((f_\bullet, s)^*, (f_\bullet, s)_*) : C(\mathcal{S}_{s(\bullet)}) \rightleftarrows C(\mathcal{T}_\bullet)$. We denote, for $(F_I, u_{IJ}) \in C(\mathcal{S}_{s(\bullet)})$ and $(G_I, v_{IJ}) \in C(\mathcal{T}_\bullet)$ by

$$\begin{aligned} \text{ad}((f_\bullet, s)^*, (f_\bullet, s)_*)((F_I, u_{IJ})) &: (F_I, u_{IJ}) \rightarrow (f_\bullet, s)_*(f_\bullet, s)^*(F_I, u_{IJ}), \\ \text{ad}((f_\bullet, s)^*, (f_\bullet, s)_*)((G_I, v_{IJ})) &: (f_\bullet, s)^*(f_\bullet, s)_*(G_I, v_{IJ}) \rightarrow (G_I, v_{IJ}) \end{aligned}$$

the adjunction maps.

- Let $\mathcal{I} \in \text{Cat}$ a small category. For $(\mathcal{S}_\bullet, O_{\mathcal{S}_\bullet}) \in \text{Fun}(\mathcal{I}, \text{RCat})$ a diagram of ringed tops, we denote by

- $\text{PSh}_{O_{\mathcal{S}_\bullet}}(\mathcal{S}_\bullet) := \text{PSh}_{O_{\Gamma\mathcal{S}_\bullet}}(\Gamma\mathcal{S}_\bullet)$ the category of presheaves of modules on $(\mathcal{S}_\bullet, O_{\mathcal{S}_\bullet})$,
 - * whose objects are $\text{PSh}_{O_{\mathcal{S}_\bullet}}(\mathcal{S}_\bullet)^0 := \{(F_I, u_{IJ})_{I \in \mathcal{I}}\}$, with $F_I \in \text{PSh}_{O_{\mathcal{S}_I}}(\mathcal{S}_I)$, and for $r_{IJ} : I \rightarrow J$ with $I, J \in \mathcal{I}$, $u_{IJ} : F_I \rightarrow r_{IJ*}F_J$ are morphism in $\text{PSh}_{O_{\mathcal{S}_I}}(\mathcal{S}_I)$, noting again $r_{IJ} : \mathcal{S}_J \rightarrow \mathcal{S}_I$ the associated morphism of presite,
 - * whose morphism are $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ*}m_J \circ u_{IJ}$ in $\text{PSh}_{O_{\mathcal{S}_I}}(\mathcal{S}_I)$,

- $C_{O_{S_\bullet}}(\mathcal{S}_\bullet) := C(\text{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet))$,
 - $K_{O_{S_\bullet}}(\mathcal{S}_\bullet) := K(\text{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet)) = \text{Ho}(C_{O_{S_\bullet}}(\mathcal{S}_\bullet))$, in particular, we have the full homotopy functor $Ho : C(\mathcal{S}_\bullet) \rightarrow K(\mathcal{S}_\bullet)$,
 - $C_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet) := C_{O_{S_\bullet}(2)fil}(\text{PSh}(\mathcal{S}_\bullet)) \subset C(\text{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet), F, W)$ the big abelian category of (bi)filtered complexes of presheaves of modules on $(\mathcal{S}_\bullet, O_{S_\bullet})$ such that the filtration is biregular, and $\text{PSh}_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet) = (\text{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet), F, W)$, by definition $C_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet)$ is the category
 - * whose objects are $C_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet)^0 := \{((F_I, F, W), u_{IJ})_{I \in \mathcal{I}}\}$, with $(F_I, F, W) \in C_{O_{S_I}(2)fil}(\mathcal{S}_I)$, and for $r_{IJ} : I \rightarrow J$ with $I, J \in \mathcal{I}$, $u_{IJ} : (F_I, F, W) \rightarrow r_{IJ*}(F_J, F, W)$ are morphism in $C_{O_{S_I}(2)fil}(\mathcal{S}_I)$, noting again $r_{IJ} : \mathcal{S}_J \rightarrow \mathcal{S}_I$ the associated morphism of presite,
 - * whose morphism are $m = (m_I) : ((F_I, F, W), u_{IJ}) \rightarrow ((G_I, F, W), v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ*}m_J \circ u_{IJ}$ in $C_{O_{S_I}(2)fil}(\mathcal{S}_I)$,
 - $K_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet) := K_{fil}(\text{PSh}_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet)) = \text{Ho}(C_{O_{S_\bullet}(2)fil}(\mathcal{S}_\bullet))$,
 - $K_{O_{S_\bullet}fil,r}(\mathcal{S}_\bullet) := K_{fil,r}(\text{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet)) = \text{Ho}_r(C_{O_{S_\bullet}fil,r}(\mathcal{S}_\bullet))$, $K_{O_{S_\bullet}fil,\infty}(\mathcal{S}_\bullet) := K_{\infty,r}(\text{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet)) = \text{Ho}_\infty(C_{O_{S_\bullet}fil,\infty}(\mathcal{S}_\bullet))$.
- Let $\mathcal{S} \in \text{Cat}$. For $\Sigma : C(\mathcal{S}) \rightarrow C(\mathcal{S})$ an endofunctor, we denote by $C_\Sigma(\mathcal{S}) = (C(\mathcal{S}), \Sigma)$ the corresponding category of spectra.
 - Denote by $\text{Sch} \subset \text{RTop}$ the full subcategory of schemes. For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$. We then denote by
 - $\text{Var}(k) \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
 - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
 - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

A morphism $h : U \rightarrow S$ with $U, S \in \text{Var}(k)$ is said to be smooth if it is flat with smooth fibers. A morphism $r : U \rightarrow X$ with $U, X \in \text{Var}(k)$ is said to be etale if it is non ramified and flat. In particular an etale morphism $r : U \rightarrow X$ with $U, X \in \text{Var}(k)$ is smooth and quasi-finite (i.e. the fibers are either the empty set or a finite subset of X)

- Denote by Top^2 the category whose set of objects is

$$(\text{Top}^2)^0 := \{(X, Z), Z \subset X \text{ closed}\} \subset \text{Top} \times \text{Top}$$

and whose set of morphism between $(X_1, Z_1), (X_2, Z_2) \in \text{Top}^2$ is

$$\text{Hom}_{\text{Top}^2}((X_1, Z_1), (X_2, Z_2)) := \{(f : X_1 \rightarrow X_2), \text{ s.t. } Z_1 \subset f^{-1}(Z_2)\} \subset \text{Hom}_{\text{Top}}(X_1, X_2)$$

For $S \in \text{Top}$, $\text{Top}^2/S := \text{Top}^2/(S, S)$ is then by definition the category whose set of objects is

$$(\text{Top}^2/S)^0 := \{((X, Z), h), h : X \rightarrow S, Z \subset X \text{ closed}\} \subset \text{Top}/S \times \text{Top}$$

and whose set of morphisms between $(X_1, Z_1)/S = ((X_1, Z_1), h_1), (X_2, Z_2)/S = ((X_2, Z_2), h_2) \in \text{Top}^2/S$ is the subset

$$\begin{aligned} & \text{Hom}_{\text{Top}^2/S}((X_1, Z_1)/S, (X_2, Z_2)/S) := \\ & \{(f : X_1 \rightarrow X_2), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\} \subset \text{Hom}_{\text{RTop}}(X_1, X_2) \end{aligned}$$

We denote by

$$\mu_S : \text{Top}^{2,pr}/S := \{((Y \times S, Z), p), p : Y \times S \rightarrow S, Z \subset Y \times S \text{ closed}\} \hookrightarrow \text{Top}^2/S$$

the full subcategory whose objects are those with $p : Y \times S \rightarrow S$ a projection, and again $\mu_S : \text{Top}^2/S \rightarrow \text{Top}^{2,pr}/S$ the corresponding morphism of sites. We denote by

$$\begin{aligned} \text{Gr}_S^{12} : \text{Top}/S &\rightarrow \text{Top}^{2,pr}/S, \quad X/S \mapsto \text{Gr}_S^{12}(X/S) := (X \times S, \bar{X})/S, \\ (g : X/S \rightarrow X'/S) &\mapsto \text{Gr}_S^{12}(g) := (g \times I_S : (X \times S, \bar{X}) \rightarrow (X' \times S, \bar{X}')) \end{aligned}$$

the graph functor, $X \hookrightarrow X \times S$ being the graph embedding (which is a closed embedding if X is separated), and again $\text{Gr}_S^{12} : \text{Top}^{2,pr}/S \rightarrow \text{Top}/S$ the corresponding morphism of sites.

- Denote by RTop^2 the category whose set of objects is

$$(\text{RTop}^2)^0 := \{((X, O_X), Z), Z \subset X \text{ closed}\} \subset \text{RTop} \times \text{Top}$$

and whose set of morphism between $((X_1, O_{X_1}), Z_1), ((X_2, O_{X_2}), Z_2) \in \text{RTop}^2$ is

$$\begin{aligned} \text{Hom}_{\text{RTop}^2}(((X_1, O_{X_1}), Z_1), ((X_2, O_{X_2}), Z_2)) &:= \\ \{(f : (X_1, O_{X_1}) \rightarrow (X_2, O_{X_2})), \text{ s.t. } Z_1 \subset f^{-1}(Z_2)\} &\subset \text{Hom}_{\text{RTop}}((X_1, O_{X_1}), (X_2, O_{X_2})) \end{aligned}$$

For $(S, O_S) \in \text{RTop}$, $\text{RTop}^2/(S, O_S) := \text{RTop}^2/((S, O_S), S)$ is then by definition the category whose set of objects is

$$\begin{aligned} (\text{RTop}^2/(S, O_S))^0 &:= \\ \{(((X, O_X), Z), h), h : (X, O_X) \rightarrow (S, O_S), Z \subset X \text{ closed}\} &\subset \text{RTop}/(S, O_S) \times \text{Top} \end{aligned}$$

and whose set of morphisms between $((X_1, O_{X_1}), Z_1, h_1), ((X_2, O_{X_2}), Z_2, h_2) \in \text{RTop}^2/(S, O_S)$ is the subset

$$\begin{aligned} \text{Hom}_{\text{RTop}^2/(S, O_S)}(((X_1, O_{X_1}), Z_1)/(S, O_S), ((X_2, O_{X_2}), Z_2)/(S, O_S)) &:= \\ \{(f : (X_1, O_{X_1}) \rightarrow (X_2, O_{X_2})), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\} &\subset \text{Hom}_{\text{RTop}}((X_1, O_{X_1}), (X_2, O_{X_2})) \end{aligned}$$

We denote by

$$\mu_S : \text{RTop}^{2,pr}/S := \{(((Y \times S, q^*O_Y \otimes p^*O_S), Z), p), p : Y \times S \rightarrow S, Z \subset Y \times S \text{ closed}\} \hookrightarrow \text{RTop}^2/S$$

the full subcategory whose objects are those with $p : Y \times S \rightarrow S$ is a projection, and again $\mu_S : \text{RTop}^2/S \rightarrow \text{RTop}^{2,pr}/S$ the corresponding morphism of sites. We denote by

$$\begin{aligned} \text{Gr}_S^{12} : \text{RTop}/S &\rightarrow \text{RTop}^{2,pr}/S, \\ (X, O_X)/(S, O_S) &\mapsto \text{Gr}_S^{12}((X, O_X)/(S, O_S)) := ((X \times S, q^*O_X \otimes p^*O_S), \bar{X})/(S, O_S), \\ (g : (X, O_X)/(S, O_S) \rightarrow (X', O_{X'})/(S, O_S)) &\mapsto \\ \text{Gr}_S^{12}(g) &:= (g \times I_S : ((X \times S, q^*O_X \otimes p^*O_S), \bar{X}) \rightarrow ((X' \times S, q^*O_{X'} \otimes p^*O_S), \bar{X}')) \end{aligned}$$

the graph functor, $X \hookrightarrow X \times S$ being the graph embedding (which is a closed embedding if X is separated), $p : X \times S \rightarrow S$, $q : X \times S \rightarrow X$ the projections, and again $\text{Gr}_S^{12} : \text{RTop}^{2,pr}/S \rightarrow \text{RTop}/S$ the corresponding morphism of sites.

- We denote by $\text{Sch}^2 \subset \text{RTop}^2$ the full subcategory such that the first factors are schemes. For a field k , we denote by $\text{Sch}^2/k := \text{Sch}^2/(\text{Spec } k, \{\text{pt}\})$ and by
 - $\text{Var}(k)^2 \subset \text{Sch}^2/k$ the full subcategory such that the first factors are algebraic varieties over k , i.e. schemes of finite type over k ,
 - $\text{PVar}(k)^2 \subset \text{QPVar}(k)^2 \subset \text{Var}(k)^2$ the full subcategories such that the first factors are quasi-projective varieties and projective varieties respectively,

- $\text{PSmVar}(k)^2 \subset \text{SmVar}(k)^2 \subset \text{Var}(k)^2$ the full subcategories such that the first factors are smooth varieties and smooth projective varieties respectively.

In particular we have, for $S \in \text{Var}(k)$, the graph functor

$$\begin{aligned} \text{Gr}_S^{12} : \text{Var}(k)/S &\rightarrow \text{Var}(k)^{2,pr}/S, \quad X/S \mapsto \text{Gr}_S^{12}(X/S) := (X \times S, X)/S, \\ (g : X/S &\rightarrow X'/S) \mapsto \text{Gr}_S^{12}(g) := (g \times I_S : (X \times S, X) \rightarrow (X' \times S, X')) \end{aligned}$$

the graph embedding $X \hookrightarrow X \times S$ is a closed embedding since X is separated in the subcategory of schemes $\text{Sch} \subset \text{RTop}$, and again $\text{Gr}_S^{12} : \text{Var}(k)^{2,pr}/S \rightarrow \text{Var}(k)/S$ the corresponding morphism of sites.

- Denote by $\text{CW} \subset \text{Top}$ the full subcategory of CW complexes, by $\text{CS} \subset \text{CW}$ the full subcategory of Δ complexes, by $\text{TM}(\mathbb{R}) \subset \text{CW}$ the full subcategory of topological (real) manifolds which admits a CW structure (a topological manifold admits a CW structure if it admits a differential structure) and by $\text{Diff}(\mathbb{R}) \subset \text{RTop}$ the full subcategory of differentiable (real) manifold. We denote by $\text{CW}^2 \subset \text{Top}^2$ the full subcategory such that the first factors are CW complexes, by $\text{TM}(\mathbb{R})^2 \subset \text{CW}^2$ the full subcategory such that the first factors are topological (real) manifolds and by $\text{Diff}(\mathbb{R})^2 \subset \text{RTop}^2$ the full subcategory such that the first factors are differentiable (real) manifold.
- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the full subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). A morphism $h : U \rightarrow S$ with $U, S \in \text{AnSp}(\mathbb{C})$ is said to be smooth if it is flat with smooth fibers. A morphism $r : U \rightarrow X$ with $U, X \in \text{AnSp}(\mathbb{C})$ is said to be etale if it is non ramified and flat. By the Weirstrass preparation theorem (or the implicit function theorem if U and X are smooth), a morphism $r : U \rightarrow X$ with $U, X \in \text{AnSp}(\mathbb{C})$ is etale if and only if it is an isomorphism local.

We denote by $\text{AnSp}(\mathbb{C})^2 \subset \text{RTop}^2$ the full subcategory such that the first factors are analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C})^2 \subset \text{AnSp}(\mathbb{C})^2$ the full subcategory such that the first factors are smooth analytic spaces (i.e. complex analytic manifold). In particular we have, for $S \in \text{AnSp}(\mathbb{C})$, the graph functor

$$\begin{aligned} \text{Gr}_S^{12} : \text{AnSp}(\mathbb{C})/S &\rightarrow \text{AnSp}(\mathbb{C})^{2,pr}/S, \quad X/S \mapsto \text{Gr}_S^{12}(X/S) := (X \times S, X)/S, \\ (g : X/S &\rightarrow X'/S) \mapsto \text{Gr}_S^{12}(g) := (g \times I_S : (X \times S, X) \rightarrow (X' \times S, X')) \end{aligned}$$

the graph embedding $X \hookrightarrow X \times S$ is a closed embedding since X is separated in RTop , and again $\text{Gr}_S^{12} : \text{AnSp}(\mathbb{C})^{2,pr}/S \rightarrow \text{AnSp}(\mathbb{C})/S$ the corresponding morphism of sites.

- For $V \in \text{Var}(\mathbb{C})$, we denote by $V^{an} \in \text{AnSp}(\mathbb{C})$ the complex analytic space associated to V with the usual topology induced by the usual topology of \mathbb{C}^N . For $W \in \text{AnSp}(\mathbb{C})$, we denote by $W^{cw} \in \text{AnSp}(\mathbb{C})$ the topological space given by W which is a CW complex. For simplicity, for $V \in \text{Var}(\mathbb{C})$, we denote by $V^{cw} := (V^{an})^{cw} \in \text{CW}$. We have then

- the analytical functor $\text{An} : \text{Var}(\mathbb{C}) \rightarrow \text{AnSp}(\mathbb{C})$, $\text{An}(V) = V^{an}$,
- the forgetful functor $\text{Cw} = tp : \text{AnSp}(\mathbb{C}) \rightarrow \text{CW}$, $\text{Cw}(W) = W^{cw}$,
- the composite of these two functors $\widetilde{\text{Cw}} = \text{Cw} \circ \text{An} : \text{Var}(\mathbb{C}) \rightarrow \text{CW}$, $\widetilde{\text{Cw}}(V) = V^{cw}$.

We have then

- the analytical functor $\text{An} : \text{Var}(\mathbb{C})^2 \rightarrow \text{AnSp}(\mathbb{C})^2$, $\text{An}((V, Z)) = (V^{an}, Z^{an})$,
- the forgetful functor $\text{Cw} = tp : \text{AnSp}(\mathbb{C})^2 \rightarrow \text{CW}^2$, $\text{Cw}((W, Z)) = (W^{cw}, Z^{cw})$,
- the composite of these two functors $\widetilde{\text{Cw}} = \text{Cw} \circ \text{An} : \text{Var}(\mathbb{C})^2 \rightarrow \text{CW}^2$, $\widetilde{\text{Cw}}((V, Z)) = (V^{cw}, Z^{cw})$.

2.2 Additive categories, abelian categories and tensor triangulated categories

Let \mathcal{A} an additive category.

- For $\phi : F^\bullet \rightarrow G^\bullet$ a morphism with $F^\bullet, G^\bullet \in C(\mathcal{A})$, we have the mapping cylinder $\text{Cyl}(\phi) := ((F^n \oplus F^{n+1} \oplus G^{n+1}, (\partial_F^n, \partial_F^{n+1}, \phi^{n+1} + \partial^n G) \in C(\mathcal{A}))$. and the mapping cone $\text{Cone}(\phi) := ((F^n \oplus G^{n+1}, (\partial_F^n, \phi^{n+1} + \partial^n G) \in C(\mathcal{A}))$.
- The category $K(\mathcal{A}) := \text{Ho}(C(\mathcal{A}))$ is a triangulated category with distinguish triangles $F^\bullet \xrightarrow{i_F} \text{Cyl}(\phi) \xrightarrow{q_F} \text{Cone}(\phi) \xrightarrow{r_F} F^\bullet[1]$.
- The category (\mathcal{A}, F) is obviously again an additive category.
- Let $\phi : F^\bullet \rightarrow G^\bullet$ a morphism with $F^\bullet, G^\bullet \in C(\mathcal{A})$. Then it is obviously a morphism of filtered complex $\phi : (F^\bullet, F_b) \rightarrow (G^\bullet, F_b)$, where we recall that F_b is the trivial filtration $(F^\bullet, F_b), (G^\bullet, F_b) \in C_{\text{fil}}(\mathcal{A})$.

We recall the following property of the internal hom functor if it exists of a tensor triangulated category and the definition of compact and cocompact object.

Proposition 2. *Let (\mathcal{T}, \otimes) a tensor triangulated category admitting countable direct sum and product compatible with the triangulation. Assume that \mathcal{T} has an internal hom (bi)functor $R\mathcal{H}om(\cdot, \cdot) : \mathcal{T}^2 \rightarrow \mathcal{T}$ which is by definition the right adjoint to $(\cdot \otimes \cdot) : \mathcal{T}^2 \rightarrow \mathcal{T}$. Then,*

- for $N \in \mathcal{T}$, the functor $R\mathcal{H}om(\cdot, N) : \mathcal{T} \rightarrow \mathcal{T}$ commutes with homotopy colimits : for $M = \text{ho} \lim_{\rightarrow i \in I} M_i$, where I is a countable category, we have

$$R\mathcal{H}om(M, N) \xrightarrow{\sim} \text{ho} \lim_{\leftarrow i \in I} R\mathcal{H}om(M_i, N).$$

- dually, for $M \in \mathcal{T}$, the functor $R\mathcal{H}om(M, \cdot) : \mathcal{T} \rightarrow \mathcal{T}$ commutes with homotopy limits : for $N = \text{ho} \lim_{\leftarrow i \in I} N_i$, where I is a countable category, we have

$$R\mathcal{H}om(M, N) \xrightarrow{\sim} \text{ho} \lim_{\leftarrow} R\mathcal{H}om(M, N_i).$$

Proof. Standard. □

Let (\mathcal{T}, \otimes) a tensor triangulated category admitting countable direct sum and product compatible with the triangulation. Assume that \mathcal{T} has an internal hom functor $R\mathcal{H}om(\cdot, \cdot) : \mathcal{T} \rightarrow \mathcal{T}$.

- For $N \in \mathcal{T}$, the functor $R\mathcal{H}om(\cdot, N) : \mathcal{T} \rightarrow \mathcal{T}$ does not commutes in general with homotopy limits : for $M = \text{ho} \lim_{\leftarrow i \in I} M_i$, where I is a countable category, the canonical map

$$\text{ho} \lim_{\rightarrow i \in I} R\mathcal{H}om(M_i, N) \rightarrow R\mathcal{H}om(M, N)$$

is not an isomorphism in general if I is infinite. It commutes if and only if N is compact.

- Dually, for $M \in \mathcal{T}$, the functor $R\mathcal{H}om(M, \cdot) : \mathcal{T} \rightarrow \mathcal{T}$ does not commutes in general with infinite homotopy colimits. It commutes if and only if M is cocompact.

Most triangulated category comes from the localization of the category of complexes of an abelian category with respect to quasi-isomorphisms. In the case where the abelian category have enough injective or projective object, the triangulated category is the homotopy category of the complexes of injective, resp. projective, objects.

Proposition 3. *Let \mathcal{A} an abelian category with enough injective and projective.*

- A quasi-isomorphism $\phi : Q^\bullet \rightarrow F^\bullet$, with $F^\bullet, Q^\bullet \in C^-(\mathcal{A})$ such that the Q^n are projective is an homotopy equivalence.
- Dually, a quasi-isomorphism $\phi : F^\bullet \rightarrow I^\bullet$, with $F^\bullet, I^\bullet \in C^+(\mathcal{A})$ such that the I^n are projective is an homotopy equivalence.

Proof. Standard. □

Proposition 4. *Let \mathcal{A} an abelian category with enough injective and projective satisfying AB3 (i.e. countable direct sum of exact sequences are exact sequence).*

- Let $K(P) \subset K(\mathcal{A})$ be the thick subcategory generated by (unbounded) complexes of projective objects. Then, $K(P) \hookrightarrow K(\mathcal{A}) \xrightarrow{D} D(\mathcal{A})$ is an equivalence of triangulated categories.
- Similarly, let $K(I) \subset K(\mathcal{A})$ be the thick subcategory generated by (unbounded) complexes of injective objects. Then $K(I) \hookrightarrow K(\mathcal{A}) \xrightarrow{D} D(\mathcal{A})$ is an equivalence of triangulated categories.

Proof. It follows from proposition 3 : see [23]. □

Let $\mathcal{A} \subset \text{Cat}$ an abelian category. Let $\phi : (M, F) \rightarrow (N, F)$ a morphism with $(M, F), (N, F) \in C_{fil}(\mathcal{A})$. Then the distinguish triangle

$$(M, F) \xrightarrow{\phi} (N, F) \xrightarrow{i_1} \text{Cone}(\phi) = ((M, F)[1] \oplus (N, F), (d, d' - \phi) \xrightarrow{p_1} (M, F)[1])$$

gives a sequence

$$\dots \rightarrow H^n(M, F) \xrightarrow{H^n(\phi)} H^n(N, F) \xrightarrow{H^n(i_1)} H^n(\text{Cone}(\phi)) \xrightarrow{H^n(p_1)} H^{n+1}(M, F) \rightarrow \dots$$

which, if we forgot filtration is a long exact sequence in \mathcal{A} ; however the morphism are NOT strict in general.

2.3 Presheaves on a site and on a ringed topos

2.3.1 Functorialities

Let $\mathcal{S} \in \text{Cat}$ a small category. For $X \in \mathcal{S}$ we denote by $\mathbb{Z}(X) \in \text{PSh}(\mathcal{S})$ the presheaf represented by X . By Yoneda lemma, a representable presheaf $\mathbb{Z}(X)$ is projective.

Proposition 5. • *Let $\mathcal{S} \in \text{Cat}$ a small category. The projective presheaves $\text{Proj}(\text{PSh}(\mathcal{S})) \subset \text{PSh}(\mathcal{S})$ are the direct summand of the representable presheaves $\mathbb{Z}(X)$ with $X \in \mathcal{S}$.*

- *More generally let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ a ringed topos. The projective presheaves $\text{Proj}(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})) \subset \text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$ of $O_{\mathcal{S}}$ modules are the direct summand of the representable presheaves $\mathbb{Z}(X) \otimes O_{\mathcal{S}}$ with $X \in \mathcal{S}$.*

Proof. Standard. □

Let $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \text{Cat}$. For $h : U \rightarrow S$ a morphism with $U, S \in \mathcal{S}$, we have $f^*\mathbb{Z}(U/S) = \mathbb{Z}(P(f)(U/S))$.

We will consider in this article filtered complexes of presheaves on a site. Let $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \text{Cat}$.

- The functor $f_* : C(\mathcal{T}) \rightarrow C(\mathcal{S})$ gives, by functoriality, the functor

$$f_* : C_{(2)fil}(\mathcal{T}) \rightarrow C_{(2)fil}(\mathcal{S}), (G, F) \mapsto f_*(G, F) := (f_*G, f_*F),$$

since f_* preserves monomorphisms.

- The functor $f^* : C(\mathcal{S}) \rightarrow C(\mathcal{T})$ gives, by functoriality, the functor

$$f^* : C_{(2)fil}(\mathcal{S}) \rightarrow C_{(2)fil}(\mathcal{T}), (G, F) \mapsto f^*(G, F), F^p(f^*(G, F)) := \text{Im}(f^* F^p G \rightarrow f^* G).$$

In the particular case where $f^* : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}(\mathcal{T})$ preserves monomorphisms, we have $f^*(G, F) = (f^*G, f^*F)$.

- The functor $f^\perp : C(\mathcal{S}) \rightarrow C(\mathcal{T})$ gives, by functoriality, the functor

$$f^\perp : C_{(2)fil}(\mathcal{T}) \rightarrow C_{(2)fil}(\mathcal{S}), (G, F) \mapsto f^\perp(G, F) := (f^\perp G, f^\perp F),$$

since $f^\perp : C(\mathcal{S}) \rightarrow C(\mathcal{T})$ preserves monomorphisms.

Let $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \text{Cat}$.

- The adjunction $(f^*, f_*) = (f^{-1}, f_*) : C(\mathcal{S}) \rightleftarrows C(\mathcal{T})$, gives an adjunction

$$(f^*, f_*) : C_{(2)fil}(\mathcal{S}) \rightleftarrows C_{(2)fil}(\mathcal{T}), (G, F) \mapsto f^*(G, F), (G, F) \mapsto f_*(G, F),$$

with adjunction maps, for $(G_1, F) \in C_{(2)fil}(\mathcal{S})$ and $(G_2, F) \in C_{(2)fil}(\mathcal{T})$

$$\text{ad}(f^*, f_*)(G_1, F) : (G_1, F) \rightarrow f_* f^*(G_1, F), \text{ad}(f^*, f_*)(G_2, F) : f^* f_*(G_2, F) \rightarrow (G_2, F).$$

- The adjunction $(f_*, f^\perp) : C(\mathcal{S}) \rightleftarrows C(\mathcal{T})$, gives an adjunction

$$(f_*, f^\perp) : C_{(2)fil}(\mathcal{T}) \rightleftarrows C_{(2)fil}(\mathcal{S}), (G, F) \mapsto f_*(G, F), (G, F) \mapsto f^\perp(G, F),$$

with adjunction maps, for $(G_1, F) \in C_{(2)fil}(\mathcal{S})$ and $(G_2, F) \in C_{(2)fil}(\mathcal{T})$

$$\text{ad}(f^*, f_*)(G_2, F) : (G_2, F) \rightarrow f^\perp f_*(G_2, F), \text{ad}(f^*, f_*)(G_1, F) : f_* f^\perp(G_1, F) \rightarrow (G_1, F).$$

Remark 1. Let $\mathcal{T}, \mathcal{S} \in \text{Cat}$ small categories and $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite. Then the functor $f^* : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}(\mathcal{T})$ preserve epimorphism but does NOT preserve monomorphism in general (the colimits involved are NOT filtered colimits). However it preserve monomorphism between projective presheaves by Yoneda and we thus set for $(Q, F) \in C_{fil}(\text{Proj}(\text{PSh}(\mathcal{S})))$, that is $F^p Q^n \in \text{Proj}(\text{PSh}(\mathcal{S}))$ for all $p, n \in \mathbb{Z}$, $f^*(Q, F) := (f^*Q, f^*F)$.

For a commutative diagram of presite :

$$D = \begin{array}{ccc} \mathcal{T}' & \xrightarrow{g_2} & \mathcal{S}' \\ \downarrow f_2 & & \downarrow f_1 \\ \mathcal{T} & \xrightarrow{g_1} & \mathcal{S} \end{array},$$

with $\mathcal{T}, \mathcal{T}', \mathcal{S}, \mathcal{S}' \in \text{Cat}$, we denote by, for $F \in C(\mathcal{S}')$,

$$T(D)(F) : g_1^* f_{1*} F \xrightarrow{g_1^* f_{1*} \text{ad}(g_2^*, g_{2*})(F)} g_1^* f_{1*} g_{2*} g_2^* F = g_1^* g_{1*} f_{2*} g_2^* F \xrightarrow{\text{ad}(g_1^* g_{1*})(f_{2*} g_2^* F)} f_{2*} g_2^* F$$

the canonical transformation map in $C(\mathcal{T})$, and for $(G, F) \in C_{fil}(\mathcal{S}')$,

$$T(D)(G, F) : g_1^* f_{1*}(G, F) \xrightarrow{g_1^* f_{1*} \text{ad}(g_2^*, g_{2*})(G, F)} g_1^* f_{1*} g_{2*} g_2^*(G, F) = g_1^* g_{1*} f_{2*} g_2^*(G, F) \xrightarrow{\text{ad}(g_1^* g_{1*})(f_{2*} g_2^*(G, F))} f_{2*} g_2^*(G, F).$$

the canonical transformation map in $C_{fil}(\mathcal{T})$ given by the adjunction maps.

We will use the internal hom functor and the tensor product for presheaves on a site or for presheaves of modules on a ringed topoi. We recall the definition in the filtered case.

- Let $(\mathcal{S}, O_S) \in \text{RCat}$. We have the tensor product bifunctor

$$(\cdot) \otimes (\cdot) : \text{PSh}(\mathcal{S})^2 \rightarrow \text{PSh}(\mathcal{S}), (F, G) \mapsto (X \in \mathcal{S} \mapsto (F \otimes G)(X) := F(X) \otimes G(X))$$

It induces a bifunctor :

$$(\cdot) \otimes (\cdot) : C(\mathcal{S}) \times C(\mathcal{S}) \rightarrow C(\mathcal{S}), (F, G) \mapsto F \otimes G := \text{Tot}(F^\bullet \otimes G^\bullet), (F \otimes G)^n = \bigoplus_{r \in \mathbb{Z}} F^r \otimes G^{n-r}$$

and a bifunctor

$$(\cdot) \otimes (\cdot) : C(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \rightarrow C_{O_S}(\mathcal{S}), \alpha.(F \otimes G) := F \otimes (\alpha.G)$$

For $(G_1, F), (G_2, F) \in C_{\text{fil}}(\mathcal{S}), G_3 \in C(\mathcal{S})$, we define (note that tensor product preserve monomorphism only after tensoring with $\mathbb{Q}_S \in \text{PSh}(\mathcal{S})$)

- $F^p((G_1, F) \otimes G_3) := \text{Im}(F^p G_1 \otimes G_3 \rightarrow G_1 \otimes G_3)$ and $F^p(G_3 \otimes (G_1, F)) := \text{Im}(G_3 \otimes F^p G_1 \rightarrow G_3 \otimes G_1)$,
- $F^p F^q((G_1, F) \otimes (G_2, F)) := \text{Im}(F^p G_1 \otimes F^q G_2 \rightarrow G_1 \otimes G_2)$ and
- $F^k((G_1, F) \otimes (G_2, F)) := F^k \text{Tot}_{FF}((G_1, F) \otimes (G_2, F)) := \bigoplus_{p \in \mathbb{Z}} \text{Im}(F^p G_1 \otimes F^{k-p} G_2 \rightarrow G_1 \otimes G_2)$

Note that in the case where $G_1^n = 0$ for $n < 0$, we have $(G_1, F_b) \otimes (G_2, F) = G_1 \otimes (G_2, F)$. We get the bifunctors

$$(-) \otimes (-) : C_{\text{fil}}(\mathcal{S})^2 \rightarrow C_{\text{fil}}(\mathcal{S}), (-) \otimes (-) : C_{\text{fil}}(\mathcal{S}) \times C_{O_S \text{fil}}(\mathcal{S}) \rightarrow C_{O_S \text{fil}}(\mathcal{S}).$$

We have the tensor product bifunctor

$$(\cdot) \otimes_{O_S} (\cdot) : \text{PSh}_{O_S}(\mathcal{S})^2 \rightarrow \text{PSh}(\mathcal{S}), (F, G) \mapsto (X \in \mathcal{S} \mapsto (F \otimes_{O_S} G)(X) := F(X) \otimes_{O_S(X)} G(X))$$

It induces a bifunctor :

$$(\cdot) \otimes_{O_S} (\cdot) : C_{O_S}(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \rightarrow C(\mathcal{S}), (F, G) \mapsto F \otimes_{O_S} G := \text{Tot}(F^\bullet \otimes_{O_S} G^\bullet)$$

For $(G_1, F), (G_2, F) \in C_{O_S \text{fil}}(\mathcal{S}), G_3 \in C_{O_S}(\mathcal{S})$, we define similarly $(G_1, F) \otimes_{O_S} G_3, G_3 \otimes_{O_S} (G_1, F)$, and

$$F^k((G_1, F) \otimes_{O_S} (G_2, F)) := F^k \text{Tot}_{FF}((G_1, F) \otimes_{O_S} (G_2, F)) := \bigoplus_{p \in \mathbb{Z}} \text{Im}(F^p G_1 \otimes_{O_S} F^{k-p} G_2 \rightarrow G_1 \otimes_{O_S} G_2)$$

Note that in the case where $G_1^n = 0$ for $n < 0$, we have $(G_1, F_b) \otimes_{O_S} (G_2, F) = G_1 \otimes_{O_S} (G_2, F)$. This gives

- in all case it gives the bifunctor $(-) \otimes_{O_S} (-) : C_{O_S^{\text{op}} \text{fil}}(\mathcal{S}) \otimes C_{O_S \text{fil}}(\mathcal{S}) \rightarrow C_{\text{fil}}(\mathcal{S})$.
- in the case O_S is commutative, it gives the bifunctor $(-) \otimes_{O_S} (-) : C_{O_S \text{fil}}(\mathcal{S})^2 \rightarrow C_{O_S \text{fil}}(\mathcal{S})$.

- Let $(\mathcal{S}, O_S) \in \text{RCat}$. We have the internal hom bifunctor

$$\begin{aligned} \mathcal{H}om(\cdot, \cdot) : \text{PSh}(\mathcal{S})^2 &\rightarrow \text{PSh}(\mathcal{S}), \\ (F, G) &\mapsto (X \in \mathcal{S} \mapsto \mathcal{H}om(F, G)(X) := \text{Hom}(r(X)_* F, r(X)_* G)) \end{aligned}$$

with $r(X) : \mathcal{S} \rightarrow \mathcal{S}/X$ (see subsection 2.1). It induces a bifunctors :

$$\mathcal{H}om(\cdot, \cdot) : C(\mathcal{S}) \times C(\mathcal{S}) \rightarrow C(\mathcal{S}), (F, G) \mapsto \mathcal{H}om^\bullet(F, G)$$

and a bifunctor

$$\mathcal{H}om(\cdot, \cdot) : C(\mathcal{S}) \times C_{O_S}(\mathcal{S}) \rightarrow C_{O_S}(\mathcal{S}), \alpha.\mathcal{H}om(F, G) := \mathcal{H}om(F, \alpha.G)$$

For $(G_1, F), (G_2, F) \in C_{\text{fil}}(\mathcal{S}), G_3 \in C(\mathcal{S})$, we define

- $F^p \text{Hom}(G_3, (G_1, F)) := \mathcal{H}om(G_3, F^p G_1) \hookrightarrow \mathcal{H}om(G_3, G_1)$, note that the functor $G \mapsto \mathcal{H}om(F, G)$ preserve monomorphism,
- the dual filtration $F^{-p} \mathcal{H}om((G_1, F), G_3) := \ker(\mathcal{H}om(G_1, G_3) \rightarrow \text{Hom}(F^p G_1, G_3))$
- $F^p F^q \mathcal{H}om((G_1, F), (G_2, F)) := \ker(\mathcal{H}om(G_1, F^p G_2) \rightarrow \text{Hom}(F^q G_1, F^p G_2))$, and

$$F^k \mathcal{H}om^\bullet((G_1, F), (G_2, F)) := \text{Tot}_{FF} \mathcal{H}om((G_1, F), (G_2, F)) := \bigoplus_{p \in \mathbb{Z}} \ker(\mathcal{H}om(G_1, F^{k+p} G_2) \rightarrow \mathcal{H}om(F^p G_1, F^{k+p} G_2))$$

We get the bifunctors

$$\mathcal{H}om(\cdot, \cdot) : C_{fil}(\mathcal{S}) \times C_{fil}(\mathcal{S}) \rightarrow C_{fil}(\mathcal{S}), \quad \mathcal{H}om(\cdot, \cdot) : C_{fil}(\mathcal{S}) \times C_{O_S fil}(\mathcal{S}) \rightarrow C_{O_S fil}(\mathcal{S}).$$

We have the internal hom bifunctor

$$\mathcal{H}om_{O_S}(\cdot, \cdot) : \text{PSh}_{O_S}(\mathcal{S}) \times \text{PSh}_{O_S}(\mathcal{S}) \rightarrow \text{PSh}(\mathcal{S})$$

$$(F, G) \longmapsto (X \in \mathcal{S} \mapsto \mathcal{H}om_{O_S}(F, G)(X) := \text{Hom}_{O_S}(r(X)_* F, r(X)_* G).$$

It gives similarly

- in all case a bifunctor $\mathcal{H}om_{O_S}(\cdot, \cdot) : C_{fil O_S}(\mathcal{S}) \times C_{fil O_S}(\mathcal{S}) \rightarrow C_{fil}(\mathcal{S})$,
- the case O_S is commutative, a bifunctor $\mathcal{H}om_{O_S}(\cdot, \cdot) : C_{fil O_S}(\mathcal{S}) \times C_{fil O_S}(\mathcal{S}) \rightarrow C_{O_S fil}(\mathcal{S})$.

Let $\phi : A \rightarrow B$ of rings.

- Let M a A module. We say that M admits a B module structure if there exists a structure of B module on the abelian group M which is compatible with ϕ together with the A module structure on M .
- For N_1 a A -module and N_2 a B module. $I(A/B)(N_1, N_2) : \text{Hom}_A(N_1, N_2) \xrightarrow{\sim} \text{Hom}_B(N_1 \otimes_A B, N_2)$ is the adjunction between the restriction of scalars and the extension of scalars.
- For N', N'' a A -modules, $ev_A(\text{hom}, \otimes)(N', N'', B) : \text{Hom}_A(N', N'') \otimes_A B \rightarrow \text{Hom}_A(N', N'' \otimes_A B)$. is the evaluation classical map.

Let $\phi : (\mathcal{S}, O_1) \rightarrow (\mathcal{S}, O_2)$ a morphism of presheaves of ring on $\mathcal{S} \in \text{Cat}$.

- Let $M \in \text{PSh}_{O_1}(\mathcal{S})$. We say that M admits an O_2 module structure if there exists a structure of O_2 module on $M \in \text{PSh}(\mathcal{S})$ which is compatible with ϕ together with the O_1 module structure on M .
- For $N_1 \in C_{O_1}(\mathcal{S})$ and $N_2 \in C_{O_2}(\mathcal{S})$,

$$I(O_1/O_2)(N_1, N_2) : \text{Hom}_{O_1}(N_1, N_2) \xrightarrow{\sim} \text{Hom}_{O_2}(N_1 \otimes_{O_1} B, N_2)$$

is the adjunction between the restriction of scalars and the extension of scalars.

- For $N', N'' \in C_{O_1}(\mathcal{S})$,

$$ev_{O_1}(\text{hom}, \otimes)(N', N'', O_2) : \mathcal{H}om_{O_1}(N', N'') \otimes_{O_1} O_2 \rightarrow \mathcal{H}om_{O_1}(N', N'' \otimes_{O_1} O_2).$$

is the classical evaluation map.

Let $(\mathcal{S}, O_S) \in \text{RCat}$.

- For $F_1, F_2, G_1, G_2 \in C(\mathcal{S})$, we denote by

$$T(\otimes, \mathcal{H}om)(F_1, F_2, G_1, G_2) : \mathcal{H}om(F_1, G_1) \otimes \mathcal{H}om(F_2, G_2) \rightarrow \mathcal{H}om(F_1 \otimes F_2, G_1 \otimes G_2)$$

the canonical map.

- For $G_3 \in C(\mathcal{S})$ and $G_1, G_2 \in C_{O_S}(\mathcal{S})$, we denote by

$$\begin{aligned} ev(hom, \otimes)(G_3, G_1, G_2) : \mathcal{H}om(G_3, G_1) \otimes_{O_S} G_2 &\rightarrow \mathcal{H}om(G_3, G_1 \otimes_{O_S} G_2) \\ \phi \otimes s &\mapsto (s' \mapsto \phi(s') \otimes s) \end{aligned}$$

- Let $\mathcal{S} \in \text{Cat}$ a small category. Let $(H_X : C(\mathcal{S}/X) \rightarrow C(\mathcal{S}/X))_{X \in \mathcal{S}}$ a family of functors which is functorial in X . We have by definition, for $F_1, F_2 \in C(\mathcal{S})$, the canonical transformation map

$$T(H, hom)(F_1, F_2) : H(\mathcal{H}om^\bullet(F_1, F_2)) \rightarrow \mathcal{H}om^\bullet(H(F_1), H(F_2)), \quad (1)$$

$$\phi \in \text{Hom}(F_1|_X, F_2|_X) \mapsto H^{F_1|_X, F_2|_X}(\phi) \in \text{Hom}(H(F_1|_X), H(F_2|_X)) \quad (2)$$

in $C(\mathcal{S})$.

Let $\mathcal{T}, \mathcal{S} \in \text{Cat}$ small categories and $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite.

- For $F_1, F_2 \in C(\mathcal{T})$ we have by definition $f_*(F_1 \otimes F_2) = f_*F_1 \otimes f_*F_2$. For $G_1, G_2 \in C(\mathcal{S})$, we have a canonical isomorphism $f^*G_1 \otimes f^*G_2 \xrightarrow{\sim} f^*(G_1 \otimes G_2)$ since the tensor product is a right exact functor, and a canonical map $f^\perp G_1 \otimes f^\perp G_2 \rightarrow f^\perp(G_1 \otimes G_2)$.
- We have for $F \in C(\mathcal{S})$ and $G \in C(\mathcal{T})$ the adjunction isomorphism,

$$I(f^*, f_*)(F, G) : f_*\mathcal{H}om^\bullet(f^*F, G) \xrightarrow{\sim} \mathcal{H}om^\bullet(F, f_*G). \quad (3)$$

- Let $O_S \in \text{PSh}(\mathcal{S}, \text{Ring})$ by a presheaf of ring so that $(\mathcal{S}, O_S), (\mathcal{T}, f^*O_S) \in \text{RCat}$. We have for $F \in C_{O_S}(\mathcal{S})$ and $G \in C_{f^*O_S}(\mathcal{T})$ the adjunction isomorphism,

$$I(f^*, f_*)(F, G) : f_*\mathcal{H}om_{f^*O_S}^\bullet(f^*F, G) \xrightarrow{\sim} \mathcal{H}om_{O_S}^\bullet(F, f_*G), \quad (4)$$

and

- the map $\text{ad}(f^*, f_*)(F) : F \rightarrow f_*f^*F$ in $C(\mathcal{S})$ is O_S linear, that is a map in $C_{O_S}(\mathcal{S})$,
- the map $\text{ad}(f^*, f_*)(G) : f^*f_*G \rightarrow G$ in $C(\mathcal{T})$ is f^*O_S linear, that is a map in $C_{f^*O_S}(\mathcal{T})$.

- For $F_1, F_2 \in C(\mathcal{T})$, we have the canonical map

$$T_*(f, hom)(F_1, F_2) := T(f_*, hom) : f_*\mathcal{H}om^\bullet(F_1, F_2) \rightarrow \mathcal{H}om^\bullet(f_*F_1, f_*F_2), \quad (5)$$

$$\text{for } X \in \mathcal{S}, \phi \in \text{Hom}(F_1|_{f^*(X)}, F_2|_{f^*(X)}) \mapsto f_*^{F_1|_{f^*(X)}, F_2|_{f^*(X)}}(\phi) \in \text{Hom}(f_*F_1|_{f^*(X)}, f_*F_2|_{f^*(X)}) \quad (6)$$

given by evaluation.

- For $G_1, G_2 \in C(\mathcal{S})$, we have the following canonical transformation in $C(\mathcal{T})$

$$T(f, hom)(G_1, G_2) := T(f^*, hom)(G_1, G_2) : \quad (7)$$

$$f^*\mathcal{H}om^\bullet(G_1, G_2) \xrightarrow{f^*\mathcal{H}om(G_1, \text{ad}(f^*, f_*)(G_2))} f^*\mathcal{H}om^\bullet(G_1, f_*f^*G_2) \xrightarrow{f^*I(f^*, f_*)(G_1, G_2)} \quad (8)$$

$$f^*f_*\mathcal{H}om^\bullet(f^*G_1, f^*G_2) \xrightarrow{\text{ad}(f^*, f_*)(\mathcal{H}om(f^*G_1, f^*G_2))} \mathcal{H}om^\bullet(f^*G_1, f^*G_2), \quad (9)$$

- Let $O_S \in \text{PSh}(\mathcal{S}, \text{Ring})$ by a presheaf of ring so that $(\mathcal{S}, O_S), (\mathcal{T}, f^*O_S) \in \text{RCat}$. For $G_1, G_2 \in C_{O_S}(\mathcal{S})$, we have the following canonical transformation in $C_{f^*O_S}(\mathcal{T})$

$$T(f, hom)(G_1, G_2) := T(f^*, hom)(G_1, G_2) : \quad (10)$$

$$f^*\mathcal{H}om_{O_S}^\bullet(G_1, G_2) \xrightarrow{f^*\mathcal{H}om_{O_S}(G_1, \text{ad}(f^*, f_*)(G_2))} f^*\mathcal{H}om_{O_S}^\bullet(G_1, f_*f^*G_2) \xrightarrow{f^*I(f^*, f_*)(G_1, G_2)} \quad (11)$$

$$f^*f_*\mathcal{H}om_{f^*O_S}^\bullet(f^*G_1, f^*G_2) \xrightarrow{\text{ad}(f^*, f_*)(\mathcal{H}om_{f^*O_S}(f^*G_1, f^*G_2))} \mathcal{H}om_{f^*O_S}^\bullet(f^*G_1, f^*G_2), \quad (12)$$

- Let $O_S \in \text{PSh}(\mathcal{S}, \text{Ring})$ by a presheaf of ring so that $(\mathcal{S}, O_S), (\mathcal{T}, f^*O_S) \in \text{RCat}$. For $M \in C_{O_S}(\mathcal{S})$ and $N \in C_{f^*O_S}(\mathcal{T})$, we denote by

$$T(f, \otimes)(M, N) : M \otimes_{O_S} f_*N \xrightarrow{\text{ad}(f^*, f^*)(M \otimes_{O_S} f_*N)} \quad (13)$$

$$f_*f^*(M \otimes_{O_S} f_*N) = f_*(f^*M \otimes_{f^*O_S} f^*f_*N) \xrightarrow{\text{ad}(f^*, f^*)(N)} f_*(f^*M \otimes_{f^*O_S} N) \quad (14)$$

the canonical transformation map.

Let $f : (\mathcal{T}, O_T) \rightarrow (\mathcal{S}, O_S)$ a morphism with $(\mathcal{S}, O_S), (\mathcal{T}, O_T) \in \text{RCat}$. We have the adjunction

$$(f^{*mod}, f_*) : C_{O_S}(\mathcal{S}) \rightleftarrows C_{O_T}(\mathcal{T})$$

with $f^{*mod}G := f^*G \otimes_{f^*O_S} O_T$. If $f^* : C(\mathcal{S}) \rightarrow C(\mathcal{T})$ preserve monomorphisms, it induces the adjunction

$$(f^{*mod}, f_*) : C_{O_S} \text{fil}(\mathcal{S}) \rightleftarrows C_{O_T} \text{fil}(\mathcal{T})$$

with $f^{*mod}(G, F) := f^*(G, F) \otimes_{f^*O_S} O_T$.

For a commutative diagram in RCat :

$$D = \begin{array}{ccc} (\mathcal{T}', O_2') & \xrightarrow{g_2} & (\mathcal{S}', O_1') \\ \downarrow f_2 & & \downarrow f_1 \\ (\mathcal{T}, O_2) & \xrightarrow{g_1} & (\mathcal{S}, O_1) \end{array}$$

we denote by, for $F \in C_{O_1'}(\mathcal{S}')$,

$$T^{mod}(D)(F) : g_1^{*mod} f_{1*} F \xrightarrow{g_1^{*mod} f_{1*} \text{ad}(g_2^{*mod}, g_{2*})(F)} g_1^{*mod} f_{1*} g_{2*} g_2^{*mod} F = g_1^{*mod} g_{1*} f_{2*} g_2^{*mod} F \xrightarrow{\text{ad}(g_1^{*mod} g_{1*})(f_{2*} g_2^{*mod} F)} f_{2*} g_2^{*mod} F$$

the canonical transformation map in $C_{O_2}(\mathcal{T})$ and, for $(G, F) \in C_{O_1'} \text{fil}(\mathcal{S}')$,

$$T^{mod}(D)(G, F) : g_1^{*mod} f_{1*}(G, F) \xrightarrow{g_1^{*mod} f_{1*} \text{ad}(g_2^{*mod}, g_{2*})(G, F)} g_1^{*mod} f_{1*} g_{2*} g_2^{*mod}(G, F) = g_1^{*mod} g_{1*} f_{2*} g_2^{*mod}(G, F) \xrightarrow{\text{ad}(g_1^{*mod} g_{1*})(f_{2*} g_2^{*mod}(G, F))} f_{2*} g_2^{*mod}(G, F)$$

the canonical transformation map in $C_{O_2} \text{fil}(\mathcal{T})$ given by the adjunction maps.

Let $f : (\mathcal{T}, O_T) \rightarrow (\mathcal{S}, O_S)$ a morphism with $(\mathcal{S}, O_S), (\mathcal{T}, O_T) \in \text{RCat}$.

- We have, for $M, N \in C_{O_S}(\mathcal{S})$ the canonical transformation map in $C_{O_T}(\mathcal{T})$

$$T^{mod}(f, \text{hom})(M, N) : f^{*mod} \text{Hom}_{O_1}(M, N) \xrightarrow{T(f, \text{hom})(M, N) \otimes_{f^*O_1} O_2} \text{Hom}_{f^*O_1}(f^*M, f^*N) \otimes_{f^*O_1} O_2 \xrightarrow{e(\text{hom}, \otimes)(f^*M, f^*N)} \text{Hom}_{f^*O_1}(f^*M, f^{*mod}N) \xrightarrow{I(f^*O_1/O_2)(f^*M, f^{*mod}N)} \text{Hom}_{O_2}(f^{*mod}M, f^{*mod}N)$$

- We have, for $M \in C_{O_S}(\mathcal{S})$ and $N \in C_{O_T}(\mathcal{T})$, the canonical transformation map in $C_{O_T}(\mathcal{T})$

$$T^{mod}(f, \otimes)(M, N) : M \otimes_{O_S} f_*N \xrightarrow{\text{ad}(f^{*mod}, f^*)(M \otimes_{O_S} f_*N)} \quad (15)$$

$$f_*f^{*mod}(M \otimes_{O_S} f_*N) = f_*(f^{*mod}M \otimes_{O_T} f^{*mod}f_*N) \xrightarrow{\text{ad}(f^{*mod}, f^*)(N)} f_*(f^{*mod}M \otimes_{O_T} N) \quad (16)$$

the canonical transformation map.

We now give some properties of the tensor product functor and hom functor given above

Proposition 6. *Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$. Then, the functors*

- $(-) \otimes (-) : C(\mathcal{S})^2 \rightarrow C(\mathcal{S}), C(\mathcal{S}) \times C_{O_{\mathcal{S}}}(\mathcal{S}) \rightarrow C_{O_{\mathcal{S}}}(\mathcal{S})$
- $(-) \otimes_{O_{\mathcal{S}}} (-) : C_{O_{\mathcal{S}}^{\text{op}}}(\mathcal{S}) \times C_{O_{\mathcal{S}}}(\mathcal{S}) \rightarrow C(\mathcal{S})$ and in case $O_{\mathcal{S}}$ is commutative $(-) \otimes_{O_{\mathcal{S}}} (-) : C_{O_{\mathcal{S}}}(\mathcal{S})^2 \rightarrow C_{O_{\mathcal{S}}}(\mathcal{S})$

are left Quillen functor for the projective model structure. In particular,

- for $L \in C(\mathcal{S})$ is such that $L^n \in \text{PSh}(\mathcal{S})$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \rightarrow G$ is a quasi-isomorphism with $F, G \in C(\mathcal{S})$, then $\phi \otimes I : F \otimes L \rightarrow G \otimes L$ is a quasi-isomorphism,
- for $L \in C_{O_{\mathcal{S}}}(\mathcal{S})$ is such that $L^n \in \text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \rightarrow G$ is a quasi-isomorphism with $F, G \in C_{O_{\mathcal{S}}}(\mathcal{S})$, then $\phi \otimes I : F \otimes_{O_{\mathcal{S}}} L \rightarrow G \otimes_{O_{\mathcal{S}}} L$ is a quasi-isomorphism.

Proof. Standard. \square

Proposition 7. *Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$. Then, the functors*

- $\text{Hom}(\cdot, \cdot) : C(\mathcal{S}) \times C_{O_{\mathcal{S}}}(\mathcal{S}) \rightarrow C_{O_{\mathcal{S}}}(\mathcal{S}), C(\mathcal{S}) \times C_{O_{\mathcal{S}}}(\mathcal{S}) \rightarrow C_{O_{\mathcal{S}}}(\mathcal{S}),$
- $\text{Hom}_{O_{\mathcal{S}}}(\cdot, \cdot) : C_{O_{\mathcal{S}}^{\text{op}}}(\mathcal{S}) \times C_{O_{\mathcal{S}}}(\mathcal{S}) \rightarrow C(\mathcal{S})$ and in the case $O_{\mathcal{S}}$ is commutative $\text{Hom}_{O_{\mathcal{S}}}(\cdot, \cdot) : C_{O_{\mathcal{S}}}(\mathcal{S}) \times C_{O_{\mathcal{S}}}(\mathcal{S}) \rightarrow C_{O_{\mathcal{S}}}(\mathcal{S}),$

are on the left hand side left Quillen functor for the projective model structure. In particular,

- for $L \in C(\mathcal{S})$ is such that $L^n \in \text{PSh}(\mathcal{S})$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \rightarrow G$ is a quasi-isomorphism with $F, G \in C(\mathcal{S})$, then $\text{Hom}(L, \phi) : \text{Hom}^{\bullet}(L, F) \rightarrow \text{Hom}^{\bullet}(L, G)$ is a quasi-isomorphism,
- for $L \in C_{O_{\mathcal{S}}}(\mathcal{S})$ is such that $L^n \in \text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \rightarrow G$ is a quasi-isomorphism with $F, G \in C_{O_{\mathcal{S}}}(\mathcal{S})$, then $\text{Hom}_{O_{\mathcal{S}}}(L, \phi) : \text{Hom}_{O_{\mathcal{S}}}^{\bullet}(L, F) \rightarrow \text{Hom}_{O_{\mathcal{S}}}^{\bullet}(L, G)$ is a quasi-isomorphism.

Proof. Standard. \square

Let $\mathcal{S} \in \text{Cat}$ a site endowed with topology τ . Denote by $a_{\tau} : \text{PSh}(\mathcal{S}) \rightarrow \text{Sh}(\mathcal{S})$ the sheafification functor A morphism $\phi : F^{\bullet} \rightarrow G^{\bullet}$ with $F^{\bullet}, G^{\bullet} \in C(\mathcal{S})$ is said to be a τ local equivalence if

$$a_{\tau}H^n(\phi) : a_{\tau}H^n(F^{\bullet}) \rightarrow a_{\tau}H^n(G^{\bullet})$$

is an isomorphism for all $n \in \mathbb{Z}$, where a_{τ} is the sheafification functor. Recall that $C_{\text{fil}}(\mathcal{S}) \subset (C(\mathcal{S}), F) = C(\text{PSh}(\mathcal{S}), F)$ denotes the category of filtered complexes of abelian presheaves on \mathcal{S} whose filtration is biregular.

- A morphism $\phi : (F^{\bullet}, F) \rightarrow (G^{\bullet}, F)$ with $(F^{\bullet}, F), (G^{\bullet}, F) \in C_{\text{fil}}(\mathcal{S})$ is said to be a filtered τ local equivalence or an 1-filtered τ local equivalence if

$$a_{\tau}H^n(\phi) : a_{\tau}H^n(\text{Gr}_F^p F^{\bullet}) \xrightarrow{\sim} a_{\tau}H^n(\text{Gr}_F^p G^{\bullet})$$

is an isomorphism for all $n, p \in \mathbb{Z}$.

- Let $r \in \mathbb{N}$. A morphism $\phi : (F^{\bullet}, F) \rightarrow (G^{\bullet}, F)$ with $(F^{\bullet}, F), (G^{\bullet}, F) \in C_{\text{fil}}(\mathcal{S})$ is said to be an r -filtered τ local equivalence if it belongs to the submonoid of arrows generated by filtered τ local equivalences and r -filtered homotopy equivalences, that is if there exists $\phi_i : (C_i^{\bullet}, F) \rightarrow (C_{i+1}^{\bullet}, F)$, $0 \leq i \leq s$, with $(C_i^{\bullet}, F) \in C_{\text{fil}}(\mathcal{S})$, $(C_0^{\bullet}, F) = (F^{\bullet}, F)$ and $(C_s^{\bullet}, F) = (G^{\bullet}, F)$, such that

$$\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : (F^{\bullet}, F) \rightarrow (G^{\bullet}, F)$$

and $\phi_i : (C_i^\bullet, F) \rightarrow (C_{i+1}^\bullet, F)$ either a filtered τ local equivalence or an r -filtered homotopy equivalence. If $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathcal{S})$ is an r -filtered τ local equivalence, then for all $p, q \in \mathbb{Z}$,

$$a_\tau E_r^{p,q}(\phi) : a_\tau E_r^{p,q}(F^\bullet, F) \xrightarrow{\sim} a_\tau E_r^{p,q}(G^\bullet, F)$$

is an isomorphism but the converse is NOT true. Note that if ϕ is an r -filtered τ local equivalence, that it is an s -filtered τ local equivalence for all $s \geq r$.

- A morphism $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be a ∞ -filtered τ local equivalence if there exists $r \in \mathbb{N}$ such that ϕ is an r -filtered τ local equivalence. If a morphism $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\mathcal{S})$ is an ∞ -filtered τ local equivalence then, for all $n \in \mathbb{Z}$,

$$a_\tau H^n(\phi) : a_\tau H^n(F^\bullet, F) \rightarrow a_\tau H^n(G^\bullet, F)$$

is an isomorphism of filtered sheaves on \mathcal{S} . Recall the converse is true in the case there exist $N_1, N_2 \in \mathbb{Z}$, such that $H^n(F^\bullet, F) = H^n(G^\bullet, F) = 0$ for $n \leq N_1$ or $n \geq N_2$.

Let (\mathcal{S}, O) a ringed topos where $\mathcal{S} \in \text{Cat}$ is a site endowed with topology τ . A morphism $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{O_S fil}(\mathcal{S})$ is said to be a filtered τ local equivalence or an 1-filtered τ local equivalence if $o\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ is one in $C_{fil}(\mathcal{S})$, i.e.

$$a_\tau H^n(\phi) : a_\tau H^n(\text{Gr}_F^p F^\bullet) \xrightarrow{\sim} a_\tau H^n(\text{Gr}_F^p G^\bullet)$$

is an isomorphism for all $n, p \in \mathbb{Z}$. Let $r \in \mathbb{N}$. A morphism $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{O_S fil}(\mathcal{S})$ is said to be an r -filtered τ local equivalence if there exists $\phi_i : (C_i^\bullet, F) \rightarrow (C_{i+1}^\bullet, F)$, $0 \leq i \leq s$, with $(C_i^\bullet, F) \in C_{O_S fil}(\mathcal{S})$, $(C_0^\bullet, F) = (F^\bullet, F)$ and $(C_s^\bullet, F) = (G^\bullet, F)$, such that

$$\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : (F^\bullet, F) \rightarrow (G^\bullet, F)$$

and $\phi_i : (C_i^\bullet, F) \rightarrow (C_{i+1}^\bullet, F)$ either a filtered τ local equivalence or an r -filtered homotopy equivalence.

Let $\mathcal{S} \in \text{Cat}$ a site which admits fiber product, endowed with topology τ . A complex of presheaves $F^\bullet \in C(\mathcal{S})$ is said to be τ fibrant if it satisfy descent for covers in \mathcal{S} , i.e. if for all $X \in \mathcal{S}$ and all τ covers $(c_i : U_i \rightarrow X)_{i \in I}$ of X , denoting $U_J := (U_{i_0} \times_S U_{i_1} \times_S \cdots \times_S U_{i_r})_{i_k \in J}$ and for $I \subset J$, $p_{IJ} : U_J \rightarrow U_I$ is the projection,

$$F^\bullet(c_i) : F^\bullet(X) \rightarrow \text{Tot}(\oplus_{\text{card} I = \bullet} F^\bullet(U_I))$$

is a quasi-isomorphism of complexes of abelian groups.

- A complex of filtered presheaves $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be filtered τ fibrant or 1-filtered τ fibrant if it satisfy descent for covers in \mathcal{S} , i.e. if for all $X \in \mathcal{S}$ and all τ covers $(c_i : U_i \rightarrow X)_{i \in I}$ of X ,

$$(F^\bullet, F)(c_i) : (F^\bullet, F)(X) \rightarrow \text{Tot}(\oplus_{\text{card} I = \bullet} (F^\bullet, F)(U_I))$$

is a filtered quasi-isomorphism of filtered complexes of abelian groups.

- Let $r \in \mathbb{N}$. A complex of filtered presheaves $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be r -filtered τ fibrant if there exist an r -filtered homotopy equivalence $m : (F^\bullet, F) \rightarrow (F'^\bullet, F)$ with $(F'^\bullet, F) \in C_{fil}(\mathcal{S})$ filtered τ fibrant. If $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is r -filtered τ fibrant, then for all $X \in \mathcal{S}$ and all τ covers $(c_i : U_i \rightarrow X)_{i \in I}$ of X ,

$$E_r^{p,q}(F^\bullet, F)(c_i) : E_r^{p,q}(F^\bullet, F)(X) \rightarrow E_r^{p,q}(\text{Tot}(\oplus_{\text{card} I = \bullet} (F^\bullet, F)(U_I)))$$

is an isomorphism for all $n, p \in \mathbb{Z}$, but the converse is NOT true. Note that if (F^\bullet, F) is r -filtered τ fibrant, then it is s -filtered τ fibrant for all $s \geq r$.

- A complex of filtered presheaves $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is said to be ∞ -filtered τ fibrant if there exist $r \in \mathbb{N}$ such that (F^\bullet, F) is r -filtered τ fibrant. If a complex of filtered presheaves $(F^\bullet, F) \in C_{fil}(\mathcal{S})$ is ∞ -filtered τ fibrant, then for all $X \in \mathcal{S}$ and all τ covers $(c_i : U_i \rightarrow X)_{i \in I}$ of X ,

$$H^n(F^\bullet, F)(c_i) : H^n(F^\bullet, F)(X) \rightarrow H^n \text{Tot}(\bigoplus_{\text{card} I = \bullet} (F^\bullet, F)(U_I))$$

is a filtered isomorphism for all $n \in \mathbb{Z}$.

Let (\mathcal{S}, O) a ringed topos where $\mathcal{S} \in \text{Cat}$ is a site endowed with topology τ . Let $r \in \mathbb{N}$.

- A complex of presheaves $F^\bullet \in C_{O_S}(\mathcal{S})$ is said to be τ fibrant if $F^\bullet = oF^\bullet \in C(\mathcal{S})$ is τ fibrant.
- A complex of presheaves $(F^\bullet, F) \in C_{O_S fil}(\mathcal{S})$ is said to be filtered τ fibrant if $(F^\bullet, F) = (oF^\bullet, F) \in C_{fil}(\mathcal{S})$ is filtered τ fibrant.
- A complex of presheaves $(F^\bullet, F) \in C_{O_S fil}(\mathcal{S})$ is said to be r -filtered τ fibrant if there exist an r -filtered homotopy equivalence $m : (F^\bullet, F) \rightarrow (F'^\bullet, F)$ with $(F'^\bullet, F) \in C_{O_S fil}(\mathcal{S})$ filtered τ fibrant.

2.3.2 Canonical flasque resolution of a presheaf on a site or a presheaf of module on a ringed topos

Let $\mathcal{S} \in \text{Cat}$ a site with topology τ . Denote $a_\tau : \text{PSh}(\mathcal{S}) \rightarrow \text{Shv}(\mathcal{S})$ the sheafification functor. There is for $F \in C(\mathcal{S})$ an explicit τ fibrant replacement :

- $k : F^\bullet \rightarrow E_\tau^\bullet(F^\bullet) := \text{Tot}(E_\tau^\bullet(F^\bullet))$, if $F^\bullet \in C^+(\mathcal{S})$,
- $k : F^\bullet \rightarrow E_\tau^\bullet(F^\bullet) := \text{holim} \text{Tot}(E_\tau^\bullet(F^{\bullet \geq n}))$, if $F^\bullet \in C(\mathcal{S})$ is not bounded below.

The bicomplex $E^\bullet(F^\bullet) := E_\tau^\bullet(F^\bullet)$ together with the map $k : F^\bullet \rightarrow E^\bullet(F^\bullet)$ is given inductively by

- considering $p_S : \mathcal{S}^\delta \rightarrow \mathcal{S}$ the morphism of site from the discrete category \mathcal{S}^τ whose objects are the points of the topos \mathcal{S} and we take

$$k_0 := \text{ad}(p_S^*, p_{S*})(F^\bullet) \rightarrow E^0(F^\bullet) := p_{S*} p_S^* F^\bullet := \bigoplus_{s \in \mathcal{S}^\tau} \lim_{X \in \mathcal{S}, s \in X} F^\bullet(X),$$

then $a_\tau k_0 : a_\tau F^\bullet \rightarrow E^0(F^\bullet)$ is injective and $E^0(F^\bullet)$ is τ fibrant,

- denote $Q^0(F^\bullet) := a_\tau \text{coker}(k_0 : F^\bullet \rightarrow E^0(F^\bullet))$ and take the composite

$$E^0(F^\bullet) \rightarrow Q^0(F^\bullet) \rightarrow E^1(F^\bullet) := E^0(Q^0(F^\bullet)).$$

Note that $k : F^\bullet \rightarrow E^\bullet(F^\bullet)$ is a τ local equivalence and that $a_\tau k : a_\tau F^\bullet \rightarrow E^\bullet(F^\bullet)$ is injective by construction.

Since E^0 is functorial, E is functorial: for $m : F^\bullet \rightarrow G^\bullet$ a morphism, with $F^\bullet, G^\bullet \in C(\mathcal{S})$, we have a canonical morphism $E(m) : E(F) \rightarrow E(G)$ such that $E(m) \circ k = k' \circ m$, with $k : F \rightarrow E(F)$ and $k' : G \rightarrow E(G)$. Note that E^0 , hence E preserve monomorphisms. This gives, for $(F^\bullet, F) \in C_{fil}(\mathcal{S})$, a filtered τ local equivalence $k : (F^\bullet, F) \rightarrow E^\bullet(F^\bullet, F)$ with $E^\bullet(F^\bullet, F)$ filtered τ fibrant.

Moreover, we have a canonical morphism $E(F) \otimes E(G) \rightarrow E(F \otimes G)$.

There is, for $g : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \text{Cat}$ two site, and $F^\bullet \in C(\mathcal{S})$, a canonical transformation

$$T(g, E)(F^\bullet) : g^* E(F^\bullet) \rightarrow E(g^* F^\bullet) \tag{17}$$

given inductively by

- $T(g, E^0)(F) := \overline{T(g, p_S)}(p_S^* F) : g^* E^0(F) = g^* p_{S*} p_S^* F \rightarrow p_{T*} g^* p_S^* F = p_{T*} p_T^* g^* F = E^0(g^* F)$,
 $T(g, Q^0)(F) := \overline{T(g, E^0)}(F) : g^* Q^0(F) = \text{coker}(g^* F \rightarrow g^* E^0(F)) \rightarrow Q^0(g^* F) = \text{coker}(g^* F \rightarrow E^0(g^* F))$

- $T(g, Q^1)(F) : g^* E^1(F) = g^* E^0(Q^0(F)) \xrightarrow{T(g, E^0)(Q^0(F))} E^0(g^* Q^0(F)) \xrightarrow{E^0(T(g, Q^0)(F))} E^0(Q^0(g^* F)) = E^1(g^* F)$.

Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ with topology τ . Then, for $F^\bullet \in C_{O_{\mathcal{S}}}(\mathcal{S})$, $E_\tau(F^\bullet)$ is naturally a complex of $O_{\mathcal{S}}$ modules such that $k : F^\bullet \rightarrow E_\tau(F^\bullet)$ is $O_{\mathcal{S}}$ linear, that is a morphism in $C_{O_{\mathcal{S}}}(\mathcal{S})$.

2.3.3 Canonical projective resolution of a presheaf of module on a ringed topos

Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$. We recall that we denote by, for $U \in \mathcal{S}$, $\mathbb{Z}(U) \in \text{PSh}(\mathcal{S})$ the presheaf represented by U : for $V \in \mathcal{S}$ $\mathbb{Z}(U)(V) = \mathbb{Z} \text{Hom}(V, U)$, and for $h : V_1 \rightarrow V_2$ a morphism in \mathcal{S} , and $h_1 : V_1 \rightarrow U$ $\mathbb{Z}(U)(h) : h_1 \rightarrow h \circ h_1$, and s is the morphism of presheaf given by $s(V_1)(h_1) = F(h_1)(s) \in F(V_1)$. There is for $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ a complex of $O_{\mathcal{S}}$ module an explicit projective replacement :

- $q : L_O^\bullet(F^\bullet) := \text{Tot}(L_O^\bullet(F^\bullet)) \rightarrow F^\bullet$, if $F^\bullet \in C^-(\mathcal{S})$,
- $q : L_O^\bullet(F^\bullet) := \text{holim Tot}(L_O^\bullet(F^{\bullet \leq n}))$ if $F^\bullet \in C(\mathcal{S})$ is not bounded above.

For $O_{\mathcal{S}} = \mathbb{Z}_{\mathcal{S}}$, we denote $L_{\mathbb{Z}_{\mathcal{S}}}^\bullet(F^\bullet) =: L(F^\bullet)$. The bicomplex $L_O^\bullet(F^\bullet)$ together with the map $q : L_O^\bullet(F^\bullet) \rightarrow F^\bullet$ is given inductively by

- considering the pairs $\{U \in \mathcal{S}, s \in F(U)\}$, where U is an object of \mathcal{S} and s a section of F over U we take

$$q_0 : L_O^0(F) := \bigoplus_{(U \in \mathcal{S}, s \in F(U))} \mathbb{Z}(U) \otimes O_{\mathcal{S}} \xrightarrow{s} F,$$

then q_0 is surjective and $L_O^0(F)$ is projective, this construction is functorial : for $m : F \rightarrow G$ a morphism in $\text{PSh}(\mathcal{S})$ the following diagram commutes

$$\begin{array}{ccc} \bigoplus_{(U \in \mathcal{S}, s \in F(U))} \mathbb{Z}(U) \otimes O_{\mathcal{S}} & \xrightarrow{q_0} & F \\ \downarrow L_O(m) & & \downarrow m \\ \bigoplus_{(U \in \mathcal{S}, s' \in G(U))} \mathbb{Z}(U) \otimes O_{\mathcal{S}} & \xrightarrow{q_0} & G \end{array}$$

where $(L_O(m)|_{(U,s)})(U, m(U)(s)) = I_{\mathbb{Z}(U)}$ and $(L_O(m)|_{(U,s)})(U, s') = 0$ if $s' \neq m(U)(s)$,

- denote $K_O^0(F) := \ker(q_0 : L_O^0(F) \rightarrow F)$ and take the composite

$$q_1 : L_O^1(F^\bullet) := L_O^0(K_O^0(F^\bullet)) \xrightarrow{q_0(K_O^0(F^\bullet))} K_O^0(F^\bullet) \hookrightarrow L_O^0(F^\bullet).$$

Note that $q = q(F) : L(F^\bullet) \rightarrow F^\bullet$ is a surjective quasi-isomorphism by construction. Since L_O^0 is functorial, L_O is functorial : for $m : F^\bullet \rightarrow G^\bullet$ a morphism, with $F^\bullet, G^\bullet \in C(\mathcal{S})$, we have a canonical morphism $L_O(m) : L_O(F) \rightarrow L_O(G)$ such that $q' \circ L_O(m) = m \circ q'$, with $q : L_O(F) \rightarrow F$ and $q' : L_O(G) \rightarrow G$. Note that L_O^0 and hence L_O preserve monomorphisms. In particular, it gives for $(F^\bullet, F) \in C_{O_{\mathcal{S}}}(\mathcal{S})$, a filtered quasi-isomorphism $q : L_O(F^\bullet, F) \rightarrow (F^\bullet, F)$. Moreover, we have a canonical morphism $L_O(F) \otimes L_O(G) \rightarrow L_O(F \otimes G)$.

Let $g : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite with $\mathcal{T}, \mathcal{S} \in \text{Cat}$ two sites.

- Let $F^\bullet \in C(\mathcal{S})$. Since $g^* L(F^\bullet)$ is projective and $q(g^* F) : L(g^* F^\bullet) \rightarrow g^* F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation

$$T(g, L)(F^\bullet) : g^* L(F^\bullet) \rightarrow L(g^* F^\bullet) \tag{18}$$

unique up to homotopy such that $q(g^* F) \circ T(g, L)(F^\bullet) = g^* q(F)$.

- Let $F^\bullet \in C(\mathcal{S})$. Since $L(g^*F^\bullet)$ is projective and $g^*q(F) : g^*L(F^\bullet) \rightarrow g^*F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation

$$T(g, L)(F^\bullet) : L(g^*F^\bullet) \rightarrow g^*L(F^\bullet) \quad (19)$$

unique up to homotopy such that $g^*q(F) \circ T(g, L)(F^\bullet) = q(g^*F)$.

- Let $F^\bullet \in C(\mathcal{T})$. Since $L(g_*F^\bullet)$ is projective and $g_*q(F) : g_*L(F^\bullet) \rightarrow g_*F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation

$$T_*(g, L)(F^\bullet) : L(g_*F^\bullet) \rightarrow g_*L(F^\bullet) \quad (20)$$

unique up to homotopy such that $g_*q(F) \circ T_*(g, L)(F^\bullet) = q(g_*F)$.

Let $g : (\mathcal{T}, O_{\mathcal{T}}) \rightarrow (\mathcal{S}, O_{\mathcal{S}})$ a morphism with $(\mathcal{T}, O_{\mathcal{T}}), (\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$. Let $F^\bullet \in C_{O_{\mathcal{S}}}(\mathcal{S})$. Since $g^{*mod}L_O(F^\bullet)$ is projective and $q(g^{*mod}F) : L_O(g^{*mod}F^\bullet) \rightarrow g^{*mod}F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation

$$T(g, L_O)(F^\bullet) : g^{*mod}L_O(F^\bullet) \rightarrow L_O(g^{*mod}F^\bullet) \quad (21)$$

unique up to homotopy such that $q(g^{*mod}F) \circ T(g, L_O)(F^\bullet) = g^{*mod}q(F)$.

Let $p : (\mathcal{S}_{12}, O_{\mathcal{S}_{12}}) \rightarrow (\mathcal{S}_1, O_{\mathcal{S}_1})$ a morphism with $(\mathcal{S}_{12}, O_{\mathcal{S}_{12}}), (\mathcal{S}_1, O_{\mathcal{S}_1}) \in \text{RCat}$, such that the structural morphism $p^*O_{\mathcal{S}_1} \rightarrow O_{\mathcal{S}_{12}}$ is flat. Let $F^\bullet \in C_{O_{\mathcal{S}}}(\mathcal{S})$. Since $L_O(p^{*mod}F^\bullet)$ is projective and $p^{*mod}q(F) : p^{*mod}L_O(F^\bullet) \rightarrow p^{*mod}F^\bullet$ is a surjective quasi-isomorphism, there is also in this case a canonical transformation

$$T(p, L_O)(F^\bullet) : L_O(p^{*mod}F^\bullet) \rightarrow p^{*mod}L_O(F^\bullet) \quad (22)$$

unique up to homotopy such that $p^{*mod}q(F) \circ T(p, L_O)(F^\bullet) = q(p^{*mod}F)$.

2.3.4 The De Rham complex of a ringed topos and functorialities

Let $A \in \text{cRing}$ a commutative ring. For $M \in \text{Mod}(A)$, we denote by

$$\text{Der}_A(A, M) \subset \text{Hom}(A, M) = \text{Hom}_{\text{Ab}}(A, M)$$

the abelian subgroup of derivation. Denote by $I_A = \ker(s_A : A \otimes A \rightarrow A) \subset A \otimes A$ the diagonal ideal with $s_A(a_1, a_2) = a_1 - a_2$. Let $\Omega_A := I_A/I_A^2 \in \text{Mod}(A)$ together with its derivation map $d = d_A : A \rightarrow \Omega_A$. Then, for $M \in \text{Mod}(A)$ the canonical map

$$w(M) : \text{Hom}_A(\Omega_A, M) \xrightarrow{\sim} \text{Der}_A(A, M), \psi \mapsto \phi \circ d$$

is an isomorphism, that is Ω_A is the universal derivation. In particular, its dual $T_A := \mathbb{D}^A(\Omega_A) = D^A(I_A/I_A^2)$ is isomorphic to the derivations group : $w(A) : T_A \xrightarrow{\sim} \text{Der}_A(A, A)$. Also note that $\text{Der}_A(A, A) \subset \text{Hom}(A, A)$ is a Lie subalgebra. If $\phi : A \rightarrow B$ is a morphism of commutative ring, we have a canonical morphism of abelian group $\Omega_{(B/A)}\phi : \Omega_A \rightarrow \Omega_B$.

Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$, with $O_{\mathcal{S}} \in \text{PSh}(\mathcal{S}, \text{cRing})$ commutative. For $G \in \text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$, we denote by

$$\text{Der}_{O_{\mathcal{S}}}(O_{\mathcal{S}}, G) \subset \mathcal{H}om(O_{\mathcal{S}}, G) = \mathcal{H}om_{\text{Ab}}(O_{\mathcal{S}}, G)$$

the abelian subsheaf of derivation. Denote by $\mathcal{I}_{\mathcal{S}} = \ker(s_{\mathcal{S}} : O_{\mathcal{S}} \otimes O_{\mathcal{S}} \rightarrow O_{\mathcal{S}}) \in \text{PSh}_{O_{\mathcal{S}} \times O_{\mathcal{S}}}(\mathcal{S})$ the diagonal ideal with $s_{\mathcal{S}}(X) = s_{O_{\mathcal{S}}}(X)$ for $X \in \mathcal{S}$. Then $\Omega_{O_{\mathcal{S}}} := \mathcal{I}_{\mathcal{S}}/\mathcal{I}_{\mathcal{S}}^2 \in \text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$ together with its derivation map $d : O_{\mathcal{S}} \rightarrow \Omega_{O_{\mathcal{S}}}$ is the universal derivation $O_{\mathcal{S}}$ -module : the canonical map

$$w(G) : \mathcal{H}om_{O_{\mathcal{S}}}(\Omega_{O_{\mathcal{S}}}, G) \xrightarrow{\sim} \text{Der}_{O_{\mathcal{S}}}(O_{\mathcal{S}}, G), \phi \mapsto \phi \circ d$$

is an isomorphism. In particular, its dual $T_{O_S} := \mathbb{D}_{O_S}^O(\Omega_{O_S}) = \mathbb{D}_S^O(\mathcal{I}_S/\mathcal{I}_S^2)$ is isomorphic to the presheaf of derivations $w(O_S) : T_{O_S} \xrightarrow{\sim} \text{Der}_{O_S}(O_S, O_S)$ and $\text{Der}_{O_S}(O_S, O_S) \subset \text{Hom}(O_S, O_S)$ is a Lie subalgebra. The universal derivation $d = d_{O_S} : O_S \rightarrow \Omega_{O_S}$ induces the De Rham complex

$$DR(O_S) : \Omega_S^\bullet := \wedge^\bullet \Omega_{O_S} \in C(\mathcal{S})$$

A morphism $\phi : O'_S \rightarrow O_S$ with $O_S, O'_S \in \text{PSh}(\mathcal{S}, \text{cRing})$ induces by the universal property canonical morphisms

$$\Omega_{O'_S/O_S} : \Omega_{O'_S} \rightarrow \Omega_{O_S}, \quad \mathbb{D}_{O_S}^O \Omega_{O'_S/O_S} : T_{O_S} \rightarrow T_{O'_S}$$

in $\text{PSh}_{O_S}(\mathcal{S})$.

- In the particular cases where $S = (S, O_S) \in \text{Var}(\mathbb{C})$ or $S = (S, O_S) \in \text{AnSp}(\mathbb{C})$, we denote as usual $\Omega_S := \Omega_{O_S/\mathbb{C}_S}$, $T_S := T_{O_S/\mathbb{C}_S}$ and $DR(S) := DR(O_S/\mathbb{C}_S) : \Omega_S^\bullet \in C(S)$.
- In the particular cases where $S = (S, O_S) \in \text{Diff}(\mathbb{R})$ is a differential manifold, we denote as usual $\mathcal{A}_S := \Omega_{O_S/\mathbb{R}_S}$, $T_S := T_{O_S/\mathbb{R}_S}$ and $DR(S) := DR(O_S/\mathbb{R}_S) : \mathcal{A}_S^\bullet \in C(S)$.

For $f : (\mathcal{X}, O_X) \rightarrow (S, O_S)$ with $(\mathcal{X}, O_X), (S, O_S) \in \text{RCat}$ such that O_X and O_S are commutative, we denote by

$$\Omega_{O_X/f^*O_S} := \text{coker}(\Omega_{O_X/f^*O_S} : \Omega_{f^*O_S} \rightarrow \Omega_{O_X}) \in \text{PSh}_{f^*O_S}(\mathcal{X})$$

the relative cotangent sheaf. The surjection $q = q_{O_X/f} : \Omega_{O_X} \rightarrow \Omega_{O_X/f^*O_S}$ gives the derivation $w(\Omega_{O_X/f^*O_S})(q) = d_{O_X/f} : O_X \rightarrow \Omega_{O_X/f^*O_S}$. It induces the surjections $q^p := \wedge^p q : \Omega_{O_X}^p \rightarrow \Omega_{O_X/f^*O_S}^p$. We then have the relative De Rham complex

$$DR(O_X/f^*O_S) := \Omega_{O_X/f^*O_S}^\bullet := \wedge^\bullet \Omega_{O_X/f^*O_S} \in C_{f^*O_S}(\mathcal{X}).$$

whose differentials are given by

$$\text{for } X^o \in \mathcal{X} \text{ and } \omega \in \Gamma(X^o, \Omega_{O_X}^p) \quad d(q^p(\omega)) := q^{p+1}(d(\omega))$$

Note that $\Omega_{O_X/f^*O_S}^\bullet \in C_{f^*O_S}(\mathcal{S})$ is a complex of f^*O_S modules, but is NOT a complex of O_X module since the differential is a derivation hence NOT O_X linear. On the other hand, the canonical map in $\text{PSh}_{f^*O_S}(\mathcal{S})$

$$T(f, \text{hom})(O_S, O_S) : f^*\mathcal{H}om(O_S, O_S) \rightarrow \mathcal{H}om(f^*O_S, f^*O_S)$$

induces morphisms

$$T(f, \text{hom})(O_S, O_S) : f^*T_{O_S} \rightarrow T_{f^*O_S} \text{ and } \mathbb{D}_{f^*O_S}^O T(f, \text{hom})(O_S, O_S) : \Omega_{f^*O_S} \rightarrow f^*\Omega_{O_S}.$$

In this article, We will be interested in the following particular cases :

- In the particular case where $O_S \in \text{PSh}(\mathcal{S}, \text{cRing})$ is a sheaf, $\Omega_{O_S}, T_{O_S} \in \text{PSh}_{O_S}(\mathcal{S})$ are sheaves. Hence,

$$T(f, \text{hom})(O_S, O_S) : a_\tau f^*T_{O_S} \xrightarrow{\sim} T_{a_\tau f^*O_S} \text{ and } \mathbb{D}_{f^*O_S}^O T(f, \text{hom})(O_S, O_S) : \Omega_{a_\tau f^*O_S} \xrightarrow{\sim} a_\tau f^*\Omega_{O_S}$$

are isomorphisms, where $a_\tau : \text{PSh}(\mathcal{S}) \rightarrow \text{Shv}(\mathcal{S})$ is the sheafification functor. We will note again in this case by abuse (as usual) $f^*O_S := a_\tau f^*O_S$, $f^*\Omega_{O_S} := a_\tau f^*\Omega_{O_S}$ and $f^*T_{O_S} := a_\tau f^*T_{O_S}$, so that

$$\Omega_{f^*O_S} = f^*\Omega_{O_S} \text{ and } f^*T_{O_S} = T_{f^*O_S}$$

- In the particular cases where $S = (S, O_S), X = (X, O_X) \in \text{Var}(\mathbb{C})$ or $S = (S, O_S), X = (X, O_X) \in \text{AnSp}(\mathbb{C})$, we denote as usual $\Omega_{X/S} := \Omega_{O_X/f^*O_S}$, $q_{X/S} := q_{O_X/S} : \Omega_X \rightarrow \Omega_{X/S}$ and $DR(X/S) := DR(O_X/f^*O_S) : \Omega_{X/S}^\bullet \in C_{f^*O_S}(S)$.
- In the particular cases where $S = (S, O_S), X = (X, O_X) \in \text{Diff}(\mathbb{R})$, we denote as usual $\mathcal{A}_{X/S} := \Omega_{O_X/f^*O_S}$, $q_{X/S} := q_{O_X/S} : \mathcal{A}_X \rightarrow \mathcal{A}_{X/S}$ and $DR(X/S) := DR(O_X/f^*O_S) : \mathcal{A}_{X/S}^\bullet \in C_{f^*O_S}(S)$.

Definition 1. For a commutative diagram in \mathbf{RCat}

$$D = \begin{array}{ccc} (\mathcal{X}, O_X) & \xrightarrow{f} & (\mathcal{S}, O_S) \\ g' \uparrow & & \uparrow g \\ (\mathcal{X}', O_{X'}) & \xrightarrow{f'} & (\mathcal{T}, O_T) \end{array}$$

whose structural presheaves are commutative sheaves, the map in $C_{g'^*O_X \text{fil}}(\mathcal{X}')$

$$\Omega_{O_{X'}/g'^*O_X} : g'^*(\Omega_{O_X}^\bullet, F_b) = (\Omega_{g'^*O_X}^\bullet, F_b) \rightarrow (\Omega_{O_{X'}}^\bullet, F_b)$$

pass to quotient to give the map in $C_{g'^*O_X \text{fil}}(\mathcal{X}')$

$$\begin{aligned} \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)} &:= (\Omega_{O_{X'}/g'^*O_X})^q : \\ g'^*(\Omega_{O_X/f^*O_S}^\bullet, F_b) &= (\Omega_{g'^*O_X/g'^*f^*O_S}^\bullet, F_b) \rightarrow (\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b) \end{aligned}$$

It is in particular given for $X'^o \in \mathcal{X}'$, $g^*(X^o) \leftarrow X'^o$ and $\hat{\omega} \in \Gamma(X^o, \Omega_{O_X/f^*O_S}^p)$,

$$\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(X'^o)(\omega) := q_{O_{X'}/f'}(\Omega_{O_{X'}/g'^*O_X}(\omega)) \in \Gamma(X'^o, \Omega_{O_{X'}/f'^*O_T}^p).$$

where $\omega \in \Gamma(X^o, \Omega_{O_X}^p)$ such that $q_{O_X/f}(\omega) = \hat{\omega}$. We then have the following canonical transformation map in $C_{O_T \text{fil}}(\mathcal{T})$

$$\begin{aligned} T_\omega^O(D) : g^{*mod} L_{O_*} E(\Omega_{O_X/f^*O_S}^\bullet, F_b) &\xrightarrow{q} g^* f_* E(\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{g^*O_S} O_T \\ &\xrightarrow{T(g', E)(-) \circ T(D)(E(\Omega_{O_X/f^*O_S}^\bullet))} (f'_* E(g'^*(\Omega_{O_X/f^*O_S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{E(\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)})} f'_* E(\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b) \otimes_{g^*O_S} O_T \xrightarrow{m} f'_* E(\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b), \end{aligned}$$

with $m(n \otimes s) = s.n$.

2.4 Presheaves on diagrams of sites or on diagrams of ringed tops

Let $\mathcal{I}, \mathcal{I}' \in \mathbf{Cat}$ and $(f_\bullet, s) : \mathcal{T}_\bullet \rightarrow \mathcal{S}_\bullet$ a morphism of diagrams of presites with $\mathcal{T}_\bullet \in \mathbf{Fun}(\mathcal{I}, \mathbf{Cat})$, $\mathcal{S}_\bullet \in \mathbf{Fun}(\mathcal{I}', \mathbf{Cat})$. Recall it is by definition given by a functor $s : \mathcal{I} \rightarrow \mathcal{I}'$ and morphism of functor $P(f_\bullet) : \mathcal{S}_{s(\bullet)} := \mathcal{S}_\bullet \circ s \rightarrow \mathcal{T}_\bullet$ and that we denote for short, $\mathcal{S}_{s(\bullet)} := \mathcal{S}_\bullet \circ s \in \mathbf{Fun}(\mathcal{I}, \mathbf{Cat})$. Recall that, for $r_{IJ} : I \rightarrow J$ a morphism, with $I, J \in \mathcal{I}$, $D_{f_{IJ}}$ is the commutative diagram in \mathbf{Cat}

$$D_{f_{IJ}} := \begin{array}{ccc} \mathcal{S}_{s(J)} & \xrightarrow{r_{IJ}^s} & \mathcal{S}_{s(I)} \\ f_J \uparrow & & \uparrow f_I \\ \mathcal{T}_J & \xrightarrow{r_{IJ}^t} & \mathcal{T}_I \end{array}$$

The adjunction

$$\begin{aligned} ((f_\bullet, s)^*, (f_\bullet, s)_*) &= ((f_\bullet, s)^{-1}, (f_\bullet, s)_*) : C(\mathcal{S}_{s(\bullet)}) \rightleftarrows C(\mathcal{T}_\bullet), \\ G = (G_I, u_{IJ}) &\mapsto (f_\bullet, s)^*(G) := (f_I^*(G_I), T(D_{f_{IJ}})(G_J) \circ f_I^* u_{IJ}) \\ G = (G_I, u_{IJ}) &\mapsto (f_\bullet, s)_*(G) := (f_{I*}(G_I), f_{I*} u_{IJ}) \end{aligned}$$

gives an adjunction

$$\begin{aligned} ((f_\bullet, s)^*, (f_\bullet, s)_*) &: C_{(2)\text{fil}}(\mathcal{S}_{s(\bullet)}) \rightleftarrows C_{(2)\text{fil}}(\mathcal{T}_\bullet), \\ (G, F) = ((G_I, F), u_{IJ}) &\mapsto (f_\bullet, s)^*(G, F) := (f_I^*(G_I, F), T(D_{f_{IJ}})(G_J, F) \circ f_I^* u_{IJ}) \\ (G, F) = ((G_I, F), u_{IJ}) &\mapsto (f_\bullet, s)_*(G, F) := (f_{I*}(G_I, F), f_{I*} u_{IJ}). \end{aligned}$$

For a commutative diagram of diagrams of presite :

$$D = \begin{array}{ccc} \mathcal{T}' & \xrightarrow{(g_2, s'_2)} & \mathcal{S}'_{s'_2(\bullet)} \\ \downarrow (f_2, s_2) & & \downarrow (f_1, s_1) \\ \mathcal{T}_{s_2(\bullet)} & \xrightarrow{(g_1, s'_1)} & \mathcal{S}_{s(\bullet)} \end{array} ,$$

with $\mathcal{I}, \mathcal{I}', \mathcal{J}, \mathcal{J}' \in \text{Cat}$ and $\mathcal{T}_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat}), \mathcal{T}'_\bullet \in \text{Fun}(\mathcal{I}', \text{Cat}), \mathcal{S}_\bullet \in \text{Fun}(\mathcal{J}, \text{Cat}), \mathcal{S}'_\bullet \in \text{Fun}(\mathcal{J}', \text{Cat})$, and $s = s_1 \circ s'_2 = s_2 \circ s'_1 : \mathcal{I}' \rightarrow \mathcal{J}$, we denote by, for $F = (F_I, u_{IJ}) \in C(\mathcal{S}'_{s'_2(\bullet)})$,

$$T(D)(F) : g_1^* f_{1*} F \xrightarrow{g_1^* f_{1*} \text{ad}(g_2^*, g_{2*})(F)} g_1^* f_{1*} g_{2*} g_2^* F = g_1^* g_{1*} f_{2*} g_2^* F \xrightarrow{\text{ad}(g_1^* g_{1*})(f_{2*} g_2^* F)} f_{2*} g_2^* F$$

the canonical transformation map in $C(\mathcal{T}_{s_2(\bullet)})$, and for $(G, F) = ((G_I, F), u_{IJ}) \in C_{fil}(\mathcal{S}'_{s'_2(\bullet)})$,

$$T(D)(G, F) : g_1^* f_{1*}(G, F) \xrightarrow{g_1^* f_{1*} \text{ad}(g_2^*, g_{2*})(G, F)} g_1^* f_{1*} g_{2*} g_2^*(G, F) = g_1^* g_{1*} f_{2*} g_2^*(G, F) \xrightarrow{\text{ad}(g_1^* g_{1*})(f_{2*} g_2^*(G, F))} f_{2*} g_2^*(G, F)$$

the canonical transformation map in $C_{fil}(\mathcal{T}_{s_2(\bullet)})$ given by the adjonction maps.

Let $\mathcal{S}_\bullet \in \text{Fun}(\mathcal{I}, \text{RCat})$ a diagram of ringed topos with $\mathcal{I} \in \text{Cat}$. We have the tensor product bifunctor

$$\begin{aligned} (\cdot) \otimes (\cdot) : \text{PSh}(\mathcal{S}_\bullet)^2 &\rightarrow \text{PSh}(\mathcal{S}_\bullet), \\ ((F_I, u_{IJ}), (G_I, u_{IJ})) &\mapsto (F_I, u_{IJ}) \otimes (G_I, u_{IJ}) := (F_I \otimes G_I, u_{IJ} \otimes v_{IJ}) \end{aligned}$$

We get the bifunctors

$$(-) \otimes (-) : C_{fil}(\mathcal{S}_\bullet)^2 \rightarrow C_{fil}(\mathcal{S}_\bullet), \quad (-) \otimes (-) : C_{fil}(\mathcal{S}_\bullet) \times C_{O_S fil}(\mathcal{S}_\bullet) \rightarrow C_{O_S fil}(\mathcal{S}_\bullet).$$

We have the tensor product bifunctor

$$\begin{aligned} (\cdot) \otimes_{O_S} (\cdot) : \text{PSh}_{O_S}(\mathcal{S}_\bullet)^2 &\rightarrow \text{PSh}(\mathcal{S}_\bullet), \\ ((F_I, u_{IJ}), (G_I, u_{IJ})) &\mapsto (F_I, u_{IJ}) \otimes_{O_S} (G_I, u_{IJ}) := (F_I \otimes_{O_S} G_I, u_{IJ} \otimes v_{IJ}) \end{aligned}$$

which gives,

- in all case it gives the bifunctor $(-) \otimes_{O_S} (-) : C_{O_S^p fil}(\mathcal{S}_\bullet) \otimes C_{O_S fil}(\mathcal{S}_\bullet) \rightarrow C_{fil}(\mathcal{S}_\bullet)$.
- in the case O_S is commutative, it gives the bifunctor $(-) \otimes_{O_S} (-) : C_{O_S fil}(\mathcal{S}_\bullet)^2 \rightarrow C_{O_S fil}(\mathcal{S}_\bullet)$.

Let $(f_\bullet, s) : (\mathcal{T}_\bullet, O_T) \rightarrow (\mathcal{S}_\bullet, O_S)$ a morphism with $(\mathcal{S}_\bullet, O_S) \in \text{Fun}(\mathcal{I}', \text{RCat}), (\mathcal{T}_\bullet, O_T) \in \text{Fun}(\mathcal{I}, \text{RCat})$ and $\mathcal{I}, \mathcal{I}' \in \text{Cat}$. which is by definition given by a functor $s : \mathcal{I} \rightarrow \mathcal{I}'$ and morphism of ringed topos $f_\bullet : (\mathcal{T}_\bullet, O_T) \rightarrow (\mathcal{S}_{s(\bullet)}, O_S)$. As before, we denote for short, $(\mathcal{S}_{s(\bullet)}, O_S) := (\mathcal{S}_\bullet, O_S) \circ s \in \text{Fun}(\mathcal{I}, \text{RCat})$. Denote as before, for $r_{IJ} : I \rightarrow J$ a morphism, with $I, J \in \mathcal{I}$, D_{fIJ} the commutative diagram in RCat

$$D_{fIJ} := \begin{array}{ccc} \mathcal{S}_{s(J)} & \xrightarrow{r_{IJ}^s} & \mathcal{S}_{s(I)} \\ f_J \uparrow & & \uparrow f_I \\ \mathcal{T}_J & \xrightarrow{r_{IJ}^t} & \mathcal{T}_I \end{array} .$$

We have then the adjonction

$$\begin{aligned} ((f_\bullet, s)^{*mod}, (f_\bullet, s)_*) : C_{O_S}(\mathcal{S}_{s(\bullet)}) &\rightleftarrows C_{O_T}(\mathcal{T}_\bullet), \\ (G_I, u_{IJ}) \mapsto (f_\bullet, s)^{*mod}(G_I, u_{IJ}) &:= (f_I^{*mod} G_I, T^{mod}(D_{fIJ})(G_J) \circ f_I^{*mod} u_{IJ}), \\ (G_I, u_{IJ}) \mapsto (f_\bullet, s)_*(G_I, u_{IJ}) &:= (f_{I*} G_I, f_{I*} u_{IJ}). \end{aligned}$$

which induces the adjunction

$$\begin{aligned} ((f_\bullet, s)^{*mod}, (f_\bullet, s)_*) : C_{O_S fil}(\mathcal{S}_s(\bullet)) &\rightleftarrows C_{O_{\mathcal{T}} fil}(\mathcal{T}_\bullet), \\ ((G_I, F), u_{IJ}) \mapsto (f_\bullet, s)^{*mod}((G_I, F), u_{IJ}) &:= (f_I^{*mod}(G_I, F), T^{mod}(D_{f_{IJ}})(G_J) \circ f_I^{*mod}u_{IJ}), \\ ((G_I, F), u_{IJ}) \mapsto (f_\bullet, s)_*((G_I, F), u_{IJ}) &:= (f_{I*}(G_I, F), f_{I*}u_{IJ}). \end{aligned}$$

For a commutative diagram of diagrams of ringed topoi, :

$$D = \begin{array}{ccc} (\mathcal{T}'_\bullet, O'_2) & \xrightarrow{(g_2, s'_2)} & (\mathcal{S}'_{s'_2}(\bullet), O'_1) \\ \downarrow (f_2, s_2) & & \downarrow (f_1, s_1) \\ (\mathcal{T}_\bullet, O_2) & \xrightarrow{(g_1, s'_1)} & (\mathcal{S}_{s_1}(\bullet), O_1) \end{array}$$

with $\mathcal{I}, \mathcal{I}', \mathcal{J}, \mathcal{J}' \in \text{Cat}$ and $\mathcal{T}_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat})$, $\mathcal{T}'_\bullet \in \text{Fun}(\mathcal{I}', \text{Cat})$, $\mathcal{S}_\bullet \in \text{Fun}(\mathcal{J}, \text{Cat})$, $\mathcal{S}'_\bullet \in \text{Fun}(\mathcal{J}', \text{Cat})$, and $s = s_1 \circ s'_2 = s'_1 \circ s_2 : \mathcal{I}' \rightarrow \mathcal{J}$, we denote by, for $F = (F_I, u_{IJ}) \in C_{O'_1}(\mathcal{S}'_{s'_2}(\bullet))$,

$$T^{mod}(D)(F) : g_1^{*mod} f_{1*} F \xrightarrow{g_1^{*mod} f_{1*} \text{ad}(g_2^{*mod}, g_{2*})(F)} g_1^{*mod} f_{1*} g_{2*} g_2^{*mod} F = g_1^{*mod} g_{1*} f_{2*} g_2^{*mod} F \xrightarrow{\text{ad}(g_1^{*mod} g_{1*})(f_{2*} g_2^{*mod} F)} f_{2*} g_2^{*mod} F$$

the canonical transformation map in $C_{O_2}(\mathcal{T}_{s_2}(\bullet))$, and for $G = ((G_I, F), u_{IJ}) \in C_{O'_1 fil}(\mathcal{S}'_{s'_2}(\bullet))$,

$$T^{mod}(D)(G, F) : g_1^{*mod} f_{1*}(G, F) \xrightarrow{g_1^{*mod} f_{1*} \text{ad}(g_2^{*mod}, g_{2*})(G, F)} g_1^{*mod} f_{1*} g_{2*} g_2^{*mod}(G, F) = g_1^{*mod} g_{1*} f_{2*} g_2^{*mod}(G, F) \xrightarrow{\text{ad}(g_1^{*mod} g_{1*})(f_{2*} g_2^{*mod}(G, F))} f_{2*} g_2^{*mod}(G, F)$$

the canonical transformation map in $C_{O_2 fil}(\mathcal{T}_{s_2}(\bullet))$ given by the adjunction maps.

Let $(\mathcal{S}_\bullet, O_S) \in \text{Fun}(\mathcal{I}, \text{RCat})$ a diagram of ringed topoi with $\mathcal{I} \in \text{Cat}$ and, for $I \in \mathcal{I}$, \mathcal{S}_I is endowed with topology τ_I and for $r : I \rightarrow J$ a morphism with $I, J \in \mathcal{I}$, $r_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ is continuous. Then the diagram category $(\Gamma \mathcal{S}_\bullet, O_S) \in \text{RCat}$ is endowed with the associated canonical topology τ , and then

- A morphism $\phi = (\phi_I) : ((F_I, F), u_{IJ}) \rightarrow ((G_I, F), u_{IJ})$ with $((F_I, F), u_{IJ}), ((G_I, F), u_{IJ}) \in C_{O_S fil}(\mathcal{S}_\bullet)$ is a filtered τ local equivalence if and only if the ϕ_I are filtered τ local equivalences for all $I \in \mathcal{I}$.
- Let $r \in \mathbb{N}$. A morphism $\phi = (\phi_I) : ((F_I, F), u_{IJ}) \rightarrow ((G_I, F), v_{IJ})$ with $((F_I, F), u_{IJ}), ((G_I, F), v_{IJ}) \in C_{O_S fil}(\mathcal{S}_\bullet)$ is an r -filtered τ local equivalence if and only if there exists $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$, $0 \leq i \leq s$, with $((C_{iI}, F), u_{iIJ}) \in C_{O_S fil}(\mathcal{S}_\bullet)$, $((C_{0I}, F), u_{0IJ}) = ((F_I, F), u_{IJ})$ and $((C_{sI}, F), u_{sIJ}) = ((G_I, F), v_{IJ})$, such that

$$\phi = \phi_s \circ \dots \circ \phi_i \circ \dots \circ \phi_0 : ((F_I, F), u_{IJ}) \rightarrow ((G_I, F), v_{IJ})$$

and $\phi_i : (C_{iI}^\bullet, F) \rightarrow (C_{(i+1)I}^\bullet, F)$ either a filtered τ local equivalence or an r -filtered homotopy equivalence. the ϕ_I are r -filtered τ local equivalences for all $I \in \mathcal{I}$.

- A complex of presheaves $((G_I, F), u_{IJ}) \in C_{O_S fil}(\mathcal{S}_\bullet)$ is filtered τ fibrant if and only if the $(G_I, F) \in C_{O_S fil}(\mathcal{S}_I)$ are filtered τ fibrant for all $I \in \mathcal{I}$.
- Let $r \in \mathbb{N}$. A complex of presheaves $((G_I, F), u_{IJ}) \in C_{O_S fil}(\mathcal{S}_\bullet)$ is r -filtered τ fibrant if there exist an r -filtered homotopy equivalence $m : ((G_I, F), u_{IJ}) \rightarrow ((G'_I, F), u_{IJ})$ with $(G'_I, F) \in C_{O_S fil}(\mathcal{S}_\bullet)$ filtered τ fibrant.

2.5 Presheaves on topological spaces and presheaves of modules on a ringed spaces

In this subsection, we will consider the particular case of presheaves on topological spaces.

Let $f : T \rightarrow S$ a continuous map with $S, T \in \text{Top}$. We denote as usual the adjunction

$$(f^*, f_*) := (P(f)^*, P(f)_*) : \text{PSh}(S) \rightleftarrows \text{PSh}(T)$$

induced by the morphism of site given by the pullback functor

$$P(f) : \text{Ouv}(S) \rightarrow \text{Ouv}(T), (S^o \subset S) \mapsto P(f)(S^o) := S^o \times_S T \xrightarrow{\sim} f^{-1}(S^o) \subset T$$

Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : \text{PSh}_{\text{fil}}(S) \rightleftarrows \text{PSh}_{\text{fil}}(T), f^*(G, F) := (f^*G, f^*F)$$

Let $f : (T, O_T) \rightarrow (S, O_S)$ a morphism with $(S, O_S), (T, O_T) \in \text{Top}$. We have then the adjunction

$$(f^{*\text{mod}}, f_*) := (P(f)^{\text{mod}}, P(f)_*) : \text{PSh}_{O_S \text{fil}}(S) \rightleftarrows \text{PSh}_{O_T \text{fil}}(T), f^{*\text{mod}}(G, F) := f^*(G, F) \otimes_{f^*O_S} O_T$$

Recall $\text{CW} \subset \text{Top}$ is the full subcategory whose objects consists of CW complexes. Denote, for $n \in \mathbb{N}$, $\mathbb{I}^n := [0, 1]^n$, $S^n := \mathbb{I}^n / \partial \mathbb{I}^n \in \text{CW}$ and $\Delta^n \subset \mathbb{I}^n$ the n dimensional simplex. We get $\mathbb{I}^*, \Delta^* \in \text{Fun}(\Delta, \text{CW})$. Denote for $S \in \text{Top}$, $\Sigma_1 S := S \times \mathbb{I}^1 / ((\{0\} \times S) \cup (\{1\} \times S)) \in \text{Top}$.

- Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$. We have the mapping cylinder $\text{Cyl}(f) := (T \times \mathbb{I}^1) \sqcup_f S \in \text{Top}$ and the mapping cone $\text{Cone}(f) := (T \times \mathbb{I}^1) \sqcup_f S \in \text{Top}$. We have then the quotient map $q_f : \text{Cyl}(f) \rightarrow \text{Cone}(f)$ and a canonical retraction $r_f : \text{Cone}(f) \rightarrow \Sigma^1 T$
- Recall two morphisms $f, g : T \rightarrow S$ with $T, S \in \text{Top}$ are homotopic if there exist $H : T \times \mathbb{I}^1 \rightarrow S$ continuous such that $H \circ (I \times i_0) = f$ and $H \circ (I \times i_1) = g$. Then $K(\text{Top}) := \text{Ho}_{O_T^1}(\text{Top})$ is a triangulated category with distinguish triangle

$$T \xrightarrow{i_T} \text{Cyl}(f) \xrightarrow{q_f} \text{Cone}(f) \xrightarrow{r_f} \Sigma^1 T.$$

- For $X \in \text{Top}$, denote for $n \in \mathbb{N}$, $\pi_n(X) : \text{Hom}_{K(\text{Top})}(S^n, X)$ the homotopy groups. For $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$, we have for $n \in \mathbb{N}$ the morphisms of abelian groups

$$f_* : \pi_n(T) \rightarrow \pi_n(S), h \mapsto f \circ h$$

Recall two morphisms $f, g : T \rightarrow S$ with $T, S \in \text{Top}$ are weakly homotopic if $f_* = g_* : \pi_n(T) \rightarrow \pi_n(S)$ for all $n \in \mathbb{N}$.

- For $X \in \text{Top}$, denote by $C_*^{\text{sing}}(X) := \mathbb{Z} \text{Hom}(\Delta^*, X) \in C^-(\mathbb{Z})$ the complex of singular chains and by $C_{\text{sing}}^*(X) := \mathbb{D}^{\mathbb{Z}} C_*^{\text{sing}}(X) := \mathbb{D}^{\mathbb{Z}} \text{Hom}(\Delta^*, X) \in C^-(\mathbb{Z})$ the complex of singular cochains. For $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$, we have

- the morphism of complexes of abelian groups

$$f_* : C_*^{\text{sing}}(T) \rightarrow C_*^{\text{sing}}(S), \sigma \mapsto f \circ \sigma,$$

- the morphism of complexes of abelian groups

$$f^* := \mathbb{D}^{\mathbb{Z}} f_* : C_{\text{sing}}^*(T) \rightarrow C_{\text{sing}}^*(S), \alpha \mapsto f^* \alpha : (\sigma \mapsto f^* \alpha(\sigma) := \alpha(f \circ \sigma))$$

We denote by $C_{X,\text{sing}}^* \in C^+(X)$ the complex of presheaves of singular cochains given by,

$$(U \subset X) \mapsto C_{X,\text{sing}}^*(U) := C_{X,\text{sing}}^*(U) := C_{\text{sing}}^*(U) := \mathbb{D}^{\mathbb{Z}} \mathbb{Z} \text{Hom}(\Delta^*, U),$$

$$(j : U_2 \hookrightarrow U_1) \mapsto (j^* : C_{\text{sing}}^*(U_1) \rightarrow C_{\text{sing}}^*(U_2))$$

and by $c_X : \mathbb{Z}_X \rightarrow C_{X,\text{sing}}^*$ the inclusion map. For $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$, we have the morphism of complexes of presheaves

$$f^* : C_{S,\text{sing}}^* \rightarrow f_* C_{T,\text{sing}}^*$$

in $C(S)$.

Theorem 7. (i) *If two morphisms $f, g : T \rightarrow S$ with $T, S \in \text{Top}$ are weakly homotopic, then*

$$H^n(f_*) = H^n(g_*) : H_{n,\text{sing}}(T, \mathbb{Z}) := H^n C_*^{\text{sing}}(T) \rightarrow H_{n,\text{sing}}(S, \mathbb{Z}) := H^n C_*^{\text{sing}}(S).$$

(ii) *For $S \in \text{Top}$ there exists $CW(S) \in \text{CW}$ together with a morphism $L_S : CW(S) \rightarrow S$ which is a weakly homotopic equivalence, that is $L_{S*} : \pi_n(CW(S)) \xrightarrow{\sim} \pi_n(S)$ are isomorphisms of abelian groups for all $n \in \mathbb{N}$.*

(ii)' *For $f : T \rightarrow S$ a morphism, with $T, S \in \text{Top}$, and $L_S : CW(S) \rightarrow S$, $L_T : CW(T) \rightarrow T$ weakly homotopy equivalence with $CW(S), CW(T) \in \text{CW}$ there exist a morphism $L(f) : CW(T) \rightarrow CW(S)$ unique up to homotopy such that the following diagram in Top commutes*

$$\begin{array}{ccc} CW(S) & \xrightarrow{L_S} & S \\ L(f) \uparrow & & \uparrow f \\ CW(T) & \xrightarrow{L_T} & T \end{array}$$

In particular, for $S \in \text{Top}$, $CW(S)$ is unique up to homotopy.

Proof. See [15]. □

We have Kunneth formula for singular cohomology :

Proposition 8. *Let $X_1, X_2 \in \text{Top}$. Denote by $p_1 : X_1 \times X_2 \rightarrow X_1$ and $p_2 : X_1 \times X_2 \rightarrow X_2$ the projections. Then*

$$p_1^* \otimes p_2^* : C_{\text{sing}}^*(X_1) \otimes C_{\text{sing}}^*(X_2) \rightarrow C_{\text{sing}}^*(X_1 \times X_2)$$

is a quasi-isomorphism.

Proof. Standard (see [15] for example): follows from the fact that for all $p \in \mathbb{N}$, $H^n C_{\text{sing}}^*(\Delta^p) = 0$ for all $n \in \mathbb{Z}$. □

Remark 2. *By definition, $X \in \text{Top}$ is locally contractile if and only if the inclusion map $c_X : \mathbb{Z}_X \rightarrow C_{X,\text{sing}}^*$ is an equivalence top local. In this case it induce, by taking injective resolutions, for $n \in \mathbb{Z}$ isomorphisms*

$$H^n c_X^k : H^n(X, \mathbb{Z}_X) \xrightarrow{\sim} \mathbb{H}^n(X, C_{X,\text{sing}}^*) = H^n C_{\text{sing}}^*(X) =: H_{\text{sing}}^n(X, \mathbb{Z}).$$

We will use the following easy propositions :

Proposition 9. (i) *Let $(S, O_S) \in \text{RTop}$. Then, if $K^\bullet \in C_{O_S}^-(S)$ is a bounded above complex such that $K^n \in \text{PSh}_{O_S}(S)$ are locally free for all $n \in \mathbb{Z}$, and $\phi : F^\bullet \rightarrow G^\bullet$ is a top local equivalence with $F, G \in C_{O_S}(S)$, then $\phi \otimes I : F^\bullet \otimes_{O_S} L^\bullet \rightarrow G^\bullet \otimes_{O_S} L^\bullet$ is an equivalence top local.*

(ii) Let $f : (T, O_T) \rightarrow (S, O_S)$ a morphism with $(T, O_T), (S, O_S) \in \text{RTop}$. Then, if $K \in C_{O_S}^b(S)$ is a bounded complex such that $K^n \in \text{PSh}_{O_S}(S)$ are locally free for all $n \in \mathbb{Z}$, and $N \in C_{O_T}(T)$

$$k \circ T^{\text{mod}}(f, \otimes)(M, E(N)) : K \otimes_{O_S} f_* E(N) \rightarrow f_*((f^{*\text{mod}} K) \otimes_{O_T} E(N)) \rightarrow f_* E((f^{*\text{mod}} K) \otimes_{O_T} E(N))$$

is an equivalence top local.

Proof. Standard. \square

Proposition 10. Let $i : (Z, O_Z) \hookrightarrow (S, O_S)$ a closed embedding of ringed spaces, with $Z, S \in \text{Top}$. Then for $M \in C_{O_S}(S)$ and $N \in C_{i^*O_S}(Z)$,

$$T(i, \otimes)(M, N) : M \otimes_{O_S} i_* N \rightarrow i_*(i^* M \otimes_{i^*O_S} N)$$

is an equivalence top local.

Proof. Standard. Follows from the fact that $j^* i_* N = 0$. \square

We note the following :

Proposition 11. Let $(S, O_S) \in \text{Sch}$ such that $O_{S,s}$ are reduced local rings for all $s \in S$. For $s \in S$ consider $q : L_{O_{S,s}}(k(s)) \rightarrow k(s)$ the canonical projective resolution of the $O_{S,s}$ module $k(s) := O_{S,s}/m_s$ (the residual field) of $s \in S$. For $s \in S$ denote by $i_s : \{s\} \hookrightarrow S$ the embedding. Let $\phi : F \rightarrow G$ a morphism with $F, G \in C_{O_{S,c}}(S)$ i.e. such that $a_{\text{zar}} H^n F, a_{\text{zar}} H^n G \in \text{Coh}(S)$. If

$$i_s^* \phi \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) : i_s^* F \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) \rightarrow i_s^* G \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))$$

is a quasi-isomorphism for all $s \in S$, then $\phi : F \rightarrow G$ is an equivalence top local.

Proof. Let $s \in S$. Since tensorizing with $L_{i_s^* O_S}(k(s))$ is an exact functor, we have canonical isomorphism $\alpha(F), \alpha(G)$ fitting in a commutative diagram

$$\begin{array}{ccc} H^n(i_s^* F \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) & \xrightarrow{H^n(i_s^* \phi \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)))} & H^n(i_s^* G \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) \\ \downarrow \alpha(F) & & \downarrow \alpha(G) \\ i_s^*(H^n F) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) & \xrightarrow{i_s^*(H^n \phi) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))} & i_s^*(H^n G) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) \end{array}$$

Let $n \in \mathbb{Z}$. By hypothesis

$$H^n(i_s^* \phi \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) : H^n(i_s^* F \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) \xrightarrow{\sim} H^n(i_s^* G \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)))$$

is an isomorphism. Hence, the diagram 2.5 implies that

$$i_s^*(H^n \phi) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) : i_s^*(H^n F) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s)) \xrightarrow{\sim} i_s^*(H^n G) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))$$

is an isomorphism. We conclude on the one hand that $i_s^* H^n \phi : i_s^* H^n F \rightarrow i_s^* H^n G$ is surjective by Nakayama lemma since $i_s^* H^n F, i_s^* H^n G$ are $O_{S,s}$ modules of finite type as $F, G \in C_{O_{S,c}}(S)$ has coherent cohomology sheaves, and on the other hand that the rows of the following commutative diagram are isomorphism

$$\begin{array}{ccc} H^0(i_s^*(H^n F) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) & \xrightarrow{H^0(i_s^*(H^n \phi) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) \sim} & H^0(i_s^*(H^n G) \otimes_{i_s^* O_S} L_{i_s^* O_S}(k(s))) \\ \downarrow = & & \downarrow = \\ i_s^*(H^n F) \otimes_{i_s^* O_S} k(s) & \xrightarrow{i_s^*(H^n \phi) \otimes_{i_s^* O_S} k(s) \sim} & i_s^*(H^n G) \otimes_{i_s^* O_S} k(s) \end{array}$$

Since

$$i_s^*(H^n \phi) \otimes_{i_s^* O_S} k(s) : i_s^*(H^n F) \otimes_{i_s^* O_S} k(s) \xrightarrow{\sim} i_s^*(H^n G) \otimes_{i_s^* O_S} k(s)$$

is an isomorphism for all $s \in S$, $O_{S,s} =: i_s^* O_S$ are reduced, and $a_{\text{zar}} H^n F, a_{\text{zar}} H^n G$ are coherent, $i_s^* H^n \phi : i_s^* H^n F \rightarrow i_s^* H^n G$ are injective. \square

Let $i : Z \hookrightarrow S$ a closed embedding, with $S, Z \in \text{Top}$. Denote by $j : S \setminus Z \hookrightarrow S$ the open embedding of the complementary subset. We have the adjunction

$$(i_*, i^!) := (i_*, i^\perp) : C(Z) \rightarrow C(S), \text{ with in this case } i^!F := \ker(F \rightarrow j_*j^*F).$$

It induces the adjunction $(i_*, i^!) : C_{(2)fil}(Z) \rightarrow C_{(2)fil}(S)$ (we recall that $i^! := i^\perp$ preserve monomorphisms).

Let $i : Z \hookrightarrow S$ a closed embedding, with $S, Z \in \text{Top}$. Denote by $j : S \setminus Z \hookrightarrow S$ the open embedding of the complementary subset. We have the support section functors :

- We have the functor

$$\Gamma_Z : C(S) \rightarrow C(S), F \mapsto \Gamma_Z(F) := \text{Cone}(\text{ad}(j^*, j_*)(F) : F \rightarrow j_*j^*F)[-1],$$

together with the canonical map $\gamma_Z(F) : \Gamma_Z F \rightarrow F$. We have the factorization

$$\text{ad}(i_*, i^!)(F) : i_*i^!F \xrightarrow{\text{ad}(i_*, i^!)(F)^\gamma} \Gamma_Z F \xrightarrow{\gamma_Z(F)} F,$$

and $\text{ad}(i_*, i^!)(F)^\gamma : i_*i^!F \rightarrow \Gamma_Z F$ is an homotopy equivalence. Since Γ_Z preserve monomorphisms, it induce the functor

$$\Gamma_Z : C_{fil}(S) \rightarrow C_{fil}(S), (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F),$$

together with the canonical map $\gamma_Z((G, F)) : \Gamma_Z(G, F) \rightarrow (G, F)$.

- We have also the functor

$$\Gamma_Z^\vee : C(S) \rightarrow C(S), F \mapsto \Gamma_Z^\vee F := \text{Cone}(\text{ad}(j_!, j^*)(F) : j_!j^*F \rightarrow F),$$

together with the canonical map $\gamma_Z^\vee(F) : F \rightarrow \Gamma_Z^\vee F$. We have the factorization

$$\text{ad}(i^*, i_*)(F) : F \xrightarrow{\gamma_Z^\vee(F)} \Gamma_Z^\vee F \xrightarrow{\text{ad}(i^*, i_*)(F)^\gamma} i_*i^*F,$$

and $\text{ad}(i^*, i_*)(F)^\gamma : \Gamma_Z^\vee F \rightarrow i_*i^*F$ is an homotopy equivalence. Since Γ_Z^\vee preserve monomorphisms, it induce the functor

$$\Gamma_Z : C_{fil}(S) \rightarrow C_{fil}(S), (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F),$$

together with the canonical map $\gamma_Z^\vee(G, F) : (G, F) \rightarrow \Gamma_Z^\vee(G, F)$.

Definition-Proposition 1. (i) Let $g : S' \rightarrow S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $S', S, Z \in \text{Top}$. Then, for $(G, F) \in C_{fil}(S)$, there is a canonical map in $C_{fil}(S')$

$$T(g, \gamma)(G, F) : g^*\Gamma_Z(G, F) \rightarrow \Gamma_{Z \times_S S'} g^*(G, F)$$

unique up to homotopy such that $\gamma_{Z \times_S S'}(g^*(G, F)) \circ T(g, \gamma)(G, F) = g^*\gamma_Z(G, F)$.

(ii) Let $i_1 : Z_1 \hookrightarrow S$, $i_2 : Z_2 \hookrightarrow Z_1$ be closed embeddings with $S, Z_1, Z_2 \in \text{Top}$. Then, for $(G, F) \in C_{fil}(S)$,

- there is a canonical map $T(Z_2/Z_1, \gamma)(G, F) : \Gamma_{Z_2}(G, F) \rightarrow \Gamma_{Z_1}(G, F)$ in $C_{fil}(S)$ unique up to homotopy such that $\gamma_{Z_1}(G, F) \circ T(Z_2/Z_1, \gamma)(G, F) = \gamma_{Z_2}(G, F)$ together with a distinguish triangle in $K_{fil}(S) := K(\text{PSh}_{fil}(S))$

$$\Gamma_{Z_2}(G, F) \xrightarrow{T(Z_2/Z_1, \gamma)(G, F)} \Gamma_{Z_1}(G, F) \xrightarrow{\text{ad}(j_2^*, j_2^*)(\Gamma_{Z_1}(G, F))} \Gamma_{Z_1 \setminus Z_2}(G, F) \rightarrow \Gamma_{Z_2}(G, F)[1]$$

- there is a canonical map $T(Z_2/Z_1, \gamma^\vee)(G, F) : \Gamma_{Z_1}^\vee(G, F) \rightarrow \Gamma_{Z_2}^\vee(G, F)$ in $C_{fil}(S)$ unique up to homotopy such that $\gamma_{Z_2}^\vee(G, F) = T(Z_2/Z_1, \gamma^\vee)(G, F) \circ \gamma_{Z_1}^\vee(G, F)$. together with a distinguished triangle in $K_{fil}(S)$

$$\Gamma_{Z_1 \setminus Z_2}^\vee(G, F) \xrightarrow{\text{ad}(j_{21}, j_2^*)(G, F)} \Gamma_{Z_1}^\vee(G, F) \xrightarrow{T(Z_2/Z_1, \gamma^\vee)(G, F)} \Gamma_{Z_2}^\vee(G, F) \rightarrow \Gamma_{Z_2 \setminus Z_1}^\vee(G, F)[1]$$

- (iii) Consider a morphism $g : (S', Z') \rightarrow (S, Z)$ with $(S', Z'), (S, Z) \in \text{Top}^2$. We denote, for $G \in C(S)$ the composite

$$T(D, \gamma^\vee)(G) : g^* \Gamma_Z^\vee G \xrightarrow{\sim} \Gamma_{Z \times_S S'}^\vee g^* G \xrightarrow{T(Z'/Z \times_S S', \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^* G$$

$$\text{and we have then the factorization } \gamma_{Z'}^\vee(g^* G) : g^* G \xrightarrow{g^* \gamma_Z^\vee(G)} g^* \Gamma_Z^\vee G \xrightarrow{T(D, \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^* G.$$

Proof. (i): We have the cartesian square

$$\begin{array}{ccc} S \setminus Z & \xrightarrow{j} & S \\ g' \uparrow & & \uparrow g \\ S' \setminus Z \times_S S' & \xrightarrow{j'} & S' \end{array}$$

and the map is given by

$$(I, T(g, j)(j^* G)) : \text{Cone}(g^* G \rightarrow g^* j_* j^* G) \rightarrow \text{Cone}(g^* G \rightarrow j'_* j'^* g^* G = j'_* g'^* j^* G).$$

- (ii): Follows from the fact that $j_1^* \Gamma_{Z_2} G = 0$ and $j_1^* \Gamma_{Z_2}^\vee G = 0$, with $j_1 : S \setminus Z_1 \hookrightarrow S$ the closed embedding.
- (iii): Obvious. \square

Let $(S, O_S) \in \text{RTop}$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding,

- For $G \in C_{O_S}(S)$, $\Gamma_Z G := \text{Cone}(\text{ad}(j^*, j_*)(G) : F \rightarrow j_* j^* G)[-1]$ has a (unique) structure of O_S module such that $\gamma_Z(G) : \Gamma_Z G \rightarrow G$ is a map in $C_{O_S}(S)$. This gives the functor

$$\Gamma_Z : C_{O_S fil}(S) \rightarrow C_{fil O_S}(S), (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F),$$

together with the canonical map $\gamma_Z((G, F)) : \Gamma_Z(G, F) \rightarrow (G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(S)$, $T(Z_2/Z, \gamma)(G) : \Gamma_{Z_2} G \rightarrow \Gamma_Z G$ is a map in $C_{O_S}(S)$ (i.e. is O_S linear).

- For $G \in C_{O_S}(S)$, $\Gamma_Z^\vee G := \text{Cone}(\text{ad}(j_!, j^*)(G) : j_! j^* G \rightarrow G)$ has a unique structure of O_S module, such that $\gamma_Z^\vee(G) : G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{O_S}(S)$. This gives the functor

$$\Gamma_Z^\vee : C_{O_S fil}(S) \rightarrow C_{fil O_S}(S), (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F),$$

together with the canonical map $\gamma_Z^\vee((G, F)) : (G, F) \rightarrow \Gamma_Z^\vee(G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(S)$, $T(Z_2/Z, \gamma^\vee)(G) : \Gamma_{Z_2}^\vee G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{O_S}(S)$ (i.e. is O_S linear).

- For $G \in C_{O_S}(S)$, we will use

$$\begin{aligned} \Gamma_Z^{\vee, h} G &:= \mathbb{D}_S^O L_O \Gamma_Z E(\mathbb{D}_S^O G) \\ &:= \text{Cone}(\mathbb{D}_S^O L_O \text{ad}(j_*, j^*)(E(\mathbb{D}_S^O G)) : \mathbb{D}_S^O L_O j_* j^* E(\mathbb{D}_S^O G) \rightarrow \mathbb{D}_S^O L_O E(\mathbb{D}_S^O G)) \end{aligned}$$

and we have the canonical map $\gamma_Z^{\vee, h}(G) : M \rightarrow \Gamma_Z^{\vee, h} G$ of O_S module. The factorization

$$\begin{aligned} \text{ad}(j_!, j^*)(L_O M) : j_! j^* L_O G &\xrightarrow{(k \circ \mathbb{D}^O I(j_!, j^*)(\mathbb{D}^O j^* L_O G) \circ \text{ad}(j_! j^* L_O G))^q} \\ &\xrightarrow{\text{ad}(j_*, j^*)(E(\mathbb{D}_S^O L_O G))} \mathbb{D}_S^O L_O E(\mathbb{D}_S^O L_O G) \end{aligned}$$

gives the factorization $\gamma_Z^{\vee,h}(L_O G) : L_O G \xrightarrow{\gamma_Z^{\vee}(L_O G)} \Gamma_Z^{\vee} L_O G \xrightarrow{(k \circ \mathbb{D}^O I(j!, j^*))(\mathbb{D}^O j^* L_O G) \circ d(j! j^* L_O G))^q} \Gamma_Z^{\vee,h} L_O G$. We get the functor

$$\Gamma_Z^{\vee,h} : C_{O_S \text{fil}}(S) \rightarrow C_{O_S \text{fil}}(S), (G, F) \mapsto \Gamma_Z^{\vee,h}(G, F) := \mathbb{D}_S^O L_O \Gamma_Z E(\mathbb{D}_S^O(G, F)),$$

together with the factorization

$$\begin{aligned} \gamma_Z^{\vee,h}(L_O(G, F)) : L_O(G, F) &\xrightarrow{\gamma_Z^{\vee}(L_O(G, F))} \Gamma_Z^{\vee} L_O(G, F) \\ &\xrightarrow{(k \circ \mathbb{D}^O I(j!, j^*))(\mathbb{D}_S^O j^* L_O(G, F)) \circ d(j! j^* L_O(G, F)))^q} \Gamma_Z^{\vee,h} L_O(G, F), \end{aligned}$$

- Consider $\mathcal{I} \subset O_S$ a right ideal of O_S such that $\mathcal{I}_Z^o \subset \mathcal{I}$, where $\mathcal{I}_Z^o \subset O_S$ is the left and right ideal consisting of section which vanish on Z .
 - For $G \in \text{PSh}_{O_S}(S)$, we consider, $S^o \subset S$ being an open subset,

$$\mathcal{I}G(S^o) = \langle \{f.m, m \in G(S^o), f \in \mathcal{I}(S^o)\} \rangle \subset G(S^o)$$

since \mathcal{I} is a right ideal, and we denote by $b_I(G) : \mathcal{I}G \rightarrow G$ the injective morphism of O_S modules and by $c_Z(G) : G \rightarrow G/\mathcal{I}G$ the quotient map. The adjunction map $\text{ad}(j!, j^*)(G) : j! j^* G \rightarrow G$ factors through $b_I(G)$:

$$\text{ad}(j!, j^*)(G) : j! j^* G \xrightarrow{b_{Z/S}^I(G)} \mathcal{I}G \xrightarrow{b_I(G)} G$$

We have then the support section functor,

$$\Gamma_Z^{\vee,O,I} : C_{O_S}(S) \rightarrow C_{O_S}(S), G \mapsto \Gamma_Z^{\vee,O,I} G := \text{Cone}(b_I(G) : \mathcal{I}G \rightarrow G)$$

together with the canonical map $\gamma_Z^{\vee,O}(G) : G \rightarrow \Gamma_Z^{\vee,O} G$ which factors through

$$\gamma_Z^{\vee,O,I}(G) : G \xrightarrow{\gamma_Z^{\vee}(G)} \Gamma_Z^{\vee} G \xrightarrow{b_{S/Z}^I(G)} \Gamma_Z^{\vee,O} G.$$

By the exact sequence $0 \rightarrow \mathcal{I}G \xrightarrow{b_I(G)} G \xrightarrow{c_I(G)} G/\mathcal{I}G \rightarrow 0$, we have an homotopy equivalence $c_I(G) : \Gamma_Z^{\vee,O,I} G \rightarrow G/\mathcal{I}G$.

- For $G \in \text{PSh}_{O_S}(S)$, we consider

$$b'_I(G) : G \rightarrow G \otimes_{O_S} \mathbb{D}_S^O(\mathcal{I}) := G \otimes_{O_S} \mathcal{H}om(\mathcal{I}, O_S)$$

The adjunction map $\text{ad}(j^*, j_*)(G) : G \rightarrow j_* j^* G$ factors through $b'_I(G)$:

$$\text{ad}(j^*, j_*)(G) : G \xrightarrow{b'_I(G)} G \otimes_{O_S} \mathbb{D}_S^O(\mathcal{I}) \xrightarrow{b'_{Z/S}(G)} j_* j^* G$$

We have then the support section functor,

$$\Gamma_Z^{O,I} : C_{O_S}(S) \rightarrow C_{O_S}(S), G \mapsto \Gamma_Z^{O,I} G := \text{Cone}(b'_I(G) : G \rightarrow G \otimes_{O_S} \mathbb{D}_S^O(\mathcal{I}))[-1]$$

together with the canonical map $\gamma_Z^O(G) : \Gamma_Z^O G \rightarrow G$ which factors through

$$\gamma_Z^{O,I}(G) : \Gamma_Z^O G \xrightarrow{b'_{S/Z}(G)} \Gamma_Z G \xrightarrow{\gamma_Z(G)} G.$$

– By definition, we have for a canonical isomorphism

$$I(D, \gamma^O)(G) : \mathbb{D}_S^O \Gamma^{\vee, O, I} G \xrightarrow{\sim} \Gamma_Z^{O, I} \mathbb{D}_S^O G$$

which gives the transformation map in $C_{O_S}(S)$

$$\begin{aligned} T(D, \gamma^O)(G) : \Gamma^{\vee, O, I} \mathbb{D}_S^O G &\xrightarrow{d(-)} \mathbb{D}_S^{O, 2} \Gamma^{\vee, O, I} \mathbb{D}_S^O G \xrightarrow{\mathbb{D}_S^O I(D, \gamma^O)(\mathbb{D}_S^O G)^{-1}} \\ &\mathbb{D}_S^O \Gamma_Z^{O, I} \mathbb{D}_S^{O, 2} G \xrightarrow{\mathbb{D}_S^O \Gamma_Z^{O, I} d(G)} \mathbb{D}_S^O \Gamma_Z^{O, I} G \end{aligned}$$

Definition-Proposition 2. (i) Let $g : (S', O_{S'}) \rightarrow (S, O_S)$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $(S', O_{S'}, (S, O_S)) \in \text{RTop}$. Then, for $(G, F) \in C_{O_S \text{fil}}(S)$, there is a canonical map in $C_{O_{S'}, \text{fil}}(S')$

$$T^{\text{mod}}(g, \gamma)(G, F) : g^{*\text{mod}} \Gamma_Z(G, F) \rightarrow \Gamma_{Z \times_S S'} g^{*\text{mod}}(G, F)$$

unique up to homotopy, such that $\gamma_{Z \times_S S'}(g^{*\text{mod}} G) \circ T^{\text{mod}}(g, \gamma)(G) = g^{*\text{mod}} \gamma_Z G$.

(ii) Let $i_1 : (Z_1, O_{Z_1}) \hookrightarrow (S, O_S)$, $i_2 : (Z_2, O_{Z_2}) \hookrightarrow (Z_1, O_{Z_1})$ be closed embeddings with $S, Z_1, Z_2 \in \text{Top}$. Then, for $(G, F) \in C_{O_S \text{fil}}(S)$, there is a canonical map in $C_{O_S \text{fil}}(S)$

$$T(Z_2/Z_1, \gamma^{\vee, O})(G, F) : \Gamma_{Z_1}^{\vee, O}(G, F) \rightarrow \Gamma_{Z_2}^{\vee, O}(G, F)$$

unique up to homotopy such that $\gamma_{Z_2}^{\vee, O}(G, F) = T(Z_2/Z_1, \gamma^{\vee, O})(G, F) \circ \gamma_{Z_1}^{\vee, O}(G, F)$.

(iii) Consider a morphism $g : ((S', O_{S'}), Z') \rightarrow ((S, O_S), Z)$ with $((S', O_{S'}), Z') \rightarrow ((S, O_S), Z) \in \text{RTop}^2$. We denote, for $M \in C_{O_S}(S)$ the composite

$$T^{\text{mod}}(D, \gamma^{\vee, O})(G) : g^{*\text{mod}} \Gamma_Z^{\vee, O} G \xrightarrow{\sim} \Gamma_{Z \times_S S'}^{\vee, O} g^{*\text{mod}} G \xrightarrow{T(Z'/Z \times_S S', \gamma^{\vee, O})(G)} \Gamma_{Z'}^{\vee, O} g^{*\text{mod}} G$$

and we have then the factorization

$$\gamma_{Z'}^{\vee, O}(g^{*\text{mod}} M) : g^{*\text{mod}} G \xrightarrow{g^{*\text{mod}} \gamma_Z^{\vee, O}(G)} g^{*\text{mod}} \Gamma_Z^{\vee, O} G \xrightarrow{T^{\text{mod}}(D, \gamma^{\vee, O})(G)} \Gamma_{Z'}^{\vee, O} g^{*\text{mod}} G$$

Proof. (i): We have the cartesian square

$$\begin{array}{ccc} S \setminus Z & \xrightarrow{j} & S \\ g' \uparrow & & \uparrow g \\ S' \setminus Z \times_S S' & \xrightarrow{j'} & S' \end{array}$$

and the map is given by

$$(I, T^{\text{mod}}(g, j)(j^* G)) : \text{Cone}(g^{*\text{mod}} G \rightarrow g^{*\text{mod}} j_* j^* G) \rightarrow \text{Cone}(g^{*\text{mod}} G \rightarrow j'_* j'^* g^{*\text{mod}} G = j'_* g'^* j^* G).$$

(ii): Obvious.

(iii): Obvious. □

Definition-Proposition 3. Consider a commutative diagram in RTop

$$\begin{array}{ccccc} D_0 = f : (X, O_X) & \xrightarrow{i} & (Y, O_Y) & \xrightarrow{p} & (S, O_S) \\ g' \uparrow & & g'' \uparrow & & \uparrow g \\ f' : (X', O_{X'}) & \xrightarrow{i'} & (Y', O_{Y'}) & \xrightarrow{p'} & (T, O_T) \end{array}$$

with i, i' being closed embeddings. Denote by D the right square of D_0 . The closed embedding $i' : X' \hookrightarrow Y'$ factors through $i' : X' \xrightarrow{i'_1} X \times_Y Y' \xrightarrow{i'_0} Y'$ where i'_1, i'_0 are closed embeddings.

(i) We have the canonical map,

$$E(\Omega_{((O_{Y'}/g''^*O_Y)/(O_T/g^*O_S))}) \circ T(g'', E)(-) \circ T(g'', \gamma)(-) : \\ g''^* \Gamma_X E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) \rightarrow \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b)$$

unique up to homotopy such that the following diagram in $C_{g''^*p^*O_S \text{ fil}}(Y') = C_{p'^*g^*O_S \text{ fil}}(Y')$ commutes

$$\begin{array}{ccc} g''^* \Gamma_X E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) & \xrightarrow{E(\Omega_{((O_{Y'}/g''^*O_Y)/(O_T/g^*O_S))}) \circ T(g'', E)(-) \circ T(g'', \gamma)(-)} & \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) . \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X \times_Y Y'}(-) \\ g''^* E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) & \xrightarrow{E(\Omega_{((O_{Y'}/g''^*O_Y)/(O_T/g^*O_S))}) \circ T(g'', E)(-)} & E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \end{array}$$

(ii) There is a canonical map,

$$T_\omega^O(D)^\gamma : g^{*\text{mod}} L_{Op^*} \Gamma_X E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) \rightarrow p'_* \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b)$$

unique up to homotopy such that the following diagram in $C_{O_T \text{ fil}}(T)$ commutes

$$\begin{array}{ccc} g^{*\text{mod}} L_{Op^*} \Gamma_X E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) & \xrightarrow{T_\omega^O(D)^\gamma} & p'_* \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) . \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X \times_Y Y'}(-) \\ g^{*\text{mod}} L_{Op^*} E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) & \xrightarrow{T_\omega^O(D)} & p'_* E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \end{array}$$

(iii) We have the canonical map in $C_{f'^*O_T}(Y')$

$$T(X'/X \times_Y Y', \gamma)(E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b)) : \Gamma_{X'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \rightarrow \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b)$$

unique up to homotopy such that $\gamma_{X \times_Y Y'}(-) \circ T(X'/X \times_Y Y', \gamma)(-) = \gamma_{X'}(-)$.

Proof. Immediate from definition. We take for the map of point (ii) the composite

$$\begin{aligned} & T_\omega^O(D)^\gamma : g^{*\text{mod}} L_{Op^*} \Gamma_X E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) \xrightarrow{q} g^* p^* \Gamma_X E(\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{g^*O_S} O_T \\ & \xrightarrow{T(g'', E)(-) \circ T(g'', \gamma)(-) \circ T(D)(E(\Omega_{O_X/p^*O_S}^\bullet))} (p'_* \Gamma_{X \times_Y Y'} E(g''^* \Omega_{O_Y/p^*O_S}^\bullet, F_b)) \otimes_{g^*O_S} O_T \\ & \xrightarrow{E(\Omega_{(O_{Y'}/g''^*O_Y)/(O_T/g^*O_S)})} p'_* \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{g^*O_S} O_T \xrightarrow{m} p'_* \Gamma_{X \times_Y Y'} E(\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b), \end{aligned}$$

with $m(n \otimes s) = s.n$. □

Definition 2. (i) Let $S \in \text{Top}$. For $Z \subset S$ a closed subset, we denote by $C_Z(S) \subset C(S)$ the full subcategory consisting of complexes of presheaves $F \in C(S)$ such that $a_{\text{top}} H^n(j^*F) = 0$ for all $n \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{top} is the sheafification functor.

(i)' More generally, let $(S, O_S) \in \text{RTop}$. For $Z \subset S$ a closed subset, we denote by

$$C_{O_S, Z}(S) \subset C_{O_S}(S), \quad \mathcal{QCoh}_Z(S) \subset \mathcal{QCoh}(S)$$

the full subcategories consisting of complexes of presheaves $G \in C_{O_S}(S)$ such that $a_{\text{top}} H^n(j^*F) = 0$ for all $n \in \mathbb{Z}$, resp. quasi-coherent sheaves $G \in \mathcal{QCoh}(S)$ such that $j^*F = 0$.

(ii) Let $S \in \text{Top}$. For $Z \subset S$ a closed subset, we denote by $C_{fil,Z}(S) \subset C_{fil}(S)$ the full subcategory consisting of filtered complexes of presheaves $(G, F) \in C_{fil}(S)$ such that there exist $r \in \mathbb{N}$ and an r -filtered homotopy equivalence $\phi : (G, F) \rightarrow (G', F)$ with $(G', F) \in C_{fil}(S)$ such that $a_{top} j^* H^n \text{Gr}_F^p(G', F) = 0$ for all $n, p \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{top} is the sheafification functor. Note that this definition say that this r does NOT depend on n and p .

(ii)' More generally, let $(S, O_S) \in \text{RTop}$. For $Z \subset S$ a closed subset, we denote by

$$C_{O_S, fil, Z}(S) \subset C_{O_S, fil}(S), \mathcal{Q}Coh_{fil, Z}(S) \subset \mathcal{Q}Coh(S)$$

the full subcategories consisting of filtered complexes of presheaves $(G, F) \in C_{O_S, fil}(S)$ such that there exist $r \in \mathbb{N}$ and an r -filtered homotopy equivalence $\phi : (G, F) \rightarrow (G', F)$ with $(G', F) \in C_{fil}(S)$ such that $a_{top} j^* H^n \text{Gr}_F^p(G', F) = 0$ for all $n, p \in \mathbb{Z}$, resp. filtered quasi-coherent sheaves $(G, F) \in \mathcal{Q}Coh(S)$ and an r -filtered homotopy equivalence $\phi : (G, F) \rightarrow (G', F)$ with $(G', F) \in C_{fil}(S)$ such that there exist $r \in \mathbb{N}$ such that $j^* H^n \text{Gr}_F^p(G, F) = 0$ for all $n, p \in \mathbb{Z}$. Note that this definition say that this r does NOT depend on n and p .

Let $(S, O_S) \in \text{RTop}$ and $Z \subset S$ a closed subset.

- For $(G, F) \in C_{fil}(S)$, we have $\Gamma_Z(G, F), \Gamma_Z^\vee(G, F) \in C_{fil, Z}(S)$.
- For $(G, F) \in C_{O_S, fil}(S)$, we have $\Gamma_Z(G, F), \Gamma_Z^\vee(G, F), \Gamma_Z^{\vee, h}(G, F), \Gamma_Z^{\vee, O}(G, F) \in C_{O_S, fil, Z}(S)$.

Proposition 12. Let $S \in \text{Top}$ and $Z \subset S$ a closed subspace. Denote by $i : Z \hookrightarrow S$ the closed embedding.

- (i) The functor $i^* : \text{Shv}_Z(S) \rightarrow \text{Shv}(Z)$ is an equivalence of category whose inverse is $i_* : \text{Shv}(Z) \rightarrow \text{Shv}_Z(S)$. More precisely $\text{ad}(i_*, i^*)(H) : i^* i_* H \rightarrow H$ is an isomorphism if $H \in \text{Shv}(Z)$ and $\text{ad}(i_*, i^*)(G) : G \rightarrow i_* i^* G$ is an isomorphism if $G \in \text{Shv}_Z(S)$.
- (ii) : The functor $i^* : \text{Shv}_{fil, Z}(S) \rightarrow \text{Shv}_{fil}(Z)$ is an equivalence of category whose inverse is $i_* : \text{Shv}_{fil}(Z) \rightarrow \text{Shv}_{fil, Z}(S)$. More precisely $\text{ad}(i_*, i^*)(H, F) : i^* i_*(H, F) \rightarrow (H, F)$ is an isomorphism if $(H, F) \in \text{Shv}(Z)$ and $\text{ad}(i_*, i^*)(G, F) : (G, F) \rightarrow i_* i^*(G, F)$ is an isomorphism if $(G, F) \in \text{Shv}_Z(S)$.
- (iii) : The functor $i^* : D_{\tau, fil, Z}(S) \rightarrow D_{\tau, fil}(Z)$ is an equivalence of category whose inverse is $i_* : D_{\tau, fil}(Z) \rightarrow D_{\tau, fil, Z}(S)$. More precisely $\text{ad}(i_*, i^*)(H, F) : i^* i_*(H, F) \rightarrow (H, F)$ is an equivalence top local if $(H, F) \in C_{fil}(Z)$ and $\text{ad}(i_*, i^*)(G, F) : (G, F) \rightarrow i_* i^*(G, F)$ is an equivalence top local if $(G, F) \in C_{fil, Z}(S)$.

Proof. (i): Standard.

(ii): Follows from (i).

(iii): Follows from (ii). □

Let $S \in \text{Top}$ and $Z \subset S$ a closed subspace. By proposition 12, if $G \in C(S)$, $\text{ad}(i_*, i^*)(\Gamma_Z G) : \Gamma_Z G \rightarrow i_* i^* \Gamma_Z G$ is an equivalence top local since $\Gamma_Z G \in C_Z(S)$.

Let $(S, O_S) \in \text{RTop}$. Let $D = \cup_i D_i \subset X$ a normal crossing divisor, denote by $j : S \setminus D \hookrightarrow S$ the open embedding, and consider $\mathcal{I}_D \subset O_S$ the ideal of vanishing function on D which is invertible. We set, for $M \in C_{O_S}(S)$,

$$M(*D) := \lim_n \mathcal{H}om_{O_S}(\mathcal{I}_D^n, M),$$

and we denote by $a_D(F) : F \rightarrow F(*D)$ the surjective morphism of presheaves. The adjonction map $\text{ad}(j^*, j_*)(F) : F \rightarrow j_* j^* F$ factors trough $a_D(F) :$

$$\text{ad}(j^*, j_*)(F) : F \xrightarrow{a_D(F)} F(*D) \xrightarrow{a_{S/D}(F)} j_* j^* F$$

Remark 3. • Let $j : U \hookrightarrow X$ an open embedding, with $(X, O_X) \in \text{RTop}$. Then if $F \in \text{Coh}_{O_U}(U)$ is a coherent sheaf of O_U module, j_*F is quasi-coherent but NOT coherent in general. In particular for $F \in C_{O_U}(U)$ whose cohomology sheaves $a_{\text{tau}}H^n F$ are coherent for all $n \in \mathbb{Z}$, the cohomology sheaves $R^n j_*F := a_{\tau}H^n j_*E(F)$ of $Rj_*F = j_*E(F)$ are quasi-coherent but NOT coherent.

- Let $j : U \hookrightarrow X$ an open embedding, with $X \in \text{Sch}$. Then if $F \in \text{Coh}(U)$ is a coherent sheaf of O_U module, j_*F is quasi-coherent but NOT coherent. However, there exist an O_X submodule $\tilde{F} \subset j_*F$ such that $j^*\tilde{F} = F$ and $\tilde{F} \in \text{Coh}(X)$.

The following propositions are true for schemes but NOT for arbitrary ringed spaces like analytic spaces :

Proposition 13. (i) Let $X = (X, O_X) \in \text{Sch}$ a noetherien scheme and $D \subset X$ a closed subset. Denote by $j : U = X \setminus D \hookrightarrow X$ an open embedding. Then for $F \in \text{QCoh}_{O_U}(U)$ a quasi coherent sheaf, $j_*F \in \text{QCoh}_{O_X}(X)$ is quasi-coherent and is the direct limit of its coherent subsheaves.

- (ii) Let $X = (X, O_X)$ a noetherien scheme and $D = \cup D_i \subset X$ a normal crossing divisor. Denote by $j : U = X \setminus D \hookrightarrow X$ an open embedding. Then for $F \in \text{QCoh}_{O_U}(U)$ a quasi coherent sheaf, the canonical map $a_{X/D}(F) : F(*D) \xrightarrow{\sim} j_*F$ is an isomorphism.

Proof. Standard. □

Proposition 14. Let $S = (S, O_S) \in \text{Sch}$ and $Z \subset S$ a closed subscheme. Denote by $i : Z \hookrightarrow S$ the closed embedding.

- (i) For $G \in \text{QCoh}_Z(S)$, i^*G has a canonical structure of O_Z module. Moreover, the functor $i^* : \text{QCoh}_Z(S) \rightarrow \text{QCoh}(Z)$ is an equivalence of category whose inverse is $i_* : \text{QCoh}(Z) \rightarrow \text{QCoh}_Z(S)$.
- (ii) : The functor $i^* : \text{QCoh}_{\text{fil},Z}(S) \rightarrow \text{QCoh}_{\text{fil}}(Z)$ is an equivalence of category whose inverse is $i_* : \text{QCoh}_{\text{fil}}(Z) \rightarrow \text{QCoh}_{\text{fil},Z}(S)$.
- (iii) : The functor $i^* : D_{O_S \text{fil},Z,\text{qc}}(S) \rightarrow D_{O_Z \text{fil},\text{qc}}(Z)$ is an equivalence of category whose inverse is $i_* : D_{O_Z \text{fil},\text{qc}}(Z) \rightarrow D_{O_S \text{fil},Z,\text{qc}}(S)$.

Proof. (i):Standard.

(ii): Follows from (i).

(iii): Follows from (ii) since i^* and i_* are exact functors. □

Definition 3. Let $(S, O_S) \in \text{RTop}$ a locally ringed space with O_S commutative. Consider an $\kappa_S \in C_{O_S}(S)$. Let $\mathcal{I} \subset O_S$ an ideal subsheaf and $Z = V(\mathcal{I}) \subset S$ the associated closed subset. For $G \in \text{PSh}_{O_S}(S)$, we denote by $\hat{G}_Z := \hat{G}_{\mathcal{I}} := \lim_k G/\mathcal{I}^k G$ the completion with respect to the ideal \mathcal{I} and by $c_Z^\infty(G) : G \rightarrow \hat{G}_Z$ the quotient map. Then, the canonical map

$$d_{\kappa_S, Z}(G) : G \xrightarrow{d(G)} \mathbb{D}_S^{O,2} G \xrightarrow{T^{\text{mod}}(\otimes_{\kappa_S, \text{hom}})(\mathbb{D}_S^O G, O_S)} \text{Hom}_{O_S}(\mathbb{D}_S^O G \otimes_{O_S} \kappa_S, \kappa_S) \xrightarrow{T^{\text{mod}}(\Gamma_Z E, \text{hom})(-, -)} \text{Hom}_{O_S}(\Gamma_Z E(\mathbb{D}_S^O G \otimes_{O_S} \kappa_S), \Gamma_Z E(\kappa_S))$$

factors through

$$d_{\kappa_S, Z}(G) : G \xrightarrow{c_Z^\infty(G)} \hat{G}_Z \xrightarrow{d_{\kappa_S, Z}(G)} \text{Hom}_{O_S}(\Gamma_Z E(\mathbb{D}_S^O G \otimes_{O_S} \kappa_S), \Gamma_Z E(\kappa_S))$$

Clearly if $G \in C_{O_S}(S)$ then $d_{\kappa_S, Z}(G)$ is a map in $C_{O_S}(S)$. On the other hand, we have a commutative diagram

$$\begin{array}{ccc} \Omega_S^p & \xrightarrow{d_{\kappa_S, Z}(\Omega_S^p)} & \text{Hom}_{O_S}(\Gamma_Z E(\mathbb{D}_S^O \Omega_S^p \otimes_{O_S} \kappa_S), \Gamma_Z E(\kappa_S)) \\ \downarrow d & & \downarrow d^{\kappa_S, \gamma} \\ \Omega_S^{p+1} & \xrightarrow{d_{\kappa_S, Z}(\Omega_S^{p+1})} & \text{Hom}_{O_S}(\Gamma_Z E(\mathbb{D}_S^O \Omega_S^{p+1} \otimes_{O_S} \kappa_S), \Gamma_Z E(\kappa_S)) \end{array}$$

so that $d_{\kappa_S, Z}(\Omega_S^\bullet) \in C(S)$.

The following theorem is the from [14]

Theorem 8. *Let $S \in \text{Var}(\mathbb{C})$. Let $Z = V(\mathcal{I}) \subset S$ a closed subset. Denote by $K_S \in \text{PSh}_{O_S}(S)$ the canonical bundle. Then, for $G \in C_{O_S, c}(S)$,*

$$d_{K_S, Z}(G) : \hat{G}_Z \rightarrow \text{Hom}_{O_S}(\Gamma_Z E(\mathbb{D}_S^O G \otimes_{O_S} K_S), \Gamma_Z E(K_S))$$

is an equivalence Zariski local.

Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(S, O_S) \in \text{RTop}$. In the particular case where O_S is a commutative sheaf of ring, $T_{O_S} \in \text{PSh}_{O_S}(S)$ and $\Omega_{O_S} = \mathbb{D}_{O_S} T_{O_S} \in \text{PSh}_{O_S}(S)$ are sheaves and the morphism in $\text{PSh}(X)$

$$T(f, \text{hom})(O_S, O_S) : f^* \text{Hom}(O_S, O_S) \rightarrow \text{Hom}(f^* O_S, f^* O_S)$$

induces isomorphisms $T(f, \text{hom})(O_S, O_S) : f^* T_{O_S} \xrightarrow{\sim} T_{f^* O_S}$ and $\mathbb{D}_{f^* O_S} T(f, \text{hom})(O_S, O_S) : \Omega_{f^* O_S} \rightarrow f^* \Omega_{O_S}$ where for $F \in \text{Shv}(S)$, we denote again (as usual) by abuse $f^* F := a_\tau f^* F \in \text{Shv}(S)$, $a_{\text{tau}} : \text{PSh}(S) \rightarrow \text{Shv}(S)$ being the sheafification functor.

Definition 4. (i) *Let $(X, O_X) \in \text{RTop}$. A foliation $(X, O_X)/\mathcal{F}$ on (X, O_X) is an O_X module $\Omega_{O_X/\mathcal{F}} \in \text{PSh}_{O_X}(X)$ together with a derivation map $d := d_{\mathcal{F}} : O_X \rightarrow \Omega_{O_X/\mathcal{F}}$ such that*

- the associated map $q := q_{\mathcal{F}} := \omega_X(d) : \Omega_{O_X} \rightarrow \Omega_{O_X/\mathcal{F}}$ is surjective
- satisfy the integrability condition $d(\ker q) \subset \ker q$ which implies that the map $d : \Omega_{O_X}^p \rightarrow \Omega_{O_X}^{p+1}$ induce factors trough

$$\begin{array}{ccc} \Omega_{O_X}^p & \xrightarrow{d} & \Omega_{O_X}^{p+1} \\ q^p := \wedge^p q \downarrow & & \downarrow q^{p+1} := \wedge^{p+1} q \\ \Omega_{O_X/\mathcal{F}}^p & \xrightarrow{d} & \Omega_{O_X/\mathcal{F}}^{p+1} \end{array}$$

and $d : \Omega_{O_X/\mathcal{F}}^p \rightarrow \Omega_{O_X/\mathcal{F}}^{p+1}$ is necessary unique by the surjectivity of $q^p : \Omega_{O_X}^p \rightarrow \Omega_{O_X/\mathcal{F}}^p$.

In the particular case where $\Omega_{O_X/\mathcal{F}} \in \text{PSh}_{O_X}(X)$ is a locally free sheaf of O_X module, $\mathbb{D}_{O_X} q : T_{O_X/\mathcal{F}} := \mathbb{D}_{O_X} \Omega_{O_X/\mathcal{F}} \rightarrow T_{O_X}$ is injective and the second condition is then equivalent to the fact that the sub O_X module $T_{O_X/\mathcal{F}} \subset T_{O_X}$ is a Lie subalgebra, that is $[T_{O_X/\mathcal{F}}, T_{O_X/\mathcal{F}}] \subset T_{O_X/\mathcal{F}}$.

(ii) *A piece of leaf a foliation $(X, O_X)/\mathcal{F}$ with $(X, O_X) \in \text{RTop}$ such that O_X is a commutative sheaf of ring is an injective morphism of ringed spaces $l : (Z, O_Z) \hookrightarrow (X, O_X)$ such that $\Omega_{i^* O_X/O_Z} :: \Omega_{i^* O_X} \rightarrow \Omega_{O_Z}$ factors trough an isomorphism*

$$\Omega_{i^* O_X/O_Z} : \Omega_{i^* O_X} \xrightarrow{\mathbb{D}_{i^* O_X} T(i, \text{hom})(O_X, O_X)} i^* \Omega_{O_X} \xrightarrow{i^* q} i^* \Omega_{O_X/\mathcal{F}} \rightarrow \Omega_{O_Z}.$$

(iii) *If $f : (X, O_X) \rightarrow (S, O_S)$ is a morphism with $(X, O_X), (S, O_S) \in \text{RTop}$, we have the foliation $(X, O_X)/(S, O_S) := ((X, O_X), f)$ on (X, O_X) given by the surjection*

$$q : \Omega_{O_X} \rightarrow \Omega_{O_X/f^* O_S} := \text{coker}(\Omega_{O_X/f^* O_S} : \Omega_{f^* O_S} \rightarrow \Omega_{O_X}).$$

The fibers $i_{X_s} : (X_s, O_{X_s}) \hookrightarrow (X, O_X)$ for each $s \in S$ are the leaves of the foliation.

(iv) *We have the category FolRTop*

- whose objects are foliated ringed spaces $(X, O_X)/\mathcal{F}$ with O_X a commutatif sheaf of ring and

- whose morphisms $f : (X, O_X)/\mathcal{F} \rightarrow (S, O_S)/\mathcal{G}$ are morphisms of ringed spaces $f : (X, O_X) \rightarrow (S, O_S)$ such that $\Omega_{O_X/f^*O_S} : \Omega_{f^*O_S} \rightarrow \Omega_{O_X}$ factors through

$$\begin{array}{ccc} f^*\Omega_{O_S} \xrightarrow{\mathbb{D}_{f^*O_S} T(f, \text{hom})(O_S, O_S)^{-1}} \Omega_{f^*O_S} \xrightarrow{\Omega_{O_X/f^*O_S}} \Omega_{O_X} & \cdot & \\ f^*q_{\mathcal{G}} \downarrow & & \downarrow q_{\mathcal{F}} \\ f^*\Omega_{O_S/\mathcal{G}} \xrightarrow{\Omega_{O_X/f^*O_S}^q} \Omega_{O_X/\mathcal{F}} & & \end{array}$$

This category admits inverse limits with $(X, O_X)/\mathcal{F} \times (Y, O_Y)/\mathcal{G} = (X \times Y, p_X^*O_X \otimes p_Y^*O_Y)/p_X^*\mathcal{F} \otimes p_Y^*\mathcal{G}$ and

$$(X, O_X)/\mathcal{F} \times_{(S, O_S)/\mathcal{H}} (Y, O_Y)/\mathcal{G} = (X \times_S Y, \delta_S^*(p_X^*O_X \otimes p_Y^*O_Y))/p_X^*\mathcal{F} \otimes p_Y^*\mathcal{G}$$

with $\delta_S : X \times_S Y \hookrightarrow X \times Y$ the embedding given by the diagonal $\delta_S : S \hookrightarrow S \times S$.

Let $S \in \text{Top}$. Let $S = \bigcup_{i=1}^l S_i$ an open cover and denote by $S_I = \bigcap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Top}$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and, for $J \subset I$, the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

where $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. and $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ the projection. This gives the diagram of topological spaces $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Top})$ which gives the diagram $(\tilde{S}_I) \in (\text{Ouv}(\tilde{S}_I)) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$ Denote $m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I$ the open embedding.

Definition 5. Let $S \in \text{Top}$. Let $S = \bigcup_{i=1}^l S_i$ an open cover and denote by $S_I = \bigcap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Top}$. We denote by $C_{\text{fil}}(S/(\tilde{S}_I)) \subset C_{\text{fil}}((\tilde{S}_I))$ the full subcategory

- whose objects are $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{\text{fil}, S_I}(\tilde{S}_I)$, and $u_{IJ} : m^*(G_I, F) \rightarrow m^*p_{IJ*}(G_J, F)$ are ∞ -filtered top local equivalences satisfying for $I \subset J \subset K$, $p_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{\text{fil}}(\tilde{S}_I)$,
- the morphisms $m : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$ being (see section 2.1) a family of morphism of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{\text{fil}}(\tilde{S}_I)$.

A morphism $m : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$ is an r -filtered top local equivalence if there exists $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$, $0 \leq i \leq s$, with $((C_{iI}, F), u_{iIJ}) \in C_{\text{fil}}(S/(\tilde{S}_I))$ $((C_{0I}, F), u_{0IJ}) = ((G_I, F), u_{IJ})$ and $((C_{sI}, F), u_{sIJ}) = ((H_I, F), v_{IJ})$, such that

$$\phi = \phi_s \circ \dots \circ \phi_1 \circ \dots \circ \phi_0 : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$$

and $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$ either a filtered τ local equivalence or an r -filtered homotopy equivalence.

Denote $L = [1, \dots, l]$ and for $I \subset L$, $p_{0(I)} : S \times \tilde{S}_I \rightarrow S$, $p_{I(0)} : S \times \tilde{S}_I \rightarrow S_I$ the projections. By definition, we have functors

- $T(S/(\tilde{S}_I)) : C_{\text{fil}}(S) \rightarrow C_{\text{fil}}(S/(\tilde{S}_I))$, $(G, F) \mapsto (i_I^*j_I^*(G, F), I)$

- $T((\tilde{S}_I)/S) : C_{fil}(S/(\tilde{S}_I)) \rightarrow C_{fil}(S), ((G_I, F), u_{IJ}) \mapsto \text{ho} \lim_{I \subset L} p_{0(0I)*} \Gamma_{\tilde{S}_I}^\vee p_{I(0I)}^*(G_I, F)$.

Note that the functors $T(S/(\tilde{S}_I))$ are embedding, since

$$\text{ad}(i_I^*, i_{I*})(j_I^* F) : i_I^* i_{I*} j_I^* F \rightarrow j_I^* F$$

are top local equivalence.

Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Top}$. Let $S = \cup_{i=1}^l S_i$ and $X = \cup_{i=1}^l X_i$ be open covers and $i_i : S_i \hookrightarrow \tilde{S}_i, i'_i : X_i \hookrightarrow \tilde{X}_i$ be closed embeddings, such that, for each $i \in [1, l]$, $f_i := f|_{X_i} : X_i \rightarrow S_i$ lift to a morphism $\tilde{f}_i : \tilde{X}_i \rightarrow \tilde{S}_i$. Then, $f_I = f|_{X_I} : X_I = \cap_{i \in I} X_i \rightarrow S_I = \cap_{i \in I} S_i$ lift to the morphism

$$\tilde{f}_I = \prod_{i \in I} \tilde{f}_i : \tilde{X}_I = \prod_{i \in I} \tilde{X}_i \rightarrow \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$$

Denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} : \tilde{X}_J \rightarrow \tilde{X}_I$ the projections. Consider for $J \subset I$ the following commutative diagrams

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}, D'_{IJ} = \begin{array}{ccc} X_I & \xrightarrow{i'_I} & \tilde{X}_I \\ j'_{IJ} \uparrow & & \uparrow p'_{IJ} \\ X_J & \xrightarrow{i'_J} & \tilde{X}_J \end{array}, D_{fI} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ f_I \uparrow & & \uparrow \tilde{f}_I \\ X_I & \xrightarrow{i'_I} & \tilde{X}_I \end{array}$$

We have then following commutative diagram

$$\begin{array}{ccccc} & & X_I & \xrightarrow{n'_I} & \tilde{X}_I & \longleftarrow & \tilde{X}_I \setminus X_I \\ & \nearrow j'_{IJ} & \uparrow p'_{IJ} & & \uparrow p'_{IJ} & & \uparrow p'_{IJ} \\ i'_J : X_J & \xrightarrow{l_{IJ}} & X_I \times X_I \times \tilde{X}_J \xrightarrow{n'_I \times I} & \tilde{X}_J & \longleftarrow & \tilde{X}_J \setminus X_J & \\ & & & \uparrow p'_{IJ} & & & \uparrow p'_{IJ} \end{array}$$

whose square are cartesian. We then have the pullback functor

$$f^* : C_{(2)fil}(S/(\tilde{S}_I)) \rightarrow C_{(2)fil}(X/(\tilde{X}_I)), \\ ((G_I, F), u_{IJ}) \mapsto f^*((G_I, F), u_{IJ}) := (\Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F), \tilde{f}_J^* u_{IJ})$$

with

$$\tilde{f}_J^* u_{IJ} : \Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F) \xrightarrow{\text{ad}(p'_{IJ}, p'_{IJ*})(-)} p'_{IJ*} p'_{IJ}^* \Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F) \xrightarrow{T_2(p_{IJ}, n'_I)(-)^{-1}} p'_{IJ*} \Gamma_{X_I \times \tilde{X}_J \setminus \tilde{X}_I}^\vee p'_{IJ}^* \tilde{f}_I^*(G_I, F) \\ \xrightarrow{p'_{IJ*} \gamma_{X_J}^\vee(-)} p'_{IJ*} \Gamma_{X_J}^\vee p'_{IJ}^* \tilde{f}_I^*(G_I, F) = p'_{IJ*} \Gamma_{X_J}^\vee \tilde{f}_J^* p'_{IJ}^*(G_I, F) \xrightarrow{\Gamma_{X_J}^\vee \tilde{f}_J^* I(p'_{IJ}, p_{IJ*})(-, -)(u_{IJ})} \Gamma_{X_J}^\vee \tilde{f}_J^*(G_J, F)$$

Let $(G, F) \in C_{fil}(S)$. Since, $j_I^* i'_{I*} j_I^* f^*(G, F) = 0$, the morphism $T(D_{fI})(j_I^*(G, F)) : \tilde{f}_I^* i_{I*} j_I^*(G, F) \rightarrow i'_{I*} j_I^* f^*(G, F)$ factors trough

$$T(D_{fI})(j_I^*(G, F)) : \tilde{f}_I^* i_{I*} j_I^*(G, F) \xrightarrow{\gamma_{X_I}^\vee(-)} \Gamma_{X_I}^\vee \tilde{f}_I^* i_{I*} j_I^*(G, F) \xrightarrow{T^\gamma(D_{fI})(j_I^*(G, F))} i'_{I*} j_I^* f^*(G, F)$$

We have then, for $(G, F) \in C_{fil}(S)$, the canonical transformation map

$$\begin{array}{ccc} f^* T(S/(\tilde{S}_I))(G, F) & \xrightarrow{T(f, T(0/I))(G, F)} & T(X/(\tilde{X}_I))(f^*(G, F)) \\ \downarrow = & & \downarrow = \\ (\Gamma_{X_I}^\vee \tilde{f}_I^* i_{I*} j_I^*(G, F), \tilde{f}_J^* I) & \xrightarrow{T^\gamma(D_{fI})(j_I^*(G, F))} & (i'_{I*} j_I^* f^*(G, F), I) \end{array}$$

Proposition 15. *Let $S \in \text{Top}$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Top}$. Denote by $D_{(2)fil,\infty}((S/(\tilde{S}_I))) = \text{Ho}_{top,\infty}(C_{(2)fil}((S/(\tilde{S}_I))))$ the localization of $C_{(2)fil}((S/(\tilde{S}_I)))$ with respect to top local equivalences. The functor $T(S/(\tilde{S}_I))$ induces an equivalence of category*

$$T(S/(\tilde{S}_I)) : D_{(2)fil,\infty}(S) \xrightarrow{\sim} D_{(2)fil,\infty}((S/(\tilde{S}_I)))$$

with inverse $T((\tilde{S}_I)/S)$

Proof. Follows from the fact that for $(G, F) \in C_{fil}(S)$,

$$\text{ho} \lim_{ICL} p_{0(0I)*} \Gamma_{S_I}^\vee p_{I(0I)}^* (i_{I*} j_I^*(G, F)) \rightarrow p_{0(0I)*} \Gamma_{S_I}^\vee j_I^*(G, F)$$

is an equivalence top local. □

For $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$ locally compact (in particular Hausdorff), e.g. $T, S \in \text{CW}$, there is also a functor $f_! : C(T) \rightarrow C(S)$ given by the section which have compact support over f , and, for $K_1, K_2 \in C(T)$, we have a canonical map

$$T_!(f, \text{hom}) : f_* \mathcal{H}om(K_1, K_2) \rightarrow \mathcal{H}om(f_! K_1, f_! K_2)$$

The main result on presheaves on locally compact spaces is the following :

Theorem 9. *Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$ locally compact.*

(i) *The derived functor $Rf_! : D(T) \rightarrow D(S)$ has a right adjoint $f^!$ (Verdier duality) and, for $K_1, K_2 \in D(T)$ and $K_3, K_4 \in D(S)$, we have canonical isomorphisms*

$$\begin{aligned} & - Rf_* R\mathcal{H}om^\bullet(Rf_! K_1, K_3) \xrightarrow{\sim} R\mathcal{H}om^\bullet(K_1, f^! K_3) \\ & - f^! R\mathcal{H}om^\bullet(K_3, K_4) \xrightarrow{\sim} R\mathcal{H}om^\bullet(f^* K_3, f^! K_4) \end{aligned}$$

(ii) *Denote, for $K \in D(S)$, $D(K) = R\mathcal{H}om^\bullet(K, a_S^! \mathbb{Z}) \in D(S)$ the Verdier dual of K . Then, if $K \in D_c(S)$, the evaluation map $ev^c(S)(K) : K \rightarrow D(D(K))$ is an isomorphism.*

(iii) *Assume we have a factorization $f : T \xrightarrow{l} Y \xrightarrow{p} S$ of f with l a closed embedding and p a smooth morphism of relative dimension d . Then $f^! K = i^! p^* K[d]$*

Proof. (i): Standard, the proof is formal (see [31]).

(ii): See [31].

(iii): The fact that $p^! K = p^* K[d]$ follows by Poincare duality for topological manifold. □

We have by theorem 9 a pair of adjoint functor

$$(Rf_!, f^!) : D(T) \rightleftarrows D(S)$$

- with $f_! = f_*$ if f is proper,
- with $f^! = f^*[d]$ if f is smooth of relative dimension d .

2.6 Presheaves on the big Zariski site or on the big etale site

For $S \in \text{Var}(\mathbb{C})$, we denote by $\rho_S : \text{Var}(\mathbb{C})^{sm}/S \hookrightarrow \text{Var}(\mathbb{C})/S$ be the full subcategory consisting of the objects $U/S = (U, h) \in \text{Var}(\mathbb{C})/S$ such that the morphism $h : U \rightarrow S$ is smooth. That is, $\text{Var}(\mathbb{C})^{sm}/S$ is the category

- whose objects are smooth morphisms $U/S = (U, h)$, $h : U \rightarrow S$ with $U \in \text{Var}(\mathbb{C})$,
- whose morphisms $g : U/S = (U, h_1) \rightarrow V/S = (V, h_2)$ is a morphism $g : U \rightarrow V$ of complex algebraic varieties such that $h_2 \circ g = h_1$.

We denote again $\rho_S : \text{Var}(\mathbb{C})/S \rightarrow \text{Var}(\mathbb{C})^{sm}/S$ the associated morphism of site. We will consider

$$r^s(S) : \text{Var}(\mathbb{C}) \xrightarrow{r(S)} \text{Var}(\mathbb{C})/S \xrightarrow{\rho_S} \text{Var}(\mathbb{C})^{sm}/S$$

the composite morphism of site. For $S \in \text{Var}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}(S/S) \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, $\mathcal{H}om(\mathbb{Z}_S, F) = F$. For $f : T \rightarrow S$ a morphism, with $T, S \in \text{Var}(\mathbb{C})$, we have the following commutative diagram of sites

$$\begin{array}{ccc} \text{Var}(\mathbb{C})/T & \xrightarrow{\rho_T} & \text{Var}(\mathbb{C})^{sm}/T \\ \downarrow P(f) & & \downarrow P(f) \\ \text{Var}(\mathbb{C})/S & \xrightarrow{\rho_S} & \text{Var}(\mathbb{C})^{sm}/S \end{array} \quad (23)$$

We denote, for $S \in \text{Var}(\mathbb{C})$, the obvious morphism of sites

$$\tilde{e}(S) : \text{Var}(\mathbb{C})/S \xrightarrow{\rho_S} \text{Var}(\mathbb{C})^{sm}/S \xrightarrow{e(S)} \text{Ouv}(S)$$

where $\text{Ouv}(S)$ is the set of the Zariski open subsets of S , given by the inclusion functors $\tilde{e}(S) : \text{Ouv}(S) \hookrightarrow \text{Var}(\mathbb{C})^{sm}/S \hookrightarrow \text{Var}(\mathbb{C})/S$. By definition, for $f : T \rightarrow S$ a morphism with $S, T \in \text{Var}(\mathbb{C})$, the commutative diagram of sites (23) extend a commutative diagram of sites :

$$\begin{array}{ccccc} \tilde{e}(T) : \text{Var}(\mathbb{C})/T & \xrightarrow{\rho_T} & \text{Var}(\mathbb{C})^{sm}/T & \xrightarrow{e(T)} & \text{Ouv}(T) \\ \downarrow P(f) & & \downarrow P(f) & & \downarrow P(f) \\ \tilde{e}(S) : \text{Var}(\mathbb{C})/S & \xrightarrow{\rho_S} & \text{Var}(\mathbb{C})^{sm}/S & \xrightarrow{e(S)} & \text{Ouv}(S) \end{array} \quad (24)$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/T)$$

the adjunction induced by $P(f) : \text{Var}(\mathbb{C})^{sm}/T \rightarrow \text{Var}(\mathbb{C})^{sm}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C_{fil}(\text{Var}(\mathbb{C})^{sm}/T), f^*(G, F) := (f^*G, f^*F)$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/T)$$

the adjunction induced by $P(f) : \text{Var}(\mathbb{C})/T \rightarrow \text{Var}(\mathbb{C})/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : C_{fil}(\text{Var}(\mathbb{C})/S) \rightleftarrows C_{fil}(\text{Var}(\mathbb{C})/T), f^*(G, F) := (f^*G, f^*F)$$

- For $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{Var}(\mathbb{C})$, the pullback functor $P(h) : \text{Var}(\mathbb{C})^{sm}/S \rightarrow \text{Var}(\mathbb{C})^{sm}/U$ admits a left adjoint $C(h)(X \rightarrow U) = (X \rightarrow U \rightarrow S)$. Hence, $h^* : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/U)$ admits a left adjoint

$$h_{\sharp} : C(\text{Var}(\mathbb{C})^{sm}/U) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S), F \mapsto ((V, h_0) \mapsto \lim_{(V', h_0 h') \rightarrow (V, h_0)} F(V', h'))$$

Note that we have for $V/U = (V, h')$ with $h' : V \rightarrow U$ a smooth morphism we have $h_{\sharp}(\mathbb{Z}(V/U)) = \mathbb{Z}(V'/S)$ with $V'/S = (V', h \circ h')$. Hence, since projective presheaves are the direct summands of

the representable presheaves, h_{\sharp} sends projective presheaves to projective presheaves. For $F^{\bullet} \in C(\text{Var}(\mathbb{C})^{sm}/S)$ and $G^{\bullet} \in C(\text{Var}(\mathbb{C})^{sm}/U)$, we have the adjunction maps

$$\text{ad}(h_{\sharp}, h^*)(G^{\bullet}) : G^{\bullet} \rightarrow h^*h_{\sharp}G^{\bullet}, \quad \text{ad}(h_{\sharp}, h^*)(F^{\bullet}) : h_{\sharp}h^*F^{\bullet} \rightarrow F^{\bullet}.$$

For a smooth morphism $h : U \rightarrow S$, with $U, S \in \text{Var}(\mathbb{C})$, we have the adjunction isomorphism, for $F \in C(\text{Var}(\mathbb{C})^{sm}/U)$ and $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$,

$$I(h_{\sharp}, h^*)(F, G) : \mathcal{H}om^{\bullet}(h_{\sharp}F, G) \xrightarrow{\sim} h_*\mathcal{H}om^{\bullet}(F, h^*G). \quad (25)$$

- For $f : T \rightarrow S$ any morphism with $T, S \in \text{Var}(\mathbb{C})$, the pullback functor $P(f) : \text{Var}(\mathbb{C})/T \rightarrow \text{Var}(\mathbb{C})/S$ admits a left adjoint $C(f)(X \rightarrow T) = (X \rightarrow T \rightarrow S)$. Hence, $f^* : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/T)$ admits a left adjoint

$$f_{\sharp} : C(\text{Var}(\mathbb{C})/T) \rightarrow C(\text{Var}(\mathbb{C})/S), \quad F \mapsto ((V, h_0) \mapsto \lim_{(V', f \circ h') \rightarrow (V, h_0)} F(V', h'))$$

Note that we have, for $(V/T) = (V, h)$, $f_{\sharp}\mathbb{Z}(V/T) = \mathbb{Z}(V/S)$ with $V/S = (V, f \circ h)$. Hence, since projective presheaves are the direct summands of the representable presheaves, h_{\sharp} sends projective presheaves to projective presheaves. For $F^{\bullet} \in C(\text{Var}(\mathbb{C})/S)$ and $G^{\bullet} \in C(\text{Var}(\mathbb{C})/T)$, we have the adjunction maps

$$\text{ad}(f_{\sharp}, f^*)(G^{\bullet}) : G^{\bullet} \rightarrow f^*f_{\sharp}G^{\bullet}, \quad \text{ad}(f_{\sharp}, f^*)(F^{\bullet}) : f_{\sharp}f^*F^{\bullet} \rightarrow F^{\bullet}.$$

For a morphism $f : T \rightarrow S$, with $T, S \in \text{Var}(\mathbb{C})$, we have the adjunction isomorphism, for $F \in C(\text{Var}(\mathbb{C})/T)$ and $G \in C(\text{Var}(\mathbb{C})/S)$,

$$I(f_{\sharp}, f^*)(F, G) : \mathcal{H}om^{\bullet}(f_{\sharp}F, G) \xrightarrow{\sim} f_*\mathcal{H}om^{\bullet}(F, f^*G). \quad (26)$$

- For a commutative diagram in $\text{Var}(\mathbb{C})$:

$$D = \begin{array}{ccc} V & \xrightarrow{g_2} & U \\ \downarrow h_2 & & \downarrow h_1 \\ T & \xrightarrow{g_1} & S \end{array},$$

where h_1 and h_2 are smooth, we denote by, for $F^{\bullet} \in C(\text{Var}(\mathbb{C})^{sm}/U)$,

$$T_{\sharp}(D)(F^{\bullet}) : h_{2\sharp}g_2^*F^{\bullet} \rightarrow g_1^*h_{1\sharp}F^{\bullet}$$

the canonical map in $C(\text{Var}(\mathbb{C})^{sm}/T)$ given by adjunction. If D is cartesian with $h_1 = h$, $g_1 = g$, $f_2 = h' : U_T \rightarrow T$, $g' : U_T \rightarrow U$,

$$T_{\sharp}(D)(F^{\bullet}) =: T_{\sharp}(g, h)(F^{\bullet}) : h'_{\sharp}g'^*F^{\bullet} \xrightarrow{\sim} g^*h_{\sharp}F^{\bullet}$$

is an isomorphism and for $G^{\bullet} \in C(\text{Var}(\mathbb{C})^{sm}/T)$

$$T(D)(G^{\bullet}) =: T(g, h)(G^{\bullet}) : g^*h_*G^{\bullet} \xrightarrow{\sim} h'_*g'^*G^{\bullet}$$

is an isomorphism.

- For a commutative diagram in $\text{Var}(\mathbb{C})$:

$$D = \begin{array}{ccc} V & \xrightarrow{g_2} & X \\ \downarrow f_2 & & \downarrow f_1 \\ T & \xrightarrow{g_1} & S \end{array},$$

we denote by, for $F^\bullet \in C(\text{Var}(\mathbb{C})/X)$,

$$T_{\sharp}(D)(F^\bullet) : f_{2\sharp}g_2^*F^\bullet \rightarrow g_1^*f_{1\sharp}F^\bullet$$

the canonical map in $C(\text{Var}(\mathbb{C})/T)$ given by adjonction. If D is cartesian with $h_1 = h$, $g_1 = g$, $f_2 = h' : X_T \rightarrow T$, $g' : X_T \rightarrow X$,

$$T_{\sharp}(D)(F^\bullet) =: T_{\sharp}(g, f)(F^\bullet) : f_{\sharp}'g'^*F^\bullet \xrightarrow{\sim} g^*f_{\sharp}F^\bullet$$

is an isomorphism and for $G^\bullet \in C(\text{Var}(\mathbb{C})/T)$

$$T(D)(G^\bullet) =: T(g, h)(G^\bullet) : f^*g_*G^\bullet \xrightarrow{\sim} g'_*f'^*G^\bullet$$

is an isomorphism.

For $f : T \rightarrow S$ a morphism with $S, T \in \text{Var}(\mathbb{C})$,

- we get for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ from the a commutative diagram of sites (24) the following canonical transformation

$$T(e, f)(F^\bullet) : f^*e(S)_*F^\bullet \rightarrow e(T)_*f^*F^\bullet,$$

which is NOT a quasi-isomorphism in general. However, for $h : U \rightarrow S$ a smooth morphism with $S, U \in \text{Var}(\mathbb{C})$, $T(e, h)(F^\bullet) : h^*e(S)_*F^\bullet \xrightarrow{\sim} e(T)_*h^*F^\bullet$ is an isomorphism.

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Let $S \in \text{Var}(\mathbb{C})$,

- We have for $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$,
 - $e(S)_*(F \otimes G) = (e(S)_*F) \otimes (e(S)_*G)$ by definition
 - the canonical forgetfull map

$$T(S, \text{hom})(F, G) : e(S)_*\mathcal{H}om^\bullet(F, G) \rightarrow \mathcal{H}om^\bullet(e(S)_*F, e(S)_*G).$$

which is NOT a quasi-isomorphism in general.

By definition, we have for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, $e(S)_*E_{zar}(F) = E_{zar}(e(S)_*F)$.

- We have for $F, G \in C(\text{Var}(\mathbb{C})/S)$,
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which is NOT a quasi-isomorphism in general.

By definition, we have for $F \in C(\text{Var}(\mathbb{C})/S)$, $e(S)_*E_{zar}(F) = E_{zar}(e(S)_*F)$.

Let $S \in \text{Var}(\mathbb{C})$.

- We have the dual functor

$$\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S), F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}_S))$$

It induces the functor

$$L\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S), F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}_S))$$

- We have the dual functor

$$\mathbb{D}_S : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/S), F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}_S))$$

It induces the functor

$$L\mathbb{D}_S : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/S), F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}_S))$$

The adjonctions

$$(\tilde{e}(S)^*, \tilde{e}(S)_*) : C(\text{Var}(\mathbb{C})/S) \rightleftarrows C(S), (e(S)^*, e(S)_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(S)$$

induce adjonctions

$$(\tilde{e}(S)^*, \tilde{e}(S)_*) : C_{fil}(\text{Var}(\mathbb{C})/S) \rightleftarrows C_{fil}(S), (e(S)^*, e(S)_*) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C_{fil}(S)$$

given by $e(S)_*(G, F) := (e(S)_*G, e(S)_*F)$, since $e(S)_*$ and $e(S)^*$ preserve monomorphisms. Note that

- for $F \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S)$, $e(S)_*F$ is simply the restriction of F to the small Zariski site of X ,
- for $F \in \text{PSh}(\text{Var}(\mathbb{C})/S)$, $\tilde{e}(S)_*F = e(S)_*\rho_{S*}F$ is simply the restriction of F to the small Zariski site of X , $\rho_{S*}F$ being the restriction of F to $\text{Var}(\mathbb{C})^{sm}/S$.

Together with the internal hom functor, we get the bifunctor,

$$e(S)_*\mathcal{H}om(\cdot, \cdot) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \times C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{2fil}(S), \quad (27)$$

$$((F, W), (G, F)) \mapsto e(S)_*\mathcal{H}om^\bullet((F^\bullet, W), (G^\bullet, F)). \quad (28)$$

For $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \text{Var}(\mathbb{C})$, we denote by

$$(i_*, i^!) := (P(i)_*, P(i)^\perp) : C(\text{Var}(\mathbb{C})^{sm}/Z) \rightleftarrows C(\text{Var}(\mathbb{C})^{sm}/S)$$

the adjunction induced by the morphism of site $P(i) : \text{Var}(\mathbb{C})^{sm}/Z \rightarrow \text{Var}(\mathbb{C})^{sm}/S$ For $i : Z \hookrightarrow S$ a closed embedding, $Z, S \in \text{Var}(\mathbb{C})$, we denote

$$\mathbb{Z}_{Z,S} := \text{Cone}(\text{ad}(i^*, i_*)(\mathbb{Z}_S) : \mathbb{Z}_S \rightarrow i_*\mathbb{Z}_Z)$$

We have the support section functors of a closed embedding $i : Z \hookrightarrow S$ for presheaves on the big Zariski site.

Definition 6. *Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \text{Var}(\mathbb{C})$ and $j : S \setminus Z \hookrightarrow S$ be the open complementary subset.*

(i) *We define the functor*

$$\Gamma_Z : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S), G^\bullet \mapsto \Gamma_Z G^\bullet := \text{Cone}(\text{ad}(j^*, j_*)(G^\bullet) : G^\bullet \rightarrow j_*j^*G^\bullet)[-1],$$

so that there is then a canonical map $\gamma_Z(G^\bullet) : \Gamma_Z G^\bullet \rightarrow G^\bullet$.

(ii) We have the dual functor of (i) :

$$\Gamma_Z^\vee : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S), F \mapsto \Gamma_Z^\vee(F^\bullet) := \text{Cone}(\text{ad}(j_\#, j^*)(G^\bullet) : j_\# j^* G^\bullet \rightarrow G^\bullet),$$

together with the canonical map $\gamma_Z^\vee(G) : F \rightarrow \Gamma_Z^\vee(G)$.

(iii) For $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we denote by

$$I(\gamma, \text{hom})(F, G) := (I, I(j_\#, j^*)(F, G)^{-1}) : \Gamma_Z \mathcal{H}om(F, G) \xrightarrow{\sim} \mathcal{H}om(\Gamma_Z^\vee F, G)$$

the canonical isomorphism given by adjunction.

Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \text{Var}(\mathbb{C})$ and $j : S \setminus Z \hookrightarrow S$ be the open complementary.

- For $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the adjunction map $\text{ad}(i_*, i^!)(G) : i_* i^! G \rightarrow G$ factor through $\gamma_Z(G) :$

$$\text{ad}(i_*, i^!)(G) : i_* i^! G \xrightarrow{\text{ad}(i_*, i^!)(G)^\gamma} \Gamma_Z(G) \xrightarrow{\gamma_Z(G)} G.$$

However, note that when dealing with the big sites $P(i) : \text{Var}(\mathbb{C})^{sm}/Z \rightarrow \text{Var}(\mathbb{C})^{sm}/S$, if $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is not \mathbb{A}_S^1 local and Zariski fibrant,

$$\text{ad}(i_*, i^!)(G)^\gamma : i_* i^! G \rightarrow \Gamma_Z(G)$$

is NOT and homotopy equivalence, and $\Gamma_Z G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is NOT in general in the image of the functor $i_* : C(\text{Var}(\mathbb{C})^{sm}/Z) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S)$.

- For $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the adjunction map $\text{ad}(i^*, i_*)(G) : G \rightarrow i_* i^* G$ factor through $\gamma_Z^\vee(G) :$

$$\text{ad}(i^*, i_*)(G) : G \xrightarrow{\gamma_Z^\vee(G)} \Gamma_Z^\vee G \xrightarrow{\text{ad}(i^*, i_*)(G)^\gamma} i_* i^* G,$$

and as in (i), $\text{ad}(i^*, i_*)(G)^\gamma : \Gamma_Z^\vee(G) \rightarrow i_* i^* G$ is NOT an homotopy equivalence but

Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \text{Var}(\mathbb{C})$.

- Since $\Gamma_Z : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor

$$\Gamma_Z : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{fil}(\text{Var}(\mathbb{C})^{sm}/S), (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F)$$

- Since $\Gamma_Z^\vee : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor

$$\Gamma_Z^\vee : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{fil}(\text{Var}(\mathbb{C})^{sm}/S), (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F)$$

Definition-Proposition 4. (i) Let $g : S' \rightarrow S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $S', S, Z \in \text{Var}(\mathbb{C})$. Then, for $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$, there exist a map in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/S')$

$$T(g, \gamma)(G, F) : g^* \Gamma_Z(G, F) \rightarrow \Gamma_{Z \times_S S'} g^*(G, F)$$

unique up to homotopy such that $\gamma_{Z \times_S S'}(g^*(G, F)) \circ T(g, \gamma)(G, F) = g^* \gamma_Z(G, F)$.

(ii) Let $i_1 : Z_1 \hookrightarrow S$, $i_2 : Z_2 \hookrightarrow Z_1$ be closed embeddings with $S, Z_1, Z_2 \in \text{Var}(\mathbb{C})$. Then, for $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$,

- there exist a canonical map $T(Z_2/Z_1, \gamma)(G, F) : \Gamma_{Z_2}(G, F) \rightarrow \Gamma_{Z_1}(G, F)$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ unique up to homotopy such that $\gamma_{Z_1}(G, F) \circ T(Z_2/Z_1, \gamma)(G, F) = \gamma_{Z_2}(G, F)$, together with a distinguish triangle

$$\Gamma_{Z_2}(G, F) \xrightarrow{T(Z_2/Z_1, \gamma)(G, F)} \Gamma_{Z_1}(G, F) \xrightarrow{\text{ad}(j_2^*, j_{2*})(\Gamma_{Z_1}(G, F))} \Gamma_{Z_1 \setminus Z_2}(G, F) \rightarrow \Gamma_{Z_2}(G, F)[1]$$

in $K_{fil}(\text{Var}(\mathbb{C})^{sm}/S) := K(\text{PSh}_{fil}(\text{Var}(\mathbb{C})^{sm}/S))$,

- there exist a map $T(Z_2/Z_1, \gamma^\vee)(G, F) : \Gamma_{Z_1}^\vee(G, F) \rightarrow \Gamma_{Z_2}^\vee(G, F)$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ unique up to homotopy such that $\gamma_{Z_2}^\vee(G, F) = T(Z_2/Z_1, \gamma^\vee)(G, F) \circ \gamma_{Z_1}^\vee(G, F)$, together with a distinguished triangle

$$\Gamma_{Z_1 \setminus Z_2}^\vee(G, F) \xrightarrow{\text{ad}(j_{2\#}, j_2^*)(\Gamma_{Z_1}^\vee(G, F))} \Gamma_{Z_1}^\vee(G, F) \xrightarrow{T(Z_2/Z_1, \gamma^\vee)(G, F)} \Gamma_{Z_2}^\vee(G, F) \rightarrow \Gamma_{Z_1 \setminus Z_2}^\vee(G, F)[1]$$

in $K_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$.

- (iii) Consider a morphism $g : (S', Z') \rightarrow (S, Z)$ with $(S', Z'), (S, Z) \in \text{Var}^2(\mathbb{C})$. We denote, for $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ the composite

$$T(D, \gamma^\vee)(G) : g^* \Gamma_Z^\vee G \xrightarrow{\sim} \Gamma_{Z \times_S S'}^\vee g^* G \xrightarrow{T(Z'/Z \times_S S', \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^* G$$

and we have then the factorization $\gamma_{Z'}^\vee(g^* G) : g^* G \xrightarrow{g^* \gamma_Z^\vee(G)} g^* \Gamma_Z^\vee G \xrightarrow{T(D, \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^* G$.

Proof. (i): We have the cartesian square

$$\begin{array}{ccc} S \setminus Z & \xrightarrow{j} & S \\ g' \uparrow & & \uparrow g \\ S' \setminus Z \times_S S' & \xrightarrow{j'} & S' \end{array}$$

and the map is given by

$$(I, T(g, j)(j^* G)) : \text{Cone}(g^* G \rightarrow g^* j_* j^* G) \rightarrow \text{Cone}(g^* G \rightarrow j'_* j'^* g^* G = j'_* g'^* j^* G).$$

- (ii): Follows from the fact that $j_1^* \Gamma_{Z_2} G = 0$ and $j_1^* \Gamma_{Z_2}^\vee G = 0$, with $j_1 : S \setminus Z_1 \hookrightarrow S$ the closed embedding.
- (iii): Obvious. \square

The following easy proposition concern the restriction from the big Zariski site to the small site Zariski site :

Proposition 16. For $f : T \rightarrow S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding, with $Z, S, T \in \text{Var}(\mathbb{C})$, we have

- (i) $e(S)_* f_* = f_* e(T)_*$ and $e(S)^* f_* = f_* e(T)^*$
- (ii) $e(S)_* \Gamma_Z = \Gamma_Z e(S)_*$.

Proof. (i): The first equality $e(S)_* f_* = f_* e(T)_*$ is given by the diagram (24). The second equality is immediate from definition after a direct computation.

- (ii) For $G^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we have the canonical equality

$$\begin{aligned} e(S)_* \Gamma_Z(G^\bullet) &= e(S)_* \text{Cone}(G \rightarrow j_* j^* G^\bullet)[-1] &= \text{Cone}(e(S)_* G^\bullet \rightarrow e(S)_* j_* j^* G^\bullet)[-1] \\ &= \text{Cone}(e(S)_* G^\bullet \rightarrow j_* j^* e(S)_* G^\bullet)[-1] \\ &= \Gamma_Z e(S)_* G^\bullet \end{aligned}$$

by (i) and since $j : S \setminus Z \hookrightarrow S$ is a smooth morphism. \square

Definition 7. For $S \in \text{Var}(\mathbb{C})$, we denote by

$$C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S) := C_{e(S)^* O_S}(\text{Var}(\mathbb{C})^{sm}/S)$$

the category of complexes of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ endowed with a structure of $e(S)^* O_S$ module, and by

$$C_{O_S fil}(\text{Var}(\mathbb{C})^{sm}/S) := C_{e(S)^* O_S fil}(\text{Var}(\mathbb{C})^{sm}/S)$$

the category of filtered complexes of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ endowed with a structure of $e(S)^* O_S$ module.

Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding,

- For $G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$, $\Gamma_Z G := \text{Cone}(\text{ad}(j^*, j_*)(G) : F \rightarrow j_* j^* G)[-1]$ has a (unique) structure of $e(S)^* O_S$ module such that $\gamma_Z(G) : \Gamma_Z G \rightarrow G$ is a map in $C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$. This gives the functor

$$\Gamma_Z : C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{\text{fil} O_S}(\text{Var}(\mathbb{C})^{sm}/S), (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F),$$

together with the canonical map $\gamma_Z((G, F) : \Gamma_Z(G, F) \rightarrow (G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$, $T(Z_2/Z, \gamma)(G) : \Gamma_{Z_2} G \rightarrow \Gamma_Z G$ is a map in $C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$ (i.e. is $e(S)^* O_S$ linear).

- For $G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$, $\Gamma_Z^\vee G := \text{Cone}(\text{ad}(j_\#^*, j^*)(G) : j_\# j^* G \rightarrow G)$ has a unique structure of $e(S)^* O_S$ module, such that $\gamma_Z^\vee(G) : G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$. This gives the the functor

$$\Gamma_Z^\vee : C_{O_S \text{fil}}(S) \rightarrow C_{\text{fil} O_S}(S), (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F),$$

together with the canonical map $\gamma_Z^\vee((G, F) : (G, F) \rightarrow \Gamma_Z^\vee(G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$, $T(Z_2/Z, \gamma^\vee)(G) : \Gamma_{Z_2}^\vee G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$ (i.e. is $e(S)^* O_S$ linear).

Definition 8. Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset.

(i) We denote by

$$C_Z(\text{Var}(\mathbb{C})^{sm}/S) \subset C(\text{Var}(\mathbb{C})^{sm}/S)$$

the full subcategory consisting of complexes of presheaves $F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $a_{et} H^n(j^* F^\bullet) = 0$ for all $n \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{et} is the sheafification functor.

(i)' We denote by

$$C_{O_S, Z}(\text{Var}(\mathbb{C})^{sm}/S) \subset C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S),$$

the full subcategory consisting of complexes of presheaves $F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $a_{et} H^n(j^* F^\bullet) = 0$ for all $n \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{et} is the sheafification functor.

(ii) We denote by

$$C_{\text{fil}, Z}(\text{Var}(\mathbb{C})^{sm}/S) \subset C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$$

the full subcategory consisting of filtered complexes of presheaves $(F^\bullet, F) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that there exist $r \in \mathbb{N}$ and an r -filtered homotopy equivalence $\phi : (F^\bullet, F) \rightarrow (F'^\bullet, F)$ with $(F'^\bullet, F) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $a_{et} j^* H^n \text{Gr}_F^p(F'^\bullet, F) = 0$ for all $n, p \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{et} is the sheafification functor. Note that by definition this r does NOT depend on n and p .

(ii)' We denote by

$$C_{O_S \text{fil}, Z}(\text{Var}(\mathbb{C})^{sm}/S) \subset C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$$

the full subcategory consisting of filtered complexes of presheaves $(F^\bullet, F) \in C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that there exist $r \in \mathbb{N}$ and an r -filtered homotopy equivalence $\phi : (F^\bullet, F) \rightarrow (F'^\bullet, F)$ with $(F'^\bullet, F) \in C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $a_{et} j^* H^n \text{Gr}_F^p(F'^\bullet, F) = 0$ for all $p, q \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{et} is the sheafification functor. Note that by definition this r does NOT depend on n and p .

Let $S \in \text{Var}(\mathbb{C})$ and $Z \subset S$ a closed subset.

- For $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$, we have $\Gamma_Z(G, F), \Gamma_Z^\vee(G, F) \in C_{fil,Z}(\text{Var}(\mathbb{C})^{sm}/S)$.
- For $(G, F) \in C_{O_S fil}(\text{Var}(\mathbb{C})^{sm}/S)$, we have $\Gamma_Z(G, F), \Gamma_Z^\vee(G, F) \in C_{O_S fil,Z}(\text{Var}(\mathbb{C})^{sm}/S)$.

Let $S_\bullet \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C}))$ with $\mathcal{I} \in \text{Cat}$, a diagram of algebraic varieties. It gives the diagram of sites $\text{Var}(\mathbb{C})^2/S_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat})$.

- Then $C_{fil}(\text{Var}(\mathbb{C})/S_\bullet)$ is the category
 - whose objects $(G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})/S_I)$, and $u_{IJ} : (G_I, F) \rightarrow r_{IJ*}(G_J, F)$ for $r_{IJ} : I \rightarrow J$, denoting again $r_{IJ} : S_I \rightarrow S_J$, are morphisms satisfying for $I \rightarrow J \rightarrow K$, $r_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{Var}(\mathbb{C})/S_I)$,
 - the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \in \mathcal{I}}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})/S_I)$.

- Then $C_{fil}(\text{Var}(\mathbb{C})^{sm}/S_\bullet)$ is the category
 - whose objects $(G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S_I)$, and $u_{IJ} : (G_I, F) \rightarrow r_{IJ*}(G_J, F)$ for $r_{IJ} : I \rightarrow J$, denoting again $r_{IJ} : S_I \rightarrow S_J$, are morphisms satisfying for $I \rightarrow J \rightarrow K$, $r_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/S_I)$,
 - the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \in \mathcal{I}}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/S_I)$.

As usual, we denote by

$$(f_\bullet^*, f_{\bullet*}) := (P(f_\bullet)^*, P(f_\bullet)_*) : C(\text{Var}(\mathbb{C})^{(sm)}/S_\bullet) \rightarrow C(\text{Var}(\mathbb{C})^{(sm)}/T_\bullet)$$

the adjunction induced by $P(f_\bullet) : \text{Var}(\mathbb{C})^{(sm)}/T_\bullet \rightarrow \text{Var}(\mathbb{C})^{(sm)}/S_\bullet$. Since the colimits involved in the definition of $f_\bullet^* = P(f_\bullet)^*$ are filtered, f_\bullet^* also preserve monomorphism. Hence, we get an adjunction

$$(f_\bullet^*, f_{\bullet*}) : C_{fil}(\text{Var}(\mathbb{C})^{(sm)}/S_\bullet) \rightleftarrows C_{fil}(\text{Var}(\mathbb{C})^{(sm)}/T_\bullet),$$

$$f_\bullet^*((G_I, F), u_{IJ}) := ((f_I^*G_I, f_I^*F), T(f_I, r_{IJ})(-) \circ f_I^*u_{IJ}).$$

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and for $J \subset I$ the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

where $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ is the projection and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. This gives the diagram of algebraic varieties $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Var}(\mathbb{C}))$ which the diagram of sites $\text{Var}(\mathbb{C})^{sm}/(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. Denote by $m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I$ the open embedding. Then $C_{fil}(\text{Var}(\mathbb{C})^{sm}/(\tilde{S}_I))$ is the category

- whose objects $(G, F) = ((G_I, F), u_{IJ})$ with $(G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$, and $u_{IJ} : (G_I, F) \rightarrow p_{IJ*}(G_J, F)$ are morphisms satisfying for $I \subset J \subset K$, $p_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$,

- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \in \mathcal{I}}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$.

Definition 9. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. We will denote by $C_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) \subset C_{fil}(\text{Var}(\mathbb{C})^{sm}/(\tilde{S}_I))$ the full subcategory

- whose objects $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil, S_I}(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$, and $u_{IJ} : m^*(G_I, F) \rightarrow m^* p_{IJ*}(G_J, F)$ for $I \subset J$, are ∞ -filtered Zariski local equivalence, satisfying for $I \subset J \subset K$, $p_{IJ*} u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$,
- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$.

A morphism $m : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$ is an r -filtered Zariski, resp. etale local, equivalence, if there exists $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$, $0 \leq i \leq s$, with $((C_{iI}, F), u_{iIJ}) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$ $((C_{0I}, F), u_{0IJ}) = ((G_I, F), u_{IJ})$ and $((C_{sI}, F), u_{sIJ}) = ((H_I, F), v_{IJ})$, such that

$$\phi = \phi_s \circ \dots \circ \phi_1 \circ \phi_0 : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$$

and $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$ either a filtered Zariski, resp. etale, local equivalence or an r -filtered homotopy equivalence.

Denote $L = [1, \dots, l]$ and for $I \subset L$, $p_{0(0I)} : S \times \tilde{S}_I \rightarrow S$, $p_{I(0I)} : S \times \tilde{S}_I \rightarrow S_I$ the projections. By definition, we have functors

- $T(S/(\tilde{S}_I)) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$, $(G, F) \mapsto (i_{I*} j_I^*(G, F), T(D_{IJ})(j_I^*(G, F)))$,
- $T((\tilde{S}_I)/S) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) \rightarrow C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$, $((G_I, F), u_{IJ}) \mapsto \text{ho} \lim_{I \subset L} p_{0(0I)*} \Gamma_{\tilde{S}_I}^\vee p_{I(0I)}^*(G_I, F)$.

Note that the functors $T(S/(\tilde{S}_I))$ are NOT embedding, since

$$\text{ad}(i_I^*, i_{I*})(j_I^* F) : i_I^* i_{I*} j_I^* F \rightarrow j_I^* F$$

are Zariski local equivalence but NOT isomorphism since we are dealing with the morphism of big sites $P(i_I) : \text{Var}(\mathbb{C})^{sm}/S_I \rightarrow \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I$. However, by theorem 16, these functors induce full embeddings

$$T(S/(\tilde{S}_I)) : D_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow D_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$$

since for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$,

$$\text{ho} \lim_{I \subset L} p_{0(0I)*} \Gamma_{\tilde{S}_I}^\vee p_{I(0I)}^*(i_{I*} j_I^* F) \rightarrow p_{0(0I)*} \Gamma_{\tilde{S}_I}^\vee j_I^* F$$

is an equivalence Zariski local.

Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ and $X = \cup_{i=1}^l X_i$ be affine open covers and $i_i : S_i \hookrightarrow \tilde{S}_i$, $i'_i : X_i \hookrightarrow \tilde{X}_i$ be closed embeddings. Let $f_i : \tilde{X}_i \rightarrow \tilde{S}_i$ be a lift of the morphism $f_i = f|_{X_i} : X_i \rightarrow S_i$. Then, $f_I = f|_{X_I} : X_I = \cap_{i \in I} X_i \rightarrow S_I = \cap_{i \in I} S_i$ lift to the morphism

$$\tilde{f}_I = \prod_{i \in I} \tilde{f}_i : \tilde{X}_I = \prod_{i \in I} \tilde{X}_i \rightarrow \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$$

Denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} : \tilde{X}_J \rightarrow \tilde{X}_I$ the projections. Consider for $J \subset I$ the following commutative diagrams

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}, \quad D'_{IJ} = \begin{array}{ccc} X_I & \xrightarrow{i'_I} & \tilde{X}_I \\ j'_{IJ} \uparrow & & \uparrow p'_{IJ} \\ X_J & \xrightarrow{i'_J} & \tilde{X}_J \end{array}, \quad D_{fI} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ f_I \uparrow & & \uparrow \tilde{f}_I \\ X_I & \xrightarrow{i'_I} & \tilde{X}_I \end{array}$$

We have then following commutative diagram

$$\begin{array}{ccccc} & & X_I & \xrightarrow{n'_I} & \tilde{X}_I & \longleftarrow & \tilde{X}_I \setminus X_I \\ & \nearrow j'_{IJ} & \uparrow p'_{IJ} & & \uparrow p'_{IJ} & & \uparrow p'_{IJ} \\ i'_J : X_J & \xrightarrow{l_{IJ}} & X_I \times X_I \times \tilde{X}_J \xrightarrow{n'_I \times I} & \tilde{X}_J & \longleftarrow & \tilde{X}_J \setminus X_J \end{array}$$

whose square are cartesian. We then have the pullback functor

$$f^* : C_{(2)fil}(\text{Var}(\mathbb{C})^{sm}/S/(\tilde{S}_I)) \rightarrow C_{(2)fil}(\text{Var}(\mathbb{C})^{sm}/X/(\tilde{X}_I)), \\ ((G_I, F), u_{IJ}) \mapsto f^*((G_I, F), u_{IJ}) := (\Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F), \tilde{f}_J^* u_{IJ})$$

with

$$\begin{aligned} \tilde{f}_J^* u_{IJ} : \Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F) &\xrightarrow{\text{ad}(p'_{IJ}, p'_{IJ*})(-)} p'_{IJ*} p'_{IJ}{}^* \Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F) \xrightarrow{T_{\sharp}(p_{IJ}, n'_I)(-)^{-1}} p'_{IJ*} \Gamma_{X_I \times \tilde{X}_J \setminus I}^\vee p'_{IJ}{}^* \tilde{f}_I^*(G_I, F) \\ &\xrightarrow{p'_{IJ*} \gamma_{X_J}^\vee(-)} p'_{IJ*} \Gamma_{X_J}^\vee p'_{IJ}{}^* \tilde{f}_I^*(G_I, F) = p'_{IJ*} \Gamma_{X_J}^\vee \tilde{f}_J^* p'_{IJ}{}^*(G_I, F) \xrightarrow{\Gamma_{X_J}^\vee \tilde{f}_J^* I(p'_{IJ}, p_{IJ*})(-, -)(u_{IJ})} \Gamma_{X_J}^\vee \tilde{f}_J^*(G_J, F) \end{aligned}$$

Let $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$. Since, $j'_I{}^* i'_{I*} j'_I{}^* f^*(G, F) = 0$, the morphism $T(D_{fI})(j'_I{}^*(G, F)) : \tilde{f}_I^* i_{I*} j'_I{}^*(G, F) \rightarrow i'_{I*} j'_I{}^* f^*(G, F)$ factors through

$$T(D_{fI})(j'_I{}^*(G, F)) : \tilde{f}_I^* i_{I*} j'_I{}^*(G, F) \xrightarrow{\gamma_{X_I}^\vee(-)} \Gamma_{X_I}^\vee \tilde{f}_I^* i_{I*} j'_I{}^*(G, F) \xrightarrow{T^\gamma(D_{fI})(j'_I{}^*(G, F))} i'_{I*} j'_I{}^* f^*(G, F)$$

We have then, for $(G, F) \in C_{fil}(S)$, the canonical transformation map

$$\begin{array}{ccc} f^* T(S/(\tilde{S}_I))(G, F) & \xrightarrow{T(f, T(0/I))(G, F)} & T(X/(\tilde{X}_I))(f^*(G, F)) \\ \downarrow = & & \downarrow = \\ (\Gamma_{X_I}^\vee \tilde{f}_I^* i_{I*} j'_I{}^*(G, F), \tilde{f}_J^* I) & \xrightarrow{T^\gamma(D_{fI})(j'_I{}^*(G, F))} & (i'_{I*} j'_I{}^* f^*(G, F), I) \end{array}$$

To show that the cohomology sheaves of the filtered De Rham realization functor of constructible motives are mixed hodge modules, we will need to take presheaves of the following form

Definition 10. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then consider

$$Q(X/S) := p_{\sharp} \Gamma_X^\vee \mathbb{Z}_{Y \times S} \in C(\text{Var}(\mathbb{C})^{sm}/S).$$

By definition $Q(X/S)$ is projective.

(ii) Let $f : X \rightarrow S$ and $g : T \rightarrow S$ two morphism with $X, S, T \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then have the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} f : X & \xrightarrow{i} & Y \times S & \xrightarrow{p} & S \\ g' \uparrow & & g'' := (I \times g) \uparrow & & g \uparrow \\ f' : X_T & \xrightarrow{i'} & Y \times T & \xrightarrow{p'} & T \end{array}$$

We then have the canonical isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/T)$

$$\begin{aligned} T(f, g, Q) : g^*Q(X/S) &:= g^*p_{\sharp}\Gamma_X^{\vee}\mathbb{Z}_{Y \times S} \xrightarrow{T_{\sharp}(g,p)(-)^{-1}} p'_{\sharp}g''^*\Gamma_X^{\vee}\mathbb{Z}_{Y \times S} \\ &\xrightarrow{p'^*T(g'', \gamma^{\vee})(-)^{-1}} p'_{\sharp}\Gamma_{X_T}^{\vee}\mathbb{Z}_{Y \times T} =: Q(X_T/T). \end{aligned}$$

(iii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then consider

$$Q^h(X/S) := p_*\Gamma_X E_{et}(\mathbb{Z}_{Y \times S}) \in C(\text{Var}(\mathbb{C})^{sm}/S).$$

(iv) Let $f : X \rightarrow S$ and $g : T \rightarrow S$ two morphism with $X, S, T \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then have the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} f : X & \xrightarrow{i} & Y \times S & \xrightarrow{p} & S \\ g' \uparrow & & g'' := (I \times g) \uparrow & & g \uparrow \\ f' : X_T & \xrightarrow{i'} & Y \times T & \xrightarrow{p'} & T \end{array}$$

We then have the canonical morphism in $C(\text{Var}(\mathbb{C})^{sm}/T)$

$$\begin{aligned} T(f, g, Q^h) : g^*Q^h(X/S) &:= g^*p_*\Gamma_X E_{et}(\mathbb{Z}_{Y \times S}) \xrightarrow{T(g,p)(-)} p'_{\sharp}g''^*\Gamma_X E_{et}(\mathbb{Z}_{Y \times S}) \\ &\xrightarrow{p'^*T(g'', \gamma)(-)} p'_{\sharp}\Gamma_{X_T} E_{et}(\mathbb{Z}_{Y \times T}) =: Q^h(X_T/T). \end{aligned}$$

We now give the definition of the \mathbb{A}^1 local property :

Denote by

$$\begin{aligned} p_a : \text{Var}(\mathbb{C})^{(sm)}/S &\rightarrow \text{Var}(\mathbb{C})^{(sm)}/S, \quad X/S = (X, h) \mapsto (X \times \mathbb{A}^1)/S = (X \times \mathbb{A}^1, h \circ p_X), \\ (g : X/S &\rightarrow X'/S) \mapsto ((g \times I_{\mathbb{A}^1}) : X \times \mathbb{A}^1/S \rightarrow X' \times \mathbb{A}^1/S) \end{aligned}$$

the projection functor and again by $p_a : \text{Var}(\mathbb{C})^{(sm)}/S \rightarrow \text{Var}(\mathbb{C})^{(sm)}/S$ the corresponding morphism of site.

Definition 11. Let $S \in \text{Var}(\mathbb{C})$. Denote for short $\text{Var}(\mathbb{C})^{(sm)}/S$ either the category $\text{Var}(\mathbb{C})/S$ or the category $\text{Var}(\mathbb{C})^{sm}/S$.

(i0) A complex $F \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ is said to be \mathbb{A}^1 homotopic if $\text{ad}(p_a^*, p_{a*})(F) : F \rightarrow p_{a*}p_a^*F$ is an homotopy equivalence.

(i) A complex $F \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ is said to be \mathbb{A}^1 invariant if for all $U/S \in \text{Var}(\mathbb{C})^{(sm)}/S$,

$$F(p_U) : F(U/S) \rightarrow F(U \times \mathbb{A}^1/S)$$

is a quasi-isomorphism, where $p_U : U \times \mathbb{A}^1 \rightarrow U$ is the projection. Obviously, if a complex $F \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ is \mathbb{A}^1 homotopic then it is \mathbb{A}^1 invariant.

(ii) Let τ a topology on $\text{Var}(\mathbb{C})$. A complex $F \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ is said to be \mathbb{A}^1 local for the topology τ , if for a (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ τ fibrant, e.g. $k : F \rightarrow E_\tau(F)$, G is \mathbb{A}^1 invariant for all $n \in \mathbb{Z}$.

(iii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ is said to an (\mathbb{A}^1, et) local equivalence if for all $H \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ which is \mathbb{A}^1 local for the etale topology

$$\text{Hom}(L(m), E_{et}(H)) : \text{Hom}(L(G), E_{et}(H)) \rightarrow \text{Hom}(L(F), E_{et}(H))$$

is a quasi-isomorphism.

Denote $\square^* := \mathbb{P}^* \setminus \{1\}$

• Let $S \in \text{Var}(\mathbb{C})$. For $U/S = (U, h) \in \text{Var}(\mathbb{C})^{sm}/S$, we consider

$$\square^* \times U/S = (\square^* \times U, h \circ p) \in \text{Fun}(\Delta, \text{Var}(\mathbb{C})^{sm}/S).$$

For $F \in C^-(\text{Var}(\mathbb{C})^{sm}/S)$, it gives the complex

$$C_*F \in C^-(\text{Var}(\mathbb{C})^{sm}/S), U/S = (U, h) \mapsto C_*F(U/S) := \text{Tot } F(\square^* \times U/S)$$

together with the canonical map $c_F := (0, I_F) : F \rightarrow C_*F$. For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we get

$$C_*F := \text{holim}_n C_*F^{\leq n} \in C(\text{Var}(\mathbb{C})^{sm}/S),$$

together with the canonical map $c_F := (0, I_F) : F \rightarrow C_*F$. For $m : F \rightarrow G$ a morphism, with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we get by functoriality the morphism $C_*m : C_*F \rightarrow C_*G$.

• Let $S \in \text{Var}(\mathbb{C})$. For $U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, we consider

$$\square^* \times U/S = (\mathbb{A}^* \times U, h \circ p) \in \text{Fun}(\Delta, \text{Var}(\mathbb{C})/S).$$

For $F \in C^-(\text{Var}(\mathbb{C})/S)$, it gives the complex

$$C_*F \in C^-(\text{Var}(\mathbb{C})/S), U/S = (U, h) \mapsto C_*F(U/S) := \text{Tot } F(\square^* \times U/S)$$

together with the canonical map $c = c(F) := (0, I_F) : F \rightarrow C_*F$. For $F \in C(\text{Var}(\mathbb{C})/S)$, we get

$$C_*F := \text{holim}_n C_*F^{\leq n} \in C(\text{Var}(\mathbb{C})/S),$$

together with the canonical map $c_F := (0, I_F) : F \rightarrow C_*F$. For $m : F \rightarrow G$ a morphism, with $F, G \in C(\text{Var}(\mathbb{C})/S)$, we get by functoriality the morphism $C_*m : C_*F \rightarrow C_*G$.

Proposition 17. (i) Let $S \in \text{Var}(\mathbb{C})$. Then for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, C_*F is \mathbb{A}^1 local for the etale topology and $c(F) : F \rightarrow C_*F$ is an equivalence (\mathbb{A}^1, et) local.

(ii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{(sm)}/S)$ is an (\mathbb{A}^1, et) local equivalence if and only if there exists

$$\{X_{1,\alpha}/S, \alpha \in \Lambda_1\}, \dots, \{X_{r,\alpha}/S, \alpha \in \Lambda_r\} \subset \text{Var}(\mathbb{C})^{(sm)}/S$$

such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{(sm)}/S))$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}(X_{1,\alpha} \times \mathbb{A}^1/S) \rightarrow \mathbb{Z}(X_{1,\alpha}/S)) \\ &\rightarrow \dots \rightarrow \bigoplus_{\alpha \in \Lambda_r} \text{Cone}(\mathbb{Z}(X_{r,\alpha} \times \mathbb{A}^1/S) \rightarrow \mathbb{Z}(X_{r,\alpha}/S))) \end{aligned}$$

Proof. Standard : see Ayoub's thesis for example. □

Definition-Proposition 5. Let $S \in \text{Var}(\mathbb{C})$.

- (i) With the weak equivalence the (\mathbb{A}^1, et) local equivalence and the fibration the epimorphism with \mathbb{A}_S^1 local and etale fibrant kernels gives a model structure on $C(\text{Var}(\mathbb{C})^{sm}/S)$: the left bousfield localization of the projective model structure of $C(\text{Var}(\mathbb{C})^{sm}/S)$. We call it the projective (\mathbb{A}^1, et) model structure.
- (ii) With the weak equivalence the (\mathbb{A}^1, et) local equivalence and the fibration the epimorphism with \mathbb{A}_S^1 local and etale fibrant kernels gives a model structure on $C(\text{Var}(\mathbb{C})/S)$: the left bousfield localization of the projective model structure of $C(\text{Var}(\mathbb{C})/S)$. We call it the projective (\mathbb{A}^1, et) model structure.

Proof. See [10]. □

Proposition 18. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$.*

- (i) The adjonction $(g^*, g_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{sm}/T)$ is a Quillen adjonction for the (\mathbb{A}^1, et) projective model structure (see definition-proposition 5).
- (i)' Let $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{Var}(\mathbb{C})$. The adjonction $(h_{\sharp}, h^*) : C(\text{Var}(\mathbb{C})^{sm}/U) \rightleftarrows C(\text{Var}(\mathbb{C})^{sm}/S)$ is a Quillen adjonction for the (\mathbb{A}^1, et) projective model structure.
- (i)'' The functor $g^* : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.
- (ii) The adjonction $(g^*, g_*) : C(\text{Var}(\mathbb{C})/S) \rightleftarrows C(\text{Var}(\mathbb{C})/T)$ is a Quillen adjonction for the (\mathbb{A}^1, et) projective model structure (see definition-proposition 5).
- (ii)' The adjonction $(g_{\sharp}, g^*) : C(\text{Var}(\mathbb{C})/T) \rightleftarrows C(\text{Var}(\mathbb{C})/S)$ is a Quillen adjonction for the (\mathbb{A}^1, et) projective model structure.
- (ii)'' The functor $g^* : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.

Proof. Standard : see [10] for example. □

Proposition 19. *Let $S \in \text{Var}(\mathbb{C})$.*

- (i) The adjonction $(\rho_S^*, \rho_{S*}) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(\text{Var}(\mathbb{C})/S)$ is a Quillen adjonction for the (\mathbb{A}^1, et) projective model structure.
- (ii) The functor $\rho_{S*} : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.

Proof. Standard : see [10] for example. □

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \bigcap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. For $(G_I, K_{IJ}) \in C(\text{Var}(\mathbb{C})^{(sm)})/(\tilde{S}_I)^{op}$ and $(H_I, T_{IJ}) \in C(\text{Var}(\mathbb{C})^{(sm)})/(\tilde{S}_I)$, we denote

$$\text{Hom}((G_I, K_{IJ}), (H_I, T_{IJ})) := (\text{Hom}(G_I, H_I), u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ}))) \in C(\text{Var}(\mathbb{C})^{(sm)})/(\tilde{S}_I)$$

with

$$\begin{aligned} & u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ})) : \text{Hom}(G_I, H_I) \\ & \xrightarrow{\text{ad}(p_{IJ}^*, p_{IJ*})(-)} p_{IJ*} p_{IJ}^* \text{Hom}(G_I, H_I) \xrightarrow{T(p_{IJ}, \text{hom})(-, -)} p_{IJ*} \text{Hom}(p_{IJ}^* G_I, p_{IJ}^* H_I) \\ & \xrightarrow{\text{Hom}(p_{IJ}^* G_I, T_{IJ})} p_{IJ*} \text{Hom}(p_{IJ}^* G_I, H_J) \xrightarrow{\text{Hom}(K_{IJ}, H_J)} p_{IJ*} \text{Hom}(G_J, H_J). \end{aligned}$$

This gives in particular the functor

$$C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I)) \rightarrow C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I)^{op}), (H_I, T_{IJ}) \mapsto (H_I, T_{IJ}).$$

Let $S \in \mathrm{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. The functor p_a extend to a functor

$$p_a : \mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I) \rightarrow \mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I), (X_I/\tilde{S}_I, s_{IJ}) \mapsto (X \times \mathbb{A}^1/\tilde{S}_I, s_{IJ} \times I),$$

$$(g = (g_I) : (X_I/\tilde{S}_I, s_{IJ}) \rightarrow (X'_I/\tilde{S}_I, s'_{IJ})) \mapsto ((g \times I_{\mathbb{A}^1}) : (X_I \times \mathbb{A}^1/\tilde{S}_I, s_{IJ}) \rightarrow (X'_I \times \mathbb{A}^1/\tilde{S}_I, s'_{IJ}))$$

the projection functor and again by $p_a : \mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I) \rightarrow \mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I)$ the corresponding morphism of site.

Definition 12. Let $S \in \mathrm{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{Var}(\mathbb{C})$.

(i0) A complex $(F_I, u_{IJ}) \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ is said to be \mathbb{A}^1 homotopic if $\mathrm{ad}(p_a^*, p_{a*})((F_I, u_{IJ})) : (F_I, u_{IJ}) \rightarrow p_{a*}p_a^*(F_I, u_{IJ})$ is an homotopy equivalence.

(i) A complex $(F_I, u_{IJ}) \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ is said to be \mathbb{A}^1 invariant if for all $(X_I/\tilde{S}_I, s_{IJ}) \in \mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I)$

$$(F_I(p_{X_I})) : (F_I(X_I/\tilde{S}_I), F_J(s_{IJ})) \circ u_{IJ}(-) \rightarrow (F_I(X_I \times \mathbb{A}^1/\tilde{S}_I), F_J(s_{IJ} \times I)) \circ u_{IJ}(-)$$

is a quasi-isomorphism, where $p_{X_I} : X_I \times \mathbb{A}^1 \rightarrow X_I$ are the projection, and $s_{IJ} : X_I \times \tilde{S}_{J \setminus I}/\tilde{S}_I \rightarrow X_J/\tilde{S}_J$. Obviously a complex $(F_I, u_{IJ}) \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ is \mathbb{A}^1 invariant if and only if all the F_I are \mathbb{A}^1 invariant.

(ii) Let τ a topology on $\mathrm{Var}(\mathbb{C})$. A complex $F = (F_I, u_{IJ}) \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ is said to be \mathbb{A}^1 local for the τ topology induced on $\mathrm{Var}(\mathbb{C})/(\tilde{S}_I)$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G = (G_I, v_{IJ}) \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ τ fibrant, e.g. $k : (F_I, u_{IJ}) \rightarrow (E_\tau(F_I), E(u_{IJ}))$, G is \mathbb{A}^1 invariant.

(iii) A morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ is said to be an $(\mathbb{A}^1, \mathrm{et})$ local equivalence if for all $(H_I, w_{IJ}) \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ which is \mathbb{A}^1 local for the etale topology

$$(\mathrm{Hom}(L(m_I), E_{\mathrm{et}}(H_I))) : \mathrm{Hom}(L(G_I, v_{IJ}), E_{\mathrm{et}}(H_I, w_{IJ})) \rightarrow \mathrm{Hom}(L(F_I, u_{IJ}), E_{\mathrm{et}}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I)^{op})$ is an $(\mathbb{A}^1, \mathrm{et})$ local equivalence, then all the $m_I : F_I \rightarrow G_I$ are $(\mathbb{A}^1, \mathrm{et})$ local equivalence.

(iv) A morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I)^{op})$ is said to be an $(\mathbb{A}^1, \mathrm{et})$ local equivalence if for all $(H_I, w_{IJ}) \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ which is \mathbb{A}^1 local for the etale topology

$$(\mathrm{Hom}(L(m_I), E_{\mathrm{et}}(H_I))) : \mathrm{Hom}(L(G_I, v_{IJ}), E_{\mathrm{et}}(H_I, w_{IJ})) \rightarrow \mathrm{Hom}(L(F_I, u_{IJ}), E_{\mathrm{et}}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\mathrm{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I)^{op})$ is an $(\mathbb{A}^1, \mathrm{et})$ local equivalence, then all the $m_I : F_I \rightarrow G_I$ are $(\mathbb{A}^1, \mathrm{et})$ local equivalence.

Proposition 20. Let $S \in \mathrm{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{Var}(\mathbb{C})$.

(i) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{(sm)})/(\tilde{S}_I)$ is an (\mathbb{A}^1, et) local equivalence if and only if there exists

$$\left\{ (X_{1,\alpha,I}/\tilde{S}_I, u_{IJ}^1), \alpha \in \Lambda_1 \right\}, \dots, \left\{ (X_{r,\alpha,I}/\tilde{S}_I, u_{IJ}^r), \alpha \in \Lambda_r \right\} \subset \text{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I)$$

with $u_{IJ}^1 : X_{1,\alpha,I} \times \tilde{S}_{J \setminus I}/\tilde{S}_J \rightarrow X_{1,\alpha,J}/\tilde{S}_J$, such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{(sm)})/(\tilde{S}_I))$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}(X_{1,\alpha,I} \times \mathbb{A}^1/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1 \times I)) \rightarrow (\mathbb{Z}(X_{1,\alpha,I}/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1))) \\ &\rightarrow \dots \rightarrow \bigoplus_{\alpha \in \Lambda_r} \text{Cone}(\mathbb{Z}(X_{r,\alpha,I} \times \mathbb{A}^1/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r \times I)) \rightarrow (\mathbb{Z}(X_{r,\alpha,I}/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r))) \end{aligned}$$

(ii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{(sm)})/(\tilde{S}_I)^{op}$ is an (\mathbb{A}^1, et) local equivalence if and only if there exists

$$\left\{ (X_{1,\alpha,I}/\tilde{S}_I, u_{IJ}^1), \alpha \in \Lambda_1 \right\}, \dots, \left\{ (X_{r,\alpha,I}/\tilde{S}_I, u_{IJ}^r), \alpha \in \Lambda_r \right\} \subset \text{Var}(\mathbb{C})^{(sm)}/(\tilde{S}_I)^{op}$$

with $u_{IJ}^1 : X_{1,\alpha,J}/\tilde{S}_J \rightarrow X_{1,\alpha,I} \times \tilde{S}_{J \setminus I}/\tilde{S}_J$, such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{(sm)})/(\tilde{S}_I)^{op})$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}(X_{1,\alpha,I} \times \mathbb{A}^1/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1 \times I)) \rightarrow (\mathbb{Z}(X_{1,\alpha,I}/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1))) \\ &\rightarrow \dots \rightarrow \bigoplus_{\alpha \in \Lambda_r} \text{Cone}(\mathbb{Z}(X_{r,\alpha,I} \times \mathbb{A}^1/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r \times I)) \rightarrow (\mathbb{Z}(X_{r,\alpha,I}/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r))) \end{aligned}$$

Proof. Standard. See Ayoub's thesis for example. \square

- For $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, we denote as usual (see [10] for example), $\mathbb{Z}^{tr}(X/S) \in \text{PSh}(\text{Var}(\mathbb{C})/S)$ the presheaf given by

- for $X'/S \in \text{Var}(\mathbb{C})/S$, with X' irreducible, $\mathbb{Z}^{tr}(X/S)(X'/S) := \mathcal{Z}^{f^s/X}(X' \times_S X) \subset \mathcal{Z}_{d_{X'}}(X' \times_S X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times_S X)$ such that, denoting $\text{supp}(\alpha) = \cup_i \alpha_i \subset X' \times_S X$ its support and $f' : X' \times_S X \rightarrow X'$ the projection, $f'|_{\text{supp}(\alpha)} : \text{supp}(\alpha) \rightarrow X'$ is finite surjective,
- for $g : X_2/S \rightarrow X_1/S$ a morphism, with $X_1/S, X_2/S \in \text{Var}(\mathbb{C})/S$,

$$\mathbb{Z}^{tr}(X/S)(g) : \mathbb{Z}^{tr}(X/S)(X_1/S) \rightarrow \mathbb{Z}^{tr}(X/S)(X_2/S), \alpha \mapsto (g \times I)^{-1}(\alpha)$$

with $g \times I : X_2 \times_S X \rightarrow X_1 \times_S X$, noting that, by base change, $f_2|_{\text{supp}((g \times I)^{-1}(\alpha))} : \text{supp}((g \times I)^{-1}(\alpha)) \rightarrow X_2$ is finite surjective, $f_2 : X_2 \times_S X \rightarrow X_2$ being the projection.

- For $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ and $r \in \mathbb{N}$, we denote as usual (see [10] for example), $\mathbb{Z}^{equiv}(X/S) \in \text{PSh}(\text{Var}(\mathbb{C})/S)$ the presheaf given by

- for $X'/S \in \text{Var}(\mathbb{C})/S$, with X' irreducible, $\mathbb{Z}^{equiv}(X/S)(X'/S) := \mathcal{Z}^{equiv/X}(X' \times_S X) \subset \mathcal{Z}_{d_{X'}}(X' \times_S X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times_S X)$ such that, denoting $\text{supp}(\alpha) = \cup_i \alpha_i \subset X' \times_S X$ its support and $f' : X' \times_S X \rightarrow X'$ the projection, $f'|_{\text{supp}(\alpha)} : \text{supp}(\alpha) \rightarrow X'$ is dominant, with fibers either empty or of dimension r ,
- for $g : X_2/S \rightarrow X_1/S$ a morphism, with $X_1/S, X_2/S \in \text{Var}(\mathbb{C})/S$,

$$\mathbb{Z}^{equiv}(X/S)(g) : \mathbb{Z}^{equiv}(X/S)(X_1/S) \rightarrow \mathbb{Z}^{equiv}(X/S)(X_2/S), \alpha \mapsto (g \times I)^{-1}(\alpha)$$

with $g \times I : X_2 \times_S X \rightarrow X_1 \times_S X$, noting that, by base change, $f_2|_{\text{supp}((g \times I)^{-1}(\alpha))} : \text{supp}((g \times I)^{-1}(\alpha)) \rightarrow X_2$ is obviously dominant, with fibers either empty or of dimension r , $f_2 : X_2 \times_S X \rightarrow X_2$ being the projection.

- Let $S \in \text{Var}(\mathbb{C})$. We denote by $\mathbb{Z}_S(d) := \mathbb{Z}^{equiv^0}(S \times \mathbb{A}^d/S)[-2d]$ the Tate twist. For $F \in C(\text{Var}(\mathbb{C})/S)$, we denote by $F(d) := F \otimes \mathbb{Z}_S(d)$.

For $S \in \text{Var}(\mathbb{C})$, let $\text{Cor}(\text{Var}(\mathbb{C})^{sm}/S)$ be the category

- whose objects are smooth morphisms $U/S = (U, h)$, $h : U \rightarrow S$ with $U \in \text{Var}(\mathbb{C})$,
- whose morphisms $\alpha : U/S = (U, h_1) \rightarrow V/S = (V, h_2)$ is finite correspondence that is $\alpha \in \bigoplus_i \mathcal{Z}^{fs}(U_i \times_S V)$, where $U = \sqcup_i U_i$, with U_i connected (hence irreducible by smoothness), and $\mathcal{Z}^{fs}(U_i \times_S V)$ is the abelian group of cycle finite and surjective over U_i .

We denote by $\text{Tr}(S) : \text{Cor}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow \text{Var}(\mathbb{C})^{sm}/S$ the morphism of site given by the inclusion functor $\text{Tr}(S) : \text{Var}(\mathbb{C})^{sm}/S \hookrightarrow \text{Cor}(\text{Var}(\mathbb{C})^{sm}/S)$ It induces an adjunction

$$(\text{Tr}(S)^* \text{Tr}(S)_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(\text{Cor}(\text{Var}(\mathbb{C})^{sm}/S))$$

A complex of preheaves $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is said to admit transferts if it is in the image of the embedding

$$\text{Tr}(S)_* : C(\text{Cor}(\text{Var}(\mathbb{C})^{sm}/S) \hookrightarrow C(\text{Var}(\mathbb{C})^{sm}/S),$$

that is $G = \text{Tr}(S)_* \text{Tr}(S)^* G$.

We will use to compute the algebraic Gauss-Manin realization functor the following

Theorem 10. *Let $\phi : F^\bullet \rightarrow G^\bullet$ an etale local equivalence with $F^\bullet, G^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$. If F^\bullet and G^\bullet are \mathbb{A}^1 local and admit transferts then $\phi : F^\bullet \rightarrow G^\bullet$ is a Zariski local equivalence. Hence if $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is \mathbb{A}^1 local and admits transfert*

$$k : E_{zar}(F) \rightarrow E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

Proof. See [10]. □

2.7 Presheaves on the big Zariski site or the big etale site of pairs

We recall the definition given in subsection 5.1 : For $S \in \text{Var}(\mathbb{C})$, $\text{Var}(\mathbb{C})^2/S := \text{Var}(\mathbb{C})^2/(S, S)$ is by definition (see subsection 2.1) the category whose set of objects is

$$(\text{Var}(\mathbb{C})^2/S)^0 := \{((X, Z), h), h : X \rightarrow S, Z \subset X \text{ closed}\} \subset \text{Var}(\mathbb{C})/S \times \text{Top}$$

and whose set of morphisms between $(X_1, Z_1)/S = ((X_1, Z_1), h_1), (X_1, Z_1)/S = ((X_2, Z_2), h_2) \in \text{Var}(\mathbb{C})^2/S$ is the subset

$$\begin{aligned} & \text{Hom}_{\text{Var}(\mathbb{C})^2/S}((X_1, Z_1)/S, (X_2, Z_2)/S) := \\ & \{(f : X_2 \rightarrow X_1), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\} \subset \text{Hom}_{\text{Var}(\mathbb{C})}(X_1, X_2) \end{aligned}$$

The category $\text{Var}(\mathbb{C})^2$ admits fiber products : $(X_1, Z_1) \times_{(S, Z)} (X_2, Z_2) = (X_1 \times_S X_2, Z_1 \times_Z Z_2)$. In particular, for $f : T \rightarrow S$ a morphism with $S, T \in \text{Var}(\mathbb{C})$, we have the pullback functor

$$P(f) : \text{Var}(\mathbb{C})^2/S \rightarrow \text{Var}(\mathbb{C})^2/T, P(f)((X, Z)/S) := (X_T, Z_T)/T, P(f)(g) := (g \times_S f)$$

and we note again $P(f) : \text{Var}(\mathbb{C})^2/T \rightarrow \text{Var}(\mathbb{C})^2/S$ the corresponding morphism of sites.

We will consider in the construction of the filtered De Rham realization functor the full subcategory $\text{Var}(\mathbb{C})^{2,sm}/S \subset \text{Var}(\mathbb{C})^2/S$ such that the first factor is a smooth morphism : We will also consider, in order to obtain a complex of D modules in the construction of the filtered De Rham realization functor, the restriction to the full subcategory $\text{Var}(\mathbb{C})^{2,pr}/S \subset \text{Var}(\mathbb{C})^2/S$ such that the first factor is a projection :

Definition 13. (i) Let $S \in \text{Var}(\mathbb{C})$. We denote by

$$\rho_S : \text{Var}(\mathbb{C})^{2,sm}/S \hookrightarrow \text{Var}(\mathbb{C})^2/S$$

the full subcategory consisting of the objects $(U, Z)/S = ((U, Z), h) \in \text{Var}(\mathbb{C})^2/S$ such that the morphism $h : U \rightarrow S$ is smooth. That is, $\text{Var}(\mathbb{C})^{2,sm}/S$ is the category

- whose objects are $(U, Z)/S = ((U, Z), h)$, with $U \in \text{Var}(\mathbb{C})$, $Z \subset U$ a closed subset, and $h : U \rightarrow S$ a smooth morphism,
- whose morphisms $g : (U, Z)/S = ((U, Z), h_1) \rightarrow (U', Z')/S = ((U', Z'), h_2)$ is a morphism $g : U \rightarrow U'$ of complex algebraic varieties such that $Z \subset g^{-1}(Z')$ and $h_2 \circ g = h_1$.

We denote again $\rho_S : \text{Var}(\mathbb{C})^2/S \rightarrow \text{Var}(\mathbb{C})^{2,sm}/S$ the associated morphism of site. We have

$$r^s(S) : \text{Var}(\mathbb{C})^2 \xrightarrow{r(S) := r(S, S)} \text{Var}(\mathbb{C})^2/S \xrightarrow{\rho_S} \text{Var}(\mathbb{C})^{2,sm}/S$$

the composite morphism of site.

(ii) Let $S \in \text{Var}(\mathbb{C})$. We will consider the full subcategory

$$\mu_S : \text{Var}(\mathbb{C})^{2,pr}/S \hookrightarrow \text{Var}(\mathbb{C})^2/S$$

whose subset of object consist of those whose morphism is a projection to S :

$$(\text{Var}(\mathbb{C})^{2,pr}/S)^0 := \{((Y \times S, X), p), Y \in \text{Var}(\mathbb{C}), p : Y \times S \rightarrow S \text{ the projection}\} \subset (\text{Var}(\mathbb{C})^2/S)^0.$$

(iii) We will consider the full subcategory

$$\mu_S : (\text{Var}(\mathbb{C})^{2,smpr}/S) \hookrightarrow \text{Var}(\mathbb{C})^{2,sm}/S$$

whose subset of object consist of those whose morphism is a smooth projection to S :

$$(\text{Var}(\mathbb{C})^{2,smpr}/S)^0 := \{((Y \times S, X), p), Y \in \text{SmVar}(\mathbb{C}), p : Y \times S \rightarrow S \text{ the projection}\} \subset (\text{Var}(\mathbb{C})^2/S)^0$$

For $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$, we have by definition, the following commutative diagram of sites

$$\begin{array}{ccc}
\text{Var}(\mathbb{C})^2/T & \xrightarrow{\mu_T} & \text{Var}(\mathbb{C})^{2,pr}/T & . & (29) \\
\downarrow P(f) & \searrow \rho_T & \downarrow \rho_T & & \\
& & \text{Var}(\mathbb{C})^{2,sm}/T & \xrightarrow{h_T} & \text{Var}(\mathbb{C})^{2,smpr}/T \\
& & \downarrow P(f) & & \downarrow P(f) \\
\text{Var}(\mathbb{C})^2/S & \xrightarrow{P(f)} & \text{Var}(\mathbb{C})^{2,pr}/S & & \text{Var}(\mathbb{C})^{2,sm}/S \\
& \searrow \rho_S & \downarrow \rho_S & & \downarrow P(f) \\
& & \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S
\end{array}$$

Recall we have (see subsection 2.1), for $S \in \text{Var}(\mathbb{C})$, the graph functor

$$\begin{aligned}
\text{Gr}_S^{12} : \text{Var}(\mathbb{C})/S &\rightarrow \text{Var}(\mathbb{C})^{2,pr}/S, \quad X/S \mapsto \text{Gr}_S^{12}(X/S) := (X \times S, X)/S, \\
(g : X/S \rightarrow X'/S) &\mapsto \text{Gr}_S^{12}(g) := (g \times I_S : (X \times S, X) \rightarrow (X' \times S, X'))
\end{aligned}$$

Note that Gr_S^{12} is fully faithful. For $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$, we have by definition, the following commutative diagram of sites

$$\begin{array}{ccccc}
\text{Var}(\mathbb{C})^{2,pr}/T & \xrightarrow{\text{Gr}_T^{12}} & \text{Var}(\mathbb{C})/T & & \\
\downarrow P(f) & \searrow \rho_T & \downarrow & \searrow \rho_T & \\
& & \text{Var}(\mathbb{C})^{2,smpr}/T & \xrightarrow{\text{Gr}_T^{12}} & \text{Var}(\mathbb{C})^{sm}/T \\
& & \downarrow & \downarrow P(f) & \downarrow P(f) \\
\text{Var}(\mathbb{C})^{2,pr}/S & \xrightarrow{P(f)} & \text{Var}(\mathbb{C})/S & & \\
& \searrow \rho_S & \downarrow & \searrow \rho_S & \\
& & \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})^{sm}/S
\end{array} \quad . \quad (30)$$

where we recall that $P(f)((X, Z)/S) := ((X_T, Z_T)/T)$, since smooth morphisms are preserved by base change.

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,(sm)}/T)$$

the adjunction induced by $P(f) : \text{Var}(\mathbb{C})^{2,(sm)}/T \rightarrow \text{Var}(\mathbb{C})^{2,(sm)}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S) \rightleftarrows C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/T), \quad f^*(G, F) := (f^*G, f^*F)$$

For $S \in \text{Var}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}((S, S)/(S, S)) \in \text{PSh}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, $\mathcal{H}om(\mathbb{Z}_S, F) = F$.

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,(sm)pr}/T)$$

the adjunction induced by $P(f) : \text{Var}(\mathbb{C})^{2,(sm)pr}/T \rightarrow \text{Var}(\mathbb{C})^{2,(sm)pr}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S) \rightleftarrows C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/T), \quad f^*(G, F) := (f^*G, f^*F)$$

For $S \in \text{Var}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}((S, S)/(S, S)) \in \text{PSh}(\text{Var}(\mathbb{C})^{2,sm}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$, $\mathcal{H}om(\mathbb{Z}_S, F) = F$.

- For $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{Var}(\mathbb{C})$, $P(h) : \text{Var}(\mathbb{C})^{2,sm}/S \rightarrow \text{Var}(\mathbb{C})^{2,sm}/U$ admits a left adjoint

$$C(h) : \text{Var}(\mathbb{C})^{2,sm}/U \rightarrow \text{Var}(\mathbb{C})^{2,sm}/S, \quad C(h)((U', Z'), h') = ((U', Z'), h \circ h').$$

Hence $h^* : C(\text{Var}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/U)$ admits a left adjoint

$$\begin{aligned}
& h_{\sharp} : C(\text{Var}(\mathbb{C})^{2,sm}/U) \rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/S), \\
F \mapsto (h_{\sharp}F : ((U, Z), h_0) \mapsto \lim_{((U', Z'), h \circ h') \rightarrow ((U, Z), h_0)} F((U', Z')/U))
\end{aligned}$$

- For $h : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, $P(h) : \text{Var}(\mathbb{C})^2/S \rightarrow \text{Var}(\mathbb{C})^2/X$ admits a left adjoint

$$C(h) : \text{Var}(\mathbb{C})^2/X \rightarrow \text{Var}(\mathbb{C})^2/S, C(h)((X', Z'), h') = ((X', Z'), h \circ h').$$

Hence $h^* : C(\text{Var}(\mathbb{C})^2/S) \rightarrow C(\text{Var}(\mathbb{C})^2/X)$ admits a left adjoint

$$h_{\sharp} : C(\text{Var}(\mathbb{C})^2/X) \rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/S),$$

$$F \mapsto (h_{\sharp}F : ((X, Z), h_0) \mapsto \lim_{((X', Z'), h \circ h') \rightarrow ((X, Z), h_0)} F((X', Z')/X))$$

- For $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{Var}(\mathbb{C})$ with Y smooth, $P(p) : \text{Var}(\mathbb{C})^{2,smpr}/S \rightarrow \text{Var}(\mathbb{C})^{2,smpr}/Y \times S$ admits a left adjoint

$$C(p) : \text{Var}(\mathbb{C})^{2,smpr}/Y \times S \rightarrow \text{Var}(\mathbb{C})^{2,smpr}/S,$$

$$C(p)((Y' \times S, Z'), p') = ((Y' \times S, Z'), p \circ p').$$

Hence $p^* : C(\text{Var}(\mathbb{C})^{2,smpr}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,smpr}/Y \times S)$ admits a left adjoint

$$p_{\sharp} : C(\text{Var}(\mathbb{C})^{2,smpr}/Y \times S) \rightarrow C(\text{Var}(\mathbb{C})^{2,smpr}/S),$$

$$F \mapsto (p_{\sharp}F : ((Y_0 \times S, Z), p_0) \mapsto \lim_{((Y' \times Y \times S, Z'), p \circ p') \rightarrow ((Y_0 \times S, Z), p_0)} F((Y' \times Y \times S, Z')/Y \times S))$$

- For $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{Var}(\mathbb{C})$, $P(p) : \text{Var}(\mathbb{C})^{2,pr}/S \rightarrow \text{Var}(\mathbb{C})^{2,pr}/Y \times S$ admits a left adjoint

$$C(p) : \text{Var}(\mathbb{C})^{2,pr}/Y \times S \rightarrow \text{Var}(\mathbb{C})^{2,pr}/S, C(p)((Y' \times S, Z'), p') = ((Y' \times S, Z'), p \circ p').$$

Hence $p^* : C(\text{Var}(\mathbb{C})^{2,pr}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,pr}/Y \times S)$ admits a left adjoint

$$p_{\sharp} : C(\text{Var}(\mathbb{C})^{2,pr}/Y \times S) \rightarrow C(\text{Var}(\mathbb{C})^{2,pr}/S),$$

$$F \mapsto (p_{\sharp}F : ((Y_0 \times S, Z), p_0) \mapsto \lim_{((Y' \times Y \times S, Z'), p \circ p') \rightarrow ((Y_0 \times S, Z), p_0)} F((Y' \times Y \times S, Z')/Y \times S))$$

Let $S \in \text{Var}(\mathbb{C})$.

- We have the dual functor

$$\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,(sm)}/S), F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}((S, S)/S)))$$

It induces the functor

$$L\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,(sm)}/S), F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}((S, S)/S)))$$

- We have the dual functor

$$\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S), F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}((S, S)/S)))$$

It induces the functor

$$L\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S), F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}((S, S)/S)))$$

Proposition 21. (i) Let $S \in \text{Var}(\mathbb{C})$. Let $h : U \rightarrow S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Then for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the canonical map in $C(\text{Var}(\mathbb{C})^{2,smpr}/S)$

$$T(\text{Gr}_S^{12}, \text{hom})(\mathbb{Z}(U/S), F) : \text{Gr}_S^{12*} \mathcal{H}om(\mathbb{Z}(U/S), F) \xrightarrow{\sim} \mathcal{H}om(\text{Gr}_S^{12*} \mathbb{Z}(U/S), \text{Gr}_S^{12*} F)$$

is an isomorphism.

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $h : U \rightarrow S$ a morphism with $U \in \text{Var}(\mathbb{C})$. Then for $F \in C(\text{Var}(\mathbb{C})/S)$, the canonical map in $C(\text{Var}(\mathbb{C})^{2,pr}/S)$

$$T(\text{Gr}_S^{12}, \text{hom})(\mathbb{Z}(U/S), F) : \text{Gr}_S^{12*} \mathcal{H}om(\mathbb{Z}(U/S), F) \xrightarrow{\sim} \mathcal{H}om(\text{Gr}_S^{12*} \mathbb{Z}(U/S), \text{Gr}_S^{12*} F)$$

is an isomorphism.

Proof. (i): We have, for $(X \times S, Z)/S \in \text{Var}(\mathbb{C})^{2,smpr}/S$ the following commutative diagram

$$\begin{array}{ccc} \text{Gr}_S^{12*} \mathcal{H}om(\mathbb{Z}(U/S), F)((X \times S, Z)/S) & \xrightarrow{T(\text{Gr}_S^{12}, \text{hom})(\mathbb{Z}(U/S), F)((X \times S, Z)/S)} & \mathcal{H}om(\text{Gr}_S^{12*} \mathbb{Z}(U/S), \text{Gr}_S^{12*} F)((X \times S, Z)/S) \\ \downarrow = & & \downarrow = \\ \lim_{(X \times S, Z)/S \rightarrow \text{Gr}_S^{12}(V/S)} F(U \times_S V) & \longrightarrow & \lim_{(X \times U, Z)/Z \times_S U \rightarrow \text{Gr}_S^{12}(W/S)} F(W) \end{array}$$

We then note that the map $\{((X \times S, Z)/S \rightarrow \text{Gr}_S^{12}(V/S))\} \rightarrow \{((X \times U, Z)/Z \times_S U \rightarrow \text{Gr}_S^{12}(W/S))\}$ obviously admits an inverse since a map $(X \times U, Z \times_S U)/S \rightarrow (W \times S, W)/S$ is uniquely determined by a map $g : X \rightarrow W$ such that $(g \times I_S)(Z) \subset W$. (ii): Similar to (i). \square

We have the support section functors of a closed embedding $i : Z \hookrightarrow S$ for presheaves on the big Zariski site of pairs.

Definition 14. Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \text{Var}(\mathbb{C})$ and $j : S \setminus Z \hookrightarrow S$ be the open complementary subset.

(i) We define the functor

$$\Gamma_Z : C(\text{Var}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/S), G^\bullet \mapsto \Gamma_Z G^\bullet := \text{Cone}(\text{ad}(j^*, j_*)(G^\bullet) : G^\bullet \rightarrow j_* j^* G^\bullet)[-1],$$

so that there is then a canonical map $\gamma_Z(G^\bullet) : \Gamma_Z G^\bullet \rightarrow G^\bullet$.

(ii) We have the dual functor of (i) :

$$\Gamma_Z^\vee : C(\text{Var}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/S), F \mapsto \Gamma_Z^\vee(F^\bullet) := \text{Cone}(\text{ad}(j_\# , j^*)(G^\bullet) : j_\# j^* G^\bullet \rightarrow G^\bullet),$$

together with the canonical map $\gamma_Z^\vee(G) : F \rightarrow \Gamma_Z^\vee(G)$.

(iii) For $F, G \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$, we denote by

$$I(\gamma, \text{hom})(F, G) := (I, I(j_\# , j^*)(F, G)^{-1}) : \Gamma_Z \mathcal{H}om(F, G) \xrightarrow{\sim} \mathcal{H}om(\Gamma_Z^\vee F, G)$$

the canonical isomorphism given by adjunction.

Note that we have similarly for $i : Z \hookrightarrow S$, $i' : Z' \hookrightarrow Z$ closed embeddings, $g : T \rightarrow S$ a morphism with $T, S, Z \in \text{Var}(\mathbb{C})$ and $F \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$, the canonical maps in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$

- $T(g, \gamma)(F) : g^* \Gamma_Z F \xrightarrow{\sim} \Gamma_{Z \times_S T} g^* F$, $T(g, \gamma^\vee)(F) : \Gamma_{Z \times_S T}^\vee g^* F \xrightarrow{\sim} g^* \Gamma_Z^\vee F$
- $T(Z'/Z, \gamma)(F) : \Gamma_{Z'} F \rightarrow \Gamma_Z F$, $T(Z'/Z, \gamma^\vee)(F) : \Gamma_Z^\vee F \rightarrow \Gamma_{Z'}^\vee F$

but we will not use them in this article.

Let $S_\bullet \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C}))$ with $\mathcal{I} \in \text{Cat}$, a diagram of algebraic varieties. It gives the diagram of sites $\text{Var}(\mathbb{C})^2/S_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat})$.

• Then $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_\bullet)$ is the category

- whose objects $(G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_I)$, and $u_{IJ} : (G_I, F) \rightarrow r_{IJ*}(G_J, F)$ for $r_{IJ} : I \rightarrow J$, denoting again $r_{IJ} : S_I \rightarrow S_J$, are morphisms satisfying for $I \rightarrow J \rightarrow K$, $r_{IJ*} u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_I)$,

- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \in \mathcal{I}}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_I)$.

- Then $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_\bullet)$ is the category

- whose objects $(G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_I)$, and $u_{IJ} : (G_I, F) \rightarrow r_{IJ*}(G_J, F)$ for $r_{IJ} : I \rightarrow J$, denoting again $r_{IJ} : S_I \rightarrow S_J$, are morphisms satisfying for $I \rightarrow J \rightarrow K$, $r_{IJ*} u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_I)$,

- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \in \mathcal{I}}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_I)$.

For $s : \mathcal{I} \rightarrow \mathcal{J}$ a functor, with $\mathcal{I}, \mathcal{J} \in \text{Cat}$, and $f_\bullet : T_\bullet \rightarrow S_{s(\bullet)}$ a morphism with $T_\bullet \in \text{Fun}(\mathcal{J}, \text{Var}(\mathbb{C}))$ and $S_\bullet \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C}))$, we have by definition, the following commutative diagrams of sites

$$\begin{array}{ccccc}
\text{Var}(\mathbb{C})^2/T_\bullet & \xrightarrow{\mu_{T_\bullet}} & \text{Var}(\mathbb{C})^{2,pr}/T_\bullet & & \\
\downarrow P(f_\bullet) & \searrow \rho_{T_\bullet} & \downarrow & \searrow \rho_{T_\bullet} & \\
& & \text{Var}(\mathbb{C})^{2,sm}/T_\bullet & \xrightarrow{\mu_{T_\bullet}^{P(f_\bullet)}} & \text{Var}(\mathbb{C})^{2,smpr}/T_\bullet \\
& & \downarrow & & \downarrow P(f_\bullet) \\
\text{Var}(\mathbb{C})^2/S_\bullet & \xrightarrow{P(f_\bullet) \mu_{S_\bullet}^{P(f_\bullet)}} & \text{Var}(\mathbb{C})^{2,pr}/S_\bullet & & \\
& \searrow \rho_{S_\bullet} & \downarrow & \searrow \rho_{S_\bullet} & \\
& & \text{Var}(\mathbb{C})^{2,sm}/S_\bullet & \xrightarrow{\mu_{S_\bullet}} & \text{Var}(\mathbb{C})^{2,smpr}/S_\bullet
\end{array} \quad (31)$$

and

$$\begin{array}{ccccc}
\text{Var}(\mathbb{C})^{2,pr}/T_\bullet & \xrightarrow{\text{Gr}_{T_\bullet}^{12}} & \text{Var}(\mathbb{C})/T_\bullet & & \\
\downarrow P(f_\bullet) & \searrow \rho_{T_\bullet} & \downarrow & \searrow \rho_{T_\bullet} & \\
& & \text{Var}(\mathbb{C})^{2,smpr}/T_\bullet & \xrightarrow{\text{Gr}_{T_\bullet}^{12, P(f_\bullet)}} & \text{Var}(\mathbb{C})^{sm}/T_\bullet \\
& & \downarrow & & \downarrow P(f_\bullet) \\
\text{Var}(\mathbb{C})^{2,pr}/S_\bullet & \xrightarrow{P(f_\bullet) \text{Gr}_{S_\bullet}^{12, P(f_\bullet)}} & \text{Var}(\mathbb{C})/S_\bullet & & \\
& \searrow \rho_{S_\bullet} & \downarrow & \searrow \rho_{S_\bullet} & \\
& & \text{Var}(\mathbb{C})^{2,sm}/S_\bullet & \xrightarrow{\text{Gr}_{S_\bullet}^{12}} & \text{Var}(\mathbb{C})^{sm}/S_\bullet
\end{array} \quad (32)$$

Let $s : \mathcal{I} \rightarrow \mathcal{J}$ a functor, with $\mathcal{I}, \mathcal{J} \in \text{Cat}$, and $f_\bullet : T_\bullet \rightarrow S_{s(\bullet)}$ a morphism with $T_\bullet \in \text{Fun}(\mathcal{J}, \text{Var}(\mathbb{C}))$ and $S_\bullet \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C}))$.

- As usual, we denote by

$$(f_\bullet^*, f_{\bullet*}) := (P(f_\bullet)^*, P(f_\bullet)_*) : C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_\bullet) \rightarrow C(\text{Var}(\mathbb{C})^{2,(sm)pr}/T_\bullet)$$

the adjunction induced by $P(f_\bullet) : \text{Var}(\mathbb{C})^{2,(sm)pr}/T_\bullet \rightarrow \text{Var}(\mathbb{C})^{2,(sm)pr}/S_\bullet$. Since the colimits involved in the definition of $f_\bullet^* = P(f_\bullet)^*$ are filtered, f_\bullet^* also preserve monomorphism. Hence, we get an

adjunction

$$(f_{\bullet}^*, f_{\bullet*}) : C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_{\bullet}) \rightleftarrows C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/T_{\bullet}),$$

$$f_{\bullet}^*((G_I, F), u_{IJ}) := ((f_I^* G_I, f_I^* F), T(f_I, r_{IJ})(-) \circ f_I^* u_{IJ})$$

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$$(f_{\bullet}^*, f_{\bullet*}) : C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_{\bullet}) \rightleftarrows C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/T_{\bullet}),$$

$$f_{\bullet}^*((G_I, F), u_{IJ}) := ((f_I^* G_I, f_I^* F), T(f_I, r_{IJ})(-) \circ f_I^* u_{IJ})$$

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and for $J \subset I$ the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

where $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ is the projection and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. This gives the diagram of algebraic varieties $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Var}(\mathbb{C}))$ which gives the diagram of sites $\text{Var}(\mathbb{C})^2/(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. This gives also the diagram of algebraic varieties $(\tilde{S}_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N})^{op}, \text{Var}(\mathbb{C}))$ which gives the diagram of sites $\text{Var}(\mathbb{C})^2/(\tilde{S}_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N})^{op}, \text{Cat})$.

- Then $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ is the category
 - whose objects $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/\tilde{S}_I)$, and $u_{IJ} : (G_I, F) \rightarrow p_{IJ*}(G_J, F)$ for $I \subset J$, are morphisms satisfying for $I \subset J \subset K$, $p_{IJ*} u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/\tilde{S}_I)$,
 - the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/\tilde{S}_I)$.

- Then $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is the category
 - whose objects $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/\tilde{S}_I)$, and $u_{IJ} : (G_I, F) \rightarrow p_{IJ*}(G_J, F)$ for $I \subset J$, are morphisms satisfying for $I \subset J \subset K$, $p_{IJ*} u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/\tilde{S}_I)$,
 - the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/\tilde{S}_I)$.

- Then $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op})$ is the category

- whose objects $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2, (sm)}/\tilde{S}_I)$, and $u_{IJ} : (G_J, F) \rightarrow p_{IJ}^*(G_I, F)$ for $I \subset J$, are morphisms satisfying for $I \subset J \subset K$, $p_{JK}^* u_{IJ} \circ u_{JK} = u_{IK}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2, (sm)}/\tilde{S}_K)$,
- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_J = p_{IJ}^* m_I \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2, (sm)}/\tilde{S}_J)$.

- Then $C_{fil}(\text{Var}(\mathbb{C})^{2, (sm)pr}/(\tilde{S}_I)^{op})$ is the category

- whose objects $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2, (sm)pr}/\tilde{S}_I)$, and $u_{IJ} : (G_J, F) \rightarrow p_{IJ}^*(G_I, F)$ for $I \subset J$, are morphisms satisfying for $I \subset J \subset K$, $p_{JK}^* u_{IJ} \circ u_{JK} = u_{IK}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2, (sm)pr}/\tilde{S}_K)$,
- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_J = p_{IJ}^* m_I \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2, (sm)pr}/\tilde{S}_J)$.

We now define the Zariski and the etale topology on $\text{Var}(\mathbb{C})^2/S$.

Definition 15. Let $S \in \text{Var}(\mathbb{C})$.

- (i) Denote by τ a topology on $\text{Var}(\mathbb{C})$, e.g. the Zariski or the etale topology. The τ covers in $\text{Var}(\mathbb{C})^2/S$ of $(X, Z)/S$ are the families of morphisms

$$\{(c_i : (U_i, Z \times_X U_i)/S \rightarrow (X, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow X)_{i \in I} \tau \text{ cover of } X \text{ in } \text{Var}(\mathbb{C})\}$$

- (ii) Denote by τ the Zariski or the etale topology on $\text{Var}(\mathbb{C})$. The τ covers in $\text{Var}(\mathbb{C})^{2, sm}/S$ of $(U, Z)/S$ are the families of morphisms

$$\{(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow U)_{i \in I} \tau \text{ cover of } U \text{ in } \text{Var}(\mathbb{C})\}$$

- (iii) Denote by τ the Zariski or the etale topology on $\text{Var}(\mathbb{C})$. The τ covers in $\text{Var}(\mathbb{C})^{2, (sm)pr}/S$ of $(Y \times S, Z)/S$ are the families of morphisms

$$\{(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow Y)_{i \in I} \tau \text{ cover of } Y \text{ in } \text{Var}(\mathbb{C})\}$$

Let $S \in \text{Var}(\mathbb{C})$. Denote by τ the Zariski or the etale topology on $\text{Var}(\mathbb{C})$. In particular, denoting $a_\tau : \text{PSh}(\text{Var}(\mathbb{C})^{2, (sm)}/S) \rightarrow \text{Shv}(\text{Var}(\mathbb{C})^{2, (sm)}/S)$ and $a_\tau : \text{PSh}(\text{Var}(\mathbb{C})^{2, (sm)pr}/S) \rightarrow \text{Shv}(\text{Var}(\mathbb{C})^{2, (sm)pr}/S)$ the sheafification functors,

- a morphism $\phi : F \rightarrow G$, with $F, G \in C(\text{Var}(\mathbb{C})^{2, (sm)}/S)$, is a τ local equivalence if $a_\tau H^n \phi : a_\tau H^n F \rightarrow a_\tau H^n G$ is an isomorphism, a morphism $\phi : F \rightarrow G$, with $F, G \in C(\text{Var}(\mathbb{C})^{2, (sm)pr}/S)$, is a τ local equivalence if $a_\tau H^n \phi : a_\tau H^n F \rightarrow a_\tau H^n G$ is an isomorphism ;
- $F^\bullet \in C(\text{Var}(\mathbb{C})^{2, (sm)}/S)$ is τ fibrant if for all $(U, Z)/S \in \text{Var}(\mathbb{C})^{2, (sm)}/S$ and all τ covers $(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}$ of $(U, Z)/S$,

$$F^\bullet(c_i) : F^\bullet((U, Z)/S) \rightarrow \text{Tot}(\oplus_{card I = \bullet} F^\bullet((U_i, Z \times_U U_i)/S))$$

is a quasi-isomorphism of complexes of abelian groups, $F^\bullet \in C(\text{Var}(\mathbb{C})^{2, (sm)pr}/S)$ is τ fibrant if for all $(Y \times S, Z)/S \in \text{Var}(\mathbb{C})^{2, (sm)pr}/S$ and all τ covers $(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}$ of $(Y \times S, Z)/S$,

$$F^\bullet(c_i \times I_S) : F^\bullet((Y \times S, Z)/S) \rightarrow \text{Tot}(\oplus_{card I = \bullet} F^\bullet((U_i \times S, Z \times_{Y \times S} U_i \times S)/S))$$

is a quasi-isomorphism of complexes of abelian groups ;

- a morphism $\phi : (G_1, F) \rightarrow (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, is an filtered τ local equivalence if for all $n, p \in \mathbb{Z}$,

$$a_\tau H^n \text{Gr}_F^p(\phi) : a_\tau H^n \text{Gr}_F^p(G_1, F) \rightarrow a_\tau H^n \text{Gr}_F^p(G_2, F)$$

is an isomorphism of sheaves on $\text{Var}(\mathbb{C})^{2,(sm)}/S$, a morphism $\phi : (G_1, F) \rightarrow (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is an filtered τ local equivalence if for all $n, p \in \mathbb{Z}$,

$$a_\tau H^n \text{Gr}_F^p(\phi) : a_\tau H^n \text{Gr}_F^p(G_1, F) \rightarrow a_\tau H^n \text{Gr}_F^p(G_2, F)$$

is an isomorphism of sheaves on $\text{Var}(\mathbb{C})^{2,(sm)pr}/S$;

- let $r \in \mathbb{N}$, a morphism $\phi : (G_1, F) \rightarrow (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, is an r -filtered τ local equivalence if there exists $\phi_i : (C_i, F) \rightarrow (C_{i+1}, F)$, $0 \leq i \leq s$, with $(C_i, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, $(C_0, F) = (G_1, F)$ and $(C_s, F) = (G_2, F)$, such that

$$\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : (G_1, F) \rightarrow (G_2, F)$$

and $\phi_i : (C_i, F) \rightarrow (C_{i+1}, F)$ either a filtered τ local equivalence or an r -filtered homotopy equivalence, a morphism $\phi : (G_1, F) \rightarrow (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is an r -filtered τ local equivalence if there exists $\phi_i : (C_i, F) \rightarrow (C_{i+1}, F)$, $0 \leq i \leq s$, with $(C_i, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, $(C_0, F) = (G_1, F)$ and $(C_s, F) = (G_2, F)$, such that

$$\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : (G_1, F) \rightarrow (G_2, F)$$

and $\phi_i : (C_i, F) \rightarrow (C_{i+1}, F)$ either a filtered τ local equivalence or an r -filtered homotopy equivalence ;

- $(F^\bullet, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is filtered τ fibrant if for all $(U, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)}/S$ and all τ covers $(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}$ of $(U, Z)/S$,

$$\begin{aligned} H^n \text{Gr}_F^p(F^\bullet, F)(c_i) : (F^\bullet, F)((U, Z)/S) &\rightarrow \\ H^n \text{Gr}_F^p(\text{Tot}(\oplus_{card I = \bullet} (F^\bullet, F)((U_i, Z \times_U U_i)/S))) & \end{aligned}$$

is an isomorphism of abelian groups for all $n, p \in \mathbb{Z}$; $(F^\bullet, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is filtered τ fibrant if for all $(Y \times S, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)pr}/S$ and all τ covers $(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}$ of $(Y \times S, Z)/S$,

$$\begin{aligned} H^n \text{Gr}_F^p(F^\bullet, F)(c_i \times I_S) : H^n \text{Gr}_F^p(F^\bullet, F)((Y \times S, Z)/S) &\xrightarrow{\sim} \\ H^n \text{Gr}_F^p(\text{Tot}(\oplus_{card I = \bullet} (F^\bullet, F)((U_i \times S, Z \times_Y U_i)/S))) & \end{aligned}$$

is an isomorphism of abelian groups for all $n, p \in \mathbb{Z}$.

Let $S_\bullet \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C}))$ with $\mathcal{I} \in \text{Cat}$.

- A morphism $m : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$ in $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_\bullet)$ is an r -filtered Zariski, resp. etale local, equivalence, if there exists $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$, $0 \leq i \leq s$, with $((C_{iI}, F), u_{iIJ}) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_\bullet)$ $((C_{0I}, F), u_{0IJ}) = ((G_I, F), u_{IJ})$ and $((C_{sI}, F), u_{sIJ}) = ((H_I, F), v_{IJ})$, such that

$$\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$$

and $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$ either a filtered Zariski, resp. etale, local equivalence or an r -filtered homotopy equivalence.

- A morphism $m : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$ in $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_\bullet)$ is an r -filtered Zariski, resp. etale local, equivalence, if there exists $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$, $0 \leq i \leq s$, with $((C_{iI}, F), u_{iIJ}) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_\bullet)$, $((C_{0I}, F), u_{0IJ}) = ((G_I, F), u_{IJ})$ and $((C_{sI}, F), u_{sIJ}) = ((H_I, F), v_{IJ})$, such that

$$\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$$

and $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$ either a filtered Zariski, resp. etale, local equivalence or an r -filtered homotopy equivalence.

Will now define the \mathbb{A}^1 local property on $\text{Var}(\mathbb{C})^2/S$.

Denote $\square^* := \mathbb{P}_{\mathbb{C}}^* \setminus \{1\}$

- Let $S \in \text{Var}(\mathbb{C})$. For $(X, Z)/S = ((X, Z), h) \in \text{Var}(\mathbb{C})^{2,(sm)}/S$, we consider

$$(\square^* \times X, \square^* \times Z)/S = ((\square^* \times X, \square^* \times Z, h \circ p) \in \text{Fun}(\Delta, \text{Var}(\mathbb{C})^{2,(sm)}/S).$$

For $F \in C^-(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, it gives the complex

$$C_*F \in C^-(\text{Var}(\mathbb{C})^{2,(sm)}/S), (X, Z)/S = ((X, Z), h) \mapsto C_*F((X, Z)/S) := \text{Tot } F((\square^* \times X, \square^* \times Z)/S)$$

together with the canonical map $c_F := (0, I_F) : F \rightarrow C_*F$. For $F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, we get

$$C_*F := \text{holim}_n C_*F^{\leq n} \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S),$$

together with the canonical map $c_F := (0, I_F) : F \rightarrow C_*F$. For $m : F \rightarrow G$ a morphism, with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, we get by functoriality the morphism $C_*m : C_*F \rightarrow C_*G$.

- Let $S \in \text{Var}(\mathbb{C})$. For $(Y \times S, Z)/S = ((Y \times S, Z), h) \in \text{Var}(\mathbb{C})^{2,(sm)pr}/S$, we consider

$$(\square^* \times Y \times S, \square^* \times Z)/S = (\square^* \times Y \times S, \square^* \times Z, h \circ p) \in \text{Fun}(\Delta, \text{Var}(\mathbb{C})/S).$$

For $F \in C^-(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, it gives the complex

$$C_*F \in C^-(\text{Var}(\mathbb{C})^{2,(sm)pr}/S), \\ (Y \times S, Z)/S = ((Y \times S, Z), h) \mapsto C_*F((Y \times S, Z)/S) := \text{Tot } F(\square^* \times Y \times S, \square^* \times Z)/S)$$

together with the canonical map $c = c(F) := (0, I_F) : F \rightarrow C_*F$. For $F \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, we get

$$C_*F := \text{holim}_n C_*F^{\leq n} \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S),$$

together with the canonical map $c = c(F) := (0, I_F) : F \rightarrow C_*F$. For $m : F \rightarrow G$ a morphism, with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, we get by functoriality the morphism $C_*m : C_*F \rightarrow C_*G$.

- Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \bigcap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. For $F = (F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$, it gives the complex

$$C_*F = (C_*F_I, C_*u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)),$$

together with the canonical map $c_F := (0, I_F) : F \rightarrow C_*F$.

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$$C_*F = (C_*F_I, C_*u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op}),$$

together with the canonical map $c_F := (0, I_F) : F \rightarrow C_*F$.

- Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \bigcap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. For $F = (F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$, it gives the complex

$$C_*F = (C_*F_I, C_*u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)),$$

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- Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \bigcap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. For $F = (F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$, it gives the complex

$$C_*F = (C_*F_I, C_*u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op}),$$

together with the canonical map $c_F := (0, I_F) : F \rightarrow C_*F$.

Let $S \in \text{Var}(\mathbb{C})$. Denote for short $\text{Var}(\mathbb{C})^{2,(sm)}/S$ either the category $\text{Var}(\mathbb{C})^2/S$ or the category $\text{Var}(\mathbb{C})^{2,sm}/S$. Denote by

$$\begin{aligned} p_a : \text{Var}(\mathbb{C})^{2,(sm)}/S &\rightarrow \text{Var}(\mathbb{C})^{2,(sm)}/S, \\ (X, Z)/S = ((X, Z), h) &\mapsto (X \times \mathbb{A}^1, Z \times \mathbb{A}^1)/S = ((X \times \mathbb{A}^1, Z \times \mathbb{A}^1), h \circ p_X), \\ (g : (X, Z)/S \rightarrow (X', Z')/S) &\mapsto ((g \times I_{\mathbb{A}^1}) : (X \times \mathbb{A}^1, Z \times \mathbb{A}^1)/S \rightarrow (X' \times \mathbb{A}^1, Z' \times \mathbb{A}^1)/S) \end{aligned}$$

the projection functor and again by $p_a : \text{Var}(\mathbb{C})^{2,(sm)}/S \rightarrow \text{Var}(\mathbb{C})^{2,(sm)}/S$ the corresponding morphism of site. Let $S \in \text{Var}(\mathbb{C})$. Denote for short $\text{Var}(\mathbb{C})^{2,(sm)}/S$ either the category $\text{Var}(\mathbb{C})^2/S$ or the category $\text{Var}(\mathbb{C})^{2,sm}/S$. Denote for short $\text{Var}(\mathbb{C})^{2,(sm)pr}/S$ either the category $\text{Var}(\mathbb{C})^{2,pr}/S$ or the category $\text{Var}(\mathbb{C})^{2,smpr}/S$. Denote by

$$\begin{aligned} p_a : \text{Var}(\mathbb{C})^{2,(sm)pr}/S &\rightarrow \text{Var}(\mathbb{C})^{2,(sm)pr}/S, \\ (Y \times S, Z)/S = ((Y \times S, Z), p_S) &\mapsto (Y \times S \times \mathbb{A}^1, Z \times \mathbb{A}^1)/S = ((Y \times S \times \mathbb{A}^1, Z \times \mathbb{A}^1), p_S \circ p_{Y \times S}), \\ (g : (Y \times S, Z)/S \rightarrow (Y' \times S, Z')/S) &\mapsto ((g \times I_{\mathbb{A}^1}) : (Y \times S \times \mathbb{A}^1, Z \times \mathbb{A}^1)/S \rightarrow (Y' \times S \times \mathbb{A}^1, Z' \times \mathbb{A}^1)/S) \end{aligned}$$

the projection functor and again by $p_a : \text{Var}(\mathbb{C})^{2,(sm)pr}/S \rightarrow \text{Var}(\mathbb{C})^{2,(sm)pr}/S$ the corresponding morphism of site.

Definition 16. (i0) A complex $F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to be \mathbb{A}^1 homotopic if $\text{ad}(p_a^*, p_{a*})(F) : F \rightarrow p_{a*}p_a^*F$ is an homotopy equivalence.

(i0)' A complex $F \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be \mathbb{A}^1 homotopic if $\text{ad}(p_a^*, p_{a*})(F) : F \rightarrow p_{a*}p_a^*F$ is an homotopy equivalence.

(i) A complex $F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, is said to be \mathbb{A}^1 invariant if for all $(X, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)}/S$

$$F(p_X) : F((X, Z)/S) \rightarrow F((X \times \mathbb{A}^1, (Z \times \mathbb{A}^1))/S)$$

is a quasi-isomorphism, where $p_X : (X \times \mathbb{A}^1, (Z \times \mathbb{A}^1)) \rightarrow (X, Z)$ is the projection. Obviously, if a complex $F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is \mathbb{A}^1 homotopic, then it is \mathbb{A}^1 invariant.

(i)' A complex $G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, is said to be \mathbb{A}^1 invariant if for all $(Y \times S, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)pr}/S$

$$G(p_{Y \times S}) : G((Y \times S, Z)/S) \rightarrow G((Y \times \mathbb{A}^1 \times S, (Z \times \mathbb{A}^1))/S)$$

is a quasi-isomorphism of abelian group. Obviously, if a complex $F \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is \mathbb{A}^1 homotopic, then it is \mathbb{A}^1 invariant.

(ii) Let τ a topology on $\text{Var}(\mathbb{C})$. A complex $F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to be \mathbb{A}^1 local for the τ topology induced on $\text{Var}(\mathbb{C})^2/S$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ τ fibrant, e.g. $k : F \rightarrow E_\tau(F)$, G is \mathbb{A}^1 invariant.

(ii)' Let τ a topology on $\text{Var}(\mathbb{C})$. A complex $F \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be \mathbb{A}^1 local for the τ topology induced on $\text{Var}(\mathbb{C})^{2,pr}/S$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ τ fibrant, e.g. $k : F \rightarrow E_\tau(F)$, G is \mathbb{A}^1 invariant.

(iii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to an (\mathbb{A}^1, et) local equivalence if for all $H \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ which is \mathbb{A}^1 local for the etale topology

$$\text{Hom}(L(m), E_{et}(H)) : \text{Hom}(L(G), E_{et}(H)) \rightarrow \text{Hom}(L(F), E_{et}(H))$$

is a quasi-isomorphism.

(iii)' A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is said to an (\mathbb{A}^1, et) local equivalence if for all $H \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ which is \mathbb{A}^1 local for the etale topology

$$\text{Hom}(L(m), E_{et}(H)) : \text{Hom}(L(G), E_{et}(H)) \rightarrow \text{Hom}(L(F), E_{et}(H))$$

is a quasi-isomorphism.

Proposition 22. (i) Let $S \in \text{Var}(\mathbb{C})$. Then for $F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$, C_*F is \mathbb{A}^1 local for the etale topology and $c(F) : F \rightarrow C_*F$ is an equivalence (\mathbb{A}^1, et) local.

(i)' Let $S \in \text{Var}(\mathbb{C})$. Then for $F \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$, C_*F is \mathbb{A}^1 local for the etale topology and $c(F) : F \rightarrow C_*F$ is an equivalence (\mathbb{A}^1, et) local.

(ii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is an (\mathbb{A}^1, et) local equivalence if and only if $a_{et}H^n C_* \text{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.

(ii)' A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is an (\mathbb{A}^1, et) local equivalence if and only if $a_{et}H^n C_* \text{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.

(iii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is an (\mathbb{A}^1, et) local equivalence if and only if there exists

$$\{(X_{1,\alpha}, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \dots, \{(X_{r,\alpha}, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \text{Var}(\mathbb{C})^{2,(sm)}/S$$

such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{2,(sm)}/S))$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}((X_{1,\alpha} \times \mathbb{A}^1, Z_{1,\alpha} \times \mathbb{A}^1)/S) \rightarrow \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \\ &\rightarrow \dots \rightarrow \bigoplus_{\alpha \in \Lambda_r} \text{Cone}(\mathbb{Z}((X_{r,\alpha} \times \mathbb{A}^1, Z_{r,\alpha} \times \mathbb{A}^1)/S) \rightarrow \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S))) \end{aligned}$$

(iii)' A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is an (\mathbb{A}^1, et) local equivalence if and only if there exists

$$\{(Y_{1,\alpha} \times S, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \dots, \{(Y_{r,\alpha} \times S, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \text{Var}(\mathbb{C})^{2,(sm)pr}/S$$

such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{2,(sm)}/S))$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}((Y_{1,\alpha} \times \mathbb{A}^1 \times S, Z_{1,\alpha} \times \mathbb{A}^1)/S) \rightarrow \mathbb{Z}((Y_{1,\alpha} \times S, Z_{1,\alpha})/S)) \\ &\rightarrow \dots \rightarrow \bigoplus_{\alpha \in \Lambda_r} \text{Cone}(\mathbb{Z}((Y_{r,\alpha} \times \mathbb{A}^1 \times S, Z_{r,\alpha} \times \mathbb{A}^1)/S) \rightarrow \mathbb{Z}((Y_{r,\alpha} \times S, Z_{r,\alpha})/S))) \end{aligned}$$

Proof. Standard : see Ayoub's thesis section 4 for example. Indeed, for (iii), by definition, if $\text{Cone}(m)$ is of the given form, then it is an equivalence (\mathbb{A}^1, et) local, on the other hand if m is an equivalence (\mathbb{A}^1, et) local, we consider the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{c(F)} & C_*F \\ \downarrow m & & \downarrow c_*m \\ G & \xrightarrow{c(G)} & C_*G \end{array}$$

to deduce that $\text{Cone}(m)$ is of the given form. □

Definition-Proposition 6. *Let $S \in \text{Var}(\mathbb{C})$.*

- (i) *With the weak equivalence the $(\mathbb{A}^1, \text{et})$ local equivalence and the fibration the epimorphism with \mathbb{A}_S^1 local and etale fibrant kernels gives a model structure on $C(\text{Var}(\mathbb{C})^{2,(\text{sm})}/S)$: the left bousfield localization of the projective model structure of $C(\text{Var}(\mathbb{C})^{2,(\text{sm})}/S)$. We call it the projective $(\mathbb{A}^1, \text{et})$ model structure.*
- (ii) *With the weak equivalence the $(\mathbb{A}^1, \text{et})$ local equivalence and the fibration the epimorphism with \mathbb{A}_S^1 local and etale fibrant kernels gives a model structure on $C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}/S)$: the left bousfield localization of the projective model structure of $C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}/S)$. We call it the projective $(\mathbb{A}^1, \text{et})$ model structure.*

Proof. Similar to the proof of proposition 5. □

We have, similarly to the case of single varieties the following :

Proposition 23. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$.*

- (i) *The adjunction $(g^*, g_*) : C(\text{Var}(\mathbb{C})^{2,(\text{sm})}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{2,(\text{sm})}/T)$ is a Quillen adjunction for the projective $(\mathbb{A}^1, \text{et})$ model structure (see definition-proposition 6)*
- (i)' *The functor $g^* : C(\text{Var}(\mathbb{C})^{2,(\text{sm})}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,(\text{sm})}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, \text{et})$ local equivalence to $(\mathbb{A}^1, \text{et})$ local equivalence.*
- (ii) *The adjunction $(g^*, g_*) : C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}/T)$ is a Quillen adjunction for the projective $(\mathbb{A}^1, \text{et})$ model structure (see definition-proposition 6)*
- (ii)' *The functor $g^* : C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, \text{et})$ local equivalence to $(\mathbb{A}^1, \text{et})$ local equivalence.*

Proof. (i): Follows immediately from definition. (i)': Since the functor g^* preserve epimorphism and also monomorphism (the colimits involved being filtered), g^* sends quasi-isomorphism to quasi-isomorphism. Hence it preserve Zariski and etale local equivalence. The fact that it preserve $(\mathbb{A}^1, \text{et})$ local equivalence then follows similarly to the single case by the fact that g_* preserve by definition \mathbb{A}^1 equivariant presheaves. (ii) and (ii)': Similar to (i) and (i)'. □

Proposition 24. *Let $S \in \text{Var}(\mathbb{C})$.*

- (i) *The adjunction $(\rho_S^*, \rho_{S*}) : C(\text{Var}(\mathbb{C})^{2,\text{sm}}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^2/S)$ is a Quillen adjunction for the $(\mathbb{A}^1, \text{et})$ projective model structure.*
- (i)' *The functor $\rho_{S*} : C(\text{Var}(\mathbb{C})^2/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,\text{sm}}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, \text{et})$ local equivalence to $(\mathbb{A}^1, \text{et})$ local equivalence.*
- (ii) *The adjunction $(\rho_S^*, \rho_{S*}) : C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{2,\text{pr}}/S)$ is a Quillen adjunction for the $(\mathbb{A}^1, \text{et})$ projective model structure.*
- (ii)' *The functor $\rho_{S*} : C(\text{Var}(\mathbb{C})^{2,\text{pr}}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, \text{et})$ local equivalence to $(\mathbb{A}^1, \text{et})$ local equivalence.*

Proof. Similar to the proof of proposition 19. □

Proposition 25. *Let $S \in \text{Var}(\mathbb{C})$.*

- (i) The adjunction $(\mu_S^*, \mu_{S*}) : C(\text{Var}(\mathbb{C})^{2,pr}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^2/S)$ is a Quillen adjunction for the (\mathbb{A}^1, et) projective model structure.
- (i)' The functor $\mu_{S*} : C(\text{Var}(\mathbb{C})^2/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,pr}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.
- (ii) The adjunction $(\mu_S^*, \mu_{S*}) : C(\text{Var}(\mathbb{C})^{2,smpr}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{2,pr}/S)$ is a Quillen adjunction for the (\mathbb{A}^1, et) projective model structure.
- (ii)' The functor $\mu_{S*} : C(\text{Var}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence.

Proof. Similar to the proof of proposition 19. Indeed, for (i)' or (ii)', if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)})$ is an equivalence (\mathbb{A}^1, et) local then (see proposition 22), there exists

$$\{(X_{1,\alpha}, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \dots, \{(X_{r,\alpha}, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \text{Var}(\mathbb{C})^{2,(sm)}/S$$

such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{2,(sm)}/S))$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}((X_{1,\alpha} \times \mathbb{A}^1, Z_{1,\alpha} \times \mathbb{A}^1)/S) \rightarrow \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \\ &\rightarrow \dots \rightarrow \bigoplus_{\alpha \in \Lambda_r} \text{Cone}(\mathbb{Z}((X_{r,\alpha} \times \mathbb{A}^1, Z_{r,\alpha} \times \mathbb{A}^1)/S) \rightarrow \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S)) \\ &\xrightarrow{\sim} \text{Cone}(\text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S) \otimes \mathbb{Z}(\mathbb{A}^1, \mathbb{A}^1)/S) \rightarrow \bigoplus_{\alpha \in \Lambda_1} \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \\ &\rightarrow \dots \rightarrow \text{Cone}(\bigoplus_{\alpha \in \Lambda_r} \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S) \otimes \mathbb{Z}(\mathbb{A}^1, \mathbb{A}^1)/S) \rightarrow \bigoplus_{\alpha \in \Lambda_r} \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S)), \end{aligned}$$

this gives in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S))$

$$\begin{aligned} &\text{Cone}(\mu_{S*}m) \xrightarrow{\sim} \text{Cone}(\text{Cone} \\ &\text{Cone}(L\mu_{S*} \bigoplus_{\alpha \in \Lambda_1} \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S) \otimes \mathbb{Z}(\mathbb{A}^1, \mathbb{A}^1)/S) \rightarrow (L\mu_{S*} \bigoplus_{\alpha \in \Lambda_1} \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \\ &\rightarrow \dots \rightarrow \text{Cone}(L\mu_{S*} \bigoplus_{\alpha \in \Lambda_r} \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S) \otimes \mathbb{Z}(\mathbb{A}^1, \mathbb{A}^1)/S) \rightarrow (L\mu_{S*} \bigoplus_{\alpha \in \Lambda_r} \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S))) \end{aligned}$$

hence $\mu_{S*}m : \mu_{S*}F \rightarrow \mu_{S*}G$ is an equivalence (\mathbb{A}^1, et) local. \square

We also have

Proposition 26. *Let $S \in \text{Var}(\mathbb{C})$.*

- (i) The adjunction $(\text{Gr}_S^{12*}, \text{Gr}_{S*}^{12}) : C(\text{Var}(\mathbb{C})/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{2,pr}/S)$ is a Quillen adjunction for the (\mathbb{A}^1, et) projective model structure.
- (ii) The adjunction $(\text{Gr}_S^{12*}, \text{Gr}_{S*}^{12}) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ is a Quillen adjunction for the (\mathbb{A}^1, et) projective model structure.

Proof. Immediate from definition. \square

- For $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ and $Z \subset X$ a closed subset, we denote $\mathbb{Z}^{tr}((X, Z)/S) \in \text{PSh}(\text{Var}(\mathbb{C})^2/S)$ the presheaf given by

– for $(X', Z')/S \in \text{Var}(\mathbb{C})^2/S$, with X' irreducible,

$$\mathbb{Z}^{tr}((X, Z)/S)((X', Z')/S) := \left\{ \alpha \in \mathcal{Z}^{fs/X}(X' \times_S X), \text{ s.t. } p_X(p_{X'}^{-1}(Z')) \subset Z \right\} \subset \mathcal{Z}_{d_{X'}}(X' \times_S X)$$

– for $g : (X_2, Z_2)/S \rightarrow (X_1, Z_1)/S$ a morphism, with $(X_1, Z_1)/S, (X_2, Z_2)/S \in \text{Var}(\mathbb{C})^2/S$,

$$\mathbb{Z}^{tr}((X, Z)/S)(g) : \mathbb{Z}^{tr}((X, Z)/S)((X_1, Z_1)/S) \rightarrow \mathbb{Z}^{tr}((X, Z)/S)((X_2, Z_2)/S), \alpha \mapsto (g \times I)^{-1}(\alpha)$$

with $g \times I : X_2 \times_S X \rightarrow X_1 \times_S X$.

- For $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, $Z \subset X$ a closed subset and $r \in \mathbb{N}$, we denote $\mathbb{Z}^{equiv}((X, Z)/S) \in \text{PSh}(\text{Var}(\mathbb{C})^2/S)$ the presheaf given by
 - for $(X', Z')/S \in \text{Var}(\mathbb{C})^2/S$, with X' irreducible,

$$\mathbb{Z}^{equiv}((X, Z)/S)((X', Z')/S) := \left\{ \alpha \in \mathcal{Z}^{equiv/X}(X' \times_S X), s.t. p_X(p_{X'}^{-1}(Z')) \right\} \subset \mathcal{Z}_{d_{X'}}(X' \times_S X)$$
 - for $g : (X_2, Z_2)/S \rightarrow (X_1, Z_1)/S$ a morphism, with $(X_1, Z_1)/S, (X_2, Z_2)/S \in \text{Var}(\mathbb{C})^2/S$,

$$\mathbb{Z}^{equiv}((X, Z)/S)(g) : \mathbb{Z}^{equiv}((X, Z)/S)((X_1, Z_1)/S) \rightarrow \mathbb{Z}^{equiv}((X, Z)/S)((X_2, Z_2)/S), \alpha \mapsto (g \times I)^{-1}(\alpha)$$
 with $g \times I : X_2 \times_S X \rightarrow X_1 \times_S X$.
- Let $S \in \text{Var}(\mathbb{C})$. We denote by $\mathbb{Z}_S(d) := \mathbb{Z}^{equiv0}((S \times \mathbb{A}^d, S \times \mathbb{A}^d)/S)[-2d]$ the Tate twist. For $F \in C(\text{Var}(\mathbb{C})^2/S)$, we denote by $F(d) := F \otimes \mathbb{Z}_S(d)$.

For $S \in \text{Var}(\mathbb{C})$, let $\text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ be the category

- whose objects are those of $\text{Var}(\mathbb{C})^{2,(sm)}/S$, i.e. $(X, Z)/S = ((X, Z), h)$, $h : X \rightarrow S$ with $X \in \text{Var}(\mathbb{C})$, $Z \subset X$ a closed subset,
- whose morphisms $\alpha : (X', Z)/S = ((X', Z), h_1) \rightarrow (X, Z)/S = ((X, Z), h_2)$ is finite correspondence that is $\alpha \in \oplus_i \mathbb{Z}^{tr}((X_i, Z)/S)((X', Z')/S)$, where $X' = \sqcup_i X'_i$, with X'_i connected, the composition being defined in the same way as the morphism $\text{Cor}(\text{Var}(\mathbb{C})^{(sm)}/S)$.

We denote by $\text{Tr}(S) : \text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)}/S) \rightarrow \text{Var}(\mathbb{C})^{2,(sm)}/S$ the morphism of site given by the inclusion functor $\text{Tr}(S) : \text{Var}(\mathbb{C})^{2,(sm)}/S \hookrightarrow \text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ It induces an adjunction

$$(\text{Tr}(S)^* \text{Tr}(S)_*) : C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \rightleftarrows C(\text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)}/S))$$

A complex of preheaves $G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to admit transferts if it is in the image of the embedding

$$\text{Tr}(S)_* : C(\text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)}/S)) \hookrightarrow C(\text{Var}(\mathbb{C})^{2,(sm)}/S),$$

that is $G = \text{Tr}(S)_* \text{Tr}(S)^* G$. We then have the full subcategory $\text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S) \subset \text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ consisting of the objects of $\text{Var}(\mathbb{C})^{2,(sm)pr}/S$. We have the adjunction

$$(\text{Tr}(S)^* \text{Tr}(S)_*) : C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S) \rightleftarrows C(\text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S))$$

A complex of preheaves $G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is said to admit transferts if it is in the image of the embedding

$$\text{Tr}(S)_* : C(\text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)) \hookrightarrow C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S),$$

that is $G = \text{Tr}(S)_* \text{Tr}(S)^* G$.

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$.

- For $(G_I, K_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op})$ and $(H_I, T_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$, we denote

$$\mathcal{H}om((G_I, K_{IJ}), (H_I, T_{IJ})) := (\mathcal{H}om(G_I, H_I), u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ}))) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$$

with

$$\begin{aligned} & u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ})) : \mathcal{H}om(G_I, H_I) \\ & \xrightarrow{\text{ad}(p_{IJ}^*, p_{IJ*})(-)} p_{IJ*} p_{IJ}^* \mathcal{H}om(G_I, H_I) \xrightarrow{T(p_{IJ, \text{hom}})(-, -)} p_{IJ*} \mathcal{H}om(p_{IJ}^* G_I, p_{IJ}^* H_I) \\ & \xrightarrow{\mathcal{H}om(p_{IJ}^* G_I, T_{IJ})} p_{IJ*} \mathcal{H}om(p_{IJ}^* G_I, H_J) \xrightarrow{\mathcal{H}om(K_{IJ}, H_J)} p_{IJ*} \mathcal{H}om(G_J, H_J). \end{aligned}$$

This gives in particular the functor

$$C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)) \rightarrow C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op}), (H_I, T_{IJ}) \mapsto (H_I, T_{IJ}).$$

- For $(G_I, K_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op})$ and $(H_I, T_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$, we denote $\mathcal{H}om((G_I, K_{IJ}), (H_I, T_{IJ})) := (\mathcal{H}om(G_I, H_I), u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ}))) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ with

$$\begin{aligned} & u_{IJ}((G_I, K_{IJ})(H_I, T_{IJ})) : \mathcal{H}om(G_I, H_I) \\ & \xrightarrow{\text{ad}(p_{IJ}^*, p_{IJ*})(-)} p_{IJ*} p_{IJ}^* \mathcal{H}om(G_I, H_I) \xrightarrow{T(p_{IJ}, \text{hom})(-, -)} p_{IJ*} \mathcal{H}om(p_{IJ}^* G_I, p_{IJ}^* H_I) \\ & \xrightarrow{\mathcal{H}om(p_{IJ}^* G_I, T_{IJ})} p_{IJ*} \mathcal{H}om(p_{IJ}^* G_I, H_J) \xrightarrow{\mathcal{H}om(K_{IJ}, H_J)} p_{IJ*} \mathcal{H}om(G_J, H_J). \end{aligned}$$

This gives in particular the functor

$$C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)) \rightarrow C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op}), (H_I, T_{IJ}) \mapsto (H_I, T_{IJ}).$$

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. The functors p_a naturally extend to functors

$$\begin{aligned} p_a : \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I) &\rightarrow \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I), \\ ((X, Z)/\tilde{S}_I, u_{IJ}) &\mapsto ((X \times \mathbb{A}^1, Z \times \mathbb{A}^1)/\tilde{S}_I, u_{IJ} \times I), \\ (g : ((X, Z)/\tilde{S}_I, u_{IJ}) &\rightarrow ((X', Z')/\tilde{S}_I, u_{IJ})) \mapsto \\ ((g \times I_{\mathbb{A}^1}) : ((X \times \mathbb{A}^1, Z \times \mathbb{A}^1)/\tilde{S}_I, u_{IJ} \times I) &\rightarrow ((X' \times \mathbb{A}^1, Z' \times \mathbb{A}^1)/\tilde{S}_I, u_{IJ} \times I)) \end{aligned}$$

the projection functor and again by $p_a : \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I) \rightarrow \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)$ the corresponding morphism of site, and

$$\begin{aligned} p_a : \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I) &\rightarrow \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I), \\ ((Y \times \tilde{S}_I, Z)/\tilde{S}_I, u_{IJ}) &\mapsto ((Y \times \tilde{S}_I \times \mathbb{A}^1, Z \times \mathbb{A}^1)/\tilde{S}_I, u_{IJ} \times I), \\ (g : ((Y \times \tilde{S}_I, Z)/\tilde{S}_I, u_{IJ}) &\rightarrow ((Y' \times \tilde{S}_I, Z')/\tilde{S}_I, u_{IJ})) \mapsto \\ ((g \times I_{\mathbb{A}^1}) : ((Y \times \tilde{S}_I \times \mathbb{A}^1, Z \times \mathbb{A}^1)/\tilde{S}_I, u_{IJ} \times I), &((Y' \times \tilde{S}_I \times \mathbb{A}^1, Z' \times \mathbb{A}^1)/\tilde{S}_I, u_{IJ} \times I)), \end{aligned}$$

the projection functor and again by $p_a : \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I) \rightarrow \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)$ the corresponding morphism of site. These functors also gives the morphisms of sites $p_a : \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op} \rightarrow \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op}$ and $p_a : \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op} \rightarrow \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op}$.

Definition 17. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$.

- (i0) A complex $(F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is said to be \mathbb{A}^1 homotopic if $\text{ad}(p_a^*, p_{a*})((F_I, u_{IJ})) : (F_I, u_{IJ}) \rightarrow p_{a*} p_a^*(F_I, u_{IJ})$ is an homotopy equivalence.
- (i0)' A complex $(F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is said to be \mathbb{A}^1 homotopic if $\text{ad}(p_a^*, p_{a*})((F_I, u_{IJ})) : (F_I, u_{IJ}) \rightarrow p_{a*} p_a^*(F_I, u_{IJ})$ is an homotopy equivalence.
- (i) A complex $(F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is said to be \mathbb{A}^1 invariant if for all $((X_I, Z_I)/\tilde{S}_I, s_{IJ}) \in \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)$

$$(F_I(p_{X_I})) : (F_I((X_I, Z_I)/\tilde{S}_I), F_J(s_{IJ}) \circ u_{IJ}(-)) \rightarrow (F_I((X_I \times \mathbb{A}^1, (Z_I \times \mathbb{A}^1))/\tilde{S}_I), F_J(s_{IJ} \times I) \circ u_{IJ}(-))$$

is a quasi-isomorphism, where $p_{X_I} : (X_I \times \mathbb{A}^1, (Z_I \times \mathbb{A}^1)) \rightarrow (X_I, Z_I)$ are the projection, and $s_{IJ} : (X_I \times \tilde{S}_{J \setminus I}, Z_I)/\tilde{S}_J \rightarrow (X_J, Z_J)/\tilde{S}_J$. Obviously a complex $(F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is \mathbb{A}^1 invariant if and only if all the F_I are \mathbb{A}^1 invariant.

(i)' A complex $(G_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is said to be \mathbb{A}^1 invariant if for all $((Y \times \tilde{S}_I, Z_I)/\tilde{S}_I, s_{IJ}) \in \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)$

$$(G_I(p_{Y \times \tilde{S}_I})) : (G_I((Y \times \tilde{S}_I, Z_I)/\tilde{S}_I), G_J(s_{IJ}) \circ u_{IJ}(-)) \rightarrow (G_I((Y \times \tilde{S}_I \times \mathbb{A}^1, (Z_I \times \mathbb{A}^1))/\tilde{S}_I), G_J(s_{IJ} \times I) \circ u_{IJ}(-))$$

is a quasi-isomorphism. Obviously a complex $(G_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is \mathbb{A}^1 invariant if and only if all the G_I are \mathbb{A}^1 invariant.

(ii) Let τ a topology on $\text{Var}(\mathbb{C})$. A complex $F = (F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ is said to be \mathbb{A}^1 local for the τ topology induced on $\text{Var}(\mathbb{C})^2/(\tilde{S}_I)$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G = (G_I, v_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ τ fibrant, e.g. $k : (F_I, u_{IJ}) \rightarrow (E_\tau(F_I), E(u_{IJ}))$, G is \mathbb{A}^1 invariant.

(ii)' Let τ a topology on $\text{Var}(\mathbb{C})$. A complex $F = (F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is said to be \mathbb{A}^1 local for the τ topology induced on $\text{Var}(\mathbb{C})^2/(\tilde{S}_I)$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G = (G_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ τ fibrant, e.g. $k : (F_I, u_{IJ}) \rightarrow (E_\tau(F_I), E(u_{IJ}))$, G is \mathbb{A}^1 invariant.

(iii) A morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ is said to be an (\mathbb{A}^1, et) local equivalence if for all $H = (H_I, w_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ which is \mathbb{A}^1 local for the etale topology

$$(\text{Hom}(L(m_I), E_{et}(H_I))) : \text{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \rightarrow \text{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ is an (\mathbb{A}^1, et) local equivalence, then all the $m_I : F_I \rightarrow G_I$ are (\mathbb{A}^1, et) local equivalence.

(iii)' A morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is said to be an (\mathbb{A}^1, et) local equivalence if for all $(H_I, w_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ which is \mathbb{A}^1 local for the etale topology

$$(\text{Hom}(L(m_I), E_{et}(H_I))) : \text{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \rightarrow \text{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is an (\mathbb{A}^1, et) local equivalence, then all the $m_I : F_I \rightarrow G_I$ are (\mathbb{A}^1, et) local equivalence.

(iv) A morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op})$ is said to be an (\mathbb{A}^1, et) local equivalence if for all $H = (H_I, w_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ which is \mathbb{A}^1 local for the etale topology

$$(\text{Hom}(L(m_I), E_{et}(H_I))) : \text{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \rightarrow \text{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op})$ is an (\mathbb{A}^1, et) local equivalence, then all the $m_I : F_I \rightarrow G_I$ are (\mathbb{A}^1, et) local equivalence.

(iv)' A morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op})$ is said to be an (\mathbb{A}^1, et) local equivalence if for all $(H_I, w_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ which is \mathbb{A}^1 local for the etale topology

$$(\text{Hom}(L(m_I), E_{et}(H_I))) : \text{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \rightarrow \text{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op})$ is an (\mathbb{A}^1, et) local equivalence, then all the $m_I : F_I \rightarrow G_I$ are (\mathbb{A}^1, et) local equivalence.

Proposition 27. *Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \bigcap_{i \in I} S_i$. Let $i_i : \tilde{S}_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$.*

(i) *Then for $F \in C(\text{Var}(\mathbb{C})^{2,(sm)})/(\tilde{S}_I)^{op}$, C_*F is \mathbb{A}^1 local for the etale topology and $c(F) : F \rightarrow C_*F$ is an equivalence (\mathbb{A}^1, et) local.*

(i)' *Then for $F \in C(\text{Var}(\mathbb{C})^{2,(sm)pr})/(\tilde{S}_I)^{op}$, C_*F is \mathbb{A}^1 local for the etale topology and $c(F) : F \rightarrow C_*F$ is an equivalence (\mathbb{A}^1, et) local.*

(ii) *A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)})/(\tilde{S}_I)^{op}$ is an (\mathbb{A}^1, et) local equivalence if and only if $a_{et}H^n C_* \text{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.*

(ii)' *A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr})/(\tilde{S}_I)^{op}$ is an (\mathbb{A}^1, et) local equivalence if and only if $a_{et}H^n C_* \text{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.*

(iii) *A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)})/(\tilde{S}_I)^{op}$ is an (\mathbb{A}^1, et) local equivalence if and only if there exists*

$$\left\{ ((X_{1,\alpha,I}, Z_{1,\alpha,I})/\tilde{S}_I, u_{IJ}^1), \alpha \in \Lambda_1 \right\}, \dots, \left\{ ((X_{r,\alpha,I}, Z_{r,\alpha,I})/\tilde{S}_I, u_{IJ}^r), \alpha \in \Lambda_r \right\} \subset \text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op}$$

with

$$u_{IJ}^l : (X_{l,\alpha,J}, Z_{l,\alpha,J})/\tilde{S}_J \rightarrow (X_{l,\alpha,I} \times \tilde{S}_{J \setminus I}, Z_{l,\alpha,I} \times \tilde{S}_{J \setminus I})/\tilde{S}_J$$

such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{2,(sm)})/(\tilde{S}_I)^{op})$

$$\begin{aligned} & \text{Cone}(m) \xrightarrow{\sim} \text{Cone} \\ & \oplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}((X_{1,\alpha,I} \times \mathbb{A}^1, Z_{1,\alpha,I} \times \mathbb{A}^1)/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1 \times I)) \rightarrow \mathbb{Z}((X_{1,\alpha,I}, Z_{1,\alpha,I})/\tilde{S}_I, \mathbb{Z}(u_{IJ}^1)) \\ & \rightarrow \dots \rightarrow \\ & \oplus_{\alpha \in \Lambda_r} \text{Cone}(\mathbb{Z}((X_{r,\alpha,I} \times \mathbb{A}^1, Z_{r,\alpha,I} \times \mathbb{A}^1)/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r \times I)) \rightarrow \mathbb{Z}((X_{r,\alpha,I}, Z_{r,\alpha,I})/\tilde{S}_I, \mathbb{Z}(u_{IJ}^r)) \end{aligned}$$

(iii)' *A morphism $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr})/(\tilde{S}_I)^{op}$ is an (\mathbb{A}^1, et) local equivalence if and only if there exists*

$$\left\{ ((Y_{1,\alpha,I} \times \tilde{S}_I, Z_{1,\alpha,I})/\tilde{S}_I, u_{IJ}^1), \alpha \in \Lambda_1 \right\}, \dots, \left\{ ((Y_{r,\alpha,I} \times \tilde{S}_I, Z_{r,\alpha,I})/\tilde{S}_I, u_{IJ}^r), \alpha \in \Lambda_r \right\} \\ \subset \text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)$$

with

$$u_{IJ}^l : (Y_{l,\alpha,J} \times \tilde{S}_J, Z_{l,\alpha,J})/\tilde{S}_J \rightarrow (Y_{l,\alpha,I} \times \tilde{S}_J, Z_{l,\alpha,I} \times \tilde{S}_{J \setminus I})/\tilde{S}_J$$

such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{2,(sm)pr})/(\tilde{S}_I)^{op})$

$$\begin{aligned} & \text{Cone}(m) \xrightarrow{\sim} \text{Cone}(\oplus_{\alpha \in \Lambda_1} \\ & \text{Cone}(\mathbb{Z}((Y_{1,\alpha,I} \times \mathbb{A}^1 \times \tilde{S}_I, Z_{1,\alpha,I} \times \mathbb{A}^1)/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1 \times I)) \rightarrow \mathbb{Z}((Y_{1,\alpha,I} \times S, Z_{1,\alpha,I})/\tilde{S}_I, \mathbb{Z}(u_{IJ}^1)) \\ & \rightarrow \dots \rightarrow \oplus_{\alpha \in \Lambda_r} \\ & \text{Cone}(\mathbb{Z}((Y_{r,\alpha,I} \times \mathbb{A}^1 \times \tilde{S}_I, Z_{r,\alpha,I} \times \mathbb{A}^1)/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r \times I)) \rightarrow \mathbb{Z}((Y_{r,\alpha,I} \times \tilde{S}_I, Z_{r,\alpha,I})/\tilde{S}_I, \mathbb{Z}(u_{IJ}^r)) \end{aligned}$$

(iv) *A similar statement then (iii) holds for equivalence (\mathbb{A}^1, et) local $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)})/(\tilde{S}_I)$*

(iv)' *A similar statement then (iii) holds for equivalence (\mathbb{A}^1, et) local $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr})/(\tilde{S}_I)$*

Proof. Similar to the proof of proposition 22. See Ayoub's thesis for example. \square

In the filtered case we also consider :

Definition 18. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

- (i) A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to be r -filtered \mathbb{A}^1 homotopic if $\text{ad}(p_a^*, p_{a*})(G, F) : (G, F) \rightarrow p_{a*}p_a^*(G, F)$ is an r -filtered homotopy equivalence.
- (i)' A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ is said to be r -filtered \mathbb{A}^1 homotopic if $\text{ad}(p_a^*, p_{a*})(G, F) : (G, F) \rightarrow p_{a*}p_a^*(G, F)$ is an r -filtered homotopy equivalence.
- (ii) A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be r -filtered \mathbb{A}^1 homotopic if $\text{ad}(p_a^*, p_{a*})(G, F) : (G, F) \rightarrow p_{a*}p_a^*(G, F)$ is an r -filtered homotopy equivalence.
- (ii)' A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is said to be r -filtered \mathbb{A}^1 homotopic if $\text{ad}(p_a^*, p_{a*})(G, F) : (G, F) \rightarrow p_{a*}p_a^*(G, F)$ is an r -filtered homotopy equivalence.

We will use to compute the algebraic De Rahm realization functor the followings

Theorem 11. Let $S \in \text{Var}(\mathbb{C})$.

- (i) Let $\phi : F^\bullet \rightarrow G^\bullet$ an etale local equivalence with $F^\bullet, G^\bullet \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$. If F^\bullet and G^\bullet are \mathbb{A}^1 local and admit transferts then $\phi : F^\bullet \rightarrow G^\bullet$ is a Zariski local equivalence. Hence if $F \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$ is \mathbb{A}^1 local and admits transfert

$$k : E_{zar}(F) \rightarrow E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

- (ii) Let $\phi : F^\bullet \rightarrow G^\bullet$ an etale local equivalence with $F^\bullet, G^\bullet \in C(\text{Var}(\mathbb{C})^{2,smpr}/S)$. If F^\bullet and G^\bullet are \mathbb{A}^1 local and admit transferts then $\phi : F^\bullet \rightarrow G^\bullet$ is a Zariski local equivalence. Hence if $F \in C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ is \mathbb{A}^1 local and admits transfert

$$k : E_{zar}(F) \rightarrow E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

Proof. Similar to the proof of theorem 10. □

Theorem 12. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$.

- (i) Let $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ a filtered etale local equivalence with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,sm}/S)$. If (F^\bullet, F) and (G^\bullet, F) are r -filtered \mathbb{A}^1 homotopic and admit transferts then $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ is an r -filtered Zariski local equivalence. Hence if $(G, F) \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$ is r -filtered \mathbb{A}^1 homotopic and admits transfert

$$k : E_{zar}(G, F) \rightarrow E_{et}(E_{zar}(G, F)) = E_{et}(G, F)$$

is an r -filtered Zariski local equivalence.

- (i)' Let $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ a filtered etale local equivalence with $((F_I^\bullet, F), u_{IJ}), ((G_I^\bullet, F), v_{IJ}) \in C_{fil}(\text{Var}(\mathbb{C})^{2,sm}/(\tilde{S}_I))$. If $((F^\bullet, F), u_{IJ})$ and $((G^\bullet, F), v_{IJ})$ are r -filtered \mathbb{A}^1 homotopic and admit transferts then $\phi : ((F^\bullet, F), u_{IJ}) \rightarrow ((G^\bullet, F), v_{IJ})$ is an r -filtered Zariski local equivalence. Hence if $((G_I, F), u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$ is r -filtered \mathbb{A}^1 homotopic and admits transfert

$$k : (E_{zar}(G_I, F), u_{IJ}) \rightarrow (E_{et}(E_{zar}(G_I, F)), u_{IJ}) = (E_{et}(G, F), u_{IJ})$$

is an r -filtered Zariski local equivalence.

(ii) Let $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ a filtered etale local equivalence with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,smpr}/S)$. If F^\bullet and G^\bullet are r -filtered \mathbb{A}^1 homotopic and admit tranfers then $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ is an r -filtered Zariski local equivalence. Hence if $(G, F) \in C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ is r -filtered \mathbb{A}^1 homotopic and admits tranfert

$$k : E_{zar}(F) \rightarrow E_{et}(E_{zar}(F)) = E_{et}(F)$$

is an r -filtered Zariski local equivalence.

(ii)' Let $\phi : (F^\bullet, F) \rightarrow (G^\bullet, F)$ a filtered etale local equivalence with $((F_I^\bullet, F), u_{IJ}), ((G_I^\bullet, F), v_{IJ}) \in C_{fil}(\text{Var}(\mathbb{C})^{2,smpr}/\tilde{S}_I)$. If $((F^\bullet, F), u_{IJ})$ and $((G^\bullet, F), v_{IJ})$ are r -filtered \mathbb{A}^1 homotopic and admit tranfers then $\phi : ((F^\bullet, F), u_{IJ}) \rightarrow ((G^\bullet, F), v_{IJ})$ is an r -filtered Zariski local equivalence. Hence if $((G_I, F), u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ is r -filtered \mathbb{A}^1 homotopic and admits tranfert

$$k : (E_{zar}(G_I, F), u_{IJ}) \rightarrow (E_{et}(E_{zar}(G_I, F)), u_{IJ}) = (E_{et}(G, F), u_{IJ})$$

is an r -filtered Zariski local equivalence.

Proof. Similar to the proof of theorem 11. □

We have the following canonical functor :

Definition 19. (i) For $S \in \text{Var}(\mathbb{C})$, we have the functor

$$\begin{aligned} (-)^\Gamma : C(\text{Var}(\mathbb{C})^{sm}/S) &\rightarrow C(\text{Var}(\mathbb{C})^{2,sm}/S), \\ F &\longmapsto F^\Gamma : (((U, Z)/S) = ((U, Z), h) \mapsto F^\Gamma((U, Z)/S) := (\Gamma_Z^\vee h^* LF)(U/U), \\ &\quad (g : ((U', Z'), h') \rightarrow ((U, Z), h)) \mapsto \\ &\quad (F^\Gamma(g) : (\Gamma_Z^\vee h^* LF)(U/U) \xrightarrow{i_{(\Gamma_Z^\vee h^* LF)(U/U)}} (g^*(\Gamma_Z^\vee h^* LF))(U'/U')) \\ &\quad \xrightarrow{T(g, \gamma^\vee)(h^* LF)(U'/U')} (\Gamma_{Z \times_U U'}^\vee g^* h^* LF)(U'/U') \\ &\quad \xrightarrow{T(Z'/Z \times_U U', \gamma^\vee)(g^* h^* LF)(U'/U')} (\Gamma_{Z'}^\vee g^* h^* LF)(U'/U')) \end{aligned}$$

where $i_{(\Gamma_Z^\vee h^* LF)(U/U)}$ is the canonical arrow of the inductive limit. Similarly, we have, for $S \in \text{Var}(\mathbb{C})$, the functor

$$\begin{aligned} (-)^\Gamma : C(\text{Var}(\mathbb{C})/S) &\rightarrow C(\text{Var}(\mathbb{C})^2/S), \\ F &\longmapsto F^\Gamma : (((X, Z)/S) = ((X, Z), h) \mapsto F^\Gamma((X, Z)/S) := (\Gamma_Z^\vee h^* F)(X/X), \\ &\quad (g : ((X', Z'), h') \rightarrow ((X, Z), h)) \mapsto (F^\Gamma(g) : (\Gamma_Z^\vee h^* LF)(X/X) \rightarrow (\Gamma_{Z'}^\vee h'^* LF)(X'/X')) \end{aligned}$$

Note that for $S \in \text{Var}(\mathbb{C})$, $I(S/S) : \mathbb{Z}((S, S)/S) \rightarrow \mathbb{Z}(S/S)^\Gamma$ given by

$$I(S/S)((U, Z), h) : \mathbb{Z}((S, S)/S)((U, Z), h) \xrightarrow{\gamma_Z^\vee(\mathbb{Z}(U/U))(U/U)} \mathbb{Z}(S/S)^\Gamma((U, Z), h) := (\Gamma_Z^\vee \mathbb{Z}(U/U))(U/U)$$

is an isomorphism.

(ii) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we have the canonical morphism in $C(\text{Var}(\mathbb{C})^{2,sm}/T)$

$$\begin{aligned} T(f, \Gamma)(F) &:= T^*(f, \Gamma)(F) : f^*(F^\Gamma) \rightarrow (f^*F)^\Gamma, \\ T(f, \Gamma)(F)((U', Z')/T = ((U', Z'), h')) &: \\ f^*(F^\Gamma)((U', Z'), h') &:= \lim_{((U', Z'), h') \xrightarrow{l} ((U_T, Z_T), h_T) \xrightarrow{f_U} ((U, Z), h)} (\Gamma_Z^\vee h^* LF)(U/U) \\ &\xrightarrow{F^\Gamma(f_U \circ l)} (\Gamma_{Z'}^\vee l^* f_U^* h^* LF)(U'/U') = (\Gamma_{Z'}^\vee h'^* f^* LF)(U'/U') \\ &\xrightarrow{(\Gamma_{Z'}^\vee h'^* T(f, L)(F))(U'/U')} (\Gamma_{Z'}^\vee h'^* L f^* F)(U'/U') =: (f^*F)^\Gamma((U', Z'), h') \end{aligned}$$

where $f_U : U_T : U \times_S T \rightarrow U$ and $h_T : U_T := U \times_S T \rightarrow T$ are the base change maps, the equality following from the fact that $h \circ f_U \circ l = f \circ h_T \circ l = f \circ h'$. For $F \in C(\text{Var}(\mathbb{C})/S)$, we have similarly the canonical morphism in $C(\text{Var}(\mathbb{C})^2/T)$

$$T(f, \Gamma)(F) : f^*(F^\Gamma) \rightarrow (f^*F)^\Gamma.$$

(iii) Let $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{Var}(\mathbb{C})$. We have, for $F \in C(\text{Var}(\mathbb{C})^{sm}/U)$, the canonical morphism in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$

$$\begin{aligned} T_{\sharp}(h, \Gamma)(F) : h_{\sharp}(F^\Gamma) &\rightarrow (h_{\sharp}LF)^\Gamma, \\ T_{\sharp}(h, \Gamma)(F)((U', Z'), h') : h_{\sharp}(F^\Gamma)((U', Z'), h') &:= \lim_{((U', Z'), h') \xrightarrow{l} ((U, U), h)} (\Gamma_{Z'}^\vee, l^*LF)(U'/U') \\ &\xrightarrow{(\Gamma_{Z'}^\vee, l^* \text{ad}(h_{\sharp}, h^*)(LF))(U'/U')} (\Gamma_{Z'}^\vee, l^* h^* h_{\sharp}LF)(U'/U') =: (h_{\sharp}LF)^\Gamma((U', Z')/h') \end{aligned}$$

(iv) Let $i : Z_0 \hookrightarrow S$ a closed embedding with $Z_0, S \in \text{Var}(\mathbb{C})$. We have the canonical morphism in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$

$$\begin{aligned} T_*(i, \Gamma)(\mathbb{Z}(Z_0/Z_0)) : i_*(\mathbb{Z}(Z_0/Z_0))^\Gamma &\rightarrow (i_*\mathbb{Z}(Z/Z))^\Gamma, \\ T_*(i, \Gamma)(\mathbb{Z}(Z_0/Z_0))((U, Z), h) : i_*(\mathbb{Z}(Z_0/Z_0))^\Gamma((U, Z), h) &:= (\Gamma_{Z \times_S Z_0}^\vee, \mathbb{Z}(Z_0/Z_0))(U \times_S Z_0) \\ &\xrightarrow{T(i_*, \gamma^\vee)(\mathbb{Z}(Z_0/Z_0))(U \times_S Z_0)} (\Gamma_Z^\vee i_* \mathbb{Z}(Z_0/Z_0))(U \times_S Z_0) =: (i_*\mathbb{Z}(Z/Z))^\Gamma((U, Z), h) \end{aligned}$$

Definition 20. Let $S \in \text{Var}(\mathbb{C})$. We have for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$\begin{aligned} \text{Gr}(F) : \text{Gr}_{S^*}^{12} \mu_{S^*} F^\Gamma &\rightarrow F, \\ \text{Gr}(F)(U/S) : \Gamma_U^\vee p^* F(U \times S/U \times S) &\xrightarrow{\text{ad}(l^*, l_*)(p^* F)(U \times S/U \times S)} h^* F(U/U) = F(U/S) \end{aligned}$$

where $h : U \rightarrow S$ is a smooth morphism with $U \in \text{Var}(\mathbb{C})$ and $h : U \xrightarrow{l} U \times S \xrightarrow{p} S$ is the graph factorization with l the graph embedding and p the projection.

Proposition 28. Let $S \in \text{Var}(\mathbb{C})$.

(i) Then,

- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is a quasi-isomorphism, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is a quasi-isomorphism in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is a Zariski local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is a Zariski local equivalence in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$, if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is an étale local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is an étale local equivalence in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is an $(\mathbb{A}^1, \text{ét})$ local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is an $(\mathbb{A}^1, \text{ét})$ local equivalence in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$.

(ii) Then,

- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})/S)$ is a quasi-isomorphism, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is a quasi-isomorphism in $C(\text{Var}(\mathbb{C})^2/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is a Zariski local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is a Zariski local equivalence in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$, if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})/S)$ is an étale local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is an étale local equivalence in $C(\text{Var}(\mathbb{C})^2/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is an $(\mathbb{A}^1, \text{ét})$ local equivalence, $m^\Gamma : F^\Gamma \rightarrow G^\Gamma$ is an $(\mathbb{A}^1, \text{ét})$ local equivalence in $C(\text{Var}(\mathbb{C})^2/S)$.

Proof. (i): Follows immediately from the fact that for $((U, Z), h) \in \text{Var}(\mathbb{C})^{2,sm}/S$,

- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is a quasi-isomorphism, $\Gamma_Z^\vee h^* LF(m) : \Gamma_Z^\vee h^* LF \rightarrow \Gamma_Z^\vee h^* LG$ is a quasi-isomorphism
- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is a Zariski or etale local equivalence, $\Gamma_Z^\vee h^* LF(m) : \Gamma_Z^\vee h^* LF \rightarrow \Gamma_Z^\vee h^* LG$ is a Zariski, resp. etale, local equivalence,
- if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is an (\mathbb{A}^1, et) local equivalence, $\Gamma_Z^\vee h^* LF(m) : \Gamma_Z^\vee h^* LF \rightarrow \Gamma_Z^\vee h^* LG$ is an (\mathbb{A}^1, et) local equivalence.

(ii): Similar to (i). \square

2.8 Presheaves on the big analytical site

For $S \in \text{AnSp}(\mathbb{C})$, we denote by $\rho_S : \text{AnSp}(\mathbb{C})^{sm}/S \hookrightarrow \text{AnSp}(\mathbb{C})/S$ be the full subcategory consisting of the objects $U/S = (U, h) \in \text{AnSp}(\mathbb{C})/S$ such that the morphism $h : U \rightarrow S$ is smooth. That is, $\text{AnSp}(\mathbb{C})^{sm}/S$ is the category

- whose objects are smooth morphisms $U/S = (U, h), h : U \rightarrow S$ with $U \in \text{AnSp}(\mathbb{C})$,
- whose morphisms $g : U/S = (U, h_1) \rightarrow V/S = (V, h_2)$ is a morphism $g : U \rightarrow V$ of complex algebraic varieties such that $h_2 \circ g = h_1$.

We denote again $\rho_S : \text{AnSp}(\mathbb{C})/S \rightarrow \text{AnSp}(\mathbb{C})^{sm}/S$ the associated morphism of site. We will consider

$$r^s(S) : \text{AnSp}(\mathbb{C}) \xrightarrow{r(S)} \text{AnSp}(\mathbb{C})/S \xrightarrow{\rho_S} \text{AnSp}(\mathbb{C})^{sm}/S$$

the composite morphism of site. For $S \in \text{AnSp}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}(S/S) \in \text{PSh}(\text{AnSp}(\mathbb{C})^{sm}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, $\text{Hom}(\mathbb{Z}_S, F) = F$. For $f : T \rightarrow S$ a morphism, with $T, S \in \text{AnSp}(\mathbb{C})$, we have the following commutative diagram of sites

$$\begin{array}{ccc} \text{AnSp}(\mathbb{C})/T & \xrightarrow{\rho_T} & \text{AnSp}(\mathbb{C})^{sm}/T \\ \downarrow P(f) & & \downarrow P(f) \\ \text{AnSp}(\mathbb{C})/S & \xrightarrow{\rho_S} & \text{AnSp}(\mathbb{C})^{sm}/S \end{array} \quad (33)$$

We denote, for $S \in \text{AnSp}(\mathbb{C})$, the obvious morphism of sites

$$\tilde{e}(S) : \text{AnSp}(\mathbb{C})/S \xrightarrow{\rho_S} \text{AnSp}(\mathbb{C})^{sm}/S \xrightarrow{e(S)} \text{Ouv}(S)$$

where $\text{Ouv}(S)$ is the set of the open subsets of S , given by the inclusion functors $\tilde{e}(S) : \text{Ouv}(S) \hookrightarrow \text{AnSp}(\mathbb{C})^{sm}/S \hookrightarrow \text{AnSp}(\mathbb{C})/S$. By definition, for $f : T \rightarrow S$ a morphism with $S, T \in \text{AnSp}(\mathbb{C})$, the commutative diagram of sites (33) extend a commutative diagram of sites :

$$\begin{array}{ccccc} \tilde{e}(T) : \text{AnSp}(\mathbb{C})/T & \xrightarrow{\rho_T} & \text{AnSp}(\mathbb{C})^{sm}/T & \xrightarrow{e(T)} & \text{Ouv}(T) \\ \downarrow P(f) & & \downarrow P(f) & & \downarrow P(f) \\ \tilde{e}(S) : \text{AnSp}(\mathbb{C})/S & \xrightarrow{\rho_S} & \text{AnSp}(\mathbb{C})^{sm}/S & \xrightarrow{e(S)} & \text{Ouv}(S) \end{array} \quad (34)$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/T)$$

the adjunction induced by $P(f) : \text{AnSp}(\mathbb{C})^{sm}/T \rightarrow \text{AnSp}(\mathbb{C})^{sm}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \rightleftarrows C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/T), \quad f^*(G, F) := (f^*G, f^*F)$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{AnSp}(\mathbb{C})/S) \rightarrow C(\text{AnSp}(\mathbb{C})/T)$$

the adjunction induced by $P(f) : \text{AnSp}(\mathbb{C})/T \rightarrow \text{AnSp}(\mathbb{C})/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : C_{fil}(\text{AnSp}(\mathbb{C})/S) \rightleftarrows C_{fil}(\text{AnSp}(\mathbb{C})/T), \quad f^*(G, F) := (f^*G, f_*F)$$

- For $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$, the pullback functor $P(h) : \text{AnSp}(\mathbb{C})^{sm}/S \rightarrow \text{AnSp}(\mathbb{C})^{sm}/U$ admits a left adjoint $C(h)(X \rightarrow U) = (X \rightarrow U \rightarrow S)$. Hence, $h^* : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/U)$ admits a left adjoint

$$h_{\sharp} : C(\text{AnSp}(\mathbb{C})^{sm}/U) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/S), \quad F \mapsto ((V, h_0) \mapsto \lim_{(V', h_0 h') \rightarrow (V, h_0)} F(V', h'))$$

Note that for $h' : V' \rightarrow V$ a smooth morphism, $V', V \in \text{AnSp}(\mathbb{C})$, we have $h_{\sharp}(\mathbb{Z}(V'/V)) = \mathbb{Z}(V'/S)$ with $V'/S = (V', h_0 h')$. Hence, since projective presheaves are the direct summands of the representable presheaves, h_{\sharp} sends projective presheaves to projective presheaves. For $F^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ and $G^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{sm}/U)$, we have the adjunction maps

$$\text{ad}(h_{\sharp}, h^*)(G^{\bullet}) : G^{\bullet} \rightarrow h^* h_{\sharp} G^{\bullet}, \quad \text{ad}(h_{\sharp}, h^*)(F^{\bullet}) : h_{\sharp} h^* F^{\bullet} \rightarrow F^{\bullet}.$$

For a smooth morphism $h : U \rightarrow S$, with $U, S \in \text{AnSp}(\mathbb{C})$, we have the adjunction isomorphism, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/U)$ and $G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$,

$$I(h_{\sharp}, h^*)(F, G) : \text{Hom}^{\bullet}(h_{\sharp}F, G) \xrightarrow{\sim} h_* \text{Hom}^{\bullet}(F, h^*G). \quad (35)$$

- For $f : T \rightarrow S$ any morphism with $T, S \in \text{AnSp}(\mathbb{C})$, the pullback functor $P(f) : \text{AnSp}(\mathbb{C})/T \rightarrow \text{AnSp}(\mathbb{C})/S$ admits a left adjoint $C(f)(X \rightarrow T) = (X \rightarrow T \rightarrow S)$. Hence, $f^* : C(\text{AnSp}(\mathbb{C})/S) \rightarrow C(\text{AnSp}(\mathbb{C})/T)$ admits a left adjoint

$$f_{\sharp} : C(\text{AnSp}(\mathbb{C})/T) \rightarrow C(\text{AnSp}(\mathbb{C})/S), \quad F \mapsto ((V, h_0) \mapsto \lim_{(V', h_0 h') \rightarrow (V, h_0)} F(V', h'))$$

Note that we have for $h' : V' \rightarrow V$ a morphism, $V', V \in \text{AnSp}(\mathbb{C})$, $h_{\sharp}(\mathbb{Z}(V'/V)) = \mathbb{Z}(V'/S)$ with $V'/S = (V', h_0 h')$. Hence, since projective presheaves are the direct summands of the representable presheaves, h_{\sharp} sends projective presheaves to projective presheaves. For $F^{\bullet} \in C(\text{AnSp}(\mathbb{C})/S)$ and $G^{\bullet} \in C(\text{AnSp}(\mathbb{C})/T)$, we have the adjunction maps

$$\text{ad}(f_{\sharp}, f^*)(G^{\bullet}) : G^{\bullet} \rightarrow f^* f_{\sharp} G^{\bullet}, \quad \text{ad}(f_{\sharp}, f^*)(F^{\bullet}) : f_{\sharp} f^* F^{\bullet} \rightarrow F^{\bullet}.$$

For a morphism $f : T \rightarrow S$, with $T, S \in \text{AnSp}(\mathbb{C})$, we have the adjunction isomorphism, for $F \in C(\text{AnSp}(\mathbb{C})/T)$ and $G \in C(\text{AnSp}(\mathbb{C})/S)$,

$$I(f_{\sharp}, f^*)(F, G) : \text{Hom}^{\bullet}(f_{\sharp}F, G) \xrightarrow{\sim} f_* \text{Hom}^{\bullet}(F, f^*G). \quad (36)$$

- For a commutative diagram in $\text{AnSp}(\mathbb{C})$:

$$D = \begin{array}{ccc} V & \xrightarrow{g_2} & U \\ \downarrow h_2 & & \downarrow h_1 \\ T & \xrightarrow{g_1} & S \end{array},$$

where h_1 and h_2 are smooth, we denote by, for $F^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{sm}/U)$,

$$T_{\sharp}(D)(F^{\bullet}) : h_{2\sharp} g_2^* F^{\bullet} \rightarrow g_1^* h_{1\sharp} F^{\bullet}$$

the canonical map in $C(\text{AnSp}(\mathbb{C})^{sm}/T)$ given by adjunction. If D is cartesian with $h_1 = h$, $g_1 = g$ $f_2 = h' : U_T \rightarrow T$, $g' : U_T \rightarrow U$,

$$T_{\sharp}(D)(F^{\bullet}) =: T_{\sharp}(g, h)(F^{\bullet}) : h'_{\sharp} g'^{*} F^{\bullet} \xrightarrow{\sim} g^* h_{\sharp} F^{\bullet}$$

is an isomorphism and for $G^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{sm}/T)$

$$T(D)(G^{\bullet}) =: T(g, h)(G^{\bullet}) : g^* h_* G^{\bullet} \xrightarrow{\sim} h'_* g'^{*} G^{\bullet}$$

is an isomorphism.

- For a commutative diagram in $\text{AnSp}(\mathbb{C})$:

$$D = \begin{array}{ccc} V & \xrightarrow{g_2} & X \\ \downarrow f_2 & & \downarrow f_1 \\ T & \xrightarrow{g_1} & S \end{array} ,$$

we denote by, for $F^{\bullet} \in C(\text{AnSp}(\mathbb{C})/X)$,

$$T_{\sharp}(D)(F^{\bullet}) : f_{2\sharp} g_2^* F^{\bullet} \rightarrow g_1^* f_{1\sharp} F^{\bullet}$$

the canonical map in $C(\text{AnSp}(\mathbb{C})/T)$ given by adjunction. If D is cartesian with $h_1 = h$, $g_1 = g$ $f_2 = h' : X_T \rightarrow T$, $g' : X_T \rightarrow X$,

$$T_{\sharp}(D)(F^{\bullet}) =: T_{\sharp}(g, f)(F^{\bullet}) : f'_{\sharp} g'^{*} F^{\bullet} \xrightarrow{\sim} g^* f_{\sharp} F^{\bullet}$$

is an isomorphism and for $G^{\bullet} \in C(\text{AnSp}(\mathbb{C})/T)$

$$T(D)(G^{\bullet}) =: T(g, h)(G^{\bullet}) : f^* g_* G^{\bullet} \xrightarrow{\sim} g'_* f'^{*} G^{\bullet}$$

is an isomorphism.

For $f : T \rightarrow S$ a morphism with $S, T \in \text{AnSp}(\mathbb{C})$,

- we get for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ from the a commutative diagram of sites (34) the following canonical transformation

$$T(e, f)(F^{\bullet}) : f^* e(S)_* F^{\bullet} \rightarrow e(T)_* f^* F^{\bullet},$$

which is NOT a quasi-isomorphism in general. However, for $h : U \rightarrow S$ a smooth morphism with $S, U \in \text{AnSp}(\mathbb{C})$, $T(e, h)(F^{\bullet}) : h^* e(S)_* F^{\bullet} \xrightarrow{\sim} e(T)_* h^* F^{\bullet}$ is an isomorphism.

- we get for $F \in C(\text{AnSp}(\mathbb{C})/S)$ from the a commutative diagram of sites (34) the following canonical transformation

$$T(e, f)(F^{\bullet}) : f^* e(S)_* F^{\bullet} \rightarrow e(T)_* f^* F^{\bullet},$$

which is NOT a quasi-isomorphism in general. However, for $h : U \rightarrow S$ a smooth morphism with $S, U \in \text{AnSp}(\mathbb{C})$, $T(e, h)(F^{\bullet}) : h^* e(S)_* F^{\bullet} \xrightarrow{\sim} e(T)_* h^* F^{\bullet}$ is an isomorphism.

Let $S \in \text{AnSp}(\mathbb{C})$,

- We have for $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$,
 - $e(S)_*(F \otimes G) = (e(S)_* F) \otimes (e(S)_* G)$ by definition
 - the canonical forgetfull map

$$T(S, \text{hom})(F, G) : e(S)_* \mathcal{H}om^{\bullet}(F, G) \rightarrow \mathcal{H}om^{\bullet}(e(S)_* F, e(S)_* G).$$

which is NOT a quasi-isomorphism in general.

By definition, we have for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, $e(S)_*E_{usu}(F) = E_{usu}(e(S)_*F)$.

- We have for $F, G \in C(\text{AnSp}(\mathbb{C})/S)$,
 - $e(S)_*(F \otimes G) = (e(S)_*F) \otimes (e(S)_*G)$ by definition
 - the canonical forgetfull map

$$T(S, \text{hom})(F, G) : e(S)_*\mathcal{H}om^\bullet(F, G) \rightarrow \mathcal{H}om^\bullet(e(S)_*F, e(S)_*G).$$

which is NOT a quasi-isomorphism in general.

By definition, we have for $F \in C(\text{AnSp}(\mathbb{C})/S)$, $e(S)_*E_{usu}(F) = E_{usu}(e(S)_*F)$.

Let $S \in \text{AnSp}(\mathbb{C})$. We have the support section functor of a closed subset $Z \subset S$ for presheaves on the big analytical site.

Definition 21. Let $S \in \text{AnSp}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ be the open complementary subset.

(i) We define the functor

$$\Gamma_Z : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/S), G^\bullet \mapsto \Gamma_Z G^\bullet := \text{Cone}(\text{ad}(j^*, j_*)(G^\bullet) : G^\bullet \rightarrow j_*j^*G^\bullet)[-1],$$

so that there is then a canonical map $\gamma_Z(G^\bullet) : \Gamma_Z G^\bullet \rightarrow G^\bullet$.

(ii) We have the dual functor of (i) :

$$\Gamma_Z^\vee : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/S), F \mapsto \Gamma_Z^\vee(F^\bullet) := \text{Cone}(\text{ad}(j_\#^*, j^*)(G^\bullet) : j_\#^*j^*G^\bullet \rightarrow G^\bullet),$$

together with the canonical map $\gamma_Z^\vee(G) : F \rightarrow \Gamma_Z^\vee(G)$.

(iii) For $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, we denote by

$$I(\gamma, \text{hom})(F, G) := (I, I(j_\#^*, j^*)(F, G)) : \Gamma_Z \mathcal{H}om(F, G) \xrightarrow{\sim} \mathcal{H}om(\Gamma_Z^\vee F, G)$$

the canonical isomorphism given by adjonction.

Let $S \in \text{AnSp}(\mathbb{C})$ and $Z \subset S$ a closed subset.

- Since $\Gamma_Z : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor

$$\Gamma_Z : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S), (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F)$$

- Since $\Gamma_Z^\vee : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor

$$\Gamma_Z^\vee : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S), (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F)$$

Definition-Proposition 7. (i) Let $g : S' \rightarrow S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $S', S, Z \in \text{AnSp}(\mathbb{C})$. Then, for $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$, there exist a map in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S')$

$$T(g, \gamma)(G, F) : g^*\Gamma_Z(G, F) \rightarrow \Gamma_{Z \times_S S'} g^*(G, F)$$

unique up to homotopy, such that $\gamma_{Z \times_S S'}(g^*(G, F)) \circ T(g, \gamma)(G, F) = g^*\gamma_Z(G, F)$.

(ii) Let $i_1 : Z_1 \hookrightarrow S$, $i_2 : Z_2 \hookrightarrow Z_1$ be closed embeddings with $S, Z_1, Z_2 \in \text{AnSp}(\mathbb{C})$. Then, for $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$,

- there exist a canonical map $T(Z_2/Z_1, \gamma)(G, F) : \Gamma_{Z_2}(G, F) \rightarrow \Gamma_{Z_1}(G, F)$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$ unique up to homotopy such that $\gamma_{Z_1}(G, F) \circ T(Z_2/Z_1, \gamma)(G, F) = \gamma_{Z_2}(G, F)$, together with a distinguish triangle

$$\Gamma_{Z_2}(G, F) \xrightarrow{T(Z_2/Z_1, \gamma)(G, F)} \Gamma_{Z_1}(G, F) \xrightarrow{\text{ad}(j_2^*, j_{2*})(\Gamma_{Z_1}(G, F))} \Gamma_{Z_1 \setminus Z_2}(G, F) \rightarrow \Gamma_{Z_2}(G, F)[1]$$

in $K_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$,

- there exist a map $T(Z_2/Z_1, \gamma^\vee)(G, F) : \Gamma_{Z_1}^\vee(G, F) \rightarrow \Gamma_{Z_2}^\vee(G, F)$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$ unique up to homotopy such that $\gamma_{Z_2}^\vee(G, F) = T(Z_2/Z_1, \gamma^\vee)(G, F) \circ \gamma_{Z_1}^\vee(G, F)$, together with a distinguish triangle

$$\Gamma_{Z_1 \setminus Z_2}^\vee(G, F) \xrightarrow{\text{ad}(j_{2\#}, j_2^*)(\Gamma_{Z_1}^\vee G)} \Gamma_{Z_1}^\vee(G, F) \xrightarrow{T(Z_2/Z_1, \gamma^\vee)(G, F)} \Gamma_{Z_2}^\vee(G, F) \rightarrow \Gamma_{Z_1 \setminus Z_2}^\vee(G, F)[1]$$

in $K_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$.

- (iii) Consider a morphism $g : (S', Z) \rightarrow (S, Z)$ with $(S', Z) \rightarrow (S, Z) \in \text{AnSp}(\mathbb{C})$. We denote, for $G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ the composite

$$T(D, \gamma^\vee)(G) : g^* \Gamma_Z^\vee G \xrightarrow{\sim} \Gamma_{Z \times_S S'}^\vee g^* G \xrightarrow{T(Z'/Z \times_S S', \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^* G$$

and we have then the factorization $\gamma_{Z'}^\vee(g^* G) : g^* G \xrightarrow{g^* \gamma_Z^\vee(G)} g^* \Gamma_Z^\vee G \xrightarrow{T(D, \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^* G$.

Proof. Similar to definition-proposition 1 or definition-proposition 4. \square

Definition 22. For $S \in \text{AnSp}(\mathbb{C})$, we denote by

$$C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S) := C_{e(S)^* O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$$

the category of complexes of presheaves on $\text{AnSp}(\mathbb{C})^{sm}/S$ endowed with a structure of $e(S)^* O_S$ module, and by

$$C_{O_S fil}(\text{AnSp}(\mathbb{C})^{sm}/S) := C_{e(S)^* O_S fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$$

the category of filtered complexes of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ endowed with a structure of $e(S)^* O_S$ module.

Let $S \in \text{AnSp}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding,

- For $G \in C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$, $\Gamma_Z G := \text{Cone}(\text{ad}(j^*, j_*)(G) : F \rightarrow j_* j^* G)[-1]$ has a (unique) structure of $e(S)^* O_S$ module such that $\gamma_Z(G) : \Gamma_Z G \rightarrow G$ is a map in $C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$. This gives the functor

$$\Gamma_Z : C_{O_S fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C_{fil O_S}(\text{AnSp}(\mathbb{C})^{sm}/S), (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F),$$

together with the canonical map $\gamma_Z((G, F)) : \Gamma_Z(G, F) \rightarrow (G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$, $T(Z_2/Z, \gamma)(G) : \Gamma_{Z_2} G \rightarrow \Gamma_Z G$ is a map in $C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$ (i.e. is $e(S)^* O_S$ linear).

- For $G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$, $\Gamma_Z^\vee G := \text{Cone}(\text{ad}(j_{\#}, j^*)(G) : j_{\#} j^* G \rightarrow G)$ has a unique structure of $e(S)^* O_S$ module, such that $\gamma_Z^\vee(G) : G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$. This gives the the functor

$$\Gamma_Z^\vee : C_{O_S fil}(S) \rightarrow C_{fil O_S}(S), (G, F) \mapsto \Gamma_Z^\vee(G, F) := (\Gamma_Z^\vee G, \Gamma_Z^\vee F),$$

together with the canonical map $\gamma_Z^\vee((G, F)) : (G, F) \rightarrow \Gamma_Z^\vee(G, F)$. Let $Z_2 \subset Z$ a closed subset. Then, for $G \in C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$, $T(Z_2/Z, \gamma^\vee)(G) : \Gamma_{Z_2}^\vee G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$ (i.e. is $e(S)^* O_S$ linear).

Definition 23. Let $S \in \text{AnSp}(\mathbb{C})$. Let $Z \subset S$ a closed subset.

(i) We denote by

$$C_Z(\text{AnSp}(\mathbb{C})^{sm}/S) \subset C(\text{AnSp}(\mathbb{C})^{sm}/S),$$

the full subcategory consisting of complexes of presheaves $F^\bullet \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ such that $a_{usu}H^n(j^*F^\bullet) = 0$ for all $n \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{usu} is the sheafification functor.

(i)' We denote by

$$C_{O_S, Z}(\text{AnSp}(\mathbb{C})^{sm}/S) \subset C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S),$$

the full subcategory consisting of complexes of presheaves $F^\bullet \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ such that $a_{usu}H^n(j^*F^\bullet) = 0$ for all $n \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{usu} is the sheafification functor.

(ii) We denote by

$$C_{fil, Z}(\text{AnSp}(\mathbb{C})^{sm}/S) \subset C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$$

the full subcategory consisting of filtered complexes of presheaves $(F^\bullet, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$ such that there exist $r \in \mathbb{N}$ and an r -filtered homotopy equivalence $\phi : (F^\bullet, F) \rightarrow (F'^\bullet, F)$ with $(F'^\bullet, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$ such that $a_{usu}j^*H^n \text{Gr}_F^p(F'^\bullet, F) = 0$ for all $n, p \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{usu} is the sheafification functor.

(ii)' We denote by

$$C_{O_S fil, Z}(\text{AnSp}(\mathbb{C})^{sm}/S) \subset C_{O_S fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$$

the full subcategory consisting of filtered complexes of presheaves $(F^\bullet, F) \in C_{O_S fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$ such that there exist $r \in \mathbb{N}$ and an r -filtered homotopy equivalence $\phi : (F^\bullet, F) \rightarrow (F'^\bullet, F)$ with $(F'^\bullet, F) \in C_{O_S fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$ such that $a_{usu}j^*H^n \text{Gr}_F^p(F^\bullet, F) = 0$ for all $n, p \in \mathbb{Z}$, where $j : S \setminus Z \hookrightarrow S$ is the complementary open embedding and a_{usu} is the sheafification functor.

Let $S \in \text{AnSp}(\mathbb{C})$ and $Z \subset S$ a closed subset.

- For $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$, we have $\Gamma_Z(G, F), \Gamma_Z^\vee(G, F) \in C_{fil, Z}(\text{AnSp}(\mathbb{C})^{sm}/S)$.
- For $(G, F) \in C_{O_S fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$, we have $\Gamma_Z(G, F), \Gamma_Z^\vee(G, F) \in C_{O_S fil, Z}(\text{AnSp}(\mathbb{C})^{sm}/S)$.

Let $S_\bullet \in \text{Fun}(\mathcal{I}, \text{AnSp}(\mathbb{C}))$ with $\mathcal{I} \in \text{Cat}$, a diagram of algebraic varieties. It gives the diagram of sites $\text{AnSp}(\mathbb{C})^2/S_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat})$.

- Then $C_{fil}(\text{AnSp}(\mathbb{C})/S_\bullet)$ is the category
 - whose objects $(G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{AnSp}(\mathbb{C})/S_I)$, and $u_{IJ} : (G_I, F) \rightarrow r_{IJ*}(G_J, F)$ for $r_{IJ} : I \rightarrow J$, denoting again $r_{IJ} : S_I \rightarrow S_J$, are morphisms satisfying for $I \rightarrow J \rightarrow K$, $r_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{AnSp}(\mathbb{C})/S_I)$,
 - the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \in \mathcal{I}}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})/S_I)$.

- Then $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S_\bullet)$ is the category
 - whose objects $(G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S_I)$, and $u_{IJ} : (G_I, F) \rightarrow r_{IJ*}(G_J, F)$ for $r_{IJ} : I \rightarrow J$, denoting again $r_{IJ} : S_I \rightarrow S_J$, are morphisms satisfying for $I \rightarrow J \rightarrow K$, $r_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S_I)$,

- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \in \mathcal{I}}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S_I)$.

As usual, we denote by

$$(f_{\bullet}^*, f_{\bullet*}) := (P(f_{\bullet})^*, P(f_{\bullet})_*) : C(\text{AnSp}(\mathbb{C})^{(sm)}/S_{\bullet}) \rightarrow C(\text{AnSp}(\mathbb{C})^{(sm)}/T_{\bullet})$$

the adjunction induced by $P(f_{\bullet}) : \text{AnSp}(\mathbb{C})^{(sm)}/T_{\bullet} \rightarrow \text{AnSp}(\mathbb{C})^{(sm)}/S_{\bullet}$. Since the colimits involved in the definition of $f_{\bullet}^* = P(f_{\bullet})^*$ are filtered, f_{\bullet}^* also preserve monomorphism. Hence, we get an adjunction

$$(f_{\bullet}^*, f_{\bullet*}) : C_{fil}(\text{AnSp}(\mathbb{C})^{(sm)}/S_{\bullet}) \rightleftarrows C_{fil}(\text{AnSp}(\mathbb{C})^{(sm)}/T_{\bullet}),$$

$$f_{\bullet}^*((G_I, F), u_{IJ}) := ((f_I^* G_I, f_I^* F), T(f_I, r_{IJ})(-) \circ f_I^* u_{IJ}).$$

Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$, and for $J \subset I$ the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

where $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ is the projection and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. This gives the diagram of analytic spaces $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{AnSp}(\mathbb{C}))$ which gives the diagram of sites $\text{AnSp}(\mathbb{C})^{sm}/(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. Denote by $m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I$ the open embedding. Then $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/(\tilde{S}_I))$ is the category

- whose objects $(G, F) = ((G_I, F), u_{IJ})$ with $(G_I, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I)$, and $u_{IJ} : (G_I, F) \rightarrow p_{IJ*}(G_J, F)$ are morphisms satisfying for $I \subset J \subset K$, $p_{IJ*} u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I)$,
- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \in \mathcal{I}}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I)$.

Definition 24. Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$. We denote by the full subcategory $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) \subset C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/(\tilde{S}_I))$ the full subcategory

- whose objects $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil, S_I}(\text{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I)$, and $u_{IJ} : m^*(G_I, F) \rightarrow m^* p_{IJ*}(G_J, F)$ for $I \subset J$, are ∞ -filtered usu local equivalence, satisfying for $I \subset J \subset K$, $p_{IJ*} u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I)$,
- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I)$.

A morphism $m : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$ is an r -filtered usu local equivalence, if there exists $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$, $0 \leq i \leq s$, with $((C_{iI}, F), u_{iIJ}) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)})/(S/(\tilde{S}_I))$, $((C_{0I}, F), u_{0IJ}) = ((G_I, F), u_{IJ})$ and $((C_{sI}, F), u_{sIJ}) = ((H_I, F), v_{IJ})$, such that

$$\phi = \phi_s \circ \cdots \circ \phi_1 \circ \cdots \circ \phi_0 : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$$

and $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$ either a filtered usu local equivalence or an r -filtered homotopy equivalence.

Denote $L = [1, \dots, l]$ and for $I \subset L$, $p_{0(0I)} : S \times \tilde{S}_I \rightarrow S$, $p_{I(0I)} : S \times \tilde{S}_I \rightarrow S_I$ the projections. By definition, we have functors

- $T(S/(\tilde{S}_I)) : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$, $(G, F) \mapsto (i_{I*}j_I^*F, T(D_{IJ})(j_I^*(G, F)))$,
- $T(\tilde{S}_I/S) : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) \rightarrow C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$, $((G_I, F), u_{IJ}) \mapsto \text{ho} \lim_{I \subset L} p_{0(0I)*} \Gamma_{S_I}^\vee p_{I(0I)}^*(G_I, F)$.

Note that the functors $T(S/(\tilde{S}_I))$ are NOT embedding, since

$$\text{ad}(i_I^*, i_{I*})(j_I^*F) : i_I^*i_{I*}j_I^*F \rightarrow j_I^*F$$

are usu local equivalence but NOT isomorphism since we are dealing with the morphism of big sites $P(i_I) : \text{AnSp}(\mathbb{C})^{sm}/S_I \rightarrow \text{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I$. However, these functors induces full embeddings

$$T(S/(\tilde{S}_I)) : D_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow D_{fil}(\text{AnSp}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$$

since for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$,

$$\text{ho} \lim_{I \subset L} p_{0(0I)*} \Gamma_{S_I} p_{I(0I)}^*(i_{I*}j_I^*F) \rightarrow p_{0(0I)*} \Gamma_{S_I} j_I^*F$$

is an equivalence usu local.

Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ and $X = \cup_{i=1}^l X_i$ be affine open covers and $i_i : S_i \hookrightarrow \tilde{S}_i$, $i'_i : X_i \hookrightarrow \tilde{X}_i$ be closed embeddings. Let $f_i : \tilde{X}_i \rightarrow \tilde{S}_i$ be a lift of the morphism $f_i = f|_{X_i} : X_i \rightarrow S_i$. Then, $f_I = f|_{X_I} : X_I \rightarrow S_I = \cap_{i \in I} S_i$ lift to the morphism

$$\tilde{f}_I = \prod_{i \in I} \tilde{f}_i : \tilde{X}_I = \prod_{i \in I} \tilde{X}_i \rightarrow \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$$

Denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} : \tilde{X}_J \rightarrow \tilde{X}_I$ the projections. Consider for $J \subset I$ the following commutative diagrams

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}, \quad D'_{IJ} = \begin{array}{ccc} X_I & \xrightarrow{i'_I} & \tilde{X}_I \\ j'_{IJ} \uparrow & & \uparrow p'_{IJ} \\ X_J & \xrightarrow{i'_J} & \tilde{X}_J \end{array}, \quad D_{fI} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ f_I \uparrow & & \uparrow \tilde{f}_I \\ X_I & \xrightarrow{i'_I} & \tilde{X}_I \end{array}$$

We have then following commutative diagram

$$\begin{array}{ccccccc} & & X_I & \xrightarrow{n'_I} & \tilde{X}_I & \longleftarrow & \tilde{X}_I \setminus X_I \\ & \nearrow j'_{IJ} & \uparrow p'_{IJ} & & \uparrow p'_{IJ} & & \uparrow p'_{IJ} \\ i'_J : X_J & \xrightarrow{l_{IJ}} & X_I \times X_I \times \tilde{X}_J^{n'_I \times I} & \xrightarrow{\quad} & \tilde{X}_J & \longleftarrow & \tilde{X}_J \setminus X_J \\ & & & & & & \uparrow p'_{IJ} \\ & & & & & & \uparrow n'_J \end{array}$$

whose square are cartesian. We then have the pullback functor

$$f^* : C_{(2)fil}(\text{AnSp}(\mathbb{C})^{sm}/S/(\tilde{S}_I)) \rightarrow C_{(2)fil}(\text{AnSp}(\mathbb{C})^{sm}/X/(\tilde{X}_I)), \\ ((G_I, F), u_{IJ}) \mapsto f^*((G_I, F), u_{IJ}) := (\Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F), \tilde{f}_J^* u_{IJ})$$

with

$$\begin{aligned} \tilde{f}_J^* u_{IJ} : \Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F) &\xrightarrow{\text{ad}(p_{IJ}^*, p'_{IJ*})(-)} p'_{IJ*} p_{IJ}^* \Gamma_{X_I}^\vee \tilde{f}_I^*(G_I, F) \xrightarrow{T_{\sharp}(p_{IJ}, n'_I)(-)^{-1}} p'_{IJ*} \Gamma_{X_I \times \tilde{X}_{J_I}}^\vee p_{IJ}^* \tilde{f}_I^*(G_I, F) \\ &\xrightarrow{p'_{IJ*} \gamma_{\tilde{X}_J}^\vee(-)} p'_{IJ*} \Gamma_{X_J}^\vee p_{IJ}^* \tilde{f}_I^*(G_I, F) = p'_{IJ*} \Gamma_{X_J}^\vee \tilde{f}_J^* p_{IJ}^*(G_I, F) \xrightarrow{\Gamma_{X_J}^\vee \tilde{f}_J^* I(p_{IJ}^*, p_{IJ*})(-, -)(u_{IJ})} \Gamma_{X_J}^\vee \tilde{f}_J^*(G_J, F) \end{aligned}$$

Let $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$. Since, $j_I^* i'_I j_I^* f^*(G, F) = 0$, the morphism $T(D_{f_I})(j_I^*(G, F)) : \tilde{f}_I^* i_I j_I^*(G, F) \rightarrow i'_I j_I^* f^*(G, F)$ factors through

$$T(D_{f_I})(j_I^*(G, F)) : \tilde{f}_I^* i_I j_I^*(G, F) \xrightarrow{\gamma_{\tilde{X}_I}^\vee(-)} \Gamma_{X_I}^\vee \tilde{f}_I^* i_I j_I^*(G, F) \xrightarrow{T^\gamma(D_{f_I})(j_I^*(G, F))} i'_I j_I^* f^*(G, F)$$

We have then, for $(G, F) \in C_{fil}(S)$, the canonical transformation map

$$\begin{array}{ccc} f^* T(S/(\tilde{S}_I))(G, F) & \xrightarrow{T(f, T(0/I))(G, F)} & T(X/(\tilde{X}_I))(f^*(G, F)) \\ \downarrow = & & \downarrow = \\ (\Gamma_{X_I}^\vee \tilde{f}_I^* i_I j_I^*(G, F), \tilde{f}_J^* I) & \xrightarrow{T^\gamma(D_{f_I})(j_I^*(G, F))} & (i'_I j_I^* f^*(G, F), I) \end{array}$$

We have similarly to the algebraic case, we have:

Definition 25. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSp}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{AnSm}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then consider

$$Q(X/S) := p_{\sharp} \Gamma_X^\vee \mathbb{Z}_{Y \times S} := \text{Cone}(\mathbb{Z}((Y \times S) \setminus X/S) \rightarrow \mathbb{Z}(Y \times S/S)) \in C(\text{AnSp}(\mathbb{C})^{sm}/S).$$

By definition $Q(X/S)$ is projective since it is a complex of two representative presheaves.

(ii) Let $f : X \rightarrow S$ and $g : T \rightarrow S$ two morphism with $X, S, T \in \text{AnSp}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{AnSm}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then have the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} f : X & \xrightarrow{i} & Y \times S & \xrightarrow{p} & S \\ g' \uparrow & & g'' := (I \times g) \uparrow & & \uparrow g \\ f' : X_T & \xrightarrow{i'} & Y \times T & \xrightarrow{p'} & T \end{array}$$

We then have the canonical isomorphism in $C(\text{AnSp}(\mathbb{C})^{sm}/T)$

$$\begin{aligned} T(f, g, Q) &:= T_{\sharp}(g, p)(-)^{-1} \circ T_{\sharp}(g'', j)(-)^{-1} : \\ g^* Q(X/S) &:= g^* p_{\sharp} \Gamma_X^\vee \mathbb{Z}_{Y \times S} \xrightarrow{\sim} p'_{\sharp} \Gamma_{X_T}^\vee \mathbb{Z}_{Y \times T} =: Q(X_T/T) \end{aligned}$$

with $j : Y \times S \setminus X \hookrightarrow Y \times S$ the closed embedding.

Let $S \in \text{AnSp}(\mathbb{C})$. Denote for short $\text{AnSp}(\mathbb{C})^{(sm)}/S$ either the category $\text{AnSp}(\mathbb{C})/S$ or the category $\text{AnSp}(\mathbb{C})^{sm}/S$. Denote by

$$\begin{aligned} p_a : \text{AnSp}(\mathbb{C})^{(sm)}/S &\rightarrow \text{AnSp}(\mathbb{C})^{(sm)}/S, \quad X/S = (X, h) \mapsto (X \times \mathbb{D}^1)/S = (X \times \mathbb{D}^1, h \circ p_X), \\ (g : X/S \rightarrow X'/S) &\mapsto ((g \times I_{\mathbb{D}^1}) : X \times \mathbb{D}^1/S \rightarrow X' \times \mathbb{D}^1/S) \end{aligned}$$

the projection functor and again by $p_a : \text{AnSp}(\mathbb{C})^{(sm)}/S \rightarrow \text{AnSp}(\mathbb{C})^{(sm)}/S$ the corresponding morphism of site.

We now define the \mathbb{D}^1 localization property :

Definition 26. Let $S \in \text{Var}(\mathbb{C})$.

- (i0) A complex $F \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to be \mathbb{D}^1 homotopic if $\text{ad}(p_a^*, p_{a*})(F) : F \rightarrow p_{a*}p_a^*F$ is an homotopy equivalence.
- (i) A complex $F \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to be \mathbb{D}^1 invariant if for all $U/S \in \text{AnSp}(\mathbb{C})^{(sm)}/S$,

$$F(p_U) : F(U/S) \rightarrow F(U \times \mathbb{D}^1/S)$$

is a quasi-isomorphism, where $p_U : U \times \mathbb{D}^1 \rightarrow U$ is the projection. Obviously, a \mathbb{D}^1 homotopic complex is \mathbb{D}^1 invariant.

- (ii) A complex $F \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to be \mathbb{D}^1 local for the usual topology, if for a (hence every) usu local equivalence $k : F \rightarrow G$ with k injective and $G \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ usu fibrant, e.g. $k : F \rightarrow E_{usu}(F)$, G is \mathbb{D}^1 invariant for all $n \in \mathbb{Z}$.
- (iii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ is said to an $(\mathbb{D}^1, \text{usu})$ local equivalence if for all $H \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ which is \mathbb{A}^1 local for the etale topology

$$\text{Hom}(L(m), E_{usu}(H)) : \text{Hom}(L(G), E_{usu}(H)) \rightarrow \text{Hom}(L(F), E_{usu}(H))$$

is a quasi-isomorphism.

Proposition 29. A morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{(sm)}/S)$ is an $(\mathbb{D}^1, \text{et})$ local equivalence if and only if there exists

$$\{X_{1,\alpha}/S, \alpha \in \Lambda_1\}, \dots, \{X_{r,\alpha}/S, \alpha \in \Lambda_r\} \subset \text{AnSp}(\mathbb{C})^{(sm)}/S$$

such that we have in $\text{Ho}_{\text{et}}(C(\text{Var}(\mathbb{C})^{(sm)}/S))$

$$\text{Cone}(m) \xrightarrow{\sim} \text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}(X_{1,\alpha} \times \mathbb{D}^1/S) \rightarrow \mathbb{Z}(X_{1,\alpha}/S)) \rightarrow \dots \rightarrow \bigoplus_{\alpha \in \Lambda_r} \text{Cone}(\mathbb{Z}(X_{r,\alpha} \times \mathbb{D}^1/S) \rightarrow \mathbb{Z}(X_{r,\alpha}/S)))$$

Proof. Standard. □

Definition-Proposition 8. Let $S \in \text{AnSp}(\mathbb{C})$

- (i) With the weak equivalence the $(\mathbb{D}^1, \text{usu})$ local equivalence and the fibration the epimorphism with \mathbb{D}_S^1 local and usu fibrant kernels gives a model structure on $C(\text{AnSp}(\mathbb{C})^{sm}/S)$: the left bousfield localization of the projective model structure of $C(\text{AnSp}(\mathbb{C})^{sm}/S)$. We call it the $(\mathbb{D}^1, \text{usu})$ projective model structure.
- (ii) With the weak equivalence the $(\mathbb{D}^1, \text{usu})$ local equivalence and the fibration the epimorphism with \mathbb{D}_S^1 local and usu fibrant kernels gives a model structure on $C(\text{AnSp}(\mathbb{C})/S)$: the left bousfield localization of the projective model structure of $C(\text{AnSp}(\mathbb{C})/S)$. We call it the $(\mathbb{D}^1, \text{usu})$ projective model structure.

Proof. Similar to the proof of definition-proposition 5. □

Proposition 30. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$.

- (i) The adjunction $(g^*, g_*) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightleftarrows C(\text{AnSp}(\mathbb{C})^{sm}/T)$ is a Quillen adjunction for the $(\mathbb{D}^1, \text{et})$ projective model structure.
- (i)' Let $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$. The adjunction $(h_{\sharp}, h^*) : C(\text{AnSp}(\mathbb{C})^{sm}/U) \rightleftarrows C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is a Quillen adjunction for the $(\mathbb{D}^1, \text{et})$ projective model structure.
- (i)'' The functor $g^* : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends $(\mathbb{D}^1, \text{et})$ local equivalence to $(\mathbb{D}^1, \text{et})$ local equivalence.

(ii) The adjunction $(g^*, g_*) : C(\text{AnSp}(\mathbb{C})/S) \rightleftarrows C(\text{AnSp}(\mathbb{C})/T)$ is a Quillen adjunction for the (\mathbb{D}^1, et) projective model structure (see definition 5).

(ii)' The adjunction $(g_{\sharp}, g^*) : C(\text{AnSp}(\mathbb{C})/T) \rightleftarrows C(\text{AnSp}(\mathbb{C})/S)$ is a Quillen adjunction for the (\mathbb{D}^1, et) projective model structure (see definition 5).

(ii)'' The functor $g^* : C(\text{AnSp}(\mathbb{C})/S) \rightarrow C(\text{AnSp}(\mathbb{C})/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends (\mathbb{D}^1, et) local equivalence to (\mathbb{D}^1, et) local equivalence.

Proof. Similar to the proof of proposition 18. \square

Proposition 31. Let $S \in \text{AnSp}(\mathbb{C})$.

(i) The adjunction $(\rho_S^*, \rho_{S*}) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightleftarrows C(\text{AnSp}(\mathbb{C})/S)$ is a Quillen adjunction for the (\mathbb{D}^1, et) projective model structure.

(ii) The functor $\rho_{S*} : C(\text{AnSp}(\mathbb{C})/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends (\mathbb{D}^1, usu) local equivalence to (\mathbb{D}^1, usu) local equivalence.

Proof. Similar to the proof of proposition 19. \square

Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affinoid cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$. For $(G_I, K_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)})/(\tilde{S}_I)^{op}$ and $(H_I, T_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)})/(\tilde{S}_I)$, we denote

$$\mathcal{H}om((G_I, K_{IJ}), (H_I, T_{IJ})) := (\mathcal{H}om(G_I, H_I), u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ}))) \in C(\text{AnSp}(\mathbb{C})^{(sm)})/(\tilde{S}_I)$$

with

$$\begin{aligned} & u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ})) : \mathcal{H}om(G_I, H_I) \\ & \xrightarrow{\text{ad}(p_{IJ}^*, p_{IJ*})(-)} p_{IJ*} p_{IJ}^* \mathcal{H}om(G_I, H_I) \xrightarrow{T(p_{IJ}, \text{hom})(-, -)} p_{IJ*} \mathcal{H}om(p_{IJ}^* G_I, p_{IJ}^* H_I) \\ & \xrightarrow{\mathcal{H}om(p_{IJ}^* G_I, T_{IJ})} p_{IJ*} \mathcal{H}om(p_{IJ}^* G_I, H_I) \xrightarrow{\mathcal{H}om(K_{IJ}, H_I)} p_{IJ*} \mathcal{H}om(G_I, H_I). \end{aligned}$$

This gives in particular the functor

$$C(\text{AnSp}(\mathbb{C})^{(sm)})/(\tilde{S}_I) \rightarrow C(\text{AnSp}(\mathbb{C})^{(sm)})/(\tilde{S}_I)^{op}, (H_I, T_{IJ}) \mapsto (H_I, T_{IJ}).$$

Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affinoid cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$. The functor p_a extend to a functor

$$\begin{aligned} p_a : \text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I) & \rightarrow \text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I), (X_I/\tilde{S}_I, s_{IJ}) \mapsto (X \times \mathbb{D}^1/\tilde{S}_I, s_{IJ} \times I), \\ (g = (g_I) : (X_I/\tilde{S}_I, s_{IJ}) & \rightarrow (X'_I/\tilde{S}_I, s'_{IJ})) \mapsto ((g \times I_{\mathbb{D}^1}) : (X_I \times \mathbb{D}^1/\tilde{S}_I, s_{IJ}) \rightarrow (X'_I \times \mathbb{D}^1/\tilde{S}_I, s'_{IJ})) \end{aligned}$$

the projection functor and again by $p_a : \text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I) \rightarrow \text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I)$ the corresponding morphism of site.

Definition 27. Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affinoid cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$.

(i0) A complex $(F_I, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)})/(\tilde{S}_I)$ is said to be \mathbb{D}^1 homotopic if $\text{ad}(p_a^*, p_{a*})((F_I, u_{IJ})) : (F_I, u_{IJ}) \rightarrow p_{a*} p_a^*(F_I, u_{IJ})$ is an homotopy equivalence.

(i) A complex $(F_I, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ is said to be \mathbb{D}^1 invariant if for all $(X_I/\tilde{S}_I, s_{IJ}) \in \text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I)$

$$(F_I(p_{X_I})) : (F_I(X_I/\tilde{S}_I), F_J(s_{IJ})) \circ u_{IJ}(-) \rightarrow (F_I(X_I \times \mathbb{A}^1/\tilde{S}_I), F_J(s_{IJ} \times I) \circ u_{IJ}(-))$$

is a quasi-isomorphism, where $p_{X_I} : X_I \times \mathbb{A}^1 \rightarrow X_I$ are the projection, and $s_{IJ} : X_I \times \tilde{S}_{J \setminus I}/\tilde{S}_J \rightarrow X_J/\tilde{S}_J$. Obviously a complex $(F_I, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ is \mathbb{D}^1 invariant if and only if all the F_I are \mathbb{A}^1 invariant.

(ii) Let τ a topology on $\text{AnSp}(\mathbb{C})$. A complex $F = (F_I, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ is said to be \mathbb{D}^1 local for the τ topology induced on $\text{AnSp}(\mathbb{C})/(\tilde{S}_I)$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G = (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ τ fibrant, e.g. $k : (F_I, u_{IJ}) \rightarrow (E_\tau(F_I), E(u_{IJ}))$, G is \mathbb{D}^1 invariant.

(iii) A morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ is said to be an (\mathbb{D}^1, usu) local equivalence if for all $(H_I, w_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ which is \mathbb{D}^1 local for the usual topology

$$(\text{Hom}(L(m_I), E_{usu}(H_I))) : \text{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \rightarrow \text{Hom}(L(F_I, u_{IJ}), E_{usu}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I)^{op})$ is an (\mathbb{D}^1, usu) local equivalence, then all the $m_I : F_I \rightarrow G_I$ are (\mathbb{D}^1, usu) local equivalence.

(iv) A morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I)^{op})$ is said to be an (\mathbb{D}^1, usu) local equivalence if for all $(H_I, w_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ which is \mathbb{D}^1 local for the etale topology

$$(\text{Hom}(L(m_I), E_{et}(H_I))) : \text{Hom}(L(G_I, v_{IJ}), E_{usu}(H_I, w_{IJ})) \rightarrow \text{Hom}(L(F_I, u_{IJ}), E_{usu}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I)^{op})$ is an (\mathbb{D}^1, usu) local equivalence, then all the $m_I : F_I \rightarrow G_I$ are (\mathbb{D}^1, usu) local equivalence.

Proposition 32. Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affinoid cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$.

(i) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I))$ is an (\mathbb{D}^1, usu) local equivalence if and only if there exists

$$\left\{ (X_{1,\alpha,I}/\tilde{S}_I, u_{IJ}^1, \alpha \in \Lambda_1) \right\}, \dots, \left\{ (X_{r,\alpha,I}/\tilde{S}_I, u_{IJ}^r, \alpha \in \Lambda_r) \right\} \subset \text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I)$$

with $u_{IJ}^1 : X_{1,\alpha,I} \times \tilde{S}_{J \setminus I}/\tilde{S}_J \rightarrow X_{1,\alpha,J}/\tilde{S}_J$, such that we have in $\text{Ho}_{usu}(C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I)))$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}(X_{1,\alpha,I} \times \mathbb{D}^1/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1 \times I)) \rightarrow \mathbb{Z}(X_{1,\alpha,I}/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1))) \\ &\rightarrow \dots \rightarrow \oplus_{\alpha \in \Lambda_r} \text{Cone}(\mathbb{Z}(X_{r,\alpha,I} \times \mathbb{D}^1/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r \times I)) \rightarrow \mathbb{Z}(X_{r,\alpha,I}/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r))) \end{aligned}$$

(ii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I)^{op})$ is an (\mathbb{D}^1, usu) local equivalence if and only if there exists

$$\left\{ (X_{1,\alpha,I}/\tilde{S}_I, u_{IJ}^1, \alpha \in \Lambda_1) \right\}, \dots, \left\{ (X_{r,\alpha,I}/\tilde{S}_I, u_{IJ}^r, \alpha \in \Lambda_r) \right\} \subset \text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I)^{op}$$

with $u_{IJ}^1 : X_{1,\alpha,I} \times \tilde{S}_{J \setminus I}/\tilde{S}_J \rightarrow X_{1,\alpha,J}/\tilde{S}_J$, such that we have in $\text{Ho}_{et}(C(\text{AnSp}(\mathbb{C})^{(sm)}/(\tilde{S}_I)^{op}))$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}(X_{1,\alpha,I} \times \mathbb{D}^1/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1 \times I)) \rightarrow \mathbb{Z}(X_{1,\alpha,I}/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1))) \\ &\rightarrow \dots \rightarrow \oplus_{\alpha \in \Lambda_r} \text{Cone}(\mathbb{Z}(X_{r,\alpha,I} \times \mathbb{D}^1/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r \times I)) \rightarrow \mathbb{Z}(X_{r,\alpha,I}/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r))) \end{aligned}$$

Proof. Standard. See Ayoub's Thesis for example. \square

2.9 Presheaves on the big analytical site of pairs

We recall the definition given in subsection 5.1 : For $S \in \text{AnSp}(\mathbb{C})$, $\text{AnSp}(\mathbb{C})^2/S := \text{AnSp}(\mathbb{C})^2/(S, S)$ is by definition (see subsection 2.1) the category whose set of objects is

$$(\text{AnSp}(\mathbb{C})^2/S)^0 := \{((X, Z), h), h : X \rightarrow S, Z \subset X \text{ closed}\} \subset \text{AnSp}(\mathbb{C})/S \times \text{Top}$$

and whose set of morphisms between $(X_1, Z_1)/S = ((X_1, Z_1), h_1), (X_1, Z_1)/S = ((X_2, Z_2), h_2) \in \text{AnSp}(\mathbb{C})^2/S$ is the subset

$$\begin{aligned} & \text{Hom}_{\text{AnSp}(\mathbb{C})^2/S}((X_1, Z_1)/S, (X_2, Z_2)/S) := \\ & \{(f : X_2 \rightarrow X_1), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\} \subset \text{Hom}_{\text{AnSp}(\mathbb{C})}(X_1, X_2) \end{aligned}$$

The category $\text{AnSp}(\mathbb{C})^2$ admits fiber products : $(X_1, Z_1) \times_{(S, Z)} (X_2, Z_2) = (X_1 \times_S X_2, Z_1 \times_Z Z_2)$. In particular, for $f : T \rightarrow S$ a morphism with $S, T \in \text{AnSp}(\mathbb{C})$, we have the pullback functor

$$P(f) : \text{AnSp}(\mathbb{C})^2/S \rightarrow \text{AnSp}(\mathbb{C})^2/T, P(f)((X, Z)/S) := (X_T, Z_T)/T, P(f)(g) := (g \times_S f)$$

and we note again $P(f) : \text{AnSp}(\mathbb{C})^2/T \rightarrow \text{AnSp}(\mathbb{C})^2/S$ the corresponding morphism of sites.

We will consider in the construction of the filtered De Rham realization functor the full subcategory $\text{AnSp}(\mathbb{C})^{2,sm}/S \subset \text{AnSp}(\mathbb{C})^2/S$ such that the first factor is a smooth morphism : We will also consider, in order to obtain a complex of D modules in the construction of the filtered De Rham realization functor, the restriction to the full subcategory $\text{AnSp}(\mathbb{C})^{2,pr}/S \subset \text{AnSp}(\mathbb{C})^2/S$ such that the first factor is a projection :

Definition 28. (i) Let $S \in \text{AnSp}(\mathbb{C})$. We denote by

$$\rho_S : \text{AnSp}(\mathbb{C})^{2,sm}/S \hookrightarrow \text{AnSp}(\mathbb{C})^2/S$$

the full subcategory consisting of the objects $(U, Z)/S = ((U, Z), h) \in \text{AnSp}(\mathbb{C})^2/S$ such that the morphism $h : U \rightarrow S$ is smooth. That is, $\text{AnSp}(\mathbb{C})^{2,sm}/S$ is the category

- whose objects are $(U, Z)/S = ((U, Z), h)$, with $U \in \text{AnSp}(\mathbb{C})$, $Z \subset U$ a closed subset, and $h : U \rightarrow S$ a smooth morphism,
- whose morphisms $g : (U, Z)/S = ((U, Z), h_1) \rightarrow (U', Z')/S = ((U', Z'), h_2)$ is a morphism $g : U \rightarrow U'$ of complex algebraic varieties such that $Z \subset g^{-1}(Z')$ and $h_2 \circ g = h_1$.

We denote again $\rho_S : \text{AnSp}(\mathbb{C})^2/S \rightarrow \text{AnSp}(\mathbb{C})^{2,sm}/S$ the associated morphism of site. We have

$$r^s(S) : \text{AnSp}(\mathbb{C})^2 \xrightarrow{r(S) := r(S, S)} \text{AnSp}(\mathbb{C})^2/S \xrightarrow{\rho_S} \text{AnSp}(\mathbb{C})^{2,sm}/S$$

the composite morphism of site.

(ii) Let $S \in \text{AnSp}(\mathbb{C})$. We will consider the full subcategory

$$\mu_S : \text{AnSp}(\mathbb{C})^{2,pr}/S \hookrightarrow \text{AnSp}(\mathbb{C})^2/S$$

whose subset of object consist of those whose morphism is a projection to S :

$$(\text{AnSp}(\mathbb{C})^{2,pr}/S)^0 := \{((Y \times S, X), p), Y \in \text{AnSp}(\mathbb{C}), p : Y \times S \rightarrow S \text{ the projection}\} \subset (\text{AnSp}(\mathbb{C})^2/S)^0.$$

(iii) We will consider the full subcategory

$$\mu_S : (\text{AnSp}(\mathbb{C})^{2,smpr}/S) \hookrightarrow \text{AnSp}(\mathbb{C})^{2,sm}/S$$

whose subset of object consist of those whose morphism is a smooth projection to S :

$$(\text{AnSp}(\mathbb{C})^{2,smpr}/S)^0 := \{((Y \times S, X), p), Y \in \text{SmVar}(\mathbb{C}), p : Y \times S \rightarrow S \text{ the projection}\} \subset (\text{AnSp}(\mathbb{C})^2/S)^0$$

For $f : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$, we have by definition, the following commutative diagram of sites

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^2/T & \xrightarrow{\mu_T} & \text{AnSp}(\mathbb{C})^{2,pr}/T & & \\
\downarrow P(f) & \searrow \rho_T & \downarrow H_T & \searrow \rho_T & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/T & \xrightarrow{P(f)} & \text{AnSp}(\mathbb{C})^{2,smpr}/T \\
& & \downarrow P(f) & & \downarrow P(f) \\
\text{AnSp}(\mathbb{C})^2/S & \xrightarrow{P(f)} & \text{AnSp}(\mathbb{C})^{2,pr}/S & & \text{AnSp}(\mathbb{C})^{2,smpr}/S \\
& \searrow \rho_S & \downarrow P(f) & \searrow \rho_S & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,smpr}/S
\end{array} \quad (37)$$

Recall we have (see subsection 2.1), for $S \in \text{Var}(\mathbb{C})$, the graph functor

$$\begin{aligned}
\text{Gr}_S^{12} : \text{AnSp}(\mathbb{C})/S &\rightarrow \text{AnSp}(\mathbb{C})^{2,pr}/S, \quad X/S \mapsto \text{Gr}_S^{12}(X/S) := (X \times S, X)/S, \\
(g : X/S \rightarrow X'/S) &\mapsto \text{Gr}_S^{12}(g) := (g \times I_S : (X \times S, X) \rightarrow (X' \times S, X'))
\end{aligned}$$

For $f : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$, we have by definition, the following commutative diagram of sites

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^{2,pr}/T & \xrightarrow{\text{Gr}_T^{12}} & \text{AnSp}(\mathbb{C})/T & & \\
\downarrow P(f) & \searrow \rho_T & \downarrow \text{Gr}_T^{12} & \searrow \rho_T & \\
& & \text{AnSp}(\mathbb{C})^{2,smpr}/T & \xrightarrow{P(f)} & \text{AnSp}(\mathbb{C})^{sm}/T \\
& & \downarrow P(f) & & \downarrow P(f) \\
\text{AnSp}(\mathbb{C})^{2,pr}/S & \xrightarrow{P(f)} & \text{AnSp}(\mathbb{C})/S & & \text{AnSp}(\mathbb{C})^{sm}/S \\
& \searrow \rho_S & \downarrow P(f) & \searrow \rho_S & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})^{sm}/S
\end{array} \quad (38)$$

where we recall that $P(f)((X, Z)/S) := ((X_T, Z_T)/T)$, since smooth morphisms are preserved by base change.

As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/T)$$

the adjunction induced by $P(f) : \text{AnSp}(\mathbb{C})^{2,sm}/T \rightarrow \text{AnSp}(\mathbb{C})^{2,sm}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*) : C_{fil}(\text{AnSp}(\mathbb{C})^{2,sm}/S) \rightleftarrows C_{fil}(\text{AnSp}(\mathbb{C})^{2,sm}/T), \quad f^*(G, F) := (f^*G, f^*F)$$

For $S \in \text{AnSp}(\mathbb{C})$, we denote by $\mathbb{Z}_S := \mathbb{Z}((S, S)/(S, S)) \in \text{PSh}(\text{AnSp}(\mathbb{C})^{2,sm}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$, $\text{Hom}(\mathbb{Z}_S, F) = F$.

For $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$, $P(h) : \text{AnSp}(\mathbb{C})^{2,sm}/S \rightarrow \text{AnSp}(\mathbb{C})^{2,sm}/U$ admits a left adjoint

$$C(h) : \text{AnSp}(\mathbb{C})^{2,sm}/U \rightarrow \text{AnSp}(\mathbb{C})^{2,sm}/S, \quad C(h)((U', Z'), h') = ((U', Z'), h \circ h').$$

Hence $h^* : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/U)$ admits a left adjoint

$$h_{\#} : C(\text{AnSp}(\mathbb{C})^{2,sm}/U) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/S), F \mapsto (h_{\#}F : ((U, Z), h_0) \mapsto \lim_{((U', Z'), h_0 h') \rightarrow ((U, Z), h_0)} F((U', Z')/U))$$

For $F^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$ and $G^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{2,sm}/U)$, we have the adjunction maps

$$\text{ad}(h_{\#}, h^*)(G^{\bullet}) : G^{\bullet} \rightarrow h^* h_{\#} G^{\bullet}, \text{ad}(h_{\#}, h^*)(F^{\bullet}) : h_{\#} h^* F^{\bullet} \rightarrow F^{\bullet}.$$

For a smooth morphism $h : U \rightarrow S$, with $U, S \in \text{AnSp}(\mathbb{C})$, we have the adjunction isomorphism, for $F \in C(\text{AnSp}(\mathbb{C})^{2,sm}/U)$ and $G \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$,

$$I(h_{\#}, h^*)(F, G) : \mathcal{H}om^{\bullet}(h_{\#}F, G) \xrightarrow{\sim} h_* \mathcal{H}om^{\bullet}(F, h^*G). \quad (39)$$

For a commutative diagram in $\text{AnSp}(\mathbb{C})$:

$$D = \begin{array}{ccc} V & \xrightarrow{g_2} & U \\ \downarrow h_2 & & \downarrow h_1 \\ T & \xrightarrow{g_1} & S \end{array},$$

where h_1 and h_2 are smooth, we denote by, for $F^{\bullet} \in C(\text{AnSp}(\mathbb{C})^{2,sm}/U)$,

$$T_{\#}(D)(F^{\bullet}) : h_{2\#} g_2^* F^{\bullet} \rightarrow g_1^* h_{1\#} F^{\bullet}$$

the canonical map given by adjunction. If D is cartesian with $h_1 = h$, $g_1 = g$, $f_2 = h' : U_T \rightarrow T$, $g' : U_T \rightarrow U$,

$$T_{\#}(D)(F^{\bullet}) =: T_{\#}(g, h)(F) : h'_* g'^* F^{\bullet} \xrightarrow{\sim} g^* h_{\#} F^{\bullet}$$

is an isomorphism.

We have the support section functors of a closed embedding $i : Z \hookrightarrow S$ for presheaves on the big analytical site of pairs.

Definition 29. Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \text{AnSp}(\mathbb{C})$ and $j : S \setminus Z \hookrightarrow S$ be the open complementary subset.

(i) We define the functor

$$\Gamma_Z : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/S), G^{\bullet} \mapsto \Gamma_Z G^{\bullet} := \text{Cone}(\text{ad}(j^*, j_*)(G^{\bullet}) : G^{\bullet} \rightarrow j_* j^* G^{\bullet})[-1],$$

so that there is then a canonical map $\gamma_Z(G^{\bullet}) : \Gamma_Z G^{\bullet} \rightarrow G^{\bullet}$.

(ii) We have the dual functor of (i) :

$$\Gamma_Z^{\vee} : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/S), F \mapsto \Gamma_Z^{\vee}(F^{\bullet}) := \text{Cone}(\text{ad}(j_{\#}, j^*)(G^{\bullet}) : j_{\#} j^* G^{\bullet} \rightarrow G^{\bullet}),$$

together with the canonical map $\gamma_Z^{\vee}(G) : F \rightarrow \Gamma_Z^{\vee}(G)$.

(iii) For $F, G \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$, we denote by

$$I(\gamma, \text{hom})(F, G) := (I, I(j_{\#}, j^*)(F, G)^{-1}) : \Gamma_Z \mathcal{H}om(F, G) \xrightarrow{\sim} \mathcal{H}om(\Gamma_Z^{\vee} F, G)$$

the canonical isomorphism given by adjunction.

Note that we have similarly for $i : Z \hookrightarrow S$, $i' : Z' \hookrightarrow Z$ closed embeddings, $g : T \rightarrow S$ a morphism with $T, S, Z \in \text{AnSp}(\mathbb{C})$ and $F \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$, the canonical maps in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$

- $T(g, \gamma)(F) : g^* \Gamma_Z F \xrightarrow{\sim} \Gamma_{Z \times_S T} g^* F$, $T(g, \gamma^{\vee})(F) : \Gamma_{Z \times_S T}^{\vee} g^* F \xrightarrow{\sim} g^* \Gamma_Z F$

- $T(Z'/Z, \gamma)(F) : \Gamma_Z F \rightarrow \Gamma_Z F$, $T(Z'/Z, \gamma^\vee)(F) : \Gamma_Z^\vee F \rightarrow \Gamma_Z^\vee F$

but we will not use them in this article.

Let $S_\bullet \in \text{Fun}(\mathcal{I}, \text{AnSp}(\mathbb{C}))$ with $\mathcal{I} \in \text{Cat}$, a diagram of algebraic varieties. It gives the diagram of sites $\text{AnSp}(\mathbb{C})^2/S_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat})$.

- Then $C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S_\bullet)$ is the category
 - whose objects $(G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S_I)$, and $u_{IJ} : (G_I, F) \rightarrow r_{IJ*}(G_J, F)$ for $r_{IJ} : I \rightarrow J$, denoting again $r_{IJ} : S_I \rightarrow S_J$, are morphisms satisfying for $I \rightarrow J \rightarrow K$, $r_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S_I)$,
 - the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \in \mathcal{I}}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_I)$.

- Then $C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S_\bullet)$ is the category
 - whose objects $(G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S_I)$, and $u_{IJ} : (G_I, F) \rightarrow r_{IJ*}(G_J, F)$ for $r_{IJ} : I \rightarrow J$, denoting again $r_{IJ} : S_I \rightarrow S_J$, are morphisms satisfying for $I \rightarrow J \rightarrow K$, $r_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S_I)$,
 - the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \in \mathcal{I}}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S_I)$.

For $s : \mathcal{I} \rightarrow \mathcal{J}$ a functor, with $\mathcal{I}, \mathcal{J} \in \text{Cat}$, and $f_\bullet : T_\bullet \rightarrow S_{s(\bullet)}$ a morphism with $T_\bullet \in \text{Fun}(\mathcal{J}, \text{AnSp}(\mathbb{C}))$ and $S_\bullet \in \text{Fun}(\mathcal{I}, \text{AnSp}(\mathbb{C}))$, we have by definition, the following commutative diagrams of sites

$$\begin{array}{ccc}
\text{AnSp}(\mathbb{C})^2/T_\bullet & \xrightarrow{\mu_{T_\bullet}} & \text{AnSp}(\mathbb{C})^{2,pr}/T_\bullet & \quad (40) \\
\downarrow P(f_\bullet) & \searrow \rho_{T_\bullet} & \downarrow & \searrow \rho_{T_\bullet} \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/T_\bullet & \xrightarrow{\mu_{T_\bullet}} & \text{AnSp}(\mathbb{C})^{2,smpr}/T_\bullet \\
& & \downarrow & \downarrow P(f_\bullet) & \downarrow P(f_\bullet) \\
\text{AnSp}(\mathbb{C})^2/S_\bullet & \xrightarrow{P(f_\bullet)} & \text{AnSp}(\mathbb{C})^{2,pr}/S_\bullet & & \text{AnSp}(\mathbb{C})^{2,smpr}/S_\bullet \\
& \searrow \rho_{S_\bullet} & \downarrow & \searrow \rho_{S_\bullet} & \downarrow P(f_\bullet) \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S_\bullet & \xrightarrow{\mu_{S_\bullet}} & \text{AnSp}(\mathbb{C})^{2,smpr}/S_\bullet
\end{array}$$

and

$$\begin{array}{ccc}
\text{AnSp}(\mathbb{C})^{2,pr}/T_\bullet & \xrightarrow{\text{Gr}_{T_\bullet}^{12}} & \text{AnSp}(\mathbb{C})/T & \quad (41) \\
\downarrow P(f_\bullet) & \searrow \rho_{T_\bullet} & \downarrow & \searrow \rho_{T_\bullet} \\
& & \text{AnSp}(\mathbb{C})^{2,smpr}/T_\bullet & \xrightarrow{\text{Gr}_{T_\bullet}^{12}} & \text{AnSp}(\mathbb{C})^{sm}/T_\bullet \\
& & \downarrow & \downarrow P(f_\bullet) & \downarrow P(f_\bullet) \\
\text{AnSp}(\mathbb{C})^{2,pr}/S_\bullet & \xrightarrow{P(f_\bullet)} & \text{AnSp}(\mathbb{C})/S_\bullet & & \text{AnSp}(\mathbb{C})^{sm}/S_\bullet \\
& \searrow \rho_{S_\bullet} & \downarrow & \searrow \rho_{S_\bullet} & \downarrow P(f_\bullet) \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S_\bullet & \xrightarrow{\text{Gr}_{S_\bullet}^{12}} & \text{AnSp}(\mathbb{C})^{sm}/S_\bullet
\end{array}$$

Let $s : \mathcal{I} \rightarrow \mathcal{J}$ a functor, with $\mathcal{I}, \mathcal{J} \in \text{Cat}$, and $f_\bullet : T_\bullet \rightarrow S_{s(\bullet)}$ a morphism with $T_\bullet \in \text{Fun}(\mathcal{J}, \text{AnSp}(\mathbb{C}))$ and $S_\bullet \in \text{Fun}(\mathcal{I}, \text{AnSp}(\mathbb{C}))$.

- As usual, we denote by

$$(f_\bullet^*, f_{\bullet*}) := (P(f_\bullet)^*, P(f_\bullet)_*) : C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S_\bullet) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,(sm)}/T_\bullet)$$

the adjunction induced by $P(f_\bullet) : \text{AnSp}(\mathbb{C})^{2,(sm)}/T_\bullet \rightarrow \text{AnSp}(\mathbb{C})^{2,(sm)}/S_\bullet$. Since the colimits involved in the definition of $f_\bullet^* = P(f_\bullet)^*$ are filtered, f_\bullet^* also preserve monomorphism. Hence, we get an adjunction

$$\begin{aligned} (f_\bullet^*, f_{\bullet*}) : C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S_\bullet) &\simeq C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/T_\bullet), \\ f_\bullet^*((G_I, F), u_{IJ}) &:= ((f_I^* G_I, f_I^* F), T(f_I, r_{IJ})(-) \circ f_I^* u_{IJ}) \end{aligned}$$

- As usual, we denote by

$$(f_\bullet^*, f_{\bullet*}) := (P(f_\bullet)^*, P(f_\bullet)_*) : C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S_\bullet) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/T_\bullet)$$

the adjunction induced by $P(f_\bullet) : \text{AnSp}(\mathbb{C})^{2,(sm)pr}/T_\bullet \rightarrow \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S_\bullet$. Since the colimits involved in the definition of $f_\bullet^* = P(f_\bullet)^*$ are filtered, f_\bullet^* also preserve monomorphism. Hence, we get an adjunction

$$\begin{aligned} (f_\bullet^*, f_{\bullet*}) : C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S_\bullet) &\simeq C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/T_\bullet), \\ f_\bullet^*((G_I, F), u_{IJ}) &:= ((f_I^* G_I, f_I^* F), T(f_I, r_{IJ})(-) \circ f_I^* u_{IJ}) \end{aligned}$$

Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \bigcup_{i=1}^l S_i$ an open affinoid cover and denote by $S_I = \bigcap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and for $J \subset I$ the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

where $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ is the projection and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. This gives the diagram of analytic spaces $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{AnSp}(\mathbb{C}))$ which gives the diagram of sites $\text{AnSp}(\mathbb{C})^2/(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. This gives in the same way the diagram of analytic spaces $(\tilde{S}_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N})^{op}, \text{AnSp}(\mathbb{C}))$ which gives the diagram of sites $\text{AnSp}(\mathbb{C})^2/(\tilde{S}_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N})^{op}, \text{Cat})$.

- Then $C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ is the category

- whose objects $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/\tilde{S}_I)$, and $u_{IJ} : (G_I, F) \rightarrow p_{IJ*}(G_J, F)$ for $I \subset J$, are morphisms satisfying for $I \subset J \subset K$, $p_{IJ*} u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/\tilde{S}_I)$,
- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/\tilde{S}_I)$.

- Then $C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is the category

- whose objects $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/\tilde{S}_I)$, and $u_{IJ} : (G_I, F) \rightarrow p_{IJ*}(G_J, F)$ for $I \subset J$, are morphisms satisfying for $I \subset J \subset K$, $p_{IJ*} u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/\tilde{S}_I)$,

- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2, (sm)pr} / \tilde{S}_I)$.

- Then $C_{fil}(\text{AnSp}(\mathbb{C})^{2, (sm)} / (\tilde{S}_I)^{op})$ is the category

- whose objects $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2, (sm)} / \tilde{S}_I)$, and $u_{IJ} : (G_J, F) \rightarrow p_{IJ}^*(G_I, F)$ for $I \subset J$, are morphisms satisfying for $I \subset J \subset K$, $p_{JK}^* u_{IJ} \circ u_{JK} = u_{IK}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2, (sm)} / \tilde{S}_K)$,

- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_J = p_{IJ}^* m_I \circ u_{IJ}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2, (sm)} / \tilde{S}_J)$.

- Then $C_{fil}(\text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I)^{op})$ is the category

- whose objects $(G, F) = ((G_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2, (sm)pr} / \tilde{S}_I)$, and $u_{IJ} : (G_J, F) \rightarrow p_{IJ}^*(G_I, F)$ for $I \subset J$, are morphisms satisfying for $I \subset J \subset K$, $p_{JK}^* u_{IJ} \circ u_{JK} = u_{IK}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2, (sm)pr} / \tilde{S}_K)$,

- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_J = p_{IJ}^* m_I \circ u_{IJ}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2, (sm)pr} / \tilde{S}_J)$.

We now define the usual topology on $\text{AnSp}(\mathbb{C})^2/S$.

Definition 30. Let $S \in \text{AnSp}(\mathbb{C})$.

- (i) Denote by τ a topology on $\text{AnSp}(\mathbb{C})$, e.g. the usual topology. The τ covers in $\text{AnSp}(\mathbb{C})^2/S$ of $(X, Z)/S$ are the families of morphisms

$$\{(c_i : (U_i, Z \times_X U_i)/S \rightarrow (X, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow X)_{i \in I} \tau \text{ cover of } X \text{ in } \text{AnSp}(\mathbb{C})\}$$

- (ii) Denote by τ the usual or the etale topology on $\text{AnSp}(\mathbb{C})$. The τ covers in $\text{AnSp}(\mathbb{C})^{2, sm}/S$ of $(U, Z)/S$ are the families of morphisms

$$\{(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow U)_{i \in I} \tau \text{ cover of } U \text{ in } \text{AnSp}(\mathbb{C})\}$$

- (iii) Denote by τ the usual or the etale topology on $\text{AnSp}(\mathbb{C})$. The τ covers in $\text{AnSp}(\mathbb{C})^{2, (sm)pr}/S$ of $(Y \times S, Z)/S$ are the families of morphisms

$$\{(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow Y)_{i \in I} \tau \text{ cover of } Y \text{ in } \text{AnSp}(\mathbb{C})\}$$

Let $S \in \text{AnSp}(\mathbb{C})$. Denote by τ the usual topology on $\text{AnSp}(\mathbb{C})$. In particular, denoting $a_\tau : \text{PSh}(\text{AnSp}(\mathbb{C})^{2, (sm)pr}/S) \rightarrow \text{Shv}(\text{AnSp}(\mathbb{C})^{2, (sm)pr}/S)$ and $a_\tau : \text{PSh}(\text{AnSp}(\mathbb{C})^{2, (sm)pr}/S) \rightarrow \text{Shv}(\text{AnSp}(\mathbb{C})^{2, (sm)pr}/S)$ the sheafification functors,

- a morphism $\phi : F \rightarrow G$, with $F, G \in C(\text{AnSp}(\mathbb{C})^{2, (sm)pr}/S)$, is a τ local equivalence if $a_\tau H^n \phi : a_\tau H^n F \rightarrow a_\tau H^n G$ is an isomorphism, a morphism $\phi : F \rightarrow G$, with $F, G \in C(\text{AnSp}(\mathbb{C})^{2, (sm)pr}/S)$, is a τ local equivalence if $a_\tau H^n \phi : a_\tau H^n F \rightarrow a_\tau H^n G$ is an isomorphism,

- $F^\bullet \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is τ fibrant if for all $(U, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)}/S$ and all τ covers $(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}$ of $(U, Z)/S$,

$$F^\bullet(c_i) : F^\bullet((U, Z)/S) \rightarrow \text{Tot}(\oplus_{\text{card} I = \bullet} F^\bullet((U_i, Z \times_U U_i)/S))$$

is a quasi-isomorphism of complexes of abelian groups ; $F^\bullet \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is τ fibrant if for all $(Y \times S, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$ and all τ covers $(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}$ of $(Y \times S, Z)/S$,

$$F^\bullet(c_i \times I_S) : F^\bullet((Y \times S, Z)/S) \rightarrow \text{Tot}(\oplus_{\text{card} I = \bullet} F^\bullet((U_i \times S, Z \times_Y U_i)/S))$$

is a quasi-isomorphism of complexes of abelian groups,

- a morphism $\phi : (G_1, F) \rightarrow (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is a filtered τ local equivalence if for all $n, p \in \mathbb{Z}$,

$$a_\tau H^n \text{Gr}_F^p \phi : a_\tau H^n \text{Gr}_F^p(G_1, F) \rightarrow a_\tau H^n \text{Gr}_F^p(G_2, F)$$

is an isomorphism of sheaves on $\text{AnSp}(\mathbb{C})^{2,(sm)}/S$; a morphism $\phi : (G_1, F) \rightarrow (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is an filtered τ local equivalence if for all $n, p \in \mathbb{Z}$

$$a_\tau H^n \text{Gr}_F^p \phi : a_\tau H^n \text{Gr}_F^p(G_1, F) \rightarrow a_\tau H^n \text{Gr}_F^p(G_2, F)$$

is an isomorphism of sheaves on $\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$;

- a morphism $\phi : (G_1, F) \rightarrow (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is an r -filtered τ local equivalence if there exists $\phi_i : (C_i, F) \rightarrow (C_{i+1}, F)$, $0 \leq i \leq s$, with $(C_i, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$, $(C_0, F) = (G_1, F)$ and $(C_s, F) = (G_2, F)$, such that

$$\phi = \phi_s \circ \dots \circ \phi_i \circ \dots \circ \phi_0 : (G_1, F) \rightarrow (G_2, F)$$

and $\phi_i : (C_i, F) \rightarrow (C_{i+1}, F)$ either a filtered τ local equivalence or an r -filtered homotopy equivalence, a morphism $\phi : (G_1, F) \rightarrow (G_2, F)$, with $(G_1, F), (G_2, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is an r -filtered τ local equivalence if there exists $\phi_i : (C_i, F) \rightarrow (C_{i+1}, F)$, $0 \leq i \leq s$, with $(C_i, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, $(C_0, F) = (G_1, F)$ and $(C_s, F) = (G_2, F)$, such that

$$\phi = \phi_s \circ \dots \circ \phi_i \circ \dots \circ \phi_0 : (G_1, F) \rightarrow (G_2, F)$$

and $\phi_i : (C_i, F) \rightarrow (C_{i+1}, F)$ either a filtered τ local equivalence or an r -filtered homotopy equivalence ;

- $(F^\bullet, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is filtered τ fibrant for all $(U, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)}/S$ and all τ covers $(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}$ of $(U, Z)/S$,

$$\begin{aligned} H^n \text{Gr}_F^p(F^\bullet, F)(c_i) : H^n \text{Gr}_F^p(F^\bullet, F)((U, Z)/S) \rightarrow \\ H^n \text{Gr}_F^p(\text{Tot}(\oplus_{\text{card} I = \bullet} (F^\bullet, F)((U_i, Z \times_U U_i)/S))) \end{aligned}$$

is an isomorphism of abelian groups for all $n, p \in \mathbb{Z}$; $(F^\bullet, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is filtered τ fibrant for all $(Y \times S, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$ and all τ covers $(c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}$ of $(Y \times S, Z)/S$,

$$\begin{aligned} H^n \text{Gr}_F^p(F^\bullet, F)(c_i \times I_S) : H^n \text{Gr}_F^p(F^\bullet, F)((Y \times S, Z)/S) \rightarrow \\ H^n \text{Gr}_F^p(\text{Tot}(\oplus_{\text{card} I = \bullet} (F^\bullet, F)((U_i \times S, Z \times_Y U_i)/S))) \end{aligned}$$

is an isomorphism of abelian groups for all $n, p \in \mathbb{Z}$.

Let $S_\bullet \in \text{Fun}(\mathcal{I}, \text{AnSp}(\mathbb{C}))$ with $\mathcal{I} \in \text{Cat}$.

- A morphism $m : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S_\bullet)$ is an r -filtered Zariski, resp. etale local, equivalence, if there exists $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$, $0 \leq i \leq s$, with $((C_{iI}, F), u_{iIJ}) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S_\bullet)$ $((C_{0I}, F), u_{0IJ}) = ((G_I, F), u_{IJ})$ and $((C_{sI}, F), u_{sIJ}) = ((H_I, F), v_{IJ})$, such that

$$\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$$

and $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$ either a filtered Zariski, resp. etale, local equivalence or an r -filtered homotopy equivalence.

- A morphism $m : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$ in $C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S_\bullet)$ is an r -filtered Zariski, resp. etale local, equivalence, if there exists $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$, $0 \leq i \leq s$, with $((C_{iI}, F), u_{iIJ}) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S_\bullet)$ $((C_{0I}, F), u_{0IJ}) = ((G_I, F), u_{IJ})$ and $((C_{sI}, F), u_{sIJ}) = ((H_I, F), v_{IJ})$, such that

$$\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : ((G_I, F), u_{IJ}) \rightarrow ((H_I, F), v_{IJ})$$

and $\phi_i : ((C_{iI}, F), u_{iIJ}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)IJ})$ either a filtered Zariski, resp. etale, local equivalence or an r -filtered homotopy equivalence.

Will now define the \mathbb{D}^1 local property on $\text{AnSp}(\mathbb{C})^2/S$. Let $S \in \text{AnSp}(\mathbb{C})$. Denote for short $\text{AnSp}(\mathbb{C})^{2,(sm)}/S$ either the category $\text{AnSp}(\mathbb{C})^2/S$ or the category $\text{AnSp}(\mathbb{C})^{2,sm}/S$. Denote by

$$\begin{aligned} p_a : \text{AnSp}(\mathbb{C})^{2,(sm)}/S &\rightarrow \text{AnSp}(\mathbb{C})^{2,(sm)}/S, \\ (X, Z)/S = ((X, Z), h) &\mapsto (X \times \mathbb{D}^1, Z \times \mathbb{D}^1)/S = ((X \times \mathbb{D}^1, Z \times \mathbb{D}^1, h \circ p_X), \\ (g : (X, Z)/S \rightarrow (X', Z')/S) &\mapsto ((g \times I_{\mathbb{D}^1}) : (X \times \mathbb{D}^1, Z \times \mathbb{D}^1)/S \rightarrow (X' \times \mathbb{D}^1, Z' \times \mathbb{D}^1)/S) \end{aligned}$$

the projection functor and again by $p_a : \text{AnSp}(\mathbb{C})^{2,(sm)}/S \rightarrow \text{AnSp}(\mathbb{C})^{2,(sm)}/S$ the corresponding morphism of site. Denote for short $\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$ either the category $\text{AnSp}(\mathbb{C})^{2,pr}/S$ or the category $\text{AnSp}(\mathbb{C})^{2,smpr}/S$. Denote by

$$\begin{aligned} p_a : \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S &\rightarrow \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S, \\ (Y \times S, Z)/S = ((Y \times S, Z), p_S) &\mapsto (Y \times S \times \mathbb{D}^1, Z \times \mathbb{D}^1)/S = ((Y \times S \times \mathbb{D}^1, Z \times \mathbb{D}^1, p_S \circ p_{Y \times S}), \\ (g : (Y \times S, Z)/S \rightarrow (Y' \times S, Z')/S) &\mapsto ((g \times I_{\mathbb{D}^1}) : (Y \times S \times \mathbb{D}^1, Z \times \mathbb{D}^1)/S \rightarrow (Y' \times S \times \mathbb{D}^1, Z' \times \mathbb{D}^1)/S) \end{aligned}$$

the projection functor and again by $p_a : \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S \rightarrow \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$ the corresponding morphism of site.

Definition 31. Let $S \in \text{AnSp}(\mathbb{C})$.

(i0) A complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is said to be \mathbb{D}^1 homotopic if $\text{ad}(p_a^*, p_{a*})(F) : F \rightarrow F$ is an homotopy equivalence.

(i) A complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is said to be \mathbb{D}^1 invariant if for all $(X, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)}/S$

$$F(p_X) : F((X, Z)/S) \rightarrow F((X \times \mathbb{D}^1, (Z \times \mathbb{D}^1))/S)$$

is a quasi-isomorphism, where $p_X : (X \times \mathbb{D}^1, (Z \times \mathbb{D}^1)) \rightarrow (X, Z)$ is the projection.

(i0)' Similarly, a complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is said to be \mathbb{D}^1 homotopic if $\text{ad}(p_a^*, p_{a*})(F) : F \rightarrow F$ is an homotopy equivalence.

(i)' Similarly, a complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be \mathbb{D}^1 invariant if for all $(Y \times S, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S$

$$F(p_{Y \times S}) : F((Y \times S, Z)/S) \rightarrow F((Y \times S \times \mathbb{D}^1, (Z \times \mathbb{D}^1))/S)$$

is a quasi-isomorphism

- (ii) A complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is said to be \mathbb{D}^1 local for the τ topology induced on $\text{AnSp}(\mathbb{C})^2/S$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ τ fibrant, (e.g. $k : F \rightarrow E_\tau(F)$), G is \mathbb{D}^1 invariant.
- (ii)' Similarly, a complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be \mathbb{D}^1 local for the τ topology induced on $\text{AnSp}(\mathbb{C})^2/S$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ τ fibrant, e.g. $k : F \rightarrow E_\tau(F)$, G is \mathbb{D}^1 invariant.
- (iii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is said to an (\mathbb{D}^1, usu) local equivalence if for all $H \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ which is \mathbb{D}^1 local for the usual topology

$$\text{Hom}(L(m), E_{usu}(H)) : \text{Hom}(L(G), E_{usu}(H)) \rightarrow \text{Hom}(L(F), E_{usu}(H))$$

is a quasi-isomorphism.

- (iii)' Similarly, a morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be an (\mathbb{D}^1, usu) local equivalence if for all $H \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ which is \mathbb{D}^1 local for the usual topology

$$\text{Hom}(L(m), E_{usu}(H)) : \text{Hom}(L(G), E_{usu}(H)) \rightarrow \text{Hom}(L(F), E_{usu}(H))$$

is a quasi-isomorphism.

Proposition 33. (i) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is an (\mathbb{D}^1, usu) local equivalence if and only if there exists

$$\{(X_{1,\alpha}, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \dots, \{(X_{r,\alpha}, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \text{AnSp}(\mathbb{C})^{2,(sm)}/S$$

such that we have in $\text{Ho}_{et}(C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S))$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}((X_{1,\alpha} \times \mathbb{D}^1, Z_{1,\alpha} \times \mathbb{D}^1)/S) \rightarrow \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \\ &\rightarrow \dots \rightarrow \bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}((X_{1,\alpha} \times \mathbb{D}^1, Z_{1,\alpha} \times \mathbb{D}^1)/S) \rightarrow \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \end{aligned}$$

- (ii) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is an (\mathbb{D}^1, usu) local equivalence if and only if there exists

$$\{(Y_{1,\alpha} \times S, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \dots, \{(Y_{r,\alpha} \times S, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \text{AnSp}(\mathbb{C})^{2,(sm)}/S$$

such that we have in $\text{Ho}_{et}(C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S))$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}((Y_{1,\alpha} \times S \times \mathbb{D}^1, Z_{1,\alpha} \times \mathbb{D}^1)/S) \rightarrow \mathbb{Z}((Y_{1,\alpha} \times S, Z_{1,\alpha})/S)) \\ &\rightarrow \dots \rightarrow \bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}((Y_{1,\alpha} \times S \times \mathbb{D}^1, Z_{1,\alpha} \times \mathbb{D}^1)/S) \rightarrow \mathbb{Z}((Y_{1,\alpha} \times S, Z_{1,\alpha})/S)) \end{aligned}$$

Proof. Standard. □

Definition-Proposition 9. Let $S \in \text{AnSp}(\mathbb{C})$.

- (i) With the weak equivalence the (\mathbb{D}^1, et) equivalence and the fibration the epimorphism with \mathbb{D}^1 local and etale fibrant kernels gives a model structure on $C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$: the left bousfield localization of the projective model structure of $C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$. We call it the projective (\mathbb{D}^1, et) model structure.
- (ii) With the weak equivalence the (\mathbb{D}^1, et) equivalence and the fibration the epimorphism with \mathbb{D}^1 local and etale fibrant kernels gives a model structure on $C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$: the left bousfield localization of the projective model structure of $C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$. We call it the projective (\mathbb{D}^1, et) model structure.

Proof. Similar to the proof of proposition 5. □

We have, similarly to the case of single varieties the following :

Proposition 34. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$.*

- (i) *The adjunction $(g^*, g_*) : C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S) \rightleftarrows C(\text{AnSp}(\mathbb{C})^{2,(sm)}/T)$ is a Quillen adjunction for the (\mathbb{D}^1, usu) model structure.*
- (i)' *The functor $g^* : C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,(sm)}/T)$ sends quasi-isomorphism to quasi-isomorphism and equivalence usu local to equivalence usu local, sends (\mathbb{D}^1, usu) local equivalence to (\mathbb{D}^1, usu) local equivalence.*
- (ii) *The adjunction $(g^*, g_*) : C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \rightleftarrows C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/T)$ is a Quillen adjunction for the (\mathbb{D}^1, usu) model structure.*
- (ii)' *The functor $g^* : C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/T)$ sends quasi-isomorphism to quasi-isomorphism and equivalence usu local to equivalence usu local, sends (\mathbb{D}^1, usu) local equivalence to (\mathbb{D}^1, usu) local equivalence.*

Proof. Similar to the proof of proposition 23. □

Proposition 35. *Let $S \in \text{Var}(\mathbb{C})$.*

- (i) *The adjunction $(\rho_S^*, \rho_{S*}) : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \rightleftarrows C(\text{AnSp}(\mathbb{C})^2/S)$ is a Quillen adjunction for the (\mathbb{A}^1, et) projective model structure.*
- (i)' *The functor $\rho_{S*} : C(\text{AnSp}(\mathbb{C})^2/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends (\mathbb{D}^1, usu) local equivalence to (\mathbb{D}^1, usu) local equivalence.*
- (ii) *The adjunction $(\rho_S^*, \rho_{S*}) : C(\text{AnSp}(\mathbb{C})^{2,smpr}/S) \rightleftarrows C(\text{AnSp}(\mathbb{C})^{2,pr}/S)$ is a Quillen adjunction for the (\mathbb{A}^1, et) projective model structure.*
- (ii)' *The functor $\rho_{S*} : C(\text{AnSp}(\mathbb{C})^{2,pr}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,smpr}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends (\mathbb{D}^1, usu) local equivalence to (\mathbb{D}^1, usu) local equivalence.*

Proof. Similar to the proof of proposition 19. □

We also have

Proposition 36. *Let $S \in \text{AnSp}(\mathbb{C})$.*

- (i) *The adjunction $(\text{Gr}_S^{12*}, \text{Gr}_{S*}^{12}) : C(\text{AnSp}(\mathbb{C})/S) \rightleftarrows C(\text{AnSp}(\mathbb{C})^{2,pr}/S)$ is a Quillen adjunction for the (\mathbb{D}^1, usu) projective model structure.*
- (ii) *The adjunction $(\text{Gr}_S^{12*}, \text{Gr}_{S*}^{12}) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightleftarrows C(\text{AnSp}(\mathbb{C})^{2,smpr}/S)$ is a Quillen adjunction for the (\mathbb{D}^1, usu) projective model structure.*

Proof. Immediate from definition. □

Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \bigcup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \bigcap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$.

- For $(G_I, K_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op})$ and $(H_I, T_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$, we denote

$$\text{Hom}((G_I, K_{IJ}), (H_I, T_{IJ})) := (\text{Hom}(G_I, H_I), u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ}))) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$$

with

$$\begin{aligned} & u_{IJ}((G_I, K_{IJ})(H_I, T_{IJ})) : \mathcal{H}om(G_I, H_I) \\ & \xrightarrow{\text{ad}(p_{IJ}^*, p_{IJ*})(-)} p_{IJ*} p_{IJ}^* \mathcal{H}om(G_I, H_I) \xrightarrow{T(p_{IJ, \text{hom}})(-, -)} p_{IJ*} \mathcal{H}om(p_{IJ}^* G_I, p_{IJ}^* H_I) \\ & \xrightarrow{\mathcal{H}om(p_{IJ}^* G_I, T_{IJ})} p_{IJ*} \mathcal{H}om(p_{IJ}^* G_I, H_J) \xrightarrow{\mathcal{H}om(K_{IJ}, H_J)} p_{IJ*} \mathcal{H}om(G_J, H_J). \end{aligned}$$

This gives in particular the functor

$$C(\text{AnSp}(\mathbb{C})^{2, (sm)} / (\tilde{S}_I)) \rightarrow C(\text{AnSp}(\mathbb{C})^{2, (sm)} / (\tilde{S}_I)^{op}), (H_I, T_{IJ}) \mapsto (H_I, T_{IJ}).$$

- For $(G_I, K_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I)^{op})$ and $(H_I, T_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I))$, we denote $\mathcal{H}om((G_I, K_{IJ}), (H_I, T_{IJ})) := (\mathcal{H}om(G_I, H_I), u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ}))) \in C(\text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I))$

with

$$\begin{aligned} & u_{IJ}((G_I, K_{IJ})(H_I, T_{IJ})) : \mathcal{H}om(G_I, H_I) \\ & \xrightarrow{\text{ad}(p_{IJ}^*, p_{IJ*})(-)} p_{IJ*} p_{IJ}^* \mathcal{H}om(G_I, H_I) \xrightarrow{T(p_{IJ, \text{hom}})(-, -)} p_{IJ*} \mathcal{H}om(p_{IJ}^* G_I, p_{IJ}^* H_I) \\ & \xrightarrow{\mathcal{H}om(p_{IJ}^* G_I, T_{IJ})} p_{IJ*} \mathcal{H}om(p_{IJ}^* G_I, H_J) \xrightarrow{\mathcal{H}om(K_{IJ}, H_J)} p_{IJ*} \mathcal{H}om(G_J, H_J). \end{aligned}$$

This gives in particular the functor

$$C(\text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I)) \rightarrow C(\text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I)^{op}), (H_I, T_{IJ}) \mapsto (H_I, T_{IJ}).$$

Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affinoid cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$. We have the functors

$$\begin{aligned} & p_a : \text{AnSp}(\mathbb{C})^{2, (sm)} / (\tilde{S}_I) \rightarrow \text{AnSp}(\mathbb{C})^{2, (sm)} / (\tilde{S}_I) \\ & ((X, Z) / \tilde{S}_I, u_{IJ}) \mapsto ((X \times \mathbb{D}^1, Z \times \mathbb{D}^1) / \tilde{S}_I, u_{IJ} \times I) \\ (g : ((X, Z) / \tilde{S}_I, u_{IJ}) \rightarrow ((X', Z') / \tilde{S}_I, u_{IJ})) & \mapsto ((g \times I_{\mathbb{D}^1}) : ((X \times \mathbb{D}^1, Z \times \mathbb{D}^1) / \tilde{S}_I, u_{IJ} \times I) \rightarrow ((X' \times \mathbb{D}^1, Z' \times \mathbb{D}^1) / \tilde{S}_I, u_{IJ} \times I)) \end{aligned}$$

the projection functor and again by $p_a : \text{AnSp}(\mathbb{C})^{2, (sm)} / (\tilde{S}_I) \rightarrow \text{Var}(\mathbb{C})^{2, (sm)} / (\tilde{S}_I)$ the corresponding morphism of site, and

$$\begin{aligned} & p_a : \text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I) \rightarrow \text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I), \\ & ((Y \times \tilde{S}_I, Z) / \tilde{S}_I, u_{IJ}) \mapsto ((Y \times \tilde{S}_I \times \mathbb{D}^1, Z \times \mathbb{D}^1) / \tilde{S}_I, u_{IJ} \times I), \\ (g : ((Y \times \tilde{S}_I, Z) / \tilde{S}_I, u_{IJ}) \rightarrow ((Y' \times \tilde{S}_I, Z') / \tilde{S}_I, u_{IJ})) & \mapsto ((g \times I_{\mathbb{D}^1}) : ((Y \times \tilde{S}_I \times \mathbb{D}^1, Z \times \mathbb{D}^1) / \tilde{S}_I, u_{IJ} \times I), \\ & ((Y' \times \tilde{S}_I \times \mathbb{D}^1, Z' \times \mathbb{D}^1) / \tilde{S}_I, u_{IJ} \times I)), \end{aligned}$$

the projection functor and again by $p_a : \text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I) \rightarrow \text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I)$ the corresponding morphism of site. These functors induces also morphism of sites $p_a : \text{AnSp}(\mathbb{C})^{2, (sm)} / (\tilde{S}_I)^{op} \rightarrow \text{AnSp}(\mathbb{C})^{2, (sm)} / (\tilde{S}_I)^{op}$ and $p_a : \text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I)^{op} \rightarrow \text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I)^{op}$.

Definition 32. Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$.

(i0) A complex $(F_I, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2, (sm)} / (\tilde{S}_I))$ is said to be \mathbb{D}^1 homotopic if $\text{ad}(p_a^*, p_{a*})((F_I, u_{IJ})) : (F_I, u_{IJ}) \rightarrow p_{a*} p_a^*(F_I, u_{IJ})$ is an homotopy equivalence.

(i0)' A complex $(F_I, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I))$ is said to be \mathbb{D}^1 homotopic if $\text{ad}(p_a^*, p_{a*})((F_I, u_{IJ})) : (F_I, u_{IJ}) \rightarrow p_{a*} p_a^*(F_I, u_{IJ})$ is an homotopy equivalence.

(i) A complex $(F_I, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ is said to be \mathbb{D}^1 invariant if for all $((X_I, Z_I)/\tilde{S}_I, s_{IJ}) \in \text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)$

$$(F_I(p_{X_I})) : (F_I((X_I, Z_I)/\tilde{S}_I), F_J(s_{IJ}) \circ u_{IJ}(-) \rightarrow (F_I((X_I \times \mathbb{D}^1, (Z_I \times \mathbb{D}^1))/\tilde{S}_I), F_J(s_{IJ} \times I) \circ u_{IJ}(-))$$

is a quasi-isomorphism, where $p_{X_I} : (X_I \times \mathbb{D}^1, (Z_I \times \mathbb{D}^1)) \rightarrow (X_I, Z_I)$ are the projection, and $s_{IJ} : (X_I \times \tilde{S}_{J,I}, Z_I)/\tilde{S}_J \rightarrow (X_J, Z_J)\tilde{S}_J$.

(i)' A complex $(G_I, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is said to be \mathbb{D}^1 invariant if for all $((Y \times \tilde{S}_I, Z_I)/\tilde{S}_I, s_{IJ}) \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)$

$$(G_I(p_{Y \times \tilde{S}_I})) : (G_I((Y \times \tilde{S}_I, Z_I)/\tilde{S}_I), G_J(s_{IJ}) \circ u_{IJ}(-) \rightarrow (G_I((Y \times \tilde{S}_I \times \mathbb{D}^1, (Z_I \times \mathbb{D}^1))/\tilde{S}_I), G_J(s_{IJ} \times I) \circ u_{IJ}(-))$$

is a quasi-isomorphism.

(ii) Let τ a topology on $\text{AnSp}(\mathbb{C})$. A complex $F = (F_I, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ is said to be \mathbb{D}^1 local for the τ topology induced on $\text{AnSp}(\mathbb{C})^2/(\tilde{S}_I)$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G = (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ τ fibrant, e.g. $k : (F_I, u_{IJ}) \rightarrow (E_\tau(F_I), E(u_{IJ}))$, G is \mathbb{D}^1 invariant.

(ii)' Let τ a topology on $\text{AnSp}(\mathbb{C})$. A complex $F = (F_I, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is said to be \mathbb{D}^1 local for the τ topology induced on $\text{AnSp}(\mathbb{C})^2/(\tilde{S}_I)$, if for an (hence every) τ local equivalence $k : F \rightarrow G$ with k injective and $G = (G_I, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ τ fibrant, e.g. $k : (F_I, u_{IJ}) \rightarrow (E_\tau(F_I), E(u_{IJ}))$, G is \mathbb{D}^1 invariant.

(iii) A morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ is said to an (\mathbb{D}^1, usu) local equivalence if for all $H = (H_I, w_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ which is \mathbb{D}^1 local for the usual topology

$$(\text{Hom}(L(m_I), E_{et}(H_I))) : \text{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \rightarrow \text{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ is an (\mathbb{D}^1, usu) local equivalence, then all the $m_I : F_I \rightarrow G_I$ are (\mathbb{D}^1, usu) local equivalence.

(iii)' A morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is said to an (\mathbb{D}^1, usu) local equivalence if for all $H = (H_I, w_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ which is \mathbb{D}^1 local for the usual topology

$$(\text{Hom}(L(m_I), E_{et}(H_I))) : \text{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \rightarrow \text{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is an (\mathbb{D}^1, usu) local equivalence, then all the $m_I : F_I \rightarrow G_I$ are (\mathbb{D}^1, usu) local equivalence.

(iv) A morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op})$ is said to an (\mathbb{D}^1, usu) local equivalence if for all $H = (H_I, w_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ which is \mathbb{D}^1 local for the usual topology

$$(\text{Hom}(L(m_I), E_{et}(H_I))) : \text{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \rightarrow \text{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op})$ is an (\mathbb{D}^1, usu) local equivalence, then all the $m_I : F_I \rightarrow G_I$ are (\mathbb{D}^1, usu) local equivalence.

(iv)' A morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op})$ is said to an (\mathbb{D}^1, usu) local equivalence if for all $(H_I, w_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ which is \mathbb{D}^1 local for the usual topology

$$(\text{Hom}(L(m_I), E_{et}(H_I))) : \text{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \rightarrow \text{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op})$ is an (\mathbb{D}^1, usu) local equivalence, then all the $m_I : F_I \rightarrow G_I$ are (\mathbb{D}^1, usu) local equivalence.

Proposition 37. Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$.

(i) A morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op})$ is an (\mathbb{D}^1, usu) local equivalence if and only if there exists

$$\left\{ ((X_{1,\alpha,I}, Z_{1,\alpha,I})/\tilde{S}_I, u_{IJ}^1), \alpha \in \Lambda_1 \right\}, \dots, \left\{ ((X_{r,\alpha,I}, Z_{r,\alpha,I})/\tilde{S}_I, u_{IJ}^r), \alpha \in \Lambda_r \right\} \subset \text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op}$$

with

$$u_{IJ}^l : (X_{l,\alpha,J}, Z_{l,\alpha,J})/\tilde{S}_J \rightarrow (X_{l,\alpha,I} \times \tilde{S}_{J \setminus I}, Z_{l,\alpha,I} \times \tilde{S}_{J \setminus I})/\tilde{S}_J$$

such that we have in $\text{Ho}_{et}(C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op}))$

$$\begin{aligned} & \text{Cone}(m) \xrightarrow{\sim} \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}((X_{1,\alpha,I} \times \mathbb{D}^1, Z_{1,\alpha,I} \times \mathbb{D}^1)/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1 \times I)) \rightarrow \mathbb{Z}((X_{1,\alpha,I}, Z_{1,\alpha,I})/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1))) \\ & \rightarrow \dots \rightarrow \\ & \oplus_{\alpha \in \Lambda_r} \text{Cone}(\mathbb{Z}((X_{r,\alpha,I} \times \mathbb{D}^1, Z_{r,\alpha,I} \times \mathbb{D}^1)/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r \times I)) \rightarrow \mathbb{Z}((X_{r,\alpha,I}, Z_{r,\alpha,I})/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r))) \end{aligned}$$

(i)' A morphism $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op})$ is an (\mathbb{D}^1, usu) local equivalence if and only if there exists

$$\left\{ ((Y_{1,\alpha,I} \times \tilde{S}_I, Z_{1,\alpha,I})/\tilde{S}_I, u_{IJ}^1), \alpha \in \Lambda_1 \right\}, \dots, \left\{ ((Y_{r,\alpha,I} \times \tilde{S}_I, Z_{r,\alpha,I})/\tilde{S}_I, u_{IJ}^r), \alpha \in \Lambda_r \right\} \subset \text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)$$

with

$$u_{IJ}^l : (Y_{l,\alpha,J} \times \tilde{S}_J, Z_{l,\alpha,J})/\tilde{S}_J \rightarrow (Y_{l,\alpha,I} \times \tilde{S}_J, Z_{l,\alpha,I} \times \tilde{S}_{J \setminus I})/\tilde{S}_J$$

such that we have in $\text{Ho}_{et}(C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op}))$

$$\begin{aligned} & \text{Cone}(m) \xrightarrow{\sim} \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}((Y_{1,\alpha,I} \times \mathbb{D}^1 \times \tilde{S}_I, Z_{1,\alpha,I} \times \mathbb{D}^1)/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1 \times I)) \rightarrow \mathbb{Z}((Y_{1,\alpha,I} \times S, Z_{1,\alpha,I})/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1))) \\ & \rightarrow \dots \rightarrow \oplus_{\alpha \in \Lambda_r} \\ & \text{Cone}(\mathbb{Z}((Y_{r,\alpha,I} \times \mathbb{D}^1 \times \tilde{S}_I, Z_{r,\alpha,I} \times \mathbb{D}^1)/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r \times I)) \rightarrow \mathbb{Z}((Y_{r,\alpha,I} \times \tilde{S}_I, Z_{r,\alpha,I})/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r))) \end{aligned}$$

(ii) A similar statement holds for (\mathbb{D}^1, usu) local equivalence $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$.

(ii)' A similar statement holds for (\mathbb{D}^1, usu) local equivalence $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$.

Proof. Standard. See Ayoub's thesis for example. \square

Definition 33. (i) A filtered complex $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is said to be r -filtered \mathbb{D}^1 homotopic if $\text{ad}(p_a^*, p_{a*})(G, F) : (G, F) \rightarrow p_{a*}p_a^*(G, F)$ is an r -filtered homotopy equivalence.

(i)' A filtered complex $((G_I, F), u_{IJ}) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ is said to be r -filtered \mathbb{D}^1 homotopic if $\text{ad}(p_a^*, p_{a*})((G_I, F), u_{IJ}) : ((G_I, F), u_{IJ}) \rightarrow p_{a*}p_a^*((G_I, F), u_{IJ})$ is an r -filtered homotopy equivalence.

(ii) A filtered complex $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be r -filtered \mathbb{D}^1 homotopic if $\text{ad}(p_a^*, p_{a*})(G, F) : (G, F) \rightarrow p_{a*}p_a^*(G, F)$ is an r -filtered homotopy equivalence.

(ii)' A filtered complex $((G_I, F), u_{IJ}) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))$ is said to be r -filtered \mathbb{D}^1 homotopic if $\text{ad}(p_a^*, p_{a*})((G_I, F), u_{IJ}) : ((G_I, F), u_{IJ}) \rightarrow p_{a*}p_a^*((G_I, F), u_{IJ})$ is an r -filtered homotopy equivalence.

We have the following canonical functor :

Definition 34. (i) For $S \in \text{AnSp}(\mathbb{C})$, we have the functor

$$\begin{aligned} (-)^\Gamma : C(\text{AnSp}(\mathbb{C})^{sm}/S) &\rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/S), \\ F \mapsto F^\Gamma : (((U, Z)/S) = ((U, Z), h) \mapsto F^\Gamma((U, Z)/S) &:= (\Gamma_Z^\vee h^* LF)(U/U), \\ (g : ((U', Z'), h') \rightarrow ((U, Z), h)) \mapsto & \\ (F^\Gamma(g) : (\Gamma_Z^\vee h^* LF)(U/U) \xrightarrow{i_{(\Gamma_Z^\vee h^* LF)(U/U)}} & (g^*(\Gamma_Z^\vee h^* LF))(U'/U') \\ \xrightarrow{T(g, \gamma^\vee)(h^* LF)(U'/U')} & (\Gamma_{Z \times_U U'}^\vee g^* h^* LF)(U'/U') \\ \xrightarrow{T(Z'/Z \times_U U', \gamma^\vee)(g^* h^* LF)(U'/U')} & (\Gamma_{Z'}^\vee g^* h^* LF)(U'/U')) \end{aligned}$$

where $i_{(\Gamma_Z^\vee h^* LF)(U/U)}$ is the canonical arrow of the inductive limit. Similarly, we have, for $S \in \text{AnSp}(\mathbb{C})$, the functor

$$\begin{aligned} (-)^\Gamma : C(\text{AnSp}(\mathbb{C})/S) &\rightarrow C(\text{AnSp}(\mathbb{C})^2/S), \\ F \mapsto F^\Gamma : (((X, Z)/S) = ((X, Z), h) \mapsto F^\Gamma((X, Z)/S) &:= (\Gamma_Z^\vee h^* F)(X/X), \\ (g : ((X', Z'), h') \rightarrow ((X, Z), h)) \mapsto (F^\Gamma(g) : & (\Gamma_Z^\vee h^* LF)(X/X) \rightarrow (\Gamma_{Z'}^\vee h'^* LF)(X'/X')) \end{aligned}$$

Note that for $S \in \text{AnSp}(\mathbb{C})$, $I(S/S) : \mathbb{Z}((S, S)/S) \rightarrow \mathbb{Z}(S/S)^\Gamma$ given by

$$I(S/S)((U, Z), h) : \mathbb{Z}((S, S)/S)((U, Z), h) \xrightarrow{\gamma_Z^\vee(\mathbb{Z}(U/U))(U/U)} \mathbb{Z}(S/S)^\Gamma((U, Z), h) := (\Gamma_Z^\vee \mathbb{Z}(U/U))(U/U)$$

is an isomorphism.

(ii) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$. For $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, we have the canonical morphism in $C(\text{AnSp}(\mathbb{C})^{2,sm}/T)$

$$\begin{aligned} T(f, \Gamma)(F) &:= T^*(f, \Gamma)(F) : f^*(F^\Gamma) \rightarrow (f^*F)^\Gamma, \\ T(f, \Gamma)(F)((U', Z')/T = ((U', Z'), h')) : & \\ f^*(F^\Gamma)((U', Z'), h') &:= \lim_{((U', Z'), h') \xrightarrow{l} ((U_T, Z_T), h_T) \xrightarrow{f_U} ((U, Z), h)} (\Gamma_Z^\vee h^* LF)(U/U) \\ &\xrightarrow{F^\Gamma(f_U \circ l)} (\Gamma_Z^\vee l^* f_U^* h^* LF)(U'/U') = (\Gamma_{Z'}^\vee h'^* f^* LF)(U'/U') \\ &\xrightarrow{(\Gamma_{Z'}^\vee h'^* T(f, L)(F))(U'/U')} (\Gamma_{Z'}^\vee h'^* Lf^* F)(U'/U') =: (f^*F)^\Gamma((U', Z'), h') \end{aligned}$$

where $f_U : U_T \rightarrow U \times_S T \rightarrow U$ and $h_T : U_T := U \times_S T \rightarrow T$ are the base change maps, the equality following from the fact that $h \circ f_U \circ l = f \circ h_T \circ l = f \circ h'$. For $F \in C(\text{AnSp}(\mathbb{C})/S)$, we have similarly the canonical morphism in $C(\text{AnSp}(\mathbb{C})^2/T)$

$$T(f, \Gamma)(F) : f^*(F^\Gamma) \rightarrow (f^*F)^\Gamma.$$

(iii) Let $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$. We have, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/U)$, the canonical morphism in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$

$$\begin{aligned} T_{\sharp}(h, \Gamma)(F) &: h_{\sharp}(F^{\Gamma}) \rightarrow (h_{\sharp}LF)^{\Gamma}, \\ T_{\sharp}(h, \Gamma)(F)((U', Z'), h') &: h_{\sharp}(F^{\Gamma})((U', Z'), h') := \lim_{((U', Z'), h') \xrightarrow{l} ((U, U), h)} (\Gamma_{Z'}^{\vee}, l^* LF)(U'/U') \\ &\xrightarrow{(\Gamma_{Z'}^{\vee}, l^* \text{ad}(h_{\sharp}, h^*)(LF))(U'/U')} (\Gamma_{Z'}^{\vee}, l^* h^* h_{\sharp} LF)(U'/U') =: (h_{\sharp}LF)^{\Gamma}((U', Z')/h') \end{aligned}$$

(iv) Let $i : Z_0 \hookrightarrow S$ a closed embedding with $Z_0, S \in \text{AnSp}(\mathbb{C})$. We have the canonical morphism in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$

$$\begin{aligned} T_*(i, \Gamma)(\mathbb{Z}(Z_0/Z_0)) &: i_*(\mathbb{Z}(Z_0/Z_0))^{\Gamma} \rightarrow (i_*\mathbb{Z}(Z/Z))^{\Gamma}, \\ T_*(i, \Gamma)(\mathbb{Z}(Z_0/Z_0))((U, Z), h) &: i_*(\mathbb{Z}(Z_0/Z_0))^{\Gamma}((U, Z), h) := (\Gamma_{Z \times_S Z_0}^{\vee} \mathbb{Z}(Z_0/Z_0))(U \times_S Z_0) \\ &\xrightarrow{T(i_*, \gamma^{\vee})(\mathbb{Z}(Z_0/Z_0))(U \times_S Z_0)} (\Gamma_Z^{\vee} i_* \mathbb{Z}(Z_0/Z_0))(U \times_S Z_0) =: (i_*\mathbb{Z}(Z/Z))^{\Gamma}((U, Z), h) \end{aligned}$$

Definition 35. Let $S \in \text{AnSp}(\mathbb{C})$. We have for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\text{AnSp}(\mathbb{C})^{sm}/S)$

$$\begin{aligned} \text{Gr}(F) &: \text{Gr}_{S^*}^{12} \mu_{S^*} F^{\Gamma} \rightarrow F, \\ \text{Gr}(F)(U/S) &: \Gamma_U^{\vee} p^* F(U \times S/U \times S) \xrightarrow{\text{ad}(l^*, l_*)(p^* F)(U \times S/U \times S)} h^* F(U/U) = F(U/S) \end{aligned}$$

where $h : U \rightarrow S$ is a smooth morphism with $U \in \text{AnSp}(\mathbb{C})$ and $h : U \xrightarrow{l} U \times S \xrightarrow{p} S$ is the graph factorization with l the graph embedding and p the projection.

Proposition 38. Let $S \in \text{AnSp}(\mathbb{C})$.

(i) Then,

- if $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is a quasi-isomorphism, $m^{\Gamma} : F^{\Gamma} \rightarrow G^{\Gamma}$ is a quasi-isomorphism in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an usu local equivalence, $m^{\Gamma} : F^{\Gamma} \rightarrow G^{\Gamma}$ is an usu local equivalence in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an (\mathbb{D}^1, usu) local equivalence, $m^{\Gamma} : F^{\Gamma} \rightarrow G^{\Gamma}$ is an (\mathbb{D}^1, usu) local equivalence in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$.

(ii) Then,

- if $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})/S)$ is a quasi-isomorphism, $m^{\Gamma} : F^{\Gamma} \rightarrow G^{\Gamma}$ is a quasi-isomorphism in $C(\text{AnSp}(\mathbb{C})^2/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an usu local equivalence, $m^{\Gamma} : F^{\Gamma} \rightarrow G^{\Gamma}$ is an usu local equivalence in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \rightarrow G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an (\mathbb{D}^1, usu) local equivalence, $m^{\Gamma} : F^{\Gamma} \rightarrow G^{\Gamma}$ is an (\mathbb{D}^1, usu) local equivalence in $C(\text{AnSp}(\mathbb{C})^2/S)$.

Proof. Similar to the proof of proposition 28. □

2.10 The analytical functor for presheaves on the big Zariski or etale site and on the big Zariski or etale site of pairs

We have for $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$ the following commutative diagram of sites

$$\begin{array}{ccccc}
 \text{Dia}(S) := \text{AnSp}(\mathbb{C})/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})/T & & \\
 \downarrow P(f) & \searrow \rho_T & \downarrow \text{An}_T & \searrow \rho_T & \\
 & \text{AnSp}(\mathbb{C})^{sm}/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^{sm}/T & \\
 & \downarrow \text{An}_S & \downarrow P(f) & \downarrow P(f) & \\
 \text{AnSp}(\mathbb{C})/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})/S & & \\
 \downarrow \rho_S & \searrow \rho_S & \downarrow \rho_S & \searrow \rho_S & \\
 & \text{AnSp}(\mathbb{C})^{sm}/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^{sm}/S &
 \end{array} \tag{42}$$

and

$$\begin{array}{ccccc}
 \text{Dia}^{12}(S) := \text{AnSp}(\mathbb{C})^2/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^2/T & & \\
 \downarrow P(f) & \searrow \rho_T & \downarrow \text{An}_T & \searrow \rho_T & \\
 & \text{AnSp}(\mathbb{C})^{2,sm}/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^{2,sm}/T & \\
 & \downarrow \text{An}_S & \downarrow P(f) & \downarrow P(f) & \\
 \text{AnSp}(\mathbb{C})^2/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^2/S & & \\
 \downarrow \rho_S & \searrow \rho_S & \downarrow \rho_S & \searrow \rho_S & \\
 & \text{AnSp}(\mathbb{C})^{2,sm}/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^{2,sm}/S &
 \end{array} \tag{43}$$

For $S \in \text{Var}(\mathbb{C})$ we have the following commutative diagrams of sites

$$\begin{array}{ccccc}
 \text{AnSp}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,pr}/S & & \\
 \downarrow \text{An}_S & \searrow \rho_S & \downarrow \mu_S & \searrow \rho_S & \\
 & \text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,smpr}/S & \\
 & \downarrow \text{An}_S & \downarrow \mu_S & \downarrow \text{An}_S & \\
 \text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S & & \\
 \downarrow \rho_S & \searrow \rho_S & \downarrow \rho_S & \searrow \rho_S & \\
 & \text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S &
 \end{array}$$

and

$$\begin{array}{ccc}
\text{AnSp}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})/S \\
\downarrow \text{An}_S & \searrow \rho_S & \downarrow \text{Gr}_S^{12} \quad \searrow \rho_S \\
& & \text{AnSp}(\mathbb{C})^{2,smpr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})^{sm}/S \\
& & \downarrow \text{An}_S & & \downarrow \text{An}_S \\
\text{Var}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})/S & & \text{Var}(\mathbb{C})^{sm}/S \\
& \searrow \rho_S & \downarrow \text{Gr}_S^{12} & \searrow \rho_S & \downarrow \text{An}_S \\
& & \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})^{sm}/S
\end{array} \quad (44)$$

For $f : T \rightarrow S$ a morphism in $\text{Var}(\mathbb{C})$ the diagramm $\text{Dia}(S)$ and $\text{Dia}(T)$ commutes with the pullback functors : we have $e(S) \circ P(f) = P(f) \circ e(T)$.

For $S \in \text{Var}(\mathbb{C})$, the analytical functor is

$$(-)^{an} : C_{O_S}(S) \rightarrow C_{O_{S^{an}}}, G \mapsto G^{an} := \text{an}_S^{*mod} G := \text{an}_S^* G \otimes_{\text{an}_S^* O_S} O_{S^{an}}$$

Let $S \in \text{Var}(\mathbb{C})$.

- As $\text{an}_S^* : \text{PSh}(S) \rightarrow \text{PSh}(S^{an})$ preserve monomorphisms (the colimits involved being filtered colimits), we define, for $(G, F) \in C_{(2)fil}(S)$, $\text{an}_S^*(G, F) := (\text{an}_S^* G, \text{an}_S^* F) \in C_{(2)fil}(S^{an})$.
- As $(-)^{an} := \text{an}_S^{*mod} : \text{PSh}_{O_S}(S) \rightarrow \text{PSh}(S^{an})$ preserve monomorphisms (an_S^* preserve monomorphism and $(-) \otimes_{O_S} O_{S^{an}}$ preserve monomorphism since $O_{S^{an}}$ is a flat O_S module), we define, for $(G, F) \in C_{(2)fil}(S)$, $(G, F)^{an} := (G^{an}, \text{an}_S^* F \otimes_{O_S} O_{S^{an}}) \in C_{(2)fil}(S^{an})$.

Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Then,

- the commutative diagrams of sites $D(\text{An}, f) := (\text{An}_S, f, \text{An}_T, f = f^{an})$ gives, for $G \in C(\text{Var}(\mathbb{C})^{sm}/T)$, the canonical map in $C(\text{AnSp}(\mathbb{C})^{am}/T)$

$$\begin{aligned}
T(\text{An}, f)(G) : \text{An}_S^* f_* G &\xrightarrow{\text{ad}(\text{An}_T^*, \text{An}_{T^*})(G)} \text{An}_S^* f_* \text{An}_{T^*} \text{An}_T^* G = \text{An}_S^* \text{An}_{S^*} f_* \text{An}_T^* G \\
&\xrightarrow{\text{ad}(\text{An}_S^*, \text{An}_{S^*})(f_* \text{An}_T^* G)} f_* \text{An}_T^* G.
\end{aligned}$$

- the commutative diagrams of sites $D(\text{an}, f) := (\text{an}_S, f, \text{an}_T, f)$ gives, for $G \in C(T)$, the canonical map in $C(T^{an})$

$$\begin{aligned}
T(\text{an}, f)(G) : \text{an}_S^* f_* G &\xrightarrow{\text{ad}(\text{an}_T^*, \text{an}_{T^*})(G)} \text{an}_S^* f_* \text{an}_{T^*} \text{an}_T^* G = \text{an}_S^* \text{an}_{S^*} f_* \text{an}_T^* G \\
&\xrightarrow{\text{ad}(\text{an}_S^*, \text{an}_{S^*})(f_* \text{an}_T^* G)} f_* \text{an}_T^* G
\end{aligned}$$

and for $G \in C_{O_T}(T)$, the canonical map in $C_{O_{T^{an}}}(T^{an})$

$$\begin{aligned}
T^{mod}(\text{an}, f)(G) : (f_* G)^{an} &:= \text{an}_S^{*mod} f_* G \xrightarrow{\text{ad}(\text{an}_T^{*mod}, \text{an}_{T^*})(G)} \\
\text{an}_S^{*mod} f_* \text{an}_{T^*} \text{an}_T^{*mod} G &= \text{an}_S^{*mod} \text{an}_{S^*} f_* \text{an}_T^{*mod} G \xrightarrow{\text{an}_S^{*mod}, \text{an}_{S^*})(f_* \text{an}_T^{*mod} G)} f_* \text{an}_T^{*mod} G =: f_* G^{an}
\end{aligned}$$

Definition-Proposition 10. Consider a closed embedding $i : Z \hookrightarrow S$ with $S, Z \in \text{Var}(\mathbb{C})$. Then, for $G^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, there exist a map in $C(\text{AnSp}(\mathbb{C})^{sm}/S)$

$$T(\text{An}, \gamma)(G) : \text{An}_S^* \Gamma_Z G \rightarrow \Gamma_Z \text{An}_S^* G$$

unique up to homotopy, such that $\gamma_Z(\text{An}_S^* G) \circ T(\text{An}, \gamma)(G) = \text{An}_S^* \gamma_Z G$.

Proof. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding. The map is given by $(I, T(\text{An}, j)(j^*G)) : \text{Cone}(\text{An}_S^* G \rightarrow \text{An}_S^* j_* j^* G) \rightarrow (\text{An}_S^* G \rightarrow j_* j^* \text{An}_S^* G)$. \square

Definition 36. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. We then have the canonical isomorphism in $C(\text{AnSp}(\mathbb{C})^{sm}/S^{an})$

$$T(f, g, Q) := T_{\sharp}(\text{An}, p)(-)^{-1} \circ T_{\sharp}(\text{An}, j)(-)^{-1} : \\ \text{An}_S^* Q(X/S) := \text{An}_S^* p_{\sharp} \Gamma_X^{\vee} \mathbb{Z}_{Y \times S}[d_Y] \xrightarrow{=} p_{\sharp} \Gamma_{X^{an}}^{\vee} \mathbb{Z}_{Y \times S}[d_Y] =: Q(X^{an}/S^{an})$$

with $j : Y \times S \setminus X \hookrightarrow Y \times S$ the closed embedding.

Definition-Proposition 11. Consider a closed embedding $i : Z \hookrightarrow S$ with $S \in \text{Var}(\mathbb{C})$. Then, for $G \in C_{O_S}(S)$, there is a canonical map in $C_{O_{S^{an}}}(S^{an})$

$$T^{mod}(an, \gamma)(G) : (\Gamma_Z G)^{an} \rightarrow \Gamma_{Z^{an}} G^{an}$$

unique up to homotopy, such that $\gamma_{Z^{an}}(G^{an}) \circ T^{mod}(an, \gamma)(G) = g^* \gamma_Z G$.

Proof. It is a particular case of definition-proposition 2(i). \square

We recall the first GAGA theorem for coherent sheaf on the projective spaces :

Theorem 13. For $X \in \text{Var}(\mathbb{C})$ and $F \in C_{O_X}(X)$ denote by

$$a(F) : \text{ad}(\text{an}_X^{*mod}, \text{an}(X)_*)(E(F)) : E(F) \rightarrow \text{an}_{X^*}(E(F))^{an} = \text{an}_{X^*} E(F^{an}),$$

the canonical morphism.

(i) Let $X \in \text{PVar}(\mathbb{C})$ a proper complex algebraic variety. For $F \in \text{Coh}_{O_X}(X)$ a coherent sheaf, the morphism

$$H^n \Gamma(X, a(F)) : H^n(X, F) = H^n \Gamma(X, E(F)) \rightarrow H^n(X, F^{an}) = H^n \Gamma(X, E(F^{an}))$$

is an isomorphism for all $n \in \mathbb{Z}$.

(ii) Let $f : X \rightarrow S$ a proper morphism with $X, S \in \text{Var}(\mathbb{C})$. For $F \in \text{Coh}_{O_X}(X)$ a coherent sheaf, the morphism

$$H^n f_* a(F) : R^n f_* F = H^n f_*(E(F)) \rightarrow R^n f_* F^{an} = H^n f_* E(F^{an})$$

is an isomorphism for all $n \in \mathbb{Z}$.

Proof. See [30]. (i) reduces to the case where X is projective and (ii) to the case where f is projective. Hence, the theorem reduce to the case of a coherent sheaf $F \in \text{Coh}_{O_{\mathbb{P}^N}}(\mathbb{P}^N)$ on \mathbb{P}^N . \square

We have for $s : \mathcal{I} \rightarrow \mathcal{J}$ a functor with $\mathcal{I}, \mathcal{J} \in \text{Cat}$ and $f : T_{\bullet} \rightarrow S_{s(\bullet)}$ a morphism of diagram of algebraic varieties with $T_{\bullet} \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C}))$, $S_{\bullet} \in \text{Fun}(\mathcal{J}, \text{Var}(\mathbb{C}))$ the following commutative diagram of sites

$$\begin{array}{ccc} \text{Dia}(S) := \text{AnSp}(\mathbb{C})/T_{\bullet}^{an} & \xrightarrow{\text{An}_{T_{\bullet}}} & \text{Var}(\mathbb{C})/T_{\bullet} \\ \downarrow P(f) & \searrow \rho_{T_{\bullet}} & \downarrow \rho_{T_{\bullet}} \\ \text{AnSp}(\mathbb{C})^{sm}/T_{\bullet}^{an} & \xrightarrow{\text{An}_{T_{\bullet}} P(f_{\bullet})} & \text{Var}(\mathbb{C})^{sm}/T_{\bullet} \\ \downarrow \text{An}_{S_{\bullet}} P(f_{\bullet}) & & \downarrow P(f_{\bullet}) \\ \text{AnSp}(\mathbb{C})/S_{\bullet}^{an} & \xrightarrow{\text{An}_{S_{\bullet}} P(f_{\bullet})} & \text{Var}(\mathbb{C})/S_{\bullet} \\ \downarrow \rho_{S_{\bullet}} & \searrow \rho_{S_{\bullet}} & \downarrow \rho_{S_{\bullet}} \\ \text{AnSp}(\mathbb{C})^{sm}/S_{\bullet}^{an} & \xrightarrow{\text{An}_{S_{\bullet}}} & \text{Var}(\mathbb{C})^{sm}/S_{\bullet} \end{array} \quad (45)$$

and

$$\begin{array}{ccc}
Dia^{12}(S) := \text{AnSp}(\mathbb{C})^2/T_{\bullet}^{an} & \xrightarrow{\text{An}_{T_{\bullet}}} & \text{Var}(\mathbb{C})^2/T_{\bullet} \\
\downarrow P(f_{\bullet}) & \searrow \rho_{T_{\bullet}} & \downarrow \rho_{T_{\bullet}} \\
& \text{AnSp}(\mathbb{C})^{2,sm}/T_{\bullet}^{an} & \xrightarrow{\text{An}_{T_{\bullet}} \circ P(f_{\bullet})} \text{Var}(\mathbb{C})^{2,sm}/T_{\bullet} \\
& \downarrow \text{An}_{S_{\bullet}} & \downarrow P(f_{\bullet}) \\
\text{AnSp}(\mathbb{C})^2/S_{\bullet}^{an} & \xrightarrow{\text{An}_{S_{\bullet}} \circ P(f_{\bullet})} & \text{Var}(\mathbb{C})^2/S_{\bullet} \\
& \searrow \rho_{S_{\bullet}} & \downarrow P(f_{\bullet}) \\
& \text{AnSp}(\mathbb{C})^{2,sm}/S_{\bullet}^{an} & \xrightarrow{\text{An}_{S_{\bullet}}} \text{Var}(\mathbb{C})^{2,sm}/S_{\bullet} \\
& & \searrow \rho_{S_{\bullet}}
\end{array} \quad . \quad (46)$$

2.11 The De Rahm complexes of algebraic varieties and analytical spaces

For $X \in \text{Var}(\mathbb{C})$, we denote by $\iota_X : \mathbb{C}_X \rightarrow \Omega_X^{\bullet} =: DR(X)$ the canonical inclusion map. More generally, for $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, we denote by $\iota_{X/S} : f^*O_S \rightarrow \Omega_{X/S}^{\bullet} =: DR(X/S)$ the canonical inclusion map.

For $X \in \text{AnSp}(\mathbb{C})$, we denote by $\iota_X : \mathbb{C}_X \rightarrow \Omega_X^{\bullet} =: DR(X)$ the canonical inclusion map. More generally, for $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSp}(\mathbb{C})$, we denote by $\iota_{X/S} : f^*O_S \rightarrow \Omega_{X/S}^{\bullet} =: DR(X/S)$ the canonical inclusion map.

Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Then, the commutative diagram of site $(an, f) =: (f, \text{An}_S, f = f^{an}, an(X))$ gives the transformation map in $C_{O_{S^{an}}}(S^{an})$ (definition 1)

$$\begin{aligned}
T_{\omega}^O(an, f) : (f_*E(\Omega_{X/S}^{\bullet}, F_b))^{an} &:= an_S^{*mod} f_*E(\Omega_{X/S}^{\bullet}, F_b) \xrightarrow{T(an(X), E)(-) \circ T(an, f)(E(\Omega_{X/S}^{\bullet}))} \\
&(f_*E(an(X)^*(\Omega_{X/S}^{\bullet}, F_b))) \otimes_{an_S^* O_S} O_{S^{an}} \xrightarrow{m \otimes E(\Omega_{(X^{an}/X)/(S^{an}/S)})} f_*E(\Omega_{X^{an}/S^{an}}^{\bullet}, F_b)
\end{aligned}$$

We will give in this paper a relative version for all smooth morphisms of the following theorem of Grothendieck

Theorem 14. *Let $U \in \text{SmVar}(\mathbb{C})$. Denote by $a_U : U \rightarrow \{\text{pt}\}$ the terminal map. Then the map*

$$T_{\omega}^O(a_U, an) : \Gamma(U, E(\Omega_U^{\bullet})) \rightarrow \Gamma(U^{an}, E(\Omega_U^{\bullet}))$$

is a quasi-isomorphism of complexes.

Proof. Take a compactification (X, D) of U , with $X \in \text{PSmVar}(\mathbb{C})$ and $D = X \setminus U$ a normal crossing divisor. The proof then use proposition 13, the first GAGA theorem (theorem 13 (i)) for the coherent sheaves $\Omega_U^p(nD)$ on X , and the fact (which is specific of the De Rahm complex) that $\Omega_{U^{an}}^{\bullet}(*D^{an}) \rightarrow j_*E(\Omega_{U^{an}})$ is a quasi-isomorphism. \square

We recall Poincare lemma for smooth morphisms of complex analytic spaces and in particular complex analytic manifold :

Proposition 39. (i) *For $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$, the inclusion map $\iota_{X/S} : h^*O_S \rightarrow \Omega_{U/S}^{\bullet} =: DR(U/S)$ is a quasi-isomorphism.*

(ii) *For $X \in \text{AnSm}(\mathbb{C})$, the inclusion map $\iota_X : \mathbb{C}_X \rightarrow \Omega_X^{\bullet}$ is a quasi-isomorphism.*

Proof. Standard. (ii) is a particular case of (i) (the absolute case $S = \{\text{pt}\}$). \square

Remark 4. We do NOT have pincare lemma in general if $h : U \rightarrow S$ is not a smooth morphism. Already in the absolute case, we can find $X \in \text{Var}(\mathbb{C})$ singular such that the inclusion map $\iota_X : \mathbb{C}_{X^{an}} \rightarrow \Omega_{X^{an}}^\bullet$ is not a quasi-isomorphism. Indeed, we can find exemple of $X \in \text{PVar}(\mathbb{C})$ projective singular where

$$H^p(c_X) : H^p(X^{an}, \mathbb{C}_{X^{an}}) \xrightarrow{\sim} H^p C_{sing}^\bullet(X^{an})$$

X^{an} being locally contractible since $X^{an} \in \text{CW}$, have not the same dimension then the De Rham cohomology

$$H^p(T_\omega^O(an, a_X)) : \mathbb{H}^p(X, E(\Omega_X^\bullet)) \xrightarrow{\sim} \mathbb{H}^p(X^{an}, E(\Omega_{X^{an}}^\bullet))$$

X being projective, that is are not isomorphic as vector spaces. Hence, in particular, the canonical map

$$H^p \iota_X : H^p(X^{an}, \mathbb{C}_{X^{an}}) \rightarrow \mathbb{H}^p(X^{an}, E(\Omega_{X^{an}}^\bullet))$$

is not an isomorphism.

Consider a commutative diagram

$$D_0 = \begin{array}{ccccc} f : X & \xrightarrow{i} & Y & \xrightarrow{p} & S \\ g' \uparrow & & g'' \uparrow & & g \uparrow \\ f' : X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & T \end{array}$$

with $X, X', Y, Y', S, T \in \text{Var}(\mathbb{C})$ or $X, X', Y, Y', S, T \in \text{AnSp}(\mathbb{C})$, i, i' being closed embeddings. Denote by D the right square of D_0 . The closed embedding $i' : X' \hookrightarrow Y'$ factors through $i' : X' \xrightarrow{i'_1} X \times_Y Y' \xrightarrow{i'_0} Y'$ where i'_1, i'_0 are closed embeddings. Then, definition-proposition 3 say that

- there is a canonical map,

$$E(\Omega_{((Y'/Y)/(T/S))} \circ T(g'', E)(-) \circ T(g'', \gamma)(-)) : g''^* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) \rightarrow \Gamma_{X \times_Y Y'} E(\Omega_{Y'/T}^\bullet, F_b)$$

unique up to homotopy such that the following diagram in $C_{g''^* p^* O_S \text{fil}}(Y') = C_{p'^* g^* O_S \text{fil}}(Y')$ commutes

$$\begin{array}{ccc} g''^* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) & \xrightarrow{E(\Omega_{((Y'/Y)/(T/S))} \circ T(g'', E)(-) \circ T(g'', \gamma)(-))} & \Gamma_{X \times_Y Y'} E(\Omega_{Y'/T}^\bullet, F_b) \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X \times_Y Y'}(-) \\ g''^* E(\Omega_{Y/S}^\bullet, F_b) & \xrightarrow{E(\Omega_{((Y'/Y)/(T/S))} \circ T(g'', E)(-))} & E(\Omega_{Y'/T}^\bullet, F_b) \end{array}$$

- there is a canonical map,

$$T_\omega^O(D)^\gamma : g^{*mod} L_{OP^*} \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) \rightarrow p'_* \Gamma_{X \times_Y Y'} E(\Omega_{Y'/T}^\bullet, F_b)$$

unique up to homotopy such that the following diagram in $C_{O_T \text{fil}}(T)$ commutes

$$\begin{array}{ccc} g^{*mod} L_{OP^*} \Gamma_X E(\Omega_{Y/S}^\bullet) & \xrightarrow{T_\omega^O(D)^\gamma} & p'_* \Gamma_{X \times_Y Y'} E(\Omega_{Y'/T}^\bullet) \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X \times_Y Y'}(-) \\ g^{*mod} L_{OP^*} E(\Omega_{Y/S}^\bullet) & \xrightarrow{T_\omega^O(D)} & p'_* E(\Omega_{Y'/T}^\bullet) \end{array}$$

- (iii) there is a map in $C_{f'^* O_T \text{fil}}(Y')$

$$T(X'/X \times_Y Y', \gamma)(E(\Omega_{Y'/T}^\bullet, F_b)) : \Gamma_{X'} E(\Omega_{Y'/T}^\bullet, F_b) \rightarrow \Gamma_{X \times_Y Y'} E(\Omega_{Y'/T}^\bullet, F_b)$$

unique up to homotopy such that $\gamma_{X \times_Y Y'}(-) \circ T(X'/X \times_Y Y', \gamma)(-) = \gamma_{X'}(-)$.

Let $h : Y \rightarrow S$ a morphism and $i : X \hookrightarrow Y$ a closed embedding with $S, Y, X \in \text{Var}(\mathbb{C})$. Then, definition-proposition 3 say that

- there is a canonical map

$$E(\Omega_{(Y^{an}/Y)/(S^{an}/S)}) \circ T(an, \gamma)(-) : an(Y)^* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) \rightarrow \Gamma_{X^{an}} E(\Omega_{Y/S}^\bullet, F_b)$$

unique up to homotopy such that the following diagram in $C_{h^*O_S \text{fil}}(Y^{an})$ commutes

$$\begin{array}{ccc} an(Y)^* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) & \xrightarrow{E(\Omega_{(Y^{an}/Y)/(S^{an}/S)}) \circ T(an, \gamma)(-)} & \Gamma_{X^{an}} E(\Omega_{Y/S}^\bullet, F_b) \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X^{an}}(-) \\ an(Y)^* E(\Omega_{Y/S}^\bullet, F_b) & \xrightarrow{E(\Omega_{(Y^{an}/Y)/(S^{an}/S)})} & E(\Omega_{Y/S}^\bullet, F_b) \end{array}$$

- there is a canonical map

$$T_\omega^O(an, h)^\gamma : (h_* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b))^{an} \rightarrow h_* \Gamma_{X^{an}} E(\Omega_{Y/S}^\bullet, F_b)$$

unique up to homotopy such that the following diagram in $C(Y)$ commutes

$$\begin{array}{ccc} (h_* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b))^{an} & \xrightarrow{T_\omega^O(an, h)^\gamma} & h_* \Gamma_{X^{an}} E(\Omega_{Y/S}^\bullet, F_b) \\ \gamma_X(-) \downarrow & & \downarrow \gamma_{X^{an}}(-) \\ (h_* E(\Omega_{Y/S}^\bullet, F_b))^{an} & \xrightarrow{T_\omega^O(an, h)} & h_* E(\Omega_{Y/S}^\bullet, F_b) \end{array}$$

2.12 The Corti-Hanamura resolution functors \hat{R}^{CH} , R^{CH} , \hat{R}^{0CH} , R^{0CH} from complexes of representable presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ with S smooth, and the functorialities of these resolutions

Definition 37. (i) Let $X_0 \in \text{Var}(\mathbb{C})$ and $Z \subset X_0$ a closed subset. A desingularization of (X_0, Z) is a pair of complex varieties $(X, D) \in \text{Var}^2(\mathbb{C})$, together with a morphism of pair of varieties $\epsilon : (X, D) \rightarrow (X_0, \Delta)$ with $Z \subset \Delta$ such that

- $X \in \text{SmVar}(\mathbb{C})$ and $D := \epsilon^{-1}(\Delta) = \epsilon^{-1}(Z) \cup (\cup_i E_i) \subset X$ is a normal crossing divisor
- $\epsilon : X \rightarrow X_0$ is a proper modification with discriminant Δ , that is $\epsilon : X \rightarrow X_0$ is proper and $\epsilon : X \setminus D \xrightarrow{\sim} X \setminus \Delta$ is an isomorphism.

(ii) Let $X_0 \in \text{Var}(\mathbb{C})$ and $Z \subset X_0$ a closed subset such that $X_0 \setminus Z$ is smooth. A strict desingularization of (X_0, Z) is a pair of complex varieties $(X, D) \in \text{Var}^2(\mathbb{C})$, together with a morphism of pair of varieties $\epsilon : (X, D) \rightarrow (X_0, Z)$ such that

- $X \in \text{SmVar}(\mathbb{C})$ and $D := \epsilon^{-1}(Z) \subset X$ is a normal crossing divisor
- $\epsilon : X \rightarrow X_0$ is a proper modification with discriminant Z , that is $\epsilon : X \rightarrow X_0$ is proper and $\epsilon : X \setminus D \xrightarrow{\sim} X \setminus Z$ is an isomorphism.

We have the following well known resolution of singularities of complex algebraic varieties and their functorialities :

Theorem 15. (i) Let $X_0 \in \text{Var}(\mathbb{C})$ and $Z \subset X_0$ a closed subset. There exists a desingularization of (X_0, Z) , that is a pair of complex varieties $(X, D) \in \text{Var}^2(\mathbb{C})$, together with a morphism of pair of varieties $\epsilon : (X, D) \rightarrow (X_0, \Delta)$ with $Z \subset \Delta$ such that

- $X \in \text{SmVar}(\mathbb{C})$ and $D := \epsilon^{-1}(\Delta) = \epsilon^{-1}(Z) \cup (\cup_i E_i) \subset X$ is a normal crossing divisor
- $\epsilon : X \rightarrow X_0$ is a proper modification with discriminant Δ , that is $\epsilon : X \rightarrow X_0$ is proper and $\epsilon : X \setminus D \xrightarrow{\sim} X \setminus \Delta$ is an isomorphism.

(ii) Let $X_0 \in \text{PVar}(\mathbb{C})$ and $Z \subset X_0$ a closed subset such that $X_0 \setminus Z$ is smooth. There exists a strict desingularization of (X_0, Z) , that is a pair of complex varieties $(X, D) \in \text{PVar}^2(\mathbb{C})$, together with a morphism of pair of varieties $\epsilon : (X, D) \rightarrow (X_0, Z)$ such that

- $X \in \text{PSmVar}(\mathbb{C})$ and $D := \epsilon^{-1}(Z) \subset X$ is a normal crossing divisor
- $\epsilon : X \rightarrow X_0$ is a proper modification with discriminant Z , that is $\epsilon : X \rightarrow X_0$ is proper and $\epsilon : X \setminus D \xrightarrow{\sim} X \setminus Z$ is an isomorphism.

Proof. (i):Standard. See [25] for example.

(ii):Follows immediately from (i). \square

We use this theorem to construct a resolution of a morphism by Corti-Hanamura morphisms, we will need these resolution in the definition of the filtered De Rham realization functor :

Definition-Proposition 12. (i) Let $h : V \rightarrow S$ a morphism, with $V, S \in \text{Var}(\mathbb{C})$. Let $\bar{S} \in \text{PVar}(\mathbb{C})$ be a compactification of S .

- There exist a compactification $\bar{X}_0 \in \text{PVar}(\mathbb{C})$ of V such that $h : V \rightarrow S$ extend to a morphism $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$. Denote by $\bar{Z} = \bar{X}_0 \setminus V$. We denote by $j : V \hookrightarrow \bar{X}_0$ the open embedding and by $i_0 : \bar{Z} \hookrightarrow \bar{X}_0$ the complementary closed embedding. We then consider $X_0 := \bar{f}_0^{-1}(S) \subset \bar{X}_0$ the open subset, $f_0 := \bar{f}_0|_{X_0} : X_0 \rightarrow S$, $Z = \bar{Z} \cap X_0$, and we denote again $j : V \hookrightarrow X_0$ the open embedding and by $i_0 : Z \hookrightarrow X_0$ the complementary closed embedding.
- In the case V is smooth, we take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) , with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. We denote by $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X} = \bar{X}_{c(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings $i_I : \bar{D}_I = \cap_{i \in I} \bar{D}_i \hookrightarrow \bar{X}$. Then the morphisms $\bar{f} := \bar{f}_0 \circ \bar{\epsilon} : \bar{X} \rightarrow \bar{S}$ and $\bar{f}_{D_\bullet} := \bar{f} \circ i_\bullet : \bar{D}_\bullet \rightarrow \bar{S}$ are projective since \bar{X} and \bar{D}_I are projective varieties. We then consider $(X, D) := \bar{\epsilon}^{-1}(X_0, Z)$, $\epsilon := \bar{\epsilon}|_X : (X, D) \rightarrow (X_0, Z)$ We denote again by $i_\bullet : D_\bullet \hookrightarrow X = X_{c(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings $i_I : D_I = \cap_{i \in I} D_i \hookrightarrow X$. Then the morphisms $f := f_0 \circ \epsilon : X \rightarrow S$ and $f_{D_\bullet} := f \circ i_\bullet : D_\bullet \rightarrow S$ are projective since $\bar{f} : \bar{X}_0 \rightarrow \bar{S}$ is projective.

(ii) Let $g : V'/S \rightarrow V/S$ a morphism, with $V'/S = (V', h')$, $V/S = (V, h) \in \text{Var}(\mathbb{C})/S$

- Take (see (i)) a compactification $\bar{X}_0 \in \text{PVar}(\mathbb{C})$ of V such that $h : V \rightarrow S$ extend to a morphism $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$. Denote by $\bar{Z} = \bar{X}_0 \setminus V$. Then, there exist a compactification $\bar{X}'_0 \in \text{PVar}(\mathbb{C})$ of V' such that $h' : V' \rightarrow S$ extend to a morphism $\bar{f}'_0 = \bar{h}'_0 : \bar{X}'_0 \rightarrow \bar{S}$, $g : V' \rightarrow V$ extend to a morphism $\bar{g}_0 : \bar{X}'_0 \rightarrow \bar{X}_0$ and $\bar{f}_0 \circ \bar{g}_0 = \bar{f}'_0$ that is \bar{g}_0 is gives a morphism $\bar{g}_0 : \bar{X}'_0/\bar{S} \rightarrow \bar{X}_0/\bar{S}$. Denote by $\bar{Z}' = \bar{X}'_0 \setminus V'$. We then have the following commutative diagram

$$\begin{array}{ccccc}
 V & \xrightarrow{j} & \bar{X}_0 & \xleftarrow{i} & \bar{Z} \\
 \bar{g} \uparrow & & \bar{g}_0 \uparrow & & \bar{g}' \uparrow \\
 V' & \xrightarrow{j'} & \bar{X}'_0 & \xleftarrow{i'} & \bar{Z}' \xleftarrow{i''_{g',0}} \bar{g}_0^{-1}(\bar{Z}) : i'_{g',0}
 \end{array}$$

It gives the following commutative diagram

$$\begin{array}{ccccc}
 V & \xrightarrow{j} & X_0 := \bar{f}_0^{-1}(S) & \xleftarrow{i} & Z \\
 \bar{g} \uparrow & & \bar{g}_0 \uparrow & & \bar{g}'_0 \uparrow \\
 V' & \xrightarrow{j'} & X'_0 := \bar{f}'_0^{-1}(S) & \xleftarrow{i'} & Z' \xleftarrow{i''_{g',0}} \bar{g}_0^{-1}(Z) : i'_{g',0}
 \end{array}$$

- In the case V and V' are smooth, we take using theorem 15 a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of (\bar{X}_0, \bar{Z}) . Then there exist a strict desingularization $\bar{\epsilon}' : (\bar{X}', \bar{D}') \rightarrow (\bar{X}'_0, \bar{Z}')$ of (\bar{X}'_0, \bar{Z}') and a morphism $\bar{g} : \bar{X}' \rightarrow \bar{X}$ such that the following diagram commutes

$$\begin{array}{ccc} \bar{X}'_0 & \xrightarrow{\bar{g}_0} & \bar{X}_0 \\ \bar{\epsilon}' \uparrow & & \uparrow \bar{\epsilon} \\ \bar{X}' & \xrightarrow{\bar{g}} & \bar{X} \end{array}$$

We then have the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccc} V = V_{c(\bullet)} & \xrightarrow{j} & \bar{X} = \bar{X}_{c(\bullet)} & \xleftarrow{i_\bullet} & \bar{D}_{s_g(\bullet)} \\ \uparrow g & & \uparrow \bar{g} & & \uparrow \bar{g}' \\ V' = V'_{c(\bullet)} & \xrightarrow{j'} & \bar{X}' = \bar{X}'_{c(\bullet)} & \xleftarrow{i'_\bullet} & \bar{D}'_\bullet \xleftarrow{i''_{g'_\bullet} \bar{g}^{-1}} (\bar{D}_{s_g(\bullet)}) : i'_{g'_\bullet} \end{array}$$

where $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X}_\bullet$ the morphism of simplicial varieties given by the closed embeddings $i_n : \bar{D}_n \hookrightarrow \bar{X}_n$, and $i'_\bullet : \bar{D}'_\bullet \hookrightarrow \bar{X}'_\bullet$ the morphism of simplicial varieties given by the closed embeddings $i'_n : \bar{D}'_n \hookrightarrow \bar{X}'_n$. It gives the commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccc} V = V_{c(\bullet)} & \xrightarrow{j} & X := \bar{\epsilon}^{-1}(X_0) = X_{c(\bullet)} & \xleftarrow{i_\bullet} & D_{s_g(\bullet)} \\ \uparrow g & & \uparrow \bar{g} & & \uparrow \bar{g}' \\ V' = V'_{c(\bullet)} & \xrightarrow{j'} & X' := \bar{\epsilon}'^{-1}(X'_0) = X'_{c(\bullet)} & \xleftarrow{i'_\bullet} & D'_\bullet \xleftarrow{i''_{g'_\bullet} \bar{g}^{-1}} (D_{s_g(\bullet)}) : i'_{g'_\bullet} \end{array}$$

Proof. (i): Let $\bar{X}_{00} \in \text{PVar}(\mathbb{C})$ be a compactification of V . Let $l_0 : \bar{X}_0 = \bar{\Gamma}_h \hookrightarrow \bar{X}_{00} \times \bar{S}$ be the closure of the graph of h and $\bar{f}_0 := p_{\bar{S}} \circ l_0 : \bar{X}_0 \hookrightarrow \bar{X}_{00} \times \bar{S} \rightarrow \bar{S}$, $\epsilon_{\bar{X}_0} := p_{\bar{X}_{00}} \circ l_0 : \bar{X}_0 \hookrightarrow \bar{X}_{00} \times \bar{S} \rightarrow \bar{X}_{00}$ be the restriction to \bar{X}_0 of the projections. Then, $\bar{X} \in \text{PVar}(\mathbb{C})$, $\epsilon_{\bar{X}_0} : \bar{X}_0 \rightarrow \bar{X}_{00}$ is a proper modification which does not affect the open subset $V \subset \bar{X}_0$, and $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ is a compactification of h .

(ii): There is two things to prove:

- Let $\bar{f}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : V \rightarrow S$ and $\bar{f}'_0 : \bar{X}'_0 \rightarrow \bar{S}$ a compactification of $h' : V' \rightarrow S$ (see (i)). Let $l_0 : \bar{X}'_0 \hookrightarrow \bar{\Gamma}_g \subset \bar{X}'_{00} \times_{\bar{S}} \bar{X}_0$ be the closure of the graph of g , $\bar{f}'_0 := (\bar{f}'_{00}, \bar{f}_0) \circ l_0 : \bar{X}'_0 \hookrightarrow \bar{X}'_{00} \times_{\bar{S}} \bar{X}_0 \rightarrow \bar{S}$ and $\bar{g}_0 := p_{\bar{X}_0} \circ l_0 : \bar{X}'_0 \hookrightarrow \bar{X}'_{00} \times_{\bar{S}} \bar{X}_0 \rightarrow \bar{X}_0$, $\epsilon_{\bar{X}'_0} := p_{\bar{X}'_{00}} \circ i : \bar{X}'_0 \hookrightarrow \bar{X}'_{00} \times_{\bar{S}} \bar{X}_0 \rightarrow \bar{X}'_{00}$ be the restriction to X of the projections. Then $\epsilon_{\bar{X}'_0} : \bar{X}'_0 \rightarrow \bar{X}'_{00}$ is a proper modification which does not affect the open subset $V' \subset \bar{X}'_0$, $\bar{f}'_0 : \bar{X}'_0 \rightarrow \bar{S}$ is an other compactification of $h' : V' \rightarrow S$ and $\bar{g}_0 : \bar{X}'_0 \rightarrow \bar{X}_0$ is a compactification of g .
- In the case V and V' are smooth, we take, using theorem 15, a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) . Take then, using theorem 15, a strict desingularization $\bar{\epsilon}' : (\bar{X}', \bar{D}') \rightarrow (\bar{X}' \times_{\bar{X}_0} \bar{X}'_0, \bar{X}' \times_{\bar{X}_0} \bar{Z}')$ of the pair $(\bar{X}' \times_{\bar{X}_0} \bar{X}'_0, \bar{X}' \times_{\bar{X}_0} \bar{Z}')$. We consider then following commutative diagram whose square is cartesian :

$$\begin{array}{ccc} & \bar{X}'_0 & \xrightarrow{\bar{g}_0} & X_0 \\ & \uparrow \bar{\epsilon}'_0 & & \uparrow \epsilon \\ \bar{X}' & \xrightarrow{\bar{g}} & \bar{X} & \\ \uparrow \epsilon' & & \uparrow \bar{g}'_0 & \\ \bar{X}' \times_{\bar{X}_0} \bar{X}'_0 & \xrightarrow{\bar{g}'_0} & \bar{X} & \\ \uparrow \epsilon'_1 & & \uparrow \bar{g} & \\ \bar{X}' & & & \end{array}$$

and $\bar{\epsilon}' := \bar{\epsilon}'_0 \circ \bar{\epsilon}'_1 : (\bar{X}', \bar{D}') \rightarrow (\bar{X}'_0, \bar{Z}')$ is a strict desingularization of the pair $(\bar{X}' \times_{\bar{X}_0} \bar{X}'_0, \bar{X}' \times_{\bar{X}_0} \bar{Z}')$.

□

Let $S \in \text{Var}(\mathbb{C})$. Recall we have the dual functor

$$\mathbb{D}_S : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/S), F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}(S/S)))$$

which induces the functor

$$L\mathbb{D}_S : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})/S), F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}(S/S))).$$

We will use the following resolutions of representable presheaves by Corti-Hanamura presheaves and their functorialities.

Definition 38. (i) Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$ and U smooth. Take, see definition-proposition 12, $f_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) , with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \epsilon^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. We denote by $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X} = \bar{X}_{c(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings $i_I : \bar{D}_I = \cap_{i \in I} \bar{D}_i \hookrightarrow \bar{X}$. We denote by $j : U \hookrightarrow \bar{X}$ the open embedding and by $p_S : \bar{X} \times S \rightarrow S$ and $p_{\bar{S}} : U \times S \rightarrow S$ the projections. Considering the graph factorization $\bar{f} : \bar{X} \xrightarrow{\bar{l}} \bar{X} \times \bar{S} \xrightarrow{p_{\bar{S}}} \bar{S}$ of $\bar{f} : \bar{X} \rightarrow \bar{S}$, where \bar{l} is the graph embedding and $p_{\bar{S}}$ the projection, we get closed embeddings $l := \bar{l} \times_{\bar{S}} S : X \hookrightarrow \bar{X} \times S$ and $l_{D_I} := \bar{D}_I \times_{\bar{X}} l : D_I \hookrightarrow \bar{D}_I \times S$. We then consider the following map in $C(\text{Var}(\mathbb{C})^2/S)$

$$\begin{aligned} & r_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) : R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) \\ & \xrightarrow{=} p_{S*} E_{et}(\text{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), u_{IJ}) \rightarrow \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))) \\ & \xrightarrow{p_{S*} E_{et}(0, k \circ \text{ad}((j \times I)^*, (j \times I)_*)(\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)))} p_{S*} E_{et}(\mathbb{Z}((U \times S, U)/U \times S))) =: \mathbb{D}_S^{12}(\mathbb{Z}(U/S)). \end{aligned}$$

Note that $\mathbb{Z}((\bar{D}_I \times S, D_I)/\bar{X} \times S)$ and $\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)$ are obviously \mathbb{A}^1 invariant. Note that $r_{(\bar{X}, \bar{D})/S}$ is NOT an equivalence (\mathbb{A}^1 , et) local by proposition 24 since $\rho_{\bar{X} \times S^*} \mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S) = 0$, whereas $\rho_{\bar{X} \times S^*} \text{ad}((j \times I)^*, (j \times I)_*)(\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))$ is not an equivalence (\mathbb{A}^1 , et) local.

(ii) Let $g : U'/S \rightarrow U/S$ a morphism, with $U'/S = (U', h'), U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, with U and U' smooth. Take, see definition-proposition 12(ii), a compactification $f_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}$ of $h : U \rightarrow S$ and a compactification $f'_0 = \bar{h}' : \bar{X}'_0 \rightarrow \bar{S}$ of $h' : U' \rightarrow S$ such that $g : U'/S \rightarrow U/S$ extend to a morphism $\bar{g}_0 : \bar{X}'_0/\bar{S} \rightarrow \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \setminus U$ and $\bar{Z}' = \bar{X}'_0 \setminus U'$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of (\bar{X}_0, \bar{Z}) , a strict desingularization $\bar{\epsilon}'_\bullet : (\bar{X}', \bar{D}') \rightarrow (\bar{X}'_0, \bar{Z}')$ of (\bar{X}'_0, \bar{Z}') and a morphism $\bar{g} : \bar{X}' \rightarrow \bar{X}$ such that the following diagram commutes

$$\begin{array}{ccc} \bar{X}'_0 & \xrightarrow{\bar{g}_0} & \bar{X}_0 \\ \bar{\epsilon}' \uparrow & & \uparrow \bar{\epsilon} \\ \bar{X}' & \xrightarrow{\bar{g}} & \bar{X} \end{array}$$

We then have, see definition-proposition 12(ii), the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} = \bar{X}_{c(\bullet)} & \xleftarrow{i_\bullet} & \bar{D}_{s_g(\bullet)} \\ \uparrow g & & \uparrow \bar{g} & & \uparrow \bar{g}' \\ U' = U'_{c(\bullet)} & \xrightarrow{j'} & \bar{X}' = \bar{X}'_{c(\bullet)} & \xleftarrow{i'_\bullet} & \bar{D}'_\bullet \xleftarrow{i''_\bullet \bar{g}'^{-1}} (\bar{D}_{s_g(\bullet)}) : i'_{g\bullet} \end{array} \quad (47)$$

Denote by $p_S : \bar{X} \times S \rightarrow S$ and $p'_S : \bar{X}' \times S \rightarrow S$ the projections. We then consider the following map in $C(\text{Var}(\mathbb{C})^2/S)$

$$\begin{aligned}
& R_S^{CH}(g) : R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) \xrightarrow{=} \\
& p_{S*} E_{et}(\text{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_{s_g(\bullet)} \times S, D_{s_g(\bullet)})/\bar{X} \times S), u_{IJ}) \rightarrow \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))) \\
& \xrightarrow{T((\bar{g} \times I), E)(-) \circ p_{S*} \text{ad}((\bar{g} \times I)^*, (\bar{g} \times I)_*)(-)} \\
& p'_{S*} E_{et}(\text{Cone}(\mathbb{Z}(i'_{g_\bullet} \times I) : \\
& (\mathbb{Z}((\bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) \times S, \bar{g}^{-1}(D_{s_g(\bullet)})/\bar{X}' \times S), u_{IJ}) \rightarrow \mathbb{Z}((\bar{X}' \times S, X')/\bar{X}' \times S))) \\
& \xrightarrow{p'_{S*} E_{et}(\mathbb{Z}(i'_{g_\bullet} \times I), I)} \\
& p'_{S*} E_{et}(\text{Cone}(\mathbb{Z}(i'_\bullet \times I) : ((\mathbb{Z}((\bar{D}'_\bullet \times S, D'_\bullet)/\bar{X}' \times S), u_{IJ}) \rightarrow \mathbb{Z}((\bar{X}' \times S, X')/\bar{X}' \times S))) \\
& \xrightarrow{=} R_{(\bar{X}', \bar{D}')/S}(\mathbb{Z}(U'/S))
\end{aligned}$$

Then by the diagram (47) and adjunction, the following diagram in $C(\text{Var}(\mathbb{C})^2/S)$ obviously commutes

$$\begin{array}{ccc}
R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) & \xrightarrow{r_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S))} p_{S*} E_{et}(\mathbb{Z}((U \times S, U)/U \times S)) =: \mathbb{D}_S^{12}(\mathbb{Z}(U/S)) & \\
R_S^{CH}(g) \downarrow & & \downarrow D_S^{12}(g) := T(g \times I, E)(-) \circ \text{ad}((g \times I)^*, (g \times I)_*)(E_{et}(\mathbb{Z}((U \times S, U)/U \times S))) \\
R_{(\bar{X}', \bar{D}')/S}(\mathbb{Z}(U'/S)) & \xrightarrow{r_{(\bar{X}', \bar{D}')/S}(\mathbb{Z}(U'/S))} p'_{S*} E_{et}(\mathbb{Z}((U' \times S, U')/U' \times S)) =: \mathbb{D}_S^{12}(\mathbb{Z}(U'/S)) &
\end{array}$$

(iii) For $g_1 : U''/S \rightarrow U'/S$, $g_2 : U'/S \rightarrow U/S$ two morphisms with $U''/S = (U', h'')$, $U'/S = (U', h')$, $U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, with U , U' and U'' smooth. We get from (i) and (ii) a compactification $\bar{f} = \bar{h} : \bar{X} \rightarrow \bar{S}$ of $h : U \rightarrow S$, a compactification $\bar{f}' = \bar{h}' : \bar{X}' \rightarrow \bar{S}$ of $h' : U' \rightarrow S$, and a compactification $\bar{f}'' = \bar{h}'' : \bar{X}'' \rightarrow \bar{S}$ of $h'' : U'' \rightarrow S$, with $\bar{X}, \bar{X}', \bar{X}'' \in \text{PSmVar}(\mathbb{C})$, $\bar{D} := \bar{X} \setminus U \subset \bar{X}$, $\bar{D}' := \bar{X}' \setminus U' \subset \bar{X}'$, and $\bar{D}'' := \bar{X}'' \setminus U'' \subset \bar{X}''$ normal crossing divisors, such that $g_1 : U''/S \rightarrow U'/S$ extend to $\bar{g}_1 : \bar{X}''/\bar{S} \rightarrow \bar{X}'/\bar{S}$, $g_2 : U'/S \rightarrow U/S$ extend to $\bar{g}_2 : \bar{X}'/\bar{S} \rightarrow \bar{X}/\bar{S}$, and

$$R_S^{CH}(g_2 \circ g_1) = R_S^{CH}(g_1) \circ R_S^{CH}(g_2) : R_{(\bar{X}, \bar{D})/S} \rightarrow R_{(\bar{X}'', \bar{D}'')/S}$$

(iv) For

$$Q^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_\alpha^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha, \beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{\beta}^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U_α^* smooth, we get from (i), (ii) and (iii) the map in $C(\text{Var}(\mathbb{C})^2/S)$

$$\begin{aligned}
R_S^{CH}(Q^*) : R^{CH}(Q^*) & := (\cdots \rightarrow \bigoplus_{\beta \in \Lambda^{n-1}} \varinjlim_{(\bar{X}_\beta^{n-1}, \bar{D}_\beta^{n-1})/S} R_{(\bar{X}_\beta^{n-1}, \bar{D}_\beta^{n-1})/S}(\mathbb{Z}(U_\beta^{n-1}/S)) \\
& \xrightarrow{(R_S^{CH}(g_{\alpha, \beta}^n))} \bigoplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S} R_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S}(\mathbb{Z}(U_\alpha^n/S)) \rightarrow \cdots) \rightarrow \mathbb{D}_S^{12}(Q^*),
\end{aligned}$$

where for $(U_\alpha^n, h_\alpha^n) \in \text{Var}(\mathbb{C})/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \rightarrow \bar{S}$ of $h_\alpha : U_\alpha \rightarrow S$ with $\bar{X}_\alpha \in \text{PSmVar}(\mathbb{C})$ and $\bar{D}_\alpha := \bar{X}_\alpha \setminus U_\alpha$ a normal crossing divisor. For $m = (m^*) : Q_1^* \rightarrow Q_2^*$ a morphism with

$$\begin{aligned}
Q_1^* & := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{1, \alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha, \beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{1, \beta}^{n-1}/S) \rightarrow \cdots), \\
Q_2^* & := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{2, \alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha, \beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{2, \beta}^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)
\end{aligned}$$

complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}^*$ and $U_{2,\alpha}^*$ smooth, we get again from (i), (ii) and (iii) a commutative diagram in $C(\text{Var}(\mathbb{C})^2/S)$

$$\begin{array}{ccc} R^{CH}(Q_2^*) & \xrightarrow{r_S^{CH}(Q_2^*)} & \mathbb{D}_S^{12}(Q_2^*) \\ \downarrow R_S^{CH}(m) := (R_S^{CH}(m^*)) & & \downarrow \mathbb{D}_S^{12}(m) := (\mathbb{D}_S^{12}(m^*)) \\ R^{CH}(Q_1^*) & \xrightarrow{r_S^{CH}(Q_1^*)} & \mathbb{D}_S^{12}(Q_1^*) \end{array} .$$

- Let $S \in \text{Var}(\mathbb{C})$ For $(h, m, m') = (h^*, m^*, m'^*) : Q_1^*[1] \rightarrow Q_2^*$ an homotopy with $Q_1^*, Q_2^* \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}^*$ and $U_{2,\alpha}^*$ smooth,

$$(R_S^{CH}(h), R_S^{CH}(m), R_S^{CH}(m')) = (R_S^{CH}(h^*), R_S^{CH}(m^*), R_S^{CH}(m'^*)) : R^{CH}(Q_2^*)[1] \rightarrow R^{CH}(Q_1^*)$$

is an homotopy in $C(\text{Var}(\mathbb{C})^2/S)$ using definition 38 (iii). In particular if $m : Q_1^* \rightarrow Q_2^*$ with $Q_1^*, Q_2^* \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}^*$ and $U_{2,\alpha}^*$ smooth is an homotopy equivalence, then $R_S^{CH}(m) : R^{CH}(Q_2^*) \rightarrow R^{CH}(Q_1^*)$ is an homotopy equivalence.

- Let $S \in \text{SmVar}(\mathbb{C})$. Let $F \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S)$. Consider

$$q : LF := (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \rightarrow \cdots) \rightarrow F$$

the canonical projective resolution given in subsection 2.3.3. Note that the U_α are smooth since S is smooth and h_α are smooth morphism. Definition 38(iv) gives in this particular case the map in $C(\text{Var}(\mathbb{C})^2/S)$

$$\begin{aligned} r_S^{CH}(\rho_S^* LF) : R^{CH}(\rho_S^* LF) &:= (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} R_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \\ &\xrightarrow{(R_S^{CH}(g_{\alpha,\beta}^n))} \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} R_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \rightarrow \cdots) \rightarrow \mathbb{D}_S^{12}(\rho_S^* LF), \end{aligned}$$

where for $(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \rightarrow \bar{S}$ of $h_\alpha : U_\alpha \rightarrow S$ with $\bar{X}_\alpha \in \text{PSmVar}(\mathbb{C})$ and $\bar{D}_\alpha := \bar{X}_\alpha \setminus U_\alpha$ a normal crossing divisor. Definition 38(iv) gives then by functoriality in particular, for $F = F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the map in $C(\text{Var}(\mathbb{C})^2/S)$

$$r_S^{CH}(\rho_S^* LF) = (r_S^{CH}(\rho_S^* LF^*)) : R^{CH}(\rho_S^* LF) \rightarrow \mathbb{D}_S^{12}(\rho_S^* LF).$$

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $h : U \rightarrow S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{array}{ccc} U_T & \xrightarrow{h'} & T \\ \downarrow g' & & \downarrow g \\ U & \xrightarrow{h} & S \end{array}$$

Note that U is smooth since S and h are smooth, and U_T is smooth since T and h' are smooth. Take, see definition-proposition 12(ii), a compactification $f_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}$ of $h : U \rightarrow S$ and a compactification $f'_0 = g \circ h' : \bar{X}'_0 \rightarrow \bar{S}$ of $g \circ h' : U' \rightarrow S$ such that $g' : U_T/S \rightarrow U/S$ extend to a morphism $\bar{g}'_0 : \bar{X}'_0/\bar{S} \rightarrow \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \setminus U$ and $\bar{Z}' = \bar{X}'_0 \setminus U_T$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{e} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of (\bar{X}_0, \bar{Z}) , a desingularization

$\bar{\epsilon}' : (\bar{X}', \bar{D}') \rightarrow (\bar{X}'_0, \bar{Z}')$ of (\bar{X}'_0, \bar{Z}') and a morphism $\bar{g}' : \bar{X}' \rightarrow \bar{X}$ such that the following diagram commutes

$$\begin{array}{ccc} \bar{X}'_0 & \xrightarrow{\bar{g}'_0} & \bar{X}_0 \\ \bar{\epsilon}' \uparrow & & \uparrow \bar{\epsilon} \\ \bar{X}' & \xrightarrow{\bar{g}'} & \bar{X} \end{array}$$

We then have, see definition-proposition 12(ii), the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} = \bar{X}_{c(\bullet)} & \xleftarrow{i_\bullet} & \bar{D}_{s_{g'}(\bullet)} \\ \uparrow g' & & \uparrow g' & & \uparrow (\bar{g}')_\bullet \\ U_T = U_{T,c(\bullet)} & \xrightarrow{j'} & \bar{X}' = X'_{c(\bullet)} & \xleftarrow{i'_\bullet} & \bar{D}'_\bullet \xleftarrow{i''_{g'_\bullet}^{-1}} (\bar{D}_{s_{g'}(\bullet)}) : i'_{g_\bullet} \end{array}$$

We then consider the following map in $C(\text{Var}(\mathbb{C})^2/T)$, see definition 38(ii)

$$\begin{aligned} & T(g, R^{CH})(\mathbb{Z}(U/S)) : g^* R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) \\ & \xrightarrow{g^* R_S^{CH}(g')} g^* R_{(\bar{X}', \bar{D}')/S}(\mathbb{Z}(U_T/S)) = g^* g_* R_{(\bar{X}', \bar{D}')/T}(\mathbb{Z}(U_T/T)) \\ & \xrightarrow{\text{ad}(g^*, g_*)(R_{(\bar{X}', \bar{D}')/T}(\mathbb{Z}(U_T/T)))} R_{(\bar{X}', \bar{D}')/T}(\mathbb{Z}(U_T/T)) \end{aligned}$$

For

$$Q^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_\alpha^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha, \beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_\beta^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with $h_\alpha^n : U_\alpha^n \rightarrow S$ smooth, we get the map in $C(\text{Var}(\mathbb{C})^2/T)$

$$\begin{aligned} & T(g, R^{CH})(Q^*) : g^* R^{CH}(Q^*) = (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S} g^* R_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S}(\mathbb{Z}(U_\alpha^n/S)) \rightarrow \cdots) \\ & \xrightarrow{(T(g, R^{CH})(\mathbb{Z}(U_\alpha^n/S)))} (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}_\alpha^{n'}, \bar{D}_\alpha^{n'})/T} R_{(\bar{X}_\alpha^{n'}, \bar{D}_\alpha^{n'})/T}(\mathbb{Z}(U_{\alpha, T}^n/S)) \rightarrow \cdots) =: R^{CH}(g^* Q^*). \end{aligned}$$

Let $F \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S)$. Consider

$$q : LF := (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \rightarrow \cdots) \rightarrow F$$

the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in $C(\text{Var}(\mathbb{C})^2/T)$

$$\begin{aligned} & T(g, R^{CH})(\rho_S^* LF) : g^* R^{CH}(\rho_S^* LF) = \\ & (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} g^* R_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \rightarrow \cdots) \xrightarrow{(T(g, R^{CH})(\mathbb{Z}(U_\alpha/S)))} \\ & (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha', \bar{D}_\alpha')/T} R_{(\bar{X}_\alpha', \bar{D}_\alpha')/T}(\mathbb{Z}(U_{\alpha, T}/S)) \rightarrow \cdots) =: R^{CH}(\rho_T^* g^* LF). \end{aligned}$$

By functoriality, we get in particular for $F = F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the map in $C(\text{Var}(\mathbb{C})^2/T)$

$$T(g, R^{CH})(\rho_S^* LF) : g^* R^{CH}(\rho_S^* LF) \rightarrow R^{CH}(\rho_T^* g^* LF).$$

- Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ and $p : S_1 \times S_2 \rightarrow S_1$ the projection. Let $h : U \rightarrow S_1$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{array}{ccc} U \times S_2 & \xrightarrow{h \times I} & S_1 \times S_2 \\ \downarrow p' & & \downarrow p \\ U & \xrightarrow{h} & S_1 \end{array}$$

Take, see definition-proposition 12(i), a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}_1$ of $h : U \rightarrow S_1$. Then $\bar{f}_0 \times I : \bar{X}_0 \times S_2 \rightarrow \bar{S}_1 \times S_2$ is a compactification of $h \times I : U \times S_2 \rightarrow S_1 \times S_2$ and $p' : U \times S_2 \rightarrow U$ extend to $\bar{p}'_0 := p_{X_0} : \bar{X}_0 \times S_2 \rightarrow \bar{X}_0$. Denote $Z = X_0 \setminus U$. Take see theorem 15(i), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) . We then have the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$ whose squares are cartesian

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} & \xleftarrow{i \bullet} & \bar{D} \bullet \\ \uparrow g & & \uparrow \bar{p}' := p_{\bar{X}} & & \uparrow \bar{p}' \bullet \\ U \times S_2 = (U \times S_2)_{c(\bullet)} & \xrightarrow{j \times I} & \bar{X} \times S_2 & \xleftarrow{i' \bullet} & \bar{D} \bullet \times S_2 \end{array} \quad (48)$$

Then the map in $C(\text{Var}(\mathbb{C})^2/S_1 \times S_2)$

$$T(p, R^{CH})(\mathbb{Z}(U/S_1)) : p^* R_{(\bar{X}, \bar{D})/S_1}(\mathbb{Z}(U/S_1)) \xrightarrow{\sim} R_{(\bar{X} \times S_2, \bar{D} \bullet \times S_2)/S_1 \times S_2}(\mathbb{Z}(U \times S_2/S_1 \times S_2))$$

is an isomorphism. Hence, for $Q^* \in C(\text{Var}(\mathbb{C})/S_1)$ a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in $C(\text{Var}(\mathbb{C})^2/S_1 \times S_2)$

$$T(p, R^{CH})(Q^*) : p^* R^{CH}(Q^*) \xrightarrow{\sim} R^{CH}(p^* Q^*)$$

is an isomorphism. In particular, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S_1)$ the map in $C(\text{Var}(\mathbb{C})^2/S_1 \times S_2)$

$$T(p, R^{CH})(\rho_{S_1}^* LF) : p^* R^{CH}(\rho_{S_1}^* LF) \xrightarrow{\sim} R^{CH}(\rho_{S_1 \times S_2}^* p^* LF)$$

is an isomorphism.

- Let $h_1 : U_1 \rightarrow S$, $h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2, S \in \text{Var}(\mathbb{C})$, U_1, U_2 smooth. Denote by $p_1 : U_1 \times_S U_2 \rightarrow U_1$ and $p_2 : U_1 \times_S U_2 \rightarrow U_2$ the projections. Take, see definition-proposition 12(i), a compactification $\bar{f}_{10} = \bar{h}_1 : \bar{X}_{10} \rightarrow \bar{S}$ of $h_1 : U_1 \rightarrow S$ and a compactification $\bar{f}_{20} = \bar{h}_2 : \bar{X}_{20} \rightarrow \bar{S}$ of $h_2 : U_2 \rightarrow S$. Then,

- $\bar{f}_{10} \times \bar{f}_{20} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \rightarrow S$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \rightarrow S$.
- $\bar{p}_{10} := p_{X_{10}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \rightarrow \bar{X}_{10}$ is a compactification of $p_1 : U_1 \times_S U_2 \rightarrow U_1$.
- $\bar{p}_{20} := p_{X_{20}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \rightarrow \bar{X}_{20}$ is a compactification of $p_2 : U_1 \times_S U_2 \rightarrow U_2$.

Denote $\bar{Z}_1 = \bar{X}_{10} \setminus U_1$ and $\bar{Z}_2 = \bar{X}_{20} \setminus U_2$. Take, see theorem 15(i), a strict desingularization $\bar{\epsilon}_1 : (\bar{X}_1, \bar{D}) \rightarrow (\bar{X}_{10}, \bar{Z}_1)$ of the pair $(\bar{X}_{10}, \bar{Z}_1)$ and a strict desingularization $\bar{\epsilon}_2 : (\bar{X}_2, \bar{E}) \rightarrow (\bar{X}_{20}, \bar{Z}_2)$ of the pair $(\bar{X}_{20}, \bar{Z}_2)$. Take then a strict desingularization

$$\bar{\epsilon}_{12} : ((\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F}) \rightarrow (\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (D \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$$

of the pair $(\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (\bar{D} \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$. We have then the following commutative diagram

$$\begin{array}{ccccc}
 & & \bar{X}_1 & \xrightarrow{\bar{f}_1} & \bar{S} \\
 & & \uparrow \bar{p}_2 & & \uparrow \bar{f}_2 \\
 & & \bar{X}_1 \times_{\bar{S}} \bar{X}_2 & \xrightarrow{\bar{p}_1} & \bar{X}_2 \\
 (\bar{p}_2)^N \nearrow & \bar{\epsilon}_{12} \nearrow & \uparrow \bar{p}_1^N & & \\
 (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N & & & &
 \end{array}$$

and

- $\bar{f}_1 \times \bar{f}_2 : \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \rightarrow \bar{S}$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \rightarrow S$.
- $(\bar{p}_1)^N := \bar{p}_1 \circ \bar{\epsilon}_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \rightarrow \bar{X}_1$ is a compactification of $p_1 : U_1 \times_S U_2 \rightarrow U_1$.
- $(\bar{p}_2)^N := \bar{p}_2 \circ \bar{\epsilon}_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \rightarrow \bar{X}_2$ is a compactification of $p_2 : U_1 \times_S U_2 \rightarrow U_2$.

We have then the morphism in $C(\text{Var}(\mathbb{C})^2/S)$

$$\begin{aligned}
 & T(\otimes, R_S^{CH})(\mathbb{Z}(U_1/S), \mathbb{Z}(U_2/S)) := R_S^{CH}(p_1) \otimes R_S^{CH}(p_2) : \\
 & R_{(\bar{X}_1, \bar{D})/S}(\mathbb{Z}(U_1/S)) \otimes R_{(\bar{X}_2, \bar{E})/S}(\mathbb{Z}(U_2/S)) \xrightarrow{\sim} R_{(\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F}}/S(\mathbb{Z}(U_1 \times_S U_2/S))
 \end{aligned}$$

For

$$Q_1^* := (\cdots \rightarrow \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{1,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{1,\beta}^{n-1}/S) \rightarrow \cdots),$$

$$Q_2^* := (\cdots \rightarrow \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{2,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{2,\beta}^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

complexes of (maybe infinite) direct sum of representable presheaves with U_α^* smooth, we get the morphism in $C(\text{Var}(\mathbb{C})^2/S)$

$$T(\otimes, R_S^{CH})(Q_1^*, Q_2^*) : R^{CH}(Q_1^*) \otimes R^{CH}(Q_2^*) \xrightarrow{(T(\otimes, R_S^{CH})(\mathbb{Z}(U_{1,\alpha}^m), \mathbb{Z}(U_{2,\beta}^n))} R^{CH}(Q_1^* \otimes Q_2^*).$$

For $F_1, F_2 \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we get in particular the morphism in $C(\text{Var}(\mathbb{C})^2/S)$

$$T(\otimes, R_S^{CH})(\rho_S^* LF_1, \rho_S^* LF_2) : R^{CH}(\rho_S^* LF_1) \otimes R^{CH}(\rho_S^* LF_2) \rightarrow R^{CH}(\rho_S^*(LF_1 \otimes LF_2)).$$

Definition 39. Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$, U irreducible. Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15, a desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \Delta)$ of the pair (\bar{X}_0, Δ) , $\bar{Z} \subset \Delta$, with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\Delta) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Denote $d_X := \dim(\bar{X}) = \dim(U)$.

- (i) The cycle $(\Delta_{\bar{D}_\bullet} \times S) \subset \bar{D}_\bullet \times \bar{D}_\bullet \times S$ induces by the diagonal $\Delta_{\bar{D}_\bullet} \subset \bar{D}_\bullet \times \bar{D}_\bullet$ gives the morphism in $C(\text{Var}(\mathbb{C})^2/S)$

$$\begin{aligned}
 [\Delta_{\bar{D}_\bullet}] \in \text{Hom}(\mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S), p_{S*} E_{et}(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S)(d_X)[2d_X])) &\xrightarrow{\sim} \\
 \text{Hom}(\mathbb{Z}((\bar{D}_\bullet \times S \times \bar{X}, D_\bullet)/\bar{X} \times S), & \\
 \mathbb{Z}^{tr}((\bar{D}_\bullet \times S \times \mathbb{P}^{d_X}, D_\bullet \times \mathbb{P}^{d_X})/\bar{X} \times S)/\mathbb{Z}^{tr}((- \times \mathbb{P}^{d_X-1}, (- \times \mathbb{P}^{d_X-1})) & \\
 \subset H^0(\mathcal{Z}_{d_{D_\bullet} + d_S}(\square^* \times \bar{D}_\bullet \times \bar{D}_\bullet \times S), s.t.\alpha_*(\times D_\bullet) = D_\bullet) &
 \end{aligned}$$

- (ii) The cycle $(\Delta_{\bar{X}} \times S) \subset \bar{X} \times \bar{X} \times S$ induces by the diagonal $\Delta_{\bar{X}} \subset \bar{X} \times \bar{X}$ gives the morphism in $C(\text{Var}(\mathbb{C})^2/S)$

$$\begin{aligned}
 [\Delta_{\bar{X}}] \in \text{Hom}(\mathbb{Z}^{tr}((\bar{X} \times S, X)/S), p_{S*} E_{et}(\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)(d_X)[2d_X])) &\xrightarrow{\sim} \\
 \text{Hom}(\mathbb{Z}((\bar{X} \times S \times \bar{X}, X)/\bar{X} \times S), & \\
 \mathbb{Z}^{tr}((\bar{X} \times S \times \mathbb{P}^{d_X}, X \times \mathbb{P}^{d_X})/\bar{X} \times S)/\mathbb{Z}^{tr}((- \times \mathbb{P}^{d_X-1}, (- \times \mathbb{P}^{d_X-1})) & \\
 \subset H^0(\mathcal{Z}_{d_X + d_S}(\square^* \times \bar{X} \times \bar{X} \times S), s.t.\alpha_*(\times X) = X) &
 \end{aligned}$$

Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$, U smooth connected (hence irreducible by smoothness). Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Denote $d_X := \dim(\bar{X}) = \dim(U)$.

(iii) We get from (i) and (ii) the morphism in $C(\text{Var}(\mathbb{C})^2/S)$

$$\begin{aligned} T(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)) &:= ([\Delta_{\bar{D}_\bullet}], [\Delta_{\bar{X}}]) : \\ \text{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}^{\text{tr}}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \rightarrow \mathbb{Z}^{\text{tr}}((\bar{X} \times S, X)/S)) &\rightarrow \\ p_{S*}E_{\text{et}}(\text{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), u_{IJ}) \rightarrow & \\ \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))) &(d_X)[2d_X] =: R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X] \end{aligned}$$

(iii)' which gives the map in $C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$

$$\begin{aligned} T^{\mu, q}(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)) &: \\ \text{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}^{\text{tr}}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \rightarrow \mathbb{Z}^{\text{tr}}((\bar{X} \times S, X)/S)) &= \\ L\rho_{S*}\mu_{S*}\text{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}^{\text{tr}}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \rightarrow \mathbb{Z}^{\text{tr}}((\bar{X} \times S, X)/S)) & \\ \xrightarrow{L\rho_{S*}\mu_{S*}T(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))} & L\rho_{S*}\mu_{S*}R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X] \end{aligned}$$

Proposition 40. Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$, U irreducible. Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \Delta)$ of the pair (\bar{X}_0, Δ) , $\bar{Z} \subset \Delta$ with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\Delta) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Denote $d_X := \dim(\bar{X}) = \dim(U)$.

(i) The morphism

$$[\Delta_{\bar{D}_\bullet}] : \mathbb{Z}^{\text{tr}}((\bar{D}_\bullet \times S, D_\bullet)/S) \rightarrow p_{S*}E_{\text{et}}(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S)(d_X)[2d_X])$$

given in definition 39(i) is an equivalence $(\mathbb{A}^1, \text{et})$ local.

(ii) The morphism

$$[\Delta_{\bar{X}}] : \mathbb{Z}^{\text{tr}}((\bar{X} \times S, X)/S) \rightarrow p_{S*}E_{\text{et}}(\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)(d_X)[2d_X])$$

given in definition 39(ii) is an equivalence $(\mathbb{A}^1, \text{et})$ local.

Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$, U smooth connected (hence irreducible by smoothness). Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) , with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor.

(iii) The morphism

$$\begin{aligned} T(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)) &:= ([\Delta_{\bar{D}_\bullet}], [\Delta_{\bar{X}}]) : \\ \text{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}^{\text{tr}}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \rightarrow \mathbb{Z}^{\text{tr}}((\bar{X} \times S, X)/S)) &\rightarrow \\ p_{S*}E_{\text{et}}(\text{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), u_{IJ}) \rightarrow & \\ \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))) &(d_X)[2d_X] =: R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X] \end{aligned}$$

given in definition 39(iii)' is an equivalence $(\mathbb{A}^1, \text{et})$ local.

(iii)' The morphism

$$\begin{aligned} & T^{\mu, q}(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}\bullet \times S, D\bullet)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)) : \\ & \text{Cone}(\mathbb{Z}(i\bullet \times I) : (\mathbb{Z}^{tr}((\bar{D}\bullet \times S, D\bullet)/S), u_{IJ}) \rightarrow \mathbb{Z}^{tr}((\bar{X} \times S, X)/S)) \\ & \rightarrow L\rho_{S*}\mu_{S*}R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X] \end{aligned}$$

given in definition 39(iii)' is an equivalence (\mathbb{A}^1, et) local.

Proof. (i): By Yoneda lemma, it is equivalent to show that for every morphism $g : T \rightarrow S$ with $T \in \text{Var}(\mathbb{C})$ and every closed subset $E \subset T$, the composition morphism

$$\begin{aligned} [\Delta_{\bar{D}\bullet}] : \text{Hom}^\bullet(\mathbb{Z}((T, E)/S), C_*\mathbb{Z}^{tr}((\bar{D}\bullet \times S, D\bullet)/S)) & \xrightarrow{\text{Hom}^\bullet(\mathbb{Z}((T, E)/S), C_*\Delta_{\bar{D}\bullet})} \\ & \text{Hom}^\bullet(\mathbb{Z}((T, E)/S), p_{S*}E_{et}(\mathbb{Z}((\bar{D}\bullet \times S, D\bullet)/\bar{X} \times S)(d_X)[2d_X])) \end{aligned}$$

is a quasi-isomorphism of abelian groups. But this map is the composite

$$\begin{aligned} & \text{Hom}^\bullet(\mathbb{Z}((T, E)/S), \mathbb{Z}^{tr}((\bar{D}\bullet \times S, D\bullet)/S)) \xrightarrow{[\Delta_{\bar{D}\bullet}]} \\ & \text{Hom}^\bullet(\mathbb{Z}((T, E)/S), p_{S*}E_{et}(\mathbb{Z}((\bar{D}\bullet \times S, D\bullet)/\bar{X} \times S)(d_X)[2d_X])) \xrightarrow{\sim} \\ & \text{Hom}^\bullet(\mathbb{Z}((T \times \bar{X}, E)/S \times \bar{X}), \\ & C_*\mathbb{Z}^{tr}((\bar{D}\bullet \times S \times \mathbb{P}^{d_X}, D\bullet \times \mathbb{P}^{d_X})/\bar{X} \times S)/\mathbb{Z}^{tr}((-) \times \mathbb{P}^{d_X-1}, (-) \times \mathbb{P}^{d_X-1})) \end{aligned}$$

which is clearly a quasi-isomorphism.

(ii): Similar to (i).

(iii): Follows from (i) and (ii).

(iii)': Follows from (iii) and the fact that μ_{S*} preserve (\mathbb{A}^1, et) local equivalence (see proposition 25) and the fact that ρ_{S*} preserve (\mathbb{A}^1, et) local equivalence (see proposition 24). \square

Definition 40. (i) Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$, U smooth. Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) , with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. We will consider the following canonical map in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$\begin{aligned} & T_{(\bar{X}, \bar{D})/S}(U/S) : \text{Gr}_{S*}^{12} L\rho_{S*}\mu_{S*}R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) \xrightarrow{q} \text{Gr}_{S*}^{12} \rho_{S*}\mu_{S*}R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) \\ & \xrightarrow{r_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S))} \text{Gr}_{S*}^{12} \rho_{S*}\mu_{S*}p_{S*}E_{et}(\mathbb{Z}((U \times S, U)/U \times S)) \xrightarrow{l(U/S)} h_*E_{et}(\mathbb{Z}(U/U)) =: \mathbb{D}_S^0(\mathbb{Z}(U/S)) \end{aligned}$$

where, for $h' : V \rightarrow S$ a smooth morphism with $V \in \text{Var}(\mathbb{C})$,

$$l^{00}(U/S)(V/S) : \mathbb{Z}((U \times S, U)/U \times S)(V \times U \times S, V \times_S U/U \times S) \rightarrow \mathbb{Z}(U/U)(V \times_S U), \alpha \mapsto \alpha|_{V \times_S U}$$

which gives

$$l^0(U/S)(V/S) : E_{et}^0(\mathbb{Z}((U \times S, U)/U \times S))(V \times U \times S, V \times_S U/U \times S) \rightarrow E_{et}^0(\mathbb{Z}(U/U))(V \times_S U),$$

and by induction

$$\tau^{\leq i} l(U/S) : \text{Gr}_{S*}^{12} \rho_{S*}\mu_{S*}p_{S*}E_{et}^{\leq i}(\mathbb{Z}((U \times S, U)/U \times S)) \rightarrow h_*E_{et}^{\leq i}(\mathbb{Z}(U/U))$$

where $\tau^{\leq i}$ is the cohomological truncation.

(ii) Let $g : U'/S \rightarrow U/S$ a morphism, with $U'/S = (U', h')$, $U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, U, U' smooth. Take, see definition-proposition 12(ii), a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}$ of $h : U \rightarrow S$ and a compactification $\bar{f}'_0 = \bar{h}' : \bar{X}'_0 \rightarrow \bar{S}$ of $h' : U' \rightarrow S$ such that $g : U'/S \rightarrow U/S$ extend to a morphism

$\bar{g}_0 : \bar{X}'_0/\bar{S} \rightarrow \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \setminus U$ and $\bar{Z}' = \bar{X}'_0 \setminus U'$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of (\bar{X}_0, \bar{Z}) , a strict desingularization $\bar{\epsilon}'_\bullet : (\bar{X}', \bar{D}') \rightarrow (\bar{X}'_0, \bar{Z}')$ of (\bar{X}'_0, \bar{Z}') and a morphism $\bar{g} : \bar{X}' \rightarrow \bar{X}$ such that the following diagram commutes

$$\begin{array}{ccc} \bar{X}'_0 & \xrightarrow{\bar{g}_0} & \bar{X}_0 \\ \bar{\epsilon}' \uparrow & & \uparrow \bar{\epsilon} \\ \bar{X}' & \xrightarrow{\bar{g}} & \bar{X} \end{array}$$

Then by the diagram given in definition 38(ii), the following diagram in $C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ obviously commutes

$$\begin{array}{ccc} \text{Gr}_{S^*}^{12} L\rho_{S^*}\mu_{S^*}R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) & \xrightarrow{T_{(\bar{X}, \bar{D})/S}(U/S)} & h_* E_{\text{et}}(\mathbb{Z}(U/U)) := \mathbb{D}_S^0(\mathbb{Z}(U/S)) \\ \downarrow R_S^{CH}(g) & & \downarrow T_{(g, E)(-)\circ \text{ad}(g^*, g^*)}(E_{\text{et}}(\mathbb{Z}(U/U))) := \mathbb{D}_S^0(g) \\ \text{Gr}_{S^*}^{12} L\rho_{S^*}\mu_{S^*}R_{(\bar{X}', \bar{D}')/S}(\mathbb{Z}(U'/S)) & \xrightarrow{T_{(\bar{X}', \bar{D}')/S}(U'/S)} & h'_* E_{\text{et}}(\mathbb{Z}(U'/U')) := \mathbb{D}_S^0(\mathbb{Z}(U'/S)) \end{array}$$

where $l(U/S)$ and $l(U'/S)$ are the maps given in (i).

(iii) Let $S \in \text{SmVar}(\mathbb{C})$. Let $F \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$. We get from (i) and (ii) morphisms in $C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$

$$\begin{aligned} & T_S^{CH}(LF) : \text{Gr}_{S^*}^{12} L\rho_{S^*}\mu_{S^*}R_{(\bar{X}^*, \bar{D}^*)/S}(\rho_S^* LF) \\ & \xrightarrow{r_S^{CH}(LF)} \text{Gr}_{S^*}^{12} L\rho_{S^*}\mu_{S^*}\mathbb{D}_S^{12}(\rho_S^* LF) \xrightarrow{l(LF)} \mathbb{D}_S^0(LF) \end{aligned}$$

We will also have the following lemma

Lemma 1. (i) Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$, U smooth. Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) , with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Then the map in $C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$

$$\begin{aligned} & \text{ad}(\text{Gr}_S^{12*}, \text{Gr}_{S^*}^{12})(L\rho_{S^*}\mu_{S^*}R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S))) \circ q : \\ & \text{Gr}_S^{12*} L \text{Gr}_{S^*}^{12} L\rho_{S^*}\mu_{S^*}R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) \rightarrow L\rho_{S^*}\mu_{S^*}R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) \end{aligned}$$

is an equivalence $(\mathbb{A}^1, \text{et})$ local.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$. Let $F \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$. Then the map in $C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$

$$\begin{aligned} & \text{ad}(\text{Gr}_S^{12*}, \text{Gr}_{S^*}^{12})(L\rho_{S^*}\mu_{S^*}R_{(\bar{X}^*, \bar{D}^*)/S}(\rho_S^* LF)) \circ q : \\ & \text{Gr}_S^{12*} L \text{Gr}_{S^*}^{12} L\rho_{S^*}\mu_{S^*}R_{(\bar{X}^*, \bar{D}^*)/S}(\rho_S^* LF) \rightarrow L\rho_{S^*}\mu_{S^*}R_{(\bar{X}^*, \bar{D}^*)/S}(\rho_S^* LF) \end{aligned}$$

is an equivalence $(\mathbb{A}^1, \text{et})$ local.

Proof. (i): Follows from proposition 40.

(ii): Follows from (i). □

Definition 41. (i) Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$ and U smooth. Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) , with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. We denote by $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X} = \bar{X}_{c(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings

$i_I : \bar{D}_I = \bigcap_{i \in I} \bar{D}_i \hookrightarrow \bar{X}$ We denote by $j : U \hookrightarrow \bar{X}$ the open embedding and by $p_S : \bar{X} \times S \rightarrow S$ and $p_{\bar{S}} : U \times S \rightarrow S$ the projections. Considering the graph factorization $\bar{f} : \bar{X} \xrightarrow{\bar{l}} \bar{X} \times \bar{S} \xrightarrow{p_{\bar{S}}} \bar{S}$ of $\bar{f} : \bar{X} \rightarrow \bar{S}$, where \bar{l} is the graph embedding and $p_{\bar{S}}$ the projection, we get closed embeddings $l := \bar{l} \times_{\bar{S}} S : X \hookrightarrow \bar{X} \times S$ and $l_{D_I} := \bar{D}_I \times_{\bar{X}} l : D_I \hookrightarrow \bar{D}_I \times S$. We then consider the map in $C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$

$$\begin{aligned} & T(\hat{R}^{CH}, R^{CH})(\mathbb{Z}(U/S)) : \hat{R}_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) \\ \xrightarrow{=} & \text{Cone}(\mathbb{Z}(i_{\bullet} \times I) : (\mathbb{Z}^{tr}((\bar{D}_{\bullet} \times S, D_{\bullet})/S), u_{IJ}) \rightarrow \mathbb{Z}^{tr}((\bar{X} \times S, X)/S))(-d_X)[-2d_X] \\ & \xrightarrow{T^{\mu, q}(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_{\bullet} \times S, D_{\bullet})/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))(-d_X)[-2d_X]} \\ & L\rho_{S*} \mu_{S*} R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)). \end{aligned}$$

given in definition 39(iii).

(ii) Let $g : U'/S \rightarrow U/S$ a morphism, with $U'/S = (U', h'), U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, with U and U' smooth. Take, see definition-proposition 12(ii), a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}$ of $h : U \rightarrow S$ and a compactification $\bar{f}'_0 = \bar{h}' : \bar{X}'_0 \rightarrow \bar{S}$ of $h' : U' \rightarrow S$ such that $g : U'/S \rightarrow U/S$ extend to a morphism $\bar{g}_0 : \bar{X}'_0/\bar{S} \rightarrow \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \setminus U$ and $\bar{Z}' = \bar{X}'_0 \setminus U'$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of (\bar{X}_0, \bar{Z}) , a strict desingularization $\bar{\epsilon}' : (\bar{X}', \bar{D}') \rightarrow (\bar{X}'_0, \bar{Z}')$ of (\bar{X}'_0, \bar{Z}') and a morphism $\bar{g} : \bar{X}' \rightarrow \bar{X}$ such that the following diagram commutes

$$\begin{array}{ccc} \bar{X}'_0 & \xrightarrow{\bar{g}_0} & \bar{X}_0 \\ \bar{\epsilon}' \uparrow & & \uparrow \bar{\epsilon} \\ \bar{X}' & \xrightarrow{\bar{g}} & \bar{X} \end{array}$$

We then have, see definition-proposition 12(ii), the diagram (47) in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} = \bar{X}_{c(\bullet)} & \xleftarrow{i_{\bullet}} & \bar{D}_{s_g(\bullet)} \\ \uparrow g & & \uparrow \bar{g} & & \uparrow \bar{g}' \\ U' = U'_{c(\bullet)} & \xrightarrow{j'} & \bar{X}' = \bar{X}'_{c(\bullet)} & \xleftarrow{i'_{\bullet}} \bar{D}' \xleftarrow{i''_{\bullet} \bar{g}'^{-1}} \bar{D}_{s_g(\bullet)} & : i'_{g_{\bullet}} \end{array}$$

Consider

$$\begin{aligned} [\Gamma_{\bar{g}}]^t & \in \text{Hom}(\mathbb{Z}^{tr}((\bar{X} \times S, X)/S)(-d_X)[-2d_X], \mathbb{Z}^{tr}((\bar{X}' \times S, X')/S)(-d_{X'})[-2d_{X'}]) \\ & \xrightarrow{\sim} \text{Hom}(\mathbb{Z}^{tr}((\bar{X} \times \mathbb{A}^{d_{X'}} \times S, X \times \mathbb{A}^{d_{X'}})/S), \\ & \mathbb{Z}_{tr}((\bar{X}' \times \mathbb{P}^{d_X} \times S, X' \times \mathbb{P}^{d_X})/S)/\mathbb{Z}_{tr}((-\times \mathbb{P}^{d_X-1}, (-\times \mathbb{P}^{d_X-1})) \end{aligned}$$

the morphism given by the transpose of the graph $\Gamma_{\bar{g}} \subset X' \times_S X$ of $\bar{g} : \bar{X}' \rightarrow \bar{X}$. Then, since $i_{\bullet} \circ \bar{g}'_{\bullet} = \bar{g} \circ i'_{g_{\bullet}} = \bar{g} \circ i'_{\bullet} \circ i'_{g_{\bullet}}$, we have the factorization

$$\begin{aligned} & [\Gamma_{\bar{g}}]^t \circ \mathbb{Z}(i_{\bullet} \times I) : (\mathbb{Z}^{tr}((\bar{D}_{s_g(\bullet)} \times S, D_{s_g(\bullet)})/S), u_{IJ})(-d_X)[-2d_X] \\ & \xrightarrow{[\Gamma_{\bar{g}'_{\bullet}}]^t} (\mathbb{Z}^{tr}((\bar{g}'^{-1}(\bar{D}_{s_g(\bullet)}) \times S, \bar{g}'^{-1}(D_{s_g(\bullet)}))/S), u_{IJ})(-d_{X'})[-2d_{X'}]) \\ & \xrightarrow{\mathbb{Z}(i'_{g_{\bullet}} \times I)} \mathbb{Z}^{tr}((\bar{X}' \times S, X')/S)(-d_{X'})[-2d_{X'}]. \end{aligned}$$

with

$$\begin{aligned} & [\Gamma_{\bar{g}'_{\bullet}}]^t \in \text{Hom}((\mathbb{Z}^{tr}((\bar{D}_{s_g(\bullet)} \times \mathbb{A}^{d_{X'}} \times S, D_{s_g(\bullet)} \times \mathbb{A}^{d_{X'}})/S), u_{IJ}), \\ & (\mathbb{Z}_{tr}((\bar{g}'^{-1}(\bar{D}_{s_g(\bullet)}) \times \mathbb{P}^{d_X} \times S, \bar{g}'^{-1}(D_{s_g(\bullet)}) \times \mathbb{P}^{d_X})/S), u_{IJ})/\mathbb{Z}^{tr}((-\times \mathbb{P}^{d_X-1}, (-\times \mathbb{P}^{d_X-1})). \end{aligned}$$

We then consider the following map in $C(\text{Var}(\mathbb{C})^{2,pr}/S)$

$$\begin{aligned}
& \hat{R}_S^{CH}(g) : \hat{R}_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) \xrightarrow{=} \\
\text{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}^{tr}((\bar{D}_{s_g(\bullet)} \times S, D_{s_g(\bullet)})/S), u_{IJ}) \rightarrow \mathbb{Z}^{tr}((\bar{X} \times S, X)/S))(-d_X)[-2d_X] \\
& \xrightarrow{([\bar{g}'_\bullet]^t, [\bar{g}]^t)} \\
& \text{Cone}(\mathbb{Z}(i'_{g_\bullet} \times I) : \\
& (\mathbb{Z}^{tr}((\bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) \times S, \bar{g}^{-1}(D_{s_g(\bullet)})/S), u_{IJ}) \rightarrow \mathbb{Z}^{tr}((\bar{X}' \times S, X')/S))(-d_{X'})[-2d_{X'}] \\
& \xrightarrow{(\mathbb{Z}(i'_{g_\bullet} \times I), I)(-d_{X'})[-2d_{X'}]} \\
& \text{Cone}(\mathbb{Z}(i'_\bullet \times I) : ((\mathbb{Z}^{tr}((\bar{D}'_\bullet \times S, D'_\bullet)/S), u_{IJ}) \rightarrow \mathbb{Z}^{tr}((\bar{X}' \times S, X')/S))(-d_{X'})[-2d_{X'}] \\
& \xrightarrow{=} \hat{R}_{(\bar{X}', \bar{D}')/S}(\mathbb{Z}(U'/S))
\end{aligned}$$

Then the following diagram in $C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ commutes by definition

$$\begin{array}{ccc}
\hat{R}_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) & \xrightarrow{T(\hat{R}^{CH}, R^{CH})(\mathbb{Z}(U/S))} & L\rho_{S*}\mu_{S*}R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) \\
\hat{R}_S^{CH}(g) \downarrow & & \downarrow L\rho_{S*}\mu_{S*}R_S^{CH}(g) \\
\hat{R}_{(\bar{X}', \bar{D}')/S}(\mathbb{Z}(U'/S)) & \xrightarrow{T(\hat{R}^{CH}, R^{CH})(\mathbb{Z}(U'/S))} & L\rho_{S*}\mu_{S*}R_{(\bar{X}', \bar{D}')/S}(\mathbb{Z}(U'/S))
\end{array}$$

(iii) For $g_1 : U''/S \rightarrow U'/S$, $g_2 : U'/S \rightarrow U/S$ two morphisms with $U''/S = (U', h'')$, $U'/S = (U', h')$, $U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, with U , U' and U'' smooth. We get from (i) and (ii) a compactification $\bar{f} = \bar{h} : \bar{X} \rightarrow \bar{S}$ of $h : U \rightarrow S$, a compactification $\bar{f}' = \bar{h}' : \bar{X}' \rightarrow \bar{S}$ of $h' : U' \rightarrow S$, and a compactification $\bar{f}'' = \bar{h}'' : \bar{X}'' \rightarrow \bar{S}$ of $h'' : U'' \rightarrow S$, with $\bar{X}, \bar{X}', \bar{X}'' \in \text{PSmVar}(\mathbb{C})$, $\bar{D} := \bar{X} \setminus U \subset \bar{X}$, $\bar{D}' := \bar{X}' \setminus U' \subset \bar{X}'$, and $\bar{D}'' := \bar{X}'' \setminus U'' \subset \bar{X}''$ normal crossing divisors, such that $g_1 : U''/S \rightarrow U'/S$ extend to $\bar{g}_1 : \bar{X}''/\bar{S} \rightarrow \bar{X}'/\bar{S}$, $g_2 : U'/S \rightarrow U/S$ extend to $\bar{g}_2 : \bar{X}'/\bar{S} \rightarrow \bar{X}/\bar{S}$, and

$$\hat{R}_S^{CH}(g_2 \circ g_1) = \hat{R}_S^{CH}(g_1) \circ \hat{R}_S^{CH}(g_2) : \hat{R}_{(\bar{X}, \bar{D})/S} \rightarrow \hat{R}_{(\bar{X}'', \bar{D}'')/S}$$

(iv) For

$$Q^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_\alpha^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha, \beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_\beta^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U_α^* smooth, we get from (i), (ii) and (iii) the map in $C(\text{Var}(\mathbb{C})^{2,smpr}/S)$

$$\begin{aligned}
T(\hat{R}^{CH}, R^{CH})(Q^*) : \hat{R}^{CH}(Q^*) & := (\cdots \rightarrow \bigoplus_{\beta \in \Lambda^{n-1}} \varinjlim_{(\bar{X}_\beta^{n-1}, \bar{D}_\beta^{n-1})/S} \hat{R}_{(\bar{X}_\beta^{n-1}, \bar{D}_\beta^{n-1})/S}(\mathbb{Z}(U_\beta^{n-1}/S)) \\
& \xrightarrow{(\hat{R}_S^{CH}(g_{\alpha, \beta}^n))} \bigoplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S} \hat{R}_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S}(\mathbb{Z}(U_\alpha^n/S)) \rightarrow \cdots) \rightarrow L\rho_{S*}\mu_{S*}R^{CH}(Q^*),
\end{aligned}$$

where for $(U_\alpha^n, h_\alpha^n) \in \text{Var}(\mathbb{C})/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \rightarrow \bar{S}$ of $h_\alpha : U_\alpha \rightarrow S$ with $\bar{X}_\alpha \in \text{PSmVar}(\mathbb{C})$ and $\bar{D}_\alpha := \bar{X}_\alpha \setminus U_\alpha$ a normal crossing divisor. For $m = (m^*) : Q_1^* \rightarrow Q_2^*$ a morphism with

$$\begin{aligned}
Q_1^* & := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{1, \alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha, \beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{1, \beta}^{n-1}/S) \rightarrow \cdots), \\
Q_2^* & := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{2, \alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha, \beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{2, \beta}^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)
\end{aligned}$$

complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}^*$ and $U_{2,\alpha}^*$ smooth, we get again from (i),(ii) and (iii) a commutative diagram in $C(\text{Var}(\mathbb{C})^{2,smpr}/S)$

$$\begin{array}{ccc} \hat{R}^{CH}(Q_2^*) & \xrightarrow{T(\hat{R}_S^{CH}, R_S^{CH})(Q_2^*)} & L\rho_{S*}\mu_{S*}R^{CH}(Q_2^*) \\ \hat{R}_S^{CH}(m) := (\hat{R}_S^{CH}(m^*)) \downarrow & & \downarrow L\rho_{S*}\mu_{S*}R_S^{CH}(m) := L\rho_{S*}\mu_{S*}(R_S^{CH}(m^*)) \\ \hat{R}^{CH}(Q_1^*) & \xrightarrow{T(\hat{R}_S^{CH}, R_S^{CH})(Q_1^*)} & L\rho_{S*}\mu_{S*}R^{CH}(Q_1^*) \end{array} .$$

- Let $S \in \text{Var}(\mathbb{C})$ For $(h, m, m') = (h^*, m^*, m'^*) : Q_1^*[1] \rightarrow Q_2^*$ an homotopy with $Q_1^*, Q_2^* \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}^*$ and $U_{2,\alpha}^*$ smooth,

$$(\hat{R}_S^{CH}(h), \hat{R}_S^{CH}(m), \hat{R}_S^{CH}(m')) = (\hat{R}_S^{CH}(h^*), \hat{R}_S^{CH}(m^*), \hat{R}_S^{CH}(m'^*)) : R^{CH}(Q_2^*[1]) \rightarrow R^{CH}(Q_1^*)$$

is an homotopy in $C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ using definition 41 (iii). In particular if $m : Q_1^* \rightarrow Q_2^*$ with $Q_1^*, Q_2^* \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}^*$ and $U_{2,\alpha}^*$ smooth is an homotopy equivalence, then $\hat{R}_S^{CH}(m) : \hat{R}^{CH}(Q_2^*) \rightarrow \hat{R}^{CH}(Q_1^*)$ is an homotopy equivalence.

- Let $S \in \text{SmVar}(\mathbb{C})$. Let $F \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S)$. Consider

$$q : LF := (\cdots \rightarrow \oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \rightarrow \cdots) \rightarrow F$$

the canonical projective resolution given in subsection 2.3.3. Note that the U_α are smooth since S is smooth and h_α are smooth morphism. Definition 41(iv) gives in this particular case the map in $C(\text{Var}(\mathbb{C})^2/S)$

$$\begin{aligned} T(\hat{R}_S^{CH}, R_S^{CH})(\rho_S^* LF) : \hat{R}^{CH}(\rho_S^* LF) &:= (\cdots \rightarrow \oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} \hat{R}_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \\ &\xrightarrow{(\hat{R}_S^{CH}(g_{\alpha,\beta}^n))} \oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} \hat{R}_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \rightarrow \cdots) \rightarrow L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^* LF), \end{aligned}$$

where for $(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \rightarrow \bar{S}$ of $h_\alpha : U_\alpha \rightarrow S$ with $\bar{X}_\alpha \in \text{PSmVar}(\mathbb{C})$ and $\bar{D}_\alpha := \bar{X}_\alpha \setminus U_\alpha$ a normal crossing divisor. Definition 41(iv) gives then by functoriality in particular, for $F = F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the map in $C(\text{Var}(\mathbb{C})^{2,smpr}/S)$

$$T(\hat{R}_S^{CH}, R_S^{CH})(\rho_S^* LF) : \hat{R}^{CH}(\rho_S^* LF) \rightarrow L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^* LF).$$

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $h : U \rightarrow S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{array}{ccc} U_T & \xrightarrow{h'} & T \\ \downarrow g' & & \downarrow g \\ U & \xrightarrow{h} & S \end{array}$$

Note that U is smooth since S and h are smooth, and U_T is smooth since T and h' are smooth. Take, see definition-proposition 12(ii), a compactification $f_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}$ of $h : U \rightarrow S$ and a compactification $f'_0 = g \circ h' : \bar{X}'_0 \rightarrow \bar{S}$ of $g \circ h' : U' \rightarrow S$ such that $g' : U_T/S \rightarrow U/S$ extend to a morphism $\bar{g}'_0 : \bar{X}'_0/\bar{S} \rightarrow \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \setminus U$ and $\bar{Z}' = \bar{X}'_0 \setminus U_T$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{e} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of (\bar{X}_0, \bar{Z}) , a desingularization

$\bar{\epsilon}'_{\bullet} : (\bar{X}', \bar{D}') \rightarrow (\bar{X}'_0, \bar{Z}')$ of (\bar{X}'_0, \bar{Z}') and a morphism $\bar{g}' : \bar{X}' \rightarrow \bar{X}$ such that the following diagram commutes

$$\begin{array}{ccc} \bar{X}'_0 & \xrightarrow{\bar{g}'_0} & \bar{X}_0 \\ \bar{\epsilon}' \uparrow & & \uparrow \bar{\epsilon} \\ \bar{X}' & \xrightarrow{\bar{g}'} & \bar{X} \end{array}$$

We then have, see definition-proposition 12(ii), the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} = \bar{X}_{c(\bullet)} & \xleftarrow{i_{\bullet}} & \bar{D}_{s_{g'}(\bullet)} \\ \uparrow g' & & \uparrow \bar{g}' & & \uparrow (\bar{g}')_{\bullet} \\ U_T = U_{T,c(\bullet)} & \xrightarrow{j'} & \bar{X}' = X'_{c(\bullet)} & \xleftarrow{i'_{\bullet}} & \bar{D}'_{\bullet} \xleftarrow{i''_{g'_{\bullet}}^{-1}} (\bar{D}_{s_{g'}(\bullet)}) : i'_{g_{\bullet}} \end{array}$$

We then consider the following map in $C(\text{Var}(\mathbb{C})^{2,pr}/T)$,

$$\begin{aligned} & T(g, \hat{R}^{CH})(\mathbb{Z}(U/S)) : g^* \hat{R}_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) \\ \xrightarrow{=} & g^* \text{Cone}(\mathbb{Z}(i_{\bullet} \times I) : (\mathbb{Z}^{tr}((\bar{D}_{\bullet} \times S, D_{\bullet})/S), u_{IJ}) \rightarrow \mathbb{Z}^{tr}((\bar{X} \times S, X)/S))(-d_X)[-2d_X] \\ & \xrightarrow{T(g,L)(-)\circ T(g,c)(-)} \\ \text{Cone}(\mathbb{Z}(i'_{g_{\bullet}} \times I) : (\mathbb{Z}^{tr}((\bar{D}_{\bullet} \times T, \bar{g}^{-1}(D_{s_g(\bullet)})/T), u_{IJ}) \rightarrow \mathbb{Z}^{tr}((\bar{X} \times T, X')/T))(-d_X)[-2d_X] \\ & \xrightarrow{([\Gamma_{\bar{g}'_{\bullet}}]^\dagger, [\Gamma_{\bar{g}}]^\dagger)} \\ & \text{Cone}(\mathbb{Z}(i'_{g_{\bullet}} \times I) : \\ & (\mathbb{Z}^{tr}((\bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) \times T, \bar{g}^{-1}(D_{s_g(\bullet)})/T), u_{IJ}) \rightarrow \mathbb{Z}^{tr}((\bar{X}' \times T, X')/T))(-d_{X'})[-2d_{X'}] \\ & \xrightarrow{(\mathbb{Z}(i''_{g_{\bullet}} \times I), I)(-d_{X'})[-2d_{X'}]} \\ \text{Cone}(\mathbb{Z}(i'_{\bullet} \times I) : ((\mathbb{Z}^{tr}((\bar{D}'_{\bullet} \times T, D'_{\bullet})/T), u_{IJ}) \rightarrow \mathbb{Z}^{tr}((\bar{X}' \times S, X')/T))(-d_{X'})[-2d_{X'}] \\ & \xrightarrow{=} \hat{R}_{(\bar{X}', \bar{D}')/T}(\mathbb{Z}(U_T/T)) \end{aligned}$$

For

$$Q^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha}^n, \beta))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{\beta}^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with $h_{\alpha}^n : U_{\alpha}^n \rightarrow S$ smooth, we get the map in $C(\text{Var}(\mathbb{C})^{2,smpr}/T)$

$$\begin{aligned} & T(g, \hat{R}^{CH})(Q^*) : g^* \hat{R}^{CH}(Q^*) = (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}_{\alpha}^n, \bar{D}_{\alpha}^n)/S} g^* \hat{R}_{(\bar{X}_{\alpha}^n, \bar{D}_{\alpha}^n)/S}(\mathbb{Z}(U_{\alpha}^n/S)) \rightarrow \cdots) \\ & \xrightarrow{(T(g, \hat{R}^{CH})(\mathbb{Z}(U_{\alpha}^n/S)))} (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}_{\alpha}^n, \bar{D}_{\alpha}^n)/T} \hat{R}_{(\bar{X}_{\alpha}^n, \bar{D}_{\alpha}^n)/T}(\mathbb{Z}(U_{\alpha}^n/S)) \rightarrow \cdots) =: \hat{R}^{CH}(g^*Q^*) \end{aligned}$$

together with the commutative diagram in $C(\text{Var}(\mathbb{C})^{2,smpr}/T)$

$$\begin{array}{ccc} g^* \hat{R}^{CH}(Q^*) & \xrightarrow{T(g, \hat{R}^{CH})(Q^*)} & \hat{R}^{CH}(g^*Q^*) \\ \downarrow g^* T(\hat{R}_S^{CH}, R_S^{CH})(Q^*) & & \downarrow T(\hat{R}_T^{CH}, R_T^{CH})(g^*Q) \\ g^* L_{\rho S^*} \mu_{S^*} \hat{R}^{CH}(Q^*) & \xrightarrow{T(g, \hat{R}^{CH})(Q^*) \circ T(g, \mu)(-) \circ T(g, \rho)(-) \circ T(g, L)(-)} & L_{\rho T^*} \mu_{T^*} \hat{R}^{CH}(g^*Q^*) \end{array}$$

Let $F \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S)$. Consider

$$q : LF := (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \rightarrow \cdots) \rightarrow F$$

the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in $C(\text{Var}(\mathbb{C})^{2,smpr}/T)$

$$\begin{aligned} T(g, \hat{R}^{CH})(\rho_S^* LF) : g^* \hat{R}^{CH}(\rho_S^* LF) = \\ (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} g^* \hat{R}_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \rightarrow \cdots) \xrightarrow{(T(g, \hat{R}^{CH}))(\mathbb{Z}(U_\alpha/S))} \\ (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}'_\alpha, \bar{D}'_\alpha)/T} \hat{R}_{(\bar{X}'_\alpha, \bar{D}'_\alpha)/T}(\mathbb{Z}(U_{\alpha,T}/S)) \rightarrow \cdots) =: \hat{R}^{CH}(\rho_T^* g^* LF), \end{aligned}$$

and by functoriality, we get in particular for $F = F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the map in $C(\text{Var}(\mathbb{C})^{2,smpr}/T)$

$$T(g, \hat{R}^{CH})(\rho_S^* LF) : g^* \hat{R}^{CH}(\rho_S^* LF) \rightarrow \hat{R}^{CH}(\rho_T^* g^* LF)$$

together with the commutative diagram in $C(\text{Var}(\mathbb{C})^{2,smpr}/T)$

$$\begin{array}{ccc} g^* \hat{R}^{CH}(\rho_S^* LF) & \xrightarrow{T(g, \hat{R}^{CH})(\rho_S^* LF)} & \hat{R}^{CH}(\rho_T^* g^* LF) \\ \downarrow g^* T(\hat{R}_S^{CH}, R_S^{CH})(\rho_S^* LF) & & \downarrow T(\hat{R}_T^{CH}, R_T^{CH})(\rho_T^* g^* LF) \\ g^* L\rho_{S*} \mu_{S*} \hat{R}^{CH}(\rho_S^* LF) & \xrightarrow{L\rho_{T*} \mu_{T*} T(g, \hat{R}^{CH})(\rho_S^* LF) \circ T(g, \mu)(-) \circ T(g, \rho)(-) \circ T(g, L)} & L\rho_{T*} \mu_{T*} \hat{R}^{CH}(\rho_T^* g^* LF) \end{array}$$

- Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ and $p : S_1 \times S_2 \rightarrow S_1$ the projection. Let $h : U \rightarrow S_1$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{array}{ccc} U \times S_2 & \xrightarrow{h \times I} & S_1 \times S_2 \\ \downarrow p' & & \downarrow p \\ U & \xrightarrow{h} & S_1 \end{array}$$

Take, see definition-proposition 12(i), a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}_1$ of $h : U \rightarrow S_1$. Then $\bar{f}_0 \times I : \bar{X}_0 \times S_2 \rightarrow \bar{S}_1 \times S_2$ is a compactification of $h \times I : U \times S_2 \rightarrow S_1 \times S_2$ and $p' : U \times S_2 \rightarrow U$ extend to $\bar{p}'_0 := p_{X_0} : \bar{X}_0 \times S_2 \rightarrow \bar{X}_0$. Denote $Z = X_0 \setminus U$. Take see theorem 15(i), a strict desingularization $\bar{e} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) . We then have the commutative diagram (48) in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$ whose squares are cartesian

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} & \xleftarrow{i_\bullet} & \bar{D}_\bullet \\ \uparrow g & & \uparrow \bar{p}' := p_{\bar{X}} & & \uparrow \bar{p}'_\bullet \\ U \times S_2 = (U \times S_2)_{c(\bullet)} & \xrightarrow{j \times I} & \bar{X} \times S_2 & \xleftarrow{i'_\bullet} & \bar{D}_\bullet \times S_2 \end{array}$$

Then the map in $C(\text{Var}(\mathbb{C})^{2,smpr}/S_1 \times S_2)$

$$T(p, \hat{R}^{CH})(\mathbb{Z}(U/S_1)) : p^* \hat{R}_{(\bar{X}, \bar{D})/S_1}(\mathbb{Z}(U/S_1)) \xrightarrow{\sim} \hat{R}_{(\bar{X} \times S_2, \bar{D}_\bullet \times S_2)/S_1 \times S_2}(\mathbb{Z}(U \times S_2/S_1 \times S_2))$$

is an isomorphism. Hence, for $Q^* \in C(\text{Var}(\mathbb{C})/S_1)$ a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in $C(\text{Var}(\mathbb{C})^{2,smpr}/S_1 \times S_2)$

$$T(p, \hat{R}^{CH})(Q^*) : p^* \hat{R}^{CH}(Q^*) \xrightarrow{\sim} \hat{R}^{CH}(p^* Q^*)$$

is an isomorphism. In particular, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S_1)$ the map in $C(\text{Var}(\mathbb{C})^{2,smpr}/S_1 \times S_2)$

$$T(p, \hat{R}^{CH})(\rho_{S_1}^* LF) : p^* \hat{R}^{CH}(\rho_{S_1}^* LF) \xrightarrow{\sim} \hat{R}^{CH}(\rho_{S_1 \times S_2}^* p^* LF)$$

is an isomorphism.

- Let $h_1 : U_1 \rightarrow S$, $h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2, S \in \text{Var}(\mathbb{C})$, U_1, U_2 smooth. Denote by $p_1 : U_1 \times_S U_2 \rightarrow U_1$ and $p_2 : U_1 \times_S U_2 \rightarrow U_2$ the projections. Take, see definition-proposition 12(i)), a compactification $\bar{f}_{10} = \bar{h}_1 : \bar{X}_{10} \rightarrow \bar{S}$ of $h_1 : U_1 \rightarrow S$ and a compactification $\bar{f}_{20} = \bar{h}_2 : \bar{X}_{20} \rightarrow \bar{S}$ of $h_2 : U_2 \rightarrow S$. Then,

- $\bar{f}_{10} \times \bar{f}_{20} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \rightarrow \bar{S}$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \rightarrow S$.
- $\bar{p}_{10} := p_{X_{10}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \rightarrow \bar{X}_{10}$ is a compactification of $p_1 : U_1 \times_S U_2 \rightarrow U_1$.
- $\bar{p}_{20} := p_{X_{20}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \rightarrow \bar{X}_{20}$ is a compactification of $p_2 : U_1 \times_S U_2 \rightarrow U_2$.

Denote $\bar{Z}_1 = \bar{X}_{10} \setminus U_1$ and $\bar{Z}_2 = \bar{X}_{20} \setminus U_2$. Take, see theorem 15(i), a strict desingularization $\bar{\epsilon}_1 : (\bar{X}_1, \bar{D}) \rightarrow (\bar{X}_{10}, \bar{Z}_1)$ of the pair $(\bar{X}_{10}, \bar{Z}_1)$ and a strict desingularization $\bar{\epsilon}_2 : (\bar{X}_2, \bar{E}) \rightarrow (\bar{X}_{20}, \bar{Z}_2)$ of the pair $(\bar{X}_{20}, \bar{Z}_2)$. Take then a strict desingularization

$$\bar{\epsilon}_{12} : ((\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F}) \rightarrow (\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (D \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$$

of the pair $(\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (D \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$. We have then the following commutative diagram

$$\begin{array}{ccccc}
 & & \bar{X}_1 & \xrightarrow{\bar{f}_1} & \bar{S} \\
 & & \uparrow \bar{p}_2 & & \uparrow \bar{f}_2 \\
 & & \bar{X}_1 \times_{\bar{S}} \bar{X}_2 & \xrightarrow{\bar{p}_1} & \bar{X}_2 \\
 & \nearrow (\bar{p}_2)^N & & & \\
 (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N & \xrightarrow{\bar{\epsilon}_{12}} & & & \\
 & \nearrow (\bar{p}_1)^N & & &
 \end{array}$$

and

- $\bar{f}_1 \times \bar{f}_2 : \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \rightarrow \bar{S}$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \rightarrow S$.
- $(\bar{p}_1)^N := \bar{p}_1 \circ \bar{\epsilon}_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \rightarrow \bar{X}_1$ is a compactification of $p_1 : U_1 \times_S U_2 \rightarrow U_1$.
- $(\bar{p}_2)^N := \bar{p}_2 \circ \bar{\epsilon}_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \rightarrow \bar{X}_2$ is a compactification of $p_2 : U_1 \times_S U_2 \rightarrow U_2$.

We have then the morphism in $C(\text{Var}(\mathbb{C})^{2,smpr}/S)$

$$T(\otimes, \hat{R}_S^{CH})(\mathbb{Z}(U_1/S), \mathbb{Z}(U_2/S)) := \hat{R}_S^{CH}(p_1) \otimes \hat{R}_S^{CH}(p_2) : \hat{R}_{(\bar{X}_1, \bar{D})/S}(\mathbb{Z}(U_1/S)) \otimes \hat{R}_{(\bar{X}_2, \bar{E})/S}(\mathbb{Z}(U_2/S)) \xrightarrow{\sim} \hat{R}_{(\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F}/S}(\mathbb{Z}(U_1 \times_S U_2/S))$$

For

$$\begin{aligned}
 Q_1^* &:= (\cdots \rightarrow \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{1,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{1,\beta}^{n-1}/S) \rightarrow \cdots), \\
 Q_2^* &:= (\cdots \rightarrow \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{2,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{2,\beta}^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)
 \end{aligned}$$

complexes of (maybe infinite) direct sum of representable presheaves with U_α^* smooth, we get the morphism in $C(\text{Var}(\mathbb{C})^{2,smpr}/S)$

$$T(\otimes, \hat{R}_S^{CH})(Q_1^*, Q_2^*) : \hat{R}^{CH}(Q_1^*) \otimes R^{CH}(Q_2^*) \xrightarrow{(T(\otimes, \hat{R}_S^{CH})(\mathbb{Z}(U_{1,\alpha}^m), \mathbb{Z}(U_{2,\beta}^n)))} \hat{R}^{CH}(Q_1^* \otimes Q_2^*)$$

, together with the commutative diagram in $C(\text{Var}(\mathbb{C})^{2,smpr}/S)$

$$\begin{array}{ccc} \hat{R}^{CH}(Q_1^*) \otimes R^{CH}(Q_2^*) & \xrightarrow{T(\otimes, \hat{R}_S^{CH})(Q_1^*, Q_2^*)} & \hat{R}^{CH}(Q_1^* \times Q_2^*) \\ \downarrow T(\hat{R}_S^{CH}, R_S^{CH})(Q_1^*) \otimes T(\hat{R}_S^{CH}, R_S^{CH})(Q_2^*) & & \downarrow T(\hat{R}_S^{CH}, R_S^{CH})(Q_1^* \otimes Q_2^*) \\ L\rho_{S*}\mu_{S*}(R^{CH}(Q_1^*) \otimes R^{CH}(Q_2^*)) & \xrightarrow{L\rho_{S*}\mu_{S*}T(\otimes, R_S^{CH})(Q_1^*, Q_2^*)} & L\rho_{S*}\mu_{S*}R^{CH}(Q_1^* \otimes Q_2^*) \end{array}$$

For $F_1, F_2 \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we get in particular the morphism in $C(\text{Var}(\mathbb{C})^2/S)$

$$T(\otimes, R_S^{CH})(\rho_S^*LF_1, \rho_S^*LF_2) : R^{CH}(\rho_S^*LF_1) \otimes R^{CH}(\rho_S^*LF_2) \rightarrow R^{CH}(\rho_S^*(LF_1 \otimes LF_2))$$

together with the commutative diagram in $C(\text{Var}(\mathbb{C})^{2,smpr}/S)$

$$\begin{array}{ccc} \hat{R}^{CH}(\rho_S^*LF_1) \otimes R^{CH}(\rho_S^*LF_2) & \xrightarrow{T(\otimes, \hat{R}_S^{CH})(\rho_S^*LF_1, \rho_S^*LF_2)} & \hat{R}^{CH}(\rho_S^*LF_1 \times \rho_S^*LF_2) \\ \downarrow T(\hat{R}_S^{CH}, R_S^{CH})(\rho_S^*LF_1) \otimes T(\hat{R}_S^{CH}, R_S^{CH})(\rho_S^*LF_2) & & \downarrow T(\hat{R}_S^{CH}, R_S^{CH})(\rho_S^*LF_1 \otimes \rho_S^*LF_2) \\ L\rho_{S*}\mu_{S*}(R^{CH}(\rho_S^*LF_1) \otimes R^{CH}(\rho_S^*LF_2)) & \xrightarrow{L\rho_{S*}\mu_{S*}T(\otimes, R_S^{CH})(\rho_S^*LF_1, \rho_S^*LF_2)} & L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*(LF_1 \times LF_2)) \end{array}$$

For $S \in \text{Var}(\mathbb{C})$, we will use rather the functors R_S^{0CH} and \hat{R}_S^{0CH} since we are working in the image of the graph functor $\text{Gr}_S^{12} : \text{Var}(\mathbb{C})/S \rightarrow \text{Var}(\mathbb{C})^2/S$. We have the full subcategory $\text{SmVar}(\mathbb{C})/S \subset \text{Var}(\mathbb{C})/S$ whose objects are morphisms $f : X \rightarrow S$ with $X \in \text{SmVar}(\mathbb{C})$. Then $\text{Gr}_S^{12}(\text{SmVar}(\mathbb{C})/S) \subset \text{Var}(\mathbb{C})^{2,smpr}/S$. If $S \in \text{SmVar}(\mathbb{C})$, we have the factorization of morphism of site

$$\text{Gr}_S^{12} : \text{Var}(\mathbb{C})^{2,smpr}/S \xrightarrow{\text{Gr}_S^{12}} \text{SmVar}(\mathbb{C})/S \xrightarrow{\rho_S} \text{Var}(\mathbb{C})^{sm}/S.$$

Definition 42. (i) Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$ and U smooth. Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) , with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \epsilon^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. We denote by $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X} = \bar{X}_{c(\bullet)}$, the morphism of simplicial varieties given by the closed embeddings $i_I : \bar{D}_I = \cap_{i \in I} \bar{D}_i \hookrightarrow \bar{X}$. We denote by $j : U \hookrightarrow \bar{X}$ the open embedding. We then consider the following map in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned} & r_{(\bar{X}, \bar{D})/S}^0(\mathbb{Z}(U/S)) : R_{(\bar{X}, \bar{D})/S}^0(\mathbb{Z}(U/S)) \\ & \xrightarrow{=} \bar{f}_* E_{et}(\text{Cone}(\mathbb{Z}(i_\bullet) : (\mathbb{Z}((\bar{D}_\bullet)/\bar{X}), u_{IJ}) \rightarrow \mathbb{Z}((\bar{X}/\bar{X})))) \\ & \xrightarrow{\bar{f}_* E_{et}(0, koad(j^*, j_*)(\mathbb{Z}(\bar{X}/\bar{X})))} h_* E_{et}(\mathbb{Z}(U/U)) =: \mathbb{D}_S^0(\mathbb{Z}(U/S)). \end{aligned}$$

Note that $\mathbb{Z}(\bar{D}_I/\bar{X})$ and $\mathbb{Z}(\bar{X}/\bar{X})$ are obviously \mathbb{A}^1 invariant. Note that $r_{(\bar{X}, \bar{D})/S}$ is NOT an equivalence (\mathbb{A}^1, et) local by proposition 19 since $\rho_{\bar{X}*}\mathbb{Z}(\bar{D}_\bullet/\bar{X}) = 0$, and $\rho_{\bar{X}*} \text{ad}(j^*, j_*)(\mathbb{Z}(\bar{X}/\bar{X}))$ is not an equivalence (\mathbb{A}^1, et) local.

(ii) Let $g : U'/S \rightarrow U/S$ a morphism, with $U'/S = (U', h'), U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, with U and U' smooth. Take, see definition-proposition 12(ii), a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}$ of $h : U \rightarrow S$ and a compactification $\bar{f}'_0 = \bar{h}' : \bar{X}'_0 \rightarrow \bar{S}$ of $h' : U' \rightarrow S$ such that $g : U'/S \rightarrow U/S$ extend to a morphism $\bar{g}_0 : \bar{X}'_0/\bar{S} \rightarrow \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \setminus U$ and $\bar{Z}' = \bar{X}'_0 \setminus U'$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of (\bar{X}_0, \bar{Z}) , a strict desingularization $\bar{\epsilon}' : (\bar{X}', \bar{D}') \rightarrow (\bar{X}'_0, \bar{Z}')$ of (\bar{X}'_0, \bar{Z}') and a morphism $\bar{g} : \bar{X}' \rightarrow \bar{X}$ such that the following diagram commutes

$$\begin{array}{ccc} \bar{X}'_0 & \xrightarrow{\bar{g}_0} & \bar{X}_0 \\ \bar{\epsilon}' \uparrow & & \uparrow \bar{\epsilon} \\ \bar{X}' & \xrightarrow{\bar{g}} & \bar{X} \end{array}$$

We then have, see definition-proposition 12(ii), the commutative diagram (47) in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} = \bar{X}_{c(\bullet)} & \xleftarrow{i_\bullet} & \bar{D}_{s_g(\bullet)} \\ \uparrow g & & \uparrow \bar{g} & & \uparrow \bar{g}' \\ U' = U'_{c(\bullet)} & \xrightarrow{j'} & \bar{X}' = \bar{X}'_{c(\bullet)} & \xleftarrow{i'_\bullet} & \bar{D}'_\bullet \xleftarrow{i''_\bullet \bar{g}^{-1}} \bar{D}_{s_g(\bullet)} : i'_{g\bullet} \end{array}$$

We then consider the following map in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned} R_S^{0CH}(g) : R_{(\bar{X}, \bar{D})/S}^0(\mathbb{Z}(U/S)) &\xrightarrow{\cong} \\ \bar{f}_* E_{et}(\text{Cone}(\mathbb{Z}(i_\bullet) : (\mathbb{Z}((\bar{D}_{s_g(\bullet)})/\bar{X}), u_{IJ}) \rightarrow \mathbb{Z}(\bar{X}/\bar{X}))) &\xrightarrow{T(\bar{g}, E)(-) \circ p_{S*} \text{ad}(\bar{g}^*, \bar{g}_*)(-)} \\ \bar{f}'_* E_{et}(\text{Cone}(\mathbb{Z}(i'_{g\bullet}) : (\mathbb{Z}((\bar{g}^{-1}(\bar{D}_{s_g(\bullet)})/\bar{X}'), u_{IJ}) \rightarrow \mathbb{Z}(\bar{X}'/\bar{X}')))) &\xrightarrow{\bar{f}'_* E_{et}(\mathbb{Z}(i''_{g\bullet}, I, I)} \\ \bar{f}'_* E_{et}(\text{Cone}(\mathbb{Z}(i'_\bullet) : (\mathbb{Z}(\bar{D}'_\bullet/\bar{X}'), u_{IJ}) \rightarrow \mathbb{Z}(\bar{X}'/\bar{X}')) &\xrightarrow{\cong} \\ R_{(\bar{X}', \bar{D}')/S}^0(\mathbb{Z}(U'/S)) \end{aligned}$$

Then by the diagram (47) and adjunction, the following diagram in $C(\text{Var}(\mathbb{C})/S)$ obviously commutes

$$\begin{array}{ccc} R_{(\bar{X}, \bar{D})/S}^0(\mathbb{Z}(U/S)) & \xrightarrow{r_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S))} & h_* E_{et}(\mathbb{Z}(U/U)) =: \mathbb{D}_S^0(\mathbb{Z}(U/S)) \\ \downarrow R_S^{0CH}(g) & & \downarrow D_S(g) := T(g, E)(-) \circ \text{ad}(g^*, g_*)(E_{et}(\mathbb{Z}(U/U))) \\ R_{(\bar{X}', \bar{D}')/S}^0(\mathbb{Z}(U'/S)) & \xrightarrow{r_{(\bar{X}', \bar{D}')/S}(\mathbb{Z}(U'/S))} & h'_* E_{et}(\mathbb{Z}(U'/U')) =: \mathbb{D}_S^0(\mathbb{Z}(U'/S)) \end{array}$$

(iii) For $g_1 : U''/S \rightarrow U'/S$, $g_2 : U'/S \rightarrow U/S$ two morphisms with $U''/S = (U', h''), U'/S = (U', h')$, $U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, with U , U' and U'' smooth. We get from (i) and (ii) a compactification $\bar{f} = \bar{h} : \bar{X} \rightarrow \bar{S}$ of $h : U \rightarrow S$, a compactification $\bar{f}' = \bar{h}' : \bar{X}' \rightarrow \bar{S}$ of $h' : U' \rightarrow S$, and a compactification $\bar{f}'' = \bar{h}'' : \bar{X}'' \rightarrow \bar{S}$ of $h'' : U'' \rightarrow S$, with $\bar{X}, \bar{X}', \bar{X}'' \in \text{PSmVar}(\mathbb{C})$, $\bar{D} := \bar{X} \setminus U \subset \bar{X}$, $\bar{D}' := \bar{X}' \setminus U' \subset \bar{X}'$, and $\bar{D}'' := \bar{X}'' \setminus U'' \subset \bar{X}''$ normal crossing divisors, such that $g_1 : U''/S \rightarrow U'/S$ extend to $\bar{g}_1 : \bar{X}''/\bar{S} \rightarrow \bar{X}'/\bar{S}$, $g_2 : U'/S \rightarrow U/S$ extend to $\bar{g}_2 : \bar{X}'/\bar{S} \rightarrow \bar{X}/\bar{S}$, and

$$R_S^{0CH}(g_2 \circ g_1) = R_S^{0CH}(g_1) \circ R_S^{0CH}(g_2) : R_{(\bar{X}, \bar{D})/S}^0 \rightarrow R_{(\bar{X}'', \bar{D}'')/S}^0$$

(iv) For

$$Q^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_\alpha^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha, \beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_\beta^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U_α^n smooth, we get from (i), (ii) and (iii) the map in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned} r_S^{0CH}(Q^*) : R^{0CH}(Q^*) &:= (\cdots \rightarrow \bigoplus_{\beta \in \Lambda^{n-1}} \varinjlim_{(\bar{X}_\beta^{n-1}, \bar{D}_\beta^{n-1})/S} R_{(\bar{X}_\beta^{n-1}, \bar{D}_\beta^{n-1})/S}^0(\mathbb{Z}(U_\beta^{n-1}/S)) \\ &\xrightarrow{(R_S^{0CH}(g_{\alpha, \beta}^n))} \bigoplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S} R_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S}^0(\mathbb{Z}(U_\alpha^n/S)) \rightarrow \cdots) \rightarrow \mathbb{D}_S(Q^*), \end{aligned}$$

where for $(U_\alpha^n, h_\alpha^n) \in \text{Var}(\mathbb{C})/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \rightarrow \bar{S}$ of $h_\alpha : U_\alpha \rightarrow S$ with $\bar{X}_\alpha \in \text{PSmVar}(\mathbb{C})$ and $\bar{D}_\alpha := \bar{X}_\alpha \setminus U_\alpha$ a normal crossing divisor. For $m = (m^*) : Q_1^* \rightarrow Q_2^*$ a morphism with

$$Q_1^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{1,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{1,\beta}^{n-1}/S) \rightarrow \cdots),$$

$$Q_2^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{2,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{2,\beta}^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}^*$ and $U_{2,\alpha}^*$ smooth, we get again from (i), (ii) and (iii) a commutative diagram in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{array}{ccc} R_S^{0CH}(Q_2^*) & \xrightarrow{r_S^{0CH}(Q_2^*)} & \mathbb{D}_S^0(Q_2^*) \\ \downarrow R_S^{0CH}(m) := (R_S^{0CH}(m^*)) & & \downarrow \mathbb{D}_S(m) := (\mathbb{D}_S^0(m^*)) \\ R_S^{0CH}(Q_1^*) & \xrightarrow{r_S^{0CH}(Q_1^*)} & \mathbb{D}_S^0(Q_1^*) \end{array} .$$

(v) Let

$$Q^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_\alpha^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_\beta^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U_α^* smooth, we have by definition

$$\text{Gr}_S^{12*} R_S^{0CH}(Q^*) = R_S^{CH}(Q^*) \in C(\text{Var}(\mathbb{C})^2/S).$$

- Let $S \in \text{Var}(\mathbb{C})$ For $(h, m, m') = (h^*, m^*, m'^*) : Q_1^*[1] \rightarrow Q_2^*$ an homotopy with $Q_1^*, Q_2^* \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}^*$ and $U_{2,\alpha}^*$ smooth,

$$(R_S^{0CH}(h), R_S^{0CH}(m), R_S^{0CH}(m')) = (R_S^{0CH}(h^*), R_S^{0CH}(m^*), R_S^{0CH}(m'^*)) : R_S^{0CH}(Q_2^*)[1] \rightarrow R_S^{0CH}(Q_1^*)$$

is an homotopy in $C(\text{Var}(\mathbb{C})/S)$ using definition 42 (iii). In particular if $m : Q_1^* \rightarrow Q_2^*$ with $Q_1^*, Q_2^* \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}^*$ and $U_{2,\alpha}^*$ smooth is an homotopy equivalence, then $R_S^{0CH}(m) : R_S^{0CH}(Q_2^*) \rightarrow R_S^{0CH}(Q_1^*)$ is an homotopy equivalence.

- Let $S \in \text{SmVar}(\mathbb{C})$. Let $F \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S)$. Consider

$$q : LF := (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \rightarrow \cdots) \rightarrow F$$

the canonical projective resolution given in subsection 2.3.3. Note that the U_α are smooth since S is smooth and h_α are smooth morphism. Definition 42(iv) gives in this particular case the map in $C(\text{Var}(\mathbb{C})/S)$

$$r_S^{0CH}(\rho_S^* LF) : R_S^{0CH}(\rho_S^* LF) := (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} R_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}^0(\mathbb{Z}(U_\alpha/S))$$

$$\xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} R_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}^0(\mathbb{Z}(U_\alpha/S)) \rightarrow \cdots) \rightarrow \mathbb{D}_S^0(\rho_S^* LF),$$

where for $(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \rightarrow \bar{S}$ of $h_\alpha : U_\alpha \rightarrow S$ with $\bar{X}_\alpha \in \text{PSmVar}(\mathbb{C})$ and $\bar{D}_\alpha := \bar{X}_\alpha \setminus U_\alpha$ a normal crossing divisor. Definition 42(iv) gives then by functoriality in particular, for $F = F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the map in $C(\text{Var}(\mathbb{C})/S)$

$$r_S^{0CH}(\rho_S^* LF) = (r_S^{0CH}(\rho_S^* LF^*)) : R_S^{0CH}(\rho_S^* LF) \rightarrow \mathbb{D}_S^0(\rho_S^* LF).$$

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $h : U \rightarrow S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{array}{ccc} U_T & \xrightarrow{h'} & T \\ \downarrow g' & & \downarrow g \\ U & \xrightarrow{h} & S \end{array}$$

Note that U is smooth since S and h are smooth, and U_T is smooth since T and h' are smooth. Take, see definition-proposition 12(ii), a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}$ of $h : U \rightarrow S$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{e} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of (\bar{X}_0, \bar{Z}) . Then $\bar{f}'_0 = g \circ \bar{h}' : \bar{X}_T \rightarrow \bar{T}$ is a compactification of $g \circ h' : U_T \rightarrow S$ such that $g' : U_T/S \rightarrow U/S$ extend to a morphism $\bar{g}'_0 : \bar{X}_T/\bar{S} \rightarrow \bar{X}/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \setminus U$ and $\bar{Z}' = \bar{X}_T \setminus U_T$. Take, see definition-proposition 12(ii), a strict desingularization $\epsilon'_\bullet : (\bar{X}', \bar{D}') \rightarrow (\bar{X}_T, \bar{Z}')$ of (\bar{X}_T, \bar{Z}') . Denote $\bar{g}' = \bar{g}'_0 \circ \epsilon'_\bullet : \bar{X}' \rightarrow \bar{X}$. We then have, see definition-proposition 12(ii), the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} = \bar{X}_{c(\bullet)} & \xleftarrow{i_\bullet} & \bar{D}_{s_{g'}(\bullet)} \\ \uparrow g' & & \uparrow \bar{g}'_0 & & \uparrow (g')_\bullet \\ U_T = U_{T,c(\bullet)} & \xrightarrow{j'} & \bar{X}_T = X_{T,c(\bullet)} & \xleftarrow{i_{g\bullet}} & \bar{g}'^{-1}(\bar{D}_{s_{g'}(\bullet)}) \\ \uparrow g' & & \uparrow \bar{g}' & & \uparrow \epsilon'_\bullet \\ U_T = U_{T,c(\bullet)} & \xrightarrow{j'} & \bar{X}' = X'_{c(\bullet)} & \xleftarrow{i'_\bullet} & \bar{D}' \xleftarrow{i''_{g'_\bullet}} \bar{g}'^{-1}(\bar{D}_{s_{g'}(\bullet)}) : i'_{g\bullet} \end{array}$$

We then consider the following map in $C(\text{Var}(\mathbb{C})/T)$, see definition 42(ii)

$$\begin{aligned} & T(g, R^{0CH})(\mathbb{Z}(U/S)) : g^* R^0_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S)) \\ & \xrightarrow{g^* R_S^{0CH}(g')} g^* R^0_{(\bar{X}', \bar{D}')/S}(\mathbb{Z}(U_T/S)) = g^* g_* R^0_{(\bar{X}', \bar{D}')/T}(\mathbb{Z}(U_T/T)) \\ & \xrightarrow{\text{ad}(g^*, g_*)(R^0_{(\bar{X}', \bar{D}')/T}(\mathbb{Z}(U_T/T)))} R^0_{(\bar{X}', \bar{D}')/T}(\mathbb{Z}(U_T/T)) \end{aligned}$$

For

$$Q^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_\alpha^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha, \beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_\beta^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with $h_\alpha^n : U_\alpha^n \rightarrow S$ smooth, we get the map in $C(\text{Var}(\mathbb{C})/T)$

$$\begin{aligned} & T(g, R^{0CH})(Q^*) : g^* R^{0CH}(Q^*) = (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S} g^* R^0_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S}(\mathbb{Z}(U_\alpha^n/S)) \rightarrow \cdots) \\ & \xrightarrow{(T(g, R^{0CH})(\mathbb{Z}(U_\alpha^n/S)))} (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}_\alpha^{n'}, \bar{D}_\alpha^{n'})/T} R^0_{(\bar{X}_\alpha^{n'}, \bar{D}_\alpha^{n'})/T}(\mathbb{Z}(U_\alpha^n/S)) \rightarrow \cdots) =: R^{CH}(g^* Q^*). \end{aligned}$$

Let $F \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S)$. Consider

$$q : LF := (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \rightarrow \cdots) \rightarrow F$$

the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in

$C(\text{Var}(\mathbb{C})/T)$

$$\begin{aligned} & T(g, R^{0CH})(\rho_S^* LF) : g^* R^{0CH}(\rho_S^* LF) = \\ & (\cdots \rightarrow \oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} g^* R_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}^0(\mathbb{Z}(U_\alpha/S)) \rightarrow \cdots) \xrightarrow{(T(g, R^{0CH})(\mathbb{Z}(U_\alpha/S)))} \\ & (\cdots \rightarrow \oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}'_\alpha, \bar{D}'_\alpha)/T} R_{(\bar{X}'_\alpha, \bar{D}'_\alpha)/T}^0(\mathbb{Z}(U_{\alpha,T}/S)) \rightarrow \cdots) =: R^{CH}(\rho_T^* g^* LF). \end{aligned}$$

By functoriality, we get in particular for $F = F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the map in $C(\text{Var}(\mathbb{C})/T)$

$$T(g, R^{0CH})(\rho_S^* LF) : g^* R^{0CH}(\rho_S^* LF) \rightarrow R^{0CH}(\rho_T^* g^* LF).$$

- Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ and $p : S_1 \times S_2 \rightarrow S_1$ the projection. Let $h : U \rightarrow S_1$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{array}{ccc} U \times S_2 & \xrightarrow{h \times I} & S_1 \times S_2 \\ \downarrow p' & & \downarrow p \\ U & \xrightarrow{h} & S_1 \end{array}$$

Take, see definition-proposition 12(i), a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}_1$ of $h : U \rightarrow S_1$. Then $\bar{f}_0 \times I : \bar{X}_0 \times S_2 \rightarrow \bar{S}_1 \times S_2$ is a compactification of $h \times I : U \times S_2 \rightarrow S_1 \times S_2$ and $p' : U \times S_2 \rightarrow U$ extend to $\bar{p}'_0 := p_{X_0} : \bar{X}_0 \times S_2 \rightarrow \bar{X}_0$. Denote $Z = X_0 \setminus U$. Take see theorem 15(i), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) . We then have the commutative diagram (48) in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$ whose squares are cartesian

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} & \xleftarrow{i_\bullet} & \bar{D}_\bullet \\ \uparrow g & & \uparrow \bar{p}' := p_{\bar{X}} & & \uparrow \bar{p}'_\bullet \\ U \times S_2 = (U \times S_2)_{c(\bullet)} & \xrightarrow{j \times I} & \bar{X} \times S_2 & \xleftarrow{i'_\bullet} & \bar{D}_\bullet \times S_2 \end{array}$$

Then the map in $C(\text{Var}(\mathbb{C})/S_1 \times S_2)$

$$T(p, R^{0CH})(\mathbb{Z}(U/S_1)) : p^* R_{(\bar{X}, \bar{D})/S_1}^0(\mathbb{Z}(U/S_1)) \xrightarrow{\sim} R_{(\bar{X} \times S_2, \bar{D}_\bullet \times S_2)/S_1 \times S_2}^0(\mathbb{Z}(U \times S_2/S_1 \times S_2))$$

is an isomorphism. Hence, for $Q^* \in C(\text{Var}(\mathbb{C})/S_1)$ a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in $C(\text{Var}(\mathbb{C})/S_1 \times S_2)$

$$T(p, R^{0CH})(Q^*) : p^* R^{0CH}(Q^*) \xrightarrow{\sim} R^{0CH}(p^* Q^*)$$

is an isomorphism. In particular, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S_1)$ the map in $C(\text{Var}(\mathbb{C})/S_1 \times S_2)$

$$T(p, R^{0CH})(\rho_{S_1}^* LF) : p^* R^{0CH}(\rho_{S_1}^* LF) \xrightarrow{\sim} R^{0CH}(\rho_{S_1 \times S_2}^* p^* LF)$$

is an isomorphism.

- Let $h_1 : U_1 \rightarrow S$, $h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2, S \in \text{Var}(\mathbb{C})$, U_1, U_2 smooth. Denote by $p_1 : U_1 \times_S U_2 \rightarrow U_1$ and $p_2 : U_1 \times_S U_2 \rightarrow U_2$ the projections. Take, see definition-proposition 12(i), a compactification $\bar{f}_{10} = \bar{h}_1 : \bar{X}_{10} \rightarrow \bar{S}$ of $h_1 : U_1 \rightarrow S$ and a compactification $\bar{f}_{20} = \bar{h}_2 : \bar{X}_{20} \rightarrow \bar{S}$ of $h_2 : U_2 \rightarrow S$. Then,

$$- \bar{f}_{10} \times \bar{f}_{20} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \rightarrow S \text{ is a compactification of } h_1 \times h_2 : U_1 \times_S U_2 \rightarrow S.$$

- $\bar{p}_{10} := p_{X_{10}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \rightarrow \bar{X}_{10}$ is a compactification of $p_1 : U_1 \times_S U_2 \rightarrow U_1$.
- $\bar{p}_{20} := p_{X_{20}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \rightarrow \bar{X}_{20}$ is a compactification of $p_2 : U_1 \times_S U_2 \rightarrow U_2$.

Denote $\bar{Z}_1 = \bar{X}_{10} \setminus U_1$ and $\bar{Z}_2 = \bar{X}_{20} \setminus U_2$. Take, see theorem 15(i), a strict desingularization $\bar{\epsilon}_1 : (\bar{X}_1, \bar{D}) \rightarrow (\bar{X}_{10}, \bar{Z}_1)$ of the pair $(\bar{X}_{10}, \bar{Z}_1)$ and a strict desingularization $\bar{\epsilon}_2 : (\bar{X}_2, \bar{E}) \rightarrow (\bar{X}_{20}, \bar{Z}_2)$ of the pair $(\bar{X}_{20}, \bar{Z}_2)$. Take then a strict desingularization

$$\bar{\epsilon}_{12} : ((\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F}) \rightarrow (\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (D \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$$

of the pair $(\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (D \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$. We have then the following commutative diagram

$$\begin{array}{ccc}
& & \bar{X}_1 \xrightarrow{\bar{f}_1} \bar{S} \\
& & \uparrow \bar{p}_2 \quad \quad \quad \uparrow \bar{f}_2 \\
& & \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \xrightarrow{\bar{p}_1} \bar{X}_2 \\
& \nearrow (\bar{p}_2)^N & \nearrow (\bar{p}_1)^N \\
& & \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \\
& \nearrow \bar{\epsilon}_{12} & \nearrow (\bar{p}_1)^N \\
& & (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N
\end{array}$$

and

- $\bar{f}_1 \times \bar{f}_2 : \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \rightarrow \bar{S}$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \rightarrow S$.
- $(\bar{p}_1)^N := \bar{p}_1 \circ \bar{\epsilon}_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \rightarrow \bar{X}_1$ is a compactification of $p_1 : U_1 \times_S U_2 \rightarrow U_1$.
- $(\bar{p}_2)^N := \bar{p}_2 \circ \bar{\epsilon}_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \rightarrow \bar{X}_2$ is a compactification of $p_2 : U_1 \times_S U_2 \rightarrow U_2$.

We have then the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$T(\otimes, R_S^{0CH})(\mathbb{Z}(U_1/S), \mathbb{Z}(U_2/S)) := R_S^{0CH}(p_1) \otimes R_S^{0CH}(p_2) : R_{(\bar{X}_1, \bar{D})/S}^0(\mathbb{Z}(U_1/S)) \otimes R_{(\bar{X}_2, \bar{E})/S}^0(\mathbb{Z}(U_2/S)) \xrightarrow{\sim} R_{(\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F}}^0(\mathbb{Z}(U_1 \times_S U_2/S))$$

For

$$Q_1^* := (\cdots \rightarrow \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{1,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{1,\beta}^{n-1}/S) \rightarrow \cdots),$$

$$Q_2^* := (\cdots \rightarrow \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{2,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{2,\beta}^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

complexes of (maybe infinite) direct sum of representable presheaves with U_α^* smooth, we get the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$T(\otimes, R_S^{0CH})(Q_1^*, Q_2^*) : R^{0CH}(Q_1^*) \otimes R^{0CH}(Q_2^*) \xrightarrow{(T(\otimes, R_S^{0CH})(\mathbb{Z}(U_{1,\alpha}^m), \mathbb{Z}(U_{2,\beta}^n))} R^{0CH}(Q_1^* \otimes Q_2^*).$$

For $F_1, F_2 \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we get in particular the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$T(\otimes, R_S^{0CH})(\rho_S^* LF_1, \rho_S^* LF_2) : R^{0CH}(\rho_S^* LF_1) \otimes R^{0CH}(\rho_S^* LF_2) \rightarrow R^{0CH}(\rho_S^*(LF_1 \otimes LF_2)).$$

Definition 43. Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$, U irreducible. Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15, a desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) , $\bar{Z} \subset \bar{D}$, with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Denote $d_X := \dim(\bar{X}) = \dim(U)$.

(i) The diagonal $\Delta_{\bar{D}_\bullet} \subset \bar{D}_\bullet \times \bar{D}_\bullet$ induces the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned}
& [\Delta_{\bar{D}_\bullet}] \in \text{Hom}(\mathbb{Z}^{tr}(\bar{D}_\bullet/S), \bar{f}_* E_{et}(\mathbb{Z}(\bar{D}_\bullet/\bar{X})(d_X)[2d_X])) \xrightarrow{\sim} \\
& \text{Hom}(\mathbb{Z}(\bar{D}_\bullet \times_S \bar{X}/\bar{X}), \mathbb{Z}^{tr}(\bar{D}_\bullet \times \mathbb{P}^{d_X}/\bar{X})/\mathbb{Z}^{tr}(\bar{D}_\bullet \times \mathbb{P}^{d_X-1}/\bar{X})) \\
& \subset H^0(\mathcal{Z}_{d_{D_\bullet}}(\square^* \times \bar{D}_\bullet \times_S \bar{D}_\bullet))
\end{aligned}$$

(ii) The cycle $\bar{\Delta}_{\bar{X}} \subset \bar{X} \times_S \bar{X}$ induces by the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned} & [\bar{\Delta}_{\bar{X}}] \in \text{Hom}(\mathbb{Z}^{\text{tr}}(\bar{X}/S), \bar{f}_* E_{\text{et}}(\mathbb{Z}(\bar{X}/\bar{X})(d_X)[2d_X])) \xrightarrow{\sim} \\ & \text{Hom}(\mathbb{Z}(\bar{X} \times_S \bar{X}/\bar{X}), \mathbb{Z}^{\text{tr}}(\bar{X} \times \mathbb{P}^{d_X}/\bar{X})/\mathbb{Z}^{\text{tr}}(\bar{X} \times \mathbb{P}^{d_X-1}/\bar{X})) \\ & \subset H^0(\mathcal{Z}_{d_X}(\square^* \times \bar{X} \times_S \bar{X})) \end{aligned}$$

Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$, U smooth connected (hence irreducible by smoothness). Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Denote $d_X := \dim(\bar{X}) = \dim(U)$. We get from (i) and (ii) the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned} & T(\bar{f}_\#, \bar{f}_*)(\mathbb{Z}(D_\bullet/\bar{X}), \mathbb{Z}(\bar{X}/\bar{X})) := ([\bar{\Delta}_{\bar{D}_\bullet}], [\bar{\Delta}_{\bar{X}}]) : \\ & \text{Cone}(\mathbb{Z}(i_\bullet) : (\mathbb{Z}^{\text{tr}}(\bar{D}_\bullet/S), u_{IJ}) \rightarrow \mathbb{Z}^{\text{tr}}(\bar{X}/S)) \rightarrow \\ & \bar{f}_* E_{\text{et}}(\text{Cone}(\mathbb{Z}(i_\bullet) : (\mathbb{Z}(\bar{D}_\bullet/\bar{X}), u_{IJ}) \rightarrow \mathbb{Z}(\bar{X}/\bar{X}))(d_X)[2d_X]) \\ & =: R_{(\bar{X}, \bar{D})/S}^0(\mathbb{Z}(U/S))(d_X)[2d_X]. \end{aligned}$$

Definition 44. (i) Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$ and U smooth. Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) , with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. We denote by $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X} = \bar{X}_{c(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings $i_I : \bar{D}_I = \cap_{i \in I} \bar{D}_i \hookrightarrow \bar{X}$. We denote by $j : U \hookrightarrow \bar{X}$ the open embedding. We then consider the map in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned} & T(\hat{R}^{0CH}, R^{0CH})(\mathbb{Z}(U/S)) : \hat{R}_{(\bar{X}, \bar{D})/S}^0(\mathbb{Z}(U/S)) \\ & \xrightarrow{=} \text{Cone}(\mathbb{Z}(i_\bullet) : (\mathbb{Z}^{\text{tr}}(D_\bullet/S), u_{IJ}) \rightarrow \mathbb{Z}^{\text{tr}}(X/S))(-d_X)[-2d_X] \\ & \quad \xrightarrow{\quad T(\bar{f}_\#, \bar{f}_*)(\mathbb{Z}(\bar{D}_\bullet/\bar{X}), \mathbb{Z}(\bar{X}/\bar{X}))(-d_X)[-2d_X] \quad} \\ & \quad \quad R_{(\bar{X}, \bar{D})/S}^0(\mathbb{Z}(U/S)). \end{aligned}$$

given in definition 39(iii).

(ii) Let $g : U'/S \rightarrow U/S$ a morphism, with $U'/S = (U', h'), U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, with U and U' smooth. Take, see definition-proposition 12(ii), a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}$ of $h : U \rightarrow S$ and a compactification $\bar{f}'_0 = \bar{h}' : \bar{X}'_0 \rightarrow \bar{S}$ of $h' : U' \rightarrow S$ such that $g : U'/S \rightarrow U/S$ extend to a morphism $\bar{g}_0 : \bar{X}'_0/\bar{S} \rightarrow \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \setminus U$ and $\bar{Z}' = \bar{X}'_0 \setminus U'$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of (\bar{X}_0, \bar{Z}) , a strict desingularization $\bar{\epsilon}' : (\bar{X}', \bar{D}') \rightarrow (\bar{X}'_0, \bar{Z}')$ of (\bar{X}'_0, \bar{Z}') and a morphism $\bar{g} : \bar{X}' \rightarrow \bar{X}$ such that the following diagram commutes

$$\begin{array}{ccc} \bar{X}'_0 & \xrightarrow{\bar{g}_0} & \bar{X}_0 \\ \bar{\epsilon}' \uparrow & & \uparrow \bar{\epsilon} \\ \bar{X}' & \xrightarrow{\bar{g}} & \bar{X} \end{array}$$

We then have, see definition-proposition 12(ii), the diagram (47) in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccccc} U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} = \bar{X}_{c(\bullet)} & \xleftarrow{i_\bullet} & \bar{D}_{s_g(\bullet)} \\ \uparrow g & & \uparrow \bar{g} & & \uparrow \bar{g}' \\ U' = U'_{c(\bullet)} & \xrightarrow{j'} & \bar{X}' = \bar{X}'_{c(\bullet)} & \xleftarrow{i'_\bullet} & \bar{D}'_\bullet & \xleftarrow{i''_{\bar{g}'^{-1}}} & \bar{D}_{s_g(\bullet)} : i'_{g_\bullet} \end{array}$$

Consider

$$\begin{aligned} [\Gamma_{\bar{g}}]^t &\in \text{Hom}(\mathbb{Z}^{\text{tr}}(\bar{X}/S)(-d_X)[-2d_X], \mathbb{Z}^{\text{tr}}(\bar{X}'/S)(-d_{X'})[-2d_{X'}]) \\ &\xrightarrow{\sim} \text{Hom}(\mathbb{Z}^{\text{tr}}(\bar{X} \times \mathbb{P}^{d_X}/S)/\mathbb{Z}_{\text{tr}}(\bar{X} \times \mathbb{P}^{d_X-1}/S), \\ &\quad \mathbb{Z}_{\text{tr}}(\bar{X}' \times \mathbb{P}^{d_{X'}/S})/\mathbb{Z}_{\text{tr}}(\bar{X}' \times \mathbb{P}^{d_{X'}-1}/S) \end{aligned}$$

the morphism given by the transpose of the graph $\Gamma_g \subset X' \times_S X$ of $\bar{g} : \bar{X}' \rightarrow \bar{X}$. Then, since $i_{\bullet} \circ \bar{g}'_{\bullet} = \bar{g} \circ i'_{\bullet} \circ i''_{g_{\bullet}}$, we have the factorization

$$\begin{aligned} [\Gamma_g]^t \circ \mathbb{Z}(i_{\bullet}) &: (\mathbb{Z}^{\text{tr}}(\bar{D}_{s_g(\bullet)}/S), u_{IJ})(-d_X)[-2d_X] \\ &\xrightarrow{[\Gamma_{\bar{g}'_{\bullet}}]^t} (\mathbb{Z}^{\text{tr}}(\bar{g}^{-1}(\bar{D}_{s_g(\bullet)})/S), u_{IJ})(-d_{X'})[-2d_{X'}] \\ &\xrightarrow{\mathbb{Z}(i'_{g_{\bullet}})} \mathbb{Z}^{\text{tr}}(\bar{X}'/S)(-d_{X'})[-2d_{X'}]. \end{aligned}$$

with

$$\begin{aligned} [\Gamma_{\bar{g}'_{\bullet}}]^t &\in \text{Hom}((\mathbb{Z}^{\text{tr}}(\bar{D}_{s_g(\bullet)} \times \mathbb{P}^{d_X}/S), u_{IJ})/(\mathbb{Z}^{\text{tr}}(\bar{D}_{s_g(\bullet)} \times \mathbb{P}^{d_X-1}/S), u_{IJ}), \\ &(\mathbb{Z}_{\text{tr}}(\bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) \times \mathbb{P}^{d_{X'}/S}), u_{IJ})/(\mathbb{Z}_{\text{tr}}(\bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) \times \mathbb{P}^{d_{X'}-1}/S), u_{IJ})). \end{aligned}$$

We then consider the following map in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned} \hat{R}_S^{0CH}(g) : \hat{R}_{(\bar{X}, \bar{D})/S}^0(\mathbb{Z}(U/S)) &\xrightarrow{=} \\ \text{Cone}(\mathbb{Z}(i_{\bullet}) : (\mathbb{Z}^{\text{tr}}(\bar{D}_{s_g(\bullet)}/S), u_{IJ}) \rightarrow \mathbb{Z}^{\text{tr}}(\bar{X}/S)(-d_X)[-2d_X]) & \\ \xrightarrow{([\Gamma_{\bar{g}'_{\bullet}}]^t, [\Gamma_{\bar{g}}]^t)} & \\ \text{Cone}(\mathbb{Z}(i'_{g_{\bullet}}) : (\mathbb{Z}^{\text{tr}}(\bar{g}^{-1}(\bar{D}_{s_g(\bullet)})/S), u_{IJ}) \rightarrow \mathbb{Z}^{\text{tr}}(\bar{X}'/S)(-d_{X'})[-2d_{X'}]) & \\ \xrightarrow{(\mathbb{Z}(i'_{g_{\bullet}}), I)(-d_{X'})[-2d_{X'}]} & \\ \text{Cone}(\mathbb{Z}(i'_{\bullet}) : ((\mathbb{Z}^{\text{tr}}(\bar{D}'_{\bullet}/S), u_{IJ}) \rightarrow \mathbb{Z}^{\text{tr}}(\bar{X}'/S)(-d_{X'})[-2d_{X'}]) & \\ \xrightarrow{=} \hat{R}_{(\bar{X}', \bar{D}')/S}^0(\mathbb{Z}(U'/S)) & \end{aligned}$$

Then the following diagram in $C(\text{Var}(\mathbb{C})/S)$ commutes by definition

$$\begin{array}{ccc} \hat{R}_{(\bar{X}, \bar{D})/S}^0(\mathbb{Z}(U/S)) & \xrightarrow{T(\hat{R}^{0CH}, R^{0CH})(\mathbb{Z}(U/S))} & \hat{R}_{(\bar{X}, \bar{D})/S}^0(\mathbb{Z}(U/S)) \\ \hat{R}_S^{0CH}(g) \downarrow & & \downarrow R_S^{0CH}(g) \\ \hat{R}_{(\bar{X}', \bar{D}')/S}^0(\mathbb{Z}(U'/S)) & \xrightarrow{T(\hat{R}^{0CH}, R^{0CH})(\mathbb{Z}(U'/S))} & \hat{R}_{(\bar{X}', \bar{D}')/S}^0(\mathbb{Z}(U'/S)) \end{array}$$

(iii) For $g_1 : U''/S \rightarrow U'/S$, $g_2 : U'/S \rightarrow U/S$ two morphisms with $U''/S = (U', h''), U'/S = (U', h')$, $U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, with U , U' and U'' smooth. We get from (i) and (ii) a compactification $\bar{f} = \bar{h} : \bar{X} \rightarrow \bar{S}$ of $h : U \rightarrow S$, a compactification $\bar{f}' = \bar{h}' : \bar{X}' \rightarrow \bar{S}$ of $h' : U' \rightarrow S$, and a compactification $\bar{f}'' = \bar{h}'' : \bar{X}'' \rightarrow \bar{S}$ of $h'' : U'' \rightarrow S$, with $\bar{X}, \bar{X}', \bar{X}'' \in \text{PSmVar}(\mathbb{C})$, $\bar{D} := \bar{X} \setminus U \subset \bar{X}$, $\bar{D}' := \bar{X}' \setminus U' \subset \bar{X}'$, and $\bar{D}'' := \bar{X}'' \setminus U'' \subset \bar{X}''$ normal crossing divisors, such that $g_1 : U''/S \rightarrow U'/S$ extend to $\bar{g}_1 : \bar{X}''/\bar{S} \rightarrow \bar{X}'/\bar{S}$, $g_2 : U'/S \rightarrow U/S$ extend to $\bar{g}_2 : \bar{X}'/\bar{S} \rightarrow \bar{X}/\bar{S}$, and

$$\hat{R}_S^{0CH}(g_2 \circ g_1) = \hat{R}_S^{0CH}(g_1) \circ \hat{R}_S^{0CH}(g_2) : \hat{R}_{(\bar{X}, \bar{D})/S}^0 \rightarrow \hat{R}_{(\bar{X}'', \bar{D}'')/S}^0$$

(iv) For

$$Q^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_\alpha^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_\beta^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U_α^* smooth, we get from (i), (ii) and (iii) the map in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{aligned} T(\hat{R}^{0CH}, R^{0CH})(Q^*) : \hat{R}^{0CH}(Q^*) &:= (\cdots \rightarrow \bigoplus_{\beta \in \Lambda^{n-1}} \varinjlim_{(\bar{X}_\beta^{n-1}, \bar{D}_\beta^{n-1})/S} \hat{R}_{(\bar{X}_\beta^{n-1}, \bar{D}_\beta^{n-1})/S}^0(\mathbb{Z}(U_\beta^{n-1}/S)) \\ &\xrightarrow{(\hat{R}_S^{0CH}(g_{\alpha,\beta}^n))} \bigoplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S} \hat{R}_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S}^0(\mathbb{Z}(U_\alpha^n/S)) \rightarrow \cdots) \rightarrow R^{0CH}(Q^*), \end{aligned}$$

where for $(U_\alpha^n, h_\alpha^n) \in \text{Var}(\mathbb{C})/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \rightarrow \bar{S}$ of $h_\alpha : U_\alpha \rightarrow S$ with $\bar{X}_\alpha \in \text{PSmVar}(\mathbb{C})$ and $\bar{D}_\alpha := \bar{X}_\alpha \setminus U_\alpha$ a normal crossing divisor. For $m = (m^*) : Q_1^* \rightarrow Q_2^*$ a morphism with

$$\begin{aligned} Q_1^* &:= (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{1,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{1,\beta}^{n-1}/S) \rightarrow \cdots), \\ Q_2^* &:= (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{2,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{2,\beta}^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S) \end{aligned}$$

complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}^*$ and $U_{2,\alpha}^*$ smooth, we get again from (i), (ii) and (iii) a commutative diagram in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{array}{ccc} \hat{R}^{0CH}(Q_2^*) & \xrightarrow{T(\hat{R}_S^{0CH}, R_S^{0CH})(Q_2^*)} & \hat{R}^{0CH}(Q_2^*) \\ \hat{R}_S^{0CH}(m) := (\hat{R}_S^{0CH}(m^*)) \downarrow & & \downarrow (R_S^{0CH}(m^*)) \\ \hat{R}^{0CH}(Q_1^*) & \xrightarrow{T(\hat{R}_S^{0CH}, R_S^{0CH})(Q_1^*)} & \hat{R}^{0CH}(Q_1^*) \end{array} .$$

(v) Let

$$Q^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_\alpha^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_\beta^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U_α^* smooth, we have by definition

$$\text{Gr}_S^{12*} \hat{R}^{0CH}(Q^*) = \hat{R}^{CH}(Q^*) \in C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S).$$

- Let $S \in \text{Var}(\mathbb{C})$ For $(h, m, m') = (h^*, m^*, m'^*) : Q_1^*[1] \rightarrow Q_2^*$ an homotopy with $Q_1^*, Q_2^* \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}^*$ and $U_{2,\alpha}^*$ smooth,

$$(\hat{R}_S^{0CH}(h), \hat{R}_S^{0CH}(m), \hat{R}_S^{0CH}(m')) = (\hat{R}_S^{0CH}(h^*), \hat{R}_S^{0CH}(m^*), \hat{R}_S^{0CH}(m'^*)) : R^{0CH}(Q_2^*)[1] \rightarrow R^{0CH}(Q_1^*)$$

is an homotopy in $C(\text{Var}(\mathbb{C})/S)$ using definition 44 (iii). In particular if $m : Q_1^* \rightarrow Q_2^*$ with $Q_1^*, Q_2^* \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}^*$ and $U_{2,\alpha}^*$ smooth is an homotopy equivalence, then $\hat{R}_S^{0CH}(m) : \hat{R}^{0CH}(Q_2^*) \rightarrow \hat{R}^{0CH}(Q_1^*)$ is an homotopy equivalence.

- Let $S \in \text{SmVar}(\mathbb{C})$. Let $F \in \text{PSh}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$. Consider

$$q : LF := (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{\text{sm}}/S} \mathbb{Z}(U_\alpha/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{\text{sm}}/S} \mathbb{Z}(U_\alpha/S) \rightarrow \cdots) \rightarrow F$$

the canonical projective resolution given in subsection 2.3.3. Note that the U_α are smooth since S is smooth and h_α are smooth morphism. Definition 44(iv) gives in this particular case the map in $C(\text{Var}(\mathbb{C})/S)$

$$T(\hat{R}_S^{0CH}, R_S^{0CH})(\rho_S^* LF) : \hat{R}^{0CH}(\rho_S^* LF) := (\cdots \rightarrow \oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} \hat{R}_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}^0(\mathbb{Z}(U_\alpha/S)) \\ \xrightarrow{(\hat{R}_S^{0CH}(g_{\alpha, \beta}^n))} \oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} \hat{R}_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}^0(\mathbb{Z}(U_\alpha/S)) \rightarrow \cdots) \rightarrow R^{0CH}(\rho_S^* LF),$$

where for $(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \rightarrow \bar{S}$ of $h_\alpha : U_\alpha \rightarrow S$ with $\bar{X}_\alpha \in \text{PSmVar}(\mathbb{C})$ and $\bar{D}_\alpha := \bar{X}_\alpha \setminus U_\alpha$ a normal crossing divisor. Definition 44(iv) gives then by functoriality in particular, for $F = F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the map in $C(\text{Var}(\mathbb{C})/S)$

$$T(\hat{R}_S^{0CH}, R_S^{0CH})(\rho_S^* LF) : \hat{R}^{0CH}(\rho_S^* LF) \rightarrow R^{0CH}(\rho_S^* LF).$$

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $h : U \rightarrow S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{array}{ccc} U_T & \xrightarrow{h'} & T \\ \downarrow g' & & \downarrow g \\ U & \xrightarrow{h} & S \end{array}$$

Note that U is smooth since S and h are smooth, and U_T is smooth since T and h' are smooth. Take, see definition-proposition 12(ii), a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}$ of $h : U \rightarrow S$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{e} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of (\bar{X}_0, \bar{Z}) . Then $\bar{f}'_0 = g \circ h' : \bar{X}_T \rightarrow \bar{T}$ is a compactification of $g \circ h' : U_T \rightarrow S$ such that $g' : U_T/S \rightarrow U/S$ extend to a morphism $\bar{g}'_0 : \bar{X}_T/\bar{S} \rightarrow \bar{X}/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \setminus U$ and $\bar{Z}' = \bar{X}_T \setminus U_T$. Take, see definition-proposition 12(ii), a strict desingularization $\epsilon'_\bullet : (\bar{X}', \bar{D}') \rightarrow (\bar{X}_T, \bar{Z}')$ of (\bar{X}_T, \bar{Z}') . Denote $\bar{g}' = \bar{g}'_0 \circ \epsilon'_\bullet : \bar{X}' \rightarrow \bar{X}$. We then have, see definition-proposition 12(ii), the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} = \bar{X}_{c(\bullet)} & \xleftarrow{i_\bullet} & \bar{D}_{s_{g'}(\bullet)} \\ \uparrow g' & & \uparrow \bar{g}'_0 & & \uparrow (\bar{g}')'_\bullet \\ U_T = U_{T,c(\bullet)} & \xrightarrow{j'} & \bar{X}_T = X_{T,c(\bullet)} & \xleftarrow{i_{g\bullet}} & \bar{g}'^{-1}(\bar{D}_{s_{g'}(\bullet)}) \\ \uparrow g' & & \uparrow \bar{g}' & & \uparrow \epsilon'_\bullet \\ U_T = U_{T,c(\bullet)} & \xrightarrow{j'} & \bar{X}' = X'_{c(\bullet)} & \xleftarrow{i'_\bullet} & \bar{D}'_\bullet \xleftarrow{i''_{g\bullet}} \bar{g}'^{-1}(\bar{D}_{s_{g'}(\bullet)}) : i'_{g\bullet} \end{array}$$

We then consider the following map in $C(\text{Var}(\mathbb{C})/T)$,

$$\begin{aligned} & T(g, \hat{R}^{0CH})(\mathbb{Z}(U/S)) : g^* \hat{R}_{(\bar{X}, \bar{D})/S}^0(\mathbb{Z}(U/S)) \\ & \xrightarrow{=} g^* \text{Cone}(\mathbb{Z}(i_\bullet)) : (\mathbb{Z}^{tr}(\bar{D}_\bullet/S), u_{IJ}) \rightarrow \mathbb{Z}^{tr}(\bar{X}/S)(-d_X)[-2d_X] \\ & \xrightarrow{=} \\ & \text{Cone}(\mathbb{Z}(i_{g\bullet})) : (\mathbb{Z}^{tr}(\bar{g}'^{-1}(\bar{D}_{s_{g'}(\bullet)})/T), u_{IJ}) \rightarrow \mathbb{Z}^{tr}(\bar{X}_T/T)(-d_X)[-2d_X] \\ & \xrightarrow{=} \\ & \text{Cone}(\mathbb{Z}(i'_\bullet)) : ((\mathbb{Z}^{tr}(\bar{D}'_\bullet/T), u_{IJ}) \rightarrow \mathbb{Z}^{tr}((\bar{X}'/T))(-d_{X'})[-2d_{X'}] \\ & \xrightarrow{=} \hat{R}_{(\bar{X}', \bar{D}')/T}^0(\mathbb{Z}(U_T/T)) \end{aligned}$$

For

$$Q^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_\alpha^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \bigoplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_\beta^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with $h_\alpha^n : U_\alpha^n \rightarrow S$ smooth, we get the map in $C(\text{Var}(\mathbb{C})/T)$

$$\begin{aligned} T(g, \hat{R}^{0CH})(Q^*) : g^* \hat{R}^{0CH}(Q^*) &= (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S} g^* \hat{R}_{(\bar{X}_\alpha^n, \bar{D}_\alpha^n)/S}^0(\mathbb{Z}(U_\alpha^n/S)) \rightarrow \cdots) \\ \xrightarrow{(T(g, \hat{R}^{0CH})(\mathbb{Z}(U_\alpha^n/S)))} & (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}_\alpha^{n'}, \bar{D}_\alpha^{n'})/T} \hat{R}_{(\bar{X}_\alpha^{n'}, \bar{D}_\alpha^{n'})/T}(U_{\alpha,T}^n) \rightarrow \cdots) =: \hat{R}^{CH}(g^* Q^*) \end{aligned}$$

together with the commutative diagram in $C(\text{Var}(\mathbb{C})/T)$

$$\begin{array}{ccc} g^* \hat{R}^{0CH}(Q^*) & \xrightarrow{T(g, \hat{R}^{0CH})(Q^*)} & \hat{R}^{0CH}(g^* Q^*) \\ \downarrow g^* T(\hat{R}_S^{0CH}, R_S^{0CH})(Q^*) & & \downarrow T(\hat{R}_T^{0CH}, R_T^{0CH})(g^* Q) \\ g^* R^{0CH}(Q^*) & \xrightarrow{T(g, R^{0CH})(Q^*)} & R^{0CH}(g^* Q^*) \end{array} .$$

Let $F \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S)$. Consider

$$q : LF := (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \rightarrow \cdots) \rightarrow F$$

the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in $C(\text{Var}(\mathbb{C})/T)$

$$\begin{aligned} T(g, \hat{R}^{0CH})(\rho_S^* LF) : g^* \hat{R}^{0CH}(\rho_S^* LF) &= \\ (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} g^* \hat{R}_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}^0(\mathbb{Z}(U_\alpha/S)) \rightarrow \cdots) &\xrightarrow{(T(g, \hat{R}^{0CH})(\mathbb{Z}(U_\alpha/S)))} \\ (\cdots \rightarrow \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \varinjlim_{(\bar{X}'_\alpha, \bar{D}'_\alpha)/T} \hat{R}_{(\bar{X}'_\alpha, \bar{D}'_\alpha)/T}(U_{\alpha,T}) \rightarrow \cdots) &=: \hat{R}^{0CH}(\rho_T^* g^* LF), \end{aligned}$$

and by functoriality, we get in particular for $F = F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the map in $C(\text{Var}(\mathbb{C})/T)$

$$T(g, \hat{R}^{0CH})(\rho_S^* LF) : g^* \hat{R}^{0CH}(\rho_S^* LF) \rightarrow \hat{R}^{0CH}(\rho_T^* g^* LF)$$

together with the commutative diagram in $C(\text{Var}(\mathbb{C})/T)$

$$\begin{array}{ccc} g^* \hat{R}^{0CH}(\rho_S^* LF) & \xrightarrow{T(g, \hat{R}^{0CH})(\rho_S^* LF)} & \hat{R}^{0CH}(\rho_T^* g^* LF) \\ \downarrow g^* T(\hat{R}_S^{0CH}, R_S^{0CH})(\rho_S^* LF) & & \downarrow T(\hat{R}_T^{0CH}, R_T^{0CH})(\rho_T^* g^* LF) \\ g^* L\rho_{S*} \mu_{S*} R^{CH}(\rho_S^* LF) & \xrightarrow{T(g, R^{0CH})(\rho_S^* LF)} & R^{CH}(\rho_T^* g^* LF) \end{array} .$$

- Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ and $p : S_1 \times S_2 \rightarrow S_1$ the projection. Let $h : U \rightarrow S_1$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{array}{ccc} U \times S_2 & \xrightarrow{h \times I} & S_1 \times S_2 \\ \downarrow p' & & \downarrow p \\ U & \xrightarrow{h} & S_1 \end{array}$$

Take, see definition-proposition 12(i), a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \rightarrow \bar{S}_1$ of $h : U \rightarrow S_1$. Then $\bar{f}_0 \times I : \bar{X}_0 \times S_2 \rightarrow \bar{S}_1 \times S_2$ is a compactification of $h \times I : U \times S_2 \rightarrow S_1 \times S_2$ and $p' : U \times S_2 \rightarrow U$ extend to $\bar{p}'_0 := p_{X_0} : \bar{X}_0 \times S_2 \rightarrow \bar{X}_0$. Denote $Z = X_0 \setminus U$. Take see theorem 15(i), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair (\bar{X}_0, \bar{Z}) . We then have the commutative diagram (48) in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$ whose squares are cartesian

$$\begin{array}{ccccc} U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} & \xleftarrow{i \bullet} & \bar{D} \bullet \\ \uparrow g & & \uparrow \bar{p}' := p_{\bar{X}} & & \uparrow \bar{p}' \bullet \\ U \times S_2 = (U \times S_2)_{c(\bullet)} & \xrightarrow{j \times I} & \bar{X} \times S_2 & \xleftarrow{i' \bullet} & \bar{D} \bullet \times S_2 \end{array}$$

Then the map in $C(\text{Var}(\mathbb{C})/S_1 \times S_2)$

$$T(p, \hat{R}^{0CH})(\mathbb{Z}(U/S_1)) : p^* \hat{R}_{(\bar{X}, \bar{D})/S_1}^0(\mathbb{Z}(U/S_1)) \xrightarrow{\sim} \hat{R}_{(\bar{X} \times S_2, \bar{D} \bullet \times S_2)/S_1 \times S_2}^0(\mathbb{Z}(U \times S_2/S_1 \times S_2))$$

is an isomorphism. Hence, for $Q^* \in C(\text{Var}(\mathbb{C})/S_1)$ a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in $C(\text{Var}(\mathbb{C})/S_1 \times S_2)$

$$T(p, \hat{R}^{0CH})(Q^*) : p^* \hat{R}^{0CH}(Q^*) \xrightarrow{\sim} \hat{R}^{0CH}(p^* Q^*)$$

is an isomorphism. In particular, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S_1)$ the map in $C(\text{Var}(\mathbb{C})/S_1 \times S_2)$

$$T(p, \hat{R}^{0CH})(\rho_{S_1}^* LF) : p^* \hat{R}^{0CH}(\rho_{S_1}^* LF) \xrightarrow{\sim} \hat{R}^{0CH}(\rho_{S_1 \times S_2}^* p^* LF)$$

is an isomorphism.

- Let $h_1 : U_1 \rightarrow S$, $h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2, S \in \text{Var}(\mathbb{C})$, U_1, U_2 smooth. Denote by $p_1 : U_1 \times_S U_2 \rightarrow U_1$ and $p_2 : U_1 \times_S U_2 \rightarrow U_2$ the projections. Take, see definition-proposition 12(i), a compactification $\bar{f}_{10} = \bar{h}_1 : \bar{X}_{10} \rightarrow \bar{S}$ of $h_1 : U_1 \rightarrow S$ and a compactification $\bar{f}_{20} = \bar{h}_2 : \bar{X}_{20} \rightarrow \bar{S}$ of $h_2 : U_2 \rightarrow S$. Then,

- $\bar{f}_{10} \times \bar{f}_{20} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \rightarrow S$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \rightarrow S$.
- $\bar{p}_{10} := p_{X_{10}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \rightarrow \bar{X}_{10}$ is a compactification of $p_1 : U_1 \times_S U_2 \rightarrow U_1$.
- $\bar{p}_{20} := p_{X_{20}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \rightarrow \bar{X}_{20}$ is a compactification of $p_2 : U_1 \times_S U_2 \rightarrow U_2$.

Denote $\bar{Z}_1 = \bar{X}_{10} \setminus U_1$ and $\bar{Z}_2 = \bar{X}_{20} \setminus U_2$. Take, see theorem 15(i), a strict desingularization $\bar{\epsilon}_1 : (\bar{X}_1, \bar{D}) \rightarrow (\bar{X}_{10}, \bar{Z}_1)$ of the pair $(\bar{X}_{10}, \bar{Z}_1)$ and a strict desingularization $\bar{\epsilon}_2 : (\bar{X}_2, \bar{E}) \rightarrow (\bar{X}_{20}, \bar{Z}_2)$ of the pair $(\bar{X}_{20}, \bar{Z}_2)$. Take then a strict desingularization

$$\bar{\epsilon}_{12} : ((\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F}) \rightarrow (\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (D \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$$

of the pair $(\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (D \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$. We have then the following commutative diagram

$$\begin{array}{ccc} & \bar{X}_1 & \xrightarrow{\bar{f}_1} & \bar{S} \\ & \uparrow \bar{p}_2 & & \uparrow \bar{f}_2 \\ & \bar{X}_1 \times_{\bar{S}} \bar{X}_2 & \xrightarrow{\bar{p}_1} & \bar{X}_2 \\ (\bar{p}_2)^N \nearrow & \uparrow \bar{\epsilon}_{12} & \nearrow (\bar{p}_1)^N & \\ (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N & & & \end{array}$$

and

- $\bar{f}_1 \times \bar{f}_2 : \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \rightarrow \bar{S}$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \rightarrow S$.
- $(\bar{p}_1)^N := \bar{p}_1 \circ \epsilon_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \rightarrow \bar{X}_1$ is a compactification of $p_1 : U_1 \times_S U_2 \rightarrow U_1$.
- $(\bar{p}_2)^N := \bar{p}_2 \circ \epsilon_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \rightarrow \bar{X}_2$ is a compactification of $p_2 : U_1 \times_S U_2 \rightarrow U_2$.

We have then the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$T(\otimes, \hat{R}_S^{0CH})(\mathbb{Z}(U_1/S), \mathbb{Z}(U_2/S)) := \hat{R}_S^{0CH}(p_1) \otimes \hat{R}_S^{0CH}(p_2) : \\ \hat{R}_{(\bar{X}_1, \bar{D})/S}^0(\mathbb{Z}(U_1/S)) \otimes \hat{R}_{(X_2, E)/S}^0(\mathbb{Z}(U_2/S)) \xrightarrow{\sim} \hat{R}_{(\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F})/S}^0(\mathbb{Z}(U_1 \times_S U_2/S))$$

For

$$Q_1^* := (\cdots \rightarrow \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{1,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{1,\beta}^{n-1}/S) \rightarrow \cdots), \\ Q_2^* := (\cdots \rightarrow \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{2,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{2,\beta}^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)$$

complexes of (maybe infinite) direct sum of representable presheaves with U_α^* smooth, we get the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$T(\otimes, \hat{R}_S^{0CH})(Q_1^*, Q_2^*) : \hat{R}^{0CH}(Q_1^*) \otimes R^{0CH}(Q_2^*) \xrightarrow{T(\otimes, \hat{R}_S^{0CH})(\mathbb{Z}(U_{1,\alpha}^n), \mathbb{Z}(U_{2,\beta}^n))} \hat{R}^{0CH}(Q_1^* \otimes Q_2^*)$$

, together with the commutative diagram in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{array}{ccc} \hat{R}^{0CH}(Q_1^*) \otimes R^{0CH}(Q_2^*) & \xrightarrow{T(\otimes, \hat{R}_S^{0CH})(Q_1^*, Q_2^*)} & \hat{R}^{0CH}(Q_1^* \times Q_2^*) \\ \downarrow T(\hat{R}_S^{0CH}, R_S^{0CH})(Q_1^*) \otimes T(\hat{R}_S^{0CH}, R_S^{0CH})(Q_2^*) & & \downarrow T(\hat{R}_S^{0CH}, R_S^{0CH})(Q_1^* \otimes Q_2^*) \\ R^{0CH}(Q_1^*) \otimes R^{0CH}(Q_2^*) & \xrightarrow{T(\otimes, R_S^{0CH})(Q_1^*, Q_2^*)} & R^{0CH}(Q_1^* \otimes Q_2^*) \end{array}$$

For $F_1, F_2 \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we get in particular the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$T(\otimes, R_S^{0CH})(\rho_S^* LF_1, \rho_S^* LF_2) : R^{0CH}(\rho_S^* LF_1) \otimes R^{0CH}(\rho_S^* LF_2) \rightarrow R^{0CH}(\rho_S^*(LF_1 \otimes LF_2))$$

together with the commutative diagram in $C(\text{Var}(\mathbb{C})/S)$

$$\begin{array}{ccc} \hat{R}^{0CH}(\rho_S^* LF_1) \otimes R^{0CH}(\rho_S^* LF_2) & \xrightarrow{T(\otimes, \hat{R}_S^{0CH})(\rho_S^* LF_1, \rho_S^* LF_2)} & \hat{R}^{0CH}(\rho_S^* LF_1 \times \rho_S^* LF_2) \\ \downarrow T(\hat{R}_S^{0CH}, R_S^{0CH})(\rho_S^* LF_1) \otimes T(\hat{R}_S^{0CH}, R_S^{0CH})(\rho_S^* LF_2) & & \downarrow T(\hat{R}_S^{0CH}, R_S^{0CH})(\rho_S^* LF_1 \otimes \rho_S^* LF_2) \\ R^{0CH}(\rho_S^* LF_1) \otimes R^{0CH}(\rho_S^* LF_2) & \xrightarrow{T(\otimes, R_S^{0CH})(\rho_S^* LF_1, \rho_S^* LF_2)} & R^{0CH}(\rho_S^* LF_1 \times \rho_S^* LF_2) \end{array}$$

2.13 The derived categories of filtered complexes of presheaves on a site or of filtered complexes of presheaves of modules on a ringed topos

Definition 45. Let $\mathcal{S} \in \text{Cat}$ a site endowed with topology τ .

- (i) We denote by $D(\mathcal{S}) := \text{Ho}_{\text{Top}} C(\mathcal{S})$ the localization of the category of complexes of presheaves on \mathcal{S} with respect to top local equivalence and by $D(\tau) : C(\mathcal{S}) \rightarrow D(\mathcal{S})$ the localization functor.
- (ii) We denote for $r = 1, \dots, \infty$, resp. $r = (1, \dots, \infty)^2$,

$$D_{fil,r}(\mathcal{S}) := K_{fil,r}(\mathcal{S})([E_1]^{-1}), \quad D_{2fil,r,r'}(\mathcal{S}) := K_{2fil,r,r'}(\mathcal{S})([E_1]^{-1}),$$

the localizations of the category of filtered complexes of presheaves on \mathcal{S} whose filtration is biregular modulo r -filtered homotopies with respect to the classes of filtered τ local equivalence $[E_1]$. Note

that the classes of filtered τ local equivalence constitute a right multiplicative system. By definition, if $m : (G_1, F) \rightarrow (G_2, F)$ with $(G_1, F), (G_2, F) \in C_{fil}(\mathcal{S})$ is an r -filtered τ local equivalence then $m := D(\tau)(m) : (G_1, F) \rightarrow (G_2, F)$ is an isomorphism in $D_{fil,r}(\mathcal{S})$. By definition, we have sequences of functors

$$C_{fil}(\mathcal{S}) \rightarrow K_{fil}(\mathcal{S}) \rightarrow D_{fil}(\mathcal{S}) \rightarrow D_{fil,2}(\mathcal{S}) \rightarrow \cdots \rightarrow D_{fil,\infty}(\mathcal{S}).$$

and commutative diagrams of functors

$$\begin{array}{ccc} K_{fil}(\mathcal{S}) & \longrightarrow & D_{fil}(\mathcal{S}) & , & K_{fil,r}(\mathcal{S}) & \longrightarrow & D_{fil,r}(\mathcal{S}) . \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_{fil,2}(\mathcal{S}) & \longrightarrow & D_{fil,2}(\mathcal{S}) & & K_{fil,r+1}(\mathcal{S}) & \longrightarrow & D_{fil,r}(\mathcal{S}) \end{array}$$

Then, for $r = 1$, $K_{fil}(\mathcal{S})$ and $D_{fil}(\mathcal{S})$ are in the canonical way triangulated categories. However, for $r > 1$, the categories $K_{fil,r}(\mathcal{S})$ and $D_{fil,r}(\mathcal{S})$ together with the canonical triangles does NOT satisfy the 2 of 3 axiom of triangulated categories.

Definition 46. Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ where $\mathcal{S} \in \text{Cat}$ is a site endowed with topology τ .

- (i) We denote by $D_{O_{\mathcal{S}}}(\mathcal{S}) := \text{Ho}_{\text{Top}} C_{O_{\mathcal{S}}}(\mathcal{S})$ the localization of the category of complexes of presheaves on \mathcal{S} with respect to top local equivalence and by $D(\tau) : C_{O_{\mathcal{S}}}(\mathcal{S}) \rightarrow D_{O_{\mathcal{S}}}(\mathcal{S})$ the localization functor.
- (ii) We denote for $r = 1, \dots, \infty$, resp. $r = (1, \dots, \infty)^2$,

$$D_{O_{\mathcal{S}}fil,r}(\mathcal{S}) := K_{O_{\mathcal{S}}fil,r}(\mathcal{S})([E_1]^{-1}) , D_{O_{\mathcal{S}}2fil,r,r'}(\mathcal{S}) := K_{O_{\mathcal{S}}2fil,r,r'}(\mathcal{S})([E_1]^{-1}),$$

the localizations of the category of filtered complexes of presheaves on \mathcal{S} whose filtration is biregular modulo r -filtered homotopies with respect to the classes of filtered τ local equivalence $[E_1]$. Note that the classes of filtered τ local equivalence constitute a right multiplicative system. By definition, if $m : (G_1, F) \rightarrow (G_2, F)$ with $(G_1, F), (G_2, F) \in C_{O_{\mathcal{S}}fil}(\mathcal{S})$ is an r -filtered τ local equivalence then $m := D(\tau)(m) : (G_1, F) \rightarrow (G_2, F)$ is an isomorphism in $D_{O_{\mathcal{S}}fil,r}(\mathcal{S})$. By definition, we have sequences of functors

$$C_{O_{\mathcal{S}}fil}(\mathcal{S}) \rightarrow K_{O_{\mathcal{S}}fil}(\mathcal{S}) \rightarrow D_{O_{\mathcal{S}}fil}(\mathcal{S}) \rightarrow D_{O_{\mathcal{S}}fil,2}(\mathcal{S}) \rightarrow \cdots \rightarrow D_{O_{\mathcal{S}}fil,\infty}(\mathcal{S}).$$

and commutative diagrams of functors

$$\begin{array}{ccc} K_{O_{\mathcal{S}}fil}(\mathcal{S}) & \longrightarrow & D_{O_{\mathcal{S}}fil}(\mathcal{S}) & , & K_{fil,r}(\mathcal{S}) & \longrightarrow & D_{O_{\mathcal{S}}fil,r}(\mathcal{S}) . \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_{O_{\mathcal{S}}fil,2}(\mathcal{S}) & \longrightarrow & D_{fil,2}(\mathcal{S}) & & K_{O_{\mathcal{S}}fil,r+1}(\mathcal{S}) & \longrightarrow & D_{O_{\mathcal{S}}fil,r}(\mathcal{S}) \end{array}$$

Then, for $r = 1$, $K_{O_{\mathcal{S}}fil}(\mathcal{S})$ and $D_{O_{\mathcal{S}}fil}(\mathcal{S})$ are in the canonical way triangulated categories. However, for $r > 1$, the categories $K_{O_{\mathcal{S}}fil,r}(\mathcal{S})$ and $D_{O_{\mathcal{S}}fil,r}(\mathcal{S})$ together with the canonical triangles does NOT satisfy the 2 of 3 axiom of triangulated categories.

Let $f : \mathcal{T} \rightarrow \mathcal{S}$ a morphism of presite, with $\mathcal{S}, \mathcal{T} \in \text{Cat}$ endowed with a topology τ . If f is a morphism of site, the adjunctions

$$(f^*, f_*) = (f^{-1}, f_*) : C(\mathcal{S}) \rightleftarrows C(\mathcal{T}), (f^*, f_*) = (f^{-1}, f_*) : C_{(2)fil}(\mathcal{S}) \rightleftarrows C_{(2)fil}(\mathcal{T}).$$

are Quillen adjunctions. They induces respectively in the derived categories, for $r = (1, \dots, \infty)$, resp. $r = (1, \dots, \infty)$ (note that f^* derive trivially)

$$(f^*, Rf_*) : D(\mathcal{S}) \rightleftarrows D(\mathcal{T}), (f^*, Rf_*) : D_{fil,r}(\mathcal{S}) \rightleftarrows D_{fil,r}(\mathcal{T}).$$

For $F^\bullet \in C(\mathcal{S})$, we have the adjunction maps

$$\mathrm{ad}(f^*, f_*)(F^\bullet) : F^\bullet \rightarrow f_* f^* F^\bullet, \quad \mathrm{ad}(f^*, f_*)(F^\bullet) : f^* f_* F^\bullet \rightarrow F^\bullet,$$

induces in the derived categories, for $(M, F) \in D_{fil}(\mathcal{S})$ and $(N, F) \in D_{fil}(\mathcal{T})$, the adjunction maps

$$\mathrm{ad}(f^*, Rf_*)(M) : (M, F) \rightarrow Rf_* f^*(M, F), \quad \mathrm{ad}(f^*, Rf_*)(N, F) : f^* Rf_*(N, F) \rightarrow (N, F).$$

For a commutative diagram of sites :

$$D = \begin{array}{ccc} \mathcal{Y} & \xrightarrow{g_2} & \mathcal{X} \\ \downarrow f_2 & & \downarrow f_1 \\ \mathcal{T} & \xrightarrow{g_1} & \mathcal{S} \end{array},$$

with $\mathcal{Y}, \mathcal{T}, \mathcal{S}, \mathcal{X} \in \mathrm{Cat}$ with topology $\tau_Y, \tau_T, \tau_S, \tau_X$, the maps, for $F \in C(\mathcal{X})$,

$$T(D)(F) : g_1^* f_{1*} F \rightarrow f_{2*} g_2^* F$$

induce in the derived category the maps in $D_{fil,r}(\mathcal{T})$, given by, for $(G, F) \in D_{fil,r}(\mathcal{X})$ with $(G, F) = D(\tau_X, \tau)((G, F))$,

$$\begin{array}{ccc} g_1^* Rf_{1*}(M, F) & \xrightarrow{T(D)(M, F)} & Rf_{2*} g_2^*(M, F) \\ \uparrow = & & \uparrow = \\ g_1^* f_{1*}(E(G, F)) & \xrightarrow{koT(D)(E(G, F))} & f_{2*} E(g_2^*(E(G, F))) \end{array}.$$

Let $\mathcal{S} \in \mathrm{Cat}$ a site with topology τ . The tensor product of complexes of abelian groups and the internal hom of presheaves on \mathcal{S}

$$((\cdot \otimes \cdot), \mathcal{H}om^\bullet(\cdot, \cdot)) : C(\mathcal{S})^2 \rightarrow C(\mathcal{S}),$$

is a Quillen adjunction which induces in the derived category

$$((\cdot \otimes^L \cdot), R\mathcal{H}om^\bullet(\cdot, \cdot)) : D_{fil,r}(\mathcal{S})^2 \rightarrow D_{fil,r}(\mathcal{S}), \quad R\mathcal{H}om^\bullet((M, W), (N, W)) = \mathcal{H}om^\bullet((Q, W), E(G, F)),$$

where, Q is projectively cofibrant such that $M = D(\tau)(Q^\bullet)$ and G such that $N = D(\tau)(G)$.

Let $i : Z \hookrightarrow S$ a closed embedding, with $S, Z \in \mathrm{Top}$. Denote by $j : S \setminus Z \hookrightarrow S$ the open embedding of the complementary subset. The adjunction

$$(i_*, i^!) := (i_*, i^\perp) : C(Z) \rightarrow C(S), \quad \text{with in this case } i^! F := \ker(F \rightarrow j_* j^* F)$$

is a Quillen adjunction. Since $i^!$ preserve monomorphisms, we have also Quillen adjunctions

$$(i_*, i^!) : C_{(2)fil}(Z) \rightarrow C_{(2)fil}(S), \quad \text{with } i^!(G, F) = (i^! G, F).$$

which induces in the derived category (i_* derive trivially)

$$(i_*, Ri^!) : D_{(2)fil}(Z) \rightarrow D_{(2)fil}(S), \quad \text{with } Ri^!(G, F) = i^! E(G, F).$$

The 2-functor $S \in \mathrm{Top} \mapsto D(S)$ obviously satisfy the localization property, that is for $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \mathrm{Top}$, denote by $j : S \setminus Z \hookrightarrow S$ the open complementary subset, we have for $K \in D(S)$ a distinguish triangle in $D(S)$

$$j_\# j^* K \xrightarrow{\mathrm{ad}(j_\#, j^*)(K)} K \xrightarrow{\mathrm{ad}(i^*, i_*)(K)} i_* i^* K \rightarrow j_\# j^* K[1]$$

equivalently,

- the functor

$$(i^*, j^*) : D(S) \xrightarrow{\sim} D(Z) \times D(S \setminus Z)$$

is conservative,

- and for $K \in C(Z)$, the adjunction map $\text{ad}(i^*, i_*)(K) : i^* i_* K \rightarrow K$ is an equivalence top local, hence for $K \in D(S)$, the induced map in the derived category

$$\text{ad}(i^*, i_*)(K) : i^* i_* K \xrightarrow{\sim} K$$

is an isomorphism.

3 Triangulated category of motives

3.1 Definition and the six functor formalism

The category of motives is obtained by inverting the (\mathbb{A}_S^1, et) equivalence. Hence the \mathbb{A}_S^1 local complexes of presheaves play a key role.

Definition 47. *The derived category of motives of complex algebraic varieties over S is the category*

$$\text{DA}(S) := \text{Ho}_{\mathbb{A}_S^1, et}(C(\text{Var}(\mathbb{C})^{sm}/S)),$$

which is the localization of the category of complexes of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ with respect to (\mathbb{A}_S^1, et) local equivalence and we denote by

$$D(\mathbb{A}_S^1, et) := D(\mathbb{A}_S^1) \circ D(et) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow \text{DA}(S)$$

the localization functor. We have $\text{DA}^-(S) := D(\mathbb{A}_S^1, et)(\text{PSh}(\text{Var}(\mathbb{C})^{sm}/S, C^-(\mathbb{Z}))) \subset \text{DA}(S)$ the full subcategory consisting of bounded above complexes.

Definition 48. *The stable derived category of motives of complex algebraic varieties over S is the category*

$$\text{DA}_{st}(S) := \text{Ho}_{\mathbb{A}_S^1, et}(C_{\Sigma}(\text{Var}(\mathbb{C})^{sm}/S)),$$

which is the localization of the category of \mathbb{G}_{mS} -spectra ($\Sigma F^{\bullet} = F^{\bullet} \otimes \mathbb{G}_{mS}$) of complexes of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ with respect to (\mathbb{A}_S^1, et) local equivalence. The functor

$$\Sigma^{\infty} : C(\text{Var}(\mathbb{C})^{sm}/S) \hookrightarrow C_{\Sigma}(\text{Var}(\mathbb{C})^{sm}/S)$$

induces the functor $\Sigma^{\infty} : \text{DA}(S) \rightarrow \text{DA}_{st}(S)$.

We have all the six functor formalism by [10]. We give a list of the operation we will use :

- For $f : T \rightarrow S$ a morphism with $S, T \in \text{Var}(\mathbb{C})$, the adjunction

$$(f^*, f_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{sm}/T)$$

is a Quillen adjunction which induces in the derived categories (f^* derives trivially), $(f^*, Rf_*) : \text{DA}(S) \rightleftarrows \text{DA}(T)$.

- For $h : V \rightarrow S$ a smooth morphism with $V, S \in \text{Var}(\mathbb{C})$, the adjunction

$$(h_{\sharp}, h^*) : C(\text{Var}(\mathbb{C})^{sm}/V) \rightleftarrows C(\text{Var}(\mathbb{C})^{sm}/S)$$

is a Quillen adjunction which induces in the derived categories (h^* derive trivially) $(Lh_{\sharp}, h^*) : \text{DA}(V) \rightleftarrows \text{DA}(S)$.

- For $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \text{Var}(\mathbb{C})$,

$$(i_*, i^!) := (i_*, i^\perp) : C(\text{Var}(\mathbb{C})^{sm}/Z) \rightleftarrows C(\text{Var}(\mathbb{C})^{sm}/S)$$

is a Quillen adjunction, which induces in the derived categories (i_* derive trivially) $(i_*, Ri^!) : \text{DA}(Z) \rightleftarrows \text{DA}(S)$. The fact that i_* derive trivially (i.e. send (\mathbb{A}^1, et) local equivalence to (\mathbb{A}^1, et) local equivalence is proved in [4].

- For $S \in \text{Var}(\mathbb{C})$, the adjunction given by the tensor product of complexes of abelian groups and the internal hom of presheaves

$$((\cdot \otimes \cdot), \mathcal{H}om^\bullet(\cdot, \cdot)) : C(\text{Var}(\mathbb{C})^{sm}/S)^2 \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S),$$

is a Quillen adjunction, which induces in the derived category

$$((\cdot \otimes^L \cdot), R\mathcal{H}om^\bullet(\cdot, \cdot)) : \text{DA}(S)^2 \rightarrow \text{DA}(S),$$

- Let $M, N \in \text{DA}(S)$, Q^\bullet projectively cofibrant such that $M = D(\mathbb{A}^1, et)(Q^\bullet)$, and G^\bullet be \mathbb{A}^1 local for the etale topology such that $N = D(\mathbb{A}^1, et)(G^\bullet)$. Then,

$$R\mathcal{H}om^\bullet(M, N) = \mathcal{H}om^\bullet(Q^\bullet, E(G^\bullet)) \in \text{DA}(S). \quad (49)$$

This is well defined since if $s : Q_1 \rightarrow Q_2$ is a etale local equivalence,

$$\mathcal{H}om(s, E(G)) : \mathcal{H}om(Q_1, E(G)) \rightarrow \mathcal{H}om(Q_2, E(G))$$

is a etale local equivalence for $1 \leq i \leq l$.

- For a commutative diagram in $\text{Var}(\mathbb{C})$:

$$D = \begin{array}{ccc} Y & \xrightarrow{g_2} & X \\ \downarrow f_2 & & \downarrow f_1 \\ T & \xrightarrow{g_1} & S \end{array},$$

and $F \in C(\text{Var}(\mathbb{C})^{sm}/X)$, the transformation map $T(D)(F) : g_1^* f_{1*} F \rightarrow f_{2*} g_2^* F$ induces in the derived category, for $M \in \text{DA}(X)$, $M = D(\mathbb{A}^1, et)(F)$ with F \mathbb{A}^1 local for the etale topology,

$$\begin{array}{ccc} g_1^* Rf_{1*} M & \xrightarrow{T(D)(M)} & Rf_{2*} g_2^* M \\ \uparrow = & & \uparrow = \\ g_1^* f_{1*} E(F) & \xrightarrow{\text{Kococ}T(D)(E(F))} & f_{2*} E(C_*(g_2^* E(F))) \end{array}$$

If D is cartesian with $f_1 = f$, $g_1 = g$, $f_2 = f' : X_T \rightarrow T$, $g' : X_T \rightarrow X$, we denote

- $T(D)(F) =: T(f, g)(F) : g^* f_* F \rightarrow f'_* g'^* F$,
- $T(D)(M) =: T(f, g)(M) : g^* Rf_* M \rightarrow Rf'_* g'^* M$.

We get from the first point 2 functors :

- The 2-functor $C(\text{Var}(\mathbb{C})^{sm}/\cdot) : \text{Var}(\mathbb{C}) \rightarrow \text{Ab Cat}$, given by

$$S \mapsto C(\text{Var}(\mathbb{C})^{sm}/S), (f : T \rightarrow S) \mapsto (f^* : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/T)).$$

- The 2-functor $\text{DA}(\cdot) : \text{Var}(\mathbb{C}) \rightarrow \text{TriCat}$, given by

$$S \mapsto \text{DA}(S), (f : T \rightarrow S) \mapsto (f^* : \text{DA}(S) \rightarrow \text{DA}(T)).$$

The main theorem is the following :

Theorem 16. [4][10] *The 2-functor $\mathrm{DA}(\cdot) : \mathrm{Var}(\mathbb{C}) \rightarrow \mathrm{TriCat}$, given by*

$$S \mapsto \mathrm{DA}(S), (f : T \rightarrow S) \mapsto (f^* : \mathrm{DA}(S) \rightarrow \mathrm{DA}(T))$$

is a 2-homotopic functor ([4])

From theorem 16, we get in particular

- For $f : T \rightarrow S$ a morphism with $T, S \in \mathrm{Var}(\mathbb{C})$, there by theorem 16 is also a pair of adjoint functor

$$(f_!, f^!) : \mathrm{DA}(S) \rightleftarrows \mathrm{DA}(T)$$

- with $f_! = Rf_*$ if f is proper,
- with $f^! = f^*[d]$ if f is smooth of relative dimension d .

For $h : U \rightarrow S$ a smooth morphism with $U, S \in \mathrm{Var}(\mathbb{C})$ irreducible, have, for $M \in \mathrm{DA}(U)$, an isomorphism

$$Lh_{\sharp}M \rightarrow h_!M[d], \quad (50)$$

in $\mathrm{DA}(S)$.

- The 2-functor $S \in \mathrm{Var}(\mathbb{C}) \mapsto \mathrm{DA}(S)$ satisfy the localization property, that is for $i : Z \hookrightarrow X$ a closed embedding with $Z, X \in \mathrm{Var}(\mathbb{C})$, denote by $j : S \setminus Z \hookrightarrow S$ the open complementary subset, we have for $M \in \mathrm{DA}(S)$ a distinguish triangle in $\mathrm{DA}(S)$

$$j_{\sharp}j^*M \xrightarrow{\mathrm{ad}(j_{\sharp}, j^*)(M)} M \xrightarrow{\mathrm{ad}(i^*, i_*)(M)} i_*i^*M \rightarrow j_{\sharp}j^*M[1]$$

equivalently,

- the functor

$$(i^*, j^*) : \mathrm{DA}(S) \xrightarrow{\sim} \mathrm{DA}(Z) \times \mathrm{DA}(S \setminus Z)$$

is conservative,

- and for $F \in C(\mathrm{Var}(\mathbb{C})^{sm}/Z)$, the adjunction map $\mathrm{ad}(i^*, i_*)(F) : i^*i_*F \rightarrow F$ is an equivalence Zariski local, hence for $M \in \mathrm{DA}(S)$, the induced map in the derived category

$$\mathrm{ad}(i^*, i_*)(M) : i^*i_*M \xrightarrow{\sim} M$$

is an isomorphism.

- For $f : X \rightarrow S$ a proper map, $g : T \rightarrow S$ a morphism, with $T, X, S \in \mathrm{Var}(\mathbb{C})$, and $M \in \mathrm{DA}(X)$,

$$T(f, g)(M) : g^*Rf_*M \rightarrow Rf'_*\tilde{g}^*M$$

is an isomorphism in $\mathrm{DA}(T)$ if f is proper.

Definition 49. *The derived category of extended motives of complex algebraic varieties over S is the category*

$$\underline{\mathrm{DA}}(S) := \mathrm{Ho}_{\mathbb{A}_S^1, et}(C(\mathrm{Var}(\mathbb{C})/S)),$$

which is the localization of the category of complexes of presheaves on $\mathrm{Var}(\mathbb{C})/S$ with respect to (\mathbb{A}_S^1, et) local equivalence and we denote by

$$D(\mathbb{A}_S^1, et) := D(\mathbb{A}_S^1) \circ D(et) : C(\mathrm{Var}(\mathbb{C})/S) \rightarrow \underline{\mathrm{DA}}(S)$$

the localization functor. We have $\underline{\mathrm{DA}}^-(S) := D(\mathbb{A}_S^1, et)(\mathrm{PSh}(\mathrm{Var}(\mathbb{C})/S, C^-(\mathbb{Z}))) \subset \underline{\mathrm{DA}}(S)$ the full subcategory consisting of bounded above complexes.

Remark 5. Let $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \text{Var}(\mathbb{C})$.

(i) By theorem 16, for $X/S = (X, h) \in \text{Var}(\mathbb{C})^{sm}/S$,

$$(0, \text{ad}(i^*, i_*)(\mathbb{Z}(X/S))) : \Gamma_Z^\vee \mathbb{Z}(X/S) \rightarrow i_* \mathbb{Z}(X_Z/Z)$$

is an equivalence (\mathbb{A}^1, et) local.

(ii) For $X/S = (X, f) \in \text{Var}(\mathbb{C})/S$,

$$(0, \text{ad}(i^*, i_*)(\mathbb{Z}(X/S))) : \Gamma_Z^\vee \mathbb{Z}(X/S) \rightarrow i_* \mathbb{Z}(X_Z/Z)$$

is NOT an equivalence (\mathbb{A}^1, et) local in general, since for example if $f(X) = Z \subset S$, $\rho_{S*} \mathbb{Z}(X/S) = 0$ but $D(\mathbb{A}^1, et)(\rho_{S*} i_* \mathbb{Z}(X_Z/Z) = i_* \rho_{S*} \mathbb{Z}(X_Z/Z)) \neq 0 \in \underline{\text{DA}}(S)$, hence it is NOT an equivalence (\mathbb{A}^1, et) local in this case by proposition 19. In particular $\underline{\text{DA}}(S)$ dos NOT satisfy the localization property.

(ii)' For $X/Z = (X, f) \in \text{Var}(\mathbb{C})/Z$, the inclusion

$$T(i_{\sharp}, i_*) : i_{\sharp} \mathbb{Z}(X/Z) \hookrightarrow i_* \mathbb{Z}(X/Z)$$

is NOT an equivalence (\mathbb{A}^1, et) local by proposition 19 since $\rho_{S*} i_{\sharp} \mathbb{Z}(X/Z) = 0$ but $D(\mathbb{A}^1, et)(\rho_{S*} i_* \mathbb{Z}(X/Z) = i_* \rho_{S*} \mathbb{Z}(X/Z)) \neq 0 \in \underline{\text{DA}}(S)$.

(iii) Let $f : X \rightarrow S$ a smooth proper morphism with $X, S \in \text{Var}(\mathbb{C})$ of relative dimension $d = d_X - d_S$ and X smooth. Then, we have then by proposition 40(i) the equivalence (\mathbb{A}^1, et) local in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T_0(f_{\sharp}, f_*)(\mathbb{Z}(X/X)) := [\Delta_X] : f_{\sharp} \mathbb{Z}(X/X) = \mathbb{Z}(X/S) \rightarrow f_* E_{et}(\mathbb{Z}(X/X))(d)[2d]$$

given by the class of the diagonal $[\Delta_X] \in \text{Hom}(f_{\sharp} \mathbb{Z}(X/X), f_* E_{et}(\mathbb{Z}(X/X))(d)[2d])$.

(iii)' Let $f : X \rightarrow S$ a proper surjective morphism with $X, S \in \text{Var}(\mathbb{C})$ with equidimensional fiber of relative dimension $d = d_X - d_S$. Assume X smooth. Then, we have then by proposition 40(i) the equivalence (\mathbb{A}^1, et) local in $C(\text{Var}(\mathbb{C})/S)$

$$T_0(f_{\sharp}, f_*)(\mathbb{Z}(X/X)) := [\Delta_X] : f_{\sharp} \mathbb{Z}(X/X) = \mathbb{Z}(X/S) \rightarrow f_* E_{et}(\mathbb{Z}(X/X))(d)[2d]$$

given by the class of the diagonal $[\Delta_X] \in \text{Hom}(f_{\sharp} \mathbb{Z}(X/X), f_* E_{et}(\mathbb{Z}(X/X))(d)[2d])$.

3.2 Constructible motives and resolution of a motive by Corti-Hanamura motives

We now give the definition of the motives of morphisms $f : X \rightarrow S$ which are constructible motives and the definition of the category of Corti-Hanamura motives.

Definition 50. Let $S \in \text{Var}(\mathbb{C})$,

- the homological motive functor is $M(/S) : \text{Var}(\mathbb{C})/S \rightarrow \text{DA}(S)$, $(f : X \rightarrow S) \mapsto M(X/S) := f_! f^! M(S/S)$,
- the cohomological motive functor is $M^\vee(/S) : \text{Var}(\mathbb{C})/S \rightarrow \text{DA}(S)$, $(f : X \rightarrow S) \mapsto M(X/S)^\vee := Rf_* M(X/X) = f_* E(\mathbb{Z}_X)$,
- the Borel-Moore motive functor is $M^{BM}(/S) : \text{Var}(\mathbb{C})/S \rightarrow \text{DA}(S)$, $(f : X \rightarrow S) \mapsto M^{BM}(X/S) := f_! M(X/X)$,
- the (homological) motive with compact support functor is $M_c(/S) : \text{Var}(\mathbb{C})/S \rightarrow \text{DA}(S)$, $(f : X \rightarrow S) \mapsto M_c(X/S) := Rf_* f^! M(S/S)$.

Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. Then,

$$Q(X/S) := p_{\sharp} \Gamma_X^{\vee} \mathbb{Z}_{Y \times S} \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$$

(see definition 10) is projective, admits transfert, and satisfy $D(\mathbb{A}_S^1, \text{et})(Q(X/S)) = M(X/S)$.

Definition 51. (i) Let $S \in \text{Var}(\mathbb{C})$. We define the full subcategory $\text{DA}_c(S) \subset \text{DA}(S)$ whose objects are constructible motives to be the thick triangulated category generated by the motives of the form $M(X/S)$, with $f : X \rightarrow S$ a morphism, $X \in \text{Var}(\mathbb{C})$.

(ii) Let $X, S \in \text{Var}(\mathbb{C})$. If $f : X \rightarrow S$ is proper (but not necessary smooth) and X is smooth, $M(X/S)$ is said to be a Corti-Hanamura motive and we have by above in this case $M(X/S) = M^{\text{BM}}(X/S)[c] = M(X/S)^{\vee}[c]$, with $c = \text{codim}(X, X \times S)$ where $f : X \hookrightarrow X \times S \rightarrow S$. We denote by

$$\mathcal{CH}(S) = \{M(X/S)\}_{\{X/S=(X,f), f \text{ pr}, X \text{ sm}\}}^{\text{pa}} \subset \text{DM}(S)$$

the full subcategory which is the pseudo-abelian completion of the full subcategory whose objects are Corti-Hanamura motives.

(iii) We denote by

$$\mathcal{CH}^0(S) \subset \mathcal{CH}(S)$$

the full subcategory which is the pseudo-abelian completion of the full subcategory whose objects are Corti-Hanamura motives $M(X/S)$ such that the morphism $f : X \rightarrow S$ is projective.

For bounded above motives, we have

Theorem 17. Let $S \in \text{Var}(\mathbb{C})$.

(i) There exists a unique weight structure ω on $\text{DA}^-(S)$ such that $\text{DA}^-(S)^{\omega=0} = \mathcal{CH}(S)$

(ii) There exist a well defined functor

$$W(S) : \text{DA}^-(S) \rightarrow K^-(\mathcal{CH}(S)), W(S)(M) = [M^{(\bullet)}]$$

where $M^{(\bullet)} \in C^-(\mathcal{CH}(S))$ is a bounded above weight complex, such that if $m \in \mathbb{Z}$ is the highest weight, we have a generalized distinguish triangle for all $i \leq m$

$$T_i : M^{(i)}[i] \rightarrow M^{(i+1)}[(i+1)] \rightarrow \cdots \rightarrow M^{(m)}[m] \rightarrow M^{w \geq i} \quad (51)$$

Moreover the maps $w(M)^{(\geq i)} : M^{\geq i} \rightarrow M$ induce an isomorphism $w(M) : \text{holim}_i M^{\geq i} \xrightarrow{\sim} M$ in $\text{DA}^-(S)$.

(iii) Denote by $\text{Chow}(S)$ the category of Chow motives, which is the pseudo-abelian completion of the category

- whose set of objects consist of the $X/S = (X, f) \in \text{Var}(\mathbb{C})/S$ such that f is proper and X is smooth,
- whose set of morphisms between X_1/S and X_2/S is $\text{CH}^{d_1}(X_1 \times_S X_2)$, and the composition law is given in [11].

We have then a canonical functor $\text{CH}_S : \text{Chow}(S) \hookrightarrow \text{DA}(S)$, with $\text{CH}_S(X/S) := M(X/S) := \text{Rf}_* \mathbb{Z}(X/X)$, which is a full embedding whose image is the category $\mathcal{CH}(S)$.

Proof. (i): The category $\text{DA}(S)$ is clearly weakly generated by $\mathcal{CH}(S)$. Moreover $\mathcal{CH}(S) \subset \text{DA}(S)$ is negative. Hence, the result follows from [6] theorem 4.3.2 III.

(ii): Follows from (i) by standard fact of weight structure on triangulated categories. See [6] theorem 3.2.2 and theorem 4.3.2 V for example.

(iii): See [13]. □

Theorem 18. *Let $S \in \text{Var}(\mathbb{C})$.*

(i) *There exists a unique weight structure ω on $\text{DA}^-(S)$ such that $\text{DA}^-(S)^{\omega=0} = \mathcal{CH}^0(S)$*

(ii) *There exist a well defined functor*

$$W(S) : \text{DA}^-(S) \rightarrow K^-(\mathcal{CH}^0(S)), \quad W(S)(M) = [M^{(\bullet)}]$$

where $M^{(\bullet)} \in C^-(\mathcal{CH}^0(S))$ is a bounded above weight complex, such that if $m \in \mathbb{Z}$ is the highest weight, we have a generalized distinguish triangle for all $i \leq m$

$$T_i : M^{(i)}[i] \rightarrow M^{(i+1)}[(i+1)] \rightarrow \cdots \rightarrow M^{(m)}[m] \rightarrow M^{w \geq i} \quad (52)$$

Moreover the maps $w(M)^{(\geq i)} : M^{\geq i} \rightarrow M$ induce an isomorphism $w(M) : \text{holim}_i M^{\geq i} \xrightarrow{\sim} M$ in $\text{DA}^-(S)$.

Proof. Similar to the proof of theorem 17. □

Corollary 1. *Let $S \in \text{Var}(\mathbb{C})$. Let $M \in \text{DA}(S)$. Then there exist $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F) = M$ and $D(\mathbb{A}^1, et)(\text{Gr}_p^W F) \in \mathcal{CH}^0(S)$.*

Proof. By theorem 18, there exist, by induction, for $i \in \mathbb{Z}$, a distinguish triangle in $\text{DA}(S)$

$$T_i : M^{(i)}[i] \xrightarrow{m_i} M^{(i+1)} \xrightarrow{m_{i+1}} \cdots \xrightarrow{m_{m-1}} M^{(m)}[m] \rightarrow M^{w \geq i} \quad (53)$$

with $M^{(j)}[j] \in \mathcal{CH}^0(S)$ and $w(M) : \text{holim}_i M^{\geq i} \xrightarrow{\sim} M$ in $\text{DA}^-(S)$. For $i \in \mathbb{Z}$, take $(F_j)_{j \geq i}, F_{w \geq i} \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F_j) = M^{(j)}[j]$, $D(\mathbb{A}^1, et)(F_{w \geq i}) = M^{w \geq i}$ and such that we have in $C(\text{Var}(\mathbb{C})^{sm}/S)$,

$$F_{w \geq i} = \text{Cone}(F_i \xrightarrow{m_i} F_{i+1} \xrightarrow{m_{i+1}} \cdots \xrightarrow{m_{m-1}} F_m) \quad (54)$$

where $m_j : F_j \rightarrow F_{j+1}$ are morphisms in $C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(m_j) = m_j$. Now set $F = \text{holim}_i F_{w \geq i} \in C(\text{Var}(\mathbb{C})^{sm}/S)$ and $W_i F := F_{w \geq i} \hookrightarrow F$, so that $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ satisfy $D(\mathbb{A}^1, et)(\text{Gr}_p^W F) = M^{(p)}[p] \in \mathcal{CH}^0(S)$. □

3.3 The restriction of relative motives to their Zariski sites

Let $S \in \text{Var}(\mathbb{C})$. The adjunction

$$(e(S)^*, e(S)_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(S)$$

is a Quillen adjunction and induces in the derived category

- $(e(S)^*, e(S)_*) : \text{Ho}_{\text{zar}}(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows D(T) := \text{Ho}_{\text{zar}} C(S)$, since $e(S)_*$ sends Zariski local equivalence on the big site $\text{Var}(\mathbb{C})^{sm}/S$ to Zariski local equivalence in the small Zariski site of S ,
- $(e(S)^*, \text{Re}(S)_*) : \text{DA}(S) \rightleftarrows D(T) := \text{Ho}_{\text{zar}} C(S)$.

We will use in the definition of the De Rahm realization functor on $\text{DA}(S)$ the following proposition concerning the restriction of the derived internal hom functor to the Zariski site :

Proposition 41. *Let $M, N \in \text{DA}(S)$ and $m : M \rightarrow N$ be a morphism. Let $F^\bullet, G^\bullet \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S, C(\mathbb{Z}))$ such that $M = D(\mathbb{A}_S^1, et)(F^\bullet)$ and $N = D(\mathbb{A}_S^1, et)(G^\bullet)$. If we take G^\bullet (\mathbb{A}_S^1, et) fibrant and admitting transfert, and F^\bullet cofibrant for the projective model structure, we have*

$$\text{Re}(S)_* \text{RHom}^\bullet(M, N) = e(S)_* \text{Hom}^\bullet(F^\bullet, G^\bullet)$$

in $D(S)$.

Proof. Since F^\bullet is projectively cofibrant and G^\bullet is (projectively) (\mathbb{A}_S^1, et) fibrant, we have $R\mathcal{H}om^\bullet(M, N) = \mathcal{H}om^\bullet(F^\bullet, G^\bullet)$. Then, $\mathcal{H}om^\bullet(F^\bullet, G^\bullet)$ is \mathbb{A}_S^1 local and admits transfert. On the other hand, we have

$$\mathcal{L}_{\mathbb{A}_S^1} D_{et}(\text{Cor Var}(\mathbb{C})^{sm}/S) = \mathcal{L}_{\mathbb{A}_S^1} D_{zar}(\text{Cor Var}(\mathbb{C})^{sm}/S) \subset D(\text{Var}(\mathbb{C})^{sm}/S)$$

by theorem 10 (ii). This gives the equality of the proposition. \square

We will also have :

Proposition 42. *For $f : T \rightarrow S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding, with $Z, S, T \in \text{Var}(\mathbb{C})$, we have*

$$(i) \quad Re(S)_* Rf_* = Rf_* Re(T)_* \text{ and } e(S)^* Rf_* = Rf_* e(T)^*$$

$$(ii) \quad Re(S)_* R\Gamma_Z = R\Gamma_Z Re(S)_*.$$

Proof. (i): Follows from proposition 16 (i) and the fact that f_* preserve (\mathbb{A}^1, et) fibrant complex of presheaves.

(ii): Follows from proposition 16 (ii) and the fact that Γ_Z preserve (\mathbb{A}^1, et) fibrant complex of presheaves. \square

3.4 Motives of complex analytic spaces

The category of motives is obtained by inverting the (\mathbb{D}_S^1, usu) local equivalence. Hence the \mathbb{D}_S^1 local complexes of presheaves plays a key role.

Definition 52. *The derived category of motives of complex algebraic varieties over S is the category*

$$\text{AnDA}(S) := \text{Ho}_{\mathbb{D}_S^1, usu}(C(\text{AnSp}(\mathbb{C})^{sm}/S)),$$

which is the localization of the category of complexes of presheaves on $\text{AnSp}(\mathbb{C})^{sm}/S$ with respect to (\mathbb{D}_S^1, usu) local equivalence and we denote by

$$D(\mathbb{D}_S^1, usu) := D(\mathbb{A}_S^1) \circ D(et) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow \text{AnDA}(S)$$

the localization functor. We have $\text{DA}^-(S) := D(\mathbb{D}_S^1, usu)(\text{PSh}(\text{AnSp}(\mathbb{C})^{sm}/S, C^-(\mathbb{Z}))) \subset \text{DA}(S)$ the full subcategory consisting of bounded above complexes.

Theorem 19. *Let $S \in \text{AnSp}(\mathbb{C})$. The adjunction $(e(S)^*, e(S)_*) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(S)$ induces an equivalence of categories*

$$(e(S)^*, Re(S)_*) : \text{AnDM}(S) \xrightarrow{\sim} D(S).$$

In particular, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, the adjunction map $\text{ad}(e(S)^, e(S)_*)(F) : e(S)^* e(S)_* F \rightarrow F$ is an equivalence (\mathbb{D}^1, usu) local.*

Proof. See [1]. \square

We deduce from this theorem the following :

Proposition 43. *Let $S \in \text{AnSp}(\mathbb{C})$. Let $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$. If G is \mathbb{D}^1 local, then the canonical map*

$$T(e, hom)(F, G) : e(S)_* \mathcal{H}om(F, G) \rightarrow \mathcal{H}om(e(S)_* F, e(S)_* G)$$

is an equivalence usu local.

Proof. The map $T(e, hom)(F, G)$ is the composite

$$\begin{aligned} T(e, hom)(F, G) : e(S)_* \mathcal{H}om(F, G) &\xrightarrow{\mathcal{H}om(\text{ad}(e(S)^*, e(S)_*)(F), G)} e(S)_* \mathcal{H}om(e(S)^* e(S)_* F, G) \\ &\xrightarrow{I(e(S)^*, e(S)_*)(F, G)} \mathcal{H}om(e(S)_* F, e(S)_* G) \end{aligned}$$

where the last map is the adjunction isomorphism. The first map is an isomorphism by theorem 19 since G is \mathbb{D}^1 local. \square

4 The category of filtered D modules on commutative ringed topos, on commutative ringed spaces, complex algebraic varieties complex analytic spaces and the functorialities

4.1 The The category of filtered D modules on commutative ringed topos, on commutative ringed spaces, and the functorialities

4.1.1 Definitions et functorialities

Let $(\mathcal{S}, O_S) \in \text{RCat}$ with O_S commutative. Recall that $\Omega_{O_S} := \mathbb{D}_S^O(\mathcal{I}_S/\mathcal{I}_S^2) \in \text{PSh}_{O_S}(\mathcal{S})$ is the universal derivation O_S -module together with its derivation map $d : O_S \rightarrow \Omega_{O_S}$, where $\mathcal{I}_S = \ker(s_S : O_S \otimes O_S \rightarrow O_S) \in \text{PSh}_{O_S \times O_S}(\mathcal{S})$ the diagonal ideal.

In the particular case of a ringed space $(S, O_S) \in \text{RTop}$, $s_S : O_S \otimes O_S = \Delta_S^*(p_1^*O_S \otimes p_2^*O_S) \rightarrow O_S$ is the structural morphism of diagonal embedding $\Delta_S : (S, O_S) \hookrightarrow (S \times S, p_1^*O_S \otimes p_2^*O_S)$, $p_1 : S \times S \rightarrow S$ and $p_2 : S \times S \rightarrow S$ being the projections. More generally, for $k \in \mathbb{N}$, $k \geq 1$ we have the sheaf of k -jets $J^k(O_S) := \Delta_S^* \mathcal{I}_S / \mathcal{I}_S^{k+1}$ with in particular $J^1(O_S) = T_S$. We have, for $s \in S$, $J^k(O_S)_s = m_s / m_s^k$ where $m_s \subset O_{S,s}$ is the maximal ideal if $O_{S,s}$ is a local ring.

Definition 53. (i) Let $(\mathcal{S}, O_S) \in \text{RCat}$ with O_S a commutative sheaf of ring and \mathcal{S} is endowed with a topology τ . We denote by

$$D(O_S) = \langle O_S, \text{Der}_{O_S}(O_S, O_S) \rangle \subset a_\tau \mathcal{H}om(O_S, O_S)$$

the subsheaf of ring generated by O_S and the subsheaf of derivations $\text{Der}_{O_S}(O_S, O_S) = T_S := \mathbb{D}_S^O \Omega_{O_S}$, $a_\tau : \text{PSh}(\mathcal{S}) \rightarrow \text{Shv}(\mathcal{S})$ being the sheaftification functor.

(ii) Let $f : \mathcal{X} \rightarrow \mathcal{S}$ be a morphism of site, with $\mathcal{X}, \mathcal{S} \in \text{Cat}$ endowed with topology τ , resp. τ' , and $O_S \in \text{PSh}(\mathcal{S}, \text{cRing})$ a commutative sheaf of ring. We will note in this case by abuse $f^*O_S := a_\tau f^*O_S$ and $f^*D(O_S) := a_\tau f^*D(O_S)$, $a_\tau : \text{PSh}(\mathcal{X}) \rightarrow \text{Shv}(\mathcal{X})$ being the sheaftification functor.

Let $f : \mathcal{X} \rightarrow \mathcal{S}$ a morphism of site, with $\mathcal{X}, \mathcal{S} \in \text{Cat}$ endowed with topology τ , resp. τ' , and $O_S \in \text{PSh}(\mathcal{S}, \text{cRing})$ a commutative sheaf of ring. Consider the ringed space $(\mathcal{X}, f^*O_S) := (\mathcal{X}, a_\tau f^*O_S) \in \text{RCat}$, $a_\tau : \text{PSh}(\mathcal{X}) \rightarrow \text{Shv}(\mathcal{X})$ being the sheaftification functor. Then, the map in $\text{PSh}(\mathcal{X})$

$$T(f, \text{hom})(O_S, O_S) : f^* \mathcal{H}om(O_S, O_S) \rightarrow \mathcal{H}om(f^*O_S, f^*O_S)$$

induces a canonical morphism of sheaf of rings

$$T(f, \text{hom})(O_S, O_S) : a_\tau f^*D(O_S) =: f^*D(O_S) \rightarrow D(a_\tau f^*O_S) =: D(f^*O_S).$$

In the special case of ringed spaces, we have then :

Proposition 44. Let $f : X \rightarrow S$ is a continous map, with $X, S \in \text{Top}$ and $O_S \in \text{PSh}(S, \text{cRing})$ a commutative sheaf of ring. Consider the ringed space $(X, f^*O_S) := (X, a_\tau f^*O_S) \in \text{RTop}$, $a_\tau : \text{PSh}(X) \rightarrow \text{Shv}(X)$ being the sheaftification functor. Then, the map in $\text{PSh}(X)$

$$T(f, \text{hom})(O_S, O_S) : f^* \mathcal{H}om(O_S, O_S) \rightarrow \mathcal{H}om(f^*O_S, f^*O_S)$$

induces a canonical isomorphism of sheaf of rings

$$T(f, \text{hom})(O_S, O_S) : f^*D(O_S) =: a_\tau f^*D(O_S) \xrightarrow{\sim} D(a_\tau f^*O_S) =: D(f^*O_S).$$

Proof. For all $x \in X$,

$$T(f, \text{hom})(O_S, O_S)_x : (f^*D(O_S))_x \xrightarrow{\sim} D(O_{S, f(x)}) \xrightarrow{\sim} (D(f^*O_S))_x.$$

Hence, since $a_\tau f^*D(O_S)$ and $D(a_\tau f^*O_S)$ are sheaves,

$$T(f, \text{hom})(O_S, O_S) : f^*D(O_S) =: a_\tau f^*D(O_S) \xrightarrow{\sim} D(a_\tau f^*O_S) =: D(f^*O_S).$$

is an isomorphism □

We will consider presheaves of $D(O_S)$ modules on a ringed topos (\mathcal{S}, O_S) :

Definition 54. Let $(\mathcal{S}, O_S) \in \text{RCat}$ with O_S a commutative sheaf of ring.

- (i) We will consider $\text{PSh}_{D(O_S)}(\mathcal{S})$ the category of presheaves of (left) $D(O_S)$ modules on \mathcal{S} and $C_{D(O_S)}(\mathcal{S}) := C(\text{PSh}_{D(O_S)}(\mathcal{S}))$ its category of complexes. We will consider $\text{PSh}_{D(O_S)^{op}}(\mathcal{S})$ the category of presheaves of right $D(O_S)$ modules on \mathcal{S} and $C_{D(O_S)^{op}}(\mathcal{S}) := C(\text{PSh}_{D(O_S)^{op}}(\mathcal{S}))$ its category of complexes. We denote again by abuse

$$\text{PSh}_{D(O_S)fil}(\mathcal{S}) = (\text{PSh}_{D(O_S)}(\mathcal{S}), F) := (\text{PSh}_{(D(O_S), F^{ord})}(\mathcal{S}), F)$$

the category of filtered $(D(O_S), F^{ord})$ -module, with, for $-p \leq 0$, $F^{ord, -p}D(O_S) = \{P \in D(O_S), \text{ord}(P) \leq p\}$ and $F^{ord, p}D(O_S) = 0$ for $p > 0$,

- whose objects are $(M, F) \in (\text{PSh}_{O_S}(\mathcal{S}), F)$ such that (M, F) is compatible with $(D(O_S), F^{ord})$ that is $F^{ord, -p}D(O_S) \cdot F^q M \subset F^{q-p}M$ (Griffithz transversality), this is to say that the structural map $md : M \otimes_{O_S} D(O_S) \rightarrow M$ induces a filtered map of presheaves (i.e a map in $(\text{PSh}_{O_S}(\mathcal{S}), F)$) $md : (M, F) \otimes_{O_S} (D(O_S), F^{ord}) \rightarrow (M, F)$,
- whose morphism $\phi : (M_1, F) \rightarrow (M_2, F)$ are as usual the morphisms of presheaves $\phi : M_1 \rightarrow M_2$ which are morphism of filtered presheaves (i.e. $\phi(F^p M_1) \subset F^p M_2$) and which are $D(O_S)$ linear (in particular O_S linear).

Note that this a NOT the category of filtered $D(O_S)$ modules in the usual sense, that is the $(M, F) \in (\text{PSh}_{O_S}(\mathcal{S}), F)$ together with a map $md : (M, F) \otimes_{O_S} D(O_S) \rightarrow (M, F)$ in $(\text{PSh}_{O_S}(\mathcal{S}), F)$, since F^{ord} is NOT the trivial filtration. More precisely the O_S submodules $F^p M \subset M$ are NOT $D(O_S)$ submodules but satisfy Griffithz transversality. We denote by

$$\text{PSh}_{D(O_S)0fil}(\mathcal{S}) \subset \text{PSh}_{D(O_S)fil}(\mathcal{S}), \text{PSh}_{D(O_S)(1,0)fil}(\mathcal{S}) \subset \text{PSh}_{D(O_S)2fil}(\mathcal{S}) := (\text{PSh}_{(D(O_S), F^{ord})}(\mathcal{S}), F, W)$$

the full subcategory consisting of filtered $D(O_S)$ module in the usual sense, resp. the full subcategory such that W is a filtration by $D(O_S)$ submodules.

- (ii) We denote again by

$$C_{D(O_S)fil}(\mathcal{S}) \subset C(\text{PSh}_{D(O_S)}(\mathcal{S}), F), C_{D(O_S)2fil}(\mathcal{S}) \subset C(\text{PSh}_{D(O_S)}(\mathcal{S}), F, W)$$

the full subcategory of complexes such that the filtration(s) is (are) regular. We will consider also

$$C_{D(O_S)0fil}(\mathcal{S}) \subset C_{D(O_S)fil}(\mathcal{S}), C_{D(O_S)(1,0)fil}(\mathcal{S}) \subset C_{D(O_S)2fil}(\mathcal{S})$$

the full subcategory consisting of complexes of filtered $D(O_S)$ modules in the usual sense (i.e. by $D(O_S)$ submodule), respectively the full subcategory consisting of complexes of bifilterd $D(O_S)$ modules such that $W^p M \subset M$ are $D(O_S)$ submodules i.e. the filtration W is a filtration in the usual sense, but NOT F wich satisfy only Griffithz transversality.

Proposition 45. Let $(\mathcal{S}, O_S) \in \text{RCat}$ with a O_S commutative sheaf of ring.

- (i) Let $M \in \text{PSh}_{O_S}(\mathcal{S})$. Then, there is a one to one correspondence between

- the $D(O_S)$ module structure on M compatible with the O_S module structure, that is the maps $md : D(O_S) \otimes_{O_S} M \rightarrow M$ in $\text{PSh}_{O_S}(\mathcal{S})$ and
- the integrable connexions on M , that is the maps $\nabla : M \rightarrow \Omega_{O_S} \otimes_{O_S} M$ satisfying $\nabla \circ \nabla = 0$ with $\nabla : \Omega_{O_S} \otimes_{O_S} M \rightarrow \Omega_{O_S}^2 \otimes_{O_S} M$ given by $\nabla(\omega \otimes m) = (d\omega) \otimes m + \omega \wedge \nabla(m)$

- (ii) Let $(M, F) \in \text{PSh}_{O_S fil}(\mathcal{S})$. Then, there is a one to one correspondence between

- the $D(O_S)$ module structure on (M, F) compatible with the O_S module structure, that is the maps $md : (D(O_S), F^{ord}) \otimes_{O_S} (M, F) \rightarrow (M, F)$ in $\text{PSh}_{O_S \text{fil}}(\mathcal{S})$ and
- the integrable connexions on M , that is the maps $\nabla : (M, F) \rightarrow \Omega_{O_S} \otimes_{O_S} (M, F)$ satisfying $\nabla \circ \nabla = 0$ and Griffith transversality (i.e. $\nabla(F^p M) \subset \Omega_{O_S} \otimes_{O_S} F^{p-1} M$).

Proof. Standard. □

The following proposition tells that the O -tensor product of D modules has a canonical structure of D module.

Definition-Proposition 13. (i) Let $f : (\mathcal{X}, O_X) \rightarrow (\mathcal{S}, O_S)$ a morphism with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RCat}$ with commutative structural sheaf of ring. For $N \in \text{PSh}_{O_X, D(f^*O_S)}(\mathcal{X})$ and $M \in \text{PSh}_{O_X, D(f^*O_S)}(\mathcal{X})$, $N \otimes_{O_X} M$ has the canonical $D(f^*O_S)$ module structure given by, for $X^o \in \mathcal{X}$,

$$\gamma \in \Gamma(X^o, D(f^*O_S)), m \in \Gamma(X^o, M), n \in \Gamma(X^o, N), \gamma.(n \otimes m) = (\gamma.n) \otimes m + n \otimes (\gamma.m).$$

This gives the functor

$$\text{PSh}_{O_X, D(f^*O_S) \text{fil}}(\mathcal{X}) \times \text{PSh}_{O_X, D(f^*O_S) \text{fil}}(\mathcal{X}) \rightarrow \text{PSh}_{O_X, D(f^*O_S) \text{fil}}(\mathcal{X}), ((M, F), (N, F)) \mapsto (M, F) \otimes_{O_X} (N, F), F^p(M, F) \otimes_{O_X} (N, F) := \text{Im}(\oplus_{q \in \mathbb{Z}} F^q M \otimes_{O_X} F^{p-q} N \rightarrow M \otimes_{O_X} N)$$

(ii) Let $f : (\mathcal{X}, O_X) \rightarrow (\mathcal{S}, O_S)$ a morphism with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RCat}$ with commutative structural sheaf of ring. For $N \in C_{D(O_X), D(f^*O_S)}(\mathcal{X})$ and $M \in C_{D(O_X)^{op}}(\mathcal{X})$, $N \otimes_{D(O_X)} M$ has the canonical $f^*D(O_S)$ module structure given by, for $X^o \in \mathcal{X}$,

$$\gamma \in \Gamma(X^o, D(f^*O_S)), m \in \Gamma(X^o, M), n \in \Gamma(X^o, N), \gamma.(n \otimes m) = (\gamma.n) \otimes m.$$

This gives the functor

$$C_{D(O_X), D(f^*O_S) \text{fil}}(\mathcal{X}) \times C_{D(O_X) \text{fil}}(\mathcal{X}) \rightarrow C_{D(f^*O_S) \text{fil}}(\mathcal{X}), ((M, F), (N, F)) \mapsto (M, F) \otimes_{D(O_X)} (N, F), F^p(M, F) \otimes_{D(O_X)} (N, F) := \text{Im}(\oplus_{q \in \mathbb{Z}} F^q M \otimes_{D(O_X)} F^{p-q} N \rightarrow M \otimes_{D(O_X)} N)$$

Note that, by definition, we have for $(M, F) \in (\text{PSh}_{D(O_S) \text{fil}}(\mathcal{S}))$, the canonical isomorphism

$$(M, F) \otimes_{D(O_S)} (D(O_S), F^{ord}) \xrightarrow{\sim} (M, F), m \otimes P \mapsto Pm, m \mapsto (m \otimes 1)$$

Proof. Immediate from definition. □

We now look at the functorialities for morphisms of ringed spaces, using proposition 44. First note that for $f : (X, O_X) \rightarrow (S, O_S)$ a morphism, with $(X, O_X), (S, O_S) \in \text{RTop}$ with structural presheaves commutative sheaves of rings, there is NO canonical morphism between $D(f^*O_S) = f^*D(O_S)$ (see proposition 44) and $D(O_X)$.

We have the pullback functor for (left) D -modules :

Definition-Proposition 14. (i) Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(X, O_X), (S, O_S) \in \text{RTop}$ with structural presheaves commutative sheaves of rings. Recall that $f^*D(O_S) = D(f^*O_S)$ in this case. Then for $(M, F) \in \text{PSh}_{D(O_S) \text{fil}}(\mathcal{S})$,

$$f^{* \text{mod}}(M, F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M, F) \in \text{PSh}_{O_X \text{fil}}(X)$$

has a canonical structure of filtered $D(O_X)$ module given by

$$\text{for } \gamma \in \Gamma(X^o, T_{O_X}), n \otimes m \in \Gamma(X^o, O_X \otimes_{f^*O_S} f^*M), \gamma.(n \otimes m) := (\gamma.n) \otimes m + n \otimes df(\gamma)(m)$$

with $df := \mathbb{D}_S^O \Omega_{O_X/f^*O_S} : T_{O_X} \rightarrow T_{f^*O_S} = f^*T_{O_S}$ and $f^*(M, F) \in \text{PSh}_{f^*D(O_S) \text{fil}}(X) = \text{PSh}_{D(f^*O_S) \text{fil}}(X)$.

(ii) More generally, let $f : (\mathcal{X}, O_X) \rightarrow (\mathcal{S}, O_S)$ a morphism with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RCat}$ with structural presheaves commutative sheaves of rings. Assume that the canonical morphism $T(f, \text{hom})(O_S, O_S) : f^*D(O_S) \rightarrow D(f^*O_S)$ is an isomorphism of sheaves. Then for $(M, F) \in \text{PSh}_{D(O_S)\text{fil}}(\mathcal{S})$,

$$f^{*\text{mod}}(M, F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M, F) \in \text{PSh}_{O_X\text{fil}}(\mathcal{X})$$

has a canonical structure of filtered $D(O_X)$ module given by

$$\text{for } \gamma \in \Gamma(X^\circ, T_{O_X}), n \otimes m \in \Gamma(X^\circ, O_X \otimes_{f^*O_S} f^*M), \gamma \cdot (n \otimes m) := (\gamma \cdot n) \otimes m + n \otimes df(\gamma)(m)$$

$$\text{with } df := \mathbb{D}_S^0 \Omega_{O_X/f^*O_S} : T_{O_X} \rightarrow T_{f^*O_S} = f^*T_{O_S} \text{ and } f^*(M, F) \in \text{PSh}_{f^*D(O_S)\text{fil}}(\mathcal{X}) = \text{PSh}_{D(f^*O_S)\text{fil}}(\mathcal{X}).$$

Proof. Standard. \square

Remark 6. • Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(X, O_X), (S, O_S) \in \text{RTop}$ with structural presheaves commutative sheaves of rings. Recall that $f^*D(O_S) = D(f^*O_S)$. Then by definition $f^{*\text{mod}}(O_S, F_b) = (O_X, F_b)$.

• More generally, let $f : (\mathcal{X}, O_X) \rightarrow (\mathcal{S}, O_S)$ a morphism with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RCat}$ with structural presheaves commutative sheaves of rings. Assume that the canonical morphism $T(f, \text{hom})(O_S, O_S) : f^*D(O_S) \rightarrow D(f^*O_S)$ is an isomorphism of sheaves. Then by definition $f^{*\text{mod}}(O_S, F_b) = (O_X, F_b)$.

For the definition of a push-forward functor for a right D module we use the transfert module

Let $f : (X, O_X) \rightarrow (S, O_S)$ be a morphism with $(X, O_X), (S, O_S) \in \text{RTop}$ with structural presheaves commutative sheaves of rings. Then, the transfer module is

$$(D(O_X \rightarrow f^*O_S), F^{\text{ord}}) := f^{*\text{mod}}(D(O_S), F^{\text{ord}}) := f^*(D(O_S), F^{\text{ord}}) \otimes_{f^*O_S} (O_X, F_b)$$

which is a left $D(O_X)$ module and a left and right $f^*D(O_S) = D(f^*O_S)$ module.

Lemma 2. Let $f_1 : (X, O_X) \rightarrow (Y, O_Y)$, $f_2 : (Y, O_Y) \rightarrow (S, O_S)$ be two morphism with $(X, O_X), (Y, O_Y), (S, O_S) \in \text{RTop}$. We have in $C_{D(O_X), (f_2 \circ f_1)^*D(O_S)\text{fil}}(X)$

$$(D_{O_X \rightarrow (f_2 \circ f_1)^*O_S}, F^{\text{ord}}) = f_1^*(D_{O_Y \rightarrow f_2^*O_S}, F^{\text{ord}}) \otimes_{f_1^*D(O_Y)} (D_{O_X \rightarrow f_1^*O_Y}, F^{\text{ord}})$$

Proof. Follows immediately from definition. \square

For right D module, we have the direct image functor :

Definition 55. Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(X, O_X), (S, O_S) \in \text{RTop}$ with structural presheaves commutative sheaves of rings. Then for $(M, F) \in C_{D(O_X)^{\text{op}}\text{fil}}(X)$, we define

$$f_{*\text{mod}}^0(M, F) = f_*((D_{O_X \rightarrow f^*O_S}, F^{\text{ord}}) \otimes_{D(O_X)} (M, F)) \in C_{D(O_S)\text{fil}}(S)$$

For a closed embedding of topological spaces, there is the V -filtration on the structural sheaf, it will play an important role in this article

Definition 56. (i) Let $(S, O_S) \in \text{RTop}$ a locally ringed space. Let $Z = V(\mathcal{I}_Z) \subset S$ a Zariski closed subset. We set, for $S^\circ \subset S$ an open subset, $p \in \mathbb{Z}$,

- $V_{Z,p}O_S(S^\circ) := O_S(S^\circ)$ if $p > 0$,
- $V_{Z,-q}O_S(S^\circ) := \mathcal{I}_Z^q(S^\circ) \subset O_S(S^\circ)$ for $p = -q \leq 0$.

We immediately check that, by definition, this filtration satisfy Griffithz transversality, that is $(O_S, V_Z) \in \text{PSh}_{D(O_S)\text{fil}}(S)$. For a morphism $g : ((T, O_T), Z') \rightarrow ((S, O_S), Z)$ with $((T, O_T), Z), ((S, O_S), Z) \in \text{RTop}^2$ locally ringed spaces, where Z and Z' are Zariski closed subsets, the structural morphism $a_g : g^*O_S \rightarrow O_T$ is a filtered morphism :

$$a_g : g^*(O_S, V_Z) \rightarrow (O_T, V_{Z'}), h \mapsto a_g(h)$$

(ii) Let $(S, O_S) \in \text{RTop}$. Let $i : Z \hookrightarrow S$ a closed embedding. The V_Z -filtration on O_S (see (i)) gives the filtration, given by for $p \in \mathbb{Z}$,

$$V_{Z,p} \text{Hom}(O_S, O_S) := \left\{ P \in \text{Hom}(O_S, O_S), \text{ s.t. } P \mathcal{I}_Z^k \subset \mathcal{I}_Z^{k-p} \right\}$$

on $\text{Hom}(O_S, O_S)$, which induces the filtration $V_{Z,p} D(O_S) := D(O_S) \cap V_{Z,p} \text{Hom}(O_S, O_S)$ on $D(O_S) \subset \text{Hom}(O_S, O_S)$. We get $(D(O_S), V_Z) \in \text{PSh}_{\text{fil}}(S, \text{Ring})$ and we call it the V_Z -filtration on $D(O_S)$.

(iii) Let $(S, O_S) \in \text{RTop}$ a locally ringed space. Let $i : Z = V(\mathcal{I}_Z) \hookrightarrow S$ a Zariski closed embedding and $O_Z := i^* O_S / \mathcal{I}_Z$. We say that $M \in \text{PSh}_{D(O_S)}(S)$ is specializable on Z if it admits an (increasing) filtration (called a V_Z -filtration) $(M, V) \in \text{PSh}_{O_S, \text{fil}}(S)$ compatible with (D_S, V_Z) , that is $V_{Z,p} D_S \cdot V_q M \subset V_{p+q} M$, this is to say that the structural map $\text{md} : M \otimes_{O_S} D(O_S) \rightarrow M$ induces filtered map of presheaves $\text{md} : (M, V) \otimes_{O_S} (D(O_S), V_Z) \rightarrow (M, V)$. For $(M, F) \in \text{PSh}_{D(O_S), \text{fil}}(S)$ such that M is specializable on Z , we thus get a filtered morphism $\text{md} : (M, F, V) \otimes_{O_S} (D(O_S), F^{\text{ord}}, V_Z) \rightarrow (M, F, V)$.

(iii)' Consider an injective morphism $m : M_1 \hookrightarrow M_2$ with $M_1, M_2 \in \text{PSh}_{D(O_S)}(S)$. If M_2 admits a V_Z filtration V_2 , then the filtration V_{21} induced on M_1 (recall $V_{21,p} M_1 := V_{2,p} M_2 \cap M_1$) is a V_Z filtration. Consider a surjective morphism $n : M_1 \rightarrow M_2$ with $M_1, M_2 \in \text{PSh}_{D(O_S)}(S)$. If M_1 admits a V_Z filtration V_1 , then the filtration V_{12} induced on M_2 (recall $V_{12,p} M_2 := n(V_{1,p} M_1)$) is a V_Z filtration.

(iv) Let $(S, O_S) \in \text{RTop}$ a locally ringed space. Let $i : Z = V(\mathcal{I}_Z) \hookrightarrow S$ a Zariski closed embedding and $O_Z := i^* O_S / \mathcal{I}_Z$. For $(M, F) \in \text{PSh}_{D(O_S), \text{fil}}(S)$ such that M admits a V_Z filtration V so that $(M, F, V) \in \text{PSh}_{O_S, 2\text{fil}}(S)$, we will consider the quotient map in $\text{PSh}_{O_S, \text{fil}}(S)$

$$q_{V,0} : (M, F) \rightarrow (M, F) / V_{-1}(M, F) =: Q_{V,0}(M, F).$$

The quotient $i^* Q_{V,0}(M, F)$ has an action of T_{O_Z} since for $S^\circ \subset S$ an open subset and $\partial_z \in \Gamma(Z \cap S^\circ, T_{O_Z}) \subset \Gamma(S^\circ, T_{O_S})$, we have $\partial_z \in \Gamma(S, V_{Z,0} D(O_S))$ since for $f = \sum_{i=1}^r t_i h_i \in \Gamma(S^\circ, \mathcal{I}_Z)$, where $(t_i) = \mathcal{I}_Z(S^\circ)$ are generators of the ideal $\mathcal{I}_Z(S^\circ) \subset O_S(S^\circ)$ and $h_i \in \Gamma(S^\circ, O_S)$, we have

$$\partial_z \left(\sum_{i=1}^r t_i h_i \right) = \sum_{i=1}^r (\partial_z(t_i) h_i + t_i \partial_z(h_i)) = \sum_{i=1}^r t_i (\partial_z(h_i)) \in \Gamma(S, \mathcal{I}_Z)$$

as $\partial_z(t_i) = 0$ (only the vector field of T_{O_S} which are transversal to $T_{O_Z} \subset T_{O_S}$ increase the grading), Then, obviously, by definition, the map in $\text{PSh}_{i^* O_S, \text{fil}}(Z)$

$$i^* q_{V,0} : i^* (M, F) / V_{-1}(M, F) =: i^* Q_{V,0}(M, F)$$

commutes with the action of $T_{O_Z} \subset i^* T_{O_S}$ and we call it the specialization map.

Definition-Proposition 15. Let $(S, O_S) \in \text{RTop}$ a locally ringed space. Consider a commutative diagram

$$\begin{array}{ccc} Z_1 = V(\mathcal{I}_1) & \xrightarrow{i_1} & (S, O_S) \\ i'_2 \uparrow & & \uparrow i_2 \\ Z = V(\mathcal{I}) & \xrightarrow{i'_1} & Z_2 = V(\mathcal{I}_2) \end{array}$$

where the maps are Zariski closed embeddings and which is cartesian (i.e. $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2)$, in particular $Z = Z_1 \cap Z_2$).

(i) Let $(M, F) \in \text{PSh}_{D(O_S), \text{fil}}(S)$ such that M admits a V_{Z_1} -filtration V_1 and a V_{Z_2} -filtration V_2 (see definition 56). Let $p, q \in \mathbb{Z}$. Then, we consider

- the filtration V_{21} on $V_{1,p}(M, F) := (V_{1,p}M, F)$ induced by V_2 ,
- the filtration V_{12} on $Q_{V_{2,p}}(M, F) := (M/V_{2,p-1}M, F)$ induced by V_1 .

The quotient map in $\text{PSh}_{i'_{1*}O_{Z_1}}\text{fil}(S)$

$$q_{V_{2,p}} : V_{1,q}(M, F) \rightarrow V_{12,q}Q_{V_{2,p}}(M, F)$$

factors through

$$q_{V_{2,p}} : V_{1,q}(M, F) \xrightarrow{q_{V_{21,p}}} Q_{V_{21,p}}V_{1,q}(M, F) \xrightarrow{Q_{V_1, V_2}^{p,q}(M, F)} V_{12,q}Q_{V_{2,p}}(M, F),$$

and the map $Q_{V_1, V_2}^{p,q}(M, F)$ in $\text{PSh}_{i'_{1*}O_{Z_1}}\text{fil}(S)$ commute with the action of T_{O_Z} .

- (ii) If $m : (M, F) \rightarrow (M', F)$ is a morphism with $(M, F), (M', F) \in \text{PSh}_{D(O_S)}\text{fil}(S)$ admitting V_{Z_1} -filtration V_1 and V'_1 respectively such that $m(V_{1,p}M) \subset V'_{1,p}M'$ and V_{Z_2} -filtration V_2 and V'_2 respectively such that $m(V_{1,p}M) \subset V'_{1,p}M'$. Then for all $p, q \in \mathbb{Z}$ the following diagram commutes

$$\begin{array}{ccc} Q_{V_{21,p}}V_{1,q}(M, F) & \xrightarrow{Q_{V_1, V_2}^{p,q}(M, F)} & V_{12,q}Q_{V_{2,p}}(M, F) \\ \downarrow m & & \downarrow m \\ Q_{V'_{21,p}}V'_{1,q}(M', F) & \xrightarrow{Q_{V'_1, V'_2}^{p,q}(M', F)} & V_{12,q}Q_{V_{2,p}}(M', F) \end{array}$$

Let $(S, O_S) \in \text{RTop}$ a locally ringed space. Consider a commutative diagram

$$\begin{array}{ccccc} Z'_1 = V(\mathcal{I}'_1) & \xrightarrow{l_1} & Z_1 = V(\mathcal{I}_1) & \xrightarrow{i_1} & (S, O_S) \\ \uparrow i''_2 & & \uparrow i'_2 & & \uparrow i_2 \\ Z' = V(\mathcal{I}') & \xrightarrow{l'_1} & Z = V(\mathcal{I}) & \xrightarrow{i'_1} & Z_2 = V(\mathcal{I}_2) \end{array}$$

where the maps are Zariski closed embeddings and whose squares are cartesian (i.e. $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2)$ and $\mathcal{I}' = (\mathcal{I}'_1, \mathcal{I})$, in particular $Z = Z_1 \cap Z_2$ and $Z' = Z'_1 \cap Z$). Let $(M, F) \in \text{PSh}_{D(O_S)}\text{fil}(S)$ such that M admits a V_{Z_1} -filtration V_1 , a $V_{Z'_1}$ -filtration V'_1 , and a V_{Z_2} -filtration V_2 (see definition 56). Then for all $p, q \in \mathbb{Z}$, denoting again V'_{12} the filtration induced by V'_1 on $Q_{V_{21,p}}V_{1,q}M$ and V'_{21} the filtration induced by V'_2 on $V_{12,q}Q_{V_{2,p}}$

- $Q_{V'_{21,p}}V'_{12,q}Q_{V_{21,p}}V_{1,q}(M, F) = Q_{V_{21,p}}V'_{1,q}(M, F)$,
- $V'_{12,q}Q_{V'_{21,p}}V_{12,q}Q_{V_{2,p}}(M, F) = V'_{12,q}Q_{V_{2,p}}(M, F)$

and

$$Q_{V'_{12}, V_{21}}^{p,q}(Q_{V_1, V_2}^{p,q}(M, F)) = Q_{V'_1, V_2}^{p,q}(M_2, F).$$

Proof. Obvious. □

We will also consider the following categories

Definition 57. Let $(\mathcal{X}, O_X) \in \text{RCat}$. We denote by $C_{O_X\text{fil}, D(O_X)}(\mathcal{X})$ the category

- whose objects $(M, F) \in C_{O_X\text{fil}, D(O_X)}(\mathcal{X})$ are filtered complexes of presheaves of O_X modules $(M, F) \in C_{O_X\text{fil}}(\mathcal{X})$ whose cohomology presheaves $H^n(M, F) \in \text{PSh}_{O_X\text{fil}}(\mathcal{X})$ are endowed with a structure of filtered $D(O_X)$ modules for all $n \in \mathbb{Z}$.
- whose set of morphisms $\text{Hom}_{C_{O_X\text{fil}, D(O_X)}(\mathcal{X})}((M, F), (N, F)) \subset \text{Hom}_{C_{O_X\text{fil}}(\mathcal{X})}((M, F), (N, F))$ between $(M, F), (N, F) \in C_{O_X\text{fil}, D(O_X)}(\mathcal{X})$ are the morphisms of filtered complexes of O_X modules $m : (M, F) \rightarrow (N, F)$ such that $H^n m : H^n(M, F) \rightarrow H^n(N, F)$ is $D(O_X)$ linear, i.e. is a morphism of (filtered) $D(O_X)$ modules, for all $n \in \mathbb{Z}$.

4.1.2 The De Rham complex of a (left) filtered D-module and the Spencer complex of a right filtered D-module

Using proposition 45, we define the filtered De Rham complex of a complex of filtered (left) D-modules :

Definition 58. (i) Let $(\mathcal{S}, O_S) \in \text{RCat}$ with O_S commutative. Let $(M, F) \in C_{D(O_S)\text{fil}}(\mathcal{S})$. By proposition 45, we have the complex

$$DR(O_S)(M, F) := (\Omega_{O_S}^\bullet, F_b) \otimes_{O_S} (M, F) \in C_{\text{fil}}(\mathcal{S})$$

whose differentials are $d(\omega \otimes m) = (d\omega) \otimes m + \omega \wedge (\nabla m)$.

(ii) More generally, let $f : (\mathcal{X}, O_X) \rightarrow (\mathcal{S}, O_S)$ with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RCat}$. The quotient map $q : \Omega_{O_X} \rightarrow \Omega_{O_X/f^*O_S}$ induce, for $G \in \text{PSh}_{O_X}(\mathcal{X})$ the quotient map

$$q^p(G) := \wedge^p q \otimes I \Omega_{O_X}^q \otimes_{O_X} G \rightarrow \Omega_{O_X/f^*O_S}^q \otimes_{O_X} G.$$

Let $(M, F) \in C_{D(O_X)\text{fil}}(\mathcal{X})$. By proposition 45, we have the relative De Rham complex

$$DR(O_X/f^*O_S)(M, F) := (\Omega_{X/S}^\bullet, F_b) \otimes_{O_X} (M, F) \in C_{f^*O_S\text{fil}}(\mathcal{X})$$

whose differentials are $d(q^p(M)(\omega \otimes m)) := q^{p+1}(M)((d\omega) \otimes m) + q^{p+1}(M)(\omega \otimes (\nabla m))$.

(iii) Let $(X, O_X)/\mathcal{F} \in \text{FolRTop}$, that is $(X, O_X) \in \text{RTop}$ endowed with a foliation with quotient map $q : \Omega_{O_X} \rightarrow \Omega_{O_X/\mathcal{F}}$. Let $(M, F) \in C_{D(O_X)\text{fil}}(X)$. By proposition 45, we have the foliated De Rham complex

$$DR(O_X/\mathcal{F})(M, F) := (\Omega_{O_X/\mathcal{F}}^\bullet, F_b) \otimes_{O_X} (M, F) \in C_{\text{fil}}(X)$$

whose differentials are $d(q(M)(\omega \otimes m)) := q(M)((d\omega) \otimes m) + q(M)(\omega \otimes (\nabla m))$.

By definition,

- with the notation of (ii) if $\phi : (M_1, F) \rightarrow (M_2, F)$ is a morphism in $C_{D(O_X)\text{fil}}(\mathcal{X})$,

$$DR(O_X/f^*O_S)(\phi) := (I \otimes \phi) : (\Omega_{X/S}^\bullet, F) \otimes_{O_X} (M_1, F) \rightarrow (\Omega_{X/S}^\bullet, F) \otimes_{O_X} (M_2, F)$$

is a morphism in $C_{f^*O_S\text{fil}}(\mathcal{X})$,

- with the notation of (ii) $DR(O_X)(O_X) = DR(O_X)$ and more generally in the relative case $DR(O_X/f^*O_S)(O_X) = DR(O_X/f^*O_S)$, and with the notation of (iii) $DR(O_X/\mathcal{F})(O_X) = DR(O_X/\mathcal{F})$.

Dually, we have the filtered Spencer complex of a complex of filtered right D-module :

Definition 59. (i) Let $(\mathcal{S}, O_S) \in \text{RCat}$ with O_S commutative. Let $(M, F) \in C_{D(O_S)^{\text{op}}\text{fil}}(\mathcal{S})$. By proposition 45, we have the complex

$$SP(O_S)(M, F) := (T_{O_S}^\bullet, F_b) \otimes_{O_S} (M, F) \in C_{\text{fil}}(\mathcal{S})$$

whose differentials are, for $X \in \mathcal{S}$, and $\partial_1 \wedge \cdots \wedge \partial_r \otimes m \in \Gamma(X, T_{O_S}^{r-1} \otimes_{O_S} M)$,

$$d(\partial_1 \wedge \cdots \wedge \partial_r \otimes m) : (\omega \in \Gamma(X, \Omega_{O_S}^{r-1})) \mapsto \sum_i \omega(\partial_1 \wedge \cdots \wedge \partial_i \cdots \wedge \partial_r) m - \sum_{i < j} \omega([\partial_i, \partial_j]) m.$$

(ii) More generally, let $f : (\mathcal{X}, O_X) \rightarrow (\mathcal{S}, O_S)$ with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RCat}$. The quotient map $q : \Omega_{O_X} \rightarrow \Omega_{O_X/f^*O_S}$ induce, for $G \in \text{PSh}_{O_X}(\mathcal{X})$ the injective map

$$q^{\vee, p}(G) := \wedge^p q^\vee \otimes I : T_{O_X/f^*O_S}^q \otimes_{O_X} G \rightarrow T_{O_X}^q \otimes_{O_X} G.$$

Let $(M, F) \in C_{D(O_X)^{\text{op}}\text{fil}}(\mathcal{X})$. By proposition 45, we have the relative Spencer complex

$$SP(O_X/f^*O_S)(M, F) := (T_{X/S}^\bullet, F_b) \otimes_{O_X} (M, F) \in C_{f^*O_S\text{fil}}(\mathcal{X})$$

whose differentials are the one of $SP(O_X)(M, F)$ given in (i) by the embedding $q^\vee : SP(O_X/f^*O_S)(M, F) \hookrightarrow SP(O_X)(M, F)$.

(iii) Let $(X, O_X)/\mathcal{F} \in \text{FolRTop}$, that is $(X, O_X) \in \text{RTop}$ endowed with a foliation with quotient map $q : \Omega_{O_X} \rightarrow \Omega_{O_X/\mathcal{F}}$. Let $(M, F) \in C_{D(O_X)^{\text{op}} \text{fil}}(X)$. By proposition 45, we have the foliated Spencer complex

$$SP(O_X/\mathcal{F})(M, F) := (T_{O_X/\mathcal{F}}^\bullet, F_b) \otimes_{O_X} (M, F) \in C_{\text{fil}}(X)$$

whose differentials are of $SP(O_X)(M, F)$ given in (i) by the embedding $q^\vee : SP(O_X/\mathcal{F})(M, F) \hookrightarrow SP(O_X)(M, F)$.

By definition, with the notation of (ii) if $\phi : (M_1, F) \rightarrow (M_2, F)$ is a morphism in $C_{D(O_X)^{\text{op}} \text{fil}}(\mathcal{X})$,

$$SP(O_X/f^*O_S)(\phi) := (I \otimes \phi) : (T_{X/S}^\bullet, F_b) \otimes_{O_X} (M_1, F) \rightarrow (T_{X/S}^\bullet, F_b) \otimes_{O_X} (M_2, F)$$

is a morphism in $C_{f^*O_S \text{fil}}(\mathcal{X})$.

Proposition 46. (i) Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(S, O_S), (X, O_X) \in \text{RTop}$. Assume that the canonical map $T(f, \text{hom})(O_X, O_X) : f^*D(O_X) \rightarrow D(f^*O_X)$ is an isomorphism of sheaves. For $(M, F) \in C_{D(O_X)^{\text{op}}, f^*D(O_S) \text{fil}}(X)$ and $(M', F), (N, F) \in C_{D(O_X) \text{fil}}(X)$, we have canonical isomorphisms in $C_{f^*D(O_S) \text{fil}}(X)$:

$$\begin{aligned} (M', F) \otimes_{O_X} (N, F) \otimes_{D(O_X)} (M, F) &= (M', F) \otimes_{D(O_X)} ((M, F) \otimes_{O_X} (N, F)) \\ &= ((M', F) \otimes_{O_X} (M, F)) \otimes_{D(O_X)} (N, F) \end{aligned}$$

(ii) Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(S, O_S), (X, O_X) \in \text{RTop}$. For $(M, F) \in C_{D(O_X) \text{fil}}(X)$, we have a canonical isomorphisms of filtered f^*O_S modules, i.e. isomorphisms in $C_{f^*O_S \text{fil}}(X)$,

$$(\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F) = ((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} D(O_X)) \otimes_{D(O_X)} (M, F)$$

Proof. These are standard fact of algebra. □

Definition-Proposition 16. Consider a commutative diagram in RCat

$$D = \begin{array}{ccc} (\mathcal{X}, O_X) & \xrightarrow{f} & (S, O_S) \\ g' \uparrow & & \uparrow g \\ (\mathcal{X}', O_{X'}) & \xrightarrow{f'} & (\mathcal{T}, O_T) \end{array}$$

with commutative structural sheaf of rings. Assume that the canonical map $T(g', \text{hom})(O_X, O_X) : g'^*D(O_X) \rightarrow D(g'^*O_X)$ is an isomorphism of sheaves.

(i) For $(M, F) \in \text{PSh}_{D(O_X) \text{fil}}(\mathcal{X})$, the graded map in $(\text{PSh}_{g'^*O_X}(\mathbb{N} \times \mathcal{X}'), F)$

$$\begin{aligned} \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(M, F) &:= m' \circ (\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)} \otimes I) : \\ g'^*((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F)) &\rightarrow (\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b) \otimes_{O_{X'}} g'^{\text{mod}}(M, F) \end{aligned}$$

given in degree $p \in \mathbb{N}$ by, for $X'^o \in \mathcal{X}'$ and $X^o \in \mathcal{X}$ such that $g'^*(X^o) \leftarrow X'^o$,

$$\begin{aligned} \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}^p(M)(X'^o) &:= m' \circ (\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}^p \otimes I)(X'^o) : \\ \omega \otimes m \in \Gamma(X^o, \Omega_{O_X}^p \otimes_{O_X} M) &\mapsto \Omega_{O_{X'}/g'^*O_X}(\omega) \otimes (m \otimes 1) \end{aligned}$$

is a map of complexes, that is a map in $C_{(f \circ g')^*O_S \text{fil}}(\mathcal{X}')$.

(ii) For $(M, F) \in C_{D(O_X) \text{fil}}(\mathcal{X})$, we get from (i) by functoriality, the map in $C_{(f \circ g')^*O_S \text{fil}}(\mathcal{X}')$

$$\begin{aligned} \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(M, F) &:= m' \circ (\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)} \otimes I) : \\ g'^*((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F)) &\rightarrow (\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b) \otimes_{O_{X'}} g'^{\text{mod}}(M, F) \end{aligned}$$

(iii) For $(M, F) \in C_{D(O_X)fil}(\mathcal{X})$, we get from (ii) the canonical transformation map in $C_{O_T fil}(\mathcal{T})$

$$\begin{aligned} T_\omega^O(D)(M, F) : g^{*mod} L_O(f_* E((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F))) &\xrightarrow{q} \\ (g^* f_* E((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F))) \otimes_{g^*O_S} O_T &\xrightarrow{T(g', E)(-) \circ T(D)(E(\Omega_{O_X/f^*O_S}^\bullet \otimes_{O_X} (M, F)))} \\ (f'_* E(g'^*((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F)))) \otimes_{g^*O_S} O_T &\xrightarrow{E(\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(M, F))} \\ (f'_* E((\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b) \otimes_{O_X} (M, F))) \otimes_{g^*O_S} O_T &\xrightarrow{m} f'_* E((\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b) \otimes_{O_{X'}} g'^{*mod}(M, F)) \end{aligned}$$

with $m(n \otimes s) = s.n$.

Proof. (i): We check that the map in $(\text{PSh}_{g'^*O_X}(\mathbb{N} \times \mathcal{X}'), F)$

$$\begin{aligned} \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(M, F) &:= m' \circ (\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)} \otimes I) : \\ g'^*((\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F)) &\rightarrow (\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b) \otimes_{O_{X'}} g'^{*mod}(M, F) \end{aligned}$$

is a map in $C_{(f \circ g')^*O_S fil}(\mathcal{X}')$. But we have, for $X'^o \in \mathcal{X}'$ the following equality in $\Gamma(X'^o, \Omega_{O_{X'}}^{p+1} \otimes_{O_{X'}} g'^{*mod} M)$

$$\begin{aligned} d(\Omega_{(O_{X'}/g'^*O_X)(O_T/g^*O_S)}^p(M)(\omega \otimes m)) &:= d(\Omega_{O_{X'}/g'^*O_X}^p(\omega) \otimes (m \otimes 1)) \\ &:= d(\Omega_{O_{X'}/g'^*O_X}^p(\omega)) \otimes (m \otimes 1) + \Omega_{O_{X'}/g'^*O_X}^p(\omega) \otimes \nabla(m \otimes 1) \\ &= \Omega_{O_{X'}/g'^*O_X}^{p+1}(d\omega) \otimes (m \otimes 1) + \Omega_{O_{X'}/g'^*O_X}^p(\omega) \otimes \nabla(m) \otimes 1 \\ &= \Omega_{O_{X'}/g'^*O_X}^{p+1}(d\omega) \otimes (m \otimes 1) + \Omega_{O_{X'}/g'^*O_X}^{p+1}(\omega \otimes \nabla(m)) \otimes 1 \\ &= : \Omega_{(O_{X'}/g'^*O_X)(O_T/g^*O_S)}^{p+1}(M)(d(\omega) \otimes m + \omega \otimes \nabla(m)) \\ &= : \Omega_{(O_{X'}/g'^*O_X)(O_T/g^*O_S)}^{p+1}(M)(d(\omega \otimes m)) \end{aligned}$$

since for $\partial' \in T_{O_{X'}}(X'^o)$,

$$\nabla_{\partial'}(m \otimes 1) = \nabla_{dg'(\partial')}(m) \otimes 1 + m \otimes \nabla_{\partial'} 1 = \nabla_{dg'(\partial')}(m) \otimes 1 :$$

see in definition-proposition 14 the definition of the $D(O_{X'})$ module structure on the $O_{X'}$ module $g'^{*mod} M := g'^* M \otimes_{g'^* O_X} O_{X'}$.

(ii) and (iii): There is nothing to prove. \square

Remark 7. Consider a commutative diagram in RCat

$$\begin{array}{ccc} D = (\mathcal{X}, O_X) & \xrightarrow{f} & (\mathcal{S}, O_S) . \\ g' \uparrow & & \uparrow g \\ (\mathcal{X}', O_{X'}) & \xrightarrow{f'} & (\mathcal{T}, O_T) \end{array}$$

Assume that the canonical map $T(g', \text{hom})(O_X, O_X) : g'^* D(O_X) \rightarrow D(g'^* O_X)$ is an isomorphism of sheaves. Under the canonical isomorphism $(-) \otimes 1 : (\Omega_{O_X/f^*O_S}^\bullet, F_b) \xrightarrow{\sim} (\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (O_X, F_b)$, we have (see definition-proposition 16 and definition 1)

- $\Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(O_X) = \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)} : g'^* \Omega_{O_X/f^*O_S}^\bullet \rightarrow \Omega_{O_{X'}/f'^*O_T}^\bullet$
- $T_\omega^O(D)(O_X) = T_\omega^O(D) : g^{*mod} L_O(f_* E(\Omega_{O_X/f^*O_S}^\bullet, F_b)) \rightarrow f'_* E(\Omega_{O_{X'}/f'^*O_T}^\bullet, F_b)$.

Definition 60. Consider a commutative diagram in \mathbf{RCat}

$$D = \begin{array}{ccc} (\mathcal{X}, O_X) & \xrightarrow{f} & (S, O_S) \\ g' \uparrow & & \uparrow g \\ (\mathcal{X}', O_{X'}) & \xrightarrow{f'} & (T, O_T) \end{array} .$$

with commutative structural sheaf of rings. Assume that the canonical map $T(g', \text{hom})(O_X, O_X) : g'^*D(O_X) \rightarrow D(g'^*O_X)$ is an isomorphism of sheaves. For $(N, F) \in C_{D(O_{X'}), g'^*D(O_X) \text{fil}}(\mathcal{X}')$, we have by definition-proposition 16 the map in $C_{f^*O_S \text{fil}}(\mathcal{X})$

$$\begin{aligned} T_\omega^O(g', \otimes)(N, F) : \Omega_{O_X/f^*O_S}^\bullet \otimes_{O_X} g'_*(N, F) &\xrightarrow{\text{ad}(g'^{*mod}, g'_*)(-)} \\ g'_*(g'^*(\Omega_{O_X/f^*O_S}^\bullet \otimes_{O_X} g'_*N) \otimes_{g'^*O_X} O_{X'}) &\xrightarrow{m \circ \Omega_{(O_{X'}/g'^*O_X)/(O_T/g^*O_S)}(g'_*(N, F))} \\ g'_*(\Omega_{O_X/f^*O_S}^\bullet \otimes_{O_X} g'^{*mod} g'_*(N, F)) &\xrightarrow{\text{ad}(g'^{*mod}, g'_*)(N, F)} g'_*(\Omega_{O_{X'}/f^*O_T}^\bullet \otimes_{O_{X'}}(N, F)) \end{aligned}$$

with $m(n \otimes s) = s.n$ and $g'_*N \in C_{D(O_X)}(\mathcal{X})$, the structure of $D(O_X)$ module being given by the canonical morphism $\text{ad}(g'^*, g'_*)(D(O_X)) : D(O_X) \rightarrow g'_*g'^*D(O_X)$ applied to $g'_*N \in C_{g'_*g'^*D(O_X)}(\mathcal{X})$.

We finish this subsection by a proposition for ringed spaces similar to proposition 9

Proposition 47. Let $f : (X, O_X) \rightarrow (S, O_S)$ a morphism with $(X, O_X), (S, O_S) \in \mathbf{RTop}$ with commutative sheaves of rings. Assume that $\Omega_{O_X/f^*O_S} \in \mathbf{PSh}_{O_X}(X)$ is a locally free O_X module of finite rank.

(i) If $\phi : (M, F) \rightarrow (N, F)$ is an r -filtered top local equivalence with $(M, F), (N, F) \in C_{D(O_X) \text{fil}}(X)$, then

$$DR(O_X/f^*O_S)(\phi) : (\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} (M, F) \rightarrow (\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_T} (N, F)$$

is an r -filtered top local equivalence.

(ii) Consider a commutative diagram in \mathbf{RTop}

$$D = \begin{array}{ccc} (X, O_X) & \xrightarrow{f} & (S, O_S) \\ g' \uparrow & & \uparrow g \\ (X', O_{X'}) & \xrightarrow{f'} & (T, O_T) \end{array} .$$

with commutative structural sheaf of rings. For $(N, F) \in C_{D(O_{X_T}) \text{fil}}(X')$, the map in $C_{f^*O_S \text{fil}}(X)$

$$k \circ T_\omega^O(g', \otimes)(E(N, F)) : (\Omega_{O_X/f^*O_S}^\bullet, F_b) \otimes_{O_X} g'_*E(N, F) \rightarrow g'_*E((\Omega_{O_{X'}/f^*O_T}^\bullet, F_b) \otimes_{O_{X'}} E(N, F))$$

is a filtered top local equivalence (see definition 60).

Proof. (i): Follows from proposition 9 (i) since $\Omega_{O_X/f^*O_S}^\bullet \in C^b(X)$ is then a bounded complex with $\Omega_{O_X/f^*O_S}^n \in \mathbf{PSh}_{O_X}(X)$ a locally free O_X module of finite rank.

(ii): Follows from proposition 9 (ii) since $\Omega_{O_X/f^*O_S}^\bullet \in C^b(X)$ is then a bounded complex with $\Omega_{O_X/f^*O_S}^n \in \mathbf{PSh}_{O_X}(X)$ a locally free O_X module of finite rank. \square

4.1.3 The support section functor for D module on ringed spaces

Let $(S, O_S) \in \text{RTop}$ with O_S commutative. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding,

- For $G \in C_{D(O_S)}(S)$, $\Gamma_Z G := \text{Cone}(\text{ad}(j^*, j_*)(G) : F \rightarrow j_* j^* G)[-1]$ has a (unique) structure of $D(O_S)$ module such that $\gamma_Z(G) : \Gamma_Z G \rightarrow G$ is a map in $C_{D(O_S)}(S)$. This gives the functor

$$\Gamma_Z : C_{D(O_S)\text{fil}}(S) \rightarrow C_{D(O_S)\text{fil}}(S), (G, F) \mapsto \Gamma_Z(G, F)$$

together with the canonical map $\gamma_Z(G, F) : \Gamma_Z(G, F) \rightarrow (G, F)$. Let $Z_2 \subset Z$ a closed subset, then for $G \in C_{D(O_S)}(S)$, $T(Z_2/Z, \gamma)(G) : \Gamma_{Z_2} G \rightarrow \Gamma_Z G$ is a map in $C_{D(O_S)}(S)$.

- For $G \in C_{O_S}(S)$, $\Gamma_Z^\vee G := \text{Cone}(\text{ad}(j!, j^*)(G) : j! j^* G \rightarrow G)$ has a unique structure of $D(O_S)$ module, such that $\gamma_Z^\vee(G) : G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{D(O_S)}(S)$. This gives the functor

$$\Gamma_Z^\vee : C_{D(O_S)\text{fil}}(S) \rightarrow C_{D(O_S)\text{fil}}(S), (G, F) \mapsto \Gamma_Z^\vee(G, F),$$

together with the canonical map $\gamma_Z^\vee(G, F) : (G, F) \rightarrow \Gamma_Z^\vee(G, F)$. Let $Z_2 \subset Z$ a closed subset, then for $G \in C_{D(O_S)}(S)$, $T(Z_2/Z, \gamma^\vee)(G) : \Gamma_{Z_2}^\vee G \rightarrow \Gamma_Z^\vee G$ is a map in $C_{D(O_S)}(S)$.

- For $G \in C_{D(O_S)}(S)$,

$$\begin{aligned} \Gamma_Z^{\vee, h} G &:= \mathbb{D}_S^O L_O \Gamma_Z E(\mathbb{D}_S^O G) \\ &:= \text{Cone}(\mathbb{D}_S^O L_O \text{ad}(j_*, j^*)(E(\mathbb{D}_S^O G)) : \mathbb{D}_S^O L_O j_* j^* E(\mathbb{D}_S^O G) \rightarrow \mathbb{D}_S^O L_O E(\mathbb{D}_S^O G)) \end{aligned}$$

has also canonical $D(O_S)$ -module structure, and $\gamma_Z^{\vee, h}(G) : G \rightarrow \Gamma_Z^{\vee, h} G$ is a map in $C_{D(O_S)}$. This gives the functor

$$\Gamma_Z^{\vee, h} : C_{D(O_S)\text{fil}}(S) \rightarrow C_{D(O_S)\text{fil}}(S), (G, F) \mapsto \Gamma_Z^{\vee, h}(G, F),$$

together with the canonical map $\gamma_Z^{\vee, h}(G, F) : (G, F) \rightarrow \Gamma_Z^{\vee, h}(G, F)$.

- Consider $\mathcal{I}_Z^O \subset O_S$ the ideal of vanishing function on Z and $\mathcal{I}_Z \subset D_S$ the right ideal of D_S generated by \mathcal{I}_Z^O . We have then $\mathcal{I}_Z^D \subset \mathcal{I}_Z$, where $\mathcal{I}_Z^D \subset D_S$ is the left and right ideal consisting of sections which vanish on Z . For $G \in \text{PSh}_{D(O_S)}(S)$, we consider, $S^o \subset S$ being an open subset,

$$\mathcal{I}_Z G(S^o) = \langle \{f.m, m \in G(S^o), f \in \mathcal{I}_Z(S^o)\} \rangle \subset G(S^o)$$

the $D(O_S)$ -submodule generated by the functions which vanish on Z (\mathcal{I}_Z is a right $D(O_S)$ ideal), This gives the functor,

$$\begin{aligned} \Gamma_Z^{\vee, O} &:= \Gamma_Z^{\vee, O, \mathcal{I}_Z} : C_{D(O_S)\text{fil}}(S) \rightarrow C_{D(O_S)\text{fil}}(S), \\ (G, F) &\mapsto \Gamma_Z^{\vee, O}(G, F) := \text{Cone}(b_Z(G, F) : \mathcal{I}_Z(G, F) \rightarrow (G, F)), \quad b_Z(-) := b_{\mathcal{I}_Z}(-) \end{aligned}$$

together with the canonical map $\gamma_Z^{\vee, O}(G, F) : (G, F) \rightarrow \Gamma_Z^{\vee, O}(G, F)$. which factors through

$$\gamma_Z^{\vee, O}(G) : G \xrightarrow{\gamma_Z^\vee(G)} \Gamma_Z^\vee G \xrightarrow{b_{S/Z}(G)} \Gamma_Z^{\vee, O} G.$$

with $b_{S/Z}(-) = b_{S/Z}^I$ and we have an homotopy equivalence $c_Z(G) := c_{\mathcal{I}_Z}(G) : \Gamma_Z^{\vee, O} G \rightarrow G/\mathcal{I}_Z G$.

Lemma 3. *Let $(Y, O_Y) \in \text{RTop}$ and $i : X \hookrightarrow Y$ a closed embedding.*

- (i) *For $(M, F) \in C_{D(O_Y)\text{fil}}(Y)$ and $(N, F) \in \text{PSh}_{D(O_Y)\text{op fil}}(Y)$ such that $a_\tau N$ is a locally free $D(O_Y)$ module of finite rank, the canonical map*

$$\begin{aligned} T(\gamma, \otimes)(E(M, F), (N, F)) &:= (I, T(j, \otimes)(E(M, F), (N, F))) : \\ (\Gamma_X E(M, F)) \otimes_{D(O_Y)} (N, F) &\rightarrow \Gamma_X E((M, F) \otimes_{D(O_Y)} (N, F)) \end{aligned}$$

is an equivalence top local.

(ii) For $(M, F) \in C_{D(O_Y)^{(op)} \text{fil}}(Y)$ and $(N, F) \in \text{PSh}_{D(O_Y)^{(op)} \text{fil}}(Y)$ such that $a_\tau N$ is a locally free O_Y module of finite rank, the canonical map

$$T(\gamma, \otimes)(E(M, F), (N, F)) := (I, T(j, \otimes)(E(M, F), (N, F))) : \\ (\Gamma_X E(M, F)) \otimes_{O_Y} (N, F) \rightarrow \Gamma_X E((M, F) \otimes_{O_Y} (N, F))$$

is a filtered top local equivalence.

Proof. Follows from proposition 9. Also note that $T(j, \otimes)(-, -) = T^{\text{mod}}(j, \otimes)(-, -)$. \square

We now look at the pullback map and the transformation map of De Rahm complexes together with the support section functor. The following is a generalization of definition-proposition 3 :

Definition-Proposition 17. Consider a commutative diagram in RTop

$$D_0 = \begin{array}{ccccc} f : (X, O_X) & \xrightarrow{i} & (Y, O_Y) & \xrightarrow{p} & (S, O_S) \\ g' \uparrow & & g'' \uparrow & & g \uparrow \\ f' : (X', O_{X'}) & \xrightarrow{i'} & (Y', O_{Y'}) & \xrightarrow{p'} & (T, O_T) \end{array}$$

with i, i' being closed embeddings. Denote by D the right square of D_0 . We have a factorization $i' : X' \xrightarrow{i'_1} X \times Y Y' \xrightarrow{i'_0} Y'$, where i'_0, i'_1 are closed embeddings.

(i) For $(M, F) \in C_{D(O_Y) \text{fil}}(Y)$, the canonical map,

$$E(\Omega_{(O_{Y'}/g''^*O_Y)/(O_T/g^*O_S)}(M, F)) \circ T(g'', E)(-) \circ T(g'', \gamma)(-) : \\ g''^* \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) \rightarrow \Gamma_{X \times Y Y'} E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} g''^{\text{mod}}(M, F))$$

unique up to homotopy such that the following diagram in $C_{g''^*p^*O_S \text{fil}}(Y') = C_{p'^*g^*O_S \text{fil}}(Y')$ commutes

$$\begin{array}{ccc} g''^* \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) & \xrightarrow{E(\Omega_{(-)/(-)}(M, F) \circ T(g'', E)(-) \circ T(g'', \gamma)(-))} & \Gamma_{X \times Y Y'} E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} (g''^{\text{mod}}(M, F))) \\ \downarrow \gamma_X(-) & & \downarrow \gamma_{X \times Y Y'}(-) \\ g''^* E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) & \xrightarrow{E(\Omega_{(-)/(-)}(M, F) \circ T(g'', E)(-))} & E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} g''^{\text{mod}}(M, F)) \end{array}$$

(ii) For $M \in C_{\mathcal{D}}(Y)$, there is a canonical map

$$T_\omega^O(D)(M, F)^\gamma : g''^{\text{mod}} \text{Lop}_* \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) \rightarrow \\ p'_* \Gamma_{X \times Y Y'} E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} g''^{\text{mod}}(M, F))$$

unique up to homotopy such that the following diagram in $C_{O_T \text{fil}}(T)$ commutes

$$\begin{array}{ccc} g''^{\text{mod}} \text{Lop}_* \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) & \xrightarrow{T_\omega^O(D)(M, F)^\gamma} & p'_* \Gamma_{X \times Y Y'} E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} (g''^{\text{mod}}(M, F))) \\ \downarrow \gamma_X(-) & & \downarrow \gamma_{X \times Y Y'}(-) \\ g''^{\text{mod}} \text{Lop}_* E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) & \xrightarrow{T_\omega^O(D)(M, F)} & p'_* E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} g''^{\text{mod}}(M, F)) \end{array}$$

(iii) For $N \in C_D(Y \times T)$, the canonical map in $C_{h'^*O_T \text{fil}}(Y')$

$$T(X'/X \times_Y Y', \gamma)(-) : \Gamma_{X'} E((\Omega_{Y'/T}^\bullet, F_b) \otimes_{O_{Y'}} (N, F)) \rightarrow \Gamma_{X \times_Y Y'} E((\Omega_{O_{Y'}/O_T}^\bullet, F_b) \otimes_{O_{Y'}} (N, F))$$

is unique up to homotopy such that $\gamma_{X \times_Y Y'}(-) \circ T(X'/X \times_Y Y', \gamma)(-) = \gamma_{X'}(-)$.

(iv) For $M = O_Y$, we have $T_\omega^O(D)(O_{Y \times S})^\gamma = T_\omega^O(D)^\gamma$ and $T_\omega^O(X \times_Y Y'/Y')(O_{Y'})^\gamma = T_\omega^O(X \times_Y Y'/Y')^\gamma$ (see definition-proposition 3).

Proof. Immediate from definition. We take for the map of point (ii) the composite

$$\begin{aligned} & T_\omega^O(D)(M, F)^\gamma : g^{* \text{mod}} L_{O p_*} \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) \xrightarrow{q} \\ & g^* p_* \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) \otimes_{g^*O_S} O_T \xrightarrow{T(g'', E)(-) \circ T(g'', \gamma)(-) \circ T(D)(E(\Omega_{O_Y/p^*O_S}^\bullet, F_b))} \\ & (p'_* \Gamma_{X \times_Y Y'} E(g''((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)))) \otimes_{g^*O_S} O_T \xrightarrow{E(\Omega_{(O_{Y'}/g''^*O_Y)/(O_T/g^*O_S)}(M, F))} \\ & p'_* \Gamma_{X \times_Y Y'} E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} g''^{* \text{mod}}(M, F)) \otimes_{g^*O_S} O_T \xrightarrow{m} \\ & p'_* \Gamma_{X \times_Y Y'} E((\Omega_{O_{Y'}/p'^*O_T}^\bullet, F_b) \otimes_{O_{Y'}} g''^{* \text{mod}}(M, F)), \end{aligned}$$

with $m(n \otimes s) = s.n$. □

Let $p : (Y, O_Y) \rightarrow (S, O_S)$ a morphism with $(Y, O_Y), (S, O_S) \in \text{RTop}$. Let $i : X \hookrightarrow Y$ a closed embedding. Denote by $j : Y \setminus X \hookrightarrow Y$ the complementary open embedding. Consider, for $(M, F) \in C_{D(O_Y) \text{fil}}(Y)$, the map in $C_{p^*O_S \text{fil}}(Y)$ (see definition 60):

$$\begin{aligned} & k \circ T_w^O(j, \otimes)(E(M, F)) : (\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} j_* j^* E(M, F) \\ & \xrightarrow{DR(O_Y/p^*O_S)(\text{ad}(j^*, j_*)(-))} \\ & j_* j^* ((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} j_* j^* E(M, F)) = j_* j^* (\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} j_* j^* E(M, F) \\ & \xrightarrow{k \circ DR(O_Y/p^*O_S)(\text{ad}(j^*, j_*)(j^* E(M)))} \\ & j_* E(j^* (\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} j^* E(M, F)) = j_* E(j^* ((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} E(M, F))) \end{aligned}$$

Definition 61. Let $p : (Y, O_Y) \rightarrow (S, O_S)$ a morphism with $(Y, O_Y), (S, O_S) \in \text{RTop}$. Let $i : X \hookrightarrow Y$ a closed embedding. Denote by $j : Y \setminus X \hookrightarrow Y$ the complementary open embedding. We consider, for $(M, F) \in C_{D(O_Y) \text{fil}}(Y)$ the canonical map in $C_{p^*O_S \text{fil}}(Y)$

$$\begin{aligned} & T_w^O(\gamma, \otimes)(M, F) := (I, k \circ T_w^O(j, \otimes)(E(M, F))) : \\ & (\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} \Gamma_X E(M, F) \rightarrow \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} E(M, F)). \end{aligned}$$

Proposition 48. Let $p : (Y, O_Y) \rightarrow (S, O_S)$ a morphism with $(Y, O_Y), (S, O_S) \in \text{RTop}$. Let $i : X \hookrightarrow Y$ a closed embedding. Then, if Ω_{O_Y/p^*O_S} is a locally free O_Y module, for $(M, F) \in C_{D(O_Y) \text{fil}}(Y)$

(i) the map

$$T_w^O(\gamma, \otimes)(M, F) : (\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} \Gamma_X E(M, F) \rightarrow \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} E(M, F))$$

is a 1-filtered top local equivalence,

(ii) the map in $D_{p^*O_S \text{fil}}(Y)$

$$\begin{aligned} & T_w^O(\gamma, \otimes) := DR(O_Y/p^*O_S)(k)^{-1} \circ T_w^O(\gamma, \otimes)(M, F) : \\ & (\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} \Gamma_X E(M, F) \rightarrow \Gamma_X E((\Omega_{O_Y/p^*O_S}^\bullet, F_b) \otimes_{O_Y} (M, F)) \end{aligned}$$

is an isomorphism.

Proof. By proposition 47,

- $\mathrm{Gr}_F^p(k \circ T_w^O(j, \otimes)(E(M, F))) : \Omega_{O_Y/p^*O_S}^\bullet \otimes_{O_Y} j_* j^* F^{p-\bullet} E(M) \rightarrow j_* E(j^*(\Omega_{O_Y/p^*O_S}^\bullet \otimes_{O_Y} F^{p-\bullet} E(M)))$ is a top local equivalence and
- $DR(O_Y/p^*O_S)(k) : \Omega_{O_Y/p^*O_S}^\bullet \otimes_{O_Y} (M, F) \rightarrow \Omega_{O_Y/p^*O_S}^\bullet \otimes_{O_Y} E(M, F)$ is a filtered top local equivalence.

□

4.2 The D-modules on smooth complex algebraic varieties and on complex analytic manifold and their functorialities in the filtered case

For convenience, we will work with and state the results for presheaves of D-modules. In this section, it is possible to assume that all the presheaves are sheaves and take the sheafification functor after the pullback functor f^* for a morphism $f : X \rightarrow S$, $X, S \in \mathrm{Var}(\mathbb{C})$ or $X, S \in \mathrm{AnSp}(\mathbb{C})$, and after the internal hom functors and tensor products of presheaves of modules on $S \in \mathrm{Var}(\mathbb{C})$ or $S \in \mathrm{AnSp}(\mathbb{C})$.

For $S = (S, O_S) \in \mathrm{SmVar}(\mathbb{C})$, resp. $S = (S, O_S) \in \mathrm{AnSm}(\mathbb{C})$, we denote by

- $D_S := D(O_S) \subset \mathcal{H}om_{\mathbb{C}_S}(O_S, O_S)$ the subsheaf consisting of differential operators. By a D_S module, we mean a left D_S module.
- we denote by

- $\mathrm{PSh}_{\mathcal{D}}(S)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of left D_S module, and by $\mathrm{PSh}_{\mathcal{D},h}(S) \subset \mathrm{PSh}_{\mathcal{D},c}(S) \subset \mathrm{PSh}_{\mathcal{D}}(S)$ the full subcategories whose objects are coherent, resp. holonomic, sheaves of left D_S modules, and by $\mathrm{PSh}_{\mathcal{D},rh}(S) \subset \mathrm{PSh}_{\mathcal{D},h}(S)$ the full subcategory of regular holonomic sheaves of left D_S modules,
- $\mathrm{PSh}_{\mathcal{D}^{op}}(S)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of right D_S module, and by $\mathrm{PSh}_{\mathcal{D}^{op},h}(S) \subset \mathrm{PSh}_{\mathcal{D}^{op},c}(S) \subset \mathrm{PSh}_{\mathcal{D}^{op}}(S)$ the full subcategories whose objects are coherent, resp. holonomic, sheaves of right D_S modules, and by $\mathrm{PSh}_{\mathcal{D}^{op},rh}(S) \subset \mathrm{PSh}_{\mathcal{D}^{op},h}(S)$ the full subcategory of regular holonomic sheaves of right D_S modules,

- we denote by

- $C_{\mathcal{D}}(S) = C(\mathrm{PSh}_{\mathcal{D}}(S))$ the category of complexes of Zariski presheaves on S with a structure of D_S module,

$$C_{\mathcal{D},rh}(S) \subset C_{\mathcal{D},h}(S) \subset C_{\mathcal{D},c}(S) \subset C_{\mathcal{D}}(S)$$

the full subcategories consisting of complexes of presheaves M such that $a_\tau H^n(M)$ are coherent, resp. holonomic, resp. regular holonomic, sheaves of D_S modules, a_τ being the sheafification functor for the Zariski, resp. usual, topology,

- $C_{\mathcal{D}^{op}}(S) = C(\mathrm{PSh}_{\mathcal{D}^{op}}(S))$ the category of complexes of Zariski presheaves on S with a structure of right D_S module,

$$C_{\mathcal{D}^{op},rh}(S) \subset C_{\mathcal{D}^{op},h}(S) \subset C_{\mathcal{D}^{op},c}(S) \subset C_{\mathcal{D}^{op}}(S)$$

the full subcategories consisting of complexes of presheaves M such that $a_\tau H^n(M)$ are coherent, resp. holonomic, resp. regular holonomic, sheaves of right D_S modules,

- in the filtered case we have

- $C_{\mathcal{D}(2)fil}(S) \subset C(\mathrm{PSh}_{\mathcal{D}}(S), F, W) := C(\mathrm{PSh}_{D(O_S)}(S), F, W)$ the category of (bi)filtered complexes of algebraic (resp. analytic) D_S modules such that the filtration is biregular (see definition 54,

$$C_{\mathcal{D}(2)fil,rh}(S) \subset C_{\mathcal{D}(2)fil,h}(S) \subset C_{\mathcal{D}(2)fil,c}(S) \subset C_{\mathcal{D}(2)fil}(S),$$

the full subcategories consisting of filtered complexes of presheaves (M, F) such that $a_\tau H^n(M, F)$ are filtered coherent, resp. filtered holonomic, resp. filtered regular holonomic, sheaves of D_S modules, that is $a_\tau H^n(M)$ are coherent, resp. holonomic, resp. regular holonomic, sheaves of D_S modules and F induces a good filtration on $a_\tau H^n(M)$ (in particular $F^p a_\tau H^n(M) \subset a_\tau H^n(M)$ are coherent sub O_S modules),

- $C_{\mathcal{D}0fil}(S) \subset C_{\mathcal{D}fil}(S)$ the full subcategory such that the filtration is a filtration by D_S submodule (which is stronger than Griffithz transversality), $C_{\mathcal{D}(1,0)fil}(S) \subset C_{\mathcal{D}2fil}(S)$ the full subcategory such that W is a filtration by D_S submodules (see definition 54),

$$C_{\mathcal{D}(1,0)fil,h}(S) = C_{\mathcal{D}2fil,h}(S) \cap C_{\mathcal{D}(1,0)fil}(S) \subset C_{\mathcal{D}2fil,h}(S),$$

the full subcategory consisting of filtered complexes of presheaves (M, F, W) such that $a_\tau H^n(M, F)$ are filtered holonomic sheaves of D_S modules and such that $W^p M \subset M$ are D_S submodules (recall that the O_S submodules $F^p M \subset M$ are NOT D_S submodules but satisfy by definition $md : F^r D_S \otimes F^p M \subset F^{p+r} M$),

$$C_{\mathcal{D}(1,0)fil,rh}(S) = C_{\mathcal{D}2fil,rh}(S) \cap C_{\mathcal{D}(1,0)fil}(S) \subset C_{\mathcal{D}2fil,rh}(S),$$

the full subcategory consisting of filtered complexes of presheaves (M, F, W) such that $a_\tau H^n(M, F)$ are filtered regular holonomic sheaves of D_S modules and such that $W^p M \subset M$ are D_S submodules

- $C_{\mathcal{D}^{op}(2)fil}(S) \subset C(\text{PSh}_{\mathcal{D}^{op}}(S), F, W) := C(\text{PSh}_{D(O_S)^{op}}(S), F, W)$ the category of (bi)filtered complexes of algebraic (resp. analytic) right D_S modules such that the filtration is biregular, as in the left case we consider the subcategories

$$C_{\mathcal{D}^{op}(2)fil,rh}(S) \subset C_{\mathcal{D}^{op}(2)fil,h}(S) \subset C_{\mathcal{D}^{op}(2)fil,c}(S) \subset C_{\mathcal{D}^{op}(2)fil}(S),$$

the full subcategories consisting of filtered complexes of presheaves (M, F) such that $a_\tau H^n(M, F)$ are filtered coherent, resp. filtered holonomic, resp. filtered regular holonomic, sheaves of right D_S modules.

For $S = (S, O_S) \in \text{AnSm}(\mathbb{C})$, we have the natural extension $D_S \subset D_S^\infty \subset \mathcal{H}om_{\mathbb{C}_S}(O_S, O_S)$ where $D_S^\infty \subset \mathcal{H}om_{\mathbb{C}_S}(O_S, O_S)$ is the subsheaf of differential operators of possibly infinite order (see [19]) for the definition of the action of a differential operator of infinite order on O_S) Similarly, we have

- $C_{\mathcal{D}^\infty(2)fil}(S) \subset C(\text{PSh}_{\mathcal{D}^\infty}(S), F, W) := C(\text{PSh}_{D_S^\infty}(S), F, W)$ the category of (bi)filtered complexes of D_S^∞ modules such that the filtration is biregular,

$$C_{\mathcal{D}^\infty(2)fil,h}(S) \subset C_{\mathcal{D}^\infty(2)fil,c}(S) \subset C_{\mathcal{D}^\infty(2)fil}(S),$$

the full subcategories consisting of filtered complexes of presheaves (M, F) such that $a_\tau H^n(M, F)$ are filtered coherent (resp. holonomic) sheaves of D_S^∞ modules that is $a_\tau H^n(M)$ are coherent (resp. holonomic) sheaves of D_S^∞ modules and F induces a good filtration on $a_\tau H^n(M)$.

- $C_{\mathcal{D}^\infty 0fil}(S) \subset C_{\mathcal{D}^\infty fil}(S)$ the full subcategory such that the filtration is a filtration by D_S^∞ submodule, $C_{\mathcal{D}^\infty(1,0)fil}(S) \subset C_{\mathcal{D}^\infty 2fil}(S)$ the full subcategory such that W is a filtration by D_S^∞ submodules,

$$C_{\mathcal{D}^\infty(1,0)fil,h}(S) = C_{\mathcal{D}^\infty 2fil,h}(S) \cap C_{\mathcal{D}^\infty(1,0)fil}(S) \subset C_{\mathcal{D}^\infty 2fil,h}(S),$$

the full subcategory consisting of filtered complexes of presheaves (M, F, W) such that $a_\tau H^n(M, F)$ are filtered holonomic sheaves of D_S^∞ modules and such that $W^p M \subset M$ are D_S submodules

- $C_{\mathcal{D}^\infty,op(2)fil}(S) \subset C(\text{PSh}_{\mathcal{D}^\infty,op}(S), F, W) := C(\text{PSh}_{D_S^\infty,op}(S), F, W)$ the category of (bi)filtered complexes of right D_S^∞ modules such that the filtration is biregular,

$$C_{\mathcal{D}^\infty,op(2)fil,h}(S) \subset C_{\mathcal{D}^\infty,op(2)fil,c}(S) \subset C_{\mathcal{D}^\infty,op(2)fil}(S),$$

the full subcategories consisting of filtered complexes of presheaves (M, F) such that $a_\tau H^n(M, F)$ are filtered coherent (resp. holonomic) sheaves of D_S modules.

For $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $(X, S) \in \text{AnSm}(\mathbb{C})$,

- we denote by
 - $\text{PSh}_{f^*\mathcal{D}}(X)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of left f^*D_S module, and $C_{f^*\mathcal{D}}(X) = C(\text{PSh}_{f^*\mathcal{D}}(X))$,
 - $\text{PSh}_{\mathcal{D}, f^*\mathcal{D}}(X)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of left f^*D_S module and left D_X module, and $C_{\mathcal{D}, f^*\mathcal{D}}(X) = C(\text{PSh}_{\mathcal{D}, f^*\mathcal{D}}(X))$,
 - $\text{PSh}_{\mathcal{D}^{op}, f^*\mathcal{D}}(X)$ the abelian category of Zariski (resp. usu) presheaves on S with a structure of left f^*D_S module and right D_X module and $C_{\mathcal{D}^{op}, f^*\mathcal{D}}(X) = C(\text{PSh}_{\mathcal{D}^{op}, f^*\mathcal{D}}(X))$,
- we denote by
 - $C_{f^*\mathcal{D}fil}(X) \subset C(\text{PSh}_{f^*\mathcal{D}}(X), F) := C(\text{PSh}_{f^*D_{(O_S)}}(X), F)$ the category of filtered complexes of algebraic (resp. analytic) f^*D_S modules such that the filtration is biregular,
 - $C_{\mathcal{D}, f^*\mathcal{D}fil}(X) \subset C(\text{PSh}_{\mathcal{D}, f^*\mathcal{D}}(X), F)$ the category of filtered complexes of algebraic (resp. analytic) (f^*D_S, D_X) modules such that the filtration is biregular,
 - $C_{\mathcal{D}^{op}, f^*\mathcal{D}fil}(X) \subset C(\text{PSh}_{\mathcal{D}^{op}, f^*\mathcal{D}}(X), F)$ the category of filtered complexes of algebraic (resp. analytic) (f^*D_S, D_X^{op}) modules such that the filtration is biregular.

For $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$, we denote by

- $C_{f^*\mathcal{D}^\infty fil}(X) \subset C(\text{PSh}_{f^*\mathcal{D}^\infty}(X), F) := C(\text{PSh}_{f^*D_S^\infty}(X), F)$ the category of filtered complexes of $f^*D_S^\infty$ modules such that the filtration is biregular,
- $C_{\mathcal{D}^\infty, f^*\mathcal{D}^\infty fil}(X) \subset C(\text{PSh}_{\mathcal{D}^\infty, f^*\mathcal{D}^\infty}(X), F)$ the category of filtered complexes of $(f^*D_S^\infty, D_X^\infty)$ modules such that the filtration is biregular,
- $C_{\mathcal{D}^\infty, op, f^*\mathcal{D}^\infty fil}(X) \subset C(\text{PSh}_{\mathcal{D}^\infty, op, f^*\mathcal{D}^\infty}(X), F)$ the category of filtered complexes of $(f^*D_S^\infty, D_X^{\infty, op})$ modules such that the filtration is biregular.

For $S \in \text{AnSm}(\mathbb{C})$, we denote by

$$J_S : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S), (M, F) \mapsto J_S(M, F) := (M, F) \otimes_{D_S} (D_S^\infty, F^{ord})$$

the natural functor. For $(M, F) \in C_{\mathcal{D}^\infty fil}(S)$, we will consider the map

$$\mathcal{J}_S(M, F) : J_S(M, F) := (M, F) \otimes_{D_S} (D_S^\infty, F^{ord}) \rightarrow (M, F), m \otimes P \mapsto Pm$$

Of course $J_S(C_{\mathcal{D}(1,0)fil}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil}(S)$. More generally, for $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$, we denote by

$$J_{X/S} : C_{f^*\mathcal{D}(2)fil}(X) \rightarrow C_{f^*\mathcal{D}^\infty(2)fil}(X), (M, F) \mapsto J_{X/S}(M, F) := (M, F) \otimes_{f^*(D_S, F)} f^*(D_S^\infty, F)$$

the natural functor, together with, for $(M, F) \in C_{f^*\mathcal{D}^\infty fil}(X)$, the map $\mathcal{J}_S(M, F) : J_S(M, F) \rightarrow (M, F)$.

Definition 62. Let $S \in \text{SmVar}(\mathbb{C})$, resp. $S \in \text{AnSm}(\mathbb{C})$. Let $Z \subset S$ a closed subset and denote by $j : S \setminus Z \hookrightarrow S$ the open embedding.

(i) We denote by

- $\text{PSh}_{\mathcal{D}, Z}(S) \subset \text{PSh}_{\mathcal{D}}(S)$, the full subcategory consisting of presheaves $M \in \text{PSh}_{\mathcal{D}}(S)$, such that $j^*M = 0$,
- $C_{\mathcal{D}, Z}(S) \subset C_{\mathcal{D}}(S)$, the full subcategory consisting of complexes presheaves $M \in C_{\mathcal{D}}(S)$ such that $a_\tau j^* H^n M = 0$ for all $n \in \mathbb{Z}$,

- $C_{\mathcal{D},Z,h}(S) := C_{\mathcal{D},Z}(S) \cap C_{\mathcal{D},h}(S) \subset C_{\mathcal{D}}(S)$ the full subcategory consisting of $M \in C_{\mathcal{D}}(S)$ such that $a_{\tau}H^n(M)$ are holonomic and $a_{\tau}j^*H^n M = 0$ for all $n \in \mathbb{Z}$,
- $C_{\mathcal{D},Z,c}(S) := C_{\mathcal{D},Z}(S) \cap C_{\mathcal{D},c}(S) \subset C_{\mathcal{D}}(S)$ the full subcategory consisting of $M \in C_{\mathcal{D}}(S)$ such that $a_{\tau}H^n(M)$ are coherent and $a_{\tau}j^*H^n M = 0$ for all $n \in \mathbb{Z}$.

(ii) We denote by

- $C_{\mathcal{D}(2)fil,Z}(S) \subset C_{\mathcal{D}(2)fil}(S)$, the full subcategory consisting of $(M, F) \in C_{\mathcal{D}(2)fil}(S)$ such that there exists $r \in \mathbb{N}$ and an r -filtered homotopy equivalence $\phi : (M, F) \rightarrow (N, F)$ with $(N, F) \in C_{\mathcal{D}(2)fil}(S)$ such that $a_{\tau}j^*H^n \text{Gr}_F^p(M, F) = 0$ for all $n, p \in \mathbb{Z}$, note that by definition this r does NOT depend on n and p ,
- $C_{\mathcal{D}(2)fil,Z,rh}(S) := C_{\mathcal{D}(2)fil,Z}(S) \cap C_{\mathcal{D}(2)fil,rh}(S) \subset C_{\mathcal{D}(2)fil}(S)$ the full subcategory consisting of (M, F) such that $a_{\tau}H^n(M, F)$ are filtered regular holonomic for all $n \in \mathbb{Z}$ and such that there exists $r \in \mathbb{N}$ and an r -filtered homotopy equivalence $\phi : (M, F) \rightarrow (N, F)$ with $(N, F) \in C_{\mathcal{D}(2)fil}(S)$ such that $a_{\tau}j^*H^n \text{Gr}_F^p(M, F) = 0$ for all $n, p \in \mathbb{Z}$,
- $C_{\mathcal{D}(2)fil,Z,h}(S) := C_{\mathcal{D}(2)fil,Z}(S) \cap C_{\mathcal{D}(2)fil,h}(S) \subset C_{\mathcal{D}(2)fil}(S)$ the full subcategory consisting of (M, F) such that $a_{\tau}H^n(M, F)$ are filtered holonomic for all $n \in \mathbb{Z}$ and such that there exists $r \in \mathbb{N}$ and an r -filtered homotopy equivalence $\phi : (M, F) \rightarrow (N, F)$ with $(N, F) \in C_{\mathcal{D}(2)fil}(S)$ such that $a_{\tau}j^*H^n \text{Gr}_F^p(M, F) = 0$ for all $n, p \in \mathbb{Z}$,
- $C_{\mathcal{D}(2)fil,Z,c}(S) := C_{\mathcal{D}(2)fil,Z}(S) \cap C_{\mathcal{D}(2)fil,c}(S) \subset C_{\mathcal{D}(2)fil}(S)$ the full subcategory consisting of (M, F) such that $a_{\tau}H^n(M, F)$ are filtered coherent for all $n \in \mathbb{Z}$ and such that there exists $r \in \mathbb{N}$ and an r -filtered homotopy equivalence $\phi : (M, F) \rightarrow (N, F)$ with $(N, F) \in C_{\mathcal{D}(2)fil}(S)$ such that $a_{\tau}j^*H^n \text{Gr}_F^p(M, F) = 0$ for all $n, p \in \mathbb{Z}$.

(iii) We have then the full subcategories

- $C_{\mathcal{D}(1,0)fil,Z}(S) = C_{\mathcal{D}(1,0)fil}(S) \cap C_{\mathcal{D}2fil,Z}(S) \subset C_{\mathcal{D}2fil}(S)$,
- $C_{\mathcal{D}(1,0)fil,Z,rh}(S) = C_{\mathcal{D}(1,0)fil}(S) \cap C_{\mathcal{D}2fil,Z,rh}(S) \subset C_{\mathcal{D}2fil}(S)$.
- $C_{\mathcal{D}(1,0)fil,Z,h}(S) = C_{\mathcal{D}(1,0)fil}(S) \cap C_{\mathcal{D}2fil,Z,h}(S) \subset C_{\mathcal{D}2fil}(S)$.

Similarly :

Definition 63. Let $S \in \text{AnSm}(\mathbb{C})$. Let $Z \subset S$ a closed subset and denote by $j : S \setminus Z \hookrightarrow S$ the open embedding.

(i) We denote by

- $C_{\mathcal{D}^{\infty}(2)fil,Z}(S) \subset C_{\mathcal{D}^{\infty}(2)fil}(S)$. the full subcategory consisting of $(M, F) \in C_{\mathcal{D}^{\infty}}(S)$ such that j^*M is acyclic
- $C_{\mathcal{D}^{\infty}(2)fil,Z,h}(S) := C_{\mathcal{D}^{\infty}(2)fil,Z}(S) \cap C_{\mathcal{D}^{\infty}(2)fil,h}(S) \subset C_{\mathcal{D}^{\infty}(2)fil}(S)$ the full subcategory consisting of (M, F) such that $a_{\tau}H^n(M)$ are holonomic and such that there exist $r \in \mathbb{Z}$ and an r -filtered homotopy equivalence $\phi : (M, F) \rightarrow (N, F)$ with $(N, F) \in C_{\mathcal{D}^{\infty}(2)fil}(S)$ such that $a_{\tau}j^*H^n \text{Gr}_F^p(M, F) = 0$.

(ii) We have then the full subcategories

- $C_{\mathcal{D}^{\infty}(1,0)fil,Z}(S) = C_{\mathcal{D}^{\infty}(1,0)fil}(S) \cap C_{\mathcal{D}^{\infty}2fil,Z}(S) \subset C_{\mathcal{D}^{\infty}2fil}(S)$,
- $C_{\mathcal{D}^{\infty}(1,0)fil,Z,h}(S) := C_{\mathcal{D}^{\infty}(1,0)fil}(S) \cap C_{\mathcal{D}^{\infty}2fil,Z,h}(S) \subset C_{\mathcal{D}^{\infty}(2)fil}(S)$.

Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. We recall (see section 2) that a morphism $m : (M, F) \rightarrow (N, F)$ with $(M, F), (N, F) \in C_{\mathcal{D}fil}(S)$ is said to be an r -filtered zariski, resp. usu, local equivalence if there exist morphisms $m_i : (C_i, F) \rightarrow (C_{i+1}, F)$ with $(C_i, F), (C_{i+1}, F) \in C_{\mathcal{D}fil}(S)$, $0 \leq i \leq s$, $(C_0, F) = (M, F)$, $(C_s, F) = (N, F)$, such that

$$m = m_s \circ \dots \circ m_i \circ \dots \circ m_0 : (M, F) \rightarrow (N, F)$$

and $m_i : (C_i, F) \rightarrow (C_{i+1}, F)$ is either a filtered zariski, resp. usu, local equivalence or an r -filtered homotopy.

Definition 64. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or with $X, S \in \text{AnSm}(\mathbb{C})$. We have, for $r = 1, \dots, \infty$, the homotopy categories

$$K_{\mathcal{D}, f^* \mathcal{D}(2) \text{fil}, r}(S) := \text{Ho}_{r,0} C_{\mathcal{D}, f^* \mathcal{D}(2) \text{fil}}(S), \quad K_{\mathcal{D}^{\text{op}}, f^* \mathcal{D}(2) \text{fil}, r}(S) := \text{Ho}_{r,0} C_{\mathcal{D}^{\text{op}}, f^* \mathcal{D}(2) \text{fil}}(S),$$

whose objects are those of $C_{\mathcal{D}, f^* \mathcal{D}(2) \text{fil}}(S)$, resp. those of $C_{\mathcal{D}^{\text{op}}, f^* \mathcal{D}(2) \text{fil}}(S)$, and whose morphisms are r -filtered for the first filtration (filtered for the second) homotopy classes of morphisms (see section 2.1). We have then

$$D_{\mathcal{D}, f^* \mathcal{D}(2) \text{fil}, r}(S) := K_{\mathcal{D}, f^* \mathcal{D}(2) \text{fil}, r}(S)([E_1]^{-1}), \quad D_{\mathcal{D}^{\text{op}}, f^* \mathcal{D}(2) \text{fil}, r}(S) := K_{\mathcal{D}^{\text{op}}, f^* \mathcal{D}(2) \text{fil}, r}(S)([E_1]^{-1}),$$

the localizations with respect to the classes of filtered Zariski, resp. usu, local equivalences (see section 2). Note that the classes of filtered τ local equivalence constitute a right multiplicative system. If $m : (M, F) \rightarrow (N, F)$ with $(M, F), (N, F) \in C_{\mathcal{D}, f^* \mathcal{D}(2) \text{fil}}(S)$ is an r -filtered zariski, resp. usu, local equivalence, then $m = D(\text{top}) : (M, F) \rightarrow (N, F)$ is an isomorphism in $D_{\mathcal{D}, f^* \mathcal{D}(2) \text{fil}, r}(S)$.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. We denote by

$$D_{\mathcal{D}(2) \text{fil}, \infty, rh}(S) \subset D_{\mathcal{D}(2) \text{fil}, \infty, h}(S) \subset D_{\mathcal{D}(2) \text{fil}, \infty}(S),$$

the full subcategories consisting of the image of $C_{\mathcal{D}(2) \text{fil}, h}(S)$, resp. $C_{\mathcal{D}(2) \text{fil}, rh}(S)$, by the localization functor

$$D(\text{top}) : C_{\mathcal{D}(2) \text{fil}}(S) \rightarrow D_{\mathcal{D}(2) \text{fil}, \infty}(S)$$

that is consisting of $(M, F) \in C_{\mathcal{D} \text{fil}}(S)$ such that $a_\tau H^n(M, F)$ are filtered holonomic, resp. filtered regular holonomic for all $n \in \mathbb{Z}$,

(iii) Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. We denote by

$$D_{\mathcal{D}(1,0) \text{fil}, \infty, rh}(S) \subset D_{\mathcal{D}(1,0) \text{fil}, \infty, h}(S) \subset D_{\mathcal{D}(2) \text{fil}, \infty}(S),$$

the full subcategories consisting of the image of $C_{\mathcal{D}(1,0) \text{fil}, h}(S)$, resp. $C_{\mathcal{D}(1,0) \text{fil}, rh}(S)$, by the localization functor

$$D(\text{top}) : C_{\mathcal{D}(2) \text{fil}}(S) \rightarrow D_{\mathcal{D}(2) \text{fil}, \infty}(S)$$

that is consisting of $(M, F, W) \in C_{\mathcal{D}2 \text{fil}}(S)$ such that $a_\tau H^n(M, F)$ are filtered holonomic, resp. filtered regular holonomic, and $W^p M^n \subset M^n$ are D_S submodules for all $n \in \mathbb{Z}$.

Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. By definition (see section 2), we have sequences of functors

$$C_{\mathcal{D}(2) \text{fil}}(S) \rightarrow K_{\mathcal{D}(2) \text{fil}}(S) \rightarrow D_{\mathcal{D}(2) \text{fil}}(S) \rightarrow D_{\mathcal{D}(2) \text{fil}, 2}(S) \rightarrow \dots \rightarrow D_{\mathcal{D}(2) \text{fil}, \infty}(S).$$

and commutative diagrams of functors

$$\begin{array}{ccc} K_{\mathcal{D}(2) \text{fil}}(S) & \longrightarrow & D_{\text{fil}}(S) & , & K_{\mathcal{D}(2) \text{fil}, r}(S) & \longrightarrow & D_{\mathcal{D}(2) \text{fil}, r}(S) & \cdot \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ K_{\mathcal{D}(2) \text{fil}, 2}(S) & \longrightarrow & D_{\mathcal{D}(2) \text{fil}, 2}(S) & & K_{\mathcal{D}(2) \text{fil}, r+1}(S) & \longrightarrow & D_{\mathcal{D}(2) \text{fil}, r+1}(S) & \end{array}$$

Then, for $r = 1$, $K_{\mathcal{D}(2) \text{fil}}(S)$ and $D_{\mathcal{D}(2) \text{fil}}(S)$ are in the canonical way triangulated categories. However, for $r > 1$, the categories $K_{\mathcal{D}(2) \text{fil}, r}(S)$ and $D_{\mathcal{D}(2) \text{fil}, r}(S)$ together with the canonical triangles does NOT satisfy the 2 of 3 axiom of triangulated categories.

Similarly,

Definition 65. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. We have, for $r = 1, \dots, \infty$, the categories

$$K_{\mathcal{D}^\infty, f^* \mathcal{D}^\infty(2) \text{fil}, r}(S) := \text{Ho}_{r,0} C_{\mathcal{D}^\infty, f^* \mathcal{D}^\infty(2) \text{fil}}(S), \quad K_{\mathcal{D}^\infty, \text{op}, f^* \mathcal{D}^\infty(2) \text{fil}, r}(S) := \text{Ho}_{r,0} C_{\mathcal{D}^\infty, \text{op}, f^* \mathcal{D}^\infty(2) \text{fil}}(S),$$

whose objects are those of $C_{\mathcal{D}^\infty, f^* \mathcal{D}^\infty(2) \text{fil}}(S)$, resp. those of $C_{\mathcal{D}^\infty, \text{op}, f^* \mathcal{D}^\infty(2) \text{fil}}(S)$, and whose morphisms are r -filtered for the first filtration (filtered for the second) homotopy classes of morphisms (see section 2.1). We have the categories

$$D_{\mathcal{D}^\infty, f^* \mathcal{D}^\infty(2) \text{fil}, r}(S) := K_{\mathcal{D}^\infty, f^* \mathcal{D}^\infty(2) \text{fil}, r}(S)([E_1]^{-1}), \quad D_{\mathcal{D}^\infty, \text{op}, f^* \mathcal{D}^\infty(2) \text{fil}, r}(S) := K_{\mathcal{D}^\infty, \text{op}, f^* \mathcal{D}^\infty(2) \text{fil}, r}(S)([E_1]^{-1}),$$

the localizations with respect to the classes of filtered usu local equivalence (see section 2.1). Note that the classes of filtered usu local equivalence constitute a right multiplicative system.

(ii) Let $S \in \text{AnSm}(\mathbb{C})$. We denote by

$$D_{\mathcal{D}^\infty(2) \text{fil}, \infty, h}(S) \subset D_{\mathcal{D}^\infty(2) \text{fil}, \infty}(S), \quad D_{\mathcal{D}^\infty(1,0) \text{fil}, \infty, h}(S) \subset D_{\mathcal{D}^\infty 2 \text{fil}, \infty}(S)$$

the full subcategories consisting of the image of $C_{\mathcal{D}^\infty(2) \text{fil}, h}(S)$, resp. $C_{\mathcal{D}^\infty(1,0) \text{fil}, h}(S)$, by the localization functor

$$D(\text{top}) : C_{\mathcal{D}^\infty(2) \text{fil}}(S) \rightarrow D_{\mathcal{D}^\infty(2) \text{fil}, \infty}(S).$$

We begin this subsection by recalling the following well known facts

Proposition 49. Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$.

- (i) The sheaf of differential operators D_S is a locally free sheaf of O_S module. Hence, a coherent D_S module $M \in \text{Coh}_{\mathcal{D}}(S)$ is a quasi-coherent sheaf of O_S modules.
- (ii) A coherent sheaf $M \in \text{Coh}_{O_S}(S)$ of O_S module admits a D_S module structure if and only if it is locally free (of finite rank by coherency) and admits an integrable connexion. In particular if $i : Z \hookrightarrow S$ is a closed embedding for the Zariski topology, then $i_* O_Z$ does NOT admit a D_S module structure since it is a coherent but not locally free O_S module.

Proof. Standard. □

In order to prove a version of the first GAGA theorem for coherent D modules, we will need to following. We start by a definition (cf. [17] definition 1.4.2) :

Definition 66. An $X \in \text{SmVar}(\mathbb{C})$ is said to be D -affine if the following two condition hold:

- (i) The global section functor $\Gamma(X, \cdot) : \mathcal{QCoh}_{\mathcal{D}}(X) \rightarrow \text{Mod}(\Gamma(X, D_X))$ is exact.
- (ii) If $\Gamma(X, M) = 0$ for $M \in \mathcal{QCoh}_{\mathcal{D}}(X)$, then $M = 0$.

Proposition 50. If $X \in \text{SmVar}(\mathbb{C})$ is D -affine, then :

- (i) Any $M \in \mathcal{QCoh}_{\mathcal{D}}(X)$ is generated by its global sections.
- (ii) The functor $\Gamma(X, \cdot) : \mathcal{QCoh}_{\mathcal{D}}(X) \rightarrow \text{Mod}(\Gamma(X, D_X))$ is an equivalence of category whose inverse is $L \in \text{Mod}(\Gamma(X, D_X)) \mapsto D_X \otimes_{\Gamma(X, D_X)} L \in \mathcal{QCoh}_{\mathcal{D}}(X)$.
- (iii) We have $\Gamma(X, \cdot)(\text{Coh}_{\mathcal{D}}(X)) = \text{Mod}(\Gamma(X, D_X))_f$, that is the global sections of a coherent D_X module is a finite module over the differential operators on X .

Proof. See [17]. □

The following proposition is from Kashiwara.

Proposition 51. Let $S \in \text{AnSm}(\mathbb{C})$.

(i) For $K \in C_c(S)$ a complex of presheaves with constructible cohomology sheaves, we have $\mathcal{H}om(L(K), E(O_S)) \in C_{\mathcal{D}^\infty, h}(S)$.

(ii) The functor $J_S : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S)$ satisfy $J_S(C_{\mathcal{D}(2)fil, rh}(S)) \subset C_{\mathcal{D}^\infty(2)fil, h}(S)$, derive trivially, and induce an equivalence of category

$$J_S : D_{\mathcal{D}(2)fil, \infty, rh}(S) \rightarrow D_{\mathcal{D}^\infty(2)fil, \infty, h}(S).$$

whose inverse satisfy, for $(M, F) \in \mathcal{H}ol_{\mathcal{D}^\infty(2)fil}(S)$ a (filtered) holonomic D_S^∞ module, that $J_S^{-1}(M, F) = (M_{reg}, F) \subset (M, F)$ is the D_S sub-module of M which is the regular part.

(iii) We have $J_S(C_{\mathcal{D}(1,0)fil, rh}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil, h}(S)$ and $J_S(D_{\mathcal{D}(1,0)fil, \infty, rh}(S)) = D_{\mathcal{D}^\infty(1,0)fil, \infty, h}(S)$.

Proof. Follows from [19]. \square

Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$, and let $i : Z \hookrightarrow S$ a closed embedding and denote by $j : S \setminus Z \hookrightarrow S$ the open complementary. For $M \in \text{PSh}_{\mathcal{D}}(S)$, we denote $\mathcal{I}_Z M \subset M$ the (left) D_S submodule given by, for $S^\circ \subset S$ an open subset, $\mathcal{I}_Z M(S^\circ) \subset M(S^\circ)$ is the (left) $D_S(S^\circ)$ submodule

$$\mathcal{I}_Z M(S^\circ) = \langle \{fm, f \in \mathcal{I}_Z(S^\circ), m \in M(S^\circ)\} \rangle \subset M(S^\circ)$$

generated by the elements of the form fm . We denote by $b_Z(M) : \mathcal{I}_Z M \rightarrow M$ the inclusion map and $c_Z(M) : M \rightarrow M/\mathcal{I}_Z M$ the quotient map of (left) D_S modules. For $M \in \text{PSh}_{\mathcal{D}}(S)$, we denote $M\mathcal{I}_Z \subset M$ the right D_S submodule given by, for $S^\circ \subset S$ an open subset, $M\mathcal{I}_Z(S^\circ) \subset M(S^\circ)$ is the right $D_S(S^\circ)$ submodule

$$\mathcal{I}_Z M(S^\circ) = \langle \{mf, f \in \mathcal{I}_Z(S^\circ), m \in M(S^\circ)\} \rangle \subset M(S^\circ)$$

generated by the elements of the form mf . We denote by $b_Z(M) : \mathcal{I}_Z M \rightarrow M$ the inclusion map and $c_Z(M) : M \rightarrow M/\mathcal{I}_Z M$ the quotient map of right D_S modules.

4.2.1 Functorialities

Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, we recall from section 4.1, the transfers modules

- $(D_{X \rightarrow S}, F^{ord}) := f^{*mod}(D_S, F^{ord}) := f^*(D_S, F^{ord}) \otimes_{f^*O_S} (O_X, F_b)$ which is a left D_X module and a left and right f^*D_S module
- $(D_{X \leftarrow S}, F^{ord}) := (K_X, F_b) \otimes_{O_X} (D_{X \rightarrow S}, F^{ord}) \otimes_{f^*O_S} f^*(K_S, F_b)$. which is a right D_X module and a left and right f^*D_S module.

Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the transfers modules of infite order are

- $(D_{X \rightarrow S}^\infty, F^{ord}) := f^{*mod}(D_S^\infty, F^{ord}) := f^*(D_S^\infty, F^{ord}) \otimes_{f^*O_S} (O_X, F_b)$ which is a left D_X^∞ module and a left and right $f^*D_S^\infty$ module
- $(D_{X \leftarrow S}^\infty, F^{ord}) := (K_X, F_b) \otimes_{O_X} (D_{X \rightarrow S}^\infty, F^{ord}) \otimes_{f^*O_S} f^*(K_S, F_b)$. which is a right D_X^∞ module and a left and right $f^*D_S^\infty$ module.

We have the following :

Lemma 4. Let $f_1 : X \rightarrow Y$, $f_2 : Y \rightarrow S$ be two morphism with $X, S, Y \in \text{SmVar}(\mathbb{C})$, or let $f_1 : X \rightarrow Y$, $f_2 : Y \rightarrow S$ be two morphism with $X, S, Y \in \text{AnSm}(\mathbb{C})$.

(i) We have $(D_{X \rightarrow S}, F^{ord}) = f_1^*(D_{Y \rightarrow S}, F^{ord}) \otimes_{f_1^*D_Y} (D_{X \rightarrow Y}, F^{ord})$ in $C_{\mathcal{D}, (f_2 \circ f_1)^* \mathcal{D}fil}(X)$ and

$$(D_{X \rightarrow S}, F^{ord}) = f_1^*(D_{Y \rightarrow S}, F^{ord}) \otimes_{f_1^*D_Y} (D_{X \rightarrow Y}, F^{ord}) = f_1^*(D_{Y \rightarrow S}, F^{ord}) \otimes_{f_1^*D_Y}^L (D_{X \rightarrow Y}, F^{ord}).$$

in $D_{\mathcal{D}, (f_2 \circ f_1)^* \mathcal{D}fil, r}(X)$.

(ii) We have $(D_{X \leftarrow S}, F^{ord}) = f_1^*(D_{Y \rightarrow S}, F^{ord}) \otimes_{f_1^* D_Y} (D_{X \leftarrow Y}, F^{ord})$ in $C_{\mathcal{D}^{op}, (f_2 \circ f_1)^* \mathcal{D}fil}(X)$ and

$$(D_{X \leftarrow S}, F^{ord}) = f_1^*(D_{Y \rightarrow S}, F^{ord}) \otimes_{f_1^* D_Y} (D_{X \leftarrow Y}, F^{ord}) = f_1^*(D_{Y \leftarrow S}, F^{ord}) \otimes_{f_1^* D_Y}^L (D_{X \leftarrow Y}, F^{ord}),$$

in $D_{\mathcal{D}^{op}, (f_2 \circ f_1)^* \mathcal{D}fil, r}(X)$.

Proof. Follows immediately from definition. The first assertions of (i) and (ii) are particular cases of lemma 2. See [17] for example. \square

In the analytical case we also have

Lemma 5. *Let $f_1 : X \rightarrow Y$, $f_2 : Y \rightarrow S$ be two morphism with $X, S, Y \in \text{AnSm}(\mathbb{C})$.*

(i) We have $(D_{X \rightarrow S}^\infty, F^{ord}) = f_1^*(D_{Y \rightarrow S}^\infty, F^{ord}) \otimes_{f_1^* D_Y^\infty} (D_{X \rightarrow Y}^\infty, F^{ord})$ in $C_{\mathcal{D}^\infty, (f_2 \circ f_1)^* \mathcal{D}^\infty fil}(X)$ and

$$(D_{X \rightarrow S}^\infty, F^{ord}) = f_1^*(D_{Y \rightarrow S}^\infty, F^{ord}) \otimes_{f_1^* D_Y^\infty} (D_{X \rightarrow Y}, F^{ord}) = f_1^*(D_{Y \rightarrow S}^\infty, F^{ord}) \otimes_{f_1^* D_Y^\infty}^L (D_{X \rightarrow Y}^\infty, F^{ord}).$$

in $D_{\mathcal{D}^\infty, (f_2 \circ f_1)^* \mathcal{D}^\infty fil, r}(X)$.

(ii) We have $(D_{X \leftarrow S}^\infty, F^{ord}) = f_1^*(D_{Y \rightarrow S}^\infty, F^{ord}) \otimes_{f_1^* D_Y^\infty} (D_{X \leftarrow Y}^\infty, F^{ord})$ in $C_{\mathcal{D}^\infty, op, (f_2 \circ f_1)^* \mathcal{D}^\infty fil}(X)$ and

$$(D_{X \leftarrow S}^\infty, F^{ord}) = f_1^*(D_{Y \rightarrow S}^\infty, F^{ord}) \otimes_{f_1^* D_Y^\infty} (D_{X \leftarrow Y}, F^{ord}) = f_1^*(D_{Y \leftarrow S}^\infty, F^{ord}) \otimes_{f_1^* D_Y^\infty}^L (D_{X \leftarrow Y}^\infty, F^{ord}),$$

in $D_{\mathcal{D}^\infty, op, (f_2 \circ f_1)^* \mathcal{D}^\infty fil, r}(X)$.

Proof. Similar to the proof of lemma 4 \square

For closed embeddings, we have :

Proposition 52. (i) *Let $i : Z \hookrightarrow S$ be a closed embedding with $Z, S \in \text{SmVar}(\mathbb{C})$. Then, $D_{Z \rightarrow S} = i^* D_S / D_S \mathcal{I}_Z$ and it is a locally free (left) D_Z module. Similarly, $D_{Z \leftarrow S} = i^* D_S / \mathcal{I}_Z D_S$ and it is a locally free right D_Z module.*

(ii) *Let $i : Z \rightarrow S$ be a closed embedding with $Z, S \in \text{AnSm}(\mathbb{C})$. Then, $D_{Z \rightarrow S} = i^* D_S / D_S \mathcal{I}_Z$ and it is a locally free (left) D_Z module. Similarly, $D_{Z \leftarrow S} = i^* D_S / \mathcal{I}_Z D_S$ and it is a locally free right D_Z module.*

(iii) *Let $i : Z \rightarrow S$ be a closed embedding with $Z, S \in \text{AnSm}(\mathbb{C})$. Then, $D_{Z \rightarrow S}^\infty = i^* D_S^\infty / D_S^\infty \mathcal{I}_Z$ and it is a locally free (left) D_Z^∞ module. Similarly, $D_{Z \leftarrow S}^\infty = i^* D_S^\infty / \mathcal{I}_Z D_S^\infty$ and it is a locally free right D_Z^∞ module.*

Proof. (i): See [17].

(ii): See [26].

(iii): Similar to (ii). \square

We now enumerate some functorialities we will use, all of them are particular case of the functoriality given in subsection 2.3 for any ringed spaces :

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{Var}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSp}(\mathbb{C})$. Then, the inverse image functor

$$f^{*mod} : \text{PSh}_{O_S}(S) \rightarrow \text{PSh}_{O_X}(X), \quad M \mapsto f^{*mod} M := O_X \otimes_{f^* O_S} f^* M$$

is a Quillen adjunction which induces in the derived category the functor

$$L f^{*mod} : D_{O_S}(S) \rightarrow D_{O_X}(X), \quad M \mapsto L f^{*mod} M := O_X \otimes_{f^* O_S}^L f^* M = O_X \otimes_{f^* O_S} f^* L_O M,$$

The adjunction $(f^{*mod}, f_*) : \text{PSh}_{O_S}(S) \rightleftarrows \text{PSh}_{O_X}(X)$ is a Quillen adjunction, the adjunction map are the maps

- for $M \in C_{O_S}(S)$, $\text{ad}(f^{*mod}, f_*)(M) : M \xrightarrow{\text{ad}(f^*, f_*)(M)} f_* f^* M \xrightarrow{f_* m} f_*(f^* M \otimes_{f^* O_S} O_X) = f_* f^{*mod} M$ where $m(m) = m \otimes 1$,
- for $M \in C_{O_X}(X)$, $\text{ad}(f^{*mod}, f_*)(M) : f^{*mod} f_* M = f^* f_* M \otimes_{f^* O_S} O_X \xrightarrow{\text{ad}(f^*, f_*)(M) \otimes_{f^* O_S} O_X} M \otimes_{f^* O_S} O_X \xrightarrow{n} M$, where $n(m \otimes h) = h.m$ is the multiplication map.

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$.

- For $M \in C_{\mathcal{D}}(S)$, we have the canonical projective resolution $q : L_D(M) \rightarrow M$ of complexes of D_S modules.
- For $M \in C_{\mathcal{D}}(S)$, there exist a unique structure of D_S module on the flasque presheaves $E^i(M)$ such that $E(M) \in C_{\mathcal{D}}(S)$ (i.e. is a complex of D_S modules) and that the map $k : M \rightarrow E(M)$ is a morphism of complexes of D_S modules.

Let $S \in \text{AnSm}(\mathbb{C})$.

- For $M \in C_{\mathcal{D}^\infty}(S)$, we have the canonical projective resolution $q : L_{D^\infty}(M) \rightarrow M$ of complexes of D_S^∞ modules.
- For $M \in C_{\mathcal{D}^\infty}(S)$, there exist a unique structure of D_S^∞ module on the flasque presheaves $E^i(M)$ such that $E(M) \in C_{\mathcal{D}^\infty}(S)$ (i.e. is a complex of D_S^∞ modules) and that the map $k : M \rightarrow E(M)$ is a morphism of complexes of D_S^∞ modules.

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}(op)}(S)$, $N \in C(S)$, we will consider the induced D module structure (right D_S module in the case one is a left D_S module and the other one is a right one) on the presheaf $M \otimes N := M \otimes_{\mathbb{Z}_S} N$ (see section 2). We get the bifunctor

$$C(S) \times C_{\mathcal{D}}(S) \rightarrow C_{\mathcal{D}}(S), (M, N) \mapsto M \otimes N$$

For $S \in \text{AnSm}(\mathbb{C})$, we also have the bifunctor $C(S) \times C_{\mathcal{D}^\infty}(S) \rightarrow C_{\mathcal{D}^\infty}(S)$, $(M, N) \mapsto M \otimes N$.

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. For $M, N \in C_{\mathcal{D}(op)}(S)$, $M \otimes_{O_S} N$ (see section 2), has a canonical structure of D_S modules (right D_S module in the case one is a left D_S module and the other one is a right one) given by (in the left case) for $S^o \subset S$ an open subset,

$$m \otimes n \in \Gamma(S^o, M \otimes_{O_S} N), \gamma \in \Gamma(S^o, D_S), \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes \gamma.n$$

This gives the bifunctor

$$C_{\mathcal{D}(op)}(S)^2 \rightarrow C_{\mathcal{D}(op)}(S), (M, N) \mapsto M \otimes_{O_S} N$$

More generally, let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$. Assume S smooth. For $M, N \in C_{f^* \mathcal{D}(op)}(X)$, $M \otimes_{f^* O_S} N$ (see section 2), has a canonical structure of $f^* D_S$ modules (right $f^* D_S$ module in the case one is a left $f^* D_S$ module and the other one is a right one) given by (in the left case) for $X^o \subset X$ an open subset,

$$m \otimes n \in \Gamma(X^o, M \otimes_{f^* O_S} N), \gamma \in \Gamma(X^o, f^* D_S), \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes \gamma.n$$

This gives the bifunctor

$$C_{f^* \mathcal{D}(op)}(X)^2 \rightarrow C_{f^* \mathcal{D}(op)}(X), (M, N) \mapsto M \otimes_{f^* O_S} N$$

For $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSp}(\mathbb{C})$ and S smooth, we also have the bifunctor $C_{f^* \mathcal{D}^\infty, (op)}(X)^2 \rightarrow C_{f^* \mathcal{D}^\infty, (op)}(X)$, $(M, N) \mapsto M \otimes_{f^* O_S} N$.

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}^{op}}(S)$ and $N \in C_{\mathcal{D}}(S)$, we have $M \otimes_{D_S} N \in C(S)$ (see section 2). This gives the bifunctor

$$C_{\mathcal{D}^{op}}(S) \times C_{\mathcal{D}}(S) \rightarrow C(S), (M, N) \mapsto M \otimes_{D_S} N$$

For $S \in \text{AnSm}(\mathbb{C})$, we also have the bifunctor $C_{\mathcal{D}^{\infty, op}}(S) \times C_{\mathcal{D}^{\infty}}(S) \rightarrow C(S), (M, N) \mapsto M \otimes_{D_S^{\infty}} N$.

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. The internal hom bifunctor

$$\mathcal{H}om(\cdot, \cdot) := \mathcal{H}om_{\mathbb{Z}_S}(\cdot, \cdot) : C(S)^2 \rightarrow C(S)$$

induces a bifunctor

$$\mathcal{H}om(\cdot, \cdot) := \mathcal{H}om_{\mathbb{Z}_S}(\cdot, \cdot) : C(S) \times C_{\mathcal{D}}(S) \rightarrow C_{\mathcal{D}}(S)$$

such that, for $F \in C(S)$ and $G \in C_{\mathcal{D}}(S)$, the D_S structure on $\mathcal{H}om^{\bullet}(F, G)$ is given by

$$\gamma \in \Gamma(S^o, D_S) \mapsto (\phi \in \text{Hom}^p(F|_{S^o}, G|_{S^o}) \mapsto (\gamma \cdot \phi : \alpha \in F^{\bullet}(S^o) \mapsto \gamma \cdot \phi^p(S^o)(\alpha))$$

where $\phi^p(S^o)(\alpha) \in \Gamma(S^o, G)$. For $S \in \text{AnSm}(\mathbb{C})$, it also induce the bifunctor

$$\mathcal{H}om(\cdot, \cdot) := \mathcal{H}om_{\mathbb{Z}_S}(\cdot, \cdot) : C(S) \times C_{\mathcal{D}^{\infty}}(S) \rightarrow C_{\mathcal{D}^{\infty}}(S)$$

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. For $M, N \in C_{\mathcal{D}}(S)$, $\mathcal{H}om_{O_S}(M, N)$, has a canonical structure of D_S modules given by for $S^o \subset S$ an open subset and $\phi \in \Gamma(S^o, \mathcal{H}om(M, O_S))$, $\gamma \in \Gamma(S^o, D_S)$, $(\gamma \cdot \phi)(m) := \gamma \cdot (\phi(m)) - \phi(\gamma \cdot m)$ This gives the bifunctor

$$\text{Hom}_{O_S}^{\bullet}(-, -) : C_{\mathcal{D}}(S)^2 \rightarrow C_{\mathcal{D}}(S)^{op}, (M, N) \mapsto \mathcal{H}om_{O_S}^{\bullet}(M, N)$$

In particular, for $M \in C_{\mathcal{D}}(S)$, we get the dual

$$\mathbb{D}_S^O(M) := \mathcal{H}om_{O_S}^{\bullet}(M, O_S) \in C_{\mathcal{D}}(S)$$

with respect to O_S , together with the canonical map $d(M) : M \rightarrow \mathbb{D}_S^{O, 2}(M)$. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. We have, for $M \in C_{\mathcal{D}}(S)$, the canonical transformation map

$$\begin{aligned} T(f, D^o)(M) : f^{*mod} \mathbb{D}_S^O M &= (f^* \mathcal{H}om_{O_S}(M, O_S)) \otimes_{f^* O_S} O_X \\ \xrightarrow{T^{mod}(f, hom)(M, O_S)} \mathcal{H}om_{O_X}(f^* M \otimes_{f^* O_S} O_X, O_X) &=: \mathbb{D}_X^O(f^{*mod} M). \end{aligned}$$

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. We have the bifunctors
 - $\text{Hom}_{D_S}^{\bullet}(-, -) : C_{\mathcal{D}}(S)^2 \rightarrow C(S)$, $(M, N) \mapsto \mathcal{H}om_{D_S}^{\bullet}(M, N)$, and if N is a bimodule (i.e. has a right D_S module structure whose opposite coincide with the left one), $\mathcal{H}om_{D_S}(M, N) \in C_{\mathcal{D}^{op}}(S)$ given by for $S^o \subset S$ an open subset and $\phi \in \Gamma(S^o, \mathcal{H}om(M, N))$, $\gamma \in \Gamma(S^o, D_S)$, $(\phi \cdot \gamma)(m) := (\phi(m)) \cdot \gamma$
 - $\text{Hom}_{D_S}(-, -) : C_{\mathcal{D}^{op}}(S)^2 \rightarrow C(S)$, $(M, N) \mapsto \mathcal{H}om_{D_S}(M, N)$ and if N is a bimodule, $\mathcal{H}om_{D_S}(M, N) \in C_{\mathcal{D}}(S)$

For $M \in C_{\mathcal{D}}(S)$, we get in particular the dual with respect \mathbb{D}_S ,

$$\mathbb{D}_S M := \mathcal{H}om_{D_S}(M, D_S) \in C_{\mathcal{D}}(S); \mathbb{D}_S^K M := \mathcal{H}om_{D_S}(M, D_S) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] \in C_{\mathcal{D}}(S)$$

and we have canonical map $d : M \rightarrow \mathbb{D}_S^2 M$. This functor induces in the derived category, for $M \in D_{\mathcal{D}}(S)$,

$$L\mathbb{D}_S M := R\mathcal{H}om_{D_S}(L_D M, D_S) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] = \mathbb{D}_S^K L_D M \in D_{\mathcal{D}}(S).$$

where $\mathbb{D}_S^O w(S) : \mathbb{D}_S^O w(K_S) \rightarrow \mathbb{D}_S^O K_S = K_S^{-1}$ is the dual of the Kozul resolution of the canonical bundle (proposition 72), and the canonical map $d : M \rightarrow L\mathbb{D}_S^2 M$. For $S \in \text{AnSm}(\mathbb{C})$, we also have the bifunctors

- $\text{Hom}_{D_S^\bullet}(-, -) : C_{\mathcal{D}^\infty}(S)^2 \rightarrow C(S)$, $(M, N) \mapsto \mathcal{H}om_{D_S^\bullet}(M, N)$, and if N is a bimodule, $\mathcal{H}om_{D_S^\infty}(M, N) \in C_{\mathcal{D}^\infty}(S)$,
- $\text{Hom}_{D_S}(-, -) : C_{\mathcal{D}^\infty, \text{op}}(S)^2 \rightarrow C(S)$, $(M, N) \mapsto \mathcal{H}om_{D_S}(M, N)$ and if N is a bimodule, $\mathcal{H}om_{D_S}(M, N) \in C_{\mathcal{D}^\infty, \text{op}}(S)$

For $M \in C_{\mathcal{D}}(S)$, we get in particular the dual with respect \mathbb{D}_S^∞ ,

$$\mathbb{D}_S^\infty M := \mathcal{H}om_{D_S^\infty}(M, D_S^\infty) \in C_{\mathcal{D}^\infty}(S), \quad \mathbb{D}_S^{\infty, K} M := \mathcal{H}om_{D_S^\infty}(M, D_S^\infty) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] \in C_{\mathcal{D}^\infty}(S)$$

and we have canonical maps $d : M \rightarrow \mathbb{D}_S^{\infty, 2} M$, $d : M \rightarrow \mathbb{D}_S^{\infty, K, 2} M$. This functor induces in the derived category, for $M \in D_{\mathcal{D}^\infty}(S)$,

$$L\mathbb{D}_S^\infty M := R\mathcal{H}om_{D_S^\infty}(M, D_S^\infty) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] = \mathbb{D}_S^{\infty, K} L_{D^\infty} M \in D_{\mathcal{D}^\infty}(S).$$

and the canonical map $d : M \rightarrow L\mathbb{D}_S^{\infty, 2} M$.

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. For $N \in C_{\mathcal{D}, f^* \mathcal{D}}(X)$ and $M \in C_{\mathcal{D}}(X)$, $N \otimes_{D_X} M$ has the canonical $f^* D_S$ module structure given by, for $X^o \subset X$ an open subset,

$$\gamma \in \Gamma(X^o, f^* D_S), m \in \Gamma(X^o, M), n \in \Gamma(X^o, N), \quad \gamma.(n \otimes m) = (\gamma.n) \otimes m.$$

This gives the functor

$$C_{\mathcal{D}, f^* \mathcal{D}}(X) \times C_{\mathcal{D}}(X) \rightarrow C_{f^* \mathcal{D}}(X), \quad (M, N) \mapsto M \otimes_{D_X} N$$

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $N \in C_{\mathcal{D}^\infty, f^* \mathcal{D}^\infty}(X)$ and $M \in C_{\mathcal{D}^\infty}(X)$, $N \otimes_{D_X^\infty} M$ has the canonical $f^* D_S^\infty$ module structure given by, for $X^o \subset X$ an open subset,

$$\gamma \in \Gamma(X^o, f^* D_S), m \in \Gamma(X^o, M), n \in \Gamma(X^o, N), \quad \gamma.(n \otimes m) = (\gamma.n) \otimes m.$$

This gives the functor

$$C_{\mathcal{D}^\infty, f^* \mathcal{D}^\infty}(X) \times C_{\mathcal{D}^\infty}(X) \rightarrow C_{f^* \mathcal{D}^\infty}(X), \quad (M, N) \mapsto M \otimes_{D_X^\infty} N$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, for $M \in C_{\mathcal{D}}(S)$, $O_X \otimes_{f^* O_S} f^* M$ has a canonical D_X module structure given by given by, for $X^o \subset X$ an open subset,

$$m \otimes n \in \Gamma(X^o, O_X \otimes_{f^* O_S} f^* M), \gamma \in \Gamma(X^o, D_X), \quad \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes d f(\gamma).n.$$

This gives the inverse image functor

$$f^{* \text{mod}} : \text{PSh}_{\mathcal{D}}(S) \rightarrow \text{PSh}_{\mathcal{D}}(X), \quad M \mapsto f^{* \text{mod}} M := O_X \otimes_{f^* O_S} f^* M = D_{X \rightarrow S} \otimes_{f^* D_S} f^* M$$

which induces in the derived category the functor

$$L f^{* \text{mod}} : D_{\mathcal{D}}(S) \rightarrow D_{\mathcal{D}}(X), \quad M \mapsto L f^{* \text{mod}} M := O_X \otimes_{f^* O_S}^L f^* M = O_X \otimes_{f^* O_S} f^* L_D M,$$

We will also consider the shifted inverse image functor

$$L f^{* \text{mod}[-]} := L f^{* \text{mod}}[d_S - d_X] : D_{\mathcal{D}}(S) \rightarrow D_{\mathcal{D}}(X).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, for $M \in C_{\mathcal{D}^\infty}(S)$, $O_X \otimes_{f^*O_S} f^*M$ has a canonical D_X^∞ module structure induced by the finite order case. This gives the inverse image functor

$$f^{*mod} : \text{PSh}_{\mathcal{D}^\infty}(S) \rightarrow \text{PSh}_{\mathcal{D}^\infty}(X), \quad M \mapsto f^{*mod}M := O_X \otimes_{f^*O_S} f^*M = D_{X \rightarrow S} \otimes_{f^*D_S^\infty} f^*M$$

which induces in the derived category the functor

$$L f^{*mod} : D_{\mathcal{D}^\infty}(S) \rightarrow D_{\mathcal{D}^\infty}(X), \quad M \mapsto L f^{*mod}M := O_X \otimes_{f^*O_S}^L f^*M = O_X \otimes_{f^*O_S} f^*L_{D_S^\infty} M,$$

We will also consider the shifted inverse image functor

$$L f^{*mod[-]} := L f^{*mod}[d_S - d_X] : D_{\mathcal{D}^\infty}(S) \rightarrow D_{\mathcal{D}^\infty}(X).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}}(X)$, $D_{X \leftarrow S} \otimes_{D_X} M$ has the canonical f^*D_S module structure given above. Then, the direct image functor

$$f_{*mod}^0 : \text{PSh}_{\mathcal{D}}(X) \rightarrow \text{PSh}_{\mathcal{D}}(S), \quad M \mapsto f_{*mod}M := f_*(D_{X \leftarrow S} \otimes_{D_X} M)$$

induces in the derived category the functor

$$\int_f = R f_{*mod} : D_{\mathcal{D}}(X) \rightarrow D_{\mathcal{D}}(S), \quad M \mapsto \int_f M = R f_*(D_{X \leftarrow S} \otimes_{D_X}^L M).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}^\infty}(X)$, $D_{X \leftarrow S} \otimes_{D_X} M$ has the canonical f^*D_S module structure given above. Then, the direct image functor

$$f_{*mod}^{00} : \text{PSh}_{\mathcal{D}^\infty}(X) \rightarrow \text{PSh}_{\mathcal{D}^\infty}(S), \quad M \mapsto f_{*mod}M := f_*(D_{X \leftarrow S}^\infty \otimes_{D_X^\infty} M)$$

induces in the derived category the functor

$$\int_f = R f_{*mod} : D_{\mathcal{D}^\infty}(X) \rightarrow D_{\mathcal{D}^\infty}(S), \quad M \mapsto \int_f M = R f_*(D_{X \leftarrow S}^\infty \otimes_{D_X^\infty}^L M).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. The direct image functor with compact support

$$f_{!mod}^{00} : \text{PSh}_{\mathcal{D}}(X) \rightarrow \text{PSh}_{\mathcal{D}}(S), \quad M \mapsto f_{!mod}M := f_!(D_{S \leftarrow X} \otimes_{D_X} M)$$

induces in the derived category the functor

$$\int_{f!} = R f_{!mod} : D_{\mathcal{D}}(X) \rightarrow D_{\mathcal{D}}(S), \quad M \mapsto \int_{f!} M = R f_!(D_{X \leftarrow S} \otimes_{D_X}^L M).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. The direct image functor with compact support

$$f_{!mod}^{00} : \text{PSh}_{\mathcal{D}^\infty}(X) \rightarrow \text{PSh}_{\mathcal{D}^\infty}(S), \quad M \mapsto f_{!mod}M := f_!(D_{S \leftarrow X}^\infty \otimes_{D_X^\infty} M)$$

induces in the derived category the functor

$$\int_{f!} = R f_{!mod} : D_{\mathcal{D}^\infty}(X) \rightarrow D_{\mathcal{D}^\infty}(S), \quad M \mapsto \int_{f!} M = R f_!(D_{X \leftarrow S}^\infty \otimes_{D_X^\infty}^L M).$$

- Let $S \in \text{SmVar}(\mathbb{C})$. The analytical functor of a D_S modules has a canonical structure of $D_{S^{an}}$ module:

$$(-)^{an} : C_{\mathcal{D}}(S) \rightarrow C_{\mathcal{D}}(S^{an}), \quad M \mapsto M^{an} := \text{an}_S^{*mod} M := M \otimes_{\text{an}_S^* O_S} O_{S^{an}}$$

which induces in the derived category

$$(-)^{an} : D_{\mathcal{D}}(S) \rightarrow D_{\mathcal{D}}(S^{an}), \quad M \mapsto M^{an} := \text{an}_S^{*mod} M$$

since an_S^{*mod} derive trivially.

The functorialities given above induce :

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{Var}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSp}(\mathbb{C})$. The adjunction map induces

– for $(M, F) \in C_{O_S \text{fil}}(S)$, the map in $D_{O_S \text{fil}}(S)$

$$\begin{aligned} \text{ad}(L f^{*mod}, R f_*)(M, F) : (M, F) &\xrightarrow{\text{koad}(f^*, f_*)(M, F)} f_* E(f^*(M, F)) = R f_* f^*(M, F) \\ &\xrightarrow{f^* m} R f_*(f^*(M, F) \otimes_{f^* O_S}^L O_X) = R f_* f^*(M, F), \end{aligned}$$

where $m(m) = m \otimes 1$,

– for $(M, F) \in C_{O_X \text{fil}}(X)$, the map in $D_{O_X \text{fil}}(X)$

$$\begin{aligned} \text{ad}(L f^{*mod}, R f_*)(M, F) : L f^{*mod} R f_*(M, F) &= f^* f_* E(M, F) \otimes_{f^* O_S}^L O_X \\ &\xrightarrow{\text{ad}(f^*, f_*)(E(M, F)) \otimes_{f^* O_S}^L O_X} (M, F) \otimes_{f^* O_S}^L O_X \xrightarrow{n} (M, F), \end{aligned}$$

where $n(m \otimes h) = h.m$ is the multiplication map.

- For a commutative diagram in $\text{Var}(\mathbb{C})$ or in $\text{AnSp}(\mathbb{C})$:

$$D = \begin{array}{ccc} Y & \xrightarrow{g_2} & X \\ \downarrow f_2 & & \downarrow f_1 \\ T & \xrightarrow{g_1} & S \end{array} ,$$

we have, for $(M, F) \in C_{O_X \text{fil}}(X)$, the canonical map in $D_{O_T \text{fil}}(T)$

$$\begin{aligned} T^{mod}(D)(M, F) : L g_1^{*mod} f_{1*}(M, F) &\xrightarrow{\text{ad}(L f_2^{*mod}, R f_{2*})(L g_1^{*mod} f_{1*} E(M, F))} \\ R f_{2*} L f_2^{*mod} L g_1^{*mod} R f_{1*}(M, F) &= R f_{2*} L g_2^{*mod} L f_1^{*mod} R f_{1*}(M, F) \\ &\xrightarrow{\text{ad}(L f_1^{*mod}, R f_1)(M, F)} R f_{2*} L g_2^{*mod}(M, F) \end{aligned}$$

the canonical transformation map given by the adjunction maps.

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\text{fil}}(S)$ and $(N, F) \in C_{\text{fil}}(S)$, recall that (see section 2)

$$F^p((M, F) \otimes (N, F)) := \text{Im}(\oplus_q F^q M \otimes F^{p-q} N \rightarrow M \otimes N)$$

This gives the functor

$$(\cdot, \cdot) : C_{\text{fil}}(S) \times C_{\mathcal{D}\text{fil}}(S) \rightarrow C_{\mathcal{D}\text{fil}}(S), ((M, F), (N, F)) \mapsto (M, F) \otimes (N, F).$$

It induces in the derived categories by taking r -projective resolutions the bifunctors, for $r = 1, \dots, \infty$,

$$(\cdot, \cdot) : D_{\mathcal{D}\text{fil}, r}(S) \times D_{\text{fil}, r}(S) \rightarrow D_{\mathcal{D}\text{fil}, r}(S), ((M, F), (N, F)) \mapsto (M, F) \otimes^L (N, F) = L_D(M, F) \otimes (N, F).$$

For $S \in \text{AnSm}(\mathbb{C})$, it gives the bifunctor

$$(\cdot, \cdot) : C_{\text{fil}}(S) \times C_{\mathcal{D}^\infty \text{fil}}(S) \rightarrow C_{\mathcal{D}^\infty \text{fil}}(S), ((M, F), (N, F)) \mapsto (M, F) \otimes (N, F),$$

and its derived functor.

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$ and $O'_S \in \text{PSh}(S, \text{cRing})$ a sheaf of commutative ring. For $(M, F) \in C_{O'_S \text{fil}}(S)$ and $(N, F) \in C_{O'_S \text{fil}}(S)$, recall that (see section 2)

$$F^p((M, F) \otimes_{O'_S} (N, F)) := \text{Im}(\oplus_q F^q M \otimes_{O'_S} F^{p-q} N \rightarrow M \otimes_{O'_S} N)$$

It induces in the derived categories by taking r-projective resolutions the bifunctors, for $r = 1, \dots, \infty$,

$$(\cdot, \cdot) : D_{\mathcal{D}\text{fil}, r}(S)^2 \rightarrow D_{\mathcal{D}\text{fil}, r}(S), ((M, F), (N, F)) \mapsto (M, F) \otimes_{O'_S}^L (N, F).$$

More generally, let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$. Assume S smooth. We have the bifunctors

$$(\cdot, \cdot) : D_{f^* \mathcal{D}\text{fil}, r}(X)^2 \rightarrow D_{f^* \mathcal{D}\text{fil}, r}(X), ((M, F), (N, F)) \mapsto (M, F) \otimes_{f^* O'_S}^L (N, F) = (M, F) \otimes_{f^* O'_S} L_{f^* D}(N, F).$$

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. The hom functor induces the bifunctor

$$\text{Hom}(-, -) : C_{\mathcal{D}\text{fil}}(S) \times C_{\text{fil}}(S) \rightarrow C_{\mathcal{D}(1,0)\text{fil}}(S), ((M, W), (N, F)) \mapsto \mathcal{H}om((M, W), (N, F)).$$

For $S \in \text{AnSm}(\mathbb{C})$, the hom functor also induces the bifunctor

$$\text{Hom}(-, -) : C_{\mathcal{D}^\infty \text{fil}}(S) \times C_{\text{fil}}(S) \rightarrow C_{\mathcal{D}^\infty(1,0)\text{fil}}(S), ((M, W), (N, F)) \mapsto \mathcal{H}om((M, W), (N, F)).$$

Note that the filtration given by W satisfy that the W^p are D_S submodule which is stronger than Griffithz transversality.

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. The hom functor induces the bifunctor

$$\text{Hom}_{O_S}(-, -) : C_{\mathcal{D}\text{fil}}(S)^2 \rightarrow C_{\mathcal{D}2\text{fil}}(S), ((M, W), (N, F)) \mapsto \mathcal{H}om_{O_S}((M, W), (N, F)).$$

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. The hom functor induces the bifunctors

$$\begin{aligned} & - \text{Hom}_{D_S}(-, -) : C_{\mathcal{D}\text{fil}}(S)^2 \rightarrow C_{2\text{fil}}(S), ((M, W), (N, F)) \mapsto \mathcal{H}om_{D_S}((M, W), (N, F)), \\ & - \text{Hom}_{D_S}(-, -) : C_{\mathcal{D}^{op}\text{fil}}(S)^2 \rightarrow C_{2\text{fil}}(S), ((M, W), (N, F)) \mapsto \mathcal{H}om_{D_S}((M, W), (N, F)). \end{aligned}$$

We get the filtered dual

$$\mathbb{D}_S^K(\cdot) : C_{\mathcal{D}(2)\text{fil}}(S) \rightarrow C_{\mathcal{D}(2)\text{fil}}(S)^{op}, (M, F) \mapsto \mathbb{D}_S^K(M, F) := \mathcal{H}om_{D_S}((M, F), D_S) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S]$$

together with the canonical map $d(M, F) : (M, F) \rightarrow \mathbb{D}_S^{2,K}(M, F)$. Of course $\mathbb{D}_S^K(\cdot)(C_{\mathcal{D}(1,0)\text{fil}}(S)) \subset C_{\mathcal{D}(1,0)\text{fil}}(S)$. It induces in the derived categories $D_{\mathcal{D}\text{fil}, r}(S)$, for $r = 1, \dots, \infty$, the functors

$$L\mathbb{D}_S(\cdot) : D_{\mathcal{D}(2)\text{fil}, r}(S) \rightarrow D_{\mathcal{D}(2)\text{fil}, r}(S)^{op}, (M, F) \mapsto L\mathbb{D}_S(M, F) := \mathbb{D}_S^K L_D(M, F).$$

together with the canonical map $d(M, F) : L_D(M, F) \rightarrow \mathbb{D}_S^{2,K} L_D(M, F)$.

- Let $S \in \text{AnSm}(\mathbb{C})$. The hom functor also induces the bifunctors

$$\begin{aligned} & - \text{Hom}_{D_S^\infty}(-, -) : C_{\mathcal{D}^\infty \text{fil}}(S)^2 \rightarrow C_{2\text{fil}}(S), ((M, W), (N, F)) \mapsto \mathcal{H}om_{D_S^\infty}((M, W), (N, F)), \\ & - \text{Hom}_{D_S}(-, -) : C_{\mathcal{D}^\infty \text{fil}}(S)^2 \rightarrow C_{2\text{fil}}(S), ((M, W), (N, F)) \mapsto \mathcal{H}om_{D_S^\infty}((M, W), (N, F)). \end{aligned}$$

We get the filtered dual

$$\mathbb{D}_S^{\infty, K}(\cdot) : C_{\mathcal{D}^\infty(2)\text{fil}}(S) \rightarrow C_{\mathcal{D}^\infty(2)\text{fil}}(S)^{op}, (M, F) \mapsto \mathbb{D}_S^{\infty, K}(M, F) := \mathcal{H}om_{D_S^\infty}((M, F), D_S^\infty) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S]$$

together with the canonical map $d(M, F) : (M, F) \rightarrow \mathbb{D}_S^{\infty, K}(M, F)$. Of course $\mathbb{D}_S^{\infty, K}(\cdot)(C_{\mathcal{D}^\infty(1,0)\text{fil}}(S)) \subset C_{\mathcal{D}^\infty(1,0)\text{fil}}(S)$. It induces in the derived categories $D_{\mathcal{D}\text{fil}, r}(S)$, for $r = 1, \dots, \infty$, the functors

$$L\mathbb{D}_S^\infty(\cdot) : D_{\mathcal{D}^\infty(2)\text{fil}, r}(S) \rightarrow D_{\mathcal{D}^\infty(2)\text{fil}, r}(S)^{op}, (M, F) \mapsto L\mathbb{D}_S^\infty(M, F) = \mathbb{D}_S^{\infty, K} L_{D^\infty}(M, F).$$

together with the canonical map $d(M, F) : (M, F) \rightarrow L\mathbb{D}_S^{\infty, K}(M, F)$.

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the inverse image functor

$$f^{*mod} : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}(2)fil}(X),$$

$$(M, F) \mapsto f^{*mod}(M, F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M, F) = (D_{X \rightarrow S}, F^{ord}) \otimes_{f^*D_S} f^*(M, F),$$

induces in the derived categories the functors, for $r = 1, \dots, \infty$ (resp. $r \in (1, \dots, \infty)^2$),

$$Lf^{*mod} : D_{\mathcal{D}(2)fil,r}(S) \rightarrow D_{\mathcal{D}(2)fil,r}(X),$$

$$(M, F) \mapsto Lf^{*mod}M := (O_X, F_b) \otimes_{f^*O_S}^L f^*(M, F) = (O_X, F_b) \otimes_{f^*O_S} f^*L_D(M, F).$$

Of course $f^{*mod}(C_{\mathcal{D}(1,0)fil}(S)) \subset C_{\mathcal{D}(1,0)fil}(X)$. Note that

- If the M is a complex of locally free O_S modules, then $Lf^{*mod}(M, F) = f^{*mod}(M, F)$ in $D_{\mathcal{D}(2)fil,\infty}(S)$.
- If the $\text{Gr}_F^p M$ are complexes of locally free O_S modules, then $Lf^{*mod}(M, F) = f^{*mod}(M, F)$ in $D_{\mathcal{D}(2)fil}(S)$.

We will consider also the shifted inverse image functors

$$Lf^{*mod[-]} := Lf^{*mod}[d_S - d_X] : D_{\mathcal{D}(2)fil,r}(S) \rightarrow D_{\mathcal{D}(2)fil,r}(X).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the inverse image functor

$$f^{*mod} : C_{\mathcal{D}^\infty(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(X), (M, F) \mapsto f^{*mod}(M, F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M, F),$$

induces in the derived categories the functors, for $r = 1, \dots, \infty$ (resp. $r \in (1, \dots, \infty)^2$),

$$Lf^{*mod} : D_{\mathcal{D}^\infty(2)fil,r}(S) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(X),$$

$$(M, F) \mapsto Lf^{*mod}M := (O_X, F_b) \otimes_{f^*O_S}^L f^*(M, F) = (O_X, F_b) \otimes_{f^*O_S} f^*L_{D^\infty}(M, F).$$

Of course $f^{*mod}(C_{\mathcal{D}^\infty(1,0)fil}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil}(X)$. Note that We will consider also the shifted inverse image functors

$$Lf^{*mod[-]} := Lf^{*mod}[d_S - d_X] : D_{\mathcal{D}^\infty(2)fil,r}(S) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(X).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the direct image functor

$$f_{*mod}^{00} : (\text{PSh}_{\mathcal{D}}(X), F) \rightarrow (\text{PSh}_{\mathcal{D}}(S), F), (M, F) \mapsto f_{*mod}(M, F) := f_*((D_{S \leftarrow X}, F^{ord}) \otimes_{D_X}(M, F))$$

induces in the derived categories by taking r -injective resolutions the functors, for $r = 1, \dots, \infty$,

$$\int_f = Rf_{*mod} : D_{\mathcal{D}(2)fil,r}(X) \rightarrow D_{\mathcal{D}(2)fil,r}(S), (M, F) \mapsto \int_f(M, F) = Rf_*((D_{S \leftarrow X}, F^{ord}) \otimes_{D_X}^L(M, F)).$$

Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$ or with $X, Y, S \in \text{AnSm}(\mathbb{C})$. We have, for $(M, F) \in C_{\mathcal{D}fil}(X)$, the canonical transformation map in $D_{\mathcal{D}(2)fil,r}(S)$

$$\begin{aligned} & T\left(\int_{f_2} \circ \int_{f_1}, \int_{f_2 \circ f_1}\right)(M, F) : \\ & \int_{f_2} \int_{f_1} (M, F) := Rf_{2*}((D_{Y \leftarrow S}, F^{ord}) \otimes_{D_Y}^L Rf_{1*}((D_{X \leftarrow Y}, F^{ord}) \otimes_{D_X}^L(M, F))) \\ & \xrightarrow{T(f_1, \otimes)(-, -)} Rf_{2*}Rf_{1*}(f_1^*(D_{Y \leftarrow S}, F^{ord}) \otimes_{D_Y}^L((D_{X \leftarrow Y}, F^{ord}) \otimes_{D_X}^L(M, F))) \\ & \xrightarrow{\sim} Rf_{2*}Rf_{1*}((f_1^*(D_{Y \leftarrow S}, F^{ord}) \otimes_{D_Y}^L(D_{X \leftarrow Y}, F^{ord})) \otimes_{D_X}^L(M, F)) \\ & \xrightarrow{\sim} Rf_{2*}Rf_{1*}((D_{X \leftarrow S}, F^{ord}) \otimes_{D_X}^L(M, F)) := \int_{f_2 \circ f_1}(M, F) \end{aligned}$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the direct image functor

$$f_{*mod}^{00} : (\text{PSh}_{\mathcal{D}^\infty}(X), F) \rightarrow (\text{PSh}_{\mathcal{D}^\infty}(S), F), \quad (M, F) \mapsto f_{*mod}(M, F) := f_*((D_{S \leftarrow X}^\infty, F^{ord}) \otimes_{D_X^\infty} (M, F))$$

induces in the derived categories by taking r -injective resolutions the functors, for $r = 1, \dots, \infty$,

$$\int_f = Rf_{*mod} : D_{\mathcal{D}^\infty(2)fil,r}(X) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(S), \quad (M, F) \mapsto \int_f(M, F) = Rf_*((D_{S \leftarrow X}^\infty, F^{ord}) \otimes_{D_X^\infty}^L (M, F)).$$

We have, similarly, for $(M, F) \in C_{\mathcal{D}^\infty fil}(X)$, the canonical transformation map in $D_{\mathcal{D}^\infty(2)fil,r}(S)$

$$T\left(\int_{f_2} \circ \int_{f_1}, \int_{f_2 \circ f_1}\right)(M, F) : \int_{f_2} \int_{f_1} (M, F) \rightarrow \int_{f_2 \circ f_1} (M, F)$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the direct image functor with compact support

$$f_{!mod}^{00} : (\text{PSh}_{\mathcal{D}}(X), F) \rightarrow (\text{PSh}_{\mathcal{D}}(S), F), \quad (M, F) \mapsto f_{!mod}^{00}(M, F) := f_!((D_{S \leftarrow X}, F^{ord}) \otimes_{D_X} (M, F))$$

induces in the derived categories by taking r -injective resolutions the functors, for $r = 1, \dots, \infty$,

$$\int_{f!} = Rf_{!mod} : D_{\mathcal{D}fil,r}(X) \rightarrow D_{\mathcal{D}fil,r}(S), \quad (M, F) \mapsto \int_{f!}(M, F) = Rf_!((D_{S \leftarrow X}, F^{ord}) \otimes_{D_X}^L (M, F)).$$

We have, similarly, for $(M, F) \in C_{\mathcal{D}fil}(X)$, the canonical transformation map in $D_{\mathcal{D}(2)fil,r}(S)$

$$T\left(\int_{f_2!} \circ \int_{f_1!}, \int_{(f_2 \circ f_1)!}\right)(M, F) : \int_{f_2!} \int_{f_1!} (M, F) \rightarrow \int_{(f_2 \circ f_1)!} (M, F)$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the direct image functor with compact support

$$f_{!mod}^{00} : (\text{PSh}_{\mathcal{D}^\infty}(X), F) \rightarrow (\text{PSh}_{\mathcal{D}^\infty}(S), F), \quad (M, F) \mapsto f_{!mod}^{00}(M, F) := f_!((D_{S \leftarrow X}^\infty, F^{ord}) \otimes_{D_X^\infty} (M, F))$$

induces in the derived categories by taking r -injective resolutions the functors, for $r = 1, \dots, \infty$,

$$\int_{f!} = Rf_{!mod} : D_{\mathcal{D}^\infty fil,r}(X) \rightarrow D_{\mathcal{D}^\infty fil,r}(S), \quad (M, F) \mapsto \int_{f!}(M, F) = Rf_!((D_{S \leftarrow X}^\infty, F^{ord}) \otimes_{D_X^\infty}^L (M, F)).$$

We have, similarly, for $(M, F) \in C_{\mathcal{D}^\infty fil}(X)$, the canonical transformation map in $D_{\mathcal{D}^\infty(2)fil,r}(S)$

$$T\left(\int_{f_2!} \circ \int_{f_1!}, \int_{(f_2 \circ f_1)!}\right)(M, F) : \int_{f_2!} \int_{f_1!} (M, F) \rightarrow \int_{(f_2 \circ f_1)!} (M, F)$$

- Let $S \in \text{SmVar}(\mathbb{C})$. The analytical functor for filtered D_S -modules is

$$(\cdot)^{an} : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}(2)fil}(S^{an}), \quad (M, F) \mapsto (M, F)^{an} := \text{an}_S^*(M, F) \otimes_{\text{an}_S^* O_S} (O_{S^{an}}, F_b).$$

It induces in the derived categories the functors, for $r = 1, \dots, \infty$,

$$(\cdot)^{an} : D_{\mathcal{D}(2)fil,r}(S) \rightarrow D_{\mathcal{D}(2)fil,r}(S^{an}), \quad (M, F) \mapsto (M, F)^{an} := \text{an}_S^*(M, F) \otimes_{\text{an}_S^* O_S}^L (O_{S^{an}}, F_b).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then the functor

$$f^{\hat{*}mod} : C_{\mathcal{D}2fil}(S) \rightarrow C_{\mathcal{D}2fil}(X), \quad (M, F) \mapsto f^{\hat{*}mod}(M, F) := \mathbb{D}_X^K L_D f^{*mod} L_D \mathbb{D}_S^K (M, F)$$

induces in the derived categories the exceptional inverse image functors, for $r = 1, \dots, \infty$ (resp. $r \in (1, \dots, \infty)^2$),

$$Lf^{\hat{*}mod} : D_{\mathcal{D}(2)fil,r}(S) \rightarrow D_{\mathcal{D}(2)fil,r}(X),$$

$$(M, F) \mapsto Lf^{\hat{*}mod}(M, F) := L\mathbb{D}_X Lf^{*mod} L\mathbb{D}_S(M, F) := f^{\hat{*}mod} L_D(M, F).$$

Of course $f^{\hat{*}mod}(C_{\mathcal{D}(1,0)fil}(S)) \subset C_{\mathcal{D}(1,0)fil}(X)$. We will also consider the shifted exceptional inverse image functors

$$Lf^{\hat{*}mod[-]} := Lf^{\hat{*}mod}[d_S - d_X] : D_{\mathcal{D}(2)fil,r}(S) \rightarrow D_{\mathcal{D}(2)fil,r}(X).$$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then the functor

$$f^{\hat{*}mod} : C_{\mathcal{D}^\infty(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(X), (M, F) \mapsto f^{\hat{*}mod}(M, F) := \mathbb{D}_X^{K,\infty} L_D f^{*mod} L_D \mathbb{D}_S^{K,\infty}(M, F)$$

induces in the derived categories the exceptional inverse image functors, for $r = 1, \dots, \infty$ (resp. $r \in (1, \dots, \infty)^2$),

$$Lf^{\hat{*}mod} : D_{\mathcal{D}^\infty(2)fil,r}(S) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(X),$$

$$(M, F) \mapsto Lf^{\hat{*}mod}(M, F) := L\mathbb{D}_X^\infty Lf^{*mod} L\mathbb{D}_S^\infty(M, F) := f^{\hat{*}mod}(M, F).$$

Of course $f^{\hat{*}mod}(C_{\mathcal{D}^\infty(1,0)fil}(S)) \subset C_{\mathcal{D}^\infty(1,0)fil}(X)$. We will also consider the shifted exceptional inverse image functors

$$Lf^{\hat{*}mod[-]} := Lf^{\hat{*}mod}[d_S - d_X] : D_{\mathcal{D}^\infty(2)fil,r}(S) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(X).$$

- Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \text{AnSm}(\mathbb{C})$. Consider $p : S_1 \times S_2 \rightarrow S_1$ the projection. Since p is a projection, we have a canonical embedding $p^* D_{S_1} \hookrightarrow D_{S_1 \times S_2}$. For $(M, F) \in C_{\mathcal{D}(2)fil}(S_1 \times S_2)$, (M, F) has a canonical $p^* D_{S_1}$ module structure. Moreover, with this structure, for $(M_1, F) \in C_{\mathcal{D}(2)fil}(S_1)$

$$\text{ad}(p^{*mod}, p)(M_1, F) : (M_1, F) \rightarrow p_* p^{*mod}(M_1, F)$$

is a map of complexes of D_{S_1} modules, and for $(M_{12}, F) \in C_{\mathcal{D}(2)fil}(S_1 \times S_2)$)

$$\text{ad}(p^{*mod}, p)(M_{12}, F) : p^{*mod} p_*(M_{12}, F) \rightarrow (M_{12}, F)$$

is a map of complexes of $D_{S_1 \times S_2}$ modules. Indeed, for the first adjunction map, we note that $p^{*mod}(M_1, F)$ has a structure of $p^* D_{S_1}$ module, hence $p_* p^{*mod}(M_1, F)$ has a structure of $p_* p^* D_{S_1}$ module, hence a structure of D_{S_1} module using the adjunction map $\text{ad}(p^*, p_*)(D_{S_1}) : D_{S_1} \rightarrow p_* p^* D_{S_1}$. For the second adjunction map, we note that (M_{12}, F) has a structure of $p^* D_{S_1}$ module, hence $p_*(M_{12}, F)$ has a structure of $p_* p^* D_{S_1}$, hence a structure of D_{S_1} module using the adjunction map $\text{ad}(p^*, p_*)(D_{S_1}) : D_{S_1} \rightarrow p_* p^* D_{S_1}$.

- Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$. Consider $p : S_1 \times S_2 \rightarrow S_1$ the projection. Since p is a projection, we have a canonical embedding $p^* D_{S_1}^\infty \hookrightarrow D_{S_1 \times S_2}^\infty$. For $(M, F) \in C_{\mathcal{D}^\infty(2)fil}(S_1 \times S_2)$, (M, F) has a canonical $p^* D_{S_1}^\infty$ module structure. Moreover, with this structure, for $(M_1, F) \in C_{\mathcal{D}^\infty(2)fil}(S_1)$

$$\text{ad}(p^{*mod}, p)(M_1, F) : (M_1, F) \rightarrow p_* p^{*mod}(M_1, F)$$

is a map of complexes of $D_{S_1}^\infty$ modules, and for $(M_{12}, F) \in C_{\mathcal{D}^\infty(2)fil}(S_1 \times S_2)$)

$$\text{ad}(p^{*mod}, p)(M_{12}, F) : p^{*mod} p_*(M_{12}, F) \rightarrow (M_{12}, F)$$

is a map of complexes of $D_{S_1 \times S_2}^\infty$ modules, similarly to the finite order case.

We following proposition concern the commutativity of the inverse images functors and the commutativity of the direct image functors.

Proposition 53. (i) Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$.

- Let $(M, F) \in C_{\mathcal{D}(2)fil,r}(S)$. Then $(f_2 \circ f_1)^{*mod}(M, F) = f_1^{*mod} f_2^{*mod}(M, F)$.
- Let $(M, F) \in D_{\mathcal{D}(2)fil,r}(S)$. Then $L(f_2 \circ f_1)^{*mod}(M, F) = L f_1^{*mod}(L f_2^{*mod}(M, F))$.

(ii) Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$. Let $M \in D_{\mathcal{D}}(X)$. Then,

$$T\left(\int_{f_2} \circ \int_{f_1}, \int_{f_2 \circ f_1}\right)(M) : \int_{f_2} \int_{f_1} (M) \xrightarrow{\sim} \int_{f_2 \circ f_1} (M)$$

is an isomorphism in $D_{\mathcal{D}}(S)$ (i.e. if we forget filtration).

(iii) Let $i_0 : Z_2 \hookrightarrow Z_1$ and $i_1 : Z_1 \hookrightarrow S$ two closed embedding, with $Z_2, Z_1, S \in \text{SmVar}(\mathbb{C})$. Let $(M, F) \in C_{\mathcal{D}(2)fil}(Z_2)$. Then, $(i_1 \circ i_0)^{*mod}(M, F) = i_1^{*mod}(i_0^{*mod}(M, F))$ in $C_{\mathcal{D}(2)fil}(S)$.

Proof. (i): Obvious : we have

- $f_1^{*mod} f_2^{*mod}(M, F) = f_1^*(f_2^*(M, F) \otimes_{f_2^* O_S} O_Y) \otimes_{f_1^* O_Y} O_X = f_1^* f_2^*(M, F) \otimes_{f_1^* f_2^* O_S} f_1^* O_Y \otimes_{f_1^* O_Y} O_X = (f_2 \circ f_1)^{*mod}(M, F)$
- $L f_1^{*mod} L f_2^{*mod}(M, F) = f_1^*(f_2^*(M, F) \otimes_{f_2^* O_S}^L O_Y) \otimes_{f_1^* O_Y}^L O_X = f_1^* f_2^*(M, F) \otimes_{f_1^* f_2^* O_S}^L f_1^* O_Y \otimes_{f_1^* O_Y}^L O_X = L(f_2 \circ f_1)^{*mod}(M, F)$

(ii): See [17] : we have by lemma 4

$$\begin{aligned} \int_{f_2 \circ f_1} M &:= Rf_{2*} Rf_{1*}(D_{X \leftarrow S} \otimes_{D_X}^L M) \\ \xrightarrow{\cong} Rf_{2*} Rf_{1*}((f_1^* D_{Y \leftarrow S} \otimes_{f_1^* D_Y}^L D_{X \leftarrow Y}) \otimes_{D_X}^L M) &\xrightarrow{Rf_{2*} T(f_1, \otimes)(D_{Y \leftarrow S}, D_{X \leftarrow Y} \otimes_{D_X}^L M)^{-1}} \\ Rf_{2*}(D_{Y \leftarrow S} \otimes_{D_Y}^L Rf_{1*}(D_{X \leftarrow Y} \otimes_{D_X}^L M)) &=: \int_{f_2} \int_{f_1} f_1 M \end{aligned}$$

where, $D_{Y \leftarrow S}$ being a quasi-coherent D_Y module, we used the fact that for $N \in C_{f_1^* \mathcal{D}}(X)$ and $N' \in C_{\mathcal{D}}(Y)$

$$T(f_1, \otimes)(N', N) : N' \otimes_{D_Y}^L Rf_{1*} N \rightarrow Rf_{1*}(f_1^* N' \otimes_{f_1^* D_Y}^L N)$$

is an isomorphism if N' is quasi-coherent, which follows from the fact that f_{1*} commutes with arbitrary (possibly infinite) direct sums (see [17]).

(iii): Denote $i_2 = i_1 \circ i_0 : Z_2 \hookrightarrow S$. We have

$$\begin{aligned} i_{2*} i_{0*}(M, F) &= i_{2*}((M, F) \otimes_{D_{Z_2}} (D_{Z_2 \leftarrow S}, F^{ord})) \xrightarrow{\cong} \\ i_{1*} i_{0*}((M, F) \otimes_{D_{Z_2}} (D_{Z_2 \leftarrow Z_1}, F^{ord}) \otimes_{i_0^* D_{Z_1}}^* i_0^*(D_{Z_1 \leftarrow S}, F^{ord})) &\xrightarrow{i_{1*} T(i_0, \otimes)(-)^{-1}} i_{1*} i_{0*} i_{0*} i_{0*}(M, F) \end{aligned}$$

using proposition 4 and proposition 10. □

Remark 8. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$. Then, for $(M, F) \in D_{\mathcal{D}(2)fil, \infty}(X)$, $\int_{f_2} \int_{f_1} (M, F)$ is NOT isomorphic to $\int_{f_2 \circ f_1} (M, F)$ in general, the filtrations on the isomorphic cohomology sheaves may be different.

Proposition 54. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$. Then,

- (i) For $(M, F) \in C_{\mathcal{D}(2)fil,h}(S)$, we have $L f^{*mod}(M, F) \in D_{\mathcal{D}(2)fil, \infty, h}(X)$.

(ii) For $M \in C_{\mathcal{D},h}(X)$, we have $\int_f M \in D_{\mathcal{D},h}(S)$.

(iii) If f is proper, for $(M, F) \in C_{\mathcal{D}(2)fil,h}(X)$, we have $\int_f(M, F) \in D_{\mathcal{D}(2)fil,\infty,h}(S)$.

Proof. See [17] for the non filtered case. The filtered case follows immediately from the non filtered case and the fact the pullback of a good filtration is a good filtration (since the pullback of a coherent O_S module is coherent) and the direct image of a good filtration by a proper morphism is a good filtration (since the pushforward of a coherent O_X module by a proper morphism is coherent). \square

The following easy proposition says that the analytical functor commutes with the pullback of D modules and the tensor product. Again it is well known in the non filtered case. Note that for $S \in \text{SmVar}(\mathbb{C})$, $D_S^{an} = D_{S^{an}}$.

Proposition 55. (i) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$.

- Let $(M, F) \in C_{\mathcal{D}(2)fil,r}(S)$. Then $(f^{*mod}(M, F))^{an} = f^{*mod}(M, F)^{an}$.
- Let $(M, F) \in D_{\mathcal{D}fil,r}(S)$, for $r = 1, \dots, \infty$. Then, $(Lf^{*mod}(M, F))^{an} = Lf^{*mod}(M, F)^{an}$.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$

- Let $(M, F), (N, F) \in C_{\mathcal{D}fil}(S)$. Then, $((M, F) \otimes_{O_S} (N, F))^{an} = (M, F)^{an} \otimes_{O_{S^{an}}} (N, F)^{an}$.
- Let $(M, F), (N, F) \in D_{\mathcal{D}fil,r}(S)$, for $r = 1, \dots, \infty$. Then, $((M, F) \otimes_{O_S}^L (N, F))^{an} = (M, F)^{an} \otimes_{O_{S^{an}}}^L (N, F)^{an}$.

Proof. (i): For $(M, F) \in C_{\mathcal{D},fil}(S)$, we have, since $f^* \text{an}_S^* = \text{an}_X^* f^{an*}$,

$$\begin{aligned} (f^{*mod}(M, F))^{an} &= \text{an}(X)^*(f^*(M, F) \otimes_{f^*O_S} O_X) \otimes_{\text{an}(X)^*O_X} O_{X^{an}} \\ &= f^{an*} \text{an}_S^*(M, F) \otimes_{f^{an*}O_{S^{an}}} O_{X^{an}} =: f^{an*mod}(M^{an}, F) \end{aligned}$$

For $(M, F) \in D_{\mathcal{D},fil,r}(S)$, we take $(M, F) \in C_{\mathcal{D},fil}(S)$ an r -projective f^*O_S module such that $D_{top,r}(M, F) = (M, F)$ so that

$$(Lf^{*mod}(M, F))^{an} = (f^{*mod}(M, F))^{an} = f^{an*mod}(M^{an}, F) = Lf^{an*mod}(M^{an}, F)$$

(ii): For $(M, F), (N, F) \in C_{\mathcal{D},fil}(S)$, we have

$$\begin{aligned} ((M, F) \otimes_{O_S} (N, F))^{an} &=: \text{an}_S^*((M, F) \otimes_{O_S} (N, F)) \otimes_{\text{an}_S^*O_S} O_{S^{an}} \\ &=: \text{an}_S^*(M, F) \otimes_{\text{an}_S^*O_S} \text{an}_S^*(N, F) \otimes_{\text{an}_S^*O_S} O_{S^{an}} \\ &=: \text{an}_S^*(M, F) \otimes_{\text{an}_S^*O_S} \otimes_{O_{S^{an}}} \otimes_{O_{S^{an}}} \text{an}_S^*(N, F) \otimes_{\text{an}_S^*O_S} O_{S^{an}} \\ &=: (M^{an}, F) \otimes_{O_{S^{an}}} (N^{an}, F) \end{aligned}$$

It implies the isomorphism in the derived category by taking an r -projective resolution of (M, F) (e.g $(L_D(M), F) = L_D(M, F)$). \square

Proposition 56. (i) Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{AnSm}(\mathbb{C})$.

- Let $(M, F) \in C_{\mathcal{D}(2)fil}(S)$ or let $(M, F) \in C_{\mathcal{D}\infty(2)fil}(S)$. Then $(f_2 \circ f_1)^{*mod}(M, F) = f_1^{*mod} f_2^{*mod}(M, F)$.
- Let $(M, F) \in D_{\mathcal{D}(2)fil,r}(S)$ or let $(M, F) \in D_{\mathcal{D}\infty(2)fil,r}(S)$. Then $L(f_2 \circ f_1)^{*mod}(M, F) = Lf_1^{*mod}(Lf_2^{*mod}(M, F))$.

(ii) Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphisms with $X, Y, S \in \text{AnSm}(\mathbb{C})$. Let $M \in D_{\mathcal{D}}(X)$. If f_1 is proper, we have $\int_{f_2 \circ f_1} M = \int_{f_2} (\int_{f_1} M)$.

(ii)' Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphisms with $X, Y, S \in \text{AnSm}(\mathbb{C})$. Let $M \in D_{\mathcal{D}\infty}(X)$. If f_1 is proper, we have $\int_{f_2 \circ f_1} M = \int_{f_2} (\int_{f_1} M)$.

(iii) Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphisms with $X, Y, S \in \text{AnSm}(\mathbb{C})$. Let $(M, F) \in D_{\mathcal{D}}(X)$. We have $\int_{(f_2 \circ f_1)!} M = \int_{f_2!} (\int_{f_1!} M)$.

(iii)' Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphisms with $X, Y, S \in \text{AnSm}(\mathbb{C})$. Let $M \in D_{\mathcal{D}^\infty}(X)$. We have $\int_{(f_2 \circ f_1)!} M = \int_{f_2!} (\int_{f_1!} M)$.

(iv) Let $i_0 : Z_2 \hookrightarrow Z_1$ and $i_1 : Z_1 \hookrightarrow S$ two closed embedding, with $Z_2, Z_1, S \in \text{AnSm}(\mathbb{C})$. Let $(M, F) \in C_{\mathcal{D}(2)\text{fil}}(Z_2)$. Then, $(i_1 \circ i_0)_{*mod}(M, F) = i_{1*mod}(i_{0*mod}(M, F))$ in $C_{\mathcal{D}(2)\text{fil}}(S)$.

(iv)' Let $i_0 : Z_2 \hookrightarrow Z_1$ and $i_1 : Z_1 \hookrightarrow S$ two closed embedding, with $Z_2, Z_1, S \in \text{AnSm}(\mathbb{C})$. Let $(M, F) \in C_{\mathcal{D}^\infty(2)\text{fil}}(Z_2)$. Then, $(i_1 \circ i_0)_{*mod}(M, F) = i_{1*mod}(i_{0*mod}(M, F))$ in $C_{\mathcal{D}^\infty(2)\text{fil}}(S)$.

Proof. (i): Similar to the proof of proposition 53(i).

(ii): Similar to the proof of proposition 53(ii) : we use lemma 4 and the fact that for $N \in C_{f_1^* \mathcal{D}}(X)$ and $N' \in C_{\mathcal{D}}(Y)$, the canonical morphism

$$T(f_{1!}, \otimes)(N', N) : N' \otimes_{D_Y}^L Rf_{1*} N \rightarrow Rf_{1*}(f_1^* N' \otimes_{f_1^* D_Y}^L N)$$

is an isomorphism if f_1 is proper (in this case $f_{1!} = f_{1*}$).

(ii)': Similar to the proof of proposition 53(ii) : we use lemma 5 and the fact that for $N \in C_{f_1^* \mathcal{D}^\infty}(X)$ and $N' \in C_{\mathcal{D}^\infty}(Y)$, the canonical morphism

$$T(f_{1!}, \otimes)(N', N) : N' \otimes_{D_Y^\infty}^L Rf_{1*} N \rightarrow Rf_{1*}(f_1^* N' \otimes_{f_1^* D_Y^\infty}^L N)$$

is an isomorphism if f_1 is proper (in this case $f_{1!} = f_{1*}$).

(iii): Similar to the proof of proposition 53(ii) : we use lemma 4 and the fact that for $N \in C_{f_1^* \mathcal{D}}(X)$ and $N' \in C_{\mathcal{D}}(Y)$, the canonical morphism

$$T(f_{1!}, \otimes)(N', N) : N' \otimes_{D_Y}^L Rf_{1!} N \rightarrow Rf_{1!}(f_1^* N' \otimes_{f_1^* D_Y}^L N)$$

is an isomorphism.

(iii)': Similar to the proof of proposition 53(ii) : we use lemma 5 and the fact that for $N \in C_{f_1^* \mathcal{D}^\infty}(X)$ and $N' \in C_{\mathcal{D}^\infty}(Y)$, the canonical morphism

$$T(f_{1!}, \otimes)(N', N) : N' \otimes_{D_Y^\infty}^L Rf_{1!} N \rightarrow Rf_{1!}(f_1^* N' \otimes_{f_1^* D_Y^\infty}^L N)$$

is an isomorphism

(iv): Similar to the proof of proposition 53(iii) : we have

$$\begin{aligned} i_{2*mod}(M, F) &= i_{2*}((M, F) \otimes_{D_{Z_2}} (D_{Z_2 \leftarrow S}, F^{ord})) \xrightarrow{=} \\ i_{1*} i_{0*}((M, F) \otimes_{D_{Z_2}} (D_{Z_2 \leftarrow Z_1}, F^{ord}) \otimes_{i_0^* D_{Z_1}} i_0^*(D_{Z_1 \leftarrow S}, F^{ord})) &\xrightarrow{i_{1*} T(i_0, \otimes)(-)^{-1}} i_{1*mod} i_{0*mod}((M, F)) \end{aligned}$$

using lemma 4 and proposition 10.

(iv)': Similar to (iv): we have

$$\begin{aligned} i_{2*mod}(M, F) &= i_{2*}((M, F) \otimes_{D_{Z_2}^\infty} (D_{Z_2 \leftarrow S}^\infty, F^{ord})) \xrightarrow{=} \\ i_{1*} i_{0*}((M, F) \otimes_{D_{Z_2}^\infty} (D_{Z_2 \leftarrow Z_1}^\infty, F^{ord}) \otimes_{i_0^* D_{Z_1}^\infty} i_0^*(D_{Z_1 \leftarrow S}^\infty, F^{ord})) &\xrightarrow{i_{1*} T(i_0, \otimes)(-)^{-1}} i_{1*mod} i_{0*mod}((M, F)) \end{aligned}$$

using lemma 5 and proposition 10. □

Proposition 57. (i) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\mathcal{D}(2)\text{fil}, h}(S)$, we have $Lf^{*mod}(M, F) \in D_{\mathcal{D}(2)\text{fil}, \infty, h}(X)$. For $(M, F) \in C_{\mathcal{D}^\infty(2)\text{fil}, h}(S)$, we have $Lf^{*mod}(M, F) \in D_{\mathcal{D}^\infty(2)\text{fil}, \infty, h}(X)$.

(ii) Let $f : X \rightarrow S$ a proper morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, for $(M, F) \in C_{\mathcal{D}(2)\text{fil},h}(X)$, we have $\int_f(M, F) \in D_{\mathcal{D}(2)\text{fil},\infty,h}(S)$.

(iii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, for $(M, F) \in C_{\mathcal{D}^\infty(2)\text{fil},h}(X)$, we have $\int_f(M, F) \in D_{\mathcal{D}^\infty(2)\text{fil},\infty,h}(S)$.

Proof. (i)and (ii):Follows imediately from the non filtered case since we look at the complex in the derived category with respect to ∞ -usu local equivalence. It says that the pullback and the proper pushforward of an holonomic D module is still holonomic. See [17] for the non filtered case.

(iii):In the case the morphism is proper, it follows from the finite order case (ii). In the case of an open embedding, it follows from proposition 51(i) : we have for $j : S^\circ \hookrightarrow S$ an open embedding,

$$j_*E(O_{S^\circ}) = j_*\mathcal{H}om(\mathbb{Z}_{S^\circ}, E(O_{S^\circ})) = \mathcal{H}om(j!\mathbb{Z}_{S^\circ}, E(O_S)) \in C_{\mathcal{D}^\infty,h}(S).$$

and on the other hand

$$\begin{aligned} T(j, \otimes)(-, -) &= T^{\text{mod}}(j, \otimes)(-, -) : \int_j(M, F) = j_*E(M, F) = j_*E(j^*O_S \otimes_{O_{S^\circ}}(M, F)) \\ &\xrightarrow{\sim} j_*E(O_{S^\circ}) \otimes_{O_S}(M, F) \end{aligned}$$

is an isomorphism by proposition 9. □

For $X, Y \in \text{SmVar}(\mathbb{C})$ or $X, Y \in \text{AnSm}(\mathbb{C})$, we denote by

- $C_{O_X}(X) \times C_{O_Y}(Y) \rightarrow C_{O_{X \times Y}}(X \times Y), (M, N) \mapsto M \cdot N := O_{X \times Y} \otimes_{p_X^*O_X \otimes p_Y^*O_Y} p_X^*M \otimes p_Y^*N,$
- $C_{\mathcal{D}}(X) \times C_{\mathcal{D}}(Y) \rightarrow C_{\mathcal{D}}(X \times Y), (M, N) \mapsto M \cdot N := O_{X \times Y} \otimes_{p_X^*O_X \otimes p_Y^*O_Y} p_X^*M \otimes p_Y^*N$

the natural functors which induces functors in the filtered cases and the derived categories, $p_X : X \times Y \rightarrow X$ and $p_Y : X \times Y \rightarrow Y$ the projections.

We have then the following easy proposition :

Proposition 58. For $X \in \text{SmVar}(\mathbb{C})$ or $X \in \text{AnSm}(\mathbb{C})$, we have for $(M, F), (N, F) \in C_{O_X, \text{fil}}(X)$ or $(M, F), (N, F) \in C_{\mathcal{D}, \text{fil}}(X)$,

$$(M, F) \otimes_{O_X}(N, F) = \Delta_X^{\text{mod}}(M, F) \cdot (N, F)$$

Proof. Standard. □

Definition 67. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. We have the canonical map in $C_{f^*\mathcal{D}, \mathcal{D}^\infty}(X)$ modules :

$$T(f, \infty) : (D_{X \rightarrow S}, F^{\text{ord}}) \otimes_{D_X}(D_X^\infty, F^{\text{ord}}) \rightarrow (D_{X \rightarrow S}^\infty, F^{\text{ord}}), (h_X \otimes P_S) \otimes P_X \mapsto (P_X \cdot h_X \otimes P_S + h_X \otimes df(P_X)P_S)$$

where $h_X \in \Gamma(X^\circ, O_X)$, $P_S \in \Gamma(X^\circ, f^*D_S)$ and $P_X \in \Gamma(X^\circ, D_X^\infty)$. This gives, for $(M, F) \in C_{\mathcal{D}(2)\text{fil}}(S)$, the following transformation map in $C_{\mathcal{D}^\infty(2)\text{fil}}(X)$

$$\begin{aligned} T(f, \infty)(M, F) : J_X(f^{*\text{mod}}(M, F)) &:= f^*(M, F) \otimes_{f^*D_S}(D_{X \rightarrow S}, F^{\text{ord}}) \otimes_{D_X}(D_X^\infty, F^{\text{ord}}) \xrightarrow{I \otimes T(f, \infty)} \\ f^*(M, F) \otimes_{f^*D_S}(D_{X \rightarrow S}^\infty, F^{\text{ord}}) &= f^*(M, F) \otimes_{f^*D_S} f^*D_S^\infty \otimes_{f^*D_S^\infty}(D_{X \rightarrow S}^\infty, F^{\text{ord}}) =: f^{*\text{mod}}J_S(M, F) \end{aligned}$$

where we recall that $J_S(M, F) = (M, F) \otimes_{D_S}(D_S^\infty, F^{\text{ord}})$.

We now look at some properties of the dual functor for D modules : For complex of D module with coherent cohomology we have the following:

Proposition 59. (i) Let $S \in \text{SmVar}(\mathbb{C})$. For $M \in C_{\mathcal{D},c}(S)$, the canonical map $d(M) : M \rightarrow \mathbb{D}_S^2 L_D M$ is an equivalence Zariski local.

(ii) Let $S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}}(S)$, the canonical map $d(M) : M \rightarrow \mathbb{D}_{\mathbb{S}}^2 L_D(M)$ is an equivalence usu local.

iii) Let $S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\mathcal{D}^\infty}(S)$, the canonical map $d(M) : M \rightarrow \mathbb{D}_{\mathbb{S}}^2 L_{D^\infty}(M)$ is an equivalence usu local.

Proof. Standard :follows from the definition of coherent sheaves. See [17] for exemple. \square

Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \rightarrow S_1$ the projection. In this case we have a canonical embedding $D_{S_1} \hookrightarrow p_* D_{S_{12}}$. This gives, for $(M, F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$, the following transformation map in $C_{\mathcal{D}fil}(S_1)$

$$\begin{aligned} T_*(p, D)(M, F) : p_* \mathbb{D}_{S_{12}}^K(M, F) &:= p_* \mathcal{H}om_{D_{S_{12}}}((M, F), D_{S_{12}}) \otimes_{O_{S_{12}}} \mathbb{D}_{S_{12}}^O w(K_{S_{12}})[d_{S_{12}}] \\ &\xrightarrow{T_*(p, \text{hom})(-, -)} \mathcal{H}om_{p_* D_{S_{12}}}(p_*(M, F), p_* D_{S_{12}}) \otimes_{p_* O_{S_{12}}} \mathbb{D}_{p_* S_{12}}^O w(p_* K_{S_{12}})[d_{S_{12}}] \\ &\xrightarrow{\sim} \mathcal{H}om_{D_{S_1}}(p_*(M, F), D_{S_1}) \otimes_{O_{S_1}} \mathbb{D}_{S_1}^O w(K_{S_1})[d_{S_1}] =: \mathbb{D}_{S_1}^K p_*(M, F) \end{aligned}$$

We have the canonical map

$$p(D) : p^{*mod} D_{S_1} = p^* D_{S_1} \otimes_{p^* O_{S_1}} O_{S_{12}} \rightarrow D_{S_{12}}, \quad \gamma \otimes f \mapsto f \cdot \gamma$$

induced by the embedding $p^* D_{S_1} \hookrightarrow D_{S_{12}}$. This gives, for $(M, F) \in C_{\mathcal{D}fil}(S_1)$, the following transformation map in $C_{\mathcal{D}fil}(S_1 \times S_2)$

$$\begin{aligned} T(p, D)(M, F) : p^{*mod} \mathbb{D}_{S_1}^K(M, F) &:= p^* \mathcal{H}om_{D_{S_1}}((M, F), D_{S_1}) \otimes_{p^* O_{S_1}} p^{*mod} \mathbb{D}_{S_1}^O w(K_{S_1})[d_{S_1}] \\ &\xrightarrow{T(p, \text{hom})(-, -) \otimes I} \mathcal{H}om_{p^* D_{S_1}}(p^*(M, F), p^* D_{S_1}) \otimes_{p^* O_{S_1}} p^{*mod} \mathbb{D}_{S_1}^O w(K_{S_1})[d_{S_1}] \\ &\xrightarrow{(\phi \mapsto \phi \otimes I_{O_{S_{12}}}) \otimes I} \mathcal{H}om_{D_{S_{12}}}(p^{*mod}(M, F), p^{*mod} D_{S_1}) \otimes_{p^* O_{S_1}} p^{*mod} \mathbb{D}_{S_1}^O w(K_{S_1})[d_{S_1}] \\ &\xrightarrow{I \otimes K^{-1}(S_1/S_{12})} \mathcal{H}om_{D_{S_{12}}}(p^{*mod}(M, F), p^{*mod} D_{S_1}) \otimes_{p^* O_{S_1}} \mathbb{D}_{S_{12}}^O w(K_{S_{12}})[d_{S_{12}}] \\ &\xrightarrow{q(p^* O_{S_1}/O_{S_{12}})} \mathcal{H}om_{D_{S_{12}}}(p^{*mod}(M, F), p^{*mod} D_{S_1}) \otimes_{O_{S_{12}}} \mathbb{D}_{S_{12}}^O w(K_{S_{12}})[d_{S_{12}}] \\ &\xrightarrow{\mathcal{H}om(p^{*mod}(M, F), p(D))} \mathcal{H}om_{D_{S_{12}}}(p^{*mod}(M, F), D_{S_{12}}) \otimes_{O_{S_{12}}} \mathbb{D}_{S_{12}}^O w(K_{S_{12}})[d_{S_{12}}] =: \mathbb{D}_{S_{12}}^K(p^{*mod}(M, F)) \end{aligned}$$

whre $K^{-1}(S_1/S_{12})$ is given by the wedge product with a generator of $\wedge^{d_{S_2}} T_{S_{12}/S_1} \xrightarrow{\sim} K_{S_2}^{-1}$.

In the case $S_1, S_2 \in \text{AnSm}(\mathbb{C})$, we also have the embedding $p^* D_{S_1}^\infty \hookrightarrow D_{S_{12}}^\infty$. This gives in the same way, for $(M, F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$, the following transformation map in $C_{\mathcal{D}fil}(S_1)$

$$T_*(p, D^\infty)(M, F) : p_* \mathbb{D}_{S_{12}}^{\infty, K}(M, F) \rightarrow \mathbb{D}_{S_1}^{\infty, K} p_*(M, F).$$

The map

$$p(D^\infty) : p^{*mod} D_{S_1} = p^* D_{S_1}^\infty \otimes_{p^* O_{S_1}} O_{S_{12}} \rightarrow D_{S_{12}}^\infty, \quad \gamma \otimes f \mapsto f \cdot \gamma$$

induced by the embedding $p^* D_{S_1}^\infty \hookrightarrow D_{S_{12}}^\infty$, gives in the same way, for $(M, F) \in C_{\mathcal{D}^\infty fil}(S_1)$, the transformation map in $C_{\mathcal{D}^\infty fil}(S_1 \times S_2)$

$$\begin{aligned} T(p, D^\infty)(M, F) : p^{*mod} \mathbb{D}_{S_1}^{\infty, K}(M, F) &:= p^{*mod} (\mathbb{D}_{S_1}^\infty(M, F) \otimes_{O_{S_1}} \mathbb{D}_{S_1}^O w(K_{S_1})[d_{S_1}]) \rightarrow \\ &\mathcal{H}om_{D_{S_{12}}^\infty}(p^{*mod}(M, F), D_{S_{12}}^\infty) \otimes_{O_{S_{12}}} \mathbb{D}_{S_{12}}^O w(K_{S_{12}}[d_{S_{12}}]) =: \mathbb{D}_{S_{12}}^{\infty, K}(p^{*mod}(M, F)), \end{aligned}$$

given in the same way then $T(p, D)(M, F)$.

Proposition 60. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. We have, for $M \in D_{\mathcal{D}}(S)$ canonical maps

- $T'(g, D)(M) : L\mathbb{D}_S Lg^{*mod}M \rightarrow Lg^{*mod}L\mathbb{D}_S M$
- $T'(g, D)(M) : Lg^{*mod}L\mathbb{D}_S M \rightarrow L\mathbb{D}_S Lg^{*mod}M$

Moreover, in the case where g is non characteristic with respect to M (e.g if g is smooth), these maps are isomorphism.

(ii) Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$, $p : S_1 \times S_2 \rightarrow S_1$ the projection. For $M \in D_{\mathcal{D}}(S_1)$, we have $T(p, D)(L_D(M)) = T'(p, D)(M)$ in $D_{\mathcal{D}fil}(S_1 \times S_2)$ (c.f.(i)).

Proof. (i):See [17] for the first map. The second one follows from the first by proposition 59(i) and (iii).
(ii):See the proof of (i) in [17] \square

We have the followings :

Proposition 61. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$. Let $M \in C_{\mathcal{D},h}(S)$. Then, we have $L(f_2 \circ f_1)^{\hat{*}mod}M = Lf_1^{\hat{*}mod}(Lf_2^{\hat{*}mod}M)$ in $D_{\mathcal{D},h}(X)$.

Proof. Follows from proposition 53 (i), proposition 54 and proposition 59. \square

Proposition 62. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{AnSm}(\mathbb{C})$.

- (i) Let $(M) \in C_{\mathcal{D},h}(S)$. Then, we have $L(f_2 \circ f_1)^{\hat{*}mod}(M) = Lf_1^{\hat{*}mod}(Lf_2^{\hat{*}mod}(M))$ in $D_{\mathcal{D},h}(X)$.
- (ii) Let $M \in C_{\mathcal{D}^\infty,h}(S)$. Then, we have $L(f_2 \circ f_1)^{\hat{*}mod}M = Lf_1^{\hat{*}mod}(Lf_2^{\hat{*}mod}M)$ in $D_{\mathcal{D}^\infty,h}(X)$.

Proof. Follows from proposition 56 (i), proposition 57 and proposition 59. \square

In the analytic case, we have the following transformation map which we will use in subsection 5.3:

Definition 68. Let $S \in \text{AnSm}(\mathbb{C})$. We have for $(M, F) \in C_{\mathcal{D}fil}(S)$ the canonical transformation map in $C_{\mathcal{D}^\infty fil}(S)$:

$$\begin{aligned} T(D, \infty)(M, F) : \\ J_S(\mathbb{D}_S^K(M, F)) := \text{Hom}_{D_S}((M, F), D_S) \otimes_{D_S} (D_S^\infty, F^{ord}) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] \xrightarrow{\text{ev}_{D_S}(\text{hom}, \otimes)(-, -, -) \otimes I} \\ \text{Hom}_{D_S}(L_D(M, F), D_S^\infty) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] \xrightarrow{I(D_S^\infty / D_S)((M, F), D_S^\infty) \otimes I} \\ \text{Hom}_{D_S^\infty}((M, F) \otimes_{D_S} (D_S^\infty, F^{ord}), D_S^\infty) \otimes_{O_S} \mathbb{D}_S^O w(K_S)[d_S] =: \mathbb{D}_S^{\infty, K} J_S(M, F). \end{aligned}$$

which is an isomorphism.

4.2.2 The (relative) De Rham of a (filtered) complex of a D-module and the filtered De Rham direct image

Recall that for $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$,

$$DR(X/S) := \Omega_{X/S}^\bullet \in C_{f^*O_S}(X)$$

denotes (see section 2) the relative De Rham complex of the morphism of ringed spaces $f : (X, O_X) \rightarrow (S, O_S)$, with $\Omega_{X/S}^p := \wedge^p \Omega_{X/S} \in \text{PSh}_{O_X}(X)$ and $\Omega_{X/S} := \text{coker}(f^* \Omega_S \rightarrow \Omega_X) \in \text{PSh}_{O_X}(X)$. Recall that $\Omega_{X/S}^\bullet \in C_{f^*O_S}(S)$ is a complex of f^*O_S modules, but is NOT a complex of O_X module since the differential is a derivation hence NOT O_X linear. Recall that (see section 4.1), for $(M, F) \in C_{D(O_X)fil}(X)$, we have the relative (filtered) De rham complex of (M, F)

$$DR(X/S)(M, F) := (\Omega_{X/S}^\bullet, F_b) \otimes_{O_X} (M, F) \in C_{f^*O_S fil}(X),$$

and that if $\phi : (M_1, F) \rightarrow (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in C_{D(O_X)fil}(X)$,

$$(I \otimes \phi) : DR(X/S)(M_1, F) := (\Omega_{X/S}^\bullet, F) \otimes_{O_X} (M_1, F) \rightarrow DR(X/S)(M_2, F) := (\Omega_{X/S}^\bullet, F) \otimes_{O_X} (M_2, F)$$

is by definition a morphism of complexes, that is a morphism in $C_{f^*O_S \text{fil}}(X)$. For $(N, F) \in C_{D(O_X)^{op} \text{fil}}(X)$, we have the relative (filtered) Spencer complex of (N, F)

$$SP(X/S)(N, F) := (T_{X/S}^\bullet, F_b) \otimes_{O_X} (N, F) \in C_{f^*O_S \text{fil}}(X),$$

and that if $\phi : (N_1, F) \rightarrow (N_2, F)$ a morphism with $(N_1, F), (N_2, F) \in C_{D(O_X)^{op} \text{fil}}(X)$,

$$(I \otimes \phi) : SP(X/S)(N_1, F) := (T_{X/S}^\bullet, F) \otimes_{O_X} (N_1, F) \rightarrow SP(X/S)(N_2, F) := (T_{X/S}^\bullet, F) \otimes_{O_X} (N_2, F)$$

is by definition a morphism of complexes, that is a morphism in $C_{f^*O_S \text{fil}}(X)$.

Proposition 63. *Let $f : X \rightarrow S$ a smooth morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$, denote $d = d_X - d_S$. The inner product gives, for $(M, F) \in C_{D(O_X) \text{fil}}(X)$, an isomorphism in $C_{f^*O_S \text{fil}}(X)$ and termwise O_X linear*

$$T(DR, SP)(M, F) : T_{X/S}^\bullet \otimes_{O_X} (M, F) \otimes_{O_X} K_{X/S} \xrightarrow{\sim} \Omega_{X/S}^{d-\bullet} \otimes_{O_X} (M, F), \partial \otimes m \otimes \kappa \mapsto \iota(\partial)\kappa \otimes m$$

Proof. Standard. □

For a commutative diagram in $\text{Var}(\mathbb{C})$ or in $\text{AnSp}(\mathbb{C})$:

$$D = \begin{array}{ccc} X & \xrightarrow{f} & S \\ g' \uparrow & & \uparrow g \\ X' & \xrightarrow{f'} & T \end{array}$$

we have (see section 2) the relative differential map of g' given by the pullback of differential forms:

$$\begin{aligned} \Omega_{(X'/X)/(T/S)} : g'^* \Omega_{X/S} &\rightarrow \Omega_{X'/T}, \text{ given by for } X'^o \subset X', X^o \supset g'(X'^o) \text{ (i.e. } g'^{-1}(X^o) \supset X'^o), \\ \omega \in \Gamma(X^o, \Omega_{X/S}^p) &\mapsto \Omega_{(X'/X)/(T/S)}(X'^o)(\omega) := [g'^* \omega] \in \Gamma(X'^o, \Omega_{X'/T}^p). \end{aligned}$$

Moreover, by definition-proposition 16 (section 4.1), for $(M, F) \in C_{D(O_X) \text{fil}}(X)$ the map

$$\Omega_{(X'/X)/(T/S)}(M, F) : g'^*(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F)) \rightarrow \Omega_{X'/T}^\bullet \otimes_{O_{X'}} g'^{*mod}(M, F)$$

given in degree (p, i) by, for $X'^o \subset X'$ an open subset and $X^o \subset X$ an open subset such that $g'^{-1}(X^o) \supset X'^o$ (i.e. $X^o \supset g'(X'^o)$), $\omega \in \Gamma(X^o, \Omega_{X/S}^p)$ and $m \in \Gamma(X^o, M^i)$,

$$\Omega_{(X'/X)/(T/S)}(M, F)(\omega \otimes m) = g'^* \omega \otimes (m \otimes 1)$$

is a map of complexes, that is a map in $C_{f^*O_S \text{fil}}(X')$. This give, for $(M, F) \in C_{D(O_X) \text{fil}}(X)$, the following transformation map in $C_{O_T \text{fil}}(T)$

$$\begin{aligned} T_\omega^O(D)(M, F) : g'^{*mod} L_O(f_* E(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F))) &\xrightarrow{T(g, L_O)(-)} \\ (g^* f_* E(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F))) \otimes_{g^* O_S} O_T &\xrightarrow{T(g', E)(-) \circ T(D)(E(\Omega_{X/S}^\bullet \otimes_{O_X} M))} \\ (f'_* E(g'^*(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F)))) \otimes_{g^* O_S} O_T &\xrightarrow{m \circ E(\Omega_{(X'/X)/(T/S)}(M))} f'_* E(\Omega_{X'/T}^\bullet \otimes_{O_{X'}} g'^{*mod}(M, F)), \end{aligned}$$

with $m'(m) = m \otimes 1$. Under the canonical isomorphism $\Omega_{X/S}^\bullet \xrightarrow{\sim} \Omega_{X/S}^\bullet \otimes_{O_X} O_X$ given by $\omega \mapsto \omega \otimes 1$, we have (see remark 7)

$$T_\omega^O(D)(O_X) = T_\omega^O(D) : g'^{*mod} L_O(f_* E(\Omega_{X/S}^\bullet)) \rightarrow f'_* E(\Omega_{X'/T}^\bullet).$$

Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Again by definition-proposition 16 (section 4.1), for $(M, F) \in C_{D(O_X)fil}(X)$ the map

$$\Omega_{(X^{an}/X)/(S^{an}/S)}(M, F) : an_X^*(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F)) \rightarrow \Omega_{X^{an}/S^{an}} \otimes_{O_{X^{an}}} M^{an}$$

given in degree (p, i) by, for $X^o \subset X$ and $X^o \supset X^{oo}$ an open subsets of X for the usual, resp. Zariski topology, $\omega \in \Gamma(X^o, \Omega_{X/S}^p)$ and $m \in \Gamma(X^o, M^i)$,

$$\Omega_{(X^{an}/X)/(S^{an}/S)}(M, F)(\omega \otimes m = \omega \otimes (m \otimes 1))$$

is a map of complexes, that is a map in $C_{f^*O_{S^{an}}fil}(X^{an})$. This gives, for $(M, F) \in C_{D(O_X)fil}(X)$, we have the following transformation map in $C_{O_{S^{an}}fil}(S^{an})$

$$\begin{aligned} T_\omega^O(an, f)(M, F) : (f_*E(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F)))^{an} &:= an_S^*(f_*E(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F))) \otimes_{an_S^*O_S} O_{S^{an}} \\ \xrightarrow{T(an(X), E)(-) \circ T(an, f)(E(\Omega_{X/S}^\bullet \otimes_{O_X} M))} &(f_*E(an_X^*(\Omega_{X/S}^\bullet \otimes_{O_X} (M, F)))) \otimes_{an_S^*O_S} O_{S^{an}} \\ \xrightarrow{m \circ E(\Omega_{(X^{an}/X)/(S^{an}/S)}(M, F))} &f_*E(\Omega_{X/S}^\bullet \otimes_{O_X^{an}} (M^{an}, F)) \end{aligned}$$

with $m(n \otimes s) = s.n$. Under the canonical isomorphism $\Omega_{X/S}^\bullet \xrightarrow{\sim} \Omega_{X/S}^\bullet \otimes_{O_X} O_X$ given by $\omega \mapsto \omega \otimes 1$, we have (see remark 7)

$$T_\omega^O(an, f)(O_X) = T_\omega^O(an, f) : (f_*E(\Omega_{X/S}^\bullet))^{an} \rightarrow f_*E(\Omega_{X^{an}/S^{an}}^\bullet)$$

Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$. In the case where X is smooth, for $(M, F) = (M^\bullet, F) \in C_{\mathcal{D}fil}(X)$, the differential of the relative De Rham complex of (M, F)

$$DR(X/S)(M, F) := (\Omega_{X/S}^\bullet, F) \otimes_{O_X} (M, F) = \text{Tot}((\Omega_{X/S}^\bullet, F) \otimes_{O_X} (M^\bullet, F)) \in C_{f^*O_Sfil}(X)$$

are given by

- $d_{p,p+1} : \Omega_{X/S}^p \otimes_{O_X} M^i \rightarrow \Omega_{X/S}^{p+1} \otimes_{O_X} M^i$, with for $X^o \subset X$ an open affine subset with (x_1, \dots, x_n) local coordinate (since X is smooth, T_X is locally free), $m \in \Gamma(X^o, M^i)$ and $\omega \in \Gamma(X^o, \Omega_{X/S}^p)$,

$$d_{p,p+1}(\omega \otimes m) := (d\omega) \otimes m + \sum_{i=1}^n (dx_i \wedge \omega) \otimes (\partial_i)m$$

- $d_{i,i+1} : \Omega_{X/S}^p \otimes_{O_X} M^i \rightarrow \Omega_{X/S}^p \otimes_{O_X} M^{i+1}$, with for $X^o \subset X$ an open subset, $m \in \Gamma(X^o, M^i)$ and $\omega \in \Gamma(X^o, \Omega_{X/S}^p)$, $d_{i,i+1}(\omega \otimes m) := (\omega \otimes dm)$.

For D_X only, the differential of its De Rahm complex $(\Omega_{X/S}^\bullet, F) \otimes_{O_X} D_X$ are right linear, so that

$$(\Omega_{X/S}^\bullet, F_b) \otimes_{O_X} (D_X, F^{ord}) \in C_{\mathcal{D}^{op}, f^*O_Sfil}(X)$$

In the particular case of a projection $p : Y \times S \rightarrow S$ with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSm}(\mathbb{C})$ we have :

Proposition 64. *Let $Y, S \in \text{SmVar}(\mathbb{C})$ or $Y, S \in \text{AnSm}(\mathbb{C})$. Let $p : Y \times S \rightarrow S$ the projection. For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$,*

$$DR(Y \times S/S)(M, F) := (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \in C_{p^*O_Sfil}(Y \times S)$$

is a naturally a complex of filtered p^*D_S modules, that is

$$DR(Y \times S/S)(M, F) := (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \in C_{p^*\mathcal{D}fil}(Y \times S),$$

where the p^*D_S module structure on $\Omega_{Y \times S/S}^p \otimes_{O_{Y \times S}} M^n$ is given by for $(Y \times S)^\circ \subset Y \times S$ an open subset,

$$(\gamma \in \Gamma((Y \times S)^\circ, T_{Y \times S}), \hat{\omega} \otimes m \in \Gamma((Y \times S)^\circ, \Omega_{Y \times S/S}^p \otimes_{O_{Y \times S}} M^n)) \mapsto \gamma.(\hat{\omega} \otimes m) := (\hat{\omega} \otimes (\gamma.m)).$$

Moreover, if $\phi : (M_1, F) \rightarrow (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in C_{\mathcal{D}fil}(Y \times S)$,

$$DR(Y \times S/S)(\phi) := (I \otimes \phi) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M_1, F) \rightarrow (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M_2, F)$$

is a morphism in $C_{p^*\mathcal{D}fil}(Y \times S)$.

Proof. Standard. □

In the analytic case, we also have

Proposition 65. *Let $Y, S \in \text{AnSm}(\mathbb{C})$. Let $p : Y \times S \rightarrow S$ the projection. For $(M, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$,*

$$DR(Y \times S/S)(M, F) := (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \in C_{p^*O_S fil}(Y \times S)$$

is naturally a complex of filtered $p^*D_S^\infty$ modules, that is

$$DR(Y \times S/S)(M, F) := (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \in C_{p^*\mathcal{D}^\infty fil}(Y \times S),$$

where the $p^*D_S^\infty$ module structure on $\Omega_{Y \times S/S}^p \otimes_{O_{Y \times S}} M^n$ is given by for $(Y \times S)^\circ \subset Y \times S$ an open subset,

$$(\gamma \in \Gamma((Y \times S)^\circ, T_{Y \times S}), \hat{\omega} \otimes m \in \Gamma((Y \times S)^\circ, \Omega_{Y \times S/S}^p \otimes_{O_{Y \times S}} M^n)) \mapsto \gamma.(\hat{\omega} \otimes m) := (\hat{\omega} \otimes (\gamma.m)).$$

Moreover, if $\phi : (M_1, F) \rightarrow (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$,

$$DR(Y \times S/S)(\phi) := (I \otimes \phi) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M_1, F) \rightarrow (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M_2, F)$$

is a morphism in $C_{p^*\mathcal{D}^\infty fil}(Y \times S)$.

Proof. Standard : follows from the finite order case (proposition 64). □

We state on the one hand the commutativity of the tensor product with respect to D_S and with respect to O_S , for $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$ in the filtered case, and on the other hand the commutativity between the tensor product with respect to D_S by D_S and the De Rahm complex :

Proposition 66. (i) *Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$.*

For $(M', F) \in C_{\mathcal{D}^{op} fil, f^\mathcal{D}}(X)$ and $(M, F), (N, F) \in C_{\mathcal{D} fil}(X)$. we have canonical isomorphisms of filtered f^*D_S modules, i.e. isomorphisms in $C_{f^*\mathcal{D}}(X)$,*

$$\begin{aligned} (M', F) \otimes_{O_X} (N, F) \otimes_{D_X} (M, F) &= (M', F) \otimes_{D_X} ((M, F) \otimes_{O_X} (N, F)) \\ &= ((M', F) \otimes_{O_X} (M, F)) \otimes_{D_X} (N, F) \end{aligned}$$

(ii) *Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$. For $(M, F) \in C_{D(O_X) fil}(X)$, we have a canonical isomorphisms of filtered f^*O_S modules, i.e. isomorphisms in $C_{f^*O_S fil}(X)$,*

$$(\Omega_{X/S}^\bullet, F_b) \otimes_{O_X} (M, F) = ((\Omega_{X/S}^\bullet, F_b) \otimes_{O_X} (D(O_X), F_b)) \otimes_{D(O_X)} (M, F)$$

(iii) *Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\mathcal{D} fil}(Y \times S)$, the isomorphisms of filtered p^*O_S modules of (ii)*

$$(\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) = ((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (D_{Y \times S}, F_b)) \otimes_{D_{Y \times S}} (M, F)$$

are isomorphisms of filtered p^*D_S modules, that is isomorphism in $C_{p^*\mathcal{D} fil}(Y \times S)$.

Proof. (i) and (ii) are particular case of proposition 46.

(iii): follows immediately by definition of the p^*D_S module structure. \square

We now look at the functorialities of the relative De Rham complex of a smooth morphisms of smooth complex algebraic varieties :

Proposition 67. *Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ or in $\text{AnSm}(\mathbb{C})$:*

$$D = \begin{array}{ccc} Y \times S & \xrightarrow{p} & S \\ g''=(g_0'' \times g) \uparrow & & \uparrow g \\ Y' \times T & \xrightarrow{p'} & T \end{array}$$

with p and p' the projections. For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$ the map in $C_{g''^*p^*O_Sfil}(Y' \times T)$

$$\Omega_{(Y' \times T / Y \times S) / (T / S)}(M, F) : g''^*((\Omega_{Y \times S / S}^\bullet, F_b) \otimes_{O_{Y \times S}}(M, F)) \rightarrow (\Omega_{Y' \times T / T}^\bullet, F_b) \otimes_{O_{Y' \times T}} g''^{*mod}(M, F)$$

given in definition-proposition 16 is a map in $C_{g''^*p^*\mathcal{D}fil}(Y' \times T)$. Hence, for $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$, the map in $C_{O_Tfil}(T)$ (with L_D instead of L_O)

$$T_\omega^O(D)(M) : g^{*mod}L_D(p_*E((\Omega_{Y \times S / S}^\bullet, F_b) \otimes_{O_{Y \times S}}(M, F))) \rightarrow p'_*E((\Omega_{Y' \times T / T}^\bullet, F_b) \otimes_{O_{Y' \times T}} g''^{*mod}(M, F)),$$

is a map in $C_{\mathcal{D}fil}(T)$.

Proof. Follows immediately by definition. \square

In the analytic case, we also have

Proposition 68. *Consider a commutative diagram in $\text{AnSm}(\mathbb{C})$:*

$$D = \begin{array}{ccc} Y \times S & \xrightarrow{p} & S \\ g''=(g_0'' \times g) \uparrow & & \uparrow g \\ Y' \times T & \xrightarrow{p'} & T \end{array}$$

with p and p' the projections. For $(M, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$ the map in $C_{g''^*p^*O_Sfil}(Y' \times T)$

$$\Omega_{(Y' \times T / Y \times S) / (T / S)}(M, F) : g''^*((\Omega_{Y \times S / S}^\bullet, F_b) \otimes_{O_{Y \times S}}(M, F)) \rightarrow (\Omega_{Y' \times T / T}^\bullet, F_b) \otimes_{O_{Y' \times T}} g''^{*mod}(M, F)$$

is a map in $C_{g''^*p^*\mathcal{D}^\infty fil}(Y' \times T)$. Hence, for $(M, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$, the map in $C_{O_Tfil}(T)$ (with L_D instead of L_O)

$$T_\omega^O(D)(M) : g^{*mod}L_D(p_*E((\Omega_{Y \times S / S}^\bullet, F_b) \otimes_{O_{Y \times S}}(M, F))) \rightarrow p'_*E((\Omega_{Y' \times T / T}^\bullet, F_b) \otimes_{O_{Y' \times T}} g''^{*mod}(M, F)),$$

is a map in $C_{\mathcal{D}^\infty fil}(T)$.

Proof. Follows immediately by definition. \square

Similarly, we have

Proposition 69. *Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$. For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$ the map in $C_{p^*O_{S^{an}}}(Y^{an} \times S^{an})$*

$$\Omega_{(Y^{an} \times S^{an} / Y \times S) / (S^{an} / S)}(M, F) : \text{an}(Y \times S)^*((\Omega_{Y \times S / S}^\bullet, F_b) \otimes_{O_{Y \times S}}(M, F)) \rightarrow (\Omega_{Y^{an} \times S^{an} / S^{an}}^\bullet, F_b) \otimes_{O_{Y \times S}^{an}}(M^{an}, F)$$

is a map in $C_{p^*\mathcal{D}fil}(Y^{an} \times S^{an})$. For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$, the map in $C_{O_{S^{an}}fil}(S^{an})$

$$T_\omega^O(\text{an}, h)(M, F) : (p_*E((\Omega_{Y \times S / S}^\bullet, F_b) \otimes_{O_{Y \times S}}(M, F)))^{an} \rightarrow p_*E((\Omega_{Y \times S / S}^\bullet, F_b) \otimes_{O_{Y \times S}^{an}}(M, F)^{an})$$

is a map in $C_{\mathcal{D}fil}(S^{an})$.

Proof. Similar to the proof of proposition 67. \square

Proposition 70. *Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSm}(\mathbb{C})$.*

(i) *If $\phi : (M, F) \rightarrow (N, F)$ is an r -filtered Zariski, resp. usu, local equivalence with $(M_1, F), (M_2, F) \in C_{\mathcal{D}fil}(Y \times S)$, then*

$$DR(Y \times S/S)(\phi) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \rightarrow \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} (N, F)$$

is an r -filtered equivalence Zariski, resp. usu, local in $C_{p^\mathcal{D}fil}(Y \times S)$.*

(ii) *Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ or in $\text{AnSm}(\mathbb{C})$*

$$D = \begin{array}{ccc} Y \times S & \xrightarrow{p} & S \\ \uparrow l & & \uparrow = \\ V & \xrightarrow{k} & S \end{array}$$

with p the projection. For $(N, F) \in C_{\mathcal{D}, l^\mathcal{D}fil}(V)$, the map in $C_{p^*O_S}(Y \times S)$ (see definition 60)*

$$\begin{aligned} k \circ T_\omega^O(l, \otimes)(E(N, F)) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} l_*E(N, F) &\rightarrow l_*((\Omega_{V/S}^\bullet, F_b) \otimes_{O_V} E(N, F)) \\ &\rightarrow l_*E((\Omega_{V/S}^\bullet, F_b) \otimes_{O_V} E(N, F)) \end{aligned}$$

is a filtered equivalence Zariski, resp. usu, local in $C_{p^\mathcal{D}fil}(Y \times S)$.*

Proof. (i): Follows from proposition 64 that it is a morphism of p^*D_S module. The fact that it is an equivalence Zariski, resp usu, local is a particular case of proposition 47(i).

(ii): Follows from proposition 64 and the first part of proposition 67 that it is a morphism of h^*D_S module. The fact that it is an equivalence Zariski, resp usu, local is a particular case of proposition 47(ii). \square

In the analytical case, we also have

Proposition 71. *Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSm}(\mathbb{C})$.*

(i) *If $\phi : (M, F) \rightarrow (N, F)$ is an r -filtered usu local equivalence with $(M_1, F), (M_2, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$, then*

$$DR(Y \times S/S)(\phi) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \rightarrow \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} (N, F)$$

is an r -filtered equivalence usu local in $C_{p^\mathcal{D}^\infty fil}(Y \times S)$.*

(ii) *Consider a commutative diagram in $\text{AnSm}(\mathbb{C})$*

$$D = \begin{array}{ccc} Y \times S & \xrightarrow{p} & S \\ \uparrow l & & \uparrow = \\ V & \xrightarrow{k} & S \end{array}$$

with p the projection. For $(N, F) \in C_{\mathcal{D}^\infty, l^\mathcal{D}^\infty fil}(V)$, the map in $C_{p^*O_S}(Y \times S)$ (see definition 60)*

$$\begin{aligned} k \circ T_\omega^O(l, \otimes)(E(N, F)) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} l_*E(N, F) &\rightarrow l_*((\Omega_{V/S}^\bullet, F_b) \otimes_{O_V} E(N, F)) \\ &\rightarrow l_*E((\Omega_{V/S}^\bullet, F_b) \otimes_{O_V} E(N, F)) \end{aligned}$$

is a filtered equivalence usu local in $C_{p^\mathcal{D}^\infty fil}(Y \times S)$.*

Proof. Follows from the finite order case : proposition 70. \square

Dually of the De Rahm complex of a D_S module M , we have the Spencer complex of M . In the particular case of D_S , we have the following:

Proposition 72. *Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$.*

- *We have the filtered resolutions of K_S by the following complex of locally free right D_S modules: $\omega(S) : \omega(K_S) := (\Omega_S^\bullet, F_b)[d_S] \otimes_{O_S} (D_S, F_b) \rightarrow (K_S, F_b)$ and $\omega(S) : \omega(K_S, F^{ord}) := (\Omega_S^\bullet, F_b)[d_S] \otimes_{O_S} (D_S, F^{ord}) \rightarrow (K_S, F^{ord})$*
- *Dually, we have the filtered resolution of O_S by the following complex of locally free (left) D_S modules: $\omega^\vee(S) : \omega(O_S) := (\wedge^\bullet T_S, F_b)[d_S] \otimes_{O_S} (D_S, F_b) \rightarrow (O_S, F_b)$ and $\omega^\vee(S) : \omega(O_S, F^{ord}) := (\wedge^\bullet T_S, F_b)[d_S] \otimes_{O_S} (D_S, F^{ord}) \rightarrow (O_S, F^{ord})$.*

Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \text{AnSm}(\mathbb{C})$. Consider the projection $p = p_1 : S_1 \times S_2 \rightarrow S_1$.

- *We have the filtered resolution of $D_{S_1 \times S_2 \rightarrow S_1}$ by the following complexes of (left) ($p^* D_{S_1}$ and right $D_{S_1 \times S_2}$) modules :*

$$\omega(S_1 \times S_2/S_1) : (\Omega_{S_1 \times S_2/S_1}^\bullet[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \rightarrow (D_{S_1 \times S_2 \leftarrow S_1}, F^{ord}).$$

- *Dually, we have the filtered resolution of $D_{S_1 \times S_2 \rightarrow S_1}$ by the following complexes of (left) ($p^* D_{S_1}, D_{S_1 \times S_2}$) modules :*

$$\omega^\vee(S_1 \times S_2/S_1) : (\wedge^\bullet T_{S_1 \times S_2/S_1}[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \rightarrow (D_{S_1 \times S_2 \rightarrow S_1}, F^{ord}),$$

Proof. See [17]. □

In the analytical case, we also have

Proposition 73. *Let $S \in \text{AnSm}(\mathbb{C})$.*

- *We have the filtered resolutions of K_S by the following complex of locally free right D_S modules: $\omega(S) : \omega(K_S) := (\Omega_S^\bullet, F_b)[d_S] \otimes_{O_S} (D_S^\infty, F^{ord}) \rightarrow (K_S, F_b)$.*
- *Dually, we have the filtered resolution of O_S by the following complex of locally free (left) D_S modules: $\omega^\vee(S) : \omega(O_S) := (\wedge^\bullet T_S, F_b)[d_S] \otimes_{O_S} (D_S^\infty, F^{ord}) \rightarrow (O_S, F_b)$.*

Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$. Consider the projection $p = p_1 : S_1 \times S_2 \rightarrow S_1$.

- *We have the filtered resolution of $D_{S_1 \times S_2 \rightarrow S_1}^\infty$ by the following complexes of (left) ($p^* D_{S_1}^\infty$ and right $D_{S_1 \times S_2}^\infty$) modules :*

$$\omega(S_1 \times S_2/S_1) : (\Omega_{S_1 \times S_2/S_1}^\bullet[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}^\infty, F^{ord}) \rightarrow (D_{S_1 \times S_2 \leftarrow S_1}^\infty, F^{ord}).$$

- *Dually, we have the filtered resolution of $D_{S_1 \times S_2 \rightarrow S_1}^\infty$ by the following complexes of (left) ($p^* D_{S_1}^\infty, D_{S_1 \times S_2}^\infty$) modules :*

$$\omega^\vee(S_1 \times S_2/S_1) : (\wedge^\bullet T_{S_1 \times S_2/S_1}[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}^\infty, F^{ord}) \rightarrow (D_{S_1 \times S_2 \rightarrow S_1}^\infty, F^{ord}),$$

Proof. Similar to the finite order case : the first map on the right is a surjection and the kernel are obtained by tensoring D_S^∞ with the kernel of the kozcul resolution of K_S (note that D_S^∞ is a locally free hence flat O_S module). □

Motivated by these resolutions, we make the following definition

Definition 69. *(i) Let $i : Z \hookrightarrow S$ be a closed embedding, with $Z, S \in \text{SmVar}(\mathbb{C})$ or with $Z, S \in \text{AnSm}(\mathbb{C})$. Then, for $(M, F) \in C_{\mathcal{D}fil}(Z)$, we set*

$$i_{*mod}(M, F) := i_{*mod}^0(M, F) := i_*((M, F) \otimes_{D_Z} (D_{Z \leftarrow S}, F^{ord})) \in C_{\mathcal{D}fil}(S)$$

(ii) Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_1 \times S_2 \rightarrow S_1$ be the projection. Then, for $(M, F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$, we set

$$\begin{aligned} - p_{*mod}^0(M, F) &:= p_*(DR(S_1 \times S_2/S_1)(M, F)) := p_*((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{\mathcal{O}_{S_1 \times S_2}} (M, F))[d_{S_2}] \in \\ &C_{\mathcal{D}fil}(S_1), \\ - p_{*mod}(M, F) &:= p_*E(DR(S_1 \times S_2/S_1)(M, F)) := p_*E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{\mathcal{O}_{S_1 \times S_2}} (M, F))[d_{S_2}] \in \\ &C_{\mathcal{D}fil}(S_1). \end{aligned}$$

(iii) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection. Then, for $(M, F) \in C_{\mathcal{D}fil}(X)$ we set

$$\begin{aligned} - f_{*mod}^{FDR}(M, F) &:= p_{S*mod}i_{*mod}(M, F) \in C_{\mathcal{D}fil}(S), \\ - \int_f^{FDR}(M, F) &:= f_{*mod}^{FDR}(M, F) := p_{S*mod}i_{*mod}(M, F) \in D_{\mathcal{D}fil, \infty}(S). \end{aligned}$$

By proposition 74 below, we have $\int_f^{FDR} M = \int_f M \in D_{\mathcal{D}}(X)$.

(iii) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection. Then, for $(M, F) \in C_{\mathcal{D}fil}(X)$ we set

$$\begin{aligned} - f_{!mod}^{FDR}(M, F) &:= \mathbb{D}_S^K L_D f_{*mod}^{FDR} \mathbb{D}_S^K L_D(M, F) := \mathbb{D}_S^K L_D p_{S*mod} i_{*mod} \mathbb{D}_{X \times S}^K L_D(M, F) \in C_{\mathcal{D}fil}(S), \\ - \int_{f!}^{FDR}(M, F) &:= f_{!mod}^{FDR}(M, F) := \mathbb{D}_S^K L_D p_{S*mod} i_{*mod} \mathbb{D}_{X \times S}^K L_D(M, F) \in D_{\mathcal{D}fil, \infty}(S). \end{aligned}$$

In the analytical case we also consider :

Definition 70. (i) Let $i : Z \hookrightarrow S$ be a closed embedding with $Z, S \in \text{AnSm}(\mathbb{C})$. Then, for $(M, F) \in C_{\mathcal{D}^\infty fil}(Z)$, we set

$$i_{*mod}(M, F) := i_{*mod}^0(M, F) := i_*((M, F) \otimes_{D_Z^\infty} (D_{Z \leftarrow S}^\infty, F^{ord})) \in C_{\mathcal{D}^\infty fil}(S)$$

(ii) Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_1 \times S_2 \rightarrow S_1$ be the projection. For $(M, F) \in C_{\mathcal{D}^\infty fil}(S_1 \times S_2)$, we consider

$$\begin{aligned} - p_{*mod}^0(M, F) &:= p_*(DR(S_1 \times S_2/S_1)(M, F)) := p_*((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{\mathcal{O}_{S_1 \times S_2}} (M, F))[d_{S_2}] \in \\ &C_{\mathcal{D}^\infty fil}(S_1), \\ - p_{*mod}(M, F) &:= p_*E(DR(S_1 \times S_2/S_1)(M, F)) := p_*E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{\mathcal{O}_{S_1 \times S_2}} (M, F))[d_{S_2}] \in \\ &C_{\mathcal{D}^\infty fil}(S_1). \end{aligned}$$

(iii) Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_1 \times S_2 \rightarrow S_1$ be the projection. For $(M, F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$ or $(M, F) \in C_{\mathcal{D}^\infty fil}(S_1 \times S_2)$, we set

$$\begin{aligned} - p_{!mod}^0(M, F) &:= p_!(DR(S_1 \times S_2/S_1)(M, F)) := p_!((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{\mathcal{O}_{S_1 \times S_2}} (M, F))[d_{S_2}] \in \\ &C_{\mathcal{D}fil}(S_1), \\ - p_{!mod}(M, F) &:= p_!E(DR(S_1 \times S_2/S_1)(M, F)) := p_!E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{\mathcal{O}_{S_1 \times S_2}} (M, F))[d_{S_2}] \in \\ &C_{\mathcal{D}fil}(S_1). \end{aligned}$$

(iv) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection. Then, for $(M, F) \in C_{\mathcal{D}^\infty fil}(X)$ we set

$$- f_{*mod}^{FDR}(M, F) := p_{S*mod}i_{*mod}(M, F) \in C_{\mathcal{D}^\infty fil}(S),$$

- $\int_f^{FDR}(M, F) := f_*^{FDR}(M, F) := p_{S*mod}i_{*mod}(M, F) \in D_{\mathcal{D}^\infty fil, \infty}(S),$
- $f_{!mod}^{FDR}(M, F) := p_{S!mod}i_{*mod}(M, F) \in C_{\mathcal{D}^\infty fil}(S),$
- $\int_{f!}^{FDR}(M, F) := f_{!*mod}^{FDR}(M, F) := p_{S!mod}i_{*mod}(M, F) \in D_{\mathcal{D}^\infty fil, \infty}(S).$

By proposition 75 below, we have $\int_{f!}^{FDR} M = \int_{f!} M \in D_{\mathcal{D}^\infty}(X)$ and $\int_f^{FDR} M = \int_f M \in D_{\mathcal{D}^\infty}(X).$

(v) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection. Then, for $(M, F) \in C_{\mathcal{D}fil}(X)$ we set

- $f_{!mod}^{FDR}(M, F) := p_{S!mod}i_{*mod}(M, F) \in C_{\mathcal{D}fil}(S),$
- $\int_{f!}^{FDR}(M, F) := f_{!*mod}^{FDR}(M, F) := p_{S!mod}i_{*mod}(M, F) \in D_{\mathcal{D}fil, \infty}(S).$

By proposition 75 below, we have $\int_{f!}^{FDR} M = \int_{f!} M \in D_{\mathcal{D}}(X).$

Proposition 74. (i) Let $i : Z \hookrightarrow S$ a closed embedding with $S, Z \in \text{SmVar}(\mathbb{C})$ or with $S, Z \in \text{AnSm}(\mathbb{C})$. Then for $(M, F) \in C_{\mathcal{D}fil}(Z)$, we have

$$\int_i (M, F) := Ri_*((M, F) \otimes_{D_Z}^L (D_{Z \leftarrow S}, F^{ord})) = i_*((M, F) \otimes_{D_Z} (D_{Z \leftarrow S}, F^{ord})) = i_{*mod}(M, F).$$

(ii) Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \rightarrow S_1$ be the projection. Then, for $(M, F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$ we have

$$\begin{aligned} \int_p (M, F) : &= Rp_*((M, F) \otimes_{D_{S_1 \times S_2}}^L (D_{S_1 \times S_2 \leftarrow S_1}, F^{ord})) \\ &= p_*E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \otimes_{D_{S_1 \times S_2}} (M, F))[d_{S_2}] \\ &= p_*E((\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F))[d_{S_2}] =: p_{*mod}(M, F). \end{aligned}$$

where the second equality follows from Griffithz transversality (the canonical isomorphism map respect by definition the filtration).

(iii) Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. Then for $M \in C_{\mathcal{D}}(X)$, we have $\int_f^{FDR} M = \int_f M$.

Proof. (i): Follows from the fact that $D_{Z \leftarrow S}$ is a locally free D_Z module and that i_* is an exact functor.
(ii): Since $\Omega_{S_{12}/S_1}^\bullet[d_{S_2}, F_b] \otimes_{O_{S_{12}}} D_{S_{12}}$ is a complex of locally free $D_{S_1 \times S_2}$ modules, we have in $D_{fil}(S_1 \times S_2)$, using proposition 72,

$$(D_{S_1 \times S_2 \leftarrow S_1}, F^{ord}) \otimes_{D_{S_1 \times S_2}}^L (M, F) = (\Omega_{S_{12}/S_1}^\bullet[d_{S_2}, F_b] \otimes_{O_{S_{12}}} (D_{S_{12}}, F^{ord}) \otimes_{D_{S_{12}}} (M, F).$$

(iii): Follows from (i) and (ii) by proposition 53 (ii) in the algebraic case and by proposition 56(ii) in the analytic case since a closed embedding is proper. \square

In the analytical case, we also have :

Proposition 75. (i) Let $i : Z \hookrightarrow S$ a closed embedding with $S, Z \in \text{AnSm}(\mathbb{C})$. Then for $(M, F) \in C_{\mathcal{D}^\infty fil}(Z)$, we have $\int_i (M, F) = i_{*mod}(M, F).$

(ii) Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \rightarrow S_1$ be the projection. Then, for $(M, F) \in C_{\mathcal{D}^\infty \text{fil}}(S_1 \times S_2)$ we have

$$\begin{aligned} \int_p (M, F) : &= R p_* ((M, F) \otimes_{D_{S_1 \times S_2}^\infty}^L (D_{S_1 \times S_2 \leftarrow S_1}^\infty, F^{\text{ord}})) \\ &= p_* E((\Omega_{S_1 \times S_2 / S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{\text{ord}}) \otimes_{D_{S_1 \times S_2}} (M, F)[d_{S_2}]) \\ &= p_* E((\Omega_{S_1 \times S_2 / S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)[d_{S_2}]) =: p_{* \text{mod}}(M, F). \end{aligned}$$

(ii)' Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \rightarrow S_1$ be the projection. Then, for $(M, F) \in C_{\mathcal{D} \text{fil}}(S_1 \times S_2)$ or $(M, F) \in C_{\mathcal{D} \text{fil}}(S_1 \times S_2)$, we have

$$\begin{aligned} \int_{p!} (M, F) : &= R p_! ((M, F) \otimes_{D_{S_1 \times S_2}^\infty}^L (D_{S_1 \times S_2 \leftarrow S_1}, F^{\text{ord}})) \\ &= p_! E((\Omega_{S_1 \times S_2 / S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{\text{ord}}) \otimes_{D_{S_1 \times S_2}} (M, F)[d_{S_2}]) \\ &= p_! E((\Omega_{S_1 \times S_2 / S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)[d_{S_2}]) =: p_{! \text{mod}}(M, F). \end{aligned}$$

(iii) Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}^\infty}(X)$, we have $\int_f^{\text{FDR}} M = \int_f M$ and $\int_{f!}^{\text{FDR}} M = \int_{f!} M$. For $M \in C_{\mathcal{D}}(X)$, we have $\int_{f!}^{\text{FDR}} M = \int_{f!} M$.

Proof. (i): Follows from the fact that $D_{Z \leftarrow S}^\infty$ is a locally free D_Z^∞ module and that i_* is an exact functor.
(ii): Similar to the proof of proposition 74(ii): follows from proposition 73.
(ii)': Similar to the proof of proposition 74(ii): follows from proposition 73.
(iii): The first assertion follows from (i), (ii) and (ii)' by proposition 56. The second one follows from proposition 74(i) and (ii)' and by proposition 56. □

Proposition 76. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$.

(i) Let $(M, F) \in C_{\mathcal{D}(2) \text{fil}}(X)$. Then $\int_{f_2 \circ f_1}^{\text{FDR}}(M, F) = \int_{f_1}^{\text{FDR}} \int_{f_2}^{\text{FDR}}(M, F) \in D_{\mathcal{D}(2) \text{fil}, \infty}(S)$.

(ii) Let $(M, F) \in C_{\mathcal{D}(2) \text{fil}, h}(X)$. Then $\int_{(f_2 \circ f_1)!}^{\text{FDR}}(M, F) = \int_{f_1!}^{\text{FDR}} \int_{f_2!}^{\text{FDR}}(M, F) \in D_{\mathcal{D}(2) \text{fil}, \infty}(S)$.

Proof. See [22]. □

Proposition 77. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{AnSm}(\mathbb{C})$.

(i) Let $(M, F) \in C_{\mathcal{D}^\infty(2) \text{fil}}(X)$. Then $\int_{(f_2 \circ f_1)!}^{\text{FDR}}(M, F) = \int_{f_1!}^{\text{FDR}} \int_{f_2!}^{\text{FDR}}(M, F)$.

(ii) Let $(M, F) \in C_{\mathcal{D}^\infty(2) \text{fil}, h}(X)$. Then $\int_{f_2 \circ f_1}^{\text{FDR}}(M, F) = \int_{f_1}^{\text{FDR}} \int_{f_2}^{\text{FDR}}(M, F)$.

Proof. Similar to proposition 76. □

Definition 71. (i) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$.

Consider the graph factorization $f : X \xrightarrow{l} X \times S \xrightarrow{p} S$, with l the graph embedding and p the projection. We have the transformation map given by, for $(M, F) \in C_{\mathcal{D} \text{fil}}(X)$,

$$T\left(\int_f^{\text{FDR}}, \int_f\right)(M, F) : \int_f^{\text{FDR}}(M, F) := \int_p \int_l (M, F) \xrightarrow{T(\int_p \circ \int_l, \int_{p \circ l})(M, F)} \int_f (M, F)$$

(ii) Let $j : S^\circ \hookrightarrow S$ an open embedding with $S \in \text{Var}(\mathbb{C})$. Consider the graph factorization $j : S^\circ \xrightarrow{l} S^\circ \times S \xrightarrow{p} S$, with l the graph embedding and p the projection. We have, for $(M, F) \in C_{\mathcal{D}fil}(S^\circ)$, the canonical map in $C_{\mathcal{D}fil}(S)$,

$$T(j_{*mod}^{FDR}, j_*)(M, F) : j_{*mod}^{FDR}(M, F) := p_*E((\Omega_{S^\circ \times S/S}^\bullet, F_b) \otimes_{O_{S^\circ \times S}} l_{*mod}(M, F)) \xrightarrow{k\omega(S^\circ \times S/S)} \\ p_*E((D_{S^\circ \times S \leftarrow S}, F^{ord}) \otimes_{D_{S^\circ \times S}} l_*(D_{S^\circ \leftarrow S^\circ \times S} \otimes_{D_{S^\circ}} E(M, F))) \xrightarrow{T(l, \otimes)(-, -)} j_*E(M, F)$$

We have, for $(M, F) \in C_{\mathcal{D}fil}(S)$, the canonical map in $C_{\mathcal{D}fil}(S)$,

$$\text{ad}(j^*, j_{*mod}^{FDR})(M, F) : (M, F) \xrightarrow{\text{ad}(p^{*mod}, p_*)(M, F)} p_*E((\Omega_{S^\circ \times S/S}^\bullet, F_b) \otimes_{O_{S^\circ \times S}} p^{*mod}(M, F))$$

4.2.3 The support section functors for D modules and the graph inverse image

Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $i : Z \hookrightarrow S$ a closed embedding and denote $j : S \setminus Z \hookrightarrow S$ the complementary open embedding. More generally, let $h : Y \rightarrow S$ a morphism with $Y, S \in \text{Var}(\mathbb{C})$ or $Y, S \in \text{AnSp}(\mathbb{C})$, S smooth, and let $i : X \hookrightarrow Y$ a closed embedding and denote by $j : Y \setminus X \hookrightarrow Y$ the open complementary. We then get from section 2 the following functors :

- We get the functor

$$\Gamma_Z : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}(2)fil}(S), \\ (M, F) \mapsto \Gamma_Z(M, F) := \text{Cone}(\text{ad}(j^*, j_*)((M, F)) : (M, F) \rightarrow j_*j^*(M, F))[-1],$$

together with the canonical map $\gamma_Z(M, F) : \Gamma_Z(M, F) \rightarrow (M, F)$, and more generally the functor

$$\Gamma_X : C_{h^*\mathcal{D}(2)fil}(Y) \rightarrow C_{h^*\mathcal{D}(2)fil}(Y), \\ (M, F) \mapsto \Gamma_X(M, F) := \text{Cone}(\text{ad}(j^*, j_*)((M, F)) : (M, F) \rightarrow j_*j^*(M, F))[-1],$$

together with the canonical map $\gamma_X(M, F) : \Gamma_X(M, F) \rightarrow (M, F)$.

- We get the functor

$$\Gamma_Z^\vee : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}(2)fil}(S), \\ (M, F) \mapsto \Gamma_Z^\vee(M, F) := \text{Cone}(\text{ad}(j_!, j^*)((M, F)) : j_!j^*(M, F) \rightarrow (M, F)),$$

together with the canonical map $\gamma_Z^\vee(M, F) : (M, F) \rightarrow \Gamma_Z^\vee(M, F)$, and more generally the functor

$$\Gamma_X^\vee : C_{h^*\mathcal{D}(2)fil}(Y) \rightarrow C_{h^*\mathcal{D}(2)fil}(Y), \\ (M, F) \mapsto \Gamma_X^\vee(M, F) := \text{Cone}(\text{ad}(j_!, j^*)((M, F)) : j_!j^*(M, F) \rightarrow (M, F)),$$

together with the canonical map $\gamma_X^\vee(M, F) : (M, F) \rightarrow \Gamma_X^\vee(M, F)$.

- We get the functor

$$\Gamma_Z^{\vee, h} : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}(2)fil}(S), (M, F) \mapsto \Gamma_Z^{\vee, h}(M, F) := \mathbb{D}_S^K L_D \Gamma_Z E(\mathbb{D}_S^K(M, F)),$$

together with the factorization

$$\gamma_Z^{\vee, h}(L_D(M, F)) : L_D(M, F) \xrightarrow{\gamma_Z^\vee(L_D(M, F))} \Gamma_Z^\vee L_D(M, F) \xrightarrow{k\circ \mathbb{D}_S^K I(j_!, j^*)(-) \circ d(-)} \Gamma_Z^{\vee, h} L_D(M, F),$$

and more generally the functor

$$\Gamma_X^{\vee, h} : C_{h^*\mathcal{D}(2)fil}(Y) \rightarrow C_{h^*\mathcal{D}(2)fil}(Y), (M, F) \mapsto \Gamma_X^{\vee, h}(M, F) := \mathbb{D}_Y^{h^*D, K} L_{h^*D} \Gamma_X E(\mathbb{D}_Y^{h^*D, K}(M, F)),$$

together with the factorization

$$\gamma_X^{\vee, h}(L_{h^*D}(M, F)) : L_{h^*D}(M, F) \xrightarrow{\gamma_X^\vee(L_{h^*D}(M, F))} \Gamma_X^\vee L_{h^*D}(M, F) \\ \xrightarrow{k\circ \mathbb{D}_Y^{h^*D, K} I(j_!, j^*)(-) \circ d(-)} \Gamma_X^{\vee, h} L_{h^*D}(M, F).$$

- We get the functor

$$\Gamma_Z^{\vee, O} : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}(2)fil}(S),$$

$$(M, F) \mapsto \Gamma_Z^{\vee, O}(M, F) := \text{Cone}(b_Z((M, F)) : \mathcal{I}_Z(M, F) \rightarrow (M, F)),$$

together we the factorization

$$\gamma_Z^{\vee, O}(M, F) : (M, F) \xrightarrow{\gamma_Z^{\vee, O}(M, F)} \Gamma_Z^{\vee, O}(M, F) \xrightarrow{b_{S/Z}(M, F)} \Gamma_Z^{\vee, O}(M, F).$$

Since $M \mapsto M/\mathcal{I}_Z M$ is a right exact functor, $M \mapsto \Gamma_Z^{\vee, O} M$ send Zariski, resp. usu, local equivalence between projective complexes of presheaves to Zariski, resp. usu local equivalence, and thus induces in the derived category

$$L\Gamma_Z^{\vee, O} : D_{\mathcal{D}fil, \infty}(S) \rightarrow D_{\mathcal{D}fil, \infty}(S),$$

$$(M, F) \mapsto \Gamma_Z^{\vee, O} L_D(M, F) := \text{Cone}(b_Z(L_D(M, F)) : \mathcal{I}_Z L_D(M, F) \rightarrow L_D(M, F)).$$

- We get the functor

$$\Gamma_Z^O : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}(2)fil}(S),$$

$$(M, F) \mapsto \Gamma_Z^O(M, F) := \text{Cone}(b'_Z((M, F)) : (M, F) \rightarrow (M, F) \otimes_{O_S} \mathbb{D}_S^K(\mathcal{I}_Z D_S)),$$

together we the factorization

$$\gamma_Z^O(M, F) : (M, F) \Gamma_Z^O \xrightarrow{b'_{S/Z}(M, F)} \Gamma_Z^O(M, F) \xrightarrow{\gamma_Z^O(M, F)} (M, F).$$

- We have, for $(M, F) \in C_{\mathcal{D}fil}(S)$, a canonical isomorphism

$$I(D, \gamma^O)(M, F) : \mathbb{D}_S^K \Gamma_Z^{\vee, O}(M, F) \xrightarrow{\sim} \Gamma_Z^O \mathbb{D}_S^K(M, F)$$

which gives the transformation map in $C_{\mathcal{D}fil}(S)$

$$T(D, \gamma^O)(M, F) : \Gamma_Z^{\vee, O} \mathbb{D}_S^K(M, F) \rightarrow \mathbb{D}_S^K \Gamma_Z^O(M, F)$$

Let $S \in \text{AnSm}(\mathbb{C})$. Let $i : Z \hookrightarrow S$ a closed embedding and denote $j : S \setminus Z \hookrightarrow S$ the complementary open embedding. More generally, let $h : Y \rightarrow S$ a morphism with $Y, S \in \text{AnSp}(\mathbb{C})$, S smooth, and let $i : X \hookrightarrow Y$ a closed embedding and denote by $j : Y \setminus X \hookrightarrow Y$ the open complementary.

- We get the functor

$$\Gamma_Z : C_{\mathcal{D}^\infty(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S),$$

$$(M, F) \mapsto \Gamma_Z(M, F) := \text{Cone}(\text{ad}(j^*, j_*)((M, F)) : (M, F) \rightarrow j_* j^*(M, F))[-1],$$

together we the canonical map $\gamma_Z(M, F) : \Gamma_Z(M, F) \rightarrow (M, F)$, and more generally the functor

$$\Gamma_X : C_{h^* \mathcal{D}^\infty(2)fil}(Y) \rightarrow C_{h^* \mathcal{D}^\infty(2)fil}(Y),$$

$$(M, F) \mapsto \Gamma_X(M, F) := \text{Cone}(\text{ad}(j^*, j_*)((M, F)) : (M, F) \rightarrow j_* j^*(M, F))[-1],$$

together we the canonical map $\gamma_X(M, F) : \Gamma_X(M, F) \rightarrow (M, F)$.

- We get the functor

$$\Gamma_Z^\vee : C_{\mathcal{D}^\infty(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S),$$

$$(M, F) \mapsto \Gamma_Z^\vee(M, F) := \text{Cone}(\text{ad}(j_!, j^*)((M, F)) : j_! j^*(M, F) \rightarrow (M, F)),$$

together we the canonical map $\gamma_Z^\vee(M, F) : (M, F) \rightarrow \Gamma_Z^\vee(M, F)$, and more generally the functor

$$\Gamma_X^\vee : C_{h^* \mathcal{D}^\infty(2)fil}(Y) \rightarrow C_{h^* \mathcal{D}^\infty(2)fil}(Y),$$

$$(M, F) \mapsto \Gamma_X^\vee(M, F) := \text{Cone}(\text{ad}(j_!, j^*)((M, F)) : j_! j^*(M, F) \rightarrow (M, F)),$$

together we the canonical map $\gamma_X^\vee(M, F) : (M, F) \rightarrow \Gamma_X^\vee(M, F)$.

- We get the functor

$$\Gamma_Z^{\vee,h} : C_{\mathcal{D}^\infty(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S), (M, F) \mapsto \Gamma_Z^{\vee,h}(M, F) := \mathbb{D}_S^{\infty,K} L_{D^\infty} \Gamma_Z E(\mathbb{D}_S^{\infty,K}(M, F)),$$

together with the factorization

$$\gamma_Z^{\vee,h}(L_{D^\infty}(M, F)) : L_{D^\infty}(M, F) \xrightarrow{\gamma_Z^{\vee}(L_{D^\infty}(M, F))} \Gamma_Z^{\vee} L_{D^\infty}(M, F) \xrightarrow{k \circ \mathbb{D}^\infty I(j!, j^*)(-) \circ d(-)} \Gamma_Z^{\vee,h} L_{D^\infty}(M, F),$$

and more generally the functor

$$\Gamma_X^{\vee,h} : C_{h^* \mathcal{D}^\infty(2)fil}(Y) \rightarrow C_{h^* \mathcal{D}^\infty(2)fil}(Y), (M, F) \mapsto \Gamma_X^{\vee,h}(M, F) := \mathbb{D}_Y^{h^* \infty, K} L_{h^* D^\infty} \Gamma_X E(\mathbb{D}_Y^{h^* \infty, K}(M, F)),$$

together with the factorization

$$\begin{aligned} \gamma_X^{\vee,h}(L_{h^* D^\infty}(M, F)) : L_{h^* D^\infty}(M, F) &\xrightarrow{\gamma_X^{\vee}(L_{h^* D^\infty}(M, F))} \Gamma_X^{\vee} L_{h^* D^\infty}(M, F) \\ &\xrightarrow{k \circ \mathbb{D}^{h^* D^\infty, K} I(j!, j^*)(-) \circ d(-)} \Gamma_X^{\vee,h} L_{h^* D^\infty}(M, F). \end{aligned}$$

- We get the functor

$$\begin{aligned} \Gamma_Z^{\vee,O} : C_{\mathcal{D}^\infty(2)fil}(S) &\rightarrow C_{\mathcal{D}^\infty(2)fil}(S), \\ (M, F) &\mapsto \Gamma_Z^{\vee,O}(M, F) := \text{Cone}(b_Z((M, F)) : \mathcal{I}_Z(M, F) \rightarrow (M, F)), \end{aligned}$$

together we the factorization

$$\gamma_Z^{\vee,O}(M, F) : (M, F) \xrightarrow{\gamma_Z^{\vee}(M, F)} \Gamma_Z^{\vee}(M, F) \xrightarrow{b_{S/Z}(M, F)} \Gamma_Z^{\vee,O}(M, F).$$

- We get the functor

$$\begin{aligned} \Gamma_Z^O : C_{\mathcal{D}^\infty(2)fil}(S) &\rightarrow C_{\mathcal{D}^\infty(2)fil}(S), \\ (M, F) &\mapsto \Gamma_Z^O(M, F) := \text{Cone}(b'_Z((M, F)) : (M, F) \rightarrow (M, F) \otimes_{O_S} \mathbb{D}_S^K(\mathcal{I}_Z D_S)), \end{aligned}$$

together we the factorization

$$\gamma_Z^O(M, F) : (M, F) \Gamma_Z^O \xrightarrow{b'_{S/Z}(M, F)} \Gamma_Z(M, F) \xrightarrow{\gamma_Z(M, F)} (M, F).$$

- We have, for $(M, F) \in C_{\mathcal{D}^\infty fil}(S)$, a canonical isomorphism

$$I(D, \gamma^O)(M, F) : \mathbb{D}_S^{K, \infty} \Gamma_Z^{\vee,O}(M, F) \xrightarrow{\sim} \Gamma_Z^O \mathbb{D}_S^{K, \infty}(M, F)$$

which gives the transformation map in $C_{\mathcal{D}^\infty fil}(S)$

$$T(D, \gamma^O)(M, F) : \Gamma_Z^{\vee,O} \mathbb{D}_S^{\infty, K}(M, F) \rightarrow \mathbb{D}_S^{\infty, K} \Gamma_Z^O(M, F)$$

In the analytic case, we have

Definition 72. Let $S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\mathcal{D}^\infty fil}(S)$, we have the map in $C_{\mathcal{D}^\infty fil}(S)$

$$\begin{aligned} T(\infty, \gamma)(M, F) &:= (I, T(j, \otimes)(-, -)) : \\ J_S(\Gamma_Z(M, F)) &:= \Gamma_Z(M, F) \otimes_{D_S} (D_S^\infty, F^{ord}) \rightarrow \Gamma_Z((M, F) \otimes_{D_S} (D_S^\infty, F^{ord})) =: \Gamma_Z J_S(M, F) \end{aligned}$$

Let $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \text{SmVar}(\mathbb{C})$ or $Z, S \in \text{AnSm}(\mathbb{C})$. We have the functor

$$i^\sharp : C_{\mathcal{D}^\infty fil}(S) \rightarrow C_{\mathcal{D}^\infty fil}(Z), (M, F) \mapsto i^\sharp(M, F) := \mathcal{H}om_{i^* D_S}((D_{S \leftarrow Z}, F^{ord}), i^*(M, F))$$

where the (left) D_Z module structure on $i^\sharp M$ comes from the right module structure on $D_{S \leftarrow Z}$, resp. O_Z . We denote by

- for $(M, F) \in C_{\mathcal{D}fil}(S)$, the canonical map in $C_{\mathcal{D}fil}(S)$

$$\begin{aligned} \text{ad}(i_{*mod}, i^\sharp)(M, F) : i_{*mod}i^\sharp(M, F) &:= i_*(\mathcal{H}om_{i^*D_S}((D_{S \leftarrow Z}, F^{ord}), i^*(M, F)) \otimes_{D_Z} (D_{S \leftarrow Z}, F^{ord})) \\ &\rightarrow (M, F), \phi \otimes P \mapsto \phi(P) \end{aligned}$$

- for $(N, F) \in C_{\mathcal{D}fil}(Z)$, the canonical map in $C_{\mathcal{D}fil}(Z)$

$$\begin{aligned} \text{ad}(i_{*mod}, i^\sharp)(N, F) : (N, F) \rightarrow i^\sharp i_{*mod}(N, F) &:= \mathcal{H}om_{i^*D_S}(D_{S \leftarrow Z}, i^*i_*(N, F) \otimes_{D_Z} (D_{S \leftarrow Z}, F^{ord})) \\ n \mapsto (P \mapsto n \otimes P) \end{aligned}$$

The functor i^\sharp induces in the derived category the functor :

$$\begin{aligned} Ri^\sharp : D_{\mathcal{D}(2)fil,r}(S) &\rightarrow D_{\mathcal{D}(2)fil,r}(Z), \\ (M, F) \mapsto Ri^\sharp(M, F) &:= R\mathcal{H}om_{i^*D_S}((D_{Z \leftarrow S}, F^{ord}), i^*(M, F)) = \mathcal{H}om_{i^*D_S}((D_{Z \leftarrow S}, F^{ord}), E(i^*(M, F))). \end{aligned}$$

Proposition 78. *Let $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \text{SmVar}(\mathbb{C})$ or $Z, S \in \text{AnSm}(\mathbb{C})$. The functor $i_{*mod} : C_{\mathcal{D}}(Z) \rightarrow C_{\mathcal{D}}(S)$ admit a right adjoint which is the functor $i^\sharp : C_{\mathcal{D}}(S) \rightarrow C_{\mathcal{D}}(Z)$ and*

$$\text{ad}(i_{*mod}, i^\sharp)(N) : N \rightarrow i^\sharp i_{*mod}N \quad \text{and} \quad \text{ad}(i_{*mod}, i^\sharp)(M) : i_{*mod}i^\sharp M \rightarrow M$$

are the adjunction maps.

Proof. See [17] for the algebraic case. The analytic case is completely analogue. \square

One of the main results in D modules is Kashiwara equivalence :

Theorem 20. (i) *Let $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \text{SmVar}(\mathbb{C})$.*

- *The functor $i_{*mod} : \mathcal{Q}Coh_{\mathcal{D}}(Z) \rightarrow \mathcal{Q}Coh_{\mathcal{D}}(S)$ is an equivalence of category whose inverse is $i^\sharp := a_\tau i^\sharp : \mathcal{Q}Coh_{\mathcal{D}}(S) \rightarrow \mathcal{Q}Coh_{\mathcal{D}}(Z)$. That is, for $M \in \mathcal{Q}Coh_{\mathcal{D}}(S)$ and $N \in \mathcal{Q}Coh_{\mathcal{D}}(Z)$, the adjunction maps*

$$\text{ad}(i_{*mod}, i^\sharp)(M) : i_{*mod}i^\sharp M \xrightarrow{\sim} M, \quad \text{ad}(i_{*mod}, i^\sharp)(N) : i^\sharp i_{*mod}N \xrightarrow{\sim} N$$

are isomorphisms.

- *The functor $\int_i i_{*mod} : D_{\mathcal{D}}(Z) \rightarrow D_{\mathcal{D}}(S)$ is an equivalence of category whose inverse is $Ri^\sharp : D_{\mathcal{D}}(S) \rightarrow D_{\mathcal{D}}(Z)$. That is, for $M \in D_{\mathcal{D}}(S)$ and $N \in D_{\mathcal{D}}(Z)$, the adjunction maps*

$$\text{ad}\left(\int_i, Ri^\sharp\right)(M) : \int_i Ri^\sharp M \xrightarrow{\sim} M, \quad \text{ad}\left(\int_i, Ri^\sharp\right)(N) : Ri^\sharp \int_i N \xrightarrow{\sim} N$$

are isomorphisms.

(ii) *Let $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \text{AnSm}(\mathbb{C})$.*

- *The functor $i_{*mod} : Coh_{\mathcal{D}}(Z) \rightarrow Coh_{\mathcal{D}}(S)$ is an equivalence of category whose inverse is $i^\sharp := a_\tau i^\sharp : Coh_{\mathcal{D}}(S) \rightarrow Coh_{\mathcal{D}}(Z)$. That is, for $M \in Coh_{\mathcal{D}}(S)$ and $N \in Coh_{\mathcal{D}}(Z)$, the adjunction maps*

$$\text{ad}(i_{*mod}, i^\sharp)(M) : i_{*mod}i^\sharp M \xrightarrow{\sim} M, \quad \text{ad}(i_{*mod}, i^\sharp)(N) : i^\sharp i_{*mod}N \xrightarrow{\sim} N$$

are isomorphisms.

- *The functor $\int_i i_{*mod} : D_{\mathcal{D},c}(Z) \rightarrow D_{\mathcal{D},c}(S)$ is an equivalence of category whose inverse is $Ri^\sharp : D_{\mathcal{D},c}(S) \rightarrow D_{\mathcal{D},c}(Z)$. That is, for $M \in D_{\mathcal{D},c}(S)$ and $N \in D_{\mathcal{D},c}(Z)$, the adjunction maps*

$$\text{ad}\left(\int_i, Ri^\sharp\right)(M) : \int_i Ri^\sharp M \xrightarrow{\sim} M, \quad \text{ad}\left(\int_i, Ri^\sharp\right)(N) : Ri^\sharp \int_i N \xrightarrow{\sim} N$$

are isomorphisms.

Proof. (i):Standard. Note that the second point follows from the first.
(ii):Standard. Note that the second point follows from the first. \square

We have a canonical embedding of rings $D_Z \hookrightarrow D_{Z \rightarrow S} := i^* D_S \otimes_{i^* O_S} O_Z$. We denote by $C_{i^* \mathcal{D}, Z}(Z)$ the category whose objects are complexes of presheaves M of $i^* D_S$ modules on Z such that the cohomology presheaves $H^n M$ have an induced structure of D_Z modules. We denote by

$$q_K : K_{O_S}(i_* O_Z) \rightarrow i_* O_Z$$

the Kozcul complex which is a resolution of the O_S module $i_* O_Z$ of length $c = \text{codim}(Z, S)$ by locally free sheaves of finite rank. The fact that it is a locally free resolution of finite rank comes from the fact that Z is a locally complete intersection in S since both Z and S are smooth. We denote again

$$q_K = i^* q_K : K_{i^* O_S}(O_Z) := i^* K_{O_S}(i_* O_Z) \rightarrow i^* i_* O_Z = O_Z$$

We denote by $K_{i^* O_S}^\vee(O_Z) := \mathcal{H}om_{i^* O_S}(K_{i^* O_S}(O_Z), i^* O_S)$ its dual, so that we have a canonical map

$$q_K^\vee : K_{i^* O_S}^\vee(O_Z) \rightarrow O_Z[-c].$$

Let $M \in C_{\mathcal{D}}(S)$. The $i^* D_S$ module structure on $\mathcal{H}om_{i^* O_S}(K_{i^* O_S}(O_Z), i^* M)$ and $K_{i^* O_S}(O_Z) \otimes_{i^* O_S} i^* M$ induce a canonical D_Z module structure on the cohomology groups $H^n \mathcal{H}om_{i^* O_S}(K_{i^* O_S}(O_Z), i^* M)$ and $H^n(K_{i^* O_S}(O_Z) \otimes_{i^* O_S} i^* M)$ for all $n \in \mathbb{Z}$.

The projection formula for ringed spaces (proposition 9) implies the following lemma :

Lemma 6. *Let $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \text{Var}(\mathbb{C})$ or with $Z, S \in \text{AnSp}(\mathbb{C})$. Denote by $j : U := S \setminus Z \hookrightarrow Z$ the open complementary embedding. Then, if i is a locally complete intersection embedding (e.g. if Z, S are smooth), we have for $M \in C_{O_U}(U)$, $Li^{*mod} Rj_* M = 0$.*

Proof. We have

$$\begin{aligned} i_* Li^{*mod} Rj_* M &:= i_*(i^* L_O(j_* E(M)) \otimes_{i^* O_S} O_Z) \xrightarrow{T(i, \otimes)(L_O(j_* E(M)), O_Z)^{-1}} L_O(j_* E(M)) \otimes_{O_S} i_* O_Z \\ &\xrightarrow{q^\circ(i_* q_K)^{-1}} (j_* E(M)) \otimes_{O_S} i_* K_{i^* O_S}(O_Z) \xrightarrow{T(j, \otimes)(E(M), K_{O_S}(i_* O_Z))} j_*(E(M) \otimes_{O_U} j^* K_{O_S}(i_* O_Z)), \end{aligned}$$

$T(i, \otimes)(L_O(j_* E(M)), O_Z)$ being an equivalence Zariski, resp. usu, local by proposition 10 and follows from the fact that $j^* K_{O_S}(i_* O_Z)$ is acyclic. But

$$T(j, \otimes)(E(M), K_{O_S}(i_* O_Z)) : (j_* E(M)) \otimes_{O_S} K_{O_S}(i_* O_Z) \rightarrow j_*(E(M) \otimes_{O_U} j^* i_* K_{i^* O_S}(O_Z))$$

is an equivalence Zariski, resp. usu, local by proposition 9 since $K_{O_S}(i_* O_Z)$ is a finite complex of locally free O_S modules of finite rank. \square

We deduce from theorem 20(i) and lemma 6 the localization for D -modules for a closed embedding of smooth algebraic varieties:

Theorem 21. *Let $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \text{SmVar}(\mathbb{C})$. Denote by $c = \text{codim}(Z, S)$. Then, for $M \in C_{\mathcal{D}}(S)$, we have by Kashiwara equivalence the following map in $C_{\mathcal{D}}(S)$:*

$$\begin{aligned} \mathcal{K}_{Z/S}(M) : \Gamma_Z E(M) &\xrightarrow{\text{ad}(i_{*mod}, i^\sharp)(-)^{-1}} i_{*mod} i^\sharp \Gamma_Z E(M) \\ \xrightarrow{\gamma_Z(-)} i_{*mod} i^\sharp(E(M)) &\xrightarrow{\mathcal{H}om(q_K, E(i^* M)) \circ \mathcal{H}om(O_Z, T(i, E)(M))} i_{*mod} K_{i^* O_S}^\vee(O_Z) \otimes_{i^* O_S} M \end{aligned}$$

which is an equivalence Zariski local. It gives the isomorphism in $D_{\mathcal{D}}(S)$

$$\mathcal{K}_{Z/S}(M) : R\Gamma_Z M \rightarrow i_{*mod} K_{i^* O_S}^\vee(O_Z) = i_{*mod} Li^{*mod} M[c]$$

Proof. Follows from theorem 20(i) and lemma 6 : see [17] for example. \square

Definition 73. Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection.

(i) Then, for $(M, F) \in C_{\mathcal{D}(2)\text{fil}}(S)$ we set

$$f^{*\text{mod}[-], \Gamma}(M, F) := \Gamma_X E(p_S^{*\text{mod}[-]}(M, F)) \in C_{\mathcal{D}(2)\text{fil}, \infty}(X \times S),$$

It induces in the derived category

$$Rf^{*\text{mod}[-], \Gamma}(M, F) := f^{*\text{mod}[-], \Gamma}(M, F) := \Gamma_X E(p_S^{*\text{mod}[-]}(M, F)) \in D_{\mathcal{D}(2)\text{fil}, \infty}(X \times S),$$

By definition-proposition 21, we have in the algebraic case $Li^{*\text{mod}} f^{*\text{mod}, \Gamma} M = Lf^{*\text{mod}} M \in D_{\mathcal{D}}(X)$.

(ii) Then, for $(M, F) \in C_{\mathcal{D}(2)\text{fil}}(S)$ we set

$$Lf^{\hat{*}\text{mod}[-], \Gamma}(M, F) := \Gamma_X^{\vee, h} L_D p_S^{*\text{mod}[-]}(M, F) := \mathbb{D}_S^K L_D \Gamma_X E(\mathbb{D}_S^K L_D p_S^{*\text{mod}[-]}(M, F)) \in D_{\mathcal{D}(2)\text{fil}, \infty}(X \times S).$$

In the analytical case we also have

Definition 74. Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection.

(i) Then, for $(M, F) \in C_{\mathcal{D}^\infty(2)\text{fil}}(S)$ we set

$$f^{*\text{mod}[-], \Gamma}(M, F) := \Gamma_X E(p_S^{*\text{mod}[-]}(M, F)) \in C_{\mathcal{D}^\infty(2)\text{fil}, \infty}(X \times S),$$

It induces in the derived category

$$Rf^{*\text{mod}[-], \Gamma}(M, F) := f^{*\text{mod}[-], \Gamma}(M, F) := \Gamma_X E(p_S^{*\text{mod}[-]}(M, F)) \in D_{\mathcal{D}^\infty(2)\text{fil}, \infty}(X \times S),$$

(ii) Then, for $(M, F) \in C_{\mathcal{D}^\infty(2)\text{fil}}(S)$ we set

$$Lf^{\hat{*}\text{mod}[-], \Gamma}(M, F) := \Gamma_X^{\vee, h} L_D p_S^{*\text{mod}[-]}(M, F) := \mathbb{D}_S^K L_D \Gamma_X E(\mathbb{D}_S^K L_D p_S^{*\text{mod}[-]}(M, F)) \in D_{\mathcal{D}^\infty(2)\text{fil}, \infty}(X \times S).$$

4.2.4 The 2 functors and transformations maps for \mathbf{D} modules on the smooth complex algebraic varieties and the complex analytic manifolds

By the definitions and the propositions 53, 54, 76, for the algebraic case, and the propositions 56, 57, 77, for the analytic case,

- we have the 2 functors on $\text{SmVar}(\mathbb{C})$:

- $C_{\mathcal{D}(2)\text{fil}}(\cdot) : \text{SmVar}(\mathbb{C}) \rightarrow C_{\mathcal{D}(2)\text{fil}}(\cdot)$, $S \mapsto C_{\mathcal{D}(2)\text{fil}}(S)$, $(f : T \rightarrow S) \mapsto f^{*\text{mod}}$, $(f : T \rightarrow S) \mapsto f^{*\text{mod}[-]}$,
- $D_{\mathcal{D}(2)\text{fil}, r}(\cdot) : \text{SmVar}(\mathbb{C}) \rightarrow D_{\mathcal{D}(2)\text{fil}, r}(\cdot)$, $S \mapsto D_{\mathcal{D}(2)\text{fil}, r}(S)$, $(f : T \rightarrow S) \mapsto Lf^{*\text{mod}}$, $(f : T \rightarrow S) \mapsto Lf^{*\text{mod}[-]}$,
- $D_{\mathcal{D}(2)\text{fil}, \infty}(\cdot) : \text{SmVar}(\mathbb{C}) \rightarrow D_{\mathcal{D}(2)\text{fil}, \infty}(\cdot)$, $S \mapsto D_{\mathcal{D}(2)\text{fil}, \infty}(S)$, $(f : T \rightarrow S) \mapsto \int_f^{\text{FDR}}$,

- we have the 2 functors on $\text{AnSm}(\mathbb{C})$:

- $C_{\mathcal{D}(2)\text{fil}}(\cdot) : \text{AnSm}(\mathbb{C}) \rightarrow C_{\mathcal{D}(2)\text{fil}}(\cdot)$, $S \mapsto C_{\mathcal{D}(2)\text{fil}}(S)$, $(f : T \rightarrow S) \mapsto f^{*\text{mod}}$, $(f : T \rightarrow S) \mapsto f^{*\text{mod}[-]}$,

- $D_{\mathcal{D}(2)fil,r}(\cdot) : \text{AnSm}(\mathbb{C}) \rightarrow D_{\mathcal{D}(2)fil,r}(\cdot)$, $S \mapsto D_{\mathcal{D}(2)fil,r}(S)$, $(f : T \rightarrow S) \mapsto Lf^{*mod}$, $(f : T \rightarrow S) \mapsto Lf^{*mod[-]}$,
- $D_{\mathcal{D}(2)fil,\infty}(\cdot) : \text{AnSm}(\mathbb{C}) \rightarrow D_{\mathcal{D}(2)fil,\infty}(\cdot)$, $S \mapsto D_{\mathcal{D}(2)fil,\infty}(S)$, $(f : T \rightarrow S) \mapsto \int_{f!}^{FDR}$,

• we have also the 2 functors on $\text{AnSm}(\mathbb{C})$:

- $C_{\mathcal{D}^\infty(2)fil}(\cdot) : \text{AnSm}(\mathbb{C}) \rightarrow C_{\mathcal{D}^\infty(2)fil}(\cdot)$, $S \mapsto C_{\mathcal{D}^\infty(2)fil}(S)$, $(f : T \rightarrow S) \mapsto f^{*mod}$, $(f : T \rightarrow S) \mapsto f^{*mod[-]}$,
- $D_{\mathcal{D}^\infty(2)fil,r}(\cdot) : \text{AnSm}(\mathbb{C}) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(\cdot)$, $S \mapsto D_{\mathcal{D}^\infty(2)fil,r}(S)$, $(f : T \rightarrow S) \mapsto Lf^{*mod}$, $(f : T \rightarrow S) \mapsto Lf^{*mod[-]}$,
- $D_{\mathcal{D}^\infty(2)fil,r}(\cdot) : \text{AnSm}(\mathbb{C}) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(\cdot)$, $S \mapsto D_{\mathcal{D}^\infty(2)fil,r}(S)$, $(f : T \rightarrow S) \mapsto \int_{f!}^{FDR}$,

inducing the following commutative diagrams of functors :

$$\begin{array}{ccc}
\text{SmVar}(\mathbb{C}) & \xrightarrow{f \mapsto f^{*mod}} & C_{\mathcal{D}(2)fil}(\cdot) \\
\downarrow \text{An} & & \downarrow an \\
\text{AnSm}(\mathbb{C}) & \xrightarrow{f \mapsto f^{*mod}} & C_{\mathcal{D}(2)fil}(\cdot) \xrightarrow{J} C_{\mathcal{D}^\infty(2)fil}(\cdot)
\end{array}
, \quad
\begin{array}{ccc}
\text{SmVar}(\mathbb{C}) & \xrightarrow{f \mapsto Lf^{*mod}} & D_{\mathcal{D}(2)fil,r}(\cdot) \\
\downarrow \text{An} & & \downarrow an \\
\text{AnSm}(\mathbb{C}) & \xrightarrow{f \mapsto Lf^{*mod}} & D_{\mathcal{D}(2)fil,r}(\cdot) \xrightarrow{J} D_{\mathcal{D}^\infty(2)fil,r}(\cdot)
\end{array}$$

where, for $S \in \text{AnSm}(\mathbb{C})$,

- $D_{\mathcal{D}(2)fil,\infty,rh}(S) \subset D_{\mathcal{D}(2)fil,\infty,h}(S)$ is the full subcategory consisting of filtered complexes of D_S module whose cohomology sheaves are regular holonomic,
- $J : C_{\mathcal{D}(2)fil}(S) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S)$ is the functor $(M, F) \mapsto J(M, F) := (M, F) \otimes_{D_S} D_S^\infty$, which derive trivially.

We first look at the pullback map and the transformation map of De Rahm complexes (see definition 16 and definition-proposition 17) together with the support section functor :

Proposition 79. *Consider a commutative diagram and a factorization*

$$D_0 = \begin{array}{ccc} X & \xrightarrow{f} & S \\ g' \uparrow & & \uparrow g \\ X' & \xrightarrow{f'} & T \end{array}
D_0 = \begin{array}{ccc} f : X & \xrightarrow{i} & Y \times S \xrightarrow{p} S \\ g' \uparrow & & \uparrow g'' = I \times g \\ f' : X' & \xrightarrow{i'} & Y \times T \xrightarrow{p'} T \end{array}$$

with $X, X', Y, S, T \in \text{Var}(\mathbb{C})$ or $X, X', Y, S, T \in \text{AnSp}(\mathbb{C})$, i, i' being closed embeddings, and p, p' the projections. Denote by D the right square of D_0 . We have a factorization $i' : X' \xrightarrow{i'_1} X_T = X \times_{Y \times S} Y \times T \xrightarrow{i'_0} Y \times T$, where i'_0, i'_1 are closed embedding. Assume S, T, Y, Y' are smooth.

(i) For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$, the canonical map in $C_{p'_* \mathcal{O}_T fil}(Y \times T)$ (c.f. definition-proposition 17),

$$E(\Omega_{((Y' \times T)/(X \times S))/(T/S)}(M, F)) \circ T(g'', E)(-) \circ T(g'', \gamma)(-) : \\
g''^* \Gamma_{X'} E((\Omega_{Y \times S}^\bullet, F_b) \otimes_{\mathcal{O}_{Y \times S}} (M, F)) \rightarrow \Gamma_{X_T} E((\Omega_{Y \times T/T}^\bullet, F_b) \otimes_{\mathcal{O}_{Y \times T}} g''^{*mod}(M, F))$$

is a map in $C_{p'_* \mathcal{D}fil}(Y \times T)$.

(ii) For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$, the canonical map in $C_{\mathcal{O}_T fil}(T)$ (c.f. definition-proposition 17 with L_D instead of L_O)

$$T_\omega^O(D)(M, F)^\gamma : g^{*mod} L_{Dp_*} \Gamma_{X'} E(\Omega_{Y \times S}^\bullet \otimes_{\mathcal{O}_{Y \times S}} (M, F)) \rightarrow p'_* \Gamma_{X_T} E(\Omega_{Y \times T/T}^\bullet \otimes_{\mathcal{O}_{Y \times T}} g''^{*mod}(M, F))$$

is a map in $C_{\mathcal{D}fil}(T)$.

(iii) For $(N, F) \in C_{\mathcal{D}fil}(Y \times T)$, the canonical map in $C_{p'^*O_T fil}(Y \times T)$

$$T(X'/X_T, \gamma)(-) : \Gamma_{X'}E((\Omega_{Y \times T/T}^\bullet, F_b) \otimes_{O_{Y \times T}} (N, F)) \rightarrow \Gamma_{X_T}E((\Omega_{Y \times T/T}^\bullet, F_b) \otimes_{O_{Y \times T}} (N, F))$$

is a map in $C_{p'^*\mathcal{D}0fil}(Y \times T)$.

(iv) For $M = O_Y$, we have $T_\omega^O(D)(O_{Y \times S})^\gamma = T_\omega^O(D)^\gamma$ as complexes of D_T modules and $T_\omega^O(X_T/Y \times T)(O_{Y \times T})^\gamma = T_\omega^O(X_T/Y \times T)^\gamma$. as complexes of p'^*D_T modules.

Proof. Follows by definition from proposition 67. \square

In the analytical case, we also have

Proposition 80. Consider a commutative diagram and a factorization

$$D_0 = \begin{array}{ccc} X & \xrightarrow{f} & S \\ g' \uparrow & & \uparrow g \\ X' & \xrightarrow{f'} & T \end{array} \quad D_0 = \begin{array}{ccccc} f : X & \xrightarrow{i} & Y \times S & \xrightarrow{p} & S \\ g' \uparrow & & \uparrow g'' = I \times g & & \uparrow g \\ f' : X' & \xrightarrow{i'} & Y \times T & \xrightarrow{p'} & T \end{array}$$

with $X, X', Y, S, T \in \text{AnSp}(\mathbb{C})$, i, i' being closed embeddings, and p, p' the projections. Denote by D the right square of D_0 . We have a factorization $i' : X' \xrightarrow{i'_1} X_T = X \times_{Y \times S} Y \times T \xrightarrow{i'_0} Y \times T$, where i'_0, i'_1 are closed embedding. Assume S, T, Y, Y' are smooth.

(i) For $(M, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$, the canonical map in $C_{p'^*O_T fil}(Y \times T)$ (c.f. definition-proposition 17),

$$E(\Omega_{((Y' \times T)/(X \times S))/(T/S)}(M, F)) \circ T(g'', E)(-) \circ T(g'', \gamma)(-) : \\ g''^* \Gamma_X E((\Omega_{Y \times S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)) \rightarrow \Gamma_{X_T} E((\Omega_{Y \times T/T}^\bullet, F_b) \otimes_{O_{Y \times T}} g''^{*mod}(M, F))$$

is a map in $C_{p'^*\mathcal{D}^\infty fil}(Y \times T)$.

(ii) For $(M, F) \in C_{\mathcal{D}^\infty fil}(Y \times S)$, the canonical map in $C_{O_T fil}(T)$ (c.f. definition-proposition 17 with L_{D^∞} instead of L_O)

$$T_\omega^O(D)(M, F)^\gamma : g^{*mod} L_{D^\infty} p_* \Gamma_X E((\Omega_{Y \times S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)) \rightarrow p'_* \Gamma_{X_T} E((\Omega_{Y \times T/T}^\bullet, F_b) \otimes_{O_{Y \times T}} g''^{*mod}(M, F))$$

is a map in $C_{\mathcal{D}^\infty fil}(T)$.

(iii) For $(N, F) \in C_{\mathcal{D}^\infty fil}(Y \times T)$, the canonical map in $C_{p'^*O_T fil}(Y \times T)$

$$T(X'/X_T, \gamma)(-) : \Gamma_{X'}E(\Omega_{Y \times T/T}^\bullet \otimes_{O_{Y \times T}} (N, F)) \rightarrow \Gamma_{X_T}E(\Omega_{Y \times T/T}^\bullet \otimes_{O_{Y \times T}} (N, F))$$

is a map in $C_{p'^*\mathcal{D}^\infty fil}(Y \times T)$.

(iv) For $M = O_Y$, we have $T_\omega^O(D)(O_{Y \times S})^\gamma = T_\omega^O(D)^\gamma$ as complexes of D_T^∞ modules and $T_\omega^O(X_T/Y \times T)(O_{Y \times T})^\gamma = T_\omega^O(X_T/Y \times T)^\gamma$. as complexes of $p'^*D_T^\infty$ modules.

Proof. Follows from proposition 79. \square

Similarly, we have :

Proposition 81. Let $p : Y \times S \rightarrow S$ a projection and $i : X \hookrightarrow Y \times S$ a closed embedding with $S, Y \in \text{SmVar}(\mathbb{C})$.

(i) For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$ the canonical map in $C_{p^*O_S fil}(Y^{an} \times S^{an})$ (see definition-proposition 17)

$$E(\Omega_{(Y^{an} \times S^{an}/Y \times S)/(S^{an}/S)}(M, F)) \circ T(an, \gamma)(-) : \\ (\Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_Y} (M, F)))^{an} \rightarrow \Gamma_{X^{an}} E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y^{an} \times S^{an}}} (M, F)^{an})$$

is a map in $C_{h^* \mathcal{D}fil}(Y^{an} \times S)$.

(ii) For $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$ the canonical map in $C_{O_S fil}(S^{an})$ (see definition-proposition 17)

$$T_\omega^O(an, p)(M, F)^\gamma : (p_* \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_Y} (M, F)))^{an} \rightarrow p_* \Gamma_{X^{an}} E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y^{an}}} (M, F)^{an})$$

is a map in $C_{\mathcal{D}fil}(S^{an})$.

(iii) For $M = O_Y$, we have $T_\omega^O(an, h)(O_Y)^\gamma = T_\omega^O(an, h)^\gamma$ as complexes of D_S modules

Proof. Follows by definition from proposition 69 □

Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSm}(\mathbb{C})$. Let $j : V \hookrightarrow Y \times S$ an open embedding. Consider (see proposition 70), for $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$, the canonical transformation map in $C_{p^* \mathcal{D}fil}(Y \times S)$

$$k \circ T_w^O(j, \otimes)(E(M, F)) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_Y} j_* j^* E(M, F) \\ \xrightarrow{DR(Y \times S/S)(\text{ad}(j^*, j_*)(-))} \\ j_* j^* ((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_Y} j_* j^* E(M, F)) = j_* j^* (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_Y} j_* j^* E(M, F) \\ \xrightarrow{k \circ DR(Y \times S/S)(\text{ad}(j^*, j_*)(E(M, F)))} \\ j_* E(j^* (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} j^* E(M, F)) = j_* E(j^* ((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)))$$

We have then :

Proposition 82. Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSm}(\mathbb{C})$. Let $i : X \hookrightarrow Y \times S$ a closed embedding. Then, for $(M, F) \in C_{\mathcal{D}fil}(Y \times S)$

(i) the canonical map in $C_{p^* \mathcal{D}fil}(Y)$ (definition 61)

$$T_w^O(\gamma, \otimes)(M, F) := (I, k \circ T_w^O(j, \otimes)(E(M, F))) : \\ (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} \Gamma_X E(M, F) \rightarrow \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)),$$

is a (1-)filtered Zariski, resp usu, local equivalence.

(ii) the map of point (i) gives the following canonical isomorphism in $D_{p^* \mathcal{D}fil}(Y)$

$$T_w^O(\gamma, \otimes)(M, F) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} \Gamma_X E(M, F) \xrightarrow{T_w^O(\gamma, \otimes)(M, F)} \\ \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)) \xrightarrow{DR(Y \times S/S)(k)^{-1}} \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)).$$

Proof. (i): By proposition 70

- $T_w^O(j, \otimes)(M, F) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} j_* j^* E(M, F) \rightarrow j_* E(j^* ((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)))$ is a filtered Zariski, resp usu, local equivalence in $C_{p^* \mathcal{D}fil}(Y \times S)$ and
- $DR(Y \times S/S)(k) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F) \rightarrow (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)$ is a filtered Zariski, resp usu, local equivalence in $C_{p^* \mathcal{D}fil}(Y \times S)$.

(ii): Follows from (i). □

In the analytic case, we also have

Proposition 83. *Let $p : Y \times S \rightarrow S$ a projection with $Y, S \in \text{AnSm}(\mathbb{C})$. Let $i : X \hookrightarrow Y$ a closed embedding. Then, for $(M, F) \in C_{\mathcal{D}^\infty \text{fil}}(Y \times S)$*

(i) *the canonical map in $C_{p^* \mathcal{D}^\infty \text{fil}}(Y)$*

$$T_w^O(\gamma, \otimes)(M, F) := (I, T_w^O(j, \otimes)(E(M, F))) : \\ (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} \Gamma_X E(M, F) \rightarrow \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)).$$

is a map in $C_{p^ \mathcal{D}^\infty \text{fil}}(Y \times S)$. Proposition 82 says that it is a filtered equivalence usu local,*

(ii) *the map of point (i) gives the following canonical isomorphism in $D_{p^* \mathcal{D}^\infty, \text{fil}}(Y \times S)$*

$$T_w^O(\gamma, \otimes)(M, F) : (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} \Gamma_X E(M, F) \xrightarrow{T_w^O(\gamma, \otimes)(M, F)} \\ \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} E(M, F)) \xrightarrow{DR(Y \times S/S)(k)^{-1}} \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (M, F)).$$

Proof. (i): By proposition 71

- $T_w^O(j, \otimes)(M) : \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} j_* j^* E(M) \rightarrow j_* E(j^*(\Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} E(M)))$ is an equivalence usu local in $C_{p^* \mathcal{D}^\infty}(Y \times S)$ and
- $DR(Y \times S/S)(k) : \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} M \rightarrow \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} E(M)$ is an equivalence usu local in $C_{p^* \mathcal{D}^\infty}(Y \times S)$.

(ii): Follows from (i). □

In the projection case, we consider the following canonical maps : Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$. Denote by $p = p_1 : S_{12} = S_1 \times S_2 \rightarrow S_1$ and $p_2 : S_{12} = S_1 \times S_2 \rightarrow S_2$ the projection. We consider

- $p(M_1, F) : (M_1, F) \rightarrow p_{* \text{mod}} p^{* \text{mod}[-1]}(M_1, F)$ in $C_{\mathcal{D}(2) \text{fil}}(S_1)$, for $(M_1, F) \in C_{\mathcal{D} \text{fil}(2)}(S_1)$, which is the composite

$$p(M_1, F) : (M_1, F) \xrightarrow{\text{ad}(p^*, p^*)(M_1)} p_* p^*(M_1, F) \xrightarrow{m_1} p_*((\Omega_{S_{12}/S_1}^\bullet, F_b) \otimes_{p^* O_{S_1}} p^*(M_1, F)) \\ \xrightarrow{=} p_*((\Omega_{S_{12}/S_1}^\bullet, F_b) \otimes_{O_{S_{12}}} p^{* \text{mod}}(M_1, F)) = p_{* \text{mod}} p^{* \text{mod}[-1]}(M_1, F)$$

where $m_1 : p^* M_1 \rightarrow p^* M_1 \otimes_{p^* O_{S_1}} \Omega_{S_{12}/S_1}^\bullet$ is given by $m_1(m) = m \otimes 1$,

- $p(M_{12}, F) : p^{* \text{mod}[-1]} p_{* \text{mod}}(M_{12}, F) \rightarrow (M_{12}, F)$ in $C_{\mathcal{D} \text{fil}}(S_1 \times S_2)$, for $(M_{12}, F) \in C_{\mathcal{D}}(S_1 \times S_2)$, which is the composite

$$p(M_{12}, F) : p^{* \text{mod}[-1]} p_{* \text{mod}}(M_{12}, F) = p^* p_*((M_{12}, F) \otimes_{O_{S_{12}}} (\Omega_{S_{12}/S_1}^\bullet, F_b)) \otimes_{p^* O_{S_1}} O_{S_{12}} \\ \xrightarrow{\text{ad}(p^*, p^*)(-) \otimes_{p^* O_{S_1}} I} (M_{12}, F) \otimes_{O_{S_{12}}} \Omega_{S_{12}/S_1}^\bullet \otimes_{p^* O_{S_1}} O_{S_{12}} = (M_{12}, F) \otimes_{p^* O_{S_1}} \Omega_{S_{12}/S_1}^\bullet \xrightarrow{m_{12}} (M_{12}, F)$$

where $m_{12} : M_{12} \otimes_{p^* O_{S_1}} \Omega_{S_{12}/S_1}^\bullet \rightarrow M_{12}$ is the multiplication map:

- $m_{12}(M_{12} \otimes_{p^* O_{S_1}} \Omega_{S_{12}/S_1}^p) = 0$ for $p \neq 0$ and
- $m_{12} : M_{12} \otimes_{p^* O_{S_1}} \Omega_{S_{12}/S_1}^0 = M_{12} \otimes_{p^* O_{S_1}} O_{S_{12}} \rightarrow M_{12}$ is given by $m_{12}(m \otimes f) = fm$

We have then $p(p^{* \text{mod}[-1]}(M_1, F)) \circ p^{* \text{mod}[-1]} p(M_1, F) = I_{p^{* \text{mod}[-1]}(M_1, F)}$. It gives the following maps

- $p!(M_{12}) : (M_{12}, F) \rightarrow p^{*mod[-]} \int_{p!} (M_{12}, F)$ in $D_{\mathcal{D}(2)fil}(S_1 \times S_2)$, for $(M_{12}, F) \in C_{\mathcal{D}fil,h}(S_1 \times S_2)$, given by

$$p!(M_{12}) : (M_{12}, F) \xrightarrow{d(M_{12}, F)} L\mathbb{D}_S^2(M_{12}, F) \xrightarrow{L\mathbb{D}_S(p(-) \circ q)} \mathbb{D}_S^K L_D p^{*mod[-]} p_{*mod} E(\mathbb{D}_S^K L_D(M_{12}, F))$$

$$\xrightarrow{T(p,D)(-)^{-1}} p^{*mod} \mathbb{D}_S^K L_D p_{*mod} E(\mathbb{D}_S^K L_D(M_{12}, F)) = p^{*mod[-]} \int_{p!} (M_{12}, F)$$

- $p!(M_1, F) : \int_{p!} p^{*mod[-]}(M_1, F) \rightarrow (M_1, F)$ in $D_{\mathcal{D}fil}(S_1)$, for $M_1 \in C_{\mathcal{D}fil,h}(S_1)$, given by

$$p!(M_1, F) : \int_{p!} p^{*mod[-]}(M_1, F) = \mathbb{D}_S^K L_D p_{*mod} E(\mathbb{D}_S^K p^{*mod[-]} L_D(M_1, F)) \xrightarrow{(\mathbb{D}_S^K k) \circ T(p,D)(-)^{-1}}$$

$$\mathbb{D}_S^K p_{*mod} p^{*mod[-]} \mathbb{D}_S^K L_D(M_1, F) \xrightarrow{\mathbb{D}_S^K p(\mathbb{D}_S^K L_D(M_1, F))} \mathbb{D}_S^{K,2} L_D(M_1, F) \xrightarrow{d(M_1, F)^{-1}} (M_1, F)$$

so that $p^{*mod[-]}(p!(M_1, F)) \circ p!(p^{*mod[-]}(M_1, F)) = I_{p^{*mod[-]}(M_1, F)}$.

Definition 75. (i) Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ which is cartesian, together with its factorization

$$D = (f, q) = \begin{array}{ccc} X \times T & \xrightarrow{f'} & T \times S \\ g' \downarrow & & \downarrow q \\ X & \xrightarrow{f} & S \end{array} \quad D = (f, q) = \begin{array}{ccccc} X \times T & \xrightarrow{i''} & X \times T \times S & \xrightarrow{p''} & T \times S \\ \downarrow q' & & \downarrow q'' = I \times q & & \downarrow q \\ f : X & \xrightarrow{i} & X \times S & \xrightarrow{p} & S \end{array}$$

where the squares are cartesian, $f = p \circ i$ being the graph factorization and q, q' the projections. We have, for $(M, F) \in C_{\mathcal{D}(2)fil,\infty}(X)$, the following transformation map in $C_{\mathcal{D}(2)fil,\infty}(T \times S)$:

$$T^{\mathcal{D}mod}(f, q)(M, F) : q^{*mod} p_* E((\Omega_{X \times S/S}, F_b) \otimes_{O_{X \times S}} i_{*mod}(M, F))$$

$$\xrightarrow{T_\omega^O(q,p)(i_{*mod}(M, F))} p''_* E((\Omega_{X \times T \times S/T \times S}, F_b) \otimes_{O_{X \times T \times S}} q''^{*mod} i_{*mod}(M, F))$$

$$\xrightarrow{p_* E(T^{\mathcal{D}mod}(i, q'')(M, F) \otimes I)} p''_* E((\Omega_{X \times T \times S/T \times S}, F_b) \otimes_{O_{X \times T \times S}} i''^{*mod} q'^{*mod}(M, F))$$

where

$$T^{\mathcal{D}mod}(i, q'')(M, F) : q''^{*mod} i_{*mod}(M, F) := q''^{*mod} i_*((M, F) \otimes_{D_X} i^{*mod}(D_{X \times S}, F^{ord}))$$

$$\xrightarrow{T^{mod}(q'', i)(-)} i_* q'^{*mod}((M, F) \otimes_{D_X} i^{*mod}(D_{X \times S}, F^{ord}))$$

$$\xrightarrow{=} i_*(q'^{*mod}(M, F) \otimes_{q'^* D_X} q'^{*mod}(D_{X \times S}, F^{ord})) \xrightarrow{=} i_*(q'^{*mod}(M, F) \otimes_{D_{X \times T}} i'^{*mod}(D_{X \times S \times T}, F^{ord})) =: i'^{*mod} q'^{*mod}(M, F)$$

(ii) Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ which is cartesian, together with its factorization

$$D = (f, g) = \begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} \quad D = (f, g) = \begin{array}{ccccc} X_T & \xrightarrow{i'} & X \times T & \xrightarrow{p'} & T \\ \downarrow l' & & \downarrow l'' = I \times l & & \downarrow l \\ f'' : X \times T & \xrightarrow{i''} & X \times T \times S & \xrightarrow{p''} & T \times S \\ \downarrow q' & & \downarrow q'' = I \times q & & \downarrow q \\ f : X & \xrightarrow{i} & X \times S & \xrightarrow{p} & S \end{array}$$

where the squares are cartesian, $f = p \circ i$, $g = q \circ l$ being the graph factorizations. We have, for $(M, F) \in D_{\mathcal{D}(2)fil, \infty}(X)$, the following transformation map in $D_{\mathcal{D}(2)fil, \infty}(T \times S)$:

$$\begin{aligned}
& T^{\mathcal{D}mod}(f, g)((M, F)) : \\
& Rg^{*mod, \Gamma}(M, F) \int_f^{FDR} (M, F) := \Gamma_T E(q^{*mod} p_* E((\Omega_{X \times S/S}, F_b) \otimes_{O_{X \times S}} i_{*mod}(M, F))) \\
& \xrightarrow{\Gamma_T E(T^{\mathcal{D}mod}(f, g)(M, F))} \Gamma_T E(p''_* E((\Omega_{X \times T \times S/T \times S}, F_b) \otimes_{O_{X \times T \times S}} i''_{*mod} q'^{*mod}(M, F))) \\
& \xrightarrow{\cong} p''_* \Gamma_{X \times T} E((\Omega_{X \times T \times S/T \times S}, F_b) \otimes_{O_{X \times T \times S}} i''_{*mod} q'^{*mod}(M, F)) \\
& \xrightarrow{T_w^O(\gamma, \otimes)(-)} p''_* E((\Omega_{X \times T \times S/T \times S}, F_b) \otimes_{O_{X \times T \times S}} \Gamma_{X \times T} E(i''_{*mod} q'^{*mod}(M, F))) \\
& \xrightarrow{\cong} p''_* E((\Omega_{X \times T \times S/T \times S}, F_b) \otimes_{O_{X \times T \times S}} (i''_{*mod} q'^{*mod} \Gamma_{X \times T} E(M, F))) =: \int_{f'}^{FDR} Rg'^{*mod, \Gamma}(M, F)
\end{aligned}$$

(ii)' We have, for $M \in D_{\mathcal{D}}(X)$, the following transformation map in $D_{\mathcal{D}}(T)$:

$$\begin{aligned}
& T^{\mathcal{D}mod}(f, g)(M) : \\
& g^{*mod}(M, F) \int_f (M) = l^{*mod} q^{*mod} \int_f M \xrightarrow{q'(M)} l^{*mod} q^{*mod} \int_f q'_{*mod} q'^{*mod} M \\
& \xrightarrow{\cong} l^{*mod} q^{*mod} q_{*mod} \int_{f''} q'^{*mod} M \xrightarrow{q(-)} l^{*mod} \int_{f''} q'^{*mod} M \\
& \xrightarrow{l^{*mod} \text{ad}(l'^{\sharp}, l'_{*mod})(-)^{-1}} l^{*mod} \int_{f''} l'_{*mod} l'^{\sharp} q'^{*mod} M \xrightarrow{\cong} l^{\sharp} l_{*mod} \int_{f'} l'^{*mod} q^{*mod} M \\
& \xrightarrow{\text{ad}(l^{\sharp}, l_{*mod})(-)} \int_{f'} l'^{*mod} q^{*mod} M =: \int_{f'} g'^{*mod}(M)
\end{aligned}$$

where $l^{*mod} \text{ad}(l'^{\sharp}, l'_{*mod})(-)$ is an isomorphism by lemma 6.

In the analytic case, we have :

Definition 76. Consider a commutative diagram in $\text{AnSm}(\mathbb{C})$ which is cartesian together with a factorization

$$D = (f, g) = \begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} \quad D = (f, g) = \begin{array}{ccccc} f' : X_T & \xrightarrow{i'} & Y \times T & \xrightarrow{p'} & T \\ \downarrow g' & & \downarrow g'' = I \times g & & \downarrow g \\ f : X & \xrightarrow{i} & Y \times S & \xrightarrow{p} & S \end{array} ,$$

where $Y \in \text{AnSm}(\mathbb{C})$, i, i' are closed embeddings and p, p' the projections.

(i) We have, for $(M, F) \in D_{\mathcal{D}(2)fil, \infty, h}(X)$, the following transformation map in $D_{\mathcal{D}(2)fil, \infty}(T \times S)$

$$T^{\mathcal{D}mod}(f, g)((M, F)) : Rg^{*mod, \Gamma} \int_f^{FDR} (M, F) \rightarrow \int_{f'}^{FDR} Rg'^{*mod, \Gamma}(M, F)$$

define in the same way as in definition 75

(ii) For $(M, F) \in D_{\mathcal{D}^\infty(2)fil, \infty}(X)$, the following transformation map in $D_{\mathcal{D}^\infty(2)fil, \infty}(T \times S)$

$$T^{\mathcal{D}mod}(f, g)((M, F)) : Rg^{*mod, \Gamma} \int_f^{FDR} (M, F) \rightarrow \int_{f'}^{FDR} Rg'^{*mod, \Gamma}(M, F)$$

is defined in the same way as in (ii) : see definition 75.

In the algebraic case, we have the following proposition:

Proposition 84. *Consider a cartesian square in $\text{SmVar}(\mathbb{C})$*

$$D = \begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ T & \xrightarrow{g} & S \end{array}$$

(i) For $(M, F) \in D_{\mathcal{D}(2)\text{fil}, \infty, c}(X)$,

$$T^{\mathcal{D}\text{mod}}(f, g)((M, F)) : Rg^{*\text{mod}, \Gamma} \int_f^{\text{FDR}} (M, F) \xrightarrow{\sim} \int_{f'}^{\text{FDR}} Rg'^{*\text{mod}, \Gamma}(M, F)$$

is an isomorphism in $D_{\mathcal{D}(2)\text{fil}, \infty}(T)$.

(ii) For $M \in D_{\mathcal{D}, c}(X)$,

$$T^{\mathcal{D}\text{mod}}(f, g)(M) : g^{*\text{mod}} \int_f M \xrightarrow{\sim} \int_{f'} g'^{*\text{mod}} M$$

is an isomorphism in $D_{\mathcal{D}}(T)$.

Proof. Follows from the projection case and the closed embedding case. □

In the analytic case, we have similarly:

Proposition 85. *Consider a cartesian square in $\text{AnSm}(\mathbb{C})$*

$$D = \begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ T & \xrightarrow{g} & S \end{array}$$

(i) Assume that f , hence f' is proper. For $(M, F) \in D_{\mathcal{D}(2)\text{fil}, \infty, h}(X)$,

$$T^{\mathcal{D}\text{mod}}(f, g)((M, F)) : Rg^{*\text{mod}, \Gamma} \int_f^{\text{FDR}} (M, F) \xrightarrow{\sim} \int_{f'}^{\text{FDR}} Rg'^{*\text{mod}, \Gamma}(M, F)$$

is an isomorphism in $D_{\mathcal{D}(2)\text{fil}, \infty}(T)$.

(ii) For $(M, F) \in D_{\mathcal{D}^\infty(2)\text{fil}, \infty, h}(X)$,

$$T^{\mathcal{D}\text{mod}}(f, g)((M, F)) : Rg^{*\text{mod}, \Gamma} \int_f^{\text{FDR}} (M, F) \xrightarrow{\sim} \int_{f'}^{\text{FDR}} Rg'^{*\text{mod}, \Gamma}(M, F)$$

is an isomorphism in $D_{\mathcal{D}^\infty(2)\text{fil}, \infty}(T)$.

Proof. (i): Similar to the proof of proposition 84.

(ii): Similar to the proof of proposition 84. □

Definition 77. *Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$.*

(i) We have, for $(M, F) \in C_{\mathcal{D}\text{fil}}(S)$ and $(N, F) \in C_{\mathcal{D}\text{fil}}(X)$, we have the map in $C_{\mathcal{D}\text{fil}}(S)$

$$\begin{aligned} & T^{\mathcal{D}\text{mod}, 0}(\otimes, f)((M, F), (N, F)) : \\ & (M, F) \otimes_{O_S} f_{*\text{mod}}^0(N, F) := (M, F) \otimes_{O_S} f_*((N, F) \otimes_{D_X} (D_{X \leftarrow S}, F^{\text{ord}})) \\ & \xrightarrow{T(\otimes, f)(-, -)} f_*(f^*(M, F) \otimes_{f^*O_S} (N, F) \otimes_{D_X} (D_{X \leftarrow S}, F^{\text{ord}})) \xrightarrow{\cong} \\ & f_*(f^{*\text{mod}}(M, F) \otimes_{O_X} (N, F) \otimes_{D_X} (D_{X \leftarrow S}, F^{\text{ord}})) =: f_{*\text{mod}}^0(f^{*\text{mod}}(M, F) \otimes_{O_X} (N, F)) \end{aligned}$$

(ii) Consider the cartesian square

$$D = \begin{array}{ccc} X & \xrightarrow{i} & X \times S \\ \downarrow f & & \downarrow f \times I_S \\ S & \xrightarrow{\Delta_S} & S \times S \end{array},$$

where $i_f = (f \times I_S) \circ \Delta_X : X \hookrightarrow X \times S$ is the graph embedding. Then, for $(M, F) \in C_{\mathcal{D}(2)fil}(S)$ and $(N, F) \in C_{\mathcal{D}fil}(X)$, we have the map in $D_{\mathcal{D}(2)fil,r}(S)$

$$\begin{aligned} T^{\mathcal{D}mod}(\otimes, f)((M, F), (N, F)) &: \int_f^{FDR} ((N, F) \otimes_{O_X} f_{FDR}^{*mod}(M, F)) = \int_f^{FDR} i_{f,FDR}^{*mod}(p_X^* N \otimes p_S^* M) \\ &\xrightarrow{T^{\mathcal{D}mod}(\Delta_S, f \times I_S)(-)} \Delta_{S,FDR}^{*mod} \int_{(f \times I_S)}^{FDR} (p_X^* N \otimes p_S^* M) = \left(\int_f (N, F) \right) \otimes_{O_S}^L (M, F). \end{aligned}$$

Clearly if $i : Z \hookrightarrow S$ is a closed embedding with $Z, S \in \text{SmVar}(\mathbb{C})$ or with $Z, S \in \text{AnSm}(\mathbb{C})$, then $T^{\mathcal{D},0}(\otimes, i)(M, N) = T^{\mathcal{D}}(\otimes, i)(M, N)$ in $D_{\mathcal{D}(2)fil,\infty}(S)$.

We have then the following :

Proposition 86. (i) Let $i : Z \hookrightarrow S$ is a closed embedding with $Z, S \in \text{SmVar}(\mathbb{C})$, then for $(M, F) \in C_{\mathcal{D}fil}(S)$ and $(N, F) \in C_{\mathcal{D}fil}(Z)$

$$T^{\mathcal{D},0}(\otimes, i)((M, F), (N, F)) : (M, F) \otimes_{O_S} i_{*mod}(N, F) \xrightarrow{\sim} i_{*mod}(i^{*mod}(M, F) \otimes_{O_Z} (N, F))$$

is an isomorphism in $C_{\mathcal{D}fil}(S)$.

(ii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$. Then, for $(M, F) \in C_{\mathcal{D}(2)fil}(X)$ and $(N, F) \in C_{\mathcal{D}(2)fil}(S)$,

$$T^{\mathcal{D}mod}(\otimes, f)((M, F), (N, F)) : \int_f^{FDR} ((M, F) \otimes_{O_X}^L f_{FDR}^{*mod}(N, F)) \xrightarrow{\sim} \left(\int_f^{FDR} (M, F) \right) \otimes_{O_Y}^L (N, F)$$

is an isomorphism in $D_{\mathcal{D}(2)fil,\infty}(S)$.

Proof. (i): Follows from proposition 10.

(ii): Follows from proposition 84(i). \square

Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$. Consider the graph embedding $f : X \xrightarrow{i} X \times S \xrightarrow{p} S$, with $X, Y, S \in \text{SmVar}(\mathbb{C})$. We have, for $(M, F) \in C_{\mathcal{D}fil}(X)$, the canonical isomorphism in $C_{\mathcal{D}(2)fil}(S^{an})$

$$\begin{aligned} \text{an}_X^{*mod} i^{*mod} L_D(p^{*mod}(M, F) \otimes_{O_{X \times S}} (O_{X \times S}, \mathcal{V}_X)) &\xrightarrow{=} \\ i^{*mod} L_D p^{*mod}((M, F)^{an} \otimes_{O_{X^{an} \times S^{an}}} (O_{X^{an} \times S^{an}}, \mathcal{V}_{X^{an}})) & \end{aligned}$$

We then define and study the transformation map between the direct image functor and the analytical functor for D-modules :

Definition 78. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$.

(i) We have for $(M, F) \in C_{\mathcal{D}(2)fil}(X)$ the canonical map in $C_{\mathcal{D}(2)fil}(S^{an})$

$$\begin{aligned} T^{\mathcal{D}mod0}(an, f)(M, F) &: \text{an}_S^{*mod}(f_* E((D_{X \leftarrow S}, F^{ord}) \otimes_{D_X} L_D(M, F))) \xrightarrow{T^{mod}(an, f)(-)} \\ & f_* (E((D_{X \leftarrow S}, F^{ord}) \otimes_{D_X} L_D(M, F)))^{an} \xrightarrow{=} f_* E(D_{X^{an} \leftarrow S^{an}} \otimes_{D_{X^{an}}} L_D(M^{an}, F)) \end{aligned}$$

(ii) Consider the graph embedding $f : X \xrightarrow{i} X \times S \xrightarrow{p} S$, with $X, Y, S \in \text{SmVar}(\mathbb{C})$. We have, for $(M, F) \in C_{\mathcal{D}fil}(X)$, the canonical map in $C_{\mathcal{D}(2)fil}(S^{an})$

$$\begin{aligned} T^{\mathcal{D}mod}(an, f)(M, F) &: an_S^{*mod}(p_*E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} i_{*mod}(M, F))) \\ &\xrightarrow{T_\omega^O(an, p)(i_{*mod}(M, F))} p_*E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y^{an} \times S^{an}}} (i_{*mod}(M, F))^{an}) \\ &\xrightarrow{p_{*mod}T^{\mathcal{D}mod0}(an, i)((M, F))} p_*E((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y^{an} \times S^{an}}} i_{*mod}((M, F)^{an})). \end{aligned}$$

In order to prove that this map gives an isomorphism in the derived category in the non filtered case if f is proper and M coherent, we will need the following (c.f.[17]):

Theorem 22. *A product $X \times S$ of a smooth projective variety X and a smooth affine variety S is D -affine.*

Proof. See [17] theorem 1.6.5. □

A main result is that we have the following version of the first GAGA theorem for coherent D -modules :

Theorem 23. *Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$. Let $M \in D_{\mathcal{D}(2)fil, c}(X)$, for $r = 1, \dots, \infty$. If f is proper,*

$$T^{\mathcal{D}mod}(an, f)(M, F) : \left(\int_f M \right)^{an} \xrightarrow{\sim} \int_{f^{an}} (M^{an})$$

is an isomorphism.

Proof. We may assume that f is projective, so that we have a factorization $f : X \xrightarrow{i} \mathbb{P}^N \times S \xrightarrow{p} S$ where i is a closed embedding and p the projection. The question being local on S , we may assume that S is affine. Since $\mathbb{P}^N \times S$ is D -affine by theorem 22, we have by proposition 50(iii) a complex $F \in C_{\mathcal{D}}(\mathbb{P}^N \times S)$ such that $i_{*mod}M = F \simeq F \in D_{\mathcal{D}, r}(\mathbb{P}^N \times S)$ and each F^n is a direct summand of a free $D_{\mathbb{P}^N \times S}$ module of finite rank. The theorem now follows from the fact that $\int_p D_{\mathbb{P}^N \times S} \simeq D_S[-N]$ and the fact that $(D_S)^{an} = D_{S^{an}}$. □

We also have

Definition 79. (i) *Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. We have, for $M, N \in C_{\mathcal{D}fil}(X)$, the canonical transformation map in $D_{\mathcal{D}fil, \infty}(S)$*

$$\begin{aligned} T^{0, \mathcal{D}}(f, hom)((M, F), (N, F)) &: Rf_* R\mathcal{H}om_{f^* D_S}((M, F), (N, F)) \rightarrow \\ Rf_* R\mathcal{H}om_{D_X}((M, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord}), (N, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord})) &\xrightarrow{T^0(f, hom)(E(-), E(-))} \\ R\mathcal{H}om_{D_X}(Rf_*((M, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord})), Rf_*((N, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord}))) &= \\ R\mathcal{H}om_{D_X} \left(\int_f (M, F), \int_f (N, F) \right) & \end{aligned}$$

(ii) *Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. We have, for $(M, F), (N, F) \in C_{\mathcal{D}fil}(X)$, the canonical transformation map in $D_{\mathcal{D}fil, \infty}(S)$*

$$\begin{aligned} T^{0, \mathcal{D}}(f_i, hom)((M, F), (N, F)) &: Rf_* \mathcal{H}om_{f^* D_S}((M, F), (N, F)) \rightarrow \\ Rf_* \mathcal{H}om_{D_X}((M, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord}), (N, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord})) &\xrightarrow{T^0(f_i, hom)(E(-), E(-))} \\ R\mathcal{H}om_{D_X}(Rf_i((M, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord})), Rf_i((N, F) \otimes_{D_X} L_D(D_{X \leftarrow S}, F^{ord}))) &= \\ R\mathcal{H}om_{D_X} \left(\int_{f!} (M, F), \int_{f!} (N, F) \right) & \end{aligned}$$

Definition 80. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. We have, for $(M, F), (N, F) \in C_{\mathcal{D}fil}(S)$, the canonical transformation map in $C_{\mathcal{D}fil}(X)$

$$\begin{aligned} T^{\mathcal{D}}(f, \text{hom})((M, F), (N, F)) &: f^* \mathcal{H}om_{D_S}((M, F), (N, F)) \\ &\xrightarrow{T(f, \text{hom})((M, F), (N, F))} \mathcal{H}om_{f^* D_S}(f^*(M, F), f^*(N, F)) \\ &\rightarrow \mathcal{H}om_{D_X}(f^*(M, F) \otimes_{f^* D_S} L_{f^* D} (D_{X \rightarrow S}, F^{ord}), f^*(N, F) \otimes_{f^* D_S} L_{f^* D} (D_{X \rightarrow S}, F^{ord})) \\ &\xrightarrow{=} \mathcal{H}om_{D_X}(f^{*mod}(M, F), f^{*mod}(N, F)) \end{aligned}$$

which is the one given by Kashiwara (see [20]).

In the algebraic case, we have, in the non filtered case, the six functor formalism for holonomic D-modules :

Theorem 24. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$.

(i) We have, for $M \in D_{\mathcal{D},h}(X)$ and $N \in D_{\mathcal{D},h}(S)$ a canonical isomorphism in $D_{\mathcal{D}}(S)$

$$I^{\mathcal{D}mod}(L_{f^*} \hat{*}mod[-], \int_f)(M, N) : Rf_* R\mathcal{H}om_{D_X}(L_{f^*} \hat{*}mod[-]N, M) \xrightarrow{\sim} R\mathcal{H}om_{D_S}(N, \int_f M).$$

(ii) We have, for $M \in D_{\mathcal{D},h}(X)$ and $N \in D_{\mathcal{D},h}(S)$ a canonical isomorphism in $D_{\mathcal{D}}(X)$

$$I^{\mathcal{D}mod}(\int_{f!}, L_{f^*} \hat{*}mod[-])(M, N) : R\mathcal{H}om_{D_X}(\int_{f!} M, N) \xrightarrow{\sim} Rf_* R\mathcal{H}om_{D_S}(M, L_{f^*} \hat{*}mod[-]N).$$

Proof. Follows from the projection case and the closed embedding case. □

Corollary 2. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$. Then,

- $(L_{f^*} \hat{*}mod[-], \int_f) : D_{\mathcal{D},h}(S) \rightarrow D_{\mathcal{D},h}(X)$ is a pair of adjoint functors.
- $(\int_{f!}, L_{f^*} \hat{*}mod[-]) : D_{\mathcal{D},h}(S) \rightarrow D_{\mathcal{D},h}(X)$ is a pair of adjoint functors.

Proof. Follows immediately from theorem 24 by taking global sections. □

Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$,

$$D = \begin{array}{ccc} X' & \xrightarrow{f'} & T \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} .$$

We have, for $M \in C_{\mathcal{D},h}(X)$, the following transformation maps

$$\begin{aligned} T_1^{\mathcal{D}mod}(D)(M) &: L_{g^*} \hat{*}mod[-] \int_f M \xrightarrow{\text{ad}(L_{f'} \hat{*}mod[-], f_{f'})(-)} \int_{f'} L_{f'} \hat{*}mod[-] L_{g^*} \hat{*}mod[-] \int_f M \xrightarrow{=} \\ &\int_{f'} L_{g_{FDR}} \hat{*}mod[-] L_{f^*} \hat{*}mod[-] \int_f M \xrightarrow{\text{ad}(L_{f^*} \hat{*}mod[-], f_f)(M)} \int_{f'} L_{g_{FDR}} \hat{*}mod[-] M \end{aligned}$$

and

$$\begin{aligned} T_2^{\mathcal{D}mod}(D)(M, F) &: \int_{f!} L_{g^*} \hat{*}mod[-] M \xrightarrow{\text{ad}(f_{f!}, L_{f^*} \hat{*}mod[-])(-)} \int_{f!} L_{g^*} \hat{*}mod[-] L_{f^*} \hat{*}mod[-] \int_{f!} M \xrightarrow{=} \\ &\int_{f!} L_{f^*} \hat{*}mod[-] L_{g^*} \hat{*}mod[-] M \int_{f!} \xrightarrow{\text{ad}(f_{f!}, L_{f^*} \hat{*}mod[-])(-)} L_{g^*} \hat{*}mod[-] \int_{f!}^{FDR} M \end{aligned}$$

Proposition 87. *Consider a cartesian square in $\text{SmVar}(\mathbb{C})$*

$$D = \begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ T & \xrightarrow{g} & S \end{array}$$

Assume that f (and hence f') is proper. Then, for $(M, F) \in D_{\mathcal{D}(2)\text{fil}, \infty, h}(X)$,

- $T_1^{\mathcal{D}\text{mod}}(f, g)(M) : Lg^{\hat{*}\text{mod}[-]} \int_f M \xrightarrow{\sim} \int_{f'} Lg'^{\hat{*}\text{mod}[-]} M$ and
- $T_2^{\mathcal{D}\text{mod}}(f, g)(M) : \int_{f'} Lg'^{\hat{*}\text{mod}[-]} M \xrightarrow{\sim} Lg^{\hat{*}\text{mod}[-]} \int_{f'} M$

are isomorphisms in $D_{\mathcal{D}}(T)$.

Proof. Follows from proposition 84 and the fact that the map $T_1^{\mathcal{D}\text{mod}}(f, g)(M)$ is given by the composite

$$\begin{aligned} T_1^{\mathcal{D}\text{mod}}(f, g)(M)[d_T - d_S] : Lg^{\hat{*}\text{mod}} \int_f (M, F) &= L\mathbb{D}_T Lg^{\hat{*}\text{mod}} L\mathbb{D}_S \int_f M \xrightarrow{T(f_*, f_!)(-)} \\ &L\mathbb{D}_T Lg^{\hat{*}\text{mod}} \int_f L\mathbb{D}_X M \xrightarrow{(L\mathbb{D}_T T^{\mathcal{D}\text{mod}}(f, g)(\mathbb{D}_X M))^{-1}} \mathbb{D}_T \int_{f'} Lg'^{\hat{*}\text{mod}} \mathbb{D}_X M \\ &\xrightarrow{T(f'_!, f'_*)(-)} \int_{f'} L\mathbb{D}_{X_T} Lg'^{\hat{*}\text{mod}} \mathbb{D}_X (M, F) = \int_{f'} Lg'^{\hat{*}\text{mod}} M \end{aligned}$$

and the map $T_2^{\mathcal{D}\text{mod}}(f, g)(M, F)$ is given by the composite

$$\begin{aligned} T_2^{\mathcal{D}\text{mod}}(f, g)(M)[d_T - d_S] : \int_{f'} Lg'^{\hat{*}\text{mod}} M &= L\mathbb{D}_T \int_{f'} \mathbb{D}_{X_T} Lg'^{\hat{*}\text{mod}} M \xrightarrow{d(-) \circ T(f_*, f_!)(-)} \\ &\int_{f'} Lg'^{\hat{*}\text{mod}} M \xrightarrow{T^{\mathcal{D}\text{mod}}(f, g)(M)^{-1}} L\mathbb{D}_T Lg^{\hat{*}\text{mod}} \int_f L\mathbb{D}_X M = Lg^{\hat{*}\text{mod}}(M, F) \int_{f'} M \end{aligned}$$

□

4.3 The D modules on singular algebraic varieties and singular complex analytic spaces

In this subsection by defining the category of complexes of filtered D-modules in the singular case and there functorialities.

4.3.1 Definition

In all this subsection, we fix the notations:

- For $S \in \text{Var}(\mathbb{C})$, we denote by $S = \cup_i S_i$ an open cover such that there exists closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have then closed embeddings $i_I : S_I := \cap_{i \in I} S_i \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Then for $I \subset J$, we denote by $j_{IJ} : S_J \hookrightarrow S_I$ the open embedding and $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ the projection, so that $p_{IJ} \circ i_J = i_I \circ j_{IJ}$. This gives the diagram of algebraic varieties $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Var}(\mathbb{C}))$ which gives the diagram of sites $(\tilde{S}_I) := \text{Ouv}(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. It also gives the diagram of sites $(\tilde{S}_I)^{op} := \text{Ouv}(\tilde{S}_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. For $I \subset J$, we denote by $m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I$ the open embedding.
- For $S \in \text{AnSp}(\mathbb{C})$ we denote by $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have then closed embeddings $i_I : S_I = \cap_{i \in I} S_i \hookrightarrow \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$.

Then for $I \subset J$, we denote by $j_{IJ} : S_J \hookrightarrow S_I$ the open embedding and $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ the projection, so that $p_{IJ} \circ i_J = i_I \circ j_{IJ}$. This gives the diagram of analytic spaces $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{AnSp}(\mathbb{C}))$ which gives the diagram of sites $(\tilde{S}_I) := \text{Ouv}(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. It also gives the diagram of sites $(\tilde{S}_I)^{op} := \text{Ouv}(\tilde{S}_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. For $I \subset J$, we denote by $m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I$ the open embedding.

The first definition is from [28] remark 2.1.20, where we give a shifted version to have compatibility with perverse sheaves.

Definition 81. *Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $\text{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \subset \text{PSh}_{\mathcal{D}(2)fil}((\tilde{S}_I))$ is the full subcategory*

- whose objects are $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, s_{IJ})$, with
 - $(M_I, F) \in \text{PSh}_{\mathcal{D}(2)fil}(\tilde{S}_I)$ such that $\mathcal{I}_{S_I} M_I = 0$, in particular $(M_I, F) \in \text{PSh}_{\mathcal{D}(2)fil, S_I}(\tilde{S}_I)$
 - $s_{IJ} : m^*(M_I, F) \xrightarrow{\sim} m^* p_{IJ*}(M_J, F)[d_{\tilde{S}_I} - d_{\tilde{S}_J}]$ for $I \subset J$, are isomorphisms, $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ*} s_{JK} \circ s_{IJ} = s_{IK}$;
- the morphisms $m : (M, F) \rightarrow (N, F)$ between $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, s_{IJ})$ and $(N, F) = ((N_I, F)_{I \subset [1, \dots, l]}, r_{IJ})$ are by definition a family of morphisms of complexes,

$$m = (m_I : (M_I, F) \rightarrow (N_I, F))_{I \subset [1, \dots, l]}$$

such that $r_{IJ} \circ m_J = p_{IJ*} m_J \circ s_{IJ}$ in $C_{\mathcal{D}, S_J}(\tilde{S}_J)$.

We denote by

$$\text{PSh}_{\mathcal{D}(2)fil, rh}(S/(\tilde{S}_I)) \subset \text{PSh}_{\mathcal{D}(2)fil, h}(S/(\tilde{S}_I)) \subset \text{PSh}_{\mathcal{D}(2)fil, c}(S/(\tilde{S}_I)) \subset \text{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

the full subcategory consisting of $((M_I, F), s_{IJ})$ such that (M_I, F) is filtered coherent, resp. filtered holonomic, resp. filtered regular holonomic, i.e. M_I are coherent, resp. holonomic, resp. filtered regular holonomic, sheaves of $D_{\tilde{S}_I}$ modules and F is a good filtration. We have the full subcategories

$$\begin{aligned} \text{PSh}_{\mathcal{D}(1,0)fil, rh}(S/(\tilde{S}_I)) \subset \text{PSh}_{\mathcal{D}2fil, rh}(S/(\tilde{S}_I)), \quad \text{PSh}_{\mathcal{D}(1,0)fil, h}(S/(\tilde{S}_I)) \subset \text{PSh}_{\mathcal{D}2fil, h}(S/(\tilde{S}_I)), \\ \text{PSh}_{\mathcal{D}(1,0)fil, h}(S/(\tilde{S}_I)) \subset \text{PSh}_{\mathcal{D}2fil, h}(S/(\tilde{S}_I)), \end{aligned}$$

consisting of $((M_I, F, W), s_{IJ})$ such that $W^p M_I$ are $D_{\tilde{S}_I}$ submodules.

We recall from section 2 the following

- A morphism $m = (m_I) : ((M_I), s_{IJ}) \rightarrow ((N_I), r_{IJ})$ in $C(\text{PSh}_{\mathcal{D}}(S/(\tilde{S}_I)))$ is a Zariski, resp. usu, local equivalence if and only if all the m_I are Zariski, resp. usu, local equivalences.
- A morphism $m = (m_I) : ((M_I, F), s_{IJ}) \rightarrow ((N_I, F), r_{IJ})$ in $C(\text{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$ is a filtered Zariski, resp. usu, local equivalence if and only if all the m_I are filtered Zariski, resp. usu, local equivalence.
- By definition, a morphism $m = (m_I) : ((M_I, F), s_{IJ}) \rightarrow ((N_I, F), r_{IJ})$ in $C(\text{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$ is an r -filtered Zariski, resp. usu, local equivalence if there exist $m_i : (C_{iI}, F), s_{iIJ}) \rightarrow (C_{(i+1)I}, F), s_{(i+1)IJ})$, $0 \leq i \leq s$, with $(C_{iI}, F), s_{iIJ}) \in C(\text{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$, $(C_{0I}, F), s_{0IJ}) = (M_I, F), s_{IJ})$, $(C_{sI}, F), s_{sIJ}) = (N_I, F), r_{IJ})$ such that

$$m = m_s \circ \dots \circ m_i \circ \dots \circ m_0 : ((M_I, F), s_{IJ}) \rightarrow ((N_I, F), r_{IJ})$$

with $m_i : (C_{iI}, F), s_{iIJ}) \rightarrow (C_{(i+1)I}, F), s_{(i+1)IJ})$ either filtered Zariski, resp. usu, local equivalence or r -filtered homotopy equivalence.

Let $S \in \text{Var}(\mathbb{C})$ or $S \in \text{AnSp}(\mathbb{C})$.

- If $S \in \text{Var}(\mathbb{C})$, let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$, and let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{SmVar}(\mathbb{C})$.
- If $S \in \text{AnSp}(\mathbb{C})$, let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$, and let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{AnSm}(\mathbb{C})$.

Denote $L = [1, \dots, l]$, $L' = [1, \dots, l']$ and $L'' := [1, \dots, l] \sqcup [1, \dots, l']$. We have then the refined open cover $S = \cup_{k \in L} S_k$ and we denote for $I \sqcup I' \subset L''$, $S_{I \sqcup I'} := \cap_{k \in I \sqcup I'} S_k$ and $\tilde{S}_{I \sqcup I'} := \prod_{k \in I \sqcup I'} \tilde{S}_k$, so that we have a closed embedding $i_{I \sqcup I'} : S_{I \sqcup I'} \hookrightarrow \tilde{S}_{I \sqcup I'}$. For $I \sqcup I' \subset J \sqcup J'$, denote by $p_{I \sqcup I', J \sqcup J'} : \tilde{S}_{J \sqcup J'} \rightarrow \tilde{S}_{I \sqcup I'}$ the projection. We then have a natural transfer map

$$T_S^{L/L'} : \text{PSh}_{\mathcal{D}fil}(S/(S_I)) \rightarrow \text{PSh}_{\mathcal{D}fil}(S/(S_{I'})),$$

$$((M_I, F), s_{IJ}) \mapsto (\text{ho} \lim_{I \in L} p_{I'(I \sqcup I')}^* (p_{I(I \sqcup I')}^{*mod} (M_I, F)) / \mathcal{I}_{S_{I \sqcup I'}}, s_{I'J'}),$$

with, in the homotopy limit, the natural transition morphisms

$$p_{I'(I \sqcup I')}^* \text{ad}(p_{IJ}^{*mod}, p_{IJ*}) (p_{I(I \sqcup I')}^{*mod[-1]} (M_I, F)) :$$

$$p_{I'(J \sqcup J')}^* (p_{J(J \sqcup J')}^{*mod[-1]} (M_J, F)) / \mathcal{I}_{S_{J \sqcup J'}} \rightarrow p_{I'(I \sqcup I')}^* (p_{I(I \sqcup I')}^{*mod[-1]} (M_I, F)) / \mathcal{I}_{S_{I \sqcup I'}}$$

for $J \subset I$, and

$$s_{I'J'} : \text{holim}_{I \in L} m^* p_{I'(I \sqcup I')}^* (p_{I(I \sqcup I')}^{*mod[-1]} (M_I, F) / \mathcal{I}_{S_{I \sqcup I'}}) \rightarrow$$

$$\text{holim}_{I \in L} p_{I'J'}^* (p_{I'J'}^{*mod[-1]} m^* p_{I'(I \sqcup I')}^* p_{I(I \sqcup I')}^{*mod[-1]} ((M_I, F) / \mathcal{I}_{S_{I \sqcup I'}})) / \mathcal{I}_{S_{J'}} \rightarrow$$

$$\text{holim}_{I \in L} p_{I'J'}^* p_{J'(I \sqcup J')}^* (p_{I(I \sqcup J')}^{*mod[-1]} (M_I, F) / \mathcal{I}_{S_{I \sqcup J'}})$$

Definition-Proposition 18. *Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then $\text{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ does not depend on the open covering of S and the closed embeddings and we set*

$$\text{PSh}_{\mathcal{D}(2)fil}(S) := \text{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

We denote by $C_{\mathcal{D}(2)fil}^0(S) := C(\text{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$ and by $D_{\mathcal{D}(2)fil,r}^0(S) := K_{\mathcal{D}(2)fil,r}^0(S)([E_1]^{-1})$ the localization of the r -filtered homotopy category with respect to the classes of filtered Zariski, resp. usu, local equivalences.

Proof. It is obvious that $T_S^{L/L'} : \text{PSh}_{\mathcal{D}fil}(S/(S_I)) \rightarrow \text{PSh}_{\mathcal{D}}(S/(S_{I'}))$ is an equivalence of category with inverse $T_S^{L'/L} : \text{PSh}_{\mathcal{D}fil}(S/(S_{I'})) \rightarrow \text{PSh}_{\mathcal{D}}(S/(S_I))$. \square

We now give the definition of our category :

Definition 82. *Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil}((\tilde{S}_I))$ is the full subcategory*

- whose objects are $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with
 - $(M_I, F) \in C_{\mathcal{D}(2)fil, S_I}(\tilde{S}_I)$ (see definition 62),

– $u_{IJ} : m^*(M_I, F) \rightarrow m^*p_{IJ*}(M_J, F)[d_{\tilde{S}_I} - d_{\tilde{S}_J}]$ for $J \subset I$, are morphisms, $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ} * u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{\mathcal{D}fil}(\tilde{S}_I)$;

- the morphisms $m : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ between $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$ and $(N, F) = ((N_I, F)_{I \subset [1, \dots, l]}, v_{IJ})$ being a family of morphisms of complexes,

$$m = (m_I : (M_I, F) \rightarrow (N_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{\mathcal{D}fil}(\tilde{S}_I)$.

We denote by $C_{\mathcal{D}(2)fil}^{\sim}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ the full subcategory consisting of objects $((M_I, F), u_{IJ})$ such that the u_{IJ} are ∞ -filtered Zariski, resp. usu, local equivalences.

Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, We denote by

$$C_{\mathcal{D}(2)fil, rh}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil, h}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil, c}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil}^{\sim}(S/(\tilde{S}_I))$$

the full subcategories consisting of those $((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)fil}^{\sim}(S/(\tilde{S}_I))$ such that $(M_I, F) \in C_{\mathcal{D}(2)fil, S_I, c}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I, F)$ are filtered coherent for all $n \in \mathbb{Z}$ and all $I \subset [1, \dots, l]$ (i.e. $a_\tau H^n(M_I)$ are coherent sheaves of $D_{\tilde{S}_I}$ modules and F induces a good filtration on $a_\tau H^n(M_I)$), resp. such that $(M_I, F) \in C_{\mathcal{D}(2)fil, S_I, h}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I, F)$ are filtered holonomic for all $n \in \mathbb{Z}$ and all $I \subset [1, \dots, l]$ (i.e. $a_\tau H^n(M_I)$ are holonomic sheaves of $D_{\tilde{S}_I}$ modules and F induces a good filtration on $a_\tau H^n(M_I)$), resp. such that $(M_I, F) \in C_{\mathcal{D}(2)fil, S_I, rh}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I, F)$ are filtered regular holonomic for all $n \in \mathbb{Z}$ and all $I \subset [1, \dots, l]$ (i.e. $a_\tau H^n(M_I)$ are regular holonomic sheaves of $D_{\tilde{S}_I}$ modules and F induces a good filtration on $a_\tau H^n(M_I)$).

We denote by

$$C_{\mathcal{D}(1,0)fil, h}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}2fil, h}(S/(\tilde{S}_I)), \quad C_{\mathcal{D}(1,0)fil, rh}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}2fil, rh}(S/(\tilde{S}_I)), \\ C_{\mathcal{D}(1,0)fil}^{\sim}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}2fil}^{\sim}(S/(\tilde{S}_I)),$$

the full subcategories consisting of those $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}2fil}^{\sim}(S/(\tilde{S}_I))$ such that $W^p M_I$ are $D_{\tilde{S}_I}$ submodules (resp. and $a_\tau H^n(M_I, F)$ are filtered holonomic).

We recall from section 2 the following

- A morphism $m = (m_I) : ((M_I), u_{IJ}) \rightarrow ((N_I), v_{IJ})$ in $C_{\mathcal{D}}(S/(\tilde{S}_I))$ is a Zariski, resp. usu, local equivalence if and only if all the m_I are Zariski, resp. usu, local equivalences.
- A morphism $m = (m_I) : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ in $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ is a filtered Zariski, resp. usu, local equivalence if and only if all the m_I are filtered Zariski, resp. usu, local equivalence.
- Let $r \in \mathbb{N}$. By definition, a morphism $m = (m_I) : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ in $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ is an r -filtered Zariski, resp. usu, local equivalence if there exist $m_i : (C_{iI}, F), u_{iIJ}) \rightarrow (C_{(i+1)I}, F), u_{(i+1)IJ})$, $0 \leq i \leq s$, with $(C_{iI}, F), u_{iIJ}) \in C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$, $(C_{0I}, F), u_{0IJ}) = (M_I, F), u_{IJ})$, $(C_{sI}, F), u_{sIJ}) = (N_I, F), v_{IJ})$ such that

$$m = m_s \circ \dots \circ m_i \circ \dots \circ m_0 : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$$

with $m_i : (C_{iI}, F), u_{iIJ}) \rightarrow (C_{(i+1)I}, F), u_{(i+1)IJ})$ either filtered Zariski, resp. usu, local equivalence or r -filtered homotopy equivalence (i.e. r -filtered for the first filtration and filtered for the second filtration).

In the analytic case, we also define in the same way :

Definition 83. Let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $C_{\mathcal{D}^\infty(2)\text{fil}}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^\infty(2)\text{fil}}(\tilde{S}_I)$ is the full subcategory

- whose objects are $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with
 - $(M_I, F) \in C_{\mathcal{D}^\infty\text{fil}, S_I}(\tilde{S}_I)$ (see definition 63),
 - $u_{IJ} : m^*(M_I, F) \rightarrow m^*p_{IJ*}(M_I, F)[d_{\tilde{S}_I} - d_{\tilde{S}_J}]$, for $J \subset I$, are morphisms, $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{\mathcal{D}^\infty\text{fil}}(\tilde{S}_I)$;
- the morphisms $m : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ between $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$ and $(N, F) = ((N_I, F)_{I \subset [1, \dots, l]}, v_{IJ})$ being a family of morphisms of complexes,

$$m = (m_I : (M_I, F) \rightarrow (N_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{\mathcal{D}^\infty\text{fil}}(\tilde{S}_I)$.

We denote by $C_{\mathcal{D}^\infty(2)\text{fil}}^\sim(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^\infty(2)\text{fil}}(S/(\tilde{S}_I))$ the full subcategory consisting of objects $((M_I, F), u_{IJ})$ such that the u_{IJ} are ∞ -filtered usu local equivalence.

Let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We denote by

$$C_{\mathcal{D}^\infty(2)\text{fil}, h}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^\infty(2)\text{fil}, c}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^\infty(2)\text{fil}}^\sim(S/(\tilde{S}_I))$$

the full subcategories consisting of $((M_I, F), u_{IJ}) \in C_{\mathcal{D}^\infty(2)\text{fil}}^\sim(S/(\tilde{S}_I))$ such that $(M_I, F) \in C_{\mathcal{D}^\infty(2)\text{fil}, S_I, c}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I, F)$ are filtered coherent for all $n \in \mathbb{Z}$ and all $I \subset [1, \dots, l]$, resp. such that $(M_I, F) \in C_{\mathcal{D}^\infty(2)\text{fil}, S_I, h}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I, F)$ are filtered holonomic for all $n \in \mathbb{Z}$ and all $I \subset [1, \dots, l]$. We denote by

$$C_{\mathcal{D}^\infty(1,0)\text{fil}}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^\infty 2\text{fil}}^\sim(S/(\tilde{S}_I)), \quad C_{\mathcal{D}^\infty(1,0)\text{fil}, h}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^\infty 2\text{fil}, h}^\sim(S/(\tilde{S}_I))$$

the full subcategories consisting of those $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}^\infty 2\text{fil}}^\sim(S/(\tilde{S}_I))$ such that $W^p M_I$ are $D_{\tilde{S}_I}$ submodules (resp. and $a_\tau H^n(M_I, F)$ filtered holonomic).

We recall from section 2 the following

- A morphism $m = (m_I) : ((M_I), u_{IJ}) \rightarrow ((N_I), v_{IJ})$ in $C_{\mathcal{D}^\infty}(S/(\tilde{S}_I))$ is a usu local equivalence if and only if all the m_I are usu local equivalences.
- A morphism $m = (m_I) : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ in $C_{\mathcal{D}^\infty(2)\text{fil}}(S/(\tilde{S}_I))$ is a filtered usu local equivalence if and only if all the m_I are filtered usu local equivalence.
- Let $r \in \mathbb{N}$. By definition, a morphism $m = (m_I) : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ in $C_{\mathcal{D}^\infty(2)\text{fil}}(S/(\tilde{S}_I))$ is an r -filtered usu local equivalence if there exist $m_i : (C_{iI}, F), u_{iIJ} \rightarrow (C_{(i+1)I}, F), u_{(i+1)IJ}$, $0 \leq i \leq s$, with $(C_{iI}, F), u_{iIJ} \in C_{\mathcal{D}^\infty(2)\text{fil}}(S/(\tilde{S}_I))$, $(C_{0I}, F), u_{0IJ} = (M_I, F), u_{IJ}$, $(C_{sI}, F), u_{sIJ} = (N_I, F), v_{IJ}$ such that

$$m = m_s \circ \dots \circ m_i \circ \dots \circ m_0 : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$$

with $m_i : (C_{iI}, F), u_{iIJ} \rightarrow (C_{(i+1)I}, F), u_{(i+1)IJ}$ either filtered usu local equivalence or r -filtered homotopy equivalence (i.e. r -filtered for the first filtration and filtered for the second filtration).

Definition 84. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$.

(i) We have, for $r = 1, \dots, \infty$, (see section 2.1) the ∞ -filtered homotopy category

$$K_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I)) := \text{Ho}_r(C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$$

whose objects are those of $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ and whose morphism are r -filtered homotopy classes of morphisms (r -filtered for the first filtration and filtered for the second), and

$$D_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I)) := K_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I))([E_1]^{-1})$$

the localization with respect to the classes of filtered Zariski, resp. usu, local equivalences, together with the localization functor

$$D(\text{top}) : C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \rightarrow K_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I)).$$

Note that the classes of filtered τ local equivalence constitute a right multiplicative system. By definition if $m : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ in $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ is an r -filtered Zariski, resp. usu, local equivalence, then $m = D(\top)(m) : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ is an isomorphism in $D_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I))$

(ii) We have

$$D_{\mathcal{D}(1,0)fil,\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}2fil,\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}2fil,\infty}(S/(\tilde{S}_I))$$

the full subcategories which are the image of $C_{\mathcal{D}2fil,h}(S/(\tilde{S}_I))$, resp. of $C_{\mathcal{D}(1,0)fil,h}(S/(\tilde{S}_I))$, by the localization functor $D(\text{top}) : C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I))$.

In the analytic case, we also have

Definition 85. Let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$.

(i) We have, for $r = 1, \dots, \infty$, (see section 2.1) the ∞ -filtered homotopy category

$$K_{\mathcal{D}^\infty(2)fil,r}(S/(\tilde{S}_I)) := \text{Ho}_r(C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)))$$

whose objects are those of $C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I))$ and whose morphism are r -filtered homotopy classes of morphisms (r -filtered for the first filtration and filtered for the second), and

$$D_{\mathcal{D}^\infty(2)fil,r}(S/(\tilde{S}_I)) := K_{\mathcal{D}^\infty(2)fil,r}(S/(\tilde{S}_I))([E_1]^{-1})$$

the localization with respect to the classes of filtered usu local equivalences, together with the localization functor

$$D(\text{usu}) : C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)) \rightarrow K_{\mathcal{D}^\infty(2)fil,r}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}^\infty(2)fil,r}(S/(\tilde{S}_I)).$$

Note that the classes of filtered usu local equivalence constitute a right multiplicative system. By definition if $m : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ in $C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I))$ is an r -filtered Zariski, resp. usu, local equivalence, then $m = D(\top)(m) : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ is an isomorphism in $D_{\mathcal{D}^\infty(2)fil,r}(S/(\tilde{S}_I))$

(ii) We have then

$$D_{\mathcal{D}^\infty(1,0)fil,\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}^\infty 2fil,\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}^\infty 2fil,\infty}(S/(\tilde{S}_I))$$

the full subcategories wich are the image of $C_{\mathcal{D}^\infty 2fil,h}(S/(\tilde{S}_I))$, resp. $C_{\mathcal{D}^\infty(1,0)fil,h}(S/(\tilde{S}_I))$, by the localization functor $D(\text{usu}) : C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}^\infty(2)fil,\infty}(S/(\tilde{S}_I))$.

Definition 86. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$.

(i) We denote by

$$C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))^0 \subset C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$$

the full subcategory consisting of $((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$ such that

$$H^n((M_I, F), u_{IJ}) = (H^n(M_I, F), H^n u_{IJ}) \in \text{PSh}_{\mathcal{D}(2)\text{fil}}^0(S/(\tilde{S}_I))$$

that is such that the $H^n u_{IJ}$ are isomorphism. We denote by $D_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))^0 := D(\text{top})(C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))^0)$ its image by the localization functor.

(ii) We have the full embedding functor

$$\begin{aligned} \iota_{S/(\tilde{S}_I)}^0 : C_{\mathcal{D}(2)\text{fil}}^0(S) &:= C_{\mathcal{D}(2)\text{fil}}^0(S/(\tilde{S}_I)) \hookrightarrow C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I)), \\ &((M_I, F), s_{IJ}) \mapsto ((M_I, F), s_{IJ}) \end{aligned}$$

By definition, $\iota_{S/(\tilde{S}_I)}^0(C_{\mathcal{D}(1,0)\text{fil}}^0(S/(\tilde{S}_I))) \subset C_{\mathcal{D}(1,0)\text{fil}}^{\sim}(S/(\tilde{S}_I))$. This full embedding induces in the derived category the functor

$$\begin{aligned} \iota_{S/(\tilde{S}_I)}^0 : D_{\mathcal{D}(2)\text{fil},\infty}^0(S) &:= D_{\mathcal{D}(2)\text{fil},\infty}^0(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}(2)\text{fil},\infty}(S/(\tilde{S}_I)), \\ &((M_I, F), s_{IJ}) \mapsto ((M_I, F), s_{IJ}). \end{aligned}$$

Proposition 88. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then,

$$\iota_{S/(\tilde{S}_I)}^0 : D_{\mathcal{D}(2)\text{fil},\infty}^0(S) \rightarrow D_{\mathcal{D}(2)\text{fil},\infty}(S/(\tilde{S}_I))$$

is a full embedding whose image is $D_{\mathcal{D}(2)\text{fil},\infty}(S/(\tilde{S}_I))^0$, that is consists of $((M_I, F), s_{IJ}) \in C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$ such that

$$H^n((M_I, F), s_{IJ}) := (H^n(M_I, F), H^n(s_{IJ})) \in \text{PSh}_{\mathcal{D}}^0(S/(\tilde{S}_I)).$$

and

$$\iota_S^0 := \iota_{S/(\tilde{S}_I)}^0 : D_{\mathcal{D}(2)\text{fil},\infty}^0(S) \xrightarrow{\sim} D_{\mathcal{D}(2)\text{fil},\infty}(S/(\tilde{S}_I))^0$$

the induced equivalence of categories.

Proof. Follows immediately from the fact that for $((M_I, F), s_{IJ}), ((N_I, F), r_{IJ}) \in C_{\mathcal{D}(2)\text{fil}}^0(S)$

$$\begin{aligned} \text{Hom}_{D_{\mathcal{D}(2)\text{fil},\infty}^0(S)}(((M_I, F), s_{IJ}), ((N_I, F), r_{IJ})) &= \text{Hom}_{K_{\mathcal{D}(2)\text{fil},\infty}(S/(\tilde{S}_I))}(L(M_I, F), s_{IJ}^q, (E(N_I, F), r_{IJ}^k)) \\ &= \text{Hom}_{D_{\mathcal{D}(2)\text{fil},\infty}(S/(\tilde{S}_I))}(((M_I, F), s_{IJ}), ((N_I, F), r_{IJ})) \end{aligned}$$

□

We finish this subsection by the statement a result of kashiwara in the singular case.

Definition 87. Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We will consider the functor

$$\begin{aligned} J_S : C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I)) &\rightarrow C_{\mathcal{D}^\infty(2)\text{fil}}(S/(\tilde{S}_I)), \\ ((M_I, F), u_{IJ}) &\mapsto J_S((M_I, F), u_{IJ}) := (J_{\tilde{S}_I}(M_I, F), J(u_{IJ})) := ((M_I \otimes_{D_S} D_S^\infty, F), J(u_{IJ})) \end{aligned}$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$,

$$J(u_{IJ}) : J(M_I, F) \xrightarrow{J(u_{IJ})} J(p_{IJ*}(M_J, F)[d_{IJ}]) \xrightarrow{T_*(p_{IJ}, J)(-)} p_{IJ*}J(M_J, F)[d_{IJ}].$$

Of course $J_S(C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))) \subset C_{\mathcal{D}^\infty(1,0)fil}(S/(\tilde{S}_I))$.

Proposition 89. *Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then the functor*

$$J_S : C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)),$$

satisfy $J_S : C_{\mathcal{D}(2)fil}^\sim(S/(\tilde{S}_I)) \subset C_{\mathcal{D}^\infty(2)fil}^\sim(S/(\tilde{S}_I))$ and induces an equivalence of category

$$J_S : D_{\mathcal{D}(2)fil, \infty, rh}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}^\infty(2)fil, \infty, h}(S/(\tilde{S}_I)).$$

and $J_S(D_{\mathcal{D}(1,0)fil, \infty, rh}(S/(\tilde{S}_I))) \subset D_{\mathcal{D}^\infty, h(1,0)fil, \infty}(S/(\tilde{S}_I))$.

Proof. Follows immediately from the smooth case (proposition 51). □

4.3.2 Duality in the singular case

The definition of Saito's category comes with a dual functor :

Definition 88. *Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup S_i$ an open cover such that there exist closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have the dual functor :*

$$\mathbb{D}_S^K : C_{\mathcal{D}fil}^0(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}fil}^0(S/(\tilde{S}_I)), ((M_I, F), s_{IJ}) \mapsto (\mathbb{D}_{\tilde{S}_I}^K(M_I, F), s_{IJ}^d),$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$,

$$u_{IJ}^q : \mathbb{D}_{\tilde{S}_I}^K(M_I, F) \xrightarrow{\mathbb{D}^K(s_{IJ}^{-1})} \mathbb{D}_{\tilde{S}_I}^K p_{IJ*}(M_J, F)[d_{IJ}] \xrightarrow{T_*(p_{IJ}, D)(-)} p_{IJ*} \mathbb{D}_{\tilde{S}_J}^K(M_J, F)[d_{IJ}]$$

It induces in the derived category the functor

$$L\mathbb{D}_S^K : D_{\mathcal{D}fil}^0(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}fil}^0(S/(\tilde{S}_I)), ((M_I, F), s_{IJ}) \mapsto \mathbb{D}_S^K Q((M_I, F), s_{IJ}),$$

with $q : Q((M_I, F), s_{IJ}) \rightarrow ((M_I, F), s_{IJ})$ a projective resolution.

In the analytic case we also define

Definition 89. *Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup S_i$ an open cover such that there exist closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have the dual functor :*

$$\mathbb{D}_S^{K, \infty} : C_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)), ((M_I, F), u_{IJ}) \mapsto (\mathbb{D}_{\tilde{S}_I}^{K, \infty}(M_I, F), u_{IJ}^d),$$

with u_{IJ}^d defined similarly as in definition 88. It induces in the derived category the functor

$$L\mathbb{D}_S^{K, \infty} : D_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)), ((M_I, F), u_{IJ}) \mapsto (\mathbb{D}_S^{K, \infty} Q((M_I, F), u_{IJ}^d),$$

with $q : Q((M_I, F), s_{IJ}) \rightarrow ((M_I, F), s_{IJ})$ a projective resolution.

4.3.3 Inverse image in the singular case

We give in this subsection the inverse image functors between our categories.

Let $n : S^o \hookrightarrow S$ be an open embedding with $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $n : S^o \hookrightarrow S$ be an open embedding with $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Denote $S_I^o := n^{-1}(S_I) = S_I \cap S^o$ and $n_I := n|_{S_I^o} : S_I^o \hookrightarrow S^o$ the open embeddings. Consider open embeddings $\tilde{n}_I : \tilde{S}_I^o \hookrightarrow \tilde{S}_I$ such that $\tilde{S}_I^o \cap S_I = S_I^o$, that is which are lift of n_I . We have the functor

$$n^* : C_{\mathcal{D}fil}(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}fil}(S^o/(\tilde{S}_I^o)),$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto n^*(M, F) := (\tilde{n}_I)^*(M, F) := (\tilde{n}_I^*(M_I, F), n^* u_{IJ})$$

which derive trivially.

Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{Var}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSp}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} : Y \times \tilde{S}_J \rightarrow Y \times \tilde{S}_I$ the projections and by

$$E_{IJ} = \begin{array}{ccc} \tilde{S}_J \setminus S_J & \xrightarrow{m_J} & \tilde{S}_J \\ \downarrow p_{IJ} & & \downarrow p_{IJ} \\ \tilde{S}_I \setminus (S_I \setminus S_J) & \xrightarrow{m_{IJ}} & \tilde{S}_I \end{array}, \quad E'_{IJ} = \begin{array}{ccc} Y \times \tilde{S}_J \setminus X_J & \xrightarrow{m'_J} & Y \times \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p'_{IJ} \\ Y \times \tilde{S}_I \setminus (X_I \setminus X_J) & \xrightarrow{m'_{IJ}} & Y \times \tilde{S}_I \end{array}, \quad E_{fIJ} = \begin{array}{ccc} \tilde{X}_J & \xrightarrow{\tilde{f}_J} & \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p_{IJ} \\ Y \times \tilde{S}_I & \xrightarrow{\tilde{f}_I} & \tilde{S}_I \end{array}$$

the commutative diagrams. The (graph) inverse image functors is :

$$f^{*mod[-], \Gamma} : C_{\mathcal{D}fil}(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}fil}(X/(Y \times \tilde{S}_I)),$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto f^{*mod[-], \Gamma}(M, F) := (\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)), \tilde{f}_J^{*mod[-]} u_{IJ})$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_I} - d_{\tilde{S}_J}$,

$$\begin{aligned} \tilde{f}_J^{*mod[-]} u_{IJ} : \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)) &\xrightarrow{\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(u_{IJ}))} \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]} p_{IJ*}(M_J, F)[d_{IJ}]) \\ &\xrightarrow{\Gamma_{X_I} E(T(p_{IJ}^{*mod}, p_{\tilde{S}_I})(-)^{-1})[d_Y + d_{IJ}]} \Gamma_{X_I} E(p'_{IJ*} p_{\tilde{S}_J}^{*mod}(M_J, F)[d_Y + d_{IJ}]) \\ &\xrightarrow{\cong} p'_{IJ*} \Gamma_{X_J} E(p_{\tilde{S}_J}^{*mod[-]}(M_J, F))[d_{IJ}]. \end{aligned}$$

It induces in the derived categories the functor

$$Rf^{*mod[-], \Gamma} : D_{\mathcal{D}(2)fil, \infty}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}(2)fil, \infty}(X/(Y \times \tilde{S}_I)),$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto Rf^{*mod[-], \Gamma}(M, F) := (\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)), \tilde{f}_J^{*mod[-]} u_{IJ}).$$

It gives by duality the functor

$$Lf^{\hat{*}mod[-], \Gamma} : D_{\mathcal{D}(2)fil, \infty}(S/(\tilde{S}_I))^0 \rightarrow D_{\mathcal{D}(2)fil, \infty}(X/(Y \times \tilde{S}_I))^0,$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto Lf^{\hat{*}mod[-], \Gamma}(M, F) := L\mathbb{D}_S^K Rf^{*mod[-], \Gamma} L\mathbb{D}_S^K \iota_S^{0, -1}(M, F).$$

where $\iota_S^0 : D_{\mathcal{D}(2)fil, \infty}^0(S/(\tilde{S}_I)) \xrightarrow{\cong} D_{\mathcal{D}(2)fil, \infty}(S/(\tilde{S}_I))^0$ is the isomorphism of definition 86.

Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSp}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We have also the functors,

$$f^{*mod[-], \Gamma} : C_{\mathcal{D}^\infty fil}(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}^\infty fil}(X/(Y \times \tilde{S}_I)),$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto f^{*mod[-], \Gamma}(M, F) := (\Gamma_{X_I}(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)), \tilde{f}_J^{*mod[-]} u_{IJ})$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$,

$$\begin{aligned} \tilde{f}_J^{*mod[-]} u_{IJ} : \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)) &\xrightarrow{\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]})(u_{IJ})} \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]} p_{IJ*}(M_J, F)[d_{IJ}]) \\ &\xrightarrow{\Gamma_{X_I} E(T(p_{IJ}^{*mod}, p_{\tilde{S}_I})(-)^{-1})[d_Y + d_{IJ}]} \Gamma_{X_I} E(p'_{IJ*} p_{\tilde{S}_J}^{*mod}(M_J, F)[d_Y + d_{IJ}]) \\ &\xrightarrow{=} p'_{IJ*} \Gamma_{X_J} E(p_{\tilde{S}_J}^{*mod[-]}(M_J, F))[d_{IJ}]. \end{aligned}$$

It induces in the derived categories, the functor

$$Rf^{*mod[-], \Gamma} : D_{\mathcal{D}^\infty(2)fil, \infty}(S/(\tilde{S}_I)) \rightarrow D_{\mathcal{D}^\infty(2)fil, \infty}(X/(Y \times \tilde{S}_I)),$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto Rf^{*mod[-], \Gamma}(M, F) := (\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)), \tilde{f}_J^{*mod[-]} u_{IJ}).$$

It gives by duality the functor

$$Lf^{\hat{*}mod[-], \Gamma} : D_{\mathcal{D}^\infty(2)fil, \infty}(S/(\tilde{S}_I))^0 \rightarrow D_{\mathcal{D}^\infty(2)fil, \infty}(X/(Y \times \tilde{S}_I))^0,$$

$$(M, F) = ((M_I, F), u_{IJ}) \mapsto Lf^{\hat{*}mod[-], \Gamma}(M, F) := L\mathbb{D}^{K, \infty} Rf^{*mod[-], \Gamma} L\mathbb{D}^{K, \infty} \iota_S^{0, -1}(M, F).$$

where $\iota_S^0 : D_{\mathcal{D}^\infty(2)fil, \infty}^0(S/(\tilde{S}_I)) \xrightarrow{\sim} D_{\mathcal{D}^\infty(2)fil, \infty}(S/(\tilde{S}_I))^0$ is the isomorphism of definition 86.

The following proposition are then easy :

Proposition 90. *Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{Var}(\mathbb{C})$. Assume there exist factorizations $f_1 : X \xrightarrow{l_1} Y' \times Y \xrightarrow{p_Y} Y$ and $f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{p_S} S$ with $Y', Y'' \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_Y the projections. We have then the factorization*

$$f_2 \circ f_1 : X \xrightarrow{(l_2 \circ I_{Y'}) \circ l_1} Y' \times Y'' \times S \xrightarrow{p_S} S.$$

We have, for $(M, F) \in C_{\mathcal{D}(2)fil}^\sim(S/(\tilde{S}_I))$, $R(f_2 \circ f_1)^{*mod[-], \Gamma}(M, F) = Rf_2^{*mod[-], \Gamma} \circ Rf_1^{*mod[-], \Gamma}(M, F)$.

Proof. Follows from the the fact that for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)fil}^\sim(S/(\tilde{S}_I))$,

$$\begin{aligned} (\Gamma_{X_I} E(\tilde{f}_{1I}^{*mod[-]} \Gamma_{Y_I} E(\tilde{f}_{2I}^{*mod[-]}(M_I, F))), \tilde{f}_{1J}^{*mod[-]}(\tilde{f}_{2J}^{*mod[-]} u_{IJ})) &\xrightarrow{=} \\ (\Gamma_{X_I} E((\tilde{f}_{1I} \circ \tilde{f}_{2I})^{*mod[-]}(M_I, F)), (\tilde{f}_{1J} \circ \tilde{f}_{2J})^{*mod[-]} u_{IJ}^q) & \end{aligned}$$

by proposition 53(i) and the fact that $X_I \subset \tilde{f}_{1I}^{-1}(Y_I)$. □

Proposition 91. *Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{Var}(\mathbb{C})$. Assume there exist factorizations $f_1 : X \xrightarrow{l_1} Y' \times Y \xrightarrow{p_Y} Y$ and $f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{p_S} S$ with $Y', Y'' \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_Y the projections. We have then the factorization*

$$f_2 \circ f_1 : X \xrightarrow{(l_2 \circ I_{Y'}) \circ l_1} Y' \times Y'' \times S \xrightarrow{p_S} S.$$

We have, for $(M, F) \in C_{\mathcal{D}(2)fil}^\sim(S/(\tilde{S}_I))$ or $(M, F) \in C_{\mathcal{D}^\infty(2)fil}^\sim(S/(\tilde{S}_I))$, $R(f_2 \circ f_1)^{*mod[-], \Gamma}(M, F) = Rf_2^{*mod[-], \Gamma} \circ Rf_1^{*mod[-], \Gamma}(M, F)$.

Proof. Similar to the proof of proposition 90. □

4.3.4 Direct image functor in the singular case

We define the direct image functors between our category.

Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{Var}(\mathbb{C})$, and assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S a the projection ; or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSp}(\mathbb{C})$, and assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S a the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; resp. let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have, for $I \subset [1, \dots, l]$, closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and the following commutative diagrams which are cartesian (we take $Y = \mathbb{P}^{N.o}$ in the algebraic case)

$$\begin{array}{ccccc} f|_{X_I} = f|_{X_I} : X_I & \xrightarrow{l_I} & Y \times S_I & \xrightarrow{p_{S_I}} & S_I & , & Y \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \\ & \searrow & \downarrow i'_I & & \downarrow i_I & & \downarrow p'_{IJ} & & \downarrow p_{IJ} \\ & & Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I & & Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \end{array}$$

with $l_I : l|_{X_I}$, $i'_I = I \times i_I$, p_{S_I} and $p_{\tilde{S}_I}$ are the projections and $p'_{IJ} = I \times p_{IJ}$. Then $\tilde{f}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $f_I = f|_{X_I}$. We define the direct image functor on our category by

$$f_{*mod}^{FDR} : C_{\mathcal{D}(2)fil}(X/(Y \times \tilde{S}_I)) \rightarrow C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)),$$

$$((M_I, F), u_{IJ}) \mapsto (\tilde{f}_{I*mod}^{FDR}(M_I, F), f^k(u_{IJ})) := (p_{\tilde{S}_I*} E((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F)[d_Y]), f^k(u_{IJ}))$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$,

$$\begin{aligned} & f^k(u_{IJ})[d_Y] : p_{\tilde{S}_I*} E((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F)) \\ & \xrightarrow{p_{\tilde{S}_I*} E(DR(Y \times \tilde{S}_I/\tilde{S}_I)(u_{IJ}))} p_{\tilde{S}_I*} E((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_I}} p'_{IJ*}(M_J, F)[d_{IJ}]) \\ & \xrightarrow{T_w^O(p_{IJ, \otimes})(M_I, F)} p_{\tilde{S}_I*} E(p_{IJ*}(\Omega_{Y \times \tilde{S}_J/\tilde{S}_J}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_J}} (M_J, F)[d_{IJ}]) \\ & \xrightarrow{=} p_{\tilde{S}_J*} E((\Omega_{Y \times \tilde{S}_J/\tilde{S}_J}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_J}} (M_J, F)[d_{IJ}]). \end{aligned}$$

It induces in the derived categories the functor

$$\int_f^{FDR} : D_{\mathcal{D}(2)fil, \infty}(X) \rightarrow D_{\mathcal{D}(2)fil, \infty}(S), ((M_I, F), u_{IJ}) \mapsto (\tilde{f}_{I*mod}^{FDR}(M_I, F), f^k(u_{IJ}))$$

Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSp}(\mathbb{C})$, and assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S a the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We also have the functors

$$f_{*mod}^{FDR} : C_{\mathcal{D}^\infty(2)fil}(X/(Y \times \tilde{S}_I)) \rightarrow C_{\mathcal{D}^\infty(2)fil}(S/(\tilde{S}_I)),$$

$$((M_I, F), u_{IJ}) \mapsto (\tilde{f}_{I*mod}^{FDR}(M_I, F), f^k(u_{IJ})) := (p_{\tilde{S}_I*} E((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F)[d_Y]), f^k(u_{IJ}))$$

where $f^k(u_{IJ})[d_Y]$ is given as above,

$$\int_f^{FDR} : D_{\mathcal{D}^\infty(2)fil, \infty}(X) \rightarrow D_{\mathcal{D}^\infty(2)fil, \infty}(S),$$

$$((M_I, F), u_{IJ}) \mapsto (\tilde{f}_{I*mod}^{FDR}(M_I, F), f^k(u_{IJ})) := (p_{\tilde{S}_I*} E((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F)[d_Y]), f^k(u_{IJ}))$$

where $f^k(u_{IJ})[d_Y]$ is given as above.

In the algebraic case, we have the followings:

Proposition 92. *Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{QPVar}(\mathbb{C})$ quasi-projective. Then there exist factorizations $f_1 : X \xrightarrow{l_1} Y' \times Y \xrightarrow{p_Y} Y$ and $f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{p_S} S$ with $Y' = \mathbb{P}^{N, \circ} \subset \mathbb{P}^N, Y'' = \mathbb{P}^{N', \circ} \subset \mathbb{P}^{N'}$ open subsets, l_1, l_2 closed embeddings and p_S, p_Y the projections. We have then the factorization $f_2 \circ f_1 : X \xrightarrow{(l_2 \circ l_{Y'}) \circ l_1} Y' \times Y'' \times S \xrightarrow{p_S} S$. Let $i : S \hookrightarrow \tilde{S}$ a closed embedding with $\tilde{S} = \mathbb{P}^{n, \circ} \subset \mathbb{P}^n$ an open subset.*

(i) *Let $(M, F) \in C_{\mathcal{D}(2)fil}(X/(Y' \times Y'' \times \tilde{S}))$. Then, we have $\int_{f_2 \circ f_1}^{FDR}(M, F) = \int_{f_2}^{FDR}(\int_{f_1}^{FDR}(M, F))$ in $D_{\mathcal{D}(2)fil, \infty}(S/(\tilde{S}_I))$.*

(ii) *Let $(M, F) \in C_{\mathcal{D}(2)fil, h}(X/(Y' \times Y'' \times \tilde{S}))$. Then, we have $\int_{(f_2 \circ f_1)!}^{FDR}(M, F) = \int_{f_2!}^{FDR}(\int_{f_1!}^{FDR}(M, F))$ in $D_{\mathcal{D}(2)fil, \infty, h}(S/(\tilde{S}_I))$.*

Proof. (i): By the smooth case : proposition 76, we have en isomorphism

$$\int_{f_2}^{FDR} \int_{f_1}^{FDR} (M, F) := \int_{p_{\tilde{S}}}^{FDR} \int_{p_{Y' \times \tilde{S}}}^{FDR} (M, F) \xrightarrow{\sim} \int_{p_{\tilde{S}}}^{FDR} (M, F) := \int_{(f_2 \circ f_1)}^{FDR} (M, F).$$

(ii): Follows from (i). □

In the analytic case, we have the followings:

Proposition 93. *Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{AnSp}(\mathbb{C})$ quasi-projective. Then there exist factorizations $f_1 : X \xrightarrow{l_1} Y' \times Y \xrightarrow{p_Y} Y$ and $f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{p_S} S$ with $Y' = \mathbb{P}^{N, \circ} \subset \mathbb{P}^N, Y'' = \mathbb{P}^{N', \circ} \subset \mathbb{P}^{N'}$ open subsets, l_1, l_2 closed embeddings and p_S, p_Y the projections. We have then the factorization $f_2 \circ f_1 : X \xrightarrow{(l_2 \circ l_{Y'}) \circ l_1} Y' \times Y'' \times S \xrightarrow{p_S} S$. Let $i : S \hookrightarrow \tilde{S}$ a closed embedding with $\tilde{S} = \mathbb{P}^{n, \circ} \subset \mathbb{P}^n$ an open subset.*

(i) *Let $(M, F) \in C_{\mathcal{D}^\infty(2)fil, h}(X/(Y' \times Y'' \times \tilde{S}))$. Then, we have $\int_{f_2 \circ f_1}^{FDR}(M, F) = \int_{f_2}^{FDR}(\int_{f_1}^{FDR}(M, F))$ in $D_{\mathcal{D}^\infty(2)fil, \infty}(S/(\tilde{S}_I))$.*

(ii) *Let $(M, F) \in C_{\mathcal{D}^\infty(2)fil, h}(X/(Y' \times Y'' \times \tilde{S}))$. Then, we have $\int_{(f_2 \circ f_1)!}^{FDR}(M, F) = \int_{f_2!}^{FDR}(\int_{f_1!}^{FDR}(M, F))$ in $D_{\mathcal{D}^\infty(2)fil, \infty}(S/(\tilde{S}_I))$.*

Proof. (i): By the smooth case : proposition 77, we have en isomorphism

$$\int_{f_2}^{FDR} \int_{f_1}^{FDR} (M, F) := \int_{p_{\tilde{S}}}^{FDR} \int_{p_{Y' \times \tilde{S}}}^{FDR} (M, F) \xrightarrow{\sim} \int_{p_{\tilde{S}}}^{FDR} (M, F) := \int_{(f_2 \circ f_1)}^{FDR} (M, F).$$

(ii): Follows from (i). □

4.3.5 Tensor product in the singular case

Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have the tensor product functors

$$\begin{aligned} & (-) \otimes_{\mathcal{O}_S}^{[-]} (-) : C_{\mathcal{D}fil}^2(S/(\tilde{S}_I)) \rightarrow C_{\mathcal{D}fil}(S/(\tilde{S}_I)), \\ & (((M_I, F), u_{IJ}), ((N_I, F), v_{IJ})) \mapsto ((M_I, F) \otimes_{\mathcal{O}_{\tilde{S}_I}} (N_I, F)[d_{\tilde{S}_I}], u_{IJ} \otimes v_{IJ}), \end{aligned}$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$ and $d_I := d_{\tilde{S}_I}$,

$$\begin{aligned} u_{IJ} \otimes v_{IJ} : (M_I, F) \otimes_{O_{\tilde{S}_I}} (N_I, F)[d_I] &\xrightarrow{T(p_{IJ}^{*mod}, p_{IJ})(-)[d_I]} p_{IJ*} p_{IJ}^{*mod}((M_I, F) \otimes_{O_{\tilde{S}_I}} (N_I, F)[d_I]) \\ &\xrightarrow{=} p_{IJ*}(p_{IJ}^{*mod}(M_I, F) \otimes_{O_{\tilde{S}_J}} p_{IJ}^{*mod}(N_I, F)[d_I]) \\ &\xrightarrow{I(p_{IJ}^{*mod}, p_{IJ})(-, -)(u_{IJ}) \otimes I(p_{IJ}^{*mod}, p_{IJ})(-, -)(v_{IJ})[d_I]} p_{IJ*}((M_J, F) \otimes_{O_{\tilde{S}_J}} (N_J, F)[d_J + d_{IJ}]). \end{aligned}$$

Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have the tensor product functors

$$\begin{aligned} (-) \otimes_{O_S}^{[-]} (-) : C_{\mathcal{D}^\infty \text{fil}}^2(S/(\tilde{S}_I)) &\rightarrow C_{\mathcal{D}^\infty \text{fil}}(S/(\tilde{S}_I)), \\ (((M_I, F), u_{IJ}), ((N_I, F), v_{IJ})) &\mapsto ((M_I, F) \otimes_{O_{\tilde{S}_I}} (N_I, F), u_{IJ} \otimes v_{IJ}), \end{aligned}$$

with $u_{IJ} \otimes v_{IJ}$ as above.

Proposition 94. *Let $S \in \text{Var}(\mathbb{C})$. Denote $\Delta_S : S \hookrightarrow S \times S$ the diagonal embedding. Let $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup S_i$ an open cover such that there exist closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have, for $((M_I, F), u_{IJ}), ((N_I, F), v_{IJ}) \in C_{\mathcal{D}^\infty \text{fil}}(S/(\tilde{S}_I))$,*

$$((M_I, F), u_{IJ}) \otimes_{O_{S_I}}^{[-]} ((N_I, F), v_{IJ}) = \Delta_S^{*mod}(((M_I, F), u_{IJ}).((N_I, F), v_{IJ}))$$

Proof. Follows from proposition 58. □

4.3.6 The 2 functors of D modules on the category of complex algebraic varieties and on the category of complex analytic spaces, and the transformation maps

Definition 90. *Consider a commutative diagram in $\text{Var}(\mathbb{C})$ which is cartesian :*

$$D = \begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} .$$

Assume there exist factorizations $f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$, with $Y_1, Y_2 \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_S the projections. Then, the above commutative diagram factors through

$$D = \begin{array}{ccccc} f' : X_T & \xrightarrow{l'_1} & Y_1 \times T & \xrightarrow{p_T} & T \\ \downarrow l'_2 & & \downarrow l''_2 = I \times l_2 & & \downarrow l_2 \\ f'' : X \times Y_2 & \xrightarrow{l'_1 = I \times l} & Y_1 \times Y_2 \times S & \xrightarrow{p_{Y_2 \times S}} & Y_2 \times S \\ \downarrow p_X & & \downarrow p_{Y_1 \times S} & & \downarrow p_S \\ f : X & \xrightarrow{l} & Y_1 \times S & \xrightarrow{p_S} & S \end{array} .$$

whose squares are cartesian. Let $S = \cup_i S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $X = \cup_i X_i$ and $T = \cup_i T_i$ with $X_i := f^{-1}(S_i)$ and $T_i := g^{-1}(S_i)$. Moreover, $f_i = f|_{X_i} : X_i \rightarrow S_i$ lift to $\tilde{f}_i := p_{\tilde{S}_i} : Y_1 \times \tilde{S}_i \rightarrow \tilde{S}_i$ and $g_i = g|_{T_i} : T_i \rightarrow S_i$ lift to

$\tilde{g}_i := p_{\tilde{S}_i} : Y_2 \times \tilde{S}_i \rightarrow \tilde{S}_i$. We then have the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc}
f' : X_{IT} & \xrightarrow{l'} & Y_1 \times T_I & \xrightarrow{p_T} & T_I \\
\downarrow l'_{2I} & & \downarrow l'_{2I} & & \downarrow l_{2I} \\
X_I \times Y_2 & \xrightarrow{l'_1 = l_1 \times l_1} & Y_1 \times Y_2 \times \tilde{S}_I & \xrightarrow{p_{Y_2 \times \tilde{S}_I}} & Y_2 \times \tilde{S}_I \\
\downarrow p_X & & \downarrow p_{Y_1 \times \tilde{S}_I} & & \downarrow \tilde{g}_I \\
X_I & \xrightarrow{i_I \circ l_I} & Y_1 \times \tilde{S}_I & \xrightarrow{\tilde{f}_I} & \tilde{S}_I
\end{array}$$

We then define, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)fil}(X/(Y_1 \times \tilde{S}_I))$, the following canonical transformation map in $D_{\mathcal{D}(2)fil, \infty}(T/(Y_2 \times \tilde{S}_I))$, using proposition 82,

$$\begin{aligned}
& T^{\mathcal{D}mod}(f, g)(M, F) : \\
Rg^{*mod, \Gamma} \int_f^{FDR} (M, F) & := (\Gamma_{T_I} E(\tilde{g}_I^{*mod} p_{\tilde{S}_I^*} E((\Omega_{Y_1 \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y_1 \times \tilde{S}_I}} (M_I, F))), \tilde{g}_J^{*mod} f^k(u_{IJ})) \\
& \xrightarrow{(T_\omega^O(p_{\tilde{S}_I}, \tilde{g}_I)(M_I, F))} \\
& (\Gamma_{T_I} E(p_{Y_2 \times \tilde{S}_I^*} E((\Omega_{Y_1 \times Y_2 \times \tilde{S}_I / Y_2 \times \tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y_1 \times Y_2 \times \tilde{S}_I}} p_{Y_1 \times \tilde{S}_I}^{*mod} (M_I, F))), f'^k(p_{Y_1 \times \tilde{S}_I}^{*mod} (u_{IJ}))) \\
& \xrightarrow{(T_\omega^O(\gamma, \otimes)(p_{Y_1 \times \tilde{S}_I}^{*mod} (M_I, F)))^{-1}} \\
& (p_{Y_2 \times \tilde{S}_I^*} E((\Omega_{Y_1 \times Y_2 \times \tilde{S}_I / Y_2 \times \tilde{S}_I}^\bullet, F_b) \otimes_{O_{Y_1 \times Y_2 \times \tilde{S}_I}} \Gamma_{Y_1 \times T_I} E(p_{Y_1 \times \tilde{S}_I}^{*mod} ((M_I, F))))), f'^k(\tilde{g}_J''^{*mod} (u_{IJ}^q))) \\
& =: \int_{f'}^{FDR} Rg'^{*mod, \Gamma}(M, F).
\end{aligned}$$

In the analytic case, we have

Definition 91. Consider a commutative diagram in $AnSp(\mathbb{C})$ which is cartesian :

$$D = (f, g) = \begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} .$$

Assume there exist factorizations $f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$, with $Y_1, Y_2 \in AnSm(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_S the projections.

(i) We have, for $(M, F) \in D_{\mathcal{D}(2)fil, \infty, h}(X/(Y_1 \times \tilde{S}_I))$, the following transformation map in $D_{\mathcal{D}(2)fil, \infty}(T/(Y_2 \times \tilde{S}_I))$

$$T^{\mathcal{D}mod}(f, g)((M, F)) : Rg^{*mod, \Gamma} \int_f^{FDR} (M, F) \rightarrow \int_{f'}^{FDR} Rg'^{*mod, \Gamma}(M, F)$$

define in the same way as in definition 90

(ii) For $(M, F) \in D_{\mathcal{D}^\infty(2)fil, \infty}(X/(Y_1 \times \tilde{S}_I))$, the following transformation map in $D_{\mathcal{D}^\infty(2)fil, \infty}(T/(Y_2 \times \tilde{S}_I))$

$$T^{\mathcal{D}mod}(f, g)((M, F)) : Rg^{*mod, \Gamma} \int_f^{FDR} (M, F) \rightarrow \int_{f'}^{FDR} Rg'^{*mod, \Gamma}(M, F)$$

is defined in the same way as in (ii) : see definition 90.

In the algebraic case, we have the following :

Proposition 95. Consider a commutative diagram in $\text{Var}(\mathbb{C})$

$$D = (f, g) = \begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} .$$

which is cartesian. Assume there exist factorizations $f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$, with $Y_1, Y_2 \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_S the projections. For $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)\text{fil},c}(X/(Y \times \tilde{S}_I))$,

$$T^{\mathcal{D}\text{mod}}(f, g) : Rg^{*\text{mod},\Gamma} \int_f^{\text{FDR}} (M, F) \rightarrow \int_{f'}^{\text{FDR}} Rg'^{*}\text{mod},\Gamma(M, F)$$

is an isomorphism in $D_{\mathcal{D}(2)\text{fil},\infty}(T/(Y_2 \times \tilde{S}_I))$.

Proof. Similar to the proof of proposition 84: the maps

$$\begin{aligned} T_\omega^O(p_{\tilde{S}_I}, \tilde{g}_I)(M_I, F) : \tilde{g}_I^{*\text{mod}} p_{\tilde{S}_I*} E((\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{\mathcal{O}_{Y \times \tilde{S}_I}} (M_I, F)) \rightarrow \\ p_{\tilde{T}_I*} E((\Omega_{Y \times \tilde{T}_I/\tilde{T}_I}, F_b) \otimes_{\mathcal{O}_{Y \times \tilde{T}_I}} \tilde{g}_I''^{*\text{mod}}(M_I, F)) \end{aligned}$$

are ∞ -filtered Zariski local equivalences since $\tilde{g}_I : Y_2 \times \tilde{S}_I \rightarrow \tilde{S}_I$ are projections. \square

Proposition 96. Consider a commutative diagram in $\text{AnSp}(\mathbb{C})$

$$D = (f, g) = \begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & S \end{array} .$$

which is cartesian. Assume that f (hence f') is proper and that there exist factorizations $f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$, with $Y_1, Y_2 \in \text{AnSm}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S, p_S the projections.

(i) For $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)\text{fil},h}(X/(Y_1 \times \tilde{S}_I))$

$$T^{\mathcal{D}\text{mod}}(f, g) : Rg^{*\text{mod},\Gamma} \int_f^{\text{FDR}} (M, F) \rightarrow \int_{f'}^{\text{FDR}} Rg'^{*}\text{mod},\Gamma(M, F)$$

is an isomorphism in $D_{\mathcal{D}(2)\text{fil},\infty}(T/Y_2 \times \tilde{S}_I)$.

(ii) For $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}^\infty(2)\text{fil},h}(X/(Y_1 \times \tilde{S}_I))$

$$T^{\mathcal{D}\text{mod}}(f, g) : Rg^{*\text{mod},\Gamma} \int_f^{\text{FDR}} (M, F) \rightarrow \int_{f'}^{\text{FDR}} Rg'^{*}\text{mod},\Gamma(M, F)$$

is an isomorphism in $D_{\mathcal{D}^\infty(2)\text{fil},\infty}(T/(Y_2 \times \tilde{S}_I))$.

Proof. (i): Similar to the proof of proposition 95.

(ii): Similar to the proof of proposition 95. \square

Definition 92. Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{Var}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We have, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$, the canonical transformation map in $D_{\mathcal{D}(2)\text{fil}}(T^{\text{an}}/(\tilde{T}_I^{\text{an}}))$

$$\begin{aligned} & T^{\text{mod}}(\text{an}, \gamma_T)(M, F) : \\ f^{*\text{mod}[-], \Gamma}(M, F)^{\text{an}} & := ((\Gamma_{T_I} E(p_{\tilde{S}_I}^{*\text{mod}[-]}(M_I, F)))^{\text{an}}, (f^{*\text{mod}[-]} u_{IJ})^{\text{an}}) \\ & \xrightarrow{(T^{\text{mod}}(\text{an}, \gamma_{T_I})(-))} (\Gamma_{T_I^{\text{an}}} E((p_{\tilde{S}_I}^{*\text{mod}[-]}(M_I, F))^{\text{an}}), f^{*\text{mod}[-]} u_{IJ}^{\text{an}}) \\ & \xrightarrow{=} (\Gamma_{T_I^{\text{an}}} E(p_{\tilde{S}_I}^{*\text{mod}[-]}(M_I^{\text{an}}, F)), f^{*\text{mod}[-]} u_{IJ}^{\text{an}}) =: f^{*\text{mod}[-], \Gamma}((M, F)^{\text{an}}) \end{aligned}$$

where the equality is obvious (see proposition 55).

Definition 93. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}\text{fil}}(X/Y \times \tilde{S}_I)$, the following transformation map in $D_{\mathcal{D}\text{fil}}(X^{\text{an}}/(Y \times \tilde{S}_I)^{\text{an}})$

$$\begin{aligned} T^{\mathcal{D}\text{mod}}(\text{an}, f)(M, F) & : \left(\int_f^{FDR} (M, F) \right)^{\text{an}} = (p_{\tilde{S}_I^*} E((\Omega_{Y \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b) \otimes_{\mathcal{O}_{Y \times \tilde{S}_I}} L_D(M_I, F)))^{\text{an}}, (f^k(u_{IJ}^q))^{\text{an}}) \\ & \xrightarrow{(T_\omega^O(p_{\tilde{S}_I}, \text{an})(M_I, F))} (p_{\tilde{T}_I^*} E((\Omega_{Y \times \tilde{T}_I / \tilde{T}_I}^\bullet, F_b) \otimes_{\mathcal{O}_{(Y \times \tilde{T}_I)^{\text{an}}} } L_D(M_I, F)^{\text{an}}), f'^k((u_{IJ}^q)^{\text{an}})) =: \int_{f^{\text{an}}}^{FDR} (M, F)^{\text{an}} \end{aligned}$$

Theorem 25. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $M \in D_{\mathcal{D}\text{fil}, c}(X/Y \times \tilde{S}_I)$. If f is proper,

$$T^{\mathcal{D}}(\text{an}, f)(M) : \left(\int_f M \right)^{\text{an}} \xrightarrow{\sim} \int_{f^{\text{an}}} (M)^{\text{an}}$$

is an isomorphism.

Proof. By theorem 23, $T_\omega^O(p_{\tilde{S}_I}, \text{an})(M_I)$ are usu local equivalences. \square

In the analytic case, we have the following canonical transformation maps

Definition 94. Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSp}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$; Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We have, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$, the canonical transformation map in $D_{\mathcal{D}^\infty\text{fil}}(T/(\tilde{T}_I))$ obtained by the canonical maps given in definition 67 and definition 72 :

$$\begin{aligned} T(f, \infty)(M, F) & : J_T(f^{*\text{mod}[-], \Gamma}(M, F)) := (J_{\tilde{T}_I}(\Gamma_{T_I} E(p_{\tilde{S}_I}^{*\text{mod}[-]}(M_I, F))), J(f^{*\text{mod}[-]} u_{IJ})) \\ & \xrightarrow{(T(\infty, \gamma_{T_I})(-))} (\Gamma_{T_I} E(J_{\tilde{T}_I}(p_{\tilde{S}_I}^{*\text{mod}[-]}(M_I, F))), N_{IJ}) \\ & \xrightarrow{(T(p_{\tilde{S}_I}, \infty)(-))} (\Gamma_{T_I} E(p_{\tilde{S}_I}^{*\text{mod}[-]} J_{\tilde{S}_I}(M_I, F)), f^{*\text{mod}[-]} J(u_{IJ})) =: f^{*\text{mod}[-], \Gamma}(J_S(M, F)) \end{aligned}$$

4.4 The category of complexes of quasi-coherent sheaves whose cohomology sheaves has a structure of D-modules

4.4.1 Definition on a smooth complex algebraic variety or smooth complex analytic space and the functorialities

Let $X \in \text{SmVar}(\mathbb{C})$ or let $X \in \text{AnSm}(\mathbb{C})$. Recall that (see definition 57 section 4.1) $C_{O_X \text{fil}, \mathcal{D}}(X)$ is the category

- whose objects $(M, F) \in C_{O_X \text{fil}, \mathcal{D}}(X)$ are filtered complexes of presheaves of O_X modules $(M, F) \in C_{O_X \text{fil}}(X)$ whose cohomology presheaves $H^n(M, F) \in \text{PSh}_{O_X \text{fil}}(X)$ are endowed with a structure of filtered D_X modules for all $n \in \mathbb{Z}$.
- whose set of morphisms $\text{Hom}_{C_{O_X \text{fil}, \mathcal{D}}(X)}((M, F), (N, F)) \subset \text{Hom}_{C_{O_X \text{fil}}(X)}((M, F), (N, F))$ between $(M, F), (N, F) \in C_{O_X \text{fil}, \mathcal{D}}(X)$ are the morphisms of filtered complexes of O_X modules $m : (M, F) \rightarrow (N, F)$ such that $H^n m : H^n(M, F) \rightarrow H^n(N, F)$ is D_X linear, i.e. is a morphism of (filtered) D_X modules, for all $n \in \mathbb{Z}$.

More generally, let $h : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. Then, $C_{h^* O_S \text{fil}, h^* \mathcal{D}}(X)$ the category

- whose objects $(M, F) \in C_{h^* O_S \text{fil}, h^* \mathcal{D}}(X)$ are filtered complexes of presheaves of $h^* O_S$ modules $(M, F) \in C_{h^* O_S \text{fil}}(X)$ whose cohomology presheaves $H^n(M, F) \in \text{PSh}_{h^* O_S \text{fil}}(X)$ are endowed with a structure of filtered $h^* D_S$ modules for all $n \in \mathbb{Z}$.
- whose set of morphisms $\text{Hom}_{C_{h^* O_S \text{fil}, h^* \mathcal{D}}(X)}((M, F), (N, F)) \subset \text{Hom}_{C_{h^* O_S \text{fil}}(X)}((M, F), (N, F))$ between $(M, F), (N, F) \in C_{h^* O_S \text{fil}, h^* \mathcal{D}}(X)$ are the morphisms of filtered complexes of $h^* O_S$ modules $m : (M, F) \rightarrow (N, F)$ such that $H^n m : H^n(M, F) \rightarrow H^n(N, F)$ is $h^* D_S$ linear, i.e. is a morphism of (filtered) $h^* D_S$ modules, for all $n \in \mathbb{Z}$.

Definition 95. Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding.

- We denote by $C_{O_S, \mathcal{D}, Z}(S) \subset C_{O_S, \mathcal{D}}(S)$ the full subcategory consisting of $M \in C_{O_S, \mathcal{D}}(S)$ such that such that $j^* H^n M = 0$ for all $n \in \mathbb{Z}$.
- We denote by $C_{O_S \text{fil}, \mathcal{D}, Z}(S) \subset C_{O_S \text{fil}, \mathcal{D}}(S)$ the full subcategory consisting of $(M, F) \in C_{O_S \text{fil}, \mathcal{D}}(S)$ such that there exist $r \in \mathbb{N}$ and an r -filtered homotopy equivalence $m : (M, F) \rightarrow (M', F)$ with $(M', F) \in C_{O_S \text{fil}, \mathcal{D}}(S)$ such that $j^* H^n \text{Gr}_F^p(M', F) = 0$ for all $n, p \in \mathbb{Z}$.

Definition 96. Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. We have then (see section 2), for $r = 1, \dots, \infty$, the homotopy category $K_{O_S \text{fil}, \mathcal{D}, r}(S) = \text{Ho}_r(C_{O_S \text{fil}, \mathcal{D}}(S))$ whose objects are those of $C_{O_S \text{fil}, \mathcal{D}}(S)$ and whose morphisms are r -filtered homotopy classes of morphism, and its localization $D_{O_S \text{fil}, \mathcal{D}, r}(S) = K_{O_S \text{fil}, \mathcal{D}, r}(S)[[E_1]^{-1}]$ with respect to filtered zariski, resp. usu local equivalence. Note that the classes of filtered τ local equivalence constitute a right multiplicative system.

We look at functoriality

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $(M, F) \in C_{O_S \text{fil}, \mathcal{D}}(S)$. Then, the canonical morphism $q : L_O(M, F) \rightarrow (M, F)$ in $C_{O_S \text{fil}}(S)$ being a quasi-isomorphism of O_S modules, we get in a unique way $L_O(M, F) \in C_{O_S \text{fil}, \mathcal{D}}(S)$ such that $q : L_O(M, F) \rightarrow (M, F)$ is a morphism in $C_{O_S \text{fil}, \mathcal{D}}(S)$
- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Let $(M, F) \in C_{O_S \text{fil}, \mathcal{D}}(S)$. Then, $f^{* \text{mod}} H^n(M, F) := (O_X, F_b) \otimes_{f^* O_S} f^* H^n(M, F)$ is canonical a filtered D_X module (see section 4.1 or 4.2). Consider the canonical surjective map $q(f) : H^n f^{* \text{mod}}(M, F) \rightarrow f^{* \text{mod}} H^n(M, F)$. Then, $q(f)$ is an isomorphism if f is smooth. Let

$h : U \rightarrow S$ be a smooth morphism with $U, S \in \text{SmVar}(\mathbb{C})$, or let $h : U \rightarrow S$ be a smooth morphism with $U, S \in \text{AnSm}(\mathbb{C})$. We get the functor

$$h^{*mod} : C_{O_S \text{fil}, \mathcal{D}}(S) \rightarrow C_{O_U \text{fil}, \mathcal{D}}(U), (M, F) \mapsto h^{*mod}(M, F),$$

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$, and let $i : Z \hookrightarrow S$ a closed embedding and denote by $j : S \setminus Z \hookrightarrow S$ the open complementary. For $M \in C_{O_S, \mathcal{D}}(S)$, the cohomology presheaves of

$$\Gamma_Z M := \text{Cone}(\text{ad}(j^*, j_*)(M) : M \rightarrow j_* j^* M)[-1]$$

has a canonical D_S -module structure (as $j^* H^n M$ is a $j^* D_S$ module, $H^n j_* j^* M = j_* j^* H^n M$ has an induced structure of D_S module), and $\gamma_Z(M) : \Gamma_Z M \rightarrow M$ is a map in $C_{O_S, \mathcal{D}}(S)$. For $Z_2 \subset Z$ a closed subset and $M \in C_{O_S, \mathcal{D}}(S)$, $T(Z_2/Z, \gamma)(M) : \Gamma_{Z_2} M \rightarrow \Gamma_Z M$ is a map in $C_{O_S, \mathcal{D}}(S)$. We get the functor

$$\begin{aligned} \Gamma_Z : C_{O_S \text{fil}, \mathcal{D}}(S) &\rightarrow C_{O_S \text{fil}, \mathcal{D}}(S), \\ (M, F) &\mapsto \Gamma_Z(M, F) := \text{Cone}(\text{ad}(j^*, j_*)((M, F)) : (M, F) \rightarrow j_* j^*(M, F))[-1], \end{aligned}$$

together we the canonical map $\gamma_Z(M, F) : \Gamma_Z(M, F) \rightarrow (M, F)$

More generally, let $h : Y \rightarrow S$ a morphism with $Y, S \in \text{Var}(\mathbb{C})$ or $Y, S \in \text{AnSp}(\mathbb{C})$, S smooth, and let $i : X \hookrightarrow Y$ a closed embedding and denote by $j : Y \setminus X \hookrightarrow Y$ the open complementary. For $M \in C_{h^* O_S, h^* \mathcal{D}}(Y)$,

$$\Gamma_X M := \text{Cone}(\text{ad}(j^*, j_*)(M) : M \rightarrow j_* j^* M)[-1]$$

has a canonical $h^* D_S$ -module structure, (as $j^* H^n M$ is a $j^* h^* D_S$ module, $H^n j_* j^* M = j_* j^* H^n M$ has an induced structure of $j^* h^* D_S$ module), and $\gamma_X(M) : \Gamma_X M \rightarrow M$ is a map in $C_{h^* O_S, h^* \mathcal{D}}(Y)$. For $X_2 \subset X$ a closed subset and $M \in C_{h^* O_S, h^* \mathcal{D}}(Y)$, $T(X_2/X, \gamma)(M) : \Gamma_{X_2} M \rightarrow \Gamma_X M$ is a map in $C_{h^* O_S, h^* \mathcal{D}}(Y)$. We get the functor

$$\begin{aligned} \Gamma_X : C_{h^* O_S \text{fil}, h^* \mathcal{D}}(Y) &\rightarrow C_{h^* O_S \text{fil}, h^* \mathcal{D}}(Y), \\ (M, F) &\mapsto \Gamma_X(M, F) := \text{Cone}(\text{ad}(j^*, j_*)((M, F)) : (M, F) \rightarrow j_* j^*(M, F))[-1], \end{aligned}$$

together we the canonical map $\gamma_X(M, F) : \Gamma_X(M, F) \rightarrow (M, F)$

- Let $f : X \rightarrow S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{l} X \times S \xrightarrow{p} S$, where l is the graph embedding and p the projection. We get from the two preceding points the functor

$$f^{*mod, \Gamma} : C_{O_S \text{fil}, \mathcal{D}}(S) \rightarrow C_{O_X \text{fil}, \mathcal{D}}(X \times S), (M, F) \mapsto f^{*mod, \Gamma}(M, F) := \Gamma_X p^{*mod}(M, F),$$

and

$$f^{*mod[-], \Gamma} : C_{O_S \text{fil}, \mathcal{D}}(S) \rightarrow C_{O_X \text{fil}, \mathcal{D}}(X \times S), (M, F) \mapsto f^{*mod[-], \Gamma}(M, F) := \Gamma_X E(p^{*mod}(M, F))[-d_X],$$

which induces in the derived categories the functor

$$Rf^{*mod[-], \Gamma} : D_{O_S \text{fil}, \mathcal{D}}(S) \rightarrow D_{O_X \text{fil}, \mathcal{D}}(X \times S), (M, F) \mapsto Rf^{*mod[-], \Gamma}(M, F) := \Gamma_X E(p^{*mod[-]}(M, F)).$$

For $(M, F) \in C_{O_S \text{fil}, \mathcal{D}}(S)$ or $(M, F) \in C_{O_S \text{fil}}(S)$, the canonical map in $C_{O_X \text{fil}}(X \times S)$

$$\text{ad}(i^{*mod}, i_*)(-) : L_O \Gamma_X E(p^{*mod}(M, F)) \rightarrow i_* i^{*mod} L_O \Gamma_X E(p^{*mod}(M, F))$$

gives in the derived category, the canonical map in $D_{O_X \text{fil}, \infty}(X \times S)$

$$\begin{aligned} I(f^{*mod, \Gamma})(M, F) : Rf^{*mod, \Gamma}(M, F) &= L_O \Gamma_X E(p^{*mod}(M, F)) \xrightarrow{\text{ad}(i^{*mod}, i_*)(-)} \\ i_* i^{*mod} L_O \Gamma_X E(p^{*mod}(M, F)) &\xrightarrow{\sim} i_* i^{*mod} L_O(p^{*mod}(M, F)) = Lf^{*mod}(M, F) \end{aligned}$$

where the isomorphism is given by lemma 6.

- Let $S \in \text{SmVar}(\mathbb{C})$. We have the analytical functor :

$$(-)^{an} : C_{O_S \text{fil}, \mathcal{D}}(S) \rightarrow C_{O_S \text{fil}, \mathcal{D}}(S^{an}), (M, F) \mapsto (M, F)^{an} := \text{an}_S^{*mod}(M, F) := (M, F) \otimes_{\text{an}_S^* O_S} O_{S^{an}}$$

which induces in the derived category

$$(-)^{an} : D_{O_S \text{fil}, \mathcal{D}, r}(S) \rightarrow D_{O_S \text{fil}, \mathcal{D}, r}(S^{an}), ((M, F) \mapsto (M, F)^{an} := \text{an}_S^{*mod}(M, F))$$

since an_S^{*mod} is an exact functor.

We have, for $f : T \rightarrow S$ with $T, S \in \text{SmVar}(\mathbb{C})$ or with $T, S \in \text{AnSm}(\mathbb{C})$, the commutative diagrams of functors

$$\begin{array}{ccc} C_{\mathcal{D} \text{fil}}(S) & \xrightarrow{o_S} & C_{O_{\mathcal{D}} \text{fil}, \mathcal{D}}(S) & , & D_{\mathcal{D} \text{fil}, r}(S) & \xrightarrow{o_S} & D_{O_{\mathcal{D}} \text{fil}, \mathcal{D}, r}(S) \\ \downarrow f^{*mod[-], \Gamma} & & \downarrow f^{*mod[-], \Gamma} & & \downarrow Rf^{*mod[-], \Gamma} & & \downarrow Rf^{*mod[-], \Gamma} \\ C_{\mathcal{D} \text{fil}}(T) & \xrightarrow{o_T} & C_{O_{\mathcal{D}} \text{fil}, \mathcal{D}}(T) & & D_{\mathcal{D} \text{fil}, r}(T) & \xrightarrow{o_T} & D_{O_{\mathcal{D}} \text{fil}, \mathcal{D}, r}(T) \end{array}$$

where o_S and o_T are the forgetfull functors.

4.4.2 Definition on a singular complex algebraic variety or singular complex analytic space and the functorialities

Definition 97. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $C_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}}(S/(\tilde{S}_I))$ is the category

- whose objects are $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$, with
 - $(M_I, F) \in C_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}, S_I}(\tilde{S}_I)$,
 - $u_{IJ} : m^*(M_I, F) \rightarrow m^*p_{IJ*}(M_J, F)[d_{\tilde{S}_J} - d_{\tilde{S}_I}]$ for $J \subset I$, are morphisms, $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}}(\tilde{S}_I)$;
- whose morphisms $m : ((M_I, F), u_{IJ}) \rightarrow ((N_I, F), v_{IJ})$ between $(M, F) = ((M_I, F)_{I \subset [1, \dots, l]}, u_{IJ})$ and $(N, F) = ((N_I, F)_{I \subset [1, \dots, l]}, v_{IJ})$ are a family of morphisms of complexes,

$$m = (m_I : (M_I, F) \rightarrow (N_I, F))_{I \subset [1, \dots, l]}$$

such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}}(\tilde{S}_I)$.

We denote by $C_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}}^{\sim}(S/(\tilde{S}_I)) \subset C_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}}(S/(\tilde{S}_I))$ the full subcategory consisting of objects $((M_I, F), u_{IJ})$ such that the u_{IJ} are ∞ -filtered Zariski, resp. usu, local equivalences.

Definition 98. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have then (see section 2), for $r = 1, \dots, \infty$, the homotopy category

$$K_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}, r}(S/(\tilde{S}_I)) := \text{Ho}_r(C_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}}(S/(\tilde{S}_I)))$$

whose objects are those of $C_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}}(S/(\tilde{S}_I))$ and whose morphisms are r -filtered homotopy classes of morphism, and its localization

$$D_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}, r}(S/(\tilde{S}_I)) := K_{O_{\tilde{S}_I} \text{fil}, \mathcal{D}, r}(S/(\tilde{S}_I))([E_1]^{-1})$$

with respect to the classes of filtered zariski, resp. usu local equivalence. Note that the classes of filtered τ local equivalence constitute a right multiplicative system.

Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{Var}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{AnSp}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} : Y \times \tilde{S}_J \rightarrow Y \times \tilde{S}_I$ the projections and by

$$E_{IJ} = \begin{array}{ccc} \tilde{S}_J \setminus S_J & \xrightarrow{m_J} & \tilde{S}_J \\ \downarrow p_{IJ} & & \downarrow p_{IJ} \\ \tilde{S}_I \setminus (S_I \setminus S_J) & \xrightarrow{m_{IJ}} & \tilde{S}_I \end{array}, \quad E'_{IJ} = \begin{array}{ccc} Y \times \tilde{S}_J \setminus X_J & \xrightarrow{m'_J} & Y \times \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p'_{IJ} \\ Y \times \tilde{S}_I \setminus (X_I \setminus X_J) & \xrightarrow{m'_{IJ}} & Y \times \tilde{S}_I \end{array}, \quad E_{fIJ} = \begin{array}{ccc} \tilde{X}_J & \xrightarrow{\tilde{f}_J} & \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p_{IJ} \\ Y \times \tilde{S}_I & \xrightarrow{\tilde{f}_I} & \tilde{S}_I \end{array}$$

the commutative diagrams. We then have the filtered De Rham the inverse image functor :

$$f^{*mod[-], \Gamma} : C_{Ofil, \mathcal{D}}(S/(\tilde{S}_I)) \rightarrow C_{Ofil, \mathcal{D}}(X/(Y \times \tilde{S}_I)), \quad (M, F) = ((M_I, F), u_{IJ}) \mapsto \\ f^{*mod[-], \Gamma}(M, F) := (\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F))), \tilde{f}_J^{*mod[-]} u_{IJ}$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$

$$\tilde{f}_J^{*mod[-]} u_{IJ} : \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)) \xrightarrow{\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(u_{IJ}))} \Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]} p_{IJ*}(M_J, F)[d_{IJ}]) \\ \xrightarrow{\Gamma_{X_I} E(T(p_{IJ}^{*mod}, p_{\tilde{S}_I})(-)^{-1})[d_Y + d_{IJ}]} \Gamma_{X_I} E(p'_{IJ*} p_{\tilde{S}_J}^{*mod}(M_J, F)[d_Y + d_{IJ}]) \\ \xrightarrow{=} p'_{IJ*} \Gamma_{X_J} E(p_{\tilde{S}_J}^{*mod[-]}(M_J, F)[d_{IJ}]).$$

It induces in the derived categories, the functor

$$Rf^{*mod[-], \Gamma} : D_{Ofil, \mathcal{D}, \infty}(S/(\tilde{S}_I)) \rightarrow D_{Ofil, \mathcal{D}, \infty}(X/(Y \times \tilde{S}_I)), \\ (M, F) = ((M_I, F), u_{IJ}) \mapsto \\ Rf^{*mod[-], \Gamma} := f^{*mod[-], \Gamma}(M, F) := (\Gamma_{X_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)), \tilde{f}_J^{*mod[-]} u_{IJ}).$$

By definition, for $f : T \rightarrow S$ with $T, S \in \text{QPVar}(\mathbb{C})$ or with $T, S \in \text{AnSp}(\mathbb{C})^{QP}$, after considering a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, the commutative diagrams of functors

$$\begin{array}{ccc} C_{\mathcal{D}fil}(S/(\tilde{S}_I)) & \xrightarrow{o_S} & C_{Ofil, \mathcal{D}}(S/(\tilde{S}_I)) & , & D_{\mathcal{D}fil, r}(S/(\tilde{S}_I)) & \xrightarrow{o_S} & D_{Ofil, \mathcal{D}, \infty}(S/(\tilde{S}_I)) \\ \downarrow f^{*mod[-], \Gamma} & & \downarrow f^{*mod[-], \Gamma} & & \downarrow Rf^{*mod[-], \Gamma} & & \downarrow Rf^{*mod[-], \Gamma} \\ C_{\mathcal{D}fil}(T/(Y \times \tilde{S}_I)) & \xrightarrow{o_T} & C_{Ofil, \mathcal{D}}(T/(Y \times \tilde{S}_I)) & & D_{\mathcal{D}fil, \infty}(T/(Y \times \tilde{S}_I)) & \xrightarrow{o_T} & D_{Ofil, \mathcal{D}, r}(T/(Y \times \tilde{S}_I)) \end{array}$$

where o_S and o_T are the forgetful functors.

Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{Var}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We have, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{Ofil, \mathcal{D}}(S/(\tilde{S}_I))^\vee$, the

canonical transformation map in $D_{Ofil, \mathcal{D}}(T^{an}/(\tilde{T}_I^{an}))^\vee$

$$\begin{aligned}
& T^{mod}(an, \gamma_T)(M, F) : \\
f^{*mod[-], \Gamma}(M, F)^{an} & := ((\Gamma_{T_I} E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F)))^{an}, (f^{*mod[-]} u_{IJ})^{an}) \\
& \xrightarrow{(T^{mod}(an, \gamma_{T_I})(-))} (\Gamma_{T_I^{an}} E((p_{\tilde{S}_I}^{*mod[-]}(M_I, F))^{an}), f^{*mod[-]} u_{IJ}^{an}) \\
& \xrightarrow{=} (\Gamma_{T_I^{an}} E(p_{\tilde{S}_I}^{*mod[-]}(M_I^{an}, F)), f^{*mod[-]} u_{IJ}^{an}) =: f^{*mod[-], \Gamma}((M, F)^{an})
\end{aligned}$$

where the equality is obvious.

5 The category of mixed Hodge modules on complex algebraic varieties and complex analytic spaces and the functorialities

For $S \in \text{Top}$ a topological space endowed with a stratification $S = \sqcup_{k=1}^d S_k$ by locally closed subsets S_k together with the perversity $p(S_k)$, we denote by $\mathcal{P}(S, W) \subset D_{fil}(S)$ the category of filtered perverse sheaves of abelian groups. For a locally compact (hence Hausdorff) topological space, we denote by $D_c(S) \subset D(S)$ the full subcategory of complexes of presheaves whose cohomology sheaves are constructible.

5.1 The De Rham functor for D modules on a complex analytic space

Let $S \in \text{AnSm}(\mathbb{C})$. Recall we have the dual functor

$$\mathbb{D}_S : C(S) \rightarrow C(S), K \mapsto \mathbb{D}_S(K) := \mathcal{H}om(K, E(\mathbb{Z}_S))$$

which induces the functor

$$L\mathbb{D}_S : D(S) \rightarrow D(S), K \mapsto L\mathbb{D}_S(K) := \mathbb{D}_S(LK) := \mathcal{H}om(LK, E_{et}(\mathbb{Z}_S)).$$

Let $S \in \text{AnSm}(\mathbb{C})$.

- The functor

$$M \in \text{PSh}_{\mathcal{D}}(S) \mapsto DR(S)(M) := \Omega_S^\bullet \otimes_{O_S} M \in C_{\mathbb{C}_S}(S)$$

which sends a D_S module to its De Rham complex (see section 4) induces, after shifting by d_S in order to send holonomic module (degree zero) to perverse sheaves, in the derived category the functor

$$DR(S)^{[-1]} : D_{\mathcal{D}}(S) \rightarrow D_{\mathbb{C}_S}(S), M \mapsto DR(S)^{[-1]}(M) := DR(S)(M)[d_S] := \Omega_S^\bullet \otimes_{O_S} M[d_S] \simeq K_S \otimes_{D_S}^L M \simeq \mathcal{H}om_{D_S}(\mathbb{D}_S L_D M, E(O_S))[d_S]$$

and, by functoriality, the functor

$$\begin{aligned}
& DR(S)^{[-1]} : D_{\mathcal{D}0fil, \infty}(S) \rightarrow D_{\mathbb{C}_S fil, \infty}(S), \\
(M, W) & \mapsto DR(S)^{[-1]}(M, W) := (\Omega_S^\bullet, F_b) \otimes_{O_S} (M, W)[d_S] = K_S \otimes_{D_S}^L (M, W)
\end{aligned}$$

- On the other hand, we have the functor

$$C_{\mathbb{C}_S}(S) \rightarrow C_{\mathcal{D}^\infty}(S), K \mapsto \mathcal{H}om_{\mathbb{C}_S}(L_{\mathbb{C}} \mathbb{D}_S(LK), E(O_S))[-d_S]$$

together with, for $K \in C_{\mathbb{C}_S}(S)$, the canonical map

$$\begin{aligned}
s(K) : K & \rightarrow DR(S)^{[-1]}(J_S^{-1} \mathcal{H}om_{\mathbb{C}_S}(L_{\mathbb{C}}(\mathbb{D}_S LK), E(O_S))[-d_S]) \\
& \xrightarrow{=} \mathcal{H}om_{D_S}(\mathbb{D}_S^K L_D J_S^{-1} \mathcal{H}om_{\mathbb{C}_S}(L_{\mathbb{C}}(\mathbb{D}_S LK), E(O_S)), E(O_S)), \\
c \in \Gamma(S^\circ, L(K)) & \mapsto s(K)(c) = (\phi \in \Gamma(S^{\circ\circ}, L_D \mathcal{H}om(L_{\mathbb{C}}(K), E(O_S))) \mapsto \phi(c)
\end{aligned}$$

where $S^{\circ\circ} \subset S^\circ \subset S$ are open subsets.

The main result is Riemann-Hilbert equivalence :

Theorem 26. *Let $S \in \text{AnSm}(\mathbb{C})$.*

- (i) *The functor $J_S : D_{\mathcal{D},rh}(S) \rightarrow D_{\mathcal{D}^\infty,h}(S)$ is an equivalence of category. Moreover, for $K \in C(S)$, we have $\mathcal{H}om(L(K), E(O_S)) \in C_{\mathcal{D}^\infty,h}(S)$.*
- (ii) *The restriction of the De Rahm functor to the full subcategory $D_{\mathcal{D},rh}(S) \subset D_{\mathcal{D}}(S)$ is an equivalence of category*

$$DR(S)^{[-1]} : D_{\mathcal{D},rh}(S) \xrightarrow{\sim} D_{C_S,c}(S)$$

whose inverse is the functor

$$K \in C_{C_S,c}(S) \mapsto J^{-1}\mathcal{H}om_{C_S}(\mathbb{D}_S L(K), E(O_S))[-d_S],$$

the map $s(K) : K \xrightarrow{\sim} DR(S)^{[-1]}(J^{-1}\mathcal{H}om_{C_S}(L_C \mathbb{D}_S L(K), E(O_S)))$ being an isomorphism.

- (iii) *The De Rahm functor $DR(S)^{[-1]}$ sends regular holonomic modules to perverse sheaves.*

Proof. See [19]. □

Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$.

- The De Rham functor is in this case

$$DR(S)^{[-1]} : D_{\mathcal{D}0fil,\infty}(S) \rightarrow D_{C_Sfil,\infty}(S), M = ((M_I, W), u_{IJ}) \mapsto DR(S)^{[-1]}(M, W) := (DR(\tilde{S}_I)^{[-1]}(M_I, W), DR^{[-1]}(u_{IJ})) := (\Omega_{\tilde{S}_I}^\bullet \otimes_{O_{\tilde{S}_I}}(M_I, W), DR^{[-1]}(u_{IJ}))$$

with, denoting for short $d_I = d_{\tilde{S}_I}$

$$\begin{aligned} DR^{[-1]}(u_{IJ}) : \Omega_{\tilde{S}_I}^\bullet \otimes_{O_{\tilde{S}_I}}(M_I, W)[d_I] &\xrightarrow{\text{ad}(p_{IJ}, p_{IJ*})(-)} p_{IJ*} p_{IJ}^* \Omega_{\tilde{S}_I}^\bullet \otimes_{O_{\tilde{S}_I}}(M_I, W)[d_I] \\ &\xrightarrow{p_{IJ*} \Omega_{\tilde{S}_J/\tilde{S}_I}[d_I]} p_{IJ*} \Omega_{\tilde{S}_J}^\bullet \otimes_{O_{\tilde{S}_J}} p_{IJ}^{*mod}(M_I, W)[d_I] \\ &\xrightarrow{p_{IJ*} I(p_{IJ}^{*mod}, p_{IJ})(-, -)(u_{IJ})[d_I]} p_{IJ*} \Omega_{\tilde{S}_J}^\bullet \otimes_{O_{\tilde{S}_J}}(M_J, W)[d_J + d_{IJ}] \end{aligned}$$

- Considering the diagrams

$$\begin{array}{ccc} D_{IJ} = & \tilde{S}_J & \xrightarrow{p_{IJ}} & \tilde{S}_I \\ & \uparrow i_J & & \uparrow i_I \\ & S_J & \xrightarrow{j_{IJ}} & S_I \end{array}$$

we get the functor

$$C_{C_Sfil}(S) \xrightarrow{T(S/(S_I))} C_{C_Sfil}(S/(S_I)) \rightarrow C_{\mathcal{D}0fil}(S/(S_I)), \\ (K, W) \mapsto (\mathcal{H}om_{C_{\tilde{S}_I}}(L_C \mathbb{D}_{\tilde{S}_I}(Li_{I*} j_I^*(K, W)), E(O_{\tilde{S}_I})))[-d_{\tilde{S}_I}], u_{IJ}(K, W))$$

where

$$\begin{aligned} u_{IJ}(K, W) : \mathcal{H}om_{C_{\tilde{S}_I}}(L_C \mathbb{D}_{\tilde{S}_I} L(i_{I*} j_I^*(K, W)), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}] \\ \xrightarrow{\text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} p_{IJ*} p_{IJ}^{*mod[-1]} \mathcal{H}om_{C_{\tilde{S}_I}}(L_C \mathbb{D}_{\tilde{S}_I} L(i_{I*} j_I^*(K, W)), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_J}] \\ \xrightarrow{\mathcal{H}om(-, Eo(p_{IJ})) \circ T(p_{IJ}, \text{hom})(-, -)} p_{IJ*} \mathcal{H}om_{C_{\tilde{S}_I}}(p_{IJ}^* L_C \mathbb{D}_{\tilde{S}_I} L(i_{I*} j_I^*(K, W)), E(O_{\tilde{S}_J}))[-d_{\tilde{S}_J}] \\ \xrightarrow{\mathcal{H}om(T(p_{IJ}, \mathbb{D})(-)^{-1}, -)} p_{IJ*} \mathcal{H}om_{C_{\tilde{S}_I}}(L_C \mathbb{D}_{\tilde{S}_J} p_{IJ}^* L(i_{I*} j_I^*(K, W)), E(O_{\tilde{S}_J}))[-d_{\tilde{S}_J}] \\ \xrightarrow{\mathcal{H}om(L_C \mathbb{D}_{\tilde{S}_J} T^a(D_{IJ})(j_I^*(K, W)), E(O_{\tilde{S}_J}))} p_{IJ*} \mathcal{H}om_{C_{\tilde{S}_J}}(L_C \mathbb{D}_{\tilde{S}_J} L(i_{J*} j_J^*(K, W)), E(O_{\tilde{S}_J}))[-d_{\tilde{S}_J}]. \end{aligned}$$

Moreover, for $(K, W) \in C_{fil}(S)$, we have

$$(\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(K, W)), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}], u_{IJ}(K, W)) \in C_{\mathcal{D}^{\infty}ofil, h}(S)^0$$

and a canonical map in $D_{fil}(S) = D_{fil}(S/(\tilde{S}_I))$

$$s(K) : T(S/(S_I))(K, W) := (L(i_{I*}j_I^*(K, W)), I) \rightarrow DR(S)^{[-1]}(J_S^{-1}\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(K, W)), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}], u_{IJ}(K, W))$$

Corollary 3. *Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$. The restriction of the De Rham functor to the full subcategory $D_{\mathcal{D}, rh}^0(S) \subset D_{\mathcal{D}}^0(S)$ is an equivalence of category*

$$DR(S)^{[-1]} : D_{\mathcal{D}, rh}^0(S) \xrightarrow{\sim} D_{\mathbb{C}_S, c}(S)$$

whose inverse is the functor

$$K \mapsto J_S^{-1}(\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*K), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}], u_{IJ}(K))$$

the map

$$s(K) : T(S/(S_I))(K, W) := (L(i_{I*}j_I^*(K, W)), I) \rightarrow DR(S)^{[-1]}(J_S^{-1}\mathcal{H}om_{\mathbb{C}_{\tilde{S}_I}}(L_{\mathbb{C}}\mathbb{D}_{\tilde{S}_I}L(i_{I*}j_I^*(K, W)), E(O_{\tilde{S}_I}))[-d_{\tilde{S}_I}], u_{IJ}(K, W))$$

being an isomorphism.

Proof. Follows from theorem 26(ii), see [28]. □

Proposition 97. (i) *Let $S \in \text{AnSm}(\mathbb{C})$. Then, for $M \in C_{\mathcal{D}, c}(S)$, there is a canonical isomorphism*

$$T(D, DR)(M) : \mathbb{D}_S^{\mathbb{C}}DR(S)^{[-1]}(M) \xrightarrow{\sim} DR(S)^{[-1]}(\mathbb{D}_S^K L_D M)$$

(ii) *Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$. Then, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D}, c}^0(S/(\tilde{S}_I))$, there is a canonical isomorphism*

$$T(D, DR)(M) : \mathbb{D}_S^{\mathbb{C}}DR(S)^{[-1]}(M) \xrightarrow{\sim} DR(S)^{[-1]}(L\mathbb{D}_S^K M)$$

Proof. (i): See [17].

(ii): Follows from (i), see [28]. □

We have the following transformation maps :

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSm}(\mathbb{C})$. We have, for $(M, W) \in C_{\mathcal{D}^0 fil}(S)$, the canonical transformation map in $D_{fil}(T)$:

$$\begin{aligned} T(g, DR)(M, W) &: g^*DR(S)^{[-1]}(M, W) := g^*(\Omega_S^{\bullet} \otimes_{O_S} L_D(M, W))[d_S] \\ &\xrightarrow{\Omega_{T/S}(L_D(M, W))[d_S]} \Omega_T^{\bullet} \otimes_{O_T} g^{*mod} L_D(M, W)[d_S] \\ &\xrightarrow{=} \Omega_T^{\bullet} \otimes_{O_T} g^{*mod[-1]} L_D(M, W)[d_S] =: DR(T)^{(-)}(Lg^{*mod[-1]}(M, W)) \end{aligned}$$

Note that this transformation map is NOT an isomorphism in general. It is an isomorphism if g is a smooth morphism. If g is a closed embedding, it is an isomorphism for M non characteristic with respect to g .

- Let $j : S^\circ \hookrightarrow S$ an open embedding with $S \in \text{AnSm}(\mathbb{C})$. We have, for $(M, W) \in C_{D0fil}(S^\circ)$, the canonical transformation map in $D_{fil}(S)$:

$$\begin{aligned} T_*(j, DR)(M, W) : DR(S)^{[-]}(j_*(M, W)) &:= \Omega_S^\bullet \otimes_{O_S} j_*(M, W)[d_S] \\ \xrightarrow{T_w^O(j, \otimes)(L_D(M, W))[d_S]} j_*(\Omega_{S^\circ}^\bullet \otimes_{O_{S^\circ}} L_D(M, W))[d_S] &=: j_* DR(S)^{[-]}(M, W) \end{aligned}$$

which is an isomorphism (see proposition 82).

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_i S_i$ an open covers such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D}}(S/(\tilde{S}_I))$, the canonical transformation map in $D_{fil}(T/(Y \times \tilde{S}_I))$

$$\begin{aligned} T^1(g, DR)(M) : T(T/(Y \times \tilde{S}_I))(g^! DR(S)^{[-]}(M, W)) \\ \xrightarrow{=} (\Gamma_{T_I} E(\tilde{g}_I^*(\Omega_{\tilde{S}_I}^\bullet \otimes_{O_{\tilde{S}_I}} L_D(M_I, W))), \tilde{g}_I^* DR(u_{IJ})) \\ \xrightarrow{(\Omega_{(Y \times \tilde{S}_I/\tilde{S}_I)}(L_D(M_I, W)))} (\Gamma_{T_I} E(\Omega_{Y \times \tilde{S}_I}^\bullet \otimes_{O_{Y \times \tilde{S}_I}} \tilde{g}_I^{*mod}(M_I, W)), DR(\tilde{g}_I^{*mod} u_{IJ})) \\ \xrightarrow{(T_w^O(\gamma, \otimes)(\tilde{g}_I^{*mod} L_D(M, W)))} (\Omega_{Y \times \tilde{S}_I}^\bullet \otimes_{O_{Y \times \tilde{S}_I}} \Gamma_{T_I} E(\tilde{g}_I^{*mod}(M_I, W)), DR(\tilde{g}_I^{*mod} u_{IJ})) \\ \xrightarrow{=} DR(T)^{[-]}(Rg^{*mod[-], \Gamma}(M, W)) \end{aligned}$$

which is an isomorphism.

- Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_i S_i$ an open covers such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D}, rh}(S/(\tilde{S}_I))$, the canonical map in $D_{fil}(T/(Y \times \tilde{S}_I))$

$$\begin{aligned} DR(T)^{[-]}(T^{mod}(an, \gamma_T)(M)) : \\ DR(T)^{[-]}((Rf^{*mod[-], \Gamma} M)^{an}) := DR(T)^{[-]}(((\Gamma_{T_I} E(\tilde{f}_I^{*mod[-]}(M_I)))^{an}, (f^{*mod[-]} u_{IJ}^q)^{an})) \\ \rightarrow DR(T)^{[-]}(Rf^{*mod[-], \Gamma}(M^{an})) := DR(T)^{[-]}((\Gamma_{T_I^{an}} E(\tilde{f}_I^{*mod[-]}(M_I^{an})), f^{*mod[-]} u_{IJ}^q)^{an}). \end{aligned}$$

Proposition 98. *Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_i S_i$ an open covers such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M = (M_I, u_{IJ}) \in C_{\mathcal{D}, rh}(S/(\tilde{S}_I))$, the map in $D_{fil}(T/(Y \times \tilde{S}_I))$*

$$\begin{aligned} DR(T)^{[-]}(T^{mod}(an, \gamma_T)(M)) : \\ DR(T)^{[-]}((Rf^{*mod[-], \Gamma} M)^{an}) := DR(T)^{[-]}(((\Gamma_{T_I} E(\tilde{f}_I^{*mod[-]}(M_I)))^{an}, (f^{*mod[-]} u_{IJ}^q)^{an})) \\ \rightarrow DR(T)^{[-]}(Rf^{*mod[-], \Gamma}(M^{an})) := DR(T)^{[-]}((\Gamma_{T_I^{an}} E(\tilde{f}_I^{*mod[-]}(M_I^{an})), f^{*mod[-]} u_{IJ}^q)^{an}) \end{aligned}$$

given above is an isomorphism.

Proof. See [17]. □

In the algebraic case, we have, by proposition 98, for complexes of D -modules whose cohomology sheaves are regular holonomic the following canonical isomorphisms:

Definition 99. (i) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M = (M_I, u_{IJ}) \in \mathcal{C}_{\mathcal{D}, rh}(S/(\tilde{S}_I))^0$, the canonical map

$$\begin{aligned} T^!(f, DR)(M) : f^! DR(S)^{[-]}(M^{an}) &\xrightarrow{T^!(f, DR)(M^{an})} DR(T)^{[-]}(Rf_*^{mod[-], \Gamma}(M^{an})) \\ &\xrightarrow{DR(T)^{[-]}(T^{mod}(an, \gamma_T)(M))} DR(T)^{[-]}((Rf_*^{mod[-], \Gamma} M)^{an}) =: DR(T)^{[-]}((Rf_*^{mod[-], \Gamma} M)^{an}). \end{aligned}$$

which is an isomorphism by proposition 98.

(ii) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M = (M_I, u_{IJ}) \in \mathcal{C}_{\mathcal{D}, rh}(S/(\tilde{S}_I))^0$, the canonical transformation map

$$\begin{aligned} T(f, DR)(M) : DR(T)^{[-]}((Lf^{\hat{*}mod[-], \Gamma} M)^{an}) &:= DR(T)^{[-]}((L\mathbb{D}_T^K Rf_*^{mod[-], \Gamma} L\mathbb{D}_S^K M)^{an}) \\ &\xrightarrow{T(D, DR)(-)} L\mathbb{D}_T^K DR(T)^{[-]}((Rf_*^{mod[-], \Gamma} L\mathbb{D}_S^K M)^{an}) \xrightarrow{L\mathbb{D}_T^K T^!(f, DR)(-)} L\mathbb{D}_T^K f^! DR(S)^{[-]}(L\mathbb{D}_S^K M^{an}) \\ &\xrightarrow{L\mathbb{D}_T^K f^! T(D, DR)(-)^{-1}} L\mathbb{D}_T^K f^! L\mathbb{D}_S^K DR(S)^{[-]}(M^{an}) = f^* DR(S)^{[-]}(M^{an}) \end{aligned}$$

which is an isomorphism by (i) and proposition 97.

(iii) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{QPVar}(\mathbb{C})$. Consider a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y = \mathbb{P}^{N, \circ} \subset \mathbb{P}^N$ an open subset, l a closed embedding, and p_S the projection. We have, for $M \in \mathcal{C}_{\mathcal{D}, rh}(T/Y \times \tilde{S})^0$, the canonical transformation map

$$\begin{aligned} T_*(f, DR)(M) : DR(S)^{[-]}(\int_f M)^{an} &\xrightarrow{\text{ad}(f^*, Rf_*)(-)} Rf_* f^* DR(S)^{[-]}(\int_f M)^{an} \xrightarrow{Rf_* T(f, DR)((\int_f M))} \\ &Rf_* DR(T)^{[-]}((Lf^{\hat{*}mod[-], \Gamma} \int_f M)^{an}) \xrightarrow{Rf_* DR(T)^{[-]}(\text{ad}(Lf^{\hat{*}mod[-], \Gamma}, \int_f)(M)^{an})} Rf_* DR(T)^{[-]}(M^{an}) \end{aligned}$$

which is an isomorphism by GAGA in the proper case and by the open embedding case (c.f. proposition 98).

(iv) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{QPVar}(\mathbb{C})$. Consider a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y = \mathbb{P}^{N, \circ} \subset \mathbb{P}^N$ an open subset, l a closed embedding, and p_S the projection. We have, for $M \in \mathcal{C}_{\mathcal{D}, rh}(T)$, the canonical transformation map

$$\begin{aligned} T_i(f, DR)(M) : Rf_i DR(T)^{[-]}(M^{an}) & \\ &\xrightarrow{Rf_i DR(T)^{[-]}(\text{ad}(\int_{f!}, Rf_*^{mod[-], \Gamma})(M)^{an})} Rf_i DR(T)^{[-]}((Rf_*^{mod[-], \Gamma} \int_{f!} (M))^{an}) \\ &\xrightarrow{T^!(f, DR)(\int_{f!} M)} Rf_i f^! DR(S)^{[-]}(\int_{f!} (M)) \xrightarrow{\text{ad}(Rf_i, f^!)(-)} DR(S)^{[-]}(\int_{f!} M) \end{aligned}$$

which is an isomorphism by (iii) and proposition 97.

In the filtered case, we will consider the weight monodromy filtration for open embeddings :

Definition 100. Let $k \subset \mathbb{C}$ a subfield.

(i) Let $S \in \text{Var}(\mathbb{C})$ and $j : S^\circ \hookrightarrow S$ an open embedding such that $D := S \setminus S^\circ = V(s) \subset S$ is a Cartier divisor.

– For $(K, W) \in P_{fil}(S^{o,an})$, we consider as in [28]

$$j_{*w}(K, W) := (Rj_*K, W) \in P_{fil}(S^{an}), \quad W_k Rj_*K := \langle Rj_*W_k K, W(N)_k K \rangle \subset Rj_*K$$

so that $j^*j_{*w}(K, W) = (K, W)$, where $W_k Rj_*K \subset Rj_*K$ is given by W and the weight monodromy filtration $W(N)$ of the universal cover $\pi : \tilde{S}^{o,an} \rightarrow S^{o,an}$. Note that a stratification of $W_k Rj_*K$ is given by the closure of a stratification of $W_k K$ and $D := S \setminus S^o$.

– For $(K, W) \in P_{fil}(S^{o,an})$, we consider

$$j_{!w}(K, W) := \mathbb{D}_S^v j_{*w} \mathbb{D}_S^v(K, W) \in P_{fil}(S^{an})$$

so that $j^*j_{!w}(K, W) = (K, W)$.

For $(K', W) \in P_{fil}(S^{an})$, there is, by construction,

- a canonical map $\text{ad}(j^*, j_{*w})(K', W) = \text{ad}(j^*, j_*)(K') : (K', W) \rightarrow j_{*w}j^*(K', W)$ in $P_{fil}(S^{an})$,
- a canonical map $\text{ad}(j_{!w}, j^*)(K', W) = \text{ad}(j_!, j^*)(K') : j_{!w}j^*(K', W) \rightarrow (K', W)$ in $P_{fil}(S^{an})$.

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $j : S^o := S \setminus Z \hookrightarrow S$ an open embedding with $Z = V(\mathcal{I}) \subset S$ an arbitrary closed subset, $\mathcal{I} \subset \mathcal{O}_S$ being an ideal subsheaf. Taking generators $\mathcal{I} = (s_1, \dots, s_r)$, we get $Z = V(s_1, \dots, s_r) = \cap_{i=1}^r Z_i \subset S$ with $Z_i = V(s_i) \subset S$, $s_i \in \Gamma(S, \mathcal{L}_i)$ and \mathcal{L}_i a line bundle. Note that Z is an arbitrary closed subset, $d_Z \geq d_X - r$ needing not be a complete intersection. Denote by $j_I : S^{o,I} := \cap_{i \in I} (S \setminus Z_i) = S \setminus (\cup_{i \in I} Z_i) \xrightarrow{j_I^o} S^o \xrightarrow{j} S$ the open complementary embeddings, where $I \subset \{1, \dots, r\}$. Denote

$$\mathcal{D}(Z/S) := \{(Z_i)_{i \in [1, \dots, r]}, Z_i \subset S, \cap Z_i = Z\}, Z'_i \subset Z_i$$

the flag category. For $(K, W) \in C(P_{fil}(S^{o,an}))$, we define by (i)

– the (bi)-filtered complex of D_S -modules

$$j_{*w}(K, W) := \varinjlim_{\mathcal{D}(Z/S)} \text{Tot}_{\text{card} I = \bullet} (j_{I^*w} j_I^{o*}(K, W)) \in C(P_{fil}(S^{an}))$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \text{ad}(j_{IJ^*}, j_{IJ^*w})(j_I^{o*}(K, W))$, $j_{IJ} : S^{o,J} \hookrightarrow S^{o,I}$ being the open embedding, and $d_{IJ} = 0$ if $I \not\subset J$,

– the (bi)-filtered complex of D_S -modules

$$j_{!w}(K, W) := \varinjlim_{\mathcal{D}(Z/S)} \text{Tot}_{\text{card} I = -\bullet} (j_{I!w} j_I^{o*}(K, W)) = \mathbb{D}_S^v j_{*w} \mathbb{D}_S^v(K, W) \in C(P_{fil}(S^{an})),$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \text{ad}(j_{I!w}, j_{IJ^*})(j_I^{o*}(K, W))$, $j_{IJ} : S^{o,J} \hookrightarrow S^{o,I}$ being the open embedding, and $d_{IJ} = 0$ if $I \not\subset J$.

By definition, we have for $(K, W) \in C(P_{fil}(S^{o,an}))$, $j^*j_{*w}(K, W) = (K, W)$ and $j^*j_{!w}(K, W) = (K, W)$. For $(K', W) \in C(P_{fil}(S^{an}))$, there is, by (i),

- a canonical map $\text{ad}(j^*, j_{*w})(K', W) : (K', W) \rightarrow j_{*w}j^*(K', W)$ in $C(P_{fil}(S^{an}))$,
- a canonical map $\text{ad}(j_{!w}, j^*)(K', W) : j_{!w}j^*(K', W) \rightarrow (K', W)$ in $C(P_{fil}(S^{an}))$.

Definition 101. Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the complementary open embedding.

(i) We define using definition 100, the filtered Hodge support section functor

$$\Gamma_Z^w : C(P_{fil}(S^{an})) \rightarrow C(P_{fil}(S^{an})),$$

$$(K, W) \mapsto \Gamma_Z^w(K, W) := \text{Cone}(\text{ad}(j^*, j_{*w})(K, W) : (K, W) \rightarrow j_{*w}j^*(K, W))[-1],$$

together with the canonical map $\gamma_Z^w(K, W) : \Gamma_Z^w(K, W) \rightarrow (K, W)$.

(i)' Since $j_{*w} : C(P_{fil}(S^{o,an})) \rightarrow C(P_{fil}(S^{an}))$ is an exact functor, Γ_Z^w induces the functor

$$\Gamma_Z^w : D_{fil,c}(S^{an}) \rightarrow D_{fil,c}(S^{an}), (K, W) \mapsto \Gamma_Z^w(K, W)$$

(ii) We define using definition 100, the dual filtered Hodge support section functor

$$\begin{aligned} \Gamma_Z^{\vee,w} &: C(P_{fil}(S^{an})) \rightarrow C(P_{fil}(S^{an})), \\ (K, W) &\mapsto \Gamma_Z^{\vee,w}(K, W) := \text{Cone}(\text{ad}(j_{!w}, j^*)(K, W) : j_{!w}, j^*(K, W) \rightarrow (K, W)), \end{aligned}$$

together we the canonical map $\gamma_Z^{\vee,Hdg}(K, W) : (K, W) \rightarrow \Gamma_Z^{\vee,w}(K, W)$.

(ii)' Since $j_{!w} : C(P_{fil}(S^{o,an})) \rightarrow C(P_{fil}(S^{an}))$ is an exact functor, $\Gamma_Z^{\vee,w}$ induces the functor

$$\Gamma_Z^{\vee,w} : D_{fil,c}(S^{an}) \rightarrow D_{fil,c}(S^{an}), (K, W) \mapsto \Gamma_Z^{\vee,w}(K, W)$$

Let $S \in \text{Var}(\mathbb{C})$ and $D = V(s) \subset S$ a Cartier divisor. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^o := S \setminus D \hookrightarrow S$ the open embedding. Let $\pi : \tilde{S}^{o,an} \rightarrow S^{o,an}$ the universal covering. We then consider, for $(K, W) \in D_{fil,c}(S^{o,an}) = \text{Ho}(C(P_{fil}(S^{o,an})))$,

- the filtered nearby cycle functor

$$\psi_D(K, W) := (\psi_D K, W) \in D_{fil,c}(D^{an}), W_k(\psi_D(K, W)) := \langle W_k \psi_D K, W(N)_k \psi_D K \rangle \subset \psi_D K,$$

- the vanishing cycle functor

$$\phi_D(K, W) := \text{Cone}(\text{ad}(j \circ \pi^*, j \circ \pi_*)(K) : i^*(K, W) \rightarrow \psi_D(K, W)) \in D_{fil,c}(D^{an}),$$

- the canonical morphisms in $D_{fil,c}(D^{an})$

$$\begin{aligned} \text{can}(K, W) &:= c(\phi_D(K, W)) : \psi_D(K, W) \rightarrow \phi_D(K, W), \\ \text{var}(K, W) &:= \mathbb{D}_S^v c(\phi_D \mathbb{D}_S^v D(K, W)) : \phi_D(K, W) \rightarrow \psi_D(K, W). \end{aligned}$$

Definition 102. Let $k \subset \mathbb{C}$ a subfield.

(i) Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Consider the graph factorization $f : X \xrightarrow{l} X \times S \xrightarrow{p} S$ of f where l the the graph closed embedding and p is the projection. We have, using definition 101,

- the inverse image functor

$$f^{*w} : D_{fil,c}(S^{an}) \rightarrow D_{fil,c}(X^{an}), (K, W) \mapsto f^{*w}(K, W) := l^* \Gamma_X^{\vee,w} p^*(K, W)$$

- the exceptional inverse image functor

$$f^{!w} : D_{fil,c}(S^{an}) \rightarrow D_{fil,c}(X^{an}), (K, W) \mapsto f^{!w}(K, W) := l^* \Gamma_X^{\vee,w} p^*(K, W).$$

(ii) Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Consider a compactification $f : X \hookrightarrow j\bar{X} \xrightarrow{\bar{f}} S$ of f with $\bar{X} \in \text{Var}(\mathbb{C})$, j an open embedding and \bar{f} a proper morphism. We have, using definition 100,

- the direct image functor

$$Rf_{*w} : D_{fil,c}(X^{an}) \rightarrow D_{fil,c}(S^{an}), (K, W) \mapsto Rf_{*w}(K, W) := R\bar{f}_* j_{*w}(K, W)$$

- the proper direct image functor

$$Rf_{!w} : D_{fil,c}(X^{an}) \rightarrow D_{fil,c}(S^{an}), (K, W) \mapsto Rf_{!w}(K, W) := R\bar{f}_* j_{!w}(K, W).$$

5.2 The filtered Hodge direct image, the filtered Hodge inverse image, and the hodge support section functors for mixed hodge modules

- Let $S \in \text{SmVar}(\mathbb{C})$. The category $C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S^{an})$ is the category

- whose set of objects is the set of triples $\{((M, F, W), (K, W), \alpha)\}$ with

$$(M, F, W) \in C_{\mathcal{D}(1,0)fil,rh}(S), (K, W) \in D_{fil}(S^{an}), \alpha : (K, W) \otimes \mathbb{C}_{S^{an}} \rightarrow DR(S)^{[-1]}((M, W)^{an})$$

where $DR(S)^{[-1]} := DR(S)^{[-1]}(S^{an}) : C_{\mathcal{D}(1,0)fil,rh}(S^{an}) \rightarrow C_{fil}(S^{an})$ is the De Rahm functor and α is an morphism in $D_{fil}(S^{an})$,

- and whose set of morphisms are

$$\phi = (\phi_D, \phi_C, [\theta]) : ((M_1, F, W), (K_1, W), \alpha_1) \rightarrow ((M_2, F, W), (K_2, W), \alpha_2)$$

where $\phi_D : (M_1, F, W) \rightarrow (M_2, F, W)$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms, and

$$\theta = (\theta^\bullet, I(DR(S)(\phi_D^{an})) \circ I(\alpha_1), I(\alpha_2) \circ I(\phi_C \otimes I)) : I(K_1, W) \otimes \mathbb{C}_{S^{an}}[1] \rightarrow I(DR(S)(M_2^{an}, W))$$

is an homotopy, i.e. for all $i \in \mathbb{Z}$,

$$\theta^i \circ \partial^i - \partial^{i+1} \circ \theta^i = (I(DR(S)(\phi_D)) \circ I(\alpha_1))^i - (I(\alpha_2) \circ I(\phi_C))^i,$$

$I : C_{fil}(S^{an}) \rightarrow K_{fil}(S^{an})$ being the injective resolution functor : for $(K, W) \in C_{fil}(S^{an})$, we take an injective resolution $k : (K, W) \rightarrow I(K, W)$ with $I(K, W) \in C_{fil}(S^{an})$ which is unique modulo homotopy, and the class $[\theta]$ of θ does NOT depend of the injective resolution ; in particular, we have

$$DR(S)^{[-1]}(\phi_D^{an}) \circ \alpha_1 = \alpha_2 \circ (\phi_C \otimes I)$$

in $D_{fil}(S^{an})$; and for

$$* \phi = (\phi_D, \phi_C, [\theta]) : ((M_1, F, W), (K_1, W), \alpha_1) \rightarrow ((M_2, F, W), (K_2, W), \alpha_2)$$

$$* \phi' = (\phi'_D, \phi'_C, [\theta']) : ((M_2, F, W), (K_2, W), \alpha_2) \rightarrow ((M_3, F, W), (K_3, W), \alpha_3)$$

the composition law is given by

$$\begin{aligned} \phi' \circ \phi := (\phi'_D \circ \phi_D, \phi'_C \circ \phi_C, I(DR(S)(\phi_D^{an})) \circ [\theta] + [\theta'] \circ I(\phi_C \otimes I)[1]) : \\ ((M_1, F, W), (K_1, W), \alpha_1) \rightarrow ((M_3, F, W), (K_3, W), \alpha_3), \end{aligned}$$

in particular for $((M, F, W), (K, W), \alpha) \in C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S^{an})$,

$$I_{((M,F,W),(K,W),\alpha)} = (I_M, I_K, 0).$$

We have then the full embedding

$$\text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S^{an})$$

where $\text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an})$ is the category

- whose set of objects is the set of triples $\{((M, F, W), (K, W), \alpha)\}$ with

$$(M, F, W) \in \text{PSh}_{\mathcal{D}(1,0)fil,rh}(S), (K, W) \in P_{fil}(S^{an}), \alpha : (K, W) \otimes \mathbb{C}_{S^{an}} \rightarrow DR(S)^{[-1]}((M, W)^{an})$$

where $DR(S)^{[-1]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S^{an})$,

- and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) = (\phi_D, \phi_C, 0) : ((M_1, F, W), (K_1, W), \alpha_1) \rightarrow ((M_2, F, W), (K_2, W), \alpha_2)$$

where $\phi_D : (M_1, F, W) \rightarrow (M_2, F, W)$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms (of filtered sheaves) and $DR(S)^{[-1]}(\phi_D^{an}) \circ \alpha_1 = \alpha_2 \circ (\phi_C \otimes I)$ in $P_{fil}(S^{an})$.

- Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \text{SmVar}(\mathbb{C})$. The category $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$ is the category

– whose set of objects is the set of triples $\{((M_I, F, W), u_{IJ}), (K, W), \alpha\}$ with

$$\begin{aligned} ((M_I, F, W), u_{IJ}) &\in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)), (K, W) \in D_{fil}(S^{an}), \\ \alpha : T(S/(\tilde{S}_I))(K, W) \otimes \mathbb{C}_{S^{an}} &\rightarrow DR(S)^{[-1]}(((M_I, W), u_{IJ})^{an}) \end{aligned}$$

where

$$DR(S)^{[-1]} := DR(S^{an})^{[-1]} : C_{\mathcal{D}(1,0)fil,rh}(S^{an}/(\tilde{S}_I^{an})) \rightarrow C_{fil}(S^{an}/(\tilde{S}_I^{an}))$$

is the De Rahm functor and α is a morphism in $D_{fil}(S^{an}/(\tilde{S}_I^{an}))$,

– and whose set of morphisms consists of

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

where $\phi_D : ((M_1, F, W), u_{IJ}) \rightarrow ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms, and

$$\begin{aligned} \theta &= (\theta^\bullet, I(DR(S)(\phi_D^{an})) \circ I(\alpha_1), I(\alpha_2) \circ I(\phi_C \otimes I)) : \\ I(T(S/(\tilde{S}_I))(K_1, W)) \otimes \mathbb{C}_{S^{an}}[1] &\rightarrow I(DR(S)((M_{2I}, W), u_{IJ})^{an}) \end{aligned}$$

is an homotopy, $I : C_{fil}(S^{an}/(\tilde{S}_I^{an})) \rightarrow K_{fil}(S^{an}/(\tilde{S}_I^{an}))$ being the injective resolution functor : for $((K_I, W), t_{IJ}) \in C_{fil}(S^{an}/(\tilde{S}_I^{an}))$, we take an injective resolution

$$k : ((K_I, W), t_{IJ}) \rightarrow I((K_I, W), t_{IJ})$$

with $I((K, W), t_{IJ}) \in C_{fil}(S^{an}/(\tilde{S}_I^{an}))$ which is unique modulo homotopy, and the class $[\theta]$ of θ does NOT depend of the injective resolution ; in particular we have

$$DR(S)^{[-1]}(\phi_D^{an}) \circ \alpha_1 = \alpha_2 \circ (\phi_C \otimes I)$$

in $D_{fil}(S^{an}/(\tilde{S}_I^{an}))$; and for

$$\begin{aligned} * \phi &= (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \\ * \phi' &= (\phi'_D, \phi'_C, [\theta']) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \rightarrow (((M_{3I}, F, W), u_{IJ}), (K_3, W), \alpha_3) \end{aligned}$$

the composition law is given by

$$\begin{aligned} \phi' \circ \phi &:= (\phi'_D \circ \phi_D, \phi'_C \circ \phi_C, I(DR(S)(\phi'_D)^{an}) \circ [\theta] + [\theta'] \circ I(\phi_C \otimes I)[1]) : \\ &(((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{3I}, F, W), u_{IJ}), (K_3, W), \alpha_3) \end{aligned}$$

in particular for $((M_I, F, W), u_{IJ}), (K, W), \alpha \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$,

$$I_{(((M_I, F, W), u_{IJ}), (K, W), \alpha)} = ((I_{M_I}), I_K, 0).$$

We have then full embeddings

$$\begin{aligned} \text{PSh}_{\mathcal{D}(1,0)fil,rh}^0(S/(\tilde{S}_I)) \times_I P_{fil}(S^{an}) &\hookrightarrow C_{\mathcal{D}(1,0)fil,rh}^0(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \\ \xrightarrow{t_{S/\tilde{S}_I}^0} C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^0 \times_I D_{fil}(S^{an}) &\hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \end{aligned}$$

where $\text{PSh}_{\mathcal{D}(1,0)fil,rh}^0(S/(\tilde{S}_I)) \times_I P_{fil}(S^{an})$ is the category

– whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with

$$\begin{aligned} ((M_I, F, W), u_{IJ}) &\in \text{PSh}_{\mathcal{D}(1,0)_{\text{fil},rh}}^0(S/(\tilde{S}_I)), (K, W) \in \text{P}_{\text{fil}}(S^{an}), \\ \alpha &: T(S/(\tilde{S}_I))(K, W) \otimes \mathbb{C}_{S^{an}} \rightarrow DR(S)^{[-1]}(((M_I, W), u_{IJ})^{an}) \end{aligned}$$

where $DR(S)^{[-1]}$ is the De Rahm functor and α is an isomorphism in $D_{\text{fil}}(S^{an}/(\tilde{S}_I^{an}))$,

– and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) = (\phi_D, \phi_C, 0) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

where $\phi_D : ((M_1, F, W), u_{IJ}) \rightarrow ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms (of filtered sheaves) such that $\phi_D^{an} \circ \alpha_1 = \alpha_2 \circ \phi_C$ in $\text{P}_{\text{fil}}(S^{an})$.

For holonomic D -modules on a smooth variety $S \in \text{SmVar}(\mathbb{C})$, there exist for a closed embedding $Z \subset S$ with Z smooth, a V_Z -filtration (see definition 56) satisfying further hypothesis so that it is unique:

Definition 103. *Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$.*

(i) *Let $D = V(s) \subset S$ be a smooth (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D . Let $M \in \text{PSh}_{\mathcal{D}}(S)$. A V_D -filtration V for M (see definition 56) is called a Kashiwara-Malgrange V_D -filtration for M if*

- $V_k M$ are coherent $V_{D^k} D_S$ modules for all $k \in \mathbb{Z}$, that is V is a good filtration,
- $sV_k M = V_{k-1} M$ for $k \ll 0$,
- all eigenvalues of $s\partial_s : \text{Gr}_{V,k} M := V_k M / V_{k-1} M \rightarrow \text{Gr}_{V,k} M := V_k M / V_{k-1} M$ have real part between $k - 1$ and k .

Almost by definition, a Kashiwara-Malgrange V_D -filtration for M if it exists is unique (see [29]) so that we denote it by $(M, V_D) \in \text{PSh}_{O_S \text{fil}}(S)$ and (M, V_D) is strict. In particular if $m : (M_1, F) \rightarrow (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in \text{PSh}_{\mathcal{D}(2)_{\text{fil}}}(S)$ such that M_1 and M_2 admit the Kashiwara-Malgrange filtration for $D \subset S$, we have $m(V_{D,q} F^p M_1) \subset V_{D,q} F^p M_2$, that is we get $m : (M_1, F, V_D) \rightarrow (M_2, F, V_D)$ a filtered morphism, and if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence, $0 \rightarrow (M', V_D) \rightarrow (M, V_D) \rightarrow (M'', V_D) \rightarrow 0$ is an exact sequence (strictness).

(ii) *More generally, let $Z = V(s_1, \dots, s_r) = D_1 \cap \dots \cap D_r \subset S$ be a smooth Zariski closed subset, where $s_i \in \Gamma(S, L_i)$ is a section of the line bundle $L = L_{D_i}$ associated to D_i . Let $M \in \text{PSh}_{\mathcal{D}}(S)$. A V_Z -filtration V for M (see definition 56) is called a Kashiwara-Malgrange V_Z -filtration for M if*

- $V_k M$ are coherent O_S modules for all $k \in \mathbb{Z}$,
- $\sum_{i=1}^r s_i V_k M = V_{k-1} M$ for $k \ll 0$,
- all eigenvalues of $\sum_{i=1}^r s_i \partial_{s_i} : \text{Gr}_{V,k} M := V_k M / V_{k-1} M \rightarrow \text{Gr}_k^V M := V_k M / V_{k-1} M$ have real part between $k - 1$ and k .

Almost by definition, a Kashiwara-Malgrange V_Z -filtration for M if it exists is unique (see [29]) so that we denote it by $(M, V_Z) \in \text{PSh}_{O_S \text{fil}}(S)$ and (M, V_Z) is strict. In particular if $m : (M_1, F) \rightarrow (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in \text{PSh}_{\mathcal{D}(2)_{\text{fil}}}(S)$ such that M_1 and M_2 admit the Kashiwara-Malgrange filtration for $D \subset S$, we have $m(V_{Z,q} F^p M_1) \subset V_{Z,q} F^p M_2$, that is we get $m : (M_1, F, V_Z) \rightarrow (M_2, F, V_Z)$ a filtered morphism, and if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence, $0 \rightarrow (M', V_Z) \rightarrow (M, V_Z) \rightarrow (M'', V_Z) \rightarrow 0$ is an exact sequence (strictness).

Proposition 99. (i) *Let $S \in \text{AnSm}(\mathbb{C})$.*

- *Let $D = V(s) \subset S$ a smooth (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D . If $M \in \text{PSh}_{\mathcal{D},rh}(S)$, the Kashiwara-Malgrange V_D -filtration for M (see definition 103) exist so that we denote it by $(M, V_D) \in \text{PSh}_{O_S \text{fil}}(S)$.*

- More generally, let $Z = V(s_1, \dots, s_r) = D_1 \cap \dots \cap D_r \subset S$ be a smooth Zariski closed subset, where $s_i \in \Gamma(S, L_i)$ is a section of the line bundle $L = L_{D_i}$ associated to D_i . If $M \in \text{PSh}_{\mathcal{D}(2)rh}(S)$, the Kashiwara-Malgrange V_Z -filtration for M (see definition 103) exist so that we denote it by $(M, V_Z) \in \text{PSh}_{O_S fil}(S)$.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$.

- Let $D = V(s) \subset S$ a smooth (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D . If $M \in \text{PSh}_{\mathcal{D}, rh}(S)$, the Kashiwara-Malgrange V_D -filtration for M (see definition 103) exist so that we denote it by $(M, V_D) \in \text{PSh}_{O_S fil}(S)$.
- More generally, let $Z = V(s_1, \dots, s_r) = D_1 \cap \dots \cap D_r \subset S$ be a smooth Zariski closed subset, where $s_i \in \Gamma(S, L_i)$ is a section of the line bundle $L = L_{D_i}$ associated to D_i . If $M \in \text{PSh}_{\mathcal{D}(2)rh}(S)$, the Kashiwara-Malgrange V_Z -filtration for M (see definition 103) exist so that we denote it by $(M, V_Z) \in \text{PSh}_{O_S fil}(S)$.

Proof. (i): Follows from the work of Kashiwara. Note that the second point is a particular case of the first by induction. (ii): Take a compactification $\bar{S} \in \text{PSmVar}(\mathbb{C})$ of S and denote by $\bar{D} \subset \bar{S}$ the closure of D . Using the closed embedding $i : \bar{S} \hookrightarrow L_{\bar{D}}$ given by the zero section, we may assume that \bar{D} is smooth. Denote by $j : \bar{S} \setminus \bar{D} \hookrightarrow \bar{S}$ the open complementary. Then, $j_*M \in \text{PSh}_{\mathcal{D}, rh}(\bar{S})$ is regular holonomic. The result then follows by (i) and GAGA for j_*M and we get $(j_*M, V_D) \in \text{PSh}_{O_{\bar{S}} fil}(\bar{S})$ and $(M, V_D) = (j^*j_*M, j^*V_D) \in \text{PSh}_{O_S fil}(S)$. We can also prove the algebraic case directly using the theory of meromorphic connexions since a simple holonomic D_S -module with support $Z \subset S$ is an integrable connexion on $Z^o = Z \cap S^o$, $S^o \subset S$ being an open subset. \square

We have from Kashiwara or Malgrange the following which relates the graded piece of the Kashiwara-Malgrange V -filtration V_D of a D_S module $M \in \text{PSh}_{\mathcal{D}, rh}(S)$ along a smooth divisor D with the nearby and vanishing cycle functors of $DR(S)(M)$ with respect to D :

Theorem 27. *Let $S \in \text{AnSm}(\mathbb{C})$. Let $D = V(s) \subset S$ be a smooth (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D . Denote by $j : S^o := S \setminus D \hookrightarrow S$ the open complementary embedding and by $k : \tilde{S}^o \xrightarrow{k} S^o \xrightarrow{j} S$ with k the universal covering of S^o . For $M \in \text{PSh}_{\mathcal{D}, rh}(S)$ a regular holonomic D_S module, consider $(M, V_D) \in \text{PSh}_{O_S fil}(S)$ it together with its V_D filtration. Then,*

- there is canonical isomorphism

$$T(\psi_D, DR)(M) : DR(S)(\oplus_{-1 \leq \alpha < 0} \text{Gr}_{V_D, \alpha} M) \xrightarrow{\sim} \psi_D(DR(S)(M)) := Rk_*k^*DR(S)(M)$$

- there is canonical isomorphism

$$\begin{aligned} T(\phi_D, DR)(M) : DR(S)(\oplus_{-1 < \alpha \leq 0} \text{Gr}_{V_D, \alpha} M) &\xrightarrow{\sim} \\ \phi_D(DR(S)(M)) := \text{Cone}(DR(S)(M) &\xrightarrow{\text{ad}(k^*, Rk_*)(-)} \psi_D DR(S)(M)) \end{aligned}$$

- $DR(S)(\partial_s) = \text{can}(DR(S)(M)) \circ T(\psi_D, DR)(M)$, with $\text{can}(DR(S)(M)) : \psi_D DR(S)(M) \rightarrow \phi_D(DR(S)(M))$ the structural embedding of complexes of the cone,
- $DR(S)(s\partial_s) = T \circ T(\psi_D, DR)(M)$, with $T : \psi_D(DR(S)(M)) \rightarrow \psi_D(DR(S)(M))$ the monodromy morphism.
- $T(\phi_D, DR)(M) : DR(S)(s) \simeq \text{var}(DR(S)(M))$ with $\text{var}(DR(S)(M)) := \mathbb{D}^v \text{can}(\mathbb{D}^v DR(S)M) : \phi_D(DR(S)(M)) \rightarrow \psi_D(DR(S)(M))$.

Proof. See [29]. \square

The main tool is the nearby and vanishing cycle functors for Cartier divisors. We need for the definition of Hodge modules on a smooth complex algebraic variety S to extend the V-filtration associated to a smooth Cartier divisor $D \subset S$ of regular holonomic D_S module M such that the monodromy morphism $T : \psi_D(DR(S)(M)) \rightarrow \psi_D(DR(S))$ is quasi-unipotent by a rational V-filtration (i.e. indexed by rational numbers).

Definition 104. *Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $D = V(s) \subset S$ a (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D . We then have the zero section embedding $i : S \hookrightarrow L$. We denote $L_0 = i(S)$ and $j : L^\circ := L \setminus L_0 \hookrightarrow L$ the open complementary subset. We denote $\text{PSh}_{\mathcal{D}(2)fil, rh}(S)^{spD^0} \subset \text{PSh}_{\mathcal{D}(2)fil, rh}(S)$ the full subcategory consisting of objects such that the monodromy operator $T : \psi_D(DR(S)(M^{(an)})) \rightarrow \psi_D(DR(S)(M^{(an)}))$ is quasi-unipotent.*

(i) *Let $(M, F) \in \text{PSh}_{\mathcal{D}(2)fil, rh}(S)^{spD^0}$. By proposition 99, we have the Kashiwara-Malgrange V_S -filtration on $i_{*mod}M$. We refine it to all rational numbers as follows : for $\alpha = k - 1 + r/q \in \mathbb{Q}$, $k, q, r \in \mathbb{Z}$, $q \leq 0$, $0 \leq r \leq q - 1$, we set*

$$V_{S, \alpha} M := q_{V, k}^{-1}(\oplus_{k-1 < \beta \leq \alpha} \text{Gr}_{k, \beta}^{V_S} M \subset V_{S, k} M$$

with $\text{Gr}_{k, \beta}^{V_S} M := \ker(\partial_s s - \beta I) \subset \text{Gr}_k^{V_S} M$ and $q_{V, k} : V_{S, k} M \rightarrow \text{Gr}_k^{V_S} M$ is the projection. We set similarly

$$V_{S, < \alpha} M := q_{V, k}^{-1}(\oplus_{k-1 < \beta < \alpha} \text{Gr}_{k, \beta}^{V_S} M \subset V_{S, k} M$$

The Hodge filtration induced on $\text{Gr}_\alpha^V M$ is

$$F^p \text{Gr}_\alpha^{V_S} M := (F^p M \cap V_{S, \alpha} M) / (F^p M \cap V_{S, < \alpha} M)$$

(ii) *we have using (i) the nearby cycle functor*

$$\psi_D : \text{PSh}_{\mathcal{D}fil, rh}(S)^{spD^0} \rightarrow \text{PSh}_{\mathcal{D}fil, rh}(D/(S)), (M, F) \mapsto \psi_D(M, F) := \oplus_{-1 \leq \alpha < 0} \text{Gr}_{V_S, \alpha} i_{*mod}(M, F),$$

and vanishing cycle functors

$$\begin{aligned} \phi_D : \text{PSh}_{\mathcal{D}(2)fil, rh}(S)^{spD^0} &\rightarrow \text{PSh}_{\mathcal{D}(2)fil, rh}(D/(S)), (M, F) \mapsto \phi_D(M, F) := \oplus_{-1 < \alpha \leq 0} \text{Gr}_{V_S, \alpha} i_{*mod}(M, F), \\ \phi_{D1} : \text{PSh}_{\mathcal{D}(2)fil, rh}(S)^{spD^0} &\rightarrow \text{PSh}_{\mathcal{D}(2)fil, rh}(D/(S)), (M, F) \mapsto \phi_{D1}(M, F) := \text{Gr}_{V_S, 0} i_{*mod}(M, F). \end{aligned}$$

(iii) *This induces, by theorem 27, the nearby cycle functor*

$$\begin{aligned} \psi_D : \text{PSh}_{\mathcal{D}(1,0)fil, rh}(S)^{spD^0} \times_I P_{fil}(S^{(an)}) &\rightarrow \text{PSh}_{\mathcal{D}(1,0)fil, rh, D}(S) \times_I P_{fil, D}(S^{(an)}), \\ ((M, F, W), (K, W), \alpha) \mapsto \psi_D((M, F, W), (K, W), \alpha) &:= (\psi_D(M, F, W), \psi_D(K, W), \psi_D(\alpha)) \end{aligned}$$

and the vanishing cycle functor

$$\begin{aligned} \phi_{D1} : \text{PSh}_{\mathcal{D}(1,0)fil, rh}(S)^{spD^0} \times_I D_{fil}(S^{(an)}) &\rightarrow \text{PSh}_{\mathcal{D}(1,0)fil, rh, D}(S) \times_I P_{fil, D}(S^{(an)}), \\ ((M, F, W), (K, W), \alpha) \mapsto \phi_{D1}((M, F, W), (K, W), \alpha) &:= (\phi_{D1}(M, F, W), \phi_D(K, W), \phi_D(\alpha)) \end{aligned}$$

We have the category of mixed Hodge modules over a complex algebraic variety or a complex analytic space S defined by, for S smooth, by induction on dimension of S , and for S singular using embeddings into smooth complex algebraic varieties, resp. smooth complex analytic spaces:

Definition 105. [28]

(i) *Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Denote $\text{PSh}_{\mathcal{D}fil, rh}(S)^{sp, ssd} \subset \text{PSh}_{\mathcal{D}fil, rh}(S)$ the full subcategory consisting of objects (M, F)*

- such that for all Cartier divisor $D = V(s) \subset S$, $s \in \Gamma(S, L)$, denoting $i : S \hookrightarrow L$ the closed embedding the monodromy morphism $T : \psi_D(DR(S)(M^{(an)})) \rightarrow \psi_D(DR(S)(M^{(an)}))$ is quasi-unipotent, $sF^p V_{S,\alpha} i_{*mod} M = F^p V_{S,\alpha-1} i_{*mod} M$ for $\alpha < 0$, $\partial_s F^p \text{Gr}_\alpha^{V_S} i_{*mod} M = \text{Gr}_{\alpha+1}^{V_S} i_{*mod} M$ for $\alpha > -1$, the filtration induced by F on $\text{Gr}_\alpha^{V_S} i_{*mod} M$ is good,
- which admits a decomposition with D_S module with strict support on closed irreducible subvarieties.

The category of Hodge modules over S of weight w is the full subcategory

$$\iota_S : HM(S, w) = \bigoplus_{d \in \mathbb{N}} HM_{\leq d}(S, w) \hookrightarrow \text{PSh}_{\mathcal{D}fil, rh}(S)^{sp, ssd} \times_I P(S^{(an)}), \hookrightarrow \text{PSh}_{\mathcal{D}fil, rh}(S) \times_I P(S^{(an)})$$

given inductively by, d being the dimension of the support of the D_S modules,

- for $i_0 : s_0 \hookrightarrow S$ a closed point, $i_{0*} \iota_{pt} : HM_{s_0}(S, w) = HS \hookrightarrow \text{PSh}_{\mathcal{D}fil, rh, s_0}(S) \times_I P_{s_0}(S^{(an)})$ consist of Hodge structures of weight w , this gives $HM_0(S, w)$
- for $Z \subset S$ an irreducible closed subvariety of dimension d , $((M, F), K, \alpha) \in \text{PSh}_{\mathcal{D}fil, rh}(S) \times_I P(S^{(an)})$ belongs to $HM_Z(S, w)$ if and only if M has strict support Z (i.e. $\text{supp}(M) = Z$ and for all non trivial subobject N or quotient of M $\text{supp}(N) = Z$), and for all proper maps $f : S^\circ \rightarrow \mathbb{A}^1$ such that $f|_{Z \cap S^\circ} \neq 0$, $j : S^\circ \hookrightarrow S$ being an open subset,

$$\text{Gr}_k^{W(N)} \psi_{f^{-1}(0)}(j^*(M, F), j^*K, j^*\alpha) \in HM_{\leq d-1}(S^\circ, w-1+k) \hookrightarrow \text{PSh}_{\mathcal{D}fil, rh, f^{-1}(0)}(S^\circ) \times_I P_{f^{-1}(0)}(S^{(an)})$$

for all $k \in \mathbb{Z}$, see definition 104, $W(N)$ being the weight filtration associated to the monodromy morphism $T : \psi_{f^{-1}(0)}(DR(S)(M^{(an)})) \rightarrow \psi_{f^{-1}(0)}(DR(S)(M^{(an)}))$, we then set $HM_{\leq d}(S, w) := \bigoplus_{Z \subset S, \dim(Z)=d} HM_Z(S, w)$.

- (ii) Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. The category of mixed Hodge modules over S is the full subcategory

$$\iota_S : MHM(S) \hookrightarrow MHW(S) \hookrightarrow \text{PSh}_{\mathcal{D}(1,0)fil, rh}(S) \times_I P_{fil}(S^{(an)}),$$

where the full subcategory $MHW(S)$ consists of objects $((M, F, W), (K, W), \alpha) \in \text{PSh}_{\mathcal{D}(1,0)fil, rh}(S) \times_I P_{fil}(S^{(an)})$ satisfy

$$(\text{Gr}_i^W((M, F, W), \text{Gr}_i^W(K, W), \text{Gr}_i^W \alpha) \in HM(S).$$

and the objects of $MHM(S)$ satisfy in addition an admissibility condition (in particular the three filtration F, W, V_Z are compatible). As usual, for $Z \subset S$ a closed subset and $j : S \setminus Z \hookrightarrow S$ the open complementary subset, we denote $MHM_Z(S) \subset MHM(S)$ the full subcategory consisting of $((M, F, W), (K, W), \alpha) \in MHM(S)$ such that

$$j^*((M, F, W), (K, W), \alpha) := (j^*(M, F, W), j^*(K, W), j^*\alpha) = 0.$$

- (iii) Let $S \in \text{Var}(\mathbb{C})$ or $S \in \text{AnSp}(\mathbb{C})$ non smooth. Take an open cover $S = \cup_i S_i$ so that there are closed embedding $S_I \hookrightarrow \tilde{S}_I$, with $S_I \in \text{SmVar}(\mathbb{C})$, resp $S_I \in \text{AnSm}(\mathbb{C})$. The category of mixed Hodge modules over S is the full subcategory

$$\iota_S : MHM(S) \hookrightarrow MHW(S) \hookrightarrow \text{PSh}_{\mathcal{D}(1,0)fil, rh}(S/(\tilde{S}_I)) \times_I P_{fil}(S^{(an)})$$

consisting of objects

$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in \text{PSh}_{\mathcal{D}(1,0)fil, rh}^0(S/(\tilde{S}_I)) \times_I P_{fil}(S^{(an)})$$

such that $((M_I, F, W), T(S/(\tilde{S}_I))(K, W), \alpha) \in (MHM_{S_I}(\tilde{S}_I))$ (see (ii)). The category $MHM(S)$ does NOT depend on the open cover and the closed embedding by proposition 100.

(iv) Let $S \in \text{Var}(\mathbb{C})$. We get from (iii) $D(\text{MHM}(S)) := (\text{Ho}_{\text{zar}}, I)(C(\text{MHM}(S)))$. By induction, using the result for mixed hodge structure and the strictness of the Kashiwara-Malgrange V -filtration for morphism of D -module, the morphism of $\text{MHM}(S)$ are strict for F and W (see [28]).

(iv)' Let $S \in \text{AnSp}(\mathbb{C})$. We get from (iii) $D(\text{MHM}(S)) := (\text{Ho}_{\text{usu}}, I)(C(\text{MHM}(S)))$. By induction, using the result for mixed hodge structure and the strictness of the Kashiwara-Malgrange V -filtration for morphism of D -module, the morphism of $\text{MHM}(S)$ are strict for F and W (see [28]).

- Let $S \in \text{SmVar}(\mathbb{C})$. We consider the canonical functor

$$\begin{aligned} \pi_S : C(\text{MHW}(S)) &\xrightarrow{\iota_S} C_{\mathcal{D}(1,0)\text{fil}}(S) \times_I D_{\text{fil}}(S^{\text{an}}) \xrightarrow{p_S} C_{\mathcal{D}(1,0)\text{fil}}(S), \\ &((M, F, W), (K, W), \alpha) \mapsto (M, F, W) \end{aligned}$$

where p_S is the projection functor. Then $\pi_S(\text{MHW}(S)) \subset \text{PSh}_{\mathcal{D}(1,0)\text{fil}}(S)$ is the subcategory consisting of $(M, F, W) \in \text{PSh}_{\mathcal{D}(1,0)\text{fil}}(S)$ such that $((M, F, W), (K, W), \alpha) \in \text{MHW}(S)$ is a W filtered Hodge module for some $(K, W) \in C_{\text{fil}}(S)$. It induces in the derived category the functor

$$\begin{aligned} \pi_S : D(\text{MHW}(S)) &\xrightarrow{\iota_S} D_{\mathcal{D}(1,0)\text{fil}}(S) \times_I D_{\text{fil}}(S^{\text{an}}) \xrightarrow{p_S} D_{\mathcal{D}(1,0)\text{fil}}(S), \\ &((M, F, W), (K, W), \alpha) \mapsto (M, F, W) \end{aligned}$$

after localization with respect to filtered Zariski and usu local equivalence.

- Let $S \in \text{Var}(\mathbb{C})$ non smooth. Take an open cover $S = \cup_i S_i$ such that there are closed embedding $S_I \hookrightarrow \tilde{S}_I$, with $S_I \in \text{SmVar}(\mathbb{C})$. We consider the canonical functor

$$\begin{aligned} \pi_S : C(\text{MHW}(S)) &\hookrightarrow C_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{\text{an}}) \xrightarrow{p_S} C_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)), \\ &(((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto ((M_I, F, W), u_{IJ}) \end{aligned}$$

where p_S is the projection functor. Then $\pi_S(\text{MHW}(S)) \subset \text{PSh}_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I))$ is the subcategory consisting of $((M, F, W), u_{IJ}) \in \text{PSh}_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I))$ such that $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in \text{MHW}(S)$ is a W filtered Hodge module for some $(K, W) \in C_{\text{fil}}(S)$. It induces in the derived category the functor

$$\begin{aligned} \pi_S : D(\text{MHW}(S)) &\xrightarrow{\iota_S} D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{\text{an}}) \xrightarrow{p_S} D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)), \\ &(((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto ((M, F, W), u_{IJ}) \end{aligned}$$

after localization with respect to filtered Zariski and usu local equivalence.

Let $S \in \text{Var}(\mathbb{C})$ or $S \in \text{AnSp}(\mathbb{C})$.

- If $S \in \text{Var}(\mathbb{C})$, let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$, and let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{SmVar}(\mathbb{C})$.
- If $S \in \text{AnSp}(\mathbb{C})$, let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$, and let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{AnSm}(\mathbb{C})$.

Denote $L = [1, \dots, l]$, $L' = [1, \dots, l']$ and $L'' := [1, \dots, l] \sqcup [1, \dots, l']$. We have then the refined open cover $S = \cup_{k \in L} S_k$ and we denote for $I \sqcup I' \subset L''$, $S_{I \sqcup I'} := \cap_{k \in I \sqcup I'} S_k$ and $\tilde{S}_{I \sqcup I'} := \prod_{k \in I \sqcup I'} \tilde{S}_k$, so that we have a closed embedding $i_{I \sqcup I'} : S_{I \sqcup I'} \hookrightarrow \tilde{S}_{I \sqcup I'}$. Consider $\pi_S^L(\text{MHM}(S)) \subset \text{PSh}_{\mathcal{D}\text{fil}}(S/(S_I))$ and $\pi_S^{L'}(\text{MHM}(S)) \subset \text{PSh}_{\mathcal{D}\text{fil}}(S/(S_{I'}))$. For $I \sqcup I' \subset J \sqcup J'$, denote by $p_{I \sqcup I', J \sqcup J'} : \tilde{S}_{J \sqcup J'} \rightarrow \tilde{S}_{I \sqcup I'}$ the projection. We then have a natural transfer map

$$\begin{aligned} T_S^{L/L'} : \pi_S^L(\text{MHM}(S)) &\rightarrow \pi_S^{L'}(\text{MHM}(S)), \\ ((M_I, F, W), s_{IJ}) &\mapsto (\text{ho} \lim_{I \in L} p_{I(I \sqcup I')}^* \text{Gr}_{V_{I \sqcup I'}}^{\text{mod}} p_{I(I \sqcup I')}^*(M_I, F)), s_{I'J'} \end{aligned}$$

with, in the homotopy limit, the natural transition morphisms

$$\begin{aligned} & p_{I'(I \sqcup I')}^* \text{ad}(p_{IJ}^{*mod}, p_{IJ*})(p_{I(I \sqcup I')}^{*mod[-1]}(M_I, F)) : \\ & p_{I'(J \sqcup I')}^*(\text{Gr}_{V_{J \sqcup I'}} p_{J(J \sqcup I')}^{*mod[-1]}(M_J, F)) \rightarrow p_{I'(I \sqcup I')}^*(\text{Gr}_{V_{I \sqcup I'}} p_{I(I \sqcup I')}^{*mod[-1]}(M_I, F)) \end{aligned}$$

for $J \subset I$, and

$$\begin{aligned} & s_{I'J'} : \text{holim}_{I \in L} m^* p_{I'(I \sqcup I')}^*(\text{Gr}_{V_{I \sqcup I'}} p_{I(I \sqcup I')}^{*mod[-1]}(M_I, F)) \rightarrow \\ & \text{holim}_{I \in L} p_{I'J'}^* \text{Gr}_{V_{J'}}(p_{I'J'}^{*mod[-1]} m^* p_{I'(I \sqcup I')}^* \text{Gr}_{V_{I \sqcup I'}} p_{I(I \sqcup I')}^{*mod[-1]}((M_I, F))) \\ & \rightarrow \text{holim}_{I \in L} p_{I'J'}^* p_{J'(I \sqcup J')}^* \text{Gr}_{V_{I \sqcup J'}} p_{I(I \sqcup J')}^{*mod[-1]}(M_I, F) \end{aligned}$$

Proposition 100. (i) Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $\pi_S(\text{MHM}(S) \subset \text{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$ does not depend on the open covering of S and the closed embeddings. More precisely, let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{SmVar}(\mathbb{C})$. Then,

$$T_S^{L/L'} : \pi_S^L(\text{MHM}(S)) \rightarrow \pi_S^{L'}(\text{MHM}(S)),$$

is an equivalence of category with inverse is $T_S^{L'/L} : \pi_S^{L'}(\text{MHM}(S)) \rightarrow \pi_S^L(\text{MHM}(S))$.

(ii) Let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then $\pi_S(\text{MHM}(S) \subset \text{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$ does not depend on the open covering of S and the closed embeddings. More precisely, let $S = \cup_{i'=1}^{l'} S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{AnSm}(\mathbb{C})$. Then,

$$T_S^{L/L'} : \pi_S^L(\text{MHM}(S)) \rightarrow \pi_S^{L'}(\text{MHM}(S)),$$

is an equivalence of category with inverse is $T_S^{L'/L} : \pi_S^{L'}(\text{MHM}(S)) \rightarrow \pi_S^L(\text{MHM}(S))$.

Proof. Follows from the definition of the Hodge filtration which use the V -filtration : see [28]. \square

The main results of Saito, which implies in the algebraic case the six functor formalism on $DMHM(-)$ are the followings

Definition 106. Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. We denote by $VMHS(S) \subset \text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an})$ the full subcategory consisting of variation of mixed Hodge structure, whose objects consist of

$$((L_S, W) \otimes O_S, F), (L_S, W), \alpha \in \text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an})$$

with

- $L_S \in \text{PSh}(S^{an})$ a local system,
- the D_S module structure on $(L_S, W) \otimes O_S$ is given by the flat connection associated to the local system L_S ,
- $F^p(W^q L_S \otimes O_S) \subset (W^q L_S \otimes O_S)$ are locally free O_S subbundle satisfying Griffiths transversality for the D_S module structure (i.e. for the flat connection).
- $\alpha : (L_S, W) \rightarrow DR(S)^{[-1]}((L_S, W) \otimes O_S)$ is the isomorphism given by theorem 26.

Theorem 28. Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$.

(i) A variation of mixed Hodge structure $((L_S, W) \otimes O_S, F), (L_S, W), \alpha \in VMHS(S)$ (see definition 106) is a mixed module. That is $VMHS(S) \subset MHM(S)$.

(ii) For $((M, F, W), (K, W), \alpha) \in \text{MHM}(S)$ a mixed Hodge module with support $\text{supp } M = Z$, there exist an open subset $j : S^\circ \hookrightarrow S$, such that $j^*((M, F, W), (K, W), \alpha) := (j^*(M, F, W), j^*(K, W), j^*\alpha) \in \text{VMHS}(Z \cap S^\circ)$. That is a mixed Hodge module is generically a variation of mixed Hodge structure.

Proof. See [28]. \square

Theorem 29. (i) Let $f : X \rightarrow S$ a projective morphism with $X, S \in \text{AnSp}(\mathbb{C})$, where projective means that there exist a factorization $f : X \xrightarrow{l} \mathbb{P}^N \times S \xrightarrow{p_S} S$ with l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embeddings $i_I : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. For $I \subset [1, \dots, s]$, recall that we denote $S_I := \cap_{i \in I} S_i$ and $X_I := f^{-1}(S_I)$. We have then the following commutative diagram

$$\begin{array}{ccccc} X_I & \xrightarrow{i_I \circ l_I} & \mathbb{P}^N \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \\ j'_{IJ} \uparrow & & p'_{IJ} \uparrow & & p_{IJ} \uparrow \\ X_J & \xrightarrow{i_J \circ l_J} & \mathbb{P}^N \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \end{array}$$

whose right square is cartesian (see section 5). Then, for

$$((M, F, W), (K, W), \alpha) = (((M_I, F, W), u_{IJ}), (K, W), \alpha) \in \text{MHM}(X),$$

where $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}2\text{fil}}(X_I/(\mathbb{P}^N \times \tilde{S}_I))$, $(K, W) \in C_{\text{fil}}(X)$, we have for all $n \in \mathbb{Z}$,

$$(H^n \int_f^{\text{FDR}} ((M_I, F, W), u_{IJ}), R^n f_*(K, W), H^n f_*(\alpha)) \in \text{MHM}(S)$$

and for all $p \in \mathbb{Z}$, the differentials of $\text{Gr}_F^p \int_f^{\text{FDR}} ((M_I, F, W), u_{IJ})$ are strict for the the Hodge filtration F .

(ii) Let $f : X \rightarrow S$ a projective morphism with $X, S \in \text{Var}(\mathbb{C})$, where projective means that there exist a factorization $f : X \xrightarrow{l} \mathbb{P}^N \times S \xrightarrow{p_S} S$ with l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embeddings $i_I : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, s]$, recall that we denote $S_I := \cap_{i \in I} S_i$ and $X_I := f^{-1}(S_I)$. We have then the following commutative diagram

$$\begin{array}{ccccc} X_I & \xrightarrow{i_I \circ l_I} & \mathbb{P}^N \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \\ j'_{IJ} \uparrow & & p'_{IJ} \uparrow & & p_{IJ} \uparrow \\ X_J & \xrightarrow{i_J \circ l_J} & \mathbb{P}^N \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \end{array}$$

whose right square is cartesian (see section 5). Then, for

$$((M, F, W), (K, W), \alpha) = (((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D(\text{MHM}(X)),$$

where $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}2\text{fil}}(X_I/(\mathbb{P}^N \times \tilde{S}_I))$, $(K, W) \in C_{\text{fil}}(X^{\text{an}})$, we have

$$H^n \left(\int_f^{\text{FDR}} ((M_I, F, W), u_{IJ}), Rf_*(K, W), f_*(\alpha) \right) \in \text{MHM}(S)$$

for all $n \in \mathbb{Z}$, and for all $p \in \mathbb{Z}$, the differentials of $\text{Gr}_F^p \int_f^{\text{FDR}} ((M_I, F, W), u_{IJ})$ are strict for the Hodge filtration F .

Proof. (i):See [28].

(ii): By (i) $(H^n \int_f ((M, F, W)^{an}), R^n f_*(K, W), H^n f_*(\alpha)) \in MHM(S^{an})$ for all $n \in \mathbb{Z}$. On the other hand, $T^{\mathcal{D}}(an, f)(M, F, W) : (\int_f (M, F, W))^{an} \xrightarrow{\sim} \int_f ((M, F, W)^{an})$ is an isomorphism since f is proper by theorem GAGA for mixed hodge modules : see [28]. \square

Theorem 30. (i) Let $S \in \text{AnSp}(\mathbb{C})$. Let $Y \in \text{AnSm}(\mathbb{C})$ and $p_S : Y \times S \rightarrow S$ the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embeddings $i_I : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. For $I \subset [1, \dots, s]$, recall that we denote $S_I := \cap_{i \in I} S_i$. We have then the following commutative diagram

$$\begin{array}{ccc} Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \\ p'_{IJ} \uparrow & & \uparrow p_{IJ} \\ Y \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \end{array}$$

which is cartesian (see section 5). Then, for

$$((M, F, W), (K, W), \alpha) = (((M_I, F, W), u_{IJ}), (K, W), \alpha) \in MHM(S),$$

where $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}2fil}(S_I/(\tilde{S}_I))$, $(K, W) \in C_{fil}(S)$,

- $(p_S^{*mod[-]}(M, F, W), p_S^*(K, W), p_S^*(\alpha)) := ((p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W), p_{\tilde{S}_J}^{*mod[-]} u_{IJ}), p_S^*(K, W), p_S^*(\alpha)) \in MHM(S)$
- $(p_S^{\hat{*}mod[-]}(M, F, W), p_S^*(K, W), p_S^*(\alpha)) := ((p_{\tilde{S}_I}^{\hat{*}mod[-]}(M_I, F, W), p_{\tilde{S}_J}^{*mod[-]} u_{IJ}), p_S^*(K, W), p_S^*(\alpha)) \in MHM(S)$

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $Y \in \text{SmVar}(\mathbb{C})$ and $p_S : Y \times S \rightarrow S$ the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embeddings $i_I : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, s]$, recall that we denote $S_I := \cap_{i \in I} S_i$. We have then the following commutative diagram

$$\begin{array}{ccc} Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \\ p'_{IJ} \uparrow & & \uparrow p_{IJ} \\ Y \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \end{array}$$

which is cartesian (see section 5). Then, for

$$((M, F, W), (K, W), \alpha) = (((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D(MHM(S))$$

where $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}2fil}(S_I/(\tilde{S}_I))$, $(K, W) \in C_{fil}(S^{an})$, we have

- $(p_S^{*mod[-]}, p_S^!)((M, F, W), (K, W), \alpha) := ((p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W), p_{\tilde{S}_J}^{*mod[-]} u_{IJ}), p_S^*(K, W), p_S^*(\alpha)) \in D(MHM(S))$
- $(p_S^{\hat{*}mod[-]}, p_S^*)((M, F, W), (K, W), \alpha) := ((p_{\tilde{S}_I}^{\hat{*}mod[-]}(M_I, F, W), p_{\tilde{S}_J}^{*mod[-]} u_{IJ}), p_S^*(K, W), p_S^*(\alpha)) \in D(MHM(S))$.

Proof. (i):See [28].

(ii):Follows immediately from (i) since $(p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W))^{an} = p_{\tilde{S}_I}^{*mod[-]}((M_I, F, W)^{an})$. \square

We have, by the results of Saito, the following key definition.

Definition 107. (i) Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $D = V(s) \subset S$ a divisor with $s \in \Gamma(S, L)$ and L a line bundle (S being smooth, D is Cartier). Denote by $j : S^\circ := S \setminus D \hookrightarrow S$ the open complementary embedding. Let $(M, F, W) \in \pi_{S^\circ}(\text{MHW}(S^\circ))$. Consider the V_S -filtration on $i_{* \text{mod}} M$ (see proposition 99). If (M, F, W) is extendable (which is always the case in the algebraic case), then,

– we have

$$j_{* \text{Hdg}}(M, F, W) := (j_* M, F, W) \in \pi_S(\text{MHM}(S)),$$

$$\text{with } F^p j_* M := \sum_{k \in \mathbb{N}} \partial_s^k F^{p+k} V_{S,0} j_* M \subset j_* M, \quad W_k j_* M := \langle j_* W_k M, W(N)_k \rangle \subset j_* M$$

which is unique such that $j^* j_{* \text{Hdg}}(M, F, W) = (M, F, W)$ and $DR(S)(j_{* \text{Hdg}}(M, F, W)) = j_* DR(S^\circ)(M, W)$,

– there exist

$$j_{! \text{Hdg}}(M, F, W) := \mathbb{D}_S^{\text{Hdg}} j_{* \text{Hdg}} \mathbb{D}_S^{\text{Hdg}}(M, F, W) \in \pi_S(\text{MHM}(S))$$

unique such that $j^* j_{! \text{Hdg}}(M, F, W) = (M, F, W)$ and $DR(S)(j_{! \text{Hdg}}(M, F, W)) = j_{! \text{Hdg}} DR(S^\circ)(M, W)$.

Moreover for $(M', F, W) \in \pi_S(\text{MHM}(S))$, by proposition 102

- there is a canonical map $\text{ad}(j^*, j_{* \text{Hdg}})(M', F, W) : (M', F, W) \rightarrow j_{* \text{Hdg}} j^*(M', F, W)$ in $\pi_S(\text{MHM}(S))$,
- there is a canonical map $\text{ad}(j_{! \text{Hdg}}, j^*)(M', F, W) : j_{! \text{Hdg}} j^*(M', F, W) \rightarrow (M', F, W)$ in $\pi_S(\text{MHM}(S))$.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$. Let $Z = V(\mathcal{I}) \subset S$ an arbitrary closed subset, $\mathcal{I} \subset \mathcal{O}_S$ being an ideal subsheaf. Taking generators $\mathcal{I} = (s_1, \dots, s_r)$, we get $Z = V(s_1, \dots, s_r) = \bigcap_{i=1}^r Z_i \subset S$ with $Z_i = V(s_i) \subset S$, $s_i \in \Gamma(S, \mathcal{L}_i)$ and \mathcal{L}_i a line bundle. Note that Z is an arbitrary closed subset, $d_Z \geq d_X - r$ needing not be a complete intersection. Denote by $j : S^\circ := S \setminus Z \hookrightarrow S$, $j_I : S^{\circ, I} := \bigcap_{i \in I} (S \setminus Z_i) = S \setminus (\bigcup_{i \in I} Z_i) \xrightarrow{j_I^o} S^\circ \xrightarrow{j} S$ the open complementary embeddings, where $I \subset \{1, \dots, r\}$. For $(M, F, W) \in \pi_{S^\circ}(C(\text{MHM}(S^\circ)))$, we define by (i)

– the (bi)-filtered complex of D_S -modules

$$j_{* \text{Hdg}}(M, F, W) := \varinjlim_{\{(Z_i)_{i \in [1, \dots, r]}, Z_i \subset S, \cap Z_i = Z, Z_i \subset Z_i\}} \text{Tot}_{\text{card } I = \bullet} (j_{I^*}^{\text{Hdg}} j_I^{o*}(M, F, W)) \in \pi_S(C(\text{MHM}(S))),$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \text{ad}(j_{IJ^*}^*, j_{IJ^*}^{\text{Hdg}})(j_I^{o*}(M, F, W))$, $j_{IJ} : S^{\circ, J} \hookrightarrow S^{\circ, I}$ being the open embedding, and $d_{IJ} = 0$ if $I \not\subset J$,

– the (bi)-filtered complex of D_S -modules

$$\begin{aligned} j_{! \text{Hdg}}(M, F, W) : &= \varprojlim_{\{(Z_i)_{i \in [1, \dots, r]}, Z_i \subset S, \cap Z_i = Z, Z_i \subset Z_i\}} \text{Tot}_{\text{card } I = -\bullet} (j_{I!}^{\text{Hdg}} j_I^{o*}(M, F, W)) \\ &= \mathbb{D}_S^{\text{Hdg}} j_{* \text{Hdg}} \mathbb{D}_S^{\text{Hdg}}(M, F, W) \in \pi_S(C(\text{MHM}(S))), \end{aligned}$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \text{ad}(j_{I!}^{\text{Hdg}}, j_{I!}^*)(j_I^{o*}(M, F, W))$, $j_{IJ} : S^{\circ, J} \hookrightarrow S^{\circ, I}$ being the open embedding, and $d_{IJ} = 0$ if $I \not\subset J$.

By definition, we have for $(M, F, W) \in \pi_{S^\circ}(C(\text{MHM}(S^\circ)))$, $j^* j_{* \text{Hdg}}(M, F, W) = (M, F, W)$ and $j^* j_{! \text{Hdg}}(M, F, W) = (M, F, W)$. For $(M', F, W) \in \pi_S(C(\text{MHM}(S)))$, there is, by construction,

- a canonical map $\text{ad}(j^*, j_{* \text{Hdg}})(M', F, W) : (M', F, W) \rightarrow j_{* \text{Hdg}} j^*(M', F, W)$,
- a canonical map $\text{ad}(j_{! \text{Hdg}}, j^*)(M', F, W) : j_{! \text{Hdg}} j^*(M', F, W) \rightarrow (M', F, W)$.

For $(M, F, W) \in \pi_{S^\circ}(C(MHM(S^\circ)))$,

– we have the canonical map in $C_{\mathcal{D}(1,0)fil}(S)$

$$T(j_{*Hdg}, j_*)(M, F, W) := k \circ \text{ad}(j^*, j_*)(j_{*Hdg}(M, F, W)) : j_{*Hdg}(M, F, W) \rightarrow j_*E(M, F, W),$$

– we have the canonical map in $C_{\mathcal{D}(1,0)fil}(S)$

$$\begin{aligned} T(j!, j!_{Hdg})(M, F, W) &:= \mathbb{D}_S^K L_D(k \circ \text{ad}(j^*, j_*)(-)) : \\ j!(M, F, W) &:= \mathbb{D}_S^K L_D j_* E(\mathbb{D}_S^K(M, F, W)) \rightarrow \mathbb{D}_S^K L_D j_{*Hdg} \mathbb{D}_S^K(M, F, W) = j!_{Hdg}(M, F, W) \end{aligned}$$

the canonical maps.

Remark 9. Let $j : S^\circ \hookrightarrow S$ an open embedding, with $S \in \text{SmVar}(\mathbb{C})$. Then, for $((M, F, W), (K, W), \alpha) \in MHM(S^\circ)$,

- the map $T(j!, j!_{Hdg})(M, W) : j!_w(M, W) \rightarrow j!_{Hdg}(M, W)$ in $C_{\mathcal{D}0fil}(S)$ is a filtered quasi-isomorphism (apply the functor $DR^{[-1]}(S^\circ)$ and use theorem 26 and theorem 99).
- the map $T(j_{*Hdg}, j_*)(M, W) : j_{*Hdg}(M, W) \rightarrow j_{*w}(M, W)$ in $C_{\mathcal{D}0fil}(S)$ is a filtered quasi-isomorphism (apply the functor $DR^{[-1]}(S^\circ)$ and use theorem 26 and theorem 99).

Hence, for $((M, F, W), (K, W), \alpha) \in MHM(S^\circ)$,

- we get, for all $p, n \in \mathbb{N}$, monomorphisms

$$F^p H^n T(j!, j!_{Hdg})(M, F, W) : F^p H^n j!_w(M, F, W) \hookrightarrow F^p H^n j!_{Hdg}(M, F, W)$$

in $\text{PSh}_{O_S}(S)$, but $F^p H^n j!_w(M, F, W) \neq F^p H^n j!_{Hdg}(M, F, W)$ (it leads to different F -filtrations), since $F^p H^n j!(M, F) \subset H^n j!M$ are sub D_S module while the F -filtration on $H^n j!_{Hdg}(M, F)$ is given by Kashiwara-Malgrange V -filtrations, hence satisfy a non trivial Griffith transversality property, thus $H^n j!(M, F)$ and $H^n j!_{Hdg}(M, F)$ are isomorphic as D_S -modules but NOT isomorphic as filtered D_S -modules.

- we get, for all $p, n \in \mathbb{N}$, monomorphisms

$$T(j_{*Hdg}, j_*)(M, F, W) : F^p H^n j_{*Hdg}(M, F, W) \hookrightarrow F^p H^n j_*E(M, F, W)$$

in $\text{PSh}_{O_S}(S)$, but $F^p H^n j_{*Hdg}(M, F, W) \neq F^p H^n j_{*w}(M, F, W)$ (it leads to different F -filtrations), since $F^p H^n j_*E(M, F) \subset H^n j_*E(M)$ are sub D_S module while the F -filtration on $H^n j_{*Hdg}(M, F)$ is given by Kashiwara-Malgrange V -filtrations, hence satisfy a non trivial Griffith transversality property, thus $H^n j_*E(M, F)$ and $H^n j_{*Hdg}(M, F)$ are isomorphic as D_S -modules but NOT isomorphic as filtered D_S -modules.

We make the following key definition

Definition 108. Let $S \in \text{SmVar}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the complementary open embedding.

(i) We define using definition 107, the filtered Hodge support section functor

$$\begin{aligned} \Gamma_Z^{Hdg} : \pi_S(C(MHM(S))) &\rightarrow \pi_S(C(MHM(S))), \\ (M, F, W) &\mapsto \Gamma_Z^{Hdg}(M, F, W) := \text{Cone}(\text{ad}(j^*, j_{*Hdg})(M, F) : (M, F) \rightarrow j_{*Hdg} j^*(M, F))[-1], \end{aligned}$$

together we the canonical map $\gamma_Z^{Hdg}(M, F, W) : \Gamma_Z^{Hdg}(M, F, W) \rightarrow (M, F, W)$. We then have the canonical map in $C_{\mathcal{D}(2)fil}(S)$

$$T(\Gamma_Z^{Hdg}, \Gamma_Z)(M, F, W) := (I, T(j_{*Hdg}, j_*)(M, F, W)) : \Gamma_Z^{Hdg}(M, F, W) \rightarrow \Gamma_Z E(M, F, W)$$

unique up to homotopy such that $\gamma_Z^{Hdg}(M, F, W) = \gamma_Z(E(M, F, W)) \circ T(\Gamma_Z^{Hdg}, \Gamma_Z)(M, F, W)$.

(i)' Since $j_{*Hdg} : \pi_{S^o}(C(MHM(S^o))) \rightarrow \pi_S(C(MHM(S)))$ is an exact functor, Γ_Z^{Hdg} induces the functor

$$\Gamma_Z^{Hdg} : \pi_S(D(MHM(S))) \rightarrow \pi_S(D(MHM(S))), (M, F, W) \mapsto \Gamma_Z^{Hdg}(M, F, W)$$

(ii) We define using definition 107, the dual filtered Hodge support section functor

$$\Gamma_Z^{\vee, Hdg} : \pi_S(C(MHM(S))) \rightarrow \pi_S(C(MHM(S))),$$

$$(M, F, W) \mapsto \Gamma_Z^{\vee, Hdg}(M, F, W) := \text{Cone}(\text{ad}(j_{!Hdg}, j^*)(M, F, W) : j_{!Hdg}, j^*(M, F, W) \rightarrow (M, F, W)),$$

together with the canonical map $\gamma_Z^{\vee, Hdg}(M, F, W) : (M, F, W) \rightarrow \Gamma_Z^{\vee, Hdg}(M, F, W)$. We then have the canonical map in $C_{\mathcal{D}(2)fil}(S)$

$$T(\Gamma_Z^{\vee, h}, \Gamma_Z^{\vee, Hdg})(M, F, W) := (I, T(j_!, j_{!Hdg})(M, F, W)) : \Gamma_Z^{\vee, h}(M, F, W) \rightarrow \Gamma_Z^{\vee, Hdg}(M, F, W)$$

unique up to homotopy such that $\gamma_Z^{\vee, Hdg}(M, F, W) = T(\Gamma_Z^{\vee, h}, \Gamma_Z^{\vee, Hdg})(M, F, W) \circ \gamma_Z^{\vee, h}(M, F, W)$.

(ii)' Since $j_{!Hdg} : \pi_{S^o}(C(MHM(S^o))) \rightarrow \pi_S(C(MHM(S)))$ is an exact functor, $\Gamma_Z^{Hdg, \vee}$ induces the functor

$$\Gamma_Z^{\vee, Hdg} : \pi_S(D(MHM(S))) \rightarrow \pi_S(D(MHM(S))), (M, F, W) \mapsto \Gamma_Z^{\vee, Hdg}(M, F, W)$$

We now give the definition of the filtered Hodge inverse image functor :

Definition 109. (i) Let $i : Z \hookrightarrow S$ be a closed embedding, with $Z, S \in \text{SmVar}(\mathbb{C})$. Then, for $(M, F, W) \in \pi_S(C(MHM(S)))$, we set

$$i_{Hdg}^{*mod}(M, F, W) := i^* \mathcal{S}_Z^{-1} \Gamma_Z^{Hdg}(M, F, W) \in \pi_Z(D(MHM(Z)))$$

and

$$i_{Hdg}^{\hat{*}mod}(M, F, W) := i^* \mathcal{S}_Z^{-1} \Gamma_Z^{\vee, Hdg}(M, F, W) \in \pi_Z(D(MHM(Z)))$$

using the fact that $\mathcal{S}_Z : \pi_Z(D(MHM(Z))) \rightarrow \pi_S(D(MHM_Z(S)))$ is an equivalence of category since $\mathcal{S}_Z : D(MHM_Z(S)) \rightarrow D(MHM_Z(S))$ is an equivalence of category by [28].

(ii) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection.

– For $(M, F, W) \in \pi_S(C(MHM(S)))$ we set

$$f_{Hdg}^{*mod}(M, F, W) := i_{Hdg}^{*mod} p_S^{*mod[-1]}(M, F, W)(d_X)[2d_X] \in \pi_X(D(MHM(X))),$$

– For $(M, F, W) \in \pi_S(C(MHM(S)))$ we set

$$f_{Hdg}^{\hat{*}mod}(M, F, W) := i_{Hdg}^{\hat{*}mod} p_S^{*mod[-1]}(M, F, W) \pi_X(D(MHM(X))),$$

If $j : S^o \hookrightarrow S$ is a closed embedding, we have (see [28]), for $(M, F, W) \in \pi_S(C(MHM(S)))$,

$$j_{Hdg}^{*mod}(M, F, W) = j_{Hdg}^{\hat{*}mod}(M, F, W) = j^*(M, F, W) \in \pi_{S^o}(D(MHM(S^o)))$$

(iii) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$, where i is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection.

– For $(M, F, W) \in \pi_S(C(MHM(S)))$ we set

$$f_{Hdg}^{*mod}(M, F, W) := \Gamma_X^{Hdg} p_S^{*mod[-]}(M, F, W)(d_X)[2d_X] \in \pi_{X \times S}(C(MHM(X \times S))),$$

We have for $(M, F, W) \in \pi_S(C(MHM(S)))$, the canonical map in $C_{\mathcal{D}(1,0)fil}(X \times S)$

$$\begin{aligned} T(f_{Hdg}^{*mod}, f^{*mod, \Gamma})(M, F, W) : f_{Hdg}^{*mod, \Gamma}(M, F, W) &:= \Gamma_X^{Hdg} p_S^{*mod[-]}(M, F, W) \\ \xrightarrow{T(\Gamma_X^{Hdg}, \Gamma_X)(-)} \Gamma_X E(p_S^{*mod[-]}(M, F, W)) &=: f^{*mod[-], \Gamma}(M, F, W) \end{aligned}$$

– For $(M, F, W) \in \pi_S(C(MHM(S)))$ we set

$$f_{Hdg}^{*mod}(M, F, W) := \Gamma_X^{\vee, Hdg} p_S^{*mod[-]}(M, F, W) \in \pi_{X \times S}(C(MHM(X \times S))),$$

We have for $(M, F, W) \in \pi_S(C(MHM(S)))$, the canonical map in $C_{\mathcal{D}(1,0)fil}(X \times S)$

$$\begin{aligned} T(f^{*mod, \Gamma}, f_{Hdg}^{*mod})(M, F, W) : f^{*mod, \Gamma}(M, F, W) &:= \Gamma_X^{\vee, h} p_S^{*mod[-]}(M, F, W) \\ \xrightarrow{T(\Gamma_X^{\vee, h}, \Gamma_X^{\vee, Hdg})(-)} \Gamma_X^{\vee, Hdg} p_S^{*mod[-]}(M, F, W) &=: f_{Hdg}^{*mod}(M, F, W) \end{aligned}$$

Definition-Proposition 19. (i) Let $g : S' \rightarrow S$ a morphism with $S', S \in \text{SmVar}(\mathbb{C})$ and $i : Z \hookrightarrow S$ a closed subset. Then, for $(M, F, W) \in \pi_S(C(MHM(S)))$, there is a canonical map in $\pi_S(C(MHM_{S'}(S' \times S)))$

$$T^{Hdg}(g, \gamma)(M, F, W) : g_{Hdg}^{*mod, \Gamma} \Gamma_Z^{Hdg}(M, F, W) \rightarrow \Gamma_{Z \times_S S'}^{Hdg} g_{Hdg}^{*mod, \Gamma}(M, F, W)$$

unique up to homotopy such that

$$\gamma_{Z \times_S S'}^{Hdg}(g_{Hdg}^{*mod, \Gamma}(M, F, W)) \circ T^{Hdg}(g, \gamma)(M, F, W) = g_{Hdg}^{*mod, \Gamma} \gamma_Z^{Hdg}(M, F, W).$$

(i)' Let $g : S' \rightarrow S$ a morphism with $S', S \in \text{SmVar}(\mathbb{C})$ and $i : Z \hookrightarrow S$ a closed subset. Then, for $(M, F, W) \in \pi_S(C(MHM(S)))$, there is a canonical isomorphism in $\pi_S(C(MHM_{S'}(S' \times S)))$

$$T^{Hdg}(g, \gamma^\vee)(M, F, W) : \Gamma_{Z \times_S S'}^{Hdg} g_{Hdg}^{*mod, \Gamma}(M, F, W) \xrightarrow{\sim} g_{Hdg}^{*mod, \Gamma} \Gamma_Z^{Hdg}(M, F, W)$$

unique up to homotopy such that

$$\gamma_{Z \times_S S'}^{\vee, Hdg}(g_{Hdg}^{*mod, \Gamma}(M, F, W)) \circ g_{Hdg}^{*mod, \Gamma} \gamma_Z^{\vee, Hdg}(M, F, W) = T^{Hdg}(g, \gamma)(M, F, W).$$

(ii) Let $S \in \text{SmVar}(\mathbb{C})$ and $i_1 : Z_1 \hookrightarrow S$, $i_2 : Z_2 \hookrightarrow Z_1$ be closed embeddings. Then, for $(M, F, W) \in \pi_S(C(MHM(S)))$,

– there is a canonical map $T(Z_2/Z_1, \gamma^{Hdg})(M, F, W) : \Gamma_{Z_2}^{Hdg}(M, F, W) \rightarrow \Gamma_{Z_1}^{Hdg}(M, F, W)$ in $\pi_S(C(MHM(S)))$ unique up to homotopy such that

$$\gamma_{Z_1}^{Hdg}(G, F) \circ T(Z_2/Z_1, \gamma^{Hdg})(G, F) = \gamma_{Z_2}^{Hdg}(G, F)$$

together with a distinguish triangle in $K(\pi_S(MHM(S)))$

$$\begin{aligned} \Gamma_{Z_2}^{Hdg}(M, F, W) &\xrightarrow{T(Z_2/Z_1, \gamma^{Hdg})(M, F, W)} \Gamma_{Z_1}^{Hdg}(M, F, W) \\ \xrightarrow{\text{ad}(j_2^*, j_{2*}^{Hdg})(\Gamma_{Z_1}^{Hdg}(G, F))} \Gamma_{Z_1/\setminus Z_2}^{Hdg}(G, F) &\rightarrow \Gamma_{Z_2}^{Hdg}(G, F)[1] \end{aligned}$$

- there is a canonical map $T(Z_2/Z_1, \gamma^{\vee, Hdg})(M, F, W) : \Gamma_{Z_1}^{\vee, Hdg}(M, F, W) \rightarrow \Gamma_{Z_2}^{\vee, Hdg}(M, F, W)$ in $\pi_S(C(MHM(S)))$ unique up to homotopy such that

$$\gamma_{Z_2}^{\vee, Hdg}(M, F, W) = T(Z_2/Z_1, \gamma^{\vee, Hdg})(M, F, W) \circ \gamma_{Z_1}^{\vee, Hdg}(M, F, W).$$

together with a distinguish triangle in $K(\pi_S((MHM(S))))$

$$\begin{array}{ccc} \Gamma_{Z_1 \setminus Z_2}^{\vee, Hdg}(M, F, W) & \xrightarrow{\text{ad}(j_{21}^{Hdg}, j_2^*)(M, F, W)} & \Gamma_{Z_1}^{\vee, Hdg}(M, F, W) \\ \xrightarrow{T(Z_2/Z_1, \gamma^{\vee, Hdg})(M, F, W)} & \Gamma_{Z_2}^{\vee, Hdg}(M, F, W) & \rightarrow \Gamma_{Z_2 \setminus Z_1}^{\vee, Hdg}(M, F, W)[1] \end{array}$$

Proof. Follows from the projection case and the closed embedding case using the adjunction maps. \square

The definitions 108 and 109 immediately extends to the non smooth case :

Definition 110. Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote $Z_I := Z \cap S_I$. Denote by $j : S \setminus Z \hookrightarrow S$ and $\tilde{j}_I : \tilde{S}_I \setminus Z_I \hookrightarrow \tilde{S}_I$ the complementary open embeddings.

- (i) We define using definition 107, the filtered Hodge support section functor

$$\begin{aligned} \Gamma_Z^{Hdg} : \pi(C(MHM(S))) &\rightarrow \pi(C(MHM(S))), ((M_I, F, W), u_{IJ}) \mapsto \Gamma_Z^{Hdg}((M_I, F, W), u_{IJ}) := \\ \text{Cone}(\text{ad}(j^*, j_{*Hdg})((M_I, F, W), u_{IJ}) : ((M_I, F, W), u_{IJ}) &\rightarrow (\tilde{j}_{I*}^{Hdg} \tilde{j}_I^*(M_I, F, W), \tilde{j}_J(u_{IJ}))[-1], \end{aligned}$$

together with the canonical map $\gamma_Z^{Hdg}((M_I, F, W), u_{IJ}) : \Gamma_Z^{Hdg}((M_I, F, W), u_{IJ}) \rightarrow ((M_I, F, W), u_{IJ})$. We then have the canonical map in $C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$

$$\begin{aligned} T(\Gamma_Z^{Hdg}, \Gamma_Z)((M_I, F, W), u_{IJ}) &:= (I, T(j_{*Hdg}, j_*)(M, F, W)) : \\ \Gamma_Z^{Hdg}((M_I, F, W), u_{IJ}) &\rightarrow (\Gamma_Z E(M_I, F, W), \Gamma(u_{IJ})) \end{aligned}$$

unique up to homotopy such that

$$\gamma_Z^{Hdg}((M_I, F, W), u_{IJ}) = (\gamma_{Z_I}(E(M_I, F, W))) \circ T(\Gamma_Z^{Hdg}, \Gamma_Z)((M_I, F, W), u_{IJ}).$$

- (i)' Since $\tilde{j}_{I*}^{Hdg} : \pi_{\tilde{S}_I}(C(MHM(\tilde{S}_I \setminus S_I))) \rightarrow \pi_{\tilde{S}_I}(C(MHM(\tilde{S}_I)))$ are exact functors, Γ_Z^{Hdg} induces the functor

$$\Gamma_Z^{Hdg} : \pi_S(D(MHM(S))) \rightarrow \pi_S(D(MHM(S))), ((M_I, F, W), u_{IJ}) \mapsto \Gamma_Z^{Hdg}((M_I, F, W), u_{IJ})$$

- (ii) We define using definition 107, the dual filtered Hodge support section functor

$$\begin{aligned} \Gamma_Z^{\vee, Hdg} : \pi(C(MHM(S))) &\rightarrow \pi(C(MHM(S))), ((M_I, F, W), u_{IJ}) \mapsto \Gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ}) := \\ \text{Cone}(\text{ad}(j_{!Hdg}, j^*)((M_I, F, W), u_{IJ}) : (\tilde{j}_{I!}^{Hdg} \tilde{j}_I^*(M_I, F, W), (\tilde{j}_J(u_{IJ}^d))^d) &\rightarrow ((M_I, F, W), u_{IJ})), \end{aligned}$$

together with the canonical map $\gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ}) : ((M_I, F, W), u_{IJ}) \rightarrow \Gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ})$. We then have the canonical map in $C_{\mathcal{D}(2)\text{fil}}(S/(\tilde{S}_I))$

$$\begin{aligned} T(\Gamma_Z^{\vee, h}, \Gamma_Z^{\vee, Hdg})((M_I, F, W), u_{IJ}) &:= (I, T(j_!, j_{!Hdg})((M_I, F, W), u_{IJ})) : \\ (\Gamma_Z^{\vee, h}(M_I, F, W), \Gamma_Z^{\vee, h}(u_{IJ})) &\rightarrow \Gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ}) \end{aligned}$$

unique up to homotopy such that

$$\gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ}) = T(\Gamma_Z^{\vee, h}, \Gamma_Z^{\vee, Hdg})((M_I, F, W), u_{IJ}) \circ (\gamma_{Z_I}^{\vee, h}(M_I, F, W)).$$

(ii)' Since $\tilde{j}_I^{Hdg} : \pi_{\tilde{S}_I}(C(MHM(\tilde{S}_I \setminus S_I))) \rightarrow \pi_{\tilde{S}_I}(C(MHM(\tilde{S}_I)))$ are exact functors, $\Gamma_Z^{Hdg, \vee}$ induces the functor

$$\Gamma_Z^{\vee, Hdg} : \pi_S(D(MHM(S))) \rightarrow \pi_S(D(MHM(S))), ((M_I, F, W), u_{IJ}) \mapsto \Gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ})$$

Definition 111. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i \in I} S_i$ an open cover such that there exist closed embeddings $i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote $X_I := f^{-1}(S_I)$. We have then $X = \cup_{i \in I} X_i$ and the commutative diagrams

$$\begin{array}{ccccc} f : X_I & \xrightarrow{l_I} & Y \times S_I & \xrightarrow{p_{S_I}} & S_I \\ & \searrow & \downarrow i'_I := (I \times i_I) & & \downarrow i_I \\ & & Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I} := \tilde{f}_I} & \tilde{S}_I \end{array}$$

(i) For $((M_I, F, W), u_{IJ}) \in \pi_S(C(MHM(S)))$ we set (see definition 110 for l)

$$f_{Hdg}^{*mod}((M_I, F, W), u_{IJ}) := \Gamma_X^{Hdg}(p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W), u_{IJ})(d_Y)[2d_Y] \in \pi_X(C(MHM(X))),$$

We have for $((M_I, F, W), u_{IJ}) \in \pi_S(C(MHM(S)))$, the canonical map in $C_{\mathcal{D}(1,0)fil}(X/(Y \times \tilde{S}_I))$

$$\begin{aligned} T(f_{Hdg}^{*mod}, f^{*mod, \Gamma})((M_I, F, W), u_{IJ}) : f_{Hdg}^{*mod}((M_I, F, W), u_{IJ}) &:= \Gamma_X^{Hdg}(p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W), p_{\tilde{S}_I}^{*mod[-]}u_{IJ}) \\ &\xrightarrow{T(\Gamma_X^{Hdg, \Gamma_X}(-))} (\Gamma_X E(p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W)), \tilde{f}_I^{*mod[-]}u_{IJ}) =: f^{*mod[-], \Gamma}(M, F, W) \end{aligned}$$

(ii) For $((M_I, F, W), u_{IJ}) \in \pi_S(C(MHM(S)))$ we set (see definition 110 for l)

$$f_{Hdg}^{\hat{*}mod}(M, F, W) := \Gamma_X^{\vee, Hdg}(p_{\tilde{S}_I}^{*mod[-]}(M_I, F, W), p_{\tilde{S}_I}^{*mod[-]}u_{IJ}) \in \pi_X(C(MHM(X))),$$

We have for $(M, F, W) \in \pi_S(C(MHM(S)))$, the canonical map in $C_{\mathcal{D}(1,0)fil}(X/(Y \times \tilde{S}_I))$

$$\begin{aligned} T(f^{\hat{*}mod, \Gamma}, f_{Hdg}^{\hat{*}mod[-]})((M_I, F, W), u_{IJ}) : f^{\hat{*}mod[-], \Gamma}(M, F, W) &:= \mathbb{D}_S^K f^{\hat{*}mod[-], \Gamma} \mathbb{D}_X^K((M_I, F, W), u_{IJ}) \\ &\xrightarrow{\mathbb{D}_S^K T(\Gamma_X^{Hdg, \Gamma_X}(-))} \Gamma_X^{\vee, Hdg}(p_S^{*mod[-]}(M_I, F, W), p_{\tilde{S}_I}^{*mod[-]}u_{IJ}) =: f_{Hdg}^{\hat{*}mod[-]}(M, F, W) \end{aligned}$$

Proposition 101. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{QPVar}(\mathbb{C})$ or with $X, Y, S \in \text{SmVar}(\mathbb{C})$.

(i) Let $(M, F, W) \in \pi_S(C(MHM(S)))$. Then,

$$(f_2 \circ f_1)_{Hdg}^{*mod}(M, F) = f_{1Hdg}^{*mod} f_{2Hdg}^{*mod}(M, F) \in \pi_X(D(MHM(X))).$$

(ii) Let $(M, F, W) \in \pi_S(C(MHM(S)))$. Then,

$$(f_2 \circ f_1)_{Hdg}^{\hat{*}mod}(M, F) = f_{1Hdg}^{\hat{*}mod} f_{2Hdg}^{\hat{*}mod}(M, F) \in \pi_X(D(MHM(X)))$$

Proof. (i): Follows from the unicity of the functor j_{*Hdg} .

(ii): Follows from the unicity of the functor $j_{!Hdg}$. □

- Let $f : X \rightarrow S$ a morphism with $S, X \in \text{SmVar}(\mathbb{C})$. Let $f : X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} S$ a compactification of f with $\bar{X} \in \text{SmVar}(\mathbb{C})$ and j the open embedding. We set, for $(M, F, W) \in \pi_X(D(\text{MHM}(X)))$, using definition 107

$$\int_f^{\text{Hdg}} (M, F, W) := \int_{\bar{f}}^{\text{FDR}} j_* \text{Hdg}(M, F, W) \in D_{\mathcal{D}(1,0)\text{fil}}(S)$$

and

$$\int_{f!}^{\text{Hdg}} (M, F, W) := \int_{\bar{f}}^{\text{FDR}} j_! \text{Hdg}(M, F, W) \in D_{\mathcal{D}(1,0)\text{fil}}(S)$$

- Let $f : X \rightarrow S$ a morphism with $S, X \in \text{QPVar}(\mathbb{C})$. Consider a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $\bar{Y} \in \text{PSmVar}(\mathbb{C})$ a smooth compactification of Y with $j : Y \hookrightarrow \bar{Y}$ the open embedding. Then $\bar{f} : \bar{X} \xrightarrow{\bar{l}} \bar{Y} \times_S \bar{p} \rightarrow S$ is a compactification of f , with $\bar{X} \subset \bar{Y} \times S$ the closure of X and \bar{l} the closed embedding. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $((M_I, F, W), u_{IJ}) \in \pi_X(C(\text{MHM}(X))) \subset C_{\mathcal{D}(1,0)\text{fil},rh}(X/(Y \times \tilde{S}_I))$, we set using definition 107

$$\int_f^{\text{Hdg}} ((M_I, F, W), u_{IJ}) := \int_{\bar{f}}^{\text{FDR}} ((I \times j)_* \text{Hdg}((M_I, F, W), u_{IJ})) \in D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I))$$

and

$$\int_{f!}^{\text{Hdg}} ((M_I, F, W), u_{IJ}) := \int_{\bar{f}}^{\text{FDR}} ((I \times j)_! \text{Hdg}((M_I, F, W), u_{IJ})) \in D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I))$$

From the D-module case on algebraic varieties and the constructible sheaves case on CW complexes, we get :

Definition 112. (i) Let $f : X \rightarrow S$ a morphism with $S, X \in \text{SmVar}(\mathbb{C})$. Let $f : X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} S$ a compactification of f with $\bar{X} \in \text{SmVar}(\mathbb{C})$ and j the open embedding. Let

$$\alpha : (K, W) \otimes \mathbb{C}_{X^{an}} \rightarrow DR(X)((M, W)^{an})$$

a morphism in $D_{\text{fil}}(X^{an})$, with $(M, F, W) \in \pi_X(C(\text{MHM}(X)))$ and $(K, W) \in D_{\text{fil}}(X^{an})$. We then consider the maps in $D_{\text{fil}}(S^{an})$

$$\begin{aligned} f_* \alpha &: Rf_{*w}(K, W) \otimes \mathbb{C}_{S^{an}} := R\bar{f}_* Rj_{*w}(K, W) \otimes \mathbb{C}_{S^{an}} \\ &\xrightarrow{R\bar{f}_* j_* \alpha} R\bar{f}_* Rj_{*w} DR(X)((M, W)^{an}) \xrightarrow{T^w(j, \otimes)(-)^{-1}} R\bar{f}_* DR(\bar{X})(j_* \text{Hdg}(M, W)^{an}) \\ &\xrightarrow{T(\bar{f}, DR)(-)^{-1}} DR(S)((\int_{\bar{f}} (j_* \text{Hdg}(M, W))^{an}) = DR(S)((\int_f^{\text{Hdg}} (M, W))^{an}) \end{aligned}$$

and

$$\begin{aligned} f_! \alpha &: Rf_{!w}(K, W) \otimes \mathbb{C}_{S^{an}} := R\bar{f}_* Rj_{!w}(K, W) \otimes \mathbb{C}_{S^{an}} \\ &\xrightarrow{R\bar{f}_* j_! \alpha} R\bar{f}_* Rj_{!w} DR(X)((M, W)^{an}) \xrightarrow{\mathbb{D}T^w(j, \otimes)(-)} R\bar{f}_* DR(\bar{X})(j_! \text{Hdg}(M, W)^{an}) \\ &\xrightarrow{T(\bar{f}, DR)(-)^{-1}} DR(S)((\int_{\bar{f}} (j_! \text{Hdg}(M, W))^{an}) = DR(S)((\int_{f!}^{\text{Hdg}} (M, W))^{an}), \end{aligned}$$

see definition 102 and definition 107 .

(ii) Let $f : X \rightarrow S$ a morphism with $S, X \in \text{QPVar}(\mathbb{C})$. Consider a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $\bar{Y} \in \text{PSmVar}(\mathbb{C})$ a smooth compactification of Y with $j : Y \hookrightarrow \bar{Y}$ the open embedding. Then $\bar{f} : \bar{X} \xrightarrow{\bar{l}} \bar{Y} \times_S \bar{S} \xrightarrow{\bar{p}} S$ is a compactification of f , with $\bar{X} \subset \bar{Y} \times S$ the closure of X and \bar{l} the closed embedding. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let

$$\alpha : T(X/(Y \times \tilde{S}_I))((K, W) \otimes \mathbb{C}_{X^{an}}) \rightarrow DR(X)((M_I, W)^{an}, u_{IJ}^{an})$$

a morphism in $D_{fil}(X^{an}/(Y \times \tilde{S}_I^{an}))$, with $((M_I, W), u_{IJ}) \in \pi_X(C(MHM(X))) \subset C_{\mathcal{D}0fil, rh}(X/(Y \times \tilde{S}_I))$ and $(K, W) \in D_{fil}(X^{an})$. We then consider the maps in $D_{fil}(S^{an}/(\tilde{S}_I^{an}))$

$$\begin{aligned} f_*\alpha &= f_*(\alpha) : T(S/\tilde{S}_I)(Rf_{*w}(K, W)) \otimes \mathbb{C}_{S^{an}} \\ \xrightarrow{=} T(S/\tilde{S}_I)(R\bar{p}_*(I \times j)_{*w}(K, W)) \otimes \mathbb{C}_{S^{an}} &\xrightarrow{=} R\bar{p}_*(I \times j)_{*w}T(X/(Y \times \tilde{S}_I))((K, W) \otimes \mathbb{C}_{X^{an}}) \\ &\xrightarrow{Rp_*\alpha} R\bar{p}_*(I \times j)_{*w}DR(X)((M_I, W), u_{IJ})^{an} \\ &\xrightarrow{(T^w(I \times j, \otimes)(-))} R\bar{p}_*DR(X)((I \times j)_{*Hdg}((M_I, W), u_{IJ})^{an}) \\ &\xrightarrow{T(\bar{f}, DR)(-)} DR(S)((\int_{\bar{f}}(I \times j)_{*Hdg}((M_I, W), u_{IJ})^{an}) = DR(S)((\int_f^{Hdg}((M_I, W), u_{IJ})^{an}) \end{aligned}$$

and

$$\begin{aligned} f_!\alpha &= f_!(\alpha) : T(S/\tilde{S}_I)(Rf_{!w}(K, W)) \otimes \mathbb{C}_{S^{an}} \\ \xrightarrow{=} T(S/\tilde{S}_I)(R\bar{p}_*(I \times j)_{!w}(K, W)) \otimes \mathbb{C}_{S^{an}} &\xrightarrow{=} R\bar{p}_*(I \times j)_{!w}T(X/(Y \times \tilde{S}_I))((K, W) \otimes \mathbb{C}_{X^{an}}) \\ &\xrightarrow{R\bar{p}_*\mathbb{D}^v R(I \times j)_*\mathbb{D}^v \alpha} R\bar{p}_*(I \times j)_{!w}DR(X)((M_I, W), u_{IJ})^{an} \\ &\xrightarrow{T(D, DR)(-)\circ(\mathbb{D}T^w(I \times j, \otimes)(-))\circ T(D, DR)(-)} R\bar{p}_*DR(X)((I \times j)_{!Hdg}((M_I, W), u_{IJ})^{an}) \\ &\xrightarrow{T(\bar{f}, DR)(-)} DR(S)((\int_{\bar{f}}(I \times j)_{!Hdg}((M_I, W), u_{IJ})^{an}) = DR(S)((\int_{f!}^{Hdg}((M_I, W), u_{IJ})^{an}), \end{aligned}$$

see definition 102 and definition 107.

(iii) Let $l : S^\circ \hookrightarrow S$ an open embedding with $S \in \text{Var}(\mathbb{C})$ and denote $Z = S \setminus S^\circ$. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $l_I : \tilde{S}_I^\circ \hookrightarrow \tilde{S}_I$ open embeddings such that $\tilde{S}_I^\circ \cap S = S^\circ \cap S_I$. Let

$$\alpha : T(S/(\tilde{S}_I))((K, W) \otimes \mathbb{C}_{S^{an}}) \rightarrow DR(S)((M_I, W), u_{IJ})^{an}$$

a morphism in $D_{fil}(S^{an}/(\tilde{S}_I^{an}))$, with $((M_I, W), u_{IJ}) \in \pi_S(CDRM(S)) \subset C_{\mathcal{D}0fil, rh}(S/(\tilde{S}_I))$ and $(K, W) \in D_{fil}(S^{an})$. We then consider the maps in $D_{fil}(S^{an}/(\tilde{S}_I^{an}))$

$$\begin{aligned} \Gamma_Z(\alpha) &: T(S/(\tilde{S}_I))(\Gamma_Z^w(K, W) \otimes \mathbb{C}_{S^{an}}) \xrightarrow{=} \Gamma_Z^w T(S/(\tilde{S}_I))((K, W) \otimes \mathbb{C}_{S^{an}}) \\ &\xrightarrow{R\Gamma_Z \alpha} \Gamma_Z^w DR(S)((M_I, W), u_{IJ})^{an} \\ &\xrightarrow{(T(\gamma_Z, DR)((M_I, W), u_{IJ})^{-1} := ((I, T^w(l_I, \otimes)(M_I, W)) \circ (T^w(an, \otimes)(M_I, W)))^{-1}} DR(S)((\Gamma_Z^{Hdg}((M_I, W), u_{IJ})^{an}) \end{aligned}$$

and

$$\begin{aligned} \Gamma_Z^\vee(\alpha) &: T(S/(\tilde{S}_I))(\Gamma_Z^{\vee, w}(K, W) \otimes \mathbb{C}_{S^{an}}) \xrightarrow{=} \Gamma_Z^\vee T(S/(\tilde{S}_I))((K, W) \otimes \mathbb{C}_{S^{an}}) \\ &\xrightarrow{\Gamma_Z^\vee \alpha} \Gamma_Z^{\vee, w} DR(S)((M_I, W), u_{IJ})^{an} \\ &\xrightarrow{T(\gamma_Z^\vee, DR)((M_I, W), u_{IJ}) := (\mathbb{D}(I, T^w(l_I, \otimes)(\mathbb{D}(M_I, W))) \circ (\mathbb{D}T^w(an, \otimes)(\mathbb{D}(M_I, W))))} DR(S)((\Gamma_Z^{\vee, w}((M_I, W), u_{IJ})^{an}), \end{aligned}$$

see definition 101 and definition 110.

(iv) Let $f : X \rightarrow S$ a morphism with $S, X \in \text{QPVar}(\mathbb{C})$. Consider a factorization $f : X \hookrightarrow Y \times S \xrightarrow{p} S$ with $Y \in \text{SmVar}(\mathbb{C})$. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let

$$\alpha : T(S/(\tilde{S}_I))((K, W) \otimes \mathbb{C}_{S^{an}}) \rightarrow DR(S)((M_I, W), u_{IJ})^{an}$$

a morphism in $D_{fil}(S^{an}/(\tilde{S}_I^{an}))$, with $((M_I, W), u_{IJ}) \in C(DRM(S)) \subset C_{\mathcal{D}0_{fil, rh}}(S/(\tilde{S}_I))$ and $(K, W) \in D_{fil}(S^{an})$. We then consider, see (iii), the maps in $D_{fil}(X^{an}/(Y \times \tilde{S}_{I, \mathbb{C}})^{an})$

$$\begin{aligned} f^! \alpha &= f^!(\alpha) : T(X/(Y \times \tilde{S}_I))(f^{!w}(K, W) \otimes \mathbb{C}_{X^{an}}) \xrightarrow{=} T(X/(Y \times \tilde{S}_I))(\Gamma_X^w p^*(K, W) \otimes \mathbb{C}_{X^{an}}) \\ &\xrightarrow{=} (\Gamma_X^w p_{\tilde{S}_I}^* T(S/(\tilde{S}_I))((K, W)_I \otimes \mathbb{C}_{S^{an}}), \Gamma_X^w p^* T(D_{IJ})(-)) \\ &\xrightarrow{R\Gamma_X p^* \alpha} (\Gamma_X^w p_{\tilde{S}_I}^* DR(S)((M_I, W), u_{IJ})_I, \Gamma_X^w p^* DR(u_{IJ})) \\ &\xrightarrow{T^!(f, DR)(-) := T(\gamma_X, DR)(-)^{-1} \circ T^!(p, DR)(-)} DR(X)(f_{Hdg}^{*mod}((M_I, W), u_{IJ})^{an}) \end{aligned}$$

and

$$\begin{aligned} f^* \alpha &= f^*(\alpha) : T(X/(Y \times \tilde{S}_I))(f^{*w}(K, W) \otimes \mathbb{C}_{X^{an}}) \xrightarrow{=} T(X/(Y \times \tilde{S}_I))(\Gamma_X^{\vee, w} p^*(K, W) \otimes \mathbb{C}_{X^{an}}) \\ &\xrightarrow{=} (\Gamma_X^{\vee, w} p_{\tilde{S}_I}^* T(S/(\tilde{S}_I))((K, W) \otimes \mathbb{C}_{S^{an}})_I, \Gamma_X^{\vee, w} p^* T(D_{IJ})(-)) \\ &\xrightarrow{\Gamma_X^{\vee, w} p^* \alpha} (\Gamma_X^{\vee, w} p_{\tilde{S}_I}^* DR(S)((M_I, W), u_{IJ})_I, \Gamma_X^{\vee, w} p^* DR(u_{IJ})) \\ &\xrightarrow{T^*(f, DR)(-) := T(\gamma_X^{\vee}, DR)(-) \circ T^*(p, DR)(-)} DR(X)(f_{Hdg}^{\hat{*}mod}((M_I, W), u_{IJ})^{an}) \end{aligned}$$

(v) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let

$$\begin{aligned} \alpha &: T(S/(\tilde{S}_I))((K, W) \otimes \mathbb{C}_{S^{an}}) \rightarrow DR(S)((M_I, W), u_{IJ})^{an}, \\ \alpha' &: T(S/(\tilde{S}_I))((K', W) \otimes \mathbb{C}_{S^{an}}) \rightarrow DR(S)((M'_I, W), v_{IJ})^{an} \end{aligned}$$

two morphism in $D_{fil}(S^{an}/(\tilde{S}_I^{an}))$, with $((M_I, W), u_{IJ}), ((M'_I, W), v_{IJ}) \in \pi_S(C(MHM(S))) \subset C_{\mathcal{D}0_{fil, rh}}(S/(\tilde{S}_I))$ and $(K, W), (K', W) \in D_{fil}(S^{an})$. We then consider the map

$$\begin{aligned} \alpha \otimes \alpha' &: T(S/(\tilde{S}_I))((K, W) \otimes (K', W) \otimes \mathbb{C}_{S^{an}}) \\ &\xrightarrow{=} T(S/(\tilde{S}_I))((K, W) \otimes \mathbb{C}_{S^{an}}) \otimes T(S/(\tilde{S}_I))((K', W) \otimes \mathbb{C}_{S^{an}}) \\ &\xrightarrow{\alpha \otimes \alpha'} DR(S)((M_I, W), u_{IJ})^{an} \otimes DR(S)((M'_I, W), v_{IJ})^{an} \\ &\xrightarrow{T(\otimes, DR)(-, -)} DR(S)((M_I, W), u_{IJ}) \otimes_{O_S}^L ((M'_I, W), v_{IJ})^{an} \\ &\xrightarrow{=} DR(S)((M_I, W), u_{IJ}) \otimes_{O_S}^{Hdg} ((M'_I, W), v_{IJ})^{an} \end{aligned}$$

in $D_{fil}(S^{an}/(\tilde{S}_I^{an}))$.

Definition 113. Let $k \subset \mathbb{C}$ a subfield. Let $S \in \text{SmVar}(\mathbb{C})$. Let $j : S^\circ \hookrightarrow S$ an open embedding. Let $Z := S \setminus S^\circ = V(\mathcal{I}) \subset S$ an the closed complementary subset, $\mathcal{I} \subset O_S$ being an ideal subsheaf. Taking generators $\mathcal{I} = (s_1, \dots, s_r)$, we get $Z = V(s_1, \dots, s_r) = \cap_{i=1}^r Z_i \subset S$ with $Z_i = V(s_i) \subset S$, $s_i \in \Gamma(S, \mathcal{L}_i)$ and L_i a line bundle. Note that Z is an arbitrary closed subset, $d_Z \geq d_X - r$ needing not be a complete intersection. Denote by $j_I : S^{\circ, I} := \cap_{i \in I} (S \setminus Z_i) = S \setminus (\cup_{i \in I} Z_i) \xrightarrow{j_I^\circ} S^\circ \xrightarrow{j} S$ the open embeddings. Let $(M, F, W) \in MHM(S^\circ)$. We then define, using definition 107 and definition 100

- the canonical extension

$$\begin{aligned} j_{*Hdg}((M, F, W), (K, W), \alpha) &:= (j_{*Hdg}(M, F, W), j_{*w}(K, W), j_* \alpha) \\ &:= \text{Tot}((j_{I^*Hdg} j_I^*(M, F, W), j_{I^*w} j_I^*(K, W), j_{I^*} \alpha)) \in MHM(S), \end{aligned}$$

so that $j^*(j_{*Hdg}((M, F, W), (K, W), \alpha)) = ((M, F, W), (K, W), \alpha)$,

- the canonical extension

$$\begin{aligned} j_{!Hdg}((M, F, W), (K, W), \alpha) &:= (j_{!Hdg}(M, F, W), j_{!w}(K, W), j_{!}\alpha) \\ &:= \text{Tot}((j_{!Hdg}j_I^*(M, F, W), j_{!w}j_I^*(K, W), j_{!}\alpha)) \in MHM(S), \end{aligned}$$

so that $j^*(j_{!Hdg}((M, F, W), (K, W), \alpha)) = ((M, F, W), (K, W), \alpha)$.

Moreover for $((M', F, W), (K', W), \alpha') \in MHM(S)$,

- there is a canonical map in $MHM(S)$

$$\text{ad}(j^*, j_{*Hdg})((M', F, W), (K', W), \alpha') : ((M', F, W), (K', W), \alpha') \rightarrow j_{*Hdg}j^*((M', F, W), (K', W), \alpha'),$$

- there is a canonical map in $MHM(S)$

$$\text{ad}(j_{!Hdg}, j^*)((M', F, W), (K', W), \alpha') : j_{!Hdg}j^*((M', F, W), (K', W), \alpha') \rightarrow ((M', F, W), (K', W), \alpha').$$

Definition 114. Let $S \in \text{SmVar}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the complementary open embedding.

- (i) We define using definition 108, definition 101 and definition 112(iii), the filtered Hodge support section functor

$$\begin{aligned} \Gamma_Z^{Hdg} : C(MHM(S)) &\rightarrow C(MHM(S)), \quad ((M, F, W), (K, W), \alpha) \mapsto \\ \Gamma_Z^{Hdg}((M, F, W), (K, W), \alpha) &:= (\Gamma_Z^{Hdg}(M, F, W), \Gamma_Z^w(K, W), \Gamma_Z(\alpha)) \\ &= \text{Cone}(\text{ad}(j^*, j_{*Hdg})(-) : j_{*Hdg}j^*((M, F, W), (K, W), \alpha) \rightarrow ((M, F, W), (K, W), \alpha)[-1]) \end{aligned}$$

see definition 113 for the last equality, together we the canonical map

$$\gamma_Z^{Hdg}((M, F, W), (K, W), \alpha) : \Gamma_Z^{Hdg}((M, F, W), (K, W), \alpha) \rightarrow ((M, F, W), (K, W), \alpha).$$

- (i)' Since $j_{*Hdg} : C(MHM(S^\circ)) \rightarrow C(MHM(S))$ is an exact functor, Γ_Z^{Hdg} induces the functor

$$\begin{aligned} \Gamma_Z^{Hdg} : D(MHM(S)) &\rightarrow D(MHM(S)), \\ ((M, F, W), (K, W), \alpha) &\mapsto \Gamma_Z^{Hdg}((M, F, W), (K, W), \alpha) \end{aligned}$$

- (ii) We define using definition 108, definition 101 and definition 112(iii) the dual filtered Hodge support section functor

$$\begin{aligned} \Gamma_Z^{\vee, Hdg} : C(MHM(S)) &\rightarrow C(MHM(S)), \quad ((M, F, W), (K, W), \alpha) \mapsto \\ \Gamma_Z^{\vee, Hdg}((M, F, W), (K, W), \alpha) &:= (\Gamma_Z^{\vee, Hdg}(M, F, W), \Gamma_Z^{\vee, w}(K, W), \Gamma_Z^{\vee}(\alpha)) \\ &= \text{Cone}(\text{ad}(j_{!Hdg}, j^*)(-) : j_{!Hdg}j^*((M, F, W), (K, W), \alpha) \rightarrow ((M, F, W), (K, W), \alpha)) \end{aligned}$$

see definition 113 for the last equality, together we the canonical map

$$\gamma_Z^{\vee, Hdg}((M, F, W), (K, W), \alpha) : ((M, F, W), (K, W), \alpha) \rightarrow \Gamma_Z^{\vee, Hdg}((M, F, W), (K, W), \alpha).$$

- (ii)' Since $j_{!Hdg} : C(MHM(S^\circ)) \rightarrow C(MHM(S))$ is an exact functor, $\Gamma_Z^{Hdg, \vee}$ induces the functor

$$\begin{aligned} \Gamma_Z^{\vee, Hdg} : D(MHM(S)) &\rightarrow D(MHM(S)), \\ ((M, F, W), (K, W), \alpha) &\mapsto \Gamma_Z^{\vee, Hdg}((M, F, W), (K, W), \alpha) \end{aligned}$$

In the singular case it gives :

Definition 115. Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote $Z_I := Z \cap S_I$. Denote by $n : S \setminus Z \hookrightarrow S$ and $\tilde{n}_I : \tilde{S}_I \setminus Z_I \hookrightarrow \tilde{S}_I$ the complementary open embeddings.

(i) We define using definition 110, definition 101 and definition 112(iii) the filtered Hodge support section functor

$$\begin{aligned} \Gamma_Z^{Hdg} : C(MHM(S)) &\rightarrow C(MHM(S)), \\ (((M_I, F, W), u_{IJ}), (K, W), \alpha) &\mapsto \Gamma_Z^{Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) := \\ &:= (\Gamma_Z^{Hdg}((M_I, F, W), u_{IJ}), \Gamma_Z^w(K, W), \Gamma_Z(\alpha)) \end{aligned}$$

together with the canonical map

$$\begin{aligned} \gamma_Z^{Hdg} &(((M_I, F, W), u_{IJ}), (K, W), \alpha) : \\ \Gamma_Z^{Hdg} &(((M_I, F, W), u_{IJ}), (K, W), \alpha) \rightarrow (((M_I, F, W), u_{IJ}), (K, W), \alpha). \end{aligned}$$

(i)' By exactness of Γ_Z^{Hdg} and Γ_Z^w it induces the functor

$$\begin{aligned} \Gamma_Z^{Hdg} : D(MHM(S)) &\rightarrow D(MHM(S)), \\ (((M_I, F, W), u_{IJ}), (K, W), \alpha) &\mapsto \Gamma_Z^{Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) \end{aligned}$$

(ii) We define using definition 110, definition 101 and definition 112(iii) the dual filtered Hodge support section functor

$$\begin{aligned} \Gamma_Z^{\vee, Hdg} : C(MHM(S)) &\rightarrow C(MHM(S)), \quad (((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto \\ \Gamma_Z^{\vee, Hdg} &(((M_I, F, W), u_{IJ}), (K, W), \alpha) := (\Gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ}), \Gamma_Z^{\vee, w}(K, W), \Gamma_Z^{\vee}(\alpha)), \end{aligned}$$

together we the canonical map

$$\begin{aligned} \gamma_Z^{\vee, Hdg} &(((M_I, F, W), u_{IJ}), (K, W), \alpha) : \\ (((M_I, F, W), u_{IJ}), (K, W), \alpha) &\rightarrow \Gamma_Z^{\vee, Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha). \end{aligned}$$

(ii)' By exactness of $\Gamma_Z^{\vee, Hdg}$ and $\Gamma_Z^{\vee, w}$, it induces the functor

$$\begin{aligned} \Gamma_Z^{\vee, Hdg} : D(MHM(S)) &\rightarrow D(MHM(S)), \\ (((M_I, F, W), u_{IJ}), (K, W), \alpha) &\mapsto \Gamma_Z^{\vee, Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) \\ &:= (\Gamma_Z^{\vee, Hdg}((M_I, F, W), u_{IJ}), \Gamma_Z^{\vee, w}(K, W), \Gamma_Z^{\vee}(\alpha)) \end{aligned}$$

For $X \in \text{SmVar}(\mathbb{C})$, we have, by definition

$$\mathbb{Z}_X^{Hdg} := a_X^{*Hdg} \mathbb{Z}_{\text{pt}}^{Hdg} := ((O_X, F_b)[d_X], \mathbb{Z}_X, \alpha(X)) \in D(MHM(X)),$$

with $\alpha(X) : \mathbb{C}_X \hookrightarrow (0 \rightarrow O_X \rightarrow \Omega_X \rightarrow \cdots \rightarrow K_X)$. If $X \in \text{SmVar}(\mathbb{C})$,

$$\mathbb{Z}_X^{Hdg} := a_X^{*Hdg} \mathbb{Z}_{\text{pt}}^{Hdg} := ((O_X, F_b)[d_X], \mathbb{Z}_{X^{an}}, \alpha(X^{an})) \in D(MHM(X)).$$

Let $X \in \text{Var}(\mathbb{C})$ non smooth. Take an open cover $X = \cup_{i=1}^l X_i$ such that there exists closed embeddings $i_i : X_i \hookrightarrow \tilde{X}_i$ with $\tilde{X}_i \in \text{SmVar}(\mathbb{C})$. Then, by definition

$$\mathbb{Z}_X^{Hdg} := a_X^{*Hdg} \mathbb{Z}_{\text{pt}}^{Hdg} := ((\Gamma_{\tilde{X}_I}^{\vee, Hdg}(O_{\tilde{X}_I}, F_b)[d_{\tilde{X}_I}], o_{\tilde{S}_I/\tilde{S}_I}), (\mathbb{Z}_{X^{an}}, W), \alpha(X/\tilde{X}_I)) \in D(MHM(X)).$$

with

$$\alpha(X/\tilde{X}_I) : (\Gamma_{\tilde{X}_I}^{\vee, w} \alpha(\tilde{X}_I)) : T(X/(\tilde{X}_I))(\mathbb{Z}_{X^{an}}) := (i_{I*} \mathbb{Z}_{X_I^{an}}, I) \rightarrow DR(X)^{[-]}((\Gamma_{\tilde{X}_I}^{\vee, Hdg}(O_{\tilde{X}_I})[d_{\tilde{X}_I}], o_{\tilde{S}_I/\tilde{S}_I}))$$

We have from [28] the following proposition which shows us how to construct inductively mixed Hodge modules, as we do for perverse sheaves :

Proposition 102. (i) Let $S \in \text{AnSm}(\mathbb{C})$. Let $D = V(s) \subset S$ a (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D . We then have the zero section embedding $i : S \hookrightarrow L$. We denote $L_0 = i(S)$ and $j : L^\circ := L \setminus L_0 \hookrightarrow L$ the open complementary subset. We denote by $MHW(S \setminus D)^{ex} \times_J MHW(D)$ the category whose set of objects consists of

$$\{(\mathcal{M}, \mathcal{N}, a, b), \mathcal{M} \in MHW(S \setminus D)^{ex}, \mathcal{N} \in MHW(D), a : \psi_{D1} \mathcal{M} \rightarrow \mathcal{N}, b : \mathcal{N} \rightarrow \psi_{D1} \mathcal{M}\}$$

where $MHW(S \setminus D)^{ex} \subset MHW(S \setminus D)$ is the full subcategory of extendable objects. The functor (see definition 104)

$$(j^*, \phi_{D1}, c, v) : MHW(S) \rightarrow MHW(S \setminus D)^{ex} \times_J MHW(D), \\ ((M, F, W), (K, W), \alpha) \mapsto ((j^*(M, F, W), j^*(K, W), j^*\alpha), \phi_{D1}((M, F, W), (K, W), \alpha), can, var)$$

is an equivalence of category.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$. Let $D = V(s) \subset S$ a (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D . We then have the zero section embedding $i : S \hookrightarrow L$. We denote $L_0 = i(S)$ and $j : L^\circ := L \setminus L_0 \hookrightarrow L$ the open complementary subset. We denote by $MHW(S \setminus D) \times_J MHW(D)$ the category whose set of objects consists of

$$\{(\mathcal{M}, \mathcal{N}, a, b), \mathcal{M} \in MHW(S \setminus D), \mathcal{N} \in MHW(D), a : \psi_{D1} \mathcal{M} \rightarrow \mathcal{N}, b : \mathcal{N} \rightarrow \psi_{D1} \mathcal{M}\}$$

The functor (see definition 104)

$$(j^*, \phi_{D1}, c, v) : MHW(S) \rightarrow MHW(S \setminus D) \times_J MHW(D), \\ ((M, F, W), (K, W), \alpha) \mapsto ((j^*(M, F, W), j^*(K, W), j^*\alpha), \phi_{D1}((M, F, W), (K, W), \alpha), can, var)$$

is an equivalence of category.

Proof. See [28]. □

Theorem 31. Let $S \in \text{Var}(\mathbb{C})$. The category of mixed Hodge modules is the full subcategory

$$\iota_S : MHM(S) \hookrightarrow MHW(S) \hookrightarrow \text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an})$$

consisting of objects

$$((M, F, W), (K, W), \alpha) = (((M_I, F, W), u_{IJ}), (K, W), \alpha) \in \text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an})$$

such that $((M, F, W)^{an}, (K, W), \alpha) := (((M_I^{an}, F, W), u_{IJ}), (K, W), \alpha) \in MHM(S^{an})$.

Proof. Follows from GAGA and the extendableness in the algebraic case (proposition 102). □

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \text{SmVar}(\mathbb{C})$. We have the category $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$

- whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with

$$((M_I, F, W), u_{IJ}) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)), (K, W) \in D_{fil}(S^{an}), \\ \alpha : T(S/(\tilde{S}_I))(K, W) \otimes \mathbb{C}^{S^{an}} \rightarrow DR(S)^{[-1]}(((M_I, W), u_{IJ})^{an})$$

where α is an morphism in $D_{fil}(S^{an}/(\tilde{S}_I^{an}))$,

- and whose set of morphisms consists of

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

where $\phi_D : ((M_1, F, W), u_{IJ}) \rightarrow ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms and

$$\begin{aligned} \theta &= (\theta^\bullet, I(DR(S)(\phi_D^{an})) \circ I(\alpha_1), I(\alpha_2) \circ I(\phi_C \otimes I)) : \\ &I(T(S/(\tilde{S}_I))(K_1, W)) \otimes \mathbb{C}_{S^{an}}[1] \rightarrow I(DR(S)((M_{2I}, W), u_{IJ})^{an}) \end{aligned}$$

is an homotopy, $I : D_{fil}(S^{an}/(\tilde{S}_I^{an})) \rightarrow K_{fil}(S^{an}/(\tilde{S}_I^{an}))$ being the injective resolution functor, and for

$$\begin{aligned} - \phi &= (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \\ - \phi' &= (\phi'_D, \phi'_C, [\theta']) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \rightarrow (((M_{3I}, F, W), u_{IJ}), (K_3, W), \alpha_3) \end{aligned}$$

the composition law is given by

$$\begin{aligned} \phi' \circ \phi &:= (\phi'_D \circ \phi_D, \phi'_C \circ \phi_C, I(DR(S)(\phi'_D)^{an}) \circ [\theta] + [\theta'] \circ I(\phi_C \otimes I)[1]) : \\ &(((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{3I}, F, W), u_{IJ}), (K_3, W), \alpha_3), \end{aligned}$$

in particular for $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$,

$$I_{(((M_I, F, W), u_{IJ}), (K, W), \alpha)} = ((I_{M_I}, I_K, 0),$$

and also the category $D_{\mathcal{D}(1,0)fil,rh,\infty}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$ defined in the same way, together with the localization functor

$$\begin{aligned} (D(zar), I) : C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) &\rightarrow D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \\ &\rightarrow D_{\mathcal{D}(1,0)fil,rh,\infty}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}). \end{aligned}$$

Moreover,

- For $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$, we set

$$(((M_I, F, W), u_{IJ}), (K, W), \alpha)[1] := (((M_I, F, W), u_{IJ})[1], (K, W)[1], \alpha[1]).$$

- For

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

a morphism in $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$, we set (see [9] definition 3.12)

$$\text{Cone}(\phi) := (\text{Cone}(\phi_D), \text{Cone}(\phi_C), ((\alpha_1, \theta), (\alpha_2, 0))) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}).$$

together with the canonical maps

$$\begin{aligned} - c_1(-) &= (c_1(\phi_D), c_1(\phi_C), 0) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \rightarrow \text{Cone}(\phi) \\ - c_2(-) &= (c_2(\phi_D), c_2(\phi_C), 0) : \text{Cone}(\phi) \rightarrow (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1)[1]. \end{aligned}$$

We now state and prove the following key theorem :

Theorem 32. (i) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then the full embedding

$$\iota_S : MHM(S) \hookrightarrow \text{PSh}_{\mathcal{D}(1,0)fil,rh}^0(S/(\tilde{S}_I)) \times_I P_{fil}(S^{an}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

induces a full embedding

$$\iota_S : D(MHM(S)) \hookrightarrow D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

whose image consists of $((M_I, F, W), u_{IJ}), (K, W), \alpha \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$ such that

$$((H^n(M_I, F, W), H^n(u_{IJ})), H^n(K, W), H^n \alpha) \in MHM(S)$$

for all $n \in \mathbb{Z}$ and such that for all $p \in \mathbb{Z}$, the differentials of $\text{Gr}_W^p(M_I, F)$ are strict for the filtrations F .

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then the full embedding

$$\iota_S : MHM(S) \hookrightarrow \text{PSh}_{\mathcal{D}(1,0)fil,rh}^0(S/(\tilde{S}_I)) \times_I P_{fil}(S^{an}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

induces a full embedding

$$\iota_S : D(MHM(S)) \hookrightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

whose image consists of $((M_I, F, W), u_{IJ}), (K, W), \alpha \in D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$ such that

$$((H^n(M_I, F, W), H^n(u_{IJ})), H^n(K, W), H^n \alpha) \in MHM(S)$$

for all $n \in \mathbb{Z}$ and such that there exist $r \in \mathbb{Z}$ and an r -filtered homotopy equivalence $((M_I, F, W), u_{IJ}) \rightarrow ((M'_I, F, W), u_{IJ})$ such that for all $p \in \mathbb{Z}$ the differentials of $\text{Gr}_W^p(M'_I, F)$ are strict for the filtrations F .

Proof. (i): We first show that ι_S is fully faithful, that is for all $\mathcal{M} = (((M_I, F, W), u_{IJ}), (K, W), \alpha), \mathcal{M}' = (((M'_I, F, W), u_{IJ}), (K', W), \alpha') \in MHM(S)$ and all $n \in \mathbb{Z}$,

$$\begin{aligned} \iota_S : \text{Ext}_{D(MHM(S))}^n(\mathcal{M}, \mathcal{M}') &:= \text{Hom}_{D(MHM(S))}(\mathcal{M}, \mathcal{M}'[n]) \\ \rightarrow \text{Ext}_{\mathcal{D}(S)}^n(\mathcal{M}, \mathcal{M}') &:= \text{Hom}_{\mathcal{D}(S) := D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})}(\mathcal{M}, \mathcal{M}'[n]) \end{aligned}$$

For this it is enough to assume S smooth. We then proceed by induction on $\max(\dim \text{supp}(M), \dim \text{supp}(M'))$.

- For $\text{supp}(M) = \text{supp}(M') = \{s\}$, it is the theorem for mixed Hodge complexes or absolute Hodge complexes, see [9]. If $\text{supp}(M) = \{s\}$ and $\text{supp}(M') = \{s'\}$, then by the localization exact sequence

$$\text{Ext}_{D(MHM(S))}^n(\mathcal{M}, \mathcal{M}') = 0 = \text{Ext}_{\mathcal{D}(S)}^n(\mathcal{M}, \mathcal{M}')$$

- Denote $\text{supp}(M) = Z \subset S$ and $\text{supp}(M') = Z' \subset S$. There exist an open subset $S^\circ \subset S$ such that $Z^\circ := Z \cap S^\circ$ and $Z'^\circ := Z' \cap S^\circ$ are smooth, and $\mathcal{M}|_{Z^\circ} := ((i^* \text{Gr}_{V_{Z^\circ}, 0} M|_{S^\circ}, F, W), i^* j^*(K, W), \alpha^*(i)) \in MHM(Z^\circ)$ and $\mathcal{M}'|_{Z'^\circ} := ((i'^* \text{Gr}_{V_{Z'^\circ}, 0} M'|_{S^\circ}, F, W), i'^* j^* K, \alpha^*(i')) \in MHM(Z'^\circ)$ are variation of mixed Hodge structure, where $j : S^\circ \hookrightarrow S$ is the open embedding, and $i : Z^\circ \hookrightarrow S^\circ, i' : Z'^\circ \hookrightarrow S^\circ$ the closed embeddings. Considering the connected components of Z° and Z'° , we may assume that Z° and Z'° are connected. Shrinking S° if necessary, we may assume that either $Z^\circ = Z'^\circ$ or $Z^\circ \cap Z'^\circ = \emptyset$. We denote $D = S \setminus S^\circ$. Shrinking S° if necessary, we may assume that D is a divisor and denote by $l : S \hookrightarrow L_D$ the zero section embedding.

- If $Z^o = Z'^o$, denote $i : Z^o \hookrightarrow S^o$ the closed embedding. We have then the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}_{D(\mathrm{MHM}(S^o))}^n(\mathcal{M}|_{S^o}, \mathcal{M}'|_{S^o}) & \xrightarrow{\iota_{S^o}} & \mathrm{Ext}_{\mathcal{D}(S^o)}^n(\mathcal{M}|_{S^o}, \mathcal{M}'|_{S^o}) \\
(i^* \mathrm{Gr}_{V_{Z^o}, 0}, i^*, \alpha^*(i)) \updownarrow (i_{* \mathrm{mod}}, i_*, \alpha_*(i)) & & (i_{* \mathrm{mod}}, i_*, \alpha_*(i)) \updownarrow (i^* \mathrm{Gr}_{V_{Z^o}, 0}, i^*, \alpha^*(i)) \\
\mathrm{Ext}_{D(\mathrm{MHM}(Z^o))}^n(\mathcal{M}|_{Z^o}, \mathcal{M}'|_{Z^o}) & \xrightarrow{\iota_{Z^o}} & \mathrm{Ext}_{\mathcal{D}(Z^o)}^n(\mathcal{M}|_{Z^o}, \mathcal{M}'|_{Z^o})
\end{array}$$

Now we prove that ι_{Z^o} is an isomorphism similarly to the proof the the generic case of 32. On the other hand the left and right column are isomorphisms. Hence ι_{S^o} is an isomorphism by the diagram.

- If $Z^o \cap Z'^o = \emptyset$, we consider the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}_{D(\mathrm{MHM}(S^o))}^n(\mathcal{M}|_{S^o}, \mathcal{M}'|_{S^o}) & \xrightarrow{\iota_{S^o}} & \mathrm{Ext}_{\mathcal{D}(S^o)}^n(\mathcal{M}|_{S^o}, \mathcal{M}'|_{S^o}) \\
(i^* \mathrm{Gr}_{V_{Z^o}, 0}, i^*, \alpha^*(i)) \updownarrow (i_{* \mathrm{mod}}, i_*, \alpha_*(i)) & & (i_{* \mathrm{mod}}, i_*, \alpha_*(i)) \updownarrow (i^* \mathrm{Gr}_{V_{Z^o}, 0}, i^*, \alpha^*(i)) \\
\mathrm{Ext}_{D(\mathrm{MHM}(Z^o))}^n(\mathcal{M}|_{Z^o}, 0) = 0 & \xrightarrow{\iota_{Z^o}} & \mathrm{Ext}_{\mathcal{D}(Z^o)}^n(\mathcal{M}|_{Z^o}, 0) = 0
\end{array}$$

where the left and right column are isomorphism by strictness of the V_{Z^o} filtration (use a bi-filtered injective resolution with respect to F and V_{Z^o} for the right column).

- We consider now the following commutative diagram in $C(\mathbb{Z})$ where we denote for short $H := D(\mathrm{MHM}(S))$

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\bullet H}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, \Gamma_D^{\mathrm{Hdg}} \mathrm{Hom}(\cdot, \gamma_D^{\mathrm{Hdg}}(\mathcal{M}')) & \xrightarrow{\iota_S} & \mathrm{Hom}_{\bullet H}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, \mathcal{M}') & \xrightarrow{\mathrm{Hom}(\cdot, \mathrm{ad}(j^*, j_{* \mathrm{Hdg}})(\mathcal{M}'))} & \mathrm{Hom}_{\bullet H}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, j_{* \mathrm{Hdg}} j^* \mathcal{M}') \longrightarrow 0 \\
& & \downarrow \iota_S & & \downarrow \iota_S & & \downarrow \iota_S \\
0 & \longrightarrow & \mathrm{Hom}_{\bullet \mathcal{D}(S)}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, \Gamma_D^{\mathrm{Hdg}} \mathrm{Hom}(\cdot, \gamma_D^{\mathrm{Hdg}}(\mathcal{M}')) & \xrightarrow{\iota_S} & \mathrm{Hom}_{\bullet \mathcal{D}(S)}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, \mathcal{M}') & \xrightarrow{\mathrm{Hom}(\cdot, \mathrm{ad}(j^*, j_{* \mathrm{Hdg}})(\mathcal{M}'))} & \mathrm{Hom}_{\bullet \mathcal{D}(S)}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, j_{* \mathrm{Hdg}} j^* \mathcal{M}') \longrightarrow 0
\end{array}$$

whose lines are exact sequence. We have on the one hand,

$$\mathrm{Hom}_{\bullet D(\mathrm{MHM}(S))}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, j_{* \mathrm{Hdg}} j^* \mathcal{M}') = 0 = \mathrm{Hom}_{\bullet \mathcal{D}(S)}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, j_{* \mathrm{Hdg}} j^* \mathcal{M}')$$

On the other hand by induction hypothesis

$$\iota_S : \mathrm{Hom}_{\bullet D(\mathrm{MHM}(S))}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, \Gamma_D^{\mathrm{Hdg}} \mathcal{M}') \rightarrow \mathrm{Hom}_{\bullet \mathcal{D}(S)}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, \Gamma_D^{\mathrm{Hdg}} \mathcal{M}')$$

is a quasi-isomorphism. Hence, by the diagram

$$\iota_S : \mathrm{Hom}_{\bullet D(\mathrm{MHM}(S))}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, \mathcal{M}') \rightarrow \mathrm{Hom}_{\bullet \mathcal{D}(S)}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, \mathcal{M}')$$

is a quasi-isomorphism.

- We consider now the following commutative diagram in $C(\mathbb{Z})$ where we denote for short $H := D(\mathrm{MHM}(S))$

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\bullet H}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, \mathrm{Hom}(\gamma_D^{\vee, \mathrm{Hdg}}(\mathcal{M}), -)) & \xrightarrow{\iota_S} & \mathrm{Hom}_{\bullet H}(\mathcal{M}, \mathrm{Hom}(\mathrm{ad}(j_{! \mathrm{Hdg}}, j^*)(\mathcal{M}'), -)) & \xrightarrow{\iota_S} & \mathrm{Hom}_{\bullet H}(j_{! \mathrm{Hdg}} j^* \mathcal{M}, \mathcal{M}') \longrightarrow 0 \\
& & \downarrow \iota_S & & \downarrow \iota_S & & \downarrow \iota_S \\
0 & \longrightarrow & \mathrm{Hom}_{\bullet \mathcal{D}(S)}(\Gamma_D^{\vee, \mathrm{Hdg}} \mathcal{M}, \mathrm{Hom}(\gamma_D^{\vee, \mathrm{Hdg}}(\mathcal{M}), -)) & \xrightarrow{\iota_S} & \mathrm{Hom}_{\bullet \mathcal{D}(S)}(\mathcal{M}, \mathrm{Hom}(\mathrm{ad}(j_{! \mathrm{Hdg}}, j^*)(\mathcal{M}'), -)) & \xrightarrow{\iota_S} & \mathrm{Hom}_{\bullet \mathcal{D}(S)}(j_{! \mathrm{Hdg}} j^* \mathcal{M}, \mathcal{M}') \longrightarrow 0
\end{array}$$

whose lines are exact sequence. On the one hand, the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{D(\mathrm{MHM}(S))}^{\bullet}(j_{!Hdg}j^*\mathcal{M}, \mathcal{M}') & \xrightarrow{j^*} & \mathrm{Hom}_{D(\mathrm{MHM}(S^{\circ}))}^{\bullet}(j^*\mathcal{M}, j^*\mathcal{M}') \\ \downarrow \iota_S & & \downarrow \iota_{S^{\circ}} \\ \mathrm{Hom}_{D(S)}^{\bullet}(j_{!Hdg}j^*\mathcal{M}, \mathcal{M}') & \xrightarrow{j^*} & \mathrm{Hom}_{D(S^{\circ})}^{\bullet}(j^*\mathcal{M}, j^*\mathcal{M}') \end{array}$$

together with the fact that the horizontal arrows j^* are quasi-isomorphism by the functoriality given the uniqueness of the V_S filtration for the embedding $l : S \hookrightarrow L_D$, (use a bi-filtered injective resolution with respect to F and V_S for the lower arrow) and the fact that $\iota_{S^{\circ}}$ is a quasi-isomorphism by the first two point, show that

$$\iota_S : \mathrm{Hom}_{D(\mathrm{MHM}(S))}^{\bullet}(j_{!Hdg}j^*\mathcal{M}, \mathcal{M}') \rightarrow \mathrm{Hom}_{D(S)}^{\bullet}(j_{!Hdg}j^*\mathcal{M}, \mathcal{M}')$$

is a quasi-isomorphism. On the other hand, by the third point

$$\iota_S : \mathrm{Hom}_{D(\mathrm{MHM}(S))}^{\bullet}(\Gamma_D^{\vee, Hdg} \mathcal{M}, \mathcal{M}') \rightarrow \mathrm{Hom}_{D(S)}^{\bullet}(\Gamma_D^{\vee, Hdg} \mathcal{M}, \mathcal{M}')$$

is a quasi-isomorphism. Hence, by the diagram

$$\iota_S : \mathrm{Hom}_{D(\mathrm{MHM}(S))}^{\bullet}(\Gamma_D^{\vee, Hdg} \mathcal{M}, \mathcal{M}') \rightarrow \mathrm{Hom}_{D(S)}^{\bullet}(\Gamma_D^{\vee, Hdg} \mathcal{M}, \mathcal{M}')$$

is a quasi-isomorphism.

This shows the fully faithfulness. We now prove the essential surjectivity : let

$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C_{D(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

such that the cohomology are mixed hodge modules and such that the differential are strict. We proceed by induction on $card\{n \in \mathbb{Z}\}$, s.t. $H^n(M_I, F, W) \neq 0$ by taking for the cohomological truncation

$$\tau^{\leq n}(((M_I, F, W), u_{IJ}), (K, W), \alpha) := ((\tau^{\leq n}(M_I, F, W), \tau^{\leq n}u_{IJ}), \tau^{\leq n}(K, W), \tau^{\leq n}\alpha)$$

and using the fact that the differential are strict for the filtration F and the fully faithfulness.

(ii): Follows from (i) and the fact that the image of the embedding given by (i) consists of classes of complexes whose differential are strict for F . \square

Definition 116. Let $f : X \rightarrow S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $\bar{Y} \in \mathrm{PSmVar}(\mathbb{C})$ a smooth compactification of Y with $n : Y \hookrightarrow \bar{Y}$ the open embedding. Then $\bar{f} : \bar{X} \xrightarrow{\bar{l}} \bar{Y} \times_S \xrightarrow{\bar{p}_S} S$ is a compactification of f , with $\bar{X} \subset \bar{Y} \times S$ the closure of X and \bar{l} the closed embedding, and we denote by $n' : X \hookrightarrow \bar{X}$ the open embedding so that $f = \bar{f} \circ n'$.

(i) For $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C(\mathrm{MHM}(X))$, we define, using definition 113 and theorem 32

$$\begin{aligned} Rf_{*Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) &:= \iota_S^{-1} \left(\int_{\bar{f}}^{FDR} (n \times I)_{*Hdg}((M_I, F, W), u_{IJ}), Rf_{*w}(K, W), f_*(\alpha) \right) \\ &\in D(\mathrm{MHM}(S)) \end{aligned}$$

where $f_*(\alpha)$ is given in definition 112, and since

– by definition

$$H^i\left(\int_{\bar{f}}^{FDR} \mathrm{Gr}_W^k(I \times n)_{Hdg^*}((M_I, F, W), u_{IJ}), R\bar{f}_* \mathrm{Gr}_W^k n'_{*w}(K, W), \bar{f}_* \mathrm{Gr}_W^k n'_* \alpha\right) \in HM(S)$$

for all $i, k \in \mathbb{Z}$, hence by the spectral sequence for the filtered complexes $\int_{\bar{f}}^{FDR} (I \times n)_{Hdg^*}((M_I, F, W), u_{IJ})$ and $R\bar{f}_*((I \times n)_{*w}(K, W))$

$$\begin{aligned} & \mathrm{Gr}_W^k H^i\left(\int_f^{Hdg} ((M_I, F, W), u_{IJ}), Rf_{*w}(K, W), f_* \alpha\right) := \\ & (\mathrm{Gr}_W^k H^i \int_{\bar{f}}^{FDR} (I \times n)_{Hdg^*}((M_I, F, W), u_{IJ}), \mathrm{Gr}_W^k H^i R\bar{f}_* n'_{*w}(K, W), \mathrm{Gr}_W^k H^i f_* \alpha) \in HM(S) \end{aligned}$$

this gives by definition $H^i(\int_f^{Hdg} ((M_I, F, W), u_{IJ}), Rf_{*w}(K, W), f_*(\alpha)) \in MHM(S)$ for all $i \in \mathbb{Z}$.

– $\int_f^{Hdg} ((M_I, F, W), u_{IJ})$ is the class of a complex such that the differential are strict for F by theorem 29.

(ii) For $((M_I, F, W), u_{IJ}), (K, W), \alpha \in C(MHM(X))$, we define, using definition 113 and theorem 32,

$$\begin{aligned} Rf_{!Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) & := \iota_S^{-1} \left(\int_{\bar{f}}^{FDR} (n \times I)_{!Hdg}((M_I, F, W), u_{IJ}), Rf_{!w}(K, W), f_!(\alpha) \right) \\ & \in D(MHM(S)) \end{aligned}$$

where $f_!(\alpha)$ is given in definition 112, and since

– by definition

$$H^i\left(\int_{\bar{f}}^{FDR} \mathrm{Gr}_W^k(n \times I)_{!Hdg}((M_I, F, W), u_{IJ}), R\bar{f}_* \mathrm{Gr}_W^k n'_{!w}(K, W), \mathrm{Gr}_W^k f_! \alpha\right) \in HM(S)$$

for all $i, k \in \mathbb{Z}$, hence by the spectral sequence for the filtered complexes $\int_{\bar{f}}^{FDR} (n \times I)_{!Hdg}((M_I, F, W), u_{IJ})$ and $R\bar{f}_*(n \times I)_{!w}(K, W)$

$$\begin{aligned} & \mathrm{Gr}_W^k H^i\left(\int_{f!}^{Hdg} ((M_I, F, W), u_{IJ}), Rf_{!w}(K, W), f_! \alpha\right) := \\ & (\mathrm{Gr}_W^k H^i \int_{\bar{f}}^{FDR} (n \times I)_{!Hdg}((M_I, F, W), u_{IJ}), \mathrm{Gr}_W^k H^i R\bar{f}_* n'_{!w}(K, W), \mathrm{Gr}_W^k H^i f_! \alpha) \in HM_{gm,k,\mathbb{C}}(S) \end{aligned}$$

this gives by definition $H^i(\int_{f!}^{Hdg} ((M_I, F, W), u_{IJ}), Rf_{!w}(K, W), f_!(\alpha)) \in MHM(S)$ for all $i \in \mathbb{Z}$.

– $\int_{f!}^{Hdg} ((M_I, F, W), u_{IJ})$ is the class of a complex such that the differential are strict for F by theorem 29.

We have the six functors formalism for mixed Hodge modules on quasi-projective varieties :

Definition 117. Let $f : X \rightarrow S$ a morphism with $X, S \in \mathrm{QPVar}(\mathbb{C})$. Then, since X is quasi-projective, there exist a factorization $f : X \xrightarrow{l} \mathbb{P}^{N,o} \times S \xrightarrow{ps} S$ with $n_0 : \mathbb{P}^{N,o} \hookrightarrow \mathbb{P}^N$ an open subset, l a closed

embedding and p_S the projection. Since S is quasi-projective, there exist a closed embedding $i : S \hookrightarrow \tilde{S}$ with $\tilde{S} \in \text{SmVar}(\mathbb{C})$. We have then the commutative diagram

$$\begin{array}{ccccc}
f : X & \xrightarrow{l} & \mathbb{P}^{N,o} \times S & \xrightarrow{p_S} & S \\
& \searrow & \downarrow i' := (I \times i) & & \downarrow i \\
& & \mathbb{P}^{N,o} \times \tilde{S} & \xrightarrow{p_{\tilde{S}}} & \tilde{S} \\
& & \downarrow n := (n_0 \times I) & & \downarrow = \\
& & \mathbb{P}^N \times \tilde{S} & \xrightarrow{\bar{p}_{\tilde{S}}} & \tilde{S}
\end{array}$$

(i) For $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(X))$, where $(M, F, W) \in C_{\mathcal{D}(1,0)\text{fil}}(X/\mathbb{P}^{N,o} \times \tilde{S})$ and $(K, W) \in C_{\text{fil}}(X^{an})$, we define, using definition 116,

$$\begin{aligned}
f_{*Hdg}((M, F, W), (K, W), \alpha) &:= (Rf_*^{Hdg}(M, F, W), Rf_{*w}(K, W), f_*(\alpha)) \\
&:= \iota_S^{-1} \left(\int_{p_S}^{FDR} n_*^{Hdg}(M, F, W), Rf_{*w}(K, W), f_*(\alpha) \right) \in D(\text{MHM}(S))
\end{aligned}$$

with $f_*(\alpha)$ given in definition 112.

(ii) For $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(X))$, where $(M, F, W) \in C_{\mathcal{D}(1,0)\text{fil}}(X/\mathbb{P}^{N,o} \times \tilde{S})$ and $(K, W) \in C_{\text{fil}}(X^{an})$, we define, using definition 116

$$\begin{aligned}
f_{!Hdg}((M, F, W), (K, W), \alpha) &:= (Rf_{!}^{Hdg}(M, F, W), Rf_{!w}(K, W), f_{!}(\alpha)) \\
&:= \iota_S^{-1} \left(\int_{p_S}^{FDR} n_{!}^{Hdg}(M, F, W), Rf_{!w}(K, W), f_{!}(\alpha) \right) \in D(\text{MHM}(S))
\end{aligned}$$

with $f_{!}(\alpha)$ given in definition 112.

(iii) For $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(S))$, where $(M, F, W) \in C_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}))$, $(K, W) \in C_{\text{fil}}(S^{an})$, we define, using definition 111 (see theorem 30(ii) for p_S and definition 108 for $i \circ l$),

$$\begin{aligned}
f^{*Hdg}((M, F, W), (K, W), \alpha) &:= (f_{Hdg}^{*\text{mod}}(M, F, W), f^{*w}(K, W), f^*\alpha) \\
&:= (\Gamma_X^{\vee, Hdg} p_S^{*\text{mod}[-]}(M, F, W), \Gamma_X^{\vee, w} p_S^*(K, W), f^*(\alpha)) \in D(\text{MHM}(X))
\end{aligned}$$

with $f^*\alpha$ given in definition 112. For $j : S^o \hookrightarrow S$ an open embedding and $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(S))$, we have (see [28])

$$j^{*Hdg}((M, F, W), (K, W), \alpha) = (j^*(M, F, W), j^*(K, W), j^*\alpha) \in D(\text{MHM}(S^o)).$$

(iv) For $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(S))$, where $(M, F, W) \in C_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}))$, $(K, W) \in C_{\text{fil}}(S^{an})$, we define, using definition 111 (see theorem 30(ii) for p_S and definition 108 for $i \circ l$),

$$\begin{aligned}
f^{\dagger Hdg}((M, F, W), (K, W), \alpha) &:= (f_{Hdg}^{*\text{mod}}(M, F, W), f^{\dagger w}(K, W), f^{\dagger} \alpha) \\
&:= (\Gamma_X^{Hdg} p_{\tilde{S}_I}^{*\text{mod}[-]}(M, F, W)(d_X)[2d_X], \Gamma_X^w p_S^*(K, W), f^{\dagger}(\alpha)) \in D(\text{MHM}(X))
\end{aligned}$$

with $f^{\dagger} \alpha$ given in definition 112. For $j : S^o \hookrightarrow S$ an open embedding and $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(S))$, we have (see [28])

$$j^{\dagger Hdg}((M, F, W), (K, W), \alpha) = (j^*(M, F, W), j^*(K, W), j^*\alpha) \in D(\text{MHM}(S^o)).$$

Using the unicity of proposition 102, we see that these definitions does NOT depends on the choice of the factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{ps} S$ of f . Moreover, using the unicity of proposition 102 and proposition 53, we see that they are 2 functors on the category of quasi-projective complex algebraic varieties $(\text{Var}(\mathbb{C}))^{QP}$.

(v) Let $S \in \text{Var}(\mathbb{C})$. Take an open cover $S = \cup_{i=1}^l S_i$ such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have the functor

$$\begin{aligned} & ((-) \otimes_{O_S}^{Hdg} (-), (-) \otimes (-)) : C(MHM(S)^2) \rightarrow C(MHM(S)), \\ & (((M_I, F, W), u_{IJ}), (K, W), \alpha), (((N_I, F, W), v_{IJ}), (K', W), \alpha')) \mapsto \\ & (((M_I, F, W), u_{IJ}) \otimes_{O_S}^{Hdg} ((N_I, F, W), v_{IJ}), (K, W) \otimes (K', W), \alpha \otimes \alpha') \end{aligned}$$

where

$$\begin{aligned} & (((M, F, W), u_{IJ}), ((N, F, W), v_{IJ})) \mapsto ((M, F, W), u_{IJ}) \otimes_{O_S}^{Hdg} ((N, F, W), v_{IJ}) := \\ & \Delta_{S, Hdg}^{*mod} (p_{1I}^{*mod}(M_I, F, W) \otimes_{O_{\tilde{S}_I \times \tilde{S}_I}} p_{2I}^{*mod}(N_I, F, W), p_{1I}^{*mod} u_{IJ} \otimes p_{2I}^{*mod} v_{IJ}) := \\ & \Delta_S^* \text{Gr}_{V_{\Delta_S, 0}} \Gamma_{\Delta_S}^{\vee, Hdg} (p_{1I}^{*mod}(M_I, F, W) \otimes_{O_{\tilde{S}_I \times \tilde{S}_I}} p_{2I}^{*mod}(N_I, F, W), p_{1I}^{*mod} u_{IJ} \otimes p_{2I}^{*mod} v_{IJ}) \end{aligned}$$

and the map $\alpha \otimes \alpha'$ is given in definition 112.

We have then the following

Theorem 33. Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, X quasi-projective. Then,

- (i) $(f^{*Hdg}, f_{*Hdg}) : D(MHM(S)) \rightarrow D(MHM(X))$ is a pair of adjoint functors,
- (ii) $(f^{*Hdg}, f_{*Hdg}) : D(MHM(S)) \rightarrow D(MHM(X))$ is a pair of adjoint functors.

Proof. For the projection case see section 4. For the open embedding see definition 107. □

Definition 117 gives the following 2 functors :

- We have the following 2 functor on the category of complex algebraic varieties

$$\begin{aligned} & D(MHW(\cdot)) : \text{Var}(\mathbb{C}) \rightarrow \text{TriCat}, S \mapsto D(MHW(S)), \\ & (f : T \rightarrow S) \mapsto (f^{*Hdg} : ((M, F, W), (K, W), \alpha) \mapsto \\ & f^{*Hdg}((M, F, W), (K, W), \alpha) := (f_{Hdg}^{*mod}(M, F, W), f^{*w}(K, W), f^*(\alpha))). \end{aligned}$$

- We have the following 2 functor on the category of complex quasi-projective algebraic varieties

$$\begin{aligned} & D(MHW(\cdot)) : \text{QPVar}(\mathbb{C}) \rightarrow \text{TriCat}, S \mapsto D(MHW(S)), \\ & (f : T \rightarrow S) \mapsto (f_{*Hdg} : ((M, F, W), (K, W), \alpha) \mapsto \\ & f_{*Hdg}((M, F, W), (K, W), \alpha) := (Rf_*^{Hdg}(M, F, W), Rf_{*w}(K, W), f_*(\alpha))). \end{aligned}$$

- We have the following 2 functor on the category of complex quasi-projective algebraic varieties

$$\begin{aligned} & D(MHW(\cdot)) : \text{QPVar}(\mathbb{C}) \rightarrow \text{TriCat}, S \mapsto D(MHW(S)), \\ & (f : T \rightarrow S) \mapsto (f_{!Hdg} : ((M, F, W), (K, W), \alpha) \mapsto \\ & f_{!Hdg}((M, F, W), (K, W), \alpha) := (Rf_{!}^{Hdg}(M, F, W), Rf_{!w}(K, W), f_!(\alpha))). \end{aligned}$$

- We have the following 2 functor on the category of complex algebraic varieties

$$\begin{aligned} & D(MHW(\cdot)) : \text{Var}(\mathbb{C}) \rightarrow \text{TriCat}, S \mapsto D(MHW(S)), \\ & (f : T \rightarrow S) \mapsto (f^{!Hdg} : ((M, F, W), (K, W), \alpha) \mapsto \\ & f^{!Hdg}((M, F, W), (K, W), \alpha) := (f_{Hdg}^{*mod}(M, F, W), f^{!w}(K, W), f^!(\alpha))). \end{aligned}$$

For a commutative diagram in $\text{Var}(\mathbb{C})$

$$D = \begin{array}{ccc} X & \xrightarrow{f} & S \\ g' \uparrow & & \uparrow g \\ X' & \xrightarrow{f'} & T \end{array}$$

with S, T, X', X quasi-projective, we have, for $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(X))$ using theorem 33, the following transformations maps

$$\begin{aligned} & T_1^{\text{Hdg}}(D)((M, F, W), (K, W), \alpha) : \\ & g^{*\text{Hdg}} f_{*\text{Hdg}}((M, F, W), (K, W), \alpha) \xrightarrow{\text{ad}(f'^{*}\text{Hdg}, f_{*\text{Hdg}})(-)} f'_{*\text{Hdg}} f'^{*}\text{Hdg} g^{*\text{Hdg}} f_{*\text{Hdg}}((M, F, W), (K, W), \alpha) \\ \xrightarrow{\cong} & f'_{*\text{Hdg}} g'^{*}\text{Hdg} f^{*\text{Hdg}} f_{*\text{Hdg}}((M, F, W), (K, W), \alpha) \xrightarrow{\text{ad}(f'^{*}\text{Hdg}, f_{*\text{Hdg}})(-)} f'_{*\text{Hdg}} g'^{*}\text{Hdg}((M, F, W), (K, W), \alpha) \end{aligned}$$

and

$$\begin{aligned} & T_2^{\text{Hdg}}(D)((M, F, W), (K, W), \alpha) : \\ & f'_{!\text{Hdg}} g'^{!\text{Hdg}}((M, F, W), (K, W), \alpha) \xrightarrow{\text{ad}(f'^{!\text{Hdg}}, f_{!\text{Hdg}})(-)} f'_{!\text{Hdg}} g'^{!\text{Hdg}} f^{!\text{Hdg}} f_{!\text{Hdg}}((M, F, W), (K, W), \alpha) \\ \xrightarrow{\cong} & f'_{!\text{Hdg}} f'^{!\text{Hdg}} g^{!\text{Hdg}} f_{!\text{Hdg}}((M, F, W), (K, W), \alpha) \xrightarrow{\text{ad}(f'^{*}\text{Hdg}, f'_{*\text{Hdg}})(-)} g^{!\text{Hdg}} f_{!\text{Hdg}}((M, F, W), (K, W), \alpha) \end{aligned}$$

One consequence of the unicity of proposition 102 is the following :

Proposition 103. *For a commutative diagram in $\text{Var}(\mathbb{C})$*

$$D = \begin{array}{ccc} X & \xrightarrow{f} & S \\ g' \uparrow & & \uparrow g \\ X_T & \xrightarrow{f'} & T \end{array}$$

which is cartesian, with S, T, X', X quasi-projective and f (hence f' proper), and $((M, F, W), (K, W), \alpha) \in D(\text{MHM}(X))$

$$\begin{aligned} & T_1^{\text{Hdg}}(f, g) : ((M, F, W), (K, W), \alpha) : \\ & g^{*\text{Hdg}} f_{*\text{Hdg}}((M, F, W), (K, W), \alpha) \xrightarrow{\cong} f'_{*\text{Hdg}} g'^{*}\text{Hdg}((M, F, W), (K, W), \alpha) \end{aligned}$$

is an isomorphism.

Proof. See [28]. □

We have the following proposition

Proposition 104. *Let $Y \in \text{PSmVar}(\mathbb{C})$ and $i : Z \hookrightarrow S$ a closed embedding with Z smooth. Denote by $j : U := S \setminus Z \hookrightarrow Y$ the complementary open subset.*

(i) *We have*

$$\begin{aligned} a_{U\text{Hdg}!} \mathbb{Z}_U^{\text{Hdg}} &:= a_{U\text{Hdg}!}((O_U, F_b), \mathbb{Z}_{U^{an}}, \alpha(U)) \xrightarrow{\cong} \left(\int_{a_Y}^{\text{FDR}} j_{!\text{Hdg}}(O_U, F_b), (Ra_{U!} \mathbb{Z}_{U^{an}}, W), a_{U*} \alpha(U) \right) \\ &\xrightarrow{\cong} \left(\int_{a_Y}^{\text{FDR}} \text{Cone}(\mathbb{D}_S^K \text{ad}(i_{*\text{mod}}, i^\sharp)(-) : (O_Y, F_b) \rightarrow i_{*\text{mod}}(O_Z, F_b)), (Ra_{U!} \mathbb{Z}_{U^{an}}, W), a_{U!} \alpha(U) \right) \\ &\xrightarrow{\cong} (\text{Cone}(E(\Omega_{D/Y})(D) : \Gamma(Y, E(\Omega_Y^\bullet, F_b)) \rightarrow \Gamma(Z, E(\Omega_Z^\bullet, F_b))), W), (Ra_{U!} \mathbb{Z}_{U^{an}}, W), a_{U!} \alpha(U) \end{aligned}$$

(ii) We have

$$\begin{aligned} a_{UHdg*} \mathbb{Z}_U^{Hdg} &:= a_{UHdg*}((O_U, F), \mathbb{Z}_{U^{an}}, \alpha(U)) \xrightarrow{\cong} \left(\int_{a_Y}^{FDR} j_{*Hdg}(O_U, F, W), (Ra_{U*} \mathbb{Z}_{U^{an}}, W), a_{U!} \alpha(U) \right) \\ &\xrightarrow{\sim} \left(\int_{a_Y}^{FDR} \text{Cone}(\text{ad}(i_{*mod}, i^\sharp)(-) : i_{*mod}(O_Z, F_b)[c] \rightarrow (O_Y, F_b)), (Ra_{U*} \mathbb{Z}_{U^{an}}, W), a_{U*} \alpha(U) \right) \\ &\xrightarrow{\sim} (\text{Cone}(i_{Z*} : \Gamma(Z, E(\Omega_Z^\bullet, F_b))(-c)[-2c] \rightarrow \Gamma(Y, E(\Omega_Y^\bullet, F_b)), (Ra_{U*} \mathbb{Z}_{U^{an}}, W), a_{U*} \alpha(U)) \end{aligned}$$

Proof. See [28]. □

In the case where $D = \cup D_i \subset Y$ is a normal crossing divisor, proposition 104 gives

$$a_{HdgU*} \mathbb{Z}_U^{Hdg} \xrightarrow{\sim} (\Gamma(Y, E(\Omega_Y^\bullet(\log D), F, W)), (Ra_{U*} \mathbb{Z}_{U^{an}}, W), a_{U*} \alpha(U))$$

and

$$a_{HdgU!} \mathbb{Z}_U^{Hdg} := (\Gamma(Y, E(\Omega_Y^\bullet(\text{nul } D), F, W)), (Ra_{U!} \mathbb{Z}_{U^{an}}, W), a_{U!} \alpha(U))$$

We recall the definition of the Deligne complex of a complex manifold and the Deligne cohomology class of an algebraic cycle of a complex algebraic variety.

Definition 118. (i) Let $X \in \text{AnSm}(\mathbb{C})$. We have for $d \in \mathbb{Z}$ the Deligne complex

$$\mathbb{Z}_{\mathcal{D}, X}(d) := (\mathbb{Z}_X(d) \hookrightarrow \tau^{\leq d} DR(X)) = (\mathbb{Z}(d) \hookrightarrow (O_X \rightarrow \cdots \rightarrow \Omega_X^{d-1}) \in C(X))$$

Let $D \subset X$ a normal crossing divisor. We have for $d \in \mathbb{Z}$ the Deligne complexes

$$\mathbb{Z}_{\mathcal{D}, (X, D)}(d) := (\mathbb{Z}_X(d) \hookrightarrow (O_X \rightarrow \cdots \rightarrow \Omega_X^{d-1}(\log D))) \in C(X)$$

and

$$\mathbb{Z}_{\mathcal{D}, (X, D)}(d)^\vee := (\mathbb{Z}_X(d) \hookrightarrow (O_X \rightarrow \cdots \rightarrow \Omega_X^{d-1}(D))) \in C(X).$$

Moreover we have (see [12]) canonical products

$$\begin{aligned} - (-) \cdot (-) &: \mathbb{Z}_{\mathcal{D}, (X, D)}(d) \otimes \mathbb{Z}_{\mathcal{D}, (X, D)}(d') \rightarrow \mathbb{Z}_{\mathcal{D}, (X, D)}(d + d') \\ - (-) \cdot (-) &: \mathbb{Z}_{\mathcal{D}, (X, D)}(d)^\vee \otimes \mathbb{Z}_{\mathcal{D}, (X, D)}(d')^\vee \rightarrow \mathbb{Z}_{\mathcal{D}, (X, D)}(d + d')^\vee \end{aligned}$$

(ii) Let $X \in \text{AnSm}(\mathbb{C})$. We have for $d \in \mathbb{Z}$ the Deligne complex

$$C_{\mathcal{D}}^\bullet(X, \mathbb{Z}(d)) := \text{Cone}(\mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\Delta^\bullet, X) \oplus \Gamma(X, F^d \mathcal{D}_X^\bullet) \hookrightarrow \Gamma(X, \mathcal{D}_X^\bullet)) \in C(\mathbb{C})$$

Let $D \subset X$ a normal crossing divisor. Denote $U := X \setminus D$. We have for $d \in \mathbb{Z}$ the Deligne complexes

$$C_{\mathcal{D}}^\bullet((X, D), \mathbb{Z}(d)) := \text{Cone}(\mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\Delta^\bullet, U) \oplus \Gamma(X, F^d \mathcal{D}_X^\bullet(\log D)) \hookrightarrow \Gamma(X, \mathcal{D}_X^\bullet(\log D))) \in C(\mathbb{C})$$

and

$$C_{\mathcal{D}}^\bullet(X, D, \mathbb{Z}(d)) := \text{Cone}(\mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\Delta^\bullet, (X, D)) \oplus \Gamma(X, F^d \mathcal{D}_X^\bullet(D)) \hookrightarrow \Gamma(X, \mathcal{D}_X^\bullet(D))) \in C(\mathbb{C}).$$

(iii) Let $X \in \text{PSmVar}(\mathbb{C})$. We have, for $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$, the Deligne cohomology

$$H_{\mathcal{D}}^k(X^{an}, \mathbb{Z}(d)) := \mathbb{H}^k(X^{an}, \mathbb{Z}_{X, \mathcal{D}}(d)) = H^k C_{\mathcal{D}}^\bullet(X^{an}, D, \mathbb{Z}(d))^\vee$$

Let $U \in \text{SmVar}(\mathbb{C})$. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of U with $D := X \setminus U$ a normal crossing divisor. We have, for $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$, the Deligne cohomology

$$H_{\mathcal{D}}^k(U^{an}, \mathbb{Z}(d)) := \mathbb{H}^k(X, \mathbb{Z}_{(X^{an}, D^{an}), \mathcal{D}}(d)) = H^k C_{\mathcal{D}}^\bullet((X^{an}, D^{an}), \mathbb{Z}(d))^\vee$$

and

$$H_{\mathcal{D}}^k(X, D, \mathbb{Z}(d)) := \mathbb{H}^k(X^{an}, \mathbb{Z}_{(X^{an}, D^{an}), \mathcal{D}}(d)^\vee) = H^k C_{\mathcal{D}}^\bullet(X^{an}, D^{an}, \mathbb{Z}(d))^\vee.$$

- (iv) Let $U \in \text{SmVar}(\mathbb{C})$. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of U with $D := X \setminus U$ a normal crossing divisor. We define the Deligne cohomology of a (higher) cycle $Z \in \mathcal{Z}^d(U, n)^{\partial=0}$ by

$$[Z]_{\mathcal{D}} := \text{Im}(H^{2d-n}(\gamma_{\text{supp}(Z)})([Z])),$$

$$H^k(\gamma_{\text{supp}(Z)}) : \mathbb{H}_{\text{supp}(Z)}^{2d-n}(X^{an}, \mathbb{Z}_{X^{an}, D^{an}}(d)) \rightarrow \mathbb{H}^{2d-n}(X^{an}, \mathbb{Z}_{X^{an}, D^{an}}(d))$$

with $\text{supp}(Z) := p_X(\text{supp}(Z)) \subset X$, where $\text{supp}(Z) \subset X \times \square^n$ is the support of Z .

- (v) Let $U \in \text{SmVar}(\mathbb{C})$. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of U with $D := X \setminus U$ a normal crossing divisor. We have for $d \in \mathbb{Z}$ the morphism of complexes

$$\mathcal{R}_U^d : \mathcal{Z}^d(U, \bullet) \rightarrow C_{\mathcal{D}}^{\bullet}(X^{an}, D^{an}, \mathbb{Z}(d)), \quad Z \mapsto \mathcal{R}_U^d(Z) := (T_{\bar{Z}}, \Omega_{\bar{Z}}, R_{\bar{Z}})$$

which gives for $Z \in \mathcal{Z}^d(U, n)^{\partial=0}$,

$$[\mathcal{R}_U^d(Z)] = [Z]_{\mathcal{D}} \in H^{2d-n}(U^{an}, \mathbb{Z}(d))$$

Theorem 34. (i) Let $U \in \text{SmVar}(\mathbb{C})$. Denote by $a_U : U \rightarrow \text{pt}$ the terminal map. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of U with $D := X \setminus U$ a normal crossing divisor. The embedding (see theorem 32)

$$\iota : D(\text{MHM}(\{\text{pt}\})) \rightarrow D_{\text{fil}}(\mathbb{C}) \times_I D(\mathbb{Z})$$

induces for $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$, canonical isomorphisms

$$\iota(a_{U!} \text{Hdg} \mathbb{Z}_U^{\text{Hdg}}) : H^k(a_{U!} \text{Hdg} \mathbb{Z}_U^{\text{Hdg}}) \xrightarrow{\sim} H_{\mathcal{D}}^k(X^{an}, D^{an}, \mathbb{Z}(d)), \text{ and}$$

$$\iota(a_{U*} \text{Hdg} \mathbb{Z}_U^{\text{Hdg}}) : H^k(a_{U*} \text{Hdg} \mathbb{Z}_U^{\text{Hdg}}) \xrightarrow{\sim} H_{\mathcal{D}}^k(U^{an}, \mathbb{Z}(d)).$$

- (ii) Let $h : U \rightarrow S$ and $h' : U' \rightarrow S$ two morphism with $S, U, U' \in \text{SmVar}(\mathbb{C})$. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of U with $D := X \setminus U$ a normal crossing divisor such that $h : U \rightarrow S$ extend to $f : X \rightarrow \bar{S}$. Let $X' \in \text{PSmVar}(\mathbb{C})$ a compactification of U' with $D' := X' \setminus U'$ a normal crossing divisor such that $h' : U' \rightarrow S$ extend to $f' : X' \rightarrow \bar{S}$. The embedding $\iota : D(\text{MHM}(\text{pt})) \rightarrow D_{\text{fil}}(k) \times_I D(\mathbb{Z})$ (see theorem 32) induces for $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$ a canonical isomorphism

$$\iota(a_{U' \times_S U!} \text{Hdg} \mathbb{Z}_{U' \times_S U}^{\text{Hdg}}) : \text{Hom}_{D(\text{MHM}(S))}(h_{U'!} \text{Hdg} \mathbb{Z}_{U'}^{\text{Hdg}}, h_{U!} \text{Hdg} \mathbb{Z}_U^{\text{Hdg}}(d)[k])$$

$$\xrightarrow{RI(-, -)} \text{Hom}_{D(\text{MHM}_{\text{gm}, k}(\text{pt}))}(\mathbb{Z}_{\text{pt}}^{\text{Hdg}}, a_{U' \times_S U!} \text{Hdg} \mathbb{Z}_{U' \times_S U}^{\text{Hdg}}(d)[k]) = H^k(a_{U' \times_S U!} \text{Hdg} \mathbb{Z}_{U' \times_S U}^{\text{Hdg}}(d))$$

$$\xrightarrow{\sim} H_{\mathcal{D}}^k((X' \times_S X)^{an}, ((X' \times_S U) \cup (U' \times_S X))^{an}, \mathbb{Z}(d)).$$

- (iii) Let $U \in \text{SmVar}(\mathbb{C})$. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of U with $D := X \setminus U$ a normal crossing divisor. For $[Z] \in \text{CH}^d(U, n)$ and $[Z'] \in \text{CH}^{d'}(U, n')$, we have

$$([Z] \cdot [Z'])_{\mathcal{D}} = [Z]_{\mathcal{D}} \cdot [Z']_{\mathcal{D}} \in H^{2d+2d'-n-n'}(U^{an}, \mathbb{Z}(d+d'))$$

where the product on the left is the intersection of higher Chow cycle which is well defined modulo boundary (they intersect properly modulo boundary) while the right product of Deligne cohomology classes is induced by the product of Deligne complexes $(-) \cdot (-) : \mathbb{Z}_{\mathcal{D}, (X, D)}(d) \otimes \mathbb{Z}_{\mathcal{D}, (X, D)}(d') \rightarrow \mathbb{Z}_{\mathcal{D}, (X, D)}(d+d')$.

- (iv) Let $h : U \rightarrow S, h' : U' \rightarrow S, h'' : U'' \rightarrow S$ three morphism with $S, U, U', U'' \in \text{SmVar}(\mathbb{C})$. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of U with $D := X \setminus U$ a normal crossing divisor such that $h : U \rightarrow S$ extend to $f : X \rightarrow \bar{S}$. Let $X' \in \text{PSmVar}(\mathbb{C})$ a compactification of U' with $D' := X' \setminus U'$ a normal crossing divisor such that $h' : U' \rightarrow S$ extend to $f' : X' \rightarrow \bar{S}$. Let $X'' \in \text{PSmVar}(\mathbb{C})$ a

compactification of U' with $D' := X' \setminus U'$ a normal crossing divisor such that $h' : U' \rightarrow S$ extend to $f' : X' \rightarrow \bar{S}$. For $[Z] \in \text{CH}^d(U \times_S U', n)$ and $[Z'] \in \text{CH}^{d'}(U' \times_S U'', n')$, we have

$$([Z] \circ [Z'])_{\mathcal{D}} = [Z]_{\mathcal{D}} \circ [Z']_{\mathcal{D}} \in H^{d''-n''}((U \times_S U'')^{an}, \mathbb{Z}(d'' - n''))$$

where the composition on the left is the composition of higher correspondence modulo boundary while the composition on the right is given by (ii).

Proof. (i):Standard.

(ii):Follows on the one hand from (i) and on the other hand the six functor formalism on the 2-functor $D(\text{MHM}(-)) : \text{SmVar}(\mathbb{C}) \rightarrow \text{TriCat}$ gives the isomorphism $RI(-, -)$.

(iii):Standard.

(iv):Follows from (iii). □

6 The algebraic and analytic filtered De Rham realizations for Voevodsky relative motives

6.1 The algebraic filtered De Rham realization functor

6.1.1 The algebraic Gauss-Manin filtered De Rham realization functor and its transformation map with pullbacks

Consider, for $S \in \text{Var}(\mathbb{C})$, the following composition of morphism in RCat (see section 2)

$$\tilde{e}(S) : (\text{Var}(\mathbb{C})/S, O_{\text{Var}(\mathbb{C})/S}) \xrightarrow{\rho_S} (\text{Var}(\mathbb{C})^{sm}/S, O_{\text{Var}(\mathbb{C})^{sm}/S}) \xrightarrow{e(S)} (S, O_S)$$

with, for $X/S = (X, h) \in \text{Var}(\mathbb{C})/S$,

- $O_{\text{Var}(\mathbb{C})/S}(X/S) := O_X(X)$,
- $(\tilde{e}(S)^* O_S(X/S) \rightarrow O_{\text{Var}(\mathbb{C})/S}(X/S)) := (h^* O_S \rightarrow O_X)$.

and $O_{\text{Var}(\mathbb{C})^{sm}/S} := \rho_{S*} O_{\text{Var}(\mathbb{C})/S}$, that is, for $U/S = (U, h) \in \text{Var}(\mathbb{C})^{sm}/S$, $O_{\text{Var}(\mathbb{C})^{sm}/S}(U/S) := O_{\text{Var}(\mathbb{C})/S}(U/S) := O_U(U)$

Definition 119. (i) For $S \in \text{Var}(\mathbb{C})$, we consider the complexes of presheaves

$$\Omega_{/S}^\bullet := \text{coker}(\Omega_{O_{\text{Var}(\mathbb{C})/S}/\tilde{e}(S)^* O_S} : \Omega_{\tilde{e}(S)^* O_S}^\bullet \rightarrow \Omega_{O_{\text{Var}(\mathbb{C})/S}}^\bullet) \in C_{O_S}(\text{Var}(\mathbb{C})/S)$$

which is by definition given by

- for X/S a morphism $\Omega_{/S}^\bullet(X/S) = \Omega_{X/S}^\bullet(X)$
- for $g : X'/S \rightarrow X/S$ a morphism,

$$\begin{aligned} \Omega_{/S}^\bullet(g) &:= \Omega_{(X'/X)/(S/S)}(X') : \Omega_{X/S}^\bullet(X) \rightarrow g^* \Omega_{X/S}(X') \rightarrow \Omega_{X'/S}^\bullet(X') \\ \omega &\mapsto \Omega_{(X'/X)/(S/S)}(X')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{X'}(X') \mapsto \omega(dg(\alpha))) \end{aligned}$$

(ii) For $S \in \text{Var}(\mathbb{C})$, we consider the complexes of presheaves

$$\Omega_{/S}^\bullet := \rho_{S*} \tilde{\Omega}_{/S}^\bullet = \text{coker}(\Omega_{O_{\text{Var}(\mathbb{C})^{sm}/S}/e(S)^* O_S} : \Omega_{e(S)^* O_S}^\bullet \rightarrow \Omega_{O_{\text{Var}(\mathbb{C})^{sm}/S}}^\bullet) \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$$

which is by definition given by

- for U/S a smooth morphism $\Omega_{/S}^\bullet(U/S) = \Omega_{U/S}^\bullet(U)$

– for $g : U'/S \rightarrow U/S$ a morphism,

$$\begin{aligned}\Omega_{/S}^\bullet(g) &:= \Omega_{(U'/U)/(S/S)}(U') : \Omega_{U'/S}^\bullet(U) \rightarrow g^* \Omega_{U/S}(U') \rightarrow \Omega_{U'/S}^\bullet(U') \\ \omega &\mapsto \Omega_{(U'/U)/(S/S)}(U')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{U'}(U') \mapsto \omega(dg(\alpha)))\end{aligned}$$

Remark 10. For $S \in \text{Var}(\mathbb{C})$, $\Omega_{/S}^\bullet \in C(\text{Var}(\mathbb{C})/S)$ is by definition a natural extension of $\Omega_{/S}^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$. However $\Omega_{/S}^\bullet \in C(\text{Var}(\mathbb{C})/S)$ does NOT satisfy cdh descent.

For a smooth morphism $h : U \rightarrow S$ with $S, U \in \text{SmVar}(\mathbb{C})$, the cohomology presheaves $H^n \Omega_{U/S}^\bullet$ of the relative De Rham complex

$$DR(U/S) := \Omega_{U/S}^\bullet := \text{coker}(h^* \Omega_S \rightarrow \Omega_U) \in C_{h^* O_S}(U)$$

for all $n \in \mathbb{Z}$, have a canonical structure of a complex of $h^* D_S$ modules given by the Gauss Manin connexion : for $S^o \subset S$ an open subset, $U^o = h^{-1}(S^o)$, $\gamma \in \Gamma(S^o, T_S)$ a vector field and $\hat{\omega} \in \Omega_{U/S}^p(U^o)^c$ a closed form, the action is given by

$$\gamma \cdot [\hat{\omega}] = [l(\tilde{\gamma})\widehat{\partial\omega}],$$

$\omega \in \Omega_{U^o}^p(U^o)$ being a representative of $\hat{\omega}$ and $\tilde{\gamma} \in \Gamma(U^o, T_U)$ a relevement of γ (h is a smooth morphism), so that

$$DR(U/S) := \Omega_{U/S}^\bullet := \text{coker}(h^* \Omega_S \rightarrow \Omega_U) \in C_{h^* O_S, h^* \mathcal{D}}(U)$$

with this $h^* D_S$ structure. Hence we get $h_* \Omega_{U/S}^\bullet \in C_{O_S, \mathcal{D}}(S)$ considering this structure. Since h is a smooth morphism, $\Omega_{U/S}^p$ are locally free O_U modules.

The point (ii) of the definition 129 above gives the object in $DA(S)$ which will, for S smooth, represent the algebraic Gauss-Manin De Rham realisation. It is the class of an explicit complex of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$.

Proposition 105. Let $S \in \text{Var}(\mathbb{C})$.

(i) For $U/S = (U, h) \in \text{Var}(\mathbb{C})^{sm}/S$, we have $e(U)_* h^* \Omega_{/S}^\bullet = \Omega_{U/S}^\bullet$.

(ii) The complex of presheaves $\Omega_{/S}^\bullet \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)$ is \mathbb{A}^1 homotopic, in particular \mathbb{A}^1 invariant. Note that however, for $p > 0$, the complexes of presheaves $\Omega_{/S}^{\bullet \geq p}$ are NOT \mathbb{A}^1 local. On the other hand, $(\Omega_{/S}^\bullet, F_b)$ admits transferts (recall that means $\text{Tr}(S)_* \text{Tr}(S)^* \Omega_{/S}^p = \Omega_{/S}^p$).

(iii) If S is smooth, we get $(\Omega_{/S}^\bullet, F_b) \in C_{O_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{sm}/S)$ with the structure given by the Gauss Manin connexion. Note that however the D_S structure on the cohomology groups given by Gauss Main connexion does NOT comes from a structure of D_S module structure on the filtered complex of O_S module. The D_S structure on the cohomology groups satisfy a non trivial Griffithz transversality (in the non projection cases), whereas the filtration on the complex is the trivial one.

Proof. (i): Let $h' : V \rightarrow U$ a smooth morphism with $V \in \text{Var}(\mathbb{C})$. We have then

$$h^* \Omega_{/S}^p(V \xrightarrow{h'} U) = \Omega_{/S}^p(V \xrightarrow{h'} U \xrightarrow{h} S).$$

Hence, if $h' : V \hookrightarrow U$ is in particular an open embedding, $h^* \Omega_{/S}^p(V \xrightarrow{h'} U) = \Omega_{U/S}^p(V)$. This proves the equality.

(ii): We prove that $E_{et}(\Omega_{/S}^\bullet, F_b) \in C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ is 2-filtered \mathbb{A}_S^1 invariant. We follow [21]. Consider the map in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$\phi := \text{ad}(p_a^*, p_{a*})(-) : \Omega_{/S}^\bullet \rightarrow p_{a*} p_a^* \Omega_{/S}^\bullet$$

which is given, for $U/S \in \text{Var}(\mathbb{C})^{sm}/S$ by

$$\text{ad}(p_a^*, p_{a*})(-)(U/S) = \Omega_{(U \times \mathbb{A}^1/U)/(S/S)}(U \times \mathbb{A}^1) : \Omega_{U/S}^\bullet(U) \rightarrow \Omega_{U \times \mathbb{A}^1/S}^\bullet(U \times \mathbb{A}^1), \omega \mapsto p^* \omega$$

where $p : U \times \mathbb{A}^1 \rightarrow U$ is the projection. On the other hand consider the map in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$\psi := I_0^* : p_{a*}p_a^*\Omega_{/S}^\bullet \rightarrow \Omega_{/S}^\bullet$$

given, for $U/S \in \text{Var}(\mathbb{C})^{sm}/S$ by

$$I_0^*(U/S) : \Omega_{U \times \mathbb{A}^1/S}^\bullet(U \times \mathbb{A}^1) \rightarrow \Omega_{U/S}^\bullet(U), \omega \mapsto i_0^*\omega$$

where $i_0 : U \hookrightarrow U \times \mathbb{A}^1$ is closed embedding given by $i_0(x) := (x, 0)$. Then,

- we have $\phi \circ \psi = I$
- considering the map in $\text{PSh}(\mathbb{N} \times \text{Var}(\mathbb{C})^{sm}/S)$

$$H : p_{a*}p_a^*\Omega_{/S}^\bullet[1] \rightarrow p_{a*}p_a^*\Omega_{/S}^\bullet$$

given for $U/S \in \text{Var}(\mathbb{C})^{sm}/S$ by

$$\begin{aligned} H(U/S)\Omega_{U \times \mathbb{A}^1/S}^p(U \times \mathbb{A}^1) &\rightarrow \Omega_{U \times \mathbb{A}^1/S}^{p-1}(U \times \mathbb{A}^1), \\ H(U/S)(p^*\omega \wedge q^*(f(s)ds)) &= \left(\int_0^t f(s)ds\right)p^*\omega, \quad H(U/S)(p^*\omega \wedge q^*f) = 0, \end{aligned}$$

note that $g(t) = \int_0^t f(s)ds$ is algebraic since $f \in O_{\mathbb{A}^1}(\mathbb{A}^1)$ is a polynomial, we have $\psi \circ \phi - I = \partial H + H\partial$.

This shows that

$$\text{ad}(p_a^*, p_{a*})(-) : \Omega_{/S}^\bullet \rightarrow p_{a*}p_a^*\Omega_{/S}^\bullet$$

is an homotopy equivalence whose inverse is I_0^* . Hence,

$$\text{ad}(p_a^*, p_{a*})(-) : (\Omega_{/S}^\bullet, F_b) \rightarrow p_{a*}p_a^*(\Omega_{/S}^\bullet, F_b)$$

is a 2-filtered homotopy equivalence whose inverse is

$$I_0^* : p_{a*}p_a^*(\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/S}^\bullet, F_b).$$

(iii): For $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{SmVar}(\mathbb{C})$, recall that the $h^*D_S(U) = D_S(h(U))$ structure on $H^p\Omega_{/S}^\bullet(U/S) := H^p\Omega_{U/S}^\bullet(U)$ is given by, for $\hat{\omega} \in \Omega_{U/S}^p(U)^c$, $\gamma \cdot [\hat{\omega}] = [\iota(\tilde{\gamma})\widehat{\partial\omega}]$, $\omega \in \Omega_U^p(U^o)$ being a representative of $\hat{\omega}$ and $\tilde{\gamma} \in \Gamma(U^o, T_U)$ a relevation of γ (h is a smooth morphism). Now, if $g : V/S \rightarrow U/S$ is a morphism, where $h' : V \rightarrow S$ is a smooth morphism with $V \in \text{SmVar}(\mathbb{C})$, we have

$$g^*(\gamma \cdot \hat{\omega}) = g^*(\iota(\tilde{\gamma})\widehat{\partial\omega}) = \iota(\tilde{\gamma})\widehat{\partial g^*\omega} = \gamma \cdot (g^*\hat{\omega})$$

that is $H^p\Omega_{/S}^\bullet(g) : H^p\Omega^\bullet(U/S) \rightarrow H^p\Omega^\bullet(V/S)$ is a map of $D_S(h(U))$ modules. \square

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Consider the following commutative diagram in RCat

:

$$\begin{array}{ccc} D(g, e) : (\text{Var}(\mathbb{C})^{sm}/T, O_{\text{Var}(\mathbb{C})^{sm}/T}) & \xrightarrow{P(g)} & (\text{Var}(\mathbb{C})^{sm}/S, O_{\text{Var}(\mathbb{C})^{sm}/S}) \\ \downarrow e(T) & & \downarrow e(S) \\ (T, O_T) & \xrightarrow{P(g)} & (S, O_S) \end{array}$$

It gives (see section 2) the canonical morphism in $C_{g^*O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/T)$

$$\begin{aligned} \Omega_{/(T/S)} &:= \Omega_{(O_{\text{Var}(\mathbb{C})^{sm}/T}/g^*O_{\text{Var}(\mathbb{C})^{sm}/S})/(O_T/g^*O_S)} : \\ g^*(\Omega_{/S}^\bullet, F_b) &= \Omega_{g^*O_{\text{Var}(\mathbb{C})^{sm}/S}/g^*e(S)^*O_S}^\bullet \rightarrow (\Omega_{/T}^\bullet, F_b) = \Omega_{O_{\text{Var}(\mathbb{C})^{sm}/T}/e(T)^*O_T}^\bullet \end{aligned}$$

which is by definition given by the pullback on differential forms : for $(V/T) = (V, h) \in \text{Var}(\mathbb{C})^{sm}/T$,

$$\begin{aligned} \Omega_{/(T/S)}(V/T) : g^*(\Omega_{/S}^\bullet)(V/T) &:= \lim_{(h':U \rightarrow \text{SSm}, g':V \rightarrow U, h, g)} \Omega_{U/S}^\bullet(U) \xrightarrow{\Omega_{(V/U)/(T/S)}(V/T)} \Omega_{V/T}^\bullet(V) =: \Omega_{/T}^\bullet(V/T) \\ \hat{\omega} &\mapsto \Omega_{(V/U)/(T/S)}(V/T)(\omega) := g'^*\hat{\omega}. \end{aligned}$$

If S and T are smooth, $\Omega_{/(T/S)} : g^*(\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/T}^\bullet, F_b)$ is a map in $C_{g^*O_S \text{fil}, g^*D_S}(\text{Var}(\mathbb{C})^{sm}/T)$ It induces the canonical morphisms in $C_{g^*O_S \text{fil}, g^*D_S}(\text{Var}(\mathbb{C})^{sm}/T)$:

$$E\Omega_{/(T/S)} : g^*E_{et}(\Omega_{/S}^\bullet, F_b) \xrightarrow{T(g, E_{et})(\Omega_{/S}^\bullet, F_b)} E_{et}(g^*(\Omega_{/S}^\bullet, F_b)) \xrightarrow{E_{et}(\Omega_{/(T/S)})} E_{et}(\Omega_{/T}^\bullet, F_b).$$

and

$$E\Omega_{/(T/S)} : g^*E_{zar}(\Omega_{/S}^\bullet, F_b) \xrightarrow{T(g, E_{zar})(\Omega_{/S}^\bullet, F_b)} E_{zar}(g^*(\Omega_{/S}^\bullet, F_b)) \xrightarrow{E_{zar}(\Omega_{/(T/S)})} E_{zar}(\Omega_{/T}^\bullet, F_b).$$

Definition 120. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. We have, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_T \text{fil}}(T)$:

$$\begin{aligned} T^O(g, \Omega_{/ \cdot})(F) &: g^{*mod}L_{Oe}(S)_*\mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b)) \\ &\xrightarrow{=} (g^*L_{Oe}(S)_*\mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{T(e, g)(-)\circ T(g, L_O)(-)} L_{Oe}(e(T)_*g^*\mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{T(g, \text{hom})(F, E_{et}(\Omega_{/S}^\bullet)) \otimes I} L_{Oe}(e(T)_*\mathcal{H}om^\bullet(g^*F, g^*E_{et}(\Omega_{/S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{ev(\text{hom}, \otimes)(-, -, -)} L_{Oe}(T)_*\mathcal{H}om^\bullet(g^*F, g^*E_{et}(\Omega_{/S}^\bullet, F_b)) \otimes_{g^*e(S)^*O_S} e(T)^*O_T \\ &\xrightarrow{\mathcal{H}om^\bullet(g^*F, E\Omega_{/(T/S)} \otimes I)} L_{Oe}(T)_*\mathcal{H}om^\bullet(g^*F, E_{et}(\Omega_{/T}^\bullet, F_b)) \otimes_{g^*e(S)^*O_S} e(T)^*O_T \\ &\xrightarrow{m} L_{Oe}(T)_*\mathcal{H}om^\bullet(g^*F, E_{et}(\Omega_{/T}^\bullet, F_b)) \end{aligned}$$

where $m(\alpha \otimes h) := h.\alpha$ is the multiplication map.

(ii) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$, S smooth. Assume there is a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. We have, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_T \text{fil}}(Y \times S)$:

$$\begin{aligned} T(g, \Omega_{/ \cdot})(F) &: g^{*mod, \Gamma}e(S)_*\mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b)) \\ &\xrightarrow{=} \Gamma_T E_{zar}(p_S^{*mod}e(S)_*\mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b))) \\ &\xrightarrow{T^O(p_S, \Omega_{/ \cdot})(F)} \Gamma_T E_{zar}(e(T \times S)_*\mathcal{H}om^\bullet(p_S^*F, E_{et}(\Omega_{/Y \times S}^\bullet, F_b))) \\ &\xrightarrow{=} e(T \times S)_*\Gamma_T(\mathcal{H}om^\bullet(p_S^*F, E_{et}(\Omega_{/Y \times S}^\bullet, F_b))) \\ &\xrightarrow{I(\gamma, \text{hom})(-, -)} e(T \times S)_*\mathcal{H}om^\bullet(\Gamma_T^\vee p_S^*F, E_{et}(\Omega_{/Y \times S}^\bullet, F_b)). \end{aligned}$$

For $Q \in \text{Proj PSh}(\text{Var}(\mathbb{C})^{sm}/S)$,

$$T(g, \Omega_{/ \cdot})(Q) : g^{*mod, \Gamma}e(S)_*\mathcal{H}om^\bullet(Q, E_{et}(\Omega_{/S}^\bullet, F_b)) \rightarrow e(T \times S)_*\mathcal{H}om^\bullet(\Gamma_T^\vee p_S^*Q, E_{et}(\Omega_{/Y \times S}^\bullet, F_b))$$

is a map in $C_{O_T \text{fil}, \mathcal{D}}(Y \times S)$.

The following easy lemma describe these transformation map on representable presheaves :

Lemma 7. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$ and $h : U \rightarrow S$ is a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider a commutative diagram whose square are cartesian :*

$$\begin{array}{ccccc} g : T & \xrightarrow{l} & S \times Y & \xrightarrow{p_S} & S \\ h' \uparrow & & h'' := h \times I \uparrow & & \uparrow h \\ g' : U_T & \xrightarrow{l'} & U \times Y & \xrightarrow{p_U} & U \end{array}$$

with l, l' the graph embeddings and p_S, p_U the projections. Then $g^*\mathbb{Z}(U/S) = \mathbb{Z}(U_T/T)$ and

(i) we have the following commutative diagram in $C_{O_T \text{fil}}(T)$ (see definition 1 and definition 120(i)) :

$$\begin{array}{ccc} g^{*mod} L_{Oe}(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{et}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(g, \Omega_{/ \cdot})(\mathbb{Z}(U/S))} & e(T)_* \mathcal{H}om^\bullet(\mathbb{Z}(U_T/T), E_{et}(\Omega_{/T}^\bullet, F_b)) \\ \uparrow k & & \uparrow k \\ g^{*mod} L_{Oe}(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{zar}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(g, \Omega_{/ \cdot})(\mathbb{Z}(U/S))} & e(T)_* \mathcal{H}om^\bullet(\mathbb{Z}(U_T/T), E_{zar}(\Omega_{/T}^\bullet, F_b)) \\ \downarrow = & & \downarrow = \\ g^{*mod} L_{Oh_*} E_{zar}(\Omega_{U/S}^\bullet, F_b) & \xrightarrow{T_\omega^{mod}(g, h)} & h'_* E_{zar}(\Omega_{U_T/T}^\bullet, F_b) \end{array}$$

(ii) if $Y, S \in \text{SmVar}(\mathbb{C})$, we have the following commutative diagram in $C_{O_T \text{fil}, \mathcal{D}}(Y \times S)$ (see definition 1 and definition 120(ii)) :

$$\begin{array}{ccc} g^{*mod, \Gamma} e(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{et}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(g, \Omega_{/ \cdot})(\mathbb{Z}(U/S))} & e(Y \times S)_* \mathcal{H}om^\bullet(\Gamma_{U_T}^\vee \mathbb{Z}(U \times Y/S \times Y), E_{et}(\Omega_{/Y \times S}^\bullet, F_b)) \\ \uparrow k & & \uparrow k \\ g^{*mod, \Gamma} L_{Oe}(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{zar}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(g, \Omega_{/ \cdot})(\mathbb{Z}(U/S))} & e(Y \times S)_* \mathcal{H}om^\bullet(\Gamma_{U_T}^\vee \mathbb{Z}(U \times Y/S \times Y), E_{zar}(\Omega_{/Y \times S}^\bullet, F_b)) \\ \downarrow = & & \downarrow = \\ g^{*mod, \Gamma} h_* E_{zar}(\Omega_{U/S}^\bullet, F_b) & \xrightarrow{T_\omega^O(p_S, h)(-)} & h''_* \Gamma_{U_T} E_{zar}(\Omega_{U \times Y/S \times Y}^\bullet, F_b) \end{array}$$

where $j : T \setminus T \times S \hookrightarrow T \times S$ is the open complementary embedding,

with

$$k : E_{zar}(h^* \Omega_{/S}^\bullet, F_b) \rightarrow E_{et}(E_{zar}(h^* \Omega_{/S}^\bullet, F_b)) = E_{et}(h^* \Omega_{/S}^\bullet, F_b).$$

which is a filtered Zariski local equivalence.

Proof. The commutative diagram follows from Yoneda lemma and proposition 105(i). On the other hand, $k : E_{zar}(\Omega_{/S}^\bullet, F_b) \rightarrow E_{et}(\Omega_{/S}^\bullet, F_b)$ is a (1-)filtered Zariski local equivalence by theorem 10 and proposition 105(ii) \square

In the projection case, we have the following :

Proposition 106. *Let $p : S_{12} \rightarrow S_1$ is a smooth morphism with $S_1, S_{12} \in \text{AnSp}(\mathbb{C})$. Then if $Q \in C(\text{Var}(\mathbb{C})^{sm}/S_1)$ is projective,*

$$T(p, \Omega_{/ \cdot})(Q) : p^{*mod} e(S_1)_* \mathcal{H}om^\bullet(Q, E_{et}(\Omega_{/S_1}^\bullet, F_b)) \rightarrow e(S_{12})_* \mathcal{H}om^\bullet(p^* Q, E_{et}(\Omega_{/S_{12}}^\bullet, F_b))$$

is an isomorphism.

Proof. Follows from lemma 7 and base change by smooth morphisms of quasi-coherent sheaves. \square

Let $S \in \text{Var}(\mathbb{C})$ and $h : U \rightarrow S$ a morphism with $U \in \text{Var}(\mathbb{C})$. We then have the canonical map given by the wedge product

$$w_{U/S} : \Omega_{U/S}^\bullet \otimes_{O_S} \Omega_{U/S}^\bullet \rightarrow \Omega_{U/S}^\bullet; \alpha \otimes \beta \mapsto \alpha \wedge \beta.$$

Let $S \in \text{Var}(\mathbb{C})$ and $h_1 : U_1 \rightarrow S$, $h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2 \in \text{Var}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \rightarrow S$ and $p_{112} : U_1 \times_S U_2 \rightarrow U_1$, $p_{212} : U_1 \times_S U_2 \rightarrow U_2$ the projections. We then have the canonical map given by the wedge product

$$w_{(U_1, U_2)/S} : p_{112}^* \Omega_{U_1/S}^\bullet \otimes_{O_S} p_{212}^* \Omega_{U_2/S}^\bullet \rightarrow \Omega_{U_{12}/S}^\bullet; \alpha \otimes \beta \mapsto p_{112}^* \alpha \wedge p_{212}^* \beta$$

which gives the map

$$\begin{aligned} & Ew_{(U_1, U_2)/S} : h_{1*} E_{zar}(\Omega_{U_1/S}^\bullet) \otimes_{O_S} h_{2*} E_{zar}(\Omega_{U_2/S}^\bullet) \\ & \xrightarrow{\text{ad}(p_{112}^*, p_{112*})(-) \otimes \text{ad}(p_{212}^*, p_{212*})(-)} (h_{1*} p_{112*} p_{112}^* E_{zar}(\Omega_{U_1/S}^\bullet)) \otimes_{O_S} (h_{2*} p_{212*} p_{212}^* E_{zar}(\Omega_{U_2/S}^\bullet)) \\ & \xrightarrow{=} h_{12*} (p_{112}^* E_{zar}(\Omega_{U_1/S}^\bullet) \otimes_{h_{12}^* O_S} p_{212}^* E_{zar}(\Omega_{U_2/S}^\bullet)) \\ & \xrightarrow{T(\otimes, E)(-) \circ (T(p_{112}, E)(-) \otimes T(p_{212}, E)(-))} h_{12*} E_{zar}(p_{112}^* \Omega_{U_1/S}^\bullet \otimes_{O_S} p_{212}^* \Omega_{U_2/S}^\bullet) \end{aligned}$$

Let $S \in \text{Var}(\mathbb{C})$. We have the canonical map in $C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$

$$w_S : (\Omega_{/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/S}^\bullet, F_b)$$

given by for $h : U \rightarrow S \in \text{Var}(\mathbb{C})^{sm}/S$

$$w_S(U/S) : (\Omega_{U/S}^\bullet, F_b) \otimes_{h^* O_S} (\Omega_{U/S}^\bullet, F_b)(U) \xrightarrow{w_{U/S}(U)} (\Omega_{U/S}^\bullet, F_b)(U)$$

It gives the map

$$Ew_S : E_{et}(\Omega_{/S}^\bullet, F_b) \otimes_{O_S} E_{et}(\Omega_{/S}^\bullet, F_b) \xrightarrow{=} E_{et}((\Omega_{/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{/S}^\bullet, F_b)) \xrightarrow{E_{et}(w_S)} E_{et}(\Omega_{/S}^\bullet, F_b)$$

If $S \in \text{SmVar}(\mathbb{C})$,

$$w_S : (\Omega_{/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/S}^\bullet, F_b)$$

is a map in $C_{O_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{sm}/S)$.

Definition 121. Let $S \in \text{Var}(\mathbb{C})$. We have, for $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_S \text{fil}}(S)$:

$$\begin{aligned} & T(\otimes, \Omega)(F, G) : e(S)_* \mathcal{H}om(F, E_{et}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} e(S)_* \mathcal{H}om(G, E_{et}(\Omega_{/S}^\bullet, F_b)) \\ & \xrightarrow{=} e(S)_* (\mathcal{H}om(F, E_{et}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} \mathcal{H}om(G, E_{et}(\Omega_{/S}^\bullet, F_b))) \\ & \xrightarrow{e(S)_* T(\mathcal{H}om, \otimes)(-)} e(S)_* \mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^\bullet, F_b) \otimes_{O_S} E_{et}(\Omega_{/S}^\bullet, F_b)) \\ & \xrightarrow{\mathcal{H}om(F \otimes G, Ew_S)} e(S)_* \mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^\bullet, F_b)) \end{aligned}$$

If $S \in \text{SmVar}(\mathbb{C})$, $T(\otimes, \Omega)(F, G)$ is a map in $C_{O_S \text{fil}, \mathcal{D}}(S)$.

Lemma 8. Let $S \in \text{Var}(\mathbb{C})$ and $h_1 : U_1 \rightarrow S$, $h_2 : U_2 \rightarrow S$ two smooth morphisms with $U_1, U_2 \in \text{Var}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \rightarrow S$ and $p_{112} : U_1 \times_S U_2 \rightarrow U_1$, $p_{212} : U_1 \times_S U_2 \rightarrow U_2$ the projections. We then have the following commutative diagram

$$\begin{array}{ccc} e(S)_* \mathcal{H}om(F, E_{et}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} e(S)_* \mathcal{H}om(G, E_{et}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(\otimes, \Omega)(F, G)} & e(S)_* \mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^\bullet, F_b)) \\ \uparrow k & & \uparrow k \\ h_{1*} E_{zar}(\Omega_{U_1/S}^\bullet, F_b) \otimes_{O_S} h_{2*} E_{zar}(\Omega_{U_2/S}^\bullet, F_b) & \xrightarrow{Ew_{(U_1, U_2)/S}} & h_{12*} E_{zar}(\Omega_{U_{12}/S}^\bullet, F_b) \end{array}$$

with

$$k : E_{zar}(\Omega_{/S}^\bullet, F_b) \rightarrow E_{et}(E_{zar}(\Omega_{/S}^\bullet, F_b)) = E_{et}(\Omega_{/S}^\bullet, F_b).$$

which is a filtered Zariski local equivalence.

Proof. Follows from Yoneda lemma. \square

Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Consider, for $I \subset J$, the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Considering the factorization of the diagram D_{IJ} by the fiber product :

$$D_{IJ} = \begin{array}{ccccc} \tilde{S}_J = \tilde{S}_I \times \tilde{S}_{J \setminus I} & \xrightarrow{p_{IJ}} & \tilde{S}_I & & \\ \uparrow & \swarrow i_I \times I & \uparrow & & \\ S_J & \xrightarrow{l_J} & S_I \times \tilde{S}_{J \setminus I} & \xrightarrow{p_{IJ}^0} & S_I \\ \uparrow i_J & \nearrow j_{IJ} & \uparrow & & \uparrow i_I \end{array}$$

the square of this factorization being cartesian, we have for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$

$$\begin{aligned} S(D_{IJ})(F) : Li_{J*}j_J^*F &\xrightarrow{q} i_{J*}j_J^*F = (i_I \times I) * l_{J*}j_J^*F \xrightarrow{(i_I \times I)_* \text{ad}(p_{IJ}^0, p_{IJ}^0)(-)} \\ &(i_I \times I)_* p_{IJ}^0 p_{IJ}^0 l_{J*}j_J^*F \xrightarrow{T(p_{IJ}, i_I)(-)^{-1}} p_{IJ}^* i_{I*} p_{IJ}^0 l_{J*}j_J^*F = p_{IJ}^* i_{I*} j_I^* F \end{aligned}$$

which factors through

$$S(D_{IJ})(F) : Li_{J*}j_J^*F \xrightarrow{S^q(D_{IJ})(F)} p_{IJ}^* Li_{I*}j_I^*F \xrightarrow{q} p_{IJ}^* i_{I*} j_I^* F$$

Definition 122. (i) Let $S \in \text{SmVar}(\mathbb{C})$. We have the functor

$$C(\text{Var}(\mathbb{C})^{sm}/S)^{op} \rightarrow C_{\text{Fil}, \mathcal{D}}(S), \quad F \mapsto e(S)_* \mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega_{/S}^\bullet, F_b))[-d_S].$$

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We have the functor

$$C(\text{Var}(\mathbb{C})^{sm}/S)^{op} \rightarrow C_{\text{Fil}, \mathcal{D}}(S/(\tilde{S}_I)), \quad F \mapsto (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], u_{IJ}^q(F))$$

where

$$\begin{aligned} u_{IJ}^q(F)[d_{\tilde{S}_I}] &: e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\ &\xrightarrow{\text{ad}(p_{IJ}^{*mod}, p_{IJ*})(-)} p_{IJ*} p_{IJ}^{*mod} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\ &\xrightarrow{p_{IJ*} T(p_{IJ}, \Omega_{/\tilde{S}_I})(L(i_{I*}j_I^*F))} p_{IJ*} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(p_{IJ}^* L(i_{I*}j_I^*F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\ &\xrightarrow{p_{IJ*} e(\tilde{S}_I)_* \mathcal{H}om(S^q(D_{IJ})(F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))} p_{IJ*} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{J*}j_J^*F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)). \end{aligned}$$

For $I \subset J \subset K$, we have obviously $p_{IJ*}u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$.

We then have the following key proposition

Proposition 107. (i) Let $S \in \text{Var}(\mathbb{C})$. Let $m : Q_1 \rightarrow Q_2$ be an equivalence (\mathbb{A}^1, et) local in $C(\text{Var}(\mathbb{C})^{sm}/S)$ with Q_1, Q_2 complexes of projective presheaves. Then,

$$e(S)_* \mathcal{H}om(m, E_{et}(\Omega_{/S}^\bullet, F_b)) : e(S)_* \mathcal{H}om^\bullet(Q_2, E_{et}(\Omega_{/S}^\bullet, F_b)) \rightarrow e(S)_* \mathcal{H}om^\bullet(Q_1, E_{et}(\Omega_{/S}^\bullet, F_b))$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{O_S \text{fil}, \mathcal{D}, \infty}(S)$ if S is smooth.

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $m = (m_I) : (Q_{1I}, s_{1I}^1) \rightarrow (Q_{2I}, s_{2I}^2)$ be an equivalence (\mathbb{A}^1, et) local in $C(\text{Var}(\mathbb{C})^{sm}/(\tilde{S}_I)^{op})$ with Q_{1I}, Q_{2I} complexes of projective presheaves. Then,

$$(e(\tilde{S}_I)_* \mathcal{H}om(m_I, E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))) : (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(Q_{2I}, E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}(Q_{2I}, s_{2I}^2)) \rightarrow (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(Q_{1I}, E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}(Q_{1I}, s_{1I}^1))$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{O_S \text{fil}, \mathcal{D}, \infty}((\tilde{S}_I))$.

Proof. (i): By definition of an (\mathbb{A}^1, et) local equivalence (see proposition 17), there exist

$$\{X_{1,\alpha}/S, \alpha \in \Lambda_1\}, \dots, \{X_{s,\alpha}/S, \alpha \in \Lambda_s\} \subset \text{Var}(\mathbb{C})^{(sm)}/S$$

such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{(sm)}/S))$

$$\begin{aligned} \text{Cone}(m) &\xrightarrow{\sim} \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}(X_{1,\alpha} \times \mathbb{A}^1/S) \rightarrow \mathbb{Z}(X_{1,\alpha}/S)) \\ &\rightarrow \dots \rightarrow \oplus_{\alpha \in \Lambda_s} \text{Cone}(\mathbb{Z}(X_{s,\alpha} \times \mathbb{A}^1/S) \rightarrow \mathbb{Z}(X_{s,\alpha}/S))) \end{aligned}$$

This gives in $D_{\text{fil}}(\mathbb{Z}) := \text{Ho}_{\text{fil}}(\mathbb{Z})$,

$$\begin{aligned} \text{Cone}(\text{Hom}(m, E_{et}(\Omega_{/S}^\bullet, F_b))) &\xrightarrow{\sim} \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(E_{et}(\Omega_{/S}^\bullet, F_b)(X_{1,\alpha}/S) \rightarrow E_{et}(\Omega_{/S}^\bullet, F_b)(X_{1,\alpha} \times \mathbb{A}^1/S)) \\ &\rightarrow \dots \rightarrow \oplus_{\alpha \in \Lambda_s} \text{Cone}(E_{et}(\Omega_{/S}^\bullet, F_b)(X_{s,\alpha}/S) \rightarrow E_{et}(\Omega_{/S}^\bullet, F_b)(X_{s,\alpha} \times \mathbb{A}^1/S))) \end{aligned}$$

Since $\Omega_{/S}^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is \mathbb{A}^1 homotopic, for all $1 \leq i \leq s$ and all $\alpha \in \Lambda_i$,

$$\text{Cone}(E_{et}(\Omega_{/S}^\bullet)(X_{i,\alpha}/S) \rightarrow E_{et}(\Omega_{/S}^\bullet)(X_{i,\alpha} \times \mathbb{A}^1/S)) \rightarrow 0$$

are homotopy equivalence. Hence $\text{Cone}(\text{Hom}(m, E_{et}(G, F))) \rightarrow 0$ is a 2-filtered quasi-isomorphism.

(ii): Similar to (i) : see proposition 20 □

Definition 123. (i) We define, using definition 122, by proposition 107, the filtered algebraic Gauss-Manin realization functor defined as

$$\mathcal{F}_S^{GM} : \text{DA}_c(S)^{op} \rightarrow D_{O_S \text{fil}, \mathcal{D}, \infty}(S), \quad M \mapsto \mathcal{F}_S^{GM}(M) := e(S)_* \mathcal{H}om^\bullet(L(F), E_{et}(\Omega_{/S}^\bullet, F_b))[-d_S]$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$,

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define, using definition 122 and proposition 107 the filtered algebraic Gauss-Manin realization functor defined as

$$\begin{aligned} \mathcal{F}_S^{GM} : \text{DA}_c(S)^{op} &\rightarrow D_{O_{\text{fil}}, \mathcal{D}, \infty}(S/(\tilde{S}_I)), \quad M \mapsto \\ \mathcal{F}_S^{GM}(M) &:= ((e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

Proposition 108. For $S \in \text{Var}(\mathbb{C})$, the functor \mathcal{F}_S^{GM} is well defined.

Proof. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Let $M \in \text{DA}(S)$. Let $F, F' \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_1, et)(F) = D(\mathbb{A}_1, et)(F')$. Then there exist by definition a sequence of morphisms in $C(\text{Var}(\mathbb{C})^{sm}/S)$:

$$F = F_1 \xrightarrow{s_1} F_2 \xleftarrow{s_2} F_3 \xrightarrow{s_3} F_4 \rightarrow \dots \xrightarrow{s_l} F' = F_s$$

where, for $1 \leq k \leq s$, and s_k are (\mathbb{A}^1, et) local equivalence. But if $s : F_1 \rightarrow F_2$ is an equivalence (\mathbb{A}^1, et) local,

$$L(i_{I*} j_I^* s) : L(i_{I*} j_I^* F_1) \rightarrow L(i_{I*} j_I^* F_2)$$

is an equivalence (\mathbb{A}^1, et) local, hence

$$\begin{aligned} \text{Hom}(L(i_{I*} j_I^* s), E_{et}(\Omega_{\tilde{S}_I}^\bullet, F_b)) &: (e(\tilde{S}_I)_* \text{Hom}(L(i_{I*} j_I^* F_2), E_{et}(\Omega_{\tilde{S}_I}^\bullet, F_b)), u_{IJ}(F_2)) \\ &\rightarrow (e(\tilde{S}_I)_* \text{Hom}(L(i_{I*} j_I^* F_1), E_{et}(\Omega_{\tilde{S}_I}^\bullet, F_b)), u_{IJ}(F_1)) \end{aligned}$$

is an ∞ -filtered quasi-isomorphism by proposition 107. \square

Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Assume that there is a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f , with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have, for $I \subset [1, \dots, l]$, closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and the following commutative diagrams which are cartesian

$$\begin{array}{ccc} f|_{X_I} : X_I \xrightarrow{l_I} Y \times S_I \xrightarrow{p_{S_I}} S_I & , & Y \times \tilde{S}_J \xrightarrow{p_{\tilde{S}_J}} \tilde{S}_J \\ & \searrow & \downarrow p'_{IJ} \quad \downarrow p_{IJ} \\ & & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I \end{array}$$

with $l_I : l|_{X_I}$, $i'_I = I \times i_I$, p_{S_I} and $p_{\tilde{S}_I}$ are the projections and $p'_{IJ} = I \times p_{IJ}$, and we recall that we denote by $j_I : \tilde{S}_I \setminus S_I \hookrightarrow \tilde{S}_I$ and $j'_I : Y \times \tilde{S}_I \setminus X_I \hookrightarrow Y \times S_I$ the open complementary embeddings. We then have the commutative diagrams

$$\begin{array}{ccc} D_{IJ} = S_J \xrightarrow{i_J} \tilde{S}_J & , & D'_{IJ} = X_J \xrightarrow{i'_J \circ l_J} Y \times \tilde{S}_J \\ \downarrow j_{IJ} \quad \downarrow p_{IJ} & & \downarrow j'_{IJ} \quad \downarrow p'_{IJ} \\ S_I \xrightarrow{i_I} \tilde{S}_I & & X_I \xrightarrow{i'_I \circ l_I} Y \times \tilde{S}_I \end{array}$$

and the factorization of D'_{IJ} by the fiber product:

$$\begin{array}{ccc} D'_{IJ} = X_J \xrightarrow{i'_I \circ l_I} Y \times \tilde{S}_J & , & D'_{IJ} = X_J \xrightarrow{i'_I \circ l_I} Y \times \tilde{S}_J \\ \downarrow j'_{IJ} \quad \downarrow p'_{IJ} & & \downarrow p'_{IJ} \\ X_I \xrightarrow{i'_I \circ l_I} Y \times \tilde{S}_I & & Y \times \tilde{S}_I \end{array}$$

$$\begin{array}{ccc} X_J & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_J \\ & \searrow \hat{l}_J & \nearrow \hat{i}_I \\ & X_I \times_{Y \times \tilde{S}_I} Y \times \tilde{S}_J = X_I \times \tilde{S}_{J \setminus I} & \\ & \nearrow \hat{p}_{IJ} & \\ X_I & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_I \end{array} \quad (55)$$

where $j'_{I,J} : X_J \hookrightarrow X_I$ is the open embedding. Consider

$$F(X/S) := p_{S,\sharp} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S) \in C(\text{Var}(\mathbb{C})^{sm}/S)$$

so that $D(\mathbb{A}^1, et)(F(X/S)) = M(X/S)$. Then, by definition,

$$\mathcal{F}_S^{GM}(M^{BM}(X/S)) := (e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*} j'_I{}^* F(X/S)), E_{et}(\Omega_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{I,J}^q(F(X/S)))$$

On the other hand, let

$$Q(X_I/\tilde{S}_I) := p_{\tilde{S}_I,\sharp} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \in C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I),$$

see definition 10. We have then for $I \subset [1, l]$ the following map in $C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$:

$$N_I(X/S) : Q(X_I/\tilde{S}_I) = p_{\tilde{S}_I,\sharp} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \xrightarrow{p_{\tilde{S}_I,\sharp} \text{ad}(i'_{I*}, i'_{I*})(-)} \quad (56)$$

$$p_{\tilde{S}_I,\sharp} i'_{I*} i'_{I*}{}^* \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I)[d_Y] \xrightarrow{p_{\tilde{S}_I,\sharp} (T(i'_{I*}, \gamma^\vee)(-))^{-1}} p_{\tilde{S}_I,\sharp} i'_{I*} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times S_I/Y \times S_I)[d_Y] \quad (57)$$

$$\xrightarrow{\hat{T}_i(p_{S_I}, i_I)(-)} i_{I*} p_{S_I,\sharp} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times S_I/Y \times S_I)[d_Y] = i_{I*} j'_I{}^* F(X/S) \quad (58)$$

We have then for $I \subset J$ the following commutative diagram in $C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$:

$$\begin{array}{ccc} p_{I,J}^* p_{\tilde{S}_I,\sharp} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) & \xrightarrow{p_{I,J}^* N_I(X/S)} & p_{I,J}^* (i_{I*} j'_I{}^* F(X/S)) \\ \uparrow H_{I,J} & & \uparrow S(D_{I,J})(F(X/S)) \\ p_{\tilde{S}_J,\sharp} \Gamma_{X_J}^\vee \mathbb{Z}(Y \times \tilde{S}_J/Y \times \tilde{S}_J) & \xrightarrow{N_J(X/S)} & i_{J*} j'_J{}^* F(X/S) \end{array} \quad (59)$$

with

$$\begin{aligned} & H_{I,J} : p_{\tilde{S}_J,\sharp} \Gamma_{X_J}^\vee \mathbb{Z}(Y \times \tilde{S}_J/Y \times \tilde{S}_J) \\ \xrightarrow{=} & p_{\tilde{S}_J,\sharp} p_{X_I,\sharp} \Gamma_{X_J}^\vee p_{I,J}^* \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \xrightarrow{\text{Cone}(\text{ad}(p'_{I,J}, p'_{I,J})(-), I)} p_{\tilde{S}_J,\sharp} \Gamma_{X_I \times \tilde{S}_{J \setminus I}}^\vee p_{I,J}^* \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \\ & \xrightarrow{T(p_{I,J}, \gamma^\vee)(-)} p_{\tilde{S}_J,\sharp} p_{I,J}^* \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \xrightarrow{T_\sharp(p_{I,J}, p_{\tilde{S}_I})(-)} p_{I,J}^* p_{\tilde{S}_I,\sharp} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I). \end{aligned}$$

This say that the maps $N_I(X/S)$ induces a map in $C(\text{Var}(\mathbb{C})^{sm}/(S/\tilde{S}_I))$

$$(N_I(X/S)) : (Q(X_I/\tilde{S}_I), H_{I,J}) \rightarrow (i_{I*} j'_I{}^* F(X/S), S(D_{I,J})(F(X/S))).$$

We denote by $v_{I,J}^q(F(X/S))$ the composite

$$\begin{aligned} & v_{I,J}^q(F(X/S))[d_{\tilde{S}_J}] : e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\ & \xrightarrow{\text{ad}(p_{I,J}^{*mod}, p_{I,J})(-)} p_{I,J*} p_{I,J}^{*mod} e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\ & \xrightarrow{p_{I,J*} T(p_{I,J}, \Omega)(Q(X_I/\tilde{S}_I))} p_{I,J*} e(\tilde{S}_J)_* \mathcal{H}om(p_{I,J}^* Q(X_I/\tilde{S}_I), E_{et}(\Omega_{/\tilde{S}_J}^\bullet, F_b)) \\ & \xrightarrow{\mathcal{H}om(H_{I,J}, E_{et}(\Omega_{/\tilde{S}_J}^\bullet, F_b))} p_{I,J*} e(\tilde{S}_J)_* \mathcal{H}om(Q(X_J/\tilde{S}_J), E_{et}(\Omega_{/\tilde{S}_J}^\bullet, F_b)). \end{aligned}$$

On the other hand, we have the following map in $C_{Ofil, \mathcal{D}, S_J}(\tilde{S}_J)$

$$\begin{aligned} & w_{I,J}(X/S)[d_{\tilde{S}_J}] : p_{\tilde{S}_I*} \Gamma_{X_I} E_{zar}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \xrightarrow{\text{ad}(p_{I,J}^{*mod}, p_{I,J})(-)} p_{I,J*} p_{I,J}^{*mod} p_{\tilde{S}_I*} \Gamma_{X_I} E_{zar}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \\ & \xrightarrow{T_w^O(p_{I,J}, p_{\tilde{S}_I})^\gamma} p_{I,J*} p_{\tilde{S}_J*} \Gamma_{X_I \times \tilde{S}_{J \setminus I}} E_{zar}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \xrightarrow{\text{Cone}(I, \text{ad}(p'_{I,J}, p_{I,J*})(-))} p_{\tilde{S}_J*} \Gamma_{X_J} E_{zar}(\Omega_{Y \times \tilde{S}_J/\tilde{S}_J}^\bullet, F_b). \end{aligned}$$

Lemma 9. (i) The map in $C(\text{Var}(\mathbb{C})^{sm}/(S/\tilde{S}_I))$

$$(N_I(X/S)) : (Q(X_I/\tilde{S}_I), H_{IJ}) \rightarrow (L(i_{I*}j_I^*F(X/S)), S^q(D_{IJ})(F(X/S))).$$

is an equivalence (\mathbb{A}^1, et) local.

(ii) The maps $(N_I(X/S))$ induces an ∞ -filtered quasi-isomorphism in $C_{Ofil, \mathcal{D}}(S/(\tilde{S}_I))$

$$\begin{aligned} & (\mathcal{H}om(N_I(X/S), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))) : \\ & (e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*}j_I^*F(X/S)), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S))) \rightarrow \\ & (e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \end{aligned}$$

(iii) The maps $(I(\gamma, \text{hom})(-, -))$ and $(k : E_{zar}(p_{\tilde{S}_I}^* \Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \rightarrow E_{et}(p_{\tilde{S}_I}^* \Omega_{/\tilde{S}_I}^\bullet, F_b))$ induce an $(1-)$ filtered Zariski local equivalence in $C_{Ofil, \mathcal{D}}(S/(\tilde{S}_I))$

$$\begin{aligned} & (k \circ I(\gamma, \text{hom})(-, -)) : (p_{\tilde{S}_I*} \Gamma_{X_I} E_{zar}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)) \\ & \rightarrow (e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \end{aligned}$$

Proof. (i): Follows from theorem 16.

(ii): These maps induce a morphism in $C_{Ofil, \mathcal{D}}(S/(\tilde{S}_I))$ by construction. The fact that it is an ∞ -filtered quasi-isomorphism follows from (i) and proposition 107.

(iii): These maps induce a morphism in $C_{Ofil, \mathcal{D}}(S/(\tilde{S}_I))$ by construction. \square

Proposition 109. Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. Assume there exist a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We then have, for $I \subset [1, \dots, l]$, the following commutative diagrams which are cartesian

$$\begin{array}{ccc} f_I = f|_{X_I} : X_I & \xrightarrow{l_I} & Y \times S_I \xrightarrow{p_{S_I}} S_I \\ & \searrow & \downarrow i'_I \quad \downarrow i_I \\ & & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I \end{array} \quad , \quad \begin{array}{ccc} Y \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p_{IJ} \\ Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \end{array}$$

Let $F(X/S) := p_{S*} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S)$. The transformations maps $(N_I(X/S) : Q(X_I/\tilde{S}_I) \rightarrow i_{I*}j_I^*F(X/S))$ and $(k \circ I(\gamma, \text{hom})(-, -))$, for $I \subset [1, \dots, l]$, induce an isomorphism in $D_{Ofil, \mathcal{D}, \infty}(S/(\tilde{S}_I))$

$$\begin{aligned} & I^{GM}(X/S) : \\ & \mathcal{F}_S^{GM}(M(X/S)) := (e(\tilde{S}_I)_* \mathcal{H}om(L(i_{I*}j_I^*F(X/S)), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S))) \\ & \xrightarrow{(e(\tilde{S}_I)_* \mathcal{H}om(LN_I(X/S), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))} (e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \\ & \xrightarrow{(k \circ I(\gamma, \text{hom})(-, -))^{-1}} (p_{\tilde{S}_I*} \Gamma_{X_I} E_{zar}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)). \end{aligned}$$

Proof. Follows from lemma 9. \square

We now define the functorialities of \mathcal{F}_S^{GM} with respect to S which makes \mathcal{F}_{GM}^- a morphism of 2-functor.

Definition 124. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Consider the factorization $g : T \xrightarrow{l} T \times S \xrightarrow{p_S} S$ where l is the graph embedding and p_S the projection. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then, $D(\mathbb{A}_T^1, et)(g^*F) = g^*M$.

(i) We have then the canonical transformation in $D_{\text{Ofil}, \mathcal{D}, \infty}(T \times S)$ (see definition 120) :

$$\begin{aligned} T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_S^{GM}(M) &:= g^{*mod, \Gamma} e(S)_* \mathcal{H}om^\bullet(LF, E_{et}(\Omega_{/S}^\bullet, F_b))[-d_T] \\ &\xrightarrow{T(g, \Omega_{/ \cdot})(LF)} \\ e(T \times S)_* \mathcal{H}om^\bullet(\Gamma_T^\vee p_S^* LF, E_{et}(\Omega_{/T \times S}^\bullet, F_b))[-d_T] &=: \mathcal{F}_{T \times S}^{GM}(l_* g^*(M, W)). \end{aligned}$$

(ii) We have then the canonical transformation in $D_{\text{Ofil}, \infty}(T)$ (see definition 120) :

$$\begin{aligned} T^O(g, \mathcal{F}^{GM})(M, W) : Lg^{*mod[-], \Gamma} \mathcal{F}_S^{GM}(M) &:= g^{*mod} e(S)_* \mathcal{H}om^\bullet(LF, E_{et}(\Omega_{/S}^\bullet, F_b))[-d_T] \\ &\xrightarrow{T^O(g, \Omega_{/ \cdot})(LF)} \\ e(T)_* \mathcal{H}om^\bullet(g^* LF, E_{et}(\Omega_{/T}^\bullet, F_b))[-d_T] &=: \mathcal{F}_T^{GM}(g^*M). \end{aligned}$$

We give now the definition in the non smooth case Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. We recall the commutative diagram :

$$\begin{array}{ccccc} E_{IJg} = (Y \times \tilde{S}_I) \setminus T_I & \xrightarrow{m'_I} & Y \times \tilde{S}_J & , & E_{IJ} = \tilde{S}_J \setminus S_J & \xrightarrow{m_J} & \tilde{S}_J & & E'_{IJ} = (Y \times \tilde{S}_J) \setminus T_J & \xrightarrow{m'_J} & Y \times \tilde{S}_J \\ \downarrow p_{\tilde{S}_I} & & \downarrow \tilde{g}_I & & \downarrow p_{IJ} & & \downarrow p_{IJ} & & \downarrow p'_{IJ} & & \downarrow p'_{IJ} \\ \tilde{S}_I \setminus S_I & \xrightarrow{m_I} & \tilde{S}_I & & \tilde{S}_I \setminus (S_I \setminus S_J) & \xrightarrow{m_{IJ}} & \tilde{S}_I & & (Y \times \tilde{S}_I) \setminus (T_I \setminus T_J) & \xrightarrow{m'_{IJ}} & Y \times \tilde{S}_I \end{array}$$

For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} := I_Y \times p_{IJ} : Y \times \tilde{S}_J \rightarrow Y \times \tilde{S}_I$ the projections, so that $\tilde{g}_I \circ p'_{IJ} = p_{IJ} \circ \tilde{g}_J$. Consider, for $I \subset J \subset [1, \dots, l]$, resp. for each $I \subset [1, \dots, l]$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$\begin{array}{ccc} D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array} & , & D'_{IJ} = \begin{array}{ccc} T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_I \\ j'_{IJ} \uparrow & & \uparrow p'_{IJ} \\ T_J & \xrightarrow{i'_J} & Y \times \tilde{S}_J \end{array} \\ D_{gI} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ g_I \uparrow & & \uparrow \tilde{g}_I \\ T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_I \end{array} & , & \end{array}$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Let $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$. Recall (see section 2) that since $j'_I{}^* i'_I{}^* j'_I{}^* g^* F = 0$, the morphism $T(D_{gI})(j'_I{}^* F) : \tilde{g}_I{}^* i_{I*} j'_I{}^* F \rightarrow i'_{I*} j'_I{}^* g^* F$ factors through

$$T(D_{gI})(j'_I{}^* F) : \tilde{g}_I{}^* i_{I*} j'_I{}^* F \xrightarrow{\gamma_{X_I}^\vee(-)} \Gamma_{X_I}^\vee \tilde{g}_I{}^* i_{I*} j'_I{}^* F \xrightarrow{T^\gamma(D_{gI})(j'_I{}^* F)} i'_{I*} j'_I{}^* g^* F$$

We then have, for each $I \subset [1, \dots, l]$, the morphism

$$\begin{aligned} T^{q, \gamma}(D_{gI})(j'_I{}^* F) : \Gamma_{T_I}^\vee \tilde{g}_I{}^* L(i_{I*} j'_I{}^* F) &\xrightarrow{T(\tilde{g}_I, L)(-)} \\ \Gamma_{T_I}^\vee L(\tilde{g}_I{}^* i_{I*} j'_I{}^* F) = L(\Gamma_{T_I}^\vee \tilde{g}_I{}^* i_{I*} j'_I{}^* F) &\xrightarrow{L(T^\gamma(D_{gI})(j'_I{}^* F))} L(i'_{I*} j'_I{}^* g^* F) \end{aligned}$$

and the following diagram in $C(\text{Var}(\mathbb{C})^{sm}/Y \times \tilde{S}_I)$ commutes

$$\begin{array}{ccc} \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F) & \xrightarrow{\Gamma_{T_I}^\vee \tilde{g}_I^* q_I(F)} & \Gamma_{T_I}^\vee \tilde{g}_I^* i_{I*} j_I^* F \\ \downarrow T^{q, \gamma}(D_{g_I})(j_I^* F) & & \downarrow T^\gamma(D_{g_I})(j_I^* F) \\ L(i'_{I*} j_I^* g^* F) & \xrightarrow{q_I(g^* F)} & i'_{I*} g_I^* j_I^* F = i'_{I*} j_I^* g^* F \end{array}$$

We have the following commutative diagram in $C(\text{Var}(\mathbb{C})^{sm}/Y \times \tilde{S}_J)$

$$\begin{array}{ccc} p'_{I*} \tilde{g}_I^* i_{I*} j_I^* F = \tilde{g}_J^* p'_{I*} i_{I*} j_I^* F & \xrightarrow{p'_{I*} T(D_{g_I})(j_I^* F)} & p'_{I*} i'_{I*} g_I^* j_I^* F = p'_{I*} i'_{I*} j_I^* g^* F \\ \uparrow \tilde{g}_J^* S(D_{I_J})(F) & & \uparrow S(D'_{I_J})(g^* F) \\ \tilde{g}_J^* i_{J*} j_{I*} j_I^* F = \tilde{g}_J^* i_{J*} j_J^* F & \xrightarrow{T(D_{g_J})(j_J^* F)} & i'_{J*} g_J^* j_J^* F = i'_{J*} j_J^* g^* F = i'_{J*} j_J^* g^* F \end{array} \quad (60)$$

This gives, after taking the functor L , the following commutative diagram in $C(\text{Var}(\mathbb{C})^{sm}/Y \times \tilde{S}_J)$

$$\begin{array}{ccc} \Gamma_{T_J}^\vee p'_{I*} \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F) = \Gamma_{T_J}^\vee \tilde{g}_J^* p'_{I*} L(i_{I*} j_I^* F) & \xrightarrow{\Gamma_{T_J}^\vee T^{q, \gamma}(D_{g_I})(j_I^* F)} & \Gamma_{T_J}^\vee p'_{I*} L(i'_{I*} j_I^* g^* F) \\ \uparrow \tilde{g}_J^* S^q(D_{I_J})(F) & & \uparrow S^q(D'_{I_J})(g^* F) \\ \Gamma_{T_J}^\vee \tilde{g}_J^* L(i_{J*} j_J^* g^* F) & \xrightarrow{T^{q, \gamma}(D_{g_J})(j_J^* F)} & L(i'_{J*} j_J^* g^* F) \end{array} \quad (61)$$

The fact that the diagrams (61) commutes says that the maps $T^{q, \gamma}(D_{g_I})(j_I^* F)$ define a morphism in $C(\text{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$

$$(T^{q, \gamma}(D_{g_I})(j_I^* F)) : (\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F), \tilde{g}_J^* S^q(D_{I_J})(F)) \rightarrow (L(i'_{I*} j_I^* g^* F), S^q(D'_{I_J})(g^* F))$$

We denote by $\tilde{g}_J^* u_{I_J}^q(F)_1$ the composite

$$\begin{aligned} & \tilde{g}_J^* u_{I_J}^q(F)_1 [d_Y + d_{\tilde{S}_I}] : e(Y \times \tilde{S}_I)_* \Gamma_{T_I} \mathcal{H}om(\tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) \\ & \xrightarrow{\text{ad}(p'_{I*}, p'_{I*})(-)} p'_{I*} p'_{I*}{}^{*mod} e(Y \times \tilde{S}_I)_* \Gamma_{T_I} \mathcal{H}om(\tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) \\ & \xrightarrow{T^{mod}(p'_{I*}, \gamma)(-)} p'_{I*} e(Y \times \tilde{S}_I)_* \Gamma_{T_I \times \tilde{S}_I} p'_{I*}{}^{*mod} \mathcal{H}om(\tilde{g}_J^* L(i_{J*} j_J^* F), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b)) \\ & \xrightarrow{\text{Cone}(\text{ad}(p'_{I*}, p'_{I*})(-), I)} p'_{I*} e(Y \times \tilde{S}_J)_* \Gamma_{T_J} p'_{I*}{}^{*mod} \mathcal{H}om(\tilde{g}_J^* L(i_{J*} j_J^* F), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b)) \\ & \xrightarrow{T(p'_{I*}, \Omega_{/ \cdot})(-)} p'_{I*} e(Y \times \tilde{S}_J)_* \Gamma_{T_J} \mathcal{H}om(\tilde{g}_J^* p'_{I*} L(i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b)) \\ & \xrightarrow{\mathcal{H}om(\tilde{g}_J^* S^q(D_{I_J})(F), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b))} p'_{I*} e(Y \times \tilde{S}_J)_* \Gamma_{T_J} \mathcal{H}om(\tilde{g}_J^* L(i_{J*} j_J^* F), E_{et}(\Omega_{Y \times \tilde{S}_J}^\bullet, F_b)) \end{aligned}$$

We denote by $\tilde{g}_J^* u_{IJ}^q(F)_2$ the composite

$$\begin{aligned}
& \tilde{g}_J^* u_{IJ}^q(F)_2 [d_Y + d_{\tilde{S}_I}] : e(\tilde{T}_I)_* \mathcal{H}om(\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) \\
& \xrightarrow{\text{ad}(p'_{IJ*} \text{mod}, p'_{IJ*})(-)} p'_{IJ*} p'_{IJ*} \text{mod} e(\tilde{T}_I)_* \mathcal{H}om(\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) \\
& \xrightarrow{T(p'_{IJ}, \Omega_{\cdot})(-)} p'_{IJ*} e(\tilde{T}_I)_* \mathcal{H}om(p'_{IJ} \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) \\
& \xrightarrow{\mathcal{H}om(T(p'_{IJ}, \gamma^\vee)(-), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b))} p'_{IJ*} e(\tilde{T}_I)_* \mathcal{H}om(\Gamma_{T_I \times \tilde{S}_I}^\vee p'_{IJ} \tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) \\
& \xrightarrow{\text{Cone}(\text{ad}(p'_{IJ_2}, p'_{IJ})(-), I)} p'_{IJ*} e(\tilde{T}_I)_* \mathcal{H}om(\Gamma_{T_I}^\vee p'_{IJ} \tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) \\
& \xrightarrow{\mathcal{H}om(\Gamma_{T_I}^\vee \tilde{g}_I^*(S^q(D_{IJ})(F)), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b))} p'_{IJ*} e(Y \times \tilde{S}_I)_* \mathcal{H}om(\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b))
\end{aligned}$$

We then have then the following lemma :

Lemma 10. (i) *The morphism in $C(\text{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$*

$$(T^{q,\gamma}(D_{g_I})(j_I^* F)) : (\Gamma_{T_I}^\vee L \tilde{g}_I^* i_{I*} j_I^* F, \tilde{g}_I^* S^q(D_{IJ})(F)) \rightarrow (i'_{I*} j_I^* g^* F, S^q(D'_{IJ})(F)(j_I^* g^* F))$$

is an equivalence (\mathbb{A}^1, et) local.

(ii) *Denote for short $d_{YI} = -d_Y - d_{\tilde{S}_I}$. The maps $\mathcal{H}om((T^{q,\gamma}(D_{g_I})(j_I^* F)), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b))$ induce an ∞ -filtered quasi-isomorphism in $C_{Ofil, \mathcal{D}}(T/(Y \times \tilde{S}_I))$*

$$\begin{aligned}
& (\mathcal{H}om(T^{q,\gamma}(D_{g_I})(j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b))) : \\
& (e(Y \times \tilde{T}_I)_* \mathcal{H}om(L(i'_{I*} j_I^* g^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) [d_{YI}], u_{IJ}^q(g^* F)) \rightarrow \\
& (e(Y \times \tilde{T}_I)_* \mathcal{H}om((\Gamma_{T_I}^\vee L \tilde{g}_I^* i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)) [d_{YI}], \tilde{g}_I^* u_{IJ}^q(F)_2)
\end{aligned}$$

(iii) *The maps $T(\tilde{g}_I, \Omega_{\cdot})(L(i_{I*} j_I^* F))$ (see definition 120) induce a morphism in $C_{Ofil, \mathcal{D}}(T/(Y \times \tilde{S}_I))$*

$$\begin{aligned}
& (T(\tilde{g}_I, \Omega_{\cdot})(L(i_{I*} j_I^* F))) : \\
& (\Gamma_{T_I} E_{zar}(\tilde{g}_I^{*mod[-1]} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), E_{et}(\Omega_{\tilde{S}_I}^\bullet, F_b))) [d_{YI}], \tilde{g}_I^{*mod} u_{IJ}^q(F)) \rightarrow \\
& (\Gamma_{T_I} (e(Y \times \tilde{S}_I)_* \mathcal{H}om(\tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b))) [d_{YI}], \tilde{g}_I^* u_{IJ}^q(F)_1).
\end{aligned}$$

Proof. (i): Follows from theorem 16.

(ii): These maps induce a morphism in $C_{Ofil, \mathcal{D}}(T/(Y \times \tilde{S}_I))$ by construction. The fact that this morphism is an ∞ -filtered equivalence Zariski local follows from (i) and proposition 107.

(iii): These maps induce a morphism in $C_{Ofil, \mathcal{D}}(T/(Y \times \tilde{S}_I))$ by construction. □

Definition 125. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Denote for short $d_{YI} := d_Y + d_{\tilde{S}_I}$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then, $D(\mathbb{A}_T^1, et)(g^* F) = g^* M$. We*

have, by lemma 10, the canonical transformation in $D_{\text{Ofil}, \mathcal{D}, \infty}(T/(Y \times \tilde{S}_I))$

$$\begin{aligned}
& T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_S^{GM}(M) := \\
& (\Gamma_{T_I} E_{zar}(\tilde{g}_I^{*mod} e(\tilde{S}_I) * \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), E_{et}(\Omega_{\tilde{S}_I}^\bullet, F_b)))[-d_Y - d_{\tilde{S}_I}], \tilde{g}_J^{*mod} u_{IJ}^q(F)) \\
& \xrightarrow{(\Gamma_{T_I} E(T(\tilde{g}_I, \Omega_{\tilde{S}_I}^\bullet)(L(i_{I*} j_I^*(F, W)))))} \\
& (\Gamma_{T_I} e(Y \times \tilde{S}_I) * \mathcal{H}om^\bullet(\tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)))[-d_Y - d_{\tilde{S}_I}], \tilde{g}_J^* u_{IJ}^q(F)_1) \\
& \xrightarrow{(I(\gamma, \text{hom}(-, -)))} \\
& (e(Y \times \tilde{S}_I) * \mathcal{H}om^\bullet(\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)))[-d_Y - d_{\tilde{S}_I}], \tilde{g}_J^* u_{IJ}^q(F)_2) \\
& \xrightarrow{(e(Y \times \tilde{S}_I) * \mathcal{H}om(T^{q, \gamma}(D_{g_I})(j_I^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)))^{-1}} \\
& (e(Y \times \tilde{S}_I) * \mathcal{H}om^\bullet(L(i'_{I*} j_I'^* g^* F), E_{et}(\Omega_{Y \times \tilde{S}_I}^\bullet, F_b)))[-d_Y - d_{\tilde{S}_I}], u_{IJ}^q(g^* F)) =: \mathcal{F}_T^{GM}(g^* M).
\end{aligned}$$

Proposition 110. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{p_S} S$ with $Y_2 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$. Assume that there is a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_S} S$, with $Y_1 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc}
f' : X_T & \longrightarrow & Y_1 \times T & \longrightarrow & T \\
\downarrow & & \downarrow & & \downarrow \\
f'' = f \times I : Y_2 \times X & \longrightarrow & Y_1 \times Y_2 \times S & \longrightarrow & Y_2 \times S \\
\downarrow & & \downarrow & & \downarrow \\
f : X & \longrightarrow & Y_1 \times S & \longrightarrow & S
\end{array}$$

Consider $F(X/S) := p_{S, \sharp} \Gamma_X^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S)$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$\begin{aligned}
T(f, g, F(X/S)) : g^* F(X/S) & := g^* p_{S, \sharp} \Gamma_X^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S) \xrightarrow{\sim} \\
& p_{T, \sharp} \Gamma_{X_T}^\vee \mathbb{Z}(Y_1 \times T/Y_1 \times T) =: F(X_T/T).
\end{aligned}$$

which gives in $\text{DA}(S)$ the isomorphism $T(f, g, F(X/S)) : g^* M(X/S) \xrightarrow{\sim} M(X_T/T)$. Then, the following diagram in $D_{\text{Ofil}, \mathcal{D}, \infty}(T/(Y_2 \times \tilde{S}_I))$ commutes

$$\begin{array}{ccc}
Rg^{*mod, \Gamma} \mathcal{F}_S^{GM}(M(X/S)) & \xrightarrow{T(g, \mathcal{F}^{GM})(M(X/S))} & \mathcal{F}_T^{GM}(M(X_T/T)) \\
\downarrow I^{GM}(X/S) & & \downarrow I^{GM}(X_T/T) \\
g^{*mod[-], \Gamma}(p_{\tilde{S}_I} * \Gamma_{X_I} E_{zar}(\Omega_{Y_1 \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)) & \xrightarrow{(T(\tilde{g}_I \times I, \gamma)(-) \circ T_w^O(\tilde{g}_I, p_{\tilde{S}_I}))} & (p_{Y_2 \times \tilde{S}_I} * \Gamma_{X_{T_I}} E_{zar}(\Omega_{Y_2 \times Y_1 \times \tilde{S}_I / Y_2 \times \tilde{S}_I}^\bullet, F_b)[-d_{Y_2} - d_{\tilde{S}_I}], w_{IJ}(X_T/T))
\end{array}$$

(ii) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$. Assume that there is a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$, with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Consider $F(X/S) := p_{S, \sharp} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S)$ and the isomorphism

in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T(f, g, F(X/S)) : g^* F(X/S) := g^* p_{S, \#} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S) \xrightarrow{\sim} \\ p_{T, \#} \Gamma_{X_T}^\vee \mathbb{Z}(Y \times T/Y \times T) =: F(X_T/T).$$

which gives in $\text{DA}(S)$ the isomorphism $T(f, g, F(X/S)) : g^* M(X/S) \xrightarrow{\sim} M(X_T/T)$. Then, the following diagram in $D_{\text{Ofil}, \infty}(T)$ commutes

$$\begin{array}{ccc} Lg^{*mod[-]} \mathcal{F}_S^{GM}(M(X/S)) & \xrightarrow{T^O(g, \mathcal{F}^{GM})(M(X/S))} & \mathcal{F}_T^{GM}(M(X_T/T)) \\ \downarrow I^{GM}(X/S) & & \downarrow I^{GM}(X_T/T) \\ g^{*mod} L_O(p_{S*} \Gamma_X E_{\text{Zar}}(\Omega_{Y \times S/S}^\bullet, F_b)[-d_T]) & \xrightarrow{(T(g \times I, \gamma)(-) \circ T_w^O(g, p_S))} & p_{Y \times T*} \Gamma_{X_T} E_{\text{Zar}}(\Omega_{Y \times T/T}^\bullet, F_b)[-d_T] \\ \downarrow T_w(\otimes, \gamma)(O_{Y \times S}) & & \downarrow T_w(\otimes, \gamma)(O_{Y \times T}) \\ Lg^{*mod} \int_{p_S}^{FDR} \Gamma_X E(O_{Y \times S}, F_b)[-d_Y - d_T] & \xrightarrow{T^{\mathcal{D}^{mod}}(g, f)(\Gamma_X E(O_{Y \times S}, F_b))} & \int_{p_T}^{FDR} \Gamma_{X_T} E(O_{Y \times T}, F_b)[-d_Y - d_T]. \end{array}$$

Proof. Follows immediately from definition. \square

We have the following theorem:

Theorem 35. (i) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S}$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_S^{GM}(M) \xrightarrow{\sim} \mathcal{F}_T^{GM}(g^* M)$$

is an isomorphism in $D_{O_T \text{fil}, \mathcal{D}, \infty}(T/(Y \times \tilde{S}_I))$.

(ii) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T^O(g, \mathcal{F}^{GM})(M) : Lg^{*mod[-]} \mathcal{F}_S^{GM}(M) \xrightarrow{\sim} \mathcal{F}_T^{GM}(g^* M)$$

is an isomorphism in $D_{O_T}(T)$.

(iii) A base change theorem for algebraic De Rham cohomology : Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $h : U \rightarrow S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Then the map (see definition 1)

$$T_w^O(g, h) : Lg^{*mod} Rh_*(\Omega_{U/S}^\bullet, F_b) \xrightarrow{\sim} Rh'_*(\Omega_{U_T/T}^\bullet, F_b)$$

is an isomorphism in $D_{O_T}(T)$.

Proof. (i): Follows from proposition 106.

(ii): Follows from proposition 110(ii) and the base change for algebraic D modules (proposition 84).

(iii): Follows from (ii) and lemma 7. \square

We finish this subsection by some remarks on the absolute case and on a particular case of the relative case:

Proposition 111. (i) Let $X \in \text{PSmVar}(\mathbb{C})$ and $D = \cup D_i \subset X$ a normal crossing divisor. Consider the open embedding $j : U := X \setminus D \hookrightarrow X$. Then,

– The map in $D_{fil,\infty}(\mathbb{C})$

$$\begin{aligned} & \text{Hom}(L\mathbb{D}(\mathbb{Z}(U)), k)^{-1} \circ \text{Hom}((0, \text{ad}(j^*, j_*)(\mathbb{Z}(X/X))), E_{et}(\Omega^\bullet, F_b)) : \\ & \mathcal{F}^{GM}(\mathbb{D}(\mathbb{Z}(U))) := \text{Hom}(L\mathbb{D}(\mathbb{Z}(U)), E_{et}(\Omega^\bullet, F_b)) \\ & \xrightarrow{\sim} \text{Hom}(\text{Cone}(\mathbb{Z}(D) \rightarrow \mathbb{Z}(X)), E_{zar}(\Omega^\bullet, F_b)) = \Gamma(X, E_{zar}(\Omega_X^\bullet(\text{nul } D), F_b)). \end{aligned}$$

is an isomorphism, where we recall $\mathbb{D}(\mathbb{Z}(U)) := a_{X*}j_*E_{et}(\mathbb{Z}(U/U)) = a_{U*}E_{et}(\mathbb{Z}(U/U))$,

– $\mathcal{F}^{GM}(\mathbb{Z}(U)) = \Gamma(U, E_{zar}(\Omega_U^\bullet, F_b)) \in D_{fil,\infty}(\mathbb{C})$ is NOT isomorphic to $\Gamma(X, E_{zar}(\Omega_X^\bullet(\log D), F_b))$ in $D_{fil,\infty}(\mathbb{C})$ in general. For exemple U is affine, then $H^n(U, \Omega_U^p) = 0$ for all $p \in \mathbb{N}$, $p \neq 0$, so that the $E_{\infty}^{p,q}(\Gamma(U, E_{zar}(\Omega_U^\bullet, F_b)))$ are NOT isomorphic to $E_{\infty}^{p,q}(\Gamma(X, E_{zar}(\Omega_X^\bullet(\log D), F_b)))$ in this case. In particular, the map,

$$j^* := \text{ad}(j^*, j_*)(-) : H^n\Gamma(X, E_{zar}(\Omega_X^\bullet(\log D))) \xrightarrow{\sim} H^n\Gamma(U, E_{zar}(\Omega_U^\bullet))$$

which is an isomorphism in $D(\mathbb{C})$ (i.e. if we forgot filtrations), gives embeddings

$$j^* := \text{ad}(j^*, j_*)(-) : F^p H^n(U, \mathbb{C}) := F^p H^n\Gamma(X, E_{zar}(\Omega_X^\bullet(\log D), F_b)) \hookrightarrow F^p H^n\Gamma(U, E_{zar}(\Omega_U^\bullet, F_b))$$

which are NOT an isomorphism in general for $n, p \in \mathbb{Z}$. Note that, since $a_U : U \rightarrow \{\text{pt}\}$ is not proper,

$$[\Delta_U] : \mathbb{Z}(U) \rightarrow a_{U*}E_{et}(\mathbb{Z}(U/U))[2d_U]$$

is NOT an equivalence (\mathbb{A}^1, et) local.

– Let $Z \subset X$ a smooth subvariety and denote $U := X \setminus Z$ the open complementary. Denote $M_Z(X) := \text{Cone}(M(U) \rightarrow M(X)) \in \text{DA}(\mathbb{C})$. The map in $D_{fil,\infty}(\mathbb{C})$

$$\begin{aligned} & \text{Hom}(G(X, Z), E_{et}(\Omega^\bullet, F_b))^{-1} \circ \text{Hom}(a_{X\sharp}\Gamma_Z^\vee\mathbb{Z}(X/X), k)^{-1} : \\ & \mathcal{F}^{GM}(M_Z(X)) := \text{Hom}(a_{X\sharp}\Gamma_Z^\vee\mathbb{Z}(X/X), E_{et}(\Omega^\bullet, F_b)) \xrightarrow{\sim} \\ & \Gamma(X, \Gamma_Z E_{zar}(\Omega_X^\bullet, F_b)) = \Gamma_Z(X, E_{zar}(\Omega_X^\bullet, F_b)) \\ & \xrightarrow{\sim} \mathcal{F}^{GM}(M(Z)(c)[2c]) = \Gamma(Z, E_{zar}(\Omega_Z^\bullet, F_b))(-c)[-2c] \end{aligned}$$

is an isomorphism, where $c = \text{codim}(Z, X)$ and $G(X, Z) : a_{X\sharp}\Gamma_Z^\vee\mathbb{Z}(X/X) \rightarrow \mathbb{Z}(Z)(c)[2c]$ is the Gysin morphism.

– Let $D \subset X$ a smooth divisor and denote $U := X \setminus Z$ the open complementary. Note that the canonical distinguish triangle in $\text{DA}(\mathbb{C})$

$$M(U) \xrightarrow{\text{ad}(j_\sharp, j^*)(\mathbb{Z}(X/X))} M(X) \xrightarrow{\gamma_Z^\vee(\mathbb{Z}(X/X))} M_D(X) \rightarrow M(U)[1]$$

give a canonical triangle in $D_{fil,\infty}(\mathbb{C})$

$$\mathcal{F}^{GM}(M_D(X)) \xrightarrow{\mathcal{F}^{GM}(\gamma_Z^\vee(\mathbb{Z}(X/X)))} \mathcal{F}^{GM}(M(X)) \xrightarrow{\mathcal{F}^{GM}(\text{ad}(j_\sharp, j^*)(\mathbb{Z}(X/X)))} \mathcal{F}^{GM}(M(U)) \rightarrow \mathcal{F}^{GM}(M_D(X))[1],$$

which is NOT the image of a distinguish triangle in $\pi(D(\text{MHM}(\mathbb{C})))$, as $\mathcal{F}^{GM}(M(U)) \notin \pi(D(\text{MHM}(\mathbb{C})))$ since the morphism

$$\text{ad}(j^*, j_\sharp) : H^n(X, E_{zar}(\Omega_X^\bullet(\log D), F_b)) \rightarrow H^n(U, E_{zar}(\Omega_U^\bullet, F_b))$$

is not strict. Note that if $U := S \setminus D$ is affine, then by the exact sequence in $C(\mathbb{Z})$

$$0 \rightarrow \Gamma_Z(X, E_{zar}(\Omega_X^p)) \rightarrow \Gamma(X, E_{zar}(\Omega_X^p)) \rightarrow \Gamma(U, E_{zar}(\Omega_U^p)) \rightarrow 0$$

we have $H^q\Gamma_Z(X, E_{zar}(\Omega_X^p)) = H^q(\Gamma(X, E_{zar}(\Omega_X^p)))$. In particular, the map,

$$j^* := \text{ad}(j^*, j_\sharp)(-) : \Gamma(X, E_{zar}(\Omega_X^\bullet(\log D), F_b)) \rightarrow \Gamma(U, E_{zar}(\Omega_U^\bullet, F_b))$$

and hence the map

$$j^* := \text{ad}(j^*, j_*)(-) : \text{Cone}(\Gamma(X, \Omega_X^\bullet, F_b) \rightarrow \Gamma(X, E_{\text{zar}}(\Omega_X^\bullet(\log D), F_b)) \rightarrow \\ \text{Cone}(\Gamma(X, \Omega_X^\bullet, F_b) \rightarrow \Gamma(U, E_{\text{zar}}(\Omega_U^\bullet, F_b))) =: \Gamma(X, \Gamma_Z E_{\text{zar}}(\Omega_X^\bullet, F_b))$$

are quasi-isomorphisms (i.e. if we forgot filtrations), but the first one is NOT an ∞ -filtered quasi-isomorphism whereas the second one is an ∞ -filtered quasi-isomorphism (recall that for $r > 1$ the r -filtered quasi-isomorphisms does NOT satisfy the 2 of 3 property for morphism of canonical triangles : see section 2.1).

(ii) More generally, let $f : X \rightarrow S$ a smooth projective morphism with $S, X \in \text{SmVar}(\mathbb{C})$. Let $D = \cup D_i \subset X$ a normal crossing divisor such that $f|_{D_i} := f \circ i_i : D_i \rightarrow S$ are SMOOTH morphisms (note that it is a very special case), with $i_i : D_i \hookrightarrow X$ the closed embeddings. Consider the open embedding $j : U := X \setminus D \hookrightarrow X$ and $h := f \circ j : U \rightarrow S$.

– The map in $D_{\mathcal{D}fil, \infty}(S)$

$$\text{Hom}(L\mathbb{D}(\mathbb{Z}(U)), k)^{-1} \circ \text{Hom}(\text{ad}(j^*, j_*)(\mathbb{Z}(X/X)), E_{\text{et}}(\Omega_{j^*}^\bullet, F_b)) : \\ \mathcal{F}_S^{GM}(\mathbb{D}(\mathbb{Z}(U/S))) := \text{Hom}(L\mathbb{D}(\mathbb{Z}(U/S)), E_{\text{et}}(\Omega_{j^*}^\bullet, F_b)) \\ \xrightarrow{\sim} \text{Hom}(\text{Cone}(\mathbb{Z}(D) \rightarrow \mathbb{Z}(X)), E_{\text{zar}}(\Omega_{j^*}^\bullet, F_b)) = f_* E_{\text{zar}}(\Omega_{X/S}^\bullet(\text{nul } D), F_b).$$

is an isomorphism, where we recall $\mathbb{D}(\mathbb{Z}(U)) := f_* j_* E_{\text{et}}(\mathbb{Z}(U/U)) = h_* E_{\text{et}}(\mathbb{Z}(U/U))$,

– $\mathcal{F}_S^{GM}(\mathbb{Z}(U/S)) = h_* E_{\text{zar}}(\Omega_{U/S}^\bullet, F_b) \in D_{\mathcal{D}fil, \infty}(S)$ is NOT isomorphic to $f_* E_{\text{zar}}(\Omega_{X/S}^\bullet(\log D), F_b)$ in $D_{\mathcal{D}fil, \infty}(S)$ in general. In particular, the map,

$$j^* := \text{ad}(j^*, j_*)(-) : H^n f_* E_{\text{zar}}(\Omega_{X/S}^\bullet(\log D)) \xrightarrow{\sim} H^n h_* E_{\text{zar}}(\Omega_{U/S}^\bullet)$$

which is an isomorphism in $D_{\mathcal{D}}(S)$ (i.e. if we forgot filtrations), gives embeddings

$$j^* := \text{ad}(j^*, j_*)(-) : F^p H^n h_* \mathbb{C}_U := F^p H^n f_* E_{\text{zar}}(\Omega_{X/S}^\bullet(\log D), F_b) \hookrightarrow F^p H^n h_* E_{\text{zar}}(\Omega_{U/S}^\bullet, F_b)$$

which are NOT an isomorphism in general for $n, p \in \mathbb{Z}$. Note that, since $a_U : U \rightarrow \{\text{pt}\}$ is not proper,

$$[\Delta_U] : \mathbb{Z}(U/S) \rightarrow h_* E_{\text{et}}(\mathbb{Z}(U/U))[2d_U]$$

is NOT an equivalence (\mathbb{A}^1, et) local.

– Let $Z \subset X$ a subvariety and denote $U := X \setminus Z$ the open complementary. Denote $M_Z(X/S) := \text{Cone}(M(U/S) \rightarrow M(X/S)) \in \text{DA}(S)$. If $f|_Z := f \circ i_Z : Z \rightarrow S$ is a SMOOTH morphism, the map in $D_{\mathcal{D}fil, \infty}(S)$

$$\text{Hom}(G(X, Z), E_{\text{et}}(\Omega^\bullet, F_b)) \circ \text{Hom}(\Gamma_Z^\vee \mathbb{Z}(X/X), k)^{-1} : \\ \mathcal{F}_S^{GM}(M_Z(X/S)) := \text{Hom}(f_{\sharp} \Gamma_Z^\vee \mathbb{Z}(X/X), E_{\text{et}}(\Omega_{j^*}^\bullet, F_b)) \xrightarrow{\sim} f_* \Gamma_Z E_{\text{zar}}(\Omega_{X/S}^\bullet, F_b) \\ \xrightarrow{\sim} \mathcal{F}_S^{GM}(M(Z/S)(c)[2c]) = f_{Z*} E_{\text{zar}}(\Omega_{Z/S}^\bullet, F_b)(-c)[-2c]$$

is an isomorphism, where $c = \text{codim}(Z, X)$ and $G(X, Z) : f_{\sharp} \Gamma_Z^\vee \mathbb{Z}(X/X) \rightarrow \mathbb{Z}(Z/S)(c)[2c]$ is the Gysin morphism.

– Let $D \subset X$ a smooth divisor and denote $U := X \setminus Z$ the open complementary. Note that the canonical distinguish triangle in $\text{DA}(S)$

$$M(U/S) \xrightarrow{\text{ad}(j_{\sharp}, j^*)(\mathbb{Z}(X/X))} M(X/S) \xrightarrow{\gamma_Z^\vee(\mathbb{Z}(X/X))} M_D(X/S) \rightarrow M(U/S)[1]$$

give a canonical triangle in $D_{\mathcal{D}fil, \infty}(S)$

$$\mathcal{F}_S^{GM}(M_D(X/S)) \xrightarrow{\mathcal{F}_S^{GM}(\gamma_Z^\vee(\mathbb{Z}(X/X)))} \mathcal{F}_S^{GM}(M(X/S)) \xrightarrow{\mathcal{F}_S^{GM}(\text{ad}(j_{\sharp}, j^*)(\mathbb{Z}(X/X)))} \mathcal{F}_S^{GM}(M(U/S)) \\ \rightarrow \mathcal{F}_S^{GM}(M_D(X/S))[1],$$

which is NOT the image of a distinguish triangle in $D(\text{MHM}(S))$.

Proof. (i):For simplicity, we may assume that $i : D \hookrightarrow X$ is a smooth divisor. Then, by theorem 16, the map

$$(0, \text{ad}(j_{\sharp}, j_*)(\mathbb{Z}(X/X)) : \mathbb{Z}(D) \rightarrow \mathbb{Z}(X)) \rightarrow \mathbb{D}(\mathbb{Z}(U/U))$$

is an equivalence (\mathbb{A}^1, et) local in $C(\text{SmVar}(\mathbb{C}))$. The result then follows from proposition 105. By theorem 16, we have an equivalence (\mathbb{A}^1, et) local in $C(\text{SmVar}(\mathbb{C}))$

$$G(X, Z) : a_{X\sharp}\Gamma^{\vee}\mathbb{Z}(X/X) \rightarrow \mathbb{Z}(Z)(c)[2c]$$

The result then follows from proposition 105.

(ii):For simplicity, we may assume that $i : D \hookrightarrow X$ is a smooth divisor. Then, by theorem 16, the map

$$(0, \text{ad}(j_{\sharp}, j_*)(\mathbb{Z}(X/X)) : \mathbb{Z}(D/S) \rightarrow \mathbb{Z}(X/S)) \rightarrow \mathbb{D}(\mathbb{Z}(U/U))$$

is an equivalence (\mathbb{A}^1, et) local in $C(\text{Var}(\mathbb{C})^{sm}/S)$. The result then follows from proposition 105. By theorem 16, we have an equivalence (\mathbb{A}^1, et) local in $C(\text{SmVar}(\mathbb{C}))$

$$G(X, Z) : f_{\sharp}\Gamma^{\vee}\mathbb{Z}(X/X) \rightarrow \mathbb{Z}(Z/S)(c)[2c]$$

The result then follows from proposition 105. \square

Definition 126. Let $S \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ projective such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{\text{Ofil}, \mathcal{D}}(S)$

$$\begin{aligned} T(\mathcal{F}_S^{GM}, \otimes)(M, N) &: \mathcal{F}_S^{GM}(M) \otimes_{O_S}^{L[-]} \mathcal{F}_S^{GM}(N) := \\ &(e(S)_*\mathcal{H}om(F, E_{et}(\Omega_{/S}^{\bullet}, F_b))) \otimes_{O_S} (e(S)_*\mathcal{H}om(G, E_{et}(\Omega_{/S}^{\bullet}, F_b)))[-d_S] \\ &\xrightarrow{T(\otimes, \Omega_{/S})(F, G)} e(S)_*\mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^{\bullet}, F_b))[-d_S] \\ &\xrightarrow{=} e(S)_*\mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^{\bullet}, F_b))[-d_S] =: \mathcal{F}_S^{GM}(M \otimes N) \end{aligned}$$

We now give the definition in the non smooth case :

Definition 127. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{\text{Ofil}, \mathcal{D}}(S/(\tilde{S}_I))$

$$\begin{aligned} T(\mathcal{F}_S^{GM}, \otimes)(M, N) &: \mathcal{F}_S^{GM}(M) \otimes_{O_S}^{L[-]} \mathcal{F}_S^{GM}(N) := \\ &(e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet}, F_b))[-d_{\tilde{S}_I}], u_{IJ}(F)) \otimes_{O_S}^{[-]} \\ &(e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*G), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet}, F_b))[-d_{\tilde{S}_I}], u_{IJ}(G)) \\ &\xrightarrow{=} ((e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet}, F_b)) \otimes_{O_{\tilde{S}_I}} \\ &e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*G), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet}, F_b)))[-d_{\tilde{S}_I}], u_{IJ}(F) \otimes u_{IJ}(G)) \\ &\xrightarrow{(T(\otimes, \Omega_{/\tilde{S}_I})(L(i_{I*}j_I^*F), L(i_{I*}j_I^*G)))} \\ &(e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F) \otimes L(i_{I*}j_I^*G), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet}, F_{DR}))[-d_{\tilde{S}_I}], v_{IJ}(F \otimes G)) \\ &\xrightarrow{=} (e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*(F \otimes G), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet}, F_b)))[-d_{\tilde{S}_I}], u_{IJ}(F \otimes G)) =: \mathcal{F}_S^{GM}(M \otimes N) \end{aligned}$$

Proposition 112. Let $f_1 : X_1 \rightarrow S$, $f_2 : X_2 \rightarrow S$ two morphism with $X_1, X_2, S \in \text{Var}(\mathbb{C})$. Assume that there exist factorizations $f_1 : X_1 \xrightarrow{l_1} Y_1 \times S \xrightarrow{ps} S$, $f_2 : X_2 \xrightarrow{l_2} Y_2 \times S \xrightarrow{ps} S$ with $Y_1, Y_2 \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S the projections. We have then the factorization

$$f_1 \times f_2 : X_{12} := X_1 \times_S X_2 \xrightarrow{l_1 \times l_2} Y_1 \times Y_2 \times S \xrightarrow{ps} S$$

Let $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ such that $M = D(\mathbb{A}^1, \text{et})(F)$ and $N = D(\mathbb{A}^1, \text{et})(G)$, the following commutative diagram in $D_{\text{Ofil}, \mathcal{D}}(S/(\tilde{S}_I))$

$$\begin{array}{ccc}
\mathcal{F}_S^{GM}(M(X_1/S)) \otimes_{\mathcal{O}_S}^L \mathcal{F}_S^{GM}(M(X_2/S)) & \xrightarrow{T(\mathcal{F}_S^{GM}, \otimes)(M(X_1/S), M(X_2/S))} & \mathcal{F}_S^{GM}(M(X_1/S) \otimes M(X_2/S)) \\
\downarrow I^{GM}(X_1/S) \otimes I^{GM}(X_2/S) & & \downarrow I^{GM}(X_{12}/S) \\
(p_{\tilde{S}_I^*} \Gamma_{X_{1I}} E_{\text{zar}}(\Omega_{Y_1 \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X_1/S)) \otimes_{\mathcal{O}_S} & \xrightarrow{(Ew_{(Y_1 \times \tilde{S}_I, Y_2 \times \tilde{S}_I) / \tilde{S}_I})} & (p_{\tilde{S}_I^*} \Gamma_{X_{12I}} E_{\text{zar}}(\Omega_{Y_1 \times Y_2 \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], \\
(p_{\tilde{S}_I^*} \Gamma_{X_{2I}} E_{\text{zar}}(\Omega_{Y_2 \times \tilde{S}_I / \tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X_2/S)) & & w_{IJ}(X_{12}/S))
\end{array}$$

Proof. Immediate from definition. \square

6.1.2 The algebraic filtered De Rham realization functor and the commutativity with the six operation

We recall (see section 2), for $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$, the commutative diagrams of sites (29) and (30)

$$\begin{array}{ccccc}
\text{Var}(\mathbb{C})^2/T & \xrightarrow{\mu_T} & \text{Var}(\mathbb{C})^{2,pr}/T & & \\
\downarrow P(f) & \searrow \rho_T & \downarrow \rho_T & & \\
& \text{Var}(\mathbb{C})^{2,sm}/T & \xrightarrow{\mu_T} & \text{Var}(\mathbb{C})^{2,smpr}/T & \\
& \downarrow P(f) & \downarrow P(f) & & \\
\text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,pr}/S & & \\
\downarrow P(f) & \searrow \rho_S & \downarrow \rho_S & & \\
& \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S & \\
& \downarrow P(f) & \downarrow P(f) & &
\end{array}$$

and

$$\begin{array}{ccccc}
\text{Var}(\mathbb{C})^{2,pr}/T & \xrightarrow{\text{Gr}_T^{12}} & \text{Var}(\mathbb{C})/T & & \\
\downarrow P(f) & \searrow \rho_T & \downarrow \rho_T & & \\
& \text{Var}(\mathbb{C})^{2,smpr}/T & \xrightarrow{\text{Gr}_T^{12}} & \text{Var}(\mathbb{C})^{sm}/T & \\
& \downarrow P(f) & \downarrow P(f) & & \\
\text{Var}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})/S & & \\
\downarrow P(f) & \searrow \rho_S & \downarrow \rho_S & & \\
& \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})^{sm}/S & \\
& \downarrow P(f) & \downarrow P(f) & &
\end{array}$$

Let $S \in \text{Var}(\mathbb{C})$. We have for $F \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ the canonical map in $C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$

$$\text{Gr}(F) : \text{Gr}_{S^*}^{12} \mu_{S^*} F^\Gamma \rightarrow F,$$

$$\text{Gr}(F)(U/S) : \Gamma_U^\vee p^* F(U \times S/U \times S) \xrightarrow{\text{ad}(l^*, l_*)(p^* F)(U \times S/U \times S)} h^* F(U/U) = F(U/S)$$

where $h : U \rightarrow S$ is a smooth morphism with $U \in \text{Var}(\mathbb{C})$ and $h : U \xrightarrow{l} U \times S \xrightarrow{p} S$ is the graph factorization with l the graph embedding and p the projection.

For $s : \mathcal{I} \rightarrow \mathcal{J}$ a functor, with $\mathcal{I}, \mathcal{J} \in \text{Cat}$, and $f_\bullet : T_\bullet \rightarrow S_{s(\bullet)}$ a morphism with $T_\bullet \in \text{Fun}(\mathcal{J}, \text{Var}(\mathbb{C}))$ and $S_\bullet \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C}))$, we have then the commutative diagrams of sites (31) and (32)

$$\begin{array}{ccccc}
\text{Var}(\mathbb{C})^2/T_\bullet & \xrightarrow{\mu_{T_\bullet}} & \text{Var}(\mathbb{C})^{2,pr}/T_\bullet & & \\
\downarrow P(f_\bullet) & \searrow \rho_{T_\bullet} & \downarrow \rho_{T_\bullet} & & \\
& \text{Var}(\mathbb{C})^{2,sm}/T_\bullet & \xrightarrow{\mu_{T_\bullet, P(f_\bullet)}} & \text{Var}(\mathbb{C})^{2,smpr}/T_\bullet & \\
& \downarrow \mu_{T_\bullet} & \downarrow P(f_\bullet) & \downarrow P(f_\bullet) & \\
\text{Var}(\mathbb{C})^2/S_\bullet & \xrightarrow{\mu_{S_\bullet}} & \text{Var}(\mathbb{C})^{2,pr}/S_\bullet & & \\
& \searrow \rho_{S_\bullet} & \downarrow \rho_{S_\bullet} & & \\
& \text{Var}(\mathbb{C})^{2,sm}/S_\bullet & \xrightarrow{\mu_{S_\bullet}} & \text{Var}(\mathbb{C})^{2,smpr}/S_\bullet &
\end{array}$$

and

$$\begin{array}{ccccc}
\text{Var}(\mathbb{C})^{2,pr}/T_\bullet & \xrightarrow{\text{Gr}_{T_\bullet}^{12}} & \text{Var}(\mathbb{C})/T & & \\
\downarrow P(f_\bullet) & \searrow \rho_{T_\bullet} & \downarrow \rho_{T_\bullet} & & \\
& \text{Var}(\mathbb{C})^{2,smpr}/T_\bullet & \xrightarrow{\text{Gr}_T^{12}} & \text{Var}(\mathbb{C})^{sm}/T_\bullet & \\
& \downarrow \text{Gr}_{T_\bullet}^{12} & \downarrow P(f_\bullet) & \downarrow P(f_\bullet) & \\
\text{Var}(\mathbb{C})^{2,pr}/S_\bullet & \xrightarrow{\text{Gr}_{S_\bullet}^{12}} & \text{Var}(\mathbb{C})/S_\bullet & & \\
& \searrow \rho_{S_\bullet} & \downarrow \rho_{S_\bullet} & & \\
& \text{Var}(\mathbb{C})^{2,sm}/S_\bullet & \xrightarrow{\text{Gr}_{S_\bullet}^{12}} & \text{Var}(\mathbb{C})^{sm}/S_\bullet &
\end{array}$$

We will use the following map from the property of mixed Hodge module (see section 5) together with the specialization map of a filtered D module for a closed embedding (see definition 56) :

Definition-Proposition 20. (i) Let $l : Z \hookrightarrow S$ a closed embedding with $S, Z \in \text{SmVar}(\mathbb{C})$. Consider an open embedding $j : S^\circ \hookrightarrow S$. We then have the cartesian square

$$\begin{array}{ccc}
S^\circ & \xrightarrow{j} & S \\
l' \uparrow & & \uparrow l \\
Z^\circ := Z \times_S S^{\circ j'} & \longrightarrow & Z
\end{array}$$

where j' is the open embedding given by base change. Using proposition 99, the morphisms $Q_{V_Z, V_D}^{p,0}(O_S, F_b)$ for $D \subset S$ a closed subset of definition-proposition 15 induces a canonical morphism in $C_{l^* O_S \text{fil}}(Z)$

$$Q(Z, j_l)(O_S, F_b) : l^* Q_{V_Z, 0} j_l \text{Hdg}(O_{S^\circ}, F_b) \rightarrow j_l' \text{Hdg}(O_{Z^\circ}, F_b),$$

where V_Z is the Kashiwara-Malgrange V_Z -filtration and V_D is the Kashiwara-Malgrange V_D -filtration, which commutes with the action of T_Z .

(ii) Let $l : Z \hookrightarrow S$ and $k : Z' \hookrightarrow Z$ be closed embeddings with $S, Z, Z' \in \text{SmVar}(\mathbb{C})$. Consider an open

embedding $j : S^\circ \hookrightarrow S$. We then have the commutative diagram whose squares are cartesian.

$$\begin{array}{ccc}
S^\circ & \xrightarrow{j} & S \\
l' \uparrow & & \uparrow l \\
Z^\circ := Z \times_S S^\circ & \xrightarrow{j'} & Z \\
k' \uparrow & & \uparrow k \\
Z'^\circ := Z' \times_S S^\circ & \xrightarrow{j''} & Z'
\end{array}$$

where j' is the open embedding given by base change. Then,

$$\begin{aligned}
Q(Z', j_1)(O_S, F_b) &= Q(Z', j'_1)(O_Z, F_b) \circ (k^* Q_{V_{Z'}, 0} Q(Z, j_1)(O_S, F_b)) : \\
k^* Q_{V_{Z'}, 0} l^* Q_{V_Z, 0} j_1! \text{Hdg}(O_{S^\circ}, F_b) &\xrightarrow{k^* Q_{V_{Z'}, 0} Q(Z, j_1)(O_S, F_b)} k^* Q_{V_{Z'}, 0} j_1'! \text{Hdg}(O_{Z^\circ}, F_b) \\
&\xrightarrow{Q(Z', j'_1)(O_Z, F_b)} j_1''! \text{Hdg}(O_{Z'^\circ}, F_b)
\end{aligned}$$

in $C_{k^* l^* O_S \text{fil}}(Z')$ which commutes with the action of $T_{Z'}$.

(iii) Consider a commutative diagram whose squares are cartesian

$$\begin{array}{ccccc}
S^{\circ\circ} & \xrightarrow{j_2} & S^\circ & \xrightarrow{j_1} & S \\
l'' \uparrow & & \uparrow l' & & \uparrow l \\
Z^{\circ\circ} := Z \times_S S^{\circ\circ} & \xrightarrow{j_2'} & Z^\circ := Z \times_S S^\circ & \xrightarrow{j_1'} & Z
\end{array}$$

where j_1, j_2 , and hence j_1', j_2' are open embeddings. We have then the following commutative diagram

$$\begin{array}{ccc}
l^* Q_{V_Z, 0} j_1! \text{Hdg}(O_{S^\circ}, F_b) & \xrightarrow{\text{ad}(j_2! \text{Hdg}, j_2^*)(O_{S^\circ}, F_b)} & l^* Q_{V_Z, 0} (j_1 \circ j_2)! \text{Hdg}(O_{S^{\circ\circ}}, F_b) \\
\downarrow Q(Z, j_1)(O_S, F_b) & & \downarrow Q(Z, (j_1 \circ j_2)_!)(O_S, F_b) \\
j_1'! \text{Hdg}(O_{Z^\circ}, F_b) & \xrightarrow{\text{ad}(j_2'! \text{Hdg}, j_2'^*)(O_{Z^\circ}, F_b)} & (j_1' \circ j_2')! \text{Hdg}(O_{Z^{\circ\circ}}, F_b)
\end{array}$$

in $C_{l^* O_S \text{fil}}(Z)$ which commutes with the action of T_Z .

Proof. (i): By definition of $j_! \text{Hdg} : \pi_{S^\circ}(MHM(S^\circ)) \rightarrow \pi_S(C(MHM(S)))$, we have to construct the isomorphism for each complement of a (Cartier) divisor $j = j_D : S^\circ = S \setminus D \hookrightarrow S$. In this case, we have the closed embedding $i : S \hookrightarrow L$ given by the zero section of the line bundle $L = L_D$ associated to D . We have then, using definition-proposition 15, the canonical morphism in $PSH_{l^* O_S \text{fil}}(Z)$ which commutes with the action of T_Z

$$Q(Z, j_!)(O_S, F_b) : l^* Q_{V_Z, 0} j_! \text{Hdg}(O_{S^\circ}, F_b) \xrightarrow{T_!(l, j)(-)^{-1}} j_! \text{Hdg} Q_{V_{Z^\circ}, 0}(O_{S^\circ}, F_b) = j_1'! \text{Hdg}(O_{Z^\circ}, F_b).$$

and $V_Z^p T_!(l, j)(-)^{-1} = Q_{V_Z, V_S}^{p, 0}(i_{* \text{mod}}(O_S, F_b))$. Now for $j : S^\circ = S \setminus R \hookrightarrow S$ an arbitrary open embedding, we set

$$Q(Z, j_!)(O_S, F_b) := \varprojlim_{(D_i), RC D_i \subset S} (Q(Z, j_{D_i}!)(j_{D_i}^*(O_S, F_b)) : l^* Q_{V_Z, 0} j_! \text{Hdg}(O_{S^\circ}, F_b) \xrightarrow{\sim} j_1'! \text{Hdg}(O_{Z^\circ}, F_b)$$

(ii): Follows from definition-proposition 15.

(iii): Follows from definition-proposition 15. □

Using definition-proposition 19 in the projection case, and the specialization map given in definition 56 and the isomorphism of definition-proposition 20, in the closed embedding case, we have the following canonical map :

Definition 128. Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ whose square are cartesian

$$\begin{array}{ccccc}
Z_T & \xrightarrow{i'} & T & \xleftarrow{j'_l} & T \setminus Z_T \\
\downarrow g' & \searrow l' & \downarrow & \searrow I \times i & \downarrow & \searrow l & \\
& & T \times_g Z & \xrightarrow{I \times i} & T \times_g S & \xleftarrow{I \times j} & T \times S \setminus (T \times Z) \\
& \swarrow pz & \downarrow & \swarrow ps & \downarrow & \swarrow ps & \\
Z & \xrightarrow{i} & S & \xleftarrow{j} & S \setminus Z & &
\end{array}$$

where i and hence $I \times i$ and i' , are closed embeddings, j , $I \times j$, j' are the complementary open embeddings and $g : T \xrightarrow{l} T \times S \xrightarrow{ps} S$ is the graph factorization, where l is the graph embedding and ps the projection. Then, the map in $C_{l^*O_{T \times S} \text{fil}}(T)$

$$\begin{aligned}
sp_{V_T}(\Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b)) &: l^* \Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b) \xrightarrow{q_{V_T, 0}} l^* Q_{V_T, 0}(\Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b)) \\
&\xrightarrow{Q(T, (I \times j)_!)(O_{T \times S}, F_b) := T_!(l, (I \times j))(-)} \Gamma_{Z_T}^{\vee, Hdg}(O_T, F_b)
\end{aligned}$$

which commutes with the action of T_T , where the first map is given in definition 56 and the last map is studied definition-proposition 20, factors through

$$\begin{aligned}
sp_{V_T}(\Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b)) &: l^* \Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b) \xrightarrow{n} l^{*mod} \Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b) \\
&\xrightarrow{\bar{sp}_{V_T}(\Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b))} \Gamma_{Z_T}^{\vee, Hdg}(O_T, F_b),
\end{aligned}$$

with for $U \subset T \times S$ an open subset, $m \in \Gamma(U, O_{T \times S})$ and $h \in \Gamma(U_T, O_T)$, $n(m) := n \otimes 1$ and $\bar{sp}_{V_T}(-)(m \otimes h) = h \cdot sp_{V_T}(m)$; see definition-proposition 19, proposition 99 and theorem 28. Then,

$$\bar{sp}_{V_T}(\Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b)) : l^{*mod} \Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b) \rightarrow \Gamma_{Z_T}^{\vee, Hdg}(O_T, F_b),$$

is a map in $C_{\mathcal{D}(1,0)\text{fil}}(T)$, i.e. is D_T linear. We then consider the canonical map in $C_{\mathcal{D}(1,0)\text{fil}}(T)$

$$\begin{aligned}
a(g, Z)(O_S, F_b) &: g^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) = l^{*mod} p_S^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) \xrightarrow{l^{*mod} \Gamma^{Hdg}(p, \gamma^\vee)(O_S, F_b)^{-1}} \\
&l^{*mod} \Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b) \xrightarrow{\bar{sp}_{V_T}(\Gamma_{T \times Z}^{\vee, Hdg}(O_{T \times S}, F_b))} \Gamma_{Z_T}^{\vee, Hdg}(O_T, F_b).
\end{aligned}$$

Lemma 11. (i) For $g : T \rightarrow S$ and $g' : T' \rightarrow T$ two morphism with $S, T, T' \in \text{SmVar}(\mathbb{C})$, considering the commutative diagram whose squares are cartesian

$$\begin{array}{ccccc}
Z_{T'} & \xrightarrow{i''} & T' & \xleftarrow{j''} & T' \setminus Z_{T'} \\
\downarrow g' & & \downarrow g' & & \downarrow g' \\
Z_T & \xrightarrow{i'} & T & \xleftarrow{j'} & T \setminus Z_T \\
\downarrow g & & \downarrow g & & \downarrow g \\
Z & \xrightarrow{i} & S & \xleftarrow{j} & S \setminus Z
\end{array}$$

we have then

$$\begin{aligned} a(g \circ g', Z)(O_S, F_b) &= a(g', Z_T)(O_T, F_b) \circ (g'^{*mod} a(g, Z)(O_S, F_b)) : \\ (g \circ g')^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) &= g'^{*mod} g^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) \xrightarrow{g'^{*mod} a(g, Z)(O_S, F_b)} g'^{*mod} \Gamma_{Z_T}^{\vee, Hdg}(O_T, F_b) \\ &\xrightarrow{a(g', Z_T)(O_T, F_b)} \Gamma_{Z_{T'}}^{\vee, Hdg}(O_{T'}, F_b). \end{aligned}$$

(ii) For $g : T \rightarrow S$ a morphism with $S, T \in \text{SmVar}(\mathbb{C})$, considering the commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} Z'_T & \xrightarrow{k'} & Z_T & \xrightarrow{i'} & T \\ \downarrow g & & \downarrow g & & \downarrow g \\ Z' & \xrightarrow{k} & Z & \xrightarrow{i} & S \end{array}$$

we have then the following commutative diagram

$$\begin{array}{ccc} g^{*mod} \Gamma_Z^{\vee, Hdg}(O_S, F_b) & \xrightarrow{g^{*mod} T(Z'/Z, \gamma^{\vee, Hdg})(O_S, F_b)} & g^{*mod} \Gamma_{Z'}^{\vee, Hdg}(O_S, F_b) \\ \downarrow a(g, Z)(O_S, F_b) & & \downarrow a(g, Z')(O_S, F_b) \\ \Gamma_{Z_T}^{\vee, Hdg}(O_T, F_b) & \xrightarrow{T(Z'_T/Z_T, \gamma^{\vee, Hdg})(O_T, F_b)} & \Gamma_{Z'_T}^{\vee, Hdg}(O_T, F_b) \end{array}$$

Proof. (i): Follows from definition-proposition 20 (ii)

(ii): Follows from definition-proposition 20 (iii) □

We can now define the main object :

Definition 129. (i) For $S \in \text{SmVar}(\mathbb{C})$, we consider the filtered complexes of presheaves

$$(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}) \in C_{D_S fil}(\text{Var}(\mathbb{C})^{2, smpr} / S)$$

given by,

– for $(Y \times S, Z)/S = ((Y \times S, Z), p) \in \text{Var}(\mathbb{C})^{2, smpr} / S$,

$$(\Omega_{/S}^{\bullet, \Gamma, pr}((Y \times S, Z)/S), F_{DR}) := ((\Omega_{Y \times S/S}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))(Y \times S)$$

with the structure of $p^* D_S$ module given by proposition 64,

– for $g : (Y_1 \times S, Z_1)/S = ((Y_1 \times S, Z_1), p_1) \rightarrow (Y \times S, Z)/S = ((Y \times S, Z), p)$ a morphism in $\text{Var}(\mathbb{C})^{2, smpr} / S$, denoting for short $\hat{Z} := Z \times_{Y \times S} (Y_1 \times S)$,

$$\begin{aligned} \Omega_{/S}^{\bullet, \Gamma, pr}(g) &: ((\Omega_{Y \times S/S}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))(Y \times S) \\ &\xrightarrow{i_-} g^*((\Omega_{Y \times S/S}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))(Y_1 \times S) \\ &\xrightarrow{\Omega_{(Y_1 \times S / Y \times S) / (S/S)}(\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))(Y_1 \times S)} (\Omega_{Y_1 \times S/S}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{Y_1 \times S}} g^{*mod} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)(Y_1 \times S) \\ &\xrightarrow{DR(Y_1 \times S/S)(a(g, Z)(O_{Y \times S}, F_b))(Y_1 \times S)} (\Omega_{Y_1 \times S/S}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{Y_1 \times S}} \Gamma_{\hat{Z}}^{\vee, Hdg}(O_{Y_1 \times S}, F_b)(Y_1 \times S) \\ &\xrightarrow{DR(Y_1 \times S/S)(T(Z_1/\hat{Z}, \gamma^{\vee, Hdg})(O_{Y_1 \times S}, F_b))(Y_1 \times S)} (\Omega_{Y_1 \times S/S}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{Y_1 \times S}} \Gamma_{Z_1}^{\vee, Hdg}(O_{Y_1 \times S}, F_b)(Y_1 \times S), \end{aligned}$$

where

* i_- is the arrow of the inductive limit,

* we recall that

$$\begin{aligned} \Omega_{(Y_1 \times S / Y \times S) / (S/S)}(\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)) &: g^*((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b)) \\ &\rightarrow (\Omega_{Y_1 \times S/S}^\bullet, F_b) \otimes_{O_{Y_1 \times S}} g^{*mod} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b) \end{aligned}$$

is the map given in definition-proposition 16, which is $p_1^* D_S$ linear by proposition 67,

* the map

$$a(g, Z)(O_{Y \times S}, F_b) : g^{*mod} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b) \rightarrow \Gamma_Z^{\vee, Hdg}(O_{Y_1 \times S}, F_b)$$

is the map given in definition 128

* the map

$$T(Z_1 / \hat{Z}, \gamma^{\vee, Hdg})(O_{Y_1 \times S}, F_b) : \Gamma_{\hat{Z}}^{\vee, Hdg}(O_{Y_1 \times S}, F_b) \rightarrow \Gamma_{Z_1}^{\vee, Hdg}(O_{Y_1 \times S}, F_b)$$

is given in definition-proposition 19.

For $g : ((Y_1 \times S, Z_1), p_1) \rightarrow ((Y \times S, Z), p)$ and $g' : ((Y_1' \times S, Z_1'), p_1) \rightarrow ((Y_1 \times S, Z_1), p)$ two morphisms in $\text{Var}(\mathbb{C})^{2, smpr} / S$, we have

$$\begin{aligned} \Omega_{/S}^{\bullet, \Gamma, pr}(g \circ g') &= \Omega_{/S}^{\bullet, \Gamma, pr}(g') \circ \Omega_{/S}^{\bullet, \Gamma, pr}(g) : ((\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))(Y \times S) \\ &\xrightarrow{\Omega_{/S}^{\bullet, \Gamma, pr}(g)} (\Omega_{Y_1 \times S/S}^\bullet, F_b) \otimes_{O_{Y_1 \times S}} \Gamma_{Z_1}^{\vee, Hdg}(O_{Y_1 \times S}, F_b)(Y_1 \times S) \\ &\xrightarrow{\Omega_{/S}^{\bullet, \Gamma, pr}(g')} (\Omega_{Y_1' \times S/S}^\bullet, F_b) \otimes_{O_{Y_1' \times S}} \Gamma_{Z_1'}^{\vee, Hdg}(O_{Y_1' \times S}, F_b)(Y_1' \times S), \end{aligned}$$

since, denoting for short $\hat{Z} := Z \times_{Y \times S} (Y_1 \times S)$ and $\hat{Z}' := Z \times_{Y \times S} (Y_1' \times S)$

– we have by lemma 11(i)

$$a(g \circ g', \hat{Z}')(O_{Y \times S}, F_b) = a(g', \hat{Z})(O_{Y_1 \times S}, F_b) \circ g'^{*mod} a(g, Z)(O_{Y \times S}, F_b),$$

– we have by lemma 11(ii)

$$\begin{aligned} &T(Z_1' / \hat{Z}', \gamma^{\vee, Hdg})(O_{Y_1' \times S}, F_b) \circ a(g', \hat{Z})(O_{Y_1 \times S}, F_b) \\ &= a(g', Z_1)(O_{Y_1 \times S}, F_b) \circ g'^{*mod} T(Z_1 / \hat{Z}, \gamma^{\vee, Hdg})(O_{Y_1 \times S}, F_b). \end{aligned}$$

(ii) For $S \in \text{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_S fil, D_S}(\text{Var}(\mathbb{C})^{sm} / S)$

$$\text{Gr}(\Omega_{/S}) : \text{Gr}_{S^*}^{12}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}) \rightarrow (\Omega_{/S}^\bullet, F_b)$$

given by, for $U/S = (U, h) \in \text{Var}(\mathbb{C})^{sm} / S$

$$\begin{aligned} \text{Gr}(\Omega_{/S})(U/S) &: \text{Gr}_{S^*}^{12}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR})(U/S) := ((\Omega_{U \times S/S}^\bullet, F_b) \otimes_{O_{U \times S}} \Gamma_U^{\vee, Hdg}(O_{U \times S}, F_b))(U \times S) \\ &\xrightarrow{\text{ad}(i_U^*, i_{U^*})(-)(U \times S)} i^*((\Omega_{U \times S/S}^\bullet, F_b) \otimes_{O_{U \times S}} \Gamma_U^{\vee, Hdg}(O_{U \times S}, F_b))(U) \\ &\xrightarrow{\Omega_{(U/U \times S)/(S/S)}(-)(U)} ((\Omega_{U/S}^\bullet, F_b) \otimes_{O_U} i_U^{*mod} \Gamma_U^{\vee, Hdg}(O_{U \times S}, F_b))(U) \\ &\xrightarrow{DR(U/S)(a(i_U, U))(U)} (\Omega_{U/S}^\bullet, F_b)(U) =: (\Omega_{/S}^\bullet, F_b)(U/S) \end{aligned}$$

where $h : U \xrightarrow{i_U} U \times S \xrightarrow{ps} S$ is the graph factorization with i_U the graph embedding and ps the projection, note that $a(i_U, U)$ is an isomorphism since for $j_U : U \times S \setminus U \hookrightarrow U \times S$ the open complementary $i_U^{*mod} j_U^{! Hdg}(M, F, W) = 0$.

Definition 130. For $S \in \text{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$

$$T(\Omega_{/S}^\Gamma) : \mu_{S^*}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})$$

given by, for $(Y \times S, X)/S = ((Y \times S, Z), p) \in \text{Var}(\mathbb{C})^{2, \text{smpr}}/S$

$$\begin{aligned} & T(\Omega_{/S}^\Gamma)((Y \times S, Z)/S) : \\ & (\Omega_{/S}^{\bullet, \Gamma}, F_b)((Y \times S, Z)/S) := \mathbb{D}_{p^*O_S} L_{p^*O} \Gamma_Z E_{zar}(\mathbb{D}_{p^*O_S} L_{p^*O}(\Omega_{Y \times S/S}^{\bullet, \Gamma}, F_b))(Y \times S) \\ & \xrightarrow{DR(Y \times S/S)(\gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}))} \\ & \mathbb{D}_{p^*O_S} L_{p^*O} \Gamma_Z E_{zar}(\mathbb{D}_{p^*O_S} L_{p^*O}((\Omega_{Y \times S/S}^{\bullet, \Gamma}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b))(Y \times S) \xrightarrow{\cong} \\ & ((\Omega_{Y \times S/S}^{\bullet, \Gamma}, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b))(Y \times S) =: (\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})((Y \times S, Z)/S). \end{aligned}$$

By definition $\text{Gr}(\Omega_{/S}) \circ \text{Gr}_{S^*} T(\Omega_{/S}^\Gamma) = \text{Gr}^O(\Omega_{/S})$.

Remark 11. (i) Let $S \in \text{Var}(\mathbb{C})$. We have by definition $o_{12^*}(\Omega_{/S}^{\bullet, \Gamma}, F_b) = (\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$.
Moreover, if $S \in \text{SmVar}(\mathbb{C})$, $o_{12^*}(\Omega_{/S}^{\bullet, \Gamma}, F_b) = (\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$.

(ii) Let $S \in \text{Var}(\mathbb{C})$. Then, $(\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^2/S)$ is a natural extension of

$$(\Omega_{/S}^{\bullet, \Gamma}, F_b) := \rho_{S^*}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \in C_{O_S \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{sm}}/S),$$

but does NOT satisfy cdh descent.

We have the following canonical transformation map given by the pullback of (relative) differential forms:

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. We have the canonical morphism in $C_{g^*O_S \text{fil}, g^*D_S}(\text{Var}(\mathbb{C})^{2, \text{sm}}/T)$

$$\Omega_{/(T/S)}^\Gamma : g^*(\Omega_{/S}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/T}^{\bullet, \Gamma}, F_b)$$

induced by the pullback of differential forms : for $((V, Z_1)/T) = ((V, Z_1), h) \in \text{Var}(\mathbb{C})^{2, \text{sm}}/T$,

$$\begin{aligned} & \Omega_{/(T/S)}^\Gamma((V, Z_1)/T) : \\ & g^*\Omega_{/S}^{\bullet, \Gamma}((V, Z_1)/T) := \lim_{(h:(U, Z) \rightarrow \text{SSm}, g_1:(V, Z_1) \rightarrow (U_T, Z_T), h, g)} \Omega_{/S}^{\bullet, \Gamma}((U, Z)/S) \\ & \xrightarrow{\Omega_{/S}^{\bullet, \Gamma}(g' \circ g_1)} \Omega_{/S}^{\bullet, \Gamma}((V, Z_1)/S) \xrightarrow{\Gamma_{Z_1}^{\vee, h} q(Y_1 \times T)} \Omega_{/T}^{\bullet, \Gamma}((V, Z_1)/T), \end{aligned}$$

where $g' : U_T := U \times_S T \rightarrow U$ is the base change map and $q : \Omega_{Y_1 \times T/S}^{\bullet, \Gamma} \rightarrow \Omega_{Y_1 \times T/T}^{\bullet, \Gamma}$ is the quotient map. If $T, S \in \text{SmVar}(\mathbb{C})$,

$$\Omega_{/(T/S)}^\Gamma : g^*(\Omega_{/S}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/T}^{\bullet, \Gamma}, F_b)$$

is a morphism in $C_{g^*O_S \text{fil}, g^*D_S}(\text{Var}(\mathbb{C})^{2, \text{sm}}/T)$ It induces the canonical morphisms in $C_{g^*O_S \text{fil}, g^*D_S}(\text{Var}(\mathbb{C})^{2, \text{sm}}/T)$:

$$E\Omega_{/(T/S)}^\Gamma : g^* E_{et}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \xrightarrow{T(g, E_{et})(\Omega_{/S}^{\bullet, \Gamma}, F_b)} E_{et}(g^*(\Omega_{/S}^{\bullet, \Gamma}, F_b)) \xrightarrow{E_{et}(\Omega_{/(T/S)}^\Gamma)} E_{et}(\Omega_{/T}^{\bullet, \Gamma}, F_b)$$

and

$$E\Omega_{/(T/S)}^\Gamma : g^* E_{zar}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \xrightarrow{T(g, E_{zar})(\Omega_{/S}^{\bullet, \Gamma}, F_b)} E_{zar}(g^*(\Omega_{/S}^{\bullet, \Gamma}, F_b)) \xrightarrow{E_{zar}(\Omega_{/(T/S)}^\Gamma)} E_{zar}(\Omega_{/T}^{\bullet, \Gamma}, F_b).$$

- Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. We have the canonical morphism in $C_{g^* D_S \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/T)$

$$\Omega_{/(T/S)}^{\Gamma, \text{pr}} : g^*(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \rightarrow (\Omega_{/T}^{\bullet, \Gamma, \text{pr}}, F_{DR})$$

induced by the pullback of differential forms : for $((Y_1 \times T, Z_1)/T) = ((Y_1 \times T, Z_1), p) \in \text{Var}(\mathbb{C})^{2, \text{smpr}}/T$,

$$\begin{aligned} & \Omega_{/(T/S)}^{\Gamma, \text{pr}}((Y_1 \times T, Z_1)/T) : \\ g^* \Omega_{/S}^{\bullet, \Gamma, \text{pr}}((Y_1 \times T, Z_1)/T) & := \lim_{(h: (Y \times S, Z) \rightarrow S, g_1: (Y_1 \times T, Z_1) \rightarrow (Y \times T, Z_T), h, g)} \Omega_{/S}^{\bullet, \Gamma, \text{pr}}((Y \times T, Z)/S) \\ & \xrightarrow{\Omega_{/S}^{\bullet, \Gamma, \text{pr}}(g' \circ g_1)} \Omega_{/S}^{\bullet, \Gamma, \text{pr}}((Y_1 \times T, Z_1)/S) \xrightarrow{q(-)(Y_1 \times T)} \Omega_{/T}^{\bullet, \Gamma, \text{pr}}((Y_1 \times T, Z_1)/T), \end{aligned}$$

where $g' = (I_Y \times g) : Y \times T \rightarrow Y \times S$ is the base change map and $q(M) : \Omega_{Y_1 \times T/S} \otimes_{\mathcal{O}_{Y_1 \times T}}(M, F) \rightarrow \Omega_{Y_1 \times T/T} \otimes_{\mathcal{O}_{Y_1 \times T}}(M, F)$ is the quotient map. It induces the canonical morphisms in $C_{g^* D_S \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/T)$:

$$E\Omega_{/(T/S)}^{\Gamma, \text{pr}} : g^* E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \xrightarrow{T(g, E)(-)} E_{et}(g^*(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \xrightarrow{E_{et}(\Omega_{/(T/S)}^{\Gamma, \text{pr}})} E_{et}(\Omega_{/T}^{\bullet, \Gamma, \text{pr}}, F_{DR})$$

and

$$E\Omega_{/(T/S)}^{\Gamma, \text{pr}} : g^* E_{zar}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \xrightarrow{T(g, E)(-)} E_{zar}(g^*(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \xrightarrow{E_{zar}(\Omega_{/(T/S)}^{\Gamma, \text{pr}})} E_{zar}(\Omega_{/T}^{\bullet, \Gamma, \text{pr}}, F_{DR}).$$

Definition 131. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. We have, for $F \in C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$, the canonical transformation in $C_{\mathcal{D}\text{fil}}(T)$:

$$\begin{aligned} T(g, \Omega_{/S}^{\Gamma, \text{pr}})(F) & : g^{* \text{mod}} L_D e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om^{\bullet}(F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \\ & \xrightarrow{=} (g^* L_D e(S)_* \mathcal{H}om^{\bullet}(F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))) \otimes_{g^* \mathcal{O}_S} \mathcal{O}_T \\ & \xrightarrow{T(g, \text{Gr}^{12})(-) \circ T(e, g)(-) \circ q} e(T)_* \text{Gr}_{T^*}^{12} g^* \mathcal{H}om^{\bullet}(F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{g^* \mathcal{O}_S} \mathcal{O}_T \\ & \xrightarrow{(T(g, \text{hom})(-, -) \otimes I)} e(T)_* \text{Gr}_{T^*}^{12} \mathcal{H}om^{\bullet}(g^* F, g^* E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{g^* \mathcal{O}_S} \mathcal{O}_T \\ & \xrightarrow{ev(\text{hom}, \otimes)(-, -, -)} e(T)_* \text{Gr}_{T^*}^{12} \mathcal{H}om^{\bullet}(g^* F, g^* E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{g^* e(S)^* \mathcal{O}_S} e(T)^* \mathcal{O}_T \\ & \xrightarrow{\mathcal{H}om^{\bullet}(g^* F, (E\Omega_{/(T/S)}^{\Gamma, \text{pr}} \otimes m))} e(T)_* \text{Gr}_{T^*}^{12} \mathcal{H}om^{\bullet}(g^* F, E_{et}(\Omega_{/T}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \end{aligned}$$

Let $S \in \text{Var}(\mathbb{C})$. Recall that for and $h : U \rightarrow S$ a morphism with $U \in \text{Var}(\mathbb{C})$, we have the canonical map given by the wedge product

$$w_{U/S} : \Omega_{U/S}^{\bullet} \otimes_{\mathcal{O}_S} \Omega_{U/S}^{\bullet} \rightarrow \Omega_{U/S}^{\bullet}; \alpha \otimes \beta \mapsto \alpha \wedge \beta.$$

Let $S \in \text{Var}(\mathbb{C})$ and $h_1 : U_1 \rightarrow S, h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2 \in \text{Var}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \rightarrow S$ and $p_{112} : U_1 \times_S U_2 \rightarrow U_1, p_{212} : U_1 \times_S U_2 \rightarrow U_2$ the projections. Recall we have the canonical map given by the wedge product

$$w_{(U_1, U_2)/S} : p_{112}^* \Omega_{U_1/S}^{\bullet} \otimes_{\mathcal{O}_S} p_{212}^* \Omega_{U_2/S}^{\bullet} \rightarrow \Omega_{U_{12}/S}^{\bullet}; \alpha \otimes \beta \mapsto p_{112}^* \alpha \wedge p_{212}^* \beta$$

which gives the map

$$Ew_{(U_1, U_2)/S} : h_{1*} E_{zar}(\Omega_{U_1/S}^{\bullet}) \otimes_{\mathcal{O}_S} h_{2*} E_{zar}(\Omega_{U_2/S}^{\bullet}) \rightarrow h_{12*} E_{zar}(p_{112}^* \Omega_{U_1/S}^{\bullet} \otimes_{\mathcal{O}_S} p_{212}^* \Omega_{U_2/S}^{\bullet})$$

Let $S \in \text{SmVar}(\mathbb{C})$.

- The complex of presheaves $(\Omega_{/S}^{\bullet,\Gamma}, F_b) \in C_{O_S \text{ fil}, D_S}(\text{Var}(\mathbb{C})^{2,sm}/S)$ have a monoidal structure given by the wedge product of differential forms: for $h : (U, Z) \rightarrow S \in \text{Var}(\mathbb{C})^2/S$, the map

$$DR(-)(\gamma_Z^{\vee,h}(-)) \circ w_{U/S} : (\Omega_{U/S}^{\bullet,\Gamma}, F_b) \otimes_{p^*O_S} (\Omega_{U/S}^{\bullet,\Gamma}, F_b) \rightarrow \Gamma_Z^{\vee,h} L_{h^*O_S}(\Omega_{U/S}^{\bullet,\Gamma}, F_b)$$

factors through

$$\begin{aligned} & DR(-)(\gamma_Z^{\vee,Hdg}(-)) \circ w_{U/S} : (\Omega_{U/S}^{\bullet,\Gamma}, F_b) \otimes_{p^*O_S} (\Omega_{U/S}^{\bullet,\Gamma}, F_b) \\ & \xrightarrow{DR(-)(\gamma_Z^{\vee,h}(-)) \otimes DR(-)(\gamma_Z^{\vee,h}(-))} \Gamma_Z^{\vee,h} L_{h^*O_S}(\Omega_{U/S}^{\bullet,\Gamma}, F_b) \otimes_{p^*O_S} \Gamma_Z^{\vee,h} L_{h^*O_S}(\Omega_{U/S}^{\bullet,\Gamma}, F_b) \\ & \xrightarrow{(DR(-)(\gamma_Z^{\vee,h}(-)) \circ w_{U/S})^\gamma} \Gamma_Z^{\vee,h} L_{h^*O_S}(\Omega_{U/S}^{\bullet,\Gamma}, F_b) \end{aligned}$$

unique up to homotopy, giving the map in $C_{O_S \text{ fil}, D_S}(\text{Var}(\mathbb{C})^{2,smpr}/S)$:

$$w_S : (\Omega_{/S}^{\bullet,\Gamma}, F_b) \otimes_{O_S} (\Omega_{/S}^{\bullet,\Gamma}, F_b) \rightarrow (\Omega_{/S}^{\bullet,\Gamma}, F_b)$$

given by for $h : (U, Z) \rightarrow S \in \text{Var}(\mathbb{C})^{2,sm}/S$,

$$\begin{aligned} w_S((U, Z)/S) : & (\Gamma_Z^{\vee,h} L_{h^*O_S}(\Omega_{U/S}^{\bullet,\Gamma}, F_b) \otimes_{p^*O_S} \Gamma_Z^{\vee,h} L_{h^*O_S}(\Omega_{U/S}^{\bullet,\Gamma}, F_b))(U) \\ & \xrightarrow{(DR(-)(\gamma_Z^{\vee,h}(-)) \circ w_{U/S})^\gamma(U)} \Gamma_Z^{\vee,h} L_{h^*O_S}(\Omega_{U/S}^{\bullet,\Gamma}, F_b)(U) \end{aligned}$$

which induces the map in $C_{O_S \text{ fil}, D_S}(\text{Var}(\mathbb{C})^{2,sm}/S)$

$$Ew_S : E_{et}(\Omega_{/S}^{\bullet,\Gamma}, F_b) \otimes_{O_S} E_{et}(\Omega_{/S}^{\bullet,\Gamma}, F_b) \xrightarrow{\cong} E_{et}((\Omega_{/S}^{\bullet,\Gamma}, F_b) \otimes_{O_S} (\Omega_{/S}^{\bullet,\Gamma}, F_b)) \xrightarrow{E_{et}(w_S)} E_{et}(\Omega_{/S}^{\bullet,\Gamma}, F_b)$$

given by the functoriality of the Godement resolution (see section 2).

- The complex of presheaves $(\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR}) \in C_{D_S \text{ fil}}(\text{Var}(\mathbb{C})^{2,smpr}/S)$ have a monoidal structure given by the wedge product of differential forms: for $p : (Y \times S, Z) \rightarrow S \in \text{Var}(\mathbb{C})^{2,smpr}/S$, the map

$$\begin{aligned} DR(-)(\gamma_Z^{\vee,Hdg}(-)) \circ w_{Y \times S/S} : & (\Omega_{Y \times S/S}^{\bullet,\Gamma,pr} \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \otimes_{p^*O_S} (\Omega_{Y \times S/S}^{\bullet,\Gamma,pr} \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \\ & \rightarrow \Omega_{Y \times S/S}^{\bullet,\Gamma,pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee,Hdg}(O_{Y \times S}, F_b) \end{aligned}$$

factors through

$$\begin{aligned} & DR(-)(\gamma_Z^{\vee,Hdg}(-)) \circ w_{Y \times S/S} : \\ & (\Omega_{Y \times S/S}^{\bullet,\Gamma,pr} \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \otimes_{p^*O_S} (\Omega_{Y \times S/S}^{\bullet,\Gamma,pr} \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \\ & \xrightarrow{DR(-)(\gamma_Z^{\vee,Hdg}(-)) \otimes DR(-)(\gamma_Z^{\vee,Hdg}(-))} \\ & (\Omega_{Y \times S/S}^{\bullet,\Gamma,pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee,Hdg}(O_{Y \times S}, F_b)) \otimes_{p^*O_S} \Omega_{Y \times S/S}^{\bullet,\Gamma,pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee,Hdg}(O_{Y \times S}, F_b) \\ & \xrightarrow{(DR(-)(\gamma_Z^{\vee,Hdg}(-)) \circ w_{Y \times S/S})^\gamma} \Omega_{Y \times S/S}^{\bullet,\Gamma,pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee,Hdg}(O_{Y \times S}, F_b) \end{aligned}$$

unique up to homotopy, giving the map in $C_{D_S \text{ fil}}(\text{Var}(\mathbb{C})^{2,smpr}/S)$:

$$w_S : (\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR}) \otimes_{O_S} (\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR}) \rightarrow (\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR})$$

given by for $p : (Y \times S, Z) \rightarrow S \in \text{Var}(\mathbb{C})^{2,smpr}/S$,

$$\begin{aligned} & w_S((Y \times S, Z)/S) : \\ & (((\Omega_{Y \times S/S}^{\bullet,\Gamma,pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee,Hdg}(O_{Y \times S}, F_b)) \otimes_{p^*O_S} (\Omega_{Y \times S/S}^{\bullet,\Gamma,pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee,Hdg}(O_{Y \times S}, F_b)))(Y \times S) \\ & \xrightarrow{(DR(-)(\gamma_Z^{\vee,Hdg}(-)) \circ w_{Y \times S/S})^\gamma(Y \times S)} (\Omega_{Y \times S/S}^{\bullet,\Gamma,pr} \otimes_{O_{Y \times S}} \Gamma_Z^{\vee,Hdg}(O_{Y \times S}, F_b))(Y \times S) \end{aligned}$$

which induces the map in $C_{D_S \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$

$$\begin{aligned} Ew_S : E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{O_S} E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) &\xrightarrow{\cong} \\ E_{et}((\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{O_S} (\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) &\xrightarrow{E_{et}(w_S)} E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \end{aligned}$$

by the functoriality of the Godement resolution (see section 2).

Definition 132. Let $S \in \text{SmVar}(\mathbb{C})$. We have, for $F, G \in C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$, the canonical transformation in $C_{\mathcal{D}\text{fil}}(S)$:

$$\begin{aligned} &T(\otimes, \Omega)(F, G) : \\ e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{O_S} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \\ &\xrightarrow{\cong} e(S)_* \text{Gr}_{S^*}^{12} (\mathcal{H}om(F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{O_S} \mathcal{H}om(G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))) \\ &\xrightarrow{T(\mathcal{H}om, \otimes)(-)} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{O_S} E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \\ &\xrightarrow{\cong} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(F \otimes G, (E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{O_S} E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))) \\ &\xrightarrow{\mathcal{H}om(F \otimes G, Ew_S)} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})). \end{aligned}$$

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and for $J \subset I$ the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

where $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ is the projection and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. This gives the diagram of algebraic varieties $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Var}(\mathbb{C}))$ which the diagram of sites $\text{Var}(\mathbb{C})^{2, \text{smpr}}/(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. This gives also the diagram of algebraic varieties $(\tilde{S}_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N})^{op}, \text{Var}(\mathbb{C}))$ which the diagram of sites $\text{Var}(\mathbb{C})^{2, \text{smpr}}/(\tilde{S}_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N})^{op}, \text{Cat})$. We then get

$$((\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, \text{pr}}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}) \in C_{D_{(\tilde{S}_I)} \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/(\tilde{S}_I))$$

with

$$\begin{aligned} T_{IJ} : (\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, \text{pr}}, F_{DR})[-d_{\tilde{S}_I}] &\xrightarrow{\text{ad}(p_{IJ}^{* \text{mod}[-]}, p_{IJ*}(-))} p_{IJ*} p_{IJ}^* (\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{p_{IJ}^* O_{\tilde{S}_I}} O_{\tilde{S}_J}[-d_{\tilde{S}_J}] \\ &\xrightarrow{m \circ p_{IJ*} \Omega_{/(\tilde{S}_J/\tilde{S}_I)}^{\Gamma, \text{pr}}[-d_{\tilde{S}_J}]} p_{IJ*} (\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, \text{pr}}, F_{DR})[-d_{\tilde{S}_J}]. \end{aligned}$$

For $(G_I, K_{IJ}) \in C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/(\tilde{S}_I)^{op})$, we denote (see section 2)

$$\begin{aligned} &e'((\tilde{S}_I)_* \mathcal{H}om((G_I, K_{IJ}), (E_{zar}(\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, \text{pr}}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) := \\ &(e'(\tilde{S}_I)_* \mathcal{H}om(G_I, E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, \text{pr}}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}((G_I, K_{IJ})))) \in C_{\mathcal{D}\text{fil}}((\tilde{S}_I)) \end{aligned}$$

with

$$\begin{aligned}
& u_{IJ}((G_I, K_{IJ})) : e'(\tilde{S}_I)_* \mathcal{H}om(G_I, E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}] \\
& \xrightarrow{\text{ad}(p_{IJ}^{*mod[-1]}, p_{IJ*})(-) \circ T(p_{IJ}, e)(-)} p_{IJ*} e'(\tilde{S}_J)_* p_{IJ}^* \mathcal{H}om(G_I, E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \otimes_{p_{IJ}^* \mathcal{O}_{\tilde{S}_I}} \mathcal{O}_{\tilde{S}_J}[-d_{\tilde{S}_J}] \\
& \xrightarrow{T(p_{IJ}, \text{hom})(-, -)} p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om(p_{IJ}^* G_I, p_{IJ}^* E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \otimes_{p_{IJ}^* \mathcal{O}_{\tilde{S}_I}} \mathcal{O}_{\tilde{S}_J}[-d_{\tilde{S}_J}] \\
& \xrightarrow{m \circ \mathcal{H}om(p_{IJ}^* G_I, T_{IJ})} p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om(p_{IJ}^* G_I, E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_J}] \\
& \xrightarrow{\mathcal{H}om(K_{IJ}, E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om(G_J, E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_J}].
\end{aligned}$$

This gives in particular

$$(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ} \in C_{D(\tilde{S}_I), \text{fil}}(\text{Var}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I)^{op}).$$

We now define the filtered De Rahm realization functor.

Definition 133. (i) Let $S \in \text{SmVar}(\mathbb{C})$. We have, using definition 129 and definition 41, the functor

$$\begin{aligned}
& \mathcal{F}_S^{FDR} : C(\text{Var}(\mathbb{C})^{sm} / S) \rightarrow C_{D\text{fil}}(S), F \mapsto \\
& \mathcal{F}_S^{FDR}(F) := e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om^{\bullet}(\hat{R}^{CH}(\rho_S^* L(F)), E_{zar}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S]
\end{aligned}$$

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Consider, for $I \subset J$, the following commutative diagram

$$\begin{array}{ccc}
D_{IJ} = & S_I & \xrightarrow{i_I} & \tilde{S}_I \\
& \uparrow j_{IJ} & & \uparrow p_{IJ} \\
& S_J & \xrightarrow{i_J} & \tilde{S}_J
\end{array}$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. We have, using definition 129 and definition 41, the functor

$$\begin{aligned}
& \mathcal{F}_S^{FDR} : C(\text{Var}(\mathbb{C})^{sm} / S) \rightarrow C_{D\text{fil}}(S / (\tilde{S}_I)), F \mapsto \\
& \mathcal{F}_S^{FDR}(F) := e'((\tilde{S}_I)_* \mathcal{H}om^{\bullet}(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), \hat{R}_{\tilde{S}_I}^{CH}(T^q(D_{IJ})(j_I^* F))), \\
& \quad (E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})) \\
& := (e'(\tilde{S}_I)_* \mathcal{H}om^{\bullet}(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F))
\end{aligned}$$

where we have denoted for short $e'(\tilde{S}_I) = e(\tilde{S}_I) \circ \text{Gr}_{\tilde{S}_I}^{1,2}$, and

$$\begin{aligned}
& u_{IJ}^q(F)[d_{\tilde{S}_J}] : e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{\text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} p_{IJ*} p_{IJ}^{*mod} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{p_{IJ*} T(p_{IJ}, \Omega^{\gamma, pr})(-)} p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(p_{IJ}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{\mathcal{H}om(T(p_{IJ}, \hat{R}^{CH})(Li_{I*} j_I^* F)^{-1}, E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_J}^* p_{IJ}^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{\mathcal{H}om(\hat{R}_{\tilde{S}_J}^{CH}(T^q(D_{IJ})(j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_J}^* L(i_{J*} j_J^* F)), E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})).
\end{aligned}$$

For $I \subset J \subset K$, we have obviously $p_{IJ*} u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$.

Recall, see section 2, that we have the projection morphisms of sites $p_a : \text{Var}(\mathbb{C})^{2, smpr} / (\tilde{S}_I)^{op} \rightarrow \text{Var}(\mathbb{C})^{2, smpr} / (\tilde{S}_I)^{op}$ given by the functor

$$\begin{aligned}
& p_a : \text{Var}(\mathbb{C})^{2, smpr} / (\tilde{S}_I)^{op} \rightarrow \text{Var}(\mathbb{C})^{2, smpr} / (\tilde{S}_I)^{op}, \\
& p_a((Y_I \times \tilde{S}_I, Z_I) / \tilde{S}_I, s_{IJ}) := ((Y_I \times \mathbb{A}^1 \times \tilde{S}_I, Z_I \times \mathbb{A}^1) / \tilde{S}_I, s_{IJ} \times I), \\
& p_a((g_I) : ((Y'_I \times \tilde{S}_I, Z'_I) / \tilde{S}_I, s'_{IJ}) \rightarrow ((Y_I \times \tilde{S}_I, Z_I) / \tilde{S}_I, s_{IJ})) = \\
& (g_I \times I) : ((Y'_I \times \mathbb{A}^1 \times \tilde{S}_I, Z'_I \times \mathbb{A}^1) / \tilde{S}_I, s'_{IJ} \times I) \rightarrow ((Y_I \times \mathbb{A}^1 \times \tilde{S}_I, Z_I \times \mathbb{A}^1) / \tilde{S}_I, s_{IJ} \times I).
\end{aligned}$$

We have the following key proposition :

Proposition 113. (i) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. The complex of presheaves $(\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, pr}, F_{DR}) \in C_{D(\tilde{S}_I)} \text{fil}(\text{Var}(\mathbb{C})^{2, smpr} / (\tilde{S}_I)^{op})$ is 2-filtered \mathbb{A}^1 homotopic, that is

$$\text{ad}(p_a^*, p_{a*})(\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, pr}, F_{DR}) : (\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}) \rightarrow p_{a*} p_a^*(\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, pr}, F_{DR})$$

is a 2-filtered homotopy.

(i2) Let $S \in \text{SmVar}(\mathbb{C})$. The complex of presheaves $(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}) \in C_{D_S} \text{fil}(\text{Var}(\mathbb{C})^{2, smpr} / S)$ admits transfers, i.e.

$$\text{Tr}(S)_* \text{Tr}(S)^*(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}) = (\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}).$$

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $m = (m_I) : (Q_{1I}, K_{1I}^1) \rightarrow (Q_{2I}, K_{1I}^2)$ be an equivalence (\mathbb{A}^1 , et) local with $(Q_{1I}, K_{1I}) \rightarrow (Q_{2I}, K_{1I}) \in C(\text{Var}(\mathbb{C})^{smpr} / (\tilde{S}_I)^{op})$ complexes of representable presheaves. Then, the map in $C_{\mathcal{D} \text{fil}}((\tilde{S}_I))$

$$\begin{aligned}
& M := (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(m_I, E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}])) : \\
& e'((\tilde{S}_I)_* \mathcal{H}om^\bullet((Q_{2I}, K_{1I}^1), (E_{zar}(\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) \\
& \rightarrow e'((\tilde{S}_I)_* \mathcal{H}om^\bullet((Q_{1I}, K_{1I}^1), (E_{zar}(\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})))
\end{aligned}$$

is a 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D} \text{fil}, \infty}((\tilde{S}_I))$.

Proof. (i1): Let $(Y \times S, Z)/S = ((Y \times S, Z), p) \in \text{Var}(\mathbb{C})^{2, \text{smpr}}/S$ so that $p_a : (Y \times \mathbb{A}^1 \times S, Z \times \mathbb{A}^1) \rightarrow (Y \times S, Z)$ is the projection, and $i_0 : (Y \times S, Z) \rightarrow (Y \times \mathbb{A}^1 \times S, Z \times \mathbb{A}^1)$ the closed embedding. Then,

$$a(p_a, Z) : p_a^{* \text{mod}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b) \rightarrow \Gamma_{Z \times \mathbb{A}^1}^{\vee, \text{Hdg}}(O_{Y \times \mathbb{A}^1 \times S}, F_b).$$

a quasi-isomorphism in $\pi_{Y \times \mathbb{A}^1 \times S}(C(\text{MHM}(Y \times \mathbb{A}^1 \times S)))$. Since a morphism of mixed Hodge module is strict for the F-filtration,

$$a(p_a, Z) : p_a^{* \text{mod}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b) \rightarrow \Gamma_{Z \times \mathbb{A}^1}^{\vee, \text{Hdg}}(O_{Y \times \mathbb{A}^1 \times S}, F_b).$$

is a filtered quasi-isomorphism in $C_{\mathcal{D} \text{fil}}(Y \times \mathbb{A}^1 \times S)$. Hence, as

$$\begin{aligned} I(p_a^*, p_{a^*})(-, -)(\Omega_{(Y \times \mathbb{A}^1 \times S/Y \times S)(S/S)}(\Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b))) : \\ (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b) \\ \rightarrow p_{a^*}((\Omega_{Y \times \mathbb{A}^1 \times S/S}^\bullet, F_b) \otimes_{O_{Y \times \mathbb{A}^1 \times S}} p_a^{* \text{mod}} \Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b)) \end{aligned}$$

is a 2-filtered homotopy equivalence whose inverse is

$$\begin{aligned} p_{a^*} I(i_0^*, i_{0^*})(-, -)(\Omega_{(Y \times S/Y \times \mathbb{A}^1 \times S)(S/S)}(p_a^{* \text{mod}}(\Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b)))) : \\ p_{a^*}((\Omega_{Y \times \mathbb{A}^1 \times S/S}^\bullet, F_b) \otimes_{O_{Y \times \mathbb{A}^1 \times S}} p_a^{* \text{mod}}(\Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b))) \\ \rightarrow p_{a^*} i_{0^*}(\Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} (i_0^{* \text{mod}} p_a^{* \text{mod}}(\Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b))) \\ \xrightarrow{\cong} (\Omega_{Y \times S/S}^\bullet, F_b) \otimes_{O_{Y \times S}} (\Gamma_Z^{\vee, \text{Hdg}}(O_{Y \times S}, F_b)) \end{aligned}$$

(see the proof of proposition 105), the map

$$\text{ad}(p_a^*, p_{a^*})(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) : (\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \rightarrow p_{a^*} p_a^*(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})$$

is an homotopy equivalence whose inverse is

$$\text{ad}(i_0^*, i_{0^*})(p_{a^*} p_a^*(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})) : p_{a^*} p_a^*(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \rightarrow (\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}).$$

(ii2): Let us show that $\Omega_{/S}^{\bullet, \Gamma, \text{pr}} \in C_{\mathcal{D} \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$ admits transferts. Let $\alpha \in \text{Cor}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)((Y_1 \times S, Z_1)/S, (Y_2 \times S, Z_2)/S)$ irreducible. Denote by $i : \alpha \hookrightarrow Y_1 \times Y_2 \times S$ the closed embedding, and $p_1 : Y_1 \times Y_2 \times S \rightarrow Y_1 \times S$, $p_2 : Y_1 \times Y_2 \times S \rightarrow Y_2 \times S$ the projections. The morphism $p_1 \circ i : \alpha \rightarrow Y_1 \times S$ is then finite surjective and $(Z_1 \times Y_2) \cap \alpha \subset Y_1 \times Z_2$ (i.e. $p_2(p_1^{-1}(Z_2) \cap \alpha) \subset Z_2$). Then, the transfert map is given by

$$\begin{aligned} \Omega_{/S}^{\bullet, \Gamma, \text{pr}}(\alpha) : ((\Omega_{Y_2 \times S/S}^\bullet, F_b) \otimes_{O_{Y_2 \times S}} \Gamma_{Z_2}^{\vee, \text{Hdg}}(O_{Y_2 \times S}, F_b))(Y_2 \times S) \\ \xrightarrow{i_-} p_2^*((\Omega_{Y_2 \times S/S}^\bullet, F_b) \otimes_{O_{Y_2 \times S}} \Gamma_{Z_2}^{\vee, \text{Hdg}}(O_{Y_2 \times S}, F_b))(Y_1 \times Y_2 \times S) \\ \xrightarrow{\Omega_{(Y_1 \times Y_2 \times S/Y_2 \times S)/(S/S)}(-)(-)} ((\Omega_{Y_1 \times Y_2 \times S/S}^\bullet, F_b) \otimes_{O_{Y_1 \times Y_2 \times S}} \Gamma_{Y_1 \times Z_2}^{\vee, \text{Hdg}}(O_{Y_1 \times Y_2 \times S}, F_b))(Y_1 \times Y_2 \times S) \\ \xrightarrow{\text{DR}(-)(T((Z_1 \times Y_2) \cap \alpha/Y_1 \times Z_2, \gamma^{\vee, \text{Hdg}})(-)(-))} \\ ((\Omega_{Y_1 \times Y_2 \times S/S}^\bullet, F_b) \otimes_{O_{Y_1 \times Y_2 \times S}} \Gamma_{(Z_1 \times Y_2) \cap \alpha}^{\vee, \text{Hdg}}(O_{Y_1 \times Y_2 \times S}, F_b))(Y_1 \times Y_2 \times S) \\ \xrightarrow{i_-} i^*((\Omega_{Y_1 \times Y_2 \times S/S}^\bullet, F_b) \otimes_{O_{Y_1 \times Y_2 \times S}} \Gamma_{(Z_1 \times Y_2) \cap \alpha}^{\vee, \text{Hdg}}(O_{Y_1 \times Y_2 \times S}, F_b))(\alpha) \\ \xrightarrow{\Omega_{(\alpha/Y_1 \times Y_2 \times S)/(S/S)}(-)(-)} ((\Omega_{\alpha/S}^\bullet, F_b) \otimes_{O_\alpha} i^{* \text{mod}} \Gamma_{(Z_1 \times Y_2) \cap \alpha}^{\vee, \text{Hdg}}(O_{Y_1 \times Y_2 \times S}, F_b))(\alpha) \\ \xrightarrow{\Omega_{(\alpha/Y_1 \times S)/(S/S)}(-)(-)^{\text{tr}}} ((\Omega_{Y_1 \times S/S}^\bullet, F_b) \otimes_{O_{Y_1 \times S}} \Gamma_{Z_1}^{\vee, \text{Hdg}}(O_{Y_1 \times S}, F_b))(Y_1 \times S). \end{aligned}$$

(ii): Two morphisms of complexes of representable presheaves in $C(\text{Var}(\mathbb{C})^{smpr}/(\tilde{S}_I)^{op})$

$$m = (m_\alpha^*), m' = (m'_\alpha) : \bigoplus_{\alpha \in \Lambda_*} (\mathbb{Z}((Y_\alpha^{1,*} \times \tilde{S}_I, Z_\alpha^{1,*})/\tilde{S}_I), s_{IJ}^{1,*}) \rightarrow \bigoplus_{\alpha \in \Lambda_*} (\mathbb{Z}((Y_\alpha^{2,*} \times \tilde{S}_I, Z_\alpha^{2,*})/\tilde{S}_I), s_{IJ}^{2,*})$$

are said to induce a twisted homotopy if there exist morphisms in $C_{D(\tilde{S}_I)fil}(\mathbb{N} \times \mathbb{Z})$

$$\begin{aligned} h = (h_\alpha^{p,*}) : & (\Omega_{Y_\alpha^{2,*} \times \tilde{S}_I / \tilde{S}_I}^p \otimes_{O_{Y_\alpha^{2,*} \times \tilde{S}_I}} \Gamma_{Z_\alpha^{2,*}}^{\vee, Hdg}(O_{Y_\alpha^{2,*} \times \tilde{S}_I}, F_b)(Y_\alpha^{2,*} \times \tilde{S}_I), u_{IJ}(-)^p) \\ & \rightarrow (\Omega_{Y_\alpha^{1,*} \times \tilde{S}_I / \tilde{S}_I}^{p-1} \otimes_{O_{Y_\alpha^{1,*} \times \tilde{S}_I}} \Gamma_{Z_\alpha^{1,*}}^{\vee, Hdg}(O_{Y_\alpha^{1,*} \times \tilde{S}_I}, F_b)(Y_\alpha^{1,*} \times \tilde{S}_I), u_{IJ}(-)^{p-1}) \end{aligned}$$

where $p \in \mathbb{N}$, such that

$$\Omega^{\Gamma, pr}(m) - \Omega^{\Gamma, pr}(m') = d_1 \circ h + h \circ d_2.$$

As for homotopy, twisted homotopy satisfy the 2 of 3 property for morphism of canonical triangles. Moreover, by definition, if two morphisms of complexes of representable presheaves in $C(\text{Var}(\mathbb{C})^{smpr}/(\tilde{S}_I)^{op})$ $m, m' : Q_{1I}, K_{IJ}^1 \rightarrow (Q_{2I}, K_{IJ}^2)$ induce a twisted homotopy, then

$$\begin{aligned} & e((\tilde{S}_I)_* \mathcal{H}om^\bullet((h, m, m'), (E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) : \\ & e'((\tilde{S}_I)_* \mathcal{H}om^\bullet((Q_{2I}, K_{IJ}^1), (E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) [1] \\ & \rightarrow e'((\tilde{S}_I)_* \mathcal{H}om^\bullet((Q_{1I}, K_{IJ}^1), (E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) \end{aligned}$$

is a 2-filtered homotopy. Now, by proposition 27, there exists

$$\begin{aligned} & \left\{ ((Y_{1I, \alpha} \times \tilde{S}_I, Z_{1I, \alpha})/\tilde{S}_I, s_{IJ}^{1, \alpha}), \alpha \in \Lambda_1 \right\}, \dots, \left\{ ((Y_{sI, \alpha} \times \tilde{S}_I, Z_{sI, \alpha})/\tilde{S}_I, s_{IJ}^{s, \alpha}), \alpha \in \Lambda_s \right\} \\ & \subset \text{Var}(\mathbb{C})^{2, (sm)pr}/(\tilde{S}_I)^{op} \end{aligned}$$

such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{2, (sm)pr}/(\tilde{S}_I)^{op}))$

$$\begin{aligned} & \text{Cone}(m) \xrightarrow{\sim} \\ & \text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}((Y_{1I, \alpha} \times \mathbb{A}^1 \times \tilde{S}_I, Z_{1I, \alpha} \times \mathbb{A}^1)/\tilde{S}_I), s_{IJ}^{1, \alpha} \times I) \rightarrow \mathbb{Z}((Y_{1I, \alpha} \times S, Z_{1I, \alpha})/\tilde{S}_I), s_{IJ}^{1, \alpha})) \\ & \rightarrow \dots \rightarrow \bigoplus_{\alpha \in \Lambda_s} \text{Cone}(\mathbb{Z}((Y_{sI, \alpha} \times \mathbb{A}^1 \times \tilde{S}_I, Z_{sI, \alpha} \times \mathbb{A}^1)/\tilde{S}_I), s_{IJ}^{s, \alpha} \times I) \rightarrow \mathbb{Z}((Y_{sI, \alpha} \times S, Z_{sI, \alpha})/\tilde{S}_I), s_{IJ}^{s, \alpha})) \end{aligned}$$

This gives in $D_{fil}((\tilde{S}_I)) := \text{Ho}_{zar, fil}((\tilde{S}_I))$,

$$\begin{aligned} & \text{Cone}(M) \xrightarrow{\sim} \\ & \text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \text{Cone}(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathbb{Z}((Y_{1I, \alpha} \times S, Z_{1I, \alpha})/\tilde{S}_I), s_{IJ}^1), (E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) \rightarrow \\ & e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathbb{Z}((Y_{1I, \alpha} \times \mathbb{A}^1 \times \tilde{S}_I, Z_{1I, \alpha} \times \mathbb{A}^1)/\tilde{S}_I), s_{IJ}^1 \times I), (E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})) \\ & \rightarrow \dots \rightarrow \bigoplus_{\alpha \in \Lambda_s} \text{Cone}(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathbb{Z}((Y_{sI, \alpha} \times \tilde{S}_I, Z_{sI, \alpha})/\tilde{S}_I), s_{IJ}^s), (E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) \rightarrow \\ & e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathbb{Z}((Y_{sI, \alpha} \times \mathbb{A}^1 \times \tilde{S}_I, Z_{sI, \alpha} \times \mathbb{A}^1)/\tilde{S}_I), s_{IJ}^s \times I), (E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})) \end{aligned}$$

Then by (i1), (i2) and theorem 12, for all $1 \leq i \leq s$ and all $\alpha \in \Lambda_i$

$$\begin{aligned} & \text{Cone}(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathbb{Z}((Y_{iI, \alpha} \times \tilde{S}_I, Z_{iI, \alpha})/\tilde{S}_I), s_{IJ}^i), (E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) \rightarrow \\ & e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathbb{Z}((Y_{iI, \alpha} \times \mathbb{A}^1 \times \tilde{S}_I, Z_{iI, \alpha} \times \mathbb{A}^1)/\tilde{S}_I), s_{IJ}^i \times I), (E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) \rightarrow 0 \end{aligned}$$

are twisted homotopy equivalence. Hence, we have in $C_{Dfil}((\tilde{S}_I))$

$$\begin{aligned} M = m_1 \circ \dots \circ m_l : & e'((\tilde{S}_I)_* \mathcal{H}om^\bullet((Q_{2I}, K_{IJ}^1), (E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) \\ & \rightarrow e'((\tilde{S}_I)_* \mathcal{H}om^\bullet((Q_{1I}, K_{IJ}^1), (E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) \end{aligned}$$

with m_i either a filtered quasi-isomorphism or a twisted homotopy equivalence. This proves (ii). \square

We deduce the following :

Proposition 114. *Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.*

- (i) *Let $m = (m_I) : (Q_{1I}, K_{IJ}^1) \rightarrow (Q_{2I}, K_{IJ}^2)$ be an etale local equivalence local with $(Q_{1I}, K_{IJ}^1), (Q_{2I}, K_{IJ}^2) \in C(\text{Var}(\mathbb{C})^{sm}/(\tilde{S}_I))$ complexes of projective presheaves. Then,*

$$\begin{aligned} & (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}_S^{CH}(m_I), E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}]) : \\ & e'(\tilde{S}_I)_* \mathcal{H}om^\bullet((\hat{R}^{CH}(\rho_S^* Q_{1I}), \hat{R}^{CH}(K_{IJ}^1)), (E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})) \\ & \rightarrow e'(\tilde{S}_I)_* \mathcal{H}om^\bullet((\hat{R}^{CH}(\rho_S^* Q_{2I}), \hat{R}^{CH}(K_{IJ}^2)), (E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})) \end{aligned}$$

is a filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil}((\tilde{S}_I))$.

- (ii) *Let $m = (m_I) : (Q_{1I}, K_{IJ}^1) \rightarrow (Q_{2I}, K_{IJ}^2)$ be an equivalence (\mathbb{A}^1, et) local equivalence local with $(Q_{1I}, K_{IJ}^1), (Q_{2I}, K_{IJ}^2) \in C(\text{Var}(\mathbb{C})^{sm}/(\tilde{S}_I))$ complexes of projective presheaves. Then,*

$$\begin{aligned} & (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}_S^{CH}(m_I), E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}]) : \\ & e'(\tilde{S}_I)_* \mathcal{H}om^\bullet((\hat{R}^{CH}(\rho_S^* Q_{1I}), \hat{R}^{CH}(K_{IJ}^1)), (E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})) \\ & \rightarrow e'(\tilde{S}_I)_* \mathcal{H}om^\bullet((\hat{R}^{CH}(\rho_S^* Q_{2I}), \hat{R}^{CH}(K_{IJ}^2)), (E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})) \end{aligned}$$

is a filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil}((\tilde{S}_I))$.

Proof. Follows from proposition 113 and the fact that the differentials of the complexes are stricts for the filtration by theorem 29. \square

Definition 134. (i) *Let $S \in \text{SmVar}(\mathbb{C})$. We define using definition 133(i) and proposition 114(ii) the filtered algebraic De Rahm realization functor defined as*

$$\begin{aligned} & \mathcal{F}_S^{FDR} : \text{DA}_c(S) \rightarrow D_{\mathcal{D}fil}(S), M \mapsto \\ & \mathcal{F}_S^{FDR}(M) := e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_S^* L(F)), E_{zar}(\Omega_{\tilde{S}}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

- (i)' *For the Corti-Hanamura weight structure W on $\text{DA}_c(S)^-$, we define using definition 133(i) and proposition 114(ii)*

$$\begin{aligned} & \mathcal{F}_S^{FDR} : \text{DA}_c^-(S) \rightarrow D_{\mathcal{D}(1,0)fil, \infty}^-(S), M \mapsto \\ & \mathcal{F}_S^{FDR}((M, W)) := e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_S^* L(F, W)), E_{zar}(\Omega_{\tilde{S}}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \end{aligned}$$

where $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)((F, W))$ using corollary 1. Note that the filtration induced by W is a filtration by sub D_S module, which is a stronger property then Griffithz transversality. Of course, the filtration induced by F satisfy only Griffithz transversality in general.

- (ii) *Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define, using definition 133(ii) and proposition 114(ii), the filtered algebraic De Rahm realization functor defined as*

$$\begin{aligned} & \mathcal{F}_S^{FDR} : \text{DA}_c(S) \rightarrow D_{\mathcal{D}fil}(S/(\tilde{S}_I)), M \mapsto \\ & \mathcal{F}_S^{FDR}(M) := (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 133.

(ii)' For the Corti-Hanamura weight structure W on $\mathrm{DA}_c^-(S)$, using definition 133(ii) and proposition 114(ii),

$$\mathcal{F}_S^{FDR} : \mathrm{DA}_c(S) \rightarrow D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)), M \mapsto \mathcal{F}_S^{FDR}((M, W)) := (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F, W))$$

where $(F, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$ using corollary 1. Note that the filtration induced by W is a filtration by sub $D_{\tilde{S}_I}$ -modules, which is a stronger property than Griffithz transversality. Of course, the filtration induced by F satisfy only Griffithz transversality in general.

Proposition 115. For $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$, the functor \mathcal{F}_S^{FDR} is well defined.

Proof. Let $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Let $M \in \mathrm{DA}(S)$. Let $F, F' \in C(\mathrm{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F) = D(\mathbb{A}^1, et)(F')$. Then there exist by definition a sequence of morphisms in $C(\mathrm{Var}(\mathbb{C})^{sm}/S)$:

$$F = F_1 \xrightarrow{s_1} F_2 \xleftarrow{s_2} F_3 \xrightarrow{s_3} F_4 \rightarrow \dots \xrightarrow{s_l} F' = F_s$$

where, for $1 \leq k \leq s$, and s_k are (\mathbb{A}^1, et) local equivalence. But if $s : F_1 \rightarrow F_2$ is an equivalence (\mathbb{A}^1, et) local,

$$L(i_{I*} j_I^* s) : L(i_{I*} j_I^* F_1) \rightarrow L(i_{I*} j_I^* F_2)$$

is an equivalence (\mathbb{A}^1, et) local, hence

$$\begin{aligned} & \mathcal{H}om(\hat{R}_{\tilde{S}_I}^{CH}(L(i_{I*} j_I^* s)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) : \\ & (e'(\tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F_1)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})), u_{IJ}^q(F_1)) \\ & \rightarrow (e'(\tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F_2)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})), u_{IJ}^q(F_2)) \end{aligned}$$

is a filtered quasi-isomorphism by proposition 114. \square

Let $f : X \rightarrow S$ a morphism with $S, X \in \mathrm{Var}(\mathbb{C})$. Assume there exists a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f , with l a closed embedding, $Y \in \mathrm{SmVar}(\mathbb{C})$ and p_S the projection. Let $\bar{Y} \in \mathrm{PSmVar}(\mathbb{C})$ a smooth compactification of Y with $\bar{Y} \setminus Y = D$ a normal crossing divisor, denote $k : D \hookrightarrow \bar{Y}$ the closed embedding and $n : Y \hookrightarrow \bar{Y}$ the open embedding. Denote $\bar{X} \subset \bar{Y} \times S$ the closure of $X \subset \bar{Y} \times S$. We have then the following commutative diagram in $\mathrm{Var}(\mathbb{C})$

$$\begin{array}{ccc} X & \xrightarrow{l} & Y \times S \\ \downarrow & & \downarrow \begin{array}{l} (n \times I) \\ p_S \end{array} \\ \bar{X} & \xrightarrow{l} & \bar{Y} \times S \xrightarrow{\bar{p}_S} S \\ \uparrow & \nearrow l_Z & \uparrow \begin{array}{l} (k \times I) \end{array} \\ Z := \bar{X} \setminus X & \longrightarrow & D \times S \end{array} .$$

Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. For

$I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \Pi_{i \in I} \tilde{S}_i$. We then have, for $I \subset [1, \dots, l]$, closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and for $I \subset J$, the following commutative diagrams which are cartesian

$$f_I = f_{|X_I} : X_I \xrightarrow{l_I} Y \times S_I \xrightarrow{p_{S_I}} S_I, \quad Y \times \tilde{S}_J \xrightarrow{p_{\tilde{S}_J}} \tilde{S}_J, \\ \begin{array}{ccc} & \downarrow i'_I & \downarrow i_I \\ & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I \end{array}, \quad \begin{array}{ccc} & \downarrow p'_{I,J} & \downarrow p_{I,J} \\ & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I \end{array}$$

with $l_I : l_{|X_I}$, $i'_I = I \times i_I$, p_{S_I} and $p_{\tilde{S}_I}$ are the projections and $p'_{I,J} = I \times p_{I,J}$, and we recall that we denote by $j_I : \tilde{S}_I \setminus S_I \hookrightarrow \tilde{S}_I$ and $j'_I : Y \times \tilde{S}_I \setminus X_I \hookrightarrow Y \times S_I$ the open complementary embeddings. We then have the commutative diagrams

$$D_{I,J} = \begin{array}{ccc} S_J & \xrightarrow{i_J} & \tilde{S}_J \\ \downarrow j_{I,J} & & \downarrow p_{I,J} \\ S_I & \xrightarrow{i_I} & \tilde{S}_I \end{array}, \quad D'_{I,J} = \begin{array}{ccc} X_J & \xrightarrow{i'_J \circ l_J} & Y \times \tilde{S}_J \\ \downarrow j'_{I,J} & & \downarrow p'_{I,J} \\ X_I & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_I \end{array}$$

and the factorization of $D'_{I,J}$ by the fiber product:

$$D'_{I,J} = \begin{array}{ccc} X_J & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_J \\ \downarrow j'_{I,J} & & \downarrow p'_{I,J} \\ X_I & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_I \end{array}, \quad D'_{I,J} = \begin{array}{ccc} X_J & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_J \\ \downarrow j'_{I,J} & \searrow \hat{l}_J & \nearrow \hat{i}_I \\ & X_I \times_{Y \times \tilde{S}_I} Y \times \tilde{S}_J = X_I \times \tilde{S}_{J \setminus I} & \\ \downarrow j'_{I,J} & \swarrow \hat{p}_{I,J} & \downarrow p'_{I,J} \\ X_I & \xrightarrow{i'_I \circ l_I} & Y \times \tilde{S}_I \end{array} \quad (62)$$

where $j'_{I,J} : X_J \hookrightarrow X_I$ is the open embedding. Consider

$$F(X/S) := p_{S,\sharp} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S) \in C(\text{Var}(\mathbb{C})^{sm}/S)$$

so that $D(\mathbb{A}^1, et)(F(X/S)) = M(X/S)$ since Y is smooth. Then, by definition,

$$\mathcal{F}_S^{FDR}(M(X/S)) := (e'(\tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F(X/S))), \\ E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{I,J}^q(F(X/S)))$$

On the other hand, let

$$Q(X_I/\tilde{S}_I) := p_{\tilde{S}_I, \sharp} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \in C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I),$$

We have then for $I \subset [1, l]$ the map (56) in $C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$:

$$N_I(X/S) : Q(X_I/\tilde{S}_I) = p_{\tilde{S}_I, \sharp} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \xrightarrow{\text{ad}(i'_I, i'_{I*})(-)} \\ p_{\tilde{S}_I, \sharp} i'_{I*} i'_I \Gamma_{X_I}^\vee \mathbb{Z}(Y \times S_I/Y \times S_I) \xrightarrow{T(i'_I, \gamma^\vee)(-)} p_{\tilde{S}_I, \sharp} i'_{I*} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times S_I/Y \times S_I) \\ \xrightarrow{\hat{T}_i(p_{S_I}, i_I)(-)} i_{I*} p_{S_I, \sharp} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times S_I/Y \times S_I) = i_{I*} j_I^* F(X/S).$$

We then have the commutative diagram in $C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_J)$

$$\begin{array}{ccc} p_{IJ}^* Q(X_I/\tilde{S}_I) & \xrightarrow{p_{IJ}^* N_I(X/S)} & p_{IJ}^* Li_I^* j_I^* F(X/S) \\ H_{IJ} \downarrow & & \downarrow T^q(D_{IJ})(j_I^* F(X/S)) \\ Q(X_J/\tilde{S}_J) & \xrightarrow{N_J(X/S)} & Li_{J^*} j_J^* F(X/S) \end{array} \quad (63)$$

with

$$\begin{aligned} H_{IJ} : p_{IJ}^* p_{\tilde{S}_I} \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) &\xrightarrow{T_{\sharp}(p_{IJ}, p_{\tilde{S}_I})(-)^{-1}} p_{\tilde{S}_J} p_{IJ}^* \Gamma_{X_I}^\vee \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \\ \xrightarrow{p_{\tilde{S}_J} T(p_{IJ}, \gamma^\vee)(-)} p_{\tilde{S}_J} \Gamma_{X_I \times \tilde{S}_J}^\vee \mathbb{Z}(Y \times \tilde{S}_J/Y \times \tilde{S}_J) &\xrightarrow{p_{\tilde{S}_J} T(X_J/X_I \times \tilde{S}_J \setminus I, \gamma^\vee)(-)} p_{\tilde{S}_J} \Gamma_{X_J}^\vee \mathbb{Z}(Y \times \tilde{S}_J/Y \times \tilde{S}_J). \end{aligned}$$

The diagram 63 say that the maps $N_I(X/S)$ induces a map in $C(\text{Var}(\mathbb{C})^{sm}/(S/\tilde{S}_I))$

$$\begin{aligned} (N_I(X/S)) : (Q(X_I/\tilde{S}_I), I(p_{IJ}^*, p_{IJ*})(-, -)(H_{IJ})) \\ \rightarrow (Li_I^* j_I^* F(X/S), I(p_{IJ}^*, p_{IJ*})(-, -)(T^q(D_{IJ})(j_I^* F(X/S)))). \end{aligned}$$

Denote $\bar{X}_I := \bar{X} \cap (\bar{Y} \times S_I) \subset \bar{Y} \times \tilde{S}_I$ the closure of $X_I \subset \bar{Y} \times \tilde{S}_I$, and $Z_I := Z \cap (\bar{Y} \times S_I) = \bar{X}_I \setminus X_I$. Consider for $I \subset [1, \dots, l]$ and $I \subset J$ the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$\begin{array}{ccc} X_I \xrightarrow{l_I} Y \times \tilde{S}_I & & \bar{Y} \times \tilde{S}_J \xrightarrow{p_{\tilde{S}_J}} \tilde{S}_J \\ \downarrow & \searrow^{p_{\tilde{S}_I}} & \downarrow p'_{IJ} & \downarrow p_{IJ} \\ \bar{X}_I \xrightarrow{l_I} \bar{Y} \times \tilde{S}_I & \xrightarrow{\bar{p}_{\tilde{S}_I}} & \tilde{S}_I & \bar{Y} \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I \\ \uparrow & \nearrow^{l_{Z_I}} & \uparrow & \uparrow \\ Z_I = \bar{X}_I \setminus X_I & \longrightarrow & D \times \tilde{S}_I & \end{array}$$

Let $\epsilon_1 : ((\bar{Y} \times \tilde{S}_I)_1, E_1) \rightarrow (\bar{Y} \times \tilde{S}_I, Z_I)$ a strict desingularization of the pair $(\bar{Y} \times \tilde{S}_I, Z_I)$, $\epsilon_2 : ((\bar{Y} \times \tilde{S}_I)_2, E_2) \rightarrow (\bar{Y} \times \tilde{S}_I, \bar{X}_I)$ a strict desingularization of the pair $(\bar{Y} \times \tilde{S}_I, \bar{X}_I)$ and a morphism $\epsilon_{12} : (\bar{Y} \times \tilde{S}_I)_2 \rightarrow (\bar{Y} \times \tilde{S}_I)_1$ such that the following diagram commutes (see definition-proposition 12) :

$$\begin{array}{ccc} (\bar{Y} \times \tilde{S}_I)_2 & \xrightarrow{\epsilon_{12}} & (\bar{Y} \times \tilde{S}_I)_1 \\ \downarrow \epsilon_2 & & \downarrow \epsilon_1 \\ \bar{Y} \times \tilde{S}_I & \xrightarrow{=} & \bar{Y} \times \tilde{S}_I \end{array}$$

and we denote by $l'_{Z_I} : E_1 \hookrightarrow (\bar{Y} \times \tilde{S}_I)_2$ $l'_I : E_2 \hookrightarrow (\bar{Y} \times \tilde{S}_I)_2$ the closed embeddings. We have then the canonical map in $C(\text{Var}(\mathbb{C})^{2,smpr}/(\tilde{S}_I)^{op})$

$$\begin{aligned} (I_\delta((\bar{X}_I, Z_I)/\tilde{S}_I)) : (\hat{R}^{CH}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), \hat{R}^{CH}(H_{IJ})) \\ := (\text{Cone}(\text{Cone}(\mathbb{Z}^{tr}((E_{1\bullet} \times \tilde{S}_I, E_{1\bullet})/(\bar{Y} \times \tilde{S}_I)_1 \times \tilde{S}_I, u_{IJ}) \rightarrow \\ \mathbb{Z}^{tr}(((\bar{Y} \times \tilde{S}_I)_1 \times \tilde{S}_I, (\bar{Y} \times \tilde{S}_I)_1)/(\bar{Y} \times \tilde{S}_I)_1) \times \tilde{S}_I)) \rightarrow \text{Cone}(\mathbb{Z}^{tr}((E_{2\bullet} \times \tilde{S}_I, E_{2\bullet})/(\bar{Y} \times \tilde{S}_I)_2 \times \tilde{S}_I, u_{IJ}) \rightarrow \\ \mathbb{Z}^{tr}(((\bar{Y} \times \tilde{S}_I)_2 \times \tilde{S}_I, (\bar{Y} \times \tilde{S}_I)_2)/(\bar{Y} \times \tilde{S}_I)_2 \times \tilde{S}_I)), \hat{R}^{CH}(H_{IJ})) \\ \xrightarrow{(([\Gamma_{\epsilon_1 \circ l'_{Z_I}}], [\Gamma_{\epsilon_1}], [\Gamma_{\epsilon_2 \circ l'_I}], [\Gamma_{\epsilon_2}]))} \\ (\text{Cone}(\mathbb{Z}^{tr}((\bar{Y} \times \tilde{S}_I, Z_I)/\tilde{S}_I) \rightarrow \mathbb{Z}^{tr}((\bar{Y} \times \tilde{S}_I, \bar{X}_I)/\tilde{S}_I))(-d_Y - d_{\tilde{S}_I})[-2d_Y - 2d_{\tilde{S}_I}], \mathbb{Z}^{tr}(I \times p_{IJ})) \end{aligned}$$

We denote by $v_{IJ}^q(F(X/S))$ the composite

$$\begin{aligned}
v_{IJ}^q(F(X/S))[d_{\tilde{S}_J}] &: e'(\tilde{S}_I)_* \mathcal{H}om(\hat{R}_{((\bar{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\
&\xrightarrow{p_{IJ*} T(p_{IJ}, \Omega^{\Gamma, pr})(-) \circ \text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} \\
&e'(\tilde{S}_J)_* \mathcal{H}om(p_{IJ}^* \hat{R}_{((\bar{Y} \times \tilde{S}_I)^*, E^*)/\tilde{S}_I}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\
&\xrightarrow{\mathcal{H}om(T(p_{IJ}, R^{CH})(Q(X_I/\tilde{S}_I))^{-1}, E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))} \\
p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om(\hat{R}_{((\bar{Y} \times \tilde{S}_I)^* \times \tilde{S}_{J \setminus I}, E^* \times \tilde{S}_{J \setminus I})/\tilde{S}_J}(\rho_{\tilde{S}_J}^* p_{IJ}^* Q(X_I/\tilde{S}_I)), E_{zar}(\Omega_{\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
&\xrightarrow{\mathcal{H}om(\hat{R}_{\tilde{S}_J}^{CH}(H_{IJ}), E_{zar}(\Omega_{\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om(\hat{R}_{((\bar{Y} \times \tilde{S}_J)^*, E'^*)/\tilde{S}_J}(\rho_{\tilde{S}_J}^* Q(X_J/\tilde{S}_J)), E_{zar}(\Omega_{\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})).
\end{aligned}$$

On the other hand, we have in $\pi_{\bar{X}}(C(MHM(\bar{X}))) \subset C_{\mathcal{D}fil}(\bar{X}/(\bar{Y} \times \tilde{S}_I))$

$$\begin{aligned}
(\text{Cone}(T(Z_I/\bar{X}_I, \gamma^{\vee, Hdg})(-))) &: (\Gamma_{\bar{X}_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b), x_{IJ}(\bar{X}/S)) \rightarrow (\Gamma_{Z_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b), x_{IJ}(Z/S)) \\
&\xrightarrow{=} (n \times I)_{!Hdg}(\Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b), x_{IJ}(X/S))
\end{aligned}$$

with

- for the closed embedding $\bar{X} \subset \bar{Y} \times S$ we consider the map in $\pi_{\bar{Y} \times \tilde{S}_J}(C(MHM(\bar{Y} \times \tilde{S}_J)))$

$$\begin{aligned}
x_{IJ}(\bar{X}/S) &: \Gamma_{\bar{X}_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b) \xrightarrow{\text{ad}(p'_{IJ*}, p'_{IJ*})(-)} p'_{IJ*} p'_{IJ*}{}^{\vee, Hdg} \Gamma_{\bar{X}_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b) \\
&\xrightarrow{T(\bar{X}_J/p'_{IJ}{}^{-1}(\bar{X}_I), \gamma^{\vee})(-) \circ T(p'_{IJ}, \gamma^{\vee})(-)} \Gamma_{\bar{X}_J}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_J}, F_b),
\end{aligned}$$

- for the closed embedding $Z \subset \bar{Y} \times S$ we consider the map in $\pi_{\bar{Y} \times \tilde{S}_J}(C(MHM(\bar{Y} \times \tilde{S}_J)))$

$$\begin{aligned}
x_{IJ}(Z/S) &: \Gamma_{Z_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b) \xrightarrow{\text{ad}(p'_{IJ*}, p'_{IJ*})(-)} p'_{IJ*} p'_{IJ*}{}^{\vee, Hdg} \Gamma_{Z_I}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_I}, F_b) \\
&\xrightarrow{T(Z_J/p'_{IJ}{}^{-1}(Z_I), \gamma^{\vee})(-) \circ T(p'_{IJ}, \gamma^{\vee})(-)} \Gamma_{Z_J}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_J}, F_b),
\end{aligned}$$

- for the closed embedding $X \subset Y \times S$ we consider the map in $\pi_{Y \times \tilde{S}_J}(C(MHM(Y \times \tilde{S}_J)))$

$$\begin{aligned}
x_{IJ}(X/S)(-d_Y)[-2d_Y] &: \Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b) \xrightarrow{\text{ad}(p'_{IJ*}, p'_{IJ*})(-)} p'_{IJ*} p'_{IJ*}{}^{\vee, Hdg} \Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b) \\
&\xrightarrow{T(X_J/X_I \times \tilde{S}_{J \setminus I}, \gamma^{\vee, Hdg})(-) \circ T(p'_{IJ}, \gamma^{\vee})(-)} \Gamma_{X_J}^{\vee, Hdg}(O_{Y \times \tilde{S}_J}, F_b).
\end{aligned}$$

The maps $x_{IJ}(X/S)$ gives the following maps in $C_{\mathcal{D}fil, S_J}(\tilde{S}_J)$

$$\begin{aligned}
w_{IJ}(X/S)(-d_Y)[-2d_Y] &: p_{\tilde{S}_I*} E_{zar}((\Omega_{\bar{Y} \times \tilde{S}_I/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{\bar{Y} \times \tilde{S}_I}} (n \times I)_{!Hdg} \Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b)) \\
&\xrightarrow{\text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} p_{IJ*} p_{IJ}^{*mod} p_{\tilde{S}_I*} E_{zar}((\Omega_{\bar{Y} \times \tilde{S}_I/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (n \times I)_{!Hdg} \Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b)) \\
&\xrightarrow{p_{IJ*} T_w^O(p_{IJ}, p_{\tilde{S}_I})(-)} p_{\tilde{S}_J*} E_{zar}((\Omega_{\bar{Y} \times \tilde{S}_J/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{\bar{Y} \times \tilde{S}_J}} p_{IJ}^{*mod} (n \times I)_{!Hdg} \Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b)) \\
&\xrightarrow{(p_{\tilde{S}_J*} E(DR(-)(x_{IJ}(X/S)))} p_{\tilde{S}_J*} E_{zar}((\Omega_{\bar{Y} \times \tilde{S}_J/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{\bar{Y} \times \tilde{S}_J}} (n \times I)_{!Hdg} \Gamma_{X_J}^{\vee, Hdg}(O_{\bar{Y} \times \tilde{S}_J}, F_b)).
\end{aligned}$$

We have then the following lemma

Lemma 12. (i) The map in $C(\text{Var}(\mathbb{C})^{sm}/(S/\tilde{S}_I))$

$$(N_I(X/S)) : (Q(X_I/\tilde{S}_I), H_{IJ}) \rightarrow (L(i_{I*}j_I^*F(X/S)), T^q(D_{IJ})(F(X/S))).$$

is an equivalence (\mathbb{A}^1, et) local.

(ii) The maps $(N_I(X/S))$ induces a filtered quasi-isomorphism in $C_{\mathcal{D}fil}(S/(\tilde{S}_I))$

$$\begin{aligned} & (\text{Hom}(\hat{R}_{\tilde{S}_I}^{CH}(N_I(X/S)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_b))) : \\ & (e'(\tilde{S}_I)_* \text{Hom}(\hat{R}_{\tilde{S}_I}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*F(X/S))), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S))) \\ & \rightarrow (e'(\tilde{S}_I)_* \text{Hom}(\hat{R}_{\tilde{S}_I}^{CH}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \end{aligned}$$

(iii) The map $(I_\delta((\bar{X}_I, Z_I)/\tilde{S}_I))$ induces a filtered Zariski local equivalence in $C_{\mathcal{D}fil}(S/(\tilde{S}_I))$

$$\begin{aligned} & (I((\bar{X}_I, Z_I)/\tilde{S}_I)) := \\ & \text{Hom}((I_\delta((\bar{X}_I, Z_I)/\tilde{S}_I)), -) \circ (DR(Y \times \tilde{S}_I/\tilde{S}_I)(\text{ad}((n \times I)_{!Hdg}, (n \times I)^{!Hdg}(O_{Y \times \tilde{S}_I}, F_b))), 0) : \\ & (p_{\tilde{S}_I*} E_{zar}((\Omega_{\bar{Y} \times \tilde{S}_I/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{\bar{Y} \times \tilde{S}_I}} (n \times I)_{!Hdg}(\Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b))(d_Y)[2d_Y + d_{\tilde{S}_I}], w_{IJ}(X/S)) \\ & \rightarrow (e'(\tilde{S}_I)_* \text{Hom}(\hat{R}_{\tilde{S}_I}^{CH}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \end{aligned}$$

Proof. (i): See lemma 9(i)

(ii): These maps induce a morphism in $C_{\mathcal{D}}(S/(\tilde{S}_I))$ by construction. It is a filtered quasi-isomorphism by (i) and proposition 113.

(iii): See section 5.2. □

Proposition 116. Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Assume there exist a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $\bar{Y} \in \text{PSmVar}(\mathbb{C})$ a compactification of Y with $\bar{Y} \setminus Y = D$ a normal crossing divisor, denote $k : D \hookrightarrow \bar{Y}$ the closed embedding and $n : Y \hookrightarrow \bar{Y}$ the open embedding. Denote $\bar{X} \subset \bar{Y} \times S$ the closure of $X \subset \bar{Y} \times S$. We have then the following commutative diagram in $\text{Var}(\mathbb{C})$

$$\begin{array}{ccccc} X & \xrightarrow{l} & Y \times S & & \\ \downarrow & & \downarrow & \searrow^{p_S} & \\ \bar{X} & \xrightarrow{l} & \bar{Y} \times S & \xrightarrow{\bar{p}_S} & S \\ \uparrow & \nearrow^{l_Z} & \uparrow & \nearrow^{(k \times I)} & \\ Z := \bar{X} \setminus X & \longrightarrow & D \times S & & \end{array}$$

Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. Denote $\bar{X}_I := \bar{X} \cap (\bar{Y} \times S_I) \subset \bar{Y} \times \tilde{S}_I$ the closure of $X_I \subset \bar{Y} \times \tilde{S}_I$, and $Z_I := Z \cap (\bar{Y} \times S_I) = \bar{X}_I \setminus X_I \subset \bar{Y} \times \tilde{S}_I$. We have then for $I \subset [1, \dots, l]$, the following commutative diagram in $\text{Var}(\mathbb{C})$

$$\begin{array}{ccccc} X_I & \xrightarrow{l_I} & Y \times \tilde{S}_I & & \\ \downarrow & & \downarrow & \searrow^{p_{\tilde{S}_I}} & \\ \bar{X}_I & \xrightarrow{l_I} & \bar{Y} \times \tilde{S}_I & \xrightarrow{\bar{p}_{\tilde{S}_I}} & \tilde{S}_I \\ \uparrow & \nearrow^{l_{Z_I}} & \uparrow & \nearrow^{(k \times I)} & \\ Z_I = \bar{X}_I \setminus X_I & \longrightarrow & D \times \tilde{S}_I & & \end{array}$$

Let $F(X/S) := p_{S,\sharp}\Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S) \in C(\text{Var}(\mathbb{C})^{sm}/S)$. We have then the following isomorphism in $D_{\mathcal{D}fil}(S/(\tilde{S}_I))$

$$\begin{aligned} & I(X/S) : \mathcal{F}_S^{FDR}(M(X/S)) \xrightarrow{\cong} \\ & (e'(\tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F(X/S))), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S))) \\ & \xrightarrow{(\mathcal{H}om(\hat{R}_{\tilde{S}_I}^{CH}(N_I(X/S)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))} \\ & (e'(\tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \\ & \xrightarrow{I((\bar{X}_I, Z_I)/\tilde{S}_I)} \\ & (p_{\tilde{S}_I*} E_{zar}((\Omega_{\bar{Y} \times \tilde{S}_I/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_b) \otimes_{O_{\bar{Y} \times \tilde{S}_I}} (n \times I)_{!Hdg} \Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b))(d_Y + d_{\tilde{S}_I})[2d_Y + d_{\tilde{S}_I}], w_{IJ}(X/S)) \\ & \xrightarrow{\cong} \iota_S Rf_{I!}^{Hdg}(\Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b)(d_Y)[2d_Y], x_{IJ}(X/S)). \xrightarrow{\cong} \iota_S Rf_{I!}^{Hdg} f_{Hdg}^{*mod} \mathbb{Z}_S^{Hdg}. \end{aligned}$$

Proof. Follows from lemma 12, proposition 113 and proposition 40. \square

Corollary 4. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F) \in \text{DA}_c(S)$,

$$\begin{aligned} H^i \mathcal{F}_S^{FDR}(M, W) & := (a_{zar} H^i e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\rho_{\tilde{S}_I*} \mu_{\tilde{S}_I*} R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), \\ & E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], H^i u_{IJ}^q(F, W)) \in \pi_S(MHM(S)) \end{aligned}$$

for all $i \in \mathbb{Z}$, and for all $p \in \mathbb{Z}$,

$$\mathcal{F}_S^{FDR}(M, W) \in D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$$

is the class of a complex $\mathcal{F}_S^{FDR}(F, W) \in C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$ such that for all $k \in \mathbb{Z}$, the differentials of $\text{Gr}_k^W \mathcal{F}_S^{FDR}(F, W)$ are strict for the filtration F .

Proof. Follows from theorem 29: Indeed, for S smooth and $g : U'/S \rightarrow U/S$ a morphism with $U/S = (U, h), U'/S = (U', h') \in \text{Var}(\mathbb{C})^{sm}/S$, U, U' connected, hence irreducible by smoothness, the complex

$$\begin{aligned} & \mathcal{F}_S^{\hat{F}DR}(g) := e'(S)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\mathbb{Z}(U'/S)), E_{zar}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \\ & \xrightarrow{\mathcal{H}om(\hat{R}^{CH}(g), -)} e'(S)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\mathbb{Z}(U/S)), E_{zar}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S] \in C_{\mathcal{D}fil}(S) \end{aligned}$$

satisfy $H^i \mathcal{F}_S^{\hat{F}DR}(g) \in \pi_S(MHM(S))$ and is the class of a complex such that the differentials are strict for F . Let $U \subset \bar{X}$ a compactification of U and $U' \subset \bar{X}'$ a compactification of U' , $S \subset \bar{S}$ a compactification of S with $\bar{X}, \bar{X}' \in \text{PSmVar}(\mathbb{C})$, $i : \bar{D} = \cup_i \bar{D}_i := \bar{X} \setminus U \hookrightarrow \bar{X}$, $i : \bar{D}' = \cup_i \bar{D}'_i := \bar{X}' \setminus U' \hookrightarrow \bar{X}'$ normal crossing divisors, such that $g : U'/S \rightarrow U/S$ extend to $\bar{g} : \bar{X}'/\bar{S} \rightarrow \bar{X}/\bar{S}$: see section 2. Denote $n : U \hookrightarrow \bar{X}$, $n' : U' \hookrightarrow \bar{X}'$ and $n'' : U' \hookrightarrow \bar{X}' \setminus \bar{g}^{-1}(\bar{D}) =: U''$ the open embeddings. We get the following map in $C_{\mathcal{D}fil}(S)$

$$\begin{aligned} & \Omega^{\Gamma, pr}(\Gamma_{\bar{g}}^t) := (p_{S*} E(\Omega_{\bar{X}' \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{\bar{X}' \times S}} (n' \times I)_{!Hdg} \Gamma_{U'}^{\vee, Hdg}(O_{U' \times S}, F_b))(d')[2d'] \\ & \xrightarrow{\Omega^{\Gamma, pr}(\Gamma_{\bar{g}}^t) \circ DR(\bar{X}' \times S/S)(\text{ad}(n''_{!Hdg}, n''_{!Hdg})(O_{U'' \times S}, F_b))} \\ & p_{S*} E(\Omega_{\bar{X} \times S/S}^{\bullet, \Gamma, pr} \otimes_{O_{\bar{X} \times S}} (n \times I)_{!Hdg} \Gamma_U^{\vee, Hdg}(O_{U \times S}, F_b))(d)[2d]) \\ & \xrightarrow{(DR(\bar{X}' \times S/S)(\text{ad}((n' \times I)_{!Hdg}, (n' \times I)_{!Hdg}))(O_{\bar{X}' \times S}, F_b), 0)(DR(\bar{X} \times S/S)(\text{ad}((n \times I)_{!Hdg}, (n \times I)_{!Hdg}))(O_{\bar{X} \times S}, F_b), 0)} \\ & e'(S)_* \mathcal{H}om((\text{Cone}(\mathbb{Z}(i_\bullet \times I) : \mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S) \rightarrow \mathbb{Z}^{tr}((\bar{X} \times S, X)/S))(-d)[-2d] \\ & \xrightarrow{([\Gamma_{\bar{g}} \times \square^*]^t, [\Gamma_{\bar{g}'_\bullet} \times \square^*]^t)} \\ & \text{Cone}(\mathbb{Z}(i_\bullet \times I) : \mathbb{Z}^{tr}((\bar{D}'_\bullet \times S, D'_\bullet)/S) \rightarrow \mathbb{Z}^{tr}((\bar{X}' \times S, X')/S))(-d')[-2d'], E_{zar}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR})) := \mathcal{F}_S^{\hat{F}DR}(g) \end{aligned}$$

which is a filtered quasi-isomorphism by proposition 113. Note that in the particular case where $\bar{g} = i : Z \hookrightarrow X$ is a closed embedding of codimension $c = \dim(X) - \dim(Z)$,

$$\begin{aligned} & p_{S*}E(\Omega_{Z \times S/S}^\bullet \otimes_{O_{Z \times S}} \Gamma_Z^{\vee, Hdg}(O_{Z \times S}, F_b))(-c)[-2c] \\ & \xrightarrow{\Omega^{\Gamma, pr}([\Gamma_i^t])=(\alpha \mapsto \alpha \wedge \gamma)} p_{S*}E(\Omega_{X \times S/S}^\bullet \otimes_{O_{X \times S}} \Gamma_X^{\vee, Hdg}(O_{X \times S}, F_b)) \end{aligned}$$

is the Gysin morphism, where $\gamma \in H^0\Gamma(X, N_{Z/X}^\bullet[2c])$ is the De Rham fundamental class of the conormal bundle, also note that denoting by $j : X \setminus Z \hookrightarrow X$ the open complementary the composite in $C_{\mathcal{D}, fil}(S)$

$$\begin{aligned} & p_{S*}E(\Omega_{Z \times S/S}^\bullet \otimes_{O_{Z \times S}} \Gamma_Z^{\vee, Hdg}(O_{Z \times S}, F_b))(-c)[-2c] \\ & \xrightarrow{\Omega^{\Gamma, pr}([\Gamma_i^t])=(\alpha \mapsto \alpha \wedge \gamma)} p_{S*}E(\Omega_{X \times S/S}^\bullet \otimes_{O_{X \times S}} \Gamma_X^{\vee, Hdg}(O_{X \times S}, F_b)) \\ & \xrightarrow{\Omega^{\Gamma, pr}(j)} p_{S*}E(\Omega_{(X \setminus Z) \times S/S}^\bullet \otimes_{O_{(X \setminus Z) \times S}} \Gamma_{(X \setminus Z)}^{\vee, Hdg}(O_{(X \setminus Z) \times S}, F_b)) \end{aligned}$$

is NOT equal to zero, and the composite in $C(\text{Var}(\mathbb{C})^{2, smpr}/S)$

$$\mathbb{Z}^{tr}(((X \setminus Z) \times S, X \setminus Z)/S) \xrightarrow{\mathbb{Z}(j \times I)} \mathbb{Z}^{tr}((X \times S, X)/S) \xrightarrow{[\Gamma_i^t]} \mathbb{Z}^{tr}((Z \times S, Z)/S)(c)[2c]$$

is NOT equal to zero. Now, for $g : U'/\tilde{S}_I \rightarrow U/\tilde{S}_I$ the commutative diagram

$$\begin{array}{ccc} p_{IJ}^{*mod} \mathcal{F}^{\hat{F}DR}_{\tilde{S}_I}(g) & \xrightarrow{T(p_{IJ}, \Omega^{\Gamma, pr})(-)} & \mathcal{F}^{\hat{F}DR}_{\tilde{S}_I}(g \times I) \\ p_{IJ}^{*mod} A_I \downarrow & & \downarrow A_J \\ p_{IJ}^{*mod} \Omega^{\Gamma, pr}(\Gamma_{\tilde{g}}^t) & \xrightarrow{T_w^O(p_{IJ}, p_{\tilde{S}_I})(-)} & \Omega^{\Gamma, pr}(\Gamma_{\tilde{g} \times I}^t) \end{array}$$

with $g \times I : U' \times \tilde{S}_{J \setminus I}/\tilde{S}_J \rightarrow U \times \tilde{S}_{J \setminus I}/\tilde{S}_J$ and

$$\begin{aligned} A_I & := ((DR(\bar{X}' \times \tilde{S}_I/\tilde{S}_I)(\text{ad}((n' \times I)_{!Hdg}, (n' \times I)_{!Hdg})(O_{\bar{X}' \times \tilde{S}_I}, F_b)), 0), \\ & (DR(\bar{X} \times \tilde{S}_I/\tilde{S}_I)(\text{ad}((n \times I)_{!Hdg}, (n \times I)_{!Hdg})(O_{\bar{X} \times \tilde{S}_I}, F_b)), 0). \end{aligned}$$

□

Proposition 117. *For $S \in \text{Var}(\mathbb{C})$ not smooth, the functor (see corollary 4)*

$$\iota_S^{-1} \mathcal{F}_S^{FDR} : \text{DA}_c^-(S)^{op} \rightarrow \pi_S(D(\text{MHM}(S)))$$

does not depend on the choice of the open cover $S = \cup_i S_i$ and the closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

Proof. Let $S = \cup_{i=l+1}^{i=l'} S_i$ is an other open cover together with closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with

$\tilde{S}_i \in \text{SmVar}(\mathbb{C})$ for $l+1 \leq i \leq l'$. Then, for $J' \subset I' \subset [l+1, \dots, l'] = L'$ and $J \subset I \subset L = [1, \dots, l]$,

$$\begin{aligned}
& T_S^{L/L'} (\iota_S^{-1} (e'(\tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}(F))) \\
& := ((\text{ho} \lim_{I \in L} p_{I'(I \sqcup I')}^* \Gamma_{S_{I \sqcup I'}}^{\vee, Hdg} p_{I(I \sqcup I')}^{*mod[-1]} (\iota_S^{-1} (e'(\tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), \\
& \quad E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], T_S^{L/L'}(u_{IJ}(F))))_I \xrightarrow{(\text{ho} \lim_{I \in L} u_{I(I \sqcup I')}(F))} \\
& (\text{ho} \lim_{I \in L} p_{I'(I \sqcup I')}^* \Gamma_{S_{I \sqcup I'}}^{\vee, Hdg} p_{I(I \sqcup I')}^{*mod[-1]} p_{I(I \sqcup I')}^* e'(\tilde{S}_{(I \sqcup I')})_* \mathcal{H}om(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{(I \sqcup I')} j_{(I \sqcup I')}^* F)), \\
& \quad E_{zar}(\Omega_{/\tilde{S}_{I \sqcup I'}}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_{I \sqcup I'}}], u_{(I \sqcup I')(I \sqcup J')}(F)) \xrightarrow{\text{ad}(p_{I(I \sqcup I')}^{*mod[-1]} p_{I(I \sqcup I')}^*)(-) \circ \gamma_{S_{I \sqcup I'}}^{\vee, Hdg}(-)} \\
& \quad (\text{ho} \lim_{I \in L} p_{I'(I \sqcup I')}^* e'(\tilde{S}_{(I \sqcup I')})_* \mathcal{H}om(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{(I \sqcup I')} j_{(I \sqcup I')}^* F)), \\
& \quad \quad E_{zar}(\Omega_{/\tilde{S}_{I \sqcup I'}}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_{I \sqcup I'}}], u_{(I \sqcup I')(I \sqcup J')}(F)) \\
& \xleftarrow{(\text{ho} \lim_{I \in L} u_{I'(I \sqcup I')}(F))} (e'(\tilde{S}_{I'})_* \mathcal{H}om(\hat{R}^{CH}(\rho_{\tilde{S}_{I'}}^* L(i_{I'*} j_{I'}^* F)), E_{zar}(\Omega_{/\tilde{S}_{I'}}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_{I'}}], u_{I'J'}(F))
\end{aligned}$$

is a filtered Zariski local equivalence, since all the morphisms are filtered Zariski local equivalences by proposition 116 and proposition 100. \square

We have the canonical transformation map between the filtered De Rham realization functor and the Gauss-Manin realization functor :

Definition 135. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$. We have, using definition 129(ii), the canonical map in $D_{O_S, \mathcal{D}, \infty}(S/(\tilde{S}_I))$

$$\begin{aligned}
& T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : \\
& \mathcal{F}_S^{GM}(L\mathbb{D}_S M) := (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* \mathbb{D}_S L F), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\
& \quad \xrightarrow{\sim} (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\mathbb{D}_{\tilde{S}_I}^0 L(i_{I*} j_I^* F), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_b))[-d_{\tilde{S}_I}], u_{IJ}^{q,d}(F)) \\
& \xrightarrow{(\mathcal{H}om(-, \text{Gr}(\Omega_{\tilde{S}_I})))^{-1}} (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\mathbb{D}_{\tilde{S}_I}^0 L(i_{I*} j_I^* F), \text{Gr}_{\tilde{S}_I}^{12} E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,d}(F)) \\
& \quad \xrightarrow{(\mathcal{H}om^\bullet(\tau_{\tilde{S}_I}^{0CH}(L(i_{I*} j_I^* F)) \circ T(\hat{R}^{0CH}, R^{0CH})(L(i_{I*} j_I^* F))), -)} \\
& \quad (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), \text{Gr}_{\tilde{S}_I}^{12} E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,d}(F)) \\
& \xrightarrow{(I(\text{Gr}_{\tilde{S}_I}^{12*}, \text{Gr}_{\tilde{S}_I}^{12*})(-, -))} (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{Gr}_{\tilde{S}_I}^{12*} \hat{R}^{0CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,d}(F)) \\
& \quad \xrightarrow{=} (e'(\tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) =: \mathcal{F}_S^{FDR}(M)
\end{aligned}$$

Proposition 118. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

(i) For $M \in \text{DA}_c(S)$ the map in $D_{O_S, \mathcal{D}}(S/(\tilde{S}_I)) = D_{O_S, \mathcal{D}}(S)$

$$o_{fil} T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : o_{fil} \mathcal{F}_S^{GM}(L\mathbb{D}_S M) \xrightarrow{\sim} o_{fil} \mathcal{F}_S^{FDR}(M)$$

given in definition 135 is an isomorphism if we forgot the Hodge filtration F .

(ii) For $M \in \text{DA}_c(S)$ and all $n, p \in \mathbb{Z}$, the map in $\text{PSh}_{O_S, \mathcal{D}}(S/(\tilde{S}_I))$

$$F^p H^n T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : F^p H^n \mathcal{F}_S^{GM}(L\mathbb{D}_S M) \hookrightarrow F^p H^n \mathcal{F}_S^{FDR}(M)$$

given in definition 135 is a monomorphism. Note that $F^p H^n T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M)$ is NOT an isomorphism in general : take for example $M(S^o/S)^\vee = D(\mathbb{A}^1, et)(j_* E_{zar}(\mathbb{Z}(S^o/S)))$ for an open embedding $j : S^o \hookrightarrow S$, then

$$H^n \mathcal{F}_S^{GM}(\mathbb{L}D_S M(S^o/S)^\vee) = \mathcal{F}_S^{GM}(\mathbb{Z}(S^o/S)) = j_* E(O_{S^o}, F_b) \notin \pi_S(MHM(S))$$

and hence is NOT isomorphic to $H^n \mathcal{F}_S^{FDR}(\mathbb{L}D_S M(S^o/S)^\vee) \in \pi_S(MHM(S))$ as filtered D_S -modules (see remark 9). It is an isomorphism in the very particular cases where $M = D(\mathbb{A}^1, et)(\mathbb{Z}(X/S))$ or $M = D(\mathbb{A}^1, et)(\mathbb{Z}(X^o/S))$ for $f : X \rightarrow S$ is a smooth proper morphism and $n : X^o \hookrightarrow X$ is an open subset such that $X \setminus X^o = \cup D_i$ is a normal crossing divisor and such that $f|_{D_i} = f \circ i_i : D_i \rightarrow X$ are SMOOTH morphism with $i_i : D_i \hookrightarrow X$ the closed embedding and considering $f|_{X^o} = f \circ n : X^o \rightarrow S$ (see proposition 111).

Proof. (i): Follows from the computation for a Borel-Moore motive.

(ii): Follows from (i). \square

We now define the functorialities of \mathcal{F}_S^{FDR} with respect to S which makes \mathcal{F}_{FDR}^- a morphism of 2 functor.

Definition 136. Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote $Z_I := Z \cap S_I$. We then have closed embeddings $Z_I \hookrightarrow S_I \hookrightarrow \tilde{S}_I$.

(i) For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we will consider the following canonical map in $\pi_S(D(MHM(S))) \subset D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$

$$\begin{aligned} & T(\Gamma_Z^{\vee, Hdg}, \Omega_{/S}^{\Gamma, pr})(F, W) : \\ & \Gamma_Z^{\vee, Hdg} \iota_S^{-1}(e'_* \text{Hom}^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F, W)) \\ & \xrightarrow{\text{Hom}^\bullet(\hat{R}_{\tilde{S}_I}^{CH}(\gamma^{\vee, Z_I}(L(i_{I*} j_I^*(F, W))), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))} \\ & \Gamma_Z^{\vee, Hdg} \iota_S^{-1}(e'_* \text{Hom}^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* \Gamma_{Z_I}^\vee L(i_{I*} j_I^*(F, W))), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F, W)) \\ & \xrightarrow{\cong} \iota_S^{-1}(e'_* \text{Hom}^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* \Gamma_{Z_I}^\vee L(i_{I*} j_I^*(F, W))), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F, W)). \end{aligned}$$

with

$$\begin{aligned} & u_{IJ}^q(F)[d_{\tilde{S}_I}] : e'(\tilde{S}_I)_* \text{Hom}^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* \Gamma_{Z_I}^\vee L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\ & \xrightarrow{p_{IJ*} T(p_{IJ}, \Omega^{\gamma, pr})(-) \circ \text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} \\ & p_{IJ*} e'(\tilde{S}_J)_* \text{Hom}^\bullet(p_{IJ}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* \Gamma_{Z_I}^\vee L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\ & \xrightarrow{\text{Hom}(\hat{T}(p_{IJ}, R^{CH})(L i_{I*} j_I^* F)^{-1}, E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\ & p_{IJ*} e'(\tilde{S}_J)_* \text{Hom}^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_J}^* p_{IJ}^* \Gamma_{Z_I}^\vee L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\ & \xrightarrow{\text{Hom}(\hat{R}_{\tilde{S}_J}^{CH}(T^q(D_{IJ})(j_I^* F) \circ T(Z_J/Z_I \times \tilde{S}_J \setminus I, \gamma^\vee)(-) \circ T(p_{IJ}, \gamma)(-)), E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\ & p_{IJ*} e'(\tilde{S}_J)_* \text{Hom}^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_J}^* \Gamma_{Z_J}^\vee L(i_{J*} j_J^* F)), E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})). \end{aligned}$$

(ii) For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we have also the following canonical map in $\pi_S(D(MHM(S))) \subset$

$D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$

$$\begin{aligned}
& T(\Gamma_Z^{Hdg}, \Omega_{/S}^{\Gamma, pr})(F, W) : \\
& \iota_S^{-1}(e'_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I} E(i_{I^*} j_I^* \mathbb{D}_S(F, W))), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q, Z, d}(F, W)) \xrightarrow{=} \\
& \Gamma_Z^{Hdg} \iota_S^{-1}(e'_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I} E(i_{I^*} j_I^* \mathbb{D}_S(F, W))), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q, Z, d}(F, W)) \\
& \xrightarrow{\mathcal{H}om^\bullet(\hat{R}_{\tilde{S}_I}^{CH}(\gamma^{Z_I}(-)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& \Gamma_Z^{Hdg} \iota_S^{-1}(e'_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I^*} j_I^* \mathbb{D}_S(F, W))), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F, W))
\end{aligned}$$

with

$$\begin{aligned}
& u_{IJ}^{q, Z}(F)[d_{\tilde{S}_I}] : e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\rho_{\tilde{S}_I^*} \mu_{\tilde{S}_I^*} R^{CH}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I} E(i_{I^*} j_I^* \mathbb{D}_S L F)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{p_{IJ^*} T(p_{IJ}, \Omega_{/S}^{\Gamma, pr})(-) \circ \text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} \\
& p_{IJ^*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(p_{IJ^*}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I} E(i_{I^*} j_I^* \mathbb{D}_S F)), E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{\mathcal{H}om(T(p_{IJ}, R^{CH})(Li_{I^*} j_I^* F)^{-1}, E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& p_{IJ^*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_J}^* L p_{IJ^*}^* \Gamma_{Z_I} E(i_{I^*} j_I^* \mathbb{D}_S L F)), E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{\mathcal{H}om(\hat{R}_{\tilde{S}_J}^{CH}(\mathbb{D}_{\tilde{S}_J} S^q(D_{IJ})(\mathbb{D}_S L F)), E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& p_{IJ^*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_J}^* \Gamma_{Z_J} E(i_{J^*} j_J^* \mathbb{D}_S L F)), E_{zar}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})).
\end{aligned}$$

This transformation map will, with the projection case, gives the transformation between the pullback functor :

Definition 137. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Consider the factorization $g : T \xrightarrow{l} T \times S \xrightarrow{ps} S$ where l is the graph embedding and ps the projection. Let $M \in \text{DA}_c(S)^-$ and $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}_S^1, et)(F, W)$. Then, $D(\mathbb{A}_T^1, et)(g^* F) = g^* M$ and there exist $(F', W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/T \times S)$ and an equivalence (\mathbb{A}^1, et) local $e : \Gamma_T^\vee p_S^*(F, W) \rightarrow (F', W)$ such that $D(\mathbb{A}_{T \times S}^1, et)(F', W) = (\Gamma_T^\vee p_S^* M, W)$. We have then the canonical transformation in $\pi_T(D(MHM(T)))$ using definition 131 and definition 136(i) :

$$\begin{aligned}
& T(g, \mathcal{F}^{FDR})(M) : g^{\hat{*}mod, Hdg} \mathcal{F}_S^{FDR}(M) := \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1}(p_S^{*mod[-1]}(e(S)_* G_{\tilde{S}_S}^{12} \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_S^* L(F, W)), E_{zar}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S])) \\
& \xrightarrow{T(p_S, \Omega_{/S}^{\Gamma, pr})(-)} \\
& \Gamma_T^{\vee, Hdg}(e'(T \times S)_* \mathcal{H}om^\bullet(p_S^* \hat{R}^{CH}(\rho_S^* L(F)), E_{zar}(\Omega_{/T \times S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S])) \\
& \xrightarrow{\mathcal{H}om(T(p_S, \hat{R}^{CH})(L(F, W))^{-1}, E_{zar}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S]} \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1}(e'(T \times S)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{T \times S}^* p_S^* L(F, W)), E_{zar}(\Omega_{/T \times S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S])) \xrightarrow{=} \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1}(e'(T \times S)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{T \times S}^* p_S^* L(F, W)), E_{zar}(\Omega_{/T \times S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S])) \\
& \xrightarrow{T(\Gamma_T^{\vee, Hdg}, \Omega_{/T \times S}^{\Gamma, pr})(F, W)} \\
& (e'(T \times S)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{T \times S}^* \Gamma_T^\vee p_S^* L(F, W)), E_{zar}(\Omega_{/T \times S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S])) \\
& \xrightarrow{\mathcal{H}om(\hat{R}_{T \times S}^{CH}(e), -)} \\
& (e'(T \times S)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{T \times S}^* L(F', W)), E_{zar}(\Omega_{/T \times S}^{\bullet, \Gamma, pr}, F_{DR}))[-d_S])) \\
& =: \mathcal{F}_{T \times S}^{FDR}(l_* g^* M) = \mathcal{F}_T^{FDR}(g^* M)
\end{aligned}$$

where the last equality follows from proposition 117.

We give now the definition in the non smooth case Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. We recall the commutative diagram :

$$E_{IJg} = \begin{array}{ccc} (Y \times \tilde{S}_I) \setminus T_I & \xrightarrow{m'_I} & Y \times \tilde{S}_J \\ \downarrow p_{\tilde{S}_I} & & \downarrow \tilde{g}_I \\ \tilde{S}_I \setminus S_I & \xrightarrow{m_I} & \tilde{S}_I \end{array}, E_{IJ} = \begin{array}{ccc} \tilde{S}_J \setminus S_J & \xrightarrow{m_J} & \tilde{S}_J \\ \downarrow p_{IJ} & & \downarrow p_{IJ} \\ \tilde{S}_I \setminus (S_I \setminus S_J) & \xrightarrow{m_I = m_{IJ}} & \tilde{S}_I \end{array} \quad E'_{IJ} = \begin{array}{ccc} (Y \times \tilde{S}_J) \setminus T_J & \xrightarrow{m'_J} & Y \times \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p'_{IJ} \\ (Y \times \tilde{S}_I) \setminus (T_I \setminus T_J) & \xrightarrow{m'_I = m'_{IJ}} & Y \times \tilde{S}_I \end{array}$$

For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} := I_Y \times p_{IJ} : Y \times \tilde{S}_J \rightarrow Y \times \tilde{S}_I$ the projections, so that $\tilde{g}_I \circ p'_{IJ} = p_{IJ} \circ \tilde{g}_J$. Consider, for $I \subset J \subset [1, \dots, l]$, resp. for each $I \subset [1, \dots, l]$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}, D'_{IJ} = \begin{array}{ccc} T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_I \\ j'_{IJ} \uparrow & & \uparrow p'_{IJ} \\ T_J & \xrightarrow{i'_J} & Y \times \tilde{S}_J \end{array} \quad D_{gI} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ g_I \uparrow & & \uparrow \tilde{g}_I \\ T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_I \end{array},$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Let $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$. Recall (see section 2) that since $j'_I{}^* i'_I{}^* j'_I{}^* g^* F = 0$, the morphism $T(D_{gI})(j'_I{}^* F) : \tilde{g}_I^* i_{I*} j'_I{}^* F \rightarrow i'_I{}^* j'_I{}^* g^* F$ factors through

$$T(D_{gI})(j'_I{}^* F) : \tilde{g}_I^* i_{I*} j'_I{}^* F \xrightarrow{\gamma_{X_I}^\vee(-)} \Gamma_{X_I}^\vee \tilde{g}_I^* i_{I*} j'_I{}^* F \xrightarrow{T^\gamma(D_{gI})(j'_I{}^* F)} i'_I{}^* j'_I{}^* g^* F.$$

and that the fact that the diagrams (61) commutes says that the maps $T^{q,\gamma}(D_{gI})(j'_I{}^* F)$ define a morphism in $C(\text{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$

$$(T^{q,\gamma}(D_{gI})(j'_I{}^* F)) : (\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j'_I{}^* F), T^q(D_{IJ})(j'_I{}^* F) \circ T(T_I/T_I \times \tilde{S}_J \setminus I, \gamma^\vee)(-) \circ T(p'_{IJ}, \gamma^\vee)(-)) \\ \rightarrow (L(i'_I{}^* j'_I{}^* g^* F), T^q(D'_{IJ})(j'_I{}^* g^* F))$$

Denote for short $d_{YI} := -d_Y - d_{\tilde{S}_I}$. We denote by $\tilde{g}_J^* u_{IJ}^q(F)$ the composite

$$\begin{aligned} & \tilde{g}_J^* u_{IJ}^q(F)[-d_{YI}] : \\ & e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*} j'_I{}^* F)), E_{zar}(\Omega_{Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\ & \xrightarrow{p'_{IJ*} T(p'_{IJ}, \Omega^{\Gamma, pr})(-) \circ \text{oad}(p'_{IJ*}{}^{mod}, p'_{IJ*})(-)} \\ & p'_{IJ*} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(p'_{IJ*} \hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*} j'_I{}^* F)), E_{zar}(\Omega_{Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\ & \xrightarrow{\mathcal{H}om(T(p'_{IJ}, \hat{R}^{CH})(-)^{-1}, E_{zar}(\Omega_{Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\ & p'_{IJ*} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_J}^* p'_{IJ*} \tilde{g}_I^* L(i_{I*} j'_I{}^* F)), E_{zar}(\Omega_{Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\ & \xrightarrow{\mathcal{H}om(\hat{R}_{Y \times \tilde{S}_J}^{CH}(T^q(D_{IJ})(j'_I{}^* F)), E_{zar}(\Omega_{Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\ & e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*} j'_I{}^* F)), E_{zar}(\Omega_{Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})). \end{aligned}$$

We denote by $\tilde{g}_J^{*,\gamma} u_{IJ}^q(F)$ the composite

$$\begin{aligned}
& \tilde{g}_J^{*,\gamma} u_{IJ}^q(F)[-d_{YJ}] : \\
& e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \quad \xrightarrow{p'_{IJ*} T(p'_{IJ, \Omega^{\Gamma, pr}}(-) \circ \text{ad}(p'_{IJ*}{}^{mod}, p'_{IJ*}(-))} \\
& p'_{IJ*} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(p'_{IJ*} \hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \quad \xrightarrow{\mathcal{H}om(T(p'_{IJ, \hat{R}^{CH}}(-)^{-1}, E_{zar}(\Omega_{/Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& p'_{IJ*} e'(Y \times \tilde{S}_I)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_J}^* p'_{IJ*} \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \quad \xrightarrow{\mathcal{H}om(\hat{R}_{Y \times \tilde{S}_J}^{CH} (T^q(D_{IJ})(j_I^* F) \circ T(T_I/T_I \times \tilde{S}_J \setminus I, \gamma^\vee)(-) \circ T(p'_{IJ, \gamma^\vee})(-)), E_{zar}(\Omega_{/Y \times \tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& e'(Y \times \tilde{S}_J)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_J}^* \Gamma_{T_I}^\vee \tilde{g}_J^* L(i_{J*} j_J^* F)), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})).
\end{aligned}$$

We then have then the following lemma :

Lemma 13. (i) *The morphism in $C(\text{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$*

$$\begin{aligned}
(T^{q,\gamma}(D_{gI})(j_I^* F)) : (\Gamma_{T_I}^\vee L \tilde{g}_I^* i_{I*} j_I^* F, T^q(D_{IJ})(j_I^* F) \circ T(T_I/T_I \times \tilde{S}_J \setminus I, \gamma^\vee)(-) \circ T(p'_{IJ, \gamma^\vee})(-)) \\
\rightarrow (Li'_{I*} j_I^* g^* F, T^q(D'_{IJ})(j_I^* g^* F))
\end{aligned}$$

is an equivalence (\mathbb{A}^1, et) local.

(ii) *The maps $\mathcal{H}om((T^{q,\gamma}(D_{gI})(j_I^* F)), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))$ induce a filtered quasi-isomorphism in $C_{\mathcal{D}fil}(T/(Y \times \tilde{S}_I))$*

$$\begin{aligned}
& (\mathcal{H}om(\hat{R}_{Y \times \tilde{S}_I}^{CH} (T^{q,\gamma}(D_{gI})(j_I^* F)), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}] : \\
& (e'(-)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}], \tilde{g}_J^{*,\gamma} u_{IJ}^q(F)) \\
& \rightarrow (e'(-)_* \mathcal{H}om(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* L(i'_{I*} j_I^* g^* F)), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}], u_{IJ}^q(g^* F))
\end{aligned}$$

(iii) *The maps $T(\tilde{g}_I, \Omega_{/Y \times \tilde{S}_I}^{\Gamma, pr})(-)$ (see definition 131), induce a morphism in $C_{\mathcal{D}fil}(T/(Y \times \tilde{S}_I))$*

$$\begin{aligned}
& T(\tilde{g}_I, \Omega_{/Y \times \tilde{S}_I}^{\Gamma, pr})(-)[d_{YI}] : \\
& (\tilde{g}_I^{*mod} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* (L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}], \tilde{g}_J^{*mod} u_{IJ}^q(F)) \\
& \rightarrow (e'(-)_* \mathcal{H}om(\tilde{g}_I^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[d_{YI}], \tilde{g}_J^* u_{IJ}^q(F)).
\end{aligned}$$

Proof. (i): Follows from theorem 16.

(ii): These morphism induce a morphism in $C_{\mathcal{D}fil}(T/(Y \times \tilde{S}_I))$ by construction. The fact that this morphism is an filtered equivalence Zariski local follows from (i) and proposition 114.

(iii): These morphism induce a morphism in $C_{\mathcal{D}fil}(T/(Y \times \tilde{S}_I))$ by construction. \square

Definition 138. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and ps the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$ Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g_{T_I} : T_I \rightarrow S_I$. Let $M \in \text{DA}_c(S)^-$ and $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}_S^1, et)(F, W)$. Then, $D(\mathbb{A}_T^1, et)(g^* F) = g^* M$ and there exist $(F', W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$*

and an equivalence (\mathbb{A}^1, et) local $e : g^*(F, W) \rightarrow (F', W)$ such that $D(\mathbb{A}_T^1, et)(F', W) = (g^*M, W)$. Denote for short $d_{YI} := -d_Y - d_{\tilde{S}_I}$. We have, using definition 131 and definition 136(i), by lemma 13, the canonical map in $\pi_T(D(MHM(T))) \subset D_{\mathcal{D}(1,0)fil}(T/(Y \times \tilde{S}_I))$

$$\begin{aligned}
& T(g, \mathcal{F}^{FDR})(M) : g_{Hdg}^{\hat{*}mod} \iota_S^{-1} \mathcal{F}_S^{FDR}(M) := \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1} (\tilde{g}_I^{*mod} (e'_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* (L(i_{I*} j_I^*(F, W))), E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))) [d_{YI}], \tilde{g}_J^{*mod} u_{IJ}^q(F, W))) \\
& \xrightarrow{(T(\tilde{g}_I, \Omega_{/\tilde{S}_I}^{\Gamma, pr})(-))} \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1} (e'_* \mathcal{H}om^\bullet(\tilde{g}_I^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) [d_{YI}], \tilde{g}_J^* u_{IJ}^q(F, W)) \\
& \xrightarrow{\mathcal{H}om(T(\tilde{g}_I, \hat{R}^{CH})(-)^{-1}, -)} \\
& \Gamma_T^{\vee, Hdg} \iota_T^{-1} (e'_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*} j_I^*(F, W))), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) [d_{YI}], \tilde{g}_J^* u_{IJ}^q(F, W)) \\
& \xrightarrow{T(\Gamma_T^{\vee, Hdg}, \Omega_{/\tilde{S}_I}^{\Gamma, pr})(F, W)} \\
& \iota_T^{-1} (e'_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^*(F, W))), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) [d_{YI}], \tilde{g}_J^{*, \gamma} u_{IJ}^q(F, W)) \\
& \xrightarrow{(\mathcal{H}om(\hat{R}_{Y \times \tilde{S}_I}^{CH}(T^{q, \gamma}(D_{gI})(j_I^*(F, W))), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) [d_{YI}])} \\
& \iota_T^{-1} (e'_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* L(i'_{I*} j'_I{}^*(F, W))), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) [d_{YI}], u_{IJ}^q(g^*(F, W))) \\
& \xrightarrow{\mathcal{H}om(\hat{R}_{Y \times \tilde{S}_I}^{CH}(L i'_{I*} j'_I{}^*(e)), -)} \\
& \iota_T^{-1} (e'_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* L(i'_{I*} j'_I{}^*(F', W))), E_{zar}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})) [d_{YI}], u_{IJ}^q(F', W)) \\
& \xrightarrow{=} \mathcal{F}_T^{FDR}(g^*M)
\end{aligned}$$

Definition 139. • Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \text{DA}_c(X)$, the following transformation map in $\pi_S(D(MHM(S)))$

$$\begin{aligned}
T_*(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_S^{FDR}(Rf_*M) & \xrightarrow{\text{ad}(f_{Hdg}^{\hat{*}mod}, Rf_*^{Hdg})(-)} Rf_*^{Hdg} f_{Hdg}^{\hat{*}mod} \mathcal{F}_S^{FDR}(Rf_*M) \\
& \xrightarrow{T(f, \mathcal{F}^{FDR})(Rf_*M)} Rf_*^{Hdg} \mathcal{F}_X^{FDR}(f^*Rf_*M) \xrightarrow{\mathcal{F}_X^{FDR}(\text{ad}(f^*, Rf_*)(M))} Rf_*^{Hdg} \mathcal{F}_X^{FDR}(M)
\end{aligned}$$

Clearly, for $p : Y \times S \rightarrow S$ a projection with $Y \in \text{PSmVar}(\mathbb{C})$, we have, for $M \in \text{DA}_c(Y \times S)$, $T_*(p, \mathcal{F}^{FDR})(M) = T_!(p, \mathcal{F}^{FDR})(M)[d_Y]$

- Let $S \in \text{Var}(\mathbb{C})$. Let $Y \in \text{SmVar}(\mathbb{C})$ and $p : Y \times S \rightarrow S$ the projection. We have then, for $M \in \text{DA}(Y \times S)$ the following transformation map in $\pi_S(D(MHM(S)))$

$$\begin{aligned}
T_!(p, \mathcal{F}^{FDR})(M) : p_!^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(M) & \xrightarrow{\mathcal{F}_{Y \times S}^{FDR}(\text{ad}(Lp_#, p^*)(M))} Rp_!^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(p^*Lp_#M) \\
& \xrightarrow{T(p, \mathcal{F}^{FDR})(Lp_#(M, W))} Rp_!^{Hdg} p^{\hat{*}mod}[-] \mathcal{F}_S^{FDR}(Lp_#M) \xrightarrow{T(p^{*mod}, p^{\hat{*}mod})(-)} p_!^{Hdg} p^{*mod}[-] \\
& \xrightarrow{\mathcal{F}_S^{FDR}(Lp_#M)} \xrightarrow{\text{ad}(Rp_!^{Hdg}, p^{*mod}[-])(\mathcal{F}_S^{FDR}(Lp_#M))} \mathcal{F}_S^{FDR}(Lp_#M)
\end{aligned}$$

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then, using the second point, for $M \in \text{DA}(X)$ the following transformation map in $\pi_S(D(MHM(S)))$

$$\begin{aligned}
T_!(f, \mathcal{F}^{FDR})(M) : Rp_!^{Hdg} \mathcal{F}_X^{FDR}(M, W) & := Rp_!^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(l_*M) \\
& \xrightarrow{T_!(p, \mathcal{F}^{FDR})(l_*M)} \mathcal{F}_S^{FDR}(Lp_#l_*M) \xrightarrow{=} \mathcal{F}_S^{FDR}(Rf_!M)
\end{aligned}$$

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, using the third point, for $M \in \text{DA}(S)$, the following transformation map in $\pi_X(D(\text{MHM}(X)))$

$$\begin{aligned} T^!(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_X^{FDR}(f^!M) &\xrightarrow{\text{ad}(Rf_!^{Hdg}, f^{*mod}_{Hdg})(\mathcal{F}_X^{FDR}(f^!M))} f^{*mod}_{Hdg} Rf_!^{Hdg} \mathcal{F}_X^{FDR}(f^!M) \\ &\xrightarrow{T_!(p_S, \mathcal{F}^{FDR})(\mathcal{F}^{FDR}(f^!M))} f^{*mod}_{Hdg} \mathcal{F}_S^{FDR}(Rf_!f^!M) \xrightarrow{\mathcal{F}_S^{FDR}(\text{ad}(Rf_!, f^!)(M))} f^{*mod}_{Hdg} \mathcal{F}_S^{FDR}(M) \end{aligned}$$

- Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, using the preceding point, denoting $\Delta_S : S \hookrightarrow S$ the diagonal closed embedding and $p_1 : S \times S \rightarrow S$, $p_2 : S \times S \rightarrow S$ the projections, for $M, N \in \text{DA}(S)$ and $(F, W), (G, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$ and $(N, W) = D(\mathbb{A}^1, et)(G, W)$, the following transformation map in $\pi_S(D(\text{MHM}(S)))$

$$\begin{aligned} T(\mathcal{F}_S^{FDR}, \otimes)(M, N) : \mathcal{F}_S^{FDR}(M) \otimes_{O_S}^{Hdg} \mathcal{F}_S^{FDR}(N) &:= \Delta_S^{!Hdg}(p_1^{*mod} \mathcal{F}_S^{FDR}(M) \otimes_{O_{S \times S}} p_2^{*mod} \mathcal{F}_S^{FDR}(N)) \\ &\xrightarrow{T^!(p_1, \mathcal{F}_S^{FDR})(M) \otimes T^!(p_2, \mathcal{F}_S^{FDR})(N)} \Delta_S^{!Hdg}(\mathcal{F}_{S \times S}^{FDR}(p_1^!M) \otimes_{O_{S \times S}} \mathcal{F}_{S \times S}^{FDR}(p_2^!N)) \\ &\xrightarrow{(T(\otimes, \Omega)(\hat{R}^{CH}(\rho_{\tilde{S}_1 \times \tilde{S}_2}^* L(i_1 \times i_2) * (j_1 \times j_2)^* p_1^! F[2d_S]), \hat{R}^{CH}(\rho_{\tilde{S}_1 \times \tilde{S}_2}^* L(i_1 \times i_2) * (j_1 \times j_2)^* p_2^! F[2d_S])))} \\ &\Delta_S^{!Hdg}(\mathcal{F}_{S \times S}^{FDR}(p_1^!M \otimes p_2^!N)) \xrightarrow{T^!(\Delta_S, \mathcal{F}^{FDR})(p_1^!M \otimes p_2^!N)} \mathcal{F}_S^{FDR}(\Delta_S^!(p_1^!M \otimes p_2^!N)) = \mathcal{F}_S^{FDR}(M \otimes N) \end{aligned}$$

where the last equality follows from the equality in $\text{DA}(S)$

$$\Delta_S^!(p_1^!M \otimes p_2^!N) = \Delta_S^!p_1^!M \otimes \Delta_S^!p_2^!N = M \otimes N$$

Proposition 119. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{p_S} S$ with $Y_2 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$ such that there exists a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_S} S$, with $Y_1 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} f' : X_T & \longrightarrow & Y_1 \times T & \longrightarrow & T \\ & \searrow & \downarrow & \searrow & \downarrow \\ & & Y_1 \times X & \longrightarrow & Y_1 \times Y_2 \times S \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ & & Y_1 \times S & \longrightarrow & S \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ f : X & \longrightarrow & Y_1 \times S & \longrightarrow & S \end{array}$$

Take a smooth compactification $\bar{Y}_1 \in \text{PSmVar}(\mathbb{C})$ of Y_1 , denote $\bar{X}_I \subset \bar{Y}_1 \times \tilde{S}_I$ the closure of X_I , and $Z_I := \bar{X}_I \setminus X_I$. Consider $F(X/S) := p_{S, \#} \Gamma_X^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S) \in C(\text{Var}(\mathbb{C})^{sm}/S)$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/T)$

$$\begin{aligned} T(f, g, F(X/S)) : g^*F(X/S) &:= g^*p_{S, \#} \Gamma_X^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S) \xrightarrow{\sim} \\ &p_{T, \#} \Gamma_{X_T}^\vee \mathbb{Z}(Y_1 \times T/Y_1 \times T) =: F(X_T/T). \end{aligned}$$

which gives in $\text{DA}(T)$ the isomorphism $T(f, g, F(X/S)) : g^*M(X/S) \xrightarrow{\sim} (X_T/T)$. Then the following diagram in $\pi_T(D(\text{MHM}(T))) \subset \mathcal{D}_{\mathcal{D}(1,0)fil}(T/(Y_2 \times \tilde{S}_I))$, where the horizontal maps are given by proposition

116, commutes

$$\begin{array}{ccc}
g_{Hdg}^{\star mod} \iota_S^{-1} \mathcal{F}_S^{FDR}(M(X/S)) & \xrightarrow{g_{Hdg}^{\star mod} I(X/S)} & g_{Hdg}^{\star mod} Rf_!^{Hdg}(\Gamma_{X_I}^{\vee, Hdg}(O_{Y_1 \times \tilde{S}_I}, F_b)(d_{Y_1})[2d_{Y_1}], x_{IJ}(X/S)) \\
\downarrow 'T(g, \mathcal{F}^{FDR})(M(X/S)) & & \downarrow T(p_{\tilde{S}_I}, \gamma^{\vee, Hdg})(-) \\
\iota_T^{-1} \mathcal{F}_T^{FDR}(M(X_T/T)) & \xrightarrow{I(X_T/T)} & Rf_!^{Hdg} g_{Hdg}^{\star mod}(\Gamma_{X_I}^{\vee, Hdg}(O_{Y_1 \times \tilde{S}_I}, F_b)(d_{Y_1})[2d_{Y_1}], x_{IJ}(X/S)) \\
& & \downarrow T(p_{Y_1 \times Y_2 \times \tilde{S}_I, Hdg}, p_{Y_1 \times Y_2 \times \tilde{S}_I, Hdg}^{\star mod})(-) \\
& & Rf_!^{Hdg}(\Gamma_{X_T}^{\vee, Hdg}(O_{Y_2 \times Y_1 \times \tilde{S}_I}, F_b)(d_{Y_{12}})[2d_{Y_{12}}], x_{IJ}(X_T/T)).
\end{array}$$

with $d_{Y_{12}} = d_{Y_1} + d_{Y_2}$.

Proof. Follows immediately from definition. \square

Proposition 120. Let $S \in \text{Var}(\mathbb{C})$. Let $Y \in \text{SmVar}(\mathbb{C})$ and $p : Y \times S \rightarrow S$ the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i^o : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, we denote by $S_I = \cap_{i \in I} S_i$, $j_I^o : S_I \hookrightarrow S$ and $j_I : Y \times S_I \hookrightarrow Y \times S$ the open embeddings. We then have closed embeddings $i_I : Y \times S_I \hookrightarrow Y \times \tilde{S}_I$. and we denote by $p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ the projections. Let $f' : X' \rightarrow Y \times S$ a morphism, with $X' \in \text{Var}(\mathbb{C})$ such that there exists a factorization $f' : X' \xrightarrow{l'} Y' \times Y \times S \xrightarrow{p'} Y \times S$ with $Y' \in \text{SmVar}(\mathbb{C})$, l' a closed embedding and p' the projection. Denoting $X'_I := f'^{-1}(Y \times S_I)$, we have closed embeddings $i'_I : X'_I \hookrightarrow Y' \times Y \times \tilde{S}_I$. Consider

$$F(X'/Y \times S) := p_{Y \times S, \#} \Gamma_{X'}^{\vee} \mathbb{Z}(Y' \times Y \times S/Y' \times Y \times S) \in C(\text{Var}(\mathbb{C})^{sm}/Y \times S)$$

and $F(X'/S) := p_{\#} F(X'/Y \times S) \in C(\text{Var}(\mathbb{C})^{sm}/S)$, so that $Lp_{\#} M(X'/Y \times S)[-2d_Y] =: M(X'/S)$. Then, the following diagram in $\pi_S(D(MHM(S))) \subset D_{\mathcal{D}(1,0)fil}(S/(Y \times \tilde{S}_I))$, where the vertical maps are given by proposition 116, commutes

$$\begin{array}{ccc}
Rp^{Hdg}! \mathcal{F}_{Y \times S}^{FDR}(M(X'/Y \times \tilde{S}_I)) & \xrightarrow{T_1(p, \mathcal{F}^{FDR})(M(X'/Y \times S))} & \mathcal{F}_S^{FDR}(M(X'/S)) \\
\uparrow T(p_{Hdg}^{\star mod}, p_{Hdg}^{\star mod})(-) \circ Rp^{Hdg}!(I(X'/Y \times S)) & & \uparrow I(X'/S) \\
Rp^{Hdg}! Rf_!^{Hdg} f'^{\star mod} \mathbb{Z}_{Y \times S}^{Hdg} & \xrightarrow{=} & Rf_!^{Hdg} f_{Hdg}^{\star mod} \mathbb{Z}_S^{Hdg}
\end{array}$$

Proof. Immediate from definition. \square

Proposition 121. Let $f_1 : X_1 \rightarrow S$, $f_2 : X_2 \rightarrow S$ two morphism with $X_1, X_2, S \in \text{Var}(\mathbb{C})$. Assume that there exist factorizations $f_1 : X_1 \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $f_2 : X_2 \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$ with $Y_1, Y_2 \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S the projections. We have then the factorization

$$f_{12} := f_1 \times f_2 : X_{12} := X_1 \times_S X_2 \xrightarrow{l_1 \times l_2} Y_1 \times Y_2 \times S \xrightarrow{p_S} S$$

Let $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have then the following commutative diagram in $\pi_S(DMHM(S)) \subset D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$ where the vertical maps are given by proposition 116

$$\begin{array}{ccc}
\mathcal{F}_S^{FDR}(M(X_1/S)) \otimes_{O_S}^{L[-]} \mathcal{F}_S^{FDR}(M(X_2/S)) & \xrightarrow{I(X_1/S) \otimes I(X_2/S)} & Rf_{1!}^{Hdg}(\Gamma_{X_{1I}}^{\vee, Hdg}(O_{Y_1 \times \tilde{S}_I}, F_b)(d_2)[2d_1], x_{IJ}(X_1/S)) \otimes_{O_S} \\
\downarrow T(\mathcal{F}_S^{FDR}, \otimes)(M(X_1/S), M(X_2/S)) & & \downarrow (E^{w_{(Y_1 \times \tilde{S}_I, Y_2 \times \tilde{S}_I)/\tilde{S}_I}}) \\
\mathcal{F}_S^{FDR}(M(X_1/S) \otimes M(X_2/S) = M(X_1 \times_S X_2/S)) & \xrightarrow{I(X_{12}/S)} & Rf_{12!}^{Hdg}(\Gamma_{X_{1I} \times_S X_{2I}}^{\vee, Hdg}(O_{Y_1 \times Y_2 \times \tilde{S}_I}, F_b)(d_{12})[2d_{12}], x_{IJ}(X_1/S))
\end{array}$$

with $d_1 = d_{Y_1}$, $d_2 = d_{Y_2}$ and $d_{12} = d_{Y_1} + d_{Y_2}$.

Proof. Immediate from definition. \square

Theorem 36. (i) Let $g : T \rightarrow S$ a morphism, with $S, T \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $M \in \text{DA}_c(S)$. Then map in $\pi_T(D(\text{MHM}(T)))$

$$T(g, \mathcal{F}^{FDR})(M) : g_{\text{Hdg}}^{\text{mod}} \mathcal{F}_S^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_T^{FDR}(g^* M)$$

given in definition 138 is an isomorphism.

(ii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \text{DA}_c(X)$, the map given in definition 139

$$T_!(f, \mathcal{F}^{FDR})(M) : Rf_!^{\text{Hdg}} \mathcal{F}_X^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_S^{FDR}(Rf_! M)$$

is an isomorphism in $\pi_S(D(\text{MHM}(S)))$.

(iii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \text{DA}_c(X)$, the map given in definition 139

$$T_*(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_S^{FDR}(Rf_* M) \xrightarrow{\sim} Rf_*^{\text{Hdg}} \mathcal{F}_X^{FDR}(M)$$

is an isomorphism in $\pi_S(D(\text{MHM}(S)))$.

(iv) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \text{DA}_c(S)$, the map given in definition 139

$$T^!(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_X^{FDR}(f^! M) \xrightarrow{\sim} f_{\text{Hdg}}^{\text{mod}} \mathcal{F}_S^{FDR}(M)$$

is an isomorphism in $\pi_X(D(\text{MHM}(X)))$.

(v) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow S$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$, the map in $\pi_S(D(\text{MHM}(S)))$

$$T(\mathcal{F}_S^{FDR}, \otimes)(M, N) : \mathcal{F}_S^{FDR}(M) \otimes_{O_S}^{\text{Hdg}} \mathcal{F}_S^{FDR}(N) \xrightarrow{\sim} \mathcal{F}_S^{FDR}(M \otimes N)$$

given in definition 139 is an isomorphism.

Proof. (i): Follows from proposition 119 and proposition 116.

(ii): By proposition 120 and proposition 116, for $S \in \text{Var}(\mathbb{C})$, $Y \in \text{SmVar}(\mathbb{C})$, $p : Y \times S \rightarrow S$ the projection and $M \in \text{DA}_c(Y \times S)$,

$$T_!(p, \mathcal{F}^{FDR})(M) : Rp_!^{\text{Hdg}} \mathcal{F}_{Y \times S}^{FDR}(M) \rightarrow \mathcal{F}_S^{FDR}(Rp_! M)$$

is an isomorphism.

(iii): Consider first an open embedding $n : S^o \hookrightarrow S$ with $S \in \text{Var}(\mathbb{C})$ so that there exist a closed embedding $i : S \hookrightarrow \tilde{S}$ with $\tilde{S} \in \text{SmVar}(\mathbb{C})$. Then, since

$$n^* : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S^o)$$

is surjective, $n^* : \mathrm{DA}(S) \rightarrow \mathrm{DA}(S^o)$ is surjective. Denote by $i : Z = S \setminus S^o \hookrightarrow S$ the complementary closed embedding. By [1], $\mathrm{DA}_c(S)$ is generated by motives of the form

$$\begin{aligned} \mathrm{DA}_c(S) = \langle M(X'/S) = f'_* E(\mathbb{Z}_{X'}), f' : X' \rightarrow S \text{ proper with } X' \in \mathrm{SmVar}(\mathbb{C}), \\ \text{s.t. } f'^{-1}(Z) = X' \text{ or } f'^{-1}(Z) = \cup D_i = D \subset X' \rangle. \end{aligned}$$

If $f'^{-1}(Z) = X'$, $n^* M(X'/S) = 0$. So let consider the case $f'^{-1}(Z) = \cup_{i=1}^l D_i = D \subset X'$ is a normal crossing divisor. Denote $f'_D : f'_D : D \rightarrow Z$, $D_I = \cap_{i \in I} D_i$ and $i'_I : D_I \hookrightarrow X'$, $n' : X'^o := X' \setminus D \hookrightarrow X'$ the complementary open embedding and $f'^o : f'_{|X'^o} : X'^o \rightarrow S^o$. Denote $L = [1, \dots, l]$. We have then a generalized distinguish triangle in $\mathrm{DA}(X')$

$$\begin{aligned} a(n', i') : n'_* n'^* E_{et}(\mathbb{Z}_{X'}) &\xrightarrow{\sim} \mathrm{Cone}(\gamma_D(-) : \Gamma_D E_{et}(\mathbb{Z}_{X'}) \rightarrow E_{et}(\mathbb{Z}_{X'})) \\ &\xrightarrow{\sim} \mathrm{Cone}(\Gamma_{D_L} E_{et}(\mathbb{Z}_{X'}) \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l E_{et}(\mathbb{Z}_{X'})) \\ &\xrightarrow{\sim} \mathrm{Cone}(i'_{L*} i'_L E_{et}(\mathbb{Z}_{X'}) \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l i'_{i*} i'_i E_{et}(\mathbb{Z}_{X'}) \xrightarrow{\mathrm{ad}(i'_{i*}, i'_i)(E_{et}(\mathbb{Z}_{X'}))} E_{et}(\mathbb{Z}_{X'})) \\ &\xrightarrow{\sim} \mathrm{Cone}(i'_{L*} \mathbb{Z}_{D_L}[-l] \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l i'_{i*} \mathbb{Z}_{D_i}[-1] \rightarrow \mathbb{Z}_{X'}) \end{aligned}$$

where the first isomorphism is the image of an homotopy equivalence by definition, the second one is the image of an homotopy equivalence by definition-proposition 4(ii), the third one follows by the localization property (see section 3, theorem 16) and the last one follows from purity since the D_I and X' are smooth (see section 3, theorem 16). Similarly, we have a generalized distinguish triangle in $D(\mathrm{MHM}(X'))$

$$\begin{aligned} a^{mod}(n', i') : n'_*{}^{Hdg} n'^*(O_{X'}, F_b) &\xrightarrow{\sim} \mathrm{Cone}(\gamma_D^{Hdg}(O_{X'}, F_b) : \Gamma_D^{Hdg}(O_{X'}, F_b) \rightarrow (O_{X'}, F_b)) \\ &\xrightarrow{\sim} \mathrm{Cone}(\Gamma_{D_L}^{Hdg}(O_{X'}, F_b) \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l \Gamma_{D_i}^{Hdg}(O_{X'}, F_b) \xrightarrow{\oplus_i \gamma_{D_i}^{Hdg}(-)} (O_{X'}, F_b)) \\ &\xrightarrow{\sim} \mathrm{Cone}(i'_{L*}{}^{mod}(O_{D_L}, F_b)[-l] \rightarrow \cdots \rightarrow \bigoplus_{i=1}^l i'_{i*}{}^{mod}(O_{D_i}, F_b)[-1] \\ &\quad \xrightarrow{\mathrm{ad}(i'_{i*}{}^{mod}, i'_i{}^\sharp)(O_{X'}, F_b)} \cdots \rightarrow (O_{X'}, F_b)) \end{aligned}$$

where the first isomorphism is the image of an homotopy equivalence by definition, the second one is the image of an homotopy equivalence by definition-proposition 19, and the third one follows by the localization property of mixed Hodge modules (see section 5). Consider $n^* M(X'/S) = M(X'^o/S^o)$. We have then the following commutative diagram in $\pi_S(\mathrm{MHM}(S))$

$$\begin{array}{ccc} \mathcal{F}_S^{FDR}(Rn_* n^* M(X'/S)) = \mathcal{F}_S^{FDR}(Rn_* n^* Rf'_{*} \mathcal{F}_{X'}^{FDR}) \xrightarrow{T_*(n, \mathcal{F}^{FDR})(n^* M(X'/S))} n_*{}^{Hdg} \mathcal{F}_{S^o}^{FDR}(n^* M(X'/S)) = n_*{}^{Hdg} \mathcal{F}_{S^o}^{FDR}(n^* Rf'_{*} \mathbb{Z}_{X'}) & & \\ \mathcal{F}_S^{FDR}(Rn_* T(n, f')(\mathbb{Z}_{X'})) \downarrow & & \downarrow n_*{}^{Hdg} \mathcal{F}_{S^o}^{FDR}(T(n, f')(\mathbb{Z}_{X'})) \\ \mathcal{F}_S^{FDR}(Rn_* Rf'_o n'^* \mathbb{Z}_{X'}) = \mathcal{F}_S^{FDR}(Rf'_o Rn'_* n'^* \mathbb{Z}_{X'}) \xrightarrow{T_*(n, \mathcal{F}^{FDR})(M(X'^o/S))} n_*{}^{Hdg} \mathcal{F}_{S^o}^{FDR}(Rf'_o n'^* \mathbb{Z}_{X'}) & & \\ T_*(f', \mathcal{F}^{FDR})(Rn'_* n'^* \mathbb{Z}_{X'}) \downarrow & & \downarrow n_*{}^{Hdg} T_*(f'^o, \mathcal{F}^{FDR})(n'^* \mathbb{Z}_{X'}) \\ Rf'_{*}{}^{Hdg} \mathcal{F}_{X'}^{FDR}(Rn'_* n'^* \mathbb{Z}_{X'}) \xrightarrow{T_*(n, \mathcal{F}^{FDR})(n'^* \mathbb{Z}_{X'})} n_*{}^{Hdg} Rf'_{*}{}^{Hdg} \mathcal{F}_{X'^o}^{FDR}(n'^* \mathbb{Z}_{X'}) = Rf'_{*}{}^{Hdg} n'_*{}^{Hdg} n'^* \mathbb{Z}_{X'}^{Hdg} & & \\ Rf'_{*}{}^{Hdg} \mathcal{F}_{X'}^{FDR}(a(n', i')) \downarrow & & \downarrow Rf'_{*}{}^{Hdg} a^{mod}(n', i') \\ Rf'_*{}^{Hdg} \mathcal{F}_{X'}^{FDR}(\mathrm{Cone}(i'_{L*} \mathbb{Z}_{D_L}[-l] \rightarrow \cdots \rightarrow \mathbb{Z}_{X'})) \xrightarrow{(T_*(i'_{i*}, \mathcal{F}^{FDR})(-) \circ T(i'_{i*}, \mathcal{F}^{FDR})(-))} Rf'_*{}^{Hdg} \mathrm{Cone}(i'_{L*}{}^{mod}(O_{D_L}, F_b)[-l] \rightarrow \cdots \rightarrow (O_{X'}, F_b)) & & \end{array}$$

Since all the morphism involved are isomorphisms, $T_*(n, \mathcal{F}^{FDR})(n^*M(X'/S))$ is an isomorphism. Hence, $T_*(n, \mathcal{F}^{FDR})(M)$ is an isomorphism for all $M \in \text{DA}(S^o)$. Consider now the case of a general morphism $f : X \rightarrow S$, $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective, which factors through $f : X \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with some $Y \in \text{SmVar}(\mathbb{C})$. By definition, for $M \in \text{DA}_c(X)$

$$\begin{aligned} T_*(f, \mathcal{F}^{FDR})(M) &: \mathcal{F}_S^{FDR}(Rf_*M) = \mathcal{F}_S^{FDR}(Rp_{S*}l_*M) \\ &\xrightarrow{T_*(p_S, \mathcal{F}^{FDR})(l_*M)} Rp_{S*}^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(l_*M) =: Rf_*^{Hdg} \mathcal{F}_X^{FDR}(M). \end{aligned}$$

Hence, we have to show that for $S \in \text{Var}(\mathbb{C})$, $Y \in \text{SmVar}(\mathbb{C})$, $p : Y \times S \rightarrow S$ the projection, and $M \in \text{DA}_c(Y \times S)$,

$$T_*(p, \mathcal{F}^{FDR})(M) : \mathcal{F}_S^{FDR}(Rp_*M) \rightarrow Rp_*^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(M)$$

is an isomorphism. Take a smooth compactification $\bar{Y} \in \text{PSmVar}(\mathbb{C})$ of Y . Denote by $n_0 : Y \hookrightarrow \bar{Y}$ and $n := n_0 \times I_S : Y \times S \hookrightarrow \bar{Y} \times S$ the open embeddings and by $\bar{p} : \bar{Y} \times S \rightarrow S$ the projection. We have $p = \bar{p} \circ n : Y \times S \rightarrow S$, which gives the factorization

$$\begin{aligned} T_*(p, \mathcal{F}^{FDR})(M) &: \mathcal{F}_S^{FDR}(Rp_*M) = \mathcal{F}_S^{FDR}(R\bar{p}_*Rn_*M) \xrightarrow{T_*(\bar{p}, \mathcal{F}^{FDR})(Rn_*M)} R\bar{p}_*^{Hdg} \mathcal{F}_{\bar{Y} \times S}^{FDR}(Rn_*M) \\ &\xrightarrow{T_*(n, \mathcal{F}^{FDR})(M)} R\bar{p}_*^{Hdg} n_*^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(M) = Rp_*^{Hdg} \mathcal{F}_{Y \times S}^{FDR}(M). \end{aligned}$$

By the open embedding case $T_*(n, \mathcal{F}^{FDR})(M)$ is an isomorphism. On the other hand, since \bar{p} is proper, $T_*(\bar{p}, \mathcal{F}^{FDR})(Rn_*M) = T_1(\bar{p}, \mathcal{F}^{FDR})(Rn_*M)$ is an isomorphism by (i).

(iv): Denote by $n : Y \times S \setminus X \hookrightarrow Y \times S$ the complementary open embedding. We have, for $M \in \text{DA}_c(S)$, the factorization

$$\begin{aligned} T^!(f, \mathcal{F}^{FDR})(M) &: \mathcal{F}_X^{FDR}(f^!M) = \mathcal{F}_{Y \times S}^{FDR}(l_*l^!p_S^!M) \xrightarrow{\mathcal{F}_{Y \times S}^{FDR}(a(n,l))} \mathcal{F}_{Y \times S}^{FDR}(\text{Cone}(p_S^!M \rightarrow Rn_*n^*p_S^!M)[-1]) \\ &\xrightarrow{\cong} \text{Cone}(\mathcal{F}_{Y \times S}^{FDR}(p_S^!M) \rightarrow \mathcal{F}_{Y \times S}^{FDR}(Rn_*n^*p_S^!M))[-1] \\ &\xrightarrow{(T(n, \mathcal{F}^{FDR})(p_S^!M) \circ T^!(p_S, \mathcal{F}^{FDR}(M)), T^!(p_S, \mathcal{F}^{FDR}(M)))} \\ &\text{Cone}(p_S^{*mod[-]} \mathcal{F}_S^{FDR}(M) \rightarrow n_*^{Hdg} n^* p_S^{*mod[-]} \mathcal{F}_S^{FDR}(M))[-1] \xrightarrow{\simeq} f_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M). \end{aligned}$$

By (ii), $T(n, \mathcal{F}^{FDR})(p_S^!M)$ is an isomorphism. On the other hand, since p_S is a smooth morphism, $T^!(p_S, \mathcal{F}^{FDR}(M)) = T(p_S, \mathbb{D}\mathcal{F}^{FDR}(M))[d_Y]$; hence, $T^!(p_S, \mathcal{F}^{FDR}(M))$ is an isomorphism by (i). \square

Lemma 14. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $M \in \text{DA}_c(S)^-$ and $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}_S^1, et)(F, W)$. Then, $g^!M = L\mathbb{D}_S g^* L\mathbb{D}_S M$, $D(\mathbb{A}_T^1, et)(g^* \mathbb{D}_S L F) = g^* L\mathbb{D}_S M$ and there exist $(F', W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ and an equivalence (\mathbb{A}^1, et) local $e : (F', W) \rightarrow (g^* \mathbb{D}_S L(F, W))$ such that $D(\mathbb{A}_T^1, et)(F', W) = g^* L\mathbb{D}_S(M, W)$ and, using definition 131 and definition 136(ii) and lemma 13, the map in $\pi_T(D(MHM(T))) \subset D_{\mathcal{D}(1,0)fil}(T/(Y \times \tilde{S}_I))$*

$$T^!(g, \mathcal{F}^{FDR})(M) : \mathcal{F}_T^{FDR}(g^!M) \rightarrow g_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M)$$

given in definition 139 is the inverse of the following map

$$\begin{aligned}
& T^{!, -1}(g, \mathcal{F}^{FDR})(M) : g_{Hdg}^{*mod} \iota_S^{-1} \mathcal{F}_S^{FDR}(M) \xrightarrow{=} \\
& (\Gamma_T^{Hdg} \iota_T^{-1} (\tilde{g}_I^{*mod} (e'_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* \mathbb{D}_S L(F, W))), E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))) [-d_{YI}], \tilde{g}_J^{*mod} u_{IJ}^q(\mathbb{D}_S L(F, W))) \\
& \xrightarrow{(T(\tilde{g}_I, \Omega_{\tilde{S}_I}^{\Gamma, pr})(-))} \\
& \Gamma_T^{Hdg} \iota_T^{-1} (e'_* \mathcal{H}om^\bullet(\tilde{g}_I^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* \mathbb{D}_S L(F, W))), E_{zar}(\Omega_{Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})), \tilde{g}_J^* u_{IJ}^q(\mathbb{D}_S L(F, W))) \\
& \xrightarrow{\mathcal{H}om(T(\tilde{g}_I, \hat{R}^{CH})(-)^{-1}, -)} \\
& \Gamma_T^{Hdg} \iota_T^{-1} (e'_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*} j_I^* \mathbb{D}_S L(F, W))), E_{zar}(\Omega_{Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))) [d_{YI}], \tilde{g}_J^* u_{IJ}^q \mathbb{D}_S L(F, W)) \\
& \xrightarrow{T(\Gamma_T^{Hdg}, \Omega_{\tilde{S}_I}^{\Gamma, pr})(F, W)^{-1}} \\
& \iota_T^{-1} (e'_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* L\Gamma_{T_I} E(\tilde{g}_I^* L(i_{I*} j_I^* \mathbb{D}_S L(F, W))), E_{zar}(\Omega_{Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))) [d_{YI}], \tilde{g}_J^{*, \gamma, d} u_{IJ}^q(F, W)) \\
& \xrightarrow{(\mathcal{H}om(\hat{R}_{Y \times \tilde{S}_I}^{CH}(\mathbb{D}_{Y \times \tilde{S}_I} T^{q, \gamma}(D_{gI})(j_I^* \mathbb{D}_S L(F, W))), E_{zar}(\Omega_{Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))) [d_{YI}]}^{-1}} \\
& \iota_T^{-1} (e'_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* L\mathbb{D}_{Y \times \tilde{S}_I} L(i'_{I*} j_I^* g^* \mathbb{D}_S L(F, W))), E_{zar}(\Omega_{Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))) [d_{YI}], u_{IJ}^{q, d}(g^* \mathbb{D}_S L(F, W))) \\
& \xrightarrow{\mathcal{H}om(\hat{R}_{Y \times \tilde{S}_I}^{CH}(\mathbb{D}_{Y \times \tilde{S}_I} L i'_{I*} j_I^*(e)), -)} \\
& \iota_T^{-1} (e'_* \mathcal{H}om(\hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* L\mathbb{D}_{Y \times \tilde{S}_I} L(i'_{I*} j_I^* g^* \mathbb{D}_S L(F, W))), E_{zar}(\Omega_{Y \times \tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))) [d_{YI}], u_{IJ}^{q, d}(L(F', W))) \\
& \xrightarrow{=} \mathcal{F}_T^{FDR}(g^! M)
\end{aligned}$$

We have the following proposition :

Proposition 122. *Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open covering such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_i} : T_i \rightarrow S_i$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then, $D(\mathbb{A}_T^1, et)(g^* F) = g^* M$. Then the following diagram in $D_{O_S \text{fil}, \mathcal{D}, \infty}(T/(Y \times \tilde{S}_I))$ commutes*

$$\begin{array}{ccc}
Rg^{*mod[-], \Gamma} \mathcal{F}_S^{GM}(\mathbb{L}\mathbb{D}_S M) & \xrightarrow{Rg^{*mod[-], \Gamma} T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M)} Rg^{*mod[-], \Gamma} \mathcal{F}_S^{FDR}(\hat{R}^{dg}(M)) & \xleftarrow{g_{Hdg}^{*mod} T(\mathcal{F}_S^{FDR}(M))} \mathcal{F}_S^{FDR}(M) \\
\downarrow T(g, \mathcal{F}^{GM})(\mathbb{L}\mathbb{D}_S M) & & \downarrow T^!(g, \mathcal{F}^{FDR})(M)^{-1} \\
\mathcal{F}_T^{GM}(g^* \mathbb{L}\mathbb{D}_S M = \mathbb{L}\mathbb{D}_T g^! M) & \xrightarrow{T(\mathcal{F}_T^{GM}, \mathcal{F}_T^{FDR})(g^* M)} & \mathcal{F}_T^{FDR}(g^! M)
\end{array}$$

Proof. Follows from lemma 14. □

We have the following easy proposition

Proposition 123. *Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following commutative diagram in $D_{O_S \text{fil}, \mathcal{D}, \infty}(S/(\tilde{S}_I))$*

$$\begin{array}{ccc}
\mathcal{F}_S^{GM}(\mathbb{L}\mathbb{D}_S M) \otimes_{\mathcal{O}_S}^L \mathcal{F}_S^{GM}(\mathbb{L}\mathbb{D}_S N) & \xrightarrow{T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) \otimes T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(N)} \mathcal{F}_S^{FDR}(M) \otimes_{\mathcal{O}_S}^{Hdg} \mathcal{F}_S^{FDR}(N) \\
\downarrow T(\mathcal{F}_S^{GM}, \otimes)(\mathbb{L}\mathbb{D}_S M, \mathbb{L}\mathbb{D}_S N) & & \downarrow T(\mathcal{F}_S^{FDR}, \otimes)(M, N) \\
\mathcal{F}_S^{GM}(\mathbb{L}\mathbb{D}_S(M \otimes N)) & \xrightarrow{T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M \otimes N)} & \mathcal{F}_S^{FDR}(M \otimes N)
\end{array}$$

Proof. Immediate from definition. □

6.2 The analytic filtered De Rahm realization functor

On $\text{AnSp}(\mathbb{C})$ the usual topology is equivalent to the etale topology since a morphism $r : U' \rightarrow U$ is etale (which means non ramified and flat, see section 2) if and only if for all $x \in U'$ there exist an open neighborhood $U'_x \subset U'$ of x such that r induces an isomorphism $r|_{U'_x} : U'_x \xrightarrow{\sim} r(U'_x)$. We note $\tau = et$ the etale topology.

6.2.1 The analytic Gauss-Manin filtered De Rham realization functor and its transformation map with pullbacks

Consider, for $S \in \text{AnSp}(\mathbb{C})$, the following composition of morphism in RCat (see section 2)

$$\tilde{e}(S) : (\text{AnSp}(\mathbb{C})/S, O_{\text{AnSp}(\mathbb{C})/S}) \xrightarrow{\rho_S} (\text{AnSp}(\mathbb{C})^{sm}/S, O_{\text{AnSp}(\mathbb{C})^{sm}/S}) \xrightarrow{e(S)} (S, O_S)$$

with, for $X/S = (X, h) \in \text{AnSp}(\mathbb{C})/S$,

- $O_{\text{AnSp}(\mathbb{C})/S}(X/S) := O_X(X)$,
- $(\tilde{e}(S)^* O_S(X/S) \rightarrow O_{\text{AnSp}(\mathbb{C})/S}(X/S)) := (h^* O_S \rightarrow O_X)$.

and $O_{\text{AnSp}(\mathbb{C})^{sm}/S} := \rho_{S*} O_{\text{AnSp}(\mathbb{C})/S}$, that is, for $U/S = (U, h) \in \text{AnSp}(\mathbb{C})^{sm}/S$, $O_{\text{AnSp}(\mathbb{C})^{sm}/S}(U/S) := O_{\text{AnSp}(\mathbb{C})/S}(U/S) := O_U(U)$

Definition 140. (i) For $S \in \text{Var}(\mathbb{C})$, we consider the complexes of presheaves

$$\Omega_{/S}^\bullet := \text{coker}(\Omega_{O_{\text{AnSp}(\mathbb{C})/S}/\tilde{e}(S)^* O_S} : \Omega_{\tilde{e}(S)^* O_S}^\bullet \rightarrow \Omega_{O_{\text{AnSp}(\mathbb{C})/S}}^\bullet) \in C_{O_S}(\text{AnSp}(\mathbb{C})/S)$$

which is by definition given by

- for X/S a morphism $\Omega_{/S}^\bullet(X/S) = \Omega_{X/S}^\bullet(X)$
- for $g : X'/S \rightarrow X/S$ a morphism,

$$\begin{aligned} \Omega_{/S}^\bullet(g) &:= \Omega_{(X'/X)/(S/S)}(X') : \Omega_{X/S}^\bullet(X) \rightarrow g^* \Omega_{X/S}(X') \rightarrow \Omega_{X'/S}^\bullet(X') \\ \omega &\mapsto \Omega_{(X'/X)/(S/S)}(X')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{X'}(X') \mapsto \omega(dg(\alpha))) \end{aligned}$$

(ii) For $S \in \text{AnSp}(\mathbb{C})$, we consider the complexes of presheaves

$$\Omega_{/S}^\bullet := \rho_{S*} \tilde{\Omega}_{/S}^\bullet = \text{coker}(\Omega_{O_{\text{AnSp}(\mathbb{C})^{sm}/S}/e(S)^* O_S} : \Omega_{e(S)^* O_S}^\bullet \rightarrow \Omega_{O_{\text{AnSp}(\mathbb{C})^{sm}/S}}^\bullet) \in C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$$

which is by definition given by

- for U/S a smooth morphism $\Omega_{/S}^\bullet(U/S) = \Omega_{U/S}^\bullet(U)$
- for $g : U'/S \rightarrow U/S$ a morphism,

$$\begin{aligned} \Omega_{/S}^\bullet(g) &:= \Omega_{(U'/U)/(S/S)}(U') : \Omega_{U/S}^\bullet(U) \rightarrow g^* \Omega_{U/S}(U') \rightarrow \Omega_{U'/S}^\bullet(U') \\ \omega &\mapsto \Omega_{(U'/U)/(S/S)}(U')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{U'}(U') \mapsto \omega(dg(\alpha))) \end{aligned}$$

Remark 12. For $S \in \text{AnSp}(\mathbb{C})$, $\Omega_{/S}^\bullet \in C(\text{AnSp}(\mathbb{C})/S)$ is by definition a natural extension of $\Omega_{/S}^\bullet \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$. However $\Omega_{/S}^\bullet \in C(\text{AnSp}(\mathbb{C})/S)$ does NOT satisfy cdh descent.

For a smooth morphism $h : U \rightarrow S$ with $S, U \in \text{AnSm}(\mathbb{C})$, the cohomology presheaves $H^n \Omega_{U/S}^\bullet$ of the relative De Rham complex

$$DR(U/S) := \Omega_{U/S}^\bullet := \text{coker}(h^* \Omega_S \rightarrow \Omega_U) \in C_{h^* O_S}(U)$$

for all $n \in \mathbb{Z}$, have a canonical structure of a complex of $h^* D_S^\infty$ modules given by the Gauss Manin connexion : for $S^o \subset S$ an open subset, $U^o = h^{-1}(S^o)$, $\gamma \in \Gamma(S^o, T_S)$ a vector field and $\hat{\omega} \in \Omega_{U/S}^p(U^o)^c$ a closed form, the action is given by

$$\gamma \cdot [\hat{\omega}] = [\iota(\tilde{\gamma}) \widehat{\partial \omega}],$$

$\omega \in \Omega_U^p(U^o)$ being a representative of $\hat{\omega}$ and $\tilde{\gamma} \in \Gamma(U^o, T_U)$ a relevation of γ (h is a smooth morphism), so that

$$DR(U/S) := \Omega_{U/S}^\bullet := \text{coker}(h^* \Omega_S \rightarrow \Omega_U) \in C_{h^* O_S, h^* \mathcal{D}^\infty}(U)$$

with this $h^* D_S^\infty$ structure. Hence we get $h_* \Omega_{U/S}^\bullet \in C_{O_S, \mathcal{D}^\infty}(S)$ considering this structure. Since h is a smooth morphism, $\Omega_{U/S}^p$ are locally free O_U modules.

The point (ii) of the definition 148 above gives the object in $\text{DA}(S)$ which will, for S smooth, represent the analytic Gauss-Manin De Rham realisation. It is the class of an explicit complex of presheaves on $\text{AnSp}(\mathbb{C})^{sm}/S$.

Proposition 124. *Let $S \in \text{Var}(\mathbb{C})$.*

(i) *For $U/S = (U, h) \in \text{AnSp}(\mathbb{C})^{sm}/S$, we have $e(U)_* h^* \Omega_{U/S}^\bullet = \Omega_{U/S}^\bullet$.*

(ii) *The complex of presheaves $\Omega_{U/S}^\bullet \in C_{O_S}(\text{AnSp}(\mathbb{C})^{sm}/S)$ is \mathbb{D}^1 homotopic. Note that however, for $p > 0$, the complexes of presheaves $\Omega_{U/S}^{\bullet \geq p}$ are NOT \mathbb{D}_S^1 local. On the other hand, $\Omega_{U/S}^\bullet$ admits transferts (recall that means $\text{Tr}(S)_* \text{Tr}(S)^* \Omega_{U/S}^p = \Omega_{U/S}^p$).*

(iii) *If S is smooth, we get $(\Omega_{U/S}^\bullet, F_b) \in C_{O_S \text{fil}, D_S^\infty}(\text{Var}(\mathbb{C})^{sm}/S)$ with the structure given by the Gauss Manin connexion. Note that however the D_S^∞ structure on the cohomology groups given by Gauss Main connexion does NOT comes from a structure of D_S^∞ module structure on the filtered complex of O_S module. The D_S structure on the cohomology groups satisfy a non trivial Griffithz transversality (in the non projection cases), whereas the filtration on the complex is the trivial one.*

Proof. Similar to the proof of proposition 105. □

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$. Consider the following commutative diagram in RCat :

$$\begin{array}{ccc} D(g, e) : (\text{AnSp}(\mathbb{C})^{sm}/T, O_{\text{AnSp}(\mathbb{C})^{sm}/T}) & \xrightarrow{P(g)} & (\text{AnSp}(\mathbb{C})^{sm}/S, O_{\text{AnSp}(\mathbb{C})^{sm}/S}) \\ \downarrow e(T) & & \downarrow e(S) \\ (T, O_T) & \xrightarrow{P(g)} & (S, O_S) \end{array}$$

It gives (see section 2) the canonical morphism in $C_{g^* O_S \text{fil}}(\text{Var}(\mathbb{C})^{sm}/T)$

$$\begin{aligned} \Omega_{(T/S)} &:= \Omega_{(O_{\text{AnSp}(\mathbb{C})^{sm}/T}/g^* O_{\text{AnSp}(\mathbb{C})^{sm}/S})/(O_T/g^* O_S)} : \\ g^*(\Omega_{U/S}^\bullet, F_b) &= \Omega_{g^* O_{\text{AnSp}(\mathbb{C})^{sm}/S}/g^* e(S)^* O_S} \rightarrow (\Omega_{T/T}^\bullet, F_b) = \Omega_{O_{\text{AnSp}(\mathbb{C})^{sm}/T}/e(T)^* O_T} \end{aligned}$$

which is by definition given by the pullback on differential forms : for $(V/T) = (V, h) \in \text{Var}(\mathbb{C})^{sm}/T$,

$$\begin{aligned} \Omega_{(T/S)}(V/T) : g^*(\Omega_{U/S}^\bullet)(V/T) &:= \lim_{(h' : U \rightarrow \text{SSm}, g' : V \rightarrow U, h, g)} \Omega_{U/S}^\bullet(U) \xrightarrow{\Omega_{(V/U)/(T/S)}(V/T)} \Omega_{V/T}^\bullet(V) =: \Omega_{V/T}^\bullet(V/T) \\ \hat{\omega} &\mapsto \Omega_{(V/U)/(T/S)}(V/T)(\omega) := g'^* \hat{\omega}. \end{aligned}$$

If S and T are smooth, $\Omega_{/}(T/S) : g^*(\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/T}^\bullet, F_b)$ is a map in $C_{g^*O_S \text{fil}, g^*D_S^\infty}(\text{AnSp}(\mathbb{C})^{sm}/T)$. It induces the canonical morphism in $C_{g^*O_S \text{fil}, g^*D_S^\infty}(\text{AnSp}(\mathbb{C})^{sm}/T)$:

$$E\Omega_{/}(T/S) : g^*E_{usu}(\Omega_{/S}^\bullet, F_b) \xrightarrow{T(g, E_{usu})(\Omega_{/S}^\bullet, F_b)} E_{usu}(g^*(\Omega_{/S}^\bullet, F_b)) \xrightarrow{E_{usu}(\Omega_{/}(T/S))} E_{usu}(\Omega_{/T}^\bullet, F_b).$$

Definition 141. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$. We have, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_T \text{fil}}(T)$:

$$\begin{aligned} T^O(g, \Omega_{/}) (F) &: g^{*mod} L_{Oe}(S) * \mathcal{H}om^\bullet(F, E_{usu}(\Omega_{/S}^\bullet, F_b)) \\ &\xrightarrow{=} (g^* L_{Oe}(S) * \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{T(e, g)(-) \circ T(g, L_O)(-)} L_{Oe}(e(T) * g^* \mathcal{H}om^\bullet(F, E_{usu}(\Omega_{/S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{T(g, \text{hom})(F, E_{et}(\Omega_{/S}^\bullet)) \otimes I} L_{Oe}(e(T) * \mathcal{H}om^\bullet(g^*F, g^*E_{usu}(\Omega_{/S}^\bullet, F_b))) \otimes_{g^*O_S} O_T \\ &\xrightarrow{ev(\text{hom}, \otimes)(-, -, -)} L_{Oe}(T) * \mathcal{H}om^\bullet(g^*F, g^*E_{usu}(\Omega_{/S}^\bullet, F_b)) \otimes_{g^*e(S)^*O_S} e(T)^*O_T \\ &\xrightarrow{\mathcal{H}om^\bullet(g^*F, E\Omega_{/}(T/S)) \otimes I} L_{Oe}(T) * \mathcal{H}om^\bullet(g^*F, E_{usu}(\Omega_{/T}^\bullet, F_b)) \otimes_{g^*e(S)^*O_S} e(T)^*O_T \\ &\xrightarrow{m} L_{Oe}(T) * \mathcal{H}om^\bullet(g^*F, E_{usu}(\Omega_{/T}^\bullet, F_b)) \end{aligned}$$

where $m(\alpha \otimes h) := h \cdot \alpha$ is the multiplication map.

(ii) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$, S smooth. Assume there is a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with l a closed embedding, $Y \in \text{AnSm}(\mathbb{C})$ and ps the projection. We have, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_T \text{fil}, \mathcal{D}^\infty}(Y \times S)$:

$$\begin{aligned} T(g, \Omega_{/}) (F) &: g^{*mod, \Gamma} e(S) * \mathcal{H}om^\bullet(F, E_{usu}(\Omega_{/S}^\bullet, F_b)) \\ &\xrightarrow{=} \Gamma_T E_{usu}(p_S^{*mod} e(S) * \mathcal{H}om^\bullet(F, E_{usu}(\Omega_{/S}^\bullet, F_b))) \\ &\xrightarrow{T^O(p_S, \Omega_{/}) (F)} \Gamma_T E_{usu}(e(T \times S) * \mathcal{H}om^\bullet(p_S^*F, E_{usu}(\Omega_{/T \times S}^\bullet, F_b))) \\ &\xrightarrow{=} e(T \times S) * \Gamma_T (\mathcal{H}om^\bullet(p_S^*F, E_{usu}(\Omega_{/T \times S}^\bullet, F_b))) \\ &\xrightarrow{I(\gamma, \text{hom})(-, -)} e(T \times S) * \mathcal{H}om^\bullet(\Gamma_T^\vee p_S^*F, E_{usu}(\Omega_{/T \times S}^\bullet, F_b)). \end{aligned}$$

For $Q \in \text{Proj PSh}(\text{AnSp}(\mathbb{C})^{sm}/S)$,

$$T(g, \Omega_{/}) (Q) : g^{*mod, \Gamma} e(S) * \mathcal{H}om^\bullet(Q, E_{usu}(\Omega_{/S}^\bullet, F_b)) \rightarrow e(T \times S) * \mathcal{H}om^\bullet(\Gamma_T^\vee p_S^*Q, E_{usu}(\Omega_{/Y \times S}^\bullet, F_b))$$

is a map in $C_{O_T \text{fil}, \mathcal{D}^\infty}(Y \times S)$.

The following easy lemma describe these transformation map on representable presheaves :

Lemma 15. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$ and $h : U \rightarrow S$ is a smooth morphism with $U \in \text{AnSp}(\mathbb{C})$. Consider a commutative diagram whose square are cartesian :

$$\begin{array}{ccccc} g : T & \xrightarrow{l} & S \times Y & \xrightarrow{ps} & S \\ h' \uparrow & & h'' := h \times I \uparrow & & h \uparrow \\ g' : U_T & \xrightarrow{l'} & U \times Y & \xrightarrow{p_U} & U \end{array}$$

with l, l' the graph embeddings and ps, p_U the projections. Then $g^*\mathbb{Z}(U/S) = \mathbb{Z}(U_T/T)$ and

(i) we have the following commutative diagram in $C_{O_T \text{fil}}(T)$ (see definition 1 and definition 141(i)) :

$$\begin{array}{ccc}
g^{*mod} L_O e(S) * \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{usu}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(g, \Omega_{/ \cdot})(\mathbb{Z}(U/S))} & e(T) * \mathcal{H}om^\bullet(\mathbb{Z}(U_T/T), E_{usu}(\Omega_{/T}^\bullet, F_b)) \\
\downarrow = & & \downarrow = \\
g^{*mod} L_O h_* E_{usu}(\Omega_{U/S}^\bullet, F_b) & \xrightarrow{T_\omega^{mod}(g, h)} & h'_* E_{usu}(\Omega_{U_T/T}^\bullet, F_b)
\end{array}$$

(ii) if $Y, S \in \text{AnSm}(\mathbb{C})$, we have the following commutative diagram in $C_{O_T \text{fil}, \mathcal{D}^\infty}(T)$ (see definition 1 and definition 141(ii)) :

$$\begin{array}{ccc}
g^{*mod, \Gamma} e(S) * \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{usu}(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(g, \Omega_{/ \cdot})(\mathbb{Z}(U/S))} & e(T) * \mathcal{H}om^\bullet(\mathbb{Z}(U_T/T), E_{usu}(\Omega_{/T}^\bullet, F_b)) \\
\downarrow = & & \downarrow = \\
g^{*mod, \Gamma} h_* E_{usu}(\Omega_{U/S}^\bullet, F_b) & \xrightarrow{T_\omega^O(\gamma, \otimes)(-) \circ T_\omega^O(p_S, h)(-)} & h'_* E_{usu}(\Omega_{U_T/T}^\bullet, F_b)
\end{array}$$

where $j : T \setminus T \times S \hookrightarrow T \times S$ is the open complementary embedding,

Proof. Obvious. □

Proposition 125. Let $p : S_{12} \rightarrow S_1$ is a smooth morphism with $S_1, S_{12} \in \text{AnSp}(\mathbb{C})$. Then if $Q \in C(\text{AnSp}(\mathbb{C})^{sm}/S_1)$ is projective,

$$T(p, \Omega_{/ \cdot})(Q) : p^{*mod} e(S_1) * \mathcal{H}om^\bullet(Q, E_{usu}(\Omega_{/S_1}^\bullet, F_b)) \rightarrow e(S_{12}) * \mathcal{H}om^\bullet(p^* Q, E_{usu}(\Omega_{/S_{12}}^\bullet, F_b))$$

is an isomorphism.

Proof. Similar to the proof of proposition 106. □

Let $S \in \text{AnSp}(\mathbb{C})$ and $h : U \rightarrow S$ a morphism with $U \in \text{AnSp}(\mathbb{C})$. We then have the canonical map given by the wedge product

$$w_{U/S} : \Omega_{U/S}^\bullet \otimes_{O_S} \Omega_{U/S}^\bullet \rightarrow \Omega_{U/S}^\bullet; \alpha \otimes \beta \mapsto \alpha \wedge \beta.$$

Let $S \in \text{Var}(\mathbb{C})$ and $h_1 : U_1 \rightarrow S, h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2 \in \text{AnSp}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \rightarrow S$ and $p_{112} : U_1 \times_S U_2 \rightarrow U_1, p_{212} : U_1 \times_S U_2 \rightarrow U_2$ the projections. We then have the canonical map given by the wedge product

$$w_{(U_1, U_2)/S} : p_{112}^* \Omega_{U_1/S}^\bullet \otimes_{O_S} p_{212}^* \Omega_{U_2/S}^\bullet \rightarrow \Omega_{U_{12}/S}^\bullet; \alpha \otimes \beta \mapsto p_{112}^* \alpha \wedge p_{212}^* \beta$$

which gives the map

$$\begin{aligned}
& Ew_{(U_1, U_2)/S} : h_{1*} E_{usu}(\Omega_{U_1/S}^\bullet) \otimes_{O_S} h_{2*} E_{usu}(\Omega_{U_2/S}^\bullet) \\
& \xrightarrow{\text{ad}(p_{112}^*, p_{112*})(-) \otimes \text{ad}(p_{212}^*, p_{212*})(-)} (h_{1*} p_{112*} p_{112}^* E_{usu}(\Omega_{U_1/S}^\bullet)) \otimes_{O_S} (h_{2*} p_{212*} p_{212}^* E_{usu}(\Omega_{U_2/S}^\bullet)) \\
& \xrightarrow{=} h_{12*} (p_{112}^* E_{usu}(\Omega_{U_1/S}^\bullet) \otimes_{h_{12}^* O_S} p_{212}^* E_{usu}(\Omega_{U_2/S}^\bullet)) \\
& \xrightarrow{T(\otimes, E)(-) \circ (T(p_{112}, E)(-) \otimes T(p_{212}, E)(-))} h_{12*} E_{zar}(p_{112}^* \Omega_{U_1/S}^\bullet \otimes_{O_S} p_{212}^* \Omega_{U_2/S}^\bullet)
\end{aligned}$$

Let $S \in \text{AnSp}(\mathbb{C})$. We have the canonical map in $C_{O_S \text{fil}}(\text{AnSp}(\mathbb{C})^{sm}/S)$

$$w_S : (\Omega_{/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/S}^\bullet, F_b)$$

given by for $h : U \rightarrow S \in \text{AnSp}(\mathbb{C})^{sm}/S$

$$w_S(U/S) : (\Omega_{U/S}^\bullet, F_b) \otimes_{h^*O_S} (\Omega_{U/S}^\bullet, F_b)(U) \xrightarrow{w_{U/S}(U)} (\Omega_{U/S}^\bullet, F_b)(U)$$

It gives the map

$$Ew_S : E_{usu}(\Omega_{/S}^\bullet, F_b) \otimes_{O_S} E_{usu}(\Omega_{/S}^\bullet, F_b) \xrightarrow{=} E_{usu}((\Omega_{/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{/S}^\bullet, F_b)) \xrightarrow{E_{usu}(w_S)} E_{usu}(\Omega_{/S}^\bullet, F_b)$$

If $S \in \text{AnSm}(\mathbb{C})$,

$$w_S : (\Omega_{/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{/S}^\bullet, F_b) \rightarrow (\Omega_{/S}^\bullet, F_b)$$

is a map in $C_{O_S \text{fil}, D^\infty}(\text{Var}(\mathbb{C})^{sm}/S)$.

Definition 142. Let $S \in \text{AnSp}(\mathbb{C})$. We have, for $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_S \text{fil}}(S)$:

$$\begin{aligned} T(\otimes, \Omega)(F, G) &: e(S)_* \mathcal{H}om(F, E_{usu}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} e(S)_* \mathcal{H}om(G, E_{usu}(\Omega_{/S}^\bullet, F_b)) \\ &\xrightarrow{=} e(S)_* (\mathcal{H}om(F, E_{usu}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} \mathcal{H}om(G, E_{usu}(\Omega_{/S}^\bullet, F_b))) \\ &\xrightarrow{e(S)_* T(\mathcal{H}om, \otimes)(-)} e(S)_* \mathcal{H}om(F \otimes G, E_{usu}(\Omega_{/S}^\bullet, F_b) \otimes_{O_S} E_{usu}(\Omega_{/S}^\bullet, F_b)) \\ &\xrightarrow{\mathcal{H}om(F \otimes G, Ew_S)} e(S)_* \mathcal{H}om(F \otimes G, E_{usu}(\Omega_{/S}^\bullet, F_b)) \end{aligned}$$

If $S \in \text{AnSm}(\mathbb{C})$, $T(\otimes, \Omega)(F, G)$ is a map in $C_{O_S \text{fil}, D^\infty}(S)$.

Lemma 16. Let $S \in \text{AnSp}(\mathbb{C})$ and $h_1 : U_1 \rightarrow S$, $h_2 : U_2 \rightarrow S$ two smooth morphisms with $U_1, U_2 \in \text{AnSp}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \rightarrow S$ and $p_{112} : U_1 \times_S U_2 \rightarrow U_1$, $p_{212} : U_1 \times_S U_2 \rightarrow U_2$ the projections. We then have the following commutative diagram

$$\begin{array}{ccc} e(S)_* \mathcal{H}om(F, E_{usu}(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} e(S)_* \mathcal{H}om(G, E_{usu}(\Omega_{/S}^\bullet, F_b)) \xrightarrow{T(\otimes, \Omega)(F, G)} e(S)_* \mathcal{H}om(F \otimes G, E_{usu}(\Omega_{/S}^\bullet, F_b)) & & \\ \downarrow = & & \downarrow = \\ h_{1*} E_{usu}(\Omega_{U_1/S}^\bullet, F_b) \otimes_{O_S} h_{2*} E_{usu}(\Omega_{U_2/S}^\bullet, F_b) \xrightarrow{Ew_{(U_1, U_2)/S}} h_{12*} E_{zar}(\Omega_{U_{12}/S}^\bullet, F_b) & & \end{array}$$

Proof. Follows from Yoneda lemma. \square

We now define the analytic Gauss Manin De Rahm realization functor.

Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$ an affine space. For $I \subset [1, \dots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Consider, for $I \subset J$, the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Considering the factorization of the diagram D_{IJ} by the fiber product :

$$D_{IJ} = \begin{array}{ccccc} \tilde{S}_J = \tilde{S}_I \times \tilde{S}_{J \setminus I} & \xrightarrow{p_{IJ}} & \tilde{S}_I & & \\ \uparrow i_J & \swarrow i_I \times I & \uparrow i_I & & \\ & S_I \times \tilde{S}_{J \setminus I} & & & \\ \uparrow j_J & \nearrow j_J & \searrow p_{IJ}^0 & & \\ S_J & \xrightarrow{j_{IJ}} & S_I & & \end{array}$$

the square of this factorization being cartesian, we have for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\text{AnSp}(\mathbb{C})^{sm}/\tilde{S}_J)$

$$\begin{aligned} S(D_{IJ})(F) : Li_{J*}j_J^*F &\xrightarrow{q} i_{J*}j_J^*F = (i_I \times I) * l_{J*}j_J^*F \xrightarrow{(i_I \times I)_* \text{ad}(p_{IJ\sharp}^0, p_{IJ}^0)(-)} \\ &(i_I \times I)_* p_{IJ}^0 p_{IJ\sharp}^0 l_{J*}j_J^*F \xrightarrow{T(p_{IJ}, i_I)(-)^{-1}} p_{IJ}^* i_{I*} p_{IJ\sharp}^0 l_{J*}j_J^*F = p_{IJ}^* i_{I*} j_I^* F \end{aligned}$$

which factors through

$$S(D_{IJ})(F) : Li_{J*}j_J^*F \xrightarrow{S^q(D_{IJ})(F)} p_{IJ}^* Li_{I*}j_I^*F \xrightarrow{q} p_{IJ}^* i_{I*} j_I^* F$$

Definition 143. (i) Let $S \in \text{AnSm}(\mathbb{C})$. We have the functor

$$\text{Hom}^\bullet(\cdot, E_{usu}(\Omega_{/S}^\bullet, F_b)) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C_{O_S \text{fil}, D_S^\infty}(S), F \mapsto e(S)_* \text{Hom}^\bullet(LF, E_{usu}(\Omega_{/S}^\bullet, F_b)).$$

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We have the functor

$$C(\text{Var}(\mathbb{C})^{sm}/S)^{op} \rightarrow C_{O_S \text{fil}, D^\infty}(S/(\tilde{S}_I)), F \mapsto (e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], u_{IJ}^q(F))$$

where

$$\begin{aligned} u_{IJ}^q(F)[d_{\tilde{S}_J}] &: e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\ &\xrightarrow{\text{ad}(p_{IJ}^{*mod}, p_{IJ*})(-)} p_{IJ*} p_{IJ}^{*mod} e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\ &\xrightarrow{p_{IJ*} T(p_{IJ}, \Omega)(L(i_{I*} j_I^* F))} p_{IJ*} e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* p_{IJ}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)) \\ &\xrightarrow{p_{IJ*} e(\tilde{S}_I)_* \text{Hom}(\text{An}_{\tilde{S}_I}^* S^q(D_{IJ})(F), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma}, F_b))} p_{IJ*} e(\tilde{S}_I)_* \text{Hom}^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)). \end{aligned}$$

For $I \subset J \subset K$, we have obviously $p_{IJ*} u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$.

Proposition 126. (i) Let $S \in \text{AnSp}(\mathbb{C})$. Let $m : Q_1 \rightarrow Q_2$ be an equivalence (\mathbb{D}^1, et) local in $C(\text{AnSp}(\mathbb{C})^{sm}/S)$ with Q_1, Q_2 complexes of projective presheaves. Then,

$$e(S)_* \text{Hom}(m, E_{et}(\Omega_{/S}^\bullet, F_b)) : e(S)_* \text{Hom}^\bullet(Q_2, E_{et}(\Omega_{/S}^\bullet, F_b)) \rightarrow e(S)_* \text{Hom}^\bullet(Q_1, E_{et}(\Omega_{/S}^\bullet, F_b))$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{O_S \text{fil}, D^\infty, \infty}(S)$ if S is smooth.

(ii) Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Let $m = (m_I) : (Q_{1I}, s_{1I}^1) \rightarrow (Q_{2I}, s_{2I}^2)$ be an equivalence (\mathbb{D}^1, usu) local in $C(\text{AnSp}(\mathbb{C})^{sm}/(\tilde{S}_I)^{op})$ with Q_{1I}, Q_{2I} complexes of projective presheaves. Then,

$$\begin{aligned} &(e(\tilde{S}_I)_* \text{Hom}(m_I, E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))) : \\ &(e(\tilde{S}_I)_* \text{Hom}^\bullet(Q_{2I}, E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}(Q_{2I}, s_{2I}^2)) \rightarrow (e(\tilde{S}_I)_* \text{Hom}^\bullet(Q_{1I}, E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}(Q_{1I}, s_{1I}^1)) \end{aligned}$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{O_S \text{fil}, D^\infty, \infty}(S/(\tilde{S}_I))$.

Proof. Similar to the proof of proposition 107. \square

Definition 144. (i) We define, according to proposition 126, the filtered analytic Gauss-Manin realization functor defined as

$$\begin{aligned} \mathcal{F}_{S, an}^{GM} : \text{DA}_c(S)^{op} &\rightarrow D_{O_S \text{fil}, D^\infty, \infty}(S), M \mapsto \\ \mathcal{F}_{S, an}^{GM}(M) &:= e(S)_* \text{Hom}^\bullet(\text{An}_S^* L(F), E_{usu}(\Omega_{/S}^\bullet, F_b))[-d_S] \\ &= e(S)_* \text{Hom}^\bullet(L(F), \text{An}_{S*} E_{usu}(\Omega_{/S}^\bullet, F_b))[-d_S] \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$,

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define the filtered analytic Gauss-Manin realization functor defined as

$$\begin{aligned} \mathcal{F}_{S,an}^{GM} : \text{DA}_c(S)^{op} &\rightarrow D_{\text{Ofil}, \mathcal{D}^\infty, \infty}(S/\tilde{S}_I), M \mapsto \\ \mathcal{F}_{S,an}^{GM}(M) &:= ((e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\ &= ((e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), \text{An}_{\tilde{S}_I}^* E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 143.

Proposition 127. For $S \in \text{Var}(\mathbb{C})$, the functor \mathcal{F}_S^{GM} is well defined. \square

Proof. Similar to the proof of proposition 108. \square

Proposition 128. Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. Assume there exist a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We then have, for $I \subset [1, \dots, l]$, the following commutative diagrams which are cartesian

$$\begin{array}{ccccc} f_I = f|_{X_I} : X_I & \xrightarrow{l_I} & Y \times S_I & \xrightarrow{p_{S_I}} & S_I \\ & \searrow & \downarrow i'_I & & \downarrow i_I \\ & & Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \end{array} \quad , \quad \begin{array}{ccc} Y \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \\ \downarrow p'_{IJ} & & \downarrow p_{IJ} \\ Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \end{array}$$

Let $F(X/S) := p_{S, \#} \Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S)$. The transformations maps $(N_I(X/S) : Q(X_I/\tilde{S}_I) \rightarrow i_{I*} j_I^* F(X/S))$ and $(k \circ I(\gamma, \text{hom})(-, -))$, for $I \subset [1, \dots, l]$, induce an isomorphism in $D_{\text{Ofil}, \mathcal{D}^\infty, \infty}(S/(\tilde{S}_I))$

$$\begin{aligned} & I^{GM}(X/S) : \\ & \mathcal{F}_{S,an}^{GM}(M(X/S)) := (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F(X/S)), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S))) \\ & \xrightarrow{(e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* N_I(X/S), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))} (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \\ & \xrightarrow{(e(\tilde{S}_I)_* \mathcal{H}om(T(\text{An}, \gamma^\vee)(-)^{-1}, E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))} (e(\tilde{S}_I)_* \mathcal{H}om(Q(X_I^{an}/\tilde{S}_I^{an}), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) \\ & \xrightarrow{(I(\gamma, \text{hom})(-, -))^{-1}} (p_{\tilde{S}_I*} \Gamma_{X_I} E_{usu}(\Omega_{Y \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], w_{IJ}(X/S)). \end{aligned}$$

Proof. Similar to the proof of proposition 109. \square

We now define the functorialities of \mathcal{F}_S^{GM} with respect to S which makes \mathcal{F}_{GM}^- a morphism of 2-functor.

Definition 145. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Consider the factorization $g : T \xrightarrow{l} T \times S \xrightarrow{p_S} S$ where l is the graph embedding and p_S the projection. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then, $D(\mathbb{A}_T^1, et)(g^* F) = g^* M$.

(i) We have then the canonical transformation in $D_{\mathcal{D}^\infty, \text{fil}, \infty}(T \times S)$ (see definition 141) :

$$\begin{aligned} T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_{S,an}^{GM}(M) &:= g^{*mod, \Gamma} e(S)_* \mathcal{H}om^\bullet(\text{An}_S^* L(F), E_{usu}(\Omega_{/S}^\bullet, F_b)))[-d_T] \\ & \xrightarrow{T(g, \Omega_{/S})(\text{An}_S^* L(F))} e(T \times S)_* \mathcal{H}om^\bullet(\Gamma_T^\vee p_S^* \text{An}_S^* L(F), E_{usu}(\Omega_{/Y \times S}^\bullet, F_b)))[-d_T] \\ & \xrightarrow{\mathcal{H}om(T(\text{An}, \gamma^\vee)(p_S^* L F)^{-1}, -)} \mathcal{F}_{T \times S, an}^{GM}(l_* g^* M). \end{aligned}$$

where the last isomorphism in the derived category comes from proposition 127.

(ii) We have then the canonical transformation in $D_{Ofil,\infty}(T)$ (see definition 141) :

$$T^O(g, \mathcal{F}^{GM})(M) : Lg^{*mod[-]} \mathcal{F}_{S,an}^{GM}(M) := g^{*mod} L_{Oe}(S) * \mathcal{H}om^\bullet(\text{An}_S^* L(F), E_{usu}(\Omega_{/S}^\bullet, F_b)))[-d_T] \\ \xrightarrow{T(g, \Omega_{/ \cdot})(\text{An}_S^* L(F))} e(T \times S) * \mathcal{H}om^\bullet(g^* \text{An}_S^* L(F), E_{usu}(\Omega_{/Y \times S}^\bullet, F_b))[-d_T] =: \mathcal{F}_{T,an}^{GM}(g^* M).$$

We give now the definition in the non smooth case Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$.

Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. We recall the commutative diagram :

$$E_{IJg} = (Y \times \tilde{S}_I) \setminus T_I \xrightarrow{m'_I} Y \times \tilde{S}_J, E_{IJ} = \tilde{S}_J \setminus S_J \xrightarrow{m_J} \tilde{S}_J \quad E'_{IJ} = (Y \times \tilde{S}_J) \setminus T_J \xrightarrow{m'_J} Y \times \tilde{S}_I \\ \downarrow p_{\tilde{S}_I} \quad \downarrow \tilde{g}_I \quad \downarrow p_{IJ} \quad \downarrow p_{IJ} \quad \downarrow p'_{IJ} \quad \downarrow p'_{IJ} \\ \tilde{S}_I \setminus S_I \xrightarrow{m_I} \tilde{S}_I \quad \tilde{S}_I \setminus (S_I \setminus S_J) \xrightarrow{m_I = m_{IJ}} \tilde{S}_I \quad (Y \times \tilde{S}_I) \setminus (T_I \setminus T'_I) \xrightarrow{m'_I = m'_{IJ}} Y \times \tilde{S}_I$$

For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p'_{IJ} := I_Y \times p_{IJ} : Y \times \tilde{S}_J \rightarrow Y \times \tilde{S}_I$ the projections, so that $\tilde{g}_I \circ p'_{IJ} = p_{IJ} \circ \tilde{g}_J$. Consider, for $I \subset J \subset [1, \dots, l]$, resp. for each $I \subset [1, \dots, l]$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}, D'_{IJ} = \begin{array}{ccc} T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_I \\ j'_{IJ} \uparrow & & \uparrow p'_{IJ} \\ T_J & \xrightarrow{i'_J} & Y \times \tilde{S}_J \end{array} \quad D_{gI} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ g_I \uparrow & & \uparrow \tilde{g}_I \\ T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_I \end{array},$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Let $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$. The fact that the diagrams (61) commutes says that the maps $T^{q,\gamma}(D_{gI})(j_I^* F)$ define a morphism in $C(\text{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$

$$(T^{q,\gamma}(D_{gI})(j_I^* F)) : (\Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*} j_I^* F), \tilde{g}_I^* S^q(D_{IJ})(F)) \rightarrow (L(i'_{I*} j'^*_I g^* F), S^q(D'_{IJ})(g^* F))$$

We then have then the following lemma :

Lemma 17. (i) The morphism in $C(\text{Var}(\mathbb{C})^{sm}/(T/(Y \times \tilde{S}_I)))$

$$(T^{q,\gamma}(D_{gI})(j_I^* F)) : (\Gamma_{T_I}^\vee L \tilde{g}_I^* i_{I*} j_I^* F, \tilde{g}_I^* S^q(D_{IJ})(F)) \rightarrow (i'_{I*} j'^*_I g^* F, S^q(D'_{IJ})(g^* F))$$

is an equivalence (\mathbb{A}^1, et) local.

(ii) Denote for short $d_{YI} := -d_Y - d_{\tilde{S}_I}$. The maps $\mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^*(T^{q,\gamma}(D_{gI})(j_I^* F)), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b))$ induce an ∞ -filtered quasi-isomorphism in $C_{Ofil, D^\infty}(T/(Y \times \tilde{S}_I))$

$$(\mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* T^{q,\gamma}(D_{gI})(j_I^* F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b))) : \\ (e(Y \times \tilde{T}_I) * \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* L(i'_{I*} j'^*_I g^* F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b))[d_{YI}], u_{IJ}^q(g^* F)) \rightarrow \\ (e(Y \times \tilde{T}_I) * \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^\vee L(\tilde{g}_I^* i_{I*} j_I^* F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b))[d_{YI}], \tilde{g}_I^* u_{IJ}^q(F)_2)$$

(iii) The maps $T(\tilde{g}_I, \Omega_\cdot)(L(i_{I*}j_I^*F))$ (see definition 141) induce a morphism in $CO_{fil, \mathcal{D}^\infty}(T/(Y \times \tilde{S}_I))$

$$\begin{aligned} & (T(\tilde{g}_I, \Omega_\cdot)(L(i_{I*}j_I^*F))) : \\ & (\Gamma_{T_I} E_{zar}(\tilde{g}_I^{*mod[-1]} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[d_{Y_I}], \tilde{g}_J^{*mod} u_{IJ}^q(F)) \rightarrow \\ & (\Gamma_{T_I}(e(Y \times \tilde{S}_I)_* \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))[d_{Y_I}], \tilde{g}_J^* u_{IJ}^q(F)_1). \end{aligned}$$

Proof. (i): Follows from theorem 16

(ii): Similar to lemma 10(ii).

(iii): Similar to lemma 10(iii). □

Definition 146. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Denote for short $d_{Y_I} := -d_Y - d_{\tilde{S}_I}$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then, $D(\mathbb{A}_T^1, et)(g^*F) = g^*M$. We have, by lemma 10, the canonical transformation in $DO_{fil, \mathcal{D}^\infty, \infty}(T/(Y \times \tilde{S}_I))$

$$\begin{aligned} & T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-], \Gamma} \mathcal{F}_{S, an}^{GM}(M) := \\ & (\Gamma_{T_I} E_{zar}(\tilde{g}_I^{*mod} e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))[d_{Y_I}], \tilde{g}_J^{*mod} u_{IJ}^q(F)) \\ & \quad \xrightarrow{(\Gamma_{T_I} E(T(\tilde{g}_I, \Omega_\cdot)(\text{An}_{\tilde{S}_I}^* L(i_{I*}j_I^*F))))} \\ & (\Gamma_{T_I} e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))[d_{Y_I}], \tilde{g}_J^* u_{IJ}^q(F)_1) \\ & \quad \xrightarrow{(I(\gamma, \text{hom}(-, -)))} \\ & (e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(\Gamma_{T_I}^\vee \text{An}_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))[d_{Y_I}], \tilde{g}_J^* u_{IJ}^q(F)_2) \\ & \quad \xrightarrow{(e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(T(\text{An}, \gamma^\vee)(\tilde{g}_I^* L(i_{I*}j_I^*F))^{-1}, E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))} \\ & (e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^\vee \tilde{g}_I^* L(i_{I*}j_I^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))[d_{Y_I}], \tilde{g}_J^* u_{IJ}^q(F)_2) \\ & \quad \xrightarrow{(e(Y \times \tilde{S}_I)_* \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* T^{q \cdot \gamma}(D_{g_I})(j_I^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))^{-1}} \\ & (e(Y \times \tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{Y \times \tilde{S}_I}^* L(i'_{I*}j_I^* g^*F), E_{usu}(\Omega_{/Y \times \tilde{S}_I}^\bullet, F_b)))[d_{Y_I}], u_{IJ}^q(g^*F)) =: \mathcal{F}_{T, an}^{GM}(g^*M). \end{aligned}$$

Proposition 129. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{p_S} S$ with $Y_2 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$. Assume that there is a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_S} S$, with $Y_1 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} f' : X_T & \longrightarrow & Y_1 \times T & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ f'' = f \times I : Y_2 \times X & \longrightarrow & Y_1 \times Y_2 \times S & \longrightarrow & Y_2 \times S \\ \downarrow & & \downarrow & & \downarrow \\ f : X & \longrightarrow & Y_1 \times S & \longrightarrow & S \end{array}$$

Consider $F(X/S) := p_{S,\sharp}\Gamma_X^\vee\mathbb{Z}(Y_1 \times S/Y_1 \times S)[d_{Y_1}]$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T(f, g, F(X/S)) : g^*F(X/S) := g^*p_{S,\sharp}\Gamma_X^\vee\mathbb{Z}(Y_1 \times S/Y_1 \times S) \xrightarrow{\sim} p_{T,\sharp}\Gamma_{X_T}^\vee\mathbb{Z}(Y_1 \times T/Y_1 \times T) =: F(X_T/T).$$

which gives in $\text{DA}(S)$ the isomorphism $T(f, g, F(X/S)) : g^*M(X/S) \xrightarrow{\sim} M(X_T/T)$. Then, the following diagram in $D_{\text{Ofil}, \mathcal{D}^\infty, \infty}(T/(Y_2 \times \tilde{S}_I))$ commutes

$$\begin{array}{ccc} Rg^{*mod, \Gamma}\mathcal{F}_{S, an}^{GM}(M(X/S)) & \xrightarrow{T(g, \mathcal{F}^{GM})(M(X/S))} & \mathcal{F}_{T, an}^{GM}(M(X_T/T)) \\ \downarrow I^{GM}(X/S) & & \downarrow I^{GM}(X_T/T) \\ g^{*mod[-], \Gamma}(p_{\tilde{S}_I^*}\Gamma_{X_I}E_{usu}(\Omega_{Y_1 \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)) & \xrightarrow{(T(\tilde{g}_I \times I, \gamma)(-) \circ T_w^O(\tilde{g}_I, p_{\tilde{S}_I}))} & (p_{Y_2 \times \tilde{S}_I^*}\Gamma_{X_{T_I}}E_{usu}(\Omega_{Y_2 \times Y_1 \times \tilde{S}_I/Y_2 \times \tilde{S}_I}^\bullet, F_b)[-d_{Y_2}], w_{IJ}(X_T/T)) \end{array}$$

(ii) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$. Assume that there is a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$, with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Consider $F(X/S) := p_{S,\sharp}\Gamma_X^\vee\mathbb{Z}(Y \times S/Y \times S)$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T(f, g, F(X/S)) : g^*F(X/S) := g^*p_{S,\sharp}\Gamma_X^\vee\mathbb{Z}(Y \times S/Y \times S) \xrightarrow{\sim} p_{T,\sharp}\Gamma_{X_T}^\vee\mathbb{Z}(Y \times T/Y \times T)[d_Y] =: F(X_T/T).$$

which gives in $\text{DA}(S)$ the isomorphism $T(f, g, F(X/S)) : g^*M(X/S) \xrightarrow{\sim} M(X_T/T)$. Then, the following diagram in $D_{\text{Ofil}, \infty}(T)$ commutes

$$\begin{array}{ccc} Lg^{*mod[-]}\mathcal{F}_{S, an}^{GM}(M(X/S)) & \xrightarrow{T^O(g, \mathcal{F}^{GM})(M(X/S))} & \mathcal{F}_{T, an}^{GM}(M(X_T/T)) \\ \downarrow I^{GM}(X/S) & & \downarrow I^{GM}(X_T/T) \\ g^{*mod}L_O(p_{S^*}\Gamma_X E_{usu}(\Omega_{Y \times S/S}^\bullet, F_b)[-d_T]) & \xrightarrow{(T(g \times I, \gamma)(-) \circ T_w^O(g, p_S))} & p_{Y \times T^*}\Gamma_{X_T} E_{usu}(\Omega_{Y \times T/T}^\bullet, F_b)[-d_T] \\ \downarrow T_w(\otimes, \gamma)(O_Y \times S) & & \downarrow T_w(\otimes, \gamma)(O_Y \times T) \\ Lg^{*mod} \int_{p_S}^{FDR}(\Gamma_X E_{usu}(O_Y \times S, F_b)[-d_Y - d_T]) & \xrightarrow{T^{\mathcal{D}^{mod}}(p_S, f)(-)} & \int_{p_T}^{FDR}(\Gamma_{X_T} E_{usu}(O_Y \times T, F_b)[-d_Y - d_T]). \end{array}$$

Proof. Follows immediately from definition. \square

We have the following theorem:

Theorem 37. (i) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S}$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, \mathcal{F}_{an}^{GM})(M) : Rg^{*mod[-], \Gamma}\mathcal{F}_{S, an}^{GM}(M) \rightarrow \mathcal{F}_{T, an}^{GM}(g^*M)$$

is an isomorphism in $D_{O_T \text{fil}, \mathcal{D}^\infty, \infty}(T/(Y \times \tilde{S}_I))$.

(ii) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, \mathcal{F}_{an}^{GM})(M) : Lg^{*mod[-]}\mathcal{F}_{S, an}^{GM}(M) \rightarrow \mathcal{F}_{T, an}^{GM}(g^*M)$$

is an isomorphism in $D_{O_T}(T)$.

Proof. (i): Follows from proposition 125.

(ii) : **First proof** : Follows from proposition 129, proposition 135 and proposition 96.

: **Second proof** : In the analytic case only, we can give a direct proof of this proposition : Indeed, let $g : T \rightarrow S$ is a morphism with $T, S \in \text{AnSp}(\mathbb{C})$ and let $h : U \rightarrow S$ a smooth morphism with $U \in \text{AnSp}(\mathbb{C})$, then,

$$T_\omega^O(g, h) : g^{*mod} L_{D^\infty} h_* E(\Omega_{U/S}^\bullet, F) \rightarrow h'_* E(\Omega_{U_T/T}^\bullet, F)$$

is an equivalence usu local : consider the following commutative diagram

$$\begin{array}{ccc} g^{*mod} L_O(h_* E(\mathbb{Z}_U) \otimes O_S) & \xrightarrow{T(g,h)(E(\mathbb{Z}_U))} & h'_* E(\mathbb{Z}_{U_T}) \otimes O_T & , \\ \downarrow g^{*mod} L_O T(h, \otimes)(-, -) & & \downarrow T(h', \otimes)(-, -) & \\ g^{*mod} L_O h_* E(h^* O_S) & \xrightarrow{T^{mod}(g,h)(h^* O_S)} & h'_* E(h'^* O_T) & \\ \downarrow g^{*mod} L_O h_* E(\iota_{U/S}) & & \downarrow h'_* E(\iota_{U_T/T}) & \\ g^{*mod} L_O h_* E(\Omega_{U/S}^\bullet) & \xrightarrow{T_\omega^O(g,h)} & h'_* E(\Omega_{U_T/T}^\bullet) & \end{array}$$

then,

- the maps $T(h', \otimes)(-, -)$ and $T(h, \otimes)(-, -)$ are usu local equivalence by proposition 9,
- since $h : U \rightarrow S$ is a smooth morphism, the inclusion $\iota_{U/S} : h^* O_S \rightarrow \Omega_{U/S}^\bullet$ is a quasi-isomorphism,
- since $h' : U_T \rightarrow T$ is a smooth morphism, the inclusion $\iota_{U_T/T} : h'^* O_T \rightarrow \Omega_{U_T/T}^\bullet$ is a quasi-isomorphism,
- since U, U_T, S, T are paracompact topological spaces (in particular Hausdorff), $T(g, h)(E(\mathbb{Z}_U)) : g^* h_* E(\mathbb{Z}_U) \rightarrow h'_* E(\mathbb{Z}_{U_T})$ is a quasi-isomorphism.

This fact, together with lemma 15, proves the proposition. \square

We finish this subsection by some remarks on the absolute case and on a particular case of the relative case:

Proposition 130. (i) *Let $X \in \text{PSmVar}(\mathbb{C})$ and $D = \cup D_i \subset X$ a normal crossing divisor. Consider the open embedding $j : U := X \setminus D \hookrightarrow X$. Then,*

- *The map in $D_{fil, \infty}(\mathbb{C})$*

$$\begin{aligned} & \text{Hom}((0, \text{ad}(j^*, j_*)(\mathbb{Z}(X/X)), E_{usu}(\Omega^\bullet, F_b)) : \\ & \mathcal{F}_{an}^{GM}(\mathbb{D}(\mathbb{Z}(U))) := \text{Hom}(L\mathbb{D}(\mathbb{Z}(U)), E_{usu}(\Omega^\bullet, F_b)) \\ & \xrightarrow{\sim} \text{Hom}(\text{Cone}(\mathbb{Z}(D) \rightarrow \mathbb{Z}(X)), E_{zar}(\Omega^\bullet, F_b)) = \Gamma(X, E_{usu}(\Omega_X^\bullet(\text{nul } D), F_b)). \end{aligned}$$

is an isomorphism, where we recall $\mathbb{D}(\mathbb{Z}(U)) := a_{X} j_* E_{et}(\mathbb{Z}(U/U)) = a_{U*} E_{et}(\mathbb{Z}(U/U))$,*

- $\mathcal{F}_{an}^{GM}(\mathbb{Z}(U)) = \Gamma(U, E_{usu}(\Omega_U^\bullet, F_b)) \in D_{fil, \infty}(\mathbb{C})$ *is NOT isomorphic to $\Gamma(X, E_{usu}(\Omega_X^\bullet(\log D), F_b))$ in $D_{fil, \infty}(\mathbb{C})$ in general. For exemple U is affine, then U^{an} is Stein so that $H^n(U, \Omega_U^p) = 0$ for all $p \in \mathbb{N}$, $p \neq 0$, so that the $E_\infty^{p,q}(\Gamma(U, E_{usu}(\Omega_U^\bullet, F_b)))$ are NOT isomorphic to $E_\infty^{p,q}(\Gamma(X, E_{usu}(\Omega_X^\bullet(\log D), F_b)))$ in this case. In particular, the map,*

$$j^* := \text{ad}(j^*, j_*)(-) : H^n \Gamma(X, E_{usu}(\Omega_X^\bullet(\log D))) \xrightarrow{\sim} H^n \Gamma(U, E_{zar}(\Omega_U^\bullet))$$

which is an isomorphism in $D(\mathbb{C})$ (i.e. if we forgot filtrations), gives embeddings

$$j^* := \text{ad}(j^*, j_*)(-) : F^p H^n(U, \mathbb{C}) := F^p H^n \Gamma(X, E_{usu}(\Omega_X^\bullet(\log D), F_b)) \hookrightarrow F^p H^n \Gamma(U, E_{usu}(\Omega_U^\bullet, F_b))$$

which are NOT an isomorphism in general for $n, p \in \mathbb{Z}$. Note that, since $a_U : U \rightarrow \{\text{pt}\}$ is not proper,

$$[\Delta_U] : \mathbb{Z}(U) \rightarrow a_{U*} E_{\text{et}}(\mathbb{Z}(U/U))[2d_U]$$

is NOT an equivalence $(\mathbb{A}^1, \text{et})$ local.

- Let $Z \subset X$ a smooth subvariety and denote $U := X \setminus Z$ the open complementary. Denote $M_Z(X) := \text{Cone}(M(U) \rightarrow M(X)) \in \text{DA}(\mathbb{C})$. The map in $D_{\text{fil}, \infty}(\mathbb{C})$

$$\begin{aligned} \text{Hom}(G(X, Z), E_{usu}(\Omega^\bullet, F_b))^{-1} : \mathcal{F}_{\text{an}}^{GM}(M_Z(X)) &:= \text{Hom}(a_{X\sharp} \Gamma_Z^\vee \mathbb{Z}(X/X), E_{usu}(\Omega^\bullet, F_b)) \xrightarrow{\sim} \\ &\Gamma(X, \Gamma_Z E_{usu}(\Omega_X^\bullet, F_b)) = \Gamma_Z(X, E_{usu}(\Omega_X^\bullet, F_b)) \\ &\xrightarrow{\sim} \mathcal{F}_{\text{an}}^{GM}(M(Z)(c)[2c]) = \Gamma(Z, E_{usu}(\Omega_Z^\bullet, F_b))(-c)[-2c] \end{aligned}$$

is an isomorphism, where $c = \text{codim}(Z, X)$ and $G(X, Z) : a_{X\sharp} \Gamma_Z^\vee \mathbb{Z}(X/X) \rightarrow \mathbb{Z}(Z)(c)[2c]$ is the Gysin morphism.

- Let $D \subset X$ a smooth divisor and denote $U := X \setminus D$ the open complementary. Note that the canonical distinguish triangle in $\text{DA}(\mathbb{C})$

$$M(U) \xrightarrow{\text{ad}(j_\sharp, j^*)(\mathbb{Z}(X/X))} M(X) \xrightarrow{\gamma_Z^\vee(\mathbb{Z}(X/X))} M_D(X) \rightarrow M(U)[1]$$

give a canonical triangle in $D_{\text{fil}, \infty}(\mathbb{C})$

$$\mathcal{F}^{GM}(M_D(X)) \xrightarrow{\mathcal{F}^{GM}(\gamma_Z^\vee(\mathbb{Z}(X/X)))} \mathcal{F}^{GM}(M(X)) \xrightarrow{\mathcal{F}^{GM}(\text{ad}(j_\sharp, j^*)(\mathbb{Z}(X/X)))} \mathcal{F}^{GM}(M(U)) \rightarrow \mathcal{F}^{GM}(M_D(X))[1],$$

which is NOT the image of a distinguish triangle in $\pi(D(\text{MHM}(\mathbb{C})))$, as $\mathcal{F}^{GM}(M(U)) \notin \pi(D(\text{MHM}(\mathbb{C})))$ since the morphism

$$j^* := \text{ad}(j^*, j_*)(-) : H^n \Gamma(X, E_{usu}(\Omega_X^\bullet(\log D), F_b)) \rightarrow H^n \Gamma(U, E_{usu}(\Omega_U^\bullet, F_b))$$

are not strict. Note that if $U := S \setminus D$ is affine, then by the exact sequence in $C(\mathbb{Z})$

$$0 \rightarrow \Gamma_Z(X, E_{usu}(\Omega_X^p)) \rightarrow \Gamma(X, E_{usu}(\Omega_X^p)) \rightarrow \Gamma(U, E_{usu}(\Omega_U^p)) \rightarrow 0$$

we have $H^q \Gamma_Z(X, E_{usu}(\Omega_X^p)) = H^q(\Gamma(X, E_{usu}(\Omega_X^p)))$.

In particular, the maps

$$j^* := \text{ad}(j^*, j_*)(-) : \Gamma(X, E_{usu}(\Omega_X^\bullet(\log D), F_b)) \rightarrow \Gamma(U, E_{usu}(\Omega_U^\bullet, F_b))$$

and

$$\begin{aligned} j^* := \text{ad}(j^*, j_*)(-) : \text{Cone}(\Gamma(X, E_{usu}(\Omega_X^\bullet, F_b)) \rightarrow \Gamma(X, E_{usu}(\Omega_X^\bullet(\log D), F_b))) \rightarrow \\ \text{Cone}(\Gamma(X, E_{usu}(\Omega_X^\bullet, F_b)) \rightarrow \Gamma(U, E_{usu}(\Omega_U^\bullet, F_b))) =: \Gamma(X, \Gamma_Z E_{usu}(\Omega_X^\bullet, F_b)) \end{aligned}$$

are quasi-isomorphism (i.e. if we forgot filtrations), but the first one is NOT an ∞ -filtered quasi-isomorphism whereas the second one is an ∞ -filtered quasi-isomorphism (recall that for $r > 1$ the r -filtered quasi-isomorphism does NOT satisfy the 2 of 3 property for morphism of canonical triangles : see section 2.1).

- (ii) More generally, let $f : X \rightarrow S$ a smooth projective morphism with $S, X \in \text{SmVar}(\mathbb{C})$. Let $D = \cup D_i \subset X$ a normal crossing divisor such that $f|_{D_i} := f \circ i_i : D_i \rightarrow S$ are SMOOTH morphisms (note that it is a very special case), with $i_i : D_i \hookrightarrow X$ the closed embeddings. Consider the open embedding $j : U := X \setminus D \hookrightarrow X$ and $h := f \circ j : U \rightarrow S$.

– The map in $D_{\mathcal{D}fil,\infty}(S)$

$$\begin{aligned} & \text{Hom}((0, \text{ad}(j^*, j_*)(\mathbb{Z}(X/X))), E_{et}(\Omega_{/S}^\bullet, F_b)) : \\ & \mathcal{F}_{S,an}^{GM}(\mathbb{D}(\mathbb{Z}(U/S))) := \text{Hom}(L\mathbb{D}(\mathbb{Z}(U/S)), E_{usu}(\Omega_{/S}^\bullet, F_b)) \\ & \xrightarrow{\sim} \text{Hom}(\text{Cone}(\mathbb{Z}(D) \rightarrow \mathbb{Z}(X)), E_{zar}(\Omega_{/S}^\bullet, F_b)) = f_* E_{usu}(\Omega_{X/S}^\bullet(\text{nul } D), F_b). \end{aligned}$$

is an isomorphism, where we recall $\mathbb{D}(\mathbb{Z}(U)) := f_* j_* E_{et}(\mathbb{Z}(U/U)) = h_* E_{usu}(\mathbb{Z}(U/U))$,

– $\mathcal{F}_{S,an}^{GM}(\mathbb{Z}(U/S)) = h_* E_{usu} \Omega_{U/S}^\bullet, F_b \in D_{\mathcal{D}fil,\infty}(S)$ is NOT isomorphic to $f_* E_{usu}(\Omega_{X/S}^\bullet(\log D), F_b)$ in $D_{\mathcal{D}fil,\infty}(S)$ in general. In particular, the map,

$$j^* := \text{ad}(j^*, j_*)(-) : H^n f_* E_{usu}(\Omega_{X/S}^\bullet(\log D)) \xrightarrow{\sim} H^n h_* E_{usu}(\Omega_{U/S}^\bullet)$$

which is an isomorphism in $D_{\mathcal{D}}(S)$ (i.e. if we forgot filtrations), gives embeddings

$$j^* := \text{ad}(j^*, j_*)(-) : F^p H^n h_* \mathbb{C}_U := F^p H^n f_* E_{usu}(\Omega_{X/S}^\bullet(\log D), F_b) \hookrightarrow F^p H^n h_* E_{usu}(\Omega_{U/S}^\bullet, F_b)$$

which are NOT an isomorphism in general for $n, p \in \mathbb{Z}$. Note that, since $a_U : U \rightarrow \{\text{pt}\}$ is not proper,

$$[\Delta_U] : \mathbb{Z}(U/S) \rightarrow h_* E_{usu}(\mathbb{Z}(U/U))[2d_U]$$

is NOT an equivalence (\mathbb{A}^1, et) local.

– Let $Z \subset X$ a subvariety and denote $U := X \setminus Z$ the open complementary. Denote $M_Z(X/S) := \text{Cone}(M(U/S) \rightarrow M(X/S)) \in \text{DA}(S)$. If $f_{|Z} := f \circ i_Z : Z \rightarrow S$ is a SMOOTH morphism, the map in $D_{\mathcal{D}fil,\infty}(S)$

$$\begin{aligned} & \text{Hom}(G(X, Z), E_{usu}(\Omega^\bullet, F_b)) : \\ & \mathcal{F}_{S,an}^{GM}(M_Z(X/S)) := \text{Hom}(f_{\sharp} \Gamma_Z^\vee \mathbb{Z}(X/X), E_{usu}(\Omega_{/S}^\bullet, F_b)) \xrightarrow{\sim} f_* \Gamma_Z E_{usu}(\Omega_{X/S}^\bullet, F_b) \\ & \xrightarrow{\sim} \mathcal{F}_S^{GM}(M(Z/S)(c)[2c]) = f_{Z*} E_{usu}(\Omega_{Z/S}^\bullet, F_b)(-c)[-2c] \end{aligned}$$

is an isomorphism, where $c = \text{codim}(Z, X)$ and $G(X, Z) : f_{\sharp} \Gamma_Z^\vee \mathbb{Z}(X/X) \rightarrow \mathbb{Z}(Z/S)(c)[2c]$ is the Gysin morphism.

– Let $D \subset X$ a smooth divisor and denote $U := X \setminus Z$ the open complementary. Note that the canonical distinguish triangle in $\text{DA}(S)$

$$M(U/S) \xrightarrow{\text{ad}(j_{\sharp}, j^*)(\mathbb{Z}(X/X))} M(X/S) \xrightarrow{\gamma_Z^\vee(\mathbb{Z}(X/X))} M_D(X/S) \rightarrow M(U/S)[1]$$

give a canonical triangle in $D_{\mathcal{D}fil,\infty}(S)$

$$\begin{aligned} \mathcal{F}_{S,an}^{GM}(M_D(X/S)) & \xrightarrow{\mathcal{F}^{GM}(\gamma_Z^\vee(\mathbb{Z}(X/X)))} \mathcal{F}_{S,an}^{GM}(M(X/S)) \xrightarrow{\mathcal{F}^{GM}(\text{ad}(j_{\sharp}, j^*)(\mathbb{Z}(X/X)))} \mathcal{F}_{S,an}^{GM}(M(U/S)) \\ & \rightarrow \mathcal{F}_{S,an}^{GM}(M_D(X/S))[1], \end{aligned}$$

which is NOT the image of a distinguish triangle in $\pi_S(D(\text{MHM}(S)))$.

Proof. Similar to the proof of theorem 111. □

Definition 147. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and

$N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{Ofil, \mathcal{D}^\infty}(S/(\tilde{S}_I))$

$$\begin{aligned}
& T(\mathcal{F}_{S,an}^{GM}, \otimes)(M, N) : \\
& \mathcal{F}_{S,an}^{GM}(M) \otimes_{O_S}^L \mathcal{F}_{S,an}^{GM}(N) := (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}(F)) \otimes_{O_S} \\
& \quad (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* G), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}(G)) \\
& \quad \xrightarrow{=} ((e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b))) \otimes_{O_{\tilde{S}_I}} \\
& \quad e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* G), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b))), u_{IJ}(F) \otimes u_{IJ}(G)) \\
& \quad \xrightarrow{(T(\otimes, \Omega_{/\tilde{S}_I})(L(i_{I*} j_I^* F), L(i_{I*} j_I^* G)))} \\
& \quad (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F) \otimes \text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* G), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), v_{IJ}(F \otimes G)) \\
& \quad \xrightarrow{=} (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^*(F \otimes G)), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b))), u_{IJ}(F \otimes G) =: \mathcal{F}_{S,an}^{GM}(M \otimes N)
\end{aligned}$$

We have in the analytical case the following :

Proposition 131. *Let $S \in \text{Var}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$*

$$T(\otimes, \mathcal{F}_{S,an}^{GM})(M, N) : \mathcal{F}_{S,an}^{GM}(M \otimes N) \xrightarrow{\sim} \mathcal{F}_{S,an}^{GM}(M) \otimes_{O_S}^L \mathcal{F}_{S,an}^{GM}(N)$$

is an isomorphism.

Proof. Assume first that S is smooth. Let $h_1 : U_1 \rightarrow S$ and $h_2 : U_2 \rightarrow S$ smooth morphisms with $U_1, U_2 \in \text{Var}(\mathbb{C})$ and consider $h_{12} : U_1 \times_S U_2 \rightarrow S$. We then have by lemma 16 the following commutative diagram

$$\begin{array}{ccc}
e(S)_* \mathcal{H}om(\mathbb{Z}(U_1/S) \otimes \mathbb{Z}(U_2/S), E(\Omega_{/S}^\bullet, F_b)) & \xrightarrow{T(\otimes, \mathcal{F}_{S,an}^{GM}(M(U_1/S), M(U_2/S)))} & e(S)_* \mathcal{H}om(\mathbb{Z}(U_1/S), E(\Omega_{/S}^\bullet, F_b)) \otimes_{O_S} e(S)_* \mathcal{H}om(\mathbb{Z}(U_2/S), E(\Omega_{/S}^\bullet, F_b)) \\
\downarrow = & & \downarrow = \\
h_{12*} E(\Omega_{U_1 \times_S U_2/S}, F) & \xrightarrow{Ew_{(U_1, U_2)/S}} & h_{1*} E(\omega_{U_1/S}, F_b) \otimes_{O_S} h_{2*} E(\Omega_{U_2/S}, F_b) \\
\downarrow \iota \otimes \iota & & \downarrow \iota \\
h_{12*} E(h_{12}^* O_S) & \xrightarrow{Ew_{(U_1, U_2)/S}} & h_{1*} E(h_1^* O_S) \otimes_{O_S} h_{2*} E(h_2^* O_S) \\
\downarrow T(h_{12}, \otimes)(O_S, \mathbb{Z}_{U_{12}}) & & \downarrow T(h_1, \otimes)(O_S, \mathbb{Z}_{U_1}) \otimes T(h_2, \otimes)(O_S, \mathbb{Z}_{U_2}) \\
h_{12*} E(\mathbb{Z}_{U_{12}}) \otimes O_S & \xrightarrow{Ew_{(U_1, U_2)/S}} & (h_{1*} E(\mathbb{Z}_{U_1}) \otimes_{O_S} h_{2*} E(\mathbb{Z}_{U_2}))
\end{array}$$

Since $U_1, U_2 \in \text{AnSp}(\mathbb{C})$ are locally contractible topological spaces, the lower row is an equivalence usu local by Kunnetth formula for topological spaces (see section 2). This proves the proposition in the case S is smooth. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. By definition, for $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$,

$$\begin{aligned}
& T(\otimes, \mathcal{F}_{S,an}^{GM})(M, N) : \\
& e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^*(F \otimes G), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b))), u_{IJ}(F \otimes G)) \\
& \quad \xrightarrow{(T(\otimes, \Omega_{/\tilde{S}_I})(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), \text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* G)))} \\
& \quad (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}(F)) \otimes_{O_S} \\
& \quad (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* G), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}(G))
\end{aligned}$$

Since $L(i_{I*} j_I^* F), L(i_{I*} j_I^* G) \in \text{DA}_c(\tilde{S}_I)$, by the smooth case applied to \tilde{S}_I for each I , $T(\otimes, \mathcal{F}_{S,an}^{FDR})(M, N)$ is an equivalence usu local. \square

6.2.2 The analytic filtered De Rham realization functor

Recall from section 2 that, for $S \in \text{Var}(\mathbb{C})$ we have the following commutative diagrams of sites

$$\begin{array}{ccccc}
 \text{AnSp}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,pr}/S & & \\
 \downarrow \text{Ans} & \searrow \rho_S & \downarrow \text{Ans} & \searrow \rho_S & \\
 & \text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,smpr}/S & \\
 \downarrow \text{Ans} & \downarrow \text{Ans} & \downarrow \text{Ans} & \downarrow \text{Ans} & \\
 \text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S & & \\
 \downarrow \text{Ans} & \searrow \rho_S & \downarrow \text{Ans} & \searrow \rho_S & \\
 & \text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S &
 \end{array}$$

and

$$\begin{array}{ccccc}
 \text{AnSp}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})/S & & \\
 \downarrow \text{Ans} & \searrow \rho_S & \downarrow \text{Ans} & \searrow \rho_S & \\
 & \text{AnSp}(\mathbb{C})^{2,smpr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})^{sm}/S & \\
 \downarrow \text{Ans} & \downarrow \text{Ans} & \downarrow \text{Ans} & \downarrow \text{Ans} & \\
 \text{Var}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})/S & & \\
 \downarrow \text{Ans} & \searrow \rho_S & \downarrow \text{Ans} & \searrow \rho_S & \\
 & \text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})^{sm}/S &
 \end{array} \tag{64}$$

and that for $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$ we have the following commutative diagrams of site,

$$\begin{array}{ccccc}
 \text{AnSp}(\mathbb{C})^2/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^2/T & & \\
 \downarrow P(f) & \searrow \rho_T & \downarrow P(f) & \searrow \rho_T & \\
 & \text{AnSp}(\mathbb{C})^{2,sm}/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^{2,sm}/T & \\
 \downarrow P(f) & \downarrow P(f) & \downarrow P(f) & \downarrow P(f) & \\
 \text{AnSp}(\mathbb{C})^2/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^2/S & & \\
 \downarrow P(f) & \searrow \rho_S & \downarrow P(f) & \searrow \rho_S & \\
 & \text{AnSp}(\mathbb{C})^{2,sm}/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^{2,sm}/S &
 \end{array}$$

For $s : \mathcal{I} \rightarrow \mathcal{J}$ a functor, with $\mathcal{I}, \mathcal{J} \in \text{Cat}$, and $f_\bullet : T_\bullet \rightarrow S_{s(\bullet)}$ a morphism with $T_\bullet \in \text{Fun}(\mathcal{J}, \text{Var}(\mathbb{C}))$

and $S_\bullet \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C}))$, we have then the commutative diagram of sites 40

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^2/T_\bullet & \xrightarrow{\mu_{T_\bullet}} & \text{AnSp}(\mathbb{C})^{2,pr}/T_\bullet & & \\
\downarrow P(f_\bullet) & \searrow \rho_{T_\bullet} & \downarrow & \searrow \rho_{T_\bullet} & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/T_\bullet & \xrightarrow{\mu_{T_\bullet}} & \text{AnSp}(\mathbb{C})^{2,smpr}/T_\bullet \\
& & \downarrow & \downarrow P(f_\bullet) & \downarrow P(f_\bullet) \\
\text{AnSp}(\mathbb{C})^2/S_\bullet & \xrightarrow{P(f_\bullet)} & \text{AnSp}(\mathbb{C})^{2,pr}/S_\bullet & & \text{AnSp}(\mathbb{C})^{2,smpr}/S_\bullet \\
& \searrow \rho_{S_\bullet} & \downarrow & \searrow \rho_{S_\bullet} & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S_\bullet & \xrightarrow{\mu_{S_\bullet}} & \text{AnSp}(\mathbb{C})^{2,smpr}/S_\bullet
\end{array}$$

and the following commutative diagrams of site,

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^2/T_\bullet^{an} & \xrightarrow{\text{An}_{T_\bullet}} & \text{Var}(\mathbb{C})^2/T_\bullet & & \\
\downarrow P(f_\bullet) & \searrow \rho_{T_\bullet} & \downarrow & \searrow \rho_{T_\bullet} & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/T_\bullet^{an} & \xrightarrow{\text{An}_{T_\bullet}} & \text{Var}(\mathbb{C})^{2,sm}/T_\bullet \\
& & \downarrow & \downarrow P(f_\bullet) & \downarrow P(f_\bullet) \\
\text{AnSp}(\mathbb{C})^2/S_\bullet^{an} & \xrightarrow{P(f_\bullet)} & \text{Var}(\mathbb{C})^2/S_\bullet & & \text{Var}(\mathbb{C})^{2,sm}/S_\bullet \\
& \searrow \rho_{S_\bullet} & \downarrow & \searrow \rho_{S_\bullet} & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S_\bullet^{an} & \xrightarrow{\text{An}_{S_\bullet}} & \text{Var}(\mathbb{C})^{2,sm}/S_\bullet
\end{array}$$

Definition 148. (i) For $S \in \text{SmVar}(\mathbb{C})$, we consider, using definition 129(i), the filtered complexes of presheaves

$$(\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR}) \in C_{D_S^\infty \text{fil}}(\text{Var}(\mathbb{C})^{2, smpr}/S)$$

given by,

– for $(Y \times S, Z)/S = ((Y \times S, Z), p) \in \text{Var}(\mathbb{C})^{2, smpr}/S$,

$$(\Omega_{/S^{an}}^{\bullet, \Gamma, pr}((Y \times S, Z)/S), F_{DR}) := ((\Omega_{(Y \times S)^{an}/S^{an}}^{\bullet}, F_b) \otimes_{O_{(Y \times S)^{an}}} (\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))^{an})((Y \times S)^{an})$$

with the structure of p^*D_S module given by proposition 64.

– for $g : (Y_1 \times S, Z_1)/S = ((Y_1 \times S, Z_1), p_1) \rightarrow (Y \times S, Z)/S = ((Y \times S, Z), p)$ a morphism in $\text{Var}(\mathbb{C})^{2, smpr}/S$,

$$\begin{aligned}
\Omega_{/S^{an}}^{\bullet, \Gamma, pr}(g) &:= (\Omega_{/S}^{\bullet, \Gamma, pr}(g))^{an} : ((\Omega_{(Y \times S)^{an}/S^{an}}^{\bullet}, F_b) \otimes_{O_{(Y \times S)^{an}}} (\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))^{an})((Y \times S)^{an}) \rightarrow \\
&((\Omega_{(Y_1 \times S)^{an}/S^{an}}^{\bullet}, F_b) \otimes_{O_{(Y_1 \times S)^{an}}} (\Gamma_{Z_1}^{\vee, Hdg}(O_{Y_1 \times S}, F_b))^{an})((Y_1 \times S)^{an}).
\end{aligned}$$

For $S \in \text{SmVar}(\mathbb{C})$, we get the filtered complexes of presheaves

$$(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an}, F_{DR}) := \text{An}_S^{*mod}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR}) := \text{An}_S^*(\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR}) \otimes_{O_S} O_{S^{an}} \in C_{D_S^\infty \text{fil}}(\text{AnSp}(\mathbb{C})^{2, smpr}/S).$$

(ii) For $S \in \text{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_S \text{fil}, D_S^\infty}(\text{Var}(\mathbb{C})^{sm}/S)$

$$\text{Gr}(\Omega_{/S^{an}}) : \text{Gr}_{S^*}^{12}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_b) \rightarrow \text{An}_{S^*}(\Omega_{/S}^{\bullet}, F_b)$$

given by

$$\begin{aligned} \text{Gr}(\Omega_{/S^{an}})(U/S) &:= (\text{Gr}(\Omega_{/S})(U/S))^{an} \otimes m : \\ J_S((\Omega_{(U \times S)^{an}/S^{an}}, F_b) \otimes_{O_{(U \times S)^{an}}} (\Gamma_U^{\vee, Hdg}(O_{U \times S}, F_b))^{an})((U \times S)^{an}) &\rightarrow (\Omega_{U^{an}/S^{an}}, F_b), \end{aligned}$$

where $\text{Gr}(\Omega_{/S^{an}})(U/S)(\omega \otimes m \otimes P) := P(\text{Gr}(\Omega_{/S})(U/S)(\omega \otimes m))$ with $P \in \Gamma(S, D_S^\infty)$, see definition 129(ii), which gives by adjonction

$$\text{Gr}(\Omega_{/S^{an}}) := I(\text{An}_S^{*mod}, \text{An}_S)(\text{Gr}(\Omega_{/S^{an}})) : J_S(\text{Gr}_{S^*}^{12}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr, an}, F_b)) \rightarrow (\Omega_{/S}^{\bullet, \Gamma}, F_b)$$

in $C_{O_S \text{fil}, D_S^\infty}(\text{AnSp}(\mathbb{C})^{sm}/S)$.

Definition 149. For $S \in \text{SmVar}(\mathbb{C})$, we have the canonical map in $C_{O_S \text{fil}, D_S^\infty}(\text{Var}(\mathbb{C})^{2, smpr}/S)$

$$T(\Omega_{/S^{an}}^\Gamma) : \text{An}_{S^*} \mu_{S^*}(\Omega_{/S^{an}}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR})$$

given by, for $(Y \times S, X)/S = ((Y \times S, Z), p) \in \text{Var}(\mathbb{C})^{2, smpr}/S$

$$\begin{aligned} T(\Omega_{/S^{an}}^\Gamma)((Y \times S, Z)/S) &:= (T(\Omega_{/S}^\Gamma)((Y \times S, Z)/S))^{an} : \\ (\Omega_{/S}^{\bullet, \Gamma}, F_b)((Y \times S)^{an}, Z^{an})/S &:= \mathbb{D}_{p^* O_S} L_{p^* O} \Gamma_Z E_{usu}(\mathbb{D}_{p^* O_S} L_{p^* O}(\Omega_{(Y \times S)^{an}/S^{an}}^{\bullet, \Gamma}, F_b))((Y \times S)^{an}) \rightarrow \\ ((\Omega_{(Y \times S)^{an}/S^{an}}^{\bullet, \Gamma}, F_b) \otimes_{O_{(Y \times S)^{an}}} (\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))^{an})((Y \times S)^{an}) &=: (\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR})((Y \times S, Z)/S), \end{aligned}$$

see definition 130. By definition we have $\text{Gr}^O(\Omega_{/S^{an}}) = \text{Gr}(\Omega_{/S^{an}}) \circ T(\Omega_{/S^{an}}^\Gamma)$.

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSm}(\mathbb{C})$.

- We have the canonical morphism in $C_{g^* O_S \text{fil}, g^* D_S^\infty}(\text{AnSp}(\mathbb{C})^{2, sm}/T)$

$$\Omega_{/(T/S)}^\Gamma : g^*(\Omega_{/S}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/T}^{\bullet, \Gamma}, F_b)$$

induced by the pullback of differential forms : for $((V, Z_1)/T) = ((V, Z_1), h) \in \text{AnSp}(\mathbb{C})^{2, sm}/T$,

$$\begin{aligned} \Omega_{/(T/S)}^\Gamma((V, Z_1)/T) &: \\ g^* \Omega_{/S}^{\bullet, \Gamma}((V, Z_1)/T) &:= \lim_{(h : (U, Z) \rightarrow \text{Ssm}, g_1 : (V, Z_1) \rightarrow (U_T, Z_T), h, g)} \Omega_{/S}^{\bullet, \Gamma}((U, Z)/S) \\ \xrightarrow{\Omega_{/S}^{\bullet, \Gamma}(g' \circ g_1)} \Omega_{/S}^{\bullet, \Gamma}((V, Z_1)/S) &\xrightarrow{\Gamma_{Z_1}^{\vee, h} q(Y_1 \times T)} \Omega_{/T}^{\bullet, \Gamma}((V, Z_1)/T), \end{aligned}$$

where $g' : U_T := U \times_S T \rightarrow U$ is the base change map and $q : \Omega_{Y_1 \times T/S}^{\bullet, \Gamma} \rightarrow \Omega_{Y_1 \times T/T}^{\bullet, \Gamma}$ is the quotient map. It induces the canonical morphisms in $C_{g^* O_S \text{fil}, g^* D_S^\infty}(\text{AnSp}(\mathbb{C})^{2, sm}/T)$:

$$E\Omega_{/(T/S)}^\Gamma : g^* E_{et}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \xrightarrow{T(g, E_{et})(\Omega_{/S}^{\bullet, \Gamma}, F_b)} E_{et}(g^*(\Omega_{/S}^{\bullet, \Gamma}, F_b)) \xrightarrow{E_{et}(\Omega_{/(T/S)}^\Gamma)} E_{et}(\Omega_{/T}^{\bullet, \Gamma}, F_b)$$

- We have the canonical morphism in $C_{g^* D_S^\infty \text{fil}}(\text{Var}(\mathbb{C})^{2, smpr}/T)$

$$\Omega_{/(T/S)^{an}}^{\Gamma, pr} : g^*(\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR}) \rightarrow (\Omega_{/T^{an}}^{\bullet, \Gamma, pr}, F_{DR})$$

induced by the pullback of differential forms : for $((Y_1 \times T, Z_1)/T) = ((Y_1 \times T, Z_1), p) \in \text{Var}(\mathbb{C})^{2, smpr}/T$,

$$\begin{aligned} \Omega_{/(T/S)^{an}}^{\Gamma, pr}((Y_1 \times T, Z_1)/T) &: \\ g^* \Omega_{/S^{an}}^{\bullet, \Gamma, pr}((Y_1 \times T, Z_1)/T) &:= \lim_{(h : (Y \times S, Z) \rightarrow S, g_1 : (Y_1 \times T, Z_1) \rightarrow (Y \times T, Z_T), h, g)} \Omega_{/S^{an}}^{\bullet, \Gamma, pr}((Y \times T, Z)/S) \\ \xrightarrow{\Omega_{/S^{an}}^{\bullet, \Gamma, pr}(g' \circ g_1)} \Omega_{/S^{an}}^{\bullet, \Gamma, pr}((Y_1 \times T, Z_1)/S) &\xrightarrow{q(-)((Y_1 \times T)^{an})} \Omega_{/T^{an}}^{\bullet, \Gamma, pr}((Y_1 \times T, Z_1)/T), \end{aligned}$$

where $g' = (I_Y \times g) : Y \times T \rightarrow Y \times S$ is the base change map and

$$q(M) : \Omega_{(Y_1 \times T)^{an}/S^{an}} \otimes_{\mathcal{O}_{(Y_1 \times T)^{an}}} (M, F) \rightarrow \Omega_{(Y_1 \times T)^{an}/T^{an}} \otimes_{\mathcal{O}_{(Y_1 \times T)^{an}}} (M, F)$$

is the quotient map. It induces the canonical morphisms in $C_{g^*D_S \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/T)$:

$$E\Omega_{(T/S)}^{\Gamma, \text{pr}} : g^* E_{et}(\Omega_{/S^{an}}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \xrightarrow{T(g, E)(-)} E_{et}(g^*(\Omega_{/S^{an}}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \xrightarrow{E_{et}(\Omega_{(T/S)^{an}}^{\Gamma, \text{pr}})} E_{et}(\Omega_{/T^{an}}^{\bullet, \Gamma, \text{pr}}, F_{DR})$$

and

$$E\Omega_{(T/S)^{an}}^{\Gamma, \text{pr}} : g^* E_{zar}(\Omega_{/S^{an}}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \xrightarrow{T(g, E)(-)} E_{zar}(g^*(\Omega_{/S^{an}}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \xrightarrow{E_{zar}(\Omega_{(T/S)^{an}}^{\Gamma, \text{pr}})} E_{zar}(\Omega_{/T^{an}}^{\bullet, \Gamma, \text{pr}}, F_{DR}).$$

Definition 150. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. We have, for $F \in C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$, the canonical transformation in $C_{\mathcal{D}^\infty \text{fil}}(T)$:

$$\begin{aligned} T(g, \Omega_{/S^{an}}^{\Gamma, \text{pr}})(F) &: g^{* \text{mod}} L_{De}(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om^\bullet(\text{An}_S^* F, E_{et}(\Omega_{/S^{an}}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR})) \\ &\stackrel{:=}{\rightarrow} (g^* L_{De}(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S^{an}}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR}))) \otimes_{g^* \mathcal{O}_S} \mathcal{O}_T \\ &\xrightarrow{T(g, \text{Gr}^{12})(-) \circ T(e, g)(-) \circ q} e(T)_* \text{Gr}_{T^*}^{12} g^* \mathcal{H}om^\bullet(\text{An}_S^* F, E_{et}(\Omega_{/S^{an}}^{\bullet, \Gamma, \text{pr}}, F_{DR})) \otimes_{g^* \mathcal{O}_S} \mathcal{O}_T \\ &\xrightarrow{(T(g, \text{hom})(-, -) \otimes I)} e(T)_* \text{Gr}_{T^*}^{12} \mathcal{H}om^\bullet(\text{An}_T^* g^* F, g^* E_{et}(\Omega_{/S^{an}}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR})) \otimes_{g^* \mathcal{O}_S} \mathcal{O}_T \\ &\xrightarrow{ev(\text{hom}, \otimes)(-, -, -)} e(T)_* \text{Gr}_{T^*}^{12} \mathcal{H}om^\bullet(\text{An}_T^* g^* F, g^* E_{et}(\Omega_{/S^{an}}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR})) \otimes_{g^* e(S)_* \mathcal{O}_S} e(T)^* \mathcal{O}_T \\ &\xrightarrow{\mathcal{H}om^\bullet(\text{An}_T^* g^* F, (E\Omega_{(T/S)^{an}}^{\Gamma, \text{pr}} \otimes m))} e(T)_* \text{Gr}_{T^*}^{12} \mathcal{H}om^\bullet(\text{An}_T^* g^* F, E_{et}(\Omega_{/T^{an}}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR})) \end{aligned}$$

- Let $S \in \text{AnSm}(\mathbb{C})$. We have the map in $C_{\mathcal{O}_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$:

$$w_S : (\Omega_{/S}^{\bullet, \Gamma}, F_b) \otimes_{\mathcal{O}_S} (\Omega_{/S}^{\bullet, \Gamma}, F_b) \rightarrow (\Omega_{/S}^{\bullet, \Gamma}, F_b) :$$

given by for $h : (U, Z) \rightarrow S \in \text{Var}(\mathbb{C})^{2, \text{sm}}/S$,

$$\begin{aligned} w_S((U, Z)/S) &: (\Gamma_Z^{\vee, h} L_{h^* \mathcal{O}_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b) \otimes_{p^* \mathcal{O}_S} \Gamma_Z^{\vee, h} L_{h^* \mathcal{O}_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b))(U) \\ &\xrightarrow{(DR(-)(\gamma_Z^{\vee, h}(-)) \circ w_{U/S})^\gamma(U)} \Gamma_Z^{\vee, h} L_{h^* \mathcal{O}_S}(\Omega_{U/S}^{\bullet, \Gamma}, F_b)(U) \end{aligned}$$

which induces the map in $C_{\mathcal{O}_S \text{fil}, D_S}(\text{Var}(\mathbb{C})^{2, \text{sm}}/S)$

$$Ew_S : E_{et}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \otimes_{\mathcal{O}_S} E_{et}(\Omega_{/S}^{\bullet, \Gamma}, F_b) \xrightarrow{=} E_{et}((\Omega_{/S}^{\bullet, \Gamma}, F_b) \otimes_{\mathcal{O}_S} (\Omega_{/S}^{\bullet, \Gamma}, F_b)) \xrightarrow{E_{et}(w_S)} E_{et}(\Omega_{/S}^{\bullet, \Gamma}, F_b).$$

- Let $S \in \text{SmVar}(\mathbb{C})$. We have the map in $C_{\mathcal{D}^\infty \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$:

$$w_S : (\Omega_{/S^{an}}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \otimes_{\mathcal{O}_S} (\Omega_{/S^{an}}^{\bullet, \Gamma, \text{pr}}, F_{DR}) \rightarrow (\Omega_{/S^{an}}^{\bullet, \Gamma, \text{pr}}, F_{DR})$$

given by for $p : (Y \times S, Z) \rightarrow S \in \text{Var}(\mathbb{C})^{2, \text{smpr}}/S$,

$$\begin{aligned} &w_S((Y \times S, Z)/S) : \\ &(((\Omega_{Y \times S/S}^{\bullet, \Gamma} \otimes_{\mathcal{O}_{Y \times S}} \Gamma_Z^{\vee, \text{Hdg}})(\mathcal{O}_{Y \times S}, F_b)) \otimes_{p^* \mathcal{O}_S} (\Omega_{Y \times S/S}^{\bullet, \Gamma} \otimes_{\mathcal{O}_{Y \times S}} \Gamma_Z^{\vee, \text{Hdg}}(\mathcal{O}_{Y \times S}, F_b)))(Y \times S) \\ &\xrightarrow{(DR(-)(\gamma_Z^{\vee, \text{Hdg}}(-)) \circ w_{Y \times S/S})^\gamma(Y \times S)} (\Omega_{Y \times S/S}^{\bullet, \Gamma} \otimes_{\mathcal{O}_{Y \times S}} \Gamma_Z^{\vee, \text{Hdg}}(\mathcal{O}_{Y \times S}, F_b))(Y \times S) \end{aligned}$$

which induces the map in $C_{D_S^\infty \text{fil}}(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$

$$\begin{aligned} Ew_S : E_{et}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR}) \otimes_{O_S} E_{et}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR}) &\xrightarrow{=} \\ E_{et}((\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR}) \otimes_{O_S} (\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR})) &\xrightarrow{E_{et}(w_S)} E_{et}(\Omega_{/S^{an}}^{\bullet, \Gamma, pr}, F_{DR}) \end{aligned}$$

by the functoriality of the Godement resolution (see section 2).

Definition 151. Let $S \in \text{SmVar}(\mathbb{C})$. We have, for $F, G \in C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$, the canonical transformation in $C_{D^\infty \text{fil}}(S^{an})$:

$$\begin{aligned} &T(\otimes, \Omega)(F, G) : \\ e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(\text{An}_S^* F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR})) \otimes_{O_S} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(\text{An}_S^* G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR})) \\ &\xrightarrow{=} e(S)_* \text{Gr}_{S^*}^{12} (\mathcal{H}om(\text{An}_S^* F, E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR})) \otimes_{O_S} \mathcal{H}om(\text{An}_S^* G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR}))) \\ &\xrightarrow{T(\mathcal{H}om, \otimes)(-)} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(\text{An}_S^* F \otimes \text{An}_S^* G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR})) \otimes_{O_S} E_{usu}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR})) \\ &\xrightarrow{=} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(\text{An}_S^*(F \otimes G), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR})) \otimes_{O_S} E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr}, F_{DR})) \\ &\xrightarrow{\mathcal{H}om(F \otimes G, \text{An}_S^{* \text{mod}} Ew_S)} e(S)_* \text{Gr}_{S^*}^{12} \mathcal{H}om(F \otimes G, E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR})). \end{aligned}$$

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $\tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and for $J \subset I$ the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

where $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ is the projection and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. This gives the diagram of algebraic varieties $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Var}(\mathbb{C}))$ which the diagram of sites $\text{Var}(\mathbb{C})^{2, \text{smpr}}/(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. This gives also the diagram of algebraic varieties $(\tilde{S}_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N})^{op}, \text{Var}(\mathbb{C}))$ which the diagram of sites $\text{Var}(\mathbb{C})^{2, \text{smpr}}/(\tilde{S}_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N})^{op}, \text{Cat})$. We then get

$$((\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}) \in C_{D_{(\tilde{S}_I)}^\infty \text{fil}}(\text{AnSp}(\mathbb{C})^{2, \text{smpr}}/(\tilde{S}_I))$$

with

$$\begin{aligned} T_{IJ} : (\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}] &\xrightarrow{\text{ad}(p_{IJ}^{* \text{mod}[-]}, p_{IJ*}(-))} p_{IJ*} p_{IJ}^* (\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}) \otimes_{p_{IJ*} O_{\tilde{S}_I}} O_{\tilde{S}_J}[-d_{\tilde{S}_J}] \\ &\xrightarrow{m \circ p_{IJ*} \Omega_{/(\tilde{S}_J/\tilde{S}_I)}^{\Gamma, pr, an}[-d_{\tilde{S}_J}]} p_{IJ*} (\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_J}]. \end{aligned}$$

For $(G_I, K_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2, \text{smpr}}/(\tilde{S}_I)^{op})$, we denote (see section 2)

$$\begin{aligned} &e'((\tilde{S}_I)_* \mathcal{H}om((G_I, K_{IJ}), (E_{usu}(\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) := \\ &(e'(\tilde{S}_I)_* \mathcal{H}om(G_I, E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}((G_I, K_{IJ}))) \in C_{D^\infty \text{fil}}((\tilde{S}_I)) \end{aligned}$$

with

$$\begin{aligned}
& u_{IJ}((G_I, K_{IJ})) : e'(\tilde{S}_I)_* \mathcal{H}om(G_I, E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}] \\
& \xrightarrow{\text{ad}(p_{IJ}^{*mod[-1]}, p_{IJ*})(-) \circ T(p_{IJ}, e)(-)} p_{IJ*} e'(\tilde{S}_J)_* p_{IJ}^* \mathcal{H}om(G_I, E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})) \otimes_{p_{IJ*} \mathcal{O}_{\tilde{S}_I}} \mathcal{O}_{\tilde{S}_J}[-d_{\tilde{S}_J}] \\
& \xrightarrow{T(p_{IJ}, hom)(-, -)} p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om(p_{IJ}^* G_I, p_{IJ}^* E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})) \otimes_{p_{IJ*} \mathcal{O}_{\tilde{S}_I}} \mathcal{O}_{\tilde{S}_J}[-d_{\tilde{S}_J}] \\
& \xrightarrow{m \circ \mathcal{H}om(p_{IJ}^* G_I, T_{IJ})} p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om(p_{IJ}^* G_I, E_{usu}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_J}] \\
& \xrightarrow{\mathcal{H}om(K_{IJ}, E_{usu}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr, an}, F_{DR}))} p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om(G_J, E_{usu}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_J}].
\end{aligned}$$

This gives in particular

$$(\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ} \in C_{D_{(\tilde{S}_I)}^\infty} \text{fil}(\text{AnSp}(\mathbb{C})^{2, (sm)pr} / (\tilde{S}_I)^{op}).$$

We now define the filtered analytic De Rahm realization functor.

Definition 152. (i) Let $S \in \text{SmVar}(\mathbb{C})$. We have, using definition 148 and definition 41, the functor

$$\begin{aligned}
\mathcal{F}_{S, an}^{FDR} : C(\text{Var}(\mathbb{C})^{sm} / S) &\rightarrow C_{D^\infty \text{fil}}(S^{an}), F \mapsto \\
\mathcal{F}_{S, an}^{FDR}(F) &:= e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \hat{R}^{CH}(\rho_S^* L(F)), E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_S] \\
&= e'(S)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_S^* L(F)), \text{An}_{S*} E_{et}(\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_S]
\end{aligned}$$

denoting for short $e'(S) = e(S) \circ \text{Gr}_S^{12}$.

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Consider, for $I \subset J$, the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. We have, using definition 148 and definition 41, the functor

$$\begin{aligned}
\mathcal{F}_{S, an}^{FDR} : C(\text{Var}(\mathbb{C})^{sm} / S) &\rightarrow C_{D^\infty \text{fil}}(S^{an} / (\tilde{S}_I^{an})), F \mapsto \\
\mathcal{F}_{S, an}^{FDR}(F) &:= e'(\tilde{S}_I)_* \mathcal{H}om^\bullet((\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F))), \hat{R}^{CH}(T^q(D_{IJ})(j_I^* F))), \\
&\quad (E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}) \\
&\quad (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F))), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\
&= (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F))), \text{An}_{\tilde{S}_I*} E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F))
\end{aligned}$$

where we have denoted for short $e'(\tilde{S}_I) = e(\tilde{S}_I) \circ \text{Gr}_{\tilde{S}_I}^{1,2}$, and

$$\begin{aligned}
& u_{IJ}^q(F)[d_{\tilde{S}_J}] : e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{et}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})) \\
& \xrightarrow{\text{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} p_{IJ*} p_{IJ}^{*mod} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})) \\
& \xrightarrow{p_{IJ*} T(p_{IJ}, \Omega^{\Gamma, pr})(-)} p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* p_{IJ}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{usu}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr, an}, F_{DR})) \\
& \xrightarrow{\mathcal{H}om(T(p_{IJ}, \hat{R}^{CH})(Li_{I*} j_I^* F)^{-1}, E_{et}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr, an}, F_{DR}))} \\
& p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_J}^* p_{IJ}^* L(i_{I*} j_I^* F)), E_{usu}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr}, F_{DR})) \\
& \xrightarrow{\mathcal{H}om(\hat{R}_{\tilde{S}_J}^{CH}(T^q(D_{IJ})(j_I^* F)), E_{usu}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr, an}, F_{DR}))} \\
& p_{IJ*} e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_J}^* L(i_{J*} j_J^* F)), E_{usu}(\Omega_{/\tilde{S}_J}^{\bullet, \Gamma, pr, an}, F_{DR})).
\end{aligned}$$

For $I \subset J \subset K$, we have obviously $p_{IJ*} u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$.

We have the following key proposition :

Recall, see section 2, that we have the projection morphisms of sites $p_a : \text{AnSp}(\mathbb{C})^{2, smpr} / (\tilde{S}_I)^{op} \rightarrow \text{AnSp}(\mathbb{C})^{2, smpr} / (\tilde{S}_I)^{op}$ given by the functor

$$\begin{aligned}
& p_a : \text{AnSp}(\mathbb{C})^{2, smpr} / (\tilde{S}_I)^{op} \rightarrow \text{AnSp}(\mathbb{C})^{2, smpr} / (\tilde{S}_I)^{op}, \\
& p_a((Y_I \times \tilde{S}_I, Z_I) / \tilde{S}_I, s_{IJ}) := ((Y_I \times \mathbb{D}^1 \times \tilde{S}_I, Z_I \times \mathbb{D}^1) / \tilde{S}_I, s_{IJ} \times I), \\
& p_a((g_I) : ((Y'_I \times \tilde{S}_I, Z'_I) / \tilde{S}_I, s'_{IJ}) \rightarrow ((Y_I \times \tilde{S}_I, Z_I) / \tilde{S}_I, s_{IJ})) = \\
& (g_I \times I) : ((Y'_I \times \mathbb{D}^1 \times \tilde{S}_I, Z'_I \times \mathbb{D}^1) / \tilde{S}_I, s'_{IJ} \times I) \rightarrow ((Y_I \times \mathbb{D}^1 \times \tilde{S}_I, Z_I \times \mathbb{D}^1) / \tilde{S}_I, s_{IJ} \times I).
\end{aligned}$$

We have the following key proposition :

Proposition 132. (i1) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. The complex of presheaves $(\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, pr, an}, F_{DR}) \in C_{D_{(\tilde{S}_I)}^\infty, fil}(\text{AnSp}(\mathbb{C})^{2, smpr} / (\tilde{S}_I)^{op})$ is 2-filtered \mathbb{D}^1 homotopic, that is

$$\text{ad}(p_a^*, p_{a*})(\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, pr, an})^{an}, F_{DR} : (\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR}) \rightarrow p_{a*} p_a^*(\Omega_{/(\tilde{S}_I)}^{\bullet, \Gamma, pr, an})^{an}, F_{DR}$$

is a 2-filtered homotopy.

(i2) Let $S \in \text{SmVar}(\mathbb{C})$. The complex of presheaves $(\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR}) \in C_{D_S^\infty, fil}(\text{AnSp}(\mathbb{C})^{2, smpr} / S)$ admits transferts, i.e.

$$\text{Tr}(S)_* \text{Tr}(S)^*(\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR}) = (\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR}).$$

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $m = (m_I) : (Q_{1I}, K_{IJ}^1) \rightarrow (Q_{2I}, K_{IJ}^2)$ be an equivalence (\mathbb{D}^1, usu) local with $(Q_{1I}, K_{IJ}) \rightarrow (Q_{2I}, K_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{smpr} / (\tilde{S}_I)^{op})$ complexes of representable presheaves. Then, the map in $C_{D^\infty, fil}((\tilde{S}_I))$

$$\begin{aligned}
& M := (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(m_I, E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}])) : \\
& e'((\tilde{S}_I)_* \mathcal{H}om^\bullet((Q_{2I}, K_{IJ}^1), (E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))) \\
& \rightarrow e'((\tilde{S}_I)_* \mathcal{H}om^\bullet((Q_{1I}, K_{IJ}^1), (E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})))
\end{aligned}$$

is a 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{D^\infty, fil, \infty}((\tilde{S}_I))$.

Proof. Similar to proposition 113. □

We deduce the following:

Proposition 133. *Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.*

(i) *Let $m = (m_I) : (Q_{1I}, K_{IJ}^1) \rightarrow (Q_{2I}, K_{IJ}^2)$ be an etale local equivalence local with $(Q_{1I}, K_{IJ}^1), (Q_{2I}, K_{IJ}^2) \in C(\text{Var}(\mathbb{C})^{sm}/(\tilde{S}_I))$ complexes of projective presheaves. Then,*

$$\begin{aligned} & (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}_S^{CH}(m_I), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}]) : \\ & e'(\tilde{S}_I)_* \mathcal{H}om^\bullet((\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_S^* Q_{1I}), R^{CH}(K_{IJ}^1)), (E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})) \\ & \rightarrow e'(\tilde{S}_I)_* \mathcal{H}om^\bullet((\text{An}_{\tilde{S}_I}^* L\hat{R}^{CH}(\rho_S^* Q_{2I}), R^{CH}(K_{IJ}^2)), (E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})) \end{aligned}$$

is a filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}^\infty fil}(\tilde{S}_I)$.

(ii) *Let $m = (m_I) : (Q_{1I}, K_{IJ}^1) \rightarrow (Q_{2I}, K_{IJ}^2)$ be an equivalence (\mathbb{A}^1, et) local equivalence local with $(Q_{1I}, K_{IJ}^1), (Q_{2I}, K_{IJ}^2) \in C(\text{Var}(\mathbb{C})^{sm}/(\tilde{S}_I))$ complexes of projective presheaves. Then,*

$$\begin{aligned} & (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}_S^{CH}(m_I), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}]) : \\ & e'(\tilde{S}_I)_* \mathcal{H}om^\bullet((\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_S^* Q_{1I}), R^{CH}(K_{IJ}^1)), (E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})) \\ & \rightarrow e'(\tilde{S}_I)_* \mathcal{H}om^\bullet((\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_S^* Q_{2I}), R^{CH}(K_{IJ}^2)), (E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ})) \end{aligned}$$

is an filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}^\infty fil}(\tilde{S}_I)$.

Proof. Similar to the proof of proposition 114. □

Definition 153. (i) *Let $S \in \text{SmVar}(\mathbb{C})$. We define using definition 152(i) and proposition 133(ii) the filtered algebraic De Rahm realization functor defined as*

$$\begin{aligned} & \mathcal{F}_{S, an}^{FDR} : \text{DA}_c(S) \rightarrow D_{\mathcal{D}^\infty fil}(S^{an}), M \mapsto \mathcal{F}_S^{FDR}(M) := \\ & e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \hat{R}^{CH}(\rho_S^* L(F)), E_{usu}(\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_S] \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(i)' *For the Corti-Hanamura weight structure W on $\text{DA}_c(S)^-$, we define using definition 152(i) and proposition 133(ii)*

$$\begin{aligned} & \mathcal{F}_{S, an}^{FDR} : \text{DA}_c^-(S) \rightarrow D_{\mathcal{D}^\infty(1,0) fil}^-(S^{an}), M \mapsto \mathcal{F}_S^{FDR}((M, W)) := \\ & e'(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \hat{R}^{CH}(\rho_S^* L(F, W)), E_{usu}(\Omega_{/S}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_S] \end{aligned}$$

where $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)((F, W))$ using corollary 1. Note that the filtration induced by W is a filtration by sub D_S module, which is a stronger property than Griffithz transversality. Of course, the filtration induced by F satisfy only Griffithz transversality in general.

(ii) *Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define, using definition 152(ii) and proposition 133(ii), the filtered algebraic De Rahm realization functor defined as*

$$\begin{aligned} & \mathcal{F}_{S, an}^{FDR} : \text{DA}_c(S) \rightarrow D_{\mathcal{D}^\infty fil}(S^{an}/(\tilde{S}_I^{an})), M \mapsto \mathcal{F}_S^{FDR}(M) := \\ & (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \end{aligned}$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 133.

(ii)' For the Corti-Hanamura weight structure W on $\mathrm{DA}_c^-(S)$, using definition 133(ii) and proposition 114(ii),

$$\mathcal{F}_{S,an}^{FDR} : \mathrm{DA}_c^-(S) \rightarrow D_{\mathcal{D}^\infty(1,0)_{fil}}^-(S^{an}/(\tilde{S}_I^{an})), M \mapsto \mathcal{F}_S^{FDR}((M, W)) := (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathrm{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} J_I^*(F, W))), E_{usu}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F, W))$$

where $(F, W) \in C_{fil}(\mathrm{Var}(\mathbb{C})^{sm}/S)$ is such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$ using corollary 1. Note that the filtration induced by W is a filtration by sub $D_{\tilde{S}_I}$ -modules, which is a stronger property than Griffithz transversality. Of course, the filtration induced by F satisfy only Griffithz transversality in general.

Proposition 134. For $S \in \mathrm{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$, the functor $\mathcal{F}_{S,an}^{FDR}$ is well defined.

Proof. Similar to the proof of proposition 115. □

Proposition 135. Let $f : X \rightarrow S$ a morphism with $S, X \in \mathrm{Var}(\mathbb{C})$. Assume there exist a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $\bar{Y} \in \mathrm{PSmVar}(\mathbb{C})$ a compactification of Y with $\bar{Y} \setminus Y = D$ a normal crossing divisor, denote $k : D \hookrightarrow \bar{Y}$ the closed embedding and $n : Y \hookrightarrow \bar{Y}$ the open embedding. Denote $\bar{X} \subset \bar{Y} \times S$ the closure of $X \subset \bar{Y} \times S$. We have then the following commutative diagram in $\mathrm{Var}(\mathbb{C})$

$$\begin{array}{ccccc} X & \xrightarrow{l} & Y \times S & & \\ \downarrow & & \downarrow & \searrow^{(n \times I) p_S} & \\ \bar{X} & \xrightarrow{l} & \bar{Y} \times S & \xrightarrow{\bar{p}_S} & S \\ \uparrow & \nearrow^{l_Z} & \uparrow & \nearrow^{(k \times I)} & \\ Z := \bar{X} \setminus X & \longrightarrow & D \times S & & \end{array} .$$

Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. Denote $\bar{X}_I := \bar{X} \cap (\bar{Y} \times S_I) \subset \bar{Y} \times \tilde{S}_I$ the closure of $X_I \subset \bar{Y} \times \tilde{S}_I$, and $Z_I := Z \cap (\bar{Y} \times S_I) = \bar{X}_I \setminus X_I \subset \bar{Y} \times \tilde{S}_I$. We have then for $I \subset [1, \dots, l]$, the following commutative diagram in $\mathrm{Var}(\mathbb{C})$

$$\begin{array}{ccccc} X_I & \xrightarrow{l_I} & Y \times \tilde{S}_I & & \\ \downarrow & & \downarrow & \searrow^{(n \times I) p_{\tilde{S}_I}} & \\ \bar{X}_I & \xrightarrow{l_I} & \bar{Y} \times \tilde{S}_I & \xrightarrow{\bar{p}_{\tilde{S}_I}} & \tilde{S}_I \\ \uparrow & \nearrow^{l_{Z_I}} & \uparrow & \nearrow^{(k \times I)} & \\ Z_I = \bar{X}_I \setminus X_I & \longrightarrow & D \times \tilde{S}_I & & \end{array} .$$

Let $F(X/S) := p_{S,\sharp}\Gamma_X^\vee\mathbb{Z}(X \times S/X \times S)$. We have then the following isomorphism in $D_{\mathcal{D}fil}(S/(\tilde{S}_I))$

$$\begin{aligned} I(X/S) : \mathcal{F}_{S,an}^{FDR}(M(X/S)) &\xrightarrow{\cong} \\ (e'_*\mathcal{H}om(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^* F(X/S))), E_{usu}(\Omega_{\tilde{S}_I}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S))) & \\ \xrightarrow{(\mathcal{H}om(\text{An}_{\tilde{S}_I}^* \hat{R}_{\tilde{S}_I}^{CH}(N_I(X/S)), E_{usu}(\Omega_{\tilde{S}_I}^{\bullet,\Gamma,pr,an}, F_{DR})))} & \\ (e'_*\mathcal{H}om(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* Q(X_I/\tilde{S}_I)), E_{usu}(\Omega_{\tilde{S}_I}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S))) & \\ \xrightarrow{I((\bar{X}_I, Z_I)/\tilde{S}_I)} & \\ (\bar{p}_{\tilde{S}_I*} E_{usu}((\Omega_{\bar{Y} \times \tilde{S}_I/\tilde{S}_I}^\bullet, F_b) \otimes_{\mathcal{O}_{Y \times \tilde{S}_I}} (n \times I)_{!} \text{Hdg} \Gamma_{X_I}^{\vee, \text{Hdg}}(\mathcal{O}_{(Y \times \tilde{S}_I)^{an}}, F_b))(d_Y + d_{\tilde{S}_I})[2d_Y + d_{\tilde{S}_I}], w_{IJ}(X/S)) & \\ \xrightarrow{\cong} \iota_S Rf_{!}^{\text{Hdg}}(\Gamma_{X_I}^{\vee, \text{Hdg}}(\mathcal{O}_{(Y \times \tilde{S}_I)^{an}}, F_b)(d_Y)[2d_Y], x_{IJ}(X/S)) \xrightarrow{\cong} \iota_S Rf_{!}^{\text{Hdg}} f_{\text{Hdg}}^{*mod} \mathbb{Z}_{S^{an}}^{\text{Hdg}} & \end{aligned}$$

Proof. Similar to the proof of proposition 116. \square

Corollary 5. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F) \in \text{DA}_c(S)$,

$$\begin{aligned} H^i \mathcal{F}_{S,an}^{FDR}(M, W) &:= (a_{usu} H^i e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F, W))), \\ E_{usu}(\Omega_{\tilde{S}_I}^{\bullet,\Gamma,pr,an}, F_{DR}))[-d_{\tilde{S}_I}], H^i u_{IJ}^q(F, W)) &\in \pi_S(\text{MHM}(S^{an})) \end{aligned}$$

for all $i \in \mathbb{Z}$, and for all $p \in \mathbb{Z}$,

$$\mathcal{F}_{S,an}^{FDR}(M, W) \in D_{\mathcal{D}^\infty(1,0)fil}(S/(\tilde{S}_I))$$

is the class of a complex $\mathcal{F}_{S,an}^{FDR}(F, W) \in C_{\mathcal{D}^\infty(1,0)fil}(S/(\tilde{S}_I))$ such that for all $k \in \mathbb{Z}$, the differentials of $\text{Gr}_k^W \mathcal{F}_{S,an}^{FDR}(F, W)$ are strict for the filtration F .

Proof. Similar to the proof of corollary 4. \square

Proposition 136. For $S \in \text{Var}(\mathbb{C})$ not smooth, the functor (see corollary 4)

$$\iota_S^{-1} \mathcal{F}_{S,an}^{FDR} : \text{DA}_c^-(S)^{op} \rightarrow \pi_S(D(\text{MHM}(S^{an})))$$

does not depend on the choice of the open cover $S = \cup_i S_i$ and the closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

Proof. Similar to the proof of proposition 117. \square

We have the canonical transformation map between the filtered analytic De Rham realization functor and the analytic Gauss-Manin realization functor :

Definition 154. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that

$M = D(\mathbb{A}^1, et)(F)$. We have, using definition 148(ii), the canonical map in $D_{O_S \text{fil}, \mathcal{D}^\infty, \infty}(S^{an}/(\tilde{S}_I^{an}))$

$$\begin{aligned}
& T(\mathcal{F}_{S,an}^{GM}, \mathcal{F}_{S,an}^{FDR})(M) : \\
& \mathcal{F}_{S,an}^{GM}(L\mathbb{D}_S M) := (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* L(i_{I*} j_I^* \mathbb{D}_S L F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}^q(F)) \\
& \quad \xrightarrow{\sim} (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* L\mathbb{D}_{\tilde{S}_I}^0(L(i_{I*} j_I^* F)), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F_b)), u_{IJ}^{q,d}(F)) \\
& \xrightarrow{\mathcal{H}om(-, \text{Gr}(\Omega_{\tilde{S}_I^{an}}))^{-1}} J_S(e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* L\mathbb{D}_{\tilde{S}_I}^0(L(i_{I*} j_I^* F)), \text{Gr}_{\tilde{S}_I}^{12} E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})), u_{IJ}^{q,d}(F)) \\
& \quad \xrightarrow{(\mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* L(\tau_{\tilde{S}_I}^{0CH}(L(i_{I*} j_I^* F))) \circ T(\hat{R}^{0CH}, R^{0CH})(L(i_{I*} j_I^* F))), -)} \\
& J_S(e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{0CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), \text{Gr}_{\tilde{S}_I}^{12} E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR})), u_{IJ}^{q,d}(F)) \\
& \quad \xrightarrow{I(\text{Gr}_{\tilde{S}_I}^{12*}, \text{Gr}_{\tilde{S}_I}^{12})(-, -)} \\
& J_S(e'(\tilde{S}_I)_* \mathcal{H}om(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) =: J_S(\mathcal{F}_{S,an}^{FDR}(M))
\end{aligned}$$

We now define the functorialities of \mathcal{F}_S^{FDR} with respect to S which makes \mathcal{F}_{FDR}^- a morphism of 2 functor.

Definition 155. Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote $Z_I := Z \cap S_I$. We then have closed embeddings $Z_I \hookrightarrow S_I \hookrightarrow \tilde{S}_I$.

(i) For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we will consider the following canonical map in $\pi_S(D(\text{MHM}(S^{an}))) \subset D_{\mathcal{D}(1,0)\text{fil}}(S^{an}/(\tilde{S}_I^{an}))$

$$\begin{aligned}
& T(\Gamma_Z^{\vee, Hdg}, \Omega_{/S}^{\Gamma, pr, an})(F, W) : \\
& \Gamma_Z^{\vee, Hdg} \iota_S^{-1}(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F, W)) \\
& \quad \xrightarrow{\mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}_{\tilde{S}_I}^{CH}(\gamma^{\vee, Z_I}(L(i_{I*} j_I^*(F, W))))), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))} \\
& \Gamma_Z^{\vee, Hdg} \iota_S^{-1}(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* \Gamma_{Z_I}^\vee L(i_{I*} j_I^*(F, W))), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr, an}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,Z}(F, W)) \\
& \quad \xrightarrow{\cong} \iota_S^{-1}(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* \Gamma_{Z_I}^\vee L(i_{I*} j_I^*(F, W))), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,Z}(F, W)),
\end{aligned}$$

with $u_{IJ}^{q,Z}(F)$ given as in definition 136(i).

(ii) For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we have also the following canonical map in $\pi_S(D(\text{MHM}(S^{an}))) \subset D_{\mathcal{D}(1,0)\text{fil}}(S^{an}/(\tilde{S}_I^{an}))$

$$\begin{aligned}
& T(\Gamma_Z^{Hdg}, \Omega_{/S}^{\Gamma, pr})(F, W) : \\
& \iota_S^{-1}(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I} E(i_{I*} j_I^* \mathbb{D}_S(F, W))), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,Z,d}(F, W)) \\
& \xrightarrow{\cong} \Gamma_Z^{Hdg} \iota_S^{-1}(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L\Gamma_{Z_I} E(i_{I*} j_I^* \mathbb{D}_S(F, W))), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^{q,Z,d}(F, W)) \\
& \quad \xrightarrow{\mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}_{\tilde{S}_I}^{CH}(\gamma^{Z_I}(-)), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))} \\
& \Gamma_Z^{Hdg} \iota_S^{-1}(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* \mathbb{D}_S(F, W))), E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F, W))
\end{aligned}$$

with $u_{IJ}^{q,Z}(F)$ given as in definition 136(ii).

Definition 156. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let

$S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g_{T_I} : T_I \rightarrow S_I$. Let $M \in \text{DA}_c(S)$ and $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ such that $(M, W) = D(\mathbb{A}_S^1, \text{et})(F, W)$. Then, $D(\mathbb{A}_T^1, \text{et})(g^*F) = g^*M$ and there exist $(F', W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ and an equivalence $(\mathbb{A}^1, \text{et})$ local $e : g^*(F, W) \rightarrow (F', W)$ such that $D(\mathbb{A}_T^1, \text{et})(F', W) = (g^*M, W)$. We have, using definition 150 and definition 155(i), the canonical map in $\pi_T(D(\text{MHM}(T^{\text{an}}))) \subset D_{\mathcal{D}(1,0)\text{fil}}(T^{\text{an}}/(Y^{\text{an}} \times \tilde{S}_I^{\text{an}}))$

$$\begin{aligned}
& T(g, \mathcal{F}_{\text{an}}^{\text{FDR}})(M) : g_{\text{Hdg}}^{\hat{*}\text{mod}} \iota_S^{-1} \mathcal{F}_{S, \text{an}}^{\text{FDR}}(M) := \\
& (\Gamma_T^{\vee, \text{Hdg}} \iota_T^{-1} (\tilde{g}_I^{\text{mod}}(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}_{\tilde{S}_I}^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^*(L(i_{I*} J_I^*(F, W))), E_{\text{usu}}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{\text{DR}}))) [d_{YI}], \tilde{g}_J^{\text{mod}} u_{IJ}^q(F, W)) \\
& \quad \xrightarrow{(T(\tilde{g}_I, \Omega_{/\tilde{S}_I}^{\Gamma, \text{pr}, \text{an}})(-))} \\
& \Gamma_T^{\vee, \text{Hdg}} \iota_T^{-1} (e'(-)_* \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* \tilde{g}_I^* \hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*} J_I^*(F, W))), E_{\text{usu}}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{\text{DR}}))) [d_{YI}], \tilde{g}_J^* u_{IJ}^q(F, W)) \\
& \quad \xrightarrow{\mathcal{H}om(T(\tilde{g}_I, \hat{R}^{CH})(-)^{-1}, -)} \\
& \Gamma_T^{\vee, \text{Hdg}} \iota_T^{-1} (e'_* \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* \hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* \tilde{g}_I^* L(i_{I*} J_I^*(F, W))), E_{\text{usu}}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{\text{DR}}))) [d_{YI}], \tilde{g}_J^* u_{IJ}^q(F, W)) \\
& \quad \xrightarrow{T(\Gamma_T^{\vee, \text{Hdg}}, \Omega_{/\tilde{S}}^{\Gamma, \text{pr}, \text{an}})(F, W)} \\
& \iota_T^{-1} (e'_* \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* \hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* \Gamma_{T_I}^{\vee} \tilde{g}_I^* L(i_{I*} J_I^*(F, W))), E_{\text{usu}}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{\text{DR}}))) [d_{YI}], \tilde{g}_J^{*, \gamma} u_{IJ}^q(F, W)) \\
& \quad \xrightarrow{(\mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* \hat{R}_{Y \times \tilde{S}_I}^{CH}(T^{q, \gamma}(D_{gI})(J_I^*(F, W))), E_{\text{usu}}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{\text{DR}}))) [d_{YI}]} \\
& \iota_T^{-1} (e'_* \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* \hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* L(i'_{I*} J_I'^* g^*(F, W))), E_{\text{et}}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{\text{DR}}))) [d_{YI}], u_{IJ}^q(g^*(F, W)) \\
& \quad \xrightarrow{\mathcal{H}om(\hat{R}_{Y \times \tilde{S}_I}^{CH}(Li'_{I*} J_I'^*(e)), \cdot)} \\
& \iota_T^{-1} (e'_* \mathcal{H}om(\text{An}_{Y \times \tilde{S}_I}^* \hat{R}^{CH}(\rho_{Y \times \tilde{S}_I}^* L(i'_{I*} J_I'^*(F', W))), E_{\text{usu}}(\Omega_{/Y \times \tilde{S}_I}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{\text{DR}}))) [d_{YI}], u_{IJ}^q(F', W)) \xrightarrow{=} \mathcal{F}_{T, \text{an}}^{\text{FDR}}(g^*M)
\end{aligned}$$

Definition 157. • Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \text{DA}_c(X)$, the following transformation map in $\pi_S(D(\text{MHM}(S^{\text{an}})))$

$$\begin{aligned}
T_*(f, \mathcal{F}_{\text{an}}^{\text{FDR}})(M) : \mathcal{F}_{S, \text{an}}^{\text{FDR}}(Rf_*M) & \xrightarrow{\text{ad}(f_{\text{Hdg}}^{\hat{*}\text{mod}}, Rf_*^{\text{Hdg}})(-)} Rf_*^{\text{Hdg}} f_{\text{Hdg}}^{\hat{*}\text{mod}} \mathcal{F}_{S, \text{an}}^{\text{FDR}}(Rf_*M) \\
& \xrightarrow{T(f, \mathcal{F}_{\text{an}}^{\text{FDR}})(Rf_*M)} Rf_*^{\text{Hdg}} \mathcal{F}_{X, \text{an}}^{\text{FDR}}(f^*Rf_*M) \xrightarrow{\mathcal{F}_X^{\text{FDR}}(\text{ad}(f^*, Rf_*)(M))} Rf_*^{\text{Hdg}} \mathcal{F}_{X, \text{an}}^{\text{FDR}}(M)
\end{aligned}$$

Clearly, for $p : Y \times S \rightarrow S$ a projection with $Y \in \text{PSmVar}(\mathbb{C})$, we have, for $M \in \text{DA}_c(Y \times S)$, $T_*(p, \mathcal{F}^{\text{FDR}})(M) = T_!(p, \mathcal{F}^{\text{FDR}})(M)[2d_Y]$

- Let $S \in \text{Var}(\mathbb{C})$. Let $Y \in \text{SmVar}(\mathbb{C})$ and $p : Y \times S \rightarrow S$ the projection. We have then, for $M \in \text{DA}(Y \times S)$ the following transformation map in $\pi_S(D(\text{MHM}(S^{\text{an}})))$

$$\begin{aligned}
T_!(p, \mathcal{F}_{\text{an}}^{\text{FDR}})(M) : p_!^{\text{Hdg}} \mathcal{F}_{Y \times S, \text{an}}^{\text{FDR}}(M) & \xrightarrow{\mathcal{F}_{Y \times S, \text{an}}^{\text{FDR}}(\text{ad}(Lp_!, p^*)(M))} Rp_!^{\text{Hdg}} \mathcal{F}_{Y \times S, \text{an}}^{\text{FDR}}(p^*Lp_!(M)) \\
& \xrightarrow{T(p, \mathcal{F}^{\text{FDR}})(Lp_!(M, W))} Rp_!^{\text{Hdg}} p^{*\text{mod}[-1]} \mathcal{F}_{S, \text{an}}^{\text{FDR}}(Lp_!M) \xrightarrow{T(p^{*\text{mod}}, p^{*\text{mod}})(-)} p_!^{\text{Hdg}} p^{*\text{mod}[-1]} \\
& \mathcal{F}_{S, \text{an}}^{\text{FDR}}(Lp_!M) \xrightarrow{\text{ad}(Rp_!^{\text{Hdg}}, p^{*\text{mod}[-1]})(\mathcal{F}_{S, \text{an}}^{\text{FDR}}(Lp_!M))} \mathcal{F}_{S, \text{an}}^{\text{FDR}}(Lp_!M)
\end{aligned}$$

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then, using

the second point, for $M \in \text{DA}(X)$ the following transformation map in $\pi_S(D(\text{MHM}(S^{an})))$

$$\begin{aligned} T_l(f, \mathcal{F}_{an}^{FDR})(M) : R p_!^{Hdg} \mathcal{F}_X^{FDR}(M) &:= R p_!^{Hdg} \mathcal{F}_{Y \times S, an}^{FDR}(l_* M) \\ &\xrightarrow{T_l(p, \mathcal{F}_{an}^{FDR})(l_* M)} \mathcal{F}_{S, an}^{FDR}(L p_{\#} l_* M) \xrightarrow{\cong} \mathcal{F}_{S, an}^{FDR}(R f_! M) \end{aligned}$$

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, using the third point, for $M \in \text{DA}(S)$, the following transformation map in $\pi_X(D(\text{MHM}(X^{an})))$

$$\begin{aligned} T^l(f, \mathcal{F}_{an}^{FDR})(M) : \mathcal{F}_{X, an}^{FDR}(f^!(M, W)) &\xrightarrow{\text{ad}(R f_!^{Hdg}, f^{*mod})(\mathcal{F}_{X, an}^{FDR}(f^! M))} f_{Hdg}^{*mod} R f_!^{Hdg} \mathcal{F}_{X, an}^{FDR}(f^! M) \\ &\xrightarrow{T^l(p_S, \mathcal{F}_{an}^{FDR})(\mathcal{F}_{an}^{FDR}(f^! M))} f_{Hdg}^{*mod} \mathcal{F}_{S, an}^{FDR}(R f_! f^!(M, W)) \xrightarrow{\mathcal{F}_{S, an}^{FDR}(\text{ad}(R f_!, f^!)(M))} f_{Hdg}^{*mod} \mathcal{F}_{S, an}^{FDR}(M) \end{aligned}$$

- Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, using the preceding point, denoting $\Delta_S : S \hookrightarrow S$ the diagonal closed embedding and $p_1 : S \times S \rightarrow S$, $p_2 : S \times S \rightarrow S$ the projections, for $M, N \in \text{DA}(S)$ and $(F, W), (G, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$ and $(N, W) = D(\mathbb{A}^1, et)(G, W)$, the following transformation map in $\pi_S(D(\text{MHM}(S)))$

$$\begin{aligned} T(\mathcal{F}_{S, an}^{FDR}, \otimes)(M, N) : \mathcal{F}_{S, an}^{FDR}(M) \otimes_{O_S}^{Hdg} \mathcal{F}_{S, an}^{FDR}(N) &:= \Delta_S^{!Hdg}(p_1^{*mod} \mathcal{F}_{S, an}^{FDR}(M) \otimes_{O_{S \times S}} p_2^{*mod} \mathcal{F}_{S, an}^{FDR}(N)) \\ &\xrightarrow{T^l(p_1, \mathcal{F}_{S, an}^{FDR})(M) \otimes T^l(p_2, \mathcal{F}_{S, an}^{FDR})(N)} \Delta_S^{!Hdg}(\mathcal{F}_{S \times S, an}^{FDR}(p_1^! M) \otimes_{O_{S \times S}} \mathcal{F}_{S \times S, an}^{FDR}(p_2^! N)) \\ &\xrightarrow{(T(\otimes, \Omega)(\hat{R}^{CH}(\rho_{\tilde{S}_I \times \tilde{S}_J}^* L(i_I \times i_J)_*(j_I \times j_J)^* p_1^* F[2d_S]), \hat{R}^{CH}(\rho_{\tilde{S}_I \times \tilde{S}_J}^* L(i_I \times i_J)_*(j_I \times j_J)^* p_2^* F[2d_S])))} \\ &\Delta_S^{!Hdg}(\mathcal{F}_{S \times S, an}^{FDR}(p_1^! M \otimes p_2^! N)) \xrightarrow{T^l(\Delta_S, \mathcal{F}_{an}^{FDR})(p_1^! M \otimes p_2^! N)} \mathcal{F}_{S, an}^{FDR}(\Delta_S^!(p_1^! M \otimes p_2^! N)) = \mathcal{F}_{S, an}^{FDR}(M \otimes N) \end{aligned}$$

Proposition 137. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{p_S} S$ with $Y_2 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(\mathbb{C})$ such that there exists a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_S} S$, with $Y_1 \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesian

$$\begin{array}{ccccc} f' : X_T & \longrightarrow & Y_1 \times T & \longrightarrow & T \\ & \searrow & \downarrow & \searrow & \downarrow \\ & & Y_1 \times X & \longrightarrow & Y_1 \times Y \times S \\ g' \downarrow & & \downarrow & & \downarrow \\ & & Y_1 \times S & \longrightarrow & S \\ f : X & \longrightarrow & Y_1 \times S & \longrightarrow & S \end{array}$$

Take a smooth compactification $\bar{Y}_1 \in \text{PSmVar}(\mathbb{C})$ of Y_1 , denote $\bar{X}_I \subset \bar{Y}_1 \times \tilde{S}_I$ the closure of X_I , and $Z_I := \bar{X}_I \setminus X_I$. Consider $F(X/S) := p_{S, \#} \Gamma_X^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S) \in C(\text{Var}(\mathbb{C})^{sm}/S)$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/T)$

$$\begin{aligned} T(f, g, F(X/S)) : g^* F(X/S) &:= g^* p_{S, \#} \Gamma_X^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S) \xrightarrow{\sim} \\ &p_{T, \#} \Gamma_{X_T}^\vee \mathbb{Z}(Y_1 \times T/Y_1 \times T) =: F(X_T/T). \end{aligned}$$

which gives in $\text{DA}(T)$ the isomorphism $T(f, g, F(X/S)) : g^*M(X/S) \xrightarrow{\sim} M(X_T/T)$. Then the following diagram in $\pi_T(D(\text{MHM}(T))) \subset D_{\mathcal{D}(1,0)\text{fil},\infty}(T/(Y_2 \times \tilde{S}_I))$, where the horizontal maps are given by proposition 135, commutes

$$\begin{array}{ccc}
g_{\text{Hdg}}^{\hat{*}\text{mod}} \iota_S^{-1} \mathcal{F}_{S,\text{an}}^{\text{FDR}}(M(X/S)) & \xrightarrow{g_{\text{Hdg}}^{\hat{*}\text{mod}} I(X/S)} & g_{\text{Hdg}}^{\hat{*}\text{mod}} Rf_!^{\text{Hdg}}(\Gamma_{X_I}^{\vee, \text{Hdg}}(O_{(Y_1 \times \tilde{S}_I)^{\text{an}}}, F_b)(d_{Y_1})[2d_{Y_1}], x_{IJ}(X/S)) \\
\downarrow 'T(g, \mathcal{F}^{\text{FDR}})(M(X/S)) & & \downarrow T(p_{\tilde{S}_I}, \gamma^{\vee, \text{Hdg}})(-) \\
& & Rf_!^{\text{Hdg}} g_{\text{Hdg}}^{\hat{*}\text{mod}}(\Gamma_{X_I}^{\vee, \text{Hdg}}(O_{(Y_1 \times \tilde{S}_I)^{\text{an}}}, F_b)(d_{Y_1})[2d_{Y_1}], x_{IJ}(X/S)) \\
& & \downarrow T(p_{Y_1 \times Y_2 \times \tilde{S}_I, \text{Hdg}}^{\hat{*}\text{mod}}, p_{Y_1 \times Y_2 \times \tilde{S}_I, \text{Hdg}}^{\hat{*}\text{mod}})(-) \\
\iota_T^{-1} \mathcal{F}_{T,\text{an}}^{\text{FDR}}(M(X_T/T)) & \xrightarrow{I(X_T/T)} & Rf_!^{\text{Hdg}}(\Gamma_{X_T}^{\vee, \text{Hdg}}(O_{(Y_2 \times Y_1 \times \tilde{S}_I)^{\text{an}}}, F_b)(d_{Y_{12}})[2d_{Y_{12}}], x_{IJ}(X_T/T)).
\end{array}$$

with $d_{Y_{12}} = d_{Y_1} + d_{Y_2}$.

Proof. Follows immediately from definition. \square

Proposition 138. Let $S \in \text{Var}(\mathbb{C})$. Let $Y \in \text{SmVar}(\mathbb{C})$ and $p : Y \times S \rightarrow S$ the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i^o : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, we denote by $S_I = \cap_{i \in I} S_i$, $j_I^o : S_I \hookrightarrow S$ and $j_I : Y \times S_I \hookrightarrow Y \times S$ the open embeddings. We then have closed embeddings $i_I : Y \times S_I \hookrightarrow Y \times S$. and we denote by $p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ the projections. Let $f' : X' \rightarrow Y \times S$ a morphism, with $X' \in \text{Var}(\mathbb{C})$ such that there exists a factorization $f' : X' \xrightarrow{l'} Y' \times Y \times S \xrightarrow{p'} Y \times S$ with $Y' \in \text{SmVar}(\mathbb{C})$, l' a closed embedding and p' the projection. Denoting $X'_I := f'^{-1}(Y \times S_I)$, we have closed embeddings $i'_I : X'_I \hookrightarrow Y' \times Y \times \tilde{S}_I$. Consider

$$F(X'/Y \times S) := p_{Y \times S, \#} \Gamma_{X'}^{\vee} \mathbb{Z}(Y' \times Y \times S/Y' \times Y \times S) \in C(\text{Var}(\mathbb{C})^{\text{sm}}/Y \times S)$$

and $F(X'/S) := p_{\#} F(X'/Y \times S) \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$, so that $Lp_{\#} M(X'/Y \times S)[-2d_Y] =: M(X'/S)$. Then, the following diagram in $\pi_S(D(\text{MHM}(S^{\text{an}}))) \subset D_{\mathcal{D}(1,0)\text{fil}}(S^{\text{an}}/(Y^{\text{an}} \times \tilde{S}_I^{\text{an}}))$, where the vertical maps are given by proposition 135, commutes

$$\begin{array}{ccc}
Rp^{\text{Hdg}}! \mathcal{F}_{Y \times S, \text{an}}^{\text{FDR}}(M(X'/Y \times S)) & \xrightarrow{T_1(p, \mathcal{F}^{\text{FDR}})(M(X'/Y \times S))} & \mathcal{F}_{S, \text{an}}^{\text{FDR}}(M(X'/S)) \\
\uparrow T(p_{\text{Hdg}}^{\hat{*}\text{mod}}, p_{\text{Hdg}}^{\hat{*}\text{mod}})(-) \circ Rp^{\text{Hdg}}!(I(X'/Y \times S)) & & \uparrow I(X'/S) \\
Rp^{\text{Hdg}}! Rf_!^{\text{Hdg}} f'^{\hat{*}\text{mod}} \mathbb{Z}_{(Y \times S)^{\text{an}}}^{\text{Hdg}} & \xrightarrow{=} & Rf_!^{\text{Hdg}} f_{\text{Hdg}}^{\hat{*}\text{mod}} \mathbb{Z}_{S^{\text{an}}}^{\text{Hdg}}
\end{array}$$

Proof. Immediate from definition. \square

Proposition 139. Let $f_1 : X_1 \rightarrow S$, $f_2 : X_2 \rightarrow S$ two morphism with $X_1, X_2, S \in \text{Var}(\mathbb{C})$. Assume that there exist factorizations $f_1 : X_1 \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $f_2 : X_2 \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$ with $Y_1, Y_2 \in \text{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S the projections. We have then the factorization

$$f_{12} := f_1 \times f_2 : X_{12} := X_1 \times_S X_2 \xrightarrow{l_1 \times l_2} Y_1 \times Y_2 \times S \xrightarrow{p_S} S$$

Let $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have then the following commutative diagram in $\pi_S(D(\text{MHM}(S^{\text{an}}))) \subset D_{\mathcal{D}(1,0)\text{fil}}(S^{\text{an}}/(\tilde{S}_I^{\text{an}}))$ where the vertical maps are given by proposition 135

$$\begin{array}{ccc}
\mathcal{F}_{S, \text{an}}^{\text{FDR}}(M(X_1/S)) \otimes_{\mathcal{O}_S}^{\text{Hdg}} \mathcal{F}_{S, \text{an}}^{\text{FDR}}(M(X_2/S)) & \xrightarrow{I(X_1/S) \otimes I(X_2/S)} & Rf_{1!}^{\text{Hdg}}(\Gamma_{X_{1I}}^{\vee, \text{Hdg}}(O_{(Y_1 \times \tilde{S}_I)^{\text{an}}}, F_b)(d_1)[2d_1], x_{IJ}(X_1/S)) \otimes_{\mathbb{Z}} Rf_{2!}^{\text{Hdg}}(\Gamma_{X_{2I}}^{\vee, \text{Hdg}}(O_{(Y_2 \times \tilde{S}_I)^{\text{an}}}, F_b)(d_2)[2d_2], x_{IJ}(X_2/S)) \\
\downarrow T(\mathcal{F}_{S, \text{an}}^{\text{FDR}}, \otimes)(M(X_1/S), M(X_2/S)) & & \downarrow (Ew_{(Y_1 \times \tilde{S}_I, Y_2 \times \tilde{S}_I)/\tilde{S}_I}) \\
\mathcal{F}_{S, \text{an}}^{\text{FDR}}(M(X_1/S) \otimes M(X_2/S) = M(X_1 \times_S X_2/S)) & \xrightarrow{I(X_{12}/S)} & Rf_{12!}^{\text{Hdg}}(\Gamma_{X_{1I} \times_S X_{2I}}^{\vee, \text{Hdg}}(O_{(Y_1 \times Y_2 \times \tilde{S}_I)^{\text{an}}}, F_b)(d_{12})[2d_{12}], x_{IJ}(X_{12}/S))
\end{array}$$

Proof. Immediate from definition. \square

Theorem 38. (i) Let $g : T \rightarrow S$ a morphism, with $S, T \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $M \in \text{DA}_c(S)$. Then map in $\pi_T(D(\text{MHM}(T^{an})))$

$$T(g, \mathcal{F}_{an}^{FDR})(M) : g_{Hdg}^{*mod} \mathcal{F}_{S,an}^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_{T,an}^{FDR}(g^*M)$$

given in definition 156 is an isomorphism.

(ii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \text{DA}_c(X)$,

$$T_!(f, \mathcal{F}_{an}^{FDR})(M) : Rf_!^{Hdg} \mathcal{F}_{X,an}^{FDR}(M) \xrightarrow{\sim} \mathcal{F}_{S,an}^{FDR}(Rf_!M)$$

is an isomorphism in $\pi_S(D(\text{MHM}(S^{an})))$

(iii) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $M \in \text{DA}_c(X)$,

$$T_*(f, \mathcal{F}_{an}^{FDR})(M) : \mathcal{F}_{S,an}^{FDR}(Rf_*M) \xrightarrow{\sim} Rf_*^{Hdg} \mathcal{F}_{X,an}^{FDR}(M)$$

is an isomorphism in $\pi_S(D(\text{MHM}(S^{an})))$.

(iv) Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Then, for $M \in \text{DA}_c(S)$

$$T^!(f, \mathcal{F}_{an}^{FDR})(M) : \mathcal{F}_{X,an}^{FDR}(f^!M) \xrightarrow{\sim} f^{*mod} \mathcal{F}_{S,an}^{FDR}(M)$$

is an isomorphism in $\pi_X(D(\text{MHM}(X^{an})))$.

(v) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$, the map in $\pi_S(D(\text{MHM}(S^{an})))$

$$T(\mathcal{F}_{S,an}^{FDR}, \otimes)(M, N) : \mathcal{F}_{S,an}^{FDR}(M) \otimes_{O_S}^{Hdg} \mathcal{F}_{S,an}^{FDR}(N) \xrightarrow{\sim} \mathcal{F}_{S,an}^{FDR}(M \otimes N)$$

given in definition is an isomorphism $\pi_S(D(\text{MHM}(S^{an})))$.

Proof. (i): Follows from proposition 137 and proposition 135.

(ii),(iii),(iv): Similar to the proof of theorem 36.

(v): Follows from proposition 139. \square

Proposition 140. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then, $D(\mathbb{A}_T^1, et)(g^*F) = g^*M$. Then the following diagram in $D_{Ofil, \mathcal{D}^{\infty, \infty}}(T^{an}/(Y^{an} \times \tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc} Rg^{*mod[-], \Gamma} \mathcal{F}_{S,an}^{GM}(\text{LD}_S M) & \xrightarrow{Rg^{*mod[-], \Gamma} T(\mathcal{F}_{S,an}^{GM}, \mathcal{F}_{S,an}^{FDR})(M)} & Rg^{*mod[-], \Gamma} \mathcal{F}_{S,an}^{FDR}(M) & \xleftarrow{Rg^{*mod[-], \Gamma} T(g^{*mod} Rg^{*mod[-], \Gamma})(\mathcal{F}_{S,an}^{FDR}(M))} & g_{Hdg}^{*mod} \mathcal{F}_{S,an}^{FDR}(M) \\ \downarrow T(g, \mathcal{F}_{an}^{GM})(\text{LD}_S M) & & \searrow T(g, \mathcal{F}_{an}^{FDR})(M) & & \downarrow T^!(g, \mathcal{F}_{an}^{FDR})(M)^{-1} \\ \mathcal{F}_{T,an}^{GM}(g^* \text{LD}_S M = \text{LD}_T g^! M) & \xrightarrow{T(\mathcal{F}_{T,an}^{GM}, \mathcal{F}_{T,an}^{FDR})(g^*M)} & & & \mathcal{F}_{T,an}^{FDR}(g^! M) \end{array}$$

Proof. Similar to the proof of proposition 122. □

We have the following easy proposition

Proposition 141. *Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \dots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following commutative diagram in $D_{O_S \text{fil}, \mathcal{D}^\infty, \infty}(S^{an}/(\tilde{S}_I^{an}))$*

$$\begin{array}{ccc}
\mathcal{F}_{S,an}^{GM}(L\mathbb{D}_S M) \otimes_{O_S}^L \mathcal{F}_{S,an}^{GM}(L\mathbb{D}_S N) & \xrightarrow{T(\mathcal{F}_{S,an}^{GM}, \mathcal{F}_{S,an}^{FDR})(M) \otimes T(\mathcal{F}_{S,an}^{GM}, \mathcal{F}_{S,an}^{FDR})(N)} & \mathcal{F}_{S,an}^{FDR}(M) \otimes_{O_S}^{Hdg} \mathcal{F}_{S,an}^{FDR}(N) \\
\downarrow T(\mathcal{F}_{S,an}^{GM}, \otimes)(L\mathbb{D}_S M, L\mathbb{D}_S N) & & \downarrow T(\mathcal{F}_{S,an}^{FDR}, \otimes)(M, N) \\
\mathcal{F}_{S,an}^{GM}(\mathbb{D}_S L(M \otimes N)) & \xrightarrow{T(\mathcal{F}_{S,an}^{GM}, \mathcal{F}_{S,an}^{FDR})(M \otimes N)} & \mathcal{F}_{S,an}^{FDR}(M \otimes N)
\end{array}$$

Proof. Immediate from definition. □

6.3 The transformation map between the analytic De Rahm functor and the analytification of the algebraic De Rahm functor

6.3.1 The transformation map between the analytic Gauss Manin realization functor and the analytification of the algebraic Gauss Manin realization functor

Recall from section 2 that, for $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$, we have the following commutative diagram of sites (42)

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})/T & & \\
\downarrow P(f) & \searrow \rho_T & \downarrow \rho_T & & \\
& \text{AnSp}(\mathbb{C})^{sm}/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^{sm}/T & \\
& \downarrow P(f) & \downarrow P(f) & & \downarrow P(f) \\
\text{AnSp}(\mathbb{C})/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})/S & & \\
& \downarrow P(f) & \downarrow P(f) & & \downarrow P(f) \\
& \text{AnSp}(\mathbb{C})^{sm}/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^{sm}/S & \\
& \downarrow \rho_S & \downarrow \rho_S & & \downarrow \rho_S
\end{array}$$

This gives for $s : \mathcal{I} \rightarrow \mathcal{J}$ a functor with $\mathcal{I}, \mathcal{J} \in \text{Cat}$ and $f : T_\bullet \rightarrow S_{s(\bullet)}$ a morphism of diagram of algebraic varieties with $T_\bullet \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C}))$, $S_\bullet \in \text{Fun}(\mathcal{J}, \text{Var}(\mathbb{C}))$ the commutative diagram of sites (45)

$$\begin{array}{ccccc}
\text{Dia}(S) := \text{AnSp}(\mathbb{C})/T_\bullet^{an} & \xrightarrow{\text{An}_{T_\bullet}} & \text{Var}(\mathbb{C})/T_\bullet & & \\
\downarrow P(f) & \searrow \rho_{T_\bullet} & \downarrow \rho_{T_\bullet} & & \\
& \text{AnSp}(\mathbb{C})^{sm}/T_\bullet^{an} & \xrightarrow{\text{An}_{T_\bullet}} & \text{Var}(\mathbb{C})^{sm}/T_\bullet & \\
& \downarrow P(f_\bullet) & \downarrow P(f_\bullet) & & \downarrow P(f_\bullet) \\
\text{AnSp}(\mathbb{C})/S_\bullet^{an} & \xrightarrow{\text{An}_{S_\bullet}} & \text{Var}(\mathbb{C})/S_\bullet & & \\
& \downarrow P(f_\bullet) & \downarrow P(f_\bullet) & & \downarrow P(f_\bullet) \\
& \text{AnSp}(\mathbb{C})^{sm}/S_\bullet^{an} & \xrightarrow{\text{An}_{S_\bullet}} & \text{Var}(\mathbb{C})^{sm}/S_\bullet & \\
& \downarrow \rho_{S_\bullet} & \downarrow \rho_{S_\bullet} & & \downarrow \rho_{S_\bullet}
\end{array}$$

We have the following canonical transformation map given by the pullback of (relative) differential forms: Let $S \in \text{Var}(\mathbb{C})$. Consider the following commutative diagram in RCat :

$$D(an, e) : (\text{AnSp}(\mathbb{C})^{sm}/S^{an}, O_{\text{AnSp}(\mathbb{C})^{sm}/T}) \xrightarrow{\text{An}_S} (\text{Var}(\mathbb{C})^{sm}/S, O_{\text{Var}(\mathbb{C})^{sm}/S})$$

$$\begin{array}{ccc} \downarrow e(T) & & \downarrow e(S) \\ (S^{an}, O_{S^{an}}) & \xrightarrow{\text{an}_S} & (S, O_S) \end{array}$$

It gives (see section 2) the canonical morphism in $C_{\text{an}_S^* O_S}(\text{AnSp}(\mathbb{C})^{sm}/S^{an})$

$$\Omega_{/(S^{an}/S)} := \Omega_{(O_{\text{AnSp}(\mathbb{C})^{sm}/S^{an}}/\text{An}_S^* O_{\text{Var}(\mathbb{C})^{sm}/S})/(O_{S^{an}}/\text{an}_S^* O_S)} :$$

$$\text{An}_S^*(\Omega_{/S}^\bullet, F_b) = (\Omega_{\text{An}_S^* O_{\text{Var}(\mathbb{C})^{sm}/S}/\text{An}_S^* e(S)^* O_S}^\bullet, F_b) \rightarrow (\Omega_{/S^{an}}^\bullet, F_b) = (\Omega_{O_{\text{AnSp}(\mathbb{C})^{sm}/S^{an}}/e(S^{an})^* O_{S^{an}}}^\bullet, F_b)$$

which is by definition given by the analytification on differential forms : for $(V/S^{an}) = (V, h) \in \text{AnSp}(\mathbb{C})^{sm}/S^{an}$,

$$\Omega_{/(S^{an}/S)}(V/S^{an}) : \hat{\omega} \in \text{An}_S^*(\Omega_{/S}^r)(V/S^{an}) := \lim_{(h':U \rightarrow \text{SSM}, g':V \rightarrow U^{an}, h, g)} \Omega_{/S}^r(U/S)$$

$$\mapsto \Omega_{(V/U)/(S^{an}/S)}(V/S^{an})(\omega) := \widehat{\text{an}_S^*(\omega)} \in \Omega_{S^{an}}^r(V/S^{an});$$

with $\omega \in \Gamma(U, \Omega_U^r)$ is such that $q(\omega) = \hat{\omega}$. If $S \in \text{SmVar}(\mathbb{C})$, the map $\Omega_{/(T/S)} : \text{An}_S^* \Omega_{/S}^\bullet \rightarrow \Omega_{/S^{an}}^\bullet$ is a map in $C_{O_S \text{fil}, \mathcal{D}}(\text{AnSp}(\mathbb{C})^{sm}/S^{an})$. It induces the canonical morphism in $C_{O_S \text{fil}, \mathcal{D}}(\text{AnSp}(\mathbb{C})^{sm}/S^{an})$:

$$E\Omega_{/(S^{an}/S)} : \text{An}_S^* E_{et}(\Omega_{/S}^\bullet, F_b) \xrightarrow{T(\text{An}_S, E)(\Omega_{/S}^\bullet, F_b)} E_{et}(\text{An}_S^*(\Omega_{/S}^\bullet, F_b)) \xrightarrow{E(\Omega_{/(S^{an}/S)})} E_{et}(\Omega_{/S^{an}}^\bullet, F_b)$$

We have the following canonical transformation map given by the analytical functor:

Definition 158. Let $S \in \text{SmVar}(\mathbb{C})$.

(i) For $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we have the canonical transformation map in $C_{O \text{fil}, \mathcal{D}}(S^{an})$

$$T(an, \Omega_{/ \cdot})(F) :$$

$$((e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b)))^{an}) := O_{S^{an}} \otimes_{\text{an}_S^* O_S} \text{an}_S^*(e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b)))$$

$$\xrightarrow{T(an, e)(-)} O_{S^{an}} \otimes_{\text{an}_S^* O_S} (e(S^{an})_* \text{An}_S^* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b)))$$

$$\xrightarrow{T(\text{An}, hom)(F, E_{et}(\Omega_{/S}, F_b))} O_{S^{an}} \otimes_{\text{an}_S^* O_S} (e(S^{an})_* \mathcal{H}om^\bullet(\text{An}_S^* F, \text{An}_S^* E_{et}(\Omega_{/S}^\bullet, F_b)))$$

$$\xrightarrow{\mathcal{H}om(\text{An}_S^* F, E\Omega_{/(S^{an}/S)} \otimes m)} e(S^{an})_* \mathcal{H}om^\bullet(\text{An}_S^* F, E_{et}(\Omega_{/S^{an}}^\bullet, F_b))$$

(ii) We get from (i), for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the canonical transformation map in $\text{PSh}_{\mathcal{D}^\infty}(S^{an})$

$$T^n(an, \Omega_{/ \cdot})(F) : J_S H^n((e(S)_* \mathcal{H}om^\bullet(F, E_{et}(\Omega_{/S}^\bullet, F_b)))^{an})$$

$$\xrightarrow{J_S(H^n T(an, \Omega_{/ \cdot})(F))} J_S H^n(e(S^{an})_* \mathcal{H}om^\bullet(\text{An}_S^* F, E_{et}(\Omega_{/S^{an}}^\bullet, F_b)))$$

$$\xrightarrow{\mathcal{J}_S(-)} H^n e(S^{an})_* \mathcal{H}om^\bullet(\text{An}_S^* F, E_{et}(\Omega_{/S^{an}}^\bullet, F_b))$$

Lemma 18. Let $S \in \text{SmVar}(\mathbb{C})$.

(i) For $h : U \rightarrow S$ a smooth morphism with $U \in \text{SmVar}(\mathbb{C})$, the following diagram commutes

$$e(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{et}(\Omega_{/S}^\bullet, F_b))^{an} \xrightarrow{T(\Omega_{/ \cdot}, an)(\mathbb{Z}(U/S))} e(S^{an})_* \mathcal{H}om^\bullet(\mathbb{Z}(U^{an}/S^{an}), E_{et}(\Omega_{/S^{an}}^\bullet, F_b)) .$$

$$\begin{array}{ccc} \uparrow k & & \uparrow = \\ (h_* E_{zar}(\Omega_{U/S}, F_b))^{an} & \xrightarrow{T_\omega^O(an, h)} & h_{an*} E_{usu}(\Omega_{U^{an}/S^{an}}, F_b) \end{array}$$

(ii) For $h : U \rightarrow S$ a smooth morphism with $U \in \text{SmVar}(\mathbb{C})$, the following diagram commutes

$$\begin{array}{ccc}
J_S H^n((e(S)_* \mathcal{H}om^\bullet(\mathbb{Z}(U/S), E_{et}(\Omega_{/S}^\bullet, F)))^{an}) & \xrightarrow{T^n(\Omega_{/.,an})(\mathbb{Z}(U/S))} & H^n e(S^{an})_* \mathcal{H}om^\bullet(\mathbb{Z}(U^{an}/S^{an}), E_{et}(\Omega_{/S^{an}}^\bullet, F_b)) . \\
\uparrow k & & \uparrow = \\
J_S H^n((h_* E_{zar}(\Omega_{U/S}, F_b))^{an}) & \xrightarrow{J_S(-) \circ J_S T_\omega^O(an, h)(O_U, F)} & H^n h_{an*} E_{usu}(\Omega_{U^{an}/S^{an}}, F_b)
\end{array}$$

Proof. Follows from Yoneda lemma. \square

By definition of the algebraic an analytic De Rahm realization functor, we have a natural transformation between them :

Definition 159. Let $S \in \text{SmVar}(\mathbb{C})$. Let $M \in \text{DA}_c(S)$ and $Q \in C(\text{Var}(\mathbb{C})^{sm}/S)$ projectively cofibrant such that $M = D(\mathbb{A}_S^1, et)(Q)$. We have the canonical transformation in $D_{Ofil, \mathcal{D}}(S^{an})$

$$\begin{aligned}
T(An, \mathcal{F}_{an}^{GM})(M) : (\mathcal{F}_{GM}^S(M))^{an} &:= (e(S)_* \mathcal{H}om^\bullet(Q, E_{et}(\Omega_{/S}^\bullet, F_b)))^{an}[-d_S] \\
&\xrightarrow{T(an, \Omega_{/.,})} e(S)_* \mathcal{H}om^\bullet(\text{An}_S^* Q, E_{et}(\Omega_{/S^{an}}^\bullet, F_b))[-d_S] \\
&\xrightarrow{=} e(S)_* \mathcal{H}om^\bullet(\text{An}_S^* Q, E_{usu}(\Omega_{/S^{an}}^\bullet, F_b))[-d_S] =: \mathcal{F}_{S,an}^{GM}(M)
\end{aligned}$$

We give now the definition in the non smooth case : Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ the projection. Consider, for $I \subset J \subset [1, \dots, l]$, resp. for each $I \subset [1, \dots, l]$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array} .$$

We then have the following lemma

Lemma 19. The maps $T(an, \Omega_{/.,})(L(i_{I*} j_I^* F))$ induce a morphism in $C_{Ofil, \mathcal{D}}(S/(\tilde{S}_I))$

$$\begin{aligned}
(T(an, \Omega_{/.,})(L(i_{I*} j_I^* F))) : (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))^{an}[-d_{\tilde{S}_I}], (u_{IJ}^q(F))^{an} \\
\rightarrow (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}(\tilde{S}_I)^* L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F))
\end{aligned}$$

Proof. Obvious. \square

Definition 160. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote $S_I = \cap_{i \in I} S_i$. We have then closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. We have, by lemma 19, the canonical transformation in $D_{Ofil, \mathcal{D}, \infty}(S^{an})$

$$\begin{aligned}
T(An, \mathcal{F}^{GM})(M) : (\mathcal{F}_S^{GM}(M))^{an} &:= (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F_b)))^{an}[-d_{\tilde{S}_I}], (u_{IJ}^q(F))^{an} \\
&\xrightarrow{T(an, \Omega_{/.,})(L(i_{I*} j_I^* F))} (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}(\tilde{S}_I)^* L(i_{I*} j_I^* F), E_{et}(\Omega_{/\tilde{S}_I}^\bullet, F))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\
&\xrightarrow{=} (e(\tilde{S}_I)_* \mathcal{H}om(\text{An}(\tilde{S}_I)^* L(i_{I*} j_I^* F), E_{usu}(\Omega_{/\tilde{S}_I}^\bullet, F))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) =: \mathcal{F}_{S,an}^{GM}(M)
\end{aligned}$$

The following proposition says this transformation map between $\mathcal{F}^{S,an}$ and $(\mathcal{F}_S^{FDR})^{an}$ is functorial in $S \in \text{Var}(\mathbb{C})$, hence define a commutative diagram of morphism of 2-functor :

Proposition 142. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We then have closed embedding $i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$ and $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$. Then, for $M \in \text{DA}_c(S)$, the following diagram in $D_{\text{Ofil}, \mathcal{D}, \infty}(T^{\text{an}}/(Y^{\text{an}} \times \tilde{S}_I^{\text{an}}))$ commutes

$$\begin{array}{ccc} Rg^{*\text{mod}[-], \Gamma}(\mathcal{F}_S^{GM}(M)) & \xrightarrow{Lg^{*\text{mod}[-], \Gamma}(T(\text{An}, \mathcal{F}_S^{FDR})(M))} & Rg^{*\text{mod}[-], \Gamma}(\mathcal{F}_{S, \text{an}}^{GM}(M)) \\ \downarrow (T(g, \mathcal{F}^{GM})(M))^{\text{an}} & & \downarrow T(g, \mathcal{F}_{\text{an}}^{GM})(M) \\ (\mathcal{F}_T^{GM}(g^*M))^{\text{an}} & \xrightarrow{(T(\text{An}, \mathcal{F}_T^{GM})(g^*M))} & (\mathcal{F}_{T, \text{an}}^{GM}(g^*M)) \end{array}$$

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$, the following diagram in $D_{\text{Ofil}, \mathcal{D}, \infty}(S^{\text{an}}/(\tilde{S}_I^{\text{an}}))$ commutes

$$\begin{array}{ccc} (\mathcal{F}_S^{GM}(M) \otimes_{O_S} \mathcal{F}_S^{GM}(N))^{\text{an}} & \xrightarrow{(T(\otimes, \mathcal{F}^{GM})(M, N))^{\text{an}}} & (\mathcal{F}_S^{GM}(M \otimes N))^{\text{an}} \\ \downarrow = & & \downarrow T(\text{An}, \mathcal{F}_S^{GM})(M \otimes N) \\ (\mathcal{F}_S^{GM}(M))^{\text{an}} \otimes_{O_{S^{\text{an}}}} \mathcal{F}_S^{GM}(N)^{\text{an}} & & \\ \downarrow T(\text{An}, \mathcal{F}_S^{GM})(M) \otimes_{O_{S^{\text{an}}}} T(\text{An}, \mathcal{F}_S^{GM})(N) & & \downarrow \\ \mathcal{F}_{S, \text{an}}^{GM}(M) \otimes_{O_{S^{\text{an}}}} \mathcal{F}_{S, \text{an}}^{GM}(N) & \xrightarrow{(T(\otimes, \mathcal{F}_{\text{an}}^{GM})(M, N))} & \mathcal{F}_{S, \text{an}}^{GM}(M \otimes N) \end{array}$$

Proof. Immediate from definition. \square

Proposition 143. Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Assume there exist a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p} S$$

with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p the projection. Let $S = \cup_i S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

(i) We have then the following commutative diagram in $D_{\text{Ofil}, \mathcal{D}, \infty}(S^{\text{an}}/(\tilde{S}_I^{\text{an}}))$,

$$\begin{array}{ccc} (\mathcal{F}_S^{GM}(M(X/S)))^{\text{an}} & \xrightarrow{T(\text{An}, \mathcal{F}_{\text{an}}^{FDR})(M(X/S))} & \mathcal{F}_{S, \text{an}}^{GM}(M(X/S)) \\ \downarrow I^{GM}(X/S)^{\text{an}} & & \downarrow I^{GM}(X/S) \\ ((p_{\tilde{S}_I^*} \Gamma_{X_I} E_{\text{zar}}(\Omega_{Y \times \tilde{S}_I / \tilde{S}_I}^\bullet))^{\text{an}}[-d_{\tilde{S}_I}], w_{IJ}(X/S)^{\text{an}}) & \xrightarrow{(T_\omega^O(\text{an}, p_{\tilde{S}_I})^\gamma)} & (p_{\tilde{S}_I^*} \Gamma_X E_{\text{usu}}(\Omega_{(Y \times \tilde{S}_I)^{\text{an}} / \tilde{S}_I^{\text{an}}}^\bullet))[-d_{\tilde{S}_I}], w_{IJ} \\ \downarrow ((p_* T_\omega^O(\otimes, \gamma)(-))^{\text{an}}) & & \downarrow (p_* T_\omega^O(\otimes, \gamma)(-)) \\ (\int_f^{FDR}(\Gamma_{X_I} E_{\text{zar}}(O_{Y \times \tilde{S}_I}, F_b)[-d_Y - d_{\tilde{S}_I}], x_{IJ}(X/S)))^{\text{an}} & \xrightarrow{(\int_f^{FDR}(\Gamma_{X_I} E_{\text{usu}}(O_{(Y \times \tilde{S}_I)^{\text{an}}, F_b))[-d_Y - d_{\tilde{S}_I}])} & \end{array}$$

(ii) We have then the following commutative diagram in $\text{PSh}_{\mathcal{D}^\infty \text{fil}}(S^{an}/(\tilde{S}_I^{an}))$,

$$\begin{array}{ccc}
J_S H^n(\mathcal{F}_S^{GM}(M^{BM}(X/S)))^{an} & \xrightarrow{\mathcal{J}_S(-) \circ H^n T(\text{An}, \mathcal{F}_{an}^{FDR})(M(X/S))} & H^n \mathcal{F}_{S,an}^{GM}(M(X/S)) \\
\downarrow H^n(I^{GM}(X/S)^{an}) & & \downarrow H^n(I^{GM}(X/S)^{an}) \\
J_S H^n((p_{\tilde{S}_I^*} \Gamma_{X_I} E_{zar}(\Omega_{Y \times \tilde{S}_I / \tilde{S}_I}^\bullet))^{an}[-d_{\tilde{S}_I}], w_{IJ}(X/S)^{an}) & \xrightarrow{H^n(T_\omega^O(an, p_{\tilde{S}_I}^\gamma))} & H^n(p_{\tilde{S}_I^*} \Gamma_X E_{usu}(\Omega_{(Y \times \tilde{S}_I)^{an} / \tilde{S}_I}^\bullet)) \\
\downarrow H^n((p_* T_\omega^O(\otimes, \gamma)(-))^{an}) & & \downarrow H^n((p_* T_\omega^O(\otimes, \gamma)(-))^{an}) \\
J_S H^n(\int_f^{FDR}(\Gamma_{X_I} E_{zar}(O_{Y \times \tilde{S}_I}, F_b)[-d_Y - d_{\tilde{S}_I}], x_{IJ}(X/S)))^{an} & \xrightarrow{H^n(T(an, \gamma)(-) \circ H^n T^{Dmod}(an, f))^{FDR}} & \int_f^{FDR}(\Gamma_{X_I} E_{usu}(O_{(Y \times \tilde{S}_I)^{an}}, F_b)[-d_Y - d_{\tilde{S}_I}])
\end{array}$$

Proof. (i): Immediate from definition.

(ii): Follows from (i). \square

We deduce from proposition 143 and theorem 23 (GAGA for D-modules) the following :

Theorem 39. (i) Let $S \in \text{Var}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$\mathcal{J}_S(-) \circ H^n T(\text{An}, \mathcal{F}_{an}^{GM})(M) : J_S(H^n(\mathcal{F}_S^{GM}(M))^{an}) \xrightarrow{\sim} H^n \mathcal{F}_{S,an}^{GM}(M)$$

is an isomorphism in $\text{PSh}_{\mathcal{D}}(S^{an}/(\tilde{S}_I^{an}))$.

(ii) A relative version of Grothendieck GAGA theorem for De Rham cohomology Let $h : U \rightarrow S$ a smooth morphism with $S, U \in \text{SmVar}(\mathbb{C})$. Then,

$$\mathcal{J}_S(-) \circ J_S T_\omega^O(an, h) : J_S((R^n h_* \Omega_{U/S}^\bullet)^{an}) \xrightarrow{\sim} R^n h_* \Omega_{U^{an}/S^{an}}^\bullet$$

is an isomorphism in $\text{PSh}_{\mathcal{D}}(S^{an})$.

Proof. (i): Follows from proposition 143(ii) and theorem 23 using a resolution by Corti-Hanamura motives.

(ii): Follows from (i) and lemma 18(ii). \square

6.3.2 The transformation map between the analytic filtered De Rham realization functor and the analytification of the filtered algebraic De Rham realization functor

Recall from section 2 that, for $S \in \text{Var}(\mathbb{C})$ we have the following commutative diagrams of sites

$$\begin{array}{ccccc}
\text{AnSp}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,pr}/S & & \\
\downarrow \text{Ans} & \searrow \rho_S & \downarrow \mu_S & \searrow \rho_S & \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,smpr}/S \\
& & \downarrow \text{Ans} & & \downarrow \text{Ans} \\
\text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S & & \\
& \searrow \rho_S & \downarrow \mu_S & \searrow \rho_S & \\
& & \text{Var}(\mathbb{C})^2/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr}/S
\end{array}$$

and

$$\begin{array}{ccc}
\text{AnSp}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{AnSp}(\mathbb{C})/S \\
\downarrow \text{Ans} & \searrow \rho_S & \downarrow \text{Gr}_S^{12} \text{Ans} \\
& & \text{AnSp}(\mathbb{C})^{2,smpr}/S \xrightarrow{\text{Gr}_S^{12}} \text{AnSp}(\mathbb{C})^{sm}/S \\
& & \downarrow \text{Ans} \\
\text{Var}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}_S^{12}} & \text{Var}(\mathbb{C})/S \\
& \searrow \rho_S & \downarrow \text{Gr}_S^{12} \\
& & \text{Var}(\mathbb{C})^{2,sm}/S \xrightarrow{\text{Gr}_S^{12}} \text{Var}(\mathbb{C})^{sm}/S \\
& & \downarrow \text{Ans}
\end{array}
, \quad (65)$$

and that for $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$ we have the commutative diagram of site (43)

$$\begin{array}{ccc}
\text{AnSp}(\mathbb{C})^2/T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^2/T \\
\downarrow P(f) & \searrow \rho_T & \downarrow \text{An}_T \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/T^{an} \xrightarrow{\text{An}_T} \text{Var}(\mathbb{C})^{2,sm}/T \\
& & \downarrow P(f) \\
\text{AnSp}(\mathbb{C})^2/S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^2/S \\
& \searrow \rho_S & \downarrow \text{An}_S \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S^{an} \xrightarrow{\text{An}_S} \text{Var}(\mathbb{C})^{2,sm}/S \\
& & \downarrow P(f)
\end{array}$$

This gives for $s : \mathcal{I} \rightarrow \mathcal{J}$ a functor with $\mathcal{I}, \mathcal{J} \in \text{Cat}$ and $f : T_\bullet \rightarrow S_{s(\bullet)}$ a morphism of diagram of algebraic varieties with $T_\bullet \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C}))$, $S_\bullet \in \text{Fun}(\mathcal{J}, \text{Var}(\mathbb{C}))$ the commutative diagram of sites (46)

$$\begin{array}{ccc}
\text{Dia}^{12}(S) := \text{AnSp}(\mathbb{C})^2/T_\bullet^{an} & \xrightarrow{\text{An}_{T_\bullet}} & \text{Var}(\mathbb{C})^2/T_\bullet \\
\downarrow P(f_\bullet) & \searrow \rho_{T_\bullet} & \downarrow \text{An}_{T_\bullet} \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/T_\bullet^{an} \xrightarrow{\text{An}_{T_\bullet}} \text{Var}(\mathbb{C})^{2,sm}/T_\bullet \\
& & \downarrow P(f_\bullet) \\
\text{AnSp}(\mathbb{C})^2/S_\bullet^{an} & \xrightarrow{\text{An}_{S_\bullet}} & \text{Var}(\mathbb{C})^2/S_\bullet \\
& \searrow \rho_{S_\bullet} & \downarrow \text{An}_{S_\bullet} \\
& & \text{AnSp}(\mathbb{C})^{2,sm}/S_\bullet^{an} \xrightarrow{\text{An}_{S_\bullet}} \text{Var}(\mathbb{C})^{2,sm}/S_\bullet \\
& & \downarrow P(f_\bullet)
\end{array}$$

Let $S \in \text{SmVar}(\mathbb{C})$. We have the canonical map in $C(\text{Var}(\mathbb{C})^{2,smpr}/S)$

$$\Omega_{(S^{an}/S)}^{\Gamma,pr} : (\Omega_{/S}^{\bullet,\Gamma,pr}, F_{DR}) \rightarrow (\Omega_{/S^{an}}^{\bullet,\Gamma,pr}, F_{DR})$$

given by for $p : (Y \times S, Z) \rightarrow S$ the projection with $Y \in \text{SmVar}(\mathbb{C})$,

$$\begin{aligned}
& \Omega_{(S^{an}/S)}^{\Gamma,pr}((Y \times S, Z)/S) : \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} \Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b) \\
& \xrightarrow{\Omega_{((Y \times S)^{an}/Y \times S)/(S^{an}/S)}(-)} \Omega_{(Y \times S)^{an}/S^{an}}^\bullet \otimes_{O_{(Y \times S)^{an}}} (\Gamma_Z^{\vee, Hdg}(O_{Y \times S}, F_b))^{an}
\end{aligned}$$

We have the following canonical transformation map given by the analytical functor:

Definition 161. Let $S \in \text{SmVar}(\mathbb{C})$. For $F \in C(\text{Var}(\mathbb{C})^{2, \text{smpr}}/S)$, we have the canonical transformation map in $C_{\mathcal{D}^\infty \text{fil}}(S^{\text{an}})$

$$\begin{aligned} & T(\text{an}, \Omega_{/S}^{\Gamma, \text{pr}})(F) : \\ & (e(S)_* \mathcal{H}om^\bullet(F, E_{\text{et}}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})))^{\text{an}} := O_{S^{\text{an}}} \otimes_{\text{an}_S^* O_S} \text{an}_S^*(e(S)_* \mathcal{H}om^\bullet(F, E_{\text{et}}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))) \\ & \xrightarrow{T(\text{an}, e)(-)} O_{S^{\text{an}}} \otimes_{\text{an}_S^* O_S} (e(S^{\text{an}})_* \text{An}_S^* \mathcal{H}om^\bullet(F, E_{\text{et}}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))) \\ & \xrightarrow{T(\text{An}, \text{hom})(-, -)} O_{S^{\text{an}}} \otimes_{\text{an}_S^* O_S} (e(S^{\text{an}})_* \mathcal{H}om^\bullet(\text{An}_S^* F, \text{An}_S^* E_{\text{et}}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR}))) \\ & \xrightarrow{\mathcal{H}om(\text{An}_S^* F, \text{An}_S^* E_{\text{et}}(\Omega_{/(S^{\text{an}}/S)}^{\Gamma, \text{pr}}) \otimes m)} e(S^{\text{an}})_* \mathcal{H}om^\bullet(\text{An}_S^* F, E_{\text{et}}(\Omega_{/S^{\text{an}}}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR})) \end{aligned}$$

By definition of the algebraic an analytic De Rham realization functor, we have a natural transformation between them :

Definition 162. Let $S \in \text{SmVar}(\mathbb{C})$. Let $M \in \text{DA}_c(S)$ and $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ such that $(M, W) = D(\mathbb{A}_S^1, \text{et})(F, W)$. We have the canonical transformation map $\pi_S(D(\text{MHM}(S))) \subset D_{\mathcal{D}(1,0)\text{fil}, \infty}(S^{\text{an}})$

$$\begin{aligned} & T(\text{An}, \mathcal{F}_{\text{an}}^{\text{FDR}})(M) : (\mathcal{F}_{\text{FDR}}^S(M))^{\text{an}} := (e(S)_* \mathcal{H}om^\bullet(\hat{R}^{\text{CH}}(\rho_S^* L(F, W)), E_{\text{zar}}(\Omega_{/S}^{\bullet, \Gamma, \text{pr}}, F_{DR})))^{\text{an}} \\ & \xrightarrow{T(\text{an}, \Omega_{/S}^{\Gamma, \text{pr}})(-)} e(S)_* \mathcal{H}om^\bullet(\text{An}_S^* \hat{R}^{\text{CH}}(\rho_S^* L(F, W)), E_{\text{usu}}(\Omega_{/S^{\text{an}}}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR})) =: \mathcal{F}_{S, \text{an}}^{\text{FDR}}(M) \end{aligned}$$

We give now the definition in the non smooth case : Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ the projection. Consider, for $I \subset J \subset [1, \dots, l]$, resp. for each $I \subset [1, \dots, l]$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ j_{IJ} \uparrow & & \uparrow p_{IJ} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

We then have the following lemma

Lemma 20. The maps $T(\text{an}, \Omega)(-)$ induce a morphism in $C_{\mathcal{D}^\infty(\text{fil})}(S/(\tilde{S}_I))$

$$\begin{aligned} & (T(\text{an}, \Omega_{/\tilde{S}_I}^{\Gamma, \text{pr}})(\hat{R}^{\text{CH}}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)))) : \\ & ((e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{\text{CH}}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{\text{zar}}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, \text{pr}}, F_{DR})))^{\text{an}}, (u_{IJ}^q(F))^{\text{an}}) \\ & \rightarrow (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}(\tilde{S}_I)^* \hat{R}^{\text{CH}}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^* F)), E_{\text{usu}}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR}))), u_{IJ}^q(F)) \end{aligned}$$

Proof. Obvious. □

Definition 163. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \dots, l]$, denote $S_I = \cap_{i \in I} S_i$. We have then closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I = \prod_{i \in I} \tilde{S}_i$. Let $M \in \text{DA}_c(S)$ and $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ such that $(M, W) = D(\mathbb{A}_S^1, \text{et})(F, W)$. We have, by lemma 20, the canonical transformation map in $\pi_S(D(\text{MHM}(S))) \subset D_{\mathcal{D}(1,0)\text{fil}, \infty}(S^{\text{an}})$

$$\begin{aligned} & T(\text{An}, \mathcal{F}_S^{\text{FDR}})(M) : (\mathcal{F}_S^{\text{FDR}}(M))^{\text{an}} \xrightarrow{=} \\ & ((e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{\text{CH}}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), E_{\text{zar}}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, \text{pr}}, F_{DR})))^{\text{an}}, (u_{IJ}^q(F, W))^{\text{an}}) \\ & \xrightarrow{(T(\text{an}, \Omega_{/\tilde{S}_I}^{\Gamma, \text{pr}})(\hat{R}^{\text{CH}}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))))} \\ & (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\text{An}(\tilde{S}_I)^* \hat{R}^{\text{CH}}(\rho_{\tilde{S}_I}^* L(i_{I*} j_I^*(F, W))), E_{\text{usu}}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, \text{pr}, \text{an}}, F_{DR}))), u_{IJ}^q(F, W)) \\ & \xrightarrow{=} \mathcal{F}_{S, \text{an}}^{\text{FDR}}(M) \end{aligned}$$

The following proposition says this transformation map between $\mathcal{F}^{S,an}$ and $(\mathcal{F}_S^{FDR})^{an}$ is functorial in $S \in \text{Var}(\mathbb{C})$, hence define a commutative diagram of morphism of 2-functor :

Proposition 144. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with, $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We then have closed embedding $i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$ and $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I := g|_{T_I} : T_I \rightarrow S_I$.

(i0) Then, for $M \in \text{DA}_c(S)$, the following diagram in $\pi_T(D(\text{MHM}(T^{an})) \subset D_{\mathcal{D}(1,0)fil}(T^{an}/(Y^{an} \times \tilde{S}_I^{an}))$, see definition 138 and definition 156 commutes

$$\begin{array}{ccc} g_{Hdg}^{*mod}((\mathcal{F}_S^{FDR}(M))^{an}) = (g_{Hdg}^{*mod}(\mathcal{F}_S^{FDR}(M))) \xrightarrow{g_{Hdg}^{*mod}(T(An, \mathcal{F}_S^{FDR})(M))} g_{Hdg}^{*mod}(\mathcal{F}_{S,an}^{FDR}(M)) & & \\ \downarrow (T(g, \mathcal{F}^{FDR})(M))^{an} & & \downarrow T(g, \mathcal{F}_{an}^{FDR})(M) \\ (\mathcal{F}_T^{FDR}(g^*M))^{an} & \xrightarrow{T(An, \mathcal{F}_T^{FDR})(g^*M)} & \mathcal{F}_{T,an}^{FDR}(g^*M) \end{array}$$

(i1) Then, for $M \in \text{DA}_c(S)$, the following diagram in $\pi_T(D(\text{MHM}(T)) \subset D_{\mathcal{D}(1,0)fil}(T^{an}/(Y^{an} \times \tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc} g_{Hdg}^{*mod}((\mathcal{F}_S^{FDR}(M))^{an}) = (g_{Hdg}^{*mod}(\mathcal{F}_S^{FDR}(M))) \xrightarrow{g_{Hdg}^{*mod}(T(An, \mathcal{F}_S^{FDR})(M))} g_{Hdg}^{*mod}(\mathcal{F}_{S,an}^{FDR}(M)) & & \\ \uparrow (T(g, \mathcal{F}^{FDR})(M))^{an} & & \uparrow T^!(g, \mathcal{F}_{an}^{FDR})(M) \\ (\mathcal{F}_T^{FDR}(g^*M))^{an} & \xrightarrow{T(An, \mathcal{F}_T^{FDR})(g^*M)} & \mathcal{F}_{T,an}^{FDR}(g^*M) \end{array}$$

(i2) Then, for $M \in \text{DA}_c(T)$, the following diagram in $\pi_S(D(\text{MHM}(S^{an})) \subset D_{\mathcal{D}(1,0)fil}(S^{an}/(\tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc} Rg_*^{Hdg}((\mathcal{F}_T^{FDR}(M))^{an}) = (Rg_*^{Hdg} * (\mathcal{F}_T^{FDR}(M))) \xrightarrow{Rg_*^{Hdg}(T(An, \mathcal{F}_S^{FDR})(M))} Rg_*^{Hdg}(\mathcal{F}_{S,an}^{FDR}(M)) & & \\ \uparrow (T_*(g, \mathcal{F}^{FDR})(M))^{an} & & \uparrow T_*(g, \mathcal{F}_{an}^{FDR})(M) \\ (\mathcal{F}_S^{FDR}(Rg_*M))^{an} & \xrightarrow{T(An, \mathcal{F}_S^{FDR})(Rg_*M)} & \mathcal{F}_{S,an}^{FDR}(Rg_*M) \end{array}$$

(i3) Then, for $M \in \text{DA}_c(T)$, the following diagram in $\pi_S(D(\text{MHM}(S^{an})) \subset D_{\mathcal{D}(1,0)fil}(S^{an}/(\tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc} Rg_!^{Hdg}((\mathcal{F}_T^{FDR}(M))^{an}) = (Rg_!^{Hdg}(\mathcal{F}_T^{FDR}(M))) \xrightarrow{Rg_!^{Hdg}(T(An, \mathcal{F}_S^{FDR})(M))} Rg_!^{Hdg}(\mathcal{F}_{S,an}^{FDR}(M)) & & \\ \downarrow (T_!(g, \mathcal{F}^{FDR})(M))^{an} & & \downarrow T_!(g, \mathcal{F}_{an}^{FDR})(M) \\ (\mathcal{F}_S^{FDR}(Rg_!M))^{an} & \xrightarrow{T(An, \mathcal{F}_S^{FDR})(Rg_!M)} & \mathcal{F}_{S,an}^{FDR}(Rg_!M) \end{array}$$

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$, the following diagram in $\pi_S(D(\text{MHM}(S))) \subset$

$D_{\mathcal{D}(1,0)}\text{fil}(S^{an}/(\tilde{S}_I^{an}))$ commutes

$$\begin{array}{ccc}
(\mathcal{F}_S^{FDR}(M) \otimes_{\mathcal{O}_S}^{Hdg} \mathcal{F}_S^{FDR}(N))^{(T(\otimes, \mathcal{F}_S^{FDR})(M,N))^{an}} & \xrightarrow{\quad} & (\mathcal{F}_S^{FDR}(M \otimes N))^{an} \\
\downarrow = & & \downarrow \\
(\mathcal{F}_S^{FDR}(M))^{an} \otimes_{\mathcal{O}_{S^{an}}} (\mathcal{F}_S^{FDR}(N))^{an} & \xrightarrow{T(\text{An}, \mathcal{F}_S^{FDR})(M \otimes N)} & \\
\downarrow T(\text{An}, \mathcal{F}_S^{FDR})(M) \otimes_{\mathcal{O}_{S^{an}}}^{Hdg} T(\text{An}, \mathcal{F}_S^{FDR})(M) & & \downarrow \\
\mathcal{F}_{S,an}^{FDR}(M) \otimes_{\mathcal{O}_{S^{an}}}^{Hdg} \mathcal{F}_{S,an}^{FDR}(N) & \xrightarrow{T(\otimes, \mathcal{F}_{S,an}^{FDR})(M,N)} & \mathcal{F}_{S,an}^{FDR}(M \otimes N)
\end{array}$$

Proof. Immediate from definition. \square

Proposition 145. Let $f : X \rightarrow S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Assume there exist a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p} S$$

with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p the projection. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have then the following commutative diagram in $D_{\mathcal{D}^\infty \text{fil}, \infty}(S^{an})$,

$$\begin{array}{ccc}
(\mathcal{F}_S^{FDR}(M(X/S)))^{an} & \xrightarrow{T(\text{An}, \mathcal{F}_S^{FDR})(M(X/S))} & \mathcal{F}_{S,an}^{FDR}(M(X/S)) \\
\downarrow I(X/S) & & \downarrow I(X/S) \\
(Rf_!^{Hdg}(\Gamma_{X_I}^{\vee, Hdg}(O_{Y \times \tilde{S}_I}, F_b)(d_Y)[2d_Y], x_{IJ}(X/S)))^{an} & \xrightarrow{T^{\text{Dmod}}(an, p_{\tilde{S}_I})^{(-)}} & Rf_!^{Hdg}((\Gamma_{X_I}^{\vee, Hdg}(O_{(Y \times \tilde{S}_I)^{an}}, F_b)(d_Y)[2d_Y], x_{IJ}(X/S)))
\end{array}$$

Proof. Immediate from definition. \square

We deduce from proposition 145 and theorem 23 (GAGA for D-modules) the following :

Theorem 40. Let $S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(S)$, the map in $\pi_S(D(\text{MHM}(S^{an}))) \subset D_{\mathcal{D}(1,0)}\text{fil}, \infty(S^{an})$

$$T(\text{An}, \mathcal{F}^{FDR})(M) : (\mathcal{F}^{FDR}(M))^{an} \xrightarrow{\sim} \mathcal{F}_{S,an}^{FDR}(M)$$

given in definition 163 is an isomorphism.

Proof. Follows from proposition 145 and theorem 23. \square

We finish this subsection by the following easy proposition :

Proposition 146. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}_S^1, et)(F)$. Then the following diagram in $D_{\mathcal{O} \text{fil}, \mathcal{D}, \infty}(S/\tilde{S}_I)$ commutes

$$\begin{array}{ccc}
(\mathcal{F}_S^{GM}(L\mathbb{D}_S M))^{an} & \xrightarrow{\mathcal{J} \circ (T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M))^{an}} & J_S((\mathcal{F}_S^{FDR}(M))^{an}) \\
\downarrow T(\text{An}, \mathcal{F}_S^{GM})(\mathbb{D}_S M) & & \downarrow J_S(T(\text{An}, \mathcal{F}_S^{FDR})(M)) \\
\mathcal{F}_{S,an}^{GM}(L\mathbb{D}_S M) & \xrightarrow{T(\mathcal{F}_{S,an}^{GM}, \mathcal{F}_{S,an}^{FDR})(M)} & J_S(\mathcal{F}_{S,an}^{FDR}(M))
\end{array}$$

Proof. Immediate from definition. \square

7 The Hodge realization functor for relative motives

7.1 The Betti realization functor

We have two definition of the Betti realization functor which coincide at least for constructible motives, one given by [1] using the analytical functor and one given in [7] by composing the analytical functor with the forgetfull functor to the topological space of a complex analytic space wich is a CW complex (see also [21] for the absolute case) .

Definition 164. *Let $S \in \text{Var}(\mathbb{C})$.*

(i) *The Ayoub's Betti realization functor is*

$$\text{Bti}_S^* : \text{DA}(S) \rightarrow D(S^{an}), M \in \text{DA}(S) \mapsto \text{Bti}_S^* M = \text{Re}(S^{an})_* \text{An}_S^* M = e(S^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_S^* F$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(ii) *In [7], we define the Betti realization functor as*

$$\widetilde{\text{Bti}}_S^* : \text{DA}(S) \rightarrow D(S^{an}) = D(S^{cw}), M \mapsto \widetilde{\text{Bti}}_S^* M = \text{Re}(S^{cw})_* \widetilde{\text{Cw}}_S^* M = e(S^{cw})_* \underline{\text{sing}}_{\mathbb{D}}^* \widetilde{\text{Cw}}_S^* F$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(iii) *For the Corti-Hanamura weight structure on $\text{DA}^-(S)$, we have by functoriality of (i) the functor*

$$\text{Bti}_S^* : \text{DA}^-(S) \rightarrow D_{fil, \infty}(S^{an}), M \mapsto \text{Bti}_S^* M = e(S^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_S^*(F, W)$$

where $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$.

Note that by [7], An_S^* and $\widetilde{\text{Cw}}_S^*$ derive trivially.

Note that, by considering the explicit \mathbb{D}_S^1 local model for presheaves on $\text{AnSp}(\mathbb{C})^{sm}/S^{an}$, $\text{Bti}_S^*(\text{DA}^-(S)) \subset D^-(S^{an})$; by considering the explicit \mathbb{I}_S^1 local model for presheaves on CW^{sm}/S^{cw} , $\widetilde{\text{Bti}}_S^*(\text{DA}^-(S)) \subset D^-(S^{an})$.

Let $f : T \rightarrow S$ a morphism, with $T, S \in \text{Var}(\mathbb{C})$. We have, for $M \in \text{DA}(S)$, $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$, and an equivalence (\mathbb{A}^1, et) local $e : f^*(F, W) \rightarrow (F', W)$ with $(F', W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(f^*M, W) = D(\mathbb{A}^1, et)(F', W)$ the following canonical transformation map in $D_{fil}(T)$:

$$T^0(f, \text{Bti})(M) : f^* \text{Bti}_S^* M := f^* e(S^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_S^*(F, W) \xrightarrow{T(f, e)(-)} e(T^{an})_* f^* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_S^*(F, W) \quad (66)$$

$$\xrightarrow{e(T^{an})_* T(f, c)(F, W)} e(T^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* f^* \text{An}_T^*(F, W) \xrightarrow{\cong} e(T^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_T^* f^*(F, W) \quad (67)$$

$$\xrightarrow{e(T^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_T^* e} e(T^{an})_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_T^*(F', W) =: \text{Bti}_T^* f^* M \quad (68)$$

Definition 165. *Let $f : T \rightarrow S$ a morphism, with $T, S \in \text{Var}(\mathbb{C})$. Consider the graph factorization $f : T \xrightarrow{l} T \times S \xrightarrow{p} S$ of f with l the graph closed embedding and p the projection. We have, for $M \in \text{DA}_c(S)$, the following canonical transformation map in $D_{fil, c}(T^{an})$:*

$$\begin{aligned} T(f, \text{Bti})(M, W) : f^{*w} \text{Bti}_S^*(M, W) &:= l^* \Gamma_T^{\vee, w} p^* \text{Bti}_S^*(F, W) \\ \xrightarrow{T^0(p, \text{Bti})(-)} l^* \Gamma_T^{\vee, w} \text{Bti}_{T \times S}^* p^*(F, W) &\xrightarrow{\gamma_T^{\vee}(p^*(F, W))} l^* \Gamma_T^{\vee, w} \text{Bti}_{T \times S}^* \Gamma_T^{\vee} p^*(F, W) \\ \xrightarrow{\cong} l^* \text{Bti}_{T \times S}^* \Gamma_T^{\vee} p^*(F, W) &\xrightarrow{T^0(l, \text{Bti})(-)} \text{Bti}_T^* l^* \Gamma_T^{\vee} p^*(F, W) = \text{Bti}_T^* f^*(M, W). \end{aligned}$$

where we use definition 102.

Definition 166. • Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. We have, for $M \in \text{DA}_c(X)$, the following transformation map in $D_{\text{fil},c}(S^{an})$

$$T_*(f, \text{Bti})(M, W) : \text{Bti}_S^*(Rf_*(M, W)) \xrightarrow{\text{ad}(f^*, Rf_{*w})(\text{Bti}_S^*(Rf_*(M, W)))} Rf_{*w} f^{*w} \text{Bti}_S^*(Rf_*(M, W)) \\ \xrightarrow{T(f, \text{Bti})(Rf_*(M, W))} Rf_{*w} \text{Bti}_X^*(f^* Rf_*(M, W)) \xrightarrow{\text{Bti}_X^*(\text{ad}(f^*, Rf_*)(M, W))} Rf_{*w} \text{Bti}_X^*(M, W)$$

Clearly if $l : Z \hookrightarrow S$ is a closed embedding, then $T_*(l, \text{Bti})(M, W)$ is an isomorphism since $\text{ad}(l^*, l_*)(-) : l^* l_*(M, W) \rightarrow (M, W)$ is an isomorphism (see section 3).

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then, for $M \in \text{DA}_c(X)$, using theorem 41 for closed embeddings, the following transformation map in $D_{\text{fil}}((Y \times S)^{an})$

$$T_l(f, \text{Bti})(M) : Rf_{!w} \text{Bti}_X^*(M, W) = Rp_{S!w} l_* \text{Bti}_X^*(M, W) \\ \xrightarrow{T_*(l, \text{Bti})(M, W)}^{-1} Rp_{S!w} \text{Bti}(Y \times S)^*(l_*(M, W)) \\ \xrightarrow{\text{Bti}(Y \times S)^* \text{ad}(Lp_{S\sharp}, p_S^*)(l_*(M, W))} Rp_{S!w} \text{Bti}(Y \times S)^*(p_S^* Lp_{S\sharp} l_*(M, W)) \xrightarrow{T(p_S, \text{Bti})(p_{S\sharp} l_*(M, W))} \\ Rp_{S!w} p_S^* \text{Bti}(Y \times S)^*(Lp_{S\sharp} l_*(M, W)) = Rp_{S!w} p_S^{!w} \text{Bti}(Y \times S)^*(Rf_l(M, W)) \\ \xrightarrow{\text{ad}(Rp_{S!w}, p_S^{!w})(-)} \text{Bti}(Y \times S)^*(Rf_l(M, W))$$

Clearly, for $f : X \rightarrow S$ a proper morphism, with $X, S \in \text{Var}(\mathbb{C})$ we have, for $M \in \text{DA}_c(Y \times S)$, $T_l(f, \text{Bti})(M, W) = T_*(f, \text{Bti})(M, W)$.

- Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. We have, using the second point, for $M \in \text{DA}(S)$, the following transformation map in $D_{\text{fil}}(X^{an})$

$$T^l(f, \text{Bti})(M, W) : \text{Bti}_X^*(f^!(M, W)) \xrightarrow{\text{ad}(f_l, Rf^l)(\text{Bti}_X^*(f^!(M, W)))} f^{!w} Rf_{!w} \text{Bti}_X^*(f^!(M, W)) \\ \xrightarrow{T_l(f, \text{Bti})(f^!(M, W))} f^{!w} \text{Bti}_S^*(f_l f^!(M, W)) \xrightarrow{\text{Bti}_S^*(\text{ad}(f_l, f^l)(M, W))} f^{!w} \text{Bti}_S^*(M, W)$$

- Let $S \in \text{Var}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, \text{et})(F)$ and $N = D(\mathbb{A}^1, \text{et})(G)$, the following transformation map in $D_{\text{fil}}(S^{an})$

$$\text{Bti}_S^*(M, W) \otimes \text{Bti}_S^*(N, W) := (e(S)_* \text{sing}_{\mathbb{D}^*} \text{An}_S^*(F, W)) \otimes (e(S)_* \text{sing}_{\mathbb{D}^*} \text{An}_S^*(G, F)) \\ \xrightarrow{T(\text{sing}_{\mathbb{D}^*}, \otimes)(\text{An}_S^*(F, W), \text{An}_S^*(G, F))} e(S)_* \text{sing}_{\mathbb{D}^*} \text{An}_S^*((F, W) \otimes (G, W)) =: \text{Bti}_S^*((M, W) \otimes (N, W))$$

Theorem 41. (i) Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(S)$,

$$T(f, \text{Bti})(M, W) : f^{*w} \text{Bti}_S^*(M, W) \xrightarrow{\sim} \text{Bti}_X^* f^*(M, W)$$

is an isomorphism in $D_{\text{fil}}(X^{an})$.

(ii) Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(X)$,

$$T_l(f, \text{Bti})(M, W) : Rf_{!w} \text{Bti}_X^*(M, W) \xrightarrow{\sim} \text{Bti}_S^* Rf_!(M, W)$$

is an isomorphism.

(iii) Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(X)$,

$$T_*(f, \text{Bti})(M, W) : Rf_{*w} \text{Bti}_X^*(M, W) \xrightarrow{\sim} \text{Bti}_S^* Rf_*(M, W)$$

is an isomorphism.

(iv) Let $f : X \rightarrow S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(S)$,

$$T^!(f, \text{Bti})(M, W) : f^{!w} \text{Bti}_S^*(M, W) \xrightarrow{\sim} \text{Bti}_X^* f^!(M, W)$$

is an isomorphism.

(v) Let $S \in \text{Var}(\mathbb{C})$. For $M, N \in \text{DA}_c(S)$,

$$T(\otimes, \text{Bti})(M, W) : \text{Bti}_S^*(M, W) \otimes \text{Bti}_S^* N \xrightarrow{\sim} \text{Bti}_X^*((M, W) \otimes (N, W))$$

is an isomorphism.

Proof. By functoriality it reduced to the case of Corti-Hanamura motives which then follows from [1]. \square

The main result on the Betti realization functor is the following

Theorem 42. (i) We have $\text{Bti}_S^* = \widetilde{\text{Bti}}_S^*$ on $\text{DA}^-(S)$

(ii) The canonical transformations $T(f, \text{Bti})$, for $f : T \rightarrow S$ a morphism in $\text{Var}(\mathbb{C})$, define a morphism of 2 functor

$$\text{Bti}_S^* : \text{DA}(\cdot) \rightarrow D(\cdot^{an}), S \in \text{Var}(\mathbb{C}) \mapsto \text{Bti}_S^* : \text{DA}(S) \rightarrow D(S^{an})$$

which is a morphism of homotopic 2 functor.

(ii)' The canonical transformations $T(f, \text{Bti})$, for $f : T \rightarrow S$ a morphism in $\text{Var}(\mathbb{C})$, define a morphism of 2 functor

$$\text{Bti}_S^* : \text{DA}(\cdot) \rightarrow D_{fil}(\cdot^{an}), S \in \text{Var}(\mathbb{C}) \mapsto \text{Bti}_S^* : \text{DA}(S) \rightarrow D_{fil}(S^{an})$$

which is a morphism of homotopic 2 functor.

Proof. (i): See [7]

(ii) and (ii)': Follows from theorem 41. \square

Remark 13. For $X \in \text{Var}(\mathbb{C})$, the quasi-isomorphisms

$$\mathbb{Z} \text{Hom}(\bar{\mathbb{D}}_{et}^\bullet, X) \xrightarrow{\text{An}^*} \mathbb{Z} \text{Hom}(\bar{\mathbb{D}}^n(0, 1), X^{an}) \xrightarrow{\text{Hom}(i, X^{cw})} \mathbb{Z} \text{Hom}([0, 1]^n, X^{cw}),$$

where,

$$\bar{\mathbb{D}}_{et}^n := (e : U \rightarrow \mathbb{A}^n, \bar{\mathbb{D}}^n(0, 1) \subset e(U)) \in \text{Fun}(\mathcal{V}_{\mathbb{A}^n}^{et}(\bar{\mathbb{D}}^n(0, 1)), \text{Var}(\mathbb{C}))$$

is the system of etale neighborhood of the closed ball $\bar{\mathbb{D}}^n(0, 1) \subset \mathbb{A}^n$, and $i : [0, 1]^n \hookrightarrow \bar{\mathbb{D}}^n(0, 1)$ is the closed embedding, shows that a closed singular chain $\alpha \in \mathbb{Z} \text{Hom}^n([0, 1]^n, X^{cw})$, is homologue to a closed singular chain

$$\beta = \alpha + \partial\gamma = \tilde{\beta}_{|[0, 1]^n} \in \mathbb{Z} \text{Hom}^n(\Delta^n, X^{cw})$$

which is the restriction by the closed embedding $[0, 1]^n \hookrightarrow U^{cw} \xrightarrow{e} \mathbb{A}^n$, where $e : U \rightarrow \mathbb{A}^n$ an etale morphism with $U \in \text{Var}(\mathbb{C})$, of a complex algebraic morphism $\tilde{\beta} : U \rightarrow X$. Hence $\beta([0, 1]^n) = \tilde{\beta}([0, 1]^n) \subset X$ is the restriction of a real algebraic subset of dimension n in $\text{Res}_{\mathbb{R}}(X)$ (after restriction a scalar that is under the identification $\mathbb{C} \simeq \mathbb{R}^2$).

Definition 167. Let $S \in \text{Var}(\mathbb{C})$ The cohomological Betti realization functor is

$$\begin{aligned} & \text{Bti}_S^\vee : \text{DA}(S) \rightarrow D(S^{cw}), \\ M \mapsto \text{Bti}_S^\vee(M) & := R\mathcal{H}om(\text{Bti}_S^* M, \mathbb{Z}_{S^{cw}}) = R\mathcal{H}om(M, \text{Bti}_{S^*} \mathbb{Z}_{S^{cw}}) \end{aligned}$$

where for $\text{Bti}_{S^*} : K \in D(S^{cw}) \mapsto R \text{Ans}_* e(S^{an})^* K \in \text{DA}(S)$ is the right ajoin to Bti_S^* .

7.2 The Hodge realization functor for relative motives

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embedding $i_i : S \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Recall (see section 5.2) that $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$ is the category

- whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with

$$\begin{aligned} ((M_I, F, W), u_{IJ}) &\in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)), (K, W) \in D_{fil}(S^{an}), \\ \alpha &: T(S/(\tilde{S}_I))(K, W) \otimes \mathbb{C}_{S^{an}} \rightarrow DR(S)^{[-1]}(((M_I, W), u_{IJ})^{an}) \end{aligned}$$

where α is an morphism in $D_{fil}(S^{an}/(\tilde{S}_I^{an}))$,

- and whose set of morphisms consists of

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

where $\phi_D : ((M_1, F, W), u_{IJ}) \rightarrow ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \rightarrow (K_2, W)$ are morphisms and

$$\begin{aligned} \theta &= (\theta^\bullet, I(DR(S)(\phi_D^{an})) \circ I(\alpha_1), I(\alpha_2) \circ I(\phi_C \otimes I)) : \\ &I(T(S/(\tilde{S}_I))(K_1, W)) \otimes \mathbb{C}_{S^{an}}[1] \rightarrow I(DR(S)((M_{2I}, W), u_{IJ})^{an}) \end{aligned}$$

is an homotopy, $I : D_{fil}(S^{an}/(\tilde{S}_I^{an})) \rightarrow K_{fil}(S^{an}/(\tilde{S}_I^{an}))$ being the injective resolution functor, and for

$$\begin{aligned} - \phi &= (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \\ - \phi' &= (\phi'_D, \phi'_C, [\theta']) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \rightarrow (((M_{3I}, F, W), u_{IJ}), (K_3, W), \alpha_3) \end{aligned}$$

the composition law is given by

$$\begin{aligned} \phi' \circ \phi &:= (\phi'_D \circ \phi_D, \phi'_C \circ \phi_C, I(DR(S)(\phi'_D{}^{an})) \circ [\theta] + [\theta'] \circ I(\phi_C \otimes I)[1]) : \\ &(((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{3I}, F, W), u_{IJ}), (K_3, W), \alpha_3), \end{aligned}$$

in particular for $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$,

$$I_{(((M_I, F, W), u_{IJ}), (K, W), \alpha)} = ((I_{M_I}, I_K, 0),$$

together with the localization functor

$$\begin{aligned} (D(zar), I) : C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) &\rightarrow D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \\ &\rightarrow D_{\mathcal{D}(1,0)fil,rh,\infty}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}). \end{aligned}$$

Moreover,

- For $((((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$, we set

$$(((M_I, F, W), u_{IJ}), (K, W), \alpha)[1] := (((M_I, F, W), u_{IJ})[1], (K, W)[1], \alpha[1]).$$

- For

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

a morphism in $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$, we set (see [9] definition 3.12)

$$\text{Cone}(\phi) := (\text{Cone}(\phi_D), \text{Cone}(\phi_C), ((\alpha_1, \theta), (\alpha_2, 0))) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}),$$

$((\alpha_1, \theta), (\alpha_2, 0))$ being the matrix given by the composition law, together with the canonical maps

- $c_1(-) = (c_1(\phi_D), c_1(\phi_C), 0) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \rightarrow \text{Cone}(\phi)$
- $c_2(-) = (c_2(\phi_D), c_2(\phi_C), 0) : \text{Cone}(\phi) \rightarrow (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1)[1]$.

Consider the category

$$(D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil}(\tilde{S}_I^{an})) \subset \text{Fun}(\Gamma(\tilde{S}_I), \text{TriCat})$$

- whose objects are $(((M_I, F, W), (K_I, W), \alpha_I), u_{IJ}) \in \text{Fun}(\Gamma(\tilde{S}_I), \text{TriCat})$ such that

$$((M_I, F, W), (K_I, W), \alpha_I) \in D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil}(\tilde{S}_I^{an}) =: \mathcal{D}(\tilde{S}_I)$$

and for $I \subset J$,

$$\begin{aligned} u_{IJ} &: ((M_I, F, W), (K_I, W), \alpha_I) \rightarrow \\ p_{IJ*}((M_J, F, W), (K_J, W), \alpha_J) &:= (p_{IJ*}(M_J, F, W), p_{IJ*}(K_J, W), p_{IJ*}\alpha_J) \end{aligned}$$

are morphisms in $\mathcal{D}(\tilde{S}_I)$,

- whose morphisms $m = (m_I) : (((M_I, F, W), (K_I, W), \alpha_I), u_{IJ}) \rightarrow (((M'_I, F, W), (K'_I, W), \alpha'_I), v_{IJ})$ is a family of morphism such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $\mathcal{D}(\tilde{S}_I)$

We have then the identity functor

$$\begin{aligned} I_S : D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))^0 \times_I D_{fil}(S^{an}) &\rightarrow (D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil}(\tilde{S}_I^{an})), \\ (((M_I, F, W), u_{IJ}), (K, W), \alpha) &\mapsto (((M_I, F, W), i_{I*}j_I^*(K, W), j_I^*\alpha), (u_{IJ}, I, 0)), \\ m = (m_I, n) &\mapsto m = (m_I, i_{I*}j_I^*n) \end{aligned}$$

which is a full embedding since for $((M_I, F, W), u_{IJ}) \in D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))^0$,

$$u_{IJ} : (M_I, F, W) \rightarrow p_{IJ*}(M_J, F, W)$$

are isomorphisms in $D_{\mathcal{D}(1,0)fil}(\tilde{S}_I)$, and hence for $((M_I, F, W), u_{IJ}), (K, W), \alpha \in D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))^0 \times_I D_{fil}(S^{an})$,

$$\begin{aligned} (u_{IJ}, I, 0) &: ((M_I, F, W), i_{I*}j_I^*(K, W), j_I^*\alpha) \rightarrow \\ p_{IJ*}((M_J, F, W), i_{J*}j_J^*(K, W), j_J^*\alpha) &= (p_{IJ*}(M_J, F, W), i_{I*}j_I^*(K, W), j_I^*\alpha) \end{aligned}$$

are isomorphisms in $\mathcal{D}(\tilde{S}_I)$.

Definition 168. For $h : U \rightarrow S$ a smooth morphism with $S, U \in \text{SmVar}(\mathbb{C})$ and $h : U \xrightarrow{n} X \xrightarrow{f} S$ a compactification of h with n an open embedding, $X \in \text{SmVar}(\mathbb{C})$ such that $D := X \setminus U = \cup_{i=1}^s D_i \subset X$ is a normal crossing divisor, we denote by, using definition 116 and definition 129

$$\begin{aligned} I(U/S) &: h!_{Hdg} h^!_{Hdg} \mathbb{Z}_S^{Hdg} \xrightarrow{=} \\ (p_{S*} E_{zar}(\Omega_{X \times S/S}^\bullet \otimes_{O_{X \times S}} (n \times I)!_{Hdg} \Gamma_U^{\vee, Hdg}(O_{U \times S}, F_b)), \mathbb{D}_S h_* E_{usu} \mathbb{Z}_{U^{an}}, h! \alpha(U)) & \\ \underline{((DR(X \times S/S)(\text{ad}((n \times I)!_{Hdg}, (n \times I)^*(-)), 0), I, 0))} & \\ (\text{Cone}((\Omega_{S/S}^{\Gamma, pr}(i_{D_i} \times I))_{i \in [1, \dots, s]} : p_{S*} E_{zar}(\Omega_{X \times S/S}^\bullet \otimes_{O_{X \times S}} \Gamma_X^{\vee, Hdg}(O_{X \times S}, F_b)) \rightarrow & \\ (\cdots \rightarrow (p_{S*} E_{zar}(\Omega_{D_i \times S/S}^\bullet \otimes_{O_{D_i \times S}} \Gamma_{D_i}^{\vee, Hdg}(O_{D_i \times S}, F_b))) \rightarrow \cdots)), \mathbb{D}_S h_* E_{usu} \mathbb{Z}_{U^{an}}, h! \alpha(U)) & \\ \xrightarrow{=} (\mathcal{F}_S^{FDR}(\mathbb{Z}(U/S)), \text{Bti}_S^* \mathbb{Z}(U/S), \alpha(\mathbb{Z}(U/S))) & \end{aligned}$$

the canonical isomorphism in $D_{\mathcal{D}fil}(S) \times_I D(S^{an})$, with

$$h^!_{Hdg} \mathbb{Z}_S^{Hdg} = (\Gamma_U^{\vee, Hdg}(O_{U \times S}, F_b), \mathbb{Z}_{U^{an}}, \alpha(U)) \in HM(U),$$

and $i_{D_i} : D_i \hookrightarrow X$ are the closed embeddings.

Lemma 21. *Let $S \in \text{SmVar}(\mathbb{C})$. Let $g : U'/S \rightarrow U/S$ a morphism with $U/S := (U, h), U'/S := (U', h) \in \text{Var}(\mathbb{C})^{\text{sm}}/S$. Let $h : U \xrightarrow{n} X \xrightarrow{f} S$ a compactification of h with n an open embedding, $X \in \text{SmVar}(\mathbb{C})$ such that $D := X \setminus U = \cup_{i=1}^s D_i \subset X$ is a normal crossing divisor, Let $h' : U \xrightarrow{n'} X' \xrightarrow{f'} S$ a compactification of h' with n' an open embedding, $X' \in \text{SmVar}(\mathbb{C})$ such that $D' := X' \setminus U = \cup_{i=1}^s D_i \subset X'$ is a normal crossing divisor and such that $g : U' \rightarrow U$ extend to $\bar{g} : X' \rightarrow X$, see definition-proposition 12. Then, using definition 168, the following diagram in $D_{\mathcal{D}fil}(S) \times_I D(S^{an})$ commutes*

$$\begin{array}{ccc} h'_{!Hdg} h'^{!Hdg} \mathbb{Z}_S^{Hdg} & \xrightarrow{I(U'/S)} & (\mathcal{F}_S^{FDR}(\mathbb{Z}(U'/S)), \text{Bti}_S^* \mathbb{Z}(U'/S), \alpha(\mathbb{Z}(U'/S))) \\ \text{ad}(g_{!Hdg}, g^{!Hdg})(h'^{!Hdg} \mathbb{Z}_S^{Hdg}) \downarrow & & \downarrow (\Omega_{\tilde{S}}^{\Gamma, pr}(\hat{R}_S^{CH}(g)), \text{Bti}_S^*(g), \theta(g)) \\ h_{!Hdg} h^{!Hdg} \mathbb{Z}_S^{Hdg} & \xrightarrow{I(U/S)} & (\mathcal{F}_S^{FDR}(\mathbb{Z}(U/S)), \text{Bti}_S^* \mathbb{Z}(U/S), \alpha(\mathbb{Z}(U/S))) \end{array}$$

where

$$\theta(g) := R_{\mathcal{D}}([\Gamma_g]) : I(\text{Bti}_S^* \mathbb{Z}(U'/S) \otimes \mathbb{C})[1] \rightarrow I(DR(S)(o_{fil} \mathcal{F}_S^{FDR}(\mathbb{Z}(U/S))^{an}))$$

is the homotopy given by the third term of the Deligne homology class of the graph $\Gamma_g \subset U' \times_S U$ (see definition 118) and $o_{fil} : C_{\mathcal{D}fil}(S) \rightarrow C_{\mathcal{D}}(S)$ is the forgetful functor and we recall (see section 5.2) that $I : C(S_C^{an}/\tilde{S}_I^{an}) \rightarrow K(S_C^{an}/\tilde{S}_I^{an})$ is the injective resolution functor.

Proof. Immediate from definition. \square

We now define the Hodge realization functor.

Definition 169. *Let $k \subset \mathbb{C}$ a subfield. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embedding $i_i : S \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We define the Hodge realization functor, using definition 133, definition 164, and lemma 21*

$$\mathcal{F}_S^{Hdg} := (\mathcal{F}_S^{FDR}, \text{Bti}_S^*) : C(\text{Var}(\mathbb{C})^{\text{sm}}/S) \rightarrow D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$$

first on objects and then on morphisms :

- for $F \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$, taking $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ such that $D(\mathbb{A}^1, et)(F, W)$ gives the weight structure on $D(\mathbb{A}^1, et)(F)$,

$$\begin{aligned} \mathcal{F}_S^{Hdg}(F) &:= (\mathcal{F}_S^{FDR}(F), \text{Bti}_S^* F, \alpha(F)) := \\ &(e(S)_* \text{Hom}((\hat{R}_{\tilde{S}_I}^{CH}(\rho_{\tilde{S}_I}^* Li_{I^*} J_I^*(F, W)), \hat{R}^{CH}(T^q(D_{IJ})(-))), (E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}), T_{IJ})), \\ &e(S)_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_S^* L(F, W), \alpha(F)) \in D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \end{aligned}$$

where $\alpha(F)$ is the map in $D_{\mathbb{D}fil}(S^{an}/(\tilde{S}_I^{an}))$, writing for short $DR(S) := DR(S)^{[-1]} := (DR(\tilde{S}_I)[-d_{\tilde{S}_I}])$

$$\begin{aligned} \alpha(F) : T(S/(\tilde{S}_I))((\text{Bti}_S^*(M, W)) \otimes \mathbb{C}_S) &:= (i_{I^*} J_I^*((e(S)_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_S^* L(F, W)) \otimes \mathbb{C}_S), I) \\ &\xrightarrow{\cong} (e(\tilde{S}_I)_* \underline{\text{sing}}_{\mathbb{D}}^* \text{An}_{\tilde{S}_I}^* Li_{I^*} J_I^*(F, W) \otimes \mathbb{C}_S, T(p_{IJ}, \text{An})(Li_{I^*} J_I^*(F, W))) \\ &\xrightarrow{\cong} (((\cdot \rightarrow \oplus_{(U_{I\alpha}, h_{I\alpha}) \in V_I} h_{I\alpha}! h_{I\alpha}^! \mathbb{Z}_{\tilde{S}_I} \xrightarrow{\text{ad}(g_{I, \alpha, \beta}^!, g_{I, \alpha, \beta}^!(-))} \oplus_{(U_{I\alpha}, h_{I\alpha}) \in V_I} h_{I\alpha}! h_{I\alpha}^! \mathbb{Z}_{\tilde{S}_I} \rightarrow \cdot), u_{IJ})) \\ &\quad \xrightarrow{(\alpha(\mathbb{Z}(U_{I\alpha}/\tilde{S}_I), \theta(g_{I, \alpha, \beta}^!))} \\ &DR(S)(o_{fil}(e(S)_* \text{Hom}((\hat{R}_{\tilde{S}_I}^{CH}(\rho_{\tilde{S}_I}^* Li_{I^*} J_I^*(F, W)), \hat{R}^{CH}(T^q(D_{IJ})(-))), (E_{zar}(\Omega_{\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{DR}), T_{IJ})))^{an}) \\ &\xrightarrow{\cong} DR(S)((o_{fil} \mathcal{F}_S^{FDR}(M, W))^{an}) \end{aligned}$$

using lemma 21, $(\alpha(\mathbb{Z}(U_{I\alpha}/S)), \theta(g_{I,\alpha,\beta}^\bullet))$ being the matrix given inductively by the composition law in $D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil}(\tilde{S}_I^{an})$, that is we have the following isomorphism in $(D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil}(\tilde{S}_I^{an}))$, denoting for short $V_I := \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I$

$$(I^\bullet(U_{I\alpha}/\tilde{S}_I)) : \\ ((\cdot \rightarrow \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in V_I} h_{I\alpha!} Hdg h_{I\alpha}^{!Hdg} \mathbb{Z}_{\tilde{S}_I}^{Hdg} \xrightarrow{\text{ad}(g_{I,\alpha,\beta}^\bullet, !Hdg, g_{I,\alpha,\beta}^\bullet)} (\cdot) \rightarrow \cdot), u_{IJ}) \\ \xrightarrow{\sim} I_S(\mathcal{F}_S^{Hdg}(F) := (\mathcal{F}_S^{FDR}(F), \text{Bti}_{\tilde{S}_I}^* Li_{I*} j_I^*(F, W), \alpha(F)))$$

where we denote by $g_{I,\alpha,\beta}^n : U_{I\alpha} \rightarrow U_{I\beta}$ which satisfy $h_{I\beta} \circ g_{I,\alpha,\beta}^n = h_{I\alpha}$ the morphisms in the canonical projective resolution

$$q : Li_{I*} j_I^* F := (\cdots \rightarrow \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I} \mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \xrightarrow{(\mathbb{Z}(g_{I,\alpha,\beta}^\bullet))} \\ \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I} \mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \rightarrow \cdots) \rightarrow i_{I*} j_I^* F,$$

- for $m : F_1 \rightarrow F_2$ a morphism in $C(\text{Var}(\mathbb{C})^{sm}/S)$, the morphism $\mathcal{F}_S^{Hdg}(m)$ in $D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})$ is given by

$$\mathcal{F}_S^{Hdg}(m) : = I_S^{-1} \circ ((I^\bullet(U_{I\alpha}/(\tilde{S}_I))) \circ (\text{ad}(l_{I,\alpha,\beta}^\bullet, !Hdg, l_{I,\alpha,\beta}^\bullet)(\mathbb{Z}_{U_{I\alpha}}^{Hdg})) \circ (I^\bullet(U_{I\alpha}/(\tilde{S}_I)))^{-1}) \\ = (\mathcal{F}_S^{FDR}(m), \text{Bti}_S^*(m), \theta(m) := (\theta(l_{I,\alpha,\beta}))) : \mathcal{F}_S^{Hdg}(F_1) \rightarrow \mathcal{F}_S^{Hdg}(F_2)$$

using lemma 21, that is we have the following commutative diagram in $(D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil}(\tilde{S}_I^{an}))$, denoting for short $V_I := \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I$,

$$\begin{array}{ccc} ((\cdot \rightarrow \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in V_I} h_{I\alpha!} Hdg h_{I\alpha}^{!Hdg} \mathbb{Z}_{\tilde{S}_I}^{Hdg} \xrightarrow{A_{g_{1I,\alpha,\beta}^\bullet}^{Hdg}} \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in V_I} h_{I\alpha!} Hdg h_{I\alpha}^{!Hdg} \mathbb{Z}_{\tilde{S}_I}^{Hdg} \rightarrow \cdot), u_{IJ}) \xrightarrow{(I^\bullet(U_{I\alpha}/\tilde{S}_I))} \mathcal{F}_S^{Hdg}(F_1) & & \\ \text{ad}(l_{I,\alpha,\beta}^\bullet, !Hdg, l_{I,\alpha,\beta}^\bullet)(-) \downarrow & & \mathcal{F}_S^{Hdg}(m) = (\mathcal{F}_S^{FDR}(m), \text{Bti}_S^*(m), (\theta(l_{I,\alpha,\beta}))) \downarrow \\ ((\cdot \rightarrow \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in V_I} h_{I\alpha!} Hdg h_{I\alpha}^{!Hdg} \mathbb{Z}_{\tilde{S}_I}^{Hdg} \xrightarrow{A_{g_{2I,\alpha,\beta}^\bullet}^{Hdg}} \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in V_I} h_{I\alpha!} Hdg h_{I\alpha}^{!Hdg} \mathbb{Z}_{\tilde{S}_I}^{Hdg} \rightarrow \cdot), u_{IJ}) \xrightarrow{(I^\bullet(U_{I\alpha}/\tilde{S}_I))} \mathcal{F}_S^{Hdg}(F_2) & & \end{array}$$

where

- we denoted for short $A_{g_{1I,\alpha,\beta}^\bullet}^{Hdg} := \text{ad}(g_{1I,\alpha,\beta}^\bullet, !Hdg, g_{1I,\alpha,\beta}^\bullet)(h_{I\alpha}^{!Hdg} \mathbb{Z}_{\tilde{S}_I}^{Hdg})$
- we denoted for short $A_{g_{2I,\alpha,\beta}^\bullet}^{Hdg} := \text{ad}(g_{2I,\alpha,\beta}^\bullet, !Hdg, g_{2I,\alpha,\beta}^\bullet)(h_{I\alpha}^{!Hdg} \mathbb{Z}_{\tilde{S}_I}^{Hdg})$
- we denote by $g_{1I,\alpha,\beta}^n : U_{I\alpha} \rightarrow U_{I\beta}$, which satisfy $h_{I\beta} \circ g_{1I,\alpha,\beta}^n = h_{I\alpha}$, the morphisms in the canonical projective resolution

$$q : Li_{I*} j_I^* F_1 := (\cdots \rightarrow \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I} \mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \xrightarrow{(\mathbb{Z}(g_{1I,\alpha,\beta}^\bullet))} \\ \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I} \mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \rightarrow \cdots) \rightarrow i_{I*} j_I^* F_1$$

- we denote by $g_{2I,\alpha,\beta}^n : U_{I\alpha} \rightarrow U_{I\beta}$, which satisfy $h_{I\beta} \circ g_{2I,\alpha,\beta}^n = h_{I\alpha}$, the morphisms in the canonical projective resolution

$$q : Li_{I*} j_I^* F_2 := (\cdots \rightarrow \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I} \mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \xrightarrow{(\mathbb{Z}(g_{2I,\alpha,\beta}^\bullet))} \\ \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I} \mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \rightarrow \cdots) \rightarrow i_{I*} j_I^* F_2$$

- we denote by $l_{I\alpha,\beta}^n : U_{I\alpha} \rightarrow U_{I\beta}$ which satisfy $h_{I\beta} \circ l_{I\alpha,\beta}^n = h_{I\alpha}$ and $l_{I\alpha,\beta}^{n+1} \circ g_{1I\alpha,\beta}^n = g_{2I\alpha,\beta}^n \circ l_{I\alpha,\beta}^n$ the morphisms in the morphism of canonical projective resolutions

$$\begin{aligned} Li_{I*}j_I^*(m) : Li_I * j_I^* F_1 &:= (\cdots \rightarrow \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I} \mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \rightarrow \cdots) \xrightarrow{(\mathbb{Z}(l_{I\alpha,\beta}^n))} \\ &(\cdots \rightarrow \bigoplus_{(U_{I\alpha}, h_{I\alpha}) \in \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I} \mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \rightarrow \cdots) =: Li_{I*}j_I^* F_2, \end{aligned}$$

- the maps $I^\bullet(U_{I\alpha})$ are given by definition 168 and lemma 21.

Obviously $\mathcal{F}_S^{Hdg}(F[1]) = \mathcal{F}_S^{Hdg}(F)[1]$ and $\mathcal{F}_S^{Hdg}(\text{Cone}(m)) = \text{Cone}(\mathcal{F}_S^{Hdg}(m))$. This functor induces by proposition 114 the functor

$$\begin{aligned} \mathcal{F}_S^{Hdg} &:= (\mathcal{F}_S^{FDR}, \text{Bti}_S^*) : \text{DA}(S) \rightarrow D_{\mathcal{D}(1,0)ful}(S/(\tilde{S}_I)) \times_I D_{ful}(S^{an}), \\ M = D(\mathbb{A}^1, et)(F) &\mapsto \mathcal{F}_S^{Hdg}(M) := \mathcal{F}_S^{Hdg}(F) = (\mathcal{F}_S^{FDR}(M), \text{Bti}_S^* M, \alpha(M)), \end{aligned}$$

with $\alpha(M) = \alpha(F)$.

We now give the functoriality with respect to the five operation using the De Rahm realization case and the Betti realization case :

Proposition 147. (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exists a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, l a closed embedding and p the projection. Let $S = \cup_{i \in I} S_i$ an open cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, $\tilde{g}_I : Y \times \tilde{S}_I \rightarrow \tilde{S}_I$ is a lift of $g_I = g|_{T_I} : T_I \rightarrow S_I$ and we have closed embeddings $i'_I := i_I \circ l \circ j'_I : T_I \hookrightarrow Y \times \tilde{S}_I$. Then, for $M \in \text{DA}_c(S)$, the following diagram commutes :

$$\begin{array}{ccc} g^*w \text{Bti}_S^* M & \xrightarrow{g^*(\alpha(M))} & DR(T)^{[-1]}((g_{Hdg}^{*mod} \mathcal{F}_S^{DR}(M))^{an}) \\ \downarrow T(g, bti)(M) & & \downarrow DR(T)^{[-1]}((T, \mathcal{F}^{FDR})(M))^{an} \\ \text{Bti}_T^* g^* M & \xrightarrow{\alpha(g^* M)} & DR(T)^{[-1]}((\mathcal{F}_T^{DR}(g^* M))^{an}) \end{array},$$

see section 5, definition 138 and definition 165

(ii) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{QPVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(T)$, the following diagram commutes :

$$\begin{array}{ccc} Rf_{*w} \text{Bti}_T^* M & \xrightarrow{f_*(\alpha(M))} & DR(S)^{[-1]}((Rf_*^{Hdg} \mathcal{F}_T^{DR}(M))^{an}) \\ \uparrow T_*(f, bti)(M) & & \uparrow DR(S)^{[-1]}((T_*(f, \mathcal{F}^{FDR})(M))^{an}) \\ \text{Bti}_S^* Rf_* M & \xrightarrow{\alpha(Rf_* M)} & DR(S)^{[-1]}((\mathcal{F}_S^{DR}(Rf_* M))^{an}) \end{array}$$

see section 5, definition 139 and definition 166

(iii) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{QPVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(T)$, the following diagram commutes :

$$\begin{array}{ccc} Rf_{!w} \text{Bti}_T^* M & \xrightarrow{f_!(\alpha(M))} & DR(S)^{[-1]}((Rf_!^{Hdg} \mathcal{F}_T^{DR}(M))^{an}) \\ \downarrow T_!(f, bti)(M) & & \downarrow DR(S)^{[-1]}((T_!(f, \mathcal{F}_{DR})(M))^{an}) \\ \text{Bti}_S^* f_! M & \xrightarrow{\alpha(Rf_! M)} & DR(S)^{[-1]}((\mathcal{F}_{DR}^S(Rf_! M))^{an}) \end{array}$$

see section 5, definition 139 and definition 166.

(iv) Let $f : T \rightarrow S$ a morphism with $T, S \in \text{QPVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$, the following diagram commutes :

$$\begin{array}{ccc} f^{!w} \text{Bti}_S^* M & \xrightarrow{f^!(\alpha(M))} & DR(T)^{[-]}((f_{Hdg}^{*mod} \mathcal{F}_S^{DR}(M))^{an}) \\ \uparrow T^!(f, bti)(M) & & \uparrow DR^{[-]}(T)((T^!(g, \mathcal{F}^{DR})(M))^{an}) \\ \text{Bti}_T^* f^! M & \xrightarrow{\alpha(f^! M)} & DR(T)^{[-]}((\mathcal{F}_T^{DR}(f^! M))^{an}) \end{array}$$

see section 5, definition 139 and definition 166.

(v) Let $S \in \text{Var}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$, the following diagram commutes :

$$\begin{array}{ccc} \text{Bti}_S^* M \otimes \text{Bti}_S^* N & \xrightarrow{\alpha(M) \otimes \alpha(N)} & DR(S)((\mathcal{F}_S^{DR}(M) \otimes_{\mathcal{O}_S} \mathcal{F}_S^{DR}(N))^{an}) \\ \downarrow T(\otimes, bti)(M, N) & & \downarrow DR(S)((T(\otimes, \mathcal{F}^{DR})(M, N))^{an}) \\ \text{Bti}_S^*(M \otimes N) & \xrightarrow{(\alpha(M \otimes N))} & DR(S)((\mathcal{F}_S^{DR}(M \otimes N))^{an}) \end{array}$$

see definition 139 and definition 166.

Proof. (i): Follows from the following commutative diagram in $(D_{\mathcal{D}(1,0)fil}(Y \times \tilde{S}_I) \times_I D_{fil}(Y \times \tilde{S}_I^{an}))$,

$$\begin{array}{ccc} ((\rightarrow \oplus_{(U_{I\alpha}, h_{I\alpha}) \in V_I} \tilde{g}_I^{*Hdg} h_{I\alpha}!Hdg h_{I\alpha}^!Hdg \mathbb{Z}_{\tilde{S}_I} \xrightarrow{A_{g_{I,\alpha,\beta}^{\bullet}}^{Hdg}} \oplus_{(U_{I\alpha}, h_{I\alpha}) \in V_I} h_{I\alpha}!Hdg h_{I\alpha}^!Hdg \mathbb{Z}_{\tilde{S}_I} \rightarrow), \tilde{g}_I^{*Hdg} \mathcal{I}^{\bullet}(U_{I\alpha}/\tilde{S}_I^{an})) & \xrightarrow{(\tilde{g}_I^{*Hdg} \mathcal{I}^{\bullet}(U_{I\alpha}/\tilde{S}_I^{an}))} & \xrightarrow{(\tilde{g}_I^{*Hdg} \mathcal{I}^{\bullet}(U_{I\alpha}/\tilde{S}_I^{an}))} \text{Bti}_S^* F, g^*(\alpha(F)) \\ \downarrow T^{Hdg}(\tilde{g}_I, h_I)(-) & & \downarrow (T(g, \mathcal{F}^{DR})(M), T(g, \text{Bti})(M), 0) \\ ((\rightarrow \oplus_{(U'_{I\alpha}, h'_{I\alpha}) \in W_I} h'_{I\alpha}!Hdg h'_{I\alpha}^!Hdg \mathbb{Z}_{Y \times \tilde{S}_I} \xrightarrow{A_{g'_{I,\alpha,\beta}^{\bullet}}^{Hdg}} \oplus_{(U'_{I\alpha}, h'_{I\alpha}) \in W_I} h'_{I\alpha}!Hdg h'_{I\alpha}^!Hdg \mathbb{Z}_{Y \times \tilde{S}_I} \rightarrow), u_{I,J}) & \xrightarrow{(\mathcal{I}^{\bullet}(U'_{\alpha}/Y \times \tilde{S}_I))} & \xrightarrow{(\mathcal{I}^{\bullet}(U'_{\alpha}/Y \times \tilde{S}_I))} \text{Bti}_T^* g^* F, \alpha(g^* F) \end{array}$$

where, we have denoted for short $V_I := \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I$ and $W_I := \text{Var}(\mathbb{C})^{sm}/Y \times \tilde{S}_I$,

- we denoted for short $A_{g_{I,\alpha,\beta}^{\bullet}}^{Hdg} := \text{ad}(g_{I,\alpha,\beta}^{\bullet, !Hdg}, g_{I,\alpha,\beta}^{\bullet, !Hdg})(h_{I\alpha}^!Hdg \mathbb{Z}_{\tilde{S}_I} \xrightarrow{Hdg} \mathbb{Z}_{\tilde{S}_I})$
- we denoted for short $A_{g'_{I,\alpha,\beta}^{\bullet}}^{Hdg} := \text{ad}(g'_{I,\alpha,\beta}^{\bullet, !Hdg}, g'_{I,\alpha,\beta}^{\bullet, !Hdg})(h'_{I\alpha}^!Hdg \mathbb{Z}_{Y \times \tilde{S}_I} \xrightarrow{Hdg} \mathbb{Z}_{Y \times \tilde{S}_I})$
- we denote by $g_{I,\alpha,\beta}^n : U_{I\alpha} \rightarrow U_{I\beta}$, which satisfy $h_{I\beta} \circ g_{I,\alpha,\beta}^n = h_{I\alpha}$, the morphisms in the canonical projective resolution

$$q : Li_I^* j_I^* F := (\cdots \rightarrow \oplus_{(U_{I\alpha}, h_{I\alpha}) \in \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I} \mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \xrightarrow{(\mathbb{Z}(g_{I,\alpha,\beta}^n))} \oplus_{(U_{I\alpha}, h_{I\alpha}) \in \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I} \mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \rightarrow \cdots) \rightarrow i_I^* j_I^* F$$

- we denote by $g'_{I,\alpha,\beta}^n : U'_{I\alpha} \rightarrow U'_{I\beta}$, which satisfy $h'_{I\beta} \circ g'_{I,\alpha,\beta}^n = h'_{I\alpha}$, the morphisms in the canonical projective resolution

$$q : Li'_I{}^* j'_I{}^* g^* F := (\cdots \rightarrow \oplus_{(U'_{I\alpha}, h'_{I\alpha}) \in \text{Var}(\mathbb{C})^{sm}/Y \times \tilde{S}_I} \mathbb{Z}(U'_{I\alpha}/Y \times \tilde{S}_I) \xrightarrow{(\mathbb{Z}(g'_{I,\alpha,\beta}^n))} \oplus_{(U'_{I\alpha}, h'_{I\alpha}) \in \text{Var}(\mathbb{C})^{sm}/Y \times \tilde{S}_I} \mathbb{Z}(U'_{I\alpha}/Y \times \tilde{S}_I) \rightarrow \cdots) \rightarrow i'_I{}^* j'_I{}^* g^* F$$

(ii): Follows from (i) by adjonction.

(iii): The closed embedding case is given by (ii) and the smooth projection case follows from (i) by adjonction.

(iv): Follows from (iii) by adjonction.

(v): Obvious

□

We can now state the following key proposition and the main theorem:

Proposition 148. *Let $k \subset \mathbb{C}$ a subfield.*

(i) *Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then we have the isomorphism in $D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{an})$*

$$\begin{aligned} \mathcal{F}_S^{\text{Hdg}}(\mathbb{Z}_S) &\xrightarrow{\cong} (\mathcal{F}_S^{\text{FDR}}(\mathbb{Z}_S), \text{Bti}_S^* \mathbb{Z}_S, \alpha(\mathbb{Z}_S)) \\ &\xrightarrow{((\Omega_{/\tilde{S}_I}^{\Gamma, pr}(\hat{R}^{\text{CH}}(\text{ad}(i_I^*, i_{I^*})(\mathbb{Z}_{\tilde{S}_I}^\vee)^\gamma))), I, 0)} \\ (e(S)_* \text{Hom}((\hat{R}^{\text{CH}}(\Gamma_{\tilde{S}_I}^\vee \mathbb{Z}_{\tilde{S}_I}), \hat{R}^{\text{CH}}(x_{IJ})), (E_{\text{zar}}(\Omega_{/\tilde{S}_I}^{\bullet, \Gamma, pr}, F_{\text{DR}}), T_{IJ})), \mathbb{Z}_{S^{an}}, \alpha(S)) & \\ \xrightarrow{\cong} \iota_S((\Gamma_{\tilde{S}_I}^{\vee, \text{Hdg}}(O_{\tilde{S}_I}, F_b), x_{IJ}), \mathbb{Z}_{S^{an}}, \alpha(S)) =: \iota_S(\mathbb{Z}_S^{\text{Hdg}}) & \end{aligned}$$

with

$$\begin{aligned} \alpha(S) : T(S/(\tilde{S}_I))(\mathbb{Z}_{S^{an}}) \otimes \mathbb{C}_S &:= (i_{I^*} j_I^* \mathbb{C}_{S^{an}}, I) \xrightarrow{(\text{ad}(i_I^*, i_{I^*})(\mathbb{C}_{\tilde{S}_I}^{\vee})^q)^{-1}} \\ (\Gamma_{\tilde{S}_I}^\vee \mathbb{C}_{\tilde{S}_I}, x_{IJ}) &\xrightarrow{(\Gamma_{\tilde{S}_I}^\vee \alpha(\tilde{S}_I))} DR(S)(o_{\text{fil}}(\Gamma_{\tilde{S}_I}^{\vee, \text{Hdg}}(O_{\tilde{S}_I}, F_b), x_{IJ})) \end{aligned}$$

(ii) *Let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, X quasi-projective. Consider a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{ps} S$ with $Y = \mathbb{P}^{N, o} \subset \mathbb{P}^N$ an open subset, l a closed embedding and ps the projection. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Recall that $S_I := \cap_{i \in I} S_i$, $X_I = f^{-1}(S_I)$, and $\tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. Then, using proposition 147(iii), the maps of definition 139 and definition 166 gives an isomorphism in $D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{an})$*

$$\begin{aligned} (T_l(f, \mathcal{F}^{\text{FDR}})(\mathbb{Z}_X), T_l(f, \text{Bti})(\mathbb{Z}_X), 0) : & \\ \mathcal{F}_S^{\text{Hdg}}(M^{\text{BM}}(X/S)) &:= (\mathcal{F}_S^{\text{FDR}}(Rf_! \mathbb{Z}_X), \text{Bti}_S^* Rf_! \mathbb{Z}_X, \alpha(Rf_! \mathbb{Z}_X)) \\ \xrightarrow{\sim} (Rf_{\text{Hdg}!}(\Gamma_{X_I}^{\vee, \text{Hdg}}(O_{Y \times \tilde{S}_I}, F_b), x_{IJ}(X/S)), Rf_{!w} \mathbb{Z}_{X^{an}}, f_!(\alpha(X))) &=: \iota_S(Rf_{! \text{Hdg}} \mathbb{Z}_X^{\text{Hdg}}). \end{aligned}$$

with

$$\mathbb{Z}_X^{\text{Hdg}} := ((\Gamma_{X_I}^{\vee, \text{Hdg}}(O_{Y \times \tilde{S}_I}, F_b), x_{IJ}(X/Y \times S)), \mathbb{Z}_{X^{an}}, \alpha(X)) \in C(\text{MHM}(X))$$

Proof. (i): Follows from proposition 114.

(ii): Follows from (i) by proposition 147(iii), theorem 36(i) and theorem 41(i).

□

The main theorem of this section is the following :

Theorem 43. *Let $k \subset \mathbb{C}$ a subfield.*

(i) *For $S \in \text{Var}(\mathbb{C})$, we have $\mathcal{F}_S^{\text{Hdg}}(\text{DA}_c(S)) \subset D(\text{MHM}(S))$,*

$$\iota_S : D(\text{MHM}(S)) \hookrightarrow D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{an})$$

being a full embedding by theorem 32.

(ii) *The Hodge realization functor $\mathcal{F}_{\text{Hdg}}(-)$ define a morphism of 2-functor on $\text{Var}(\mathbb{C})$*

$$\mathcal{F}_-^{\text{Hdg}} : \text{Var}(\mathbb{C}) \rightarrow (\text{DA}_c(-) \rightarrow D(\text{MHM}(-)))$$

whose restriction to $\text{QPVar}(\mathbb{C})$ is an homotopic 2-functor in sense of Ayoub. More precisely,

(ii0) for $g : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(S)$, the the maps of definition 138 and of definition 165 induce an isomorphism in $D(\text{MHM}(T))$

$$\begin{aligned} T(g, \mathcal{F}^{\text{Hdg}})(M) &:= (T(g, \mathcal{F}^{\text{FDR}})(M), T(g, \text{bti})(M), 0) : \\ g^{\hat{*}\text{Hdg}} \mathcal{F}_S^{\text{Hdg}}(M) &:= \iota_T^{-1}(g^{\hat{*}\text{mod}} \mathcal{F}_S^{\text{FDR}}(M), g^{*w} \text{Bti}_S(M), g^*(\alpha(M))) \\ &\xrightarrow{\sim} \iota_T^{-1}(\mathcal{F}_T^{\text{FDR}}(g^*M), \text{Bti}_T^*(g^*M), \alpha(g^*M)) =: \mathcal{F}_T^{\text{Hdg}}(g^*M), \end{aligned}$$

(ii1) for $f : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(T)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(\text{MHM}(S))$

$$\begin{aligned} T_*(f, \mathcal{F}^{\text{Hdg}})(M) &:= (T_*(f, \mathcal{F}^{\text{FDR}})(M), T_*(f, \text{bti})(M), 0) : \\ Rf_{\text{Hdg}*} \mathcal{F}_T^{\text{Hdg}}(M) &:= \iota_S^{-1}(Rf_*^{\text{Hdg}} \mathcal{F}_T^{\text{FDR}}(M), Rf_{**} \text{Bti}_T(M), f_*(\alpha(M))) \\ &\xrightarrow{\sim} \iota_S^{-1}(\mathcal{F}_S^{\text{FDR}}(Rf_*M), \text{Bti}_S^*(Rf_*M), \alpha(Rf_*M)) =: \mathcal{F}_S^{\text{Hdg}}(Rf_*M), \end{aligned}$$

(ii2) for $f : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(T)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(\text{MHM}(S))$

$$\begin{aligned} T_!(f, \mathcal{F}^{\text{Hdg}})(M) &:= (T_!(f, \mathcal{F}^{\text{FDR}})(M), T_!(f, \text{bti})(M), 0) : \\ Rf_{!\text{Hdg}} \mathcal{F}_T^{\text{Hdg}}(M) &:= \iota_S^{-1}(Rf_{!}^{\text{Hdg}} \mathcal{F}_T^{\text{FDR}}(M), Rf_{!w} \text{Bti}_T^*(M), f_!(\alpha(M))) \\ &\xrightarrow{\sim} \iota_S^{-1}(\mathcal{F}_S^{\text{FDR}}(Rf_!M), \text{Bti}_S^*(Rf_!M), \alpha(f_!M)) =: \mathcal{F}_S^{\text{Hdg}}(Rf_!M), \end{aligned}$$

(ii3) for $f : T \rightarrow S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(S)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(\text{MHM}(T))$

$$\begin{aligned} T^!(f, \mathcal{F}^{\text{Hdg}})(M) &:= (T^!(f, \mathcal{F}^{\text{FDR}})(M), T^!(f, \text{bti})(M), 0) : \\ f^{*\text{Hdg}} \mathcal{F}_S^{\text{Hdg}}(M) &:= \iota_T^{-1}(f^{*\text{mod}} \mathcal{F}_S^{\text{FDR}}(M), f^{!w} \text{Bti}_S(M), f^!(\alpha(M))) \\ &\xrightarrow{\sim} \iota_T^{-1}(\mathcal{F}_T^{\text{FDR}}(f^!M), \text{Bti}_T^*(f^!M), \alpha(f^!M)) =: \mathcal{F}_T^{\text{Hdg}}(f^!M), \end{aligned}$$

(ii4) for $S \in \text{Var}(\mathbb{C})$, and $M, N \in \text{DA}_c(S)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(\text{MHM}(S))$

$$\begin{aligned} T(\otimes, \mathcal{F}^{\text{Hdg}})(M, N) &:= (T(\otimes, \mathcal{F}_S^{\text{FDR}})(M, N), T(\otimes, \text{bti})(M, N), 0) : \\ \iota_S^{-1}(\mathcal{F}_S^{\text{FDR}}(M) \otimes_{\mathcal{O}_S}^{\text{Hdg}} \mathcal{F}_S^{\text{FDR}}(N), \text{Bti}_S(M) \otimes \text{Bti}_S(N), \alpha(M) \otimes \alpha(N)) \\ &\xrightarrow{\sim} \mathcal{F}_S^{\text{Hdg}}(M \otimes N) := \iota_S^{-1}(\mathcal{F}_S^{\text{FDR}}(M \otimes N), \text{Bti}_S(M \otimes N), \alpha(M \otimes N)). \end{aligned}$$

(iii) For $S \in \text{Var}(\mathbb{C})$, the following diagram commutes :

$$\begin{array}{ccc} \text{Var}(\mathbb{C})/S & \xrightarrow{\text{MH}/S} & D(\text{MHM}(S)) \\ \downarrow M(/S) & & \downarrow \iota^S \\ \text{DA}(S) & \xrightarrow{\mathcal{F}_S^{\text{Hdg}}} & D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{\text{an}}) \end{array}$$

Proof. (i): Let $M \in \text{DA}_c(S)$. There exist by definition of constructible motives an isomorphism in $\text{DA}(S)$

$$w(M) : M \xrightarrow{\sim} \text{Cone}(M(X_0/S)[d_0] \xrightarrow{m_1} \dots \xrightarrow{m_m} M(X_m/S)[d_m]),$$

with $f_n : X_n \rightarrow S$ morphisms and $X_n \in \text{QPVar}(\mathbb{C})$. This gives the isomorphism in $D_{\mathcal{D}(1,0)\text{fil}}(S/(\tilde{S}_I)) \times_I D_{\text{fil}}(S^{\text{an}})$

$$\mathcal{F}_S^{\text{Hdg}}(w(M)) : \mathcal{F}_S^{\text{Hdg}}(M) \xrightarrow{\sim} \text{Cone}(\mathcal{F}_S^{\text{Hdg}}(M(X_0/S))[d_0] \xrightarrow{\mathcal{F}_S^{\text{Hdg}}(m_1)} \dots \xrightarrow{\mathcal{F}_S^{\text{Hdg}}(m_m)} \mathcal{F}_S^{\text{Hdg}}(M(X_m/S))[d_m]),$$

On the other hand, by proposition 148(i), we have

$$\mathcal{F}_S^{Hdg}(M(X_n/S)) \xrightarrow{\sim} Rf_{*Hdg}\mathbb{Z}_X^{Hdg} \in D(MHM(S)).$$

This prove (i).

(ii0): Follows from theorem 36(i), proposition 147(i) and theorem 41.

(ii1): Follows from theorem 36(iii), proposition 147(ii), and theorem 41(iii).

(ii2): Follows from theorem 36(ii), proposition 147(iii), and theorem 41(ii).

(ii3): Follows from theorem 36(iv), proposition 147(iv), and theorem 41(iv).

(ii4): Follows from theorem 36(v), proposition 147(v) and theorem 41(v).

(iii): By (ii), for $g : X'/S \rightarrow X/S$ a morphism, with $X', X, S \in \text{Var}(\mathbb{C})$ and $X/S = (X, f)$, $X'/S = (X', f')$, we have by adjonction the following commutative diagram

$$\begin{array}{ccc} \mathcal{F}_S^{Hdg}(M(X'/S) = f'_! f'^! \mathbb{Z}_S = f'_! g'_! f'^! \mathbb{Z}_S) & \xrightarrow{\mathcal{F}_S^{Hdg}(M(/S)(g)=f_! \text{ad}(g_!, g^!)(f'^! \mathbb{Z}_S))} & \mathcal{F}_S^{Hdg}(M(X/S) = f'_! f'^! \mathbb{Z}_S) \\ \downarrow T_1(f', \mathcal{F}^{Hdg})(f'^! M(X'/S)) \circ T^1(f', \mathcal{F}^{Hdg})(M(X'/S)) & & \downarrow T_1(f, \mathcal{F}^{Hdg})(f^! M(X/S)) \circ T^1(f, \mathcal{F}^{Hdg})(M(X/S)) \\ MH(X'/S) := Rf'_{!Hdg} f'^! {}^1Hdg \mathbb{Z}_S^{Hdg} = f_! {}_!Hdg g'_! {}_!Hdg f'^! {}^1Hdg \mathbb{Z}_S^{Hdg} & \xrightarrow{f_! {}_!Hdg g'_! {}_!Hdg f'^! {}^1Hdg \mathbb{Z}_S^{Hdg}} & MH(X/S) := f_! {}_!Hdg f'^! {}^1Hdg \mathbb{Z}_S^{Hdg} \end{array}$$

where the left and right columns are isomorphisms by (ii). This proves (iii). \square

The theorem 43 gives immediately the following :

Corollary 6. *Let $f : U \rightarrow S$, $f' : U' \rightarrow S$ morphisms, with $U, U', S \in \text{Var}(\mathbb{C})$ irreducible, U' smooth. Let $\bar{S} \in \text{PVar}(\mathbb{C})$ a compactification of S . Let $\bar{X}, \bar{X}' \in \text{PVar}(\mathbb{C})$ compactification of U and U' respectively, such that f (resp. f') extend to a morphism $\bar{f} : \bar{X} \rightarrow \bar{S}$, resp. $\bar{f}' : \bar{X}' \rightarrow \bar{S}$. Denote $\bar{D} = \bar{X} \setminus U$ and $\bar{D}' = \bar{X}' \setminus U'$ and $\bar{E} = (\bar{D} \times_{\bar{S}} \bar{X}') \cup (\bar{X} \times_{\bar{S}} \bar{D}')$. Denote $i : \bar{D} \hookrightarrow \bar{X}$, $i' : \bar{D}' \hookrightarrow \bar{X}'$ denote the closed embeddings and $j : U \hookrightarrow \bar{X}$, $j' : U' \hookrightarrow \bar{X}'$ the open embeddings. Denote $d = \dim(U)$ and $d' = \dim(U')$. We have the following commutative diagram in $D(\mathbb{Z})$*

$$\begin{array}{ccc} RHom_{\text{DA}(\bar{S})}^{\bullet}(M(U'/\bar{S}), M((\bar{X}, \bar{D})/\bar{S})) & \xrightarrow{\mathcal{F}_S^{FDR}(-, -)} & RHom_{DMHM(\bar{S})}^{\bullet}(f'_{!Hdg} \mathbb{Z}_{U'}^{Hdg}, f_{*Hdg} \mathbb{Z}_U^{Hdg}) \\ \downarrow RI(-, -) & & \downarrow RI(-, -) \\ RHom^{\bullet}(M(\text{pt}), M(\bar{X}' \times_{\bar{S}} \bar{X}, \bar{E})(d')[2d']) & \xrightarrow{\mathcal{F}_{FDR}^{\text{pt}}(-, -)} & RHom^{\bullet}(\mathbb{Z}_{\text{pt}}^{Hdg}, a_{U' \times_S U'} \mathbb{Z}_{U' \times_S U'}^{Hdg}(d')[2d']) \\ \downarrow l & & \downarrow l \\ \mathcal{Z}_d(\bar{X}' \times_{\bar{S}} \bar{X}, E, \bullet) & \xrightarrow{\mathcal{R}_{\bar{X}' \times_{\bar{S}} \bar{X}}^d} & C_{2d+}^{\mathcal{D}}(\bar{X}' \times_{\bar{S}} \bar{X}, E, Z(d)) \end{array}$$

where

$$M((\bar{X}, \bar{D})/\bar{S}) := \text{Cone}(\text{ad}(i_*, i^!) : M(\bar{D}/\bar{S}) \rightarrow M(\bar{X}/\bar{S})) = \bar{f}_* j_* E_{et}(\mathbb{Z}(U/U)) \in \text{DA}(\bar{S})$$

and l the isomorphisms given by canonical embedding of complexes.

Proof. The upper square of this diagram follows from theorem 43(ii). On the other side, the lower square follows from the absolute case. \square

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