The Hodge realization functor on the derived category of relative motives

Johann Bouali

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Abstract
We give, for a complex algebraic variety $S$, a Hodge realization functor $F_{Hdg}^S$ from the (unbounded) derived category of constructible motives $DA_c(S)$ over $S$ to the (undounded) derived category $D(MHM(S))$ of algebraic mixed Hodge modules over $S$. Moreover, for $f : T \to S$ a morphism of complex quasi-projective algebraic varieties, $F_{Hdg}^S$ commutes with the four operations $f^*, f_*, f_! , f_!$ on $DA_c(-)$ and $D(MHM(-))$, making in particular the Hodge realization functor a morphism of 2-functor on the category of complex quasi-projective algebraic varieties which for a given $S$ sends $DA_c(S)$ to $D(MHM(S))$, moreover $F_{Hdg}^S$ commutes with tensor product. We also give an algebraic and analytic Gauss-Manin realization functor from which we obtain a base change theorem for algebraic De Rham cohomology and for all smooth morphisms a relative version of the comparaison theorem of Grothendieck between the algebraic De Rham cohomology and the analytic De Rham cohomology.

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1 Introduction

Saito’s theory of mixed Hodge modules associate to each complex algebraic variety \( S \) a category \( \text{MHM}(S) \) which is a full subcategory of \( \text{PSh}_D(\text{fil}, \text{rh})(S/\tilde{S}_I)) \times_I C_{fil}(S) \) which extend variations of mixed Hodge structure and admits a canonical monoidal structure given by tensor product, and associate to each morphism of complex algebraic varieties \( f : X \to S \), four functor \( Rf^*_Hdg, Rf^!_{Hdg}, F^*_{Hdg}, F^!_{Hdg} \). In the case of a smooth proper morphism \( f : X \to S \) with \( S \) and \( X \) smooth, \( H^nRf^!_{Hdg}Z_X^{Hdg} \) is the variation of Hodge structure given by the Gauss-Manin connexion and the local system \( H^nRf^*_hZ_X \). Moreover, these functors induce the six functor formalism of Grothendieck. We thus have, for a complex algebraic variety \( S \) a canonical functor

\[
\text{MH}(/S) : \text{Var}(\mathbb{C})/S \to D(\text{MHM}(S)), (f : X \to S) \mapsto Rf^!_{Hdg}Z_X^{Hdg}
\]

and

\[
\text{MH}(/-) : \text{Var}(\mathbb{C}) \to \text{TriCat}, S \mapsto (\text{MH}(S) : \text{Var}(\mathbb{C})/S \to D(\text{MHM}(S))),
\]

is a morphism of 2-functor. In this work, we extend \( \text{MH}(/-) \) to motives by constructing, for each complex algebraic variety \( S \), a canonical functor \( F^!_{Hdg} : \text{DA}(S) \to D(\text{MHM}(S)) \) which is monoidal, that is commutes with tensor product, together with, for each morphism of complex algebraic varieties \( g : T \to S \) a canonical transformation map \( T(g, F^!_{Hdg}) \), which make

\[
F^!_{Hdg} : \text{Var}(\mathbb{C}) \to \text{TriCat}, S \mapsto (F^!_{Hdg} : \text{DA}(S) \to D(\text{MHM}(S))),
\]

is a morphism of 2-functor: this is the contain of theorem 43. A partial result in this direction has been obtained by Ivorra in [18] using a different approach. We already have a Betti realization functor

\[
\text{Bti}_- : \text{Var}(\mathbb{C}) \to \text{TriCat}, S \mapsto (\text{Bti}^*_S : \text{DA}(S) \to D(S)),
\]

which extend the Betti realization. The functor \( F^!_{Hdg} := (F^!_{FDR}, \text{Bti}_-) \) is obtained by constructing the De Rham part

\[
F^!_{FDR} : \text{Var}(\mathbb{C}) \to \text{TriCat}, S \mapsto (F^!_{FDR} : \text{DA}(S) \to D_{D(1,0)fil, \infty}(S/\tilde{S}_I)),
\]

which takes values in the derived category of filtered algebraic \( D \)-modules \( D_{D(1,0)fil, \infty}(S/\tilde{S}_I)) := K_{D(1,0)fil, \infty}(S/\tilde{S}_I))/(\text{[E]}_1^{-1}) \) obtained by inverting the classes filtered Zariski local equivalences \( \text{[E]}_1 \) modulo \( \infty \)-filtered homotopy and then using the following key theorem (theorem 32)

**Theorem 1.** \( (i) \) Let \( S \in \text{Var}(\mathbb{C}) \). Let \( S = \cup_i \tilde{S}_i \) an open cover such that there exists closed embedding \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). Then the full embedding

\[
i_S : \text{MHM}(S) \hookrightarrow C_{D(1,0)fil, \infty}(S/\tilde{S}_I)) \times_I D_{fil}(S^{an})
\]
induces a full embedding

\[ \iota_S : D(MHM(S)) \hookrightarrow D_{D(1,0)f\text{-}fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \]

whose image consists of \(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{D(1,0)f\text{-}fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})\) such that

\[ (H^n(M_I, F, W), H^n(u_{IJ})), H^n(K, W), H^n\alpha) \in MHM(S) \]

for all \(n \in \mathbb{Z}\) and such that for all \(p \in \mathbb{Z}\), the differential of \((\text{Gr}_p^W M_I, F)\) are strict for the filtration \(F\) (in particular, the differentials of \((M_I, F, W)\) are strict for the filtration \(F\)).

(ii) Let \(S \in \text{Var}(\mathbb{C})\). Let \(S = \bigcup_{i \in I} S_i\) an open cover such that there exists closed embedding \(i_i : S_i \hookrightarrow S_i\) with \(S_i \in \text{SmVar}(\mathbb{C})\). Then the full embedding

\[ \iota_S : MHM(S) \hookrightarrow C_{D(1,0)f\text{-}fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \]

induces a full embedding

\[ \iota_S : D(MHM(S)) \hookrightarrow D_{D(1,0)f\text{-}fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \]

whose image consists of \(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{D(1,0)f\text{-}fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})\) such that

\[ (H^n(M_I, F, W), H^n(u_{IJ})), H^n(K, W), H^n\alpha) \in MHM(S) \]

for all \(n \in \mathbb{Z}\) and such that there exist \(r \in \mathbb{N}\) and an \(r\)-homotopy equivalence \((M_I, F, W), u_{IJ}) \rightarrow ((M^r_I, F, W), u_{IJ})\) such that for all \(p \in \mathbb{Z}\), the differential of \((\text{Gr}_p^W M^r_I, F)\) are strict for the filtration \(F\) (in particular, the differentials of \((M^r_I, F, W)\) are strict for the filtration \(F\)).

Note that the category \(D_{D(1,0)f\text{-}fil,\infty,rh}(S/(\tilde{S}_I))\) is NOT triangulated. More precisely the canonical triangles of \(D_{D(1,0)f\text{-}fil,\infty,rh}(S/(\tilde{S}_I))\) does NOT satisfy the 2 of 3 axiom of a triangulated category. Moreover there exist canonical triangles of \(D_{D(1,0)f\text{-}fil,\infty,rh}(S/(\tilde{S}_I))\) which are NOT the image of distinguish triangles of \(\pi_S(D(MHM(S)))\). This method can be seen as a relative version of the construction of F.Lecomte and N.Wach in [21].

In section 6.1.1 and 6.2.1, we construct an algebraic and analytic Gauss-Manin realization functor, but this functor does NOT give a complex of filtered \(D\)-module, BUT a complex of filtered \(O\)-modules whose cohomology sheaves have a structure of filtered \(D_S\) modules. Hence, it does NOT get to the desired category. Moreover the Hodge filtration is NOT the right one: see proposition 118 and proposition 111. However this functor gives some interesting results. Let \(S \in \text{Var}(\mathbb{C})\) and \(S = \bigcup_{i \in I} S_i\) an open cover such that there exist closed embeddings \(i_i : S_i \hookrightarrow S_i\) with \(S_i \in \text{SmVar}(\mathbb{C})\) smooth. For \(I \subset [1, \cdots, l]\), denote by \(S_I = \cap_{i \in I} S_i\) and \(j_I : S_I \hookrightarrow S\) the open embedding. We then have closed embeddings \(i_I : S_I \hookrightarrow S_I := \Pi_{i \in I} S_i\). We define the filtered algebraic Gauss-Manin realization functor defined as

\[ \mathcal{F}_S^{GM} : C(\text{Var}(\mathbb{C}))^{sm}/S \rightarrow C_{O_{fil,D}(S/(\tilde{S}_I))^\vee}, M \mapsto \mathcal{F}_S^{GM}(F) := ((e(\tilde{S}_I)_*\text{Hom}^\bullet(L(i_*j^*_IF), E_{et}(\Omega^*_j/F)),-d_{\tilde{S}_I}, u_{iJ}^*(F)), \]

see definition 122. Note that the canonical triangles of \(D_{O_{fil,D}(S/(\tilde{S}_I))}\) does NOT satisfy the 2 of 3 axiom of a triangulated category. The filtered algebraic Gauss-Manin realization functor induces by proposition 107

\[ \mathcal{F}_S^{GM} : DA_{et}(S)^{op} \rightarrow D_{O_{fil,D}(S/(\tilde{S}_I))}, M \mapsto \mathcal{F}_S^{GM}(M) := ((e(\tilde{S}_I)_*\text{Hom}^\bullet(L(i_*j^*_IF), E_{et}(\Omega^*_j/F)),-d_{\tilde{S}_I}, u_{iJ}^*(F)), \]

where \(F \in C(\text{Var}(\mathbb{C}))^{sm}/S\) is such that \(M = D(\mathbb{A}^1, et)(F)\). We then prove (theorem 35):
Theorem 2. (i) Let \( g : T \to S \) is a morphism with \( T, S \in \text{Var}(\mathbb{C}) \). Assume there exist a factorization \( g : T \xrightarrow{\eta} Y \times S \xrightarrow{p_S} \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. Let \( S = \bigcup_{i=1}^{l} S_i \) be an open cover such that there exists closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). Then, for \( M \in \text{DA}_c(S) \)

\[
T(g, \mathcal{F}^{GM})(M) : Rg^* \mathcal{F}^{GM}_S(M) \to \mathcal{F}^{GM}_T(g^* M)
\]

is an isomorphism in \( D_{O_{T \text{fil}, D, \infty}}(T/(Y \times \tilde{S}_1)) \).

(ii) Let \( g : T \to S \) is a morphism with \( T, S \in \text{SmVar}(\mathbb{C}) \). Then, for \( M \in \text{DA}_c(S) \)

\[
T^O(g, \mathcal{F}^{GM})(M) : Lg^* \mathcal{F}^{GM}_S(M) \to \mathcal{F}^{GM}_T(g^* M)
\]

is an isomorphism in \( D_{O_{T}}(T) \).

(iii) A base change theorem for algebraic De Rham cohomology: Let \( g : T \to S \) is a morphism with \( T, S \in \text{SmVar}(\mathbb{C}) \). Let \( h : U \to S \) a smooth morphism with \( U \in \text{Var}(\mathbb{C}) \). Then the map (see definition 1)

\[
T^O_w(g, h) : Lg^* \mathcal{H}^n(\Omega^*_S/F_S) \to \mathcal{H}^n(\Omega^*_{U/T}/F_U)
\]

is an isomorphism in \( D_{O_{T}}(T) \).

Let \( S \in \text{Var}(\mathbb{C}) \) and \( S = \bigcup_{i=1}^{l} S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). For \( I \subset [1, \ldots, l] \), denote by \( S_I = \cap_{i \in I} S_i \) and \( j_I : S_I \hookrightarrow S \) the open embedding. We then have closed embeddings \( i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i \). We define the filtered analytic Gauss-Manin realization functor defined as

\[
\mathcal{F}^{GM}_S^\vee : \text{DA}_c(S)^{\text{op}} \to D_{O_{T \text{fil}, D, \infty}}(S/\tilde{S}_I)^{\vee}, M \mapsto \mathcal{F}^{GM}_S(M) := ((e(\tilde{S}_I), \text{Hom}(\text{An}_{S} \cdot L(i_{*}j_{!}^*F), \text{Et}(\Omega^*_{\tilde{S}_I}/F_{\tilde{S}_I}))[−d_{\tilde{S}_I}], u_{\tilde{S}_I}^*(F))
\]

where \( F \in C(\text{Var}(\mathbb{C})^{sm}/S) \) is such that \( M = D(K^1, \text{et})(F) \), see definition 143. We then prove (theorem 37):

Theorem 3. (i) Let \( g : T \to S \) is a morphism with \( T, S \in \text{Var}(\mathbb{C}) \). Assume there exist a factorization \( g : T \xrightarrow{\eta} Y \times S \xrightarrow{p_S} \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. Let \( S = \bigcup_{i=1}^{l} S_i \) be an open cover such that there exists closed embeddings \( i_i : S_i \hookrightarrow S_i \) with \( S_i \in \text{SmVar}(\mathbb{C}) \). Then, for \( M \in \text{DA}_c(S) \)

\[
T(g, \mathcal{F}^{GM}_{an})(M) : Rg^* \mathcal{F}^{GM}_{S, an}(M) \to \mathcal{F}^{GM}_{T, an}(g^* M)
\]

is an isomorphism in \( D_{O_{T \text{fil}, D, \infty}}(T/(Y \times \tilde{S}_1)) \).

(ii) Let \( g : T \to S \) is a morphism with \( T, S \in \text{SmVar}(\mathbb{C}) \). Then, for \( M \in \text{DA}_c(S) \)

\[
T(g, \mathcal{F}^{GM}_{an})(M) : Lg^* \mathcal{F}^{GM}_{S, an}(M) \to \mathcal{F}^{GM}_{T, an}(g^* M)
\]

is an isomorphism in \( D_{O_{T}}(T) \).

A consequence of the construction of the transformation map between the algebraic and analytic Gauss-Manin realization functor is the following (theorem 39):

Theorem 4. (i) Let \( S \in \text{Var}(\mathbb{C}) \). Then, for \( M \in \text{DA}_c(S) \)

\[
\mathcal{J}^n_S(-) \circ H^n(\text{An}_{\mathcal{F}^{GM}_{an}}(M)) : J_S(H^n(\mathcal{F}^{GM}_S(M))^{an}) \rightarrow H^n(\mathcal{F}^{GM}_{S, an}(M))
\]

is an isomorphism in \( PSh_{\mathbb{P}}(S^{an}/\tilde{S}_I^{an}) \).
(ii) A relative version of Grothendieck GAGA theorem for De Rham cohomology Let \( h : U \to S \) a smooth morphism with \( S, U \in \text{SmVar}(\mathbb{C}) \). Then,

\[
\mathcal{J}_S(-) \circ J_{ST}^O(an, h) : J_S((R^n h_* \Omega^*_{U/S})^a) \xrightarrow{\sim} R^n h_* \Omega^*_{U\times S/S^n}
\]

is an isomorphism in \( \text{PSh}_{\mathcal{D}}(S^n) \).

In section 6.1.2, using results of sections 2, 4 and 5, we construct the algebraic filtered De Rham realization functor \( \mathcal{F}^\text{FDR}_x \). We construct it via a larger category and use theorem 32: Let \( S \in \text{Var}(\mathbb{C}) \) and \( S = \bigcup_{i=1}^{l} S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). For \( i \in [1, \cdots l] \), denote by \( S_I = \bigcap_{i \in I} S_i \) and \( j_i : S_I \hookrightarrow S \) the open embedding. We then have closed embeddings \( j_i : S_I \hookrightarrow \tilde{S}_I := \Pi_{i \in I} \tilde{S}_i \). We define in definition 138(ii) which use definition 129 and definition 41, the filtered algebraic De Rham realization functor defined as

\[
\mathcal{F}^\text{FDR}_S : C(\text{Var}(\mathbb{C})^m/S) \to C_{\text{Dfil}}(S/(\tilde{S}_I)), F \mapsto \mathcal{F}^\text{FDR}_S(M) :=
\]

\[
(e'_{(n)}(\hat{\text{Hom}}^*(\tilde{R}C^{\text{H}}(\rho_{S_I}^* L(i_{I+} j_I^* F)), E_{\text{zar}}(\Omega^*_S, F_{\text{DR}}))[-d_{\tilde{S}_I}], u_{I+}^j(F)).
\]

By proposition 114(ii), it induces

\[
\mathcal{F}_S^\text{FDR} : D\text{A}_{\text{c}}(S) \to D_{\text{Dfil}}(S/(\tilde{S}_I)), M \mapsto \mathcal{F}^\text{FDR}_S(M) :=
\]

\[
(e'_{(n)}(\hat{\text{Hom}}^*(\tilde{R}C^{\text{H}}(\rho_{S_I}^* L(i_{I+} j_I^* F)), E_{\text{zar}}(\Omega^*_S, F_{\text{DR}}))[-d_{\tilde{S}_I}], u_{I+}^j(F)).
\]

where \( F \in C(\text{Var}(\mathbb{C})^m/S) \) is such that \( M = \mathcal{D}(\hat{\text{A}}^n, e_l)(F) \). We compute this functor for an homological motive in proposition 116 and we get by corollary 4, for \( S \in \text{Var}(\mathbb{C}) \) and \( M \in D\text{A}_{\text{c}}(S) \), \( \mathcal{F}_S^\text{FDR}(M) \in \pi_S(D(MHM(S))) \), and the following (theorem 36):

**Theorem 5.**

(i) Let \( g : T \to S \) a morphism, with \( S, T \in \text{Var}(\mathbb{C}) \). Assume we have a factorization \( g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. Let \( M \in D\text{A}_{\text{c}}(S) \). Then map in \( \pi_T(D(MHM(T))) \)

\[
T(g, \mathcal{F}^\text{FDR}_T)(M) : g^*_{\text{Hdg}} \mathcal{F}^\text{FDR}_S(M) \xrightarrow{\sim} \mathcal{F}^\text{FDR}_T(g^* M)
\]

is given in definition 138 is an isomorphism.

(ii) Let \( f : X \to S \) a morphism with \( X, S \in \text{Var}(\mathbb{C}) \). Assume there exist a factorization \( f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. Then, for \( M \in D\text{A}_{\text{c}}(X) \), the map given in definition 139

\[
T(f, \mathcal{F}^\text{FDR}_X)(M) : Rf^!_{\text{Hdg}} \mathcal{F}^\text{FDR}_X(M) \xrightarrow{\sim} \mathcal{F}^\text{FDR}_X(Rf_* M)
\]

is an isomorphism in \( \pi_S(D(MHM(S))) \).

(iii) Let \( f : X \to S \) a morphism with \( X, S \in \text{Var}(\mathbb{C}) \), \( S \) quasi-projective. Assume there exist a factorization \( f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. We have, for \( M \in D\text{A}_{\text{c}}(X) \), the map given in definition 139

\[
T_* (f, \mathcal{F}^\text{FDR}_X)(M) : \mathcal{F}^\text{FDR}_S(Rf_* M) \xrightarrow{\sim} Rf^*_{\text{Hdg}} \mathcal{F}^\text{FDR}_X(M)
\]

is an isomorphism in \( \pi_S(D(MHM(S))) \).

(iv) Let \( f : X \to S \) a morphism with \( X, S \in \text{Var}(\mathbb{C}) \), \( S \) quasi-projective. Assume there exist a factorization \( f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. Then, for \( M \in D\text{A}_{\text{c}}(S) \), the map given in definition 139

\[
T^1 (f, \mathcal{F}^\text{FDR}_X)(M) : \mathcal{F}^\text{FDR}_X(f^! M) \xrightarrow{\sim} f^*_{\text{Hdg}} \mathcal{F}^\text{FDR}_S(M)
\]

is an isomorphism in \( \pi_X(D(MHM(X))) \).
(v) Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^{l} S_i$ an open affine covering and denote, for $I \subseteq \{1, \ldots, l\}$, $S_I = \cap_{i \in I} S_i$ and $i_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M, N \in \text{DA}_{c}(S)$, the map in $\pi_S(D(MHM(S)))$

$$T(F_{S}^{\text{FD}}, \times)(M, N) : F_{S}^{\text{FD}}(M) \otimes_{O_{S}} F_{S}^{\text{FD}}(N) \sim F_{S}^{\text{FD}}(M \otimes N)$$

given in definition 139 is an isomorphism.

We also have a canonical transformation map between the Gauss-Manin and the De Rham functor given in definition 135 which satisfy (see proposition 118):

**Proposition 1.** Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

(i) For $M \in \text{DA}_{c}(S)$ the map in $D_{O_{S}, D}(S/(\tilde{S}_I)) = D_{O_{S}, D}(S)$

$$o_{D}T(F_{S}^{\text{GM}}, F_{S}^{\text{FD}})(M) : o_{D}F_{S}^{\text{GM}}(L_{D}S_{M}) \sim o_{D}F_{S}^{\text{FD}}(M)$$

given in definition 135 is an isomorphism if we forgot the Hodge filtration $F$.

(ii) For $M \in \text{DA}_{c}(S)$ and all $n, p \in \mathbb{Z}$, the map in $\text{PSh}_{O_{S}, D}(S/(\tilde{S}_i))$

$$F^{p}H^{n}T(F_{S}^{\text{GM}}, F_{S}^{\text{FD}})(M) : F^{p}H^{n}F_{S}^{\text{GM}}(L_{D}S_{M}) \hookrightarrow F^{p}H^{n}F_{S}^{\text{FD}}(M)$$

given in definition 135 is a monomorphism. Note that $F^{p}H^{n}T(F_{S}^{\text{GM}}, F_{S}^{\text{FD}})(M)$ is NOT an isomorphism in general: take for example $M(S^o/S)' = D(\mathbb{A}^1, et)(j_{*, \text{et}}(\mathbb{Z}(S^o/S)))$ for an open embedding $j : S^o \hookrightarrow S$; then

$$F_{S}^{\text{GM}}(L_{D}S_{M}(S^o/S)') = F_{S}^{\text{GM}}(\mathbb{Z}(S^o/S)) = j_{*}E(O_{S}, F_{b}) \notin \pi_S(MHM(S))$$

and hence NOT isomorphic to $F_{S}^{\text{GM}}(L_{D}S_{M}(S^o/S)') \in \pi_S(MHM(S))$, see remark 9. It is an isomorphism in the very particular cases where $M = D(\mathbb{A}^1, et)(\mathbb{Z}(X/S))$ or $M = D(\mathbb{A}^1, et)(\mathbb{Z}(X^o/S))$ for $f : X \to S$ a smooth proper morphism and $n : X^o \to X$ is an open subset such that $X \backslash X^o = \cup D_{i}$ is a normal crossing divisor and such that $f_{|D_{i}} = f \circ i_{i} : D_{i} \to X$ are SMOOTH morphism with $i_{i} : D_{i} \hookrightarrow X$ the closed embedding and considering $f_{|X^o} = f \circ n : X^o \to S$ (see proposition 111).

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i} S_i$ an open cover such that there exists closed embedding $i_i : S \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We define the Hodge realization functor as, using definition 133,

$$F_{S}^{Hd} := (F_{S}^{\text{FD}}, \text{Bti}^{*}_{S}) : C(\text{Var}(\mathbb{C})^{sm}/S) \to D_{D(1, 0), \text{fil}}(S/(\tilde{S}_I)) \times_1 D_{\text{fil}}(S^{an})$$

which is on objects given by, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, taking $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F, W)$ gives the weight structure on $D(\mathbb{A}^1, et)(F)$,

$$F_{S}^{Hd}(F) := (F_{S}^{\text{FD}}(F), \text{Bti}^{*}_{S}(F, \alpha(F))) = (e(S), \text{Hom}(\tilde{F}_{S}^{CH}(\rho_{S}^{*}, C_{i}^{*}(F^{*}(W))), \tilde{R}_{S}^{CH}(T^{q}(D_{I, j}(-))),(E_{\text{zar}}(\Omega_{I}^{*, r, \text{pr}}, F_{DR}, T_{I, j})),

\text{e}(S), \text{Sing}_{S}^{\text{an}}, L_{S}(F, W), \alpha(F)) \in D_{D(1, 0), \text{fil}}(S/(\tilde{S}_I)) \times_1 D_{\text{fil}}(S^{an})$$

where $\alpha(F)$ is given in definition 169 and on morphism, for $m : F_1 \to F_2$ with $F_1, F_2 \in C(\text{Var}(\mathbb{C})^{sm}/S)$,

$$F_{S}^{Hd}(m) := (F_{S}^{\text{FD}}(m), \text{Bti}^{*}_{S}(m), \theta(m)),$$

where $\theta(m)$ is given in definition 169. This functor induces by proposition 114 the functor

$$F_{S}^{Hd} := (F_{S}^{\text{FD}}, \text{Bti}^{*}_{S}) : \text{DA}(S) \to D_{D(1, 0), \text{fil}}(S/(\tilde{S}_I)) \times_1 D_{\text{fil}}(S^{an}),

M = D(\mathbb{A}^1, et)(F) \mapsto F_{S}^{Hd}(M) := F_{S}^{Hd}(F) = (F_{S}^{\text{FD}}(M), \text{Bti}^{*}_{S}M, \alpha(M)),$$

with $\alpha(M) = \alpha(F)$.

The main result of this article is the following (theorem 43) :
Theorem 6. Let $k \subset \mathbb{C}$ be a subfield.

(i) For $S \in \text{Var}(\mathbb{C})$, we have $F_{\text{Hdg}}^S(\text{DA}_c(S)) \subset D(MHM(S))$,

$\iota_S : D(MHM(S)) \hookrightarrow D_{D(1,0)\text{ful}}(S/\bar{S}_T) \times I D_{\text{ful}}(S^{an})$

being a full embedding by theorem 32.

(ii) The Hodge realization functor $F_{\text{Hdg}}^S(-)$ define a morphism of 2-functor on $\text{Var}(\mathbb{C})$

$F_{\text{Hdg}}^S : \text{Var}(\mathbb{C}) \to (\text{DA}_c(-) \to D(MHM(-)))$

whose restriction to $\text{QPVar}(\mathbb{C})$ is an homotopic 2-functor in sense of Ayoub. More precisely,

(iii0) for $g : T \to S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(T)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(MHM(T))$

$\iota_T : D(MHM(T)) \hookrightarrow D_{D(1,0)\text{ful}}(T/\bar{T}_T) \times I D_{\text{ful}}(T^{an})$

(iii1) for $f : T \to S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(T)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(MHM(S))$

$\iota_T : D(MHM(S)) \hookrightarrow D_{D(1,0)\text{ful}}(S/\bar{S}_T) \times I D_{\text{ful}}(S^{an})$

(iii2) for $f : T \to S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(T)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(MHM(T))$

$\iota_T : D(MHM(T)) \hookrightarrow D_{D(1,0)\text{ful}}(T/\bar{T}_T) \times I D_{\text{ful}}(T^{an})$

(iii3) for $f : T \to S$ a morphism, with $T, S \in \text{QPVar}(\mathbb{C})$, and $M \in \text{DA}_c(T)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(MHM(S))$

$\iota_T : D(MHM(S)) \hookrightarrow D_{D(1,0)\text{ful}}(S/\bar{S}_T) \times I D_{\text{ful}}(S^{an})$

(iii4) for $S \in \text{Var}(\mathbb{C})$, and $M, N \in \text{DA}_c(S)$, the maps of definition 139 and of definition 166 induce an isomorphism in $D(MHM(S))$

$\iota_T : D(MHM(S)) \hookrightarrow D_{D(1,0)\text{ful}}(S/\bar{S}_T) \times I D_{\text{ful}}(S^{an})$
We obtain theorem 6 from theorem 5 and from the result on the Betti factor after checking the compatibility of these transformation maps with the isomorphisms \( \alpha(M) \).

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## 2 Generality and Notations

### 2.1 Notations

- After fixing a universe, we denote by
  - Set the category of sets,
  - Top the category of topological spaces,
  - Ring the category of rings and cRing \( \subset \) Ring the full subcategory of commutative rings,
  - RTop the category of ringed spaces,
    - whose set of objects is \( \text{RTop} := \{(X, O_X), X \in \text{Top}, O_X \in \text{PSh}(X, \text{Ring})\} \)
    - whose set of morphism is \( \text{Hom}((T, O_T), (S, O_S)) := \{(f : T \to S), (a_f : f^*O_S \to O_T)\} \)
  and by \( ts : \text{RTop} \to \text{Top} \) the forgetful functor.
- Cat the category of small categories which comes with the forgetful functor \( - \) \( \in \) \( \text{Cat} \).
- For \( \mathcal{C} \in \text{Cat} \), we denote by \( \mathcal{C}^{op} \) the opposite category whose set of objects is the one of \( \mathcal{C} \) : \( (\mathcal{C}^{op})^0 = \mathcal{C}^0 \), and whose morphisms are the morphisms of \( \mathcal{C} \) with reversed arrows.
- Let \( F : \mathcal{C} \to \mathcal{C}' \) be a functor with \( \mathcal{C}, \mathcal{C}' \in \text{Cat} \). For \( X \in \mathcal{C} \), we denote by \( F(X) \in \mathcal{C}' \) the image of \( X \), and for \( X, Y \in \mathcal{C} \), we denote by \( F^{X,Y} : \text{Hom}(X, Y) \to \text{Hom}(F(X), F(Y)) \) the corresponding map.
- Let \( \mathcal{C} \in \text{Cat} \), we denote by \( \mathcal{C}^{op} \in \text{Cat} \) the opposite category whose set of object is the one of \( \mathcal{C} \) : \( (\mathcal{C}^{op})^0 = \mathcal{C}^0 \), and whose morphisms are the morphisms of \( \mathcal{C} \) with reversed arrows.
- For \( S \in \mathcal{C} \), we denote by \( \mathcal{C}/S \) the category
  - whose set of objects \( (\mathcal{C}/S)^0 = \{X/S = (X, h)\} \) consist of the morphisms \( h : X \to S \) with \( X \in \mathcal{C} \),
  - whose set of morphism \( \text{Hom}(X'/S, X/S) \) between \( X'/S = (X', h'), X/S = (X, h) \in \mathcal{C}/S \) consists of the morphisms \( (g : X' \to X) \in \text{Hom}(X', X) \) such that \( h \circ g = h' \).

We have then, for \( S \in \mathcal{C} \), the canonical forgetful functor

\[
  r(S) : \mathcal{C}/S \to \mathcal{C}, \quad X/S \mapsto r(S)(X/S) = X, \quad (g : X'/S \to X/S) \mapsto r(S)(g) = g
\]

and we denote again \( r(S) : \mathcal{C} \to \mathcal{C}/S \) the corresponding morphism of (pre)sites.

- Let \( F : \mathcal{C} \to \mathcal{C}' \) be a functor with \( \mathcal{C}, \mathcal{C}' \in \text{Cat} \). Then for \( S \in \mathcal{C} \), we have the canonical functor

\[
  F_S : \mathcal{C}/S \to \mathcal{C}' / F(S), \quad X/S \mapsto F(X/S) = F(X)/F(S), \\
  (g : X'/S \to X/S) \mapsto (F(g) : F(X')/F(S) \to F(X)/F(S))
\]
Let $\mathcal{S} \in \text{Cat}$. Then, for a morphism $f : X' \to X$ with $X, X' \in \mathcal{S}$ we have the functor

$$C(f) : \mathcal{S}/X' \to \mathcal{S}/X, \ \ Y/X' = (Y, f_1) \mapsto C(f)(Y/X') := (Y, f \circ f_1) \in \mathcal{S}/X,$$

$$(g : Y_1/X' \to Y_2/X') \mapsto (C(f)(g) := g : Y_1/X \to Y_2/X)$$

Let $\mathcal{S}$ be a category which admits fiber products. Then, for a morphism $f : X' \to X$ with $X, X' \in \mathcal{S}$, we have the pullback functor

$$P(f) : \mathcal{S}/X \to \mathcal{S}/X', \ \ Y/X \mapsto P(f)(Y/X) := Y \times_X X'/X' \in \mathcal{S}/X',$$

$$(g : Y_1/X \to Y_2/X) \mapsto (P(f)(g) := (g \times I) : Y_1 \times_X X' \to Y_2 \times_X X')$$

which is right adjoint to $C(f) : \mathcal{S}/X' \to \mathcal{S}/X$, and we denote again $P(f) : \mathcal{S}/X' \to \mathcal{S}/X$ the corresponding morphism of (pre)sites.

- Let $\mathcal{C}, \mathcal{I} \in \text{Cat}$. Assume that $\mathcal{C}$ admits fiber products. For $(\mathcal{S}_i) \in \text{Fun}(\mathcal{I}^{\text{op}}, \mathcal{C})$, we denote by $\mathcal{C}/(\mathcal{S}_i) \in \text{Fun}(\mathcal{I}, \text{Cat})$ the diagram of category given by
  - for $I \in \mathcal{I}$, $\mathcal{C}/(\mathcal{S}_i)(I) := \mathcal{C}/\mathcal{S}_i$,
  - for $r_{IJ} : I \to J$, $\mathcal{C}/(\mathcal{S}_i)(r_{IJ}) := P(r_{IJ}) : \mathcal{C}/\mathcal{S}_i \to \mathcal{C}/\mathcal{S}_j$, where we denoted again $r_{IJ} : \mathcal{S}_i \to \mathcal{S}_j$ the associated morphism in $\mathcal{C}$.

- Let $(F, G) : C \to C'$ an adjunction between two categories.
  - For $X \in C$ and $Y \in C'$, we consider the adjunction isomorphisms
    * $I(F,G)(X, Y) : \text{Hom}(F(X), Y) \to \text{Hom}(X, G(Y)), \ (u : F(X) \to Y) \mapsto (I(F,G)(X,Y)(u) : X \to G(Y))$
    * $I(F,G)(X, Y) : \text{Hom}(X, G(Y)) \to \text{Hom}(F(X), Y), \ (v : X \to G(Y)) \mapsto (I(F,G)(X,Y)(v) : F(X) \to Y)$.
  - For $X \in C$, we denote by $\text{ad}(F,G)(X) := I(F,G)(X, F(X))(I_{F(X)}) : X \to G \circ F(X)$.
  - For $Y \in C'$ we denote also by $\text{ad}(F,G)(Y) := I(F,G)(G(Y), Y)(I_{G(Y)}) : F \circ G(Y) \to Y$.

Hence,

- for $u : F(X) \to Y$ a morphism with $X \in C$ and $Y \in C'$, we have $I(F,G)(X,Y)(u) = G(u) \circ \text{ad}(F,G)(X)$,
- for $v : X \to G(Y)$ a morphism with $X \in C$ and $Y \in C'$, we have $I(F,G)(X,Y)(v) = \text{ad}(F,G)(Y) \circ F(v)$.

- Let $\mathcal{C}$ a category.
  - We denote by $(\mathcal{C}, F)$ the category of filtered objects : $(X, F) \in (\mathcal{C}, F)$ is a sequence $(F^m X)_{m \in \mathbb{Z}}$ indexed by $\mathbb{Z}$ in $\mathcal{C}$ together with monomorphisms $a_p : F^p X \hookrightarrow F^{p-1} X \hookleftarrow X$.
  - We denote by $(\mathcal{C}, F, W)$ the category of bifiltered objects : $(X, F, W) \in (\mathcal{C}, F, W)$ is a sequence $(W^q F^p X)_{m, n} \in \mathbb{Z}^2$ indexed by $\mathbb{Z}^2$ in $\mathcal{C}$ together with monomorphisms $W^q F^p X \hookrightarrow F^{p-1} X$, $W^q F^p X \hookrightarrow W^{q-1} F^p X$.

- For $\mathcal{C}$ a category and $\Sigma : \mathcal{C} \to \mathcal{C}$ an endofunctor, we denote by $(\mathcal{C}, \Sigma)$ the corresponding category of spectra, whose objects are sequence of objects of $\mathcal{C}$ $(T_i)_{i \in \mathbb{Z}} \in \text{Fun}(\mathbb{Z}, \mathcal{C})$ together with morphisms $s_i : T_i \to \Sigma T_{i+1}$, and whose morphism from $(T_i)$ to $(T'_i)$ are sequence of morphisms $T_i \to T'_i$ which commutes with the $s_i$.

- Let $\mathcal{A}$ an additive category.
We denote by $C(A) := \text{Fun}(\mathbb{Z}, A)$ the category of (unbounded) complexes with value in $A$, where we have denoted $\mathbb{Z}$ the category whose set of objects is $\mathbb{Z}$, and whose set of morphism between $m, n \in \mathbb{Z}$ consists of one element (identity) if $n = m$, of one element if $n = m + 1$ and is $\emptyset$ in the other cases.

We have the full subcategories $C^b(A), C^-(A), C^+(A)$ of $C(A)$ consisting of bounded, resp. bounded above, resp. bounded below complexes.

We denote by $K(A) := \text{Ho}(C(A))$ the homotopy category of $C(A)$ whose morphisms are equivalent homotopic classes of morphism and by $Ho : C(A) \to K(A)$ the full homotopy functor. The category $K(A)$ is in the standard way a triangulated category.

Let $A$ an additive category.

We denote by $C_{fil}(A) \subset (C(A), F) = C(A, F)$ the full additive subcategory of filtered complexes of $A$ such that the filtration is biregular: for $(A^\bullet, F) \in (C(A), F)$, we say that $F$ is biregular if $F^r A^r$ is finite for all $r \in \mathbb{Z}$.

We denote by $C_{2fil}(A) \subset (C(A), F, W) = C(A, F, W)$ the full subcategory of bifiltered complexes of $A$ such that the filtration is biregular.

For $A^\bullet \in C(A)$, we denote by $(A^\bullet, F_\bullet) \in (C(A), F)$ the complex endowed with the trivial filtration (filtration bete): $F^r A^n = 0$ if $p \geq n + 1$ and $F^p A^n = A^n$ if $p \leq n$. Obviously, a morphism $\phi : A^\bullet \to B^\bullet$, with $A^\bullet, B^\bullet \in C(A)$ induces a morphism $\phi : (A^\bullet, F_\bullet) \to (B^\bullet, F_\bullet)$.

For $(A^\bullet, F) \in C(A, F)$, we denote by $(A^\bullet, F(r)) \in C(A, F)$ the filtered complex where the filtration is given by $F^p (A^\bullet, F(r)) := F^{p+r}(A^\bullet, F)$.

Two morphisms $\phi_1, \phi_2 : (M, F) \to (N, F)$ with $(M, F), (N, F) \in C(A, F)$ are said to be r-filtered homotopic if there exist a morphism in $\text{Fun}(\mathbb{Z}, (A, F))$

$$h : (M, F(r-1))[1] \to (N, F), h := (h^n : (M^{n+1}, F(r-1)) \to (N^n, F))_{n \in \mathbb{Z}},$$

where $\mathbb{Z}$ have only trivial morphism (i.e. $h$ is a graded morphism but not a morphism of complexes) such that $d'h + hd = \phi_1 - \phi_2$, where $d$ is the differential of $M$ and $d'$ is the differential of $N$, and we have $h(F^n M^{n+1}) \subset F^{p-r+1} N^n$, note that by definition $r$ does NOT depend on $p$ and $n$; we then say that

$$(h, \phi_1, \phi_2) : (M, F)[1] \to (N, F)$$

is an r-filtered homotopy. By definition, an r-filtered homotopy $(h, \phi_1, \phi_2) : (M, F)[1] \to (N, F)$ is an r'-filtered homotopy for all $r' \geq r$, and a 1-filtered homotopy is an homotopy of $C(A, F)$. By definition, an r-filtered homotopy $(h, \phi_1, \phi_2) : (M, F)[1] \to (N, F)$ gives if we forgot filtration an homotopy $(h, \phi_1, \phi_2) : M[1] \to N$ in $C(A)$.

Two morphisms $\phi_1, \phi_2 : (M, F) \to (N, F)$ with $(M, F), (N, F) \in C(A, F)$ are said to be $\infty$-filtered homotopic if there exist $r \in \mathbb{N}$ such that $\phi_1, \phi_2 : (M, F) \to (N, F)$ are r-filtered homotopic. Hence, if $\phi_1, \phi_2 : (M, F) \to (N, F)$ are $\infty$-filtered homotopic, then $\phi_1, \phi_2 : M \to N$ are homotopic: of course the converse is NOT true since $r$ does NOT depend on $p, n \in \mathbb{Z}$.

We will use the fact that by definition if $\phi : M \to N$ with $M, N \in C(A)$ is an homotopy equivalence, then $\phi : (M, F_\bullet) \to (N, F_\bullet)$ is a 2-filtered homotopy equivalence.

A morphism $\phi : (M, F) \to (N, F)$ with $(M, F), (N, F) \in C(A, F)$ is said to be an r-filtered homotopy equivalence if there exist a morphism $\phi' : (N, F) \to (M, F)$ such that

* $\phi' \circ \phi : (M, F) \to (M, F)$ is r-filtered homotopic to $I_M$ and
* $\phi \circ \phi' : (N, F) \to (N, F)$ is r-filtered homotopic to $I_N$.

If $\phi : (M, F) \to (N, F)$ is an r-filtered homotopy equivalence, then it is an s-filtered homotopy equivalence for $s \geq r$. If $\phi : (M, F) \to (N, F)$ is an r-filtered homotopy equivalence, $\phi : M \to N$ is an homotopy equivalence.
A morphism \( \phi : (M, F) \to (N, F) \) with \((M, F), (N, F) \in C(\mathcal{A}, F)\) is said to be an infinite filtered homotopy equivalence if there exist \( r \in \mathbb{Z} \) such that \( \phi : (M, F) \to (N, F) \) with \((M, F), (N, F) \in C(\mathcal{A}, F)\) is an \( r \)-filtered homotopy equivalence. If \( \phi : (M, F) \to (N, F) \) is an infinite filtered homotopy equivalence, \( \phi : M \to N \) is an homotopy equivalence; the converse is not true since \( r \) does not depend on \( p, n \in \mathbb{Z} \).

We denote by \( K_r(\mathcal{A}, F) := \text{Ho}_r(C(\mathcal{A}, F)) \) the homotopy category of \( C(\mathcal{A}, F) \) whose objects are those of \( C(\mathcal{A}, F) \) and whose morphisms are the morphisms of \( C(\mathcal{A}, F) \) modulo \( r \)-filtered homotopies, and by \( \text{Ho}_r : C(\mathcal{A}, F) \to K_r(\mathcal{A}, F) \) the full homotopy functor. However, for \( r > 1 \) the category \( K_r(\mathcal{A}, F) \) with the canonical triangles the standard ones does not satisfy the 2 of 3 axiom of a triangulated category. Indeed, for \( r > 1 \), a commutative diagram in \( K_r(\mathcal{A}, F) \)

\[
\begin{array}{ccc}
(A^\bullet, F) & \xrightarrow{m} & (B^\bullet, F) \\
\downarrow & & \downarrow \\
(A^\bullet, F) & \xrightarrow{m'} & (B^\bullet, F)
\end{array}
\]

where

\[
\text{Cone}(m) = ((A^\bullet, F)[1] \oplus (B^\bullet, F), d, d'^{p_1} - m) \to (A^\bullet, F)[1],
\]

i.e. which commutes modulo \( r \)-filtered homotopy can NOT be completed with a third vertical arrow. In particular, for \( r > 1 \) and commutative diagram in \( C(\mathcal{A}, F) \)

\[
\begin{array}{ccc}
(A^\bullet, F) & \xrightarrow{m} & (B^\bullet, F) \\
\downarrow & & \downarrow \\
(A^\bullet, F) & \xrightarrow{m'} & (B^\bullet, F)
\end{array}
\]

\[
\text{Cone}(m') = ((A^\bullet, F)[1] \oplus (B^\bullet, F), d, d'^{p_1} - m') \to (A^\bullet, F)[1]
\]

if \( \psi \) and \( \phi \) are \( r \)-filtered homotopy equivalence, then \( (\phi[1], \psi) \) is NOT necessary an \( r \)-filtered homotopy equivalence.

We denote by \( K_{fil,r}(\mathcal{A}) := \text{Ho}_r(C_{fil}(\mathcal{A})) \) the homotopy category of \( C_{fil}(\mathcal{A}) \) whose objects are those of \( C_{fil}(\mathcal{A}) \) and whose morphisms are the morphisms of \( C_{fil}(\mathcal{A}) \) modulo \( r \)-filtered homotopies, and by \( \text{Ho}_r : C_{fil}(\mathcal{A}) \to K_{fil,r}(\mathcal{A}) \) the full homotopy functor. However, for \( r > 1 \) the category \( K_{fil,r}(\mathcal{A}) \) with the canonical triangles the standard ones does not satisfy the 2 of 3 axiom of a triangulated category.

We denote by \( K_{fil,\infty}(\mathcal{A}) := \text{Ho}_\infty(C_{fil}(\mathcal{A})) \) the homotopy category of \( C_{fil}(\mathcal{A}) \) whose objects are those of \( C_{fil}(\mathcal{A}) \) and whose morphisms are the morphisms of \( C_{fil}(\mathcal{A}) \) modulo infinite filtered homotopies, and by \( \text{Ho}_\infty : C_{fil}(\mathcal{A}) \to K_{fil,\infty}(\mathcal{A}) \) the full homotopy functor. However, the category \( K_{fil,\infty}(\mathcal{A}) \) with the canonical triangles the standard ones does not satisfy the 2 of 3 axiom of a triangulated category.

We have the Deligne decalage functor
\[
\text{Dec} : C(\mathcal{A}, F) \to C(\mathcal{A}, F), \quad (M, F) \mapsto \text{Dec}(M, F) := (M, \text{Dec} F),
\]
\[
\text{Dec} F^p M^n := F^{p+n} M^n \cap d^{-1}(F^{p+n+1} M^{n+1})
\]

It is the right adjoint of the shift functor
\[
S : C(\mathcal{A}, F) \to C(\mathcal{A}, F), \quad (M, F) \mapsto S(M, F) := (M, SF), \quad SF^p M^n := F^{p-n} M^n
\]

The dual decalage functor
\[
\text{Dec}^\vee : C(\mathcal{A}, F) \to C(\mathcal{A}, F), \quad (M, F) \mapsto \text{Dec}^\vee(M, F) := (M, \text{Dec}^\vee F),
\]
\[
\text{Dec}^\vee F^p M^n := F^{p+n} M^n + d(F^{p+n-1} M^{n+1})
\]

is the left adjoint of the shift functor. Note that \( \text{Dec}((M, F)[1]) \neq (\text{Dec}(M, F))[1], \text{Dec}^\vee((M, F)[1]) \neq (\text{Dec}^\vee(M, F))[1] \) and \( \text{Dec}((M, F)[1]) \neq (S(M, F))[1] \).

\[12\]
Let \( \mathcal{A} \) be an abelian category. Then the additive category \((\mathcal{A}, F)\) is an exact category which admits kernel and cokernel (but is NOT an abelian category). A morphism \( \phi : (M, F) \to (N, F) \) with \((M, F) \in (\mathcal{A}, F)\) is strict if the inclusion \( \phi(F^n M) \subset F^n N \cap \text{Im}(\phi) \) is an equality, i.e. if 
\[
\phi(F^n M) = F^n N \cap \text{Im}(\phi).
\]

Let \( \mathcal{A} \) be an abelian category.

- For \((A^*, F) \in C(\mathcal{A}, F)\), considering \(a_p : F^p A^* \to A^*\) the structural monomorphism of the filtration, we denote by, for \( n \in \mathbb{N}\),
\[
H^n(\mathcal{A}^*, F) \in (\mathcal{A}, F),\quad F^p H^n(\mathcal{A}^*, F) := \text{Im}(H^n(a_p) : H^n(F^p A^*) \to H^n(A^*)) \subset H^n(\mathcal{A}^*)
\]
the filtration induced on the cohomology objects of the complex. In the case \((A^*, F) \in C_{fil}(\mathcal{A})\), the spectral sequence \( E^{p,q}_{\bullet}(\mathcal{A}^*, F) \) associated to \((A^*, F)\) converge to \(\text{Gr}_{F^p} H^{p+q}(\mathcal{A}^*, F)\), that is for all \( p, q \in \mathbb{Z} \), there exist \( r_{p+q} \in \mathbb{N} \), such that \( E^{p,q}_{r}(\mathcal{A}^*, F) = \text{Gr}_{F^p} H^{p+q}(\mathcal{A}^*, F) \) for all \( s \leq r_{p+q} \).

- A morphism \( m : (A^*, F) \to (B^*, F) \) with \((A^*, F), (B^*, F) \in C(\mathcal{A}, F)\) is said to be a filtered quasi-isomorphism if for all \( n, p \in \mathbb{Z}\),
\[
H^n(\text{Gr}_{F^p}(m) : H^n(\text{Gr}_{F^p} A^*) \cong H^n(\text{Gr}_{F^p} B^*)
\]
is an isomorphism in \( \mathcal{A} \). Consider a commutative diagram in \( C(\mathcal{A}, F) \)
\[
\begin{array}{ccc}
(A^*, F) & \xrightarrow{m} & (B^*, F) \\
\phi \downarrow & & \downarrow \psi \\
(A^*, F) & \xrightarrow{m'} & (B^*, F)
\end{array}
\]
\[
\text{Con}(m) = ((A^*, F)[1] \oplus (B^*, F), d, d' m) \quad \text{and} \quad \text{Con}(m') = ((A^*, F)[1] \oplus (B^*, F), d, d' m')
\]
If \( \phi \) and \( \psi \) are filtered quasi-isomorphisms, then \( (\phi[1], \psi) \) is an filtered quasi-isomorphism.

- If two morphisms \( \phi_1, \phi_2 : (M, F) \to (N, F) \) with \((M, F), (N, F) \in C(\mathcal{A}, F)\) are r-filtered homotopic, then for all \( p, q \in \mathbb{Z} \) and \( s \geq r \),
\[
E^{p,q}_{s}(\phi_1) = E^{p,q}_{s}(\phi_2) : E^{p,q}_{s}(M, F) \to E^{p,q}_{s}(M, F).
\]
Hence if \( \phi : (M, F) \to (N, F) \) with \((M, F), (N, F) \in C(\mathcal{A}, F)\) is an r-filtered homotopy equivalence then for all \( p, q \in \mathbb{Z} \) and \( s \geq r \),
\[
E^{p,q}_{s}(\phi) : E^{p,q}_{s}(M, F) \cong E^{p,q}_{s}(N, F)
\]
is an isomorphism in \( \mathcal{A} \).

- Let \( r \in \mathbb{N} \). A morphism \( m : (A^*, F) \to (B^*, F) \) with \((A^*, F), (B^*, F) \in C(\mathcal{A}, F)\) is said to be an r-filtered quasi-isomorphism if it belongs to the submonoid of arrows generated by filtered quasi-isomorphism and r-filtered homotopy equivalence, that is if there exists \( m_i : (C_{s+i}^*, F) \to (C_{s+i+1}^*, F) \), \( 0 \leq i \leq s \), with \((C_i^*, F) \in C(\mathcal{A}, F)\) \((C_0^*, F) = (A^*, F)\) and \((C_s^*, F) = (B^*, F)\), such that
\[
m = m_s \circ \cdots \circ m_i \circ \cdots \circ m_0 : (A^*, F) \to (B^*, F)
\]
and \( m_i : (C_{s+i}^*, F) \to (C_{s+i+1}^*, F) \) either a filtered quasi-isomorphism or an r-filtered homotopy equivalence. Note that our definition is stronger then the one given in [9] in order to get a multiplicative system. Indeed, if \( m : (A^*, F) \to (B^*, F) \) with \((A^*, F), (B^*, F) \in C(\mathcal{A}, F)\) is an r-filtered quasi-isomorphism then for all \( p, q \in \mathbb{Z} \) and \( s \geq r \),
\[
E^{p,q}_{r}(m) : E^{p,q}_{r}(A^*, F) \cong E^{p,q}_{r}(B^*, F)
\]
is an isomorphism in $\mathcal{A}$, but the converse is NOT true. If a morphism $m : (A^\bullet, F) \to (B^\bullet, F)$, with $(A^\bullet, F), (B^\bullet, F) \in C_{fil}(\mathcal{A})$ is an $r$-filtered quasi-isomorphism, then for all $n \in \mathbb{Z}$

$$H^n(m) : H^n(A^\bullet, F) \xrightarrow{\sim} H^n(B^\bullet, F)$$

is a filtered isomorphism, i.e. an isomorphism in $(\mathcal{A}, F)$. The converse is true if there exist $N_1, N_2 \in \mathbb{Z}$ such that $H^n(A^\bullet) = H^n(B^\bullet) = 0$ for $n \leq N_1$ or $n \geq N_2$. A filtered quasi-isomorphism is obviously a 1-filtered quasi-isomorphism. However for $r > 1$, the $r$-filtered quasi-isomorphism does NOT satisfy the 2 of 3 property for morphisms of canonical triangles.

- A morphism $m : (A^\bullet, F) \to (B^\bullet, F)$ with $(A^\bullet, F), (B^\bullet, F) \in C(\mathcal{A}, F)$ is said to be an $\infty$-filtered quasi-isomorphism if there exist $n \in \mathbb{N}$ such that $m : (A^\bullet, F) \to (B^\bullet, F)$ an $r$-filtered quasi-isomorphism. If a morphism $m : (A^\bullet, F) \to (B^\bullet, F)$, with $(A^\bullet, F), (B^\bullet, F) \in C_{fil}(\mathcal{A})$ is an $\infty$-filtered quasi-isomorphism, then for all $n \in \mathbb{Z}$

$$H^n(m) : H^n(A^\bullet, F) \xrightarrow{\sim} H^n(B^\bullet, F)$$

is a filtered isomorphism. The converse is true if there exist $N_1, N_2 \in \mathbb{Z}$ such that $H^n(A^\bullet) = H^n(B^\bullet) = 0$ for $n \leq N_1$ or $n \geq N_2$. The $\infty$-filtered quasi-isomorphism does NOT satisfy the 2 of 3 property for morphisms of canonical triangles. If $m : (A^\bullet, F) \to (B^\bullet, F)$ is such that $m : (A^\bullet, F) \to (B^\bullet, F)$ is a quasi-isomorphism but $m : (A^\bullet, F) \to (B^\bullet, F)$ is not an $\infty$-filtered quasi-isomorphism, then it induces an isomorphisms $H^n(m) : H^n(A^\bullet) \xrightarrow{\sim} H^n(B^\bullet)$, hence injective maps

$$H^n(m) : F^n H^n(A^\bullet, F) \hookrightarrow F^n H^n(B^\bullet, F)$$

which are not isomorphism (the non strict case). If $m : (A^\bullet, F) \to (B^\bullet, F)$ is such that $m : (A^\bullet, F) \to (B^\bullet, F)$ is a quasi-isomorphism but $m : (A^\bullet, F) \to (B^\bullet, F)$ is not an $\infty$-filtered quasi-isomorphism (the non strict case), then $H^n \text{Cone}(m) = 0$ for all $n \in \mathbb{N}$, hence $\text{Cone}(m) \to 0$ is an $\infty$-filtered quasi-isomorphism ; this shows that the $\infty$-filtered quasi-isomorphism does NOT satisfy the 2 of 3 property for morphism of canonical triangles.

- Let $\mathcal{A}$ be an abelian category.

  - We denote by $D(\mathcal{A})$ the localization of $K(\mathcal{A})$ with respect to the quasi-isomorphisms and by $D : K(\mathcal{A}) \to D(\mathcal{A})$ the localization functor. The category $D(\mathcal{A})$ is a triangulated category in the unique way such that $D$ a triangulated functor.

  - We denote by $D_{fil}(\mathcal{A})$ the localization of $K_{fil}(\mathcal{A})$ with respect to the filtered quasi-isomorphisms and by $D : K_{fil}(\mathcal{A}) \to D_{fil}(\mathcal{A})$ the localization functor.

- Let $\mathcal{A}$ be an abelian category. We denote by $\text{Inj}(\mathcal{A}) \subset A$ the full subcategory of injective objects, and by $\text{Proj}(\mathcal{A}) \subset A$ the full subcategory of projective objects.

- For $\mathcal{S} \in \text{Cat}$ a small category, we denote by

  - $\text{PSh}(\mathcal{S}) := \text{PSh}(\mathcal{S}, \text{Ab}) := \text{Fun}(\mathcal{S}, \text{Ab})$ the category of presheaves on $\mathcal{S}$, i.e. the category of presheaves of abelian groups on $\mathcal{S}$,

  - $\text{PSh}(\mathcal{S}, \text{Ring}) := \text{Fun}(\mathcal{S}, \text{Ring})$ the category of presheaves of ring on $\mathcal{S}$, and $\text{PSh}(\mathcal{S}, \text{cRing}) \subset \text{PSh}(\mathcal{S}, \text{Ring})$ the full subcategory of presheaves of commutative ring.

  - for $F \in \text{PSh}(\mathcal{S})$ and $X \in \mathcal{S}$, $F(X) = \Gamma(X, F)$ the sections on $X$ and for $h : X' \to X$ a morphism with $X, X' \in \mathcal{S}$, $F(h) := F^{X, X'}(h) : F(X) \to F(X')$ the morphism of abelian groups,

  - $C(\mathcal{S}) = \text{PSh}(\mathcal{S}, C(\mathbb{Z})) = \text{C}(\text{PSh}(\mathcal{S})) = \text{PSh}(\mathcal{S} \times \mathbb{Z})$ the big abelian category of complexes of presheaves on $\mathcal{S}$ with value in abelian groups,

  - $K(\mathcal{S}) := K(\text{PSh}(\mathcal{S})) = \text{Ho}(C(\mathcal{S}))$ In particular, we have the full homotopy functor $\text{Ho} : C(\mathcal{S}) \to K(\mathcal{S})$, 

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Consider the adjunctions given by the direct and inverse image functors:

$$C_{2,fil}(S) := C_{2,fil}(PSh(S)) \subset C(PSh(S), F, W)$$

the big abelian category of (bi)filtered complexes of presheaves on $S$ with value in abelian groups such that the filtration is biregular, and $PSh_{2,fil}(S) = (PSh(S), F, W)$,

$$K_{fil}(S) := K_{fil}(PSh(S)) = Ho(C_{fil}(S)),$$

$$K_{fil,r}(S) := K_{fil,r}(PSh(S)) = Ho_r(C_{fil}(S)), K_{fil,\infty}(S) := K_{fil,\infty}(PSh(S)) = Ho_\infty(C_{fil}(S)).$$

For $f : T \to S$ a morphism a presite with $T, S \in \text{Cat}$, given by the functor $P(f) : S \to T$, we will consider the adjunctions given by the direct and inverse image functors:

$$\{f^*, f_*\} = \{f^{-1}, f_*\} : PSh(S) \rightleftharpoons PSh(T),$$

which induces $(f^*, f_* : C(S) \rightleftharpoons C(T)$, we denote, for $F \in C(S)$ and $G \in C(T)$ by

$$\text{ad}(f^*, f_*)(F) : F \to f_* f^* F, \text{ ad}(f^*, f_*)(G) : f^* f_* G \to G$$

the adjunction maps,

$$(f_*, f^+) : PSh(T) \rightleftharpoons PSh(S),$$

which induces $(f_*, f^+) : C(T) \rightleftharpoons C(S)$, we denote for $F \in C(S)$ and $G \in C(T)$ by

$$\text{ad}(f_*, f^+)(F) : G \to f^+ f_* G, \text{ ad}(f_*, f^+)(G) : f_* f^+ F \to F$$

the adjunction maps.

- For $(S, O_S) \in \text{RCat}$ a ringed topos, we denote by

  $$\text{PSh}_{O_S}(S)$$

  the category of presheaves of $O_S$ modules on $S$, whose objects are $\text{PSh}_{O_S}(S)^0 := \{(M, m), M \in \text{PSh}(S), m : M \otimes O_S \to M\}$, together with the forgetful functor $o : \text{PSh}(S) \to \text{PSh}_{O_S}(S),$

  $$C_{O_S}(S) = C(\text{PSh}_{O_S}(S))$$

  the big abelian category of complexes of presheaves of $O_S$ modules on $S$,

  $$K_{O_S}(S) := K(\text{PSh}_{O_S}(S)) = Ho(C_{O_S}(S)),$$

  in particular, we have the full homotopy functor $Ho : C_{O_S}(S) \to K_{O_S}(S),$

  $$C_{O_S(2,fil)}(S) := C_{2,fil}(\text{PSh}_{O_S}(S)) \subset C(\text{PSh}_{O_S}(S), F, W),$$

  the big abelian category of (bi)filtered complexes of presheaves of $O_S$ modules on $S$ such that the filtration is biregular and $\text{PSh}_{O_S(2,fil)}(S) = (\text{PSh}_{O_S}(S), F, W),$.

  $$K_{O_S(2,fil)}(S) := K_{fil}(\text{PSh}_{O_S}(S)) = Ho(C_{O_S(2,fil)}(S)),$$

  $$K_{O_S(2,fil,r)}(S) := K_{fil,r}(\text{PSh}_{O_S}(S)) = Ho_r(C_{O_S(2,fil)}(S)), K_{O_S(2,fil,\infty)}(S) := K_{fil,\infty}(\text{PSh}_{O_S}(S)) = Ho_\infty(C_{O_S(2,fil)}(S)).$$

- For $S_* \in \text{Fun}(I, \text{Cat})$ a diagram of (pre)sites, with $I \in \text{Cat}$ a small category, we denote by

  $$\Gamma S_* \in \text{Cat}$$

  the associated diagram category

  * whose objects are $\Gamma S_*^0 := \{(X_I, u_{IJ})_{I \in I}\},$ with $X_I \in S_I,$ and for $r_{IJ} : I \to J$ with $I, J \in I,$ $u_{IJ} : X_J \to r_{IJ}(X_I)$ are morphism in $S_J$ notig again $r_{IJ} : S_I \to S_J$ the associated functor,

  * whose morphism are $m = (m_I) : (X_I, u_{IJ}) \to (X'_I, v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ}(m_I) \circ u_{IJ}$ in $S_J,$

  $$\text{PSh}(S_*) := \text{PSh}(\Gamma S_*, \text{Ab})$$

  the category of presheaves on $S_*,$

  * whose objects are $\text{PSh}(S_*)^0 := \{(F_I, u_{IJ})_{I \in I}\},$ with $F_I \in \text{PSh}(S_I),$ and for $r_{IJ} : I \to J$ with $I, J \in I,$ $u_{IJ} : F_J \to r_{IJ}(F_I)$ are morphism in $\text{PSh}(S_I),$ noting again $r_{IJ} : S_J \to S_I$ the associated morphism of presite,

  * whose morphism are $m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ}m_I \circ u_{IJ}$ in $\text{PSh}(S_I),$
- $\text{PSh}(\mathcal{S}_\bullet, \text{Ring}) := \text{PSh}(\Gamma \mathcal{S}_\bullet, \text{Ring})$ the category of presheaves of ring on $\mathcal{S}_\bullet$ given in the same way, and $\text{PSh}(\mathcal{S}_\bullet, c\text{Ring}) \subset \text{PSh}(\mathcal{S}_\bullet, \text{Ring})$ the full subcategory of presheaves of commutative ring.

- $C(\mathcal{S}_\bullet) := C(\text{PSh}(\mathcal{S}_\bullet))$ the big abelian category of complexes of presheaves on $\mathcal{S}_\bullet$ with value in abelian groups.

- $K(\mathcal{S}_\bullet) := K(\text{PSh}(\mathcal{S}_\bullet)) = \text{Ho}(C(\mathcal{S}_\bullet))$, in particular, we have the full homotopy functor $\text{Ho} : C(\mathcal{S}_\bullet) \to K(\mathcal{S}_\bullet)$.

- $C(\mathcal{S}_\bullet) := C(\text{PSh}(\mathcal{S}_\bullet), F, W)$ the big abelian category of (bi)filtered complexes of presheaves on $\mathcal{S}_\bullet$ with value in abelian groups such that the filtration is biregular, and $\text{PSh}(\mathcal{S}_\bullet) = (\text{PSh}(\mathcal{S}_\bullet), F, W)$, by definition $C(\mathcal{S}_\bullet)$ is the category

  - whose objects are $C(\mathcal{S}_\bullet)^0 := \{(F, u_{ij})\}_{i,j \in I}$, with $(F, u_{ij}) \in C(\mathcal{S}_\bullet)$, and for $r_{ij} : I \to J$ with $I, J \in \mathcal{I}$, $u_{ij} : (F, u_{ij}) \to r_{ij} = r_{ij}(F, u_{ij})$ are morphism in $C(\mathcal{S}_\bullet)$, noting again $r_{ij} : F \to F$ the associated morphism of presite, whose morphism are $m = (m_l) : ((F, u_{ij}), u_{ij}) \to ((G, v_{ij}), v_{ij})$ satisfying $v_{ij} \circ m_l = r_{ij} \circ m_l \circ u_{ij}$ in $C(\mathcal{S}_\bullet)$.

- $K(\mathcal{S}_\bullet) := K(\text{PSh}(\mathcal{S}_\bullet)) = \text{Ho}(C(\mathcal{S}_\bullet))$, $K(\mathcal{S}_\bullet) := K(\text{PSh}(\mathcal{S}_\bullet)) = K(\mathcal{S}_\bullet)$.

Let $\mathcal{I}, \mathcal{I}' \in \text{Cat}$ be small categories. Let $(f_0, s) : \mathcal{T}_0 \to \mathcal{S}_\bullet$ a morphism a diagrams of (pre)site with $\mathcal{T}_0 \in \text{Fun}(\mathcal{I}, \text{Cat})$, $\mathcal{S}_\bullet \in \text{Fun}(\mathcal{I}', \text{Cat})$, which is by definition given by a functor $s : \mathcal{I} \to \mathcal{I}'$ and morphism of functor $P(f_0) : \mathcal{S}_\bullet(s) := \mathcal{S}_\bullet \circ s \to \mathcal{T}_0$. Here, we denote for short, $\mathcal{S}_\bullet(s) := \mathcal{S}_\bullet \circ s \in \text{Fun}(\mathcal{I}, \text{Cat})$. We have then, for $r_{ij} : I \to J$ a morphism, with $I, J \in \mathcal{I}$, a commutative diagram in $\text{Cat}$

$$
\begin{array}{ccc}
D_{IJ} & \xrightarrow{r_{ij}} & S_{s(I)} \\
\downarrow f_I & & \downarrow f_I \\
\mathcal{T}_I & \xrightarrow{r_{ij}} & \mathcal{T}_I
\end{array}
$$

We will consider the the adjunction given by the direct and inverse image functors:

$$
((f_0, s)^*, (f_0, s)_*) = ((f_0, s)^{-1}, (f_0, s)_*) : \text{PSh}(\mathcal{S}_\bullet(s)) \Rightarrow \text{PSh}(\mathcal{T}_0),
$$

$$
F = (F, u_{ij}) \mapsto (f_0, s)^* F := (f_0, s)^* F = (f_0, s)^* F, G = (G, v_{ij}) \mapsto (f_0, s)_* G := (f_0, s)_* G.
$$

It induces the adjunction $((f_0, s)^*, (f_0, s)_*) : C(\mathcal{S}_\bullet(s)) \Rightarrow C(\mathcal{T}_0)$. We denote, for $(F, u_{ij}) \in C(\mathcal{S}_\bullet(s))$ and $(G, v_{ij}) \in C(\mathcal{T}_0)$ by

$$
\text{ad}((f_0, s)^*, (f_0, s)_*)((F, u_{ij})) : (F, u_{ij}) \to (f_0, s)_*(f_0, s)^* (F, u_{ij}),
$$

$$
\text{ad}((f_0, s)^*, (f_0, s)_*)((G, v_{ij})) : (f_0, s)^* (f_0, s)_*(G, v_{ij}) \to (G, v_{ij})
$$

the adjunction maps.

- Let $\mathcal{I} \in \text{Cat}$ a small category. For $(\mathcal{S}_\bullet, O_{\mathcal{S}_\bullet}) \in \text{Fun}(\mathcal{I}, \text{RCat})$ a diagram of ringed topos, we denote by

$$
\text{PSh}_{O_{\mathcal{S}_\bullet}}(\mathcal{S}_\bullet) := \text{PSh}_{O_{\mathcal{S}_\bullet}}(\Gamma \mathcal{S}_\bullet) \text{ the category of presheaves of modules on } (\mathcal{S}_\bullet, O_{\mathcal{S}_\bullet}),
$$

- whose objects are $\text{PSh}_{O_{\mathcal{S}_\bullet}}(\mathcal{S}_\bullet)^0 := \{(F, u_{ij})\}_{i,j \in I}$, with $F \in \text{PSh}_{O_{\mathcal{S}_\bullet}}(\mathcal{S}_\bullet)$, and for $r_{ij} : I \to J$ with $I, J \in \mathcal{I}$, $u_{ij} : F \to r_{ij} F$ are morphism in $\text{PSh}_{O_{\mathcal{S}_\bullet}}(\mathcal{S}_\bullet)$, noting again $r_{ij} : S \to S$ the associated morphism of presite,

- whose morphism are $m = (m_l) : (F, u_{ij}) \to (G, v_{ij})$ satisfying $v_{ij} \circ m_l = r_{ij} \circ m_l \circ u_{ij}$ in $\text{PSh}_{O_{\mathcal{S}_\bullet}}(\mathcal{S}_\bullet)$. 

particular an étale morphism $X \rightarrow Y$, whose set of morphisms between $(X, O_X)$ is the subset of morphisms in $C_{O_\bullet}(S_\bullet) := K(PSh_{O_\bullet}(S_\bullet)) = K(PSh_{O_\bullet}(S_\bullet), F, W)$, we consider $Sch/k := Sch / Spec k$ the category of schemes over $Spec k$.  We then denote by

- $Var(k) \subset Sch / k$ the full subcategory consisting of algebraic varieties over $k$, i.e. schemes of finite type over $k$,
- $PVar(k) \subset QPVar(k) \subset Var(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
- $PSmVar(k) \subset SmVar(k) \subset Var(k)$ the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

A morphism $h : U \rightarrow S$ with $U, S \in Var(k)$ is said to be smooth if it is flat with smooth fibers. A morphism $r : U \rightarrow X$ with $U, X \in Var(k)$ is said to be étale if it is non ramified and flat. In particular an étale morphism $r : U \rightarrow X$ with $U, X \in Var(k)$ is smooth and quasi-finite (i.e. the fibers are either the empty set or a finite subset of $X$)

- Denote by $Top^2$ the category whose set of objects is

$$(Top^2)^0 := \{(X, Z), Z \subset X \text{ closed}\} \subset Top \times Top$$

and whose set of morphisms between $(X_1, Z_1), (X_2, Z_2) \in Top^2$ is

$$\text{Hom}_{Top^2}((X_1, Z_1), (X_2, Z_2)) := \{ (f : X_1 \rightarrow X_2), \text{ s.t. } Z_1 \subset f^{-1}(Z_2) \} \subset \text{Hom}_{Top}(X_1, X_2)$$

For $S \in Top$, $Top^2 / S := Top^2 / (S, S)$ is then by definition the category whose set of objects is

$$(Top^2 / S)^0 := \{(X, Z), h : X \rightarrow S, Z \subset X \text{ closed}\} \subset Top / S \times Top$$

and whose set of morphisms between $(X_1, Z_1)/S = ((X_1, Z_1), h_1), (X_2, Z_2)/S = ((X_2, Z_2), h_2) \in Top^2 / S$ is the subset

$$\text{Hom}_{Top^2 / S}((X_1, Z_1)/S, (X_2, Z_2)/S) := \{(f : X_1 \rightarrow X_2), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2) \} \subset \text{Hom}_{RTop}(X_1, X_2)$$

We denote by

$$\mu_S : Top^2_{pr} / S := \{((Y \times S, Z), p), p : Y \times S \rightarrow S, Z \subset Y \times S \text{ closed}\} \hookrightarrow Top^2 / S$$
the full subcategory whose objects are those with \( p : Y \times S \to S \) a projection, and again \( \mu_S : \text{Top}^2 / S \to \text{Top}^{2,pr} / S \) the corresponding morphism of sites. We denote by

\[
\text{Gr}^0_S : \text{Top} / S \to \text{Top}^{2,pr} / S, \quad X/S \mapsto \text{Gr}^0_S (X/S) := (X \times S, \tilde{X}) / S,
\]

\[
(g : X/S \to X'/S) \mapsto \text{Gr}^0_S (g) := (g \times I_S : (X \times S, \tilde{X}) \to (X' \times S, \tilde{X}')
\]

the graph functor, \( X \to X \times S \) being the graph embedding (which is a closed embedding if \( X \) is separated), and again \( \text{Gr}^0_S : \text{Top}^{2,pr} / S \to \text{Top} / S \) the corresponding morphism of sites.

- Denote by \( \text{RTop}^2 \) the category whose set of objects is

\[
(\text{RTop}^2)^0 := \{(X, O_X), Z) : Z \subset X \text{ closed}\} \subset \text{RTop} \times \text{Top}
\]

and whose set of morphisms between \( (X, O_X), (Y, O_Y), (Z, O_Z) \in \text{RTop}^2 \) is

\[
\text{Hom}_{\text{RTop}^2} (((X, O_X), (Y, O_Y), (Z, O_Z)) := \{(f : (X, O_X) \to (Y, O_Y), \text{ s.t. } Z \subset f^{-1}(Z)\} \subset \text{Hom}_{\text{RTop}} ((X, O_X), (Y, O_Y))
\]

For \( (S, O_S) \in \text{RTop} \), \( \text{RTop}^2 / (S, O_S) := \text{RTop}^2 / ((S, O_S), S) \) is then by definition the category whose set of objects is

\[
(\text{RTop}^2 / (S, O_S))^0 := \{(X, O_X, Z), h) : (X, O_X) \to (S, O_S), Z \subset X \text{ closed}\} \subset \text{RTop} / (S, O_S) \times \text{Top}
\]

and whose set of morphisms between \( (((X, O_X), (Y, O_Y), (Z, O_Z)), h_1), (((X', O_X'), (Y', O_Y'), (Z', O_Z')), h_2) \in \text{RTop}^2 / (S, O_S) \) is the subset

\[
\text{Hom}_{\text{RTop}^2 / (S, O_S)} (((X, O_X), (Y, O_Y), (Z, O_Z)), (X', O_X'), (Y', O_Y'), (Z', O_Z')) := \{(f : (X, O_X) \to (Y', O_Y'), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z \subset f^{-1}(Z)\} \subset \text{Hom}_{\text{RTop}} ((X, O_X), (X', O_X'))
\]

We denote by

\[
\mu_S : \text{RTop}^{2,pr} / S := \{(((Y \times S, q^* O_Y \otimes p^* O_S), Z), p) : Y \times S \to S, Z \subset Y \times S \text{ closed}\} \to \text{RTop}^2 / S
\]

the full subcategory whose objects are those with \( p : Y \times S \to S \) is a projection, and again \( \mu_S : \text{RTop}^2 / S \to \text{RTop}^{2,pr} / S \) the corresponding morphism of sites. We denote by

\[
\text{Gr}^1_S : \text{RTop} / S \to \text{RTop}^{2,pr} / S, \quad (X, O_X) / (S, O_S) \mapsto \text{Gr}^1_S ((X, O_X) / (S, O_S)) := ((X \times S, q^* O_X \otimes \pi^* O_S), \tilde{X}) / (S, O_S),
\]

\[
(g : (X, O_X) / (S, O_S) \to (X', O_X') / (S, O_S)) \mapsto \text{Gr}^1_S (g) := (g \times I_S : (X \times S, q^* O_X \otimes \pi^* O_S), \tilde{X}) \to ((X' \times S, q^* O_X \otimes \pi^* O_S), \tilde{X}')
\]

the graph functor, \( X \to X \times S \) being the graph embedding (which is a closed embedding if \( X \) is separated), \( p : X \to S, q : X \times S \to X \) the projections, and again \( \text{Gr}^1_S : \text{RTop}^{2,pr} / S \to \text{RTop} / S \) the corresponding morphism of sites.

- We denote by \( \text{Sch}^2 \subset \text{RTop}^2 \) the full subcategory such that the first factors are schemes. For a field \( k \), we denote by \( \text{Sch}^2 / k := \text{Sch}^2 / (\text{Spec } k, \{ \text{pt} \}) \) and by

- \( \text{Var}(k)^2 \subset \text{Sch}^2 / k \) the full subcategory such that the first factors are algebraic varieties over \( k \), i.e. schemes of finite type over \( k \),

- \( \text{PVar}(k)^2 \subset \text{QPVar}(k)^2 \subset \text{Var}(k)^2 \) the full subcategories such that the first factors are quasi-projective varieties and projective varieties respectively,
– \text{PSmVar}(k)^2 \subset \text{SmVar}(k)^2 \subset \text{Var}(k)^2 \text{ the full subcategories such that the first factors are smooth varieties and smooth projective varieties respectively.}

In particular we have, for \( S \in \text{Var}(k) \), the graph functor
\[
\text{Gr}^{12}_S : \text{Var}(k)/S \to \text{Var}(k)^2_{\text{pr}}/S, \ X/S \mapsto \text{Gr}^{12}_S(X/S) := (X \times S, X)/S,
\]
\[
(g : X/S \to X'/S) \mapsto \text{Gr}^{12}_S(g) := (g \times I_S : (X \times S, X) \to (X' \times S, X'))
\]
the graph embedding \( X \hookrightarrow X \times S \) is a closed embedding since \( X \) is separated in the subcategory of schemes \( \text{Sch} \subset \text{RTop} \), and again \( \text{Gr}^{12}_S : \text{Var}(k)^2_{\text{pr}}/S \to \text{Var}(k)/S \) the corresponding morphism of sites.

• Denote by \( \text{CW} \subset \text{Top} \) the full subcategory of \( \text{CW} \) complexes, by \( \text{CS} \subset \text{CW} \) the full subcategory of topological (real) manifolds which admits a \( \text{CW} \) structure (a topological manifold admits a \( \text{CW} \) structure if it admits a differential structure) and by \( \text{Diff}(\mathbb{R}) \subset \text{RTop} \) the full subcategory of differentiable (real) manifold. We denote by \( \text{CW}^2 \subset \text{Top}^2 \) the full subcategory such that the first factors are \( \text{CW} \) complexes, by \( \text{TM}(\mathbb{R})^2 \subset \text{CW}^2 \) the full subcategory such that the first factors are topological (real) manifolds and by \( \text{Diff}(\mathbb{R})^2 \subset \text{RTop}^2 \) the full subcategory such that the first factors are differentiable (real) manifold.

• Denote by \( \text{AnSp}(\mathbb{C}) \subset \text{RTop} \) the full subcategory of analytic spaces over \( \mathbb{C} \), and by \( \text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C}) \) the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). A morphism \( h : U \to S \) with \( U, S \in \text{AnSp}(\mathbb{C}) \) is said to be smooth if it is flat with smooth fibers. A morphism \( r : U \to X \) with \( U, X \in \text{AnSp}(\mathbb{C}) \) is said to be etale if it is non ramified and flat. By the Weirstrass preparation theorem (or the implicit function theorem if \( U \) and \( X \) are smooth), a morphism \( r : U \to X \) with \( U, X \in \text{AnSp}(\mathbb{C}) \) is etale if and only if it is an isomorphism local.

We denote by \( \text{AnSp}(\mathbb{C})^2 \subset \text{RTop}^2 \) the full subcategory such that the first factors are analytic spaces over \( \mathbb{C} \), and by \( \text{AnSm}(\mathbb{C})^2 \subset \text{AnSp}(\mathbb{C})^2 \) the full subcategory such that the first factors are smooth analytic spaces (i.e. complex analytic manifold). In particular we have, for \( S \in \text{AnSp}(\mathbb{C}) \), the graph functor
\[
\text{Gr}^{12}_S : \text{AnSp}(\mathbb{C})/S \to \text{AnSp}(\mathbb{C})^2_{\text{pr}}/S, \ X/S \mapsto \text{Gr}^{12}_S(X/S) := (X \times S, X)/S,
\]
\[
(g : X/S \to X'/S) \mapsto \text{Gr}^{12}_S(g) := (g \times I_S : (X \times S, X) \to (X' \times S, X'))
\]
the graph embedding \( X \hookrightarrow X \times S \) is a closed embedding since \( X \) is separated in \( \text{RTop} \), and again \( \text{Gr}^{12}_S : \text{AnSp}(\mathbb{C})^2_{\text{pr}}/S \to \text{AnSp}(\mathbb{C})/S \) the corresponding morphism of sites.

• For \( V \in \text{Var}(\mathbb{C}) \), we denote by \( V^{\text{an}} \in \text{AnSp}(\mathbb{C}) \) the complex analytic space associated to \( V \) with the usual topology induced by the usual topology of \( \mathbb{C}^N \). For \( W \in \text{AnSp}(\mathbb{C}) \), we denote by \( W^{\text{cw}} \in \text{AnSp}(\mathbb{C}) \) the topological space given by \( W \) which is a \( \text{CW} \) complex. For simplicity, for \( V \in \text{Var}(\mathbb{C}) \), we denote by \( V^{\text{cw}} := (V^{\text{an}})^{\text{cw}} \in \text{CW} \). We have then
  – the analytical functor \( \text{An} : \text{Var}(\mathbb{C}) \to \text{AnSp}(\mathbb{C}) \), \( \text{An}(V) = V^{\text{an}} \),
  – the forgetful functor \( \text{Cw} = \text{tp} : \text{AnSp}(\mathbb{C}) \to \text{CW} \), \( \text{Cw}(W) = W^{\text{cw}} \),
  – the composite of these two functors \( \text{Cw} = \text{Cw} \circ \text{An} : \text{Var}(\mathbb{C}) \to \text{CW} \), \( \text{Cw}(V) = V^{\text{cw}} \).

We have then
  – the analytical functor \( \text{An} : \text{Var}(\mathbb{C})^2 \to \text{AnSp}(\mathbb{C})^2 \), \( \text{An}((V,Z)) = (V^{\text{an}}, Z^{\text{an}}) \),
  – the forgetful functor \( \text{Cw} = \text{tp} : \text{AnSp}(\mathbb{C})^2 \to \text{CW}^2 \), \( \text{Cw}((W,Z)) = (W^{\text{cw}}, Z^{\text{cw}}) \),
  – the composite of these two functors \( \text{Cw} = \text{Cw} \circ \text{An} : \text{Var}(\mathbb{C})^2 \to \text{CW}^2 \), \( \text{Cw}((V,Z)) = (V^{\text{cw}}, Z^{\text{cw}}) \).
2.2 Additive categories, abelian categories and tensor triangulated categories

Let $\mathcal{A}$ an additive category.

- For $\phi : F^\bullet \to G^\bullet$ a morphism with $F^\bullet, G^\bullet \in C(\mathcal{A})$, we have the mapping cylinder $\text{Cyl}(\phi) := ((F^n \oplus F^{n+1} \oplus G^{n+1}, (\partial_F^n, \partial_F^{n+1}, \phi^{n+1} + \partial^n G) \in C(\mathcal{A})$. and the mapping cone $\text{Cone}(\phi) := ((F^n \oplus G^{n+1}, (\partial^n F, \phi^{n+1} + \partial^n G) \in C(\mathcal{A})$.

- The category $K(\mathcal{A}) := \text{Ho}(C(\mathcal{A}))$ is a triangulated category with distinguish triangles $F^\bullet \xrightarrow{i_{F^\bullet}} \text{Cyl}(\phi) \xrightarrow{\pi_{F^\bullet}} \text{Cone}(\phi) \xrightarrow{r_{F^\bullet}} F^\bullet[1]$.

- The category $(\mathcal{A}, F)$ is obviously again an additive category.

- Let $\phi : F^\bullet \to G^\bullet$ a morphism with $F^\bullet, G^\bullet \in C(\mathcal{A})$. Then it is obviously a morphism of filtered complex $\phi : (F^\bullet, F_b) \to (G^\bullet, F_b)$, where we recall that $F_b$ is the trivial filtration $(F^\bullet, F_b), (G^\bullet, F_b) \in C_{fil}(\mathcal{A})$.

We recall the following property of the internal hom functor if it exists of a tensor triangulated category and the definition of compact and cocompact object.

**Proposition 2.** Let $(\mathcal{T}, \otimes)$ a tensor triangulated category admitting countable direct sum and product compatible with the triangulation. Assume that $\mathcal{T}$ has an internal hom (bi)functor $\text{RHom}(...): \mathcal{T}^2 \to \mathcal{T}$ which is by definition the right adjoint to $(\cdot \otimes \cdot): \mathcal{T}^2 \to \mathcal{T}$. Then,

- for $N \in \mathcal{T}$, the functor $\text{RHom}(\cdot, N): \mathcal{T} \to \mathcal{T}$ commutes with homotopy colimits : for $M = \text{holim}_{i \in I} M_i$, where $I$ is a countable category, we have
  $$\text{RHom}(M, N) \xrightarrow{\sim} \text{ho lim}_{i \in I} \text{RHom}(M_i, N).$$

- dually, for $M \in \mathcal{T}$, the functor $\text{RHom}(M, \cdot): \mathcal{T} \to \mathcal{T}$ commutes with homotopy limits : for $N = \text{holim}_{i \in I} M_i$, where $I$ is a countable category, we have
  $$\text{RHom}(M, N) \xrightarrow{\sim} \text{ho lim}_{i \in I} \text{RHom}(M, M_i).$$

**Proof.** Standard. □

Let $(\mathcal{T}, \otimes)$ a tensor triangulated category admitting countable direct sum and product compatible with the triangulation. Assume that $\mathcal{T}$ has an internal hom functor $\text{RHom}(\cdot, \cdot): \mathcal{T} \to \mathcal{T}$.

- For $N \in \mathcal{T}$, the functor $\text{RHom}(\cdot, N): \mathcal{T} \to \mathcal{T}$ does not commutes in general with homotopy limits : for $M = \text{holim}_{i \in I} M_i$, where $I$ is a countable category, the canonical map
  $$\text{ho lim}_{\rightarrow i \in I} \text{RHom}(M_i, N) \to \text{RHom}(M, N)$$

is not an isomorphism in general if $I$ is infinite. It commutes if and only if $N$ is compact.

- Dually, for $M \in \mathcal{T}$, the functor $\text{RHom}(M, \cdot): \mathcal{T} \to \mathcal{T}$ does not commutes in general with infinite homotopy colimits. It commutes if and only if $M$ is cocompact.

Most triangulated category comes from the localization of the category of complexes of an abelian category with respect to quasi-isomorphisms. In the case where the abelian category have enough injective or projective object, the triangulated category is the homotopy category of the complexes of injective, resp. projective, objects.

**Proposition 3.** Let $\mathcal{A}$ an abelian category with enough injective and projective.
A quasi-isomorphism \( \phi : Q^\bullet \to F^\bullet \), with \( F^\bullet, Q^\bullet \in C^{-}(A) \) such that the \( Q^n \) are projective is an homotopy equivalence.

Dually, a quasi-isomorphism \( \phi : F^\bullet \to I^\bullet \), with \( F^\bullet, I^\bullet \in C^+(A) \) such that the \( I^n \) are projective is an homotopy equivalence.

**Proof.** Standard. \( \square \)

**Proposition 4.** Let \( A \) be an abelian category with enough injective and projective satisfying AB3 (i.e. countable direct sum of exact sequences are exact sequence).

- Let \( K(P) \subset K(A) \) be the thick subcategory generated by (unbounded) complexes of projective objects. Then, \( K(P) \hookrightarrow K(A) \xrightarrow{D} D(A) \) is an equivalence of triangulated categories.
- Similarly, let \( K(I) \subset K(A) \) be the thick subcategory generated by (unbounded) complexes of injective objects. Then \( K(I) \hookrightarrow K(A) \xrightarrow{D} D(A) \) is an equivalence of triangulated categories.

**Proof.** It follows from proposition 3 : see [23]. \( \square \)

Let \( A \subset \text{Cat} \) be an abelian category. Let \( \phi : (M, F) \to (N, F) \) a morphism with \( (M, F), (N, F) \in \text{C}_\text{fil}(A) \). Then the distinguish triangle

\[
(M, F) \xrightarrow{\phi} (N, F) \xrightarrow{1} \text{Cone}(\phi) = ((M, F)[1] \oplus (N, F), (d, d' - \phi) \xrightarrow{p_1} (M, F)[1])
\]
gives a sequence

\[
\cdots \to H^n(M, F) \xrightarrow{H^n(\phi)} H^n(N, F) \xrightarrow{H^n(1)} H^n(\text{Cone}(\phi)) \xrightarrow{H^n(p_1)} H^{n+1}(M, F) \to \cdots
\]
which, if we forgot filtration is a long exact sequence in \( A \) : however the morphism are NOT strict in general.

### 2.3 Presheaves on a site and on a ringed topos

#### 2.3.1 Functorialities

Let \( S \subset \text{Cat} \) be a small category. For \( X \in S \) we denote by \( Z(X) \in \text{PSh}(S) \) the presheaf represented by \( X \). By Yoneda lemma, a representable presheaf \( Z(X) \) is projective.

**Proposition 5.**  
- Let \( S \subset \text{Cat} \) be a small category. The projective presheaves \( \text{Proj}(\text{PSh}(S)) \subset \text{PSh}(S) \) are the direct summand of the representable presheaves \( Z(X) \) with \( X \in S \).
- More generally let \( (S, O_S) \subset \text{RCat} \) a ringed topos. The projective presheaves \( \text{Proj}(\text{PSh}_{O_S}(S)) \subset \text{PSh}_{O_S}(S) \) of \( O_S \) modules are the direct summand of the representable presheaves \( Z(X) \otimes O_S \) with \( X \in S \).

**Proof.** Standard. \( \square \)

Let \( f : T \to S \) a morphism of presite with \( T, S \subset \text{Cat} \). For \( h : U \to S \) a morphism with \( U, S \in S \), we have \( f^*Z(U/S) = Z(P(f)(U/S)) \).

We will consider in this article filtered complexes of presheaves on a site. Let \( f : T \to S \) a morphism of presite with \( T, S \subset \text{Cat} \).

- The functor \( f_* : C(T) \to C(S) \) gives, by functoriality, the functor

\[
f_* : C(\text{fil}(T)) \to C(\text{fil}(S)), \ (G, F) \mapsto f_*(G, F) := (f_*G, f_*F),
\]

since \( f_* \) preserves monomorphisms.
• The functor \( f^* : C(S) \to C(T) \) gives, by functoriality, the functor
\[
f^* : C_{fil}(S) \to C_{fil}(T), \quad (G,F) \mapsto f^*(G,F), \quad F^p(f^*(G,F)) := \text{Im}(f^*F^pG \to f^*G).
\]
In the particular case where \( f^* : \text{PSh}(S) \to \text{PSh}(T) \) preserves monomorphisms, we have \( f^*(G,F) = (f^*G, f^*F) \).

• The functor \( f^\perp : C(S) \to C(T) \) gives, by functoriality, the functor
\[
f^\perp : C_{fil}(T) \to C_{fil}(S), \quad (G,F) \mapsto f^\perp(G,F) := (f^\perp G, f^\perp F),
\]
since \( f^\perp : C(S) \to C(T) \) preserves monomorphisms.

Let \( f : T \to S \) a morphism of presite with \( T, S \in \text{Cat} \).

• The adjunction \((f^*, f_*) = (f^{-1}, f_*) : C(S) \leftrightarrows C(T)\), gives an adjunction
\[
(f^*, f_*): C_{fil}(S) \leftrightarrows C_{fil}(T), \quad (G,F) \mapsto f^*(G,F), \quad (G,F) \mapsto f_*(G,F),
\]
with adjunction maps, for \((G_1, F) \in C_{fil}(S)\) and \((G_2, F) \in C_{fil}(T)\)
\[
ad(f^*, f_*)(G_1, F) : (G_1, F) \to f_*f^*(G_1, F), \quad \text{ad}(f^*, f_*)(G_2, F) : f^*f_*(G_2, F) \to (G_2, F).
\]

• The adjunction \((f_*, f^\perp) : C(S) \leftrightarrows C(T)\), gives an adjunction
\[
(f_*, f^\perp) : C_{fil}(T) \leftrightarrows C_{fil}(S), \quad (G,F) \mapsto f^\perp(G,F), \quad (G,F) \mapsto f_*(G,F),
\]
with adjunction maps, for \((G_1, F) \in C_{fil}(S)\) and \((G_2, F) \in C_{fil}(T)\)
\[
ad(f^*, f_*)(G_2, F) : (G_2, F) \to f^\perp f_*(G_2, F), \quad \text{ad}(f^*, f_*)(G_1, F) : f^\perp f_*(G_1, F) \to (G_1, F).
\]

Remark 1. Let \( T, S \in \text{Cat} \) small categories and \( f : T \to S \) a morphism of presite. Then the functor \( f^* : \text{PSh}(S) \to \text{PSh}(T) \) preserve epimorphism but does NOT preserve monomorphism in general (the colimits involved are NOT filtered colimits). However it preserve monomorphism between projective presheaves by Yoneda and we thus set for \((Q, F) \in C_{fil}(\text{Proj}(\text{PSh}(S)))\), that is \( F^pQ^n \in \text{Proj}(\text{PSh}(S)) \) for all \( p, n \in \mathbb{Z} \), \( f^*(Q, F) := (f^*Q, f^*F) \).

For a commutative diagram of presite :
\[
D = T' \xrightarrow{g_2} S', \quad \xymatrix{ T \ar[r]^{g_1} \ar[dr]_{f_2} & S' \ar[dl]^{f_1} \\ & S }
\]
with \( T, T', S, S' \in \text{Cat} \), we denote by, for \( F \in C(S')\),
\[
T(D)(F) : g_{1}^*f_1^*F \xrightarrow{g_{1}^*f_1^*\text{ad}(g_{2}^*g_{2}^*)(F)} g_{1}^*f_1^*g_2^*F = g_{1}^*g_1^*f_2^*F \xrightarrow{\text{ad}(g_{1}^*g_1^*)(f_2^*g_2^*)} f_2^*g_2^*F.
\]
the canonical transformation map in \( C(T) \), and for \((G,F) \in C_{fil}(S')\),
\[
T(D)(G,F) : g_{1}^*f_1^*(G,F) \xrightarrow{g_{1}^*f_1^*\text{ad}(g_{2}^*g_{2}^*)(G,F)} g_{1}^*f_1^*g_2^*(G,F) = g_{1}^*g_1^*f_2^*(G,F) \xrightarrow{\text{ad}(g_{1}^*g_1^*)(f_2^*g_2^*(G,F))} f_2^*g_2^*(G,F).
\]
the canonical transformation map in \( C_{fil}(T) \) given by the adjunction maps.

We will use the internal hom functor and the tensor product for presheaves on a site or for presheaves of modules on a ringed topos. We recall the definition in the filtered case.
Let \((S, O_S) \in \mathbf{RCat}\). We have the tensor product bifunctor
\[
(\cdot \otimes \cdot) : \mathbf{PSh}(S)^2 \rightarrow \mathbf{PSh}(S), (F, G) \mapsto (X \in S \mapsto (F \otimes G)(X) := F(X) \otimes G(X)
\]
It induces a bifunctor:
\[
(\cdot \otimes \cdot) : C(S) \times C(S) \rightarrow C(S), (F, G) \mapsto F \otimes G := \text{Tot}(F^\bullet \otimes G^\bullet), (F \otimes G)^n = \bigoplus_{r \in \mathbb{Z}} F^r \otimes G^{n-r}
\]
and a bifunctor
\[
(\cdot \otimes \cdot) : C(S) \times C_{O_S}(S) \rightarrow C_{O_S}(S), \alpha.(F \otimes G) := F \otimes (\alpha.G)
\]
For \((G_1, F), (G_2, F) \in C^f_{fil}(S), G_3 \in C(S)\), we define (note that tensor product preserve monomorphism only after tensoring with \(Q_S \in \mathbf{PSh}(S)\))
\[
\begin{align*}
F^p((G_1, F) \otimes G_3) &:= \text{Im}(F^pG_1 \otimes G_3 \rightarrow G_1 \otimes G_3) \text{ and } F^p(G_3 \otimes (G_1, F)) := \text{Im}(G_3 \otimes F^pG_1 \rightarrow G_3 \otimes G_1), \\
F^pF^q((G_1, F) \otimes (G_2, F)) &:= \text{Im}(F^pG_1 \otimes F^qG_2 \rightarrow G_1 \otimes G_2) \text{ and }
F^k((G_1, F) \otimes (G_2, F)) := F^k \text{Tot}_{FF}((G_1, F) \otimes (G_2, F)) := \bigoplus_{p \in \mathbb{Z}} \text{Im}(F^pG_1 \otimes F^k-qG_2 \rightarrow G_1 \otimes G_2)
\end{align*}
\]
Note that in the case where \(G^n_1 = 0\) for \(n < 0\), we have \((G_1, F_b) \otimes (G_2, F) = G_1 \otimes (G_2, F)\). We get the bifunctors
\[
(\cdot \otimes (\cdot)) : C_{fil}(S)^2 \rightarrow C_{fil}(S), (\cdot \otimes (\cdot)) : C_{fil}(S) \times C_{O_Sfil}(S) \rightarrow C_{O_Sfil}(S).
\]
We have the tensor product bifunctor
\[
(\cdot \otimes_{O_S} \cdot) : \mathbf{PSh}_{O_S}(S)^2 \rightarrow \mathbf{PSh}(S), (F, G) \mapsto (X \in S \mapsto (F \otimes_{O_S} G)(X) := F(X) \otimes_{O_S(X)} G(X)
\]
It induces a bifunctor:
\[
(\cdot \otimes_{O_S} \cdot) : C_{O_S}(S) \times C_{O_S}(S) \rightarrow C(S), (F, G) \mapsto F \otimes_{O_S} G := \text{Tot}(F^\bullet \otimes_{O_S} G^\bullet)
\]
For \((G_1, F), (G_2, F) \in C_{O_Sfil}(S), G_3 \in C_{O_S}(S)\), we define similarly \((G_1, F) \otimes_{O_S} G_3, G_3 \otimes_{O_S} (G_1, F)\), and
\[
F^k((G_1, F) \otimes_{O_S} G_2, G_3) := F^k \text{Tot}_{FF}((G_1, F) \otimes_{O_S} G_2, G_3) := \bigoplus_{p \in \mathbb{Z}} \text{Im}(F^pG_1 \otimes F^k-qG_2 \rightarrow G_1 \otimes_{O_S} G_2)
\]
Note that in the case where \(G^n_1 = 0\) for \(n < 0\), we have \((G_1, F_b) \otimes_{O_S} (G_2, F) = G_1 \otimes_{O_S} (G_2, F)\). This gives
- in all case it gives the bifunctor \((-) \otimes_{O_S} (-) : C_{O_Sfil}(S) \otimes C_{O_Sfil}(S) \rightarrow C_{fil}(S)\).
- in the case \(O_S\) is commutative, it gives the bifunctor \((-) \otimes_{O_S} (-) : C_{O_Sfil}(S)^2 \rightarrow C_{O_Sfil}(S)\).

Let \((S, O_S) \in \mathbf{RCat}\). We have the internal hom bifunctor
\[
\mathcal{H}om(\cdot, \cdot) : \mathbf{PSh}(S)^2 \rightarrow \mathbf{PSh}(S),
\]
\[
(F, G) \mapsto (X \in S \mapsto \mathcal{H}om(F, G)(X) := \text{Hom}(r(X)_*, F, r(X)_*, G)
\]
with \(r(X) : S \rightarrow S/X\) (see subsection 2.1). It induces a bifunctors:
\[
\mathcal{H}om(\cdot, \cdot) : C(S) \times C(S) \rightarrow C(S), (F, G) \mapsto \mathcal{H}om^*(F, G)
\]
and a bifunctor
\[
\mathcal{H}om(\cdot, \cdot) : C(S) \times C_{O_S}(S) \rightarrow C_{O_S}(S), \alpha.\mathcal{H}om(F, G) := \mathcal{H}om(F, \alpha.G)
\]
For \((G_1, F), (G_2, F) \in C_{fil}(S), G_3 \in C(S)\), we define
Let $\phi$ be the canonical map.

$M$ is the adjunction between the restriction of scalars and the extension of scalars.

$F$ is the classical evaluation map.

We get the bifunctors

$$\mathcal{H}om((\cdot, \cdot) : C_\text{fil}(S) \times C_\text{fil}(S) \rightarrow C_\text{fil}(S), \quad \mathcal{H}om(\cdot, \cdot) : C_\text{fil}(S) \times C_\text{O}\_S^\text{fil}(S) \rightarrow C_\text{O}\_S^\text{fil}(S).$$

We have the internal hom bifunctor

$$\mathcal{H}om_{O_S}(\cdot, \cdot) : PSh_{O_S}(S) \times PSh_{O_S}(S) \rightarrow PSh(S)$$

$$(F, G) \mapsto (X \in S \mapsto \mathcal{H}om_{O_S}(F, G)(X) := \mathcal{H}om_{O_S}(r(X)_*, F, r(X)_* G).$$

It gives similarly

- in all cases a bifunctor $\mathcal{H}om_{O_S}(\cdot, \cdot) : C_\text{fil}O_S(S) \times C_\text{fil}O_S(S) \rightarrow C_\text{fil}(S),$
- the case $O_S$ is commutative, a bifunctor $\mathcal{H}om_{O_S}(\cdot, \cdot) : C_\text{fil}O_S(S) \times C_\text{fil}O_S(S) \rightarrow C_\text{O}_S^\text{fil}(S).$

Let $\phi : A \rightarrow B$ of rings.

- Let $M$ be a $A$ module. We say that $M$ admits a $B$ module structure if there exists a structure of $B$ module on the abelian group $M$ which is compatible with $\phi$ together with the $A$ module structure on $M.$

- For $N_1$ a $A$-module and $N_2$ a $B$ module. $I(A/B)(N_1, N_2) : \text{Hom}_A(N_1, N_2) \quad \mapsto \quad \text{Hom}_B(N_1 \otimes_A B, N_2)$ is the adjunction between the restriction of scalars and the extension of scalars.

- For $N', N''$ a $A$-modules, $ev_A(\text{hom}(\otimes))(N', N'', B) : \text{Hom}_A(N', N'') \otimes_A B \rightarrow \text{Hom}_A(N', N'' \otimes_A B),$ is the evaluation classical map.

Let $\phi : (S, O_1) \rightarrow (S, O_2)$ a morphism of presheaves of ring on $S \in \text{Cat}.$

- Let $M \in PSh_{O_1}(S).$ We say that $M$ admits an $O_2$ module structure if there exits a structure of $O_2$ module on $M \in PSh(S)$ which is compatible with $\phi$ together with the $O_1$ module structure on $M.$

- For $N_1 \in C_{O_1}(S)$ and $N_2 \in C_{O_2}(S),$

$$I(O_1/O_2)(N_1, N_2) : \text{Hom}_{O_1}(N_1, N_2) \quad \mapsto \quad \text{Hom}_{O_2}(N_1 \otimes_{O_1} B, N_2)$$

is the adjunction between the restriction of scalars and the extension of scalars.

- For $N', N'' \in C_{O_1}(S),$

$$ev_{O_1}(\text{hom}(\otimes))(N', N'', O_2) : \text{Hom}_{O_1}(N', N'') \otimes_{O_1} O_2 \rightarrow \text{Hom}_{O_1}(N', N'' \otimes_{O_1} O_2).$$

is the classical evaluation map.

Let $(S, O_S) \in \text{RCat}.$

- For $F_1, F_2, G_1, G_2 \in C(S),$ we denote by

$$T(\otimes, \mathcal{H}om)(F_1, F_2, G_1, G_2) : \mathcal{H}om(F_1, G_1) \otimes \mathcal{H}om(F_2, G_2) \rightarrow \mathcal{H}om(F_1 \otimes F_2, G_1 \otimes G_2)$$

the canonical map.

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• For $G_1, G_2 \in C(S)$, we denote by
\[
ev(hom, \otimes)(G_1, G_2) : H\text{om}(G_1, G_2) \otimes_{O_S} G_2 \to H\text{om}(G_3, G_1 \otimes_{O_S} G_2)
\phi \otimes s \mapsto (s' \mapsto \phi(s') \otimes s)
\]

• Let $S \in \text{Cat}$ a small category. Let $(H_X : C(S/X) \to C(S/X))_{X \in S}$ a family of functors which is functorial in $X$. We have by definition, for $F_1, F_2 \in C(S)$, the canonical transformation map
\[T(H, hom)(F_1, F_2) : H(\text{Hom}^\bullet(F_1, F_2)) \to \text{Hom}^\bullet(H(F_1), H(F_2)), \tag{1}\]
\[\phi \in \text{Hom}(F_{1,X}, F_{2,X}) \mapsto H^\bullet(F_{1,X}, F_{2,X}) (\phi) \in \text{Hom}(H(F_{1,X}), H(F_{2,X})) \tag{2}\]
in $C(S)$.

Let $T, S \in \text{Cat}$ small categories and $f : T \to S$ a morphism of presite.

• For $F_1, F_2 \in C(T)$ we have by definition $f_*(F_1 \otimes F_2) = f_!F_1 \otimes f_!F_2$. For $G_1, G_2 \in C(S)$, we have a canonical isomorphism $f^*G_1 \otimes f^*G_2 \simeq f^*(G_1 \otimes G_2)$ since the tensor product is a right exact functor, and a canonical map $f^!G_1 \otimes f^!G_2 \to f^!(G_1 \otimes G_2)$.

• We have for $F \in C(S)$ and $G \in C(T)$ the adjunction isomorphism,
\[I(f^*, f_*)(F, G) : f_*\text{Hom}^\bullet(f^*F, G) \simeq \text{Hom}^\bullet(F, f_*G). \tag{3}\]

• Let $O_S \in \text{PSh}(S, \text{Ring})$ by a presheaf of ring so that $(S, O_S), (T, f^*O_S) \in \text{RCat}$. We have for $F \in C_{O_S}(S)$ and $G \in C_{f^*O_S}(T)$ the adjunction isomorphism,
\[I(f^*, f_*)(F, G) : f_*\text{Hom}_{f^*O_S}(f^*F, G) \simeq \text{Hom}_{O_S}(F, f_*G), \tag{4}\]
and
- the map $\text{ad}(f^*, f_*)(F) : F \to f_*f^*F$ in $C(S)$ is $O_S$ linear, that is is a map in $C_{O_S}(S)$,
- the map $\text{ad}(f^*, f_*)(G) : f^*f_*G \to G$ in $C(T)$ is $f^*O_S$ linear, that is is a map in $C_{f^*O_S}(T)$.

• For $F_1, F_2 \in C(T)$, we have the canonical map
\[T_*(f, hom)(F_1, F_2) := T(f_*, hom) : f_*\text{Hom}^\bullet(F_1, F_2) \to \text{Hom}^\bullet(f_!F_1, f_!F_2), \tag{5}\]
for $X \in S$, $\phi \in \text{Hom}(F_{1,X}, F_{2,X}) \mapsto f_{1,X}^!(f^*F_{1,X}) \cdot f_{2,X}^!(f^*F_{2,X}) (\phi) \in \text{Hom}(f_{1,X}^!f^*(X), f_{2,X}^!f^*(X)) \tag{6}\]
given by evaluation.

• For $G_1, G_2 \in C(S)$, we have the following canonical transformation in $C(T)$
\[T(f, hom)(G_1, G_2) := T(f^*, hom)(G_1, G_2) : \tag{7}\]
\[f^*\text{Hom}^\bullet(G_1, G_2) \xrightarrow{f^!\text{Hom}^\bullet(G_1, G_2) \cdot f^!f_*(G_1, G_2)} f^*\text{Hom}^\bullet(G_1, f_*f^*G_2) \xrightarrow{f^!f_*(G_1, G_2)} \text{Hom}^\bullet(f_!G_1, f_!f^*G_2), \tag{8}\]
\[f^*f_*\text{Hom}^\bullet(G_1, G_2) \xrightarrow{\text{ad}(f^*F_1) \cdot \text{Hom}^\bullet(G_1, G_2)} \text{Hom}^\bullet(f_!G_1, f_!f^*G_2), \tag{9}\]

• Let $O_S \in \text{PSh}(S, \text{Ring})$ by a presheaf of ring so that $(S, O_S), (T, f^*O_S) \in \text{RCat}$. For $G_1, G_2 \in C_{O_S}(S)$, we have the following canonical transformation in $C_{f^*O_S}(T)$
\[T(f, hom)(G_1, G_2) := T(f^*, hom)(G_1, G_2) : \tag{10}\]
\[f^*\text{Hom}_{O_S}^\bullet(G_1, G_2) \xrightarrow{f^!\text{Hom}_{O_S}^\bullet(G_1, G_2) \cdot f^!f_*(G_1, G_2)} f^*\text{Hom}_{O_S}^\bullet(G_1, f_*f^*G_2) \xrightarrow{f^!f_*(G_1, G_2)} \text{Hom}_{f^*O_S}^\bullet(f_!G_1, f_!f^*G_2), \tag{11}\]
\[f^*f_*\text{Hom}_{f^*O_S}^\bullet(G_1, G_2) \xrightarrow{\text{ad}(f^*F_1) \cdot \text{Hom}^\bullet(G_1, G_2)} \text{Hom}_{f^*O_S}^\bullet(f_!G_1, f_!f^*G_2), \tag{12}\]
Let $O_S \in \text{PSh}(S, \text{Ring})$ by a presheaf of ring so that $(S, O_S), (T, f^* O_S) \in \text{RCat}$. For $M \in C_{O_S}(S)$ and $N \in C_{T,O_S}(T)$, we denote by

$$T(f, \otimes)(M, N) : M \otimes_{O_S} f_* N \xrightarrow{\text{ad}(f^*, f_*)(M \otimes_{O_S} f_* N)} f_*(f^* M \otimes_{O_S} f_* N)$$

$$f_* f^*(M \otimes_{O_S} f_* N) = f_*(f^* M \otimes_{T} f_* O_S)$$

the canonical transformation map.

Let $f : (T, O_T) \to (S, O_S)$ a morphism with $(S, O_S), (T, O_T) \in \text{RCat}$. We have the adjonction

$$(f^{* \text{mod}}, f_*) : C_{O_S}(S) \rightleftarrows \text{RCat}.$$ 

with $f^{* \text{mod}}G := f^* G \otimes_{T} O_T$. If $f^* : C(S) \to C(T)$ preserve monomorphisms, it induces the adjonction

$$(f^{* \text{mod}}, f_*) : C_{O_S} \rightarrow \text{RCat}.$$ 

For a commutative diagram in $\text{RCat}$:

$$D = (T', O'_2) \xrightarrow{g_2} (S', O'_1),$$ 

$$\downarrow f_2 \quad \downarrow f_1$$

$$(T, O_2) \xrightarrow{g_1} (S, O_1)$$

we denote by, for $F \in C_{O'_1}(S')$

$$T^{* \text{mod}}(D)(F) : g_1^{* \text{mod}} f_1, F \xrightarrow{g_2^{* \text{mod}} f_1, \text{ad}(g_2, g_2^*)} g_1^{* \text{mod}} f_1, g_2^{* \text{mod}} F = g_1^{* \text{mod}} f_1, g_2^{* \text{mod}} F$$

the canonical transformation map in $C_{O_2}(T)$ and, for $(G, F) \in C_{O'_1}(S')$

$$T^{* \text{mod}}(D)(G, F) : g_1^{* \text{mod}} f_1, (G, F) \xrightarrow{g_2^{* \text{mod}} f_1, \text{ad}(g_2, g_2^*)} g_1^{* \text{mod}} f_1, g_2^{* \text{mod}} (G, F) = g_1^{* \text{mod}} f_1, g_2^{* \text{mod}} (G, F)$$

the canonical transformation map in $C_{O_2} \rightarrow \text{RCat}$ given by the adjonction maps.

Let $f : (T, O_T) \to (S, O_S)$ a morphism with $(S, O_S), (T, O_T) \in \text{RCat}$.

- We have, for $M, N \in C_{O_S}(S)$ the canonical transformation map in $C_{O_T}(T)$

$$T^{* \text{mod}}(f, \text{hom})(M, N) : f^{* \text{mod}} \text{Hom}_O(M, N) \xrightarrow{T(f, \text{hom})(M, N) \otimes f_! O_2} \text{Hom}_O(f^* M, f^* N) \otimes f_! O_2$$

- We have, for $M \in C_{O_S}(S)$ and $N \in C_{O_T}(T)$, the canonical transformation map in $C_{O_T}(T)$

$$T^{* \text{mod}}(f, \otimes)(M, N) : M \otimes_{O_S} f_* N \xrightarrow{\text{ad}(f^{* \text{mod}}, f_*)(M \otimes_{O_S} f_* N)} f_*(f^{* \text{mod}} M \otimes_{O_T} N)$$

$$f_* f^{* \text{mod}}(M \otimes_{O_S} f_* N) = f_*(f^{* \text{mod}} M \otimes_{O_T} f^{* \text{mod}} f_* N) \xrightarrow{\text{ad}(f^{* \text{mod}}, f_*)(N)} f_*(f^{* \text{mod}} M \otimes_{O_T} N)$$

the canonical transformation map.
We now give some properties of the tensor product functor and hom functor given above

**Proposition 6.** Let $(S, O_S) \in R\text{Cat}$. Then, the functors

- $(-) \otimes (-) : C(S)^2 \to C(S)$, $C(S) \times C_{O_S}(S) \to C_{O_S}(S)$
- $(-) \otimes_{O_S} (-) : C_{O_S}^\text{proj}(S) \times C_{O_S}(S) \to C(S)$ and in case $O_S$ is commutative $(-) \otimes_{O_S} (-) : C_{O_S}(S)^2 \to C_{O_S}(S)$

are left Quillen functor for the projective model structure. In particular,

- for $L \in C(S)$ is such that $L^n \in \text{PSh}(S)$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \to G$ is a quasi-isomorphism with $F, G \in C(S)$, then $\phi \otimes I : F \otimes L \to G \otimes L$ is a quasi-isomorphism,
- for $L \in C_{O_S}(S)$ is such that $L^n \in \text{PSh}_{O_S}(S)$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \to G$ is a quasi-isomorphism with $F, G \in C_{O_S}(S)$, then $\phi \otimes I : F \otimes_{O_S} L \to G \otimes_{O_S} L$ is a quasi-isomorphism.

**Proof.** Standard. □

**Proposition 7.** Let $(S, O_S) \in R\text{Cat}$. Then, the functors

- $\text{Hom}(\cdot, \cdot) : C(S) \times C_{O_S}(S) \to C_{O_S}(S)$, $C(S) \times C_{O_S}(S) \to C_{O_S}(S)$,
- $\text{Hom}_{O_S}(\cdot, \cdot) : C_{O_S}^\text{proj}(S) \times C_{O_S}(S) \to C(S)$ and in the case $O_S$ is commutative $\text{Hom}_{O_S}(\cdot, \cdot) : C_{O_S}(S) \times C_{O_S}(S) \to C_{O_S}(S)$,

are on the left hand side left Quillen functor for the projective model structure. In particular,

- for $L \in C(S)$ is such that $L^n \in \text{PSh}(S)$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \to G$ is a quasi-isomorphism with $F, G \in C(S)$, then $\text{Hom}(L, \phi) : \text{Hom}^\bullet(L, F) \to \text{Hom}^\bullet(L, G)$ is a quasi-isomorphism,
- for $L \in C_{O_S}(S)$ is such that $L^n \in \text{PSh}_{O_S}(S)$ are projective for all $n \in \mathbb{Z}$, and $\phi : F \to G$ is a quasi-isomorphism with $F, G \in C_{O_S}(S)$, then $\text{Hom}_{O_S}(L, \phi) : \text{Hom}^\bullet_{O_S}(L, F) \to \text{Hom}^\bullet_{O_S}(L, G)$ is a quasi-isomorphism.

**Proof.** Standard. □

Let $S \in \text{Cat}$ a site endowed with topology $\tau$. Denote by $a_\tau : \text{PSh}(S) \to \text{Sh}(S)$ the sheafification functor. A morphism $\phi : F^\bullet \to G^\bullet$ with $F^\bullet, G^\bullet \in C(S)$) is said to be a $\tau$ local equivalence if

$$a_\tau H^n(\phi) : a_\tau H^n(F^\bullet) \to a_\tau H^n(G^\bullet)$$

is an isomorphism for all $n \in \mathbb{Z}$, where $a_\tau$ is the sheafification functor. Recall that $C_{fil}(S) \subset (C(S), F) = C(\text{PSh}(S), F)$ denotes the category of filtered complexes of abelian presheaves on $S$ whose filtration is biregular.

- A morphism $\phi : (F^\bullet, F) \to (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(S)$ is said to be a filtered $\tau$ local equivalence or an 1-filtered $\tau$ local equivalence if

  $$a_\tau H^n(\phi) : a_\tau H^n(\text{Gr}_F^p F^\bullet) \to a_\tau H^n(\text{Gr}_F^p G^\bullet)$$

  is an isomorphism for all $n, p \in \mathbb{Z}$.

- Let $r \in \mathbb{N}$. A morphism $\phi : (F^\bullet, F) \to (G^\bullet, F)$ with $(F^\bullet, F), (G^\bullet, F) \in C_{fil}(S)$ is said to be an $r$-filtered $\tau$ local equivalence if it belongs to the submonoid of arrows generated by filtered $\tau$ local equivalences and $r$-filtered homotopy equivalences, that is if there exists $\phi : (C_0^\bullet, F) \to (C_{i+1}^\bullet, F)$, $0 \leq i \leq s$, with $(C_i^\bullet, F) \in C_{fil}(S)$, $(C_0^\bullet, F) = (F^\bullet, F)$ and $(C_s^\bullet, F) = (G^\bullet, F)$, such that

  $$\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : (F^\bullet, F) \to (G^\bullet, F)$$
and \( \phi_i : (C_i^\bullet, F) \to (C_{i+1}^\bullet, F) \) either a filtered \( \tau \) local equivalence or an \( r \)-filtered homotopy equivalence. If \( \phi : (F^\bullet, F) \to (G^\bullet, F) \) with \((F^\bullet, F) \in C_{fil}(S)\) is an \( r \)-filtered \( \tau \) local equivalence, then for all \( p, q \in \mathbb{Z} \),

\[
  a_* E^{p,q}_r(\phi) : a_* E^{p,q}_r(F^\bullet, F) \xrightarrow{\sim} a_* E^{p,q}_r(G^\bullet, F)
\]

is an isomorphism but the converse is NOT true. Note that if \( \phi \) is an \( r \)-filtered \( \tau \) local equivalence, that it is an \( s \)-filtered \( \tau \) local equivalence for all \( s \geq r \).

- A morphism \( \phi : (F^\bullet, F) \to (G^\bullet, F) \) with \((F^\bullet, F), (G^\bullet, F) \in C_{O_{fil}}(S)\) is said to be a \( \infty \)-filtered \( \tau \) local equivalence if there exists \( r \in \mathbb{N} \) such that \( \phi \) is an \( r \)-filtered \( \tau \) local equivalence. If a morphism \( \phi : (F^\bullet, F) \to (G^\bullet, F) \) with \((F^\bullet, F), (G^\bullet, F) \in C_{fil}(S)\) is an \( \infty \)-filtered \( \tau \) local equivalence, then for all \( n \in \mathbb{Z} \),

\[
  a_* H^n(\phi) : a_* H^n(F^\bullet, F) \to a_* H^n(G^\bullet, F)
\]

is an isomorphism of filtered sheaves on \( S \). Recall the converse is true in the case there exist \( N_1, N_2 \in \mathbb{Z} \), such that \( H^n(F^\bullet, F) = H^n(G^\bullet, F) = 0 \) for \( n \leq N_1 \) or \( n \geq N_2 \).

Let \((S, O)\) a ringed topos where \( S \in \text{Cat} \) is a site endowed with topology \( \tau \). A morphism \( \phi : (F^\bullet, F) \to (G^\bullet, F) \) with \((F^\bullet, F), (G^\bullet, F) \in C_{O_{fil}}(S)\) is said to be a filtered \( \tau \) local equivalence or an \( 1 \)-filtered \( \tau \) local equivalence if \( \phi \) is \( 1 \)-filtered \( \tau \) local equivalence, i.e.

\[
  a_* H^n(\phi) : a_* H^n(Gr^p F) \xrightarrow{\sim} a_* H^n(Gr^p G)
\]

is an isomorphism for all \( n, p \in \mathbb{Z} \). Let \( r \in \mathbb{N} \). A morphism \( \phi : (F^\bullet, F) \to (G^\bullet, F) \) with \((F^\bullet, F), (G^\bullet, F) \in C_{O_{fil}}(S)\) is said to be an \( r \)-filtered \( \tau \) local equivalence if there exists \( \phi_i : (C_i^\bullet, F) \to (C_{i+1}^\bullet, F) \), \( 0 \leq i \leq s \), such that

\[
  \phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : (F^\bullet, F) \to (G^\bullet, F)
\]

and \( \phi_i : (C_i^\bullet, F) \to (C_{i+1}^\bullet, F) \) either a filtered \( \tau \) local equivalence or an \( r \)-filtered homotopy equivalence.

Let \( S \in \text{Cat} \) a site which admits fiber product, endowed with topology \( \tau \). A complex of presheaves \( F^\bullet \in C(S) \) is said to be \( \tau \) fibrant if it satisfy descent for covers in \( S \), i.e. if for all \( X \in S \) and all \( \tau \) covers \((c_i : U_i \to X)_{i \in I}\) of \( X \), denoting \( U_J := (U_{i_0} \times_S U_{i_1} \times_S \cdots U_{i_k})_{i_k \in J} \) and for \( I \subseteq J \), \( p_{IJ} : U_J \to U_I \) is the projection,

\[
  F^\bullet(c_i) : F^\bullet(X) \to \text{Tot}(\oplus_{\text{card } I = \bullet} F^\bullet(U_I))
\]

is a quasi-isomorphism of complexes of abelian groups.

- A complex of filtered presheaves \((F^\bullet, F) \in C_{fil}(S)\) is said to be \( \tau \) fibrant or \( 1 \)-filtered \( \tau \) fibrant if it satisfy descent for covers in \( S \), i.e. if for all \( X \in S \) and all \( \tau \) covers \((c_i : U_i \to X)_{i \in I}\) of \( X \),

\[
  (F^\bullet, F)(c_i) : (F^\bullet, F)(X) \to \text{Tot}(\oplus_{\text{card } I = \bullet} (F^\bullet, F)(U_I))
\]

is a filtered quasi-isomorphism of filtered complexes of abelian groups.

- Let \( r \in \mathbb{N} \). A complex of filtered presheaves \((F^\bullet, F) \in C_{fil}(S)\) is said to be \( r \)-filtered \( \tau \) fibrant if there exist an \( r \)-filtered homotopy equivalence \( m : (F^\bullet, F) \to (F^\bullet, F) \) with \((F^\bullet, F) \in C_{fil}(S)\) filtered \( \tau \) fibrant. If \((F^\bullet, F) \in C_{fil}(S)\) is \( r \)-filtered \( \tau \) fibrant, then for all \( X \in S \) and all \( \tau \) covers \((c_i : U_i \to X)_{i \in I}\) of \( X \),

\[
  E^{p,q}_r(F^\bullet, F)(c_i) : E^{p,q}_r(F^\bullet, F)(X) \to E^{p,q}_r(\text{Tot}(\oplus_{\text{card } I = \bullet} (F^\bullet, F)(U_I)))
\]

is an isomorphism for all \( n, p \in \mathbb{Z} \), but the converse is NOT true. Note that if \((F^\bullet, F) \) is \( r \)-filtered \( \tau \) fibrant, then it is \( s \)-filtered \( \tau \) fibrant for all \( s \geq r \).
• A complex of filtered presheaves \((F^\bullet, F) \in C_{fil}(S)\) is said to be \(\infty\)-filtered \(\tau\) fibrant if there exist \(r \in \mathbb{N}\) such that \((F^r, F)\) is \(r\)-filtered \(\tau\) fibrant. If a complex of filtered presheaves \((F^\bullet, F) \in C_{fil}(S)\) is \(\infty\)-filtered \(\tau\) fibrant, then for all \(X \in S\) and all \(\tau\) covers \((c_i : U_i \rightarrow X)_{i \in I}\) of \(X\),

\[ H^n(F^\bullet, F)(c_i) : H^n(F^\bullet, F)(X) \rightarrow H^n \text{Tot}(\oplus_{\text{card} I = \bullet}(F^\bullet, F)(U_I)) \]

is a filtered isomorphism for all \(n \in \mathbb{Z}\).

Let \((S, O)\) a ringed topos where \(S \in \text{Cat}\) is a site endowed with topology \(\tau\). Let \(r \in \mathbb{N}\).

• A complex of presheaves \(F^\bullet \in C_{O_S}(S)\) is said to be \(\tau\) fibrant if \(F^\bullet = oF^\bullet \in C(S)\) is \(\tau\) fibrant.

• A complex of presheaves \((F^\bullet, F) \in C_{O_S, fil}(S)\) is said to be filtered \(\tau\) fibrant if \((F^\bullet, F) = (oF^\bullet, F) \in C_{fil}(S)\) is filtered \(\tau\) fibrant.

• A complex of presheaves \((F^\bullet, F) \in C_{O_S, fil}(S)\) is said to be \(r\)-filtered \(\tau\) fibrant if there exist an \(r\)-filtered homotopy equivalence \(m : (F^\bullet, F) \rightarrow (F^\bullet, F)\) with \((F^\bullet, F) \in C_{O_S, fil}(S)\) filtered \(\tau\) fibrant.

### 2.3.2 Canonical flasque resolution of a presheaf on a site or a presheaf of module on a ringed topos

Let \(S \in \text{Cat}\) a site with topology \(\tau\). Denote \(a_{\tau} : \text{PSh}(S) \rightarrow \text{Shv}(S)\) the sheafification functor. There is for \(F \in C(S)\) an explicit \(\tau\) fibrant replacement :

- \(k : F^\bullet \rightarrow E^\bullet_{\tau}(F^\bullet) := \text{Tot}(E^\bullet_{\tau}(F^\bullet))\), if \(F^\bullet \in \mathcal{C}^+(S)\),
- \(k : F^\bullet \rightarrow E^\bullet_{\tau}(F^\bullet) := \text{holim } \text{Tot}(E^\bullet_{\tau}(F^{\ast \geq n}))\), if \(F^\bullet \in C(S)\) is not bounded below.

The bicomplex \(E^\bullet_{\tau}(F^\bullet) := E^\bullet(\tau^\ast F^\bullet)\) together with the map \(k : F^\bullet \rightarrow E^\bullet_{\tau}(F^\bullet)\) is given inductively by

- considering \(ps : S^\delta \rightarrow S\) the morphism of site from the discrete category \(S^\tau\) whose objects are the points of the topos \(S\) and we take

\[
  k_0 := \text{ad}(p_S^*, p_{S*})(F^\bullet) \rightarrow E^0(F^\bullet) := p_{S*} p_S^* F^\bullet := \bigoplus_{s \in S^\tau} \lim_{X \in S, s \in X} F^\bullet(X),
\]

then \(a_{\tau} k_0 : a_{\tau} F^\bullet \rightarrow E^0(F^\bullet)\) is injective and \(E^0(F^\bullet)\) is \(\tau\) fibrant,

- denote \(Q^0(F^\bullet) := a_{\tau} \text{coker}(k_0 : F^\bullet \rightarrow E^0(F^\bullet))\) and take the composite

\[
  E^0(F^\bullet) \rightarrow Q^0(F^\bullet) \rightarrow E^1(F^\bullet) := E^0(Q^0(F^\bullet)).
\]

Note that \(k : F^\bullet \rightarrow E^\bullet_{\tau}(F^\bullet)\) is a \(\tau\) local equivalence and that \(a_{\tau} k : a_{\tau} F^\bullet \rightarrow E^\bullet_{\tau}(F^\bullet)\) is injective by construction.

Since \(E^0\) is functorial, \(E\) is functorial: for \(m : F^\bullet \rightarrow G^\bullet\) a morphism, with \(F^\bullet, G^\bullet \in C(S)\), we have a canonical morphism \(E(m) : E(F) \rightarrow E(G)\) such that \(E(m) \circ k = k' \circ m\), with \(k : F \rightarrow E(F)\) and \(k' : G \rightarrow E(G)\). Note that \(E^0\), hence \(E\) preserve monomorphisms. This gives, for \((F^\bullet, F) \in C_{fil}(S)\), a filtered \(\tau\) local equivalence \(k : (F^\bullet, F) \rightarrow E^\bullet_{\tau}(F^\bullet, F)\) with \(E^\bullet_{\tau}(F^\bullet, F)\) filtered \(\tau\) fibrant.

Moreover, we have a canonical morphism \(E(F) \otimes E(G) \rightarrow E(F \otimes G)\).

There is, for \(g : T \rightarrow S\) a morphism of presite with \(T, S \in \text{Cat}\) two site, and \(F^\bullet \in C(S)\), a canonical transformation

\[
  T(g, E)(F^\bullet) : g^* E(F^\bullet) \rightarrow E(g^* F^\bullet)
\]

given inductively by

- \(T(g, E^0)(F) := T(g, p_S)(p_S^* F) : g^* E^0(F) = g^* p_{S*} p_S^* F \rightarrow p_{T*} g^* p_S^* F = p_{T*} p_T^* g^* F = E^0(g^* F),\)
- \(T(g, Q^0)(F) := T(g, E^0)(F) : g^* Q^0(F) = \text{coker}(g^* F \rightarrow g^* E^0(F)) \rightarrow Q^0(g^* F) = \text{coker}(g^* F \rightarrow E^0(g^* F))\)
2.3.3 Canonical projective resolution of a presheaf of modules on a ringed topos

Let \((\mathcal{S}, O_\mathcal{S})\) be a ringed topos with topology \(\tau\). Then, for \(F^\bullet \in C_{O_\mathcal{S}}(\mathcal{S})\), \(E_r(F^\bullet)\) is naturally a complex of \(O_\mathcal{S}\) modules such that \(k : F^\bullet \to E_r(F^\bullet)\) is \(O_\mathcal{S}\) linear, that is is a morphism in \(C_{O_\mathcal{S}}(\mathcal{S})\).

### Canonical projective resolution of a presheaf of module on a ringed topos

Let \((\mathcal{S}, O_\mathcal{S})\) be a ringed topos. We recall that we denote by, for \(U \in \mathcal{S}, Z(U) \in \text{PSh}(\mathcal{S})\) the presheaf represented by \(U\) : for \(V \in \mathcal{S}\) \(Z(U)(V) = \text{Hom}(V, U)\), and for \(h : V_1 \to V_2\) a morphism in \(\mathcal{S}\), and \(h_1 : V_1 \to U\) \(Z(U)(h) : h_1 \to h \circ h_1\), and \(s\) is the morphism of presheaf given by \(s(V_1)(h_1) = F(h_1)(s) \in F(V_1)\). There is for \(F \in C_{O_\mathcal{S}}(\mathcal{S})\) a complex of \(O_\mathcal{S}\) module an explicit projective replacement:

- \(q : L^0(D) := \text{Tot}(L^0(D)) \to F^\bullet\), if \(F^\bullet \in C^-\mathcal{S}\),
- \(q : L^0(D) := \text{holim}(L^0(D))\) if \(F^\bullet \in C\mathcal{S}\) is not bounded above.

For \(O_\mathcal{S} = \mathbb{Z}\), we denote \(L^0_\mathcal{S}(F) := L(F)\). The bicomplex \(L^0_\mathcal{S}(F)\) together with the map \(q : L^0_\mathcal{S}(F) \to F^\bullet\) is given inductively by:

- considering the pairs \(\{U \in \mathcal{S}, s \in F(U)\}\), where \(U\) is an object of \(\mathcal{S}\) and \(s\) a section of \(F\) over \(U\) we take

\[
q_0 : L^0_\mathcal{S}(F) := \bigoplus_{(U \in \mathcal{S}, s \in F(U))} \mathbb{Z}(U) \otimes O_\mathcal{S} \twoheadrightarrow F,
\]

then \(q_0\) is surjective and \(L^0_\mathcal{S}(F)\) is projective, this construction is functorial : for \(m : F \to G\) a morphism in \(\text{PSh}(\mathcal{S})\) the following diagram commutes

\[
\begin{array}{ccc}
\bigoplus_{(U \in \mathcal{S}, s \in F(U))} \mathbb{Z}(U) \otimes O_\mathcal{S} & \xrightarrow{q_0} & F \\
\| & & m \downarrow \\
\bigoplus_{(U \in \mathcal{S}, s \in G(U))} \mathbb{Z}(U) \otimes O_\mathcal{S} & \xrightarrow{q_0} & G
\end{array}
\]

where \((L_\mathcal{S}(m))_{(U, s)}(U, m(U)(s)) = I_{\mathbb{Z}(U)}\) and \((L_\mathcal{S}(m))_{(U, s)}(U, s') = 0\) if \(s' \neq m(U)(s)\),
- denote \(K^0_\mathcal{S}(F) := \ker(q_0 : L^0_\mathcal{S}(F) \to F)\) and take the composite

\[
q_1 : L^0_\mathcal{S}(F^\bullet) := L^0_\mathcal{S}(K^0_\mathcal{S}(F^\bullet)) \xrightarrow{q_0(K^0_\mathcal{S}(F^\bullet))} K^0_\mathcal{S}(F^\bullet) \xrightarrow{m} L^0_\mathcal{S}(F^\bullet).
\]

Note that \(q = q(F) : L(F^\bullet) \to F^\bullet\) is a surjective quasi-isomorphism by construction. Since \(L^0_\mathcal{S}\) is functorial, \(L_{O_\mathcal{S}}\) is functorial : for \(m : F^\bullet \to G^\bullet\) a morphism, with \(F^\bullet, G^\bullet \in C(\mathcal{S})\), we have a canonical morphism \(L_{O_\mathcal{S}}(m) : L_{O_\mathcal{S}}(F) \to L_{O_\mathcal{S}}(G)\) such that \(q' \circ L_{O_\mathcal{S}}(m) = m \circ q'\), with \(q : L_{O_\mathcal{S}}(F) \to F\) and \(q' : L_{O_\mathcal{S}}(G) \to G\). Note that \(L^0_\mathcal{S}\) and hence \(L_{O_\mathcal{S}}\) preserve monomorphisms. In particular, it gives for \((F^\bullet, F) \in C_{O_\mathcal{S}}(\mathcal{S})\), a filtered quasi-isomorphism \(q : L_{O_\mathcal{S}}(F^\bullet, F) \to (F^\bullet, F)\). Moreover, we have a canonical morphism \(L_{O_\mathcal{S}}(F) \otimes L_{O_\mathcal{S}}(G) \to L_{O_\mathcal{S}}(F \otimes G)\).

Let \(g : \mathcal{T} \to \mathcal{S}\) a morphism of pre-site with \(\mathcal{T}, \mathcal{S} \in \text{Cat}\) two sites.

- Let \(F^\bullet \in C(\mathcal{S})\). Since \(g^*L(F^\bullet)\) is projective and \(q(g^*F) : L(g^*F^\bullet) \to g^*F^\bullet\) is a surjective quasi-isomorphism, there is a canonical transformation

\[
T(g, L)(F^\bullet) : g^*L(F^\bullet) \to L(g^*F^\bullet)
\]

unique up to homotopy such that \(q(g^*F) \circ T(g, L)(F^\bullet) = g^*q(F)\).

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\begin{itemize}
\item Let $F^\bullet \in C(S)$. Since $L(g^*F^\bullet)$ is projective and $g^*q(F) : g^*L(F^\bullet) \to g^*F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation
\[ T(g,L)(F^\bullet) : L(g^*F^\bullet) \to g^*L(F^\bullet) \] (19)
unique up to homotopy such that $g^*q(F) \circ T(g,L)(F^\bullet) = q(g^*F).
\item Let $F^\bullet \in C(T)$. Since $L(g,F^\bullet)$ is projective and $g_*q(F) : g_*L(F^\bullet) \to g_*F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation
\[ T_*(g,L)(F^\bullet) : L(g,F^\bullet) \to g_*L(F^\bullet) \] (20)
unique up to homotopy such that $g_*q(F) \circ T_*(g,L)(F^\bullet) = q(g_*F).
\end{itemize}

Let $g : (T,O_T) \to (S,O_S)$ a morphism with $(T,O_T),(S,O_S) \in \text{RCat}$. Let $F^\bullet \in C_{O_S}(S)$. Since $g^{\text{mod}}L_O(F^\bullet)$ is projective and $q(g^{\text{mod}}F) : L_O(g^{\text{mod}}F^\bullet) \to g^{\text{mod}}F^\bullet$ is a surjective quasi-isomorphism, there is a canonical transformation
\[ T(g,L_O)(F^\bullet) : g^{\text{mod}}L_O(F^\bullet) \to L_O(g^{\text{mod}}F^\bullet) \] (21)
unique up to homotopy such that $q(g^{\text{mod}}F) \circ T(g,L_O)(F^\bullet) = g^{\text{mod}}q(F).

Let $p : (S_{12},O_{S_{12}}) \to (S_1,O_{S_1})$ a morphism with $(S_{12},O_{S_{12}}),(S_1,O_{S_1}) \in \text{RCat}$, such that the structural morphism $p^*O_{S_1} \to O_{S_{12}}$ is flat. Let $F^\bullet \in C_{O_S}(S)$. Since $L_O(p^{\text{mod}}F^\bullet)$ is projective and $p^{\text{mod}}q(F) : p^{\text{mod}}L_O(F^\bullet) \to p^{\text{mod}}F^\bullet$ is a surjective quasi-isomorphism, there is also in this case a canonical transformation
\[ T(p,L_O)(F^\bullet) : L_O(p^{\text{mod}}F^\bullet) \to p^{\text{mod}}L_O(F^\bullet) \] (22)
unique up to homotopy such that $p^{\text{mod}}q(F) \circ T(p,L_O)(F^\bullet) = q(p^{\text{mod}}F).

### 2.3.4 The De Rham complex of a ringed topos and functorialities

Let $A \in \text{cRing}$ a commutative ring. For $M \in \text{Mod}(A)$, we denote by
\[ \text{Der}_A(A,M) \subset \text{Hom}(A,M) = \text{Hom}_{\text{Ab}}(A,M) \]
the abelian subgroup of derivation. Denote by $I_A = \ker(s_A : A \otimes A \to A) \subset A \otimes A$ the diagonal ideal with $s_A(a_1,a_2) = a_1 - a_2$. Let $\Omega_A := I_A/I_A^3 \in \text{Mod}(A)$ together with its derivation map $d = d_A : A \to \Omega_A$. Then, for $M \in \text{Mod}(A)$ the canonical map
\[ w(M) : \text{Hom}_A(\Omega_A,M) \xrightarrow{\sim} \text{Der}_A(A,M), \psi \mapsto \phi \circ d \]
is an isomorphism, that is $\Omega_A$ is the universal derivation. In particular, its dual $T_A := D^A(\Omega_A) = D^A(I_A/I_A^3)$ is isomorphic to the derivations group : $w(A) : T_A \xrightarrow{\sim} \text{Der}_A(A,A)$. Also note that $\text{Der}_A(A,A) \subset \text{Hom}(A,A)$ is a Lie subalgebra. If $\phi : A \to B$ is a morphism of commutative ring, we have a canonical morphism of abelian group $\Omega_{(B/A)}\phi : \Omega_A \to \Omega_B$.

Let $(S,O_S) \in \text{RCat}$, with $O_S \in \text{PSh}(S,\text{cRing})$ commutative. For $G \in \text{PSh}_{O_S}(S)$, we denote by
\[ \text{Der}_{O_S}(O_S,G) \subset \text{Hom}(O_S,G) = \text{Hom}_{\text{Ab}}(O_S,G) \]
the abelian subsheaf of derivation. Denote by $I_S = \ker(s_S : O_S \otimes O_S \to O_S) \in \text{PSh}_{O_S \times O_S}(S)$ the diagonal ideal with $s_S(X) = s_{O_S}(X)$ for $X \in S$. Then $\Omega_{O_S} := I_S/I_S^2 \in \text{PSh}_{O_S}(S)$ together with its derivation map $d : O_S \to \Omega_{O_S}$ is the universal derivation $O_S$-module : the canonical map
\[ w(G) : \text{Hom}_{O_S}(\Omega_{O_S},G) \xrightarrow{\sim} \text{Der}_{O_S}(O_S,G), \phi \mapsto \phi \circ d \]
is an isomorphism. In particular, its dual $T_{O_S} := \mathcal{D}^O_{O_S}(\Omega_{O_S}) = \mathcal{D}^O_S(\mathcal{I}_S/\mathcal{I}_S^2)$ is isomorphic to the presheaf of derivations: $w(O_S) : T_{O_S} \cong \text{Der}_{O_S}(O_S, O_S)$ and $\text{Der}_{O_S}(O_S, O_S) \subset \text{Hom}(O_S, O_S)$ is a Lie subalgebra. The universal derivation $d = d_{O_S} : O_S \to \Omega_{O_S}$ induces the De Rham complex

$$DR(O_S) : \Omega^*_S := \wedge^* \Omega_{O_S} \in C(S)$$

A morphism $\phi : O'_S \to O_S$ with $O_S, O'_S \text{PSh}(S, \text{eRing})$ induces by the universal property canonical morphisms

$$\Omega_{O'_S/O_S} : \Omega_{O'_S} \to \Omega_{O_S},\ \mathcal{D}^O_{O_S} \Omega_{O'_S/O_S} : T_{O_S} \to T_{O'_S}$$
in $\text{PSh}_{O_S}(S)$.

- In the particular cases where $S = (S, O_S) \in \text{Var}(\mathbb{C})$ or $S = (S, O_S) \in \text{AnSp}(\mathbb{C})$, we denote as usual $\Omega_S := \Omega_{O_S/C_S}, T_S := T_{O_S/C_S}$ and $DR(S) := DR(O_S/C_S) : \Omega^*_S = \mathcal{C}$. 

- In the particular cases where $S = (S, O_S) \in \text{Diff}(\mathbb{R})$ is a differential manifold, we denote as usual $A_S := \Omega_{O_S/B_S}, T_S := T_{O_S/B_S}$ and $DR(S) := DR(O_S/R_S) : A^*_S = \mathcal{C}$. 

For $f : (\mathcal{X}, O_X) \to (S, O_S)$ with $(\mathcal{X}, O_X), (S, O_S) \in \text{RCat}$ such that $O_X$ and $O_S$ are commutative, we denote by

$$\Omega_{O_X/f^*O_S} := \text{coker}(\Omega_{O_X/f^*O_S} : \Omega_{f^*O_S} \to \Omega_{O_X}) \in \text{PSh}_{f^*O_S}(\mathcal{X})$$

the relative cotangent sheaf. The surjection $q = q_{O_X/f^*O_S} : \Omega_{O_X} \to \Omega_{O_X/f^*O_S}$ gives the derivation $w(\Omega_{O_X/f^*O_S})(q) = d_{O_X/f^*O_S} : O_X \to \Omega_{O_X/f^*O_S}$. It induces the surjections $q^p := \wedge^p q : \Omega^p_{O_X} \to \Omega^O_{O_X/f^*O_S}$. We then have the relative De Rham complex

$$DR(O_X/f^*O_S) := \Omega^*_O \in C_{f^*O_S}(\mathcal{X})$$

whose differentials are given by

$$d q^p(\omega) := q^{p+1}(d(\omega))$$

Note that $\Omega^*_O \in C_{f^*O_S}(\mathcal{X})$ is a complex of $f^*O_S$ modules, but is NOT a complex of $O_X$ module since the differential is a derivation hence NOT $O_X$ linear. On the other hand, the canonical map in $\text{PSh}_{f^*O_S}(\mathcal{X})$

$$T(f, \text{hom})(O_S, O_S) : f^*\text{Hom}(O_S, O_S) \to \text{Hom}(f^*O_S, f^*O_S)$$

induces morphisms

$$T(f, \text{hom})(O_S, O_S) : f^*T_{O_S} \to T_{f^*O_S}$$

In this article, we will be interested in the following particular cases:

- In the particular case where $O_S \text{PSh}(S, \text{eRing})$ is a sheaf, $\Omega_{O_S}, T_{O_S} \in \text{PSh}_{O_S}(S)$ are sheaves. Hence, $T(f, \text{hom})(O_S, O_S) : a_{\tau} f^*T_{O_S} \cong T_{a_{\tau} f^*O_S}$ and $\mathcal{D}^O_{f^*O_S} T(f, \text{hom})(O_S, O_S) : a_{\tau} f^*O_S \cong a_{\tau} f^*\Omega_{O_S}$ are isomorphisms, where $a_{\tau} : \text{PSh}(S) \to \text{Shv}(S)$ is the sheafification functor. We will note again in this case by abuse (as usual) $f^*O_S := a_{\tau} f^*O_S, f^*\Omega_{O_S} := a_{\tau} f^*\Omega_{O_S}$ and $f^*T_{O_S} := a_{\tau} f^*T_{O_S}$, so that

$$\Omega_{f^*O_S} = f^*\Omega_{O_S} \text{ and } f^*T_{O_S} = T_{f^*O_S}$$

- In the particular cases where $S = (S, O_S), X = (X, O_X) \in \text{Var}(\mathbb{C})$ or $S = (S, O_S), X = (X, O_X) \in \text{AnSp}(\mathbb{C})$, we denote as usual $\Omega_X/S := \Omega_{O_X/f^*O_S}, q_{X/S} := q_{O_X/S} : \Omega_X \to \Omega_{X/S}$ and $DR(X/S) := DR(O_X/f^*O_S) : \Omega^*_X/S \in C_{f^*O_S}(S)$.

- In the particular cases where $S = (S, O_S), X = (X, O_X) \in \text{Diff}(\mathbb{R})$, we denote as usual $A_X/S := \Omega_{O_X/f^*O_S}, q_{X/S} := q_{O_X/S} : A_X \to A_{X/S}$ and $DR(X/S) := DR(O_X/f^*O_S) : A^*_X/S \in C_{f^*O_S}(S)$.
Definition 1. For a commutative diagram in $\text{RCat}$

\[ D = (\mathcal{X}, O_X) \xrightarrow{f} (\mathcal{S}, O_S), \]

\[ g' \downarrow \quad g \downarrow \]

\[ (\mathcal{X}', O_{X'}) \xrightarrow{f'} (\mathcal{T}, O_T) \]

whose structural presheaves are commutative sheaves, the map in $C_{g'}O_X(\mathcal{X}')$

\[ \Omega_{O_X'/g'*O_X} : g'^* (\Omega_{O_X}^*, F_b) = (\Omega_{g'*O_X}^*, F_b) \rightarrow (\Omega_{O_X'}^*, F_b) \]

pass to quotient to give the map in $C_{g'}O_X(\mathcal{X}')$

\[ \Omega_{(O_Y/g'*O_X)/(O_T/g'*O_S)} := (\Omega_{O_Y/g'*O_X})^g : \]

\[ g'^* (\Omega_{O_X'/g'*O_X}^*, F_b) = (\Omega_{g'*O_X'/g'*f'*O_S}^*, F_b) \rightarrow (\Omega_{O_Y/f'*O_T}^*, F_b) \]

It is in particular given for $X^\circ \in \mathcal{X}', g'^* (X^\circ) \rightarrow X^\circ$ and $\hat{\omega} \in \Gamma(X^\circ, \Omega^p_{O_X'/f'*O_T})$,

\[ (\Omega_{O_Y/g'*O_X}/(O_T/g'*O_S)(X^\circ)(\omega) := qo_{X'}(\omega) \rightarrow \Omega_{O_Y/X'}(g'^* (X^\circ), \omega)) \in \Gamma(X^\circ, \Omega^p_{O_X'/f'*O_T}), \]

where $\omega \in \Gamma(X^\circ, \Omega^p_{O_X})$ such that $qo_{X'/f}(\omega) = \hat{\omega}$. We then have the following canonical transformation in $C_{O_T}(\mathcal{T})$

\[ T^D_O(D) := \frac{g*E(\Omega_{O_X'/f'*O_S}^*, F_b)}{T(g', E)(\Omega_{O_Y/g'*O_X}^*, F_b)} \rightarrow \frac{g*E(\Omega_{O_X'/f'*O_S}^*, F_b)}{\Omega_{O_Y/f'*O_T}^*, F_b} \otimes g*O_S O_T \]

\[ \frac{E(\Omega_{O_Y/g'*O_X}/(O_T/g'*O_S))}{T^D_O(D)} \rightarrow \frac{g*E(\Omega_{O_X'/f'*O_S}^*, F_b)}{\Omega_{O_Y/f'*O_T}^*, F_b} \otimes g*O_S O_T \rightarrow \frac{m}{m} \rightarrow \]

with $m(n \otimes s) = s.n.$

### 2.4 Presheaves on diagrams of sites or on diagrams of ringed topos

Let $\mathcal{I}, \mathcal{I}' \in \text{Cat}$ and $(f_\bullet, s) : T_\bullet \rightarrow S_\bullet$, a morphism of diagrams of presites with $T_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat}), S_\bullet \in \text{Fun}(\mathcal{I}', \text{Cat})$. Recall it is by definition given by a functor $s : \mathcal{I} \rightarrow \mathcal{I}'$ and morphism of functor $P(f_\bullet) : S_{s(\bullet)} := S_\circ o s \rightarrow T_\bullet$, and that we denote for short, $S_{s(\bullet)} := S_\circ o s \in \text{Fun}(\mathcal{I}, \text{Cat})$. Recall that, for $r_{IJ} : I \rightarrow J$ a morphism, with $I, J \in \mathcal{I}$, $D_{fIJ}$ is the commutative diagram in $\text{Cat}$

\[ D_{fIJ} := s_{s(\bullet)} \xrightarrow{r_{IJ}} s_{s(\bullet)}. \]

The adjonction

\[ (f_\bullet, s)^*(f_\bullet, s)_* = ((f_\bullet, s)^*(f_\bullet, s)_*) : C(S_{s(\bullet)}) \Rightarrow C(T_\bullet), \]

\[ G = (G_I, u_{IJ}) \rightarrow (f_\bullet, s)^*(G) := (f_\bullet^T(G_I), T(D_{fIJ}))(G_J) \circ f_\bullet^T u_{IJ} \]

\[ G = (G_I, u_{IJ}) \rightarrow (f_\bullet, s)_*(G) := (f_\bullet(G_I), f_\bullet u_{IJ}) \]

gives an adjonction

\[ ((f_\bullet, s)^*(f_\bullet, s)_*) : C(2)_{fI}(S_{s(\bullet)}) \Rightarrow C(2)_{fI}(T_\bullet), \]

\[ (G, F) = ((G_I, F), u_{IJ}) \rightarrow (f_\bullet, s)^*(G, F) := (f_\bullet^T(G_I), T(D_{fIJ}))(G_J, F) \circ f_\bullet^T u_{IJ} \]

\[ (G, F) = ((G_I, F), u_{IJ}) \rightarrow (f_\bullet, s)_*(G, F) := (f_\bullet(G_I, F), f_\bullet u_{IJ}). \]
For a commutative diagram of diagrams of presite:

\[
\begin{array}{c}
D = T'_{s_2'}(g_2, s'_2) \\
\downarrow (f_2, s_2) \\
T_{s_2}(g_1, s'_1) \\
\end{array}
\]

with \(I, I', J, J'\) \in \text{Cat} and \(T_{s_2} \in \text{Fun}(I, \text{Cat}), T'_{s_2'} \in \text{Fun}(I', \text{Cat}), S_{s_2} \in \text{Fun}(J, \text{Cat}), S'_{s_2'} \in \text{Fun}(J', \text{Cat}),\) and \(s = s_1 \circ s'_2 = s_2 \circ s'_1 : I' \to J\), we denote by, for \(F = (F_I, u_{IJ}) \in C(S'_{s_2'}(\bullet))\),

\[
T(D)(F) : g_1^* f_1^* F \xrightarrow{g_1^* f_1^* \cdot \text{ad}(g_2^* g_2^*)} g_1^* f_1^* g_2^* g_2^* F = g_1^* f_1^* f_2^* g_2^* g_2^* F \xrightarrow{\text{ad}(g_1^* g_1^*)} f_2^* g_2^* F
\]

the canonical transformation map in \(C(T_{s_2}(\bullet))\), and for \((G, F) = ((G_I, F), u_{IJ}) \in C_{fil}(S'_{s_2'}(\bullet))\),

\[
T(D)(G, F) : g_1^* f_1^* (G, F) \xrightarrow{g_1^* f_1^* \cdot \text{ad}(g_2^* g_2^*)} g_1^* f_1^* g_2^* g_2^*(G, F) = g_1^* f_1^* f_2^* g_2^* g_2^*(G, F) \xrightarrow{\text{ad}(g_1^* g_1^*)} f_2^* g_2^*(G, F)
\]

the canonical transformation map in \(C_{fil}(T_{s_2}(\bullet))\) given by the adjunction maps.

Let \(S_{\bullet} \in \text{Fun}(I, \text{RCat})\) a diagram of ringed topos with \(I \in \text{Cat}\). We have the tensor product bifunctor

\[
((-) \otimes (-) : \text{PSh}(S_{\bullet})^2 \to \text{PSh}(S_{\bullet}), \quad ((F_I, u_{IJ}, (G_I, u_{IJ})) \mapsto (F_I \otimes G_I, u_{IJ}) := (F_I \otimes G_I, u_{IJ} \otimes v_{IJ})
\]

We get the bifunctors

\[
(-) \otimes (-) : C_{fil}(S_{\bullet})^2 \to C_{fil}(S_{\bullet}), \quad (-) \otimes (-) : C_{fil}(S_{\bullet}) \times C_{O_S,fil}(S_{\bullet}) \to C_{O_S,fil}(S_{\bullet}).
\]

We have the tensor product bifunctor

\[
((-) \otimes_{O_S} (-) : \text{PSh}_{O_S}(S_{\bullet})^2 \to \text{PSh}(S_{\bullet}), \quad ((F_I, u_{IJ}, (G_I, u_{IJ})) \mapsto (F_I \otimes_{O_S} G_I, u_{IJ}) := (F_I \otimes_{O_S} G_I, u_{IJ} \otimes v_{IJ})
\]

which gives,

- in all case it gives the bifunctor \((-) \otimes_{O_S} (-) : C_{O_S,fil}(S_{\bullet}) \times C_{O_S,fil}(S_{\bullet}) \to C_{fil}(S_{\bullet}).\)
- in the case \(O_S\) is commutative, it gives the bifunctor \((-) \otimes_{O_S} (-) : C_{O_S,fil}(S_{\bullet})^2 \to C_{O_S,fil}(S_{\bullet}).\)

Let \((f, s) : (T_{\bullet}, O_T) \to (S_{\bullet}, O_S)\) a morphism with \((S_{\bullet}, O_S) \in \text{Fun}(I', \text{RCat}), (T_{\bullet}, O_T) \in \text{Fun}(I, \text{RCat})\) and \(I, I' \in \text{Cat}\). which is by definition given by a functor \(s : I \to I'\) and morphism of ringed topos \(f : (T_{\bullet}, O_T) \to (S_{\bullet}(\bullet), O_S)\). As before, we denote for short, \((S_{\bullet}(\bullet), O_S) := (S_{\bullet}, O_S) \circ s \in \text{Fun}(I, \text{RCat}).\)

Denote as before, for \(r_{IJ} : I \to J\) a morphism, with \(I, J \in I, D_{IJ}\) the commutative diagram in \(\text{RCat}\)

\[
\begin{array}{c}
D_{IJ} := S_{s(J)} \xrightarrow{r_{IJ}^*} S_{s(I)} \\
\downarrow f_J \quad \downarrow f_I \\
T_J \quad T_I
\end{array}
\]

We have then the adjunction

\[
((f, s)^{s_{mod}}, (f, s)_s) : C_{O_S}(S_{s(\bullet)}) \Rightarrow C_{O_T}(T_{\bullet}),
\]

\[
(G_I, u_{IJ}) \mapsto (f, s)^{s_{mod}}(G_I, u_{IJ}) := (f_I^{s_{mod}} G_I, T^{s_{mod}}(D_{IJ})(G_I) \circ f_I^{s_{mod}} u_{IJ}),
\]

\[
(G_I, u_{IJ}) \mapsto (f, s)_s (G_I, u_{IJ}) := (f_{s(\bullet)} G_I, f_{s(\bullet)} u_{IJ}).
\]

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which induces the adjunction

\[ ((f_s, s)^{\text{mod}} , (f_s, s)_\ast) : C_{O \subset \text{fil}}(S_{\bullet}^1) \Rightarrow C_{O \subset \text{fil}}(T_{\bullet}) \]

\[ ((G, F), u_{I, J}) \mapsto (f_s, s)^{\text{mod}}((G, F), u_{I, J}) := (f_I^{\text{mod}}(G, F), T^{\text{mod}}(D_{f, s})(G, F) \circ f_I^{\text{mod}}u_{I, J}), \]

\[ ((G, F), u_{I, J}) \mapsto (f_s, s)_\ast((G, F), u_{I, J}) := (f_{I, \ast}(G, F), f_{I, \ast}u_{I, J}). \]

For a commutative diagram of rings of topoi, :

\[
\begin{array}{ccc}
D = (T', O_2') & \xrightarrow{(g_2, s_2')} & (S_{s_2(\bullet)}')O_1' \\
\_ & \_ & \_ \\
(T, O_2) & \xrightarrow{(g_1, s_1)} & (S_{s(\bullet)}, O_1)
\end{array}
\]

with \( I, T', J, J' \in \text{Cat} \) and \( T_{\bullet} \in \text{Fun}(I, \text{Cat}), T'_{\bullet} \in \text{Fun}(I, \text{Cat}), S_{\bullet} \in \text{Fun}(J, \text{Cat}), S'_{\bullet} \in \text{Fun}(J', \text{Cat}), \) and \( s = s_1 \circ s_2' = s'_1 \circ s_2 : T' \to J, \) we denote by, for \( F = (F_I, u_{I, J}) \in C_{O'_{s_2(\bullet)}}, \)

\[ T^{\text{mod}}(D)(F) : g_1^{\text{mod}}f_{I, \ast}F \xrightarrow{g_2^{\text{mod}}f_{s, \ast}} \xrightarrow{\text{ad}(g_1^{\text{mod}})} g_1^{\text{mod}}f_{I, \ast}g_2^{\text{mod}}F = g_1^{\text{mod}}f_{I, \ast}g_2^{\ast}g_2^{\text{mod}}F \]

the canonical transformation map in \( C_{O_2}(T_{s_2(\bullet)}), \) and for \( G = ((G, F), u_{I, J}) \in C_{O'_{s_2(\bullet)}}, \)

\[ T^{\text{mod}}(D)(G, F) : g_1^{\text{mod}}f_{I, \ast}(G, F) \xrightarrow{g_2^{\text{mod}}f_{s, \ast}(G, F)} \xrightarrow{\text{ad}(g_1^{\text{mod}}g_2^{\text{mod}})} g_1^{\text{mod}}f_{I, \ast}g_2^{\text{mod}}(G, F) = g_1^{\text{mod}}f_{I, \ast}g_2^{\ast}g_2^{\text{mod}}(G, F) \]

the canonical transformation map in \( C_{O_2}(T_{s_2(\bullet)}), \) given by the adjunction maps.

Let \((S_{\bullet}, O_S) \in \text{Fun}(I, \text{RCat})\) a diagram of ringed topoi with \( I \in \text{Cat} \) and, for \( I \in I, S_I \) is endowed with topology \( \tau_I \) and for \( r : I \to J \) a morphism with \( I, J \in I, r_{I, J} : S_I \to S_J \) is continuous. Then the diagram category \((I, S_{\bullet}, O_S) \in \text{RCat} \) is endowed with the associated canonical topology \( \tau \), and then

- A morphism \( \varphi = (\varphi_I) : ((F_I, F), u_{I, J}) \to ((G_I, F), u_{I, J}) \) with \( ((F_I, F), u_{I, J}), ((G_I, F), u_{I, J}) \in C_{O_{\subset \text{fil}}}(S_{\bullet}) \) is a filtered \( \tau \) local equivalence if and only if the \( \varphi_I \) are filtered \( \tau \) local equivalences for all \( I \in I \).

- Let \( r \in \mathbb{N} \). A morphism \( \varphi = (\varphi_I) : ((F_I, F), u_{I, J}) \to ((G_I, F), u_{I, J}) \) with \( ((F_I, F), u_{I, J}), ((G_I, F), u_{I, J}) \in C_{O_{\subset \text{fil}}}(S_{\bullet}) \) is an \( r \)-filtered \( \tau \) local equivalence if and only if there exists \( \varphi_i = ((C_{i, I} F), u_{i, I}) \to ((C_{i+1} F), u_{i+1, I}), 0 \leq i \leq s, \) with \( ((C_{i, I} F), u_{i, I}) \in C_{O_{\subset \text{fil}}}(S_{\bullet}), ((C_{i, I} F), u_{i, I}) = ((F_I, F), u_{I, J}) \) and \( ((C_{s, I} F), u_{s, I}) = ((G_I, F), u_{I, J}) \), such that

\[ \varphi = \varphi_s \circ \cdots \circ \varphi_1 \circ \cdots \circ \varphi_0 : ((F_I, F), u_{I, J}) \to ((G_I, F), u_{I, J}) \]

and \( \varphi_i : (C_{i, I} F) \to (C_{i+1, I} F) \) either a filtered \( \tau \) local equivalence or an \( r \)-filtered homotopy equivalence. the \( \varphi_i \) are \( r \)-filtered \( \tau \) local equivalences for all \( I \in I \).

- A complex of presheaves \( ((G_I, F), u_{I, J}) \in C_{O_{\subset \text{fil}}}(S_{\bullet}) \) is filtered \( \tau \) fibrant if and only if the \( (G_I, F) \in C_{O_{\subset \text{fil}}}(S_I) \) are filtered \( \tau \) fibrant for all \( I \in I \).

- Let \( r \in \mathbb{N} \). A complex of presheaves \( ((G_I, F), u_{I, J}) \in C_{O_{\subset \text{fil}}}(S_{\bullet}) \) is \( r \)-filtered \( \tau \) fibrant if there exist an \( r \)-filtered homotopy equivalence \( m : ((G_I, F), u_{I, J}) \to ((G'_{I}, F), u_{I, J}) \) with \( (G'_{I}, F) \in C_{O_{\subset \text{fil}}}(S_{\bullet}) \) filtered \( \tau \) fibrant.
2.5 Presheaves on topological spaces and presheaves of modules on a ringed spaces

In this subsection, we will consider the particular case of presheaves on topological spaces. Let \( f : T \to S \) a continous map with \( S, T \in \text{Top} \). We denote as usual the adjonction

\[
(f^*, f_*) := (P(f)^*, P(f)_*) : \text{PSh}(S) \rightleftarrows \text{PSh}(T)
\]

induced by the morphism of site given by the pullback functor

\[
P(f) : \text{Ouv}(S) \to \text{Ouv}(T), (S^o \subset S) \mapsto P(f)(S^o) := S^o \times_S T \xrightarrow{f} f^{-1}(S^o) \subset T
\]

Since the colimits involved in the definition of \( f^* = P(f)^* \) are filtered, \( f^* \) also preserve monomorphism. Hence, we get an adjonction

\[
(f^*, f_*) : \text{PSh}_{fil}(S) \rightleftarrows \text{PSh}_{fil}(T), f^*(G, F) := (f^*G, f^*F)
\]

Let \( f : (T, O_T) \to (S, O_S) \) a morphism with \((S, O_S), (T, O_T) \in \text{Top} \). We have then the adjonction

\[
(f_{*\text{mod}}, f_*) := (P(f)^{*\text{mod}}, P(f)_*) : \text{PSh}_{O_S,fil}(S) \rightleftarrows \text{PSh}_{O_T,fil}(T), f^{*\text{mod}}(G, F) := f^*(G, F) \otimes f_! O_S \otimes O_T
\]

Recall \( \text{CW} \subset \text{Top} \) is the full subcategory whose objects consist of \( CW \) complexes. Denote, for \( n \in \mathbb{N} \), \( \mathbb{I}^n := [0, 1]^n, S^n := \mathbb{I}^n / \partial \mathbb{I}^n \in \text{CW} \) and \( \Delta^n \subset \mathbb{I}^n \) the \( n \) dimensional simplex. We get \( \mathbb{I}^*, \Delta^* \in \text{Fun}(\Delta, \text{CW}) \). Denote for \( S \in \text{Top} \), \( \Sigma_1 S := S \times \mathbb{I}^1 / (\{0\} \times S) \cup (\{1\} \times S) \) to \( \text{Top} \).

- Let \( f : T \to S \) a morphism with \( T, S \in \text{Top} \). We have the mapping cylinder \( \text{Cyl}(f) := (T \times [0, 1]) \cup f S \in \text{Top} \) and the mapping cone \( \text{Cone}(f) := (T \times [0, 1]) \cup f S \in \text{Top} \). We have then the quotient map \( q_f : \text{Cyl}(f) \to \text{Cone}(f) \) and a canonical retraction \( r_f : \text{Cone}(f) \to \Sigma^1 T \).

- Recall two morphisms \( f, g : T \to S \) with \( T, S \in \text{Top} \) are homotopic if there exist \( H : T \times [0, 1] \to S \) continous such that \( H \circ (I \times i_0) = f \) and \( H \circ (I \times i_1) = g \). Then \( K(\text{Top}) := \text{Ho}_{\mathbb{I}}(\text{Top}) \) is a triangulated category with distinguish triangle

\[
T \xrightarrow{f} \text{Cyl}(f) \xrightarrow{q_f} \text{Cone}(f) \xrightarrow{r_f} \Sigma^1 T.
\]

- For \( X \in \text{Top} \), denote for \( n \in \mathbb{N} \), \( \pi_n(X) : \text{Hom}_{K(\text{Top})}(S^n, X) \) the homotopy groups. For \( f : T \to S \) a morphism with \( T, S \in \text{Top} \), we have for \( n \in \mathbb{N} \) the morphisms of abelian groups

\[
f_* : \pi_n(T) \to \pi_n(S), h \mapsto f \circ h
\]

Recall two morphisms \( f, g : T \to S \) with \( T, S \in \text{Top} \) are weakly homotopic if \( f_* = g_* : \pi_n(T) \to \pi_n(S) \) for all \( n \in \mathbb{N} \).

- For \( X \in \text{Top} \), denote by \( C^*_\text{sing}(X) := \mathbb{Z} \text{Hom}(\Delta^*, X) \subset C^-(\mathbb{Z}) \) the complex of singular chains and by \( C^\text{sing}_*(X) := \mathbb{D}C^\text{sing}_*(X) \subset \mathbb{D}\mathbb{Z} \text{Hom}(\Delta^*, X) \subset C^-(\mathbb{Z}) \) the complex of singular cochains. For \( f : T \to S \) a morphism with \( T, S \in \text{Top} \), we have

- the morphism of complexes of abelian groups

\[
f_* : C^\text{sing}_*(T) \to C^\text{sing}_*(S), \sigma \mapsto f \circ \sigma,
\]

- the morphism of complexes of abelian groups

\[
f^* := \mathbb{D}f_* : C^\text{sing}_*(T) \to C^\text{sing}_*(S), \alpha \mapsto f^* \alpha : (\sigma \mapsto f^* \alpha(\sigma) := \alpha(f \circ \sigma))
\]
We denote by $C_{X,sing}^* \in C^+(X)$ the complex of presheaves of singular cochains given by,

$$(U \subset X) \mapsto C^*_{X,sing}(U) := C^*_{X,sing}(U) := C^*_{sing}(U) := D^* \mathbb{Z} \text{Hom}(\Delta^*, U),$$

and by $c_X : \mathbb{Z}_X \rightarrow C^*_{X,sing}$ the inclusion map. For $f : T \rightarrow S$ a morphism with $T, S \in \text{Top}$, we have the morphism of complexes of presheaves

$$f^* : C_{S,sing}^* \rightarrow f_*C_{T,sing}^*$$

in $C(S)$.

**Theorem 7.** (i) If two morphisms $f, g : T \rightarrow S$ with $T, S \in \text{Top}$ are weakly homotopic, then

$$H^n(f_*) = H^n(g_*) : H_n(sing)(T, \mathbb{Z}) := H^nC^*_{sing}(T) \rightarrow H_n(sing)(S, \mathbb{Z}) := H^nC^*_{sing}(S).$$

(ii) For $S \in \text{Top}$ there exists $CW(S) \in CW$ together with a morphism $L_S : CW(S) \rightarrow S$ which is a weakly homotopic equivalence, that is $L_{S*} : \pi_n(CW(S)) \xrightarrow{\sim} \pi_n(S)$ are isomorphisms of abelian groups for all $n \in \mathbb{N}$.

(ii)' For $f : T \rightarrow S$ a morphism, with $T, S \in \text{Top}$, and $L_S : CW(S) \rightarrow S$, $L_S : CW(T) \rightarrow T$ weakly homotopy equivalence with $CW(S), CW(T) \in \text{CW}$ there exist a morphism $L(f) : CW(T) \rightarrow CW(S)$ unique up to homotopy such that the following diagram in $\text{Top}$ commutes

$$\begin{array}{ccc}
CW(S) & \xrightarrow{L_S} & S \\
\downarrow{L(f)} & & \uparrow{f} \\
CW(T) & \xrightarrow{L_T} & T
\end{array}$$

In particular, for $S \in \text{Top}$, $CW(S)$ is unique up to homotopy.

**Proof.** See [15].

We have Kunneth formula for singular cohomology :

**Proposition 8.** Let $X_1, X_2 \in \text{Top}$. Denote by $p_1 : X_1 \times X_2 \rightarrow X_1$ and $p_2 : X_1 \times X_2 \rightarrow X_2$ the projections. Then

$$p_1^* \otimes p_2^* : C^*_{sing}(X_1) \otimes C^*_{sing}(X_2) \rightarrow C^*_{sing}(X_1 \times X_2)$$

is a quasi-isomorphism.

**Proof.** Standard (see [15] for example): follows from the fact that for all $p \in \mathbb{N}$, $H^nC^*_{sing}(\Delta^p) = 0$ for all $n \in \mathbb{Z}$.

**Remark 2.** By definition, $X \in \text{Top}$ is locally contractile if an only if the inclusion map $c_X : \mathbb{Z}_X \rightarrow C^*_{X,sing}$ is an equivalence top local. In this case it induce, by taking injective resolutions, for $n \in \mathbb{Z}$ isomorphisms

$$H^n c_X^* : H^n(X, \mathbb{Z}, X) \xrightarrow{\sim} \mathbb{H}^n(X, C^*_{X,sing}) = H^n C^*_{sing}(X) =: H^n_{sing}(X, \mathbb{Z}).$$

We will use the following easy propositions :

**Proposition 9.** (i) Let $(S, O_S) \in \text{RTop}$. Then, if $K^* \in C^0(S)$ is a bounded above complex such that $K_n^* \in \text{PSh}_{O_S}(S)$ are locally free for all $n \in \mathbb{Z}$, and $\phi : F^* \rightarrow G^*$ is a top local equivalence with $F, G \in C_{O_S}(S)$, then $\phi \otimes I : F^* \otimes_{O_S} L^* \rightarrow G^* \otimes_{O_S} L^*$ is an equivalence top local.
(ii) Let \( f : (T, O_T) \to (S, O_S) \) a morphism with \((T, O_T), (S, O_S) \in \text{RTop}\). Then, if \( K \in C^0_{O_S}(S) \) is a bounded complex such that \( K^n \in \text{PSh}_{O_S}(S) \) are locally free for all \( n \in \mathbb{Z} \), and \( N \in C_{O_T}(T) \)

\[
k \circ T_{\text{mod}} f_{*, \otimes}(M, E(N)) : K \otimes_{O_S} f_* E(N) \to f_* ((f_{**})_{\text{mod}} K) \otimes_{O_T} E(N) \to f_* E((f_{**})_{\text{mod}} K) \otimes_{O_T} E(N)
\]

is an equivalence top local.

Proof. Standard.

**Proposition 10.** Let \( i : (Z, O_Z) \to (S, O_S) \) a closed embedding of ringed spaces, with \( Z, S \in \text{Top} \). Then for \( M \in C_{O_S}(S) \) and \( M \in C_{O_Z}(Z) \),

\[
T(i, \otimes)(M, N) : M \otimes_{O_S} i_* N \to i_* (i^* M \otimes_{O_Z} N)
\]

is an equivalence top local.

Proof. Standard. Follows form the fact that \( j^* i_* N = 0 \).

We note the following :

**Proposition 11.** Let \((S, O_S) \in \text{Sch} \) such that \( O_{S,s} \) are reduced local rings for all \( s \in S \). For \( s \in S \) consider \( q : L_{O_S,s}(k(s)) \to k(s) \) the canonical projective resolution of the \( O_{S,s} \) module \( k(s) := O_{S,s}/m_s \) (the residual field) of \( s \in S \). For \( s \in S \) denote by \( i_s : \{s\} \to S \) the embedding. Let \( \phi : F \to G \) a morphism with \( F, G \in C_{O_{S,c}}(S) \) i.e. such that \( a_{zar} H^n F, a_{zar} H^n G \in \text{Coh}(S) \). If \( i_s^* \phi \otimes i_{s*} O_S L_i^* O_S(k(s)) : i_s^* F \otimes i_{s*} O_S L_i^* O_S(k(s)) \to i_s^* G \otimes i_{s*} O_S L_i^* O_S(k(s)) \) is a quasi-isomorphism for all \( s \in S \), then \( \phi : F \to G \) is an equivalence top local.

Proof. Let \( s \in S \). Since tensorizing with \( L_i^* O_S(k(s)) \) is an exact functor, we have canonical isomorphism \( \alpha(F), \alpha(G) \) fitting in a commutative diagram

\[
\begin{array}{ccc}
H^n(i_s^* F \otimes i_{s*} O_S L_i^* O_S(k(s))) & \\ \alpha(F) & \\ i_s^*(H^n F) \otimes i_{s*} L_i^* O_S(k(s)) & \alpha(G)
\end{array}
\]

Let \( n \in \mathbb{Z} \). By hypothesis

\[
H^n(i_s^* \phi \otimes i_{s*} O_S L_i^* O_S(k(s))) : H^n(i_s^* F \otimes i_{s*} O_S L_i^* O_S(k(s))) \cong H^n(i_s^* G \otimes i_{s*} O_S L_i^* O_S(k(s)))
\]

is an isomorphism. Hence, the diagram 2.5 implies that

\[
i_s^*(H^n \phi) \otimes i_{s*} O_S L_i^* O_S(k(s)) : i_s^*(H^n F) \otimes i_{s*} O_S L_i^* O_S(k(s)) \cong i_s^*(H^n G) \otimes i_{s*} O_S L_i^* O_S(k(s))
\]

is an isomorphism. We conclude on the one hand that \( i_s^* H^n \phi : i_s^* H^n F \to i_s^* H^n G \) is surjective by Nakayama lemma since \( i_s^* H^n F, i_s^* H^n G \) are \( O_{S,s} \) modules of finite type as \( F, G \in C_{O_{S,c}}(S) \) has coherent cohomology sheaves, and on the other hand that the rows of the following commutative diagram are isomorphism

\[
\begin{array}{ccc}
H^0(i_s^*(H^n F) \otimes i_{s*} O_S L_i^* O_S(k(s))) & \\ \alpha & \\ i_s^*(H^n F) \otimes i_{s*} O_S k(s) & \alpha
\end{array}
\]

Since

\[
i_s^*(H^n \phi) \otimes i_{s*} O_S k(s) : i_s^*(H^n F) \otimes i_{s*} O_S k(s) \cong i_s^*(H^n F) \otimes i_{s*} O_S k(s)
\]

is an isomorphism for all \( s \in S, O_{S,s} =: i_s^* O_S \) are reduced, and \( a_{zar} H^n F, a_{zar} H^n G \) are coherent, \( i_s^* H^n \phi : i_s^* H^n F \to i_s^* H^n G \) are injective.

\[\blacksquare\]
Let $i : S' \to S$ a closed embedding, with $S, S' \in \text{Top}$. Denote by $j : S \setminus Z \to S$ the open embedding of the complementary subset. We have the adjunction

$$(i_*, i^! : \mathbb{C}(Z) \to \mathbb{C}(S), \text{ with in this case } i^! := \ker(F \to j_* j^* F)).$$

It induces the adjunction $(i_*, i^! : \mathbb{C}(Z) \to \mathbb{C}(S))$ (we recall that $i^! := i^!$ preserve monomorphisms).

Let $i : Z \to S$ a closed embedding, with $S, Z \in \text{Top}$. Denote by $j : S \setminus Z \to S$ the open embedding of the complementary subset. We have the support section functors:

- We have the functor

  $$\Gamma_Z : \mathbb{C}(S) \to \mathbb{C}(S), \text{ } F \mapsto \Gamma_Z(F) := \text{Cone}(\text{ad}(j^*, j_*)(F) : F \to j_* j^* F)[-1],$$

  together with the canonical map $\gamma_Z(F) : \Gamma_Z F \to F$. We have the factorization

  $$\text{ad}(i_*, i^!)(F) : i_* i^! F \xrightarrow{\text{ad}(i_*, i^!)(F)\gamma} \Gamma_Z F \xrightarrow{\gamma_Z(F)} F,$$

  and $\text{ad}(i_*, i^!)(F) \gamma : i_* i^! F \to \Gamma_Z F$ is an homotopy equivalence. Since $\Gamma_Z$ preserve monomorphisms, it induce the functor

  $$\Gamma_Z : C_{fil}(S) \to C_{fil}(S), \text{ } (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F),$$

  together with the canonical map $\gamma_Z((G, F) : \Gamma_Z(G, F) \to (G, F)$.

- We have also the functor

  $$\Gamma'_Z : \mathbb{C}(S) \to \mathbb{C}(S), \text{ } F \mapsto \Gamma'_Z(F) := \text{Cone}(\text{ad}(j_!, j^*)(F) : j_* j^* F \to F),$$

  together with the canonical map $\gamma'_Z(F) : F \to \Gamma'_Z F$. We have the factorization

  $$\text{ad}(i^!, i_*)(F) : F \xrightarrow{\gamma'_Z(F)} \Gamma'_Z F \xrightarrow{\text{ad}(i^!, i_*)(F)\gamma} i_* i^* F,$$

  and $\text{ad}(i^!, i_*)(F) \gamma : \Gamma'_Z F \to i_* i^* F$ is an homotopy equivalence. Since $\Gamma'_Z$ preserve monomorphisms, it induce the functor

  $$\Gamma_Z : C_{fil}(S) \to C_{fil}(S), \text{ } (G, F) \mapsto \Gamma'_Z(G, F) := (\Gamma'_Z G, \Gamma'_Z F),$$

  together with the canonical map $\gamma'_Z(G, F) : (G, F) \to \Gamma'_Z(G, F)$.

**Definition-Proposition 1.**

(i) Let $g : S' \to S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $S', S, Z \in \text{Top}$. Then, for $(G, F) \in C_{fil}(S)$, there is a canonical map in $C_{fil}(S')$

$$T(g, \gamma)(G, F) : g^* \Gamma_Z(G, F) \to \Gamma_{Z \times S' S} g^* (G, F)$$

unique up to homotopy such that $\gamma_{Z \times S' S}(g^* (G, F)) \circ T(g, \gamma)(G, F) = g^* \gamma_Z(G, F)$.

(ii) Let $i_1 : Z_1 \to S$, $i_2 : Z_2 \to Z_1$ be closed embeddings with $S, Z_1, Z_2 \in \text{Top}$. Then, for $(G, F) \in C_{fil}(S)$,

- there is a canonical map $T(Z_2/Z_1, \gamma)(G, F) : \Gamma_{Z_2}(G, F) \to \Gamma_{Z_1}(G, F)$ in $C_{fil}(S)$ unique up to homotopy such that $\gamma_{Z_2}(G, F) \circ T(Z_2/Z_1, \gamma)(G, F) = \gamma_{Z_1}(G, F)$ together with a distinguish triangle in $K_{fil}(S) := K(PSh_{fil}(S))$

$$\Gamma_{Z_2}(G, F) \xrightarrow{T(Z_2/Z_1, \gamma)(G, F)} \Gamma_{Z_1}(G, F) \xrightarrow{\text{ad}(j_{Z_2/Z_1}^!, j_{Z_2}(G, F))} \Gamma_{Z_1 \setminus Z_2}(G, F) \to \Gamma_{Z_2}(G, F)[1]$$
(i) We have the cartesian square

\[
\begin{array}{ccc}
S \setminus Z & \xrightarrow{j} & S \\
\downarrow{g'} & & \downarrow{g} \\
S' \setminus Z \times S & \xrightarrow{g} & S'
\end{array}
\]

and the map is given by

\[(I, T(g, j)(j^*G)) : \text{Cone}(g^*G \rightarrow g^*z_1 j^*G) \rightarrow \text{Cone}(g^*G \rightarrow j_1^* j^*G = j_1^* g^*G).
\]

(ii) Follows from the fact that \(j_1^* \Gamma Z_2 G = 0\) and \(j_1^* \Gamma^\vee Z_2 G = 0\), with \(j_1 : S \setminus Z_1 \hookrightarrow S\) the closed embedding. (iii): Obvious.

Let \((S, O_S) \in RTop\). Let \(Z \subset S\) a closed subset. Denote by \(j : S \setminus Z \hookrightarrow S\) the open complementary embedding.

- For \(G \in C_{O_S}(S)\), \(\Gamma Z G := \text{Cone}(\text{ad}(j_1^*, j_1^*)(G) : F \rightarrow j_1^* j_1^* G)[-1]\) has a (unique) structure of \(O_S\) module such that \(\gamma Z(G) : \Gamma Z G \rightarrow G\) is a map in \(C_{O_S}(S)\). This gives the functor

\[
\Gamma Z : C_{O_S(fu)}(S) \rightarrow C_{fuO_S}(S), \ (G, F) \mapsto \Gamma Z(G, F) := (\Gamma Z G, \Gamma Z F),
\]

together with the canonical map \(\gamma Z((G, F) : \Gamma Z(G, F) \rightarrow (G, F)\). Let \(Z_2 \subset Z\) a closed subset. Then, for \(G \in C_{O_S}(S), T(Z_2/Z, \gamma)(G) : \Gamma Z_2 G \rightarrow \Gamma Z G\) is a map in \(C_{O_S}(S)\) (i.e. is \(O_S\) linear).

- For \(G \in C_{O_S}(S), \Gamma Z^\vee G := \text{Cone}(\text{ad}(j_2^*, j_2^*)(G) : j_2^* j_2^* G \rightarrow G)\) has a unique structure of \(O_S\) module, such that \(\gamma Z^\vee(G) : G \rightarrow \Gamma Z^\vee G\) is a map in \(C_{O_S}(S)\). This gives the functor

\[
\Gamma Z^\vee : C_{O_S(fu)}(S) \rightarrow C_{fuO_S}(S), \ (G, F) \mapsto \Gamma Z^\vee(G, F) := (\Gamma Z^\vee G, \Gamma Z^\vee F),
\]

together with the canonical map \(\gamma Z^\vee((G, F) : (G, F) \rightarrow \Gamma Z^\vee(G, F)\). Let \(Z_2 \subset Z\) a closed subset. Then, for \(G \in C_{O_S}(S), T(Z_2/Z, \gamma^\vee)(G) : \Gamma Z^\vee G \rightarrow \Gamma Z_2^\vee G\) is a map in \(C_{O_S}(S)\) (i.e. is \(O_S\) linear).

- For \(G \in C_{O_S}(S)\), we will use

\[
\Gamma Z^\vee, h G := \text{D}_S^Q L_O \Gamma Z E(D_S^Q G) = \text{Cone}(\text{D}_S^Q L_O \text{ad}(j_1^*, j_1^*)(E(D_S^Q G)) : \text{D}_S^Q L_O j_1^* E(D_S^Q G) \rightarrow \text{D}_S^Q L_O E(D_S^Q G))
\]

and we have the canonical map \(\gamma Z^\vee h(G) : M \rightarrow \Gamma Z^\vee h G\) of \(O_S\) module. The factorization

\[
\text{ad}(j_1^*, j_1^*)(L_O M) : j_1^* L_O G \rightarrow \text{D}_S^Q L_O j_1^* E(D_S^Q L_O G) \xrightarrow{\text{ad}(j_1^*, j_1^*)(E(D_S^Q L_O G))} \text{D}_S^Q L_O E(D_S^Q L_O G)
\]
Consider the section which vanish on \( Z \), together with the factorization
\[
\Gamma_Z^\vee h L_O G. \text{ We get the functor }
\]
\[
\Gamma_Z^\vee h : C_{Os_{fil}}(S) \to C_{Os_{fil}}(S), (G, F) \mapsto \Gamma_Z^\vee h (G, F) := D_S^\vee L_O \Gamma_Z E(D_S^\vee (G, F)),
\]

\[
\text{together with the factorization }
\]
\[
\gamma_Z^\vee h (L_O (G, F)) : L_O (G, F) \xrightarrow{\gamma_Z^\vee (L_O (G, F))} \Gamma_Z^\vee h L_O (G, F)
\]
\[
\xrightarrow{(ko D^\vee(j, j^*) (D^\vee j^* L_O (G, F)) = d(j, j^* L_O (G, F)))^q} \Gamma_Z^\vee h L_O (G, F),
\]

- Consider \( I \subset O_S \) a right ideal of \( O_S \) such that \( I^o_2 \subset I \), where \( I^o_2 \subset O_S \) is the left and right ideal consisting of section which vanish on \( Z \).

  - For \( G \in \text{PSh}_{O_S}(S) \), we consider, \( S^o \subset S \) being an open subset,
\[
IG(S^o) := \langle f, m \in G(S^o), f \in I(S^o) \rangle > G(S^o)
\]
since \( I \) is a right ideal, and we denote by \( b_I(G) : IG \to G \) the injective morphism of \( O_S \) modules and by \( c_I(G) : G \to G/IG \) the quotient map. The adjonction map \( ad(j, j^*)(G) : j j^* G \to G \) factors trough \( b_I(G) : \)
\[
ad(j, j^*)(G) : j j^* G \xrightarrow{b_I(G)} IG \xrightarrow{b_I(G)} G
\]

We have then the support section functor,
\[
\Gamma_Z^{V, I} : C_{Os}(S) \to C_{Os}(S), G \mapsto \Gamma_Z^{V, I} G := \text{Cone}(b_I(G) : IG \to G)
\]

\[
\text{together with the canonical map } \gamma_Z^{V, I} (G) : G \to \Gamma_Z^{V, I} G \text{ which factors through }
\]
\[
\gamma_Z^{V, I} (G) : G \xrightarrow{\gamma_Z^0} \Gamma_Z G \xrightarrow{b_I^0(G)} \Gamma_Z^{V, I} G.
\]

By the exact sequence
\[
0 \to IG \xrightarrow{b_I(G)} G \xrightarrow{c_I(G)} G/IG \to 0,
\]
we have an homotopy equivalence \( c_I(G) : \Gamma_Z^{V, I} G \to G/IG \).

- For \( G \in \text{PSh}_{O_S}(S) \), we consider
\[
b_I^0(G) : G \to G \otimes_{O_S} D_S^0(I) := G \otimes_{O_S} \text{Hom}(I, O_S)
\]
The adjonction map \( ad(j^*, j_*)(G) : G \to j_* j^* G \) factors trough \( b_I^0(G) : \)
\[
ad(j^*, j_*)(G) : G \xrightarrow{b_I^0(G)} G \otimes_{O_S} D_S^0(I) \xrightarrow{b_I^0(j^*)(G)} j_* j^* G
\]
We have then the support section functor,
\[
\Gamma_Z^{O, I} : C_{Os}(S) \to C_{Os}(S), G \mapsto \Gamma_Z^{O, I} G := \text{Cone}(b_I^0(G) : G \to G \otimes_{O_S} D_S^0(I))[-1]
\]

\[
\text{together with the canonical map } \gamma_Z^{O, I} (G) : \Gamma_Z^{O, I} G \to G \text{ which factors through }
\]
\[
\gamma_Z^{O, I} (G) : \Gamma_Z^{O, I} G \xrightarrow{b_I^0(G)} \Gamma_Z G \xrightarrow{\gamma_Z^0(G)} G.
\]

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– By definition, we have for a canonical isomorphism

\[ I(D, \gamma^O)(G) : D^O_S \Gamma^\gamma^O_G \to \Gamma^O_Z D^O_S G \]

which gives the transformation map in \( C_{O_{S,fit}}(S) \)

\[ T(D, \gamma^O)(G) : \Gamma^\gamma^O_G D^O_S G \to D^O_S \Gamma^\gamma^O_G \]

\[ \xrightarrow{d(\gamma)} D^O_S \Gamma^\gamma^O_G D^O_S G \]

\[ \xrightarrow{\text{def.}} D^O_S \Gamma^\gamma^O_G D^O_S G \]

**Definition-Proposition 2.** (i) Let \( g : (S', O_{S'}) \to (S, O_S) \) a morphism and \( i : Z \to S \) a closed embedding with \((S', O_{S'}, (S, O_S)) \in \text{RTop}\). Then, for \((G, F) \in C_{O_{S,fit}}(S)\), there is a canonical map in \( C_{O_{S,fit}}(S')\)

\[ T^\text{mod}(g, \gamma)(G, F) : g^\text{mod} \Gamma^\gamma Z G, F \to \Gamma^\gamma_{Z'} g^\text{mod} G, F \]

unique up to homotopy, such that \( \gamma_{Z \times S'} \Gamma^\gamma Z G, F \circ T^\text{mod}(g, \gamma)(G, F) = g^\text{mod} \gamma Z G \).

(ii) Let \( i_1 : (Z_1, O_{Z_1}) \to (S, O_S), i_2 : (Z_2, O_{Z_2}) \to (Z_1, O_{Z_1}) \) be closed embeddings with \( S, Z_1, Z_2 \in \text{Top}\). Then, for \((G, F) \in C_{O_{S,fit}}(S)\), there is a canonical map in \( C_{O_{S,fit}}(S)\)

\[ T(Z_2/Z_1, \gamma^\gamma Z)(G, F) : \Gamma^\gamma Z_1 G, F \to \Gamma^\gamma Z_2 G, F \]

unique up to homotopy such that \( \gamma_{Z_2} \Gamma^\gamma Z G, F = T(Z_2/Z_1, \gamma^\gamma Z)(G, F) \circ \gamma_{Z_1}^\gamma Z G, F \).

(iii) Consider a morphism \( g : ((S', O_{S'}), Z') \to ((S, O_S), Z) \) with \(((S', O_{S'}), Z') \to ((S, O_S), Z) \in \text{RTop}^2\). We denote, for \( M \in C_{O_{S}}(S) \) the composite

\[ T^\text{mod}(D, \gamma^\gamma Z)(G) : g^\text{mod} \Gamma^\gamma Z G \to \Gamma^\gamma_{Z \times S'} g^\text{mod} \]

and we have then the factorization

\[ \gamma^\gamma Z (g^\text{mod} M) : g^\text{mod} G \to \Gamma^\gamma Z \Gamma^\gamma Z G \to \Gamma^\gamma Z g^\text{mod} G \]

**Proof.** (i): We have the cartesian square

\[ \begin{array}{ccc}
S \setminus Z & \xrightarrow{j} & S \\
\downarrow{g} & \uparrow{g} & \\
S' \setminus Z \times S & \xrightarrow{f'} & S'
\end{array} \]

and the map is given by

\[ (I, T^\text{mod}(g, j)(j^* G)) : \text{Cone}(g^\text{mod} G \to g^\text{mod} j^* j^* G) \to \text{Cone}(g^\text{mod} G \to j^* j^* G, j^* G) = j^* j^* G). \]

(ii): Obvious.

(iii): Obvious. \( \square \)

**Definition-Proposition 3.** Consider a commutative diagram in \( \text{RTop} \)

\[ D_0 = f : (X, O_X) \xrightarrow{i} (Y, O_Y) \xrightarrow{p} (S, O_S) \]

\[ \xrightarrow{g} \]

\[ f' : (X', O_{X'}) \xrightarrow{i'} (Y', O_{Y'}) \xrightarrow{p'} (T, O_T) \]

with \( i, i' \) being closed embeddings. Denote by \( D \) the right square of \( D_0 \). The closed embedding \( i' \) factors through \( i' : X' \xrightarrow{i_1} X \times_Y Y' \xrightarrow{i_{10}} Y' \) where \( i_1, i_{10} \) are closed embeddings.
(i) We have the canonical map,

\[ E(\Omega_{(O_Y/g''\cdot O_Y)/(O_T/g''\cdot O_S)}) \circ T(g'', E)(- \circ T(g'', \gamma)(-): \]

\[ g'' \cdot \Gamma_X E(\Omega^\bullet_{O_Y/p''\cdot O_S}, F_b) \rightarrow \Gamma_X \times_Y \cdot E(\Omega^\bullet_{O_Y/p''\cdot O_T}, F_b) \]

unique up to homotopy such that the following diagram in \( C_{g''\cdot O_S, f\cdot Y'} = C_{p', g\cdot O_S, f\cdot Y'} \) commutes

\[ \begin{array}{ccc}
E(\Omega_{(O_Y/g''\cdot O_Y)/(O_T/g''\cdot O_S)}) & \rightarrow & T(g'', E)(- \circ T(g'', \gamma)(-)) \\
\gamma_X(-) & & \gamma_X \times_Y (\cdot)(-)
\end{array} \]

(ii) There is a canonical map,

\[ T^\Omega_\omega (D)\gamma : g'' \cdot \text{mod} LOP_E \cdot \Gamma_X E(\Omega^\bullet_{O_Y/p''\cdot O_S}, F_b) \rightarrow \Gamma_X \times_Y \cdot E(\Omega^\bullet_{O_Y/p''\cdot O_T}, F_b) \]

unique up to homotopy such that the following diagram in \( C_{O_T, f\cdot Y'} \) commutes

\[ \begin{array}{ccc}
g'' \cdot \Gamma_X E(\Omega^\bullet_{O_Y/p''\cdot O_S}, F_b) & \rightarrow & \Gamma_X \times_Y \cdot E(\Omega^\bullet_{O_Y/p''\cdot O_T}, F_b) \\
\gamma_X(-) & & \gamma_X \times_Y (\cdot)(-)
\end{array} \]

(iii) We have the canonical map in \( C_{g''\cdot O_T}(Y') \)

\[ T(X'/X \times_Y Y', \gamma)(E(\Omega^\bullet_{O_Y'/p''\cdot O_T}, F_b)) : \Gamma_X \cdot E(\Omega^\bullet_{O_Y'/p''\cdot O_T}, F_b) \rightarrow \Gamma_X \times_Y \cdot E(\Omega^\bullet_{O_Y'/p''\cdot O_T}, F_b) \]

unique up to homotopy such that \( \gamma_X \times_Y (\cdot)(-) \circ T(X'/X \times_Y Y', \gamma)(-)(-) = \gamma_X(-). \)

Proof. Immediate from definition. We take for the map of point (ii) the composite

\[ \begin{array}{ccc}
E(\Omega_{(O_Y/g''\cdot O_Y)/(O_T/s\cdot O_S)}) & \rightarrow & g'' \cdot \text{mod} LOP_E \cdot \Gamma_X E(\Omega^\bullet_{O_Y/p''\cdot O_S}, F_b) \\
g'' \cdot \text{mod} LOP_E \cdot \Gamma_X E(\Omega^\bullet_{O_Y/p''\cdot O_S}, F_b) & \rightarrow & \Gamma_X \times_Y \cdot E(\Omega^\bullet_{O_Y/p''\cdot O_T}, F_b) \\
\end{array} \]

with \( m(n \otimes s) = s.n. \)

\[ \square \]

**Definition 2.**

(i) Let \( S \in \text{Top} \). For \( Z \subset S \) a closed subset, we denote by \( C_S(Z) \subset C_S \) the full subcategory consisting of complexes of presheaves \( F \in C_S \) such that \( a_{\text{top}}H^n(j^*F) = 0 \) for all \( n \in \mathbb{Z} \), where \( j : S \setminus Z \hookrightarrow S \) is the complementary open embedding and \( a_{\text{top}} \) is the sheafification functor.

(ii) More generally, let \( (S, O_S) \in R\text{Top} \). For \( Z \subset S \) a closed subset, we denote by

\[ C_{O_S, Z}(S) \subset C_{O_S}(S), \text{QCoh}_Z(S) \subset \text{QCoh}(S) \]

the full subcategories consisting of complexes of presheaves \( G \in C_{O_S}(S) \) such that \( a_{\text{top}}H^n(j^*F) = 0 \) for all \( n \in \mathbb{Z} \), resp. quasi-coherent sheaves \( G \in \text{QCoh}(S) \) such that \( j^*F = 0 \).

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(ii) Let $S \in \text{Top}$. For $Z \subset S$ a closed subset, we denote by $C_{\text{fil},Z}(S) \subset C_{\text{fil}}(S)$ the full subcategory consisting of filtered complexes of presheaves $(G,F) \in C_{\text{fil}}(S)$ such that there exist $r \in \mathbb{N}$ and an $r$-filtered homotopy equivalence $\phi : (G,F) \to (G',F)$ with $(G',F) \in C_{\text{fil}}(S)$ such that $a_{\text{top}}j^*H^n\text{Gr}^p_F(G',F) = 0$ for all $n,p \in \mathbb{Z}$, where $j : S\setminus Z \to S$ is the complementary open embedding and $a_{\text{top}}$ is the sheafification functor. Note that this definition say that this $r$ does NOT depend on $n$ and $p$.

(ii)' More generally, let $(S,O_S) \in \text{RTop}$. For $Z \subset S$ a closed subset, we denote by

$$C_{O_S,\text{fil},Z}(S) \subset C_{O_S,\text{fil}}(S), \text{Qcoh}_{\text{fil},Z}(S) \subset \text{Qcoh}(S)$$

the full subcategories consisting of filtered complexes of presheaves $(G,F) \in C_{O_S,\text{fil}}(S)$ such that there exist $r \in \mathbb{N}$ and an $r$-filtered homotopy equivalence $\phi : (G,F) \to (G',F)$ with $(G',F) \in C_{\text{fil}}(S)$ such that $a_{\text{top}}j^*H^n\text{Gr}^p_F(G',F) = 0$ for all $n,p \in \mathbb{Z}$, resp. filtered quasi-coherent sheaves $(G,F) \in \text{Qcoh}(S)$ and an $r$-filtered homotopy equivalence $\phi : (G,F) \to (G',F)$ with $(G',F) \in C_{\text{fil}}(S)$ such that there exist $r \in \mathbb{N}$ such that $j^*H^n\text{Gr}^p_F(G',F) = 0$ for all $n,p \in \mathbb{Z}$. Note that this definition say that this $r$ does NOT depend on $n$ and $p$.

Let $(S,O_S) \in \text{RTop}$ and $Z \subset S$ a closed subset.

- For $(G,F) \in C_{\text{fil}}(S)$, we have $\Gamma_Z(G,F), \Gamma_Z'(G,F) \in C_{\text{fil},Z}(S)$.
- For $(G,F) \in C_{O_S,\text{fil}}(S)$, we have $\Gamma_Z(G,F), \Gamma_Z'(G,F), \Gamma^{h,\text{fil}}_Z(G,F), \Gamma^{\text{fil},O}_Z(G,F) \in C_{O_S,\text{fil},Z}(S)$.

**Proposition 12.** Let $S \in \text{Top}$ and $Z \subset S$ a closed subspace. Denote by $i : Z \hookrightarrow S$ the closed embedding.

(i) The functor $i^* : \text{Shv}_Z(S) \to \text{Shv}(Z)$ is an equivalence of category whose inverse is $i_* : \text{Shv}(Z) \to \text{Shv}_Z(S)$. More precisely $\text{ad}(i_*, i^*)(H) : i^*i_*H \to H$ is an isomorphism if $H \in \text{Shv}(Z)$ and $\text{ad}(i_*, i^*)(G) : G \to i_*i^*G$ is an isomorphism if $G \in \text{Shv}_Z(S)$.

(ii) : The functor $i^* : \text{Shv}_{\text{fil},Z}(S) \to \text{Shv}_{\text{fil}}(Z)$ is an equivalence of category whose inverse is $i_* : \text{Shv}_{\text{fil}}(Z) \to \text{Shv}_{\text{fil},Z}(S)$. More precisely $\text{ad}(i_*, i^*)(H,F) : i^*i_*H \to (H,F)$ is an isomorphism if $(H,F) \in \text{Shv}(Z)$ and $\text{ad}(i_*, i^*)(G,F) : (G,F) \to i_*i^*(G,F)$ is an isomorphism if $(G,F) \in \text{Shv}_Z(S)$.

(iii) : The functor $i^* : D_{\text{r,fil},Z}(S) \to D_{\text{r,fil}}(Z)$ is an equivalence of category whose inverse is $i_* : D_{\text{r,fil}}(Z) \to D_{\text{r,fil},Z}(S)$. More precisely $\text{ad}(i_*, i^*)(H,F) : i^*i_*H \to (H,F)$ is an equivalence top local if $(H,F) \in C_{\text{fil}}(Z)$ and $\text{ad}(i_*, i^*)(G,F) : (G,F) \to i_*i^*(G,F)$ is an equivalence top local if $(G,F) \in C_{\text{fil},Z}(S)$.

**Proof.**

(i): Standard.

(ii): Follows from (i).

(iii): Follows from (ii). \qed 

Let $S \in \text{Top}$ and $Z \subset S$ a closed subspace. By proposition 12, if $G \in \text{C}(S)$, $\text{ad}(i_*, i^*)(\Gamma_Z G) : \Gamma_Z G \to i_*i^*\Gamma_Z G$ is an equivalence top local since $\Gamma_Z G \in C_Z(S)$.

Let $(S,O_S) \in \text{RTop}$. Let $D = \bigcup_i D_i \subset X$ a normal crossing divisor, denote by $j : S\setminus D \hookrightarrow S$ the open embedding, and consider $\mathcal{I}_D \subset O_S$ the ideal of vanishing function on $D$ which is invertible. We set, for $M \in C_{O_S}(S)$,

$$M(*D) := \lim_n \text{Hom}_{O_S}(\mathcal{I}^n, M),$$

and we denote by $a_D(F) : F \to F(*D)$ the surjective morphism of presheaves. The adjonction map $\text{ad}(j^*, j_*)(F) : F \to j_*j^*F$ factors through $a_D(F)$:

$$\text{ad}(j^*, j_*)(F) : F \xrightarrow{a_D(F)} F(*D) \xrightarrow{a_{S/D}(F)} j_*j^*F$$
Remark 3. \( \bullet \) Let \( j : U \hookrightarrow X \) be an open embedding, with \((X, O_X) \in \text{RTop}\). Then if \( F \in \text{Coh}_{O_U}(U) \) is a coherent sheaf of \( O_U \) module, \( j_* F \) is quasi-coherent but NOT coherent in general. In particular for \( F \in \text{Coh}_{O_U}(U) \) whose cohomology sheaves \( a^n F \) are cohenent for all \( n \in \mathbb{Z} \), the cohomology sheaves \( R^n j_* F := a^n H^n j_* E(F) \) for \( Rj_* F = j_* E(F) \) are quasi-coherent but NOT coherent.

\( \bullet \) Let \( j : U \hookrightarrow X \) be an open embedding, with \( X \in \text{Sch} \). If \( F \in \text{Coh}(U) \) is a coherent sheaf of \( O_U \) module, \( j_* F \) is quasi-coherent but NOT coherent. However, there exist an \( O_X \) submodule \( F \subset j_* F \) such that \( j^* F = F \) and \( F \in \text{Coh}(X) \).

The following propositions are true for schemes but NOT for arbitrary ringed spaces like analytic spaces:

**Proposition 13.**
(i) Let \( X = (X, O_X) \in \text{Sch} \) be a noetherian scheme and \( D \subset X \) a closed subset. Denote by \( j : U = X \setminus D \hookrightarrow X \) an open embedding. Then for \( F \in \text{Qcoh}_{O_U}(U) \) a quasi coherent sheaf, \( j_* F \in \text{Qcoh}_{O_X}(X) \) is quasi-coherent and is the direct limit of its coherent subsheaves.

(ii) Let \( X = (X, O_X) \) be a noetherian scheme and \( D = \cup D_i \subset X \) a normal crossing divisor. Denote by \( j : U = X \setminus D \hookrightarrow X \) an open embedding. Then for \( F \in \text{Qcoh}_{O_U}(U) \) a quasi coherent sheaf, the canonical map \( a_{X/D}(F) : F(*D) \xrightarrow{\sim} j_* F \) is an isomorphism.

\[ \square \]

**Proposition 14.** Let \( S = (S, O_S) \in \text{Sch} \) and \( Z \subset S \) a closed subscheme. Denote by \( i : Z \hookrightarrow S \) the closed embedding.

(i) For \( G \in \text{Qcoh}_{S}(S) \), \( i^* G \) has a canonical structure of \( O_Z \) module. Moreover, the functor \( i^* : \text{Qcoh}_{S}(S) \rightarrow \text{Qcoh}(Z) \) is an equivalence of category whose inverse is \( i_* : \text{Qcoh}(Z) \rightarrow \text{Qcoh}_{S}(S) \).

(ii) The functor \( i^* : \text{Qcoh}_{fil,Z}(S) \rightarrow \text{Qcoh}_{fil,Z}(Z) \) is an equivalence of category whose inverse is \( i_* : \text{Qcoh}_{fil,Z}(Z) \rightarrow \text{Qcoh}_{fil,Z}(S) \).

(iii) The functor \( i^* : \text{D}_{OZ,fil,Z,qc}(S) \rightarrow \text{D}_{OZ,fil,Z,qc}(Z) \) is an equivalence of category whose inverse is \( i_* : \text{D}_{OZ,fil,Z,qc}(Z) \rightarrow \text{D}_{OZ,fil,Z,qc}(S) \).

**Proof.** (i): Standard.

(ii), (iii): Follows from (i).

**Definition 3.** Let \( (S, O_S) \in \text{RTop} \) a locally ringed space with \( O_S \) commutative. Consider an \( \kappa_S \in C_{O_S}(S) \). Let \( \mathcal{I} \subset O_S \) an ideal subsheaf and \( Z = V(\mathcal{I}) \subset S \) the associated closed subset. For \( G \in \text{PSh}_{O_S}(S) \), we denote by \( G_{\hat{\kappa}_S} := G_{\hat{\kappa}_S} := \lim_k G/I^k G \) the completion with respect to the ideal \( \mathcal{I} \) and by \( c^2_{\hat{\kappa}_S}(G) : G \rightarrow G_{\hat{\kappa}_S} \) the quotient map. Then, the canonical map

\[ d_{\kappa_S,Z}(G) : G \xrightarrow{\text{d}(G)} D_S^{O_S}G \xrightarrow{T_{\text{mod}}(\delta_{\kappa_S,Z})} \mathcal{H}_{\text{hom}_{O_S}}(\mathbb{D}_S^O G \otimes_{O_S} \kappa_S, \mathbb{D}_S^O G \otimes_{O_S} \kappa_S, \Gamma_Z E(\kappa_S)) \]

factors through

\[ d_{\kappa_S,Z}(G) : G \xrightarrow{c^2_{\hat{\kappa}_S}(G)} \hat{G}_Z \xrightarrow{d_{\kappa_S,Z}(G)} \mathcal{H}_{\text{hom}_{O_S}}(\Gamma_Z E(\mathbb{D}_S^O G \otimes_{O_S} \kappa_S), \Gamma_Z E(\kappa_S)) \]

Clearly if \( G \in C_{O_S}(S) \) then \( d_{\kappa_S,Z}(G) \) is a map in \( C_{O_S}(S) \). On the other hand, we have a commutative diagram

\[ \begin{array}{ccc}
\Omega^p_{O_S} & \xrightarrow{d_{\kappa_S,Z}(\Omega^p_{O_S})} & \mathcal{H}_{\text{hom}_{O_S}}(\Gamma_Z E(\mathbb{D}_S^O O_{O_S} \otimes_{O_S} \kappa_S), \Gamma_Z E(\kappa_S)) \\
\quad d \downarrow & & \quad d_{\kappa_S,Z}(\Omega^{p+1}_{O_S}) \downarrow d^{\kappa_S,Z} \\
\Omega^1_{O_S} & \xrightarrow{d_{\kappa_S,Z}(\Omega^{p+1}_{O_S})} & \mathcal{H}_{\text{hom}_{O_S}}(\Gamma_Z E(\mathbb{D}_S^O O_{O_S} \otimes_{O_S} \kappa_S), \Gamma_Z E(\kappa_S))
\end{array} \]
so that \( d_{\kappa,s} \Omega^1_S \in C(S) \).

The following theorem is the from [14]

**Theorem 8.** Let \( S \in \text{Var}(\mathbb{C}) \). Let \( Z = V(I) \subset S \) a closed subset. Denote by \( K_S \in \text{Psh}_{O_S}(S) \) the canonical bundle. Then, for \( G \in \text{C}O_{s,c}(S) \),

\[
d_{K_S,Z}(G) : \tilde{G}_Z \to \mathcal{H}om_{O_S}(\Gamma_Z E(\mathbb{D}_S^O G \otimes_{O_S} K_S), \Gamma_Z E(K_S))
\]

is an equivalence Zariski local.

Let \( f : (X,O_X) \to (S,O_S) \) a morphism with \( (S,O_S) \in \text{RTop} \). In the particular case where \( O_S \) is a commutative sheaf of ring, \( T_O \in \text{Psh}_{O_S}(S) \) and \( \Omega_{O_S} = \mathbb{D}_O T_O \in \text{Psh}_{O_S}(S) \) are sheaves and the morphism in \( \text{Psh}(X) \)

\[
T(f, \text{hom})(O_S,O_S) : f^* \mathcal{H}om(O_S,O_S) \to \mathcal{H}om(f^* O_S,f^* O_S)
\]

induces isomorphisms \( T(f, \text{hom})(O_S,O_S) : f^* T_O \Rightarrow T_{f^* O_S} \) and \( \mathbb{D}_{f^* O_S} T(f, \text{hom})(O_S,O_S) : \Omega_{f^* O_S} \to f^* \Omega_{O_S} \) where for \( F \in \text{Shv}(S) \), we denote again (as usual) by abuse \( f^* F := a_{f^*} f^* F \in \text{Shv}(S) \), \( a_{\text{tau}} : \text{Psh}(S) \to \text{Shv}(S) \) being the sheafification functor.

**Definition 4.**

(i) Let \( (X,O_X) \in \text{RTop} \). A foliation \( (X,O_X)/\mathcal{F} \) on \( (X,O_X) \) is an \( O_X \) module \( \Omega_{O_X}/\mathcal{F} \in \text{Psh}_{O_X}(X) \) together with a derivation map \( d := d_X : O_X \to \Omega_{O_X}/\mathcal{F} \) such that

- the associated map \( q := q_X := \omega_X(d) : \Omega_{O_X} \to \Omega_{O_X}/\mathcal{F} \) is surjective

- satisfy the integrability condition \( d(ker q) \subset \ker q \) which implies that the map \( d : \Omega^p_{O_X} \to \Omega^{p+1}_{O_X} \) induce factors trough

\[
\begin{array}{ccc}
\Omega^p_{O_X} & \xrightarrow{d} & \Omega^{p+1}_{O_X} \\
q^p := \land^p q & \downarrow & q^{p+1} := \land^p q \\
\Omega^p_{O_X}/\mathcal{F} & \xrightarrow{d} & \Omega^{p+1}_{O_X}/\mathcal{F}
\end{array}
\]

and \( d : \Omega^p_{O_X}/\mathcal{F} \to \Omega^{p+1}_{O_X}/\mathcal{F} \) is necessary unique by the surjectivity of \( q^p : \Omega^p_{O_X} \to \Omega^p_{O_X}/\mathcal{F} \).

In the particular case where \( \Omega_{O_X}/\mathcal{F} \in \text{Psh}_{O_X}(X) \) is a locally free sheaf of \( O_X \) module, \( \mathbb{D}_O X \Omega_{O_X}/\mathcal{F} \to \mathcal{T}_{O_X} \) is injective and the second condition is then equivalent to the fact that the sub \( O_X \) module \( \mathbb{D}_O X \mathcal{T}_{O_X}/\mathcal{F} \subset \mathcal{T}_{O_X} \) is a Lie subalgebra, that is \( [\mathcal{T}_{O_X}/\mathcal{F}, \mathcal{T}_{O_X}/\mathcal{F}] \subset \mathcal{T}_{O_X}/\mathcal{F} \).

(ii) A piece of leaf a foliation \( (X,O_X)/\mathcal{F} \) with \( (X,O_X) \in \text{RTop} \) such that \( O_X \) is a commutative sheaf of ring is an injective morphism of ringed spaces \( l : (Z,O_Z) \to (X,O_X) \) such that \( \Omega_{l^* O_X/O_Z} : \Omega_{l^* O_X} \to \Omega_{O_Z} \) factors trough an isomorphism

\[
\Omega_{l^* O_X/O_Z} : \Omega_{l^* O_X} \xrightarrow{\mathbb{D}_{l^* O_X} T(l, \text{hom})(O_X,O_X)} i^* \Omega_{O_X} \overset{i^* q}{\longrightarrow} i^* \Omega_{O_X}/\mathcal{F} \rightarrow \Omega_{O_X}.
\]

(iii) If \( f : (X,O_X) \to (S,O_S) \) is a morphism with \( (X,O_X),(S,O_S) \in \text{RTop} \), we have the foliation \( (X,O_X)/(S,O_S) := ((X,O_X),f) \) on \( (X,O_X) \) given by the surjection

\[ q : \Omega_{O_X} \to \Omega_{O_X}/f^* O_S := \text{coker}(\Omega_{O_X}/f^* O_S : \Omega_{f^* O_S} \rightarrow \Omega_{O_X}). \]

The fibers \( i_X : (X_s,O_X) \to (X,O_X) \) for each \( s \in S \) are the leaves of the foliation.

(iv) We have the category Fol\text{RTop}

- whose objects are foliated ringed spaces \( (X,O_X)/\mathcal{F} \) with \( O_X \) a commutative sheaf of ring and
whose morphisms $f : (X, O_X)/F \to (S, O_S)/G$ are morphisms of ringed spaces $f : (X, O_X) \to (S, O_S)$ such that $\Omega_{O_X/F \times O_S} : \Omega_{F \times O_S} \to \Omega_{O_X}$ factors through

\[
\begin{array}{ccc}
\Omega_{O_X/F} & \xrightarrow{\phi} & \Omega_{F \times O_S} \\
\downarrow & & \downarrow \\
\Omega_{O_X/F} & \xrightarrow{q_F} & \Omega_{O_X/F}
\end{array}
\]

This category admits inverse limits with $(X, O_X)/F \times (Y, O_Y)/G = (X \times Y, p_X^*O_X \otimes p_Y^*O_Y)/p_X^*F \otimes p_Y^*G$ and

\[
(X, O_X)/F \times (S, O_S)/H, (Y, O_Y)/G = (X \times S, \delta_S(p_X^*O_X \otimes p_Y^*O_Y))/p_X^*F \otimes p_Y^*G
\]

with $\delta_S : X \times S Y \to X \times Y$ the embedding given by the diagonal $\delta_S : S \to S \times S$.

Let $S \in \text{Top}$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \to \tilde{S}_j$ closed embeddings, with $\tilde{S}_j \in \text{Top}$ For $I \subseteq [1, \ldots, l]$, denote by $S_I = \Pi_{i \in I} S_i$. We then have closed embeddings $i_1 : S_I \to \tilde{S}_I$ and, for $J \subset I$, the following commutative diagram

\[
\begin{array}{ccc}
D_{IJ} & \xrightarrow{i_1} & \tilde{S}_I \\
\downarrow & & \downarrow \\
S_J & \xrightarrow{i_J} & \tilde{S}_J
\end{array}
\]

where $j_{IJ} : S_J \to S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$ and $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ the projection. This gives diagram of topological spaces $(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Top})$ which gives the diagram $(\tilde{S}_I) \in (\text{Ouv}(\tilde{S}_I)) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$ Denote $m : \tilde{S}_I \setminus (\tilde{S}_I \setminus S_J) \to \tilde{S}_I$ the open embedding.

**Definition 5.** Let $S \in \text{Top}$. Let $S = \cup_{i=1}^l S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \to \tilde{S}_j$ closed embeddings, with $\tilde{S}_j \in \text{Top}$. We denote by $C_{fil}(S)((\tilde{S}_I)) \subseteq C_{fil}(\tilde{S}_I)$ the full subcategory

- whose objects are $(G, F) = ((G_I, F)_{I \subseteq [1, \ldots, l]}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\tilde{S}_I)$, and $u_{IJ} : m^*(G_I, F) \to m^*(G_J, F)$ are $\tau$-filtered top local equivalences satisfying for $I \subset J \subset K$, $p_{IJ}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\tilde{S}_I)$,

- the morphisms $m : ((G_I, F), u_{IJ}) \to ((H_I, F), v_{IJ})$ being (see section 2.1) a family of morphism of complexes,

\[
m = (m_I : (G_I, F) \to (H_I, F))_{I \subseteq [1, \ldots, l]}
\]

such that $v_{IJ} \circ m_I = p_{IJ}m_J \circ u_{IJ}$ in $C_{fil}(\tilde{S}_I)$.

A morphism $m : ((G_I, F), u_{IJ}) \to ((H_I, F), v_{IJ})$ is an $\tau$-filtered top local equivalence if there exists $\phi_i : \left((C_{(i+1)I}, F), u_{i+1, iJ}\right) \to \left((C_{(i+1)I}, F), u_{i+1, iJ}\right)$, $0 \leq i \leq s$, with $((C_I, F), u_{IJ}) \in C_{fil}(S/(\tilde{S}_I)) ((C_{0I}, F), u_{0IJ}) = ((G_I, F), u_{IJ})$ and $((C_{sI}, F), u_{sIJ}) = ((H_I, F), v_{IJ})$, such that

\[
\phi = \phi_s \circ \cdots \phi_i \circ \cdots \phi_0 : ((G_I, F), u_{IJ}) \to ((H_I, F), v_{IJ})
\]

and $\phi_i : ((C_{iI}, F), u_{ij}) \to ((C_{(i+1)I}, F), u_{(i+1)j})$ either a filtered $\tau$ local equivalence or an $\tau$-filtered homotopy equivalence.

Denote $L = [1, \ldots, l]$ and for $I \subseteq L$, $p_0(0I) : S \times \tilde{S}_I \to S$, $p_I(0I) : S \times \tilde{S}_I \to S_I$ the projections. By definition, we have functors

- $T(S/(\tilde{S}_I)) : C_{fil}(S) \to C_{fil}(S/(\tilde{S}_I))$, $(G, F) \mapsto (i_Ij_I^*(G, F), I)$
• \( T((\tilde{S}_t)/S) : C_{fil}(S/(\tilde{S}_t)) \to C_{fil}(S), ((G_I, F), u_{IJ}) \mapsto \text{holim}_{I \subseteq L} p_{0(0)}^* \Gamma^\gamma_I p_{1(0)}^* (G_I, F). \)

Note that the functors \( T(S/(\tilde{S}_t)) \) are embedding, since

\[
\text{ad}(i_I^*, i_{IJ}^*): i_I^* i_{IJ}^* F \to j_I^* F
\]

are top local equivalence.

Let \( f : X \to S \) be a morphism, with \( X, S \in \text{Top} \). Let \( S = \cup_{i=1}^l S_i \) and \( X = \cup_{i=1}^l X_i \) be open covers and \( i_i : S_i \to \tilde{S}_t \), \( i_i : X_i \to \tilde{X}_i \) be closed embeddings, such that, for each \( i \in [1, l] \), \( f_i : X_i \to S_i \) lift to a morphism \( \tilde{f}_i : \tilde{X}_i \to \tilde{S}_i \). Then, \( f_I = f_{|X_I} : X_I = \cap_{i \in I} X_i \to S_I = \cap_{i \in I} S_i \) lift to the morphism

\[
\tilde{f}_I = \prod_{i \in I} \tilde{f}_i : \tilde{X}_I = \prod_{i \in I} \tilde{X}_i \to \tilde{S}_I = \prod_{i \in I} \tilde{S}_i
\]

Denote by \( p_{IJ} : \tilde{S}_J \to \tilde{S}_I \) and \( p_{IJ}^*: \tilde{X}_J \to \tilde{X}_I \) the projections. Consider for \( J \subseteq I \) the following commutative diagrams

\[
D_{IJ} = S_I \leftarrow \tilde{S}_I , \quad D_{IJ}' = X_I \leftarrow \tilde{X}_I , \quad D_{II} = S_I \leftarrow \tilde{S}_I
\]

\[
i_i : S_I \to \tilde{S}_I , \quad i_I : X_I \to \tilde{X}_I , \quad i_1 : S_I \to \tilde{S}_I
\]

\[
i_{IJ} : X_I \times X_J \leftarrow X_I \times \tilde{X}_J, \quad i_{IJ}^* : X_I \times \tilde{X}_J \leftarrow \tilde{X}_J
\]

\[
i_I^* : X_I \leftarrow \tilde{X}_I
\]

\[
i_I'^* : \tilde{X}_I \leftarrow \tilde{X}_I
\]

\[
i_{IJ}' : \tilde{X}_J \leftarrow \tilde{X}_J
\]

\[
i_J : X_J \leftarrow X_J
\]

We have then the pullback functor

\[
f^*: C_{(2)fil}(S/(\tilde{S}_t)) \to C_{(2)fil}(X/(\tilde{X}_I)),
\]

\[
((G_I, F), u_{IJ}) \mapsto f^*((G_I, F), u_{IJ}) := (\Gamma^\gamma_{X_I} \tilde{f}_I^*(G_I, F), \tilde{f}_J^* u_{IJ})
\]

with

\[
\tilde{f}_J^* u_{IJ} : \Gamma^\gamma_{X_I} \tilde{f}_I (G_I, F) \overset{\text{ad}(p_{IJ}^*, p_{IJ}^*, (-)(-))}{\longrightarrow} \Gamma^\gamma_{X_I} \Gamma^\gamma_{X_I} \tilde{f}_I (G_I, F) \overset{T_{(p_{IJ}^*, n_{IJ}^*)}(-)^{-1}}{\longrightarrow} \Gamma^\gamma_{X_I} \Gamma^\gamma_{X_I} \tilde{f}_I (G_I, F) \overset{p_{IJ}^*}{\longrightarrow} \Gamma^\gamma_{X_I} \tilde{f}_I (G_I, F)
\]

\[
\tilde{f}_J^* i_{IJ}^* f^*(G, F) = 0,
\]

the morphism \( T(D_{IJ})(j_I^*(G, F)) : \tilde{j}_I^* i_{IJ}^* f^*(G, F) \to i_{IJ}^* j_I^* f^*(G, F) \) factors through

\[
T(D_{IJ})(j_I^*(G, F)) : \tilde{j}_I^* i_{IJ}^* f^*(G, F) \overset{\gamma_{X_I}}{\longrightarrow} \Gamma^\gamma_{X_I} \tilde{f}_I^* i_{IJ}^* j_I^* f^*(G, F) \overset{T_{(D_{IJ})(j_I^*(G, F))}}{\longrightarrow} i_{IJ}^* j_I^* f^*(G, F)
\]

We have then, for \( (G, F) \in C_{fil}(S) \), the canonical transformation map

\[
\text{ad}(i_I^*, i_{IJ}^*): i_I^* i_{IJ}^* F \to j_I^* F
\]
Proposition 15. Let $S \in \text{Top}$. Let $S = \bigcup_{i=1}^{j} S_i$ an open cover and denote by $S_i = \cap_{i \in I} S_i$. Let $i_i : S_i \to \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Top}$. Denote by $D_{(2)fil,\infty}((S_i/\tilde{S}_i)) = \text{Ho}_{\text{top,\infty}}(C_{(2)fil}((S_i/\tilde{S}_i)))$ the localization of $C_{(2)fil}((S_i/\tilde{S}_i))$ with respect to top local equivalences. The functor $T(S_i/\tilde{S}_i)$ induces an equivalence of category $T(S_i/\tilde{S}_i) : D_{(2)fil,\infty}(S) \sim \sim D_{(2)fil,\infty}((S_i/\tilde{S}_i))$

with inverse $T((\tilde{S}_i)/S)$

Proof. Follows from the fact that for $(G,F) \in C_{fil}(S)$, $\text{ho lim}_{i \in L} p_{0(\partial T)} \Gamma_{\tilde{S}_i}^* p^{*}_{i(\partial T)}(i_1 \cdot j^*_I(G,F)) \to p_{0(\partial T)} \Gamma_{\tilde{S}_i}^* j^*_I(G,F)$

is an equivalence top local.

For $f : T \to S$ a morphism with $T, S \in \text{Top}$ locally compact (in particular Hausdorff), e.g. $T, S \in \text{CW}$, there is also a functor $f_! : C(T) \to C(S)$ given by the section which have compact support over $f$, and, for $K_1, K_2 \in C(T)$, we have a canonical map $T_!(f_!, \text{Hom}) : f_! \text{Hom}(K_1, K_2) \to \text{Hom}(f_! K_1, f_! K_2)$

The main result on presheaves on locally compact spaces is the following:

Theorem 9. Let $f : T \to S$ a morphism with $T, S \in \text{Top}$ locally compact.

(i) The derived functor $Rf_! : D(T) \to D(S)$ has a right adjoint $f^!$ (Verdier duality) and, for $K_1, K_2 \in D(T)$ and $K_3, K_4 \in D(S)$, we have canonical isomorphisms

- $Rf_* R\text{Hom}^*(Rf_! K_1, K_3) \sim \sim R\text{Hom}^*(K_1, f^! K_3)$
- $f^! R\text{Hom}^*(K_3, K_4) \sim \sim R\text{Hom}^*(f^* K_3, f^! K_4)$

(ii) Denote, for $K \in D(S)$, $D(K) = R\text{Hom}^*(K, a^!_2 \mathbb{Z}) \in D(S)$ the Verdier dual of $K$. Then, if $K \in D_c(S)$, the evaluation map $\text{ev}^c(S)(K) : K \to D(D(K)$ is an isomorphism.

(iii) Assume we have a factorization $f : T \xrightarrow{i} Y \xrightarrow{p} S$ of $f$ with $l$ a closed embedding and $p$ a smooth morphism of relative dimension $d$. Then $f^! K = i^! p^* K[d]$.

Proof. (i):Standard, the proof is formal (see [31]).

(ii): See [31].

(iii): The fact that $p^! K = p^* K[d]$ follows by Poincaré duality for topological manifold.  

We have by theorem 9 a pair of adjoint functors $(Rf_!, f^!) : D(T) \rightleftarrows D(S)$

- with $f_! = f_*$ if $f$ is proper,
- with $f^! = f^*[d]$ if $f$ is smooth of relative dimension $d$.

2.6 Presheaves on the big Zariski site or on the big etale site

For $S \in \text{Var}(\mathbb{C})$, we denote by $\rho_S : \text{Var}(\mathbb{C})^{sm}/S \hookrightarrow \text{Var}(\mathbb{C})/S$ be the full subcategory consisting of the objects $U/S = (U, h) \in \text{Var}(\mathbb{C})/S$ such that the morphism $h : U \to S$ is smooth. That is, $\text{Var}(\mathbb{C})^{sm}/S$ is the category

- whose objects are smooth morphisms $U/S = (U, h), h : U \to S$ with $U \in \text{Var}(\mathbb{C})$,
- whose morphisms $g : U/S = (U, h_1) \to V/S = (V, h_2)$ is a morphism $g : U \to V$ of complex algebraic varieties such that $h_2 \circ g = h_1$.  

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We denote again \( \rho_S : \text{Var}(\mathcal{C})/S \to \text{Var}(\mathcal{C})^{\text{sm}}/S \) the associated morphism of site. We will consider

\[
r^*(S) : \text{Var}(\mathcal{C}) \xrightarrow{r(S)} \text{Var}(\mathcal{C})/S \xrightarrow{\rho_S} \text{Var}(\mathcal{C})^{\text{sm}}/S
\]

the composite morphism of site. For \( S \in \text{Var}(\mathcal{C}) \), we denote by \( Z_S := Z(S/S) \in \text{PSh}(\text{Var}(\mathcal{C})^{\text{sm}}/S) \) the constant presheaf By Yoneda lemma, we have for \( S \) the composite morphism of site. For \( S \in \text{Var}(\mathcal{C}) \), we have for \( F \in C(\text{Var}(\mathcal{C})^{\text{sm}}/S) \), \( \text{Hom}(Z_S, F) = F \). For \( f : T \to S \) a morphism, with \( T, S \in \text{Var}(\mathcal{C}) \), we have the following commutative diagram of sites

\[
\begin{array}{c}
\text{Var}(\mathcal{C})/T \xrightarrow{\rho_T} \text{Var}(\mathcal{C})^{\text{sm}}/T \\
\downarrow P(f) \downarrow \downarrow P(f) \\
\text{Var}(\mathcal{C})/S \xrightarrow{\rho_S} \text{Var}(\mathcal{C})^{\text{sm}}/S
\end{array}
\]

We denote, for \( S \in \text{Var}(\mathcal{C}) \), the obvious morphism of sites

\[
\tilde{\epsilon}(S) : \text{Var}(\mathcal{C})/S \xrightarrow{\rho_S} \text{Var}(\mathcal{C})^{\text{sm}}/S \xrightarrow{\epsilon(S)} \text{Ouv}(S)
\]

where \( \text{Ouv}(S) \) is the set of the Zariski open subsets of \( S \), given by the inclusion functors \( \tilde{\epsilon}(S) : \text{Ouv}(S) \to \text{Var}(\mathcal{C})^{\text{sm}}/S \to \text{Var}(\mathcal{C})/S \). By definition, for \( f : T \to S \) a morphism with \( S, T \in \text{Var}(\mathcal{C}) \), the commutative diagram of sites (23) extend a commutative diagram of sites:

\[
\begin{array}{c}
\tilde{\epsilon}(T) : \text{Var}(\mathcal{C})/T \xrightarrow{\rho_T} \text{Var}(\mathcal{C})^{\text{sm}}/T \xrightarrow{\epsilon(T)} \text{Ouv}(T) \\
\downarrow P(f) \downarrow \downarrow P(f) \\
\tilde{\epsilon}(S) : \text{Var}(\mathcal{C})/S \xrightarrow{\rho_S} \text{Var}(\mathcal{C})^{\text{sm}}/S \xrightarrow{\epsilon(S)} \text{Ouv}(S)
\end{array}
\]

- As usual, we denote by
  \[
  (f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{Var}(\mathcal{C})^{\text{sm}}/S) \to C(\text{Var}(\mathcal{C})^{\text{sm}}/T)
  \]
  the adjonction induced by \( P(f) : \text{Var}(\mathcal{C})^{\text{sm}}/T \to \text{Var}(\mathcal{C})^{\text{sm}}/S \). Since the colimits involved in the definition of \( f^* = P(f)^* \) are filtered, \( f^* \) also preserve monomorphism. Hence, we get an adjonction
  \[
  (f^*, f_*) : C_{fil}(\text{Var}(\mathcal{C})^{\text{sm}}/S) \leftrightarrows C_{fil}(\text{Var}(\mathcal{C})^{\text{sm}}/T), \ f^*(G, F) := (f^*G, f^*F)
  \]

- As usual, we denote by
  \[
  (f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{Var}(\mathcal{C})/S) \to C(\text{Var}(\mathcal{C})/T)
  \]
  the adjonction induced by \( P(f) : \text{Var}(\mathcal{C})/T \to \text{Var}(\mathcal{C})/S \). Since the colimits involved in the definition of \( f^* = P(f)^* \) are filtered, \( f^* \) also preserve monomorphism. Hence, we get an adjonction
  \[
  (f^*, f_*) : C_{fil}(\text{Var}(\mathcal{C})/S) \leftrightarrows C_{fil}(\text{Var}(\mathcal{C})/T), \ f^*(G, F) := (f^*G, f^*F)
  \]

- For \( h : U \to S \) a smooth morphism with \( U, S \in \text{Var}(\mathcal{C}) \), the pullback functor \( P(h) : \text{Var}(\mathcal{C})^{\text{sm}}/S \to \text{Var}(\mathcal{C})^{\text{sm}}/U \) admits a left adjoint \( C(h)(X \to U) = (X \to U \to S) \). Hence, \( h^* : C(\text{Var}(\mathcal{C})^{\text{sm}}/S) \to C(\text{Var}(\mathcal{C})^{\text{sm}}/U) \) admits a left adjoint
  \[
  h^*_t : C(\text{Var}(\mathcal{C})^{\text{sm}}/U) \to C(\text{Var}(\mathcal{C})^{\text{sm}}/S), \ F \mapsto \lim_{(V, h_0) \to (V', h_0 \circ h')} F(V', h')
  \]
  Note that we have for \( V/U = (V, h') \) with \( h' : V \to U \) a smooth morphism we have \( h_0(Z(V/U)) = Z(V'/S) \) with \( V'/S = (V', h \circ h') \). Hence, since projective sheaves are the direct summands of
the representable presheaves, \( h_2 \) sends projective presheaves to projective presheves. For \( F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S) \) and \( G^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/U) \), we have the adjonction maps

\[
\text{ad}(h_2, h^\ast)(G^\bullet) : G^\bullet \to h^\ast h_2 G^\bullet, \quad \text{ad}(h_2, h^\ast)(F^\bullet) : h_2 h^\ast F^\bullet \to F^\bullet.
\]

For a smooth morphism \( h : U \to S \), with \( U, S \in \text{Var}(\mathbb{C}) \), we have the adjonction isomorphism, for \( F \in C(\text{Var}(\mathbb{C})^{sm}/U) \) and \( G \in C(\text{Var}(\mathbb{C})^{sm}/S) \),

\[
I(h_2, h^\ast)(F, G) : \mathcal{H}\text{om}^\ast(h_2 F, G) \isom h_* \mathcal{H}\text{om}^\ast(F, h^\ast G).
\]  

(25)

- For \( f : T \to S \) any morphism with \( T, S \in \text{Var}(\mathbb{C}) \), the pullback functor \( P(f) : \text{Var}(\mathbb{C})/T \to \text{Var}(\mathbb{C})/S \) admits a left adjoint \( C(f)(X \to T) = (X \to T \to S) \). Hence, \( f^\ast : C(\text{Var}(\mathbb{C})/S) \to C(\text{Var}(\mathbb{C})/T) \) admits a left adjoint

\[
f_2 : C(\text{Var}(\mathbb{C})/T) \to C(\text{Var}(\mathbb{C})/S), \quad F \mapsto \lim_{(V', f_0^\ast V) \to (V, h_0^\ast V)} F(V', f_0^\ast V)
\]

Note that we have, for \( (V/T) = (V, h) \), \( f_2^\ast Z(V/T) = Z(V/S) \) with \( V/S = (V, f \circ h) \). Hence, since projective presheves are the direct summands of the representable presheves, \( h_2 \) sends projective presheves to projective presheves. For \( F^\bullet \in C(\text{Var}(\mathbb{C})/S) \) and \( G^\bullet \in C(\text{Var}(\mathbb{C})/T) \), we have the adjonction maps

\[
\text{ad}(f_2, f^\ast)(G^\bullet) : G^\bullet \to f^\ast f_2 G^\bullet, \quad \text{ad}(f_2, f^\ast)(F^\bullet) : f_2 f^\ast F^\bullet \to F^\bullet.
\]

For a morphism \( f : T \to S \), with \( T, S \in \text{Var}(\mathbb{C}) \), we have the adjonction isomorphism, for \( F \in C(\text{Var}(\mathbb{C})/T) \) and \( G \in C(\text{Var}(\mathbb{C})/S) \),

\[
I(f_2, f^\ast)(F, G) : \mathcal{H}\text{om}^\ast(f_2 F, G) \isom f_* \mathcal{H}\text{om}^\ast(F, f^\ast G).
\]  

(26)

- For a commutative diagram in \( \text{Var}(\mathbb{C}) \):

\[
\begin{array}{ccc}
D = V & \xrightarrow{g_2} & U \\
\downarrow h_2 & & \downarrow h_1 \\
T & \xleftarrow{g_1} & S
\end{array}
\]

where \( h_1 \) and \( h_2 \) are smooth, we denote by, for \( F^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/U) \),

\[
T_2(D)(F^\bullet) : h_2 g_2^\ast F^\bullet \to g_1^\ast h_1^\ast F^\bullet
\]

the canonical map in \( C(\text{Var}(\mathbb{C})^{sm}/T) \) given by adjonction. If \( D \) is cartesian with \( h_1 = h, \ g_1 = g \)

\[
f_2 = h' : U_T \to T, \quad g' : U_T \to U,
\]

\[
T_2(D)(F^\bullet) = T_2(g, h)(F^\bullet) : h_2^\ast g'^\ast F^\bullet \isom g^\ast h_2 F^\bullet
\]

is an isomorphism and for \( G^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/T) \)

\[
T(D)(G^\bullet) = T(g, h)(G^\bullet) : g^\ast h_2^\ast G^\bullet \isom h_2^\ast g'^\ast G^\bullet
\]

is an isomorphism.

- For a commutative diagram in \( \text{Var}(\mathbb{C}) \):

\[
\begin{array}{ccc}
D = V & \xrightarrow{g_2} & X \\
\downarrow f_2 & & \downarrow f_1 \\
T & \xleftarrow{g_1} & S
\end{array}
\]
we denote by, for \( F^\bullet \in C(\Var(\mathbb{C})/X) \),

\[
T_2(D)(F^\bullet) : f_2^*g_2^*F^\bullet \to g_2^*f_1^*F^\bullet
\]

the canonical map in \( C(\Var(\mathbb{C})/T) \) given by adjunction. If \( D \) is cartesian with \( h_1 = h, \ g_1 = g \)
\( f_2 = h' : X_T \to T, \ g' : X_T \to X \),

\[
T_2(D)(F^\bullet) =: T_2(g,f)(F^\bullet) : f_2^*g^*F^\bullet \cong g^*f_2^*F^\bullet
\]
is an isomorphism and for \( G^\bullet \in C(\Var(\mathbb{C})/T) \)

\[
T(D)(G^\bullet) =: T(g,h)(G^\bullet) : f^*g_*G^\bullet \cong g_*f^*G^\bullet
\]
is an isomorphism.

For \( f : T \to S \) a morphism with \( S, T \in \Var(\mathbb{C}) \),

- we get for \( F \in C(\Var(\mathbb{C})^{sm}/S) \) from the a commutative diagram of sites (24) the following canonical transformation

\[
T(e,f)(F^\bullet) : f^*e(S)_*F^\bullet \to e(T)_*f^*F^\bullet,
\]

which is NOT a quasi-isomorphism in general. However, for \( h : U \to S \) a smooth morphism with \( S, U \in \Var(\mathbb{C}) \), \( T(e,h)(F^\bullet) : h^*e(S)_*F^\bullet \cong e(T)_*h^*F^\bullet \) is an isomorphism.

- we get for \( F \in C(\Var(\mathbb{C})/S) \) from the a commutative diagram of sites (24) the following canonical transformation

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Let \( S \in \Var(\mathbb{C}) \),

- We have for \( F, G \in C(\Var(\mathbb{C})^{sm}/S) \),
  - \( e(S)_*(F \otimes G) = (e(S)_*F) \otimes (e(S)_*G) \) by definition
  - the canonical forgetful map

\[
T(S, \Hom)(F,G) : e(S)_*\Hom^\bullet(F,G) \to \Hom^\bullet(e(S)_*F, e(S)_*G).
\]

which is NOT a quasi-isomorphism in general.

By definition, we have for \( F \in C(\Var(\mathbb{C})^{sm}/S) \), \( e(S)_*E_{zar}(F) = E_{zar}(e(S)_*F) \).

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By definition, we have for \( F \in C(\Var(\mathbb{C})/S) \), \( e(S)_*E_{zar}(F) = E_{zar}(e(S)_*F) \).

Let \( S \in \Var(\mathbb{C}) \).
We have the dual functor
\[ \mathbb{D}_S : C(\text{Var}(\mathbb{C})^{sm}/S) \to C(\text{Var}(\mathbb{C})^{sm}/S), \ F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}_S)) \]
It induces the functor
\[ L\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{sm}/S) \to C(\text{Var}(\mathbb{C})^{sm}/S), \ F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}_S)) \]
We have the dual functor
\[ \mathbb{D}_S : C(\text{Var}(\mathbb{C})/S) \to C(\text{Var}(\mathbb{C})/S), \ F \mapsto \mathbb{D}_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}_S)) \]
It induces the functor
\[ L\mathbb{D}_S : C(\text{Var}(\mathbb{C})/S) \to C(\text{Var}(\mathbb{C})/S), \ F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}_S)) \]

The adjunctions
\[ (\tilde{e}(S)^*, \tilde{e}(S)_*) : C(\text{Var}(\mathbb{C})/S) \rightleftharpoons C(S), \ (e(S)^*, e(S)_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftharpoons C(S) \]
induce adjunctions
\[ (\tilde{e}(S)^*, \tilde{e}(S)_*) : C_{fil}(\text{Var}(\mathbb{C})/S) \rightleftharpoons C_{fil}(S), \ (e(S)^*, e(S)_*) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \rightleftharpoons C_{fil}(S) \]
given by \( e(S)_*G, F := (e(S)_*G, e(S)_*F), \) since \( e(S)_* \) and \( e(S)^* \) preserve monomorphisms. Note that
- for \( F \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S) \), \( e(S)_*F \) is simply the restriction of \( F \) to the small Zariski site of \( X \),
- for \( F \in \text{PSh}(\text{Var}(\mathbb{C})/S) \), \( \tilde{e}(S)_*F = e(S)_*\rho_{S*}F \) is simply the restriction of \( F \) to the small Zariski site of \( X \), \( \rho_{S*}F \) being the restriction of \( F \) to \( \text{Var}(\mathbb{C})^{sm}/S \).

Together with the internal hom functor, we get the bifunctor,
\[ e(S)_*\mathcal{H}om(\cdot, \cdot) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \times C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \to C_{2fil}(S), \]
\[ ((F, W), (G, F)) \mapsto e(S)_*\mathcal{H}om^*((F^*, W), (G^*, F)). \]

For \( i : Z \hookrightarrow S \) a closed embedding, with \( Z, S \in \text{Var}(\mathbb{C}) \), we denote by
\[ (i_*, i^!): = (P(i)_*, P(i)^!): C(\text{Var}(\mathbb{C})^{sm}/Z) \rightleftharpoons C(\text{Var}(\mathbb{C})^{sm}/S) \]
the adjunction induced by the morphism of site \( P(i) : \text{Var}(\mathbb{C})^{sm}/Z \to \text{Var}(\mathbb{C})^{sm}/S \) For \( i : Z \hookrightarrow S \) a closed embedding, \( Z, S \in \text{Var}(\mathbb{C}) \), we denote
\[ Z_{Z,S} := \text{Cone}(\text{ad}(i^*, i_*)(Z_S)) : Z_S \to i_*Z_Z \]
We have the support section functors of a closed embedding \( i : Z \hookrightarrow S \) for presheaves on the big Zariski site.

**Definition 6.** Let \( i : Z \hookrightarrow S \) be a closed embedding with \( S, Z \in \text{Var}(\mathbb{C}) \) and \( j : S\setminus Z \hookrightarrow S \) be the open complementary subset.

(i) We define the functor
\[ \Gamma_Z : C(\text{Var}(\mathbb{C})^{sm}/S) \to C(\text{Var}(\mathbb{C})^{sm}/S), \ G^* \mapsto \Gamma_ZG^* := \text{Cone}(\text{ad}(j_*, j^!)(G^*): G^* \to j_*j^*G^*)[-1], \]
so that there is then a canonical map \( \gamma_Z(G^*) : \Gamma_ZG^* \to G^* \).
(ii) We have the dual functor of (i):
\[ \Gamma_Z^\vee : C(\Var(\mathbb{C})^{sm}/S) \to C(\Var(\mathbb{C})^{sm}/S), \ F \mapsto \Gamma_Z^\vee(F^\bullet) := \text{Cone}(\text{ad}(j_Z,j^\bullet)(G^\bullet) : j_Z^*G^\bullet \to G^\bullet), \]
together with the canonical map \( \gamma_Z^\vee(G) : F \to \Gamma_Z^\vee(G) \).

(iii) For \( F,G \in C(\Var(\mathbb{C})^{sm}/S) \), we denote by
\[ \begin{aligned}
I(\gamma,\text{hom})(F,G) := (1,1(j_Z,j^\bullet)(F,G)^{-1}) : \Gamma_Z\text{Hom}(F,G) \xrightarrow{\sim} \text{Hom}(\Gamma_Z^\vee F,G)
\end{aligned} \]
the canonical isomorphism given by adjunction.

Let \( i : Z \hookrightarrow S \) be a closed embedding with \( S,Z \in \Var(\mathbb{C}) \) and \( j : S \setminus Z \to S \) be the open complement.

- For \( G \in C(\Var(\mathbb{C})^{sm}/S) \), the adjunction map \( \text{ad}(i_*,i^!)(G) : i_*i^!G \to G \) factor through \( \gamma_Z(G) \):
  \[ \text{ad}(i_*,i^!)(G) : i_*i^!G \xrightarrow{\text{ad}(i_*,i^!)\gamma(G)} \Gamma_Z(G) \xrightarrow{\gamma_Z(G)} G. \]
  However, note that when dealing with the big sites \( P(i) : \Var(\mathbb{C})^{sm}/Z \to \Var(\mathbb{C})^{sm}/S \), if \( G \in C(\Var(\mathbb{C})^{sm}/S) \) is NOT in general in the image of the functor \( i_* : C(\Var(\mathbb{C})^{sm}/Z) \to C(\Var(\mathbb{C})^{sm}/S) \).

- For \( G \in C(\Var(\mathbb{C})^{sm}/S) \), the adjunction map \( \text{ad}(i^*,i_*)(G) : G \to i_*i^*G \) factor through \( \gamma_Z^\vee(G) \):
  \[ \text{ad}(i^*,i_*)(G) : G \xrightarrow{\gamma_Z^\vee(G)} \Gamma_Z^\vee G \xrightarrow{\text{ad}(i^*,i_!)(G)} i_*i^*G, \]
and as in (i), \( \text{ad}(i^*,i_!)(G) \gamma(G) : \Gamma_Z^\vee(G) \to i_*i^*G \) is NOT an homotopy equivalence but

Let \( i : Z \hookrightarrow S \) be a closed embedding with \( S,Z \in \Var(\mathbb{C}) \).

- Since \( \Gamma_Z : C(\Var(\mathbb{C})^{sm}/S) \to C(\Var(\mathbb{C})^{sm}/S) \) preserve monomorphism, it induces a functor
  \[ \Gamma_Z : C_{fil}(\Var(\mathbb{C})^{sm}/S) \to C_{fil}(\Var(\mathbb{C})^{sm}/S), \ (G,F) \mapsto \Gamma_Z(G,F) := (\Gamma_ZG,\Gamma_ZF) \]
- Since \( \Gamma_Z^\vee : C(\Var(\mathbb{C})^{sm}/S) \to C(\Var(\mathbb{C})^{sm}/S) \) preserve monomorphism, it induces a functor
  \[ \Gamma_Z^\vee : C_{fil}(\Var(\mathbb{C})^{sm}/S) \to C_{fil}(\Var(\mathbb{C})^{sm}/S), \ (G,F) \mapsto \Gamma_Z^\vee(G,F) := (\Gamma_Z^\vee G,\Gamma_Z^\vee F) \]

**Definition-Proposition 4.**

(i) Let \( g : S' \to S \) a morphism and \( i : Z \hookrightarrow S \) a closed embedding with \( S',S,Z \in \Var(\mathbb{C}) \). Then, for \( (G,F) \in C_{fil}(\Var(\mathbb{C})^{sm}/S') \), there exist a map in \( C_{fil}(\Var(\mathbb{C})^{sm}/S') \)
\[ T(g,\gamma)(G,F) : g^*\Gamma_Z(G,F) \to \Gamma_{Z \times_S S'}g^*(G,F) \]
unique up to homotopy such that \( \gamma_{Z \times_S S'}(g^*(G,F)) \circ T(g,\gamma)(G,F) = g^*\gamma_Z(G,F) \).

(ii) Let \( i_1 : Z_1 \hookrightarrow S, \ i_2 : Z_2 \hookrightarrow Z_1 \) be closed embeddings with \( S,Z_1,Z_2 \in \Var(\mathbb{C}) \). Then, for \( (G,F) \in C_{fil}(\Var(\mathbb{C})^{sm}/S) \),
- there exist a canonical map \( T(Z_2/Z_1,\gamma_1)(G,F) : \Gamma_{Z_2/Z_1}(G,F) \to \Gamma_{Z_1}(G,F) \) in \( C_{fil}(\Var(\mathbb{C})^{sm}/S) \)
unique up to homotopy such that \( \gamma_{Z_2}(G,F) \circ T(Z_2/Z_1,\gamma_1)(G,F) = \gamma_{Z_1}(G,F) \), together with a distinguish triangle
\[ \begin{aligned}
\Gamma_{Z_1}(G,F) \xrightarrow{T(Z_2/Z_1,\gamma_1)(G,F)} \Gamma_{Z_1}(G,F) \xrightarrow{\text{ad}(j_{Z_2/Z_1})\gamma_{Z_2}(G,F)} \Gamma_{Z_1\setminus Z_2}(G,F) \to \Gamma_{Z_2}(G,F)[1]
\end{aligned} \]
in \( K_{fil}(\Var(\mathbb{C})^{sm}/S) := K(\mathbb{P}Sh_{fil}(\Var(\mathbb{C})^{sm}/S)) \),

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there exist a map $T(Z_2/Z_1, \gamma^\vee)(G, F) : \Gamma_{Z_2}^\vee(G, F) \to \Gamma_{Z_1}^\vee(G, F)$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ unique up to homotopy such that $\gamma_{Z_2}^\vee(G, F) = T(Z_2/Z_1, \gamma^\vee)(G, F) \circ \gamma_{Z_1}^\vee(G, F)$, together with a distinguished triangle

\[
\Gamma_{Z_1\setminus Z_2}^\vee(G, F) \xrightarrow{\text{add}(j_2, j_2^*)/(\Gamma_{Z_1}^\vee(G, F))} \Gamma_{Z_1}^\vee(G, F) \xrightarrow{T(Z_2/Z_1, \gamma^\vee)(G, F)} \Gamma_{Z_2}^\vee(G, F) \to \Gamma_{Z_1\setminus Z_2}^\vee(G, F)[1]
\]

in $K_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$.

(iii) Consider a morphism $g : (S', Z') \to (S, Z)$ with $(S', Z'), (S, Z) \in \text{Var}^2(\mathbb{C})$ We denote, for $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ the composite

\[
T(D, \gamma^\vee)(G) : g^*\Gamma_{Z}^\vee \xrightarrow{\sim} \Gamma_{Z\times S\times S'}^\vee \xrightarrow{T(X/Z\times S', \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^*G
\]

and we have then the factorization $\gamma_{Z'}^\vee(g^*G) : g^*G \xrightarrow{g^*\gamma_{Z}^\vee(G)} g^*\Gamma_{Z}^\vee \xrightarrow{T(D, \gamma^\vee)(G)} \Gamma_{Z'}^\vee g^*G$.

Proof. (i): We have the cartesian square

\[
\begin{array}{ccc}
S' & \xrightarrow{g} & S \\
\downarrow{g} & & \downarrow{g} \\
S\setminus Z \times S & \xrightarrow{j} & S
\end{array}
\]

and the map is given by

\[(I, T(g, j)(j^*G)) : \text{Cone}(g^*G \to g^*j_*j^*G) \to \text{Cone}(g^*G \to j_1^*j_1^*G = j_*j^*G).
\]

(ii): Follows from the fact that $j_1^1\Gamma_{Z_2}G = 0$ and $j_1^*\Gamma_{Z_2}^\vee G = 0$, with $j_1 : S\setminus Z_1 \hookrightarrow S$ the closed embedding.

(iii): Obvious.

The following easy proposition concern the restriction from the big Zariski site to the small site Zariski site:

**Proposition 16.** For $f : T \to S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding, with $Z, S, T \in \text{Var}(\mathbb{C})$, we have

(i) $e(S)_*f_* = f_*e(T)_*$ and $e(S)^*f_* = f_*e(T)^*$

(ii) $e(S)_*\Gamma_Z = \Gamma_Z e(S)_*$.

Proof. (i): The first equality $e(S)_*f_* = f_*e(T)_*$ is given by the diagram (24). The second equality is immediate from definition after a direct computation.

(ii) For $G^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$, we have the canonical equality

\[
e(S)_*\Gamma_Z(G^\bullet) = e(S)_*\text{Cone}(G \to j_*j^*G^\bullet)[-1] = \text{Cone}(e(S)_*G^\bullet \to e(S)_*j_*j^*G^\bullet)[-1] = \text{Cone}(e(S)_*G^\bullet \to j_*j^*e(S)_*G^\bullet)[-1] = \Gamma_Z e(S)_*G^\bullet
\]

by (i) and since $j : S\setminus Z \hookrightarrow S$ is a smooth morphism.

**Definition 7.** For $S \in \text{Var}(\mathbb{C})$, we denote by

\[
C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S) := C_{e(S)^*O_S}(\text{Var}(\mathbb{C})^{sm}/S)
\]

the category of complexes of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ endowed with a structure of $e(S)^*O_S$ module, and by

\[
C_{O_S}^{fil}(\text{Var}(\mathbb{C})^{sm}/S) := C_{e(S)^*O_S}^{fil}(\text{Var}(\mathbb{C})^{sm}/S)
\]

the category of filtered complexes of presheaves on $\text{Var}(\mathbb{C})^{sm}/S$ endowed with a structure of $e(S)^*O_S$ module.
Let \( S \in \text{Var}(\mathbb{C}) \). Let \( Z \subset S \) a closed subset. Denote by \( j : S \setminus Z \rightarrow S \) the open complementary embedding.

- For \( G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S) \), \( \Gamma_Z G := \text{Cone}(\text{ad}(j^*,j_*)(G) : F \rightarrow j_*j^*G)[-1] \) has a (unique) structure of \( e(S)^*O_S \) module such that \( \gamma_Z(G) : \Gamma_Z G \rightarrow G \) is a map in \( C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S) \). This gives the functor

\[
\Gamma_Z : C_{O_S}^{\text{filt}}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{\text{filt}O_S}(\text{Var}(\mathbb{C})^{sm}/S), \quad (G,F) \mapsto \Gamma_Z(G,F) := (\Gamma_Z G, \Gamma_Z F),
\]

together with the canonical map \( \gamma_Z((G,F) : \Gamma_Z(G,F) \rightarrow (G,F) \). Let \( Z_2 \subset Z \) a closed subset. Then, for \( G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S) \), \( T(Z_2/Z,\gamma)(G) : \Gamma_{Z_2}G \rightarrow \Gamma_Z G \) is a map in \( C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S) \) (i.e. is \( e(S)^*O_S \) linear).

- For \( G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S) \), \( \Gamma_Z' G := \text{Cone}(\text{ad}(j_*,j^*)(G) : j_*j^*G \rightarrow G) \) has a unique structure of \( e(S)^*O_S \) module, such that \( \gamma_Z'(G) : G \rightarrow \Gamma_Z' G \) is a map in \( C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S) \). This gives the functor

\[
\Gamma_Z' : C_{O_S}^{\text{filt}}(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{O_S}^{\text{filt}O_S}(\text{Var}(\mathbb{C})^{sm}/S), \quad (G,F) \mapsto \Gamma_Z'(G,F) := (\Gamma_Z' G, \Gamma_Z' F),
\]

together with the canonical map \( \gamma_Z'((G,F) : (G,F) \rightarrow \Gamma_Z' G, \Gamma_Z' F) \). Let \( Z_2 \subset Z \) a closed subset. Then, for \( G \in C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S) \), \( T(Z_2/Z,\gamma')(G) : \Gamma_Z' G \rightarrow \Gamma_{Z_2}' G \) is a map in \( C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S) \) (i.e. is \( e(S)^*O_S \) linear).

**Definition 8.** Let \( S \in \text{Var}(\mathbb{C}) \). Let \( Z \subset S \) a closed subset.

(i) We denote by

\[
C_Z(\text{Var}(\mathbb{C})^{sm}/S) \subset C(\text{Var}(\mathbb{C})^{sm}/S)
\]

the full subcategory consisting of complexes of presheaves \( F^* \in C(\text{Var}(\mathbb{C})^{sm}/S) \) such that \( a_{\text{et}}H^n(j^*F^*) = 0 \) for all \( n \in \mathbb{Z} \), where \( j : S \setminus Z \rightarrow S \) is the complementary open embedding and \( a_{\text{et}} \) is the sheafification functor.

(i)' We denote by

\[
C_{O_S,Z}(\text{Var}(\mathbb{C})^{sm}/S) \subset C_{O_S}(\text{Var}(\mathbb{C})^{sm}/S),
\]

the full subcategory consisting of complexes of presheaves \( F^* \in C(\text{Var}(\mathbb{C})^{sm}/S) \) such that \( a_{\text{et}}H^n(j^*F^*) = 0 \) for all \( n \in \mathbb{Z} \), where \( j : S \setminus Z \rightarrow S \) is the complementary open embedding and \( a_{\text{et}} \) is the sheafification functor.

(ii) We denote by

\[
C_{\text{filt}Z}(\text{Var}(\mathbb{C})^{sm}/S) \subset C_{\text{filt}}(\text{Var}(\mathbb{C})^{sm}/S)
\]

the full subcategory consisting of filtered complexes of presheaves \( (F^*,F) \subset C_{\text{filt}}(\text{Var}(\mathbb{C})^{sm}/S) \) such that there exist \( r \in \mathbb{N} \) and an \( r \)-filtered homotopy equivalence \( \phi : (F^*,F) \rightarrow (F^*,F) \) with \( (F^*,F) \in C_{\text{filt}}(\text{Var}(\mathbb{C})^{sm}/S) \) such that \( a_{\text{et}}j^*H^n\text{Gr}_F^p(F^*,F) = 0 \) for all \( n,p \in \mathbb{Z} \), where \( j : S \setminus Z \rightarrow S \) is the complementary open embedding and \( a_{\text{et}} \) is the sheafification functor. Note that by definition this \( r \) does NOT depend on \( n \) and \( p \).

(ii)' We denote by

\[
C_{O_S,\text{filt}Z}(\text{Var}(\mathbb{C})^{sm}/S) \subset C_{O_S,\text{filt}}(\text{Var}(\mathbb{C})^{sm}/S)
\]

the full subcategory consisting of filtered complexes of presheaves \( (F^*,F) \subset C_{O_S,\text{filt}}(\text{Var}(\mathbb{C})^{sm}/S) \) such that there exist \( r \in \mathbb{N} \) and an \( r \)-filtered homotopy equivalence \( \phi : (F^*,F) \rightarrow (F^*,F) \) with \( (F^*,F) \subset C_{O_S,\text{filt}}(\text{Var}(\mathbb{C})^{sm}/S) \) such that \( a_{\text{et}}j^*H^n\text{Gr}_F^p(F^*,F) = 0 \) for all \( n,p \in \mathbb{Z} \), where \( j : S \setminus Z \rightarrow S \) is the complementary open embedding and \( a_{\text{et}} \) is the sheafification functor. Note that by definition this \( r \) does NOT depend on \( n \) and \( p \).

Let \( S \in \text{Var}(\mathbb{C}) \) and \( Z \subset S \) a closed subset.
• For \((G, F) \in C_{fil}(\text{Var}(\mathbb{C})^\text{sm}/S)\), we have \(\Gamma_Z(G, F), \Gamma_Z^X(G, F) \in C_{fil,Z}(\text{Var}(\mathbb{C})^\text{sm}/S)\).

• For \((G, F) \in C_{O_{S}fil}(\text{Var}(\mathbb{C})^\text{sm}/S)\), we have \(\Gamma_Z(G, F), \Gamma_Z^X(G, F) \in C_{O_{S}fil,Z}(\text{Var}(\mathbb{C})^\text{sm}/S)\).

Let \(S_\bullet \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C}))\) with \(\mathcal{I} \in \text{Cat}\), a diagram of algebraic varieties. It gives the diagram of sites \(\text{Var}(\mathbb{C})^2/S_\bullet \in \text{Fun}(\mathcal{I}, \text{Cat})\).

• Then \(C_{fil}(\text{Var}(\mathbb{C})/S_\bullet)\) is the category
  
  whose objects \((G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ}),\) with \((G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})/S_I),\) and \(u_{IJ} : (G_I, F) \to r_{IJ}^*(G_J, F)\) for \(r_{IJ} : I \to J,\) denoting again \(r_{IJ} : S_I \to S_J,\) are morphisms satisfying for \(I \to J \to K, r_{IJ} \circ u_{JK} \circ u_{IJ} = u_{IK}\) in \(C_{fil}(\text{Var}(\mathbb{C})/S_I),\)

  the morphisms \(m : ((G, F), u_{IJ}) \to (H, F), v_{IJ})\) being (see section 2.1) a family of morphisms of complexes,

  \[
  m = (m_I : (G_I, F) \to (H_I, F))_{I \in \mathcal{I}}
  \]

  such that \(v_{IJ} \circ m_I = p_{IJ} \circ m_{IJ} \circ u_{IJ}\) in \(C_{fil}(\text{Var}(\mathbb{C})/S_I).\)

• Then \(C_{fil}(\text{Var}(\mathbb{C})^\text{sm}/S_\bullet)\) is the category
  
  whose objects \((G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ}),\) with \((G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^\text{sm}/S_I),\) and \(u_{IJ} : (G_I, F) \to r_{IJ}^*(G_J, F)\) for \(r_{IJ} : I \to J,\) denoting again \(r_{IJ} : S_I \to S_J,\) are morphisms satisfying for \(I \to J \to K, r_{IJ} \circ u_{JK} \circ u_{IJ} = u_{IK}\) in \(C_{fil}(\text{Var}(\mathbb{C})^\text{sm}/S_I),\)

  the morphisms \(m : ((G, F), u_{IJ}) \to (H, F), v_{IJ})\) being (see section 2.1) a family of morphisms of complexes,

  \[
  m = (m_I : (G_I, F) \to (H_I, F))_{I \in \mathcal{I}}
  \]

  such that \(v_{IJ} \circ m_I = p_{IJ} \circ m_{IJ} \circ u_{IJ}\) in \(C_{fil}(\text{Var}(\mathbb{C})^\text{sm}/S_I).\)

As usual, we denote by

\[
(f_\bullet^*, f_\bullet^*) := (P(f_\bullet)^*, P(f_\bullet)^*) : C(\text{Var}(\mathbb{C})^{\text{sm}}/S_\bullet) \to C(\text{Var}(\mathbb{C})^{\text{sm}}/T_\bullet)
\]

the adjunction induced by \(P(f_\bullet) : \text{Var}(\mathbb{C})^{\text{sm}}/T_\bullet \to \text{Var}(\mathbb{C})^{\text{sm}}/S_\bullet.\) Since the colimits involved in the definition of \(f_\bullet^* = P(f_\bullet)^*\) are filtered, \(f_\bullet^*\) also preserve monomorphism. Hence, we get an adjunction

\[
(f_\bullet^*, f_\bullet^*) : C_{fil}(\text{Var}(\mathbb{C})^{\text{sm}}/S_\bullet) \rightleftarrows C_{fil}(\text{Var}(\mathbb{C})^{\text{sm}}/T_\bullet),
\]

\[
f_\bullet^*((G_I, F), u_{IJ}) := ((f_I^*G_I, f_I^*F), T(f_I, r_{IJ})((-) \circ f_I^*u_{IJ})).
\]

Let \(S \in \text{Var}(\mathbb{C}).\) Let \(S = \bigcup_{i=1}^l S_i\) an open affine cover and denote by \(S_I = \cap_{i \in I} S_i.\) Let \(i_i : S_i \hookrightarrow \tilde{S}_i\) closed embeddings, with \(\tilde{S}_i \in \text{Var}(\mathbb{C}).\) For \(I \subset \{1, \ldots, l\},\) denote by \(\tilde{S}_I = \Pi_{i \in I} \tilde{S}_i.\) We then have closed embeddings \(i_I : S_I \hookrightarrow \tilde{S}_I\) and for \(J \subset I\) the following commutative diagram

\[
D_{IJ} = \begin{array}{ccc}
S_I & \xrightarrow{ii} & \tilde{S}_I \\
\downarrow j_{IJ} & & \downarrow p_{IJ} \\
S_J & \xrightarrow{ij} & \tilde{S}_J
\end{array}
\]

where \(p_{IJ} : \tilde{S}_J \hookrightarrow \tilde{S}_I\) is the projection and \(j_{IJ} : S_J \hookrightarrow S_I\) is the open embedding so that \(j_{IJ} \circ j_{IJ} = j_{IJ}.
\)

This gives the diagram of algebraic varieties \((\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Var}(\mathbb{C}))\) which the diagram of sites \(\text{Var}(\mathbb{C})^{\text{sm}}/(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat}).\) Denote by \(m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I\) the open embedding. Then \(C_{fil}(\text{Var}(\mathbb{C})^{\text{sm}}/(\tilde{S}_I))\) is the category

• whose objects \((G, F) = ((G_I, F), u_{IJ}),\) with \((G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{\text{sm}}/(\tilde{S}_I)),\) and \(u_{IJ} : (G_I, F) \to p_{IJ}^*(G_J, F)\) are morphisms satisfying for \(I \subset J \subset K, p_{IJ} \circ u_{JK} \circ u_{IJ} = u_{IK}\) in \(C_{fil}(\text{Var}(\mathbb{C})^{\text{sm}}/(\tilde{S}_I)),\)
• the morphisms $m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ})$ being a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \to (H_I, F))_{I \in I}$$

such that $v_{IJ} \circ m_I = p_{IJ} \circ m \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$.

**Definition 9.** Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^d S_i$ an open cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$, closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. We will denote by $C_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) \subset C_{fil}(\text{Var}(\mathbb{C})^{sm}/(\tilde{S}_I))$ the full subcategory

- whose objects $(G, F) = ((G_I, F)_{I \subset [1, \ldots, d]}, u_{IJ})$, with $(G_I, F) \in C_{fil}(S_i(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I))$, and $u_{IJ} : m^*(G_I, F) \to m^*p_{IJ}(G_I, F)$ for $I \subset J$, are $\infty$-filtered Zariski local equivalence, satisfying for $I \subset J \subset K$, $p_{JL}u_{JK} \circ u_{IJ} = u_{JK}$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$,

- the morphisms $m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \to (H_I, F))_{I \subset [1, \ldots, d]}$$

such that $v_{IJ} \circ m_I = p_{IJ} \circ m \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$.

A morphism $m : ((G_I, F), u_{IJ}) \to ((H_I, F), v_{IJ})$ is an $r$-filtered Zariski, resp. etale, local equivalence, if there exists $\phi_i : ((C_I, F), u_{IJ}) \to ((C_{i+1}, F), u_{i+1\mid J})$, $0 \leq i \leq s$, with $((C_I, F), u_{IJ}) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) (C_{i+1}, F, u_{i+1\mid J}) = ((G_I, F), u_{IJ})$ and $((C_{i+1}, F), u_{i+1\mid J}) = ((H_I, F), v_{IJ})$, such that

$$\phi = \phi_0 \circ \cdots \circ \phi_i : ((G_I, F), u_{IJ}) \to ((H_I, F), v_{IJ})$$

and $\phi_i : ((C_I, F), u_{IJ}) \to ((C_{i+1\mid J}), u_{i+1\mid J})$ either a filtered Zariski, resp. etale, local equivalence or an $r$-filtered homotopy equivalence.

Denote $L = [1, \ldots, d]$ and for $I \subset L$, $p_{0(I)} : S \times \tilde{S}_I \to S$, $p_{l(I)} : S \times \tilde{S}_L \to S_I$ the projections. By definition, we have functors

- $T(S/(\tilde{S}_I)) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \to C_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$, $(G, F) \to (i_{I*}j_{I*}^*(G, F), T(D_{I*}(j_{I*}^*(G, F)))$,

- $T((\tilde{S}_I)/S) : C_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I))) \to C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$, $((G_I, F), u_{IJ}) \to \text{holim}_{I \subset L} p_{0(0)} \Gamma_{\tilde{S}_I}^* p_{l(0)}^* (G_I, F)$.

Note that the functors $T(S/(\tilde{S}_I))$ are NOT embedding, since

$$\text{ad}(i_{I*}, j_{I*}^*) : i_{I*}j_{I*}^* F \to j_{I*} F$$

are Zariski local equivalence but NOT isomorphism since we are dealing with the morphism of big sites $P(i_I) : \text{Var}(\mathbb{C})^{sm}/S \to \text{Var}(\mathbb{C})^{sm}/\tilde{S}_I$. However, by theorem 16, these functors induce full embeddings

$$T(S/(\tilde{S}_I)) : D_{fil}(\text{Var}(\mathbb{C})^{sm}/S) \to D_{fil}(\text{Var}(\mathbb{C})^{sm}/(S/(\tilde{S}_I)))$$

since for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$,

$$\text{holim}_{I \subset L} p_{0(0)} \Gamma_{\tilde{S}_I}^* p_{l(0)}^* (i_{I*}j_{I*}^* F) \to \text{holim}_{I \subset L} \Gamma_{\tilde{S}_I}^* j_{I*} F$$

is an equivalence Zariski local.

Let $f : X \to S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^d S_i$ and $X = \cup_{i=1}^d X_i$ be affine open covers and $i_i : S_i \hookrightarrow \tilde{S}_i$, $i'_i : X_i \hookrightarrow \tilde{X}_i$ be closed embeddings. Let $\tilde{f}_i : \tilde{X}_i \to \tilde{S}_i$ be a lift of the morphism $f_i = f|X_i : X_i \to S_i$. Then, $f_I = f|X_I : X_I = \cap_{i \in I} X_i \to S_I = \cap_{i \in I} S_i$ lift to the morphism

$$\tilde{f}_I = \Pi_{i \in I} \tilde{f}_i : \tilde{X}_I = \Pi_{i \in I} \tilde{X}_i \to \tilde{S}_I = \Pi_{i \in I} \tilde{S}_i$$
Denote by \( p_{IJ} : \tilde{S}_J \to \tilde{S}_I \) and \( p'_{IJ} : \tilde{X}_J \to \tilde{X}_I \) the projections. Consider for \( J \subset I \) the following commutative diagrams

\[
\begin{array}{ccc}
D_{IJ} = S_I \xrightarrow{ii} S_I \xrightarrow{i} \tilde{S}_I, & \quad & D'_{IJ} = X_I \xrightarrow{i'} \tilde{X}_I \xrightarrow{ii} X_I \xrightarrow{i} \tilde{X}_I = S_I \xrightarrow{ii} S_I,
\end{array}
\]

We have then following commutative diagram

\[
\begin{array}{ccc}
X_I & \xrightarrow{n'_j} & \tilde{X}_I \setminus X_I, \\
\downarrow{p'_{IJ}} & & \downarrow{p'_{IJ}} \\
X_J \times X_I \times \tilde{X}_J \setminus I & \xrightarrow{n_j} & \tilde{X}_J \setminus X_J
\end{array}
\]

whose square are cartesian. We then have the pullback functor

\[
f^* : C(2)_{fil}(\text{Var}(\mathbb{C})^{sm}/S/\langle \tilde{S}_I \rangle) \to C(2)_{fil}(\text{Var}(\mathbb{C})^{sm}/X/\langle \tilde{X}_I \rangle),
\]

\[
((G, F), u_{IJ}) \to f^*((G, F), u_{IJ}) := (\Gamma_{X_I}^\vee f_I^* (G, F), f_I^* u_{IJ})
\]

with

\[
f^*_I u_{IJ} : \Gamma_{X_I}^\vee f_I^* (G, F) \xrightarrow{\text{ad}(p'_{IJ}, \gamma_{X_I})^{-1}} \Gamma_{X_I}^\vee f_I^* (G, F) \xrightarrow{f_I^*(p_{IJ}, u_I)^(-1)} \Gamma_{X_I}^\vee f_I^* (G, F)
\]

Let \((G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)\). Since, \( j_I^* i'_I j_I^* f^*(G, F) = 0 \), the morphism \( T(D_{IJ})/(j_I^* G, F) : j_I^* i'_I j_I^* f^*(G, F) \to i'_I j_I^* f^*(G, F) \) factors trough

\[
T(D_{IJ})/(j_I^* G, F) : j_I^* i'_I j_I^* f^*(G, F) \xrightarrow{\gamma_{X_I}^\vee} \Gamma_{X_I}^\vee f_I^* (G, F) \xrightarrow{T^*(D_{IJ})/(j_I^* G, F)} j'_I j_I^* f^*(G, F)
\]

We have then, for \((G, F) \in C_{fil}(S)\), the canonical transformation map

\[
f^* T(S/\langle \tilde{S}_I \rangle)(G, F) \xrightarrow{T(f, f(0)/I)(G, F)} T(X/\langle \tilde{X}_I \rangle)(f^*(G, F))
\]

\[
= (\Gamma_{X_I}^\vee f_I^* (G, F), f_I^* j_I^* f^*(G, F), I)
\]

To show that the cohomology sheaves of the filtered De Rham realization functor of constructible motives are mixed hodge modules, we will need to take presheaves of the following form

**Definition 10.** (i) Let \( f : X \to S \) a morphism with \( X, S \in \text{Var}(\mathbb{C}) \). Assume that there exist a factorization \( f : X \xrightarrow{i} Y \times S \xrightarrow{p} S \), with \( Y \in \text{SmVar}(\mathbb{C}) \), \( i : X \hookrightarrow Y \) is a closed embedding and \( p \) the projection. We then consider

\[
Q(X/S) := p_! \Gamma_{X}^\vee \mathbb{Z}_{Y \times S} \in C(\text{Var}(\mathbb{C})^{sm}/S).
\]

By definition \( Q(X/S) \) is projective.
(ii) Let $f : X \to S$ and $g : T \to S$ two morphism with $X, S, T \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \dashrightarrow Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ a closed embedding and $p$ the projection. We then have the following commutative diagram whose squares are cartesian

$$
\begin{array}{ccc}
X & \xrightarrow{i} & Y \times S \\
\downarrow g' & & \downarrow p \\
X_T & \xrightarrow{i'} & Y \times T \\
\end{array}
$$

We then have the canonical isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$
T(f,g,Q) : g^{*}Q(X/S) := g^{*}p_{\ast} \Gamma_{X}(Z_{Y \times S}) \xrightarrow{T_{1}(g,p)(\ast)^{-1}} p'_{\ast}g''^{*} \Gamma_{X}(Z_{Y \times S})
$$

(iii) Let $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ a closed embedding and $p$ the projection. We then consider

$$
Q^{h}(X/S) := p_{\ast} \Gamma_{X} E_{et}(Z_{Y \times S}) \in C(\text{Var}(\mathbb{C})^{sm}/S).
$$

(iv) Let $f : X \to S$ and $g : T \to S$ two morphism with $X, S, T \in \text{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{h} Y \times S \xrightarrow{p} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $i : X \hookrightarrow Y$ a closed embedding and $p$ the projection. We then have the following commutative diagram whose squares are cartesian

$$
\begin{array}{ccc}
X & \xrightarrow{i} & Y \times S \\
\downarrow g' & & \downarrow g \\
X_T & \xrightarrow{i'} & Y \times T \\
\end{array}
$$

We then have the canonical morphism in $C(\text{Var}(\mathbb{C})^{sm}/T)$

$$
T(f,g,Q^{h}) : g^{*}Q^{h}(X/S) := g^{*}p_{\ast} \Gamma_{X} E_{et}(Z_{Y \times S}) \xrightarrow{T(g,f)(\ast)} p'_{\ast}g''^{*} \Gamma_{X} E_{et}(Z_{Y \times S})
$$

We now give the definition of the $\mathbb{A}^{1}$ local property :

Denote by

$$
p_{a} : \text{Var}(\mathbb{C})^{sm}/S \to \text{Var}(\mathbb{C})^{sm}/X, X/S = (X,h) \mapsto (X \times \mathbb{A}^{1})/S = (X \times \mathbb{A}^{1}, h \circ p_{X}),
$$

$$(g : X/S \to X'/S) \mapsto ((g \times I_{\mathbb{A}^{1}}) : X \times \mathbb{A}^{1}/S \to X' \times \mathbb{A}^{1}/S)$$

the projection functor and again by $p_{a} : \text{Var}(\mathbb{C})^{sm}/S \to \text{Var}(\mathbb{C})^{sm}/S$ the corresponding morphism of site.

**Definition 11.** Let $S \in \text{Var}(\mathbb{C})$. Denote for short $\text{Var}(\mathbb{C})^{sm}/S$ either the category $\text{Var}(\mathbb{C})/S$ or the category $\text{Var}(\mathbb{C})^{sm}/S$.

(i) A complex $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is said to be $\mathbb{A}^{1}$ homotopic if ad($p_{a}^{*}p_{a}\ast$)($F$) : $F \to p_{a}^{*}p_{a}\ast F$ is an homotopy equivalence.

(i) A complex $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is said to be $\mathbb{A}^{1}$ invariant if for all $U/S \in \text{Var}(\mathbb{C})^{sm}/S$,

$$
F(p_{U}) : F(U/S) \to F(U \times \mathbb{A}^{1}/S)
$$

is a quasi-isomorphism, where $p_{U} : U \times \mathbb{A}^{1} \to U$ is the projection. Obviously, if a complex $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is $\mathbb{A}^{1}$ homotopic then it is $\mathbb{A}^{1}$ invariant.
(ii) Let $\tau$ a topology on $\Var(\mathbb{C})$. A complex $F \in C(\Var(\mathbb{C})^{sm}/S)$ is said to be $A^1$ local for the topology $\tau$, if for a (hence every) $\tau$ local equivalence $k : F \to G$ with $k$ injective and $G \in C(\Var(\mathbb{C})^{sm}/S)$ $\tau$ fibrant, e.g. $k : F \to E_\tau(F)$, $G$ is $A^1$ invariant for all $n \in \mathbb{Z}$.

(iii) A morphism $m : F \to G$ with $F,G \in C(\Var(\mathbb{C})^{sm}/S)$ is said to an $(A^1, \text{et})$ local equivalence if for all $H \in C(\Var(\mathbb{C})^{sm}/S)$ which is $A^1$ local for the etale topology

$$\Hom(L(m), E_{et}(H)) : \Hom(L(G), E_{et}(H)) \to \Hom(L(F), E_{et}(H))$$

is a quasi-isomorphism.

Denote $\square^* := \mathbb{P}^* \backslash \{1\}$

- Let $S \in \Var(\mathbb{C})$. For $U/S = (U, h) \in \Var(\mathbb{C})^{sm}/S$, we consider

$$\square^* \times U/S = (\square^* \times U, h \circ p) \in \Fun(\Delta, \Var(\mathbb{C})^{sm}/S).$$

For $F \in C^-(\Var(\mathbb{C})^{sm}/S)$, it gives the complex

$$C_* F \in C^- (\Var(\mathbb{C})^{sm}/S), U/S = (U, h) \mapsto C_* F(U/S) := \Tot F(\square^* \times U/S)$$

together with the canonical map $c_F := (0, I_F) : F \to C_* F$. For $F \in C(\Var(\mathbb{C})^{sm}/S)$, we get

$$C_* F := \holim_n C_* F^{\leq n} \in C(\Var(\mathbb{C})^{sm}/S),$$

together with the canonical map $c_F := (0, I_F) : F \to C_* F$. For $m : F \to G$ a morphism, with $F,G \in C(\Var(\mathbb{C})^{sm}/S)$, we get by functoriality the morphism $C_* m : C_* F \to C_* G$.

- Let $S \in \Var(\mathbb{C})$. For $U/S = (U, h) \in \Var(\mathbb{C})/S$, we consider

$$\square^* \times U/S = (\mathbb{A}^* \times U, h \circ p) \in \Fun(\Delta, \Var(\mathbb{C})/S).$$

For $F \in C^-(\Var(\mathbb{C})/S)$, it gives the complex

$$C_* F \in C^- (\Var(\mathbb{C})/S), U/S = (U, h) \mapsto C_* F(U/S) := \Tot F(\square^* \times U/S)$$

together with the canonical map $c = c(F) := (0, I_F) : F \to C_* F$. For $F \in C(\Var(\mathbb{C})/S)$, we get

$$C_* F := \holim_n C_* F^{\leq n} \in C(\Var(\mathbb{C})/S),$$

together with the canonical map $c_F := (0, I_F) : F \to C_* F$. For $m : F \to G$ a morphism, with $F,G \in C(\Var(\mathbb{C})/S)$, we get by functoriality the morphism $C_* m : C_* F \to C_* G$.

**Proposition 17.** (i) Let $S \in \Var(\mathbb{C})$. Then for $F \in C(\Var(\mathbb{C})^{sm}/S)$, $C_* F$ is $A^1$ local for the etale topology and $c(F) : F \to C_* F$ is an equivalence $(A^1, \text{et})$ local.

(ii) A morphism $m : F \to G$ with $F,G \in C(\Var(\mathbb{C})^{sm}/S)$ is an $(A^1, \text{et})$ local equivalence if and only if there exists

$$\{X_{1,\alpha}/S, \alpha \in A_1\}, \ldots, \{X_{r,\alpha}/S, \alpha \in A_r\} \subset \Var(\mathbb{C})^{sm}/S$$

such that we have in $\Ho_{et}(C(\Var(\mathbb{C})^{sm}/S))$

$$\Cone(m) \xrightarrow{\sim} \Cone(\oplus_{\alpha \in A_1} \Cone(\mathbb{Z}(X_{1,\alpha} \times A^1/S) \to \mathbb{Z}(X_{1,\alpha}/S)))$$

$$\to \cdots \to \oplus_{\alpha \in A_r} \Cone(\mathbb{Z}(X_{r,\alpha} \times A^1/S) \to \mathbb{Z}(X_{r,\alpha}/S)))$$

**Proof.** Standard : see Ayoub’s thesis for example. 

**Definition-Proposition 5.** Let $S \in \Var(\mathbb{C})$. 

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(i) With the weak equivalence the \((A^1, et)\) local equivalence and the fibration the epimorphism with \(A^1_S\) local and etale fibrant kernels gives a model structure on \(C(\text{Var}(\mathbb{C})^{sm}/S)\) : the left bousfield localization of the projective model structure of \(C(\text{Var}(\mathbb{C})^{sm}/S)\). We call it the projective \((A^1, et)\) model structure.

(ii) With the weak equivalence the \((A^1, et)\) local equivalence and the fibration the epimorphism with \(A^1_S\) local and etale fibrant kernels gives a model structure on \(C(\text{Var}(\mathbb{C})/S)\) : the left bousfield localization of the projective model structure of \(C(\text{Var}(\mathbb{C})/S)\). We call it the projective \((A^1, et)\) model structure.

Proof. See [10].

Proposition 18. Let \(g : T \rightarrow S\) a morphism with \(T, S \in \text{Var}(\mathbb{C})\).

(i) The adjunction \((g^*, g_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \Rightarrow C(\text{Var}(\mathbb{C})^{sm}/T)\) is a Quillen adjunction for the \((A^1, et)\) projective model structure (see definition-proposition 5).

(ii) Let \(h : U \rightarrow S\) a smooth morphism with \(U, S \in \text{Var}(\mathbb{C})\). The adjunction \((h_*, h^*) : C(\text{Var}(\mathbb{C})^{sm}/U) \Rightarrow C(\text{Var}(\mathbb{C})^{sm}/S)\) is a Quillen adjunction for the \((A^1, et)\) projective model structure.

Proposition 19. Let \(S \in \text{Var}(\mathbb{C})\).

(i) The adjunction \((p^*_S, p_{S*}) : C(\text{Var}(\mathbb{C})^{sm}/S) \Rightarrow C(\text{Var}(\mathbb{C})/S)\) is a Quillen adjunction for the \((A^1, et)\) projective model structure.

(ii) The functor \(p_{S*} : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})^{sm}/S)\) sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends \((A^1, et)\) local equivalence to \((A^1, et)\) local equivalence.

This gives in particular the functor
\[ C(\text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)) \to C(\text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)^{\text{op}}), (H_I, T_{IJ}) \mapsto (H_I, T_{IJ}) \].

Let \( S \in \text{Var}(\mathbb{C}) \). Let \( S = \bigcup_{i=1}^d S_i \) an open affine cover and denote by \( S_I = \bigcap_{i \in I} S_i \). Let \( i_i : S_i \to \tilde{S}_i \) closed embeddings, with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). The functor \( p_a \) extend to a functor
\[ p_a : \text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I) \to \text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I), \ (X_I/\tilde{S}_I, s_{IJ}) \to (X \times \mathbb{A}^1/\tilde{S}_I, s_{IJ} \times I), \]
\[ (g = (g_I) : (X_I/\tilde{S}_I, s_{IJ}) \to (X'_I/\tilde{S}_I, s'_{IJ})) \to \left( (g \times \mathbb{A}^1_i) : (X_I \times \mathbb{A}^1/\tilde{S}_I, s_{IJ}) \to (X'_I \times \mathbb{A}^1/\tilde{S}_I, s'_{IJ}) \right) \]
the projection functor and again by \( p_a : \text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I) \to \text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I) \) the corresponding morphism of site.

**Definition 12.** Let \( S \in \text{Var}(\mathbb{C}) \). Let \( S = \bigcup_{i=1}^d S_i \) an open affine cover and denote by \( S_I = \bigcap_{i \in I} S_i \). Let \( i_i : S_i \to \tilde{S}_i \) closed embeddings, with \( \tilde{S}_i \in \text{Var}(\mathbb{C}) \).

(iii) A morphism \( m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ}) \) with \( (F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)) \) is said to be \( \mathbb{A}^1 \)-local equivalence if for all \( (H_I, w_{IJ}) \in C(\text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)) \) is \( \mathbb{A}^1 \)-local equivalence if only if all the \( F_I \) are \( \mathbb{A}^1 \)-local.

(ii) Let \( \tau \) a topology on \( \text{Var}(\mathbb{C}) \). A complex \( F = (F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)) \) is said to be \( \mathbb{A}^1 \)-local for the \( \tau \) topology induced on \( \text{Var}(\mathbb{C})/(\tilde{S}_I) \), if for any \( \tau \) local equivalence \( k : F \to G \) with \( k \) injective and \( G = (G_I, v_{IJ}) \in C(\text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)) \) \( \tau \) fibrant, e.g. \( k : (F_I, u_{IJ}) \to (E_*(F_I), E(u_{IJ})) \), \( G \) is \( \mathbb{A}^1 \)-local.

(iii) A morphism \( m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ}) \) with \( (F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)) \) is said to be an \( (\mathbb{A}^1, \text{et}) \) local equivalence if for all \( (H_I, w_{IJ}) \in C(\text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)) \) which is \( \mathbb{A}^1 \)-local.

**Proposition 20.** Let \( S \in \text{Var}(\mathbb{C}) \). Let \( S = \bigcup_{i=1}^d S_i \) an open affine cover and denote by \( S_I = \bigcap_{i \in I} S_i \). Let \( i_i : S_i \to \tilde{S}_i \) closed embeddings, with \( \tilde{S}_i \in \text{Var}(\mathbb{C}) \).
(i) A morphism $m : F \to G$ with $F, G \in C(\text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I))$ is an $(\mathbb{A}^1, \text{et})$ local equivalence if and only if there exists

$$\{ (X_{1, \alpha, I}/\tilde{S}_I, u_{IJ}^1), \alpha \in \Lambda_I \}, \ldots, \{ (X_{r, \alpha, I}/\tilde{S}_I, u_{IJ}^r), \alpha \in \Lambda_r \} \subset \text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)$$

with $u_{IJ}^I : X_{1, \alpha, I} \times \tilde{S}_{I,J}/\tilde{S}_I \to X_{1, \alpha, J}/\tilde{S}_I$, such that we have in $\text{Ho}_{\text{et}}(C(\text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)))$

$$\text{Cone}(m) \to \text{Cone}(\bigoplus_{\alpha \in \Lambda_I} \text{Cone}((\mathbb{Z}(X_{1, \alpha, I} \times \mathbb{A}^1/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1 \times I))) \to (\mathbb{Z}(X_{1, \alpha, I}/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1)))$$

$$\to \cdots \to \bigoplus_{\alpha \in \Lambda_r} \text{Cone}((\mathbb{Z}(X_{r, \alpha, I} \times \mathbb{A}^1/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r \times I))) \to (\mathbb{Z}(X_{r, \alpha, I}/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r)))$$

(ii) A morphism $m : F \to G$ with $F, G \in C(\text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)^{\text{op}})$ is an $(\mathbb{A}^1, \text{et})$ local equivalence if and only if there exists

$$\{ (X_{1, \alpha, I}/\tilde{S}_I, u_{IJ}^1), \alpha \in \Lambda_I \}, \ldots, \{ (X_{r, \alpha, I}/\tilde{S}_I, u_{IJ}^r), \alpha \in \Lambda_r \} \subset \text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)^{\text{op}}$$

with $u_{IJ}^I : X_{1, \alpha, I} \times \tilde{S}_{I,J}/\tilde{S}_I \to X_{1, \alpha, J}/\tilde{S}_I$, such that we have in $\text{Ho}_{\text{et}}(C(\text{Var}(\mathbb{C})^{(\text{sm})}/(\tilde{S}_I)^{\text{op}}))$

$$\text{Cone}(m) \to \text{Cone}(\bigoplus_{\alpha \in \Lambda_I} \text{Cone}((\mathbb{Z}(X_{1, \alpha, I} \times \mathbb{A}^1/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1 \times I))) \to (\mathbb{Z}(X_{1, \alpha, I}/\tilde{S}_I), \mathbb{Z}(u_{IJ}^1)))$$

$$\to \cdots \to \bigoplus_{\alpha \in \Lambda_r} \text{Cone}((\mathbb{Z}(X_{r, \alpha, I} \times \mathbb{A}^1/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r \times I))) \to (\mathbb{Z}(X_{r, \alpha, I}/\tilde{S}_I), \mathbb{Z}(u_{IJ}^r)))$$

Proof. Standard. See Ayoub’s thesis for example. □

• For $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, we denote as usual (see [10] for example), $Z^{tr}(X/S) \in \text{PSh}(\text{Var}(\mathbb{C})/S)$ the presheaf given by

- for $X'/S \in \text{Var}(\mathbb{C})/S$, with $X'$ irreducible, $Z^{tr}(X/S)(X'/S) := Z^{fs}(X'/S \times S) \subset Z_{d_{vt}}(X'/S \times S)$ which consist of algebraic cycles $\alpha = \sum n_i \alpha_i \in Z_{d_{vt}}(X'/S \times S)$ such that, denoting $\text{supp}(\alpha) = \cup_{i} \alpha_i \subset X' \times S$ $X$ its support and $f' : X' \times S \to X'$ the projection, $f'|_{\text{supp}(\alpha)} : \text{supp}(\alpha) \to X'$ is finite surjective,

- for $g : X_2/S \to X_1/S$ a morphism, with $X_1/S, X_2/S \in \text{Var}(\mathbb{C})/S$,

$$Z^{tr}(X/S)(g) : Z^{tr}(X/S)(X_1/S) \to Z^{tr}(X/S)(X_2/S), \quad \alpha \mapsto \alpha f|_{\text{supp}(\alpha)} : \text{supp}(\alpha) \to X_2$$

with $g \times I : X_2 \times S \to X_1 \times S$, noting that, by base change, $f_2|_{\text{supp}((g \times I)^{-1}(\alpha))} : \text{supp}((g \times I)^{-1}(\alpha)) \to X_2$ is finite surjective, $f_2 : X_2 \times S \to X_2$ being the projection.

• For $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ and $r \in \mathbb{N}$, we denote as usual (see [10] for example), $Z^{\text{equr}}(X/S) \in \text{PSh}(\text{Var}(\mathbb{C})/S)$ the presheaf given by

- for $X'/S \in \text{Var}(\mathbb{C})/S$, with $X'$ irreducible, $Z^{\text{equr}}(X/S)(X'/S) := Z^{\text{equr}}(X'/S \times S) \subset Z_{d_{vt}}(X'/S \times S)$ which consist of algebraic cycles $\alpha = \sum n_i \alpha_i \in Z_{d_{vt}}(X'/S \times S)$ such that, denoting $\text{supp}(\alpha) = \cup_{i} \alpha_i \subset X' \times S$ $X$ its support and $f' : X' \times S \to X'$ the projection, $f'|_{\text{supp}(\alpha)} : \text{supp}(\alpha) \to X'$ is dominant, with fibers either empty or of dimension $r$,

- for $g : X_2/S \to X_1/S$ a morphism, with $X_1/S, X_2/S \in \text{Var}(\mathbb{C})/S$,

$$Z^{\text{equr}}(X/S)(g) : Z^{\text{equr}}(X/S)(X_1/S) \to Z^{\text{equr}}(X/S)(X_2/S), \quad \alpha \mapsto (g \times I)^{-1}(\alpha)$$

with $g \times I : X_2 \times S \to X_1 \times S$, noting that, by base change, $f_2|_{\text{supp}((g \times I)^{-1}(\alpha))} : \text{supp}((g \times I)^{-1}(\alpha)) \to X_2$ is obviously dominant, with fibers either empty or of dimension $r$, $f_2 : X_2 \times S \to X_2$ being the projection.

• Let $S \in \text{Var}(\mathbb{C})$. We denote by $Z_S(d) := Z^{\text{equr}}(S \times \mathbb{A}^d/S)[-2d]$ the Tate twist. For $F \in C(\text{Var}(\mathbb{C})/S)$, we denote by $F(d) := F \otimes Z_S(d)$. 

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For $S \in \text{Var}(\mathbb{C})$, let $\text{Cor}(\text{Var}(\mathbb{C})^{sm}/S)$ be the category

- whose objects are smooth morphisms $U/S = (U, h)$, $h : U \to S$ with $U \in \text{Var}(\mathbb{C})$,
- whose morphisms $\alpha : U/S = (U, h_1) \to V/S = (V, h_2)$ is finite correspondence that is $\alpha \in \mathcal{H}_Z(U \times_S V)$, where $U = \sqcup_i U_i$, with $U_i$ connected (hence irreducible by smoothness), and $\mathcal{H}_Z(U \times_S V)$ is the abelian group of cycle finite and surjective over $U_i$.

We denote by $\text{Tr}(S) : \text{Cor}(\text{Var}(\mathbb{C})^{sm}/S) \to \text{Var}(\mathbb{C})^{sm}/S$ the morphism of site given by the inclusion functor $\text{Tr}(S) : \text{Var}(\mathbb{C})^{sm}/S \hookrightarrow \text{Cor}(\text{Var}(\mathbb{C})^{sm}/S)$. It induces an adjunction

$$(\text{Tr}(S)^* \text{Tr}(S)_*) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(\text{Cor}(\text{Var}(\mathbb{C})^{sm}/S))$$

A complex of preheaves $G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is said to admit transferts if it is in the image of the embedding

$$\text{Tr}(S)_* : C(\text{Cor}(\text{Var}(\mathbb{C})^{sm}/S)) \to C(\text{Var}(\mathbb{C})^{sm}/S),$$

that is $G = \text{Tr}(S)_* \text{Tr}(S)^* G$.

We will use to compute the algebraic Gauss-Manin realization functor the following

**Theorem 10.** Let $\phi : F^\bullet \to G^\bullet$ an etale local equivalence with $F^\bullet, G^\bullet \in C(\text{Var}(\mathbb{C})^{sm}/S)$. If $F^\bullet$ and $G^\bullet$ are $\mathbb{A}^1$ local and admit transferts then $\phi : F^\bullet \to G^\bullet$ is a Zariski local equivalence. Hence if $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is $\mathbb{A}^1$ local and admits transfert

$$k : E_{zar}(F) \to E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

**Proof.** See [10]. \hfill \Box

### 2.7 Presheaves on the big Zariski site or the big etale site of pairs

We recall the definition given in subsection 5.1 : For $S \in \text{Var}(\mathbb{C})$, $\text{Var}(\mathbb{C})^2/S := \text{Var}(\mathbb{C})^2/(S, S)$ is by definition (see subsection 2.1) the category whose set of objects is

$$(\text{Var}(\mathbb{C})^2/S)^0 := \{(X, Z, h), h : X \to S, Z \subset X \text{ closed } \} \subset \text{Var}(\mathbb{C})/S \times \text{Top}$$

and whose set of morphisms between $(X_1, Z_1)/S = ((X_1, Z_1), h_1), (X_1, Z_1)/S = ((X_2, Z_2), h_2) \in \text{Var}(\mathbb{C})^2/S$ is the subset

$$\text{Hom}_{\text{Var}(\mathbb{C})^2/S}((X_1, Z_1)/S, (X_2, Z_2)/S) := \{(f : X_2 \to X_2), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\} \subset \text{Hom}_{\text{Var}(\mathbb{C})}(X_1, X_2)$$

The category $\text{Var}(\mathbb{C})^2$ admits fiber products : $(X_1, Z_1) \times_{(S, Z)} (X_2, Z_2) = (X_1 \times_S X_2, Z_1 \times_Z Z_2)$. In particular, for $f : T \to S$ a morphism with $S, T \in \text{Var}(\mathbb{C})$, we have the pullback functor

$$P(f) : \text{Var}(\mathbb{C})^2/S \to \text{Var}(\mathbb{C})^2/T, P(f)((X, Z)/S) := (X_T, Z_T)/T, P(f)(g) := (g \times_S f)$$

and we note again $P(f) : \text{Var}(\mathbb{C})^2/T \to \text{Var}(\mathbb{C})^2/S$ the corresponding morphism of sites.

We will consider in the construction of the filtered De Rham realization functor the full subcategory $\text{Var}(\mathbb{C})^{2-sm}/S \subset \text{Var}(\mathbb{C})^2/S$ such that the first factor is a smooth morphism : We will also consider, in order to obtain a complex of D modules in the construction of the filtered De Rham realization functor, the restriction to the full subcategory $\text{Var}(\mathbb{C})^{2-pr}/S \subset \text{Var}(\mathbb{C})^2/S$ such that the first factor is a projection :
Definition 13.  

(i) Let $S \in \text{Var}(\mathbb{C})$. We denote by  
\[ \rho_S : \text{Var}(\mathbb{C})^{2, sm}/S \hookrightarrow \text{Var}(\mathbb{C})^2/S \]
the full subcategory consisting of the objects $(U, Z)/S = ((U, Z), h) \in \text{Var}(\mathbb{C})^2/S$ such that the morphism $h : U \to S$ is smooth. That is, $\text{Var}(\mathbb{C})^{2, sm}/S$ is the category

- whose objects are $(U, Z)/S = ((U, Z), h)$, with $U \in \text{Var}(\mathbb{C})$, $Z \subset U$ a closed subset, and $h : U \to S$ a smooth morphism,

- whose morphisms $g : (U, Z)/S = ((U, Z), h_1) \to (U', Z')/S = ((U', Z'), h_2)$ is a morphism $g : U \to U'$ of complex algebraic varieties such that $Z \subset g^{-1}(Z')$ and $h_2 \circ g = h_1$.

We denote again $\rho_S : \text{Var}(\mathbb{C})^2/S \to \text{Var}(\mathbb{C})^{2, sm}/S$ the associated morphism of site. We have  
\[ r^*(S) : \text{Var}(\mathbb{C})^2 \xrightarrow{r(S) = r(S, S)} \text{Var}(\mathbb{C})^2/S \xrightarrow{\rho_S} \text{Var}(\mathbb{C})^{2, sm}/S \]
the composite morphism of site.

(ii) Let $S \in \text{Var}(\mathbb{C})$. We will consider the full subcategory  
\[ \mu_S : \text{Var}(\mathbb{C})^{2, pr}/S \hookrightarrow \text{Var}(\mathbb{C})^2/S \]
whose subset of object consist of those whose morphism is a projection to $S$:

\[ (\text{Var}(\mathbb{C})^{2, pr}/S)^0 := \{((Y \times S, X), p), Y \in \text{Var}(\mathbb{C}), p : Y \times S \to S \text{ the projection} \} \subset (\text{Var}(\mathbb{C})^2/S)^0. \]

(iii) We will consider the full subcategory  
\[ \mu_S : (\text{Var}(\mathbb{C})^{2, smpr}/S) \hookrightarrow \text{Var}(\mathbb{C})^{2, sm}/S \]
whose subset of object consist of those whose morphism is a smooth projection to $S$:

\[ (\text{Var}(\mathbb{C})^{2, smpr}/S)^0 := \{((Y \times S, X), p), Y \in \text{SmVar}(\mathbb{C}), p : Y \times S \to S \text{ the projection} \} \subset (\text{Var}(\mathbb{C})^2/S)^0. \]

For $f : T \to S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$, we have by definition, the following commutative diagram of sites:

\[ \begin{array}{ccc}
\text{Var}(\mathbb{C})^2/T & \xrightarrow{\mu_T} & \text{Var}(\mathbb{C})^{2, pr}/T \\
\rho_T & \downarrow & \rho_T \\
\text{Var}(\mathbb{C})^{2, sm}/T & \xrightarrow{\rho_S} & \text{Var}(\mathbb{C})^{2, smpr}/T \\
\rho_S & \downarrow & \rho_S \\
\text{Var}(\mathbb{C})^{2, sm}/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2, smpr}/S \\
\rho_S & \downarrow & \rho_S \\
\text{Var}(\mathbb{C})^{2, sm}/S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2, smpr}/S \\
\end{array} \]

Recall we have (see subsection 2.1), for $S \in \text{Var}(\mathbb{C})$, the graph functor

\[ \text{Gr}^2_S : \text{Var}(\mathbb{C})/S \to \text{Var}(\mathbb{C})^{2, pr}/S, \quad X/S \mapsto \text{Gr}^2_S(X/S) := (X \times S, X)/S, \]

\[ (g : X/S \to X'/S) \mapsto \text{Gr}^2_S(g) := (g \times I_S : (X \times S, X) \to (X' \times S, X')) \]

\[ \text{Gr}^{12}_S : \text{Var}(\mathbb{C})/S \to \text{Var}(\mathbb{C})^{2, sm}/S, \quad X/S \mapsto \text{Gr}^{12}_S(X/S) := (X \times S, X)/S, \]

\[ (g : X/S \to X'/S) \mapsto \text{Gr}^{12}_S(g) := (g \times I_S : (X \times S, X) \to (X' \times S, X')) \]
Note that $\text{Gr}^1_2$ is fully faithful. For $f : T \to S$ a morphism with $T, S \in \text{Var}(C)$, we have by definition, the following commutative diagram of sites

\[
\begin{array}{ccc}
\text{Var}(C)^{2,pr}/T & \xrightarrow{\text{Gr}^1_2} & \text{Var}(C)/T \\
\downarrow P(f) & & \downarrow P(f) \\
\text{Var}(C)^{2,sm}/S & \xrightarrow{\text{Gr}^2} & \text{Var}(C)^{sm}/T \\
\downarrow P(f) & & \downarrow P(f) \\
\text{Var}(C)^{2,sm}/S & \xrightarrow{\text{Gr}^1_2} & \text{Var}(C)^{sm}/S
\end{array}
\]

where we recall that $P(f)((X, Z)/S) := ((X_T, Z_T)/T)$, since smooth morphisms are preserved by base change.

- As usual, we denote by
  \[
  (f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{Var}(C)^{2,(sm)}/S) \to C(\text{Var}(C)^{2,(sm)}/T)
  \]
  the adjonction induced by $P(f) : \text{Var}(C)^{2,(sm)}/T \to \text{Var}(C)^{2,(sm)}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, $f^*$ also preserve monomorphism. Hence, we get an adjonction
  \[
  (f^*, f_*) : C_{fil}(\text{Var}(C)^{2,(sm)}/S) \leftrightarrows C_{fil}(\text{Var}(C)^{2,(sm)}/T), f^*(G, F) := (f^*G, f^*F)
  \]
  For $S \in \text{Var}(C)$, we denote by $Z_S := Z((S, S)/(S, S)) \in \text{PSh}(\text{Var}(C)^{2,(sm)}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\text{Var}(C)^{2,(sm)}/S)$, $\text{Hom}(Z_S, F) = F$.

- As usual, we denote by
  \[
  (f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{Var}(C)^{2,(sm)pr}/S) \to C(\text{Var}(C)^{2,(sm)pr}/T)
  \]
  the adjonction induced by $P(f) : \text{Var}(C)^{2,(sm)pr}/T \to \text{Var}(C)^{2,(sm)pr}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, $f^*$ also preserve monomorphism. Hence, we get an adjonction
  \[
  (f^*, f_*) : C_{fil}(\text{Var}(C)^{2,(sm)pr}/S) \leftrightarrows C_{fil}(\text{Var}(C)^{2,(sm)pr}/T), f^*(G, F) := (f^*G, f^*F)
  \]
  For $S \in \text{Var}(C)$, we denote by $Z_S := Z((S, S)/(S, S)) \in \text{PSh}(\text{Var}(C)^{2,sm}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\text{Var}(C)^{2,sm}/S)$, $\text{Hom}(Z_S, F) = F$.

- For $h : U \to S$ a smooth morphism with $U, S \in \text{Var}(C)$, $P(h) : \text{Var}(C)^{2,sm}/S \to \text{Var}(C)^{2,sm}/U$ admits a left adjoint
  \[
  C(h) : \text{Var}(C)^{2,sm}/U \to \text{Var}(C)^{2,sm}/S, C(h)((U', Z'), h') = ((U', Z'), h \circ h').
  \]
  Hence $h^* : C(\text{Var}(C)^{2,sm}/S) \to C(\text{Var}(C)^{2,sm}/U)$ admits a left adjoint
  \[
  h_2 : C(\text{Var}(C)^{2,sm}/U) \to C(\text{Var}(C)^{2,sm}/S),
  F \mapsto (h_2F : ((U, Z), h_0) \mapsto \lim_{((U', Z'), h_0 h') \to ((U, Z), h_0)} F((U', Z')/U))
  \]

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• For \( h : X \to S \) a morphism with \( X, S \in \text{Var}(\mathbb{C}) \), \( P(h) : \text{Var}(\mathbb{C})^2/S \to \text{Var}(\mathbb{C})^2/X \) admits a left adjoint

\[
C(h) : \text{Var}(\mathbb{C})^2/X \to \text{Var}(\mathbb{C})^2/S, \quad C(h)((X', Z'), h') = ((X', Z'), h \circ h').
\]

Hence \( h^* : C(\text{Var}(\mathbb{C})^2/S) \to C(\text{Var}(\mathbb{C})^2/X) \) admits a left adjoint

\[
h_\sharp : C(\text{Var}(\mathbb{C})^2/X) \to C(\text{Var}(\mathbb{C})^{2,\text{sm}}/S), \quad F \mapsto (h_\sharp F : ((X, Z), h_0) \mapsto \lim_{((X', Z'), h_0) \to ((X, Z), h_0)} F((X', Z')/X))
\]

• For \( p : Y \times S \to S \) a projection with \( Y, S \in \text{Var}(\mathbb{C}) \) with \( Y \) smooth, \( P(p) : \text{Var}(\mathbb{C})^{2,\text{smpr}}/Y \times S \to \text{Var}(\mathbb{C})^{2,\text{smpr}}/Y \times S \) admits a left adjoint

\[
C(p) : \text{Var}(\mathbb{C})^{2,\text{smpr}}/Y \times S \to \text{Var}(\mathbb{C})^{2,\text{smpr}}/S, \quad C(p)((Y' \times S, Z'), p') = ((Y' \times S, Z'), p \circ p').
\]

Hence \( p^* : C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/S) \to C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/Y \times S) \) admits a left adjoint

\[
p_\sharp : C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/Y \times S) \to C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/S), \quad F \mapsto (p_\sharp F : ((Y_0 \times S, Z), p_0) \mapsto \lim_{((Y' \times Y \times S, Z'), p_0) \to ((Y_0 \times S, Z), p_0)} F((Y' \times Y \times S, Z')/Y \times S))
\]

• For \( p : Y \times S \to S \) a projection with \( Y, S \in \text{Var}(\mathbb{C}) \), \( P(p) : \text{Var}(\mathbb{C})^{2,\text{pr}}/Y \times S \to \text{Var}(\mathbb{C})^{2,\text{pr}}/Y \times S \) admits a left adjoint

\[
C(p) : \text{Var}(\mathbb{C})^{2,\text{pr}}/Y \times S \to \text{Var}(\mathbb{C})^{2,\text{pr}}/S, \quad C(p)((Y' \times S, Z'), p') = ((Y' \times S, Z'), p \circ p').
\]

Hence \( p^* : C(\text{Var}(\mathbb{C})^{2,\text{pr}}/S) \to C(\text{Var}(\mathbb{C})^{2,\text{pr}}/Y \times S) \) admits a left adjoint

\[
p_\sharp : C(\text{Var}(\mathbb{C})^{2,\text{pr}}/Y \times S) \to C(\text{Var}(\mathbb{C})^{2,\text{pr}}/S), \quad F \mapsto (p_\sharp F : ((Y_0 \times S, Z), p_0) \mapsto \lim_{((Y' \times Y \times S, Z'), p_0) \to ((Y_0 \times S, Z), p_0)} F((Y' \times Y \times S, Z')/Y \times S))
\]

Let \( S \in \text{Var}(\mathbb{C}) \).

• We have the dual functor

\[
\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{2,(\text{sm})}/S) \to C(\text{Var}(\mathbb{C})^{2,(\text{sm})}/S), \quad F \mapsto \mathbb{D}_S(F) := \text{Hom}(F, E_{\text{et}}(\mathbb{Z}((S, S)/S)))
\]

which induces the functor

\[
L\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{2,(\text{sm})}/S) \to C(\text{Var}(\mathbb{C})^{2,(\text{sm})}/S), \quad F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \text{Hom}(LF, E_{\text{et}}(\mathbb{Z}((S, S)/S)))
\]

• We have the dual functor

\[
\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}/S) \to C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}/S), \quad F \mapsto \mathbb{D}_S(F) := \text{Hom}(F, E_{\text{et}}(\mathbb{Z}((S, S)/S)))
\]

which induces the functor

\[
L\mathbb{D}_S : C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}/S) \to C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}/S), \quad F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \text{Hom}(LF, E_{\text{et}}(\mathbb{Z}((S, S)/S)))
\]

**Proposition 21.**

(i) Let \( S \in \text{Var}(\mathbb{C}) \). Let \( h : U \to S \) a smooth morphism with \( U \in \text{Var}(\mathbb{C}) \). Then for \( F \in C(\text{Var}(\mathbb{C})^{2,\text{sm}}/S) \), the canonical map in \( C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/S) \)

\[
T(\text{Gr}_S^{12}, \text{hom})(\mathbb{Z}(U/S), F) : \text{Gr}_S^{12} \text{Hom}((\mathbb{Z}(U/S), F) \xrightarrow{\sim} \text{Hom}(\text{Gr}_S^{12}, \mathbb{Z}(U/S), \text{Gr}_S^{12}F)
\]

is an isomorphism.
(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $h : U \to S$ a morphism with $U \in \text{Var}(\mathbb{C})$. Then for $F \in C(\text{Var}(\mathbb{C})/S)$, the canonical map in $C(\text{Var}(\mathbb{C})^{2,pr}/S)$

$$T(\text{Gr}_S^{12}, \text{hom})(\mathbb{Z}(U/S), F) : \text{Gr}_S^{12} \text{Hom}(\mathbb{Z}(U/S), F) \xrightarrow{\sim} \text{Hom}(\text{Gr}_S^{12} \mathbb{Z}(U/S), \text{Gr}_S^{12} F)$$

is an isomorphism.

**Proof.** (i): We have, for $(X \times S, Z)/S \in \text{Var}(\mathbb{C})^{2,sm}/S$ the following commutative diagram

$$\begin{array}{ccc}
\text{Gr}_S^{12} \text{Hom}(\mathbb{Z}(U/S), F)((X \times S, Z)/S) & \xrightarrow{T(\text{Gr}_S^{12}, \text{hom})(\mathbb{Z}(U/S), F)((X \times S, Z)/S)} & \text{Hom}(\text{Gr}_S^{12} \mathbb{Z}(U/S), \text{Gr}_S^{12} F)((X \times S, Z)/S) \\
\text{lim}_{((X \times S, Z)/S) \to \text{Gr}_S^{12}(V/S)} F(U \times S V) & = & \text{lim}_{((X \times U, Z)/Z \times S U) \to \text{Gr}_S^{12}(W/S)} F(W)
\end{array}$$

We then note that the map $\{(((X \times S, Z)/S) \to \text{Gr}_S^{12}(V/S))\} \to \{((X \times U, Z)/Z \times S U) \to \text{Gr}_S^{12}(W/S)\}$ obviously admits an inverse since a map $(X \times U, Z \times S U)/S \to (W \times S, W)/S)$ is uniquely determined by a map $g : X \to W$ such that $(g \times I_S)(Z) \subset W$. (ii):Similar to (i).

We have the support section functors of a closed embedding $i : Z \hookrightarrow S$ for presheaves on the big Zariski site of pairs.

**Definition 14.** Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \text{Var}(\mathbb{C})$ and $j : S \setminus Z \to S$ be the open complementary subset.

(i) We define the functor

$$\Gamma_Z : C(\text{Var}(\mathbb{C})^{2,sm}/S) \to C(\text{Var}(\mathbb{C})^{2,sm}/S), \ G \mapsto \Gamma_Z G^* := \text{Cone}(\text{ad}(j^*, j_*)(G^*) : G^* \to j_* j^* G^*)[-1],$$

so that there is then a canonical map $\gamma_Z(G^*) : \Gamma_Z G^* \to G^*$.

(ii) We have the dual functor of (i):

$$\Gamma_Z^* : C(\text{Var}(\mathbb{C})^{2,sm}/S) \to C(\text{Var}(\mathbb{C})^{2,sm}/S), \ F \mapsto \Gamma_Z^*(F^*) := \text{Cone}(\text{ad}(j_!, j^!)(F^*) : j_! j^! F^* \to F^*),$$

together with the canonical map $\gamma_Z^*(F) : F \to \Gamma_Z^*(F)$.

(iii) For $F, G \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$, we denote by

$$I(\gamma, \text{hom})(F, G) := (I, \text{I}(j_!, j^!)(F, G)^{-1}) : \Gamma_Z \text{Hom}(F, G) \xrightarrow{\sim} \text{Hom}(\Gamma_Z^* F, G)$$

the canonical isomorphism given by adjunction.

Note that we have similarly for $i : Z \hookrightarrow S$, $i' : Z' \hookrightarrow Z$ closed embeddings, $g : T \to S$ a morphism with $T, S, Z \in \text{Var}(\mathbb{C})$ and $F \in C(\text{Var}(\mathbb{C})^{2,sm}/S)$, the canonical maps in $C(\text{Var}(\mathbb{C})^{2,sm}/S)$

$$- \ T(g, \gamma)(F) : g^* \Gamma_Z F \xrightarrow{\sim} \Gamma_Z g^* T F, \ T(g, \gamma^\vee)(F) : \Gamma_Z^* g^* T F \xrightarrow{\sim} g^* \Gamma_Z F$$
$$- \ T(Z'/Z, \gamma)(F) : \Gamma_Z^* F \xrightarrow{\sim} \Gamma_Z F, \ T(Z'/Z, \gamma^\vee)(F) : \Gamma_Z^* F \xrightarrow{\sim} \Gamma_Z^* F$$

but we will not use them in this article.

Let $S_* \subset \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C}))$ with $\mathcal{I} \in \text{Cat}$, a diagram of algebraic varieties. It gives the diagram of sites $\text{Var}(\mathbb{C})^{2}/S_* \subset \text{Fun}(\mathcal{I}, \text{Cat})$.

- Then $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_*)$ is the category

  - whose objects $(G, F) = ((G_I, F_I)_{I \in \mathcal{I}}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_I)$, and $u_{IJ} : (G_I, F) \to r_{IJ,*}(G_J, F)$ for $r_{IJ} : I \to J$, denoting again $r_{IJ} : S_I \to S_J$, are morphisms satisfying for $I \to J \to K$, $r_{IJ} u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_I)$.  

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the morphisms \( m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ}) \) being (see section 2.1) a family of morphisms of complexes,
\[
m = (m_I : (G_I, F) \to (H_I, F))_{I \in I}
\]
such that \( v_{IJ} \circ m_I = p_{IJ}, m_J \circ u_{IJ} \) in \( C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_I) \).

- Then \( C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_*) \) is the category
- whose objects \( (G, F) = ((G_I, F)_{I \in I}, u_{IJ}) \), with \((G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_I)\), and \( u_{IJ} : (G_I, F) \to r_{IJ}G_J, F) \) for \( r_{IJ} \) denoting again \( r_{IJ} : S_I \to S_J \), are morphisms satisfying for \( I \to J \to K \), \( r_{IJ}r_{JK}u_{IJ}K = u_{IK} \) in \( C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_I) \),
- the morphisms \( m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ}) \) being (see section 2.1) a family of morphisms of complexes,
\[
m = (m_I : (G_I, F) \to (H_I, F))_{I \in I}
\]
such that \( v_{IJ} \circ m_I = p_{IJ}, m_J \circ u_{IJ} \) in \( C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_I) \).

For \( s : I \to J \) a functor, with \( I, J \in \text{Cat} \), and \( f_* : T_* \to S_* \) a morphism with \( T_* \in \text{Fun}(\mathcal{J}, \text{Var}(\mathbb{C})) \) and \( S_* \in \text{Fun}(I, \text{Var}(\mathbb{C})) \), we have by definition, the following commutative diagrams of sites

\[
\begin{array}{c}
\text{Var}(\mathbb{C})^2/T_* & \xrightarrow{\rho^{T_*}} & \text{Var}(\mathbb{C})^{2,pr}/T_* \\
\downarrow P(f_*) & & \downarrow P(f_*) \\
\text{Var}(\mathbb{C})^{2,sm}/T_* & \xrightarrow{\rho^{T_*}_{(f_*)}} & \text{Var}(\mathbb{C})^{2,smpr}/T_* \\
\downarrow \rho_{S_*} & & \downarrow \rho_{S_*} \\
\text{Var}(\mathbb{C})^{2,sm}/S_* & \xrightarrow{\rho_{S_*}} & \text{Var}(\mathbb{C})^{2,smpr}/S_* \\
\end{array}
\]

and

\[
\begin{array}{c}
\text{Var}(\mathbb{C})^{2,pr}/T_* & \xrightarrow{\text{Gr}_{\mathcal{J}_*}^2} & \text{Var}(\mathbb{C})/T \\
\downarrow P(f_*) & & \downarrow P(f_*) \\
\text{Var}(\mathbb{C})^{2,smpr}/T_* & \xrightarrow{\text{Gr}_{\mathcal{J}_*}^2_{(f_*)}} & \text{Var}(\mathbb{C})^{sm}/T_* \\
\downarrow \rho_{S_*} & & \downarrow \rho_{S_*} \\
\text{Var}(\mathbb{C})^{2,sm}/S_* & \xrightarrow{\text{Gr}_{\mathcal{J}_*}^2} & \text{Var}(\mathbb{C})^{sm}/S_* \\
\end{array}
\]

Let \( s : I \to J \) a functor, with \( I, J \in \text{Cat} \), and \( f_* : T_* \to S_* \) a morphism with \( T_* \in \text{Fun}(\mathcal{J}, \text{Var}(\mathbb{C})) \) and \( S_* \in \text{Fun}(I, \text{Var}(\mathbb{C})) \).

- As usual, we denote by

\[
(f^*, f_*) := (P(f_*), P(f_*)_* : C(\text{Var}(\mathbb{C})^{2,(sm)}/S_*) \to C(\text{Var}(\mathbb{C})^{2,(sm)}/T_*)
\]

the adjunction induced by \( P(f_* : \text{Var}(\mathbb{C})^{2,(sm)}/T_* \to \text{Var}(\mathbb{C})^{2,(sm)}/S_* \). Since the colimits involved in the definition of \( f_* = P(f_*)^* \) are filtered, \( f_* \) also preserve monomorphism. Hence, we get an
adjonction

\[(f^*_\bullet, f_\bullet) : C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/S_\bullet) \Rightarrow C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/T_\bullet),\]

\[f^*_\bullet((G_I, F), u_{II}) := ((f_I^* G_I, f_I^* F), T(f_I, r_{II})(-) \circ f_I^* u_{II})\]

- As usual, we denote by

\[(f^*_\bullet, f_\bullet) := (P(f^*_\bullet), P(f_\bullet)) : C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_\bullet) \Rightarrow C(\text{Var}(\mathbb{C})^{2,(sm)pr}/T_\bullet)\]

the adjonction induced by \(P(f^*_\bullet) : \text{Var}(\mathbb{C})^{2,(sm)pr}/T_\bullet \rightarrow \text{Var}(\mathbb{C})^{2,(sm)pr}/S_\bullet\). Since the colimits involved in the definition of \(f^*_\bullet = P(f^*_\bullet)\) are filtered, \(f^*_\bullet\) also preserve monomorphism. Hence, we get an adjonction

\[(f^*_\bullet, f_\bullet) : C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/S_\bullet) \Rightarrow C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/T_\bullet),\]

\[f^*_\bullet((G_I, F), u_{II}) := ((f_I^* G_I, f_I^* F), T(f_I, r_{II})(-) \circ f_I^* u_{II})\]

Let \(S \in \text{Var}(\mathbb{C})\). Let \(S = \bigcup_{i=1}^l S_i\) an open affine cover and denote by \(S_I = \cap_{i \in I} S_i\). Let \(i_i : S_i \hookrightarrow \tilde{S}_i\) closed embeddings, with \(\tilde{S}_i \in \text{Var}(\mathbb{C})\). For \(I \subset [1, \ldots, l]\), denote by \(\tilde{S}_I = \Pi_{i \in I} \tilde{S}_i\). We then have closed embeddings \(i_I : S_I \hookrightarrow \tilde{S}_I\) and for \(J \subset I\) the following commutative diagram

\[
\begin{array}{ccc}
D_{IJ} & \xrightarrow{i_I} & \tilde{S}_I \\
\downarrow j_{IJ} & & \downarrow p_{IJ} \\
S_J & \xrightarrow{i_J} & \tilde{S}_J
\end{array}
\]

where \(p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I\) is the projection and \(j_{IJ} : S_J \hookrightarrow S_I\) is the open embedding so that \(j_I \circ j_{IJ} = j_J\). This gives the diagram of algebraic varieties \((\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Var}(\mathbb{C}))\) which gives the diagram of sites \(\text{Var}(\mathbb{C})^2/(\tilde{S}_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})\). This gives also the diagram of algebraic varieties \((\tilde{S}_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N})^{op}, \text{Var}(\mathbb{C}))\) which gives the diagram of sites \(\text{Var}(\mathbb{C})^2/(\tilde{S}_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N})^{op}, \text{Cat})\).

- Then \(C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))\) is the category

- whose objects \((G, F) = ((G_I, F)_{I \subset [1, \ldots, l]}, u_{II})\), with \((G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/\tilde{S}_I)\), and \(u_{II} : (G_I, F) \rightarrow p_{II,J}(G_J, F)\) for \(I \subset J\), are morphisms satisfying for \(I \subset J \subset K\), \(p_{IJ,K} u_{II,J} \circ u_{IJ} = u_{IK}\) in \(C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/\tilde{S}_I)\),

- the morphisms \(m : ((G, F), u_{II}) \rightarrow ((H, F), v_{II})\) being (see section 2.1) a family of morphisms of complexes,

\[
m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \ldots, l]}\]

such that \(v_{II} \circ m_I = p_{IJ,K} m_J \circ u_{IJ}\) in \(C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/\tilde{S}_I)\).

- Then \(C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I))\) is the category

- whose objects \((G, F) = ((G_I, F)_{I \subset [1, \ldots, l]}, u_{II})\), with \((G_I, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/\tilde{S}_I)\), and \(u_{II} : (G_I, F) \rightarrow p_{II,J}(G_J, F)\) for \(I \subset J\), are morphisms satisfying for \(I \subset J \subset K\), \(p_{IJ,K} u_{II,J} \circ u_{IJ} = u_{IK}\) in \(C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/\tilde{S}_I)\),

- the morphisms \(m : ((G, F), u_{II}) \rightarrow ((H, F), v_{II})\) being (see section 2.1) a family of morphisms of complexes,

\[
m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset [1, \ldots, l]}\]

such that \(v_{II} \circ m_I = p_{IJ,K} m_J \circ u_{IJ}\) in \(C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/\tilde{S}_I)\).

- Then \(C_{fil}(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)^{op})\) is the category
— whose objects \((G, F) = ((G_I, F)_{i \in [1, \cdots, n]}, u_{IJ})\), with \((G_I, F) \in C_{fil}(\mathbb{C}^2, (\tilde{S}_I))\), and \(u_{IJ} : (G_J, F) \to p^*_IJ(G_I, F)\) for \(I \subseteq J\), are morphisms satisfying for \(I \subseteq J \subseteq K\), 
\[ p^*_JK u_{IJ} \circ u_{IK} = u_{IK} \in C_{fil}(\mathbb{C}^2, (\tilde{S}_K)) \],

— the morphisms \(m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ})\) being (see section 2.1) a family of morphisms of complexes,

\[ m = (m_I : (G_I, F) \to (H_I, F))_{i \in [1, \cdots, n]} \]

such that \(v_{IJ} \circ m_I = p^*_IJ m_I \circ u_{IJ} \in C_{fil}(\mathbb{C}^2, (\tilde{S}_J))\).

• Then \(C_{fil}(\mathbb{C}^2, (\tilde{S}_I))^{op}\) is the category

— whose objects \((G, F) = ((G_I, F)_{i \in [1, \cdots, n]}, u_{IJ})\), with \((G_I, F) \in C_{fil}(\mathbb{C}^2, (\tilde{S}_I))\), and \(u_{IJ} : (G_J, F) \to p^*_IJ(G_I, F)\) for \(I \subseteq J\), are morphisms satisfying for \(I \subseteq J \subseteq K\), 
\[ p^*_JK u_{IJ} \circ u_{IK} = u_{IK} \in C_{fil}(\mathbb{C}^2, (\tilde{S}_K)) \],

— the morphisms \(m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ})\) being (see section 2.1) a family of morphisms of complexes,

\[ m = (m_I : (G_I, F) \to (H_I, F))_{i \in [1, \cdots, n]} \]

such that \(v_{IJ} \circ m_I = p^*_IJ m_I \circ u_{IJ} \in C_{fil}(\mathbb{C}^2, (\tilde{S}_J))\).

We now define the Zariski and the etale topology on \(\mathbb{C}^2/S\).

**Definition 15.** Let \(S \in Var(\mathbb{C})\).

(i) Denote by \(\tau\) a topology on \(Var(\mathbb{C})\), e.g. the Zariski or the etale topology. The \(\tau\) covers in \(\mathbb{C}^2/S\) of \((X, Z)/S\) are the families of morphisms

\[ \{(c_i : (U_i, Z \times_X U_i)/S \to (X, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \to X)_{i \in I} \text{ \tau cover of } X \text{ in } Var(\mathbb{C}) \} \]

(ii) Denote by \(\tau\) the Zariski or the etale topology on \(Var(\mathbb{C})\). The \(\tau\) covers in \(\mathbb{C}^{2,sm}/S\) of \((U, Z)/S\) are the families of morphisms

\[ \{(c_i : (U_i, Z \times_U U_i)/S \to (U, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \to U)_{i \in I} \text{ \tau cover of } U \text{ in } Var(\mathbb{C}) \} \]

(iii) Denote by \(\tau\) the Zariski or the etale topology on \(Var(\mathbb{C})\). The \(\tau\) covers in \(\mathbb{C}^{2,smpr}/S\) of \((Y \times S, Z)/S\) are the families of morphisms

\[ \{(c_i \times I_S : (U_i \times S, Z \times_Y S U_i \times S)/S \to (Y \times S, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \to Y)_{i \in I} \text{ \tau cover of } Y \text{ in } Var(\mathbb{C}) \} \]

Let \(S \in Var(\mathbb{C})\). Denote by \(\tau\) the Zariski or the etale topology on \(Var(\mathbb{C})\). In particular, denoting \(a_\tau : \text{PSh}(\mathbb{C}^{2,sm}/S) \to \text{Shv}(\mathbb{C}^{2,sm}/S)\) and \(a_\tau : \text{PSh}(\mathbb{C}^{2,smpr}/S) \to \text{Shv}(\mathbb{C}^{2,smpr}/S)\) the sheafification functors,

• a morphism \(\phi : F \to G\), with \(F, G \in C(\mathbb{C}^{2,sm}/S)\), is a \(\tau\) local equivalence if \(a_\tau H^n \phi : a_\tau H^n F \to a_\tau H^n G\) is an isomorphism, \(\phi : F \to G\), with \(F, G \in C(\mathbb{C}^{2,smpr}/S)\), is a \(\tau\) local equivalence if \(a_\tau H^n \phi : a_\tau H^n F \to a_\tau H^n G\) is an isomorphism;

• \(F^\bullet \in C(\mathbb{C}^{2,sm}/S)\) is \(\tau\) taut if for all \((U, Z)/S \in \mathbb{C}^{2,sm}/S\) and all \(\tau\) covers \((c_i : (U_i, Z \times_U U_i)/S \to (U, Z)/S)_{i \in I}\) of \((U, Z)/S\),

\[ F^\bullet(c_i) : F^\bullet((U, Z)/S) \to \text{Tot}(\oplus_{c_i \tau \text{ cover}} F^\bullet((U_i, Z \times_U U_i)/S)) \]

is a quasi-isomorphism of complexes of abelian groups, \(F^\bullet \in C(\mathbb{C}^{2,smpr}/S)\) is \(\tau\) taut if for all \((Y \times S, Z)/S \in \mathbb{C}^{2,smpr}/S\) and all \(\tau\) covers \((c_i \times I_S : (U_i \times S, Z \times_Y S U_i \times S)/S \to (Y \times S, Z)/S)_{i \in I}\) of \((Y \times S, Z)/S\),

\[ F^\bullet(c_i \times I_S) : F^\bullet((Y \times S, Z)/S) \to \text{Tot}(\oplus_{\tau \text{ cover}} F^\bullet((U_i \times S, Z \times_U U_i)/S)) \]

is a quasi-isomorphism of complexes of abelian groups;
• a morphism \( \phi : (G_1, F) \to (G_2, F) \), with \((G_1, F), (G_2, F) \in C_{fil}(Var(\mathbb{C})^{2,(sm)}/S) \), is an filtered \( \tau \)
local equivalence if for all \( n \), \( p \in \mathbb{Z} \),

\[
a_{r}H^n Gr_{p}^{U}(\phi) : a_{r}H^n Gr_{p}^{U}(G_1, F) \to a_{r}H^n Gr_{p}^{U}(G_2, F)
\]
is an isomorphism of sheaves on \( Var(\mathbb{C})^{2,(sm)}/S \), a morphism \( \phi : (G_1, F) \to (G_2, F) \), with \((G_1, F), (G_2, F) \in C_{fil}(Var(\mathbb{C})^{2,(sm)pr}/S) \), is an filtered \( \tau \) local equivalence if for all \( n \), \( p \in \mathbb{Z} \),

\[
a_{r}H^n Gr_{p}^{U}(\phi) : a_{r}H^n Gr_{p}^{U}(G_1, F) \to a_{r}H^n Gr_{p}^{U}(G_2, F)
\]
is an isomorphism of sheaves on \( Var(\mathbb{C})^{2,(sm)pr}/S \);

• let \( r \in \mathbb{N} \), a morphism \( \phi : (G_1, F) \to (G_2, F) \), with \((G_1, F), (G_2, F) \in C_{fil}(Var(\mathbb{C})^{2,(sm)}/S) \),
is an \( r \)-filtered \( \tau \) local equivalence if there exists \( \phi_i : (C_i, F) \to (C_{i+1}, F), 0 \leq i \leq s \), with \((C_i, F) \in C_{fil}(Var(\mathbb{C})^{2,(sm)}/S), (C_0, F) = (G_1, F) \) and \((C_s, F) = (G_2, F) \), such that

\[
\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : (G_1, F) \to (G_2, F)
\]
and \( \phi_i : (C_i, F) \to (C_{i+1}, F) \) either a filtered \( \tau \) local equivalence or an \( r \)-filtered homotopy equivalence, a morphism \( \phi : (G_1, F) \to (G_2, F) \), with \((G_1, F), (G_2, F) \in C_{fil}(Var(\mathbb{C})^{2,(sm)pr}/S) \),
is an \( r \)-filtered \( \tau \) local equivalence if there exists \( \phi_i : (C_i, F) \to (C_{i+1}, F), 0 \leq i \leq s \), with \((C_i, F) \in C_{fil}(Var(\mathbb{C})^{2,(sm)pr}/S), (C_0, F) = (G_1, F) \) and \((C_s, F) = (G_2, F) \), such that

\[
\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : (G_1, F) \to (G_2, F)
\]
and \( \phi_i : (C_i, F) \to (C_{i+1}, F) \) either a filtered \( \tau \) local equivalence or an \( r \)-filtered homotopy equivalence;

• \((F^\bullet, F) \in C_{fil}(Var(\mathbb{C})^{2,(sm)}/S) \) is filtered \( \tau \) fibrant if for all \((U, Z)/S \in Var(\mathbb{C})^{2,(sm)}/S \) and all \( \tau \) covers \((c_i : (U_i, Z \times_U U_i)/S \to (U, Z)/S)_{i \in I} \) of \((U, Z)/S \),

\[
H^n Gr_{p}^{U}(F^\bullet, F)(c_i) : (F^\bullet, F)((U, Z)/S) \to
H^n Gr_{p}^{U}(Tot(\oplus_{\text{card}I=s}(F^\bullet, F)((U_1, Z \times_U U_1)/S))
\]
is an isomorphism of of abelian groups for all \( n \), \( p \in \mathbb{Z} \); \((F^\bullet, F) \in C_{fil}(Var(\mathbb{C})^{2,(sm)pr}/S) \) is filtered \( \tau \) fibrant if for all \((Y \times S, Z)/S \in Var(\mathbb{C})^{2,(sm)pr}/S \) and all \( \tau \) covers \((c_i \times I_S : (U_i \times S, Z \times_Y S U_i \times S)/S \to (Y \times S, Z)/S)_{i \in I} \) of \((Y \times S, Z)/S \),

\[
H^n Gr_{p}^{U}(F^\bullet, F)(c_i \times I_S) : H^n Gr_{p}^{U}(F^\bullet, F)((Y \times S, Z)/S) \to
H^n Gr_{p}^{U}(Tot(\oplus_{\text{card}I=s}(F^\bullet, F)((U_1 \times S, Z \times_Y U_1)/S))
\]
is an isomorphism of abelian groups for all \( n \), \( p \in \mathbb{Z} \).

Let \( S_\bullet \in \text{Fun}(\mathcal{I}, Var(\mathbb{C})) \) with \( \mathcal{I} \in \text{Cat} \).

• A morphism \( m : ((G_1, F), u_{I_1}) \to ((H_1, F), v_{I_1}) \) in \( C_{fil}(Var(\mathbb{C})^{2,(sm)}/S_\bullet) \) is an \( r \)-filtered Zariski, resp. etale local, equivalence, if there exists \( \phi_i : ((C_i, F), u_{i+1}) \to ((C_{i+1}, F), u_{i+1}), 0 \leq i \leq s \), with \((C_i, F), u_{i+1}) \in C_{fil}(Var(\mathbb{C})^{2,(sm)}/S_\bullet) \((C_0, F), u_0) = ((G_1, F), u_I) \) and \((C_s, F), u_s) = ((H_1, F), v_I) \), such that

\[
\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : ((G_1, F), u_I) \to ((H_1, F), v_I)
\]
and \( \phi_i : ((C_i, F), u_{i+1}) \to ((C_{i+1}, F), u_{i+1}) \) either a filtered Zariski, resp. etale equivalence or an \( r \)-filtered homotopy equivalence.
A morphism $m : ((G_I, F), u_{iI}) \rightarrow ((H_I, F), v_{iI})$ in $\text{C}_{\text{fil}}(\text{Var}(\mathbb{C}))^{2,(\text{sm})}_{\mathbb{C}/S}$ is an r-filtered Zariski, resp. etale local, equivalence, if there exists $\phi_i : ((C_{iI}, F), u_{iI}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)I})$, $0 \leq i \leq s$, with $((C_{iI}, F), u_{iI}) \in \text{C}_{\text{fil}}(\text{Var}(\mathbb{C}))^{2,(\text{sm})}_{\mathbb{C}/S}$, $(C_{0l}, F), u_{0l}) = ((G_I, F), u_{iI})$ and $((C_{sI}, F), u_{sI}) = ((H_I, F), v_{iI})$, such that

$$
\phi = \phi_s \circ \cdots \phi_1 \circ \cdots \phi_0 : ((G_I, F), u_{iI}) \rightarrow ((H_I, F), v_{iI})
$$

and $\phi_i : ((C_{iI}, F), u_{iI}) \rightarrow ((C_{(i+1)I}, F), u_{(i+1)I})$ either a filtered Zariski, resp. etale, local equivalence or an r-filtered homotopy equivalence.

Will now define the $A^1$ local property on $\text{Var}(\mathbb{C})^2/S$.
Denote $\square^* := \mathbb{P}^n_{\mathbb{C}} \setminus \{0\}$

Let $S \in \text{Var}(\mathbb{C})$. For $(X, Z)/S = ((X, Z), h) \in \text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S}$, we consider

$$(\square^* \times X, \square^* \times Z)/S = ((\square^* \times X, \square^* \times Z, h \circ p) \in \text{Fun}(\Delta, \text{Var}(\mathbb{C})^{2,(\text{sm})}_{\mathbb{C}/S}).$$

For $F \in C^-(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S})$, it gives the complex

$$C_* F \in C^-(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S}), (X, Z)/S = ((X, Z), h) \mapsto C_* F((X, Z) / S) := \text{Tot} F((\square^* \times X, \square^* \times Z) / S)$$

together with the canonical map $c_F := (0, I_F) : F \rightarrow C_* F$. For $F \in C(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S})$, we get

$$C_* F := \text{holim}_n C_* F^{\leq n} \in C(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S}),$$

together with the canonical map $c_F := (0, I_F) : F \rightarrow C_* F$. For $m : F \rightarrow G$ a morphism, with $F, G \in C(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S})$, we get by functoriality the morphism $C_* m : C_* F \rightarrow C_* G$.

Let $S \in \text{Var}(\mathbb{C})$. For $(Y \times S, Z)/S = ((Y \times S, Z), h) \in \text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S}$, we consider

$$(\square^* \times Y \times S, \square^* \times Z)/S = ((\square^* \times Y \times S, \square^* \times Z, h \circ p) \in \text{Fun}(\Delta, \text{Var}(\mathbb{C})/S).$$

For $F \in C^-(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S})$, it gives the complex

$$C_* F \in C^-(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S}), (Y \times S, Z)/S = ((Y \times S, Z), h) \mapsto C_* F((Y \times S, Z) / S) := \text{Tot} F((\square^* \times Y \times S, \square^* \times Z) / S)$$

together with the canonical map $c = c(F) := (0, I_F) : F \rightarrow C_* F$. For $F \in C(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S})$, we get

$$C_* F := \text{holim}_n C_* F^{\leq n} \in C(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S}),$$

together with the canonical map $c = c(F) := (0, I_F) : F \rightarrow C_* F$. For $m : F \rightarrow G$ a morphism, with $F, G \in C(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S})$, we get by functoriality the morphism $C_* m : C_* F \rightarrow C_* G$.

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^s S_i$ an open affine cover and denote by $S_i = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. For $F = (F_i, u_{iI}) \in C(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S})$, it gives the complex

$$C_* F = (C_* F_i, C_* u_{iI}) \in C(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S})$$

together with the canonical map $c_F := (0, I_F) : F \rightarrow C_* F$.

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^s S_i$ an open affine cover and denote by $S_i = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$. For $F = (F_i, u_{iI}) \in C(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S})$, it gives the complex

$$C_* F = (C_* F_i, C_* u_{iI}) \in C(\text{Var}(\mathbb{C})^2,(\text{sm})_{\mathbb{C}/S})$$

together with the canonical map $c_F := (0, I_F) : F \rightarrow C_* F$. 

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Let \( S \in \text{Var}(\mathbb{C}) \). Let \( S = \bigcup_{i=1}^{l} S_i \) an open affine cover and denote by \( S_I = \bigcap_{i \in I} S_i \). Let \( i : S_i \hookrightarrow \tilde{S}_i \) closed embeddings, with \( \tilde{S}_i \in \text{Var}(\mathbb{C}) \). For \( F = (F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)) \), it gives the complex

\[
C_*F = (C_*F_I, C_*u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)),
\]

together with the canonical map \( c_F := (0, I_F) : F \to C_*F \).

Let \( S \in \text{Var}(\mathbb{C}) \). Let \( S = \bigcup_{i=1}^{l} S_i \) an open affine cover and denote by \( S_I = \bigcap_{i \in I} S_i \). Let \( i : S_i \hookrightarrow \tilde{S}_i \) closed embeddings, with \( \tilde{S}_i \in \text{Var}(\mathbb{C}) \). For \( F = (F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)) \), it gives the complex

\[
C_*F = (C_*F_I, C_*u_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)^{op}),
\]

together with the canonical map \( c_F := (0, I_F) : F \to C_*F \).

Let \( S \in \text{Var}(\mathbb{C}) \). Denote for short \( \text{Var}(\mathbb{C})^{2,(sm)}/S \) either the category \( \text{Var}(\mathbb{C})^{2}/S \) or the category \( \text{Var}(\mathbb{C})^{2,(sm)}/S \). Denote by

\[
p_a : \text{Var}(\mathbb{C})^{2,(sm)}/S \to \text{Var}(\mathbb{C})^{2,(sm)}/S,
\]

\[
(X, Z)/S \mapsto ((X, Z), h) \mapsto (X \times A^1, Z \times A^1)/S = ((X \times A^1, Z \times A^1, h \circ p_X),
\]

\[
(g : (X, Z)/S \to (X', Z')/S) \mapsto ((g \times I_{A^1}) : (X \times A^1, Z \times A^1)/S \to (X' \times A^1, Z' \times A^1)/S)
\]

the projection functor and again by \( p_a : \text{Var}(\mathbb{C})^{2,(sm)}/S \to \text{Var}(\mathbb{C})^{2,(sm)}/S \) the corresponding morphism of site. Let \( S \in \text{Var}(\mathbb{C}) \). Denote for short \( \text{Var}(\mathbb{C})^{2,(sm)}/S \) either the category \( \text{Var}(\mathbb{C})^{2}/S \) or the category \( \text{Var}(\mathbb{C})^{2,(sm)}/S \). Denote for short \( \text{Var}(\mathbb{C})^{2,(sm)pr}/S \) either the category \( \text{Var}(\mathbb{C})^{2,pr}/S \) or the category \( \text{Var}(\mathbb{C})^{2,(sm)pr}/S \). Denote by

\[
p_a : \text{Var}(\mathbb{C})^{2,(sm)pr}/S \to \text{Var}(\mathbb{C})^{2,(sm)pr}/S,
\]

\[
(Y \times S, Z)/S \mapsto ((Y \times S, Z), p_S) \mapsto (Y \times S \times A^1, Z \times A^1)/S = ((Y \times S \times A^1, Z \times A^1, p_S \circ p_{Y \times S}),
\]

\[
(g : (Y \times S, Z)/S \to (Y' \times S, Z')/S) \mapsto ((g \times I_{A^1}) : (Y \times S \times A^1, Z \times A^1)/S \to (Y' \times S \times A^1, Z' \times A^1)/S)
\]

the projection functor and again by \( p_a : \text{Var}(\mathbb{C})^{2,(sm)pr}/S \to \text{Var}(\mathbb{C})^{2,(sm)pr}/S \) the corresponding morphism of site.

**Definition 16.** (i0) A complex \( F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \) is said to be \( A^1 \) homotopic if \( \text{ad}(p_a, p_a)(F) : F \to p_a p_a F \) is an homotopy equivalence.

(i0)' A complex \( F \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S) \) is said to be \( A^1 \) homotopic if \( \text{ad}(p_a, p_a)(F) : F \to p_a p_a F \) is an homotopy equivalence.

(i) A complex \( F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \), is said to be \( A^1 \) invariant if for all \((X, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)}/S \)

\[
F(p_X) : F((X, Z)/S) \to F((X \times A^1, (Z \times A^1))/S)
\]

is a quasi-isomorphism, where \( p_X : (X \times A^1, (Z \times A^1)) \to (X, Z) \) is the projection. Obviously, if a complex \( F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \) is \( A^1 \) homotopic, then it is \( A^1 \) invariant.

(i)' A complex \( G \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S) \), is said to be \( A^1 \) invariant if for all \((Y \times S, Z)/S \in \text{Var}(\mathbb{C})^{2,(sm)pr}/S \)

\[
G(p_{Y \times S}) : G((Y \times S, Z)/S) \to G((Y \times A^1 \times S, (Z \times A^1))/S)
\]

is a quasi-isomorphism of abelian group. Obviously, if a complex \( F \in C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S) \) is \( A^1 \) homotopic, then it is \( A^1 \) invariant.

(ii) Let \( \tau \) a topology on \( \text{Var}(\mathbb{C}) \). A complex \( F \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \) is said to be \( A^1 \) local for the \( \tau \) topology induced on \( \text{Var}(\mathbb{C})^{2}/S \), if for an (hence every) \( \tau \) local equivalence \( k : F \to G \) with \( k \) injective and \( G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \) \( \tau \) fibrant, e.g. \( k : F \to E_\tau(F) \), \( G \) is \( A^1 \) invariant.
(ii) Let $\tau$ be a topology on $\text{Var}(\mathbb{C})$. A complex $F \in C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}) / S$ is said to be $\mathbb{A}^1$-local for the $\tau$ topology induced on $\text{Var}(\mathbb{C})^{2,\text{pr}}$ if for any (hence every) $\tau$-local equivalence $k : F \to G$ with $k$ injective and $G \in C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}) / S$ $\tau$ fibrant, e.g. $k : F \to E_\tau(F)$, $G$ is $\mathbb{A}^1$-invariant.

(iii) A morphism $m : F \to G$ with $F,G \in C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}})$ is said to an $(\mathbb{A}^1,\text{et})$ local equivalence if for all $H \in C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}})$ which is $\mathbb{A}^1$-local for the etale topology

$$\text{Hom}(L(m), E_{\text{et}}(H)) : \text{Hom}(L(G), E_{\text{et}}(H)) \to \text{Hom}(L(F), E_{\text{et}}(H))$$

is a quasi-isomorphism.

(iii) A morphism $m : F \to G$ with $F,G \in C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}})$ is said to an $(\mathbb{A}^1,\text{et})$ local equivalence if for all $H \in C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}})$ which is $\mathbb{A}^1$-local for the etale topology

$$\text{Hom}(L(m), E_{\text{et}}(H)) : \text{Hom}(L(G), E_{\text{et}}(H)) \to \text{Hom}(L(F), E_{\text{et}}(H))$$

is a quasi-isomorphism.

**Proposition 22.**

(i) Let $S \in \text{Var}(\mathbb{C})$. Then for $F \in C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}) / S$, $C_* F$ is $\mathbb{A}^1$-local for the etale topology and $c(F) : F \to C_* F$ is an equivalence $(\mathbb{A}^1,\text{et})$ local.

(i) Let $S \in \text{Var}(\mathbb{C})$. Then for $F \in C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}})$, $C_* F$ is $\mathbb{A}^1$-local for the etale topology and $c(F) : F \to C_* F$ is an equivalence $(\mathbb{A}^1,\text{et})$ local.

(ii) A morphism $m : F \to G$ with $F,G \in C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}) / S$ is an $(\mathbb{A}^1,\text{et})$ local equivalence if and only if $a_{\text{et}} H^n C_* \text{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.

(ii) A morphism $m : F \to G$ with $F,G \in C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}})$ is an $(\mathbb{A}^1,\text{et})$ local equivalence if and only if $a_{\text{et}} H^n C_* \text{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.

(iii) A morphism $m : F \to G$ with $F,G \in C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}) / S$ is an $(\mathbb{A}^1,\text{et})$ local equivalence if and only if there exists

$$\{(X_{1,\alpha}, Z_{1,\alpha}) / S, \alpha \in \Lambda_1 \}, \ldots, \{(X_{r,\alpha}, Z_{r,\alpha}) / S, \alpha \in \Lambda_r \} \subset \text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}} / S$$

such that we have in $\text{Ho}_{\text{et}}(C(\text{Var}(\mathbb{C})^{2,(\text{sm})}\text{pr}) / S)$

$$\text{Cone}(m) \Rightarrow \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(\text{Z}(\{(X_{1,\alpha} \times \mathbb{A}^1) \times Z_{1,\alpha}) / S\}) \to \text{Z}(\{(X_{1,\alpha}, Z_{1,\alpha}) / S\}))$$

$$\Rightarrow \cdots \Rightarrow \oplus_{\alpha \in \Lambda_r} \text{Cone}(\text{Z}(\{(X_{r,\alpha} \times \mathbb{A}^1) \times Z_{r,\alpha}) / S\}) \to \text{Z}(\{(X_{r,\alpha}, Z_{r,\alpha}) / S\}))$$

(iii) A morphism $m : F \to G$ with $F,G \in C(\text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}}) / S$ is an $(\mathbb{A}^1,\text{et})$ local equivalence if and only if there exists

$$\{(Y_{1,\alpha} \times S, Z_{1,\alpha}) / S, \alpha \in \Lambda_1 \}, \ldots, \{(Y_{r,\alpha} \times S, Z_{r,\alpha}) / S, \alpha \in \Lambda_r \} \subset \text{Var}(\mathbb{C})^{2,(\text{sm})\text{pr}} / S$$

such that we have in $\text{Ho}_{\text{et}}(C(\text{Var}(\mathbb{C})^{2,(\text{sm})}\text{pr}) / S)$

$$\text{Cone}(m) \Rightarrow \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(\text{Z}(\{(Y_{1,\alpha} \times S, Z_{1,\alpha}) \times \mathbb{A}^1) / S\}) \to \text{Z}(\{(Y_{1,\alpha} \times S, Z_{1,\alpha}) / S\}))$$

$$\Rightarrow \cdots \Rightarrow \oplus_{\alpha \in \Lambda_r} \text{Cone}(\text{Z}(\{(Y_{r,\alpha} \times S, Z_{r,\alpha}) \times \mathbb{A}^1) / S\}) \to \text{Z}(\{(Y_{r,\alpha} \times S, Z_{r,\alpha}) / S\}))$$

Proof. Standard: see Ayoub’s thesis section 4 for example. Indeed, for (iii), by definition, if $\text{Cone}(m)$ is of the given form, then it is an equivalence $(\mathbb{A}^1,\text{et})$ local, on the other hand if $m$ is an equivalence $(\mathbb{A}^1,\text{et})$ local, we consider the commutative diagram

$$\begin{array}{c}
F \xrightarrow{c(F)} C_* F \\
\downarrow^m \downarrow \downarrow^{c_* m} \\
G \xrightarrow{c(G)} C_* G
\end{array}$$

To deduce that $\text{Cone}(m)$ is of the given form. □
Definition-Proposition 6. Let $S \in \text{Var}(\mathbb{C})$.

(i) With the weak equivalence the $(\mathbb{A}^1, \text{et})$ local equivalence and the fibration the epimorphism with $\mathbb{A}^1_1$ local and etale fibrant kernels gives a model structure on $C(\text{Var}(\mathbb{C})^{2,(sm)}/S) : \text{the left bousfield localization of the projective model structure of } C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$. We call it the projective $(\mathbb{A}^1, \text{et})$ model structure.

(ii) With the weak equivalence the $(\mathbb{A}^1, \text{et})$ local equivalence and the fibration the epimorphism with $\mathbb{A}^1_1$ local and etale fibrant kernels gives a model structure on $C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S) : \text{the left bousfield localization of the projective model structure of } C(\text{Var}(\mathbb{C})^{2,(sm)pr}/S)$. We call it the projective $(\mathbb{A}^1, \text{et})$ model structure.

Proof. Similar to the proof of proposition 5.

We have, similarly to the case of single varieties the following:

Proposition 23. Let $g : T \to S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$.

(i) The adjunction $(g^*, g_*): C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{2,(sm)}/T)$ is a Quillen adjunction for the projective $(\mathbb{A}^1, \text{et})$ model structure (see definition-proposition 6)

(ii) The functor $g^*: C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \to C(\text{Var}(\mathbb{C})^{2,(sm)}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local.

Proposition 24. Let $S \in \text{Var}(\mathbb{C})$.

(i) The adjunction $(\rho_S^*, \rho_S_*): C(\text{Var}(\mathbb{C})^{2,sm}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{2}/S)$ is a Quillen adjunction for the $(\mathbb{A}^1, \text{et})$ projective model structure.

(ii) The functor $\rho_S^*: C(\text{Var}(\mathbb{C})^{2}/S) \to C(\text{Var}(\mathbb{C})^{2,sm}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local.

Proposition 25. Let $S \in \text{Var}(\mathbb{C})$. 

Proof. Similar to the proof of proposition 19.

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(i) The adjunction $(\mu_{S*}^2, \mu_{S*}) : C(\text{Var}(\mathbb{C})^{2pr}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{2}/S)$ is a Quillen adjunction for the $(\mathbb{A}^{1}, et)$ projective model structure.

(ii) The functor $(\mu_{S*}) : C(\text{Var}(\mathbb{C})^{2}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2pr}/S)$ sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^{1}, et)$ local equivalence to $(\mathbb{A}^{1}, et)$ local equivalence.

Proof. Similar to the proof of proposition 19. Indeed, for (i)' or (ii)', if $m : F \rightarrow G$ with $F, G \in C(\text{Var}(\mathbb{C})^{2,sm})$ is an equivalence $(\mathbb{A}^{1}, et)$ local then (see proposition 22), there exists

\[
\{ (X_{1,\alpha}, Z_{1,\alpha})/S, \alpha \in A_{1} \}, \ldots, \{ (X_{r,\alpha}, Z_{r,\alpha})/S, \alpha \in A_{r} \} \subset \text{Var}(\mathbb{C})^{2,sm}/S
\]

such that we have in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{2,sm}/S))$

\[
\text{Cone}(m) \xrightarrow{\sim} \text{Cone}(\bigoplus_{\alpha \in A_{1}} \text{Cone}(\mathbb{Z}((X_{1,\alpha} \times \mathbb{A}^{1}, Z_{1,\alpha} \times \mathbb{A}^{1}))/S) \rightarrow \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha}))/S)
\rightarrow \cdots \rightarrow \text{Cone}(\bigoplus_{\alpha \in A_{n}} \mathbb{Z}((X_{r,\alpha} \times \mathbb{A}^{1}, Z_{r,\alpha} \times \mathbb{A}^{1}))/S) \rightarrow \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha}))/S),
\]

this gives in $\text{Ho}_{et}(C(\text{Var}(\mathbb{C})^{2,sm}/S))$

\[
\text{Cone}(m_{S*}m) \xrightarrow{\sim} \text{Cone}(\bigoplus_{\alpha \in A_{1}} \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha}))/S) \otimes \mathbb{Z}((\mathbb{A}^{1}, \mathbb{A}^{1}))/S \rightarrow (L_{\mu_{S*}} \oplus_{\alpha \in A_{1}} \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha}))/S)
\rightarrow \cdots \rightarrow \text{Cone}(\bigoplus_{\alpha \in A_{n}} \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha}))/S) \otimes \mathbb{Z}((\mathbb{A}^{1}, \mathbb{A}^{1}))/S) \rightarrow (L_{\mu_{S*}} \oplus_{\alpha \in A_{n}} \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha}))/S)),
\]

hence $\mu_{S*}m : \mu_{S*} F \rightarrow \mu_{S*} G$ is an equivalence $(\mathbb{A}^{1}, et)$ local.

We also have

**Proposition 26.** Let $S \in \text{Var}(\mathbb{C})$.

(i) The adjunction $(\text{Gr}^{2pr}_{S}, \text{Gr}_{S}^{2}) : C(\text{Var}(\mathbb{C})/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{2pr}/S)$ is a Quillen adjunction for the $(\mathbb{A}^{1}, et)$ projective model structure.

(ii) The adjunction $(\text{Gr}^{2,sm}_{S}, \text{Gr}_{S}^{12}) : C(\text{Var}(\mathbb{C})^{sm}/S) \rightleftarrows C(\text{Var}(\mathbb{C})^{2,smpr}/S)$ is a Quillen adjunction for the $(\mathbb{A}^{1}, et)$ projective model structure.

Proof. Immediate from definition.

- For $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ and $Z \subset X$ a closed subset, we denote $\mathbb{Z}^{tr}((X, Z)/S) \subset \text{PSh}(\text{Var}(\mathbb{C})^{2}/S)$ the presheaf given by
  
  \[
  \mathbb{Z}^{tr}((X, Z)/S)((X', Z')/S) := \{ \alpha \in \mathbb{Z}^{fs/X}(X' \times_{S} X), s.t.p_{X}(p_{X}^{-1}(Z')) \subset Z \} \subset \mathbb{Z}_{d_{X'}}(X' \times_{S} X)
  \]
  
  \[
  \mathbb{Z}^{tr}((X, Z)/S) : \mathbb{Z}^{tr}((X, Z)/S)((X, Z)/S) \rightarrow \mathbb{Z}^{tr}((X, Z)/S)((X, Z)/S), \alpha \mapsto (g \times I)^{-1}(\alpha)
  \]
  
  with $g \times I : X_{2} \times_{S} X \rightarrow X_{1} \times_{S} X$. 


For $f : X \to S$ a morphism with $X,S \in \text{Var}(\mathbb{C})$, $Z \subset X$ a closed subset and $r \in \mathbb{N}$, we denote $\mathbb{Z}^{\text{equir}}((X,Z)/S) \in \mathbb{PSh}(\text{Var}(\mathbb{C})^2/S)$ the presheaf given by

- for $(X',Z')/S \in \text{Var}(\mathbb{C})^2/S$, with $X'$ irreducible,
  $$\mathbb{Z}^{\text{equir}}((X,Z)/S)((X',Z')/S) := \left\{ \alpha \in \mathbb{Z}^{\text{equir}/X}(X' \times_S X), s.t. p_X^{-1}(Z') \right\} \subset Z_{d_X}(X' \times_S X)$$

- for $g : (X_2,Z_2)/S \to (X_1,Z_1)/S$ a morphism, with $(X_1,Z_1)/S, (X_2,Z_2)/S \in \text{Var}(\mathbb{C})^2/S$,
  $$\mathbb{Z}^{\text{equir}}((X,Z)/S)(g) : \mathbb{Z}^{\text{equir}}((X,Z)/S)((X_1,Z_1)/S) \to \mathbb{Z}^{\text{equir}}((X,Z)/S)((X_2,Z_2)/S), \alpha \mapsto (g \times I)^{-1}(\alpha)$$

with $g \times I : X_2 \times_S X \to X_1 \times_S X$.

Let $S \in \text{Var}(\mathbb{C})$. We denote by $\mathbb{Z}_S(d) := \mathbb{Z}^{\text{equir}}((S \times \mathbb{A}^d,S \times \mathbb{A}^d)/S)[-2d]$ the Tate twist. For $F \in C(\text{Var}(\mathbb{C})^2/S)$, we denote by $F(d) := F \otimes \mathbb{Z}_S(d)$.

For $S \in \text{Var}(\mathbb{C})$, let $\text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ be the category

- whose objects are those of $\text{Var}(\mathbb{C})^{2,(sm)}/S$, i.e. $(X,Z)/S = ((X,Z), h), h : X \to S$ with $X \in \text{Var}(\mathbb{C})$, $Z \subset X$ a closed subset,

- whose morphisms $\alpha : (X',Z)/S = ((X',Z), h_1) \to (X,Z)/S = ((X,Z), h_2)$ is finite correspondence that is $\alpha \in \oplus_i [Z'^i_s]/(X_i, Z)/S)((X',Z')/S)$, where $X' = \cup X_i'$, with $X_i'$ connected, the composition being defined in the same way as the morphism $\text{Cor}(\text{Var}(\mathbb{C})^{(sm)}/S)$.

We denote by $\text{Tr}(S) : \text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)}/S) \to \text{Var}(\mathbb{C})^{2,(sm)}/S$ the morphism of site given by the inclusion functor $\text{Tr}(S) : \text{Var}(\mathbb{C})^{2,(sm)}/S \to \text{Cor}(\text{Var}(\mathbb{C})^{(sm)}/S)$ It induces an adjunction

$$(\text{Tr}(S)^* \text{Tr}(S)_*) : C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \Rightarrow C(\text{Cor}(\text{Var}(\mathbb{C})^{(sm)}/S))$$

A complex of preheaves $G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to admit transferts if it is in the image of the embedding

$$\text{Tr}(S)_* : C(\text{Cor}(\mathbb{C})^{2,(sm)}/S) \hookrightarrow C(\mathbb{C})^{2,(sm)}/S,$$

that is $G = \text{Tr}(S)_* \text{Tr}(S)^* G$. We then have the full subcategory $\text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)}/S) \subset \text{Cor}(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ consisting of the objects of $\text{Var}(\mathbb{C})^{2,(sm)}/S$.

We have the adjonction

$$(\text{Tr}(S)^* \text{Tr}(S)_*) : C(\text{Var}(\mathbb{C})^{2,(sm)}/S) \Rightarrow C(\text{Cor}(\text{Var}(\mathbb{C})^{(sm)}/S))$$

A complex of preheaves $G \in C(\text{Var}(\mathbb{C})^{2,(sm)}/S)$ is said to admit transferts if it is in the image of the embedding

$$\text{Tr}(S)_* : C(\text{Cor}(\mathbb{C})^{2,(sm)}/S) \hookrightarrow C(\mathbb{C})^{2,(sm)}/S,$$

that is $G = \text{Tr}(S)_* \text{Tr}(S)^* G$.

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^I S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{Var}(\mathbb{C})$.

- For $(G_I, K_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$ and $(H_I, T_{IJ}) \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$, we denote

  $\text{Hom}((G_I, K_{IJ}),(H_I, T_{IJ})) := \langle \text{Hom}(G_I, H_I), u_{IJ}((G_I, K_{IJ}),(H_I, T_{IJ})) \rangle \in C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))$

  with

  $$u_{IJ}((G_I, K_{IJ}),(H_I, T_{IJ})) \xrightarrow{\text{ad}(p_{IJ, p_{IJ}}^* \text{Hom} G_I, H_I)^{T(p_{IJ, \text{hom}})^{(-1)}}} p_{IJ} \text{Hom}(p_{IJ}^* G_I, p_{IJ}^* H_I) \xrightarrow{\text{Hom}(p_{IJ}^* G_I, T_{IJ})} p_{IJ} \text{Hom}(p_{IJ}^* G_I, H_J) \xrightarrow{\text{Hom}(K_{IJ}, H_J)} p_{IJ} \text{Hom}(G_J, H_J).$$

This gives in particular the functor

$$C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)) \to C(\text{Var}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I))^{op}, (H_I, T_{IJ}) \mapsto (H_I, T_{IJ}).$$
• For \((G_I, K_{IJ}) \in C(\text{Var}(\mathbb{C}))^{2, (sm)pr}/(\tilde{S}_I)^{op}\) and \((H_I, T_{IJ}) \in C(\text{Var}(\mathbb{C}))^{2, (sm)pr}/(\tilde{S}_I)^{op}\), we denote
\[
\text{Hom}((G_I, K_{IJ}), (H_I, T_{IJ})) := (\text{Hom}(G_I, H_I), u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ}))) \in C(\text{Var}(\mathbb{C}))^{2, (sm)pr}/(\tilde{S}_I)^{op}
\]
with
\[
\begin{aligned}
\text{Hom}(G_I, H_I) &\xrightarrow{\text{ad}(p^*_{IJ}, p_{I*})(-)} p_{IJ}^* p_{I*}^* \text{Hom}(G_I, H_I) \\
&\xrightarrow{T(p^*_{IJ})} p_{IJ*} \text{Hom}(p_{IJ}^* G_I, p_{IJ}^* H_I) \\
&\xrightarrow{\text{Hom}(p_{IJ*}, H_I)} p_{IJ*} \text{Hom}(p_{IJ}^* G_I, H_I) \xrightarrow{\text{Hom}(K_{IJ}, H_I)} p_{IJ*} \text{Hom}(G_I, H_I).
\end{aligned}
\]
This gives in particular the functor
\[
C(\text{Var}(\mathbb{C}))^{2, (sm)pr}/(\tilde{S}_I)^{op} \to C(\text{Var}(\mathbb{C}))^{2, (sm)pr}/(\tilde{S}_I)^{op}, (H_I, T_{IJ}) \mapsto (H_I, T_{IJ}).
\]

Let \(S \in \text{Var}(\mathbb{C})\). Let \(S = \bigcup_{i=1}^k S_i\) an open affine cover and denote by \(S_I = \cap_{i \in I} S_i\). Let \(i : S_i \hookrightarrow \tilde{S}_i\) closed embeddings, with \(\tilde{S}_i \in \text{Var}(\mathbb{C})\). The functors \(p_a\) naturally extend to functors
\[
p_a : \text{Var}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I) \to \text{Var}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I),
\]
\[
((X, Z)/\tilde{S}_I, u_{IJ}) \mapsto ((X \times \mathbb{A}^1, Z \times \mathbb{A}^1)/\tilde{S}_I, u_{IJ} \times \mathbb{I}),
\]
\[
g : ((X, Z)/\tilde{S}_I, u_{IJ}) \mapsto ((X' \times \mathbb{A}^1, Z' \times \mathbb{A}^1)/\tilde{S}_I, u_{IJ} \times \mathbb{I})
\]
the projection functor and again by \(p_a : \text{Var}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I) \to \text{Var}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)\) the corresponding morphism of site, and
\[
p_a : \text{Var}(\mathbb{C})^{2, (sm)pr}/(\tilde{S}_I) \to \text{Var}(\mathbb{C})^{2, (sm)pr}/(\tilde{S}_I),
\]
\[
((Y \times \tilde{S}_I, Z)/\tilde{S}_I, u_{IJ}) \mapsto ((Y \times \tilde{S}_I, Z \times \mathbb{A}^1)/\tilde{S}_I, u_{IJ} \times \mathbb{I}),
\]
\[
g : ((Y \times \tilde{S}_I, Z)/\tilde{S}_I, u_{IJ}) \mapsto ((Y' \times \tilde{S}_I, Z' \times \mathbb{A}^1)/\tilde{S}_I, u_{IJ} \times \mathbb{I})
\]
the projection functor and again by \(p_a : \text{Var}(\mathbb{C})^{2, (sm)pr}/(\tilde{S}_I) \to \text{Var}(\mathbb{C})^{2, (sm)pr}/(\tilde{S}_I)\) the corresponding morphism of site. These functors also gives the morphisms of sites \(p_a : \text{Var}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)^{op} \to \text{Var}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)^{op}\) and \(p_a : \text{Var}(\mathbb{C})^{2, (sm)pr}/(\tilde{S}_I)^{op} \to \text{Var}(\mathbb{C})^{2, (sm)pr}/(\tilde{S}_I)^{op}\).

**Definition 17.** Let \(S \in \text{Var}(\mathbb{C})\). Let \(S = \bigcup_{i=1}^k S_i\) an open affine cover and denote by \(S_I = \cap_{i \in I} S_i\). Let \(i : S_i \hookrightarrow \tilde{S}_i\) closed embeddings, with \(\tilde{S}_i \in \text{Var}(\mathbb{C})\).

(i) A complex \((F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C}))^{2, (sm)}/(\tilde{S}_I)\) is said to be \(\mathbb{A}^1\) homotopic if \(\text{ad}(p^*_{IJ}, p_{I*})(F_I, u_{IJ})) : (F_I, u_{IJ}) \to p_{a*}p_{I*}^*(F_I, u_{IJ})\) is a homotopy equivalence.

(ii) A complex \((F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C}))^{2, (sm)pr}/(\tilde{S}_I)\) is said to be \(\mathbb{A}^1\) homotopic if \(\text{ad}(p^*_{IJ}, p_{I*})(F_I, u_{IJ})) : (F_I, u_{IJ}) \to p_{a*}p_{I*}^*(F_I, u_{IJ})\) is a homotopy equivalence.

(i) A complex \((F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C}))^{2, (sm)}/(\tilde{S}_I)\) is said to be \(\mathbb{A}^1\) invariant if for all \(((X_I, Z_I)/\tilde{S}_I, s_{IJ}) \in \text{Var}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)\)

\[
(F_I(p_{X_I})) : (F_I((X_I, Z_I)/\tilde{S}_I), F_I(s_{IJ}) \circ u_{IJ})(-) \to (F_I((X_I \times \mathbb{A}^1, (Z_I \times \mathbb{A}^1))/\tilde{S}_I), F_I(s_{IJ} \times \\ \times \mathbb{I}) \circ u_{IJ})(-)
\]
is a quasi-isomorphism, where \(p_{X_I} : (X_I \times \mathbb{A}^1, (Z_I \times \mathbb{A}^1)) \to (X_I, Z_I)\) are the projection, and \(s_{IJ} : (X_I \times_S P_{I*}, Z_I) \to (X_I, Z_I)\). Obviously a complex \((F_I, u_{IJ}) \in C(\text{Var}(\mathbb{C}))^{2, (sm)}/(\tilde{S}_I)\) is \(\mathbb{A}^1\) invariant if and only if all the \(F_I\) are \(\mathbb{A}^1\) invariant.
(i) A morphism \( \tau : (Y \times \tilde{S}_1, \tilde{S}_1, s_{1J}) \rightarrow (G_1([Y \times \tilde{S}_1]) : (G_1([Y \times \tilde{S}_1, Z_1]/\tilde{S}_1), G_J(s_{1J}) \circ u_{1J}(-) \rightarrow (G_1([Y \times \tilde{S}_1 \times \mathbb{A}^1, (Z_1 \times \mathbb{A}^1)]/\tilde{S}_1), G_J(s_{1J} \times I) \circ u_{1J}(-)) \)

is a quasi-isomorphism. Obviously a complex \((G_1, u_{1J}) \in C(Var(\mathbb{C}2,(sm)pr)/(\tilde{S}_1))\) is \( \mathbb{A}^1 \) invariant if and only if all the \( G_1 \) are \( \mathbb{A}^1 \) invariant.

(ii) Let \( \tau \) be a topology on \( Var(\mathbb{C}) \). A complex \( F = (F_1, u_{1J}) \in C(Var(\mathbb{C}2,(sm))/(\tilde{S}_1)) \) is said to be \( \mathbb{A}^1 \) local for the \( \tau \) topology induced on \( Var(\mathbb{C}2)/(\tilde{S}_1) \), if for all \( \tau \) local equivalence \( k : F \rightarrow G \) with \( k \) injective and \( G = (G_1, v_{1J}) \in C(Var(\mathbb{C}2,(sm))/(\tilde{S}_1)) \) \( \tau \) fibrant, e.g. \( k : (F_1, u_{1J}) \rightarrow (E_r(F_1), E(u_{1J})), G \) is \( \mathbb{A}^1 \) invariant.

(iii) A morphism \( m = (m_1) : (F_1, u_{1J}) \rightarrow (G_1, v_{1J}) \) with \( (F_1, u_{1J}), (G_1, v_{1J}) \in C(Var(\mathbb{C}2,(sm))/(\tilde{S}_1)) \) is said to be an \( (\mathbb{A}^1, et) \) local equivalence if for all \( H = (H_1, w_{1J}) \in C(Var(\mathbb{C}2,(sm))/(\tilde{S}_1)) \) which is \( \mathbb{A}^1 \) local for the etale topology

\[
(\text{Hom}(L(m_1), E_{et}(H_1))) : \text{Hom}(L(G_1, v_{1J}), E_{et}(H_1, w_{1J})) \rightarrow \text{Hom}(L(F_1, u_{1J}), E_{et}(H_1, w_{1J}))
\]

is a quasi-isomorphism. Obviously, if a morphism \( m = (m_1) : (F_1, u_{1J}) \rightarrow (G_1, v_{1J}) \) with \( (F_1, u_{1J}), (G_1, v_{1J}) \in C(Var(\mathbb{C}2,(sm))/(\tilde{S}_1)) \) is an \( (\mathbb{A}^1, et) \) local equivalence, then all the \( m_1 : F_1 \rightarrow G_1 \) are \( (\mathbb{A}^1, et) \) local equivalence.

(iv) A morphism \( m = (m_1) : (F_1, u_{1J}) \rightarrow (G_1, v_{1J}) \) with \( (F_1, u_{1J}), (G_1, v_{1J}) \in C(Var(\mathbb{C}2,(sm))/(\tilde{S}_1)) \) is said to be an \( (\mathbb{A}^1, et) \) local equivalence if for all \( H = (H_1, w_{1J}) \in C(Var(\mathbb{C}2,(sm))/(\tilde{S}_1)) \) which is \( \mathbb{A}^1 \) local for the etale topology

\[
(\text{Hom}(L(m_1), E_{et}(H_1))) : \text{Hom}(L(G_1, v_{1J}), E_{et}(H_1, w_{1J})) \rightarrow \text{Hom}(L(F_1, u_{1J}), E_{et}(H_1, w_{1J}))
\]

is a quasi-isomorphism. Obviously, if a morphism \( m = (m_1) : (F_1, u_{1J}) \rightarrow (G_1, v_{1J}) \) with \( (F_1, u_{1J}), (G_1, v_{1J}) \in C(Var(\mathbb{C}2,(sm))/(\tilde{S}_1)) \) is an \( (\mathbb{A}^1, et) \) local equivalence, then all the \( m_1 : F_1 \rightarrow G_1 \) are \( (\mathbb{A}^1, et) \) local equivalence.
Proposition 27. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^d S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_I : S_I \hookrightarrow \tilde{S}_I$ closed embeddings, with $\tilde{S}_I \in \text{Var}(\mathbb{C})$.

(i) Then for $F \in \text{C}(\text{Var}(\mathbb{C})^{2,(sm)})/(\tilde{S}_I)^{op}$, $C_s F$ is $\mathbb{A}^1$ local for the etale topology and $c(F) : F \to C_s F$ is an equivalence $(\mathbb{A}^1, et)$ local.

(ii) Then for $F \in \text{C}(\text{Var}(\mathbb{C})^{2,(sm)pr})/(\tilde{S}_I)^{op}$, $C_s F$ is $\mathbb{A}^1$ local for the etale topology and $c(F) : F \to C_s F$ is an equivalence $(\mathbb{A}^1, et)$ local.

(iii) A morphism $m : F \to G$ with $F, G \in \text{C}(\text{Var}(\mathbb{C})^{2,(sm)})/(\tilde{S}_I)^{op}$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if $a_c H^n C_s \text{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.

(iv) A morphism $m : F \to G$ with $F, G \in \text{C}(\text{Var}(\mathbb{C})^{2,(sm)pr})/(\tilde{S}_I)^{op}$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if $a_c H^n C_s \text{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.

Proof. Similar to the proof of proposition 22. See Ayoub’s thesis for example. 

In the filtered case we also consider :
Definition 18. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^{d} S_i$ an open affine cover and denote by $S_1 = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \hat{S}_i$ closed embeddings, with $\hat{S}_i \in \text{SmVar}(\mathbb{C})$.

(i) A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2, (sm)}/S)$ is said to be $\mathbb{A}^1$ filtered homotopy equivalence if $\text{adh}(p_a^* , p_\alpha^*) (G, F) : (G, F) \rightarrow p_\alpha^* p_a^* (G, F)$ is an $\mathbb{A}^1$ filtered homotopy equivalence.

(ii) A filtered complex $(G, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2, (sm)}/(\hat{S}_I))$ is said to be $\mathbb{A}^1$ filtered homotopy equivalence if $\text{adh}(p_a^* , p_\alpha^*) (G, F) : (G, F) \rightarrow p_\alpha^* p_a^* (G, F)$ is an $\mathbb{A}^1$ filtered homotopy equivalence.

Theorem 11. Let $S \in \text{Var}(\mathbb{C})$.

(i) Let $\phi : F^* \rightarrow G^*$ an etale local equivalence with $F^*, G^* \in C(\text{Var}(\mathbb{C})^{2, sm}/S)$. If $F^*$ and $G^*$ are $\mathbb{A}^1$ local and admit tramsfers then $\phi : F^* \rightarrow G^*$ is a Zariski local equivalence. Hence if $F \in C(\text{Var}(\mathbb{C})^{2, sm}/S)$ is $\mathbb{A}^1$ local and admits transfers

$$k : E_{zar}(F) \rightarrow E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

(ii) Let $\phi : F^* \rightarrow G^*$ an etale local equivalence with $F^*, G^* \in C(\text{Var}(\mathbb{C})^{2, smpr}/S)$. If $F^*$ and $G^*$ are $\mathbb{A}^1$ local and admit transfers then $\phi : F^* \rightarrow G^*$ is a Zariski local equivalence. Hence if $F \in C(\text{Var}(\mathbb{C})^{2, smpr}/S)$ is $\mathbb{A}^1$ local and admits transfers

$$k : E_{zar}(F) \rightarrow E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

Proof. Similar to the proof of theorem 10. \(\square\)

Theorem 12. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^{d} S_i$ an open affine cover and denote by $S_1 = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \hat{S}_i$ closed embeddings, with $\hat{S}_i \in \text{Var}(\mathbb{C})$.

(i) Let $\phi : (F^*, F) \rightarrow (G^*, F)$ a filtered etale local equivalence with $(F^*, F), (G^*, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2, sm}/S)$. If $(F^*, F)$ and $(G^*, F)$ are $\mathbb{A}^1$ filtered homotopy and admit transfers then $\phi : (F^*, F) \rightarrow (G^*, F)$ is an $\mathbb{A}^1$ filtered Zariski local equivalence. Hence if $(G, F) \in C(\text{Var}(\mathbb{C})^{2, sm}/S)$ is $\mathbb{A}^1$ filtered homotopy and admits transfers

$$k : E_{zar}(G, F) \rightarrow E_{et}(E_{zar}(G, F)) = E_{et}(G, F)$$

is an $\mathbb{A}^1$ filtered Zariski local equivalence.

(i)’ Let $\phi : (F^*, F) \rightarrow (G^*, F)$ a filtered etale local equivalence with $((F^*_1, F), u_{1I}), ((G^*_1, F), v_{1I}) \in C_{fil}(\text{Var}(\mathbb{C})^{2, sm}/(\hat{S}_I))$. If $((F^*, F), u_{1I})$ and $((G^*, F), v_{1I})$ are $\mathbb{A}^1$ filtered homotopy and admit transfers then $\phi : ((F^*, F), u_{1I}) \rightarrow ((G^*, F), v_{1I})$ is an $\mathbb{A}^1$ filtered Zariski local equivalence. Hence if $((G_1, F), u_{1I}) \in C(\text{Var}(\mathbb{C})^{2, sm}/S)$ is $\mathbb{A}^1$ filtered homotopy and admits transfers

$$k : (E_{zar}(G_1, F), u_{1I}) \rightarrow (E_{et}(E_{zar}(G_1, F), u_{1I}) = (E_{et}(G, F), u_{1I})$$

is an $\mathbb{A}^1$ filtered Zariski local equivalence.
(ii) Let \( \phi : (F^*, F) \rightarrow (G^*, F) \) a filtered étale local equivalence with \( (F^*, F), (G^*, F) \in C_{fil}(\text{Var}(\mathbb{C})^{2, smpr}/S) \). If \( F^* \) and \( G^* \) are \( r \)-filtered \( \mathbb{A}^1 \) homotopic and admit transferts then \( \phi : (F^*, F) \rightarrow (G^*, F) \) is an \( r \)-filtered Zariski local equivalence. Hence if \( (G, F) \in C(\text{Var}(\mathbb{C})^{2, smpr}/S) \) is \( r \)-filtered \( \mathbb{A}^1 \) homotopic and admits transfert

\[
k : E_{zar}(F) \rightarrow E_{et}(E_{zar}(F)) = E_{et}(F)
\]

is an \( r \)-filtered Zariski local equivalence.

(ii)' Let \( \phi : (F^*, F) \rightarrow (G^*, F) \) a filtered étale local equivalence with \( ((F^*_i, F), u_{ij}), ((G^*_i, F), v_{ij}) \in C_{fil}(\text{Var}(\mathbb{C})^{2, smpr}/S) \). If \( ((F^*, F), u_{ij}) \) and \( ((G^*, F), v_{ij}) \) are \( r \)-filtered \( \mathbb{A}^1 \) homotopic and admit transferts then \( \phi : ((F^*, F), u_{ij}) \rightarrow ((G^*, F), v_{ij}) \) is an \( r \)-filtered Zariski local equivalence. Hence if \( ((G_1, F), u_{ij}) \in C(\text{Var}(\mathbb{C})^{2, smpr}/S) \) is \( r \)-filtered \( \mathbb{A}^1 \) homotopic and admits transfert

\[
k : (E_{zar}(G_1, F), u_{ij}) \rightarrow (E_{et}(E_{zar}(G_1, F)), u_{ij}) = (E_{et}(G, F), u_{ij})
\]

is an \( r \)-filtered Zariski local equivalence.

Proof. Similar to the proof of theorem 11.

We have the following canonical functor:

**Definition 19.**  
(i) For \( S \in \text{Var}(\mathbb{C}) \), we have the functor

\[
(-)^F : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C(\text{Var}(\mathbb{C})^{2, sm}/S),
\]

\[
F \mapsto F^T : (((U, Z)/S) = ((U, Z), h) \mapsto F^T(U, Z, h) := ((\Gamma_Z^h)h^*LF(U/U), (g : ((U', Z'), h') \mapsto ((U, Z), h)) \mapsto \Gamma_Z^h(U/U)
\]

where \( i((\Gamma_Z^h)h^*LF(U/U)) \) is the canonical arrow of the inductive limit. Similarly, we have, for \( S \in \text{Var}(\mathbb{C}) \), the functor

\[
(-)^F : C(\text{Var}(\mathbb{C})/S) \rightarrow C(\text{Var}(\mathbb{C})^2/S),
\]

\[
F \mapsto F^T : (((X, Z)/S) = ((X, Z), h) \mapsto F^T((X, Z), h) := ((\Gamma_Z^h)h^*F)(X/X), (g : ((X', Z'), h') \mapsto ((X, Z), h)) \mapsto (F^T(g) : (\Gamma_Z^h)h^*LF((X', Z') \mapsto (\Gamma_Z^h)h^*LF((X'/X')))\)
\]

Note that for \( S \in \text{Var}(\mathbb{C}) \), \( I(S/S) : \mathbb{Z}((S, S)/S) \rightarrow \mathbb{Z}(S/S)^F \) given by

\[
I(S/S)((U, Z), h) \mapsto \mathbb{Z}((S, S)/S)((U, Z), h) \mapsto \Gamma_Z^h(U/U)(U'/U)
\]

is an isomorphism.

(ii) Let \( f : T \rightarrow S \) a morphism with \( T, S \in \text{Var}(\mathbb{C}) \). For \( F \in C(\text{Var}(\mathbb{C})^{sm}/S) \), we have the canonical morphism in \( C(\text{Var}(\mathbb{C})^{2, sm}/T) \)

\[
T(f, \Gamma)(F) := T^*(f, \Gamma)(F) : f^*(F^T) \rightarrow (f^*F)^T,
\]

\[
f^*(F^T)((U', Z'), h') := \lim_{f(U', Z')/T \rightarrow ((U', Z'), h')} \Gamma_Z^h(U/U)
\]

\[
(T^*(f, \Gamma))(F) = (\Gamma_Z^h)h^*LF(U'/U') \rightarrow (\Gamma_Z^h)h^*LF(U/U')
\]

\[
(f^*F)^T(U'/U') = (f^*F)((U', Z'), h')
\]

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where \( f_U: U_T: U \times_S T \to U \) and \( h_T: U_T := U \times_S T \to T \) are the base change maps, the equality following from the fact that \( h \circ f_U \circ l = f \circ h_T \circ l = f \circ h \). For \( F \in \operatorname{C(Var(C)/S)} \), we have similarly the canonical morphism in \( \operatorname{C(Var(C)^{2,sm}/T)} \)

\[
T(f, \Gamma)(F) : f^*(F^\Gamma) \to (f^*F)^\Gamma.
\]

(iii) Let \( h: U \to S \) a smooth morphism with \( U, S \in \operatorname{Var(C)} \). We have, for \( F \in \operatorname{C(Var(C)^{sm}/U)} \), the canonical morphism in \( \operatorname{C(Var(C)^{2,sm}/S)} \)

\[
T(h, \Gamma)(F)((U', Z'), h') : h'_*(F^\Gamma)((U', Z'), h') := \lim_{\to \downarrow (U', Z'), h'} (\Gamma_{Z', h'}^\gamma)^*(h_*LF)(U'/U')
\]

\[
\to (\Gamma_{Z', h'}^\gamma)^*(h'_*h_1LF)(U'/U') =: (h_1LF)^\Gamma((U', Z'), h')
\]

(iv) Let \( i: Z_0 \hookrightarrow S \) a closed embedding with \( Z_0, S \in \operatorname{Var(C)} \). We have the canonical morphism in \( \operatorname{C(Var(C)^{2,sm}/S)} \)

\[
T(i, \Gamma)(Z(Z_0/Z_0)) : i_*((Z(Z_0/Z_0))^\Gamma) \to (i_*Z(Z/Z))^\Gamma,
\]

\[
T(i, \Gamma)(Z(Z_0/Z_0))(U, Z, h) := i_*((Z(Z_0/Z_0))^\Gamma)(U, Z, h) := (\Gamma_{Z, h}^\gamma)^*(h_*Z(Z_0/Z_0))(U \times_S Z_0)
\]

\[
\to (\Gamma_{Z, h}^\gamma)^*(h_*Z(Z_0/Z_0))(U \times_S Z_0) =: (i_*Z(Z/Z))^\Gamma((U, Z, h)
\]

**Definition 20.** Let \( S \in \operatorname{Var(C)} \). We have for \( F \in \operatorname{C(Var(C)^{sm}/S)} \) the canonical map in \( \operatorname{C(Var(C)^{sm}/S)} \)

\[
\operatorname{Gr}(F) : \operatorname{Gr}_{S*}^\mu \mu_S F^\Gamma \to F,
\]

\[
\operatorname{Gr}(F)(U/S) : \Gamma_{U*}^\gamma p^*(F(U \times_S U \times S)) \xrightarrow{\operatorname{ad}(t^*, \Gamma(t^*F)(U \times S/U \times S))} h^*(F(U/U)) = F(U/S)
\]

where \( h: U \to S \) is a smooth morphism with \( U \in \operatorname{Var(C)} \) and \( h: U \xrightarrow{\downarrow} U \times S \xrightarrow{\uparrow} S \) is the graph factorization with \( l \) the graph embedding and \( p \) the projection.

**Proposition 28.** Let \( S \in \operatorname{Var(C)} \).

(i) Then,

- if \( m: F \to G \) with \( F, G \in \operatorname{C(Var(C)^{sm}/S)} \) is a quasi-isomorphism, \( m^\Gamma : F^\Gamma \to G^\Gamma \) is a quasi-isomorphism in \( \operatorname{C(Var(C)^{2,sm}/S)} \),
- if \( m: F \to G \) with \( F, G \in \operatorname{C(Var(C)^{sm}/S)} \) is a Zariski local equivalence, \( m^\Gamma : F^\Gamma \to G^\Gamma \) is a Zariski local equivalence in \( \operatorname{C(Var(C)^{2,sm}/S)} \), if \( m: F \to G \) with \( F, G \in \operatorname{C(Var(C)^{sm}/S)} \) is an etale local equivalence, \( m^\Gamma : F^\Gamma \to G^\Gamma \) is an etale local equivalence in \( \operatorname{C(Var(C)^{2,sm}/S)} \),
- if \( m: F \to G \) with \( F, G \in \operatorname{C(Var(C)^{sm}/S)} \) is an \((\mathbb{A}^1, \text{et})\) local equivalence, \( m^\Gamma : F^\Gamma \to G^\Gamma \) is an \((\mathbb{A}^1, \text{et})\) local equivalence in \( \operatorname{C(Var(C)^{2,sm}/S)} \).

(ii) Then,

- if \( m: F \to G \) with \( F, G \in \operatorname{C(Var(C)/S)} \) is a quasi-isomorphism, \( m^\Gamma : F^\Gamma \to G^\Gamma \) is a quasi-isomorphism in \( \operatorname{C(Var(C)^{2}/S)} \),
- if \( m: F \to G \) with \( F, G \in \operatorname{C(Var(C)^{sm}/S)} \) is a Zariski local equivalence, \( m^\Gamma : F^\Gamma \to G^\Gamma \) is a Zariski local equivalence in \( \operatorname{C(Var(C)^{2,sm}/S)} \), if \( m: F \to G \) with \( F, G \in \operatorname{C(Var(C)^{sm}/S)} \) is an etale local equivalence, \( m^\Gamma : F^\Gamma \to G^\Gamma \) is an etale local equivalence in \( \operatorname{C(Var(C)^{2,sm}/S)} \),
- if \( m: F \to G \) with \( F, G \in \operatorname{C(Var(C)^{sm}/S)} \) is an \((\mathbb{A}^1, \text{et})\) local equivalence, \( m^\Gamma : F^\Gamma \to G^\Gamma \) is an \((\mathbb{A}^1, \text{et})\) local equivalence in \( \operatorname{C(Var(C)^{2}/S)} \).
Proof. (i): Follows immediately from the fact that for \((U, Z), h) \in \text{Var}(\mathbb{C})^{2, sm}/S,\)

- if \(m : F \rightarrow G\) with \(F, G \in \text{C}(\text{Var}(\mathbb{C})^{sm}/S)\) is a quasi-isomorphism, \(\Gamma_Z^* h^* LF(m) : \Gamma_Z^* h^* LF \rightarrow \Gamma_Z^* h^* LG\) is a quasi-isomorphism.

- if \(m : F \rightarrow G\) with \(F, G \in \text{C}(\text{Var}(\mathbb{C})^{sm}/S)\) is a a Zariski or etale local equivalence, \(\Gamma_Z^* h^* LF(m) : \Gamma_Z^* h^* LF \rightarrow \Gamma_Z^* h^* LG\) is a Zariski, resp. etale, local equivalence.

- if \(m : F \rightarrow G\) with \(F, G \in \text{C}(\text{Var}(\mathbb{C})^{sm}/S)\) is an \(\mathbb{A}_1, et\) equivalence, \(\Gamma_Z^* h^* LF(m) : \Gamma_Z^* h^* LF \rightarrow \Gamma_Z^* h^* LG\) is an \(\mathbb{A}_1, et\) equivalence.

(ii): Similar to (i). □

2.8 Presheaves on the big analytical site

For \(S \in \text{AnSp}(\mathbb{C}),\) we denote by \(\rho_S : \text{AnSp}(\mathbb{C})^{sm}/S \rightarrow \text{AnSp}(\mathbb{C})/S\) be the full subcategory consisting of the objects \(U/S = (U, h) \in \text{AnSp}(\mathbb{C})/S\) such that the morphism \(h : U \rightarrow S\) is smooth. That is, \(\text{AnSp}(\mathbb{C})^{sm}/S\) is the category

- whose objects are smooth morphisms \(U/S = (U, h) \in \text{AnSp}(\mathbb{C}),\)

- whose morphisms \(g : U/S = (U, h_1) \rightarrow V/S = (V, h_2)\) is a morphism \(g : U \rightarrow V\) of complex algebraic varieties such that \(h_2 \circ g = h_1.\)

We denote again \(\rho_S : \text{AnSp}(\mathbb{C})/S \rightarrow \text{AnSp}(\mathbb{C})^{sm}/S\) the associated morphism of site. We will consider

\[
\begin{array}{ccc}
\text{AnSp}(\mathbb{C})/T & \xrightarrow{\rho_T} & \text{AnSp}(\mathbb{C})^{sm}/T \\
\downarrow P(f) & & \downarrow P(f) \\
\text{AnSp}(\mathbb{C})/S & \xrightarrow{\rho_S} & \text{AnSp}(\mathbb{C})^{sm}/S
\end{array}
\]  

(33)

We denote, for \(S \in \text{AnSp}(\mathbb{C}),\) the obvious morphism of sites

\[
\tilde{e}(S) : \text{AnSp}(\mathbb{C})/S \xrightarrow{\rho_S} \text{AnSp}(\mathbb{C})^{sm}/S \xrightarrow{e(S)} \text{Ouv}(S)
\]

where \(\text{Ouv}(S)\) is the set of the open subsets of \(S,\) given by the inclusion functors \(\tilde{e}(S) : \text{Ouv}(S) \hookrightarrow \text{AnSp}(\mathbb{C})^{sm}/S \rightarrow \text{AnSp}(\mathbb{C})/S.\) By definition, for \(f : T \rightarrow S\) a morphism with \(S, T \in \text{AnSp}(\mathbb{C}),\) the commutative diagram of sites (33) extend a commutative diagram of sites:

\[
\begin{array}{ccc}
\text{AnSp}(\mathbb{C})/T & \xrightarrow{\rho_T} & \text{AnSp}(\mathbb{C})^{sm}/T \\
\downarrow P(f) & & \downarrow P(f) \\
\text{AnSp}(\mathbb{C})/S & \xrightarrow{\rho_S} & \text{AnSp}(\mathbb{C})^{sm}/S
\end{array}
\]

(34)

\[
\text{Ouv}(T) \xrightarrow{e(T)} \text{Ouv}(S)
\]

- As usual, we denote by

\[
(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{sm}/T)
\]

the adjonction induced by \(P(f) : \text{AnSp}(\mathbb{C})^{sm}/T \rightarrow \text{AnSp}(\mathbb{C})^{sm}/S.\) Since the colimits involved in the definition of \(f^* = P(f)^*\) are filtered, \(f^*\) also preserve monomorphism. Hence, we get an adjonction

\[
(f^*, f_*) : C_{fd}(\text{AnSp}(\mathbb{C})^{sm}/S) \leftrightarrows C_{fd}(\text{AnSp}(\mathbb{C})^{sm}/T), f^*(G, F) := (f^*G, f^*F)
\]
• As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*): C(\text{AnSp}(\mathbb{C})/S) \to C(\text{AnSp}(\mathbb{C})/T)$$

the adjunction induced by $P(f): \text{AnSp}(\mathbb{C})/T \to \text{AnSp}(\mathbb{C})/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, $f^*$ also preserve monomorphism. Hence, we get an adjunction

$$(f^*, f_*): C_{fil}(\text{AnSp}(\mathbb{C})/S) \rightleftarrows C_{fil}(\text{AnSp}(\mathbb{C})/T), \quad f^*(G, F) := (f^*G, f^*F)$$

• For $h: U \to S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$, the pullback functor $P(h): \text{AnSp}(\mathbb{C})^{sm}/S \to \text{AnSp}(\mathbb{C})^{sm}/U$ admits a left adjoint $C(h)(X \to U) = (X \to U \to S)$. Hence, $h^*: C(\text{AnSp}(\mathbb{C})^{sm}/S) \to C(\text{AnSp}(\mathbb{C})^{sm}/U)$ admits a left adjoint

$$h^*_t: C(\text{AnSp}(\mathbb{C})^{sm}/U) \to C(\text{AnSp}(\mathbb{C})^{sm}/S), \quad F \mapsto ((V, h_0) \mapsto \lim_{(V', h_0') \to (V, h_0)} F(V', h'))$$

Note that for $h': V' \to V$ a smooth morphism, $V', V \in \text{AnSp}(\mathbb{C})$, we have $h^*_t(\mathcal{Z}(V'/V)) = \mathcal{Z}(V'/S)$ with $V'/S = (V', h \circ h')$. Hence, since projective presheaves are the direct summands of the representable presheaves, $h^*_t$ sends projective presheaves to projective presheaves. For $F^* \in C(\text{AnSp}(\mathbb{C})^{sm}/U)$ and $G^* \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$,

$$I(h^*_t, h^*)(F, G): \mathcal{H}om^*(_h^*_tF, G) \cong h^*_t\mathcal{H}om^*(F, h^*G). \quad (35)$$

• For $f: T \to S$ any morphism with $T, S \in \text{AnSp}(\mathbb{C})$, the pullback functor $P(f): \text{AnSp}(\mathbb{C})/T \to \text{AnSp}(\mathbb{C})/S$ admits a left adjoint $C(f)(X \to T) = (X \to T \to S)$. Hence, $f^*: C(\text{AnSp}(\mathbb{C})/S) \to C(\text{AnSp}(\mathbb{C})/T)$ admits a left adjoint

$$f^*_t: C(\text{AnSp}(\mathbb{C})/T) \to C(\text{AnSp}(\mathbb{C})/S), \quad F \mapsto ((V, h_0) \mapsto \lim_{(V', h_0') \to (V, h_0)} F(V', h'))$$

Note that we have for $h': V' \to V$ a morphism, $V', V \in \text{AnSp}(\mathbb{C})$, $h^*_t(\mathcal{Z}(V'/V)) = \mathcal{Z}(V'/S)$ with $V'/S = (V', h \circ h')$. Hence, since projective presheaves are the direct summands of the representable presheaves, $h^*_t$ sends projective presheaves to projective presheaves. For $F^* \in C(\text{AnSp}(\mathbb{C})/S)$ and $G^* \in C(\text{AnSp}(\mathbb{C})/T)$, we have the adjunction maps

$$\text{ad}(f^*_t, f^*)(G^*): G^* \to f^*f^*_tG^*, \quad \text{ad}(f^*_t, f^*)(F^*): f^*_tF^* \to F^*.$$

For a morphism $f: T \to S$, with $T, S \in \text{AnSp}(\mathbb{C})$, we have the adjunction isomorphism, for $F \in C(\text{AnSp}(\mathbb{C})/T)$ and $G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$,

$$I(f^*_t, f^*)(F, G): \mathcal{H}om^*(f^*_tF, G) \cong f^*_t\mathcal{H}om^*(F, f^*G). \quad (36)$$

• For a commutative diagram in $\text{AnSp}(\mathbb{C})$:

$$D = V \xrightarrow{g_2} U, \quad T \xrightarrow{g_1} S$$

where $h_1$ and $h_2$ are smooth, we denote by, for $F^* \in C(\text{AnSp}(\mathbb{C})^{sm}/U)$,

$$T^*_2(D)(F^*): h_{2g_2}^*F^* \to g_1^*h_{1g_1}^*F^*$$

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the canonical map in \( C(\text{AnSp}(\mathbb{C})^{sm}/T) \) given by adjonction. If \( D \) is cartesian with \( h_1 = h, g_1 = g \)
\[ f_2 = h': U_T \to T, \ g': U_T \to U, \]

\[ T\sharp(D)(F^\bullet) =: T\sharp(g,h)(F^\bullet) : h'_2g'^\ast F^\bullet \sim \rightarrow g'^\ast h_2 F^\bullet \]
is an isomorphism and for \( G^\bullet \in C(\text{AnSp}(\mathbb{C})^{sm}/T) \)
\[ T(D)(G^\bullet) =: T(g,h)(G^\bullet) : g'^\ast h_\ast G^\bullet \sim \rightarrow h'_\ast g'^\ast G^\bullet \]
is an isomorphism.

- For a commutative diagram in \( \text{AnSp}(\mathbb{C}) \):
  \[
  D = \begin{array}{c}
  V \\
  T
  \end{array} \]
  \[
  \xrightarrow{g_2} X, \quad \xrightarrow{f_2} \]
  \[
  \xrightarrow{f_1} S
  \]
  we denote by, for \( F^\bullet \in C(\text{AnSp}(\mathbb{C})/X) \),

\[ T\sharp(D)(F^\bullet) =: T\sharp(g,f)(F^\bullet) : f_2g'_2 F^\bullet \rightarrow g'_1 f_1 F^\bullet \]

the canonical map in \( C(\text{AnSp}(\mathbb{C})/T) \) given by adjonction. If \( D \) is cartesian with \( h_1 = h, g_1 = g \)
\[ f_2 = h': X_T \to T, \ g': X_T \to X, \]

\[ T\sharp(D)(F^\bullet) =: T\sharp(g,f)(F^\bullet) : f'_2g'^\ast F^\bullet \sim \rightarrow g'^\ast f_2' F^\bullet \]
is an isomorphism and for \( G^\bullet \in C(\text{AnSp}(\mathbb{C})/T) \)
\[ T(D)(G^\bullet) =: T(g,h)(G^\bullet) : f'^\ast g_\ast G^\bullet \sim \rightarrow g'_\ast f'^\ast G^\bullet \]
is an isomorphism.

For \( f : T \to S \) a morphism with \( S,T \in \text{AnSp}(\mathbb{C}) \),
- we get for \( F \in C(\text{AnSp}(\mathbb{C})^{sm}/S) \) from the a commutative diagram of sites (34) the following canonical transformation

\[ T(e,f)(F^\bullet) : f^\ast e(S)_\ast F^\bullet \rightarrow e(T)_\ast f'^\ast F^\bullet, \]

which is NOT a quasi-isomorphism in general. However, for \( h : U \to S \) a smooth morphism with \( S,U \in \text{AnSp}(\mathbb{C}) \), \( T(e,h)(F^\bullet) : h^\ast e(S)_\ast F^\bullet \rightarrow e(T)_\ast h'^\ast F^\bullet \) is an isomorphism.
- we get for \( F \in C(\text{AnSp}(\mathbb{C})/S) \) from the a commutative diagram of sites (34) the following canonical transformation

\[ T(e,f)(F^\bullet) : f^\ast e(S)_\ast F^\bullet \rightarrow e(T)_\ast f'^\ast F^\bullet, \]

which is NOT a quasi-isomorphism in general. However, for \( h : U \to S \) a smooth morphism with \( S,U \in \text{AnSp}(\mathbb{C}) \), \( T(e,h)(F^\bullet) : h^\ast e(S)_\ast F^\bullet \rightarrow e(T)_\ast h'^\ast F^\bullet \) is an isomorphism.

Let \( S \in \text{AnSp}(\mathbb{C}) \),
- We have for \( F,G \in C(\text{AnSp}(\mathbb{C})^{sm}/S) \),
  \( e(S)_\ast (F \otimes G) = (e(S)_\ast F) \otimes (e(S)_\ast G) \) by definition
  \( e(S)_\ast (F \otimes G) \) by definition
- the canonical forgetful map

\[ T(S,\text{hom})(F,G) : e(S)_\ast \mathcal{H}om^\bullet(F,G) \rightarrow \mathcal{H}om^\bullet(e(S)_\ast F,e(S)_\ast G) \]

which is NOT a quasi-isomorphism in general.
By definition, we have for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, $e(S)_{*}E_{usu}(F) = E_{usu}(e(S)_{*}F)$.

- We have for $F, G \in C(\text{AnSp}(\mathbb{C})/S)$,
  - $e(S)_{*}(F \otimes G) = (e(S)_{*}F) \otimes (e(S)_{*}G)$ by definition
  - the canonical forgetful map
    \[ T(S, \text{hom})(F, G) : e(S)_{*}\text{Hom}^{*}(F, G) \to \text{Hom}^{*}(e(S)_{*}F, e(S)_{*}G). \]
    which is NOT a quasi-isomorphism in general.

By definition, we have for $F \in C(\text{AnSp}(\mathbb{C})/S)$, $e(S)_{*}E_{usu}(F) = E_{usu}(e(S)_{*}F)$.

Let $S \in \text{AnSp}(\mathbb{C})$. We have the support section functor of a closed subset $Z \subset S$ for presheaves on the big analytical site.

**Definition 21.** Let $S \in \text{AnSp}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ be the open complementary subset.

(i) We define the functor
\[ \Gamma_{Z} : C(\text{AnSp}(\mathbb{C})^{sm}/S) \to C(\text{AnSp}(\mathbb{C})^{sm}/S), \quad G^{*} \mapsto \Gamma_{Z}G^{*} := \text{Cone}(\text{ad}(j^{*}, j_{*})(G^{*}) : G^{*} \to j_{*}j^{*}G^{*})[-1], \]
so that there is then a canonical map $\gamma_{Z}(G^{*}) : \Gamma_{Z}G^{*} \to G^{*}$.

(ii) We have the dual functor of (i):
\[ \Gamma_{Z}^{\vee} : C(\text{AnSp}(\mathbb{C})^{sm}/S) \to C(\text{AnSp}(\mathbb{C})^{sm}/S), \quad F \mapsto \Gamma_{Z}^{\vee}(F^{*}) := \text{Cone}(\text{ad}(j_{*}, j^{*})(G^{*}) : j_{*}j^{*}G^{*} \to G^{*}), \]
together with the canonical map $\gamma_{Z}^{\vee}(G) : F \to \Gamma_{Z}^{\vee}(G)$.

(iii) For $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, we denote by
\[ I(\gamma, \text{hom})(F, G) := (I, I(j_{Z}, j^{*})(F, G)) : \Gamma_{Z}\text{Hom}(F, G) \xrightarrow{\sim} \text{Hom}(\Gamma_{Z}^{\vee}F, G) \]
the canonical isomorphism given by adjunction.

Let $S \in \text{AnSp}(\mathbb{C})$ and $Z \subset S$ a closed subset.

- Since $\Gamma_{Z} : C(\text{AnSp}(\mathbb{C})^{sm}/S) \to C(\text{AnSp}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor
  \[ \Gamma_{Z} : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \to C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S), \quad (G, F) \mapsto \Gamma_{Z}(G, F) := (\Gamma_{Z}G, \Gamma_{Z}F) \]
- Since $\Gamma_{Z}^{\vee} : C(\text{AnSp}(\mathbb{C})^{sm}/S) \to C(\text{AnSp}(\mathbb{C})^{sm}/S)$ preserve monomorphism, it induces a functor
  \[ \Gamma_{Z}^{\vee} : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \to C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S), \quad (G, F) \mapsto \Gamma_{Z}^{\vee}(G, F) := (\Gamma_{Z}^{\vee}G, \Gamma_{Z}^{\vee}F) \]

**Definition-Proposition 7.** (i) Let $g : S' \to S$ a morphism and $i : Z \hookrightarrow S$ a closed embedding with $S', S, Z \in \text{AnSp}(\mathbb{C})$. Then, for $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S')$, there exist a map in $C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S')$
\[ T(g, \gamma)(G, F) : g^{*}\Gamma_{Z}(G, F) \to \Gamma_{Z_{\times S}g^{*}}(G, F) \]
unique up to homotopy, such that $\gamma_{Z_{\times S}g^{*}}(G, F) \circ T(g, \gamma)(G, F) = g^{*}\gamma_{Z}(G, F)$.

(ii) Let $i_{1} : Z_{1} \hookrightarrow S$, $i_{2} : Z_{2} \hookrightarrow Z_{1}$ be closed embeddings with $S, Z_{1}, Z_{2} \in \text{AnSp}(\mathbb{C})$. Then, for $(G, F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S)$,
Proof. Similar to definition-proposition 1 or definition-proposition 4.

(iii) Consider a morphism \( g : (S', Z) \to (S, Z) \) with \((S', Z) \to (S, Z) \in \text{AnSp}(\mathbb{C}) \). We denote, for \( G \in C(\text{AnSp}(\mathbb{C})^\text{sm}/S) \) the composite

\[
T(D, \gamma^\vee)(G) : g^*\Gamma^\vee_Z G \xrightarrow{\sim} \Gamma^\vee_{Z \times S'} g^*G \xrightarrow{T(Z_2/Z_1, \gamma^\vee)} \Gamma^\vee_{Z_2} g^*G
\]

and we have then the factorization \( \gamma^\vee_{Z_2}(g^*G) : g^*G \xrightarrow{g^*\gamma^\vee_Z(G)} g^*\Gamma^\vee_{Z_2} g^*G \xrightarrow{T(D, \gamma^\vee)} \Gamma^\vee_{Z_2} g^*G \).

\[\blacksquare\]

**Definition 22.** For \( S \in \text{AnSp}(\mathbb{C}) \), we denote by

\[C_{O_S}(\text{AnSp}(\mathbb{C})^\text{sm}/S) := C_{e(S)\cdot O_S}(\text{AnSp}(\mathbb{C})^\text{sm}/S)\]

the category of complexes of presheaves on \( \text{AnSp}(\mathbb{C})^\text{sm}/S \) endowed with a structure of \( e(S)^* O_S \) module, and by

\[C_{O_S}fil(\text{AnSp}(\mathbb{C})^\text{sm}/S) := C_{e(S)\cdot O_S fil}(\text{AnSp}(\mathbb{C})^\text{sm}/S)\]

the category of filtered complexes of presheaves on \( \text{Var}(\mathbb{C})^\text{sm}/S \) endowed with a structure of \( e(S)^* O_S \) module.

Let \( S \in \text{AnSp}(\mathbb{C}) \). Let \( Z \subset S \) a closed subset. Denote by \( j : S \setminus Z \to S \) the open complementary embedding.

- For \( G \in C_{O_S}(\text{AnSp}(\mathbb{C})^\text{sm}/S) \), \( \Gamma_Z G := \text{Cone}(ad(j^*, j_*)(G) : F \to j_* j^* G)[-1] \) has a (unique) structure of \( e(S)^* O_S \) module such that \( \gamma_Z(G) : \Gamma_Z G \to G \) is a map in \( C_{O_S}(\text{AnSp}(\mathbb{C})^\text{sm}/S) \). This gives the functor

\[\Gamma_Z : C_{O_S}(\text{AnSp}(\mathbb{C})^\text{sm}/S) \to C_{filO_S}(\text{AnSp}(\mathbb{C})^\text{sm}/S), \quad (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F),\]

together with the canonical map \( \gamma_Z((G, F) : \Gamma_Z(G, F) \to (G, F)) \). Let \( Z_2 \subset Z \) a closed subset. Then, for \( G \in C_{O_S}(\text{AnSp}(\mathbb{C})^\text{sm}/S) \), \( T(Z_2/Z_1, \gamma)(G) : \Gamma_{Z_2} G \to \Gamma_{Z_2} G \) is a map in \( C_{O_S}(\text{AnSp}(\mathbb{C})^\text{sm}/S) \) (i.e. is \( e(S)^* O_S \) linear).

- For \( G \in C_{O_S}(\text{Var}(\mathbb{C})^\text{sm}/S) \), \( \Gamma_Z G := \text{Cone}(ad(j_2^*, j_{2*})(G) : j_2 j^* G \to G) \) has a unique structure of \( e(S)^* O_S \) module, such that \( \gamma_Z(G) : G \to \Gamma_Z G \) is a map in \( C_{O_S}(\text{AnSp}(\mathbb{C})^\text{sm}/S) \). This gives the the functor

\[\Gamma_Z : C_{O_S}(\text{Var}(\mathbb{C})^\text{sm}/S) \to C_{filO_S}(\text{Var}(\mathbb{C})^\text{sm}/S), \quad (G, F) \mapsto \Gamma_Z(G, F) := (\Gamma_Z G, \Gamma_Z F),\]

together with the canonical map \( \gamma_Z((G, F) : (G, F) \to \Gamma_Z G, \Gamma_Z F) \). Let \( Z_2 \subset Z \) a closed subset. Then, for \( G \in C_{O_S}(\text{AnSp}(\mathbb{C})^\text{sm}/S) \), \( T(Z_2/Z_1, \gamma)(G) : \Gamma_Z G \to \Gamma_{Z_2} G \) is a map in \( C_{O_S}(\text{AnSp}(\mathbb{C})^\text{sm}/S) \) (i.e. is \( e(S)^* O_S \) linear).
Definition 23. Let $S \in \text{AnSp}(\mathbb{C})$. Let $Z \subset S$ a closed subset.

(i) We denote by

$$C_Z(\text{AnSp}(\mathbb{C})^m/S) \subset C(\text{AnSp}(\mathbb{C})^m/S),$$

the full subcategory consisting of complexes of presheaves $F^* \in C(\text{AnSp}(\mathbb{C})^m/S)$ such that $a_{\text{usu}}H^n(j^*F^*) = 0$ for all $n \in \mathbb{Z}$, where $j : S \setminus Z \rightarrow S$ is the complementary open embedding and $a_{\text{usu}}$ is the sheafification functor.

(ii) We denote by

$$C_{O_S,Z}(\text{AnSp}(\mathbb{C})^m/S) \subset C_{O_S}(\text{AnSp}(\mathbb{C})^m/S),$$

the full subcategory consisting of complexes of presheaves $F^* \in C(\text{AnSp}(\mathbb{C})^m/S)$ such that $a_{\text{usu}}H^n(j^*F^*) = 0$ for all $n \in \mathbb{Z}$, where $j : S \setminus Z \rightarrow S$ is the complementary open embedding and $a_{\text{usu}}$ is the sheafification functor.

(iii) We denote by

$$C_{fil,Z}(\text{AnSp}(\mathbb{C})^m/S) \subset C_{fil}(\text{AnSp}(\mathbb{C})^m/S)$$

the full subcategory consisting of filtered complexes of presheaves $(F^*,F) \in C_{\text{fil}}(\text{AnSp}(\mathbb{C})^m/S)$ such that there exist $r \in \mathbb{N}$ and an $r$-filtered homotopy equivalence $\phi : (F^*,F) \rightarrow (F^*,F)$ with $(F^*,F) \in C_{fil}(\text{AnSp}(\mathbb{C})^m/S)$ such that $a_{\text{usu}}H^n(\text{Gr}^{p}_F(F^*,F)) = 0$ for all $n,p \in \mathbb{Z}$, where $j : S \setminus Z \rightarrow S$ is the complementary open embedding and $a_{\text{usu}}$ is the sheafification functor.

Let $S \in \text{AnSp}(\mathbb{C})$ and $Z \subset S$ a closed subset.

- For $(G,F) \in C_{\text{fil}}(\text{AnSp}(\mathbb{C})^m/S)$, we have $\Gamma_Z(G,F), \Gamma_{Z'}^*(G,F) \in C_{fil,Z}(\text{AnSp}(\mathbb{C})^m/S)$.
- For $(G,F) \in C_{O_S,fil}(\text{AnSp}(\mathbb{C})^m/S)$, we have $\Gamma_Z(G,F), \Gamma_{Z'}^*(G,F) \in C_{O_S,fil,Z}(\text{AnSp}(\mathbb{C})^m/S)$.

Let $S_* \in \text{Fun}(\mathcal{I}, \text{AnSp}(\mathbb{C}))$ with $\mathcal{I} \in \text{Cat}$, a diagram of algebraic varieties. It gives the diagram of sites $\text{AnSp}(\mathbb{C})^2/S_* \in \text{Fun}(\mathcal{I}, \text{Cat})$.

- Then $C_{fil}(\text{AnSp}(\mathbb{C})/S_*)$ is the category

  - whose objects $(G,F) = \{(G_I,F)_{I \in \mathcal{I}}, u_{IJ}\}$, with $(G_I,F) \in C_{fil}(\text{AnSp}(\mathbb{C})/S_I)$, and $u_{IJ} : (G_I,F) \rightarrow r_{IJ*}(G_J,F)$ for $r_{IJ} : I \rightarrow J$, denoting again $r_{IJ} : S_I \rightarrow S_J$, are morphisms satisfying for $I \rightarrow J \rightarrow K$, $r_{IJ}, u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{AnSp}(\mathbb{C})/S_I)$,

  - the morphisms $m : ((G,F),u_{IJ}) \rightarrow ((H,F),v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

  $$m = (m_I : (G_I,F) \rightarrow (H_I,F))_{I \in \mathcal{I}}$$

  such that $v_{IJ} \circ m_I = p_{IJ}, m_J \circ u_{IJ}$ in $C_{fil}(\text{Var}(\mathbb{C})/S_I)$.

- Then $C_{fil}(\text{AnSp}(\mathbb{C})^m/S_*)$ is the category

  - whose objects $(G,F) = \{(G_I,F)_{I \in \mathcal{I}}, u_{IJ}\}$, with $(G_I,F) \in C_{fil}(\text{AnSp}(\mathbb{C})^m/S_I)$, and $u_{IJ} : (G_I,F) \rightarrow r_{IJ*}(G_J,F)$ for $r_{IJ} : I \rightarrow J$, denoting again $r_{IJ} : S_I \rightarrow S_J$, are morphisms satisfying for $I \rightarrow J \rightarrow K$, $r_{IJ}, u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\text{AnSp}(\mathbb{C})^m/S_I)$,
the adjonction induced by the definition of AnSp(C)fil 
whose objects (such that \( u \in \mathcal{I} \)) are filtered, \( f^* \) also preserve monomorphism. Hence, we get an adjonction 
\[
(f^*, f_*) : C_{fil}(AnSp(C)^{sm}/S) \rightleftharpoons C_{fil}(AnSp(C)^{sm}/T)
\]
the adjonction induced by \( P(f_*): AnSp(C)^{sm}/T \rightarrow AnSp(C)^{sm}/S \). Since the colimits involved in the definition of \( f_* = P(f_*) \) are filtered, \( f_* \) also preserve monomorphism. Hence, we get an adjonction 
\[
(f^*, f_*): C_{fil}(AnSp(C)^{sm}/S) \rightleftharpoons C_{fil}(AnSp(C)^{sm}/T) \]
\[
f^*((G_1, F), u_{IJ}) := (f^*_1 G_1, f^*_1 F), T(f_1, r_{IJ})((-) \circ f_1^* u_{IJ}).
\]
Let \( S \in AnSp(C) \). Let \( S = \bigcup_{i=1}^{l} S_i \) an open cover and denote by \( S_I = \cap_{i \in I} S_i \). Let \( i_i : S_i \hookrightarrow \tilde{S}_i \) closed embeddings, with \( \tilde{S}_i \in AnSp(C) \). For \( I \subset \{1, \ldots, l\} \), denote by \( \tilde{S}_I = \Pi_{i \in I} \tilde{S}_i \). We then have closed embeddings \( i_I : S_I \hookrightarrow \tilde{S}_I \), and for \( J \subset I \) the following commutative diagram

\[
D_{IJ} = \begin{array}{c}
S_I \\
\downarrow j_{IJ} \quad \downarrow i_I \\
S_J \\
\end{array} \begin{array}{c}
\tilde{S}_I \\
\downarrow p_{IJ} \\
\tilde{S}_J \\
\end{array}
\]
where \( p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I \) is the projection and \( j_{IJ} : S_I \hookrightarrow S_I \) is the open embedding so that \( j_I \circ j_{IJ} = j_J \).
This gives the diagram of analytic spaces \( (\tilde{S}_I)_{I \in \mathcal{I}} \in Fun(\mathcal{P}(\mathbb{N}), AnSp(C)) \) which gives the diagram of sites \( AnSp(C)^{sm}/(\tilde{S}_I) \in Fun(\mathcal{P}(\mathbb{N}), \text{Cat}) \). Denote by \( m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I \) the open embedding. Then \( C_{fil}(AnSp(C)^{sm}/(\tilde{S}_I)) \) is the category

\[
\begin{itemize}
  \item whose objects \( (G, F) = ((G_I, F), u_{IJ}) \) with \( (G_I, F) \in C_{fil}(AnSp(C)^{sm}/\tilde{S}_I) \), and \( u_{IJ} : (G_I, F) \rightarrow p_{IJ*}(G_J, F) \) are morphisms satisfying for \( I \subset J \subset K \), \( p_{IJ*} u_{JK} \circ u_{IJ} = u_{IK} \) in \( C_{fil}(AnSp(C)^{sm}/\tilde{S}_I) \),
  \item the morphisms \( m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ}) \) being a family of morphisms of complexes,
\end{itemize}
\[
m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \in \mathcal{I}}
\]
such that \( v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ} \) in \( C_{fil}(AnSp(C)^{sm}/\tilde{S}_I) \).

**Definition 24.** Let \( S \in AnSp(C) \). Let \( S = \bigcup_{i=1}^{l} S_i \) an open cover and denote by \( S_I = \cap_{i \in I} S_i \). Let \( i_i : S_i \hookrightarrow \tilde{S}_i \) closed embeddings, with \( \tilde{S}_i \in AnSp(C) \). We denote by the full subcategory \( C_{fil}(AnSp(C)^{sm}/(S/(\tilde{S}_I))) \subset C_{fil}(AnSp(C)^{sm}/(\tilde{S}_I)) \) the full subcategory

\[
\begin{itemize}
  \item whose objects \( (G, F) = ((G_I, F)_{I \subset \{1, \ldots, l\}}, u_{IJ}) \), with \( (G_I, F) \in C_{fil, S_I}(AnSp(C)^{sm}/\tilde{S}_I) \), and \( u_{IJ} : m^*(G_I, F) \rightarrow m^* p_{IJ*}(G_J, F) \) for \( I \subset J \), are \( \infty \)-filtered usu local equivalence, satisfying for \( I \subset J \subset K \), \( p_{IJ*} u_{JK} \circ u_{IJ} = u_{IK} \) in \( C_{fil}(AnSp(C)^{sm}/\tilde{S}_I) \),
  \item the morphisms \( m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ}) \) being (see section 2.1) a family of morphisms of complexes,
\end{itemize}
\[
m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subset \{1, \ldots, l\}}
\]
such that \( v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ} \) in \( C_{fil}(AnSp(C)^{sm}/\tilde{S}_I) \).
A morphism \( m : ((G_1, F), u_{1J}) \to ((H_1, F), v_{1J}) \) is an \( r \)-filtered usu local equivalence, if there exists \( \phi_i : ((C_{i1}, F), u_{i1J}) \to ((C_{i(1+1)}, F), u_{i(1+1)J}), 0 \leq i \leq s \), with \( ((C_{i1}, F), u_{i1J}) \in C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/(S/\tilde{S}_I)) \) \( ((C_{i0}, F), u_{i0J}) = ((G_1, F), v_{1J}) \) and \( ((C_{s1}, F), u_{s1J}) = ((H_1, F), v_{1J}) \), such that
\[
\phi_0 = \phi_1 \circ \cdots \circ \phi_s : ((G_1, F), u_{1J}) \to ((H_1, F), v_{1J})
\]
and \( \phi_i : ((C_{i1}, F), u_{i1J}) \to ((C_{i(1+1)}, F), u_{i(1+1)J}) \) either a filtered usu local equivalence or an \( r \)-filtered homotopy equivalence.

Denote \( L = \{1, \ldots, l\} \) and for \( I \subset L, p_{(0I)} : S \times \tilde{S}_I \to S, p_{I(0)} : S \times \tilde{S}_I \to S_I \) the projections. By definition, we have functors
- \( T(S/\tilde{S}_I) : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \to C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/(S/\tilde{S}_I)), (G, F) \mapsto (i_{l*}j_{l*}F, T(D_{ij}(j_{l}^*(G, F)))\rangle \)

\( T((\tilde{S}_I)/S) : C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/(S/\tilde{S}_I)) \to C_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S), ((G, F), u_{1J}) \mapsto \text{ho } \lim_{I \subset L} p_{(0I)}^* \Gamma_{\tilde{S}_I}^J p_{I(0)}^*(G_1, F) \). Note that the functors \( T(S/\tilde{S}_I) \) are NOT embedding, since
\[
\text{ad}(i_{l*}^*, i_{l*}) (j_{l*}^* F) : i_{l*}^* i_{l*} j_{l*}^* F \to j_{l*}^* F
\]
are usu local equivalence but NOT isomorphism since we are dealing with the morphism of big sites \( P(i) : \text{AnSp}(\mathbb{C})^{sm}/S_I \to \text{AnSp}(\mathbb{C})^{sm}/\tilde{S}_I \). However, these functors induces full embeddings
\[
T(S/\tilde{S}_I) : D_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S) \to D_{fil}(\text{AnSp}(\mathbb{C})^{sm}/(S/\tilde{S}_I))
\]
since for \( F \in C(\text{AnSp}(\mathbb{C})^{sm}/S) \),
\[
\text{ho } \lim_{I \subset L} p_{(0I)}^* \Gamma_{S_I}^J p_{I(0)}^*(i_{l*} j_{l*} F) \to p_{(0I)}^* \Gamma_{S_I}^J j_{l*} F
\]
is an equivalence usu local.

Let \( f : X \to S \) a morphism, with \( X, S \in \text{AnSp}(\mathbb{C}) \). Let \( S = \bigcup_{i \in I} S_i \) and \( X = \bigcup_{i \in I} X_i \) be affine open covers and \( i_i : S_i \to \tilde{S}_I, i'_i : X_i \to \tilde{X}_I \) be closed embeddings. Let \( f_i : \tilde{X}_I \to \tilde{S}_I \) be a lift of the morphism \( f_i = f_i|_{X_i} : X_i \to S_i \). Then, \( f_I = f_i|_{X_I} : X_I = \cap_{i \in I} X_i \to S_I = \cap_{i \in I} S_i \) lift to the morphism
\[
\tilde{f}_I = \Pi_{i \in J} \tilde{f}_i : \tilde{X}_I = \Pi_{i \in J} \tilde{X}_i \to \tilde{S}_I = \Pi_{i \in I} \tilde{S}_i
\]
Denote by \( p_{IJ} : \tilde{S}_I \to \tilde{S}_J \) and \( p_I'_{IJ} : \tilde{X}_I \to \tilde{X}_J \) the projections. Consider for \( J \subset I \) the following commutative diagrams

\[
\begin{align*}
D_{IJ} = \quad & S_I \xto{i_J} \tilde{S}_I, \quad D'_IJ = \quad & X_I \xto{i'_J} \tilde{X}_I, \quad D_{II} = \quad & S_I \xto{i_I} \tilde{S}_I
\end{align*}
\]

We then have following commutative diagram

\[
\begin{align*}
& X_I \xto{n'_J} \tilde{X}_I \xto{\tilde{n}'_J} \tilde{X}_I \xto{\tilde{n}'_J} \tilde{X}_I \xto{n'_J} \tilde{X}_I
\end{align*}
\]
whose square are cartesian. We then have the pullback functor
\[
f^* : C_{(2)}_{fil}(\text{AnSp}(\mathbb{C})^{sm}/S/(\tilde{S}_I)) \to C_{(2)}_{fil}(\text{AnSp}(\mathbb{C})^{sm}/X/(\tilde{X}_I)),
((G_1, F), u_{1J}) \mapsto f^*((G_1, F), u_{1J}) := (\Gamma_{\tilde{X}_I}^J \tilde{f}_I^*(G_1, F), \tilde{f}_J^* u_{1J})
\]
with
\[ f^*_{IJ} : \Gamma_X f^*_I(G, F) \xrightarrow{\text{ad}(p^*_I \to p^*_J)} p^*_I \to p^*_J \to \Gamma_X f^*_J(G, F) \]

\[ p^*_I \to \Gamma_X f^*_J(G, F) \xrightarrow{\text{ad}(p^*_J \to p^*_I)} p^*_J \to p^*_I \to \Gamma_X f^*_I(G, F) \]

Let \((G, F) \in C_{fil}(\text{AnSp}(C)^{sm}/S)\). Since, \(j^*_I j^*_I f^*(G, F) = 0\), the morphism \(T(D_{fil})(j^*_I(G, F)) : f^*_I f^*_I f^*(G, F) \to j^*_I j^*_I f^*(G, F)\) factors through

\[ T(D_{fil})(j^*_I(G, F)) : f^*_I f^*_I f^*(G, F) \xrightarrow{\gamma_{X_I}} \Gamma_X f^*_I f^*_I f^*_I(G, F) \]

\[ T(D_{fil})(j^*_I(G, F)) : f^*_I f^*_I f^*(G, F) \xrightarrow{\gamma_{X_I}} \Gamma_X f^*_I f^*_I f^*_I(G, F) \]

We have then, for \((G, F) \in C_{fil}(S)\), the canonical transformation map

\[ f^*T(S/\tilde{S}_I)(G, F) \xrightarrow{T(f, T(0/I))(G, F)} T(X/(\tilde{X}_I))(f^*(G, F)) \]

\[ (\Gamma_X f^*_I f^*_I f^*_I(G, F), f^*_I f^*_I f^*_I) \xrightarrow{T^*(D_{fil})(j^*_I(G, F))} (j^*_I j^*_I f^*(G, F), I) \]

We have similarly to the algebraic case, we have:

**Definition 25.**

(i) Let \(f : X \to S\) a morphism with \(X, S \in \text{AnSp}(C)\). Assume that there exist a factorization \(f : X \xrightarrow{i} Y \times S \xrightarrow{p} S\), with \(Y \in \text{AnSm}(C)\), \(i : X \to Y\) is a closed embedding and \(p\) the projection. We then consider

\[ Q(X/S) := p_Y \Gamma_X Z_{Y \times S} := \text{Cone}(\mathbb{Z}((Y \times S) \setminus X/S) \to \mathbb{Z}(Y \times S/S)) \in C(\text{AnSp}(C)^{sm}/S). \]

By definition \(Q(X/S)\) is projective since it is a complex of two representative presheaves.

(ii) Let \(f : X \to S\) and \(g : T \to S\) two morphism with \(X, S, T \in \text{AnSp}(C)\). Assume that there exist a factorization \(f : X \xrightarrow{i} Y \times S \xrightarrow{p} S\), with \(Y \in \text{AnSm}(C)\), \(i : X \to Y\) is a closed embedding and \(p\) the projection. We then have the following commutative diagram whose squares are cartesian

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \times S \\
\uparrow{g'} & & \uparrow{p}
\end{array}
\]

\[
\begin{array}{ccc}
Y \times T & \xrightarrow{\psi} & T \\
\downarrow{g'' := (1 \times g)} & & \downarrow{g}
\end{array}
\]

We then have the canonical isomorphism in \(C(\text{AnSp}(C)^{sm}/T)\)

\[ T(f, g, Q) := \mathcal{T}_2(g, p)(-1) \circ \mathcal{T}_2(g'', j)(-1) \]

\[ g^* Q(X/S) : = g^* p_Y \Gamma_X Z_{Y \times S} \to p_Y \Gamma_{X_T} Z_{Y \times T} =: Q(X_T/T) \]

with \(j : Y \times S \setminus X \to Y \times S\) the closed embedding.

Let \(S \in \text{AnSp}(C)\). Denote for short \(\text{AnSp}(C)^{sm}/S\) either the category \(\text{AnSp}(C)/S\) or the category \(\text{AnSp}(C)^{sm}/S\). Denote by

\[ p_a : \text{AnSp}(C)^{sm}/S \to \text{AnSp}(C)^{sm}/S, X/S = (X, h) \mapsto (X \times \mathbb{D}^1)/S = (X \times \mathbb{D}^1, h \circ p_X), \]

\[ (g : X/S \to X'/S) \mapsto ((g \times j_{D^1}) : X \times \mathbb{D}^1/S \to X' \times \mathbb{D}^1/S) \]

the projection functor and again by \(p_a : \text{AnSp}(C)^{sm}/S \to \text{AnSp}(C)^{sm}/S\) the corresponding morphism of site.

We now define the \(\mathbb{D}^1\) localization property:
Definition 26. Let $S \in \text{Var}(\mathbb{C})$.

(iii) A morphism $F : F \rightarrow G$ with $F$, $G \in C(\text{AnSp}(\mathbb{C})(\text{sm})/S)$ is said to be $D^1$ invariant if for all $U/S \in \text{AnSp}(\mathbb{C})(\text{sm})/S$,

$$F(p_U) : F(U/S) \rightarrow F(U \times D^1/S)$$

is a quasi-isomorphism, where $p_U : U \times D^1 \rightarrow U$ is the projection. Obviously, a $D^1$ homotopic complex is $D^1$ invariant.

(ii) A complex $F \in C(\text{AnSp}(\mathbb{C})(\text{sm})/S)$ is said to be $D^1$ local for the usual topology, if for a (hence every) usu local equivalence $k : F \rightarrow G$ with $k$ injective and $G \in C(\text{AnSp}(\mathbb{C})(\text{sm})/S)$ usu fibrant, e.g. $k : F \rightarrow E_{\text{usu}}(F)$, $G$ is $D^1$ invariant for all $n \in \mathbb{Z}$.

(iii) A morphism $m : F \rightarrow G$ with $F$, $G \in C(\text{AnSp}(\mathbb{C})(\text{sm})/S)$ is said to an $(D^1, \text{usu})$ local equivalence if for all $H \in C(\text{AnSp}(\mathbb{C})(\text{sm})/S)$ which is $D^1$ local for the étale topology

$$\text{Hom}(L(m), E_{\text{usu}}(H)) : \text{Hom}(L(G), E_{\text{usu}}(H)) \rightarrow \text{Hom}(L(F), E_{\text{usu}}(H))$$

is a quasi-isomorphism.

Proposition 29. A morphism $m : F \rightarrow G$ with $F$, $G \in C(\text{AnSp}(\mathbb{C})(\text{sm})/S)$ is an $(D^1, \text{usu})$ local equivalence if and only if there exists

$$\{X_{1,\alpha}/S, \alpha \in \Lambda_1\}, \ldots, \{X_{r,\alpha}/S, \alpha \in \Lambda_r\} \subset \text{AnSp}(\mathbb{C})(\text{sm})/S$$

such that we have in $\text{Ho}_{\text{et}}(C(\text{Var}(\mathbb{C})(\text{sm})/S))$

$$\text{Cone}(m) \cong \text{Cone}(\bigoplus_{\alpha \in \Lambda_1} \text{Cone}(\mathbb{Z}(X_{1,\alpha} \times D^1/S) \rightarrow \mathbb{Z}(X_{1,\alpha}/S)) \rightarrow \cdots \rightarrow \bigoplus_{\alpha \in \Lambda_r} \text{Cone}(\mathbb{Z}(X_{r,\alpha} \times D^1/S) \rightarrow \mathbb{Z}(X_{r,\alpha}/S)))$$

Proof. Standard.

Definition-Proposition 8. Let $S \in \text{AnSp}(\mathbb{C})$

(i) With the weak equivalence the $(D^1, \text{usu})$ local equivalence and the fibration the epimorphism with $D^1$ local and usu fibrant kernels gives a model structure on $C(\text{AnSp}(\mathbb{C})(\text{sm})/S)$ : the left bousfield localization of the projective model structure of $C(\text{AnSp}(\mathbb{C})(\text{sm})/S)$. We call it the $(D^1, \text{usu})$ projective model structure.

(ii) With the weak equivalence the $(D^1, \text{usu})$ local equivalence and the fibration the epimorphism with $D^1$ local and usu fibrant kernels gives a model structure on $C(\text{AnSp}(\mathbb{C})/S)$ : the left bousfield localization of the projective model structure of $C(\text{AnSp}(\mathbb{C})/S)$. We call it the $(D^1, \text{usu})$ projective model structure.

Proof. Similar to the proof of definition-proposition 5.

Proposition 30. Let $g : T \rightarrow S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$.

(i) The adjunction $(g^*, g_*): C(\text{AnSp}(\mathbb{C})(\text{sm})/S) \rightleftarrows C(\text{AnSp}(\mathbb{C})(\text{sm})/T)$ is a Quillen adjonction for the $(D^1, \text{et})$ projective model structure.

(i') Let $h : U \rightarrow S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$. The adjunction $(h_!, h^*) : C(\text{AnSp}(\mathbb{C})(\text{sm})/U) \rightleftarrows C(\text{AnSp}(\mathbb{C})(\text{sm})/S)$ is a Quillen adjonction for the $(D^1, \text{et})$ projective model structure.

(i'') The functor $g^* : C(\text{AnSp}(\mathbb{C})(\text{sm})/S) \rightarrow C(\text{AnSp}(\mathbb{C})(\text{sm})/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends $(D^1, \text{et})$ local equivalence to $(D^1, \text{et})$ local equivalence.
Proof. Similar to the proof of proposition 18. 

**Proposition 31.** Let $S \in \text{AnSp}(\mathbb{C})$.

(i) The adjunction $(\rho^*_S, \rho_S) : C(\text{AnSp}(\mathbb{C})^{sm}/S) \Rightarrow C(\text{AnSp}(\mathbb{C})/S)$ is a Quillen adjunction for the $(\mathbb{D}^1, \text{et})$-projective model structure.

(ii) The functor $\rho_S : C(\text{AnSp}(\mathbb{C})/S) \to C(\text{AnSp}(\mathbb{C})^{sm}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends $(\mathbb{D}^1, \text{ usu})$ local equivalence to $(\mathbb{D}^1, \text{ usu})$ local equivalence.

Proof. Similar to the proof of proposition 19. 

Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \bigcup_{i=1}^I S_i$ an open affinoid cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow S_i$ closed embeddings, with $\bar{S}_i \in \text{AnSp}(\mathbb{C})$. For $(G_I, K_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{sm}/(\bar{S}_I)\text{op})$ and $(H_I, T_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{sm}/(\bar{S}_I))$, we denote

$$\mathcal{H}om((G_I, K_{IJ}), (H_I, T_{IJ})) := (\mathcal{H}om(G_I, H_I), u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ}))) \in C(\text{AnSp}(\mathbb{C})^{sm}/(\bar{S}_I))$$

with

$$u_{IJ}((G_I, K_{IJ})(H_I, T_{IJ})) : \mathcal{H}om(G_I, H_I) \xrightarrow{Ad(p^*_{IJ}, p_{IJ}, \cdot)} p_{IJ}^* \mathcal{H}om(G_I, H_I) \xrightarrow{T(p_{IJ}, \mathcal{H}om)(\cdot, -)} p_{IJ}^* \mathcal{H}om(p^*_{IJ} G_I, p^*_{IJ} H_I) \xrightarrow{\mathcal{H}om(p^*_{IJ} G_I, H_I)} p_{IJ}^* \mathcal{H}om(G_I, H_I).$$

This gives in particular the functor

$$C(\text{AnSp}(\mathbb{C})^{sm}/(\bar{S}_I)) \to C(\text{AnSp}(\mathbb{C})^{sm}/(\bar{S}_I)\text{op}), (H_I, T_{IJ}) \mapsto (H_I, T_{IJ}).$$

Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \bigcup_{i=1}^I S_i$ an open affinoid cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow S_i$ closed embeddings, with $\bar{S}_i \in \text{AnSp}(\mathbb{C})$. The functor $p_a$ extend to a functor

$$p_a : \text{AnSp}(\mathbb{C})^{sm}/(\bar{S}_I) \to \text{AnSp}(\mathbb{C})^{sm}/(\bar{S}_I), (X_I/\bar{S}_I, s_{IJ}) \mapsto (X \times \mathbb{D}^1/\bar{S}_I, s_{IJ} \times I),$$

$$(g : (g_I) : (X_I/\bar{S}_I, s_{IJ}) \to (X'_I/\bar{S}_I, s'_{IJ})) \mapsto ((g \times I_{\mathbb{D}^1}) : (X_I \times \mathbb{D}^1/\bar{S}_I, s_{IJ}) \to (X'_I \times \mathbb{D}^1/\bar{S}_I, s'_{IJ}))$$

the projection functor and again by $p_a : \text{AnSp}(\mathbb{C})^{sm}/(\bar{S}_I) \to \text{AnSp}(\mathbb{C})^{sm}/(\bar{S}_I)$ the corresponding morphism of site.

**Definition 27.** Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \bigcup_{i=1}^I S_i$ an open affinoid cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow S_i$ closed embeddings, with $\bar{S}_i \in \text{AnSp}(\mathbb{C})$.

(i0) A complex $(F_I, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{sm}/(\bar{S}_I))$ is said to be $\mathbb{D}^1$ homotopic if $Ad(p^*_a, p_{a*})((F_I, u_{IJ})) : (F_I, u_{IJ}) \to p_{a*}p^*_a(F_I, u_{IJ})$ is an homotopy equivalence.
(i) A complex \((F_I, u_{IJ}) \in \text{C(AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\) is said to be \(D^1\) invariant if for all \((X_I/\tilde{S}_I, s_{IJ}) \in \text{AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\)
\[(F_I(p_{X_I})): (F_I(X_I/\tilde{S}_I), F_J(s_{IJ})) \circ u_{IJ}(-) \to (F_I(X_I \times \mathbb{A}^1/\tilde{S}_I), F_J(s_{IJ} \times I) \circ u_{IJ}(-))\]
is a quasi-isomorphism, where \(p_{X_I}: X_I \times \mathbb{A}^1 \to X_I\) are the projection, and \(s_{IJ}: X_I \times \tilde{S}_{IJ} \to X_I/\tilde{S}_J\). Obviously a complex \((F_I, u_{IJ}) \in \text{C(AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\) is \(D^1\) invariant if and only if all the \(F_I\) are \(\mathbb{A}^1\) invariant.

(ii) Let \(\tau\) a topology on \(\text{AnSp}(\mathbb{C})\). A complex \(F = (F_I, u_{IJ}) \in \text{C(AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\) is said to be \(D^1\) local for the \(\tau\) topology induced on \(\text{AnSp}(\mathbb{C})/\tilde{S}_I\), if for an (hence every) \(\tau\) local equivalence \(k : F \to G\) with \(k\) injective and \(G = (G_I, v_{IJ}) \in \text{C(AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\) \(\tau\) fibrant, e.g. \(k : (F_I, u_{IJ}) \to (E_r(F_I), E(u_{IJ})), G \in \mathbb{A}^1\) invariant.

(iii) A morphism \(m = (m_I): (F_I, u_{IJ}) \to (G_I, v_{IJ})\) with \((F_I, u_{IJ}), (G_I, v_{IJ}) \in \text{C(AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\) is said to be an \((D^1, usu)\) local equivalence if for all \((H_I, w_{IJ}) \in \text{C(AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\) which is \(D^1\) local for the usual topology
\[
\text{Hom}(L(m_I), E_{usu}(H_I)) : \text{Hom}(L(G_I, v_{IJ}), E_{usu}(H_I, w_{IJ})) \to \text{Hom}(L(F_I, u_{IJ}), E_{usu}(H_I, w_{IJ}))
\]
is a quasi-isomorphism. Obviously, if a morphism \(m = (m_I): (F_I, u_{IJ}) \to (G_I, v_{IJ})\) with \((F_I, u_{IJ}), (G_I, v_{IJ}) \in \text{C(AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\) is an \((D^1, usu)\) local equivalence, then all the \(m_I: F_I \to G_I\) are \((D^1, usu)\) local equivalence.

(iv) A morphism \(m = (m_I): (F_I, u_{IJ}) \to (G_I, v_{IJ})\) with \((F_I, u_{IJ}), (G_I, v_{IJ}) \in \text{C(AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\) is said to be an \((D^1, usu)\) local equivalence if for all \((H_I, w_{IJ}) \in \text{C(AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\) which is \(D^1\) local for the etale topology
\[
\text{Hom}(L(m_I), E_{usu}(H_I)) : \text{Hom}(L(G_I, v_{IJ}), E_{usu}(H_I, w_{IJ})) \to \text{Hom}(L(F_I, u_{IJ}), E_{usu}(H_I, w_{IJ}))
\]
is a quasi-isomorphism. Obviously, if a morphism \(m = (m_I): (F_I, u_{IJ}) \to (G_I, v_{IJ})\) with \((F_I, u_{IJ}), (G_I, v_{IJ}) \in \text{C(AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\) is an \((D^1, usu)\) local equivalence, then all the \(m_I: F_I \to G_I\) are \((D^1, usu)\) local equivalence.

**Proposition 32.** Let \(S \in \text{AnSp}(\mathbb{C})\). Let \(S = \bigcup_{i \in I} S_i\) an open affinoid cover and denote by \(S_I = \cap_{i \in I} S_i\).
Let \(i_s : S_i \hookrightarrow \tilde{S}_i\) closed embeddings, with \(\tilde{S}_i \in \text{AnSp}(\mathbb{C})\).

(i) A morphism \(m : F \to G\) with \(F, G \in \text{C(AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\) is an \((D^1, usu)\) local equivalence if and only if there exists
\[
\{(X_{I, \alpha_1}/\tilde{S}_I, u_{IJ}^{\alpha_1}), \alpha \in \Lambda_1\}, \ldots, \{(X_{R, \alpha_r}/\tilde{S}_I, u_{IJ}^{\alpha_r}), \alpha \in \Lambda_r\} \subset \text{AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I
\]
with \(u_{IJ}^{\alpha_1} : X_{I, \alpha_1} \times \tilde{S}_{IJ, \alpha} \to X_{I, \alpha_1}/\tilde{S}_J\), such that we have in \(\text{Ho}_{usu}(C(\text{AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I)\)
\[
\text{Cone}(m) \xrightarrow{\sim} \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(\text{C}(X_{I, \alpha_1} \times \mathbb{D}^1/\tilde{S}_I, Z(u_{IJ}^{\alpha_1} \times I)) \to (Z(X_{I, \alpha_1}/\tilde{S}_I), Z(u_{IJ}^{\alpha_1})))
\]
\[
\to \cdots \to \oplus_{\alpha \in \Lambda_r} \text{Cone}(\text{C}(X_{R, \alpha_r} \times \mathbb{D}^1/\tilde{S}_I, Z(u_{IJ}^{\alpha_r} \times I)) \to (Z(X_{R, \alpha_r}/\tilde{S}_I), Z(u_{IJ}^{\alpha_r})))
\]

(ii) A morphism \(m : F \to G\) with \(F, G \in \text{C(AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I\) is an \((D^1, usu)\) local equivalence if and only if there exists
\[
\{(X_{I, \alpha_1}/\tilde{S}_I, u_{IJ}^{\alpha_1}), \alpha \in \Lambda_1\}, \ldots, \{(X_{R, \alpha_r}/\tilde{S}_I, u_{IJ}^{\alpha_r}), \alpha \in \Lambda_r\} \subset \text{AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I^{\text{op}}
\]
with \(u_{IJ}^{\alpha_1} : X_{I, \alpha_1} / \tilde{S}_J \to X_{I, \alpha_1} \times \tilde{S}_{IJ, \alpha} / \tilde{S}_J\), such that we have in \(\text{Ho}_{\text{et}}(C(\text{AnSp}(\mathbb{C})^{\text{sm}})/\tilde{S}_I^{\text{op}})\)
\[
\text{Cone}(m) \xrightarrow{\sim} \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(\text{C}(X_{I, \alpha_1} \times \mathbb{D}^1/\tilde{S}_I, Z(u_{IJ}^{\alpha_1} \times I)) \to (Z(X_{I, \alpha_1}/\tilde{S}_I), Z(u_{IJ}^{\alpha_1})))
\]
\[
\to \cdots \to \oplus_{\alpha \in \Lambda_r} \text{Cone}(\text{C}(X_{R, \alpha_r} \times \mathbb{D}^1/\tilde{S}_I, Z(u_{IJ}^{\alpha_r} \times I)) \to (Z(X_{R, \alpha_r}/\tilde{S}_I), Z(u_{IJ}^{\alpha_r})))
\]

**Proof.** Standard. See Ayoub's Thesis for example. \(\blacksquare\)
2.9 Presheaves on the big analytical site of pairs

We recall the definition given in subsection 5.1: For $S \in \mathbf{AnSp}(\mathbb{C})$, $\mathbf{AnSp}(\mathbb{C})^2/S := \mathbf{AnSp}(\mathbb{C})^2/(S,S)$ is by definition (see subsection 2.1) the category whose set of objects is

$$(\mathbf{AnSp}(\mathbb{C})^2/S)^0 := \{(X,Z), h : X \to S, Z \subset X \text{ closed} \} \subset \mathbf{AnSp}(\mathbb{C})/S \times \mathbf{Top}$$

and whose set of morphisms between $(X_1, Z_1)/S = ((X_1, Z_1), h_1), (X_1, Z_1)/S = ((X_2, Z_2), h_2) \in \mathbf{AnSp}(\mathbb{C})^2/S$ is the subset

$$\text{Hom}_{\mathbf{AnSp}(\mathbb{C})^2/S}((X_1, Z_1)/S, (X_2, Z_2)/S) := \{(f : X_1 \to X_2), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\} \subset \text{Hom}_{\mathbf{AnSp}(\mathbb{C})}(X_1, X_2)$$

The category $\mathbf{AnSp}(\mathbb{C})^2$ admits fiber products: $((X_1, Z_1) \times_{(S,Z)} (X_2, Z_2)) = (X_1 \times_S X_2, Z_1 \times_Z Z_2)$. In particular, for $f : T \to S$ a morphism with $S,T \in \mathbf{AnSp}(\mathbb{C})$, we have the pullback functor

$$P(f) : \mathbf{AnSp}(\mathbb{C})^2/S \to \mathbf{AnSp}(\mathbb{C})^2/T, P(f)((X,Z)/S) := (X_T, Z_T)/T, P(f)(g) := (g \times_S f)$$

and we note again $P(f) : \mathbf{AnSp}(\mathbb{C})^2/T \to \mathbf{AnSp}(\mathbb{C})^2/S$ the corresponding morphism of sites.

We will consider in the construction of the filtered De Rham realization functor the full subcategory $\mathbf{AnSp}(\mathbb{C})^{2,sm}/S \subset \mathbf{AnSp}(\mathbb{C})^2/S$ such that the first factor is a smooth morphism: We will also consider, in order to obtain a complex of D-modules in the construction of the filtered De Rham realization functor, the restriction to the full subcategory $\mathbf{AnSp}(\mathbb{C})^{2,pr}/S \subset \mathbf{AnSp}(\mathbb{C})^2/S$ such that the first factor is a projection:

**Definition 28.** (i) Let $S \in \mathbf{AnSp}(\mathbb{C})$. We denote by

$$\rho_S : \mathbf{AnSp}(\mathbb{C})^{2,sm}/S \hookrightarrow \mathbf{AnSp}(\mathbb{C})^2/S$$

the full subcategory consisting of the objects $(U,Z)/S = ((U,Z), h) \in \mathbf{AnSp}(\mathbb{C})^2/S$ such that the morphism $h : U \to S$ is smooth. That is, $\mathbf{AnSp}(\mathbb{C})^{2,sm}/S$ is the category

- whose objects are $(U,Z)/S = ((U,Z), h)$, with $U \in \mathbf{AnSp}(\mathbb{C})$, $Z \subset U$ a closed subset, and $h : U \to S$ a smooth morphism,

- whose morphisms $g : (U,Z)/S = ((U,Z), h_1) \to (U',Z')/S = ((U',Z'), h_2)$ is a morphism $g : U \to U'$ of complex algebraic varieties such that $Z \subset g^{-1}(Z')$ and $h_2 \circ g = h_1$.

We denote again $\rho_S : \mathbf{AnSp}(\mathbb{C})^2/S \to \mathbf{AnSp}(\mathbb{C})^{2,sm}/S$ the associated morphism of site. We have

$$r^s(S) : \mathbf{AnSp}(\mathbb{C})^2 \xrightarrow{r(S) := r(S,S)} \mathbf{AnSp}(\mathbb{C})^2/S \xrightarrow{\rho_S} \mathbf{AnSp}(\mathbb{C})^{2,sm}/S$$

the composite morphism of site.

(ii) Let $S \in \mathbf{AnSp}(\mathbb{C})$. We will consider the full subcategory

$$\mu_S : \mathbf{AnSp}(\mathbb{C})^{2,pr}/S \hookrightarrow \mathbf{AnSp}(\mathbb{C})^2/S$$

whose subset of object consist of those whose morphism is a projection to $S$:

$$(\mathbf{AnSp}(\mathbb{C})^{2,pr}/S)^0 := \{((Y \times S,X), p), Y \in \mathbf{AnSp}(\mathbb{C}), p : Y \times S \to S \text{ the projection} \} \subset (\mathbf{AnSp}(\mathbb{C})^2/S)^0.$$ 

(iii) We will consider the full subcategory

$$\mu_S : (\mathbf{AnSp}(\mathbb{C})^{2,smpr}/S) \hookrightarrow \mathbf{AnSp}(\mathbb{C})^{2,sm}/S$$

whose subset of object consist of those whose morphism is a smooth projection to $S$:

$$(\mathbf{AnSp}(\mathbb{C})^{2,smpr}/S)^0 := \{((Y \times S,X), p), Y \in \mathbf{SmVar}(\mathbb{C}), p : Y \times S \to S \text{ the projection} \} \subset (\mathbf{AnSp}(\mathbb{C})^2/S)^0.$$
For \( f : T \rightarrow S \) a morphism with \( T, S \in \text{AnSp}(\mathbb{C}) \), we have by definition, the following commutative diagram of sites

\[
\begin{array}{ccc}
\text{AnSp}(\mathbb{C})^2/T & \xrightarrow{\mu_T} & \text{AnSp}(\mathbb{C})^{2,pr}/T \\
\downarrow{\rho_T} & & \downarrow{\rho_T} \\
\text{AnSp}(\mathbb{C})^{2,sm}/T & \xrightarrow{\mu_2} & \text{AnSp}(\mathbb{C})^{2,smpr}/T \\
\downarrow{\rho_2} & & \downarrow{\rho_2} \\
\text{AnSp}(\mathbb{C})^2/S & \xrightarrow{\rho_1} & \text{AnSp}(\mathbb{C})^{2}/S \\
\end{array}
\]  \hspace{1cm} (37)

\[
\begin{array}{ccc}
\text{AnSp}(\mathbb{C})^{2,pr}/T & \xrightarrow{\text{Gr}^2} & \text{AnSp}(\mathbb{C})/T \\
\downarrow{\rho_T} & & \downarrow{\rho_T} \\
\text{AnSp}(\mathbb{C})^{2,smpr}/T & \xrightarrow{\text{Gr}^2} & \text{AnSp}(\mathbb{C})^{sm}/T \\
\downarrow{\rho_2} & & \downarrow{\rho_2} \\
\text{AnSp}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}^2} & \text{AnSp}(\mathbb{C})/S \\
\downarrow{\rho_1} & & \downarrow{\rho_1} \\
\text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}^2} & \text{AnSp}(\mathbb{C})^{sm}/S \\
\end{array}
\]  \hspace{1cm} (38)

Recall we have (see subsection 2.1), for \( S \in \text{Var}(\mathbb{C}) \), the graph functor

\[
\text{Gr}^2_S : \text{AnSp}(\mathbb{C})/S \rightarrow \text{AnSp}(\mathbb{C})^{2,pr}/S, \ X/S \mapsto \text{Gr}^2_S(X/S) := (X \times S, X)/S,
\]

\[
(g : X/S \rightarrow X'/S) \mapsto \text{Gr}^2_S(g) := (g \times \text{Id}_S : (X \times S, X) \rightarrow (X' \times S, X'))
\]

For \( f : T \rightarrow S \) a morphism with \( T, S \in \text{AnSp}(\mathbb{C}) \), we have by definition, the following commutative diagram of sites

\[
\begin{array}{ccc}
\text{AnSp}(\mathbb{C})^{2,pr}/T & \xrightarrow{\text{Gr}^2} & \text{AnSp}(\mathbb{C})/T \\
\downarrow{\rho_T} & & \downarrow{\rho_T} \\
\text{AnSp}(\mathbb{C})^{2,smpr}/T & \xrightarrow{\text{Gr}^2} & \text{AnSp}(\mathbb{C})^{sm}/T \\
\downarrow{\rho_2} & & \downarrow{\rho_2} \\
\text{AnSp}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}^2} & \text{AnSp}(\mathbb{C})/S \\
\downarrow{\rho_1} & & \downarrow{\rho_1} \\
\text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{Gr}^2} & \text{AnSp}(\mathbb{C})^{sm}/S \\
\end{array}
\]

where we recall that \( P(f)((X, Z)/S) := ((X_T, Z_T)/T) \), since smooth morphisms are preserved by base change.

As usual, we denote by

\[
(f^*, f_*) := (P(f)^*, P(f)_*) : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \rightarrow C(\text{AnSp}(\mathbb{C})^{2,sm}/T)
\]

the adjunction induced by \( P(f) : \text{AnSp}(\mathbb{C})^{2,sm}/T \rightarrow \text{AnSp}(\mathbb{C})^{2,sm}/S \). Since the colimits involved in the definition of \( f^* = P(f)^* \) are filtered, \( f^* \) also preserve monomorphism. Hence, we get an adjonction

\[
(f^*, F) : C_{\text{fd}}(\text{AnSp}(\mathbb{C})^{2,sm}/S) \Rightarrow C_{\text{fd}}(\text{AnSp}(\mathbb{C})^{2,sm}/T), \ F^*(G, F) := (f^*G, f^*F)
\]

For \( S \in \text{AnSp}(\mathbb{C}) \), we denote by \( \mathbb{Z}_S := \mathbb{Z}((S, S)/(S, S)) \in \text{PSh}(\text{AnSp}(\mathbb{C})^{2,sm}/S) \) the constant presheaf
By Yoneda lemma, we have for \( F \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \), \( \text{Hom}(\mathbb{Z}_S, F) = F \).

For \( h : U \rightarrow S \) a smooth morphism with \( U, S \in \text{AnSp}(\mathbb{C}) \), \( P(h) : \text{AnSp}(\mathbb{C})^{2,sm}/S \rightarrow \text{AnSp}(\mathbb{C})^{2,sm}/U \)
admits a left adjoint

\[
C(h) : \text{AnSp}(\mathbb{C})^{2,sm}/U \rightarrow \text{AnSp}(\mathbb{C})^{2,sm}/S, \ C(h)((U', Z'), h') = ((U', Z'), h \circ h').
\]
Hence \( h^* : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \to C(\text{AnSp}(\mathbb{C})^{2,sm}/U) \) admits a left adjoint
\[
h_2 : C(\text{AnSp}(\mathbb{C})^{2,sm}/U) \to C(\text{AnSp}(\mathbb{C})^{2,sm}/S), \quad F \mapsto (h_2 F : ((U, Z), h_0) \mapsto \lim_{((U', Z'), h_{0'}) \to (U, Z), h_0} F((U', Z')/U))
\]
For \( F^* \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \) and \( G^* \in C(\text{AnSp}(\mathbb{C})^{2,sm}/U) \), we have the adjonction maps
\[
\text{ad}(h_2, h^*)(G^*) : G^* \to h^* h_2 G^*, \quad \text{ad}(h_2, h^*)(F^*) : h_2 h^* F^* \to F^*.
\]
For a smooth morphism \( h : U \to S \), with \( U, S \in \text{AnSp}(\mathbb{C}) \), we have the adjonction isomorphism, for \( F \in C(\text{AnSp}(\mathbb{C})^{2,sm}/U) \) and \( G \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \),
\[
I(h_2, h^*)(F, G) : \text{Hom}^* (h_2 F, G) \xrightarrow{\sim} h_2 \text{Hom}^*(F, h^* G).
\]
For a commutative diagram in \( \text{AnSp}(\mathbb{C}) \):
\[
D = V \xrightarrow{g_1} U, \quad T \xrightarrow{g_2} \quad \xrightarrow{h_2} \quad \xrightarrow{h_1} \quad S
\]
where \( h_1 \) and \( h_2 \) are smooth, we denote by, for \( F^* \in C(\text{AnSp}(\mathbb{C})^{2,sm}/U) \),
\[
T_{g_2} (D)(F^*) : h_2 g_2^* F^* \to g_1^* h_1 F^*
\]
the canonical map given by adjonction. If \( D \) is cartesian with \( h_1 = h, \ g_1 = g, \ f_2 = h' : U \to T, \ g' : U' \to U \),
\[
T_{g_2} (D)(F^*) = T_g (g, h)(F) : h' g'^* F^* \xrightarrow{\sim} g^* h_2 F^*
\]
is an isomorphism.

We have the support section functors of a closed embedding \( i : Z \hookrightarrow S \) for presheaves on the big analytical site of pairs.

**Definition 29.** Let \( i : Z \hookrightarrow S \) be a closed embedding with \( S, Z \in \text{AnSp}(\mathbb{C}) \) and \( j : S \setminus Z \hookrightarrow S \) be the open complementary subset.

(i) We define the functor
\[
\Gamma_Z : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \to C(\text{AnSp}(\mathbb{C})^{2,sm}/S), \quad G^* \mapsto \Gamma_Z G^* := \text{Cone}(\text{ad}(j^*, j_*)(G^*) : G^* \to j_* j^* G^*)[-1],
\]
so that there is then a canonical map \( \gamma_Z(G^*) : \Gamma_Z G^* \to G^* \).

(ii) We have the dual functor of (i):
\[
\Gamma_Z^\vee : C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \to C(\text{AnSp}(\mathbb{C})^{2,sm}/S), \quad F \mapsto \Gamma_Z^\vee (F^*) := \text{Cone}(\text{ad}(j_*, j^*)(G^*) : j_! j^* G^* \to G^*),
\]
together with the canonical map \( \gamma_Z^\vee(G) : F \to \Gamma_Z^\vee(G) \).

(iii) For \( F, G \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \), we denote by
\[
I(\gamma, \text{hom})(F, G) := (I, I(j_* j^*)(F, G)^{-1}) : \Gamma_Z \text{Hom}(F, G) \xrightarrow{\sim} \text{Hom}(\Gamma_Z^\vee F, G)
\]
the canonical isomorphism given by adjonction.

Note that we have similarly for \( i : Z \hookrightarrow S, \ i' : Z' \hookrightarrow Z \) closed embeddings, \( g : T \to S \) a morphism with \( T, S, Z \in \text{AnSp}(\mathbb{C}) \) and \( F \in C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \), the canonical maps in \( C(\text{AnSp}(\mathbb{C})^{2,sm}/S) \)
\[
\bullet \ T(g, \gamma)(F) : g^* \Gamma_Z F \xrightarrow{\sim} \Gamma_{Z \times_S T} g^* F, \quad T(g, \gamma)(F) : \Gamma_Z^\vee \times_S T g^* F \xrightarrow{\sim} g^* \Gamma_Z F
\]
\[ T(Z'/Z, \gamma)(F) : \Gamma_{Z'} F \to \Gamma_Z F, \quad T(Z'/Z, \gamma^\vee)(F) : \Gamma_{Z'}^\vee F \to \Gamma_Z^\vee F \]

but we will not use them in this article.

Let \( S_* \in \text{Fun}(\mathcal{I}, \text{AnSp} (\mathbb{C})) \) with \( \mathcal{I} \in \text{Cat} \), a diagram of algebraic varieties. It gives the diagram of

\[ \text{AnSp} (\mathbb{C})^2 / S_* \in \text{Fun}(\mathcal{I}, \text{AnSp} (\mathbb{C})) \]

\[ \text{•} \quad \text{Then } C_{fil} (\text{AnSp} (\mathbb{C})^{2,(sm)}/ S_*) \]

is the category

\[ \text{•} \quad \text{whose objects } (G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ}), \text{ with } (G_I, F) \in C_{fil} (\text{AnSp} (\mathbb{C})^{2,(sm)}/ S_I), \text{ and } u_{IJ} : (G_I, F) \to r_{IJ, s}(G_J, F) \]

for \( r_{IJ, s} : I \to J, \) denoting again \( r_{IJ, s} : S_I \to S_J, \) are morphisms satisfying for \( I \to J \to K, r_{IJ, s} \circ u_{IJ} = u_{IK} \) in \( C_{fil} (\text{AnSp} (\mathbb{C})^{2,(sm)}/ S_I), \)

- the morphisms \( m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ}) \) being (see section 2.1) a family of morphisms of complexes,

\[ m = (m_I : (G_I, F) \to (H_I, F))_{I \in \mathcal{I}} \]

such that \( v_{IJ} \circ m_I = p_{IJ, s}, m_J \circ u_{IJ} \) in \( C_{fil} (\text{Var} (\mathbb{C})^{2,(sm)}/ S_I). \)

- Then \( C_{fil} (\text{AnSp} (\mathbb{C})^{2,(sm)pr}/ S_*) \) is the category

\[ \text{•} \quad \text{whose objects } (G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ}), \text{ with } (G_I, F) \in C_{fil} (\text{AnSp} (\mathbb{C})^{2,(sm)pr}/ S_I), \text{ and } u_{IJ} : (G_I, F) \to r_{IJ, s}(G_J, F) \]

for \( r_{IJ, s} : I \to J, \) denoting again \( r_{IJ, s} : S_I \to S_J, \) are morphisms satisfying for \( I \to J \to K, r_{IJ, s} \circ u_{IJ} = u_{IK} \) in \( C_{fil} (\text{AnSp} (\mathbb{C})^{2,(sm)}/ S_I), \)

- the morphisms \( m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ}) \) being (see section 2.1) a family of morphisms of complexes,

\[ m = (m_I : (G_I, F) \to (H_I, F))_{I \in \mathcal{I}} \]

such that \( v_{IJ} \circ m_I = p_{IJ, s}, m_J \circ u_{IJ} \) in \( C_{fil} (\text{AnSp} (\mathbb{C})^{2,(sm)pr}/ S_I). \)

For \( s : \mathcal{I} \to \mathcal{J} \) a functor, with \( \mathcal{I}, \mathcal{J} \in \text{Cat} \), and \( f_* : T_* \to S_* \) a morphism with \( T_* \in \text{Fun}(\mathcal{J}, \text{AnSp} (\mathbb{C})) \)

\[ \text{and } S_* \in \text{Fun}(\mathcal{I}, \text{AnSp} (\mathbb{C})) \], we have by definition, the following commutative diagrams of sites

\[ \text{AnSp} (\mathbb{C})^2 / T_* \xrightarrow{\mu T_\bullet} \text{AnSp} (\mathbb{C})^{2,pr} / T_* \]

\[ \text{AnSp} (\mathbb{C})^2 / T_* \xrightarrow{\rho T_\bullet} \text{AnSp} (\mathbb{C})^{2,sm} / T_* \xrightarrow{\mu T_\bullet} \text{AnSp} (\mathbb{C})^{2,smpr} / T_* \]

\[ \text{AnSp} (\mathbb{C})^{2,pr} / S_* \xrightarrow{\rho S_\bullet} \text{AnSp} (\mathbb{C})^{2,sm} / S_* \xrightarrow{\mu S_\bullet} \text{AnSp} (\mathbb{C})^{2,smpr} / S_* \]

\[ \text{AnSp} (\mathbb{C})^{2,pr} / T_* \xrightarrow{Gr^{12}} \text{AnSp} (\mathbb{C}) / T \]

\[ \text{AnSp} (\mathbb{C})^{2,smpr} / T_* \xrightarrow{Gr^E} \text{AnSp} (\mathbb{C})^E / T_* \]

\[ \text{AnSp} (\mathbb{C})^{2,pr} / S_* \xrightarrow{Gr^{12}} \text{AnSp} (\mathbb{C}) / S_* \]

\[ \text{AnSp} (\mathbb{C})^{2,sm} / S_* \xrightarrow{Gr^{12}} \text{AnSp} (\mathbb{C})^{sm} / S_* \]
Let \( s : \mathcal{I} \to \mathcal{J} \) a functor, with \( \mathcal{I}, \mathcal{J} \in \operatorname{Cat} \), and \( f_\bullet : \mathcal{T}_\bullet \to S_{\bullet(\bullet)} \) a morphism with \( T_\bullet \in \operatorname{Fun}(\mathcal{J}, \operatorname{AnSp}(\mathbb{C})) \) and \( S_{\bullet} \in \operatorname{Fun}(\mathcal{I}, \operatorname{AnSp}(\mathbb{C})) \).

- As usual, we denote by 
  \[
  (f^*_\bullet, f_\bullet) := (P(f_\bullet)^*, P(f_\bullet)_\bullet) : C(\operatorname{AnSp}(\mathbb{C})^{2,(sm)}/S_{\bullet}) \to C(\operatorname{AnSp}(\mathbb{C})^{2,(sm)}/T_\bullet)
  \]
  \[f^*_\bullet((G_1, F), u_{1,j}) := ((f^1_1 G_1, f^1_1 F), T(f_1, r_{1,j})(-)) \circ f^1_1 u_{1,j})
  \]

- As usual, we denote by 
  \[
  (f^*_\bullet, f_\bullet) := (P(f_\bullet)^*, P(f_\bullet)_\bullet) : C(\operatorname{AnSp}(\mathbb{C})^{2,(sm)pr}/S_{\bullet}) \to C(\operatorname{AnSp}(\mathbb{C})^{2,(sm)pr}/T_\bullet)
  \]
  \[f^*_\bullet((G_1, F), u_{1,j}) := ((f^1_1 G_1, f^1_1 F), T(f_1, r_{1,j})(-)) \circ f^1_1 u_{1,j})
  \]

Let \( S \in \operatorname{AnSp}(\mathbb{C}) \). Let \( S = \bigcup_{i=1}^l S_i \) an open affinoid cover and denote by \( S_I = \bigcap_{i \in I} S_i \). Let \( i : S_I \hookrightarrow \tilde{S}_I \) closed embeddings, with \( \tilde{S}_I \in \operatorname{AnSp}(\mathbb{C}) \). For \( I \subseteq [1, \cdots l] \), denote by \( \tilde{S}_I = \Pi_{i \in I} \tilde{S}_i \). We then have closed embeddings \( i_I : S_I \hookrightarrow \tilde{S}_I \) and for \( J \subseteq I \) the following commutative diagram

\[
\begin{array}{ccc}
D_{IJ} & = & S_I \\
\downarrow{i_I} & & \downarrow{\pi_I} \\
S_J & = & \tilde{S}_J
\end{array}
\]

where \( \pi_I : \tilde{S}_J \to \tilde{S}_I \) is the projection and \( j_{IJ} : S_J \hookrightarrow \tilde{S}_I \) is the open embedding so that \( j_I \circ j_{IJ} = j_I \). This gives the diagram of analytic spaces \( (\tilde{S}_I) \in \operatorname{Fun}(\mathbb{P}(\mathbb{N}), \operatorname{AnSp}(\mathbb{C})) \) which gives the diagram of sites \( \operatorname{AnSp}(\mathbb{C})^{2,op}/(\tilde{S}_I) \in \operatorname{Fun}(\mathbb{P}(\mathbb{N})^{op}, \operatorname{AnSp}(\mathbb{C})) \) which gives the diagram of sites \( \operatorname{AnSp}(\mathbb{C})^{2,op}/(\tilde{S}_I) \in \operatorname{Fun}(\mathbb{P}(\mathbb{N})^{op}, \operatorname{Cat}) \).

- Then \( C_{fil}(\operatorname{AnSp}(\mathbb{C})^{2,(sm)}/(\tilde{S}_I)) \) is the category
  - whose objects \( (G, F) = ((G_1, F)_{I \subseteq [1, \cdots l]}, u_{IJ}) \), with \((G_1, F) \in C_{fil}(\operatorname{AnSp}(\mathbb{C})^{2,(sm)}/\tilde{S}_I) \), and \( u_{IJ} : (G_1, F) \to p_{IJ*(}G_1, F) \) for \( I \subseteq J \), are morphisms satisfying for \( I \subseteq J \subseteq K \), \( p_{IJ*} u_{JK} \circ u_{IJ} = u_{IK} \) in \( C_{fil}(\operatorname{AnSp}(\mathbb{C})^{2,(sm)}/\tilde{S}_I) \),
  - the morphisms \( m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ}) \) being (see section 2.1) a family of morphisms of complexes,
    \[
    m = (m_I : (G_1, F) \to (H_1, F))_{I \subseteq [1, \cdots l]}
    \]
    such that \( v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ} \) in \( C_{fil}(\operatorname{AnSp}(\mathbb{C})^{2,(sm)}/\tilde{S}_I) \).

- Then \( C_{fil}(\operatorname{AnSp}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_I)) \) is the category
  - whose objects \( (G, F) = ((G_1, F)_{I \subseteq [1, \cdots l]}, u_{IJ}) \), with \((G_1, F) \in C_{fil}(\operatorname{AnSp}(\mathbb{C})^{2,(sm)pr}/\tilde{S}_I) \), and \( u_{IJ} : (G_1, F) \to p_{IJ*} G_1, F) \) for \( I \subseteq J \), are morphisms satisfying for \( I \subseteq J \subseteq K \), \( p_{IJ*} u_{JK} \circ u_{IJ} = u_{IK} \) in \( C_{fil}(\operatorname{AnSp}(\mathbb{C})^{2,(sm)pr}/\tilde{S}_I) \),
the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subseteq [1 \cdots q]}$$

such that $\phi_I \circ m_I = p^*_I \circ m_I \circ u_{IJ}$ in $\text{C}_{\text{fil}}(\text{AnSp}(\mathbb{C})^{2,(\text{sm})}/\tilde{S}_I)$.

- Then $\text{C}_{\text{fil}}(\text{AnSp}(\mathbb{C})^{2,(\text{sm})}/(\tilde{S}_I)^{\text{op}})$ is the category

  - whose objects $(G, F) = ((G_I, F)_{I \subseteq [1 \cdots q]}, u_{IJ})$, with $(G_I, F) \in \text{C}_{\text{fil}}(\text{AnSp}(\mathbb{C})^{2,(\text{sm})}/\tilde{S}_I)$, and $u_{IJ} : (G_J, F) \rightarrow p^*_I (G_I, F)$ for $I \subseteq J$, are morphisms satisfying for $I \subseteq J \subseteq K$, $p^*_K \circ u_{IJ} \circ u_{JK} = u_{IK}$ in $\text{C}_{\text{fil}}(\text{AnSp}(\mathbb{C})^{2,(\text{sm})}/\tilde{S}_K)$,

- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subseteq [1 \cdots q]}$$

such that $\phi_I \circ m_I = p^*_I \circ m_I \circ u_{IJ}$ in $\text{C}_{\text{fil}}(\text{AnSp}(\mathbb{C})^{2,(\text{sm})}/\tilde{S}_I)$.

- Then $\text{C}_{\text{fil}}(\text{AnSp}(\mathbb{C})^{2,(\text{sm})}pr/(\tilde{S}_I)^{\text{op}})$ is the category

  - whose objects $(G, F) = ((G_I, F)_{I \subseteq [1 \cdots q]}, u_{IJ})$, with $(G_I, F) \in \text{C}_{\text{fil}}(\text{AnSp}(\mathbb{C})^{2,(\text{sm})}pr/\tilde{S}_I)$, and $u_{IJ} : (G_J, F) \rightarrow p^*_I (G_I, F)$ for $I \subseteq J$, are morphisms satisfying for $I \subseteq J \subseteq K$, $p^*_K \circ u_{IJ} \circ u_{JK} = u_{IK}$ in $\text{C}_{\text{fil}}(\text{AnSp}(\mathbb{C})^{2,(\text{sm})}pr/\tilde{S}_K)$,

- the morphisms $m : ((G, F), u_{IJ}) \rightarrow ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m = (m_I : (G_I, F) \rightarrow (H_I, F))_{I \subseteq [1 \cdots q]}$$

such that $\phi_I \circ m_I = p^*_I \circ m_I \circ u_{IJ}$ in $\text{C}_{\text{fil}}(\text{AnSp}(\mathbb{C})^{2,(\text{sm})}pr/\tilde{S}_I)$.

We now define the usual topology on $\text{AnSp}(\mathbb{C})^2/S$.

**Definition 30.** Let $S \in \text{AnSp}(\mathbb{C})$.

(i) Denote by $\tau$ a topology on $\text{AnSp}(\mathbb{C})$, e.g. the usual topology. The $\tau$ covers in $\text{AnSp}(\mathbb{C})^2/S$ of $(X, Z)/S$ are the families of morphisms

$$\{(c_i : (U_i, Z \times_X U_i)/S \rightarrow (X, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow X)_{i \in I} \text{ \tau cover of } X \text{ in } \text{AnSp}(\mathbb{C})\}$$

(ii) Denote by $\tau$ the usual or the etale topology on $\text{AnSp}(\mathbb{C})$. The $\tau$ covers in $\text{AnSp}(\mathbb{C})^{2, sm}/S$ of $(U, Z)/S$ are the families of morphisms

$$\{(c_i : (U_i, Z \times_U U_i)/S \rightarrow (U, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow U)_{i \in I} \text{ \tau cover of } U \text{ in } \text{AnSp}(\mathbb{C})\}$$

(iii) Denote by $\tau$ the usual or the etale topology on $\text{AnSp}(\mathbb{C})$. The $\tau$ covers in $\text{AnSp}(\mathbb{C})^{2, (sm)pr}/S$ of $(Y \times S, Z)/S$ are the families of morphisms

$$\{(c_i \times IS : (U_i \times S, Z \times_Y S U_i \times S)/S \rightarrow (Y \times S, Z)/S)_{i \in I}, \text{ with } (c_i : U_i \rightarrow Y)_{i \in I} \text{ \tau cover of } Y \text{ in } \text{AnSp}(\mathbb{C})\}$$

Let $S \in \text{AnSp}(\mathbb{C})$. Denote by $\tau$ the usual topology on $\text{AnSp}(\mathbb{C})$. In particular, denoting $a_{\tau} : \text{PSh}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S) \rightarrow \text{Shv}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ and $a_{\tau} : \text{PSh}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \rightarrow \text{Shv}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ the sheafification functors,

- a morphism $\phi : F \rightarrow G$, with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is a $\tau$ local equivalence if $a_{\tau}H^n \phi : a_{\tau}H^n F \rightarrow a_{\tau}H^n G$ is an isomorphism, a morphism $\phi : F \rightarrow G$, with $F, G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$, is a $\tau$ local equivalence if $a_{\tau}H^n \phi : a_{\tau}H^n F \rightarrow a_{\tau}H^n G$ is an isomorphism,
• \( F^* \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S) \) is \( \tau \) fibrant if for all \( (U,Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)}/S \) and all \( \tau \) covers \( (c_i : (U_i, Z \times_U U_i)/S \to (U,Z)/S)_{i \in I} \) of \( (U,Z)/S \),
\[
F^*(c_i) : F^*((U,Z)/S) \to \text{Tot}(\oplus_{\text{card}I=\bullet} F^*((U_1, Z \times_U U_1)/S))
\]
is a quasi-isomorphism of complexes of abelian groups; \( F^* \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \) is \( \tau \) fibrant if for all \( (Y \times S,Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S \) and all \( \tau \) covers \( (c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \to (Y \times S,Z)/S)_{i \in I} \) of \( (Y \times S,Z)/S \),
\[
F^*(c_i \times I_S) : F^*((Y \times S,Z)/S) \to \text{Tot}(\oplus_{\text{card}I=\bullet} F^*((U_1 \times S, Z \times_Y U_1)/S))
\]
is a quasi-isomorphism of complexes of abelian groups,

• a morphism \( \phi : (G_1,F) \to (G_2,F) \), with \( (G_1,F),(G_2,F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S) \), is a filtered \( \tau \) local equivalence if for all \( n,p \in \mathbb{Z} \),
\[
a_\tau H^n \text{Gr}_F \phi : a_\tau H^n \text{Gr}_F(G_1,F) \to a_\tau H^n \text{Gr}_F(G_2,F)
\]
is an isomorphism of sheaves on \( \text{AnSp}(\mathbb{C})^{2,(sm)}/S \); a morphism \( \phi : (G_1,F) \to (G_2,F) \), with \( (G_1,F),(G_2,F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \), is an \( \tau \) local equivalence if for all \( n,p \in \mathbb{Z} \)
\[
a_\tau H^n \text{Gr}_F \phi : a_\tau H^n \text{Gr}_F(G_1,F) \to a_\tau H^n \text{Gr}_F(G_2,F)
\]
is an isomorphism of sheaves on \( \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S \);

• a morphism \( \phi : (G_1,F) \to (G_2,F) \), with \( (G_1,F),(G_2,F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S) \), is an \( r \)-filtered \( \tau \) local equivalence if there exists \( \phi_i : (C_i,F) \to (C_{i+1},F), 0 \leq i \leq s \), with \( (C_i,F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S) \), \( C_0,F) = (G_1,F) \) and \( (C_s,F) = (G_2,F) \), such that
\[
\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : (G_1,F) \to (G_2,F)
\]
and \( \phi_i : (C_i,F) \to (C_{i+1},F) \) either a \( \tau \) local equivalence or an \( r \)-filtered homotopy equivalence, a morphism \( \phi : (G_1,F) \to (G_2,F) \), with \( (G_1,F),(G_2,F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \), is an \( r \)-filtered \( \tau \) local equivalence if there exists \( \phi_i : (C_i,F) \to (C_{i+1},F), 0 \leq i \leq s \), with \( (C_i,F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \), \( C_0,F) = (G_1,F) \) and \( (C_s,F) = (G_2,F) \), such that
\[
\phi = \phi_s \circ \cdots \circ \phi_i \circ \cdots \circ \phi_0 : (G_1,F) \to (G_2,F)
\]
and \( \phi_i : (C_i,F) \to (C_{i+1},F) \) either a \( \tau \) local equivalence or an \( r \)-filtered homotopy equivalence;

• \((F^*,F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)\) is filtered \( \tau \) fibrant for all \( (U,Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)}/S \) and all \( \tau \) covers \( (c_i : (U_i, Z \times_U U_i)/S \to (U,Z)/S)_{i \in I} \) of \( (U,Z)/S \),
\[
H^n \text{Gr}_F(F^*,F)(c_i) : H^n \text{Gr}_F(F^*,F)((U,Z)/S) \to H^n \text{Gr}_F((\text{Tot}(\oplus_{\text{card}I=\bullet} F^*,F)((U_1, Z \times_U U_1)/S)))
\]
is an isomorphism of abelian groups for all \( n,p \in \mathbb{Z} ; (F^*,F) \in C_{fil}(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S) \) is filtered \( \tau \) fibrant for all \( (Y \times S,Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)pr}/S \) and all \( \tau \) covers \( (c_i \times I_S : (U_i \times S, Z \times_{Y \times S} U_i \times S)/S \to (Y \times S,Z)/S)_{i \in I} \) of \( (Y \times S,Z)/S \),
\[
H^n \text{Gr}_F(F^*,F)(c_i \times I_S) : H^n \text{Gr}_F(F^*,F)((Y \times S,Z)/S) \to H^n \text{Gr}_F((\text{Tot}(\oplus_{\text{card}I=\bullet} F^*,F)((U_1 \times S, Z \times_Y U_1)/S)))
\]
is an isomorphism of abelian groups for all \( n,p \in \mathbb{Z} \).

Let \( S_\bullet \in \text{Fun}(I, \text{AnSp}(\mathbb{C})) \) with \( I \in \text{Cat} \).
A morphism $m : ((G_t, F), u_{t}) \rightarrow ((H_t, F), v_{t})$ in $C_{filt}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S_\bullet)$ is an $r$-filtered Zariski, resp. etale local, equivalence, if there exists $\phi_t : ((C_{t}, F), u_{t}) \rightarrow ((C_{t+1}, F), u_{t+1})$, $0 \leq t \leq s$, with $((C_t, F), u_{t}) \in C_{filt}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S_\bullet)$ and $((C_{t+1}, F), u_{t+1}) = ((H_t, F), v_{t})$, such that

$$\phi = \phi_t \circ \cdots \circ \phi_1 : ((G_t, F), u_t) \rightarrow ((H_t, F), v_t)$$

and $\phi_t : ((C_t, F), u_t) \rightarrow ((C_{t+1}, F), u_{t+1})$ either a filtered Zariski, resp. etale, local equivalence or an $r$-filtered homotopy equivalence.

A morphism $m : ((G_t, F), u_{t}) \rightarrow ((H_t, F), v_{t})$ in $C_{filt}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S_\bullet)$ is an $r$-filtered Zariski, resp. etale local, equivalence, if there exists $\phi_t : ((C_{t}, F), u_{t}) \rightarrow ((C_{t+1}, F), u_{t+1})$, $0 \leq t \leq s$, with $((C_t, F), u_{t}) \in C_{filt}(\text{AnSp}(\mathbb{C})^{2,(sm)}/S_\bullet)$ and $((C_{t+1}, F), u_{t+1}) = ((H_t, F), v_{t})$, such that

$$\phi = \phi_t \circ \cdots \circ \phi_1 : ((G_t, F), u_t) \rightarrow ((H_t, F), v_t)$$

and $\phi_t : ((C_t, F), u_t) \rightarrow ((C_{t+1}, F), u_{t+1})$ either a filtered Zariski, resp. etale, local equivalence or an $r$-filtered homotopy equivalence.

Will now define the $\mathbb{D}^1$ local property on $\text{AnSp}(\mathbb{C})^{2}/S$. Let $S \in \text{AnSp}(\mathbb{C})$. Denote for short $\text{AnSp}(\mathbb{C})^{2,(sm)}/S$ either the category $\text{AnSp}(\mathbb{C})^{2}/S$ or the category $\text{AnSp}(\mathbb{C})^{2,sm}/S$. Denote by

$$p_a : \text{AnSp}(\mathbb{C})^{2,(sm)}/S \rightarrow \text{AnSp}(\mathbb{C})^{2,(sm)}/S,$$

$$(X, Z)/S = ((X, Z), h) \mapsto (X \times \mathbb{D}^1, Z \times \mathbb{D}^1)/S = ((X \times \mathbb{D}^1, Z \times \mathbb{D}^1, h \circ p_X),$$

$$(g : (X, Z)/S \rightarrow (X', Z')/S) \mapsto ((g \circ I_{01}) : (X \times \mathbb{D}^1, Z \times \mathbb{D}^1)/S \rightarrow (X' \times \mathbb{D}^1, Z' \times \mathbb{D}^1)/S)$$

the projection functor and again by $p_a : \text{AnSp}(\mathbb{C})^{2,(sm)}/S \rightarrow \text{AnSp}(\mathbb{C})^{2,(sm)}/S$ the corresponding morphism of site. Denote for short $\text{AnSp}(\mathbb{C})^{2,(sm)}/S$ either the category $\text{AnSp}(\mathbb{C})^{2,pr}/S$ or the category $\text{AnSp}(\mathbb{C})^{2,sm,pr}/S$. Denote by

$$p_a : \text{AnSp}(\mathbb{C})^{2,(sm)}/S \rightarrow \text{AnSp}(\mathbb{C})^{2,(sm)}/S,$$

$$(Y \times S, Z)/S = ((Y \times S, Z), p_S) \mapsto (Y \times S \times \mathbb{D}^1, Z \times \mathbb{D}^1)/S = ((Y \times S \times \mathbb{D}^1, Z \times \mathbb{D}^1, p_S \circ p_Y \times S),$$

$$(g : (Y \times S, Z)/S \rightarrow (Y' \times S, Z')/S) \mapsto ((g \circ I_{01}) : (Y \times S \times \mathbb{D}^1, Z \times \mathbb{D}^1)/S \rightarrow (Y' \times S \times \mathbb{D}^1, Z' \times \mathbb{D}^1)/S)$$

the projection functor and again by $p_a : \text{AnSp}(\mathbb{C})^{2,(sm)}/S \rightarrow \text{AnSp}(\mathbb{C})^{2,(sm)}/S$ the corresponding morphism of site.

**Definition 31.** Let $S \in \text{AnSp}(\mathbb{C})$.

(i0) A complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is said to be $\mathbb{D}^1$ homotopic if $\text{ad}(p_a^*, p_a)(F) : F \rightarrow F$ is an homotopy equivalence.

(i) A complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$, is said to be $\mathbb{D}^1$ invariant if for all $(X, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)}/S$

$$F(p_X) : F((X, Z)/S) \rightarrow F((X \times \mathbb{D}^1, (Z \times \mathbb{D}^1))/S)$$

is a quasi-isomorphism, where $p_X : (X \times \mathbb{D}^1, (Z \times \mathbb{D}^1)) \rightarrow (X, Z)$ is the projection.

(i0)' Similarly, a complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S), \text{is said to be}\ \mathbb{D}^1 \text{ homotopic if } \text{ad}(p_a^*, p_a)(F) : F \rightarrow F \text{ is an homotopy equivalence.}$

(i)' Similarly, a complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is said to be $\mathbb{D}^1$ invariant if for all $(Y \times S, Z)/S \in \text{AnSp}(\mathbb{C})^{2,(sm)}/S$

$$F(p_Y \times S) : F((Y \times S, Z)/S) \rightarrow F((Y \times S \times \mathbb{D}^1, (Z \times \mathbb{D}^1))/S)$$

is a quasi-isomorphism.
(ii) A complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is said to be $\mathbb{D}^1$ local for the $\tau$ topology induced on $\text{AnSp}(\mathbb{C})^{2}/S$, if for an (hence every) $\tau$ local equivalence $k : F \to G$ with $k$ injective and $G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ $\tau$ fibrant, (e.g. $k : F \to E_r(F)$), $G$ is $\mathbb{D}^1$ invariant.

(ii) A complex $F \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be $\mathbb{D}^1$ local for the $\tau$ topology induced on $\text{AnSp}(\mathbb{C})^{2}/S$, if for an (hence every) $\tau$ local equivalence $k : F \to G$ with $k$ injective and $G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ $\tau$ fibrant, e.g. $k : F \to E_r(F)$, $G$ is $\mathbb{D}^1$ invariant.

(iii) A morphism $m : F \to G$ with $F,G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is said to be an ($\mathbb{D}^1$, usu) local equivalence if for all $H \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ which is $\mathbb{D}^1$ local for the usual topology

$$\text{Hom}(L(m), E_{usu}(H)) : \text{Hom}(L(G), E_{usu}(H)) \to \text{Hom}(L(F), E_{usu}(H))$$

is a quasi-isomorphism.

(iii) A morphism $m : F \to G$ with $F,G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is said to be an ($\mathbb{D}^1$, usu) local equivalence if for all $H \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ which is $\mathbb{D}^1$ local for the usual topology

$$\text{Hom}(L(m), E_{usu}(H)) : \text{Hom}(L(G), E_{usu}(H)) \to \text{Hom}(L(F), E_{usu}(H))$$

is a quasi-isomorphism.

**Proposition 33.** (i) A morphism $m : F \to G$ with $F,G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ is an ($\mathbb{D}^1$, usu) local equivalence if and only if there exists

$$\{(X_{1,\alpha}, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \ldots, \{(X_{r,\alpha}, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \text{AnSp}(\mathbb{C})^{2,(sm)}/S$$

such that we have in $\text{Ho}_{et}(C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S))$

$$\text{Cone}(m) \simeq \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(\text{Z}(\text{Z}(X_{1,\alpha} \times \mathbb{D}^1, Z_{1,\alpha} \times \mathbb{D}^1)/S) \to \text{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)))$$

$$\to \cdots \to \oplus_{\alpha \in \Lambda_r} \text{Cone}(\text{Z}(\text{Z}(X_{r,\alpha} \times \mathbb{D}^1, Z_{r,\alpha} \times \mathbb{D}^1)/S) \to \text{Z}((X_{r,\alpha}, Z_{r,\alpha})/S)))$$

(ii) A morphism $m : F \to G$ with $F,G \in C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ is an ($\mathbb{D}^1$, usu) local equivalence if and only if there exists

$$\{(Y_{1,\alpha} \times S, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \ldots, \{(Y_{r,\alpha} \times S, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \text{AnSp}(\mathbb{C})^{2,(sm)}/S$$

such that we have in $\text{Ho}_{et}(C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S))$

$$\text{Cone}(m) \simeq \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(\text{Z}(\text{Z}(Y_{1,\alpha} \times S \times \mathbb{D}^1, Z_{1,\alpha} \times \mathbb{D}^1)/S) \to \text{Z}((Y_{1,\alpha} \times S, Z_{1,\alpha})/S)))$$

$$\to \cdots \to \oplus_{\alpha \in \Lambda_r} \text{Cone}(\text{Z}(\text{Z}(Y_{r,\alpha} \times S \times \mathbb{D}^1, Z_{r,\alpha} \times \mathbb{D}^1)/S) \to \text{Z}((Y_{r,\alpha} \times S, Z_{r,\alpha})/S)))$$

Proof. Standard. □

**Definition-Proposition 9.** Let $S \in \text{AnSp}(\mathbb{C})$.

(i) With the weak equivalence the ($\mathbb{D}^1$, et) equivalence and the fibration the epimorphism with $\mathbb{D}^1$ local and etale fibrant kernels gives a model structure on $C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$ : the left bousfield localization of the projective model structure of $C(\text{AnSp}(\mathbb{C})^{2,(sm)}/S)$. We call it the projective ($\mathbb{D}^1$, et) model structure.

(ii) With the weak equivalence the ($\mathbb{D}^1$, et) equivalence and the fibration the epimorphism with $\mathbb{D}^1$ local and etale fibrant kernels gives a model structure on $C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$ : the left bousfield localization of the projective model structure of $C(\text{AnSp}(\mathbb{C})^{2,(sm)pr}/S)$. We call it the projective ($\mathbb{D}^1$, et) model structure.

Proof. Similar to the proof of proposition 5. □
We have, similarly to the case of single varieties the following:

**Proposition 34.** Let $g : T \to S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$.

1. The adjunction $(g^*, g_*) : C(\text{AnSp}(\mathbb{C})^2 / T) \leftrightarrow C(\text{AnSp}(\mathbb{C})^2 / S)$ is a Quillen adjunction for the $(\mathbb{D}^1, \text{usu})$ model structure.

2. The functor $g^* : C(\text{AnSp}(\mathbb{C})^2 / S) \to C(\text{AnSp}(\mathbb{C})^2 / T)$ sends quasi-isomorphism to quasi-isomorphism and equivalence usu local to equivalence usu local, sends $(\mathbb{D}^1, \text{usu})$ local equivalence to $(\mathbb{D}^1, \text{usu})$ local equivalence.

3. The adjunction $(g^*, g_*) : C(\text{AnSp}(\mathbb{C})^2 / S) \leftrightarrow C(\text{AnSp}(\mathbb{C})^2 / T)$ is a Quillen adjunction for the $(\mathbb{D}^1, \text{usu})$ model structure.

4. The functor $g^* : C(\text{AnSp}(\mathbb{C})^2 / S) \to C(\text{AnSp}(\mathbb{C})^2 / T)$ sends quasi-isomorphism to quasi-isomorphism and equivalence usu local to equivalence usu local, sends $(\mathbb{D}^1, \text{usu})$ local equivalence to $(\mathbb{D}^1, \text{usu})$ local equivalence.

*Proof.* Similar to the proof of proposition 23. □

**Proposition 35.** Let $S \in \text{Var}(\mathbb{C})$.

1. The adjunction $(\rho_{S^*}^*, \rho_{S*}) : C(\text{AnSp}(\mathbb{C})^2 / S) \leftrightarrow C(\text{AnSp}(\mathbb{C})^2 / S)$ is a Quillen adjunction for the $(\mathbb{A}^1, \text{et})$ projective model structure.

2. The functor $\rho_{S*} : C(\text{AnSp}(\mathbb{C})^2 / S) \to C(\text{AnSp}(\mathbb{C})^2 / S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends $(\mathbb{D}^1, \text{usu})$ local equivalence to $(\mathbb{D}^1, \text{usu})$ local equivalence.

3. The adjunction $(\rho_{S*}^*, \rho_{S*}) : C(\text{AnSp}(\mathbb{C})^2 / S) \leftrightarrow C(\text{AnSp}(\mathbb{C})^2 / S)$ is a Quillen adjunction for the $(\mathbb{A}^1, \text{et})$ projective model structure.

4. The functor $\rho_{S*} : C(\text{AnSp}(\mathbb{C})^2 / S) \to C(\text{AnSp}(\mathbb{C})^2 / S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence usu local to equivalence usu local, sends $(\mathbb{D}^1, \text{usu})$ local equivalence to $(\mathbb{D}^1, \text{usu})$ local equivalence.

*Proof.* Similar to the proof of proposition 19. □

We also have

**Proposition 36.** Let $S \in \text{AnSp}(\mathbb{C})$.

1. The adjunction $(\text{Gr}_S^{12}, \text{Gr}_S^{12}) : C(\text{AnSp}(\mathbb{C}) / S) \leftrightarrow C(\text{AnSp}(\mathbb{C})^2 / S)$ is a Quillen adjunction for the $(\mathbb{D}^1, \text{usu})$ projective model structure.

2. The adjunction $(\text{Gr}_S^{12}, \text{Gr}_S^{12}) : C(\text{AnSp}(\mathbb{C})^2 / S) \leftrightarrow C(\text{AnSp}(\mathbb{C})^2 / S)$ is a Quillen adjunction for the $(\mathbb{D}^1, \text{usu})$ projective model structure.

*Proof.* Immediate from definition. □

Let $S \in \text{AnSp}(\mathbb{C})$. Let $S = \bigcup_{i=1}^d S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow S_I$ closed embeddings, with $S_i \in \text{AnSp}(\mathbb{C})$.

- For $(G_I, K_{I,J}) \in C(\text{AnSp}(\mathbb{C})^{2,(\text{sm})}/(\tilde{S}_I)^{pr})$ and $(H_I,T_{I,J}) \in C(\text{AnSp}(\mathbb{C})^{2,(\text{sm})}/(\tilde{S}_I))$, we denote

  \[
  \text{Hom}((G_I,K_{I,J}),(H_I,T_{I,J})) := (\text{Hom}(G_I,H_I), u_{I,J}((G_I,K_{I,J}),(H_I,T_{I,J}))) \in C(\text{AnSp}(\mathbb{C})^{2,(\text{sm})}/(\tilde{S}_I))
  \]
with
\[ u_{IJ}((G_1, K_{IJ}),(H_1, T_{IJ})) : \mathcal{H}om(G_1, H_1) \xrightarrow{\text{adj}(p_{IJ}^*, p_{IJ}^*)^-} p_{IJ}^* \mathcal{H}om(G_1, H_1) \xrightarrow{T(p_{IJ}^*, \mathcal{H}om)(-, -)} p_{IJ}^* \mathcal{H}om(p_{IJ}^* G_1, p_{IJ}^* H_1) \xrightarrow{\mathcal{H}om(K_{IJ}, H_1)} p_{IJ}^* \mathcal{H}om(G_1, H_1) \xrightarrow{\text{adj}(p_{IJ}^*, p_{IJ}^*)^-} p_{IJ}^* \mathcal{H}om(G_1, H_1) \xrightarrow{T(p_{IJ}^*, \mathcal{H}om)(-, -)} p_{IJ}^* \mathcal{H}om(p_{IJ}^* G_1, p_{IJ}^* H_1) \xrightarrow{\mathcal{H}om(K_{IJ}, H_1)} p_{IJ}^* \mathcal{H}om(G_1, H_1). \]

This gives in particular the functor
\[ C(\text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)) \rightarrow C(\text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)^{op}), (H_1, T_{IJ}) \mapsto (H_1, T_{IJ}). \]

- For \((G_1, K_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)^{op})\) and \((H_1, T_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)^{op})\), we denote \(\mathcal{H}om((G_1, K_{IJ}), (H_1, T_{IJ})) := (\mathcal{H}om(G_1, H_1), u_{IJ}((G_1, K_{IJ}), (H_1, T_{IJ}))) \in C(\text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)^{op})\) with
\[ u_{IJ}((G_1, K_{IJ}),(H_1, T_{IJ})) : \mathcal{H}om(G_1, H_1) \xrightarrow{\text{adj}(p_{IJ}^*, p_{IJ}^*)^-} p_{IJ}^* \mathcal{H}om(G_1, H_1) \xrightarrow{T(p_{IJ}^*, \mathcal{H}om)(-, -)} p_{IJ}^* \mathcal{H}om(p_{IJ}^* G_1, p_{IJ}^* H_1) \xrightarrow{\mathcal{H}om(K_{IJ}, H_1)} p_{IJ}^* \mathcal{H}om(G_1, H_1) \xrightarrow{\text{adj}(p_{IJ}^*, p_{IJ}^*)^-} p_{IJ}^* \mathcal{H}om(G_1, H_1) \xrightarrow{T(p_{IJ}^*, \mathcal{H}om)(-, -)} p_{IJ}^* \mathcal{H}om(p_{IJ}^* G_1, p_{IJ}^* H_1) \xrightarrow{\mathcal{H}om(K_{IJ}, H_1)} p_{IJ}^* \mathcal{H}om(G_1, H_1). \]

This gives in particular the functor
\[ C(\text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)) \rightarrow C(\text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)^{op}), (H_1, T_{IJ}) \mapsto (H_1, T_{IJ}). \]

Let \(S \in \text{AnSp}(\mathbb{C})\). Let \(S = \bigcup_{i=1}^d S_i\) an open affinoid cover and denote by \(S_I = \cap_{i \in I} S_i\). Let \(i_i : S_i \hookrightarrow \tilde{S}_i\) closed embeddings, with \(\tilde{S}_i \in \text{AnSp}(\mathbb{C})\). We have the functors
\[ p_a : \text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I) \rightarrow \text{Var}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I) \]
\[ ((X, Z)/\tilde{S}_I, u_{IJ}) \mapsto (\mathcal{F}(X \times \mathbb{D}^1, Z \times \mathbb{D}^1)/\tilde{S}_I, u_{IJ} \times I) \]
the projection functor and again by \(p_a : \text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I) \rightarrow \text{Var}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)\) the corresponding morphism of site, and
\[ p_a : \text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I) \rightarrow \text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I), \]
\[ ((Y \times \tilde{S}_I, Z)/\tilde{S}_I, u_{IJ}) \mapsto ((Y \times \tilde{S}_I \times \mathbb{D}^1, Z \times \mathbb{D}^1)/\tilde{S}_I, u_{IJ} \times I), \]
\[ (g : ((X, Z)/\tilde{S}_I, u_{IJ}) \rightarrow ((X', Z')/\tilde{S}_I, u_{IJ})) \rightarrow ((g \times I_{\mathbb{D}^1}) : ((X \times \mathbb{D}^1, Z \times \mathbb{D}^1)/\tilde{S}_I, u_{IJ} \times I) \rightarrow ((X' \times \mathbb{D}^1, Z' \times \mathbb{D}^1)/\tilde{S}_I, u_{IJ} \times I), \]
the projection functor and again by \(p_a : \text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I) \rightarrow \text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)^{op}\) and \(p_a : \text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)^{op} \rightarrow \text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)^{op}\) the corresponding morphism of site. These functors induces also morphism of sites \(p_a : \text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)^{op} \rightarrow \text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I)^{op}\)

**Definition 32.** Let \(S \in \text{AnSp}(\mathbb{C})\). Let \(S = \bigcup_{i=1}^d S_i\) an open affinoid cover and denote by \(S_I = \cap_{i \in I} S_i\). Let \(i_i : S_i \hookrightarrow \tilde{S}_i\) closed embeddings, with \(\tilde{S}_i \in \text{AnSp}(\mathbb{C})\).

(i) A complex \((F_1, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I))\) is said to be \(\mathbb{D}^1\) homotopic if \(\text{adj}(p_a^*, p_a^*)((F_1, u_{IJ})) : (F_1, u_{IJ}) \rightarrow p_a^* p_a^*(F_1, u_{IJ})\) is an homotopy equivalence.

(ii) A complex \((F_1, u_{IJ}) \in C(\text{AnSp}(\mathbb{C})^{2, (sm)}/(\tilde{S}_I))\) is said to be \(\mathbb{D}^1\) homotopic if \(\text{adj}(p_a^*, p_a^*)((F_1, u_{IJ})) : (F_1, u_{IJ}) \rightarrow p_a^* p_a^*(F_1, u_{IJ})\) is an homotopy equivalence.
(i) A complex $(F_I, u_{IJ}) \in C(\text{AnSp}(C)^{2,\text{sm}})/(S_I)$ is said to be $\mathbb{D}^1$ invariant if for all $((X_I, Z_I)/\tilde{S}_I, s_{IJ}) \in \text{AnSp}(C)^{2,\text{sm}})/S_I$ 

$$(F_I(p_{X_I})) : (F_I((X_I, Z_I)/\tilde{S}_I), F_J(s_{IJ}) \circ u_{IJ}(-) \rightarrow (F_I((X_I \times \mathbb{D}^1, (Z_I \times \mathbb{D}^1))/\tilde{S}_I), F_J(s_{IJ} \times 1) \circ u_{IJ}(-))$$

is a quasi-isomorphism, where $p_{X_I} : (X_I \times \mathbb{D}^1, (Z_I \times \mathbb{D}^1)) \rightarrow (X_I, Z_I)$ are the projection, and $s_{IJ} : (X_I \times \tilde{S}_J, Z_I)/\tilde{S}_J \rightarrow (X_J, Z_J)\tilde{S}_J$.

(ii) Let $\tau$ be a topology on $\text{AnSp}(C)$. A complex $F = (F_I, u_{IJ}) \in C(\text{AnSp}(C)^{2,\text{sm}})/S_I$ is said to be $\mathbb{D}^1$ local for the $\tau$ topology induced on $\text{AnSp}(C)^{2,\text{sm}})/S_I$, if for any (hence every) $\tau$ local equivalence $k : F \rightarrow G$ with $k$ injective and $G = (G_I, v_{IJ}) \in C(\text{AnSp}(C)^{2,\text{sm}})/S_I$ $\tau$ fibrant, e.g. $k : (F_I, u_{IJ}) \rightarrow (E_{\tau}(F_I), E_{\tau}(u_{IJ}))$, $G$ is $\mathbb{D}^1$ invariant.

(iii) A morphism $\mathfrak{m} = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(C)^{2,\text{sm}})/S_I$ is said to an $(\mathbb{D}^1, \text{usu})$ local equivalence if for all $H = (H_I, w_{IJ}) \in C(\text{AnSp}(C)^{2,\text{sm}})/S_I$ which is $\mathbb{D}^1$ local for the usual topology 

$$(\text{Hom}(L(m_I), E_{\text{ct}}(H_I))) : \text{Hom}(L(G_I, v_{IJ}), E_{\text{ct}}(H_I, w_{IJ})) \rightarrow \text{Hom}(L(F_I, u_{IJ}), E_{\text{ct}}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $\mathfrak{m} = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(C)^{2,\text{sm}})/S_I$ is an $(\mathbb{D}^1, \text{usu})$ local equivalence, then all the $m_I : F_I \rightarrow G_I$ are $(\mathbb{D}^1, \text{usu})$ local equivalence.

(iv) A morphism $\mathfrak{m} = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(C)^{2,\text{sm}})/S_I$ is said to an $(\mathbb{D}^1, \text{usu})$ local equivalence if for all $H = (H_I, w_{IJ}) \in C(\text{AnSp}(C)^{2,\text{sm}})/S_I$ which is $\mathbb{D}^1$ local for the usual topology 

$$(\text{Hom}(L(m_I), E_{\text{ct}}(H_I))) : \text{Hom}(L(G_I, v_{IJ}), E_{\text{ct}}(H_I, w_{IJ})) \rightarrow \text{Hom}(L(F_I, u_{IJ}), E_{\text{ct}}(H_I, w_{IJ}))$$

is a quasi-isomorphism. Obviously, if a morphism $\mathfrak{m} = (m_I) : (F_I, u_{IJ}) \rightarrow (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(C)^{2,\text{sm}})/S_I$ is an $(\mathbb{D}^1, \text{usu})$ local equivalence, then all the $m_I : F_I \rightarrow G_I$ are $(\mathbb{D}^1, \text{usu})$ local equivalence.
(iv) A morphism \( m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ}) \) with \((F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(C)_{2,(sm)pr}/(\tilde{S}_I)^{op}) \) is said to an \((\mathbb{D}^1, \text{usu})\) local equivalence if for all \((H_I, w_{IJ}) \in C(\text{AnSp}(C)_{2,(sm)pr}/(\tilde{S}_I)^{op}) \) which is \(\mathbb{D}^1\) local for the usual topology

\[
\text{Hom}(L(m_I), E_{ct}(H_I)) : \text{Hom}(L(G_I, v_{IJ}), E_{ct}(H_I, w_{IJ})) \to \text{Hom}(L(F_I, u_{IJ}), E_{ct}(H_I, w_{IJ}))
\]

is a quasi-isomorphism. Obviously, if a morphism \( m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ}) \) with \((F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\text{AnSp}(C)_{2,(sm)pr}/(\tilde{S}_I)^{op}) \) is an \((\mathbb{D}^1, \text{usu})\) local equivalence, then all the \(m_I : F_I \to G_I\) are \((\mathbb{D}^1, \text{usu})\) local equivalence.

**Proposition 37.** Let \( S \in \text{AnSp}(C) \). Let \( S = \bigcup_{i=1}^l S_i \) an open affine cover and denote by \( S_i = \cap_{j \in I} S_i \). Let \( i : S_i \hookrightarrow \tilde{S}_i \) closed embeddings, with \( \tilde{S}_i \in \text{AnSp}(C) \).

(i) A morphism \( m : F \to G \) with \( F, G \in C(\text{AnSp}(C)_{2,(sm)pr}/(\tilde{S}_I)^{op}) \) is an \((\mathbb{D}^1, \text{usu})\) local equivalence if and only if there exists

\[
\{(X_{1, \alpha, I}, Z_{1, \alpha, I})/\tilde{S}_I, u^1_{IJ}, \alpha \in \Lambda_1 \}, \ldots, \{(X_{r, \alpha, I}, Z_{r, \alpha, I})/\tilde{S}_I, u^r_{IJ}, \alpha \in \Lambda_r \} \subset \text{AnSp}(C)_{2,(sm)pr}/(\tilde{S}_I)^{op}
\]

with

\[
u^1_{IJ} : (X_{1, \alpha, I}, Z_{1, \alpha, I})/\tilde{S}_I \to (X_{1, \alpha, I} \times \tilde{S}_{J \setminus I}, Z_{1, \alpha, I} \times \tilde{S}_{J \setminus I})/\tilde{S}_J\]

such that we have in \( \text{Ho}_{ct}(C(\text{AnSp}(C)_{2,(sm)pr}/(\tilde{S}_I)^{op})) \)

\[
\text{Cone}(m) \sim \text{Cone} \left( \bigoplus_{\alpha \in \Lambda_1} \text{Cone}((\mathbb{Z}(X_{1, \alpha, I} \times \mathbb{D}^1, Z_{1, \alpha, I} \times \mathbb{D}^1)/\tilde{S}_I, \mathbb{Z}(u^1_{IJ} \times I)) \to (\mathbb{Z}(X_{1, \alpha, I} \times \tilde{S}_I, Z_{1, \alpha, I} \times \tilde{S}_I), \mathbb{Z}(u^1_{IJ}))\right) \to \cdots
\]

\[
\bigoplus_{\alpha \in \Lambda_r} \text{Cone}((\mathbb{Z}(X_{r, \alpha, I} \times \mathbb{D}^1, Z_{r, \alpha, I} \times \mathbb{D}^1)/\tilde{S}_I, \mathbb{Z}(u^r_{IJ} \times I)) \to (\mathbb{Z}(X_{r, \alpha, I} \times \tilde{S}_I, Z_{r, \alpha, I} \times \tilde{S}_I), \mathbb{Z}(u^r_{IJ}))\right)
\]

(ii) A similar statement holds for \((\mathbb{D}^1, \text{usu})\) local equivalence \( m : F \to G \) with \( F, G \in C(\text{AnSp}(C)_{2,(sm)pr}/(\tilde{S}_I)^{op}) \).

Proof. Standard. See Ayoub's thesis for example.

**Definition 33.** (i) A filtered complex \((G, F) \in C_{fil}(\text{AnSp}(C)_{2,(sm)}/S)\) is said to be \(r\)-filtered \(\mathbb{D}^1\) homotopic if \(\text{ad}(p^*_a, p^*_a)(G, F) : (G, F) \to p^*_a p^*_a(G, F)\) is an \(r\)-filtered homotopy equivalence.
(i) A filtered complex $((G_1, F), u_{11}) \in C_{fil}(AnSp(\mathbb{C})^{2, (sm)}/(\mathcal{S}_1))$ is said to be $r$-filtered $\mathbb{D}^1$ homotopic if $ad(p_a^*, p_a)((G_1, F), u_{11}) : ((G_1, F), u_{11}) \to p_a p_a^*((G_1, F), u_{11})$ is an $r$-filtered homotopy equivalence.

(ii) A filtered complex $(G, F) \in C_{fil}(AnSp(\mathbb{C})^{2, (sm)pr}/S)$ is said to be $r$-filtered $\mathbb{D}^1$ homotopic if $ad(p_a^*, p_a)((G, F)) : (G, F) \to p_a p_a^*((G, F))$ is an $r$-filtered homotopy equivalence.

(iii) A filtered complex $((G_1, F), u_{11}) \in C_{fil}(AnSp(\mathbb{C})^{2, (sm)pr}/(\mathcal{S}_1))$ is said to be $r$-filtered $\mathbb{D}^1$ homotopic if $ad(p_a^*, p_a)((G_1, F), u_{11}) : ((G_1, F), u_{11}) \to p_a p_a^*((G_1, F), u_{11})$ is an $r$-filtered homotopy equivalence.

We have the following canonical functor:

**Definition 34.** (i) For $S \in AnSp(\mathbb{C})$, we have the functor

$$(-)^\Gamma : C(AnSp(\mathbb{C})^{sm}/S) \to C(AnSp(\mathbb{C})^{2, sm}/S),$$

$$F \mapsto F^\Gamma : (((U, Z)/S) = ((U, Z), h) \mapsto F^\Gamma((U, Z)/S) := (\Gamma^\vee_Z h^* LF)(U/U),$$

$$(g : ((U, Z), h') \to ((U, Z), h)) \mapsto (F^\Gamma(g) : (\Gamma^\vee_Z h^* LF)(U/U) \xrightarrow{i(f_{(U, Z), h'}^*(\Gamma^\vee_Z h^* LF)(U/U))} (g^*(\Gamma^\vee_Z h^* LF))(U'/U'))$$

where $i(f_{(U, Z), h'}^*(\Gamma^\vee_Z h^* LF)(U/U))$ is the canonical arrow of the inductive limit. Similarly, we have, for $S \in AnSp(\mathbb{C})$, the functor

$$(-)^\Gamma : C(AnSp(\mathbb{C})/S) \to C(AnSp(\mathbb{C})^2/S),$$

$$F \mapsto F^\Gamma : (((X, Z)/S) = ((X, Z), h) \mapsto F^\Gamma((X, Z)/S) := (\Gamma^\vee_Z h^* F)(X/X),$$

$$(g : ((X, Z), h') \to ((X, Z), h)) \mapsto (F^\Gamma(g) : (\Gamma^\vee_Z h^* F)(X/X) \to (\Gamma^\vee_Z h^* F)(X'/X'))$$

Note that for $S \in AnSp(\mathbb{C})$, $I(S/S) : \mathbb{Z}((S, S)/S) \to \mathbb{Z}(S/S)^\Gamma$ given by

$$I(S/S)((U, Z), h) : \mathbb{Z}((S, S)/S)(((U, Z), h)) \xrightarrow{\gamma^\vee(S(U/Z)(U/U))} \mathbb{Z}(S/S)^\Gamma((U, Z), h) := (\Gamma^\vee_Z h^* F(U/U))$$

is an isomorphism.

(ii) Let $f : T \to S$ a morphism with $T, S \in AnSp(\mathbb{C})$. For $F \in C(AnSp(\mathbb{C})^{sm}/S)$, we have the canonical morphism in $C(AnSp(\mathbb{C})^{2, sm}/T)$

$$T(f, \Gamma)(F) := (f^*(F^\Gamma)) \to (f^* F)^\Gamma,$$

$$T(f, \Gamma)(F)((U', Z'), h') := \lim_{(U', Z'), h') \xrightarrow{(U, Z), h)} (\Gamma^\vee_Z h^* LF)(U/U)$$

$$(f^*(F^\Gamma)((U', Z'), h') := \lim_{(U', Z'), h') \xrightarrow{(U, Z), h)} (\Gamma^\vee_Z h^* LF)(U/U)$$

$$(\Gamma^\vee_Z h^* F(U'/U')) \xrightarrow{(f^* F)^\Gamma((U', Z'), h')}$$

where $f_U : U_T \times_S T \to U$ and $h_T : U_T : U \times_S T \to T$ are the base change maps, the equality following from the fact that $h \circ f_U \circ l = f \circ h_T \circ l = f \circ h'$. For $F \in C(AnSp(\mathbb{C})/S)$, we have similarly the canonical morphism in $C(AnSp(\mathbb{C})^{2}/T)$

$$T(f, \Gamma)(F) : f^*(F^\Gamma) \to (f^* F)^\Gamma.$$
(iii) Let $h : U \to S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$. We have, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/U)$, the canonical morphism in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$

$$T_2(h, \Gamma)(F)((U', Z'), h') : h_2(F^\Gamma)((U', Z'), h') := \lim_{((U', Z'), h') \to ((U, U), h)} (\Gamma_2^\vee, l^*LF)((U'/U')$$

$$\xrightarrow{(\Gamma_2^\vee, l^*h^*h_2LF)((U'/U'))} (\Gamma_2^\vee, l^*h^*h_2LF)((U'/U')) := (h_2LF)^\Gamma((U', Z')/h')$$

(iv) Let $i : Z_0 \hookrightarrow S$ a closed embedding with $Z_0, S \in \text{AnSp}(\mathbb{C})$. We have the canonical morphism in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$

$$T_2(i, \Gamma)(Z(Z_0/Z_0)) : i_*((Z(Z_0/Z_0))^\Gamma \to (i_*Z(Z/Z))^\Gamma,$$

$$T_2(i, \Gamma)(Z(Z_0/Z_0))((U, Z), h) : i_*((Z(Z_0/Z_0))^\Gamma(U, Z), h) := (\Gamma_2^\vee Z \times_S Z_0 Z(Z_0/Z_0))(U \times_S Z 0)$$

$$\xrightarrow{T(i_*\gamma_0)(Z(Z_0/Z_0))((U \times_S Z 0))} (\Gamma_2^\vee i_*Z(Z_0/Z_0))(U \times_S Z 0) := (i_*Z(Z/Z))^\Gamma((U, Z), h)$$

**Definition 35.** Let $S \in \text{AnSp}(\mathbb{C})$. We have for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\text{AnSp}(\mathbb{C})^{sm}/S)$

$$\text{Gr}(F) : \text{Gr}_{S*} \mu_{S*} F^\Gamma \to F,$$

$$\text{Gr}(F)(U/S) : \Gamma_2^\vee p^*F(U \times_S U \times_S) \xrightarrow{ad(I^\vee \gamma_0)(+F)(U \times_S U \times_S)} h^*F(U/U) = F(U/S)$$

where $h : U \to S$ is a smooth morphism with $U \in \text{AnSp}(\mathbb{C})$ and $h : U \xrightarrow{1} U \times_S S \xrightarrow{p} S$ is the graph factorization with $l$ the graph embedding and $p$ the projection.

**Proposition 38.** Let $S \in \text{AnSp}(\mathbb{C})$.

(i) Then,

- if $m : F \to G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is a quasi-isomorphism, $m^\Gamma : F^\Gamma \to G^\Gamma$ is a quasi-isomorphism in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \to G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an usu local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is an usu local equivalence in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \to G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an $(\mathcal{D}^1, \text{usu})$ local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is an $(\mathcal{D}^1, \text{usu})$ local equivalence in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$.

(ii) Then,

- if $m : F \to G$ with $F, G \in C(\text{AnSp}(\mathbb{C})/S)$ is a quasi-isomorphism, $m^\Gamma : F^\Gamma \to G^\Gamma$ is a quasi-isomorphism in $C(\text{AnSp}(\mathbb{C})^{2}/S)$,
- if $m : F \to G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an usu local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is an usu local equivalence in $C(\text{AnSp}(\mathbb{C})^{2,sm}/S)$,
- if $m : F \to G$ with $F, G \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$ is an $(\mathcal{D}^1, \text{usu})$ local equivalence, $m^\Gamma : F^\Gamma \to G^\Gamma$ is an $(\mathcal{D}^1, \text{usu})$ local equivalence in $C(\text{AnSp}(\mathbb{C})^{2}/S)$.

**Proof.** Similar to the proof of proposition 28. \qed

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2.10 The analytical functor for presheaves on the big Zariski or etale site and on the big Zariski or etale site of pairs

We have for \( f : T \to S \) a morphism with \( T, S \in \text{Var}(\mathbb{C}) \) the following commutative diagram of sites

\[
\begin{array}{ccc}
\text{Dia}(S) := \text{AnSp}(\mathbb{C}) / T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C}) / T \\
\rho_T & \swarrow & \searrow \rho_T \\
\text{AnSp}(\mathbb{C})^{sm} / T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^{sm} / T \\
P(f) & \swarrow & \searrow P(f) \\
\text{AnSp}(\mathbb{C}) / S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C}) / S \\
\rho_S & \swarrow & \searrow \rho_S \\
\text{AnSp}(\mathbb{C})^{sm} / S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^{sm} / S \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{Dia}^{12}(S) := \text{AnSp}(\mathbb{C})^2 / T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^2 / T \\
\rho_T & \swarrow & \searrow \rho_T \\
\text{AnSp}(\mathbb{C})^{2,sm} / T^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathbb{C})^{2,sm} / T \\
P(f) & \swarrow & \searrow P(f) \\
\text{AnSp}(\mathbb{C})^2 / S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^2 / S \\
\rho_S & \swarrow & \searrow \rho_S \\
\text{AnSp}(\mathbb{C})^{2,sm} / S^{an} & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^{2,sm} / S \\
\end{array}
\]

For \( S \in \text{Var}(\mathbb{C}) \) we have the following commutative diagrams of sites

\[
\begin{array}{ccc}
\text{AnSp}(\mathbb{C})^2 / S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,pr} / S \\
\rho_S & \swarrow & \searrow \rho_S \\
\text{AnSp}(\mathbb{C})^{2,sm} / S & \xrightarrow{\mu_S} & \text{AnSp}(\mathbb{C})^{2,smpr} / S \\
\text{Var}(\mathbb{C})^2 / S & \xrightarrow{\text{An}_S} & \text{Var}(\mathbb{C})^{2,smpr} / S \\
\rho_S & \swarrow & \searrow \rho_S \\
\text{Var}(\mathbb{C})^2 / S & \xrightarrow{\mu_S} & \text{Var}(\mathbb{C})^{2,smpr} / S \\
\end{array}
\]
Definition-Proposition 10. Consider a closed embedding \( i : Z \hookrightarrow S \) with \( S, Z \in \text{Var}(\mathbb{C}) \). Then, for \( G^\bullet \in C(\text{Var}(\mathbb{C})^\text{sm}/S) \), there exist a map in \( C(\text{AnSp}(\mathbb{C})^\text{sm}/S) \)

\[
T(\text{An}, \gamma)(G) : \text{An}^*_S \Gamma_Z G \to \Gamma_Z \text{An}^*_S G
\]

unique up to homotopy, such that \( \gamma_Z(\text{An}^*_S G) \circ T(\text{An}, \gamma)(G) = \text{An}^*_S \gamma_Z G \).
**Proof.** Denote by \( j : S \setminus Z \to S \) the open complementary embedding. The map is given by \((I, T(\text{An}, j)(j^*G)) : \text{Cone}(\text{An}_S^*G \to \text{An}_S^*j_*j^*G) \to (\text{An}_S^*G \to j_*j^*\text{An}_Z^*G) \).

**Definition 36.** Let \( f : X \to S \) a morphism with \( X, S \in \text{Var}(\mathbb{C}) \). Assume that there exist a factorization \( f : X \dashrightarrow Y \times S \xrightarrow{p} S \), with \( Y \in \text{SmVar}(\mathbb{C}) \), \( i : X \hookrightarrow Y \) is a closed embedding and \( p \) the projection. We then have the canonical isomorphism in \( \text{C}(\text{AnSp}(\mathbb{C})^{sm}/S^{an}) \)

\[
T(f, g, Q) := T_1(\text{An}, p)(-)^{-1} \circ T_1(\text{An}, j)(-)^{-1} : \text{An}_S^*Q(X/S) := \text{An}_S^*p_1\Gamma_Y^\vee Z_{Y \times S}[dv] \xrightarrow{\cong} p_1\Gamma_{Z^{an}}Z_{Y \times S}[dv] =: Q(X^{an}/S^{an})
\]
with \( j : Y \times S \setminus X \hookrightarrow Y \times S \) the closed embedding.

**Definition-Proposition 11.** Consider a closed embedding \( i : Z \hookrightarrow S \) with \( S \in \text{Var}(\mathbb{C}) \). Then, for \( G \in \text{C}_{O_S}(S) \), there is a canonical map in \( \text{C}(\text{AnSp}(\mathbb{C})^{an}/S^{an}) \)

\[
T^{\text{mod}}(an, \gamma)(G) : (\Gamma_Z G)^{an} \to \Gamma_{Z^{an}}G^{an}
\]
unique up to homotopy, such that \( \gamma_{Z^{an}}(G^{an}) \circ T^{\text{mod}}(an, \gamma)(G) = g^*\gamma ZG \).

**Proof.** It is a particular case of definition-proposition 2(i).

We recall the first GAGA theorem for coherent sheaf on the projective spaces:

**Theorem 13.** For \( X \in \text{Var}(\mathbb{C}) \) and \( F \in \text{C}_{O_X}(X) \) denote by

\[
a(F) : \text{ad}(\text{an}_X^{\text{mod}}, \text{an}(X)_*)(E(F)) : E(F) \to \text{an}_X^*(E(F))^{an} = \text{an}_X^*E(F^{an}),
\]
the canonical morphism.

(i) Let \( X \in \text{PVar}(\mathbb{C}) \) a proper complex algebraic variety. For \( F \in \text{Coh}_{O_X}(X) \) a coherent sheaf, the morphism

\[
H^n\Gamma(X, a(F)) : H^n(X, F) = H^n\Gamma(X, E(F)) \to H^n(X, F^{an}) = H^n\Gamma(X, E(F^{an}))
\]
is an isomorphism for all \( n \in \mathbb{Z} \).

(ii) Let \( f : X \to S \) a proper morphism with \( X, S \in \text{Var}(\mathbb{C}) \). For \( F \in \text{Coh}_{O_X}(X) \) a coherent sheaf, the morphism

\[
H^n f_\ast a(F) : R^n f_\ast F = H^n f_\ast (E(F)) \to R^n f_\ast (F^{an}) = H^n f_\ast (E(F^{an})
\]
is an isomorphism for all \( n \in \mathbb{Z} \).

**Proof.** See [30]. (i) reduces to the case where \( X \) is projective and (ii) to the case where \( f \) is projective. Hence, the theorem reduce to the case of a coherent sheaf \( F \in \text{Coh}_{O_X^{an}}(\mathbb{P}^N) \) on \( \mathbb{P}^N \).

We have for \( s : \mathcal{I} \to \mathcal{J} \) a functor with \( \mathcal{I}, \mathcal{J} \in \text{Cat} \) and \( f : T_\bullet \to S_s(\bullet) \) a morphism of diagram of algebraic varieties with \( T_\bullet \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C})) \), \( S_\bullet \in \text{Fun}(\mathcal{J}, \text{Var}(\mathbb{C})) \) the following commutative diagram of sites

\[
\text{Dia}(S) := \text{AnSp}(\mathbb{C})/T^{an}_\bullet \xrightarrow{\text{An}_T^\ast} \text{Var}(\mathbb{C})/T_\bullet \xrightarrow{P(f)} \text{Var}(\mathbb{C})^{sm}/T^{an}_\bullet \xrightarrow{\text{An}_T^\ast} \text{Var}(\mathbb{C})^{sm}/T_\bullet \xrightarrow{P(f)_\ast} \text{AnSp}(\mathbb{C})^{sm}/S^{an}_\bullet \xrightarrow{\text{An}_S^\ast} \text{Var}(\mathbb{C})^{sm}/S_\bullet \xrightarrow{P(f)_\ast}
\]

(45)
2.11 The De Rahm complexes of algebraic varieties and analytical spaces

For $X \in \text{Var}(\mathbb{C})$, we denote by $\iota_X : \mathbb{C}_X \to \Omega^*_X := DR(X)$ the canonical inclusion map. More generally, for $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, we denote by $\iota_{X/S} : f^*\Omega_S \to \Omega^*_{X/S} := DR(X/S)$ the canonical inclusion map.

For $X \in \text{AnSp}(\mathbb{C})$, we denote by $\iota_X : \mathbb{C}_X \to \Omega^*_X := DR(X)$ the canonical inclusion map. More generally, for $f : X \to S$ a morphism with $X, S \in \text{AnSp}(\mathbb{C})$, we denote by $\iota_{X/S} : f^*\Omega_S \to \Omega^*_{X/S} := DR(X/S)$ the canonical inclusion map.

Let $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Then, the commutative diagram of site $(an,f) := (f, \text{an}_S, f = f^\text{an}, \text{an}(X))$ gives the transformation map in $\text{C}^{\text{an}}_{\text{Sm}}(S^\text{an})$ (definition 1)

$$T^\text{O}_\omega(a_n,f) : (f_*\text{an}_S \Omega^*_{X/S}, F_b)^\text{an} \to \text{an}^\text{mod}_S f_* \text{an}_S \Omega^*_{X/S}, F_b) \xrightarrow{T(\text{an}(X),\text{E}(\text{an}(X)\text{E}(\text{an}^\text{mod}_S \Omega^*_{X/S}, F_b))}$$

We will give in this paper a relative version for all smooth morphisms of the following theorem of Grothendieck

**Theorem 14.** Let $U \in \text{SmVar}(\mathbb{C})$. Denote by $a_U : U \to \{\text{pt}\}$ the terminal map. Then the map

$$T^\text{O}_\omega(a_U, a_n) : \Gamma(U, E(\Omega^*_U)) \to \Gamma(U^\text{an}, E(\Omega^*_U))$$

is a quasi-isomorphism of complexes.

**Proof.** Take a compactification $(X,D)$ of $U$, with $X \in \text{PSmVar}(\mathbb{C})$ and $D = X \setminus U$ a normal crossing divisor. The proof then use proposition 13, the first GAGA theorem (theorem 13 (i)) for the coherent sheaves $\Omega^*_U(nD)$ on $X$, and the fact (which is specific of the De Rahm complex) that $\Omega^*_{U^\text{an}}(\ast D^\text{an}) \to j_* E(\Omega^*_{U^\text{an}})$ is a quasi-isomorphism.

We recall Poincare lemma for smooth morphisms of complex analytic spaces and in particular complex analytic manifold:

**Proposition 39.** (i) For $h : U \to S$ a smooth morphism with $U, S \in \text{AnSp}(\mathbb{C})$, the inclusion map $\iota_{X/S} : h^*\Omega_S \to \Omega^*_{U/S} := DR(U/S)$ is a quasi-isomorphism.

(ii) For $X \in \text{AnSm}(\mathbb{C})$, the inclusion map $\iota_X : \mathbb{C}_X \to \Omega^*_X$ is a quasi-isomorphism.

**Proof.** Standard. (ii) is a particular case of (i) (the absolute case $S = \{\text{pt}\}$).
Remark 4. We do NOT have Poincaré lemma in general if $h : U \to S$ is not a smooth morphism. Already in the absolute case, we can find $X \in \text{Var}(\mathbb{C})$ singular such that the inclusion map $i_X : \mathbb{C} \to \Omega_{X}^{\text{an}}$ is not a quasi-isomorphism. Indeed, we can find example of $X \in \text{PVar}(\mathbb{C})$ projective singular where

$$H^p(C_X) : H^p(X_{\text{an}}, \mathbb{C}) \sim H^pC_{\text{sing}}(X_{\text{an}})$$

$X_{\text{an}}$ being locally contractible since $X_{\text{an}} \in \text{CW}$, have not the same dimension then the De Rham cohomology

$$H^p(T^G_\omega(an, a_X)) : \mathbb{H}^p(X, \mathbb{E}(\Omega_{X}^{\text{an}})) \sim \mathbb{H}^p(X_{\text{an}}, \mathbb{E}(\Omega_{X}^{\text{an}}))$$

$X$ being projective, that is are not isomorphic as vector spaces. Hence, in particular, the canonical map

$$H^p_{\text{ls}} : H^p(X_{\text{an}}, \mathbb{C}) \to \mathbb{H}^p(X_{\text{an}}, \mathbb{E}(\Omega_{X}^{\text{an}}))$$

is not an isomorphism.

Consider a commutative diagram

$$D_0 = \begin{array}{ccc} X & \xrightarrow{i} & Y & \xrightarrow{p} & S \\ \downarrow{g'} & & \downarrow{g} & & \downarrow{h} \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & T \\ \end{array}$$

with $X, X', Y, Y', S, T \in \text{Var}(\mathbb{C})$ or $X, X', Y, Y', S, T \in \text{AnSp}(\mathbb{C})$, $i, i'$ being closed embeddings. Denote by $D$ the right square of $D_0$. The closed embedding $i' : X' \to Y'$ factors through $i'' : X' \xrightarrow{i''_1} \Gamma_X \xrightarrow{i''_0} Y'$ where $i''_1, i''_0$ are closed embeddings. Then, definition-proposition 3 say that

• there is a canonical map,

$$\Gamma_{X(Y/Y')} \circ T(g'', E)(-(-) \circ T(g'', \gamma)(-) : g''^*\Gamma_X E(\Omega_{X/Y/S}, F_b) \to \Gamma_{X \times Y/Y'} E(\Omega_{X/Y'/T}, F_b)$$

unique up to homotopy such that the following diagram in $C_{g''^*p^{\ast}O_{\text{fil}}Y'}(Y') = C_{p'^*g^*O_{\text{fil}}Y'}(Y')$

commutes

$$\begin{array}{ccc}
\gamma_X(\cdot) & \downarrow & \gamma_{X \times Y/Y'}(\cdot) \\
g''^*\Gamma_X E(\Omega_{X/Y/S}, F_b) & \xrightarrow{E(\Omega_{(Y/Y')/T/S}) \circ T(g'', E)(-(-) \circ T(g'', \gamma)(\cdot))} & \Gamma_{X \times Y/Y'} E(\Omega_{X/Y'/T}, F_b) \\
g''^* E(\Omega_{X/Y/S}, F_b) & \xrightarrow{E(\Omega_{(Y/Y')/T/S}) \circ T(g'', E)(-(-)} & \Gamma_{X/Y'/T}, F_b \\
\end{array}$$

• there is a canonical map,

$$T_{\omega}^G(D) : g'^{\text{mod}} L_{O_{\text{fil}}} \Gamma_X E(\Omega_{X/Y/S}, F_b) \to p'_* \Gamma_{X \times Y/Y'} E(\Omega_{X/Y'/T}, F_b)$$

unique up to homotopy such that the following diagram in $C_{O_{\text{fil}}}(T)$ commutes

$$\begin{array}{ccc} 
\gamma_X(\cdot) & \downarrow & \gamma_{X \times Y/Y'}(\cdot) \\
g'^{\text{mod}} L_{O_{\text{fil}}} \Gamma_X E(\Omega_{X/Y/S}) & \xrightarrow{T_{\omega}^G(D)} & p'_* \Gamma_{X \times Y/Y'} E(\Omega_{X/Y'/T}) \\
g'^{\text{mod}} L_{O_{\text{fil}}} \Gamma_X E(\Omega_{X/Y/S}) & \xrightarrow{T_{\omega}^G(D)} & p'_* E(\Omega_{X/Y'/T}) \\
\end{array}$$

(iii) there is a map in $C_{p'^*O_{\text{fil}}Y'}(Y')$

$$T(X'/X \times Y/Y', a)(E(\Omega_{X/Y'/T}, F_b)) : \Gamma_X E(\Omega_{X/Y'/T}, F_b) \to \Gamma_{X \times Y/Y'} E(\Omega_{X/Y'/T}, F_b)$$

unique up to homotopy such that $\gamma_{X \times Y/Y'}(\cdot) \circ T(X'/X \times Y/Y', a)(\cdot) = \gamma_X(\cdot)$. 

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Theorem 15. functorialities:

2.12 The Corti-Hanamura resolution functors

Definition 37. There is a canonical map

\[ E(\Omega_{Y^{an}/Y}/(S^{an}/S)) \circ T(an, \gamma)(-): \text{an}(Y)^* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) \to \Gamma_{X^{an}} E(\Omega_{Y/S}^\bullet, F_b) \]

unique up to homotopy such that the following diagram in \( C_{h^*O_S f!} (Y^{an}) \) commutes

\[
\begin{array}{ccc}
\text{an}(Y)^* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b) & \xrightarrow{E(\Omega_{Y^{an}/Y}/(S^{an}/S)) \circ T(an, \gamma)(-)} & \Gamma_{X^{an}} E(\Omega_{Y/S}^\bullet, F_b) \\
\gamma_X(-) & \downarrow & \gamma_{X^{an}}(-) \\
\text{an}(Y)^* E(\Omega_{Y/S}^\bullet, F_b) & \xrightarrow{E(\Omega_{Y^{an}/Y}/(S^{an}/S))} & E(\Omega_{Y/S}^\bullet, F_b)
\end{array}
\]

there is a canonical map

\[ T^O(\alpha, h) \gamma: (h_* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b))^{an} \to h_* \Gamma_{X^{an}} E(\Omega_{Y/S}^\bullet, F_b) \]

unique up to homotopy such that the following diagram in \( C(Y) \) commutes

\[
\begin{array}{ccc}
(h_* \Gamma_X E(\Omega_{Y/S}^\bullet, F_b))^{an} & \xrightarrow{T^O(\alpha, h) \gamma} & h_* \Gamma_{X^{an}} E(\Omega_{Y/S}^\bullet, F_b) \\
\gamma_X(-) & \downarrow & \gamma_{X^{an}}(-) \\
(h_* E(\Omega_{Y/S}^\bullet, F_b))^{an} & \xrightarrow{T^O(\alpha, h)} & h_* E(\Omega_{Y/S}^\bullet, F_b)
\end{array}
\]

2.12 The Corti-Hanamura resolution functors \( \hat{R}^{CH}, R^{CH}, \hat{R}^{CGCH}, R^{CGCH} \) from complexes of representable presheaves on \( \text{Var}(\mathbb{C})^m/S \) with \( S \) smooth, and the functorialities of these resolutions

Definition 37. (i) Let \( X_0 \in \text{Var}(\mathbb{C}) \) and \( Z \subset X_0 \) a closed subset. A desingularization of \((X_0, Z)\) is a pair of complex varieties \((X, D) \in \text{Var}^2(\mathbb{C}))\), together with a morphism of pair of varieties \( \epsilon : (X, D) \to (X_0, \Delta) \) with \( Z \subset \Delta \) such that

- \( X \in \text{SmVar}(\mathbb{C}) \) and \( D := \epsilon^{-1}(\Delta) = \epsilon^{-1}(Z) \cup (\cup_i E_i) \subset X \) is a normal crossing divisor
- \( \epsilon : X \to X_0 \) is a proper modification with discriminant \( \Delta \), that is \( \epsilon : X \to X_0 \) is proper and \( \epsilon : X \backslash D \xrightarrow{\sim} X \backslash \Delta \) is an isomorphism.

(ii) Let \( X_0 \in \text{Var}(\mathbb{C}) \) and \( Z \subset X_0 \) a closed subset such that \( X_0 \backslash Z \) is smooth. A strict desingularization of \((X_0, Z)\) is a pair of complex varieties \((X, D) \in \text{Var}^2(\mathbb{C}))\), together with a morphism of pair of varieties \( \epsilon : (X, D) \to (X_0, Z) \) such that

- \( X \in \text{SmVar}(\mathbb{C}) \) and \( D := \epsilon^{-1}(Z) \subset X \) is a normal crossing divisor
- \( \epsilon : X \to X_0 \) is a proper modification with discriminant \( Z \), that is \( \epsilon : X \to X_0 \) is proper and \( \epsilon : X \backslash D \xrightarrow{\sim} X \backslash Z \) is an isomorphism.

We have the following well known resolution of singularities of complex algebraic varieties and their functorialities:

Theorem 15. (i) Let \( X_0 \in \text{Var}(\mathbb{C}) \) and \( Z \subset X_0 \) a closed subset. There exists a desingularization of \((X_0, Z)\), that is a pair of complex varieties \((X, D) \in \text{Var}^2(\mathbb{C}))\), together with a morphism of pair of varieties \( \epsilon : (X, D) \to (X_0, \Delta) \) with \( Z \subset \Delta \) such that

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- $X \in \text{SmVar}(\mathbb{C})$ and $D := e^{-1}(\Delta) = e^{-1}(Z) \cup (\cup_i E_i) \subset X$ is a normal crossing divisor
- $\epsilon : X \to X_0$ is a proper modification with discriminant $\Delta$, that is $\epsilon : X \to X_0$ is proper and $\epsilon : X \setminus D \cong X \setminus \Delta$ is an isomorphism.

(ii) Let $X_0 \in \text{PVar}(\mathbb{C})$ and $Z \subset X_0$ a closed subset such that $X_0 \setminus Z$ is smooth. There exists a strict desingularization of $(X_0, Z)$, that is a pair of complex varieties $(X, D) \in \text{PVar}^2(\mathbb{C})$, together with a morphism of pair of varieties $\epsilon : (X, D) \to (X_0, Z)$ such that

- $X \in \text{PSmVar}(\mathbb{C})$ and $D := e^{-1}(Z) \subset X$ is a normal crossing divisor
- $\epsilon : X \to X_0$ is a proper modification with discriminant $Z$, that is $\epsilon : X \to X_0$ is proper and $\epsilon : X \setminus D \cong X \setminus Z$ is an isomorphism.

Proof. (i): Standard. See [25] for example. (ii): Follows immediately from (i). \hfill \Box

We use this theorem to construct a resolution of a morphism by Corti-Hanamura morphisms, we will need these resolution in the definition of the filtered De Rham realization functor:

**Definition-Proposition 12.** (i) Let $h : V \to S$ a morphism, with $V, S \in \text{Var}(\mathbb{C})$. Let $\tilde{S} \in \text{PVar}(\mathbb{C})$ be a compactification of $S$.

- There exist a compactification $\tilde{X}_0 \in \text{PVar}(\mathbb{C})$ of $V$ such that $h : V \to S$ extend to a morphism $\tilde{f}_0 = \tilde{h}_0 : \tilde{X}_0 \to \tilde{S}$. Denote by $\tilde{Z} = \tilde{X}_0 \setminus V$. We denote by $j : V \to \tilde{X}_0$ the open embedding and by $i_0 : Z \hookrightarrow \tilde{X}_0$ the complementary closed embedding. We then consider $X_0 := \tilde{f}_0^{-1}(S) \subset \tilde{X}_0$ the open subset, $f_0 := \tilde{f}_0|_{X_0} : X_0 \to S$, $Z = \tilde{Z} \cap X_0$, and we denote again $j : V \to X_0$ the open embedding and by $i_0 : Z \to X_0$ the complementary closed embedding.

- In the case $V$ is smooth, we take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (X_0, Z)$ of the pair $(X_0, Z)$, with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} = \bigcup_{i \in I} D_i \subset \bar{X}$ a normal crossing divisor. We denote by $\bar{i}_* : D_\bullet \to \bar{X} = \bar{X}_{\bar{i}(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings $i_1 : D_1 = \bigcap_{i \in I} D_i \hookrightarrow \bar{X}$. Then the morphisms $f := f_0 \circ \bar{\epsilon} : \bar{X} \to \bar{S}$ and $f_{D_\bullet} := f \circ \bar{i}_* : D_\bullet \to \bar{S}$ are projective since $\bar{X}$ and $D_1$ are projective varieties. We then consider $(X, D) := \bar{\epsilon}^{-1}(X_0, Z)$, $\epsilon := \bar{\epsilon}|_X : (X, D) \to (X_0, Z)$. We denote again by $i_* : D_\bullet \to X = X_c(\bullet)$ the morphism of simplicial varieties given by the closed embeddings $i_1 : D_1 = \bigcap_{i \in I} D_i \hookrightarrow X$. Then the morphisms $f := f_0 \circ \epsilon : X \to S$ and $f_{D_\bullet} := f \circ i_* : D_\bullet \to S$ are projective since $f : X_0 \to S$ is projective.

(ii) Let $g : V' / S \to V / S$ a morphism, with $V' / S = (V', h')$, $V / S = (V, h) \in \text{Var}(\mathbb{C}) / S$.

- Take (see (i)) a compactification $\tilde{X}_0 \in \text{PVar}(\mathbb{C})$ of $V$ such that $h : V \to S$ extend to a morphism $\tilde{f}_0 = \tilde{h}_0 : \tilde{X}_0 \to \tilde{S}$. Denote by $\tilde{Z} = \tilde{X}_0 \setminus V$. Then, there exist a compactification $\tilde{X}_0' \in \text{PVar}(\mathbb{C})$ of $V'$ such that $h' : V' \to S$ extend to a morphism $\tilde{f}_0' = \tilde{h}_0' : \tilde{X}_0' \to \tilde{S}$, $g : V' \to V$ extend to a morphism $\tilde{g}_0 : \tilde{X}_0' \to \tilde{X}_0$ and $f_0 \circ \tilde{g}_0 = f_0'$ that is $\tilde{g}_0$ is gives a morphism $\tilde{g}_0 : \tilde{X}_0' / S \to \tilde{X}_0 / S$. Denote by $Z' = \tilde{X}_0' \setminus V'$. We then have the following commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
V \xrightarrow{j} \tilde{X}_0 \xrightarrow{i} \tilde{Z} \\
\downarrow g \quad \downarrow \tilde{g}_0 \quad \downarrow g' \\
V' \xrightarrow{j'} \tilde{X}_0' \xrightarrow{i'} Z' \xrightarrow{i''_{g,0}} \tilde{Z} \xrightarrow{g_0^{-1}(Z)} \tilde{Z} \xrightarrow{i''_{g,0}} \tilde{Z}
\end{array}
\end{array}
\]

It gives the following commutative diagram:

\[
\begin{array}{c}
\begin{array}{c}
V \xrightarrow{j} X_0 := \tilde{f}_0^{-1}(S) \xrightarrow{i} Z \\
\downarrow g \quad \downarrow \tilde{g}_0 \\
V' \xrightarrow{j'} X_0' := \tilde{f}_0'^{-1}(S') \xrightarrow{i'} Z' \xrightarrow{i''_{g,0}} \tilde{Z}' \xrightarrow{g_0^{-1}(Z)} \tilde{Z}' \xrightarrow{i''_{g,0}} \tilde{Z}'
\end{array}
\end{array}
\]
In the case $V$ and $V'$ are smooth, we take using theorem 15 a strict desingularization $\tilde{\varepsilon} : (\bar{X}, D) \to (X_0, \bar{Z})$. Then there exist a strict desingularization $\tilde{\varepsilon}' : (\bar{X}', \bar{D}') \to (\bar{X}_0', \bar{Z}')$ of $(\bar{X}_0', \bar{Z}')$ and a morphism $\tilde{g} : \bar{X}' \to \bar{X}$ such that the following diagram commutes:

```
\[
\begin{array}{ccc}
\bar{X}_0' & \xrightarrow{\tilde{g}_0} & \bar{X}_0 \\
\downarrow{\tilde{\varepsilon}} & & \downarrow{\varepsilon} \\
\bar{X}' & \xrightarrow{\tilde{g}} & \bar{X}
\end{array}
\]
```

We then have the following commutative diagram in Fun($\Delta, \text{Var}(\mathbb{C})$)

```
\[
\begin{array}{ccc}
V = V_c(\bullet) & \xrightarrow{j} & \bar{X} = \bar{X}_c(\bullet) & \xrightarrow{i^*} & D_{s_0}(\bullet) \\
\downarrow{g} & & \downarrow{\bar{g}} & & \downarrow{\bar{g}^*} \\
V' = V_c'(\bullet) & \xrightarrow{j'} & \bar{X}' = \bar{X}_c'(\bullet) & \xrightarrow{i^*_0} & \bar{D}' \quad \bar{g}^* \circ (D_{s_0}(\bullet)) : i^*_0
\end{array}
\]
```

where $i_0 : \bar{D}_0 \to \bar{X}_n$ the morphism of simplicial varieties given by the closed embeddings $i_n : D_n \hookrightarrow X_n$, and $i_0' : \bar{D}_n' \hookrightarrow \bar{X}_n'$ the morphism of simplicial varieties given by the closed embeddings $i_n' : D_n' \hookrightarrow X_n'$. It gives the commutative diagram in Fun($\Delta, \text{Var}(\mathbb{C})$)

```
\[
\begin{array}{ccc}
V = V_c(\bullet) & \xrightarrow{j} X := \bar{\varepsilon}^{-1}(X_0) = X_c(\bullet) & \xrightarrow{i^*} & D_{s_0}(\bullet) \\
\downarrow{g} & & \downarrow{\bar{g}} & & \downarrow{\bar{g}^*} \\
V' = V_c'(\bullet) & \xrightarrow{j'} X' := \bar{\varepsilon}'^{-1}(X_0') = X_c'(\bullet) & \xrightarrow{i^*_0} & \bar{D}' \quad \bar{g}^* \circ (D_{s_0}(\bullet)) : i^*_0
\end{array}
\]
```

Proof. (i): Let $X_{00} \in \text{PVar}(\mathbb{C})$ be a compactification of $V$. Let $l_0 : \bar{X}_0 = \Gamma_n \hookrightarrow X_{00} \times \bar{S}$ be the closure of the graph of $h$ and $f_0 := p_{\bar{S}} \circ l_0 : \bar{X}_0 \hookrightarrow X_{00} \times \bar{S} \to \bar{S}, \epsilon_{\bar{X}_0} := p_{\bar{S}} \circ l_0 : \bar{X}_0 \hookrightarrow X_{00} \times \bar{S} \to X_{00}$ be the restriction to $X_0$ of the projections. Then, $\bar{X} \in \text{PVar}(\mathbb{C}), \epsilon_{\bar{X}_0} : \bar{X}_0 \to X_{00}$ is a proper modification which does not affect the open subset $V \subset \bar{X}_0$, and $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ is a compactification of $h$.

(ii): There are two things to prove:

- Let $\bar{f}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : V \to S$ and $\bar{f}_{00} : \bar{X}_{00}' \to \bar{S}$ a compactification of $h' : \bar{V}' \to \bar{S}$ (see (i)). Let $l_0 : \bar{X}_0' \hookrightarrow \bar{X}_{00}' \times \bar{S} \bar{X}_0$ be the closure of the graph of $g$, $\bar{f}_0' := (\bar{f}_{00}, \bar{f}_0) \circ l_0 : \bar{X}_0' \hookrightarrow \bar{X}_{00}' \times \bar{S} \bar{X}_0 \to \bar{S}$ and $\bar{g}_0 := p_{\bar{S}} \circ l_0 : \bar{X}_0' \hookrightarrow \bar{X}_{00}' \times \bar{S} \bar{X}_0 \to \bar{X}_0, \epsilon_{\bar{X}_0} := p_{\bar{S}} \circ \bar{i} : \bar{X}_0' \hookrightarrow \bar{X}_{00}' \times \bar{S} \bar{X}_0 \to \bar{X}_{00}$ be the restriction to $X$ of the projections. Then $\epsilon_{\bar{X}_0} : \bar{X}_0' \to \bar{X}_{00}'$ is a proper modification which does not affect the open subset $V' \subset \bar{X}_{00}'$, $\bar{f}'_0 : \bar{X}_0' \to \bar{S}$ is an other compactification of $h' : \bar{V}' \to \bar{S}$ and $\bar{g}_0 : \bar{X}_0' \to \bar{X}_0$ is a compactification of $g$.

- In the case $V$ and $V'$ are smooth, we take, using theorem 15, a strict desingularization $\bar{\varepsilon} : (\bar{X}, \bar{D}) \to (X_0, \bar{Z})$ of the pair $(X_0, \bar{Z})$. Take then, using theorem 15, a strict desingularization $\bar{\varepsilon}_1' : (\bar{X}', \bar{D}') \to (\bar{X} \times_{X_0} \bar{X}_0', \bar{X} \times_{X_0} \bar{Z}')$ of the pair $(\bar{X} \times_{X_0} \bar{X}_0', \bar{X} \times_{X_0} \bar{Z}')$. We consider then following commutative diagram whose square is cartesian:

```
\[
\begin{array}{ccc}
\bar{X}_0 & \xrightarrow{\bar{g}_0} & \bar{X}_0 \\
\downarrow{\bar{\varepsilon}} & & \downarrow{\varepsilon} \\
\bar{X}' & \xrightarrow{\bar{g}} & \bar{X}
\end{array}
\]
```

and $\bar{\varepsilon} : = \bar{\varepsilon}_0 \circ \bar{\varepsilon}_1' : (\bar{X}', \bar{D}') \to (\bar{X}_0', \bar{Z})'$ is a strict desingularization of the pair $(\bar{X} \times_{X_0} \bar{X}_0', \bar{X} \times_{X_0} \bar{Z}')$. 

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Let $S \in \text{Var}(\mathbb{C})$. Recall we have the dual functor

$$
\mathbb{D}_S : C(\text{Var}(\mathbb{C})/S) \to C(\text{Var}(\mathbb{C})/S), \; F \mapsto \mathbb{D}_S(F) := \text{Hom}(F, E_{et}(\mathbb{Z}/S/S))
$$

which induces the functor

$$
L\mathbb{D}_S : C(\text{Var}(\mathbb{C})/S) \to C(\text{Var}(\mathbb{C})/S), \; F \mapsto L\mathbb{D}_S(F) := \mathbb{D}_S(LF) := \text{Hom}(LF, E_{et}(\mathbb{Z}/S/S)).
$$

We will use the following resolutions of representable presheaves by Corti-Hanamura presheaves and their the functorialities.

**Definition 38.**

(i) Let $h : U \to S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$ and $U$ smooth. Take, see definition-proposition 12, $f_0 = h_0 : X_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $Z = X_0 \setminus U$. Take, using theorem 15(ii), a strict desingularization $\bar{e} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$, with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := e^{-1}(\bar{Z}) = \bigcup_{i=1}^n \bar{D}_i \subset \bar{X}$ a normal crossing divisor. We denote by $i_\bullet : D_\bullet \hookrightarrow \bar{X} = \bar{X}_c(\bullet)$ the morphism of simplicial varieties given by the closed embeddings $i_1 : D_1 = \cap_{i \in I} D_i \hookrightarrow \bar{X}$ We denote by $j : U \hookrightarrow \bar{X}$ the open embedding and by $p_S : \bar{X} \times S \to S$ and $p_S : U \times S \to S$ the projections. Considering the graph factorization $\bar{f} : \bar{X} \xrightarrow{\bar{f}} \bar{X} \times \bar{S} \xrightarrow{p_S} \bar{S}$ of $\bar{f} : \bar{X} \to \bar{S}$, where $\bar{l}$ is the graph embedding and $p_S$ the projection, we get closed embeddings $l := l \times \bar{S} : X \hookrightarrow \bar{X} \times S$ and $l_{D_1} : D_1 \times X \hookrightarrow \bar{D}_1 \times S$. We then consider the following map in $C(\text{Var}(\mathbb{C})^2/S)$

$$
\mathbb{r}_{(\bar{X}, \bar{D})/S}(\mathbb{Z}/U/S)) := R_{(\bar{X}, \bar{D})/S}(U/S)
$$

Note that $\mathbb{Z}((\bar{D}_1 \times S, D_1)/\bar{X} \times S)$ and $\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))$ are obviously $\mathbb{A}^1$ invariant. Note that $r_{(X, D)/S}$ is NOT an equivalence ($\mathbb{A}^1$, et) local by proposition 27 since $\rho_{X \times S, Z}(\mathbb{D}_\bullet \times S, D_\bullet)/X \times S) = 0$, whereas $\rho_{X \times S, \text{ad}(j \times I)})((\bar{X} \times S, X)/\bar{X} \times S))$ is not an equivalence ($\mathbb{A}^1$, et) local.

(ii) Let $g : U'/S \to U/S$ a morphism, with $U'/S = (U', h')$, $U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, with $U$ and $U'$ smooth. Take, see definition-proposition 12(ii), a compactification $f_0 = h : X_0 \to \bar{S}$ of $h : U \to S$ and a compactification $f_0 = h' : X_0 \to \bar{S}$ of $h' : U' \to S$ such that $g : U'/S \to U/S$ extend to a morphism $\bar{g}_0 : \bar{X}_0 \setminus \bar{S} \to \bar{X}_0 \setminus \bar{S}$. Denote $Z = X_0 \setminus U$ and $Z' = X_0' \setminus U'$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{e} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$, a strict desingularization $\bar{e}' : (\bar{X}', \bar{D}') \to (\bar{X}_0', \bar{Z}')$ of $(\bar{X}_0', \bar{Z}')$ and a morphism $\bar{g} : \bar{X}' \to \bar{X}$ such that the following diagram commutes

$$
\begin{array}{ccc}
\bar{X}_0' & \xrightarrow{\bar{g}_0} & \bar{X}_0 \\
\bar{e}' \downarrow & & \bar{e} \downarrow \\
\bar{X}' & \xrightarrow{\bar{g}} & \bar{X}
\end{array}
$$

We then have, see definition-proposition 12(ii), the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$
\begin{array}{ccc}
U = U_c(\bullet) & \xrightarrow{j} & \bar{X} = \bar{X}_c(\bullet) \xleftarrow{i} \bar{D}_{s_0}(\bullet) \\
\downarrow{g} & & \downarrow{\bar{g}} \\
U' = U'_c(\bullet) & \xrightarrow{j'} & X' = X'_c(\bullet) \xleftarrow{i'} D'_s(\bullet) \xleftarrow{\bar{g}^{-1}(D_{s_0}(\bullet)) : i'_{\bullet}}
\end{array}
$$

(47)
Denote by \( p_S : \tilde{X} \times S \to S \) and \( p'_S : \tilde{X}' \times S \to S \) the projections. We then consider the following map in \( C(\text{Var}(\mathbb{C})^2)/S \):

\[
R^C_S(g) : R_{(\tilde{X},D)/S}(\mathbb{Z}(U/S)) \xrightarrow{\sim} \]

\[
p_{S,\text{et}}(\text{Cone}(\mathbb{Z}(\bullet \times I) : (\mathbb{Z}(\tilde{D}_s(\bullet) \times S, D_{s_p}(\bullet)\mathbb{Z}(\tilde{X} \times S), u_{1,J})) \to \mathbb{Z}((\tilde{X} \times S, X/\tilde{X} \times S)))

\[
T((\tilde{g} \times I), E)(-\circ p_{S,\text{et}} + \text{ad}((\tilde{g} \times I)^* (\tilde{g} \times I)\cdot (-)),)
\]

\[
p'_{S,\text{et}}(\text{Cone}(\mathbb{Z}(\bullet' \times I) : (\mathbb{Z}(\tilde{D}_s(\bullet) \times S, D_{s_p}(\bullet)\mathbb{Z}(\tilde{X}' \times S), u_{1,J})) \to \mathbb{Z}((\tilde{X}' \times S, X'/\tilde{X}' \times S)))
\]

\[
p'_{S,\text{et}}(\text{Cone}(\mathbb{Z}(\bullet' \times I) : (\mathbb{Z}(\tilde{D}_s(\bullet) \times S, D_{s_p}(\bullet)\mathbb{Z}(\tilde{X}' \times S), u_{1,J})) \to \mathbb{Z}((\tilde{X}' \times S, X'/\tilde{X}' \times S)))
\]

Then by the diagram (47) and adjunction, the following diagram in \( C(\text{Var}(\mathbb{C})^2)/S \) obviously commutes:

\[
\begin{array}{c}
R_{(X,D)/S}(\mathbb{Z}(U/S)) \\
\xrightarrow{r_{(X,D)/S}(\mathbb{Z}(U'/S))} \xrightarrow{p_{S,\text{et}}(\mathbb{Z}(U \times S, U/U \times S))} \xrightarrow{D_2^S(\mathbb{Z}(U'/S))} \xrightarrow{p'_{S,\text{et}}(\mathbb{Z}(U' \times S, U'/U' \times S))} \xrightarrow{D_2^S(\mathbb{Z}(U'/S))}
\end{array}
\]

(iii) For \( g_1 : U''/S \to U'/S, g_2 : U'/S \to U/S \) two morphisms with \( U''/S = (U', h''), U'/S = (U', h'), U/S = (U, h) \in \text{Var}(\mathbb{C})/S \), with \( U, U' \) and \( U'' \) smooth. We get from (i) and (ii) a compactification \( \overline{f} = h : \overline{X} \to S \) of \( h : U \to S \), a compactification \( \overline{f}' = h' : \overline{X}' \to S \) of \( h' : U' \to S \), and a compactification \( \overline{f}'' = h'' : \overline{X}'' \to S \) of \( h'' : U'' \to S \), with \( \overline{X}, \overline{X}', \overline{X}'' \in \text{PSmVar}(\mathbb{C}), D := \overline{X} \setminus U \subset \overline{X}, D' := \overline{X}' \setminus U' \subset \overline{X}', \) and \( D'' := \overline{X}'' \setminus U'' \subset \overline{X}'' \) normal crossing divisors, such that \( g_1 : U''/S \to U'/S \) extend to \( \overline{g}_1 : \overline{X}''/\overline{S} \to \overline{X}'/\overline{S} \), \( g_2 : U'/S \to U/S \) extend to \( \overline{g}_2 : \overline{X}'/\overline{S} \to \overline{X}/\overline{S} \), and

\[
R^C_S(g_2 \circ g_1) = R^C_S(g_1) \circ R^C_S(g_2) : R_{(X,D)/S} \to R_{(X',D')/S}
\]

(iv) For

\[
Q^* := (\cdots \to \oplus_{\alpha \in \Lambda^*} Z(U_{\alpha}^n/S) \xrightarrow{(Z(g_{\alpha}^n))} \oplus_{\beta \in \Lambda^{-1} \alpha} Z(U_{\beta}^{n-1}/S) \to \cdots) \in C(\text{Var}(\mathbb{C})/S)
\]

a complex of (maybe infinite) direct sum of representable presheaves with \( U_{\alpha} \) smooth, we get from (i), (ii) and (iii) the map in \( C(\text{Var}(\mathbb{C})^2)/S \):

\[
r^C_S(Q^*) : R^C_S(Q^*) := (\cdots \to \oplus_{\beta \in \Lambda^{n-1}} \lim_{(X_{\beta}^{n-1}, D_{\beta}^{n-1})/S} R_{(X_{\beta}^{n-1}, D_{\beta}^{n-1})/S}(Z(U_{\beta}^{n-1}/S))
\]

\[
\xrightarrow{(R^C_S(g_{\alpha}^n))} \oplus_{\alpha \in \Lambda^*} \lim_{(X_{\alpha}^{n}, D_{\alpha}^{n})/S} R_{(X_{\alpha}^{n}, D_{\alpha}^{n})/S}(Z(U_{\alpha}^n/S)) \to \cdots \to D_2^S(Q^*),
\]

where for \((U_{\alpha}^n, h_{\alpha}^n) \in \text{Var}(\mathbb{C})/S \), the inductive limit run over all the compactifications \( \overline{f}_{\alpha} : \overline{X}_{\alpha} \to \overline{S} \) of \( h_{\alpha} : U_{\alpha} \to S \) with \( \overline{X}_{\alpha} \in \text{PSmVar}(\mathbb{C}) \) and \( D_{\alpha} := \overline{X}_{\alpha} \setminus U_{\alpha} \) a normal crossing divisor. For \( m = (m^*) : Q_1^* \to Q_2^* \) a morphism with

\[
Q_1^* := (\cdots \to \oplus_{\alpha \in \Lambda^*} Z(U_{1,\alpha}^n/S) \xrightarrow{(Z(g_{\alpha}^n))} \oplus_{\beta \in \Lambda^{-1} \alpha} Z(U_{1,\beta}^{n-1}/S) \to \cdots),
\]

\[
Q_2^* := (\cdots \to \oplus_{\alpha \in \Lambda^*} Z(U_{2,\alpha}^n/S) \xrightarrow{(Z(g_{\alpha}^n))} \oplus_{\beta \in \Lambda^{-1} \alpha} Z(U_{2,\beta}^{n-1}/S) \to \cdots) \in C(\text{Var}(\mathbb{C})/S)
\]

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complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}$ and $U_{2,\alpha}$ smooth, we get again from (i), (ii) and (iii) a commutative diagram in $C(\text{Var}(\mathbb{C})^2/S)$

\[
\begin{array}{ccc}
R^{CH}(Q_2^1) & \xrightarrow{r_S^{CH}(Q_2^1)} & \mathbb{D}_S^{12}(Q_2^1) \\
R^{CH}(Q_2^0) & \xrightarrow{r_S^{CH}(Q_2^0)} & \mathbb{D}_S^{12}(Q_2^0)
\end{array}
\]

- Let $S \in \text{Var}(\mathbb{C})$. For $(h, m, m') = (h^*, m^*, m'^*) : Q_1^* \to Q_2^*$ an homotopy with $Q_1^*, Q_2^* \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}$ and $U_{2,\alpha}$ smooth,

\[R^{CH}(h), R^{CH}(m), R^{CH}(m') = (R^{CH}(h^*), R^{CH}(m^*), R^{CH}(m'^*)) : R^{CH}(Q_2^*)[1] \to R^{CH}(Q_1^*)\]

is an homotopy in $C(\text{Var}(\mathbb{C})^2/S)$ using definition 38 (iii). In particular if $m : Q_1^* \to Q_2$ with $Q_1^*, Q_2^* \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}$ and $U_{2,\alpha}$ smooth is an homotopy equivalence, then $R^{CH}(m) : R^{CH}(Q_2^*) \to R^{CH}(Q_1^*)$ is an homotopy equivalence.

- Let $S \in \text{SmVar}(\mathbb{C})$. Let $F \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S)$. Consider

\[q : LF := (\cdots \to \oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \xrightarrow{(\mathbb{Z}g_{\alpha}^{\alpha})} \oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \to \cdots) \to F\]

the canonical projective resolution given in subsection 2.3.3. Note that the $U_\alpha$ are smooth since $S$ is smooth and $h_\alpha$ are smooth morphism. Definition 38(iv) gives in this particular case the map in $C(\text{Var}(\mathbb{C})^2/S)$

\[r_S^{CH}(\rho_S^L F) : R^{CH}(\rho_S^L F) := (\cdots \to \oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \lim_{(X_\alpha, D_\alpha)/S} R(\mathcal{X}_\alpha, D_\alpha)/(\mathbb{Z}(U_\alpha/S)))
\]

\[(\mathbb{Z}g_{\alpha}^{\alpha}) \xrightarrow{\oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \lim_{(X_\alpha, D_\alpha)/S} R(\mathcal{X}_\alpha, D_\alpha)/(\mathbb{Z}(U_\alpha/S))) \to \cdots) \to \mathbb{D}_S^{12}(\rho_S^{L F}),\]

where for $(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S$, the inductive limit run over all the compactifications $\tilde{f}_\alpha : \tilde{X}_\alpha \to \tilde{S}$ of $h_\alpha : U_\alpha \to S$ with $\tilde{X}_\alpha \in \text{PShVar}(\mathbb{C})$ and $D_\alpha := \tilde{X}_\alpha \setminus U_\alpha$ a normal crossing divisor. Definition 38(iv) gives then by functoriality in particular, for $F = F^* \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the map in $C(\text{Var}(\mathbb{C})^2/S)$

\[r_S^{CH}(\rho_S^L F) = (r_S^{CH}(\rho_S^L F^*)) : R^{CH}(\rho_S^L F^*) \to \mathbb{D}_S^{12}(\rho_S^{L F}).\]

- Let $g : T \to S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $h : U \to S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

\[
\begin{array}{ccc}
U_T & \xrightarrow{h'} & T \\
\downarrow g' & & \downarrow g \\
U & \xrightarrow{h} & S
\end{array}
\]

Note that $U$ is smooth since $S$ and $h$ are smooth, and $U_T$ is smooth since $T$ and $h'$ are smooth. Take, see definition-proposition 12(ii), a compactification $\tilde{f}_0 = \tilde{h} : \tilde{X}_0 \to \tilde{S}$ of $h : U \to S$ and a compactification $\tilde{f}_0 = g \circ h' : \tilde{X}_0' \to S$ of $g \circ h' : U' \to S$ such that $g' : U_T/S \to U/S$ extend to a morphism $g'_0 : \tilde{X}_0'/\tilde{S} \to \tilde{X}_0/\tilde{S}$. Denote $Z = X_0 \setminus U$ and $Z' = X_0' \setminus U_T$. Take, see definition-proposition 12(ii), a strict desingularization $\tilde{\epsilon} : (\tilde{X}, D) \to (\tilde{X}_0, \tilde{Z})$ of $(\tilde{X}_0, \tilde{Z})$, a desingularization
\( \varphi : (X', D') \to (X'_0, Z') \) of \((X'_0, Z')\) and a morphism \( \bar{g}' : X' \to X \) such that the following diagram commutes

\[
\begin{array}{ccc}
X'_0 & \xrightarrow{\bar{g}_0} & X_0 \\
\uparrow \varphi & & \downarrow \\
X' & \xrightarrow{\bar{g}'} & X
\end{array}
\]

We then have, see definition-proposition 12(ii), the following commutative diagram in \( \text{Fun}(\Delta, \text{Var}(\mathbb{C})) \)

\[
\begin{array}{ccc}
U = U_{c(\bullet)} & \xrightarrow{j} & X = X_{c(\bullet)} \\
\uparrow g' & & \uparrow i' \\
U_T = U_{T,c(\bullet)} & \xrightarrow{j'} & X' = X'_{c(\bullet)} \\
\downarrow \bar{g}' & & \downarrow \bar{i}' \\
& \xrightarrow{i''} & \bar{D}' = \bar{D}'_{s'(-)} \\
& \downarrow (\bar{g}')_T & \\
& \xrightarrow{(\bar{g}')_T^{-1}} \bar{D}'_{s'(-)} =: i'_{\bar{g}'}
\end{array}
\]

We then consider the following map in \( C(\text{Var}(\mathbb{C}^2/T)) \), see definition 38(ii)

\[
T(g, R^{CH})(Z(U/S)) : g^* R_{(X, D)/S}(Z(U/S))
\]

\[
\xrightarrow{g'^* R_{s'H}(g')}
\]

\[
g'^* R_{s'H}(g') \to g^* R_{(X', D')/S}(Z(U_T/S)) = g'^* g_* R_{(X', D')/T}(Z(U_T/T))
\]

\[
\xrightarrow{\text{add}(g'^*, g_*)(R_{(X', D')/T}(Z(U_T/T)))}
\]

\[
R_{(X', D')/T}(Z(U_T/T))
\]

For

\[
Q^* := (\cdots \to \bigoplus_{\alpha \in \Lambda^n} Z(U_{\alpha}^n/S) \xrightarrow{(Z(g^n_{\alpha, \beta}))} \bigoplus_{\beta \in \Lambda^{n-1}} Z(U_{\beta}^{n-1}/S) \to \cdots) \subseteq C(\text{Var}(\mathbb{C}/S))
\]

a complex of (maybe infinite) direct sum of representable presheaves with \( h^n_{\alpha} : U_{\alpha}^n \to S \) smooth, we get the map in \( C(\text{Var}(\mathbb{C}^2/T)) \)

\[
T(g, R^{CH})(Q^*) : g^* R^{CH}(Q^*) = (\cdots \to \bigoplus_{\alpha \in \Lambda^n} \lim_{(X_{\alpha}, D_{\alpha})/S} g^* R_{(X_{\alpha}, D_{\alpha})/S}(Z(U_{\alpha}^n/S)) \to \cdots)
\]

\[
\xrightarrow{T(g, R^{CH})(Z(U_{\alpha}^n/S))}
\]

\[
(\cdots \to \bigoplus_{\alpha \in \Lambda^n} \lim_{(X_{\alpha}^{n'}, D_{\alpha}^{n'})/T} R_{(X_{\alpha}^{n'}, D_{\alpha}^{n'})/T}(Z(U_{\alpha}^{n', T}/S)) \to \cdots) =: R^{CH}(g'^* Q^*).
\]

Let \( F \in \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S) \). Consider

\[
q : LF := (\cdots \to \bigoplus_{(U_{\alpha}, h_{\alpha}) \in \text{Var}(\mathbb{C})^{sm}/S} Z(U_{\alpha}/S) \to \cdots) \to F
\]

the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in \( C(\text{Var}(\mathbb{C}^2/T)) \)

\[
T(g, R^{CH})(\rho_{S}^* LF) : g^* R^{CH}(\rho_{S}^* LF) =
\]

\[
(\cdots \to \bigoplus_{(U_{\alpha}, h_{\alpha}) \in \text{Var}(\mathbb{C})^{sm}/S} \lim_{(X_{\alpha}, D_{\alpha})/S} g^* R_{(X_{\alpha}, D_{\alpha})/S}(Z(U_{\alpha}/S)) \to \cdots)
\]

\[
\xrightarrow{T(g, R^{CH})(Z(U_{\alpha}/S))}
\]

\[
(\cdots \to \bigoplus_{(U_{\alpha}, h_{\alpha}) \in \text{Var}(\mathbb{C})^{sm}/S} \lim_{(X_{\alpha}^{n'}, D_{\alpha}^{n'})/T} R_{(X_{\alpha}^{n'}, D_{\alpha}^{n'})/T}(Z(U_{\alpha}^{n', T}/S)) \to \cdots) =: R^{CH}(\rho_{T}^* g^* LF).
\]

By functoriality, we get in particular for \( F = F^* \in C(\text{Var}(\mathbb{C})^{sm}/S) \), the map in \( C(\text{Var}(\mathbb{C}^2/T)) \)

\[
T(g, R^{CH})(\rho_{S}^* LF) : g^* R^{CH}(\rho_{S}^* LF) \to R^{CH}(\rho_{T}^* g^* LF).
\]
• Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ and $p : S_1 \times S_2 \to S_1$ the projection. Let $h : U \to S_1$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$
\begin{array}{ccc}
U \times S_2 & \xrightarrow{h \times I} & S_1 \times S_2 \\
\downarrow p' & & \downarrow p \\
U & \xrightarrow{h} & S_1
\end{array}
$$

Take, see definition-proposition 12(i), a compactification $\tilde{f}_0 = \tilde{h} : \tilde{X}_0 \to \tilde{S}_1$ of $h : U \to S_1$. Then $\tilde{f}_0 \times I : \tilde{X}_0 \times S_2 \to \tilde{S}_1 \times S_2$ is a compactification of $h \times I : U \times S_2 \to S_1 \times S_2$ and $p' : U \times S_2 \to U$ extend to $\tilde{p}_0 := p_{X_0} : \tilde{X}_0 \times S_2 \to \tilde{X}_0$. Denote $Z = X_0 \setminus U$. Take see theorem 15(i), a strict desingularization $\tilde{\epsilon} : (\tilde{X}, D) \to (X_0, Z)$ of the pair $(\tilde{X}_0, \tilde{Z})$. Then we have the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$ whose squares are cartesian

$$
U = U_{c(\bullet)} \xrightarrow{j} \tilde{X} \xrightarrow{i} D_{\bullet} \\
\ \\
U \times S_2 = (U \times S_2)_{c(\bullet)} \xrightarrow{j \times I} \tilde{X} \times S_2 \xrightarrow{i \times I} D_{\bullet} \times S_2
$$

Then the map in $C(\text{Var}(\mathbb{C})^2/S_1 \times S_2)$

$$
T(p, R^{CH})(\mathbb{Z}(U/S_1)) : p^* R_{(\tilde{X}, D)/S_1}(\mathbb{Z}(U/S_1)) \xrightarrow{\sim} R_{(\tilde{X} \times S_2, D_{\bullet} \times S_2)/S_1 \times S_2}(\mathbb{Z}(U \times S_2/S_1 \times S_2))
$$

is an isomorphism. Hence, for $Q^* \in C(\text{Var}(\mathbb{C})/S_1)$ a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in $C(\text{Var}(\mathbb{C})^2/S_1 \times S_2)$

$$
T(p, R^{CH})(Q^*) : p^* R^{CH}(Q^*) \xrightarrow{\sim} R^{CH}(p^* Q^*)
$$

is an isomorphism. In particular, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S_1)$ the map in $C(\text{Var}(\mathbb{C})^2/S_1 \times S_2)$

$$
T(p, R^{CH})(\rho_{S_1}^* LF) : p^* R^{CH}(\rho_{S_1}^* LF) \xrightarrow{\sim} R^{CH}(\rho_{S_1 \times S_2}^* p^* LF)
$$

is an isomorphism.

• Let $h_1 : U_1 \to S, h_2 : U_2 \to S$ two morphisms with $U_1, U_2, S \in \text{Var}(\mathbb{C}), U_1, U_2$ smooth. Denote by $p_1 : U_1 \times S U_2 \to U_1$ and $p_2 : U_1 \times S U_2 \to U_2$ the projections. Take, see definition-proposition 12(i)), a compactification $f_{10} = h_1 : X_{10} \to S$ of $h_1 : U_1 \to S$ and a compactification $f_{20} = h_2 : X_{20} \to S$ of $h_2 : U_2 \to S$. Then,

- $\tilde{f}_{10} \times \tilde{f}_{20} : \tilde{X}_{10} \times S \tilde{X}_{20} \to S$ is a compactification of $h_1 \times h_2 : U_1 \times S U_2 \to S$.
- $\tilde{p}_{10} := p_{X_{10}} : \tilde{X}_{10} \times S \tilde{X}_{20} \to \tilde{X}_{10}$ is a compactification of $p_1 : U_1 \times S U_2 \to U_1$.
- $\tilde{p}_{20} := p_{X_{20}} : \tilde{X}_{10} \times S \tilde{X}_{20} \to \tilde{X}_{20}$ is a compactification of $p_2 : U_1 \times S U_2 \to U_2$.

Denote $\tilde{Z}_1 = \tilde{X}_{10} \setminus U_1$ and $\tilde{Z}_2 = \tilde{X}_{20} \setminus U_2$. Take, see theorem 15(i), a strict desingularization $\tilde{\epsilon}_1 : (\tilde{X}_1, D) \to (X_{10}, Z_1)$ of the pair $(\tilde{X}_{10}, \tilde{Z}_1)$ and a strict desingularization $\tilde{\epsilon}_2 : (\tilde{X}_2, E) \to (\tilde{X}_{20}, Z_2)$ of the pair $(\tilde{X}_{20}, \tilde{Z}_2)$. Take then a strict desingularization

$$
\tilde{\epsilon}_{12} : ((\tilde{X}_1 \times S \tilde{X}_2)^N, \tilde{F}) \to (\tilde{X}_1 \times S \tilde{X}_2, (D \times S \tilde{X}_2) \cup (\tilde{X}_1 \times S \tilde{E}))
$$

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of the pair \((\bar{X}_1 \times_S \bar{X}_2, (\bar{D} \times_S \bar{X}_2) \cup (\bar{X}_1 \times_S \bar{E}))\). We have then the following commutative diagram

\[
\begin{array}{c}
\bar{X}_1 \\
\downarrow \bar{f}_2 \\
\bar{X}_1 \times_S \bar{X}_2 \downarrow \bar{p}_2 \\
\downarrow \bar{f}_1 \\
\bar{S}
\end{array}
\]

and

\[
\begin{align*}
& (\bar{f}_1 \times \bar{f}_2) : \bar{X}_1 \times_S \bar{X}_2 \to \bar{S} \text{ is a compactification of } h_1 \times h_2 : U_1 \times_S U_2 \to S. \\
& (\bar{p}_1)^N := \bar{p}_1 \circ \epsilon_{12} : (\bar{X}_1 \times_S \bar{X}_2)^N \to \bar{X}_1 \text{ is a compactification of } p_1 : U_1 \times_S U_2 \to U_1. \\
& (\bar{p}_2)^N := \bar{p}_2 \circ \epsilon_{12} : (\bar{X}_1 \times_S \bar{X}_2)^N \to \bar{X}_2 \text{ is a compactification of } p_2 : U_1 \times_S U_2 \to U_2.
\end{align*}
\]

We have then the morphism in \(C(\text{Var}(\mathbb{C})^2/S)\)

\[
T(\otimes, R_S^{CS}(\mathbb{Z}(U_1/S), \mathbb{Z}(U_2/S)) : R(\bar{X}_1, \bar{D})/S(\mathbb{Z}(U_1/S)) \otimes R(\bar{X}_1, \bar{D})/(\mathbb{Z}(U_2/S)) \cong R(\bar{X}_1 \times_S \bar{X}_2, (\bar{D} \times_S \bar{X}_2) \cup (\bar{X}_1 \times_S \bar{E}))/S(\mathbb{Z}(U_1 \times_S U_2/S))
\]

For

\[
\begin{align*}
Q_1^* & := (\cdots \otimes_{\alpha \in A^n} \mathbb{Z}(U_{1, \alpha}^n/S)) \otimes_{\beta \in A^{n-1}} \mathbb{Z}(U_{2, \beta}^{n-1}/S) \cdots, \\
Q_2^* & := (\cdots \otimes_{\alpha \in A^n} \mathbb{Z}(U_{1, \alpha}^n/S)) \otimes_{\beta \in A^{n-1}} \mathbb{Z}(U_{2, \beta}^{n-1}/S) \cdots) \in C(\text{Var}(\mathbb{C})/S)
\end{align*}
\]

complexes of (maybe infinite) direct sum of representable presheaves with \(U_\alpha^n\) smooth, we get the morphism in \(C(\text{Var}(\mathbb{C})^2/S)\)

\[
T(\otimes, R_S^{CS})(Q_1^*, Q_2^*) : R^{CH}(Q_1^*) \otimes R^{CH}(Q_2^*) \cong R^{CH}(Q_1^* \otimes Q_2^*)
\]

For \(F_1, F_2 \in C(\text{Var}(\mathbb{C})^{m'/m}/S)\), we get in particular the morphism in \(C(\text{Var}(\mathbb{C})^2/S)\)

\[
T(\otimes, R_S^{CS})(\rho_S^* \mathcal{L}F_1, \rho_S^* \mathcal{L}F_2) : R^{CH}(\rho_S^* \mathcal{L}F_1) \otimes R^{CH}(\rho_S^* \mathcal{L}F_2) \to R^{CH}(\rho_S^* (\mathcal{L}F_1 \otimes \mathcal{L}F_2)).
\]

**Definition 39.** Let \(h : U \to S\) a morphism, with \(U, S \in \text{Var}(\mathbb{C})\), \(U\) irreducible. Take, see definition-proposition 12, \(\bar{h}_0 = h_0 : \bar{X}_0 \to \bar{S}\) a compactification of \(h : U \to S\) and denote by \(\bar{Z} = \bar{X}_0 \times \bar{U}\). Take, using theorem 15, a desingularization \(\bar{e} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \Delta)\) of the pair \((\bar{X}_0, \Delta), \bar{Z} \subset \bar{\Delta}\), with \(\bar{X} \in \text{PShmVar}(\mathbb{C})\) and \(\bar{D} := \bar{e}^{-1}(\Delta) = \bigcup_{\epsilon=1}^n \bar{D}_\epsilon \subset \bar{X}\) a normal crossing divisor. Denote \(d_X := \dim(\bar{X}) = \dim(U)\).

(i) The cycle \((\Delta_{\bar{D}_1} \times S) \subset \bar{D}_1 \times \bar{D}_1 \times S\) induces by the diagonal \(\Delta_{\bar{D}_1} \subset \bar{D}_1 \times \bar{D}_1\) the morphism in \(C(\text{Var}(\mathbb{C})^2/S)\)

\[
[\Delta_{\bar{D}_1}] \in \text{Hom}(\mathbb{Z}^{tr}(\bar{D}_1 \times S, D_1)/(\bar{D}_1 \times \bar{S}, D_1), \mathbb{Z}^{tr}(\bar{X} \times S, X)/(\bar{X} \times \bar{S}, X)) \cong \text{Hom}(\mathbb{Z}((\bar{D}_1 \times S \times \bar{D}_1)/(\bar{D}_1 \times S, D_1), \mathbb{Z}^{tr}(\bar{X} \times S, X)/(\bar{X} \times \bar{S}, X))
\]

(ii) The cycle \((\Delta_{\bar{X}} \times S) \subset \bar{X} \times \bar{X} \times S\) induces by the diagonal \(\Delta_{\bar{X}} \subset \bar{X} \times \bar{X}\) the morphism in \(C(\text{Var}(\mathbb{C})^2/S)\)

\[
[\Delta_{\bar{X}}] \in \text{Hom}(\mathbb{Z}^{tr}((\bar{X} \times S, X)/(\bar{X} \times S, X)) \cong \text{Hom}(\mathbb{Z}((\bar{X} \times S \times \bar{X} \times S), \mathbb{Z}^{tr}(\bar{X} \times S, X)/(\bar{X} \times \bar{S}, X))
\]

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Let $h : U \to S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$, $U$ smooth connected (hence irreducible by smoothness). Take, see definition-proposition 12, $f_0 = h_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$ with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \bigcup_{i=1}^r \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Denote $d_X := \dim(\bar{X}) = \dim(U)$.

(iii) We get from (i) and (ii) the morphism in $C(\text{Var}(\mathbb{C})^2 / S)$

$$T(p_{S\cdot _1}, p_{S\cdot _2})[\mathbb{Z}((\bar{D}_* \times S, D_*)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)] := ([\Delta_{D_\cdot _1}], [\Delta_{\bar{X}}]) :$$

$$\text{Cone}(\mathbb{Z}(i_\cdot _1 \times I) : (\mathbb{Z}^I((\bar{D}_* \times S, D_*)/S), u_{IJ}) \to \mathbb{Z}^I((\bar{X} \times S, X)/S)) \to$$

$$p_{S\cdot _1, E_{et}}(\text{Cone}(\mathbb{Z}(i_\cdot _1 \times I) : (\mathbb{Z}((\bar{D}_* \times S, D_*)/\bar{X} \times S), u_{IJ}) \to$$

$$\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)))(d_X)[2d_X] := R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X]$$

(iii)' which gives the map in $C(\text{Var}(\mathbb{C})^{2,\text{smpr}} / S)$

$$T^{I_{et}}(p_{S\cdot _1}, p_{S\cdot _2})[\mathbb{Z}((\bar{D}_* \times S, D_*)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)] :$$

$$\text{Cone}(\mathbb{Z}(i_\cdot _1 \times I) : (\mathbb{Z}^I((\bar{D}_* \times S, D_*)/S), u_{IJ}) \to \mathbb{Z}^I((\bar{X} \times S, X)/S)) =$$

$$L_{p_{S\cdot _1, S\cdot _2}} \text{Cone}(\mathbb{Z}(i_\cdot _1 \times I) : (\mathbb{Z}((\bar{D}_* \times S, D_*)/S), u_{IJ}) \to \mathbb{Z}^I((\bar{X} \times S, X)/S))$$

$$L_{p_{S\cdot _1, S\cdot _2}} R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X]$$

Proposition 40. Let $h : U \to S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$, $U$ irreducible. Take, see definition-proposition 12, $f_0 = h_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$, $\bar{Z} \subset \Delta$ with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \bigcup_{i=1}^r \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Denote $d_X := \dim(\bar{X}) = \dim(U)$.

(i) The morphism

$$[\Delta_{D_\cdot _1}] : \mathbb{Z}^I((\bar{D}_* \times S, D_*)/S), \to p_{S\cdot _1, E_{et}}(\mathbb{Z}((\bar{D}_* \times S, D_*)/\bar{X} \times S))(d_X)[2d_X]$$

given in definition 39(i) is an equivalence $(\mathbb{A}^1, \text{et})$ local.

(ii) The morphism

$$[\Delta_{\bar{X}}] : \mathbb{Z}^I((\bar{X} \times S, X)/S), \to p_{S\cdot _1, E_{et}}(\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))(d_X)[2d_X]$$

given in definition 39(ii) is an equivalence $(\mathbb{A}^1, \text{et})$ local.

Let $h : U \to S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$, $U$ smooth connected (hence irreducible by smoothness). Take, see definition-proposition 12, $f_0 = h_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \setminus U$. Take, using theorem 15(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$, with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \bigcup_{i=1}^r \bar{D}_i \subset \bar{X}$ a normal crossing divisor.

(iii) The morphism

$$T(p_{S\cdot _1}, p_{S\cdot _2})[\mathbb{Z}((\bar{D}_* \times S, D_*)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)] := ([\Delta_{D_\cdot _1}], [\Delta_{\bar{X}}]) :$$

$$\text{Cone}(\mathbb{Z}(i_\cdot _1 \times I) : (\mathbb{Z}^I((\bar{D}_* \times S, D_*)/S), u_{IJ}) \to \mathbb{Z}^I((\bar{X} \times S, X)/S)) \to$$

$$p_{S\cdot _1, E_{et}}(\text{Cone}(\mathbb{Z}(i_\cdot _1 \times I) : (\mathbb{Z}((\bar{D}_* \times S, D_*)/\bar{X} \times S), u_{IJ}) \to$$

$$\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))(d_X)[2d_X] =: R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X]$$

given in definition 39(iii)' is an equivalence $(\mathbb{A}^1, \text{et})$ local.
(iii') The morphism

\[ T^{\mu_q}(p_{S*}, p_{S*})(\mathbb{Z}((\tilde{D}_\bullet \times S, D_{\bullet})/\tilde{X} \times S), \mathbb{Z}((\tilde{X} \times S, X)/\tilde{X} \times S)) : \]

\[ \text{Cone}(\mathbb{Z}(i_* \times I) : (\mathbb{Z}^{tr}((\tilde{D}_\bullet \times S, D_{\bullet})/S), u_{11}) \to \mathbb{Z}^{tr}((\tilde{X} \times S, X)/S)) \]

\[ \to L\rho_{S*} \mu_{S*} R_i(\tilde{X}, \tilde{D})/S(\mathbb{Z}(U/S))(d_X)[2d_X] \]

given in definition 39(iii)' is an equivalence \((\mathbb{A}^1, et)\) local.

Proof. (i): By Yoneda lemma, it is equivalent to show that for every morphism \(g : T \to S\) with \(T \in \text{Var}(\mathbb{C})\) and every closed subset \(E \subset T\), the composition morphism

\[ [\Delta_{\tilde{D}_\bullet}] : \text{Hom}^* (\mathbb{Z}((T, E)/S), C_\bullet \mathbb{Z}^{tr}((\tilde{D}_\bullet \times S, D_{\bullet})/S)) \to \text{Hom}^* (\mathbb{Z}((T, E)/S), p_{S*} E_{et} (\mathbb{Z}((\tilde{D}_\bullet \times S, D_{\bullet})/\tilde{X} \times S)(d_X)[2d_X])) \]

is a quasi-isomorphism of abelian groups. But this map is the composite

\[ \text{Hom}^* (\mathbb{Z}((T, E)/S), C_\bullet \mathbb{Z}^{tr}((\tilde{D}_\bullet \times S, D_{\bullet})/S)) \xrightarrow{[\Delta_{\tilde{D}_\bullet}]} \text{Hom}^* (\mathbb{Z}((T, E)/S), p_{S*} E_{et} (\mathbb{Z}((\tilde{D}_\bullet \times S, D_{\bullet})/\tilde{X} \times S)(d_X)[2d_X])) \]

which is clearly a quasi-isomorphism.

(ii): Similar to (i).

(iii'): Follows from (i) and (ii).

(iii''): Follows from (iii) and the fact that \(\mu_{S*}\) preserve \((\mathbb{A}^1, et)\) local equivalence (see proposition 25) and the fact that \(p_{S*}\) preserve \((\mathbb{A}^1, et)\) local equivalence (see proposition 24).

\[ \square \]

Definition 40. (i) Let \(h : U \to S\) a morphism, with \(U, S \in \text{Var}(\mathbb{C}), U\) smooth. Take, see definition-proposition 12, \(\tilde{h}_0 : \tilde{X}_0 \to \tilde{S}\) a compactification of \(h : U \to S\) and denote by \(\tilde{Z} = \tilde{X}_0 \setminus U\).

Take, using theorem 15(ii), a strict desingularization \(\tilde{c} : (\tilde{X}, \tilde{D}) \to (\tilde{X}_0, \tilde{Z})\) of the pair \((\tilde{X}_0, \tilde{Z})\), with \(\tilde{X} \in \text{PSmVar}(\mathbb{C})\) and \(\tilde{D} := c^{-1}(\tilde{Z}) = \bigcup_{i=1}^d \tilde{D}_i \subset \tilde{X}\) a normal crossing divisor. We will consider the following canonical map in \(C(\text{Var}(\mathbb{C}^{sm})/S)\)

\[ T_{(\tilde{X}, \tilde{D})/S}(U/S) : \text{Gr}^{12}_{S\mathbb{C}} L\rho_{S*} \mu_{S*} R_i(\tilde{X}, \tilde{D})/S(\mathbb{Z}(U/S)) \rightarrow \text{Gr}^{12}_{S\mathbb{C}} \rho_{S*} \mu_{S*} R_i(\tilde{X}, \tilde{D})/S(\mathbb{Z}(U/S)) \]

\[ \tau_{(\tilde{X}, \tilde{D})/S}(U/S) \]

where, for \(h' : V \to S\) a smooth morphism with \(V \in \text{Var}(\mathbb{C})\),

\[ i^{00}(U/S)(V/S) : \mathbb{Z}((U \times S, U)/U \times S)(V \times U \times S, V \times U \times S) \to \mathbb{Z}(U/U)(V \times S) \to \mathbb{Z}(U/U)(V \times S) \]

which gives

\[ i^0(U/S)(V/S) : E_{et}^0(\mathbb{Z}((U \times S, U)/U \times S))(V \times U \times S, V \times S U \times U) \to E_{et}^0(\mathbb{Z}(U/U))(V \times S) \]

and by induction

\[ \tau^\leq(i)(U/S) : \text{Gr}^{12}_{S\mathbb{C}} \rho_{S*} \mu_{S*} E_{et}^i(\mathbb{Z}((U \times S, U)/U \times S)) \to h_* E_{et}^i(\mathbb{Z}(U/U)) \]

where \(\tau^\leq\) is the cohomological truncation.

(ii) Let \(g : U' \to U/S\) a morphism, with \(U'/S = (U', h'), U/S = (U, h) \in \text{Var}(\mathbb{C})/S, U, U'\) smooth. Take, see definition-proposition 12(ii), a compactification \(\tilde{h}_0 : \tilde{X}_0 \to \tilde{S}\) of \(h : U \to S\) and a compactification \(\tilde{h}_0' = h' : \tilde{X}'_0 \to \tilde{S}\) of \(h' : U' \to S\) such that \(g : U'/S \to U/S\) extend to a morphism
\[ \tilde{\gamma}_0 : X'_0 / \tilde{S} \to \tilde{X}_0 / \tilde{S}. \] Denote \( \bar{Z} = X_0 \setminus U \) and \( \bar{Z}' = X'_0 \setminus U' \). Take, see definition-proposition 12(ii), a strict desingularization \( \tilde{\epsilon} : (\tilde{X}, \tilde{D}) \to (\tilde{X}_0, \tilde{Z}) \) of \( (X_0, Z) \), a strict desingularization \( \tilde{\epsilon}' : (X'_0, \tilde{Z}') \to (\tilde{X}_0, \tilde{Z}') \) and a morphism \( \tilde{g} : \tilde{X}' \to \tilde{X} \) such that the following diagram commutes

\[
\begin{array}{ccc}
X'_0 & \xrightarrow{\tilde{g}_0} & \tilde{X}_0 \\
\tilde{\epsilon} \downarrow & & \downarrow \iota \\
X' & \xrightarrow{\tilde{g}} & \tilde{X}
\end{array}
\]

Then by the diagram given in definition 38(ii), the following diagram in \( C(\text{Var}(\mathbb{C})^{sm}/S) \) obviously commutes

\[
\begin{array}{ccc}
\text{Gr}_S^{12} L_{\rho_S, \mu_S} R_{(\tilde{X}, \tilde{D})/S}(\mathbb{Z}(U/S)) & \xrightarrow{T_{(X, D)}/S(U/S)} & h_* E_{et}(\mathbb{Z}(U/U)) := D^0_S(\mathbb{Z}(U/S)) \\
R^H_S ((g)) & & T((g), \tilde{E}) := D^0_S((g))
\end{array}
\]

\[
\begin{array}{ccc}
\text{Gr}_S^{12} L_{\rho_S, \mu_S} R_{(X', D')/S}(\mathbb{Z}(U'/S)) & \xrightarrow{T_{(X', D')}/S(U'/S)} & h'_* E_{et}(\mathbb{Z}(U'/U')) := D^0_S(\mathbb{Z}(U'/S))
\end{array}
\]

where \( l(U/S) \) are \( l(U'/S) \) are the maps given in (i).

(iii) Let \( S \in \text{SmVar}(\mathbb{C}) \). Let \( F \in C(\text{Var}(\mathbb{C})^{sm}/S) \). We get from (i) and (ii) morphisms in \( C(\text{Var}(\mathbb{C})^{sm}/S) \)

\[
T^{CH}_S (LF) : \text{Gr}_S^{12} L_{\rho_S, \mu_S} R_{(\tilde{X}, \tilde{D})/S}(\rho_S^* LF) \xrightarrow{\iota(LF)} D^0_S (L(F))
\]

We will also have the following lemma

**Lemma 1.** (i) Let \( h : U \to S \) a morphism, with \( U, S \in \text{Var}(\mathbb{C}) \), \( U \) smooth. Take, see definition-proposition 12, \( f_0 = h_0 : X_0 \to S \) a compactification of \( h : U \to S \) and denote by \( \bar{Z} = X_0 \setminus U \). Take, using theorem 15(ii), a strict desingularization \( \tilde{\epsilon} : (\tilde{X}, \tilde{D}) \to (\tilde{X}_0, \tilde{Z}) \) of the pair \( (X_0, Z) \), with \( \tilde{X} \in \text{PSmVar}(\mathbb{C}) \) and \( \tilde{D} := \tilde{\epsilon}^{-1}(Z) = \sqcup_{i=1}^s \tilde{D}_i \subset \tilde{X} \) a normal crossing divisor. Then the map in \( C(\text{Var}(\mathbb{C})^{2, smpr}/S) \)

\[
\text{ad}(\text{Gr}_S^{12*}, \text{Gr}_S^{12})(L_{\rho_S, \mu_S} R_{(\tilde{X}, \tilde{D})/S}(\mathbb{Z}(U/S))) \circ q : \text{Gr}_S^{12*} L \text{Gr}_S^{12} L_{\rho_S, \mu_S} R_{(\tilde{X}, \tilde{D})/S}(\mathbb{Z}(U/S)) \to L_{\rho_S, \mu_S} R_{(\tilde{X}, \tilde{D})/S}(\mathbb{Z}(U/S))
\]

is an equivalence \((A^1, \text{et})\) local.

(ii) Let \( S \in \text{SmVar}(\mathbb{C}) \). Let \( F \in C(\text{Var}(\mathbb{C})^{sm}/S) \). Then the map in \( C(\text{Var}(\mathbb{C})^{2, smpr}/S) \)

\[
\text{ad}(\text{Gr}_S^{12*}, \text{Gr}_S^{12})(L_{\rho_S, \mu_S} R_{(\tilde{X}, \tilde{D})/S}(\rho_S^* LF)) \circ q : \text{Gr}_S^{12*} L \text{Gr}_S^{12} L_{\rho_S, \mu_S} R_{(\tilde{X}, \tilde{D})/S}(\rho_S^* LF) \to L_{\rho_S, \mu_S} R_{(\tilde{X}, \tilde{D})/S}(\rho_S^* LF)
\]

is an equivalence \((A^1, \text{et})\) local.

**Proof.** (i): Follows from proposition 40.

(ii): Follows from (i). \(\square\)

**Definition 41.** (i) Let \( h : U \to S \) a morphism, with \( U, S \in \text{Var}(\mathbb{C}) \) and \( U \) smooth. Take, see definition-proposition 12, \( f_0 = h_0 : X_0 \to S \) a compactification of \( h : U \to S \) and denote by \( \bar{Z} = X_0 \setminus U \). Take, using theorem 15(ii), a strict desingularization \( \tilde{\epsilon} : (\tilde{X}, \tilde{D}) \to (\tilde{X}_0, \tilde{Z}) \) of the pair \( (X_0, Z) \), with \( \tilde{X} \in \text{PSmVar}(\mathbb{C}) \) and \( \tilde{D} := \tilde{\epsilon}^{-1}(Z) = \sqcup_{i=1}^s \tilde{D}_i \subset \tilde{X} \) a normal crossing divisor. We denote by \( i_* : \tilde{D}_* \leftrightarrow \tilde{X} = X_c(\bullet) \) the morphism of simplicial varieties given by the closed embeddings
Let $i : D_I = \cap_{i \in I} D_i \hookrightarrow \bar{X}$. We denote by $j : U \hookrightarrow \bar{X}$ the open embedding and by $p_S : \bar{X} \times S \rightarrow S$ and $p_S : U \times S \rightarrow S$ the projections. Considering the graph factorization $\bar{f} : X \xto{\bar{f}} X \times S \xto{p_S} S$ of $\bar{f} : \bar{X} \rightarrow S$, where $\bar{I}$ is the graph embedding and $p_S$ the projection, we get closed embeddings $I := \bar{I} 	imes_{\bar{S}} S : X \hookrightarrow \bar{X} \times S$ and $I_{D_I} := D_I \times_{\bar{X}} \bar{I} : D_I \rightarrow D_I \times S$. We then consider the map in $C(\text{Var}(\mathbb{C})^{d, smpr}, S)$

$$T(\tilde{R}^{CH}, R^{CH})(\mathbb{Z}(U/S)) : \tilde{R}(\mathbb{X}, \mathbb{D})/S(\mathbb{Z}(U/S))$$

$$\cong \text{Cone}(\mathbb{Z}(i_* \times I) : (\mathbb{Z}^{tr}((\bar{D}_s \times S, D_s)_, S), u_{IJK}) \rightarrow \mathbb{Z}^{tr}((\bar{X} \times S, X)/S)(-d_X)[-2d_X]$$

$$T^{\alpha,q}(p_S, p_S')((\mathbb{Z}^{tr}((\bar{D}_s \times S, D_s)_, S), S, (\mathbb{Z}(X \times S, X, X)/S)(-d_X)[-2d_X])$$

$$L_{p_S + p_{S'}} R(\mathbb{X}, \mathbb{D})/S(\mathbb{Z}(U/S)).$$

given in definition 39(ii).

(ii) Let $g : U' \rightarrow S$ a morphism, with $U' \rightarrow S = (U', h'), U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, with $U$ and $U'$ smooth. Take, see definition-proposition 12(ii), a compactification $f_0 = h' : X_0 \rightarrow S$ of $h : U \rightarrow S$ and a compactification $f'_0 = h' : X'_0 \rightarrow S$ of $h' : U' \rightarrow S$ such that $g : U'/S \rightarrow U/S$ extend to a morphism $\bar{g}_0 : X'_0/S \rightarrow X_0/S$. Denote $\bar{Z} = X_0 \setminus U$ and $\bar{Z}' = X_0 \setminus U'$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{e} : (X, \mathbb{D}) \rightarrow (X_0, \bar{Z})$ of $(X_0, \bar{Z})$, a strict desingularization $\bar{e} : (X', \mathbb{D}') \rightarrow (X'_0, \bar{Z}')$ of $(X'_0, \bar{Z}')$ and a morphism $\bar{g} : X' \rightarrow \bar{X}$ such that the following diagram commutes

$$\begin{array}{ccc}
X'_0 & \xrightarrow{\bar{g}_0} & X_0 \\
\downarrow i' & \downarrow \bar{e} & \downarrow \bar{e} \\
X' & \xrightarrow{\bar{g}} & \bar{X}
\end{array}$$

We then have, see definition-proposition 12(ii), the diagram (47) in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccc}
U = U_c(\bullet) & \xrightarrow{j} & X = X_c(\bullet) & \xleftarrow{i} & D_{s_D}(\bullet) \\
g \downarrow & \downarrow & \bar{g} \downarrow & \downarrow & i' \bar{g}' \downarrow \\
U' = U_c(\bullet) & \xrightarrow{j'} & X' = X_c(\bullet) & \xleftarrow{i'} & D_{s_D}(\bullet) \xrightarrow{i'' \bar{g}^{-1}(D_{s_D}(\bullet))} i''_{\bar{g}'}. \\
\end{array}$$

Consider

$$[\Gamma'_g]' \in \text{Hom}(\mathbb{Z}^{tr}((\bar{X} \times S, X)/S)(-d_X)[-2d_X], \mathbb{Z}^{tr}((\bar{X}' \times S, X')/S)(-d_X)[-2d_X]).$$

$$\cong \text{Hom}(\mathbb{Z}^{tr}((\bar{X} \times \mathbb{A}^{d_X'} \times S, X \times \mathbb{A}^{d_X'})/S), \mathbb{Z}^{tr}((\bar{X}' \times \mathbb{A}^{d_X'} \times S, X' \times \mathbb{A}^{d_X'})/S)/\mathbb{Z}^{tr}((-) \times \mathbb{P}^{d_X-1}, (-) \times \mathbb{P}^{d_X-1})).$$

the morphism given by the transpose of the graph $\Gamma_g \subset X' \times S$ of $\bar{g} : X' \rightarrow \bar{X}$. Then, since $i_* \bar{g}' = \bar{g} \circ i''_{\bar{g}'}$, we have the factorization

$$[\Gamma'_g]' \circ Z(i_* \times I) : (\mathbb{Z}^{tr}((\bar{D}_{s_D}(\bullet) \times S, D_{s_D}(\bullet))/S), u_{IJK})(-d_X)[-2d_X]$$

$$\xrightarrow{[\Gamma'_g]' \circ \bar{Z}(\bar{g}^{-1}(\bar{D}_{s_D}(\bullet))) \times S, (\bar{g}^{-1}(\bar{D}_{s_D}(\bullet)))/S, u_{IJK})(-d_X)[-2d_X].$$

with

$$[\Gamma'_g]' \in \text{Hom}(\mathbb{Z}^{tr}((\bar{D}_{s_D}(\bullet) \times \mathbb{A}^{d_X'} \times S, D_{s_D}(\bullet) \times \mathbb{A}^{d_X'})/S), u_{IJK}), \mathbb{Z}^{tr}((\bar{X} \times S, X)/S)(-d_X)[-2d_X]).$$

$$\mathbb{Z}^{tr}((\bar{g}^{-1}(\bar{D}_{s_D}(\bullet)) \times \mathbb{P}^{d_X} \times S, \bar{g}^{-1}(\bar{D}_{s_D}(\bullet)) \times \mathbb{P}^{d_X})/S, u_{IJK})/\mathbb{Z}^{tr}((-) \times \mathbb{P}^{d_X-1}, (-) \times \mathbb{P}^{d_X-1}).$$

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We then consider the following map in $C(\mathrm{Var}(C)^{2pr}/S)$

$$\hat{R}_S^{CH}(g) : \hat{R}(X, D)/S(Z(U/S)) \rightarrow \hat{R}(X, D)/S(Z(U/S))$$

$$(\mathrm{Cone}(Z(i \times I) : (Z^t((\bar{D}_{s_1}(\bullet) \times S, D_{s_2}(\bullet))/S, u_{1,1})) \rightarrow Z^t((\bar{X} \times S, X)/S))(-dx)[-2dx] \rightarrow \hat{R}(X, D)/S(Z(U/S))$$

$$\Rightarrow \quad \hat{R}_S^{CH}(g) : \hat{R}(X, D)/S(Z(U/S)) \rightarrow \hat{R}(X, D)/S(Z(U/S))$$

Then the following diagram in $C(\mathrm{Var}(C)^{2,\mathrm{smpr}}/S)$ commutes by definition

$$\begin{array}{ccc}
\hat{R}(X, D)/S(Z(U/S)) & \xrightarrow{T(\hat{R}_S^{CH}, R_S^{CH})(Z(U/S)) \Rightarrow L_{\rho_S + \mu_S} R(\bar{X}, D)/S(Z(U/S))} \\
\hat{R}_S^{CH}(g) & \Rightarrow & L_{\rho_S + \mu_S} R_S^{CH}(g)
\end{array}$$

(iii) For $g_1 : U''/S \rightarrow U'/S$, $g_2 : U'/S \rightarrow U/S$ two morphisms with $U''/S = (U', h')$, $U'/S = (U', h)$, $U/S = (U, h) \in \mathrm{Var}(C)/S$, with $U$, $U'$ and $U''$ smooth. We get from (i) and (ii) a compactification $\hat{h} = h : \bar{X} \rightarrow \bar{S}$ of $h : U \rightarrow S$, a compactification $\hat{f} = f : \bar{X} \rightarrow \bar{S}$ of $h' : U' \rightarrow S$, and a compactification $\hat{f}'' = h'' : X'' \rightarrow \bar{S}$ of $h'' : U'' \rightarrow S$, with $X, X', X'' \in \mathrm{PSmVar}(C)$, $D := X \setminus X' \subset X$, $D' := X' \setminus U' \subset X'$, and $D'' := X'' \setminus U'' \subset X''$ normal crossing divisors, such that $g_1 : U''/S \rightarrow U'/S$ extend to $\bar{g}_1 : \bar{X}/\bar{S} \rightarrow \bar{X}'/\bar{S}$, $g_2 : U'/S \rightarrow U/S$ extend to $\bar{g}_2 : \bar{X}'/\bar{S} \rightarrow \bar{X}/\bar{S}$, and

$$\hat{R}_S^{CH}(g_2 \circ g_1) = \hat{R}_S^{CH}(g_1) \circ \hat{R}_S^{CH}(g_2) : \hat{R}(\bar{X}, \bar{D})/S \rightarrow \hat{R}(\bar{X}'', \bar{D}'')/S$$

(iv) For

$$Q^* := (\cdots \rightarrow \oplus_{\alpha \in \Lambda^n} Z(U^n_{\alpha}/S) \xrightarrow{(Z(g_{n, \alpha}))} \oplus_{\beta \in \Lambda^{n-1}} Z(U^{n-1}_{\beta}/S) \rightarrow \cdots) \in C(\mathrm{Var}(C)/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with $U_{\alpha}^*$ smooth, we get from (i),(ii) and (iii) the map in $C(\mathrm{Var}(C)^{2,\mathrm{smpr}}/S)$

$$T(\hat{R}_S^{CH}, R_S^{CH})(Q^*) : \hat{R}_S^{CH}(Q^*) := (\cdots \rightarrow \oplus_{\beta \in \Lambda^{n-1}} \lim_{(X^{n-1}_{\beta}, D^{n-1}_{\beta})/S} \hat{R}((X^{n-1}_{\beta}, D^{n-1}_{\beta})/S) \rightarrow \cdots) \rightarrow L_{\rho_S + \mu_S} R_S^{CH}(Q^*),$$

where for $(U^n_{\alpha}, h^n_{\alpha}) \in \mathrm{Var}(C)/S$, the inductive limit run over all the compactifications $\hat{f}_\alpha : \bar{X}_\alpha \rightarrow \bar{S}$ of $h_{\alpha} : U_{\alpha} \rightarrow S$ with $X_\alpha \in \mathrm{PSmVar}(C)$ and $D_\alpha := X_\alpha \setminus U_{\alpha}$ a normal crossing divisor. For $m = (m^*) : Q_1^* \rightarrow Q_2^*$ a morphism with

$$Q_1^* := (\cdots \rightarrow \oplus_{\alpha \in \Lambda^n} Z(U_{1, \alpha}/S) \xrightarrow{(Z(g_{1, \alpha}))} \oplus_{\beta \in \Lambda^{n-1}} Z(U^{n-1}_{1, \beta}/S) \rightarrow \cdots),$$

$$Q_2^* := (\cdots \rightarrow \oplus_{\alpha \in \Lambda^n} Z(U_{2, \alpha}/S) \xrightarrow{(Z(g_{2, \alpha}))} \oplus_{\beta \in \Lambda^{n-1}} Z(U^{n-1}_{2, \beta}/S) \rightarrow \cdots) \in C(\mathrm{Var}(C)/S)$$
Let $q$ be an homotopy in $U$ to a compactification $\bar{\mathcal{C}}$ of (maybe infinite) direct sum of representable presheaves with $\mathcal{Q}_1, \mathcal{Q}_2 \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}$ and $U_{2,\alpha}$ smooth,

$$\begin{array}{ccc}
\hat{R}^{CH}(Q_2^*) & \xrightarrow{\iota (\hat{R}_S^{CH}, \hat{R}_S^{CH})(Q_2^*)} & L_{P\mathcal{S}_{\mu S}\mathcal{R}^{CH}}(Q_2^*) \\
\hat{R}_S^{CH}(m) := (\hat{R}_S^{CH}(m^*)) & \xrightarrow{T (\hat{R}_S^{CH}, \hat{R}_S^{CH})(Q_1^*)} & L_{P\mathcal{S}_{\mu S}\mathcal{R}^{CH}}(Q_1^*)
\end{array}$$

- Let $S \in \text{Var}(\mathbb{C})$. For $(h, m, m') = (h^*, m^*, m'^*) : Q_1^*[1] \to Q_2^*$ an homotopy with $Q_1^*, Q_2^* \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}$ and $U_{2,\alpha}$ smooth,

$$(\hat{R}_S^{CH}(h), \hat{R}_S^{CH}(m), \hat{R}_S^{CH}(m')) = (\hat{R}_S^{CH}(h^*), \hat{R}_S^{CH}(m^*), \hat{R}_S^{CH}(m'^*)) : R^{CH}(Q_2^*)[1] \to R^{CH}(Q_1^*)$$

is an homotopy in $C(\text{Var}(\mathbb{C})/S)$ using definition 41 (iii). In particular if $m : Q_1^* \to Q_2^*$ with $Q_1^*, Q_2^* \in C(\text{Var}(\mathbb{C})/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U_{1,\alpha}$ and $U_{2,\alpha}$ smooth is an homotopy equivalence, then $\hat{R}_S^{CH}(m) : R^{CH}(Q_2^*) \to R^{CH}(Q_1^*)$ is an homotopy equivalence.

- Let $S \in \text{SmVar}(\mathbb{C})$. Let $F \in \text{PSh}(\text{Var}(\mathbb{C})/S)$. Consider

$$q : LF := (\cdots \to \oplus (U_{\alpha}, h_\alpha) \in \text{Var}(\mathbb{C})/S \mathbb{Z}(U_\alpha/S) \xrightarrow{(g_{a,b}^\alpha)} \oplus (U_{\alpha}, h_\alpha) \in \text{Var}(\mathbb{C})/S \mathbb{Z}(U_\alpha/S) \to \cdots) \to F$$

the canonical projective resolution given in subsection 2.3.3. Note that the $U_{\alpha}$ are smooth since $S$ is smooth and $h_\alpha$ are smooth morphism. Definition 41(iv) gives in this particular case the map in $C(\text{Var}(\mathbb{C})/S)$

$$T(\hat{R}_S^{CH}, \hat{R}_S^{CH})(\rho_S^{*}LF) : \hat{R}^{CH}(\rho_S^{*}LF) := (\cdots \to \oplus (U_{\alpha}, h_\alpha) \in \text{Var}(\mathbb{C})/S \lim_{(X_{\alpha}, D_{\alpha})/S} \hat{R}(\bar{X}_{\alpha}, h_{\alpha}))/S(\mathbb{Z}(U_\alpha/S)) \to \cdots) \to L_{P\mathcal{S}_{\mu S}\mathcal{R}^{CH}}(\rho_S^{*}LF),$$

where for $(U_{\alpha}, h_\alpha) \in \text{Var}(\mathbb{C})/S$, the inductive limit run over all the compactifications $\bar{h}_{\alpha} : \bar{X}_{\alpha} \to \bar{S}$ of $h_\alpha : U_\alpha \to S$ with $\bar{X}_{\alpha} \in \text{PSmVar}(\mathbb{C})$ and $D_{\alpha} := \bar{X}_{\alpha} \setminus U_\alpha$ a normal crossing divisor. Definition 41(iv) gives then by functoriality in particular, for $F = F^* \in C(\text{Var}(\mathbb{C})^{sm}/S)$, the map in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T(\hat{R}_S^{CH}, \hat{R}_S^{CH})(\rho_S^{*}LF) : \hat{R}^{CH}(\rho_S^{*}LF) \to L_{P\mathcal{S}_{\mu S}\mathcal{R}^{CH}}(\rho_S^{*}LF).$$

- Let $g : T \to S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $h : U \to S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{array}{ccc}
U_T & \xrightarrow{h'} & T \\
\downarrow g' & & \downarrow g \\
U & \xrightarrow{h} & S
\end{array}$$

Note that $U$ is smooth since $S$ and $h$ are smooth, and $U_T$ is smooth since $T$ and $h'$ are smooth. Take, see definition-proposition 12(ii), a compactification $\bar{h}_0 : \bar{X}_0 \to \bar{S}$ of $h : U \to S$ and a compactification $\bar{h}_1 = \bar{g} \circ h' : \bar{X}_1 \to \bar{S}$ of $g \circ h' : U' \to S$ such that $g' : U_T/S \to U/S$ extend to a morphism $\bar{g}_0' : \bar{X}_0 \to \bar{S}$ to $\bar{S}$. Denote $Z = \bar{X}_0 \setminus U$ and $\bar{Z}' = \bar{X}_0 \setminus U_T$. Take, see definition-proposition 12(ii), a strict desingularization $\hat{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$, a desingularization
We then have, see definition-proposition 12(ii), the following commutative diagram in $\text{Fun}(\Delta^C)$:

$$
\begin{array}{ccc}
\bar{X}_0 & \xrightarrow{\bar{g}_0} & \bar{X}_0 \\
\downarrow{\bar{\iota}} & & \downarrow{\bar{\iota}} \\
\bar{X}' & \xrightarrow{\bar{g}'} & \bar{X}
\end{array}
$$

We then consider the following map in $C(\text{Var}(\mathbb{C})^2, \text{spr}/T)$:

$$
U = U_c(\bullet) \xrightarrow{j} \bar{X} = \bar{X}_c(\bullet) \xrightarrow{\iota^*} \bar{D}_{s'(\bullet)}
$$

$$
U_T = U_{T,c}(\bullet) \xrightarrow{j^*} \bar{X}' = \bar{X}'_{c(\bullet)} \xrightarrow{\iota'_{\bullet}} \bar{D}' = \bar{D}'_{s'(\bullet)} : \iota'_{g_{\bullet}}^{-1}(\bar{D}_{s'(\bullet)})
$$

We then consider the following map in $C(\text{Var}(\mathbb{C})^2, \text{spr}/T)$:

$$
T(g, \hat{R}^{CH}((\mathbb{Z}/U/S)) : g^*\hat{R}((\bar{X},D)/S)) \xrightarrow{\iota_{\bullet}} \text{Cone}(\mathbb{Z}_{(i_{\bullet} \times I)}(\mathbb{Z}^r((\bar{D} \times S, D_{\bullet}))/S) \cup \cup_{u_{1,J}}) \xrightarrow{T_{(g,L)}(-) \circ T(g,\iota)(-)} \mathbb{Z}^r((\bar{X} \times S, X)/S) \xrightarrow{(-d_X)[-2d_X]} \mathbb{Z}^r((\bar{X} \times X)/S) \xrightarrow{T_{(g,L)}(-) \circ T(g,\iota)(-)} \mathbb{Z}^r((\bar{X} \times X)/S) \xrightarrow{(-d_X)[-2d_X]} \mathbb{Z}^r((\bar{X} \times X)/S) \xrightarrow{T_{(g,L)}(-) \circ T(g,\iota)(-)} \mathbb{Z}^r((\bar{X} \times X)/S)
$$

For

$$
Q^* := (\cdots \rightarrow \bigoplus_{\alpha \in \Lambda_n} \mathbb{Z}(U_{\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha}^n))} \bigoplus_{\beta \in \Lambda_{n-1}} \mathbb{Z}(U_{\beta}^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)
$$

a complex of (maybe infinite) direct sum of representable presheaves with $h_n^\alpha : U_{\alpha}^n \to S$ smooth, we get the map in $C(\text{Var}(\mathbb{C})^2, \text{spr}/T)$:

$$
T(g, \hat{R}^{CH})(Q^*) : g^*\hat{R}^{CH}((\mathbb{Z}/U/S)) \xrightarrow{T_{(g,L)}(-) \circ T(g,\iota)(-)} \mathbb{Z}^r((\bar{X} \times X)/S) \xrightarrow{(-d_X)[-2d_X]} \mathbb{Z}^r((\bar{X} \times X)/S) \xrightarrow{(-d_X)[-2d_X]} \mathbb{Z}^r((\bar{X} \times X)/S) \xrightarrow{(-d_X)[-2d_X]} \mathbb{Z}^r((\bar{X} \times X)/S)
$$

$$
T(g, \hat{R}^{CH})(Q^*) \xrightarrow{T_{(g,L)}(-) \circ T(g,\iota)(-)} \mathbb{Z}^r((\bar{X} \times X)/S) \xrightarrow{(-d_X)[-2d_X]} \mathbb{Z}^r((\bar{X} \times X)/S) \xrightarrow{(-d_X)[-2d_X]} \mathbb{Z}^r((\bar{X} \times X)/S) \xrightarrow{(-d_X)[-2d_X]} \mathbb{Z}^r((\bar{X} \times X)/S)
$$

$$
\xrightarrow{T(g, \hat{R}^{CH})(Q^*)} \xrightarrow{T(g, \hat{R}^{CH})(Q^*)} \xrightarrow{T(g, \hat{R}^{CH})(Q^*)} \xrightarrow{T(g, \hat{R}^{CH})(Q^*)}
$$

$$
\begin{array}{ccc}
g^*\hat{R}^{CH}(Q^*) & \xrightarrow{T(g, \hat{R}^{CH})(Q^*)} & \hat{R}^{CH}(g^*Q^*)
\end{array}
$$

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Let $F \in \mathbf{PSh}((\text{Var}(\mathbb{C}))^{\text{sm}})/S)$. Consider
\[ q : LF := (\cdots \to \oplus_{(U_\alpha, h_\alpha)} \in \text{Var}(\mathbb{C})^{\text{sm}}/S} \mathbb{Z}(U_\alpha/S) \to \cdots) \to F \]
the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in $C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/T)$
\[ T(g, \hat{R}^{CH})(\rho_2^* LF) : g^* \hat{R}^{CH}(\rho_2^* LF) = \]
\[ (\cdots \to \oplus_{(U_\alpha, h_\alpha)} \in \text{Var}(\mathbb{C})^{\text{sm}}/S} \lim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} g^* \hat{R}(\bar{X}_\alpha, \bar{D}_\alpha)/S(Z(U_\alpha/S)) \to \cdots) \to (T(g, \hat{R}^{CH})(Z(U_\alpha/S))) \]
and by functoriality, we get in particular for $F = F^* \in C(\text{Var}(\mathbb{C})^{\text{sm}})/S)$, the map in $C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/T)$
\[ T(g, \hat{R}^{CH})(\rho_2^* LF) : g^* \hat{R}^{CH}(\rho_2^* LF) \to \hat{R}^{CH}(\rho_2^* g^* LF) \]

- Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ and $p : S_1 \times S_2 \to S_1$ the projection. Let $h : U \to S_1$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square
\[
\begin{array}{ccc}
U \times S_2 & \xrightarrow{h \times I} & S_1 \times S_2 \\
\downarrow{p'} & & \downarrow{p} \\
U & \xrightarrow{h} & S_1 \\
\end{array}
\]

Take, see definition-proposition 12(i), a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}_1$ of $h : U \to S_1$. Then $\bar{f}_0 \times I : \bar{X}_0 \times S_2 \to \bar{S}_1 \times S_2$ is a compactification of $h \times I : U \times S_2 \to S_1 \times S_2$ and $p' : U \times S_2 \to U$ extend to $\bar{p}_0 = p_{X_0} : \bar{X}_0 \times S_2 \to \bar{X}_0$. Denote $Z = X_0 \setminus U$. Take see theorem 15(i), a strict desingularization $\epsilon : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$. We then have the commutative diagram (48) in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$ whose squares are cartesian

\[
\begin{array}{ccc}
U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} \\
\downarrow{g} & & \downarrow{i} \\
U \times S_2 = (U \times S_2)_{c(\bullet)} & \xrightarrow{j \times I} & \bar{X} \times S_2 \\
\end{array}
\]

Then the map in $C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/S_1 \times S_2)$
\[ T(p, \hat{R}^{CH})(\mathbb{Z}(U/S_1)) : p^* \hat{R}(\bar{X}_\alpha, \bar{D}_\alpha)/S_1(\mathbb{Z}(U/S_1)) \cong \hat{R}(\bar{X} \times S_2, \bar{D} \times S_2)/S_1 \times S_2(\mathbb{Z}(U \times S_2/S_1 \times S_2)) \]
is an isomorphism. Hence, for $Q^* \in C(\text{Var}(\mathbb{C})/S_1)$ a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in $C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/S_1 \times S_2)$
\[ T(p, \hat{R}^{CH})(Q^*) : p^* \hat{R}^{CH}(Q^*) \cong \hat{R}^{CH}(p^* Q^*) \]
is an isomorphism. In particular, for $F \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S_1)$ the map in $C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/S_1 \times S_2)$

$$T(p, \hat{R}^{CH}(\rho_{S_1}, LF)) : p* \hat{R}^{CH}(\rho_{S_1}, LF) \xrightarrow{\sim} \hat{R}^{CH}(\rho_{S_1 \times S_2}, p^* LF)$$

is an isomorphism.

- Let $h_1 : U_1 \rightarrow S$, $h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2, S \in \text{Var}(\mathbb{C})$, $U_1, U_2$ smooth. Denote by $p_1 : U_1 \times_S U_2 \rightarrow U_1$ and $p_2 : U_1 \times_S U_2 \rightarrow U_2$ the projections. Take, see definition-proposition 12(i)), a compactification $\bar{f}_{10} = \hat{h}_1 : \bar{X}_{10} \rightarrow \bar{S}$ of $h_1 : U_1 \rightarrow S$ and a compactification $\bar{f}_{20} = \hat{h}_2 : \bar{X}_{20} \rightarrow \bar{S}$ of $h_2 : U_2 \rightarrow S$. Then,

$$- \bar{f}_{10} \times \bar{f}_{20} : \bar{X}_{10} \times_S \bar{X}_{20} \rightarrow \bar{S}$$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \rightarrow S$.

- $\bar{p}_{10} := p_{X_{10}} : \bar{X}_{10} \times_S \bar{X}_{20} \rightarrow \bar{X}_{10}$ is a compactification of $p_1 : U_1 \times_S U_2 \rightarrow U_1$.

- $\bar{p}_{20} := p_{X_{20}} : \bar{X}_{10} \times_S \bar{X}_{20} \rightarrow \bar{X}_{20}$ is a compactification of $p_2 : U_1 \times_S U_2 \rightarrow U_2$.

Denote $\bar{Z}_1 = \bar{X}_{10} \setminus U_1$ and $\bar{Z}_2 = \bar{X}_{20} \setminus U_2$. Take, see theorem 15(i), a strict desingularization $\tilde{\epsilon}_1 : (\bar{X}_1, \bar{D}) \rightarrow (\bar{X}_{10}, \bar{Z}_1)$ of the pair $(\bar{X}_{10}, \bar{Z}_1)$ and a strict desingularization $\tilde{\epsilon}_2 : (\bar{X}_2, \bar{E}) \rightarrow (\bar{X}_{20}, \bar{Z}_2)$ of the pair $(\bar{X}_{20}, \bar{Z}_2)$. Take then a strict desingularization

$$\tilde{\epsilon}_{12} : ((\bar{X}_1 \times_S \bar{X}_2)^N, \bar{F}) \rightarrow (\bar{X}_1 \times_S \bar{X}_2, (\bar{D} \times_S \bar{X}_2) \cup (\bar{X}_1 \times_S \bar{E}))$$

of the pair $(\bar{X}_1 \times_S \bar{X}_2, (\bar{D} \times_S \bar{X}_2) \cup (\bar{X}_1 \times_S \bar{E}))$. We have then the following commutative diagram

$$\begin{array}{ccc}
\bar{X}_1 & \xrightarrow{\bar{f}_1} & \bar{S} \\
\downarrow{\bar{p}_2} & & \downarrow{\bar{f}_2} \\
\bar{X}_1 \times_S \bar{X}_2 & \xrightarrow{(\bar{p}_2)^N} & \bar{X}_2 \\
\downarrow{\tilde{\epsilon}_{12}} & & \downarrow{\tilde{\epsilon}_{12}} \\
(\bar{X}_1 \times_S \bar{X}_2)^N & \xrightarrow{(\tilde{\epsilon}_{12})^N} & (\bar{X}_1 \times_S \bar{X}_2)^N
\end{array}$$

and

$$- \bar{f}_1 \times \bar{f}_2 : \bar{X}_1 \times_S \bar{X}_2 \rightarrow \bar{S}$$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \rightarrow S$.

- $(\bar{p}_1)^N := \bar{p}_1 \circ \tilde{\epsilon}_{12} : (\bar{X}_1 \times_S \bar{X}_2)^N \rightarrow \bar{X}_1$ is a compactification of $p_1 : U_1 \times_S U_2 \rightarrow U_1$.

- $(\bar{p}_2)^N := \bar{p}_2 \circ \tilde{\epsilon}_{12} : (\bar{X}_1 \times_S \bar{X}_2)^N \rightarrow \bar{X}_2$ is a compactification of $p_2 : U_1 \times_S U_2 \rightarrow U_2$.

We have then the morphism in $C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/S)$

$$T(\otimes, \hat{R}^{CH}_S(\mathbb{Z}(U_1/S), \mathbb{Z}(U_2/S)) := \hat{R}^{CH}_S(p_1) \otimes \hat{R}^{CH}_S(p_2) : \hat{R}(\bar{X}_1, \bar{D})_{/S}(\mathbb{Z}(U_1/S)) \otimes \hat{R}(\bar{X}_2, \bar{E})_{/S}(\mathbb{Z}(U_2/S)) \xrightarrow{\sim} \hat{R}(\bar{X}_1 \times_S \bar{X}_2)^N_{/S}(\mathbb{F})_{/S}(\mathbb{Z}(U_1 \times_S U_2/S))$$

For

$$Q_1^* := (\cdots \oplus \mathbb{A}^n \otimes \mathbb{Z}(U_{1, \alpha}/S)^{2n(n^2-1)}_{\otimes_{\beta}} S)_{\oplus_{\beta}} \otimes \mathbb{Z}(U_{1, \alpha}/S) \rightarrow \cdots,$$

$$Q_2^* := (\cdots \oplus \mathbb{A}^n \otimes \mathbb{Z}(U_{2, \alpha}/S)^{2n(n^2-1)}_{\otimes_{\beta}} S)_{\oplus_{\beta}} \otimes \mathbb{Z}(U_{2, \beta}/S) \rightarrow \cdots \in C(\text{Var}(\mathbb{C})/\mathbb{C})$$

direct sum of representable presheaves with $U^*_\alpha$ smooth, we get the morphism in $C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/S)$

$$T(\otimes, \hat{R}^{CH}_S(Q_1^*, Q_2^*) : \hat{R}^{CH}_S(Q_1^*) \otimes \hat{R}^{CH}_S(Q_2^*)^{(T(\otimes, \hat{R}^{CH}_S(\mathbb{Z}(U_{1, \alpha}), \mathbb{Z}(U_{2, \beta})))} \rightarrow \hat{R}^{CH}_S(Q_1^* \otimes Q_2^*))$$

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Definition 42. Let \( h : U \to S \) be a morphism, with \( U, S \in \text{Var}(\mathbb{C}) \) and \( U \) smooth. Take, see definition-proposition 12, \( f_0 = h_0 : \hat{X}_0 \to \hat{S} \) a compactification of \( h : U \to S \) and denote by \( Z = \hat{X}_0 \setminus \hat{U} \).

Take, using theorem 15(ii), a strict desingularization \( \hat{\epsilon} : (\hat{X}, \hat{D}) \to (\hat{X}_0, \hat{Z}) \) of the pair \((\hat{X}_0, \hat{Z})\), with \( \hat{X} \in \text{PSmVar}(\mathbb{C}) \) and \( \hat{D} \) such that \( \epsilon^{-1}(\hat{Z}) = \bigcup_{i \in I} \hat{D}_i \subset \hat{X} \) a normal crossing divisor. We denote by \( i_* : \hat{D}_{\bullet} \hookrightarrow \hat{X} = \hat{X}_{\hat{\epsilon}([\bullet])} \) the morphism of simplicial varieties given by the closed embeddings \( i_I : \hat{D}_I = \cap_{j \in I} \hat{D}_j \hookrightarrow \hat{X} \). We denote by \( j : U \to \hat{X} \) the open embedding. We then consider the following map in \( \text{Var}(\mathbb{C}) \)

\[
\rho_{(\hat{X}, \hat{D})/S}(Z(U/S)) : R^0_{(\hat{X}, \hat{D})/S}(Z(U/S)) \xrightarrow{\epsilon_*} f_* \text{E}_{et}((Z_{\hat{\epsilon}})_{(\hat{D}_{\bullet})/\hat{X}}, u_{11}) \to Z((\hat{X}/\hat{X}))
\]

\[
\rho_{(\hat{X}, \hat{D})/S}(Z(U/U)) = : D_0^0(Z(U/U))
\]

Note that \( Z(\hat{D}_I/\hat{X}) \) and \( Z(\hat{X}/\hat{X}) \) are obviously \( A^1 \) invariant. Note that \( \rho_{(\hat{X}, \hat{D})/S} \) is NOT an equivalence \( (A^1, \text{et}) \) local by proposition 19 since \( \rho_{\hat{X}, Z(\hat{D}_{\bullet}/\hat{X})} = 0 \), and \( \rho_{\hat{X}, \text{ad}(j^*, j)}(Z(\hat{X}/\hat{X})) \) is not an equivalence \( (A^1, \text{et}) \) local.

(ii) Let \( g : U'/S \to U/S \) a morphism, with \( U' = U'h' \), \( U/S = (U, h) \in \text{Var}(\mathbb{C}) \), with \( U \) and \( U' \) smooth. Take, see definition-proposition 12(ii), a compactification \( f_0 = h_0 : \hat{X}_0 \to \hat{S} \) of \( h : U \to S \) and a compactification \( f_0' = h_0' : \hat{X}_0' \to \hat{S} \) of \( h' : U' \to S \) such that \( g : U'/S \to U/S \) extend to a morphism \( g_0 : \hat{X}_0'/\hat{S} \to \hat{X}_0/\hat{S} \). Denote \( Z = \hat{X}_0 \setminus U \) and \( Z' = \hat{X}_0'/U' \). Take, see definition-proposition 12(ii), a strict desingularization \( \hat{\epsilon} : (\hat{X}, \hat{D}) \to (\hat{X}_0, \hat{Z}) \) of \( (\hat{X}_0, \hat{Z}) \), a strict desingularization \( \hat{\epsilon}' : (\hat{X}', \hat{D}') \to (\hat{X}_0, \hat{Z}') \) of \( (\hat{X}_0, \hat{Z}') \) and a morphism \( \hat{g} : \hat{X}' \to \hat{X} \) such that the following diagram commutes

\[
\begin{array}{ccc}
\hat{X}_0' & \xrightarrow{\hat{g}_0} & \hat{X}_0 \\
\hat{\epsilon}' \downarrow & & \downarrow \hat{\epsilon} \\
\hat{X}' & \xrightarrow{\hat{g}} & \hat{X} \\
\end{array}
\]
We then have, see definition-proposition 12(ii), the commutative diagram (47) in Fun($\Delta$, Var($\mathbb{C}$))

$$
\begin{array}{c}
U = U(c) \xrightarrow{j} \bar{X} = X(c) \xrightarrow{i} D_{s}(c) \\
\downarrow g \hspace{1cm} \downarrow \bar{g} \hspace{1cm} \downarrow h \\
U' = U'(c) \xrightarrow{j'} \bar{X}' = X'(c) \xrightarrow{i'} D'(c) \xrightarrow{h'} \bar{D}'(c) = (\bar{D}_{s}(c))^{\alpha} \end{array}
$$

We then consider the following map in $\text{C}(\text{Var}(\mathbb{C})/S$)

$$
R^{0}_{SCH}(g) : R_{(X,D)/S}^{0}(Z(U/S)) \xrightarrow{\sim} \check{f}_{*}C_{et}(\text{Cone}(Z(i_{*}(:Z((\bar{D}_{s}(c))/\bar{X}), u_{1,1}) \rightarrow Z(\bar{X}/\bar{X})))
\quad T(\bar{g},E)(-)=\text{Ker}(\bar{g}^{*},g)(-))
\check{f}_{*}C_{et}(\text{Cone}(Z(i_{*}(:Z(\bar{D}_{s}(c))/\bar{X}), u_{1,1}) \rightarrow Z(\bar{X}/\bar{X})))
\check{f}_{*}C_{et}(\text{Cone}(Z(i_{*}(:Z(\bar{D}_{s}(c))/\bar{X}), u_{1,1}) \rightarrow Z(\bar{X}/\bar{X})))
\overset{\sim}{\longrightarrow} R_{(\bar{X}',D')/S}^{0}(Z(U'/S))
$$

Then by the diagram (47) and adjunction, the following diagram in $\text{C}(\text{Var}(\mathbb{C})/S$) obviously commutes

$$
\begin{array}{c}
R_{(X,D)/S}^{0}(Z(U/S)) \xrightarrow{r_{(X,D)/S}(Z(U/S))} h_{*}C_{et}(Z(U/U)) =: D_{S}^{0}(Z(U/S)) \\
R_{S}^{0}(g) \mid R_{S}^{0}(g) =: D_{S}(g) = \text{Ker}(\bar{g}^{*},g)(E_{et}(Z(U/U)))
\end{array}
$$

(iii) For $g_{1} : U''/S \rightarrow U'/S$, $g_{2} : U'/S \rightarrow U/S$ two morphisms with $U''/S = (U', h''), U'/S = (U', h')$, $U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, with $U$, $U'$, and $U''$ smooth. We get from (i) and (ii) a compactification $\bar{h} : \bar{X} \rightarrow \bar{S}$ of $h : U \rightarrow S$, a compactification $\bar{h} : \bar{X}' \rightarrow \bar{S}$ of $h' : U' \rightarrow S$, and a compactification $\bar{h}'' : \bar{X}'' \rightarrow \bar{S}$ of $h'' : U'' \rightarrow S$, with $X, X', X'' \in \text{PSmVar}(\mathbb{C})$, $D := X \times U \subseteq X \times D' := X \times U' \subseteq X'$, and $D' := X'' \times U \subseteq X''$ normal crossing divisors, such that $g_{1} : U''/S \rightarrow U'/S$ extend to $\bar{g}_{1} : \bar{X}/\bar{S} \rightarrow \bar{X}'/\bar{S}$, $g_{2} : U'/S \rightarrow U/S$ extend to $\bar{g}_{2} : \bar{X}'/\bar{S} \rightarrow \bar{X}/\bar{S}$, and

$$
R_{S}^{0}(g_{2} \circ g_{1}) = R_{S}^{0}(g_{1}) \circ R_{S}^{0}(g_{2}) : R_{(X,D)/S}^{0}(Z(U'/S)) \rightarrow R_{(X'',D'')/S}^{0}(Z(U''/S))
$$

(iv) For

$$
Q^{*} := (\cdots \rightarrow \oplus_{\alpha \in \Lambda^{n}} Z(U_{\alpha}^{n}/S) \xrightarrow{(Z(g_{\alpha}^{n}))} \oplus_{\beta \in \Lambda^{n-1}} Z(U_{\beta}^{n-1}/S) \rightarrow \cdots) \in C(\text{Var}(\mathbb{C})/S)
$$

a complex of (maybe infinite) direct sum of representable presheaves with $U_{\alpha}^{*}$ smooth, we get from (i), (ii) and (iii) the map in $C(\text{Var}(\mathbb{C})/S$)

$$
R_{S}^{0}(Q^{*}) := (\cdots \rightarrow \oplus_{\beta \in \Lambda^{n-1}} \lim_{(X_{\beta}^{n-1},D_{\beta}^{n-1})/S} R_{S}^{0}(X_{\beta}^{n-1},D_{\beta}^{n-1}/S)(Z(U_{\beta}^{n-1}/S))
\quad (R_{S}(g_{\alpha}^{n}))
\lim_{(X_{\alpha}^{n-1},D_{\alpha}^{n-1})/S} R_{S}(X_{\alpha}^{n-1},D_{\alpha}^{n-1}/S)(Z(U_{\alpha}^{n}/S)) \rightarrow \cdots) \rightarrow D_{S}(Q^{*})
$$

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where for \((U^n_\alpha, h^n_\alpha)\) in \(\text{Var}(\mathbb{C})/S\), the inductive limit run over all the compactifications \(\bar{f}_\alpha : \bar{X}_\alpha \to \bar{S}\) of \(h_\alpha : U_\alpha \to S\) with \(\bar{X}_\alpha \in \text{PSh}(\mathbb{C})\) and \(\bar{D}_\alpha := \bar{X}_\alpha \setminus U_\alpha\) a normal crossing divisor. For \(m = (m^*) : Q^*_1 \to Q^*_2\) a morphism with

\[
Q^*_1 := \cdots \to \bigoplus_{\alpha \in \Lambda^*} Z(U^*_{1,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^*_\alpha, \bar{\alpha}))} \bigoplus_{\beta \in \Lambda^*} Z(U^*_{2,\beta}/S) \to \cdots,
\]

\[
Q^*_2 := \cdots \to \bigoplus_{\alpha \in \Lambda^*} Z(U^*_{1,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^*_\alpha, \bar{\alpha}))} \bigoplus_{\beta \in \Lambda^*} Z(U^*_{2,\beta}/S) \to \cdots \in C(\text{Var}(\mathbb{C})/S)
\]

complexes of \((\text{maybe infinite})\) direct sum of representable presheaves with \(U^*_1, \alpha\) and \(U^*_2, \alpha\) smooth, we get again from (i), (ii) and (iii) a commutative diagram in \(C(\text{Var}(\mathbb{C})/S)\)

\[
\begin{array}{ccc}
R^{\partial CH}(Q^*_2) & \xrightarrow{r^{\partial CH}(Q^*_2)} & D^0_S(Q^*_2) \\
R^{\partial CH}(m) := (R^{\partial CH}(m^*)) & \downarrow \quad \downarrow & D_S(m) := (D^0_S(m^*)) \\
R^{\partial CH}(Q^*_1) & \xrightarrow{r^{\partial CH}(Q^*_1)} & D^0_S(Q^*_1)
\end{array}
\]

(v) Let

\[
Q^* := \cdots \to \bigoplus_{\alpha \in \Lambda^*} Z(U^*_{\alpha}/S) \xrightarrow{(\mathbb{Z}(g^*_\alpha, \bar{\alpha}))} \bigoplus_{\beta \in \Lambda^*} Z(U^*_{\beta}/S) \to \cdots \in C(\text{Var}(\mathbb{C})/S)
\]

a complex of \((\text{maybe infinite})\) direct sum of representable presheaves with \(U^*_\alpha\) smooth, we have by definition

\[Gr^1_{S^{12}} R^{\partial CH}(Q^*) = R^{CH}(Q^*) \in C(\text{Var}(\mathbb{C})^2/S).
\]

- Let \(S \in \text{Var}(\mathbb{C})\). For \((h^*, m^*, m^*) : (Q^*_1, Q^*_2) \to (Q^*_1, Q^*_2)\) an homotopy with \(Q^*_1, Q^*_2 \in C(\text{Var}(\mathbb{C})/S)\) complexes of \((\text{maybe infinite})\) direct sum of representable presheaves with \(U^*_1, \alpha\) and \(U^*_2, \alpha\) smooth,

\[
(R^{\partial CH}(h), R^{\partial CH}(m), R^{\partial CH}(m^*)) = (R^{\partial CH}(h^*), R^{\partial CH}(m^*), R^{\partial CH}(m^*)) : R^{\partial CH}(Q^*_1)[1] \to R^{\partial CH}(Q^*_1)
\]

is an homotopy in \(C(\text{Var}(\mathbb{C})/S)\) using definition 42 (iii). In particular if \(m : Q^*_1 \to Q^*_2\) with \(Q^*_1, Q^*_2 \in C(\text{Var}(\mathbb{C})/S)\) complexes of \((\text{maybe infinite})\) direct sum of representable presheaves with \(U^*_1, \alpha\) and \(U^*_2, \alpha\) smooth is an homotopy equivalence, then \(R^{\partial CH}(m) : R^{\partial CH}(Q^*_1) \to R^{\partial CH}(Q^*_1)\) is an homotopy equivalence.

- Let \(S \in \text{SmVar}(\mathbb{C})\). Let \(F \in \text{PSh}(\text{Var}(\mathbb{C})^m/S)\). Consider

\[
q : LF := \cdots \to \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^m/S} Z(U_\alpha/S) \xrightarrow{(2(g^*_\alpha, \bar{\alpha}))} \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^m/S} Z(U_\alpha/S) \to \cdots \to F
\]

the canonical projective resolution given in subsection 2.3.3. Note that the \(U_\alpha\) are smooth since \(S\) is smooth and \(h_\alpha\) are smooth morphism. Definition 42(iv) gives in this particular case the map in \(C(\text{Var}(\mathbb{C})/S)\)

\[
r^S_{\partial CH}(\rho^*_S LF) : R^{\partial CH}(\rho^*_S LF) := (\cdots \to \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^m/S} \varinjlim_{(X_\alpha, D_\alpha)/S} R^0_{(X_\alpha, D_\alpha)/S} Z(U_\alpha/S) \to \cdots \to D^0_S(\rho^*_S LF),
\]

where for \((U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^m/S\), the inductive limit run over all the compactifications \(\bar{f}_\alpha : \bar{X}_\alpha \to \bar{S}\) of \(h_\alpha : U_\alpha \to S\) with \(\bar{X}_\alpha \in \text{PSh}(\mathbb{C})\) and \(\bar{D}_\alpha := \bar{X}_\alpha \setminus U_\alpha\) a normal crossing divisor. Definition 42(iv) gives then by functoriality in particular, for \(F = F^* \in C(\text{Var}(\mathbb{C})^m/S)\), the map in \(C(\text{Var}(\mathbb{C})/S)\)

\[
r^S_{\partial CH}(\rho^*_S LF) = (r^S_{\partial CH}(\rho^*_S LF^*)) : R^{\partial CH}(\rho^*_S LF) \to D^0_S(\rho^*_S LF).
\]
• Let \(g : T \to S\) a morphism with \(T,S \in \text{SmVar}(\mathbb{C})\). Let \(h : U \to S\) a smooth morphism with \(U \in \text{Var}(\mathbb{C})\). Consider the cartesian square

\[
\begin{array}{ccc}
U_T & \xrightarrow{h'} & T \\
\downarrow{g'} & & \downarrow{g} \\
U & \xrightarrow{h} & S
\end{array}
\]

Note that \(U\) is smooth since \(S\) and \(h\) are smooth, and \(U_T\) is smooth since \(T\) and \(h'\) are smooth. Take, see definition-proposition 12(ii), a compactification \(\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}\) of \(h : U \to S\). Take, see definition-proposition 12(ii), a strict desingularization \(\bar{h} : \bar{X}_T \to \bar{T}\) a compactification of \(g \circ h' : U_T \to S\) such that \(g' : U_T/S \to U/S\) extend to a morphism \(\bar{g}' : \bar{X}_T/\bar{S} \to \bar{X}/\bar{S}\). Denote \(\bar{Z} = \bar{X}_0\setminus U\) and \(\bar{Z}' = \bar{X}_T\setminus U_T\). Take, see definition-proposition 12(ii), a strict desingularization \(\bar{\epsilon}' : (\bar{X}', \bar{D}') \to (\bar{X}_T, \bar{Z}')\) of \((\bar{X}_T, \bar{Z}')\). Denote \(\bar{g}' = \bar{g}' \circ \bar{\epsilon}' : \bar{X}' \to \bar{X}\). We then have, see definition-proposition 12(ii), the following commutative diagram in \(\text{Fun}(\Delta, \text{Var}(\mathbb{C}))\)

\[
\begin{array}{cccc}
U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} = \bar{X}_{c(\bullet)} & \xrightarrow{i} \bar{D}_{s,p}(\bullet) \\
g' \downarrow & & \bar{g}' \downarrow & \downarrow (\bar{g}')' \\
U_T = U_{T,c(\bullet)} & \xrightarrow{j'} & \bar{X}_T = \bar{X}_{c(\bullet)} & \xrightarrow{i'} \bar{D}'_{s,p}(\bullet) \\
g' \downarrow & & \bar{g}' \downarrow & \downarrow (\bar{g}')' \\
U_T = U_{T,c(\bullet)} & \xrightarrow{j'} & \bar{X}' = \bar{X}'_{c(\bullet)} & \xrightarrow{i'} \bar{D}'_{s,p}(\bullet) : i'_p \\
\end{array}
\]

We then consider the following map in \(\text{C(Var(\mathbb{C}))/T}\), see definition 42(ii)

\[
T(g, R'^{\text{CH}}(\mathbb{Z}(U/S))) = g'^{*}R_0(\bar{X}', \bar{D}')/T(\mathbb{Z}(U_T/T))
\]

\[
\xrightarrow{t(g'^{*}R_0(\bar{X}', \bar{D}')/T(\mathbb{Z}(U_T/T)))} R_0(\bar{X}', \bar{D}')/T(\mathbb{Z}(U_T/T))
\]

For

\[
Q^* := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U^n_{\alpha}/S) \xrightarrow{(\mathbb{Z}(s^n_{\alpha}))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U^n_{\beta}/S) \to \cdots) \in \text{C(Var(\mathbb{C}))/S}
\]

a complex of (maybe infinite) direct sum of representable presheaves with \(h^n_{\alpha} : U^n_{\alpha} \to S\) smooth, we get the map in \(\text{C(Var(\mathbb{C}))/T}\)

\[
T(g, R'^{\text{CH}}(Q^*)) = (\cdots \to \oplus_{\alpha \in \Lambda^n} \lim_{(\bar{X}_n, \bar{D}_n)} g'^{*}R_0(\bar{X}_n, \bar{D}_n)/S(\mathbb{Z}(U^n_{\alpha}/S)) \to \cdots)
\]

\[
\xrightarrow{(T(g, R'^{\text{CH}}(\mathbb{Z}(U^n_{\alpha}/S))))} (\cdots \to \oplus_{\alpha \in \Lambda^n} \lim_{(\bar{X}_n, \bar{D}_n)} R_0(\bar{X}_n, \bar{D}_n)/T(\mathbb{Z}(U^n_{\alpha}/T)) \to \cdots) =: R'^{\text{CH}}(g^*Q^*).
\]

Let \(F \in \text{PSh(Var(\mathbb{C}))}^{sm}/S\). Consider

\[
q : LF := (\cdots \to \oplus_{(U_n, h_n) \in \text{Var(\mathbb{C}))}^{sm}/S} \mathbb{Z}(U_n/S) \to \cdots) \to F
\]

the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in
C(Var(\mathbb{C})/T) \quad T(g, R^{0\overline{\text{CH}}})(\rho_S^*LF) = g^*R^{0\overline{\text{CH}}} (\rho_S^*LF) = \\
(\cdots \to \oplus_{(U, h, a) \in \text{Var}(\mathbb{C})^{sm}/S} \lim_{(X_a, \tilde{D}_a)/S} g^*R^0_{(X_a, \tilde{D}_a)/S}(\mathbb{Z}(U_a/S)) \to \cdots) \xrightarrow{\left( T(g, R^{0\overline{\text{CH}}})(\mathbb{Z}(U_a/S)) \right)} \\
(\cdots \to \oplus_{(U, h, a) \in \text{Var}(\mathbb{C})^{sm}/S} \lim_{(X_a, \tilde{D}_a)/T} R^0_{(X_a, \tilde{D}_a)/T}(\mathbb{Z}(U_a,T/S)) \to \cdots) =: R^{\overline{\text{CH}}}(\rho_T^*g^*LF).

By functoriality, we get in particular for \( F = F^* \in C(\text{Var}(\mathbb{C})^{sm}/S) \), the map in \( C(\text{Var}(\mathbb{C})/T) \)

\[ T(g, R^{0\overline{\text{CH}}})(\rho_S^*LF) : g^*R^{0\overline{\text{CH}}} (\rho_S^*LF) \to R^{0\overline{\text{CH}}} (\rho_T^*g^*LF). \]

- Let \( S_1, S_2 \in \text{SmVar}(\mathbb{C}) \) and \( p : S_1 \times S_2 \to S_1 \) the projection. Let \( h : U \to S_1 \) a smooth morphism with \( U \in \text{Var}(\mathbb{C}) \). Consider the cartesian square

\[
\begin{array}{ccc}
U \times S_2 & \xrightarrow{h \times I} & S_1 \times S_2 \\
\downarrow \quad \downarrow p' & & \downarrow p \\
U & \xrightarrow{h} & S_1
\end{array}
\]

Take, see definition-proposition 12(i), a compactification \( \bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}_1 \) of \( h : U \to S_1 \). Then \( \bar{f}_0 \times I : \bar{X}_0 \times S_2 \to \bar{S}_1 \times S_2 \) is a compactification of \( h \times I : U \times S_2 \to S_1 \times S_2 \) and \( p' : U \times S_2 \to U \) extend to \( \bar{p}'_0 := p_{X_0} : \bar{X}_0 \times S_2 \to \bar{X}_0 \). Denote \( Z = X_0 \setminus U \). Take see theorem 15(i), a strict desingularization \( \varepsilon : (\bar{X}, D) \to (\bar{X}_0, \bar{Z}) \) of the pair \( (\bar{X}_0, \bar{Z}) \). We then have the commutative diagram (48) in \( \text{Fun}(\Delta, \text{Var}(\mathbb{C})) \) whose squares are cartesian

\[
\begin{array}{ccc}
U = U_{c(\bullet)} & \xrightarrow{j} & \bar{X} \\
\downarrow g & & \downarrow i \\
U \times S_2 = (U \times S_2, i(\bullet)) & \xrightarrow{j \times I} & \bar{X} \times S_2
\end{array}
\]

Then the map in \( C(\text{Var}(\mathbb{C})/S_1 \times S_2) \)

\[ T(p, R^{0\overline{\text{CH}}})(\mathbb{Z}(U/S_1)) : p^*R^0_{(\bar{X}, D)/S_1}(\mathbb{Z}(U/S_1)) \xrightarrow{\sim} R^0_{(\bar{X} \times S_2, \bar{D} \times S_2)/S_1 \times S_2}(\mathbb{Z}(U \times S_2/S_1 \times S_2)) \]

is an isomorphism. Hence, for \( Q^* \in C(\text{Var}(\mathbb{C})/S_1) \) a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in \( C(\text{Var}(\mathbb{C})/S_1 \times S_2) \)

\[ T(p, R^{0\overline{\text{CH}}})(Q^*) : p^*R^{0\overline{\text{CH}}}(Q^*) \xrightarrow{\sim} R^{0\overline{\text{CH}}}(p^*Q^*) \]

is an isomorphism. In particular, for \( F \in C(\text{Var}(\mathbb{C})^{sm}/S_1) \) the map in \( C(\text{Var}(\mathbb{C})/S_1 \times S_2) \)

\[ T(p, R^{0\overline{\text{CH}}})(\rho_S^*LF) : p^*R^{0\overline{\text{CH}}} (\rho_S^*LF) \xrightarrow{\sim} R^{0\overline{\text{CH}}} (\rho_S^*XS_2\rho_T^*LF) \]

is an isomorphism.

- Let \( h_1 : U_1 \to S, h_2 : U_2 \to S \) two morphisms with \( U_1, U_2, S \in \text{Var}(\mathbb{C}) \), \( U_1, U_2 \) smooth. Denote by \( p_1 : U_1 \times U_2 \to U_1 \) and \( p_2 : U_1 \times U_2 \to U_2 \) the projections. Take, see definition-proposition 12(i)), a compactification \( f_{10} = h_1 : \bar{X}_{10} \to S \) of \( h_1 : U_1 \to S \) and a compactification \( f_{20} = h_2 : \bar{X}_{20} \to S \) of \( h_2 : U_2 \to S \). Then,

\[ f_{10} \times f_{20} : \bar{X}_{10} \times S \bar{X}_{20} \to S \]

is a compactification of \( h_1 \times h_2 : U_1 \times S U_2 \to S \).

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Definition 43. Let \( h : U \to S \) a morphism, with \( U, S \in \text{Var}(\mathbb{C}) \), \( U \) irreducible. Take, see definition-proposition 12, \( f_0 = h_0 : X_0 \to S \) a compactification of \( h : U \to S \) and denote by \( \bar{Z} = X_0 \setminus U \). Take, using theorem 15, a desingularization \( \bar{c} : (\bar{X}, \bar{D}) \to (X_0, \Delta) \) of the pair \( (X_0, \Delta) \), \( \bar{Z} \subset \Delta \), with \( \bar{X} \in \text{PSmVar}(\mathbb{C}) \) and \( D := \bar{c}^{-1}(\Delta) = \sqcup_{i=1}^r D_i \subset \bar{X} \) a normal crossing divisor. Denote \( d_X := \dim(\bar{X}) = \dim(U) \).

(i) The diagonal \( \Delta_{\bar{D}_*} \subset \bar{D}_* \times \bar{D}_* \) induces the morphism in \( \text{Var}(\mathbb{C})/S \)

\[
[\Delta_{\bar{D}_*}] \in \text{Hom}(\mathbb{Z}^r(\bar{D}_* \times S), \mathfrak{f}_*\mathcal{E}_{\bar{D}_*}(\bar{D}_* \times \bar{X}/\bar{X})(d_X)[2d_X]) \cong \\
\text{Hom}(\mathbb{Z}(\bar{D}_* \times S \bar{X}/\bar{X}), \mathbb{Z}^r(\bar{D}_* \times S \mathfrak{p}^d_{\bar{X}}/\bar{X})/\mathbb{Z}^r(\bar{D}_* \times S \mathfrak{p}^{d_X-1}/\bar{X})) \subset H^0(\mathbb{Z}_{d_{\bar{D}_*}}(\square^* \to \bar{D}_* \times S \bar{D}_*))
\]
(ii) The cycle $\Delta X \subset \bar{X} \times_S \bar{X}$ induces by the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$[\Delta X] \in \text{Hom}(\mathbb{Z}^r(\bar{X}/S), \bar{f}_*E_{ct}(\mathbb{Z}(\bar{X}/\bar{X})(d_X)[2d_X])) \xrightarrow{\sim} \text{Hom}(\mathbb{Z}(\bar{X} \times_S \bar{X}/\bar{X}), \mathbb{Z}^r(\bar{X} \times \mathbb{P}^{d_X}/\bar{X})/\mathbb{Z}^r(\bar{X} \times \mathbb{P}^{d_X-1}/\bar{X}))$$

$$\subset H^0(\mathbb{Z}_{d_X}(\square^* \times_S \bar{X} \times_S \bar{X}))$$

Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$, $U$ smooth connected (hence irreducible by smoothness). Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \backslash U$. Take, using theorem 15(ii), a strict desingularization $\bar{e} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$ with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $D := \bar{e}^{-1}(\bar{Z}) = \cup_{i=1}^p D_i \subset \bar{X}$ a normal crossing divisor. Denote $d_X := \dim(\bar{X}) = \dim(U)$. We get from (i) and (ii) the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$T(\bar{f}_*, \bar{f}_*)((\mathbb{Z}(D_*/\bar{X}), \mathbb{Z}(\bar{X}/\bar{X}))) := ([\Delta \bar{D}^\bullet], [\Delta \bar{X}]) :$$

$$\text{Cone}(\mathbb{Z}(i_*) : (\mathbb{Z}^r(\bar{D}_*/S), u_{1j}) \rightarrow \mathbb{Z}^r(\bar{X}/S)) \rightarrow$$

$$\bar{f}_*E_{ct}(\text{Cone}(\mathbb{Z}(i_*) : (\mathbb{Z}(\bar{D}_*/\bar{X}), u_{1j}) \rightarrow \mathbb{Z}(\bar{X}/\bar{X})))\{d_X, \mathbb{Z}^r(\bar{X}/\bar{X})\}[2d_X]$$

$$=: R^0(\mathbb{Z}_{\bar{X}/\bar{D}}/S)(\mathbb{Z}(\bar{U}/S))\{d_X\}[2d_X].$$

**Definition 44.**

(i) Let $h : U \rightarrow S$ a morphism, with $U, S \in \text{Var}(\mathbb{C})$ and $U$ smooth. Take, see definition-proposition 12, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ a compactification of $h : U \rightarrow S$ and denote by $\bar{Z} = \bar{X}_0 \backslash U$. Take, using theorem 15(ii), a strict desingularization $\bar{e} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$, with $\bar{X} \in \text{PSmVar}(\mathbb{C})$ and $D := \bar{e}^{-1}(\bar{Z}) = \cup_{i=1}^p D_i \subset \bar{X}$ a normal crossing divisor. We denote by $i_* : D_0 \rightarrow \bar{X} = \bar{X}_c(\bullet)$ the morphism of simplicial varieties given by the closed embeddings

$$i_1 : D_1 = \cap_{i \in I} D_i \rightarrow \bar{X} \text{ We denote by } j : U \hookrightarrow \bar{X} \text{ the open embedding. We then consider the map in } C(\text{Var}(\mathbb{C})/S)$$

$$T(\tilde{R}^0(\mathbb{Z}(U/S)), R^0(\mathbb{Z}(U/S))) : \tilde{R}^0(\mathbb{Z}(U/S))$$

$$\xrightarrow{\sim} \text{Cone}(\mathbb{Z}(i_*) : (\mathbb{Z}^r(\bar{D}_*/S), u_{1j}) \rightarrow \mathbb{Z}^r(\bar{X}/S))\{d_X, \mathbb{Z}^r(\bar{X}/\bar{X})\}[-2d_X]$$

$$=: R^0(\mathbb{Z}_{\bar{X}/\bar{D}}/S)(\mathbb{Z}(U/S))\{d_X\}[-2d_X].$$

given in definition 39(iii).

(ii) Let $g : U'/S \rightarrow U/S$ a morphism, with $U'/S = (U', h'), U/S = (U, h) \in \text{Var}(\mathbb{C})/S$, with $U$ and $U'$ smooth. Take, see definition-proposition 12(ii),a compactification $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ of $h : U \rightarrow S$ and a compactification $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \rightarrow \bar{S}$ of $h' : U' \rightarrow S$ such that $g : U'/S \rightarrow U/S$ extend to a morphism $\bar{g}_0 : \bar{X}_0'/\bar{S} \rightarrow \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \backslash U$ and $\bar{Z}' = \bar{X}_0'/\bar{U}'$. Take, see definition-proposition 12(ii), a strict desingularization $\bar{e} : (\bar{X}, \bar{D}) \rightarrow (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$, a strict desingularization $\bar{e}' : (\bar{X}', \bar{D}') \rightarrow (\bar{X}_0', \bar{Z}')$ of $(\bar{X}_0', \bar{Z}')$ and a morphism $\bar{g} : \bar{X}' \rightarrow \bar{X}$ such that the following diagram commutes

We then have, see definition-proposition 12(ii), the diagram (47) in Fun($\Delta, \text{Var}(\mathbb{C})$)

$$U = U_c(\bullet) \xrightarrow{j} \bar{X} = \bar{X}_c(\bullet) \xrightarrow{i^*} \bar{D}_{s_0}(\bullet)$$

$$\xrightarrow{g} \bar{g} \xrightarrow{\bar{g}^*} \bar{D}_{s_0}(\bullet)$$

$$U' = U_c'(\bullet) \xrightarrow{j'} \bar{X}' = \bar{X}_c'(\bullet) \xrightarrow{i'^*} \bar{D}' \xrightarrow{\bar{g}^*} \bar{D}_{s_0}'(\bullet).$$
Consider
\[ [\Gamma_g]^t \in \text{Hom}(\mathbb{Z}^{tr}(\bar{X}/S)(-d_X)[-2d_X], \mathbb{Z}^{tr}(\bar{X}'/S)(-d_X)[-2d_X]) \]
\[ \xrightarrow{\sim} \text{Hom}(\mathbb{Z}^{tr}(\bar{X} \times \mathbb{P}^d_S)/S, \mathbb{Z}^{tr}(\bar{X} \times \mathbb{P}^{d-1}_d)/S), \]
\[ \mathbb{Z}^{tr}(\bar{X}' \times \mathbb{P}^{d-1}_d)/S) \]
the morphism given by the transpose of the graph \( \Gamma_g \subset X' \times S X \) of \( \bar{g} : \bar{X}' \to \bar{X} \). Then, since \( i_* \circ \bar{g}^t = \bar{g} \circ i_* \), we have the factorization
\[ [\Gamma_g]^t \circ i_*(\mathbb{Z}(\bar{D}_{s_0}(\mathbb{Z}), u_{1J}))/(-d_X)[-2d_X] \]
\[ \xrightarrow{[\mathbb{F}_g]^t} (\mathbb{Z}^{tr}(\bar{D}_{s_0}(\mathbb{Z}), u_{1J}))/(-d_X)[-2d_X] \]
\[ \xrightarrow{\mathbb{Z}(\bar{g}_*)} \mathbb{Z}^{tr}(\bar{X}'/S)(-d_X)[-2d_X] \]
with
\[ [\mathbb{F}_g]^t \in \text{Hom}(\mathbb{Z}^{tr}(\bar{D}_{s_0}(\mathbb{Z}) \times \mathbb{P}^d_S/S), u_{1J})/(\mathbb{Z}^{tr}(\bar{D}_{s_0}(\mathbb{Z}) \times \mathbb{P}^{d-1}_d/S), u_{1J}), \]
\[ (\mathbb{Z}^{tr}(\bar{g}^{-1}(\bar{D}_{s_0}(\mathbb{Z})) \times \mathbb{P}^{d-1}_d/S), u_{1J})/(\mathbb{Z}^{tr}(\bar{g}^{-1}(\bar{D}_{s_0}(\mathbb{Z})) \times \mathbb{P}^{d-1}_d/S), u_{1J}) \].

We then consider the following map in \( C(\text{Var}(\mathbb{C})/S) \)
\[ \hat{R}^{0CH}_S(g) : \hat{R}^{0CH}_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \xrightarrow{\sim} \text{Cone}(\mathbb{Z}(\bar{g}_*) : (\mathbb{Z}^{tr}(\bar{D}_{s_0}(\mathbb{Z}), u_{1J}) \to \mathbb{Z}^{tr}(\bar{X}/S)(-d_X)[-2d_X] \]
\[ \xrightarrow{([\mathbb{F}_g]^t \circ i_*)} \text{Cone}(\mathbb{Z}(\bar{g}^{-1}(\bar{D}_{s_0}(\mathbb{Z}))) \times \mathbb{P}^{d-1}_d/S), u_{1J}))/(-d_X][-2d_X] \]
\[ \xrightarrow{\mathbb{Z}(\bar{g}_*)} \mathbb{Z}^{tr}(\bar{X}'/S)(-d_X)[-2d_X] \]
\[ \xrightarrow{\mathbb{Z}(\bar{g}_*)} \mathbb{Z}^{tr}(\bar{X}'/S)(-d_X)[-2d_X] \]
\[ \xrightarrow{\mathbb{Z}(\bar{g}_*)} \mathbb{Z}^{tr}(\bar{X}'/S)(-d_X)[-2d_X] \]
Then the following diagram in \( C(\text{Var}(\mathbb{C})/S) \) commutes by definition
\[ \hat{R}^{0CH}_{(X',\bar{D}')/S}(\mathbb{Z}(U'/S)) \xrightarrow{\hat{R}^{0CH}_{(X',\bar{D}')/S}(\mathbb{Z}(U'/S))} \hat{R}^{0CH}_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \xrightarrow{\hat{R}^{0CH}_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))} \hat{R}^{0CH}_{(X',\bar{D}')/S}(\mathbb{Z}(U'/S)) \]

(iii) For \( g_1 : U''/S \to U'/S, g_2 : U'/S \to U/S \) two morphisms with \( U''/S = (U', h''), U'/S = (U', h'), U/S = (U, h) \) \( \in \text{Var}(\mathbb{C})/S \), with \( U, U' \) and \( U'' \) smooth. We get from (i) and (ii) a compactification \( \bar{h} : \bar{X} \to \bar{S} \) of \( h : U \to S \), a compactification \( \bar{h}' : \bar{X}' \to \bar{S}' \) of \( h' : U' \to S \), and a compactification \( \bar{h}'' : \bar{X}'' : \bar{X}' \to \bar{S} '' \) of \( h'' : U'' \to S \), with \( \bar{X}, \bar{X}', \bar{X}'' \in \text{PSmVar}(\mathbb{C}, D) := \bar{X} \setminus U \times \bar{D}' := \bar{X}' \setminus U' \subset \bar{X}', \) and \( D'' := \bar{X}'' \setminus U'' \subset \bar{X}'' \) normal crossing divisors, such that \( g_1 : U''/S \to U'/S \) extend to \( \bar{g}_1 : \bar{X}''/S \to \bar{X}'/S \), \( g_2 : U'/S \to U/S \) extend to \( \bar{g}_2 : \bar{X}'/S \to \bar{X}/S \), and
\[ \hat{R}^{0CH}_S(g_2 \circ g_1) = \hat{R}^{0CH}_S(g_2) \circ \hat{R}^{0CH}_S(g_1) : \hat{R}^{0CH}_{(X',\bar{D}')/S} \to \hat{R}^{0CH}_{(\bar{X},\bar{D})/S} \to \hat{R}^{0CH}_{(X',\bar{D}')/S} \]
(iv) For
\[ Q^* := (\cdots \to \bigoplus_{\alpha \in \Lambda^*} \mathbb{Z}(U^*_\alpha/S) \xrightarrow{(\mathbb{Z}(g^*_\alpha))} \bigoplus_{\beta \in \Lambda^*} \mathbb{Z}(U^*_\beta/S) \to \cdots) \in C(\text{Var}(\mathbb{C})/S) \]
a complex of (maybe infinite) direct sum of representable presheaves with \( U^*_\alpha \) smooth, we get from (i),(ii) and (iii) the map in \( C(\text{Var}(\mathbb{C})/S) \)
\[ T(\hat{R}^{0CH}, R^{0CH})(Q^*) : \hat{R}^{0CH}(Q^*) := (\cdots \to \bigoplus_{\beta \in \Lambda^*} \lim_{(X^*_{\beta - 1}, D^*_\beta - 1)/S} \hat{R}^0_{(X^*_\beta - 1, D^*_\beta - 1)/S}(\mathbb{Z}(U^*_\beta/S)) \to \cdots) \to R^{0CH}(Q^*), \]
where for \( (U^*_\alpha, h^*_\alpha) \in \text{Var}(\mathbb{C})/S \), the inductive limit run over all the compactifications \( \bar{f}_\alpha : \bar{X}_\alpha \to \bar{S} \)
of \( h_\alpha : U_\alpha \to S \) with \( \bar{X}_\alpha \in \text{PShVar}(\mathbb{C}) \) and \( \bar{D}_\alpha := \bar{X}_\alpha \setminus U_\alpha \) a normal crossing divisor. For \( m = (m^*) : Q^*_1 \to Q^*_2 \) a morphism with
\[ Q^*_1 := (\cdots \to \bigoplus_{\alpha \in \Lambda^*} \mathbb{Z}(U^*_1\alpha/S) \xrightarrow{(\mathbb{Z}(g^*_\alpha))} \bigoplus_{\beta \in \Lambda^*} \mathbb{Z}(U^*_1\beta/S) \to \cdots), \]
\[ Q^*_2 := (\cdots \to \bigoplus_{\alpha \in \Lambda^*} \mathbb{Z}(U^*_2\alpha/S) \xrightarrow{(\mathbb{Z}(g^*_\alpha))} \bigoplus_{\beta \in \Lambda^*} \mathbb{Z}(U^*_2\beta/S) \to \cdots) \in C(\text{Var}(\mathbb{C})/S) \]
complexes of (maybe infinite) direct sum of representable presheaves with \( U^*_1\alpha \) and \( U^*_2\alpha \) smooth, we get again from (i),(ii) and (iii) a commutative diagram in \( C(\text{Var}(\mathbb{C})/S) \)
\[ \hat{R}^{0CH}(Q^*_2) \xrightarrow{T(\hat{R}^{0CH}, R^{0CH})(Q^*_1)} \hat{R}^{0CH}(Q^*_1) \]
\[ R^{0CH}(Q^*_2) \xrightarrow{T(\hat{R}^{0CH}, R^{0CH})(Q^*_1)} R^{0CH}(Q^*_1). \]

(v) Let
\[ Q^* := (\cdots \to \bigoplus_{\alpha \in \Lambda^*} \mathbb{Z}(U^*_\alpha/S) \xrightarrow{(\mathbb{Z}(g^*_\alpha))} \bigoplus_{\beta \in \Lambda^*} \mathbb{Z}(U^*_\beta/S) \to \cdots) \in C(\text{Var}(\mathbb{C})/S) \]
a complex of (maybe infinite) direct sum of representable presheaves with \( U^*_\alpha \) smooth, we have by definition
\[ \text{Gr}^{12*}_S \hat{R}^{0CH}(Q^*) = \hat{R}^{0CH}(Q^*) \in C(\text{Var}(\mathbb{C})^{2, smpr}/S). \]

- Let \( S \in \text{Var}(\mathbb{C}) \) For \( (h, m, m') = (h^*, m^*, m'^*) : Q^*_1[1] \to Q^*_2 \) an homotopy with \( Q^*_1, Q^*_2 \in C(\text{Var}(\mathbb{C})/S) \) complexes of (maybe infinite) direct sum of representable presheaves with \( U^*_1\alpha \) and \( U^*_2\alpha \) smooth,
\[ (\hat{R}^{0CH}_S(h), \hat{R}^{0CH}_S(m), \hat{R}^{0CH}_S(m')) = (\hat{R}^{0CH}_S(h^*), \hat{R}^{0CH}_S(m^*), \hat{R}^{0CH}_S(m'^*)) : \hat{R}^{0CH}(Q^*_2)[1] \to \hat{R}^{0CH}(Q^*_1) \]
is an homotopy in \( C(\text{Var}(\mathbb{C})/S) \) using definition 44 (iii). In particular if \( m : Q^*_1 \to Q^*_2 \) with \( Q^*_1, Q^*_2 \in C(\text{Var}(\mathbb{C})/S) \) complexes of (maybe infinite) direct sum of representable presheaves with \( U^*_1\alpha \) and \( U^*_2\alpha \) smooth is an homotopy equivalence, then \( \hat{R}^{0CH}_S(m) : \hat{R}^{0CH}(Q^*_2) \to \hat{R}^{0CH}(Q^*_1) \) is an homotopy equivalence.

- Let \( S \in \text{SmVar}(\mathbb{C}) \). Let \( F \in \text{PShVar}(\mathbb{C})^{sm}/S \). Consider
\[ q : LF := (\cdots \to \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \xrightarrow{(\mathbb{Z}(g_\alpha, h_\alpha))} \bigoplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^{sm}/S} \mathbb{Z}(U_\alpha/S) \to \cdots) \to F \]
the canonical projective resolution given in subsection 2.3.3. Note that the $U_\alpha$ are smooth since $S$ is smooth and $h_\alpha$ are smooth morphism. Definition 44(iv) gives in this particular case the map in $C(\text{Var}(\mathbb{C})/S)$

$$T(\check{R}^0_{S,C}, \check{R}^0_{S,C})(\rho_\mu^*LF) : (\cdots \to \oplus_{(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^m/S} \lim_{(X_\alpha, \mathcal{D}_\alpha)/S} \check{R}^0_{(X_\alpha, \mathcal{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S))$$

where for $(U_\alpha, h_\alpha) \in \text{Var}(\mathbb{C})^m/S$, the inductive limit run over all the compactifications $\check{f}_\alpha : \check{X}_\alpha \to \check{S}$ of $h_\alpha : U_\alpha \to S$ with $\check{X}_\alpha \in \text{PSmVar}(\mathbb{C})$ and $\mathcal{D}_\alpha := \check{X}_\alpha \setminus U_\alpha$ a normal crossing divisor. Definition 44(iv) gives then by functoriality in particular, for $F = F^* \in C(\text{Var}(\mathbb{C})^m/S)$, the map in $C(\text{Var}(\mathbb{C})/S)$

$$T(\check{R}^0_{S,C}, \check{R}^0_{S,C})(\rho_\mu^*LF) : \check{R}^0_{S,C}(\rho_\mu^*LF) \to \check{R}^0_{S,C}(\rho_\mu^*LF).$$

- Let $g : T \to S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $h : U \to S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Consider the cartesian square

$$\begin{array}{ccc}
U_T & \xrightarrow{h'} & T \\
\downarrow{g'} & & \downarrow{g} \\
U & \xrightarrow{h} & S
\end{array}$$

Note that $U$ is smooth since $S$ and $h$ are smooth, and $U_T$ is smooth since $T$ and $h'$ are smooth. Take, see definition-proposition 12(ii), a compactification $\check{f}_0 = \check{h} : \check{X}_0 \to \check{S}$ of $h : U \to S$. Take, see definition-proposition 12(ii), a strict desingularization $\check{\epsilon} : (\check{X}, \check{D}) \to (\check{X}_0, \check{Z})$ of $(\check{X}_0, \check{Z})$. Then $f'_0 = g \circ h' : X_T \to T$ is a compactification of $g \circ h' : U_T \to S$ such that $g' : U_T/S \to U/S$ extend to a morphism $\check{g}'_0 : \check{X}_T/S \to \check{X}/S$. Denote $\check{Z} = \check{X}_0 \setminus U$ and $\check{Z}' = \check{X}_T \setminus U_T$. Take, see definition-proposition 12(ii), a strict desingularization $\check{\epsilon}' : (\check{X}', \check{D}') \to (\check{X}_T, \check{Z}')$ of $(\check{X}_T, \check{Z}')$. Denote $\check{g}' = \check{g}'_0 \circ \check{\epsilon}' : \check{X}' \to \check{X}$. We then have, see definition-proposition 12(ii), the following commutative diagram in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$

$$\begin{array}{ccc}
U & \xrightarrow{j} & \check{X} = \check{X}_c(\bullet) \xleftarrow{i} & \check{D}_{s_i(\bullet)} \\
\downarrow{g} & & \downarrow{\check{g}} & \\
U_T = U_{T,c(\bullet)} & \xrightarrow{j'} & \check{X}_T = X_{T,c(\bullet)} \xleftarrow{i_{s_i(\bullet)}} & \check{D}_{s_i(\bullet)} \\
\downarrow{g'} & & \downarrow{\check{g}'} & \\
U_T' = U_{T,c(\bullet)} & \xrightarrow{j'} & \check{X}' = X_{c(\bullet)} \xleftarrow{i_{s_i(\bullet)}'} & \check{D}_{s_i(\bullet)} \\
\end{array}$$

We then consider the following map in $C(\text{Var}(\mathbb{C})/T)$

$$T(g, \check{R}^{0CH}(\mathbb{Z}(U/S)) : g^* \check{R}^0_{(X, D)/S}(\mathbb{Z}(U/S))$$

$$\xrightarrow{\cong} g^* \text{Cone}(\mathbb{Z}(i_\bullet) : (Z^{tr}(\check{D}_{s_i(\bullet)}/T), u_{1,J}) \to \mathbb{Z}^{tr}(\check{X}/T)(-d_X)[-2d_X]$$

$$\xrightarrow{\cong} \text{Cone}(\mathbb{Z}(i_\bullet) : (Z^{tr}(\check{D}_{s_i(\bullet)}/T), u_{1,J}) \to \mathbb{Z}^{tr}(\check{X}/T)(-d_X)[-2d_X]$$

$$\xrightarrow{\cong} \text{Cone}(\mathbb{Z}(i_\bullet) : (Z^{tr}(\check{D}_{s_i(\bullet)}/T), u_{1,J}) \to \mathbb{Z}^{tr}(\check{X}/T)(-d_X)[-2d_X]$$

$$\xrightarrow{\cong} \check{R}^0_{(\check{X}, \check{D})/T}(\mathbb{Z}(U/T))$$
Let \( F \) together with the commutative diagram in 
\[
\begin{array}{c}
\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U^n_{\alpha}/S) \xrightarrow{(Z(s_{\alpha}^n, s))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U^{n-1}_{\beta}/S) \to \cdots \in C(\text{Var}(\mathbb{C})/S)
\end{array}
\]
a complex of (maybe infinite) direct sum of representable presheaves with \( h^n_{\alpha} : U^n_{\alpha} \to S \) smooth, we get the map in \( C(\text{Var}(\mathbb{C})/T) \)
\[
T(g, \hat{R}\hat{p}^n CH)(Q^*) = (\cdots \to \oplus_{\alpha \in \Lambda^n} \varinjlim \left. g^* \hat{R}(X^n_{\alpha}, D_{\alpha})/S(Z(U^n_{\alpha}/S)) \right) \to \cdots)
\]
and by functoriality, we get in particular for \( F^* \in C(\text{Var}(\mathbb{C})/\smash{T}(g, \hat{R}\hat{p}^n CH)(Q^*)) \)
\[
\begin{array}{c}
\cdots \to \oplus_{\alpha \in \Lambda^n} \varinjlim \left. \hat{R}(X^n_{\alpha}, D_{\alpha})/T(Z(U^n_{\alpha}, T/S)) \right) \to \cdots =: \hat{R}CH(g^* Q^*)
\end{array}
\]
together with the commutative diagram in \( C(\text{Var}(\mathbb{C})/T) \)
\[
\begin{array}{c}
g^* \hat{R}CH(Q^*) \xrightarrow{T(g, \hat{R}\hat{p}^n CH)(Q^*)} \hat{R}\hat{p}^n CH(g^* Q^*)
\end{array}
\]
Let \( F \in \text{PSh}(\text{Var}(\mathbb{C})^{\smash{\op}}/S) \). Consider
\[
q : LF := (\cdots \to \oplus_{(U_{\alpha}, h_{\alpha})} \in \text{Var}(\mathbb{C})^{\smash{\op}}/S \mathbb{Z}(U_{\alpha}/S) \to \cdots) \to F
\]
the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in \( C(\text{Var}(\mathbb{C})/T) \)
\[
T(g, \hat{R}\hat{p}^n CH)(\rho_{\alpha}^n LF) : g^* \hat{R}CH(\rho_{\alpha}^n LF) = (\cdots \to \oplus_{(U_{\alpha}, h_{\alpha})} \in \text{Var}(\mathbb{C})^{\smash{\op}}/S \varinjlim \left. g^* \hat{R}(X^n_{\alpha}, D_{\alpha})/S(Z(U^n_{\alpha}/S)) \right) \to \cdots) 
\]
and by functoriality, we get in particular for \( F = F^* \in C(\text{Var}(\mathbb{C})^{\smash{\op}}/S) \), the map in \( C(\text{Var}(\mathbb{C})/T) \)
\[
T(g, \hat{R}\hat{p}^n CH)(\rho_{\alpha}^n LF) : g^* \hat{R}CH(\rho_{\alpha}^n LF) \to \hat{R}CH(\rho_{\alpha}^n g^* LF)
\]
together with the commutative diagram in \( C(\text{Var}(\mathbb{C})/T) \)
\[
\begin{array}{c}
g^* \hat{R}CH(\rho_{\alpha}^n LF) \xrightarrow{T(g, \hat{R}\hat{p}^n CH)(\rho_{\alpha}^n LF)} \hat{R}CH(\rho_{\alpha}^n g^* LF)
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{ccc}
g^* T(\hat{R}\hat{p}^n CH)(\rho_{\alpha}^n LF) & \xrightarrow{T(g, \hat{R}\hat{p}^n CH)(\rho_{\alpha}^n LF)} & \hat{R}CH(\rho_{\alpha}^n g^* LF) \\
\end{array}
\end{array}
\]
\bullet Let \( S_1, S_2 \in \text{SmVar}(\mathbb{C}) \) and \( p : S_1 \times S_2 \to S_1 \) the projection. Let \( h : U \to S_1 \) a smooth morphism with \( U \in \text{Var}(\mathbb{C}) \). Consider the cartesian square
\[
\begin{array}{ccc}
U \times S_2 & \xrightarrow{h \times I} & S_1 \times S_2 \\
\downarrow p' & & \downarrow p \\
U & \xrightarrow{h} & S_1
\end{array}
\]

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Take, see definition-proposition 12(i), a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}_1$ of $h : U \to S_1$. Then $\bar{f}_0 \times I : \bar{X}_0 \times S_2 \to \bar{S}_1 \times S_2$ is a compactification of $h \times I : U \times S_2 \to S_1 \times S_2$ and $p' : U \times S_2 \to U$ extend to $\bar{p}_0 := p_{X_0} : \bar{X}_0 \times S_2 \to \bar{X}_0$. Denote $Z = X_0 \setminus U$. Take see theorem 15(i), a strict desingularization $\epsilon : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$. We then have the commutative diagram (48) in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$ whose squares are cartesian

![Diagram](https://via.placeholder.com/150)

Then the map in $C(\text{Var}(\mathbb{C})/S_1 \times S_2)$

$$T(p, \bar{R}^{\mathcal{O}CH}(\mathbb{Z}(U/S_1))) : p^* \bar{R}^{0}_{\mathcal{O}(\mathcal{X}, \mathcal{D})/S_1}(\mathbb{Z}(U/S_1)) \cong \bar{R}^{0}_{\mathcal{O}(\bar{X}, \bar{S}_2, \mathcal{D}_2 \times S_2)/S_1 \times S_2}(\mathbb{Z}(U \times S_2/S_1 \times S_2))$$

is an isomorphism. Hence, for $Q^* \in C(\text{Var}(\mathbb{C})/S_1)$ a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in $C(\text{Var}(\mathbb{C})/S_1 \times S_2)$

$$T(p, \bar{R}^{\mathcal{O}CH}(Q^*)) : p^* \bar{R}^{\mathcal{O}CH}(Q^*) \cong \bar{R}^{\mathcal{O}CH}(p^* Q^*)$$

is an isomorphism. In particular, for $F \in C(\text{Var}(\mathbb{C})^{sm}/S_1)$ the map in $C(\text{Var}(\mathbb{C})/S_1 \times S_2)$

$$T(p, \bar{R}^{\mathcal{O}CH}(\rho_{S_1}^* LF)) : p^* \bar{R}^{\mathcal{O}CH}(\rho_{S_1}^* LF) \cong \bar{R}^{\mathcal{O}CH}(\rho_{S_1 \times S_2}^* p^* LF)$$

is an isomorphism.

**•** Let $h_1 : U_1 \to S$, $h_2 : U_2 \to S$ two morphisms with $U_1, U_2, S \in \text{Var}(\mathbb{C})$, $U_1, U_2$ smooth. Denote by $p_1 : U_1 \times S U_2 \to U_1$ and $p_2 : U_1 \times S U_2 \to U_2$ the projections. Take, see definition-proposition 12(i)), a compactification $\bar{f}_{10} = \bar{h}_1 : \bar{X}_{10} \to \bar{S}_1$ of $h_1 : U_1 \to S$ and a compactification $\bar{f}_{20} = \bar{h}_2 : \bar{X}_{20} \to S$ of $h_2 : U_2 \to S$. Then,

- $\bar{f}_{10} \times \bar{f}_{20} : \bar{X}_{10} \times S \bar{X}_{20} \to S$ is a compactification of $h_1 \times h_2 : U_1 \times S U_2 \to S$.
- $\bar{p}_{10} := p_{X_{10}} : \bar{X}_{10} \times S \bar{X}_{20} \to \bar{X}_{10}$ is a compactification of $p_1 : U_1 \times S U_2 \to U_1$.
- $\bar{p}_{20} := p_{X_{20}} : \bar{X}_{10} \times S \bar{X}_{20} \to \bar{X}_{20}$ is a compactification of $p_2 : U_1 \times S U_2 \to U_2$.

Denote $\bar{Z}_1 = \bar{X}_{10}/U_1$ and $\bar{Z}_2 = \bar{X}_{20}/U_2$. Take, see theorem 15(i), a strict desingularization $\bar{\epsilon}_1 : (\bar{X}_1, \bar{D}) \to (\bar{X}_{10}, \bar{Z}_1)$ of the pair $(\bar{X}_{10}, \bar{Z}_1)$ and a strict desingularization $\bar{\epsilon}_2 : (\bar{X}_2, \bar{E}) \to (\bar{X}_{20}, \bar{Z}_2)$ of the pair $(\bar{X}_{20}, \bar{Z}_2)$. Take then a strict desingularization

$$\bar{\epsilon}_{12} : ((\bar{X}_1 \times S \bar{X}_2)^N, \bar{F}) \to (\bar{X}_1 \times S \bar{X}_2, (\bar{D} \times S \bar{X}_2) \cup (\bar{X}_1 \times S \bar{E}))$$

of the pair $(\bar{X}_1 \times S \bar{X}_2, (\bar{D} \times S \bar{X}_2) \cup (\bar{X}_1 \times S \bar{E}))$. We have then the following commutative diagram

![Diagram](https://via.placeholder.com/150)

and

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The derived categories of filtered complexes of presheaves on a site or

(ii) We denote for $\rho_2 : (\breve{X}_1 \times S \breve{X}_2)^N \to \breve{X}_2$ is a compactification of $p_2 : U_1 \times_S U_2 \to U_2$.

We have then the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$T(\otimes, R_{S}^{CH})(Q_1^*, Q_2^*): R_{S}^{CH}(Q_1^*) \otimes R_{S}^{CH}(Q_2^*) \xrightarrow{T(\otimes, R_{S}^{CH})(Z(U_{1,1}^n), Z(U_{2,1}^n))} R_{S}^{CH}(Q_1^* \otimes Q_2^*)$$

, together with the commutative diagram in $C(\text{Var}(\mathbb{C})/S)$

$$\xymatrix{ R_{S}^{CH}(Q_1^*) \otimes R_{S}^{CH}(Q_2^*) \ar[r]^{T(\otimes, R_{S}^{CH})(Q_1^*, Q_2^*)} & R_{S}^{CH}(Q_1^* \times Q_2^*) \ar[d]^{T(\otimes, R_{S}^{CH}, R_{S}^{CH})(Q_1^* \otimes Q_2^*)} \\
R_{S}^{CH}(Q_1^*) \otimes R_{S}^{CH}(Q_2^*) \ar[r]^{T(\otimes, R_{S}^{CH})(Q_1^*, Q_2^*)} & R_{S}^{CH}(Q_1^* \otimes Q_2^*) }$$

For $F_1, F_2 \in C(\text{Var}(\mathbb{C})/S)$, we get in particular the morphism in $C(\text{Var}(\mathbb{C})/S)$

$$T(\otimes, R_{S}^{CH})(\rho_2^*LF_1, \rho_2^*LF_2): R_{S}^{CH}(\rho_2^*LF_1) \otimes R_{S}^{CH}(\rho_2^*LF_2) \to R_{S}^{CH}(\rho_2^*(LF_1 \otimes LF_2))$$

together with the commutative diagram in $C(\text{Var}(\mathbb{C})/S)$

$$\xymatrix{ R_{S}^{CH}(\rho_2^*LF_1) \otimes R_{S}^{CH}(\rho_2^*LF_2) \ar[r]^{T(\otimes, R_{S}^{CH})(\rho_2^*LF_1, \rho_2^*LF_2)} & R_{S}^{CH}(\rho_2^*LF_1 \times \rho_2^*LF_2) \ar[d]^{T(\otimes, R_{S}^{CH}, R_{S}^{CH})(\rho_2^*LF_1 \otimes \rho_2^*LF_2)} \\
R_{S}^{CH}(\rho_2^*LF_1) \otimes R_{S}^{CH}(\rho_2^*LF_2) \ar[r]^{T(\otimes, R_{S}^{CH})(\rho_2^*LF_1, \rho_2^*LF_2)} & R_{S}^{CH}(\rho_2^*LF_1 \times \rho_2^*LF_2) }$$

### 2.13 The derived categories of filtered complexes of presheaves on a site or of filtered complexes of presheaves of modules on a ringed topos

**Definition 45.** Let $S \in \text{Cat}$ a site endowed with topology $\tau$.

(i) We denote by $D(S) := \text{HoTop} C(S)$ the localization of the category of complexes of presheaves on $S$ with respect to top local equivalence and by $D(\tau) : C(S) \to D(S)$ the localization functor.

(ii) We denote for $r = 1, \ldots, \infty$, resp. $r = (1, \ldots, \infty)^2$,

$$D_{fil,r}(S) := K_{fil,r}(S)[(E_1)^{-1}], D_{2fil,r,r'}(S) := K_{2fil,r,r'}(S)[(E_1)^{-1}],$$

the localizations of the category of filtered complexes of presheaves on $S$ whose filtration is irregular modulo $r$-filtered homotopy with respect to the classes of filtered $\tau$ local equivalence $[E_1]$. Note
that the classes of filtered $\tau$ local equivalence constitute a right multiplicative system. By definition, if $m : (G_1, F) \to (G_2, F)$ with $(G_1, F), (G_2, F) \in C_{fil}(S)$ is an $r$-filtered $\tau$ local equivalence then $m := D(\tau)(m) : (G_1, F) \to (G_2, F)$ is an isomorphism in $D_{fil,r}(S)$. By definition, we have sequences of functors

$$C_{fil}(S) \to K_{fil}(S) \to D_{fil}(S) \to D_{fil,2}(S) \to \cdots \to D_{fil,\infty}(S).$$

and commutative diagrams of functors

$$
\begin{array}{ccc}
K_{fil}(S) & \longrightarrow & D_{fil}(S) \\
\downarrow & & \downarrow \\
K_{fil,r}(S) & \longrightarrow & D_{fil,r}(S).
\end{array}
\begin{array}{ccc}
K_{fil,2}(S) & \longrightarrow & D_{fil,2}(S) \\
\downarrow & & \downarrow \\
K_{fil,r+1}(S) & \longrightarrow & D_{fil,r}(S).
\end{array}
$$

Then, for $r = 1$, $K_{fil}(S)$ and $D_{fil}(S)$ are in the canonical way triangulated categories. However, for $r > 1$, the categories $K_{fil,r}(S)$ and $D_{fil,r}(S)$ together with the canonical triangles does NOT satisfy the 2 of 3 axiom of triangulated categories.

**Definition 46.** Let $(S, O_S) \in R\text{Cat}$ where $S \in \text{Cat}$ is a site endowed with topology $\tau$.

(i) We denote by $D_{O_S}(S) := \text{Ho}_{\text{Top}} C_{O_S}(S)$ the localization of the category of complexes of presheaves on $S$ with respect to top local equivalence and by $D(\tau) : C_{O_S}(S) \to D_{O_S}(S)$ the localization functor.

(ii) We denote for $r = 1, \ldots, \infty$, resp. $r = (1, \ldots, \infty)^2$,

$$D_{O_S,fil,r}(S) := K_{O_S,fil,r}(S)[[E_1]^{-1}], \; D_{O_S,fil,r,r'}(S) := K_{O_S,fil,r,r'}(S)[[E_1]^{-1}],$$

the localizations of the category of filtered complexes of presheaves on $S$ whose filtration is birregular modulo $r$-filtered homotopy with respect to the classes of filtered $\tau$ local equivalence $[E_1]$. Note that the classes of filtered $\tau$ local equivalence constitute a right multiplicative system. By definition, if $m : (G_1, F) \to (G_2, F)$ with $(G_1, F), (G_2, F) \in C_{O_S,fil}(S)$ is an $r$-filtered $\tau$ local equivalence then $m := D(\tau)(m) : (G_1, F) \to (G_2, F)$ is an isomorphism in $D_{O_S,fil,r}(S)$. By definition, we have sequences of functors

$$C_{O_S,fil}(S) \to K_{O_S,fil}(S) \to D_{O_S,fil}(S) \to D_{O_S,fil,2}(S) \to \cdots \to D_{O_S,fil,\infty}(S).$$

and commutative diagrams of functors

$$
\begin{array}{ccc}
K_{O_S,fil}(S) & \longrightarrow & D_{O_S,fil}(S) \\
\downarrow & & \downarrow \\
K_{O_S,fil,r}(S) & \longrightarrow & D_{O_S,fil,r}(S).
\end{array}
\begin{array}{ccc}
K_{O_S,fil,2}(S) & \longrightarrow & D_{fil,2}(S) \\
\downarrow & & \downarrow \\
K_{O_S,fil,r+1}(S) & \longrightarrow & D_{O_S,fil,r}(S).
\end{array}
$$

Then, for $r = 1$, $K_{O_S,fil}(S)$ and $D_{O_S,fil}(S)$ are in the canonical way triangulated categories. However, for $r > 1$, the categories $K_{O_S,fil,r}(S)$ and $D_{O_S,fil,r}(S)$ together with the canonical triangles does NOT satisfy the 2 of 3 axiom of triangulated categories.

Let $f : T \to S$ a morphism of presite, with $S, T \in \text{Cat}$ endowed with a topology $\tau$. If $f$ is a morphism of site, the adjunctions

$$(f^*, f_*) = (f^{-1}, f_*) : C(S) \leftrightarrows C(T), \; (f^*, f_*^t) = (f^{-1}, f_*^t) : C_{(2)fil}(S) \leftrightarrows C_{(2)fil}(T).$$

are Quillen adjunctions. They induces respectively in the derived categories, for $r = (1, \ldots, \infty)$, resp. $r = (1, \ldots, \infty)$ (note that $f^*$ derive trivially)

$$(f^*, Rf_*^t) : D(S) \leftrightarrows D(T), \; (f^*, Rf_*) : D_{fil,r}(S) \leftrightarrows D_{fil,r}(T).$$
For $F^\bullet \in C(S)$, we have the adjunction maps
\[
\text{ad}(f^*, f_*)(F^\bullet) : F^\bullet \to f_* f^* F^\bullet, \quad \text{ad}(f^*, f_*)(F^\bullet) : f^* f_* F^\bullet \to F^\bullet,
\]
induces in the derived categories, for $(M, F) \in D_{fil}(S)$ and $(N, F) \in D_{fil}(T)$, the adjunction maps
\[
\text{ad}(f^*, Rf_*)(M) : (M, F) \to Rf_* f^*(M, F), \quad \text{ad}(f^*, Rf_*)(N, F) : f^* Rf_*(N, F) \to (N, F).
\]
For a commutative diagram of sites:

\[\begin{array}{ccc}
D & \xrightarrow{g_2} & X \\
\downarrow f_2 & & \downarrow f_1 \\
T & \xrightarrow{g_1} & S
\end{array}\]

with $\mathcal{Y}, \mathcal{T}, \mathcal{S}, \mathcal{X} \in \text{Cat}$ with topology $\tau_{\mathcal{Y}}, \tau_{\mathcal{T}}, \tau_{\mathcal{S}}, \tau_{\mathcal{X}}$, the maps, for $F \in C(\mathcal{X})$,
\[
T(D)(F) : g_1^* f_1^* F \to f_2^* g_2^* F
\]
induce in the derived category the maps in $D_{fil, r}(\mathcal{T})$, given by, for $(G, F) \in D_{fil, r}(\mathcal{X})$ with $(G, F) = D(\tau_{\mathcal{X}}, r)((G, F))$,
\[
g_1^* Rf_1^*(M, F) \xrightarrow{T(D)(M, F)} Rf_2^* g_2^*(M, F).
\]

Let $S \in \text{Cat}$ with topology $\tau$. The tensor product of complexes of abelian groups and the internal hom of presheaves on $S$
\[(\cdot \otimes \cdot, \mathcal{H}om^\bullet(\cdot, \cdot)) : C(S)^2 \to C(S),\]
is a Quillen adjunction which induces in the derived category
\[(\cdot \otimes^L \cdot, R\mathcal{H}om^\bullet(\cdot, \cdot)) : D_{fil, r}(S)^2 \to D_{fil, r}(S), \quad R\mathcal{H}om^\bullet((M, W), (N, W)) = \mathcal{H}om^\bullet((Q, W), E(G, F)),\]
where, $Q$ is projectively cofibrant such that $M = D(\tau)(Q^\bullet)$ and $G$ such that $N = D(\tau)(G)$.

Let $i : Z \hookrightarrow S$ a closed embedding, with $S, Z \in \text{Top}$. Denote by $j : S \setminus Z \hookrightarrow S$ the open embedding of the complementary subset. The adjunction
\[(i_*, i^!): C(Z, F) \to C(S), \text{ with in this case } i^! F := \ker(F \to j_* j^* F)\]
is a Quillen adjunction. Since $i^!$ preserve monomorphisms, we have also Quillen adjunctions
\[(i_*, i^!): C_{(2)fil}(Z) \to C_{(2)fil}(S), \text{ with } i^! (G, F) = (i^! G, F).\]
which induces in the derived category $(i_*$ derive trivially)
\[(i_*, Ri^!): D_{(2)fil}(Z) \to D_{(2)fil}(S), \text{ with } Ri^! (G, F) = i^! E(G, F).\]

The 2-functor $S \in \text{Top} \to D(S)$ obviously satisfy the localization property, that is for $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \text{Top}$, denote by $j : S \setminus Z \hookrightarrow S$ the open complementary subset, we have for $K \in D(S)$ a distinguish triangle in $D(S)$
\[j_{2!} j^* K \xrightarrow{\text{ad}(j_2, j^*)(K)} K \xrightarrow{\text{ad}(i^*, i_*)(K)} \rightarrow j_2^* j^* K \to j_{2!} j^* K[1]\]
equivalently,
• the functor
\[(i^*, j^*): D(S) \xrightarrow{\sim} D(Z) \times D(S \setminus Z)\]
is conservative,
• and for \(K \in C(Z)\), the adjunction map \(\text{ad}(i^*, i_*)(K): i^*i_*K \to K\) is an equivalence top local, hence for \(K \in D(S)\), the induced map in the derived category
\[\text{ad}(i^*, i_*)(K): i^*i_*K \xrightarrow{\sim} K\]
is an isomorphism.

3 Triangulated category of motives

3.1 Definition and the six functor formalism

The category of motives is obtained by inverting the \((\mathbb{A}^1_S, \text{et})\) equivalence. Hence the \(\mathbb{A}^1_S\) local complexes of presheaves plays a key role.

**Definition 47.** The derived category of motives of complex algebraic varieties over \(S\) is the category
\[DA(S) := \text{Ho}_{\mathbb{A}^1_S, \text{et}}(C(\text{Var}(\mathbb{C})^{\text{sm}}/S)),\]
which is the localization of the category of complexes of presheaves on \(\text{Var}(\mathbb{C})^{\text{sm}}/S\) with respect to \((\mathbb{A}^1_S, \text{et})\) local equivalence and we denote by
\[D(\mathbb{A}^1_S, \text{et}) := D(\mathbb{A}^1_S) \circ D(\text{et}) : C(\text{Var}(\mathbb{C})^{\text{sm}}/S) \to DA(S)\]
the localization functor. We have \(DA^{-}(S) := D(\mathbb{A}^1_S, \text{et})(\text{PSh}(\text{Var}(\mathbb{C})^{\text{sm}}/S, C^{-}(\mathbb{Z}))) \subset DA(S)\) the full subcategory consisting of bounded above complexes.

**Definition 48.** The stable derived category of motives of complex algebraic varieties over \(S\) is the category
\[DA_{st}(S) := \text{Ho}_{\mathbb{A}^1_S, \text{et}}(C_{\Sigma}(\text{Var}(\mathbb{C})^{\text{sm}}/S)),\]
which is the localization of the category of \(\mathbb{G}_{m,S}\)-spectra \((\Sigma F^\bullet = F^\bullet \otimes \mathbb{G}_{m,S})\) of complexes of presheaves on \(\text{Var}(\mathbb{C})^{\text{sm}}/S\) with respect to \((\mathbb{A}^1_S, \text{et})\) local equivalence. The functor
\[\Sigma^\infty : C(\text{Var}(\mathbb{C})^{\text{sm}}/S) \hookrightarrow C_{\Sigma}(\text{Var}(\mathbb{C})^{\text{sm}}/S)\]
induces the functor \(\Sigma^\infty : DA(S) \to DA_{st}(S)\).

We have all the six functor formalism by [10]. We give a list of the operation we will use:

- For \(f: T \to S\) a morphism with \(S, T \in \text{Var}(\mathbb{C})\), the adjunction
  \[(f^*, f_*): C(\text{Var}(\mathbb{C})^{\text{sm}}/S) \xrightarrow{\sim} C(\text{Var}(\mathbb{C})^{\text{sm}}/T)\]
is a Quillen adjunction which induces in the derived categories \((f^* \text{ derives trivially}), (f^*, Rf_*): DA(S) \xrightarrow{\sim} DA(T)\).
- For \(h: V \to S\) a smooth morphism with \(V, S \in \text{Var}(\mathbb{C})\), the adjunction
  \[(h^*, h_*): C(\text{Var}(\mathbb{C})^{\text{sm}}/V) \xrightarrow{\sim} C(\text{Var}(\mathbb{C})^{\text{sm}}/S)\]
is a Quillen adjunction which induces in the derived categories \((h^* \text{ derive trivially}) (Lh^*_2, h^*) =: DA(V) \xrightarrow{\sim} DA(S)\).
• For \( i : Z \hookrightarrow S \) a closed embedding, with \( Z, S \in \text{Var}(\mathbb{C}) \),
\[
(i_+ i^!)(1) := (i_+ i^!) : C(\text{Var}(\mathbb{C})^{\text{sm}}/Z) \Rightarrow C(\text{Var}(\mathbb{C})^{\text{sm}}/S)
\]
is a Quillen adjunction, which induces in the derived categories \((i_+ \text{ derive trivially}) \) \((i_+, Ri^! : \text{DA}(Z) \Rightarrow \text{DA}(S)) \). The fact that \( i_+ \text{ derive trivially (i.e. send } (\mathbb{A}^1, et) \text{ local equivalence to } (\mathbb{A}^1, et)) \) local equivalence is proved in [4].

• For \( S \in \text{Var}(\mathbb{C}) \), the adjunction given by the tensor product of complexes of abelian groups and the internal hom of presheaves
\[
((\cdot \otimes \cdot), \text{Hom}^\bullet(\cdot, \cdot)) : C(\text{Var}(\mathbb{C})^{\text{sm}}/S)^2 \rightarrow C(\text{Var}(\mathbb{C})^{\text{sm}}/S),
\]
is a Quillen adjunction, which induces in the derived category
\[
((\cdot \otimes L \cdot), R\text{Hom}^\bullet(\cdot, \cdot)) : \text{DA}(S)^2 \rightarrow \text{DA}(S),
\]
– Let \( M, N \in \text{DA}(S), Q^\bullet \) projectively cofibrant such that \( M = D(\mathbb{A}^1, et)(Q^\bullet) \), and \( G^\bullet \) be \( \mathbb{A}^1 \) local for the etale topology such that \( N = D(\mathbb{A}^1, et)(G^\bullet) \). Then,
\[
R\text{Hom}^\bullet(M, N) = \text{Hom}^\bullet(Q^\bullet, E(G^\bullet)) \in \text{DA}(S).
\]
This is well defined since if \( s : Q_1 \rightarrow Q_2 \) is a etale local equivalence,
\[
\text{Hom}(s, E(G)) : \text{Hom}(Q_1, E(G)) \rightarrow \text{Hom}(Q_2, E(G))
\]
is an etale local equivalence for \( 1 \leq i \leq l \).

• For a commutative diagram in \( \text{Var}(\mathbb{C}) \):

\[
D = \begin{array}{ccc}
Y & \xrightarrow{g_2} & X \\
\downarrow{f_2} & & \downarrow{f_1} \\
T & \xrightarrow{g_1} & S
\end{array}
\]
and \( F \in C(\text{Var}(\mathbb{C})^{\text{sm}}/X) \), the transformation map \( T(D)(F) : g_1^* f_1_+ F \rightarrow f_2_+ g_2^* F \) induces in the derived category, for \( M \in \text{DA}(X), M = D(\mathbb{A}^1, et)(F) \) with \( \mathbb{A}^1 \) local for the etale topology,
\[
g_1^* Rf_1_+ M \xrightarrow{T(D)(M)} Rf_2_+ g_2^* M
\]
If \( D \) is cartesian with \( f_1 = f, g_1 = g, f_2 = f' : X_T \rightarrow T, g' : X_T \rightarrow X \), we denote
\[
\begin{align*}
T(D)(F) &= T(f, g)(F) : g^* f_1 F \rightarrow f_2 g' F, \\
T(D)(M) &= T(f, g)(M) : g^* Rf_1 M \rightarrow Rf_2 g' M.
\end{align*}
\]
We get from the first point 2 functors:

• The 2-functor \( C(\text{Var}(\mathbb{C})^{\text{sm}}/\cdot) : \text{Var}(\mathbb{C}) \rightarrow \text{AbCat} \), given by
\[
S \mapsto C(\text{Var}(\mathbb{C})^{\text{sm}}/S), \ (f : T \rightarrow S) \mapsto (f^* : C(\text{Var}(\mathbb{C})^{\text{sm}}/S) \rightarrow C(\text{Var}(\mathbb{C})^{\text{sm}}/T)).
\]

• The 2-functor \( \text{DA}(\cdot) : \text{Var}(\mathbb{C}) \rightarrow \text{TriCat} \), given by
\[
S \mapsto \text{DA}(S), \ (f : T \rightarrow S) \mapsto (f^* : \text{DA}(S) \rightarrow \text{DA}(T)).
\]
The main theorem is the following:

**Theorem 16.** [4]/[10] The 2-functor \( DA(\cdot) : \text{Var}(\mathbb{C}) \to \text{TriCat} \), given by

\[
S \mapsto DA(S), \quad (f : T \to S) \mapsto (f^* : DA(S) \to DA(T))
\]

is a 2-homotopic functor ([4])

From Theorem 16, we get in particular

- For \( f : T \to S \) a morphism with \( T, S \in \text{Var}(\mathbb{C}) \), there by Theorem 16 is also a pair of adjoint functor \((f_!, f^!) : DA(S) \rightleftarrows DA(T)\)

  - with \( f_! = Rf_* \) if \( f \) is proper,
  - with \( f^! = f^*[d] \) if \( f \) is smooth of relative dimension \( d \).

For \( h : U \to S \) a smooth morphism with \( U, S \in \text{Var}(\mathbb{C}) \) irreducible, have, for \( M \in DA(U) \), an isomorphism

\[
Lh_* M \to h_! [d][50]
\]

in \( DA(S) \).

- The 2-functor \( S \in \text{Var}(\mathbb{C}) \mapsto DA(S) \) satisfy the localization property, that is for \( i : Z \hookrightarrow X \) a closed embedding with \( Z, X \in \text{Var}(\mathbb{C}) \), denote by \( j : S \setminus Z \hookrightarrow S \) the open complementary subset, we have for \( M \in DA(S) \) a distinguish triangle in \( DA(S) \)

\[
j_2 j^* M \xrightarrow{\text{ad}(j_!, j^*)(M)} M \xrightarrow{\text{ad}(i^!, i_*)(M)} i_* i^* M \to j_2 j^* M[1]
\]
equivalently,

  - the functor

\[
(i^!, j^!) : DA(S) \xrightarrow{\sim} DA(Z) \times DA(S \setminus Z)
\]
is conservative,

  - and for \( F \in C(\text{Var}(\mathbb{C})^{\text{sm}}/Z) \), the adjonction map \( \text{ad}(i^!, i_*)(M) : i^* i_* F \to F \) is an equivalence Zariski local, hence for \( M \in DA(S) \), the induced map in the derived category

\[
\text{ad}(i^!, i_*)(M) : i^* i_* M \xrightarrow{\sim} M
\]
is an isomorphism.

- For \( f : X \to S \) a proper map, \( g : T \to S \) a morphism, with \( T, X, S \in \text{Var}(\mathbb{C}) \), and \( M \in DA(X) \),

\[
T(f, g)(M) : g^* Rf_* M \to Rf_* g^* M
\]
is an isomorphism in \( DA(T) \) if \( f \) is proper.

**Definition 49.** The derived category of extended motives of complex algebraic varieties over \( S \) is the category

\[
DA(S) := \text{Ho}_{\mathbb{A}^1_S, \text{et}}(C(\text{Var}(\mathbb{C})/S)),
\]

which is the localization of the category of complexes of presheaves on \( \text{Var}(\mathbb{C})/S \) with respect to \( (\mathbb{A}^1_S, \text{et}) \) local equivalence and we denote by

\[
D(\mathbb{A}^1_S, \text{et}) := D(\mathbb{A}^1_S) \circ D(\text{et}) : C(\text{Var}(\mathbb{C})/S) \to DA(S)
\]
the localization functor. We have \( DA^{-}(S) := D(\mathbb{A}^1_S, \text{et})(\text{PSh}(\text{Var}(\mathbb{C})/S, C^{-}(\mathbb{Z}))) \subset DA(S) \) the full subcategory consisting of bounded above complexes.
Remark 5. Let \( i : Z \to S \) a closed embedding, with \( Z,S \in \text{Var}(\mathbb{C}) \).

(i) By theorem 16, for \( X/S = (X,h) \in \text{Var}(\mathbb{C})^{sm}/S \),

\[
(0, \text{ad}(i^*, i_*)(\mathbb{Z}(X/S))) : \Gamma^Z_2\mathbb{Z}(X/S) \to i_*\mathbb{Z}(X_Z/Z)
\]

is an equivalence \((A^1, et)\) local.

(ii) For \( X/S = (X,f) \in \text{Var}(\mathbb{C})/S \),

\[
(0, \text{ad}(i^*, i_*)(\mathbb{Z}(X/S))) : \Gamma^Z_2\mathbb{Z}(X/S) \to i_*\mathbb{Z}(X_Z/Z)
\]

is NOT an equivalence \((A^1, et)\) local in general, since for example if \( f(X) = Z \subset S \), \( \rho_{S*}\mathbb{Z}(X/S) = 0 \) but \( D(A^1, et)(\rho_{S*}i_*\mathbb{Z}(X_Z/Z) = i_*\rho_{S*}\mathbb{Z}(X_Z/Z)) \neq 0 \in DA(S) \), hence it is NOT an equivalence \((A^1, et)\) local in this case by proposition 19. In particular \( DA(S) \) does NOT satisfy the localization property.

(ii)' For \( X/Z = (X,f) \in \text{Var}(\mathbb{C})/Z \), the inclusion

\[
T(i^*, i_*) : i_2\mathbb{Z}(X/Z) \to i_*\mathbb{Z}(X/Z)
\]

is NOT an equivalence \((A^1, et)\) local by proposition 19 since \( \rho_{S*}i_2\mathbb{Z}(X/Z) = 0 \) but \( D(A^1, et)(\rho_{S*}i_*\mathbb{Z}(X_Z/Z) = i_*\rho_{S*}\mathbb{Z}(X_Z/Z)) \neq 0 \in DA(S) \).

(iii) Let \( f : X \to S \) a smooth proper morphism with \( X,S \in \text{Var}(\mathbb{C}) \) of relative dimension \( d = d_X - d_S \) and \( X \) smooth. Then, we have then by proposition 40(i) the equivalence \((A^1, et)\) local in \( C(\text{Var}(\mathbb{C})^{sm}/S) \)

\[
T_0(f_2, f_3)((\mathbb{Z}(X/X)) := [\Delta_X] : f_2\mathbb{Z}(X/X) = \mathbb{Z}(X/S) \to f_*E_{et}(\mathbb{Z}(X/X))(d)[2d]
\]

given by the class of the diagonal \([\Delta_X] \in \text{Hom}(f_2\mathbb{Z}(X/X), f_*E_{et}(\mathbb{Z}(X/X))(d)[2d]) \).

(iii)' Let \( f : X \to S \) a proper surjective morphism with \( X,S \in \text{Var}(\mathbb{C}) \) with equidimensional fiber of relative dimension \( d = d_X - d_S \). Assume \( X \) smooth. Then, we have then by proposition 40(i) the equivalence \((A^1, et)\) local in \( C(\text{Var}(\mathbb{C})/S) \)

\[
T_0(f_2, f_3)((\mathbb{Z}(X/X)) := [\Delta_X] : f_2\mathbb{Z}(X/X) = \mathbb{Z}(X/S) \to f_*E_{et}(\mathbb{Z}(X/X))(d)[2d]
\]

given by the class of the diagonal \([\Delta_X] \in \text{Hom}(f_2\mathbb{Z}(X/X), f_*E_{et}(\mathbb{Z}(X/X))(d)[2d]) \).

3.2 Constructible motives and resolution of a motive by Corti-Hanamura motives

We now give the definition of the motives of morphisms \( f : X \to S \) which are constructible motives and the definition of the category of Corti-Hanamura motives.

Definition 50. Let \( S \in \text{Var}(\mathbb{C}) \),

- the homological motive functor is \( M(/S) : \text{Var}(\mathbb{C})/S \to DA(S) \), \( (f : X \to S) \mapsto M(X/S) := f_*f^!M(S/S) \),

- the cohomological motive functor is \( M^v(/S) : \text{Var}(\mathbb{C})/S \to DA(S) \), \( (f : X \to S) \mapsto M(X/S)^v := Rf_*M(X/X) = f_*E(Z_X) \),

- the Borel-Moore motive functor is \( M^{BM}(/S) : \text{Var}(\mathbb{C})/S \to DA(S) \), \( (f : X \to S) \mapsto M^{BM}(X/S) := f_*M(X/X) \),

- the (homological) motive with compact support functor is \( M_c(/S) : \text{Var}(\mathbb{C})/S \to DA(S) \), \( (f : X \to S) \mapsto M_c(X/S) := Rf_*f^!M(S/S) \).
Let $f : X \to S$ a morphism with $X, S \in \operatorname{Var}(\mathbb{C})$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \xrightarrow{p} S$, with $Y \in \operatorname{SmVar}(\mathbb{C})$, $i : X \to Y$ is a closed embedding and $p$ the projection. Then,

$$Q(X/S) := p_1 \Gamma_Y^* Z_{Y \times S} \in C(\operatorname{Var}(\mathbb{C})^{sm}/S)$$

(see definition 10) is projective, admits transfert, and satisfy $D(\mathbb{A}^1_{\mathbb{Z}}, \text{et})(Q(X/S)) = M(X/S)$.

**Definition 51.**

(i) Let $S \in \operatorname{Var}(\mathbb{C})$. We define the full subcategory $\mathcal{DA}_c(S) \subset \mathcal{DA}(S)$ whose objects are constructible motives to be the thick triangulated category generated by the motives of the form $M(X/S)$, with $f : X \to S$ a morphism, $X \in \operatorname{Var}(\mathbb{C})$.

(ii) Let $X, S \in \operatorname{Var}(\mathbb{C})$. If $f : X \to S$ is proper (but not necessary smooth) and $X$ is smooth, $M(X/S)$ is said to be a Corti-Hanamura motive and we have by above in this case $M(X/S) = M^{BM}(X/S)[e] = M(X/S)^{\vee}[e]$, with $e = \operatorname{codim}(X, X \times S)$ where $f : X \hookrightarrow X \times S \to S$. We denote by

$$\mathcal{CH}(S) = \{M(X/S)\}_{(X/S=(X,f), \text{pr}, \text{sm})}^{\text{pt}} \subset DM(S)$$

the full subcategory which is the pseudo-abelian completion of the full subcategory whose objects are Corti-Hanamura motives.

(iii) We denote by

$$\mathcal{CH}^0(S) \subset \mathcal{CH}(S)$$

the full subcategory which is the pseudo-abelian completion of the full subcategory whose objects are Corti-Hanamura motives $M(X/S)$ such that the morphism $f : X \to S$ is projective.

For bounded above motives, we have

**Theorem 17.** Let $S \in \operatorname{Var}(\mathbb{C})$.

(i) There exists a unique weight structure $\omega$ on $\mathcal{DA}^-(S)$ such that $\mathcal{DA}^-(S)^{\omega=0} = \mathcal{CH}(S)$

(ii) There exist a well defined functor

$$W(S) : \mathcal{DA}^-(S) \to K^-(\mathcal{CH}(S)) , \quad W(S)(M) = [M^\bullet]$$

where $M^\bullet \in C^-(\mathcal{CH}(S))$ is a bounded above weight complex, such that if $m \in \mathbb{Z}$ is the highest weight, we have a generalized distinguish triangle for all $i \leq m$

$$T_i : M^{(i)}[i] \to M^{(i+1)}[(i+1)] \to \cdots \to M^{(m)}[m] \to M^{\geq i+1}$$

Moreover the maps $w(M)^{\geq i} : M^{\geq i} \to M$ induce an isomorphism $w(M) : \operatorname{holim}_{i} M^{\geq i} \simto M$ in $\mathcal{DA}^-(S)$.  

(iii) Denote by $\operatorname{Chow}(S)$ the category of Chow motives, which is the pseudo-abelian completion of the category

\begin{itemize}
  \item whose set of objects consist of the $X/S = (X, f) \in \operatorname{Var}(\mathbb{C})/S$ such that $f$ is proper and $X$ is smooth,
  \item whose set of morphisms between $X_1/S$ and $X_2/S$ is $\mathcal{CH}^d(X_1 \times_S X_2)$, and the composition law is given in [11].
\end{itemize}

We have then a canonical functor $CH_S : \operatorname{Chow}(S) \to \mathcal{DA}(S)$, with $CH_S(X/S) := M(X/S) := Rf_* Z(X/X)$, which is a full embedding whose image is the category $\mathcal{CH}(S)$.

**Proof.** (i): The category $\mathcal{DA}(S)$ is clearly weakly generated by $\mathcal{CH}(S)$. Moreover $\mathcal{CH}(S) \subset \mathcal{DA}(S)$ is negative. Hence, the result follows from [6] theorem 4.3.2 III.

(ii): Follows from (i) by standard fact of weight structure on triangulated categories. See [6] theorem 3.2.2 and theorem 4.3.2 V for example.

(iii): See [13].
Theorem 18. Let $S \in \operatorname{Var}(\mathbb{C})$.

(i) There exists a unique weight structure $\omega$ on $\operatorname{DA}^{-}(S)$ such that $\operatorname{DA}^{-}(S)^{\omega=0} = \mathcal{H}^{0}(S)$.

(ii) There exists a well-defined functor

$$W(S) : \operatorname{DA}^{-}(S) \to K^{-}(\mathcal{H}^{0}(S)), W(S)(M) = [M^{(\bullet)}]$$

where $M^{(\bullet)} \in C^{-}(\mathcal{H}^{0}(S))$ is a bounded above weight complex, such that if $m \in \mathbb{Z}$ is the highest weight, we have a generalized distinguished triangle for all $i \leq m$

$$T_{i} : M^{(i)}[i] \to M^{(i+1)}[(i+1)] \to \cdots \to M^{(m)}[m] \to M^{w \geq i}$$

Moreover the maps $w(M)^{(\geq i)} : M^{\geq i} \to M$ induce an isomorphism $w(M) : \operatorname{holim}_{i} M^{\geq i} \sim M$ in $\operatorname{DA}^{-}(S)$.

Proof. Similar to the proof of theorem 17. \qed

Corollary 1. Let $S \in \operatorname{Var}(\mathbb{C})$. Let $M \in \operatorname{DA}(S)$. Then there exist $(F,W) \in \operatorname{Cfd}(\operatorname{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^{1}, et)(F) = M$ and $D(\mathbb{A}^{1}, et)(\operatorname{Gr}^{W}_{p} F) \in \mathcal{H}^{0}(S)$.

Proof. By theorem 18, there exist, by induction, for $i \in \mathbb{Z}$, a distinguished triangle in $\operatorname{DA}(S)$

$$T_{i} : M^{(i)}[i] \to M^{(i+1)}[m-1] \to \cdots \to M^{(m)}[m] \to M^{w \geq i}$$

with $M^{(j)}[j] \in \mathcal{H}^{0}(S)$ and $w(M) : \operatorname{holim}_{i} M^{\geq i} \sim M$ in $\operatorname{DA}^{-}(S)$. For $i \in \mathbb{Z}$, take $(F_{j})_{j \geq i}, F_{w \geq i} \in C(\operatorname{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^{1}, et)(F_{j}) = M^{(j)}[j], D(\mathbb{A}^{1}, et)(F_{w \geq i}) = M^{w \geq i}$ and such that we have in $C(\operatorname{Var}(\mathbb{C})^{sm}/S),$

$$F_{w \geq i} = \operatorname{Cone}(F_{i} \to F_{i+1} \to \cdots \to F_{m})$$

(54)

where $m_{j} : F_{j} \to F_{j+1}$ are morphisms in $C(\operatorname{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^{1}, et)(m_{j}) = m_{j}$.

Now set $F = \operatorname{holim}_{i} F_{w \geq i} \in C(\operatorname{Var}(\mathbb{C})^{sm}/S)$ and $W_{i} F := F_{w \geq i} \sim F$, so that $(F,W) \in \operatorname{Cfd}(\operatorname{Var}(\mathbb{C})^{sm}/S)$ satisfy $D(\mathbb{A}^{1}, et)(\operatorname{Gr}^{W}_{p} F) = M^{(p)}[p] \in \mathcal{H}^{0}(S)$. \qed

3.3 The restriction of relative motives to their Zariski sites

Let $S \in \operatorname{Var}(\mathbb{C})$. The adjunction

$$(e(S)^{*}, e(S)_{*}) : C(\operatorname{Var}(\mathbb{C})^{sm}/S) \Rightarrow C(S)$$

is a Quillen adjunction and induces in the derived category

- $(e(S)^{*}, e(S)_{*}) : \operatorname{Ho}_{zar}(\operatorname{Var}(\mathbb{C})^{sm}/S) \Rightarrow D(T) := \operatorname{Ho}_{zar} C(S)$, since $e(S)_{*}$ sends Zariski local equivalence on the big site $\operatorname{Var}(\mathbb{C})^{sm}/S$ to Zariski local equivalence in the small Zariski site of $S$,

- $(e(S)^{*}, \operatorname{Re}(S)_{*}) : \operatorname{DA}(S) \Rightarrow D(T) := \operatorname{Ho}_{zar} C(S)$.

We will use in the definition of the De Rahm realization functor on $\operatorname{DA}(S)$ the following proposition concerning the restriction of the derived internal hom functor to the Zariski site:

Proposition 41. Let $M, N \in \operatorname{DA}(S)$ and $m : M \to N$ be a morphism. Let $F^{*}, G^{*} \in \operatorname{PSh}(\operatorname{Var}(\mathbb{C})^{sm}/S, C(\mathbb{Z}))$ such that $M = D(\mathbb{A}^{1}_{S}, et)(F^{*})$ and $N = D(\mathbb{A}^{1}_{S}, et)(G^{*})$. If we take $G^{*} (\mathbb{A}^{1}_{S}, et)$ fibrant and admitting transfer, and $F^{*}$ cofibrant for the projective model structure, we have

$$\operatorname{Re}(S)_{*} \operatorname{RHom}^{*}(M, N) = e(S)_{*} \operatorname{Hom}^{*}(F^{*}, G^{*})$$

in $D(S)$.
Proof. Since $F^\bullet$ is projectively cofibrant and $G^\bullet$ is (projectively) $(\mathbb{A}^1_k, et)$ fibrant, we have $R\hom^\bullet(M, N) = \hom^\bullet(F^\bullet, G^\bullet)$. Then, $\hom^\bullet(F^\bullet, G^\bullet)$ is $\mathbb{A}^1_k$ local and admits transfert. On the other hand, we have

$$L_{\mathbb{A}^1_k}D_{et}(\cor \var(C)^{sm}/S) = L_{\mathbb{A}^1_k}D_{zar}(\cor \var(C)^{sm}/S) \subset D(\var(C)^{sm}/S)$$

by theorem 10 (ii). This gives the equality of the proposition.

We will also have:

**Proposition 42.** For $f : T \to S$ a morphism and $i : Z \to S$ a closed embedding, with $Z, S, T \in \var(C)$, we have

(i) $\re(S)_*Rf_* = Rf_*\re(T)_*$ and $e(S)^*Rf_* = Rf_*e(T)^*$

(ii) $\re(S)_*\re Z = \re Z \re(S)_*$.

*Proof.* (i):Follows from proposition 16 (i) and the fact that $f_*$ preserve $(\mathbb{A}^1_k, et)$ fibrant complex of presheaves.

(ii):Follows from proposition 16 (ii) and the fact that $\re Z$ preserve $(\mathbb{A}^1_k, et)$ fibrant complex of presheaves.

3.4 Motives of complex analytic spaces

The category of motives is obtained by inverting the $(\mathbb{D}^1_S, usu)$ local equivalence. Hence the $\mathbb{D}^1_S$ local complexes of presheaves plays a key role.

**Definition 52.** The derived category of motives of complex algebraic varieties over $S$ is the category

$$\An DA(S) := \hom_{\mathbb{D}^1_S, usu}(\AnSp(C)^{sm}/S),$$

which is the localization of the category of complexes of presheaves on $\AnSp(C)^{sm}/S$ with respect to $(\mathbb{D}^1_S, usu)$ local equivalence and we denote by

$$D(\mathbb{D}^1_S, usu) := D(\mathbb{A}^1_k) \circ D(et) : C(\AnSp(C)^{sm}/S) \to \An DA(S)$$

the localization functor. We have $\DA^{-}(S) := D(\mathbb{D}^1_S, usu)((\PSh(\AnSp(C)^{sm}/S, C^{−}(\mathbb{Z}))) \subset \DA(S)$ the full subcategory consisting of bounded above complexes.

**Theorem 19.** Let $S \in \AnSp(C)$. The adjonction $(e(S)^*, e(S)_*) : C(\AnSp(C)^{sm}/S) \to C(S)$ induces an equivalence of categories

$$(e(S)^*, e(S)_*) : \An DM(S) \xrightarrow{\sim} D(S).$$

In particular, for $F \in C(\AnSp(C)^{sm}/S)$, the adjonction map $ad(e(S)^*, e(S)_*)(F) : e(S)^*e(S)_*F \to F$ is an equivalence $(\mathbb{D}^1, usu)$ local.

*Proof.* See [1].

We deduce from this theorem the following :

**Proposition 43.** Let $S \in \AnSp(C)$. Let $F, G \in C(\AnSp(C)^{sm}/S)$. If $G$ is $\mathbb{D}^1$ local, then the canonical map

$$(e, \hom)(F, G) : e(S)_*\hom(F, G) \to \hom(e(S)_*F, e(S)_*G)$$

is an equivalence $usu$ local.

*Proof.* The map $(e, \hom)(F, G)$ is the composite

$$T(e, \hom)(F, G) : e(S)_*\hom(F, G) \xrightarrow{\hom(ad(e(S)^*, e(S)_*)(F), G)} e(S)^*e(S)_*\hom(F, G) \xrightarrow{f(e(S)^*, e(S)_*)(F, G)} \hom(e(S)_*F, e(S)_*G)$$

where the last map is the adjonction isomorphism. The first map is an isomorphism by theorem 19 since $G$ is $\mathbb{D}^1$ local.
4 The category of filtered D modules on commutative ringed topos, on commutative ringed spaces, complex algebraic varieties complex analytic spaces and the functorialities

4.1 The category of filtered D modules on commutative ringed topos, on commutative ringed spaces, and the functorialities

4.1.1 Definitions et functorialities

Let $(S,O_S) \in \text{RCat}$ with $O_S$ commutative. Recall that $\Omega_{O_S} := \mathcal{D}^0(S/\mathbb{Z}) \in \text{PSh}_{O_S}(S)$ is the universal derivation $O_S$-module together with its derivation map $d : O_S \to \Omega_{O_S}$, where $\mathcal{I}_S = \ker(s_S : O_S \otimes O_S \to O_S) \in \text{PSh}_{O_X \otimes O_S}(S)$ the diagonal ideal.

In the particular case of a ringed space $(S,O_S) \in \text{RTop}$, $s_S : O_S \otimes O_S = \Delta_S(p_1^* O_S \otimes p_2^* O_S) \to O_S$ is the structural morphism of diagonal embedding $\Delta_S : (S,O_S) \hookrightarrow (S \times S, p_1^* O_S \otimes p_2^* O_S)$, $p_1 : S \times S \to S$ and $p_2 : S \times S \to S$ being the projections. More generally, for $k \in \mathbb{N}$, $k \geq 1$ we have the sheaf of $k$-jets $J^k(O_S) := \Delta_{S}^{*} \mathcal{I}_S / \mathcal{I}_S^{k+1}$ with in particular $J^1(O_S) = \mathcal{I}_S$. We have, for $s \in S$, $J^k(O_S)_s = m_s / m_s^k$ where $m_s \subset O_{S,s}$ is the maximal ideal if $O_{S,s}$ is a local ring.

**Definition 53.** (i) Let $(S,O_S) \in \text{RCat}$ with $O_S$ a commutative sheaf of ring and $S$ is endowed with a topology $\tau$. We denote by

$$D(O_S) = (O_S, \text{Der}_{O_S}(O_S, O_S) > \subset a_\tau \text{Hom}(O_S, O_S)$$

the subsheaf of generated by $O_S$ and the subsheaf of derivations $\text{Der}_{O_S}(O_S, O_S) = T_S := \mathcal{D}^0_{\Omega_{O_S}}$, $a_\tau : \text{PSh}(S) \to \text{Shv}(S)$ being the sheafification functor.

(ii) Let $f : \mathcal{X} \to S$ be a morphism of site, with $\mathcal{X}, S \in \text{Cat}$ endowed with topology $\tau$, resp. $\tau'$, and $O_S \in \text{PSh}(S, \text{cRing})$ a commutative sheaf of ring. We will note in this case by abuse $f^* O_S := a_{\tau'} f^* O_S$ and $f^* D(O_S) := a_{\tau'} f^* D(O_S)$, $a_{\tau'} : \text{PSh}(\mathcal{X}) \to \text{Shv}(\mathcal{X})$ being the sheafification functor.

Let $f : \mathcal{X} \to S$ a morphism of site, with $\mathcal{X}, S \in \text{Cat}$ endowed with topology $\tau$, resp. $\tau'$, and $O_S \in \text{PSh}(S, \text{cRing})$ a commutative sheaf of ring. Consider the ringed space $(\mathcal{X}, f^* O_S) := (\mathcal{X}, a_{\tau'} f^* O_S) \in \text{RCat}$, $a_{\tau'} : \text{PSh}(\mathcal{X}) \to \text{Shv}(\mathcal{X})$ being the sheafification functor. Then, the map in $\text{PSh}(\mathcal{X})$

$$T(f, \text{hom})(O_S, O_S) : f^* \text{Hom}(O_S, O_S) \to \text{Hom}(f^* O_S, f^* O_S)$$

induces a canonical morphism of sheaf of rings

$$T(f, \text{hom})(O_S, O_S) : a_{\tau'} f^* D(O_S) =: f^* D(O_S) \to D(a_{\tau'} f^* O_S) =: D(f^* O_S).$$

In the special case of ringed spaces, we have then:

**Proposition 44.** Let $f : X \to S$ a continuous map, with $X, S \in \text{Top}$ and $O_S \in \text{PSh}(S, \text{cRing})$ a commutative sheaf of ring. Consider the ringed space $(X, f^* O_S) := (X, a_{\tau} f^* O_S) \in \text{RTop}$, $a_{\tau} : \text{PSh}(X) \to \text{Shv}(X)$ being the sheafification functor. Then, the map in $\text{PSh}(X)$

$$T(f, \text{hom})(O_S, O_S) : f^* \text{Hom}(O_S, O_S) \to \text{Hom}(f^* O_S, f^* O_S)$$

induces a canonical isomorphism of sheaf of rings

$$T(f, \text{hom})(O_S, O_S) : f^* D(O_S) := a_{\tau} f^* D(O_S) \sim D(a_{\tau} f^* O_S) =: D(f^* O_S).$$

**Proof.** For all $x \in X$,

$$T(f, \text{hom})(O_S, O_S)_x : (f^* D(O_S))_x \sim D(O_{S,f(x)}) \sim (D(f^* O_S))_x.$$

Hence, since $a_{\tau} f^* D(O_S)$ and $D(a_{\tau} f^* O_S)$ are sheaves,

$$T(f, \text{hom})(O_S, O_S) : f^* D(O_S) := a_{\tau} f^* D(O_S) \sim D(a_{\tau} f^* O_S) =: D(f^* O_S)$$

is an isomorphism. 

\[\square\]
We will consider presheaves of \( D(O_S) \) modules on a ringed topos \((S, O_S)\):

**Definition 54.** Let \((S, O_S) \in \text{RCat} \) with \( O_S \) a commutative sheaf of ring.

(i) We will consider \( \text{PSh}_{D(O_S)}(S) \) the category of presheaves of (left) \( D(O_S) \) modules on \( S \) and \( C_{D(O_S)}(S) := C(\text{PSh}_{D(O_S)}(S)) \) its category of complexes. We will consider \( \text{PSh}_{D(O_S)^{op}}(S) \) the category of presheaves of right \( D(O_S) \) modules on \( S \) and \( C_{D(O_S)^{op}}(S) := C(\text{PSh}_{D(O_S)^{op}}(S)) \) its category of complexes. We denote again by abuse

\[
\text{PSh}_{D(O_S)}(S) = (\text{PSh}_{D(O_S)}(S), F) := (\text{PSh}_{D(O_S), F^{\text{ord}}}(S), F)
\]

the category of filtered \( (D(O_S), F^{\text{ord}}) \)-module, with, for \(-p \leq 0, F^{\text{ord}, -p}(D(O_S)) = \{p \in D(O_S), \text{ord}(P) \leq p\} \) and \( F^{\text{ord}, p}(D(O_S)) = 0 \) for \( p > 0 \),

- whose objects are \((M, F) \in (\text{PSh}_{O_S}(S), F)\) such that \((M, F)\) is compatible with \((D(O_S), F^{\text{ord}})\) that is \( F^{\text{ord}, -p}(D(O_S)) \cdot F^q M \subset F^{q-p} M\) (Griffiths transversality), this is to say that the structural map \( m_d : M \otimes_{O_S} D(O_S) \to M \) induces a filtered map of presheaves (i.e. a map in \( \text{PSh}_{O_S}(S, F) \)) \( m_d : (M, F) \otimes_{O_S} (D(O_S), F^{\text{ord}}) \to (M, F) \),

- whose morphism \( \phi : (M_1, F) \to (M_2, F)\) are as usual the morphisms of presheaves \( \phi : M_1 \to M_2\) which are morphism of filtered presheaves (i.e. \( \phi(F^p M_1) \subset F^p M_2\) and which are \( D(O_S) \) linear (in particular \( O_S \) linear).

Note that this a NOT the category of filtered \( D(O_S) \) modules in the usual sense, that is the \((M, F) \in (\text{PSh}_{O_S}(S), F)\) together with a map \( m_d : (M, F) \otimes_{O_S} D(O_S) \to (M, F) \) in \( (\text{PSh}_{O_S}(S), F) \), since \( F^{\text{ord}} \) is NOT the trivial filtration. More precisely the \( O_S \) submodules \( F^p M \subset M \) are NOT \( D(O_S) \) submodules but satisfy Griffiths transversality. We denote by

\[
\text{PSh}_{D(O_S)}(S) \subset \text{PSh}_{D(O_S), F^{\text{ord}}}(S), \text{PSh}_{D(O_S), (1,0)}(S) \subset \text{PSh}_{D(O_S), (2,0)}(S) := (\text{PSh}_{D(O_S), F^{\text{ord}}}(S), F, W)
\]

the full subcategory consisting of filtered \( D(O_S) \) module in the usual sense, resp. the full subcategory such that \( W \) is a filtration by \( D(O_S) \) submodules.

(ii) We denote again by

\[
C_{D(O_S)}(S) \subset C(\text{PSh}_{D(O_S)}(S), F), C_{D(O_S), (2,0)}(S) \subset C(\text{PSh}_{D(O_S)}(S), F, W)
\]

the full subcategory of complexes such that the filtration(s) is (are) regular. We will consider also

\[
C_{D(O_S), (0,1)}(S) \subset C_{D(O_S)}(S), C_{D(O_S), (1,0)}(S) \subset C_{D(O_S), (2,0)}(S)
\]

the full subcategory consisting of complexes of filtered \( D(O_S) \) modules in the usual sense (i.e. by \( D(O_S) \) submodule), respectively the full subcategory consisting of complexes of bifiltered \( D(O_S) \) modules such that \( W^p M \subset M \) are \( D(O_S) \) submodules i.e. the filtration \( W \) is a filtration in the usual sense, but NOT \( F \) which satisfy only Griffiths transversality.

**Proposition 45.** Let \((S, O_S) \in \text{RCat} \) with \( O_S \) commutative sheaf of ring.

(i) Let \( M \in \text{PSh}_{O_S}(S) \). Then, there is a one to one correspondence between

- the \( D(O_S) \) module structure on \( M \) compatible with the \( O_S \) module structure, that is the maps \( m_d : D(O_S) \otimes_{O_S} M \to M \) in \( \text{PSh}_{O_S}(S) \) and

- the integrable connections on \( M \), that is the maps \( \nabla : M \to \Omega_{O_S} \otimes_{O_S} M \) satisfying \( \nabla \circ \nabla = 0 \) with \( \nabla : \Omega_{O_S} \otimes_{O_S} M \to \Omega_{O_S} \otimes_{O_S} M \) given by \( \nabla(\omega \otimes m) = (d\omega) \otimes m + \omega \wedge \nabla(m) \)

(ii) Let \((M, F) \in \text{PSh}_{O_S}(S) \). Then, there is a one to one correspondence between
– the $D(O_S)$ module structure on $(M, F)$ compatible with the $O_S$ module structure, that is the maps $md : (D(O_S), F^\text{ord}) \otimes_{O_S} (M, F) \to (M, F)$ in $\text{PSh}_{O_S}fil(\mathcal{S})$ and
– the integrable connexions on $M$, that is the maps $\nabla : (M, F) \to \Omega_{O_S} \otimes_{O_S} (M, F)$ satisfying $\nabla \circ \nabla = 0$ and Griffith transversality (i.e. $\nabla(F^p M) \subset \Omega_{O_S} \otimes_{O_S} F^{p-1} M$).

Proof. Standard. \hfill $\square$

The following proposition tells that the $O$-tensor product of $D$ modules has a canonical structure of $D$ module.

**Definition-Proposition 13.** (i) Let $f : (\mathcal{X}, O_X) \to (\mathcal{S}, O_S) \in \text{RCat}$ with commutative structural sheaf of ring. For $N \in \text{PSh}_{O_X}D(f^*O_S)(\mathcal{X})$ and $M \in \text{PSh}_{O_X}D(f^*O_S)(\mathcal{X})$, $N \otimes_{O_X} M$ has the canonical $D(f^*O_S)$ module structure given by, for $X^0 \in \mathcal{X}$,

$$\gamma \in \Gamma(X^0, D(f^*O_S)), m \in \Gamma(X^0, M), n \in \Gamma(X^0, N), \gamma.(n \otimes m) = (\gamma.n) \otimes m + n \otimes (\gamma.m).$$

This gives the functor

$$\text{PSh}_{O_X}D(f^*O_S)fil(\mathcal{X}) \times \text{PSh}_{O_X}D(f^*O_S)fil(\mathcal{X}) \to \text{PSh}_{O_X}D(f^*O_S)fil(\mathcal{X}), ((M, F), (N, F)) \mapsto (M, F) \otimes_{O_X} (N, F), F^p(M, F) \otimes_{O_X} (N, F) := \text{Im}(\oplus_{q \in \mathbb{Z}} F^q M \otimes_{O_X} F^{p-q} N \to M \otimes_{O_X} N)$$

(ii) Let $f : (\mathcal{X}, O_X) \to (\mathcal{S}, O_S) \in \text{RCat}$ with commutative structural sheaf of ring. For $N \in C_D(O_X), D(f^*O_S)(\mathcal{X})$ and $M \in C_D(O_X), f^*D(O_S)(\mathcal{X})$, $N \otimes_{D(O_X)} M$ has the canonical $f^*D(O_S)$ module structure given by, for $X^0 \in \mathcal{X}$,

$$\gamma \in \Gamma(X^0, D(f^*O_S)), m \in \Gamma(X^0, M), n \in \Gamma(X^0, N), \gamma.(n \otimes m) = (\gamma.n) \otimes m.$$  

This gives the functor

$$C_D(O_X), D(f^*O_S)fil(\mathcal{X}) \times C_D(O_X), f^*D(O_S)fil(\mathcal{X}) \to C_D(O_X), f^*D(O_S)fil(\mathcal{X}), ((M, F), (N, F)) \mapsto (M, F) \otimes_{D(O_X)} (N, F), F^p(M, F) \otimes_{D(O_X)} (N, F) := \text{Im}(\oplus_{q \in \mathbb{Z}} F^q M \otimes_{D(O_X)} F^{p-q} N \to M \otimes_{D(O_X)} N)$$

Note that, by definition, we have for $(M, F) \in (\text{PSh}_{D(O_S)fil(\mathcal{S})})$, the canonical isomorphism

$$(M, F) \otimes_{D(O_S)} (D(O_S), F^\text{ord}) \cong (M, F), m \otimes P \mapsto Pm, m \mapsto (m \otimes 1)$$

Proof. Immediate from definition. \hfill $\square$

We now look at the functorialities for morphisms of ringed spaces, using proposition 44. First note that for $f : (\mathcal{X}, O_X) \to (\mathcal{S}, O_S)$ a morphism, with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RTop}$ with structural presheaves commutative sheaves of rings, there is NO canonical morphism between $D(f^*O_S) = f^*D(O_S)$ (see proposition 44) and $D(O_X)$.

We have the pullback functor for (left) $D$-modules :

**Definition-Proposition 14.** (i) Let $f : (\mathcal{X}, O_X) \to (\mathcal{S}, O_S)$ a morphism with $(\mathcal{X}, O_X), (\mathcal{S}, O_S) \in \text{RTop}$ with structural presheaves commutative sheaves of rings. Recall that $f^*D(O_S) = D(f^*O_S)$ in this case. Then for $(M, F) \in \text{PSh}_{D(O_S)fil(\mathcal{S})}$,

$$f^*\text{mod}(M, F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M, F) \in \text{PSh}_{O_X}fil(\mathcal{X})$$

has a canonical structure of filtered $D(O_X)$ module given by

for $\gamma \in \Gamma(X^0, T_{O_X}), n \otimes m \in \Gamma(X^0, O_X \otimes_{f^*O_S} f^*M), \gamma.(n \otimes m) := (\gamma.n) \otimes m + n \otimes df(\gamma)(m)$

with $df := D_{O_X}^g \Omega_{O_X/f^*O_S} : T_{O_X} \to T_{f^*O_S} = f^*T_{O_S}$ and $f^*(M, F) \in \text{PSh}_{f^*D(O_S)fil(\mathcal{X})} = \text{PSh}_{D(f^*O_S)fil(\mathcal{X})}$.  

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(ii) More generally, let \( f : (X, O_X) \to (S, O_S) \) a morphism with \((X, O_X), (S, O_S) \in \text{RCat} \) with structural presheaves commutative sheaves of rings. Assume that the canonical morphism \( T(f, \text{hom})(O_S, O_S) : f^* D(O_S) \to D(f^* O_S) \) is an isomorphism of sheaves. Then for \((M, F) \in PSh_{D(O_S)fil}(S)\),

\[
f^{\text{smod}}(M, F) := (O_X, F_b) \otimes_{f^* O_S} f^*(M, F) \in PSh_{O_X fil}(X)
\]

has a canonical structure of filtered \( D(O_X) \) module given by

\[
for \gamma \in \Gamma(X^\circ, T_{O_X}), n \otimes m \in \Gamma(X^\circ, O_X \otimes_{f^* O_S} f^* M), \gamma.(n \otimes m) := (\gamma.n) \otimes m + n \otimes df(\gamma)(m)
\]

with \( df := D^0 (O_X / f^* O_S) : T_{O_X} \to T_{f^* O_S} = f^* T_{O_S} \) and \( f^*(M, F) \in PSh_{f^* D(O_S)fil}(X) = PSh_{D(f^* O_S)fil}(X). \)

Proof. Standard. \( \square \)

Remark 6. • Let \( f : (X, O_X) \to (S, O_S) \) a morphism with \((X, O_X), (S, O_S) \in \text{RTop} \) with structural presheaves commutative sheaves of rings. Recall that \( f^* D(O_S) = D(f^* O_S) \). Then by definition \( f^{\text{smod}}(O_S, F_b) = (O_X, F_b) \).

• More generally, let \( f : (X, O_X) \to (S, O_S) \) a morphism with \((X, O_X), (S, O_S) \in \text{RCat} \) with structural presheaves commutative sheaves of rings. Assume that the canonical morphism \( T(f, \text{hom})(O_S, O_S) : f^* D(O_S) \to D(f^* O_S) \) is an isomorphism of sheaves. Then by definition \( f^{\text{smod}}(O_S, F_b) = (O_X, F_b) \).

For the definition of a push-forward functor for a right D module we use the transfert module

Let \( f : (X, O_X) \to (S, O_S) \) be a morphism with \((X, O_X), (S, O_S) \in \text{RTop} \) with structural presheaves commutative sheaves of rings. Then, the transfer module is

\[
(D(O_X \to f^* O_S), F^{\text{ord}}) := f^{\text{smod}}(D(O_S), F^{\text{ord}}) := f^*(D(O_S), F^{\text{ord}}) \otimes_{f^* O_S} (O_X, F_b)
\]

which is a left \( D(O_X) \) module and a left and right \( f^* D(O_S) = D(f^* O_S) \) module.

Lemma 2. Let \( f_1 : (X, O_X) \to (Y, O_Y), f_2 : (Y, O_Y) \to (S, O_S) \) be two morphism with \((X, O_X), (Y, O_Y),(S, O_S) \in \text{RTop} \). We have in \( C_{D(O_X)fil}(X) \)

\[
(D_{O_X \to f_2 f_1^* O_S}, F^{\text{ord}}) = f_1^*(D_{O_Y \to f_2^* O_S}, F^{\text{ord}}) \otimes_{f_2^* D(O_Y)} (D_{O_X \to f_1^* O_Y}, F^{\text{ord}})
\]

Proof. Follows immediately from definition. \( \square \)

For right D module, we have the direct image functor :

Definition 55. Let \( f : (X, O_X) \to (S, O_S) \) a morphism with \((X, O_X), (S, O_S) \in \text{RTop} \) with structural presheaves commutative sheaves of rings. Then for \((M, F) \in C_{D(O_X)fil}(X) \), we define

\[
f_{\text{smod}}^{00}(M, F) = f_* ((D_{O_X \to f^* O_S}, F^{\text{ord}}) \otimes_{D(O_X)} (M, F)) \in C_{D(O_S)fil}(S)
\]

For a closed embedding of topological spaces, there is the \( V \)-filtration on the structural sheaf, it will play an important role in this article

Definition 56. (i) Let \((S, O_S) \in \text{RTop} \) a locally ringed space. Let \( Z = V(I_Z) \subset S \) a Zariski closed subset. We set, for \( S^p \subset S \) an open subset, \( p < 0, \)

\[
- V_{Z, p} O_S(S^p) := O_S(S^p) \text{ if } p > 0,
- V_{Z, -q} O_S(S^p) := I_Z^p(S^p) \subset O_S(S^p) \text{ for } p = -q \leq 0.
\]

We immediately check that, by definition, this filtration satisfy Giffirtz transversality, that is \((O_S, V_Z) \in PSh_{D(O_S)fil}(S) \). For a morphism \( g : ((T, O_T), Z') \to ((S, O_S), Z) \) with \(((T, O_T), Z), ((S, O_S), Z) \in \text{RTop}^2 \) locally ringed spaces, where \( Z \) and \( Z' \) are Zariski closed subsets, the structural morphism \( a_g : g^* O_S \to O_T \) is a filtered morphism :

\[
a_g : g^* (O_S, V_Z) \to (O_T, V_{Z'}), h \mapsto a_g(h)
\]
(ii) Let \((S, O_S) \in \text{RTop}\). Let \(i : Z \hookrightarrow S\) a closed embedding. The \(V_Z\)-filtration on \(O_S\) (see (i)) gives the filtration, given by for \(p \in \mathbb{Z}\),

\[
V_{Z,p} \text{Hom}(O_S, O_S) := \left\{ P \in \text{Hom}(O_S, O_S), \text{s.t.} \cdot \mathbb{T}_Z^p \subset I_Z^{p-1} \right\}
\]
on \(\text{Hom}(O_S, O_S)\), which induces the filtration \(V_{Z,p} D(O_S) := D(O_S) \cap V_{Z,p} \text{Hom}(O_S, O_S)\) on \(D(O_S) \subset \text{Hom}(O_S, O_S)\). We get \((D(O_S), V_Z) \in \text{PSh}_{fil}(S, \text{Ring})\) and we call it the \(V_Z\)-filtration on \(D(O_S)\).

(iii) Let \((S, O_S) \in \text{RTop}\) a locally ringed space. Let \(i : Z = V(\mathcal{I}_Z) \hookrightarrow S\) a Zariski closed embedding and \(O_Z := i^* O_S / \mathcal{I}_Z\). We say that \(M \in \text{PSh}_{D(O_S)}(S)\) is specializable on \(Z\) if it admits an (increasing) filtration (called a \(V_Z\)-filtration) \((M, V) \in \text{PSh}_{O_S,fil}(S)\) compatible with \((D_S, V_Z)\), that is \(V_{Z,p} D_S \ni V_{q} M \subset V_{p+q} M\). This is to say that the structural map \(m_d : M \otimes_{O_S} D(O_S) \to M\) induces filtered map of presheaves \(m_d : (M, V) \otimes_{O_S} D(O_S), V_Z) \to (M, V)\). For \((M, F) \in \text{PSh}_{D(O_S),fil}(S)\) such that \(M\) is specializable on \(Z\), we thus get a filtered morphism \(m_d : (M, F, V) \otimes_{O_S} (D(O_S), F^{ord}, V_Z) \to (M, F, V)\).

(iii') Consider an injective morphism \(m : M_1 \hookrightarrow M_2\) with \(M_1, M_2 \in \text{PSh}_{D(O_S)}(S)\). If \(M_2\) admits a \(V_Z\) filtration \(V_2\), then the filtration \(V_{21}\) induced on \(M_1\) (recall \(V_{31,p} M := V_{2,p} M_2 \cap M_1\)) is a \(V_Z\) filtration. Consider a surjective morphism \(n : M_1 \to M_2\) with \(M_1, M_2 \in \text{PSh}_{D(O_S)}(S)\). If \(M_1\) admits a \(V_Z\) filtration \(V_1\), then the filtration \(V_{12}\) induced on \(M_2\) (recall \(V_{12,p} M := n(V_{1,p} M_1)\)) is a \(V_Z\) filtration.

(iv) Let \((S, O_S) \in \text{RTop}\) a locally ringed space. Let \(i : Z = V(\mathcal{I}_Z) \hookrightarrow S\) a Zariski closed embedding and \(O_Z := i^* O_S / \mathcal{I}_Z\). For \((M, F) \in \text{PSh}_{D(O_S),fil}(S)\) such that \(M\) admits a \(V_Z\) filtration \(V\) so that \((M, F, V) \in \text{PSh}_{O_S,2,fil}(S)\), we will consider the quotient map in \(\text{PSh}_{O_S,fil}(S)\)

\[
q_{V0} : (M, F) \to (M, F) / V_{-1}(M, F) =: Q_{V,0}(M, F).\]

The quotient \(i^* Q_{V,0}(M, F)\) has an action of \(T_{O_S}\) since for \(S^o \subset S\) an open subset and \(\partial_z \in \Gamma(Z \cap S^o, T_{O_S}) \subset \Gamma(S^o, T_{O_S})\), we have \(\partial_z \in \Gamma(S, V_{2,0} D(O_S))\) since for \(f = \sum_{i=1}^r t_i h_i \in \Gamma(S^o, \mathcal{I}_Z)\), where \(t_i = \mathcal{I}_Z(S^o)\) are generators of the ideal \(\mathcal{I}_Z(S^o) \subset O_S(S^o)\) and \(h_i \in \Gamma(S^o, O_S)\), we have

\[
\partial_z \left( \sum_{i=1}^r t_i h_i \right) = \sum_{i=1}^r (\partial_z(t_i) h_i + t_i \partial_z(h_i)) = \sum_{i=1}^r t_i (\partial_z(h_i)) \in \Gamma(S, I_Z)
\]
as \(\partial_z(t_i) = 0\) (only the vector field of \(T_{O_S}\) which are transversal to \(T_{O_S} \subset T_{O_S}\) increase the grading). Then, obviously, by definition, the map in \(\text{PSh}_{O_S,fil}(Z)\)

\[
i^* q_{V0} : i^* (M, F) / V_{-1}(M, F) =: i^* Q_{V,0}(M, F)
\]
commutes with the action of \(T_{O_S} \subset i^* T_{O_S}\) and we call it the specialization map.

**Definition-Proposition 15.** Let \((S, O_S) \in \text{RTop}\) a locally ringed space. Consider a commutative diagram

\[
\begin{array}{ccc}
Z_1 = V(\mathcal{I}_1) & \xrightarrow{i_1} & (S, O_S) \\
\downarrow{i_1'} & & \downarrow{i_2}\ \\
Z = V(\mathcal{I}) & \xrightarrow{i'} & Z_2 = V(\mathcal{I}_2)
\end{array}
\]

where the maps are Zariski closed embeddings and which is cartesian (i.e. \(\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2)\), in particular \(Z = Z_1 \cap Z_2\)).

(i) Let \((M, F) \in \text{PSh}_{D(O_S),fil}(S)\) such that \(M\) admits a \(V_{Z_1}\)-filtration \(V_1\) and a \(V_{Z_2}\)-filtration \(V_2\) (see definition 56). Let \(p, q \in \mathbb{Z}\). Then, we consider...
– the filtration $V_{21}$ on $V_{1,p}(M, F) := (V_{1,p}M, F)$ induced by $V_2$,
– the filtration $V_{12}$ on $Q_{V_{2,p}}(M, F) := (M/V_{2,p-1}M, F)$ induced by $V_1$.

The quotient map in $\text{PSh}_{\mathcal{I}_{\mathcal{I}}, S_{\mathcal{O}_S}}(\mathcal{S})$

$$q_{V_{2,p}} : V_{1,q}(M, F) \to V_{12,q}Q_{V_{2,p}}(M, F)$$

factors through

$$q_{V_{2,p}} : V_{1,q}(M, F) \xrightarrow{q_{V_{21},p}} Q_{V_{21},p}V_{1,q}(M, F) \xrightarrow{Q_{V_{12},V_{2}}^{p,q}(M, F)} V_{12,q}Q_{V_{2,p}}(M, F),$$

and the map $Q_{V_{12},V_{2}}^{p,q}(M, F)$ in $\text{PSh}_{\mathcal{I}_{\mathcal{I}}, S_{\mathcal{O}_S}}(\mathcal{S})$ commute with the action of $T_{O_S}$.

(ii) If $m : (M, F) \to (M', F)$ is a morphism with $(M, F), (M', F) \in \text{PSh}_{\mathcal{D}_{(O_S)}}(\mathcal{S})$ admitting $V_{Z_1}$-filtration $V_1$ and $V'_1$ respectively such that $m(V_{1,p}M) \subset V'_{1,p}M'$ and $V_{Z_2}$-filtration $V_2$ and $V'_2$ respectively such that $m(V_{1,p}M) \subset V'_{1,p}M'$. Then for all $p, q \in \mathbb{Z}$ the following diagram commutes

$$\xymatrix{ Q_{V_{21},p}V_{1,q}(M, F) \ar[r]^{Q_{V_{12},V_{2}}^{p,q}(M, F)} \ar[d]^m & V_{12,q}Q_{V_{2,p}}(M, F) \ar[d]^m \\
Q_{V_{21},p}V'_{1,q}(M', F) \ar[r]_{Q_{V_{12},V_{2}}^{p,q}(M', F)} & V_{12,q}Q_{V_{2,p}}(M', F) }$$

Let $(S, O_S) \in \text{RTop}$ a locally ringed space. Consider a commutative diagram

$$\xymatrix{ Z'_1 = V(I'_1) \ar[r]^{i'_1} \ar[d]_{\iota'_2} & Z_1 = V(I_1) \ar[r]^{i_1} \ar[d]_{\iota_2} & (S, O_S) \ar[d] \\\nZ' = V(I') \ar[r]^{i'_1} & Z = V(I) \ar[r]^{i_1} & Z_2 = V(I_2) }$$

where the maps are Zariski closed embeddings and whose squares are cartesian (i.e. $\mathcal{I} = (I_1, I_2)$ and $\mathcal{I}' = (I'_1, I'_2)$, in particular $Z = Z_1 \cap Z_2$ and $Z' = Z'_1 \cap Z$). Let $(M, F) \in \text{PSh}_{\mathcal{D}_{(O_S)}}(\mathcal{S})$ such that $M$ admits a $V_{Z_1}$-filtration $V_1$, a $V_{Z'_1}$-filtration $V'_1$, and a $V_{Z_2}$-filtration $V_2$ (see definition 56). Then for all $p, q \in \mathbb{Z}$, denoting again $V'_{12}$ the filtration induced by $V'_1$ on $Q_{V_{21},p}V_{1,q}M$ and $V_{21}$ the filtration induced by $V'_2$ on $V_{12,q}Q_{V_{2,p}}$

- $Q_{V_{21},p}V_{12,q}Q_{V_{21},p}V_{1,q}(M, F) = Q_{V_{21},p}V'_{1,q}(M, F),$ 
- $V'_{12,q}Q_{V_{21},p}V_{12,q}Q_{V_{21},p}(M, F) = V'_{12,q}Q_{V_{21},p}(M, F),$ 

and

$$Q_{V_{12},V_{2}}^{p,q}(M, F) = Q_{V_{12},V_{2}}^{p,q}(M, F).$$

Proof. Obvious. \[\square\]

We will also consider the following categories

**Definition 57.** Let $(\mathcal{X}, O_X) \in \text{RCat}$. We denote by $C_{O_X, \mathcal{D}_{(O_X)}}(\mathcal{X})$ the category

- whose objects $(M, F) \in C_{O_X, \mathcal{D}_{(O_X)}}(\mathcal{X})$ are filtered complexes of presheaves of $O_X$ modules $(M, F) \in C_{O_X, \mathcal{D}_{(O_X)}}(\mathcal{X})$ whose cohomology presheaves $H^n(M, F) \in \text{PSh}_{O_X}(\mathcal{X})$ are endowed with a structure of filtered $D(O_X)$ modules for all $n \in \mathbb{Z}$.

- whose set of morphisms $\text{Hom}_{C_{O_X, \mathcal{D}_{(O_X)}}(\mathcal{X})}((M, F), (N, F)) \subset \text{Hom}_{C_{O_X, \mathcal{D}_{(O_X)}}(\mathcal{X})}((M, F), (N, F))$ between $(M, F), (N, F) \in C_{O_X, \mathcal{D}_{(O_X)}}(\mathcal{X})$ are the morphisms of filtered complexes of $O_X$ modules $m : (M, F) \to (N, F)$ such that $H^n m : H^n(M, F) \to H^n(N, F)$ is $D(O_X)$ linear, i.e. is a morphism of (filtered) $D(O_X)$ modules, for all $n \in \mathbb{Z}$.  

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### 4.1.2 The De Rham complex of a (left) filtered D-module and the Spencer complex of a right filtered D-module

Using proposition 45, we define the filtered De Rham complex of a complex of filtered (left) D-modules:

**Definition 58.**

(i) Let $(S, O_S) \in \text{RCat}$ with $O_S$ commutative. Let $(M, F) \in C_{D(O_S)fil}(S)$. By proposition 45, we have the complex

$$DR(O_S)(M, F) := (\Omega^{\bullet}_{O_S}, F_b) \otimes_{O_S} (M, F) \in C_{fil}(S)$$

whose differentials are $d(\omega \otimes m) = (d\omega) \otimes m + \omega \wedge (\nabla m)$.

(ii) More generally, let $f : (X, O_X) \to (S, O_S)$ with $(X, O_X), (S, O_S) \in \text{RCat}$. The quotient map $q : \Omega_{O_X} \to \Omega_{O_X/f^{\ast}O_S}$ induce, for $G \in \text{PSh}_{O_X}(X)$ the quotient map

$$q^p(G) := \wedge^p q \otimes \Omega^0_{O_X} \otimes_{O_X} G \to \Omega^q_{O_X/f^{\ast}O_S} \otimes_{O_X} G.$$

Let $(M, F) \in C_{D(O_X)fil}(X)$. By proposition 45, we have the relative De Rham complex

$$DR(O_X/f^{\ast}O_S)(M, F) := (\Omega^{\bullet}_{O_X}, F_b) \otimes_{O_X} (M, F) \in C_{f^{\ast}O_Sfil}(X)$$

whose differentials are $d(q^p(M)(\omega \otimes m)) := q^{p+1}(M)((d\omega) \otimes m) + q(M)(\omega \otimes (\nabla m))$.

(iii) Let $(X, O_X)/F \in \text{FolRTop}$, that is $(X, O_X) \in \text{RTop}$ endowed with a foliation with quotient map $q : \Omega_{O_X} \to \Omega_{O_X}/F$. Let $(M, F) \in C_{D(O_X)fil}(X)$. By proposition 45, we have the foliated De Rham complex

$$DR(O_X/F)(M, F) := (\Omega^{\bullet}_{O_X/F}, F_b) \otimes_{O_X} (M, F) \in C_{fil}(X)$$

whose differentials are $d(q(M)(\omega \otimes m)) := q(M)((d\omega) \otimes m) + q(M)(\omega \otimes (\nabla m))$.

By definition,

- with the notation of (ii) if $\phi : (M_1, F) \to (M_2, F)$ is a morphism in $C_{D(O_X)fil}(X)$,

$$DR(O_X/f^{\ast}O_S)(\phi) := (I \otimes \phi) : (\Omega^{\bullet}_{X/S}, F_b) \otimes_{O_X} (M_1, F) \to (\Omega^{\bullet}_{X/S}, F_b) \otimes_{O_X} (M_2, F)$$

is a morphism in $C_{f^{\ast}O_Sfil}(X)$,

- with the notation of (ii) $DR(O_X)(O_X) = DR(O_X)$ and more generally in the relative case $DR(O_X/f^{\ast}O_S)(O_X) = DR(O_X/f^{\ast}O_S)$, and with the notation of (iii) $DR(O_X/F)(O_X) = DR(O_X/F)$.

Dually, we have the filtered Spencer complex of a complex of filtered right D-module:

**Definition 59.**

(i) Let $(S, O_S) \in \text{RCat}$ with $O_S$ commutative. Let $(M, F) \in C_{D(O_S)^{\text{op}}fil}(S)$. By proposition 45, we have the complex

$$SP(O_S)(M, F) := (T_{O_S}^{\bullet}, F_b) \otimes_{O_S} (M, F) \in C_{fil}(S)$$

whose differentials are, for $X \in S$, and $\partial_1 \wedge \cdots \wedge \partial_r \otimes m \in \Gamma(X, T_{O_S}^{r-1} \otimes_{O_S} M)$,

$$d(\partial_1 \wedge \cdots \wedge \partial_r \otimes m) : (\omega \in \Gamma(X, T_{O_S}^{r-1})) \mapsto \sum_i \omega(\partial_1 \wedge \cdots \wedge \partial_i \wedge \partial_r \otimes m) - \sum_{i<j} \omega([\partial_i, \partial_j])m).$$

(ii) More generally, let $f : (X, O_X) \to (S, O_S)$ with $(X, O_X), (S, O_S) \in \text{RCat}$. The quotient map $q : \Omega_{O_X} \to \Omega_{O_X/f^{\ast}O_S}$ induce, for $G \in \text{PSh}_{O_X}(X)$ the injective map

$$q^{\ast}q^p(G) := \wedge^p q^{\ast} \otimes I : T^q_{O_X/f^{\ast}O_S} \otimes_{O_X} G \to T^q_{O_X} \otimes_{O_X} G.$$

Let $(M, F) \in C_{D(O_X)^{\text{op}}fil}(X)$. By proposition 45, we have the relative Spencer complex

$$SP(O_X/f^{\ast}O_S)(M, F) := (T_{X/S}^{\bullet}, F_b) \otimes_{O_X} (M, F) \in C_{f^{\ast}O_Sfil}(X)$$

whose differentials are the one of $SP(O_X)(M, F)$ given in (i) by the embedding $q^{\ast} : SP(O_X/f^{\ast}O_S)(M, F) \hookrightarrow SP(O_X)(M, F)$.

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(iii) Let \((X, O_X)/\mathcal{F} \in \text{FolRTop},\) that is \((X, O_X) \in \text{RTop}\) endowed with a foliation with quotient map \(q : \Omega_{O_X} \to \Omega_{O_X/\mathcal{F}}.\) Let \((M, F) \in C_{D(O_X)_{\text{fil}}} \mathcal{X}.\) By proposition 45, we have the foliated Spencer complex

\[
SP(O_X/\mathcal{F})(M, F) := (T^*_{O_X/\mathcal{F}}, F_b) \otimes_{O_X} (M, F) \in C_{\text{fil}}(X)
\]

whose differentials are of \(SP(O_X)(M, F)\) given in (i) by the embedding \(q^\vee : SP(O_X/\mathcal{F})(M, F) \hookrightarrow SP(O_X)(M, F).\)

By definition, with the notation of (ii) if \(\phi : (M_1, F) \to (M_2, F)\) is a morphism in \(C_{D(O_X)_{\text{fil}}} \mathcal{X},\)

\[
SP(O_X/f^*O_S)(\phi) := (I \otimes \phi) : (T^*_{X/S}, F_b) \otimes_{O_X} (M_1, F) \to (T^*_{X/S}, F_b) \otimes_{O_X} (M_2, F)
\]
is a morphism in \(C_{f^*O_Sfil}(\mathcal{X}).\)

**Proposition 46.** (i) Let \(f : (X, O_X) \to (S, O_S)\) a morphism with \((S, O_S), (X, O_X) \in \text{RTop}.\) Assume that the canonical map \(T(f, \text{hom})(O_X, O_X) : f^*D(O_X) \to D(f^*O_X)\) is an isomorphism of sheaves. For \((M, F) \in C_{D(O_X)_{\text{fil}}} \mathcal{X}\) and \((M', F), (N, F) \in C_{D(O_X)_{\text{fil}}} \mathcal{X},\) we have canonical isomorphisms in \(C_{f^*O_Sfil}(\mathcal{X}) :\)

\[
(M', F) \otimes_{O_X} (N, F) \otimes_{D(O_X)} (M, F) = (M', F) \otimes_{D(O_X)} ((M, F) \otimes_{O_X} (N, F))
\]

(ii) Let \(f : (X, O_X) \to (S, O_S)\) a morphism with \((S, O_S), (X, O_X) \in \text{RTop}.\) For \((M, F) \in C_{D(O_X)_{\text{fil}}} \mathcal{X},\)

\[
(\Omega^*_{O_X/f^*O_S}, F_b) \otimes_{O_X} (M, F) = ((\Omega_{O_X/f^*O_S}^*, F_b) \otimes_{O_X} D(O_X)) \otimes_{D(O_X)} (M, F)
\]

**Proof.** These are standard facts of algebra.

**Definition-Proposition 16.** Consider a commutative diagram in \(\text{RCat}\)

\[
\begin{array}{ccc}
D = (\mathcal{X}, O_X) & \xrightarrow{f} & (S, O_S) \\
g' \downarrow & & \downarrow g \\
(\mathcal{X}', O_{\mathcal{X}'}) & \xrightarrow{f'} & (T', O_T)
\end{array}
\]

with commutative structural sheaf of rings. Assume that the canonical map \(T(g', \text{hom})(O_X, O_X) : g'^*D(O_X) \to D(g'^*O_X)\) is an isomorphism of sheaves.

(i) For \((M, F) \in PSh_{D(O_X)_{\text{fil}}} \mathcal{X},\) the graded map in \((\text{PSh}_{g'^*O_X} (\mathbb{N} \times \mathcal{X}'), F)\)

\[
\Omega^p_{(O_X, g'^*O_X)/(O_T, g^*O_S)}(M, F) := m' \circ (\Omega_{(O_X, g'^*O_X)/(O_T, g^*O_S)} \otimes I) : g'^*((\Omega^*_{O_X/f^*O_S}, F_b) \otimes_{O_X} (M, F)) \to (\Omega^*_{O_X/f^*O_S}, F_b) \otimes_{O_X} g'^*\text{mod}(M, F)
\]
given in degree \(p \in \mathbb{N}\) by, for \(X' \in \mathcal{X}'\) and \(X^o \in \mathcal{X}\) such that \(g'((X'^o) \hookrightarrow X' \in \mathcal{X}),\)

\[
\Omega^p_{(O_X, g'^*O_X)/(O_T, g^*O_S)}(M)(X'^o) := m' \circ (\Omega^p_{(O_X, g'^*O_X)/(O_T, g^*O_S)} \otimes I)(X'^o) : \omega \otimes m \in \Gamma(\mathcal{X}', \Omega^p_{O_X} \otimes_{O_X} M) \mapsto \Omega_{O_X/g^*O_S}(\omega) \otimes (m \otimes 1)
\]
is a map of complexes, that is a map in \(C_{(f'g'^*)O_Sfil}(\mathcal{X}').\)

(ii) For \((M, F) \in C_{D(O_X)_{\text{fil}}} \mathcal{X},\) we get from (i) by functoriality, the map in \(C_{(f'g'^*)O_Sfil}(\mathcal{X}'),\)

\[
\Omega_{(O_X, g'^*O_X)/(O_T, g^*O_S)}(M, F) := m' \circ (\Omega_{(O_X, g'^*O_X)/(O_T, g^*O_S)} \otimes I) : g'^*((\Omega^*_{O_X/f^*O_S}, F_b) \otimes_{O_X} (M, F)) \to (\Omega^*_{O_X/f^*O_S}, F_b) \otimes_{O_X} g'^*\text{mod}(M, F)
\]
(iii) For \((M, F) \in C_{D(O_X)fil}(\mathcal{X})\), we get from (ii) the canonical transformation map in \(C_{O_Tfil}(\mathcal{T})\)

\[ T^O_\omega(D)(M, F) : g^{*\text{mod}L_0}(f_*E((\Omega^*_{O_X/f^*O_T}, F_b) \otimes O_X (M, F))) \twoheadrightarrow (g^*f_*E((\Omega^*_{O_X/f^*O_T}, F_b) \otimes O_X (M, F))) \otimes g^{*O_S} O_T \]

\[ (f'_*E((\Omega^*_{O_X'/f'^*O_T}, F_b) \otimes O_X (M, F))) \otimes g^{*O_S} O_T \]

with \(m(n \otimes s) = s.n\).

**Proof.** (i): We check that the map in \((\text{PSh}_{g^{*}O_X}(\mathbb{N} \times \mathcal{X}'), F)\)

\[ \Omega((O_{X'}/g^{*}O_X)/(O_T/g^{*}O_S))(M, F) := m' \circ (\Omega((O_{X'}/g^{*}O_X)/(O_T/g^{*}O_S)) \otimes I) : \]

\[ g^{*}((\Omega^*_{O_X'/f'^*O_T}, F_b) \otimes O_X (M, F)) \rightarrow (\Omega^*_{O_X'/f'^*O_T}, F_b) \otimes O_X \]

is a map in \(C_{(fg')O_Tfil}(\mathcal{X}')\). But we have, for \(X' \in \mathcal{X}'\) the following equality in \(\Gamma(X' , \Omega_{O_X'}^{p+1} \otimes O_{X'}, g^{*\text{mod}M})\)

\[ d((\Omega^p_{(O_{X'}/g^{*}O_X)/(O_T/g^{*}O_S)}(M)(\omega \otimes m)) : = d((\Omega^p_{O_X'/g^{*}O_X}(\omega) \otimes (m \otimes 1)) \]

\[ = d((\Omega^p_{O_X'/g^{*}O_X}(\omega))) \otimes (m \otimes 1) + \Omega^p_{O_X'/g^{*}O_X}(\omega) \otimes \nabla(m \otimes 1) \]

\[ = \Omega^p_{O_X'/g^{*}O_X}(d(\omega) \otimes (m \otimes 1) + \Omega^p_{O_X'/g^{*}O_X}(\omega) \otimes \nabla(m) \otimes 1 \]

\[ = \Omega^p_{O_X'/g^{*}O_X}(d(\omega) \otimes (m \otimes 1) + \Omega^{p+1}_{O_X'/g^{*}O_X}(\omega) \otimes \nabla(m)) \otimes 1 \]

\[ = : \Omega^{p+1}_{(O_{X'}/g^{*}O_X)/(O_T/g^{*}O_S)}(M)(d(\omega) \otimes m + \omega \otimes \nabla(m)) \]

since for \(\partial' \in T_{O_X'}(X''),\)

\[ \nabla_{\partial'}(m \otimes 1) = \nabla_{d\partial'}(\partial')(m) \otimes 1 + m \otimes \nabla_{\partial'}1 = \nabla_{d\partial'}(\partial')(m) \otimes 1 : \]

see in definition-proposition 14 the definition of the \(D(O_X')\) module structure on the \(O_{X'}\) module \(g^{*\text{mod}M} := g^{*}M \otimes g^{*}O_X O_{X'}\).

(ii) and (iii): There is nothing to prove. \(\square\)

**Remark 7.** Consider a commutative diagram in \(\text{RCat}\)

\[ D = (\mathcal{X}, O_X) \twoheadrightarrow (\mathcal{S}, O_S) \]

\[ (\mathcal{X}', O_{X'}) \twoheadrightarrow (\mathcal{T}, O_T) \]

Assume that the canonical map \(T(g', \text{hom})(O_X, O_X) : g^{*}D(O_X) \rightarrow D(g^{*}O_X)\) is an isomorphism of sheaves. Under the canonical isomorphism \(\sim : \Omega_{O_X/x}^{*} \rightarrow \Omega_{O_X/x}^{*} \otimes O_X (O_X, F_b)\), we have (see definition-proposition 16 and definition 1)

- \(\Omega((O_{X'}/g^{*}O_X)/(O_T/g^{*}O_S))(O_X) = \Omega((O_{X'}/g^{*}O_X)/(O_T/g^{*}O_S)) : g^{*}\Omega_{O_X/f^{*}O_T} \rightarrow \Omega_{O_X/f'^*O_T} \)

- \(T^O_\omega(D)(O_X) = T^O_\omega(D) : g^{*\text{mod}L_0}(f_*E(\Omega_{O_X/f^{*}O_T}^{*}, F_b)) \rightarrow f'_*E(\Omega_{O_X'/f'^*O_T}^{*}, F_b)\).
Definition 60. Consider a commutative diagram in RCat

\[
D = (X, O_X) \xrightarrow{f} (S, O_S).
\]

\[
g' \downarrow \quad g \downarrow
\]

\[
(X', O_{X'}) \xrightarrow{f'} (T, O_T)
\]

with commutative structural sheaf of rings. Assume that the canonical map \( T(g', \text{hom})(O_X, O_X) \to D(g^* O_X) \) is an isomorphism of sheaves. For \((N, F) \in C_{D(O_X), g^* D(O_X)}(X')\), we have by definition-proposition 16 the map in \( C_{T, O_S} PSh(X')\)

\[
T_\omega(g', \otimes)(N, F) : \Omega_{O_X/f^* O_S} \otimes O_X g_*(N, F) \xrightarrow{\text{ad}(g^* \text{mod} g_*)} g'_*(\Omega_{O_{X'}/f^* O_T} \otimes O_{X'}(N, F))
\]

with \( g_S N \in C_{D(O_X)}(X) \), the structure of \( D(O_X) \) module being given by the canonical morphism \( \text{ad}(g^*: g'_*)(D(O_X)) : D(O_X) \to g'_* g^* D(O_X) \) applied to \( g_S N \in C_{g^* g^* D(O_X)}(X) \).

We finish this subsection by a proposition for ringed spaces similar to proposition 9

Proposition 47. Let \( f : (X, O_X) \to (S, O_S) \) a morphism with \( (X, O_X), (S, O_S) \in \text{RTop} \) with commutative sheaves of rings. Assume that \( \Omega_{O_X/f^* O_S} \in \text{PSh}_{O_X}(X) \) is a locally free \( O_X \) module of finite rank.

(i) If \( \phi : (M, F) \to (N, F) \) an \( r \)-filtered top local equivalence with \( (M, F), (N, F) \in C_{D(O_X)}(X) \), then

\[
\text{DR}(O_X/f^* O_S)(\phi) : (\Omega^*_{O_X/f^* O_S}, F_b) \otimes O_X (M, F) \to (\Omega^*_{O_X/f^* O_S}, F_b) \otimes O_T (N, F)
\]

is an \( r \)-filtered top local equivalence.

(ii) Consider a commutative diagram in \( \text{RTop} \)

\[
D = (X, O_X) \xrightarrow{f} (S, O_S).
\]

\[
g' \downarrow \quad g \downarrow
\]

\[
(X', O_{X'}) \xrightarrow{f'} (T, O_T)
\]

with commutative structural sheaf of rings. For \((N, F) \in C_{D(O_X), f^* D(O_X)}(X')\), the map in \( C_{T, O_S} \text{PSh}(X')\)

\[
k \circ T_\omega(g', \otimes)(E(N, F)) : (\Omega^*_{O_X/f^* O_S}, F_b) \otimes O_X g'_* E(N, F) \to g'_* E)((\Omega^*_{O_X/f^* O_T}, F_b) \otimes O_{X'}, E(N, F))
\]

is a filtered top local equivalence (see definition 60).

Proof. (i): Follows from proposition 9 (i) since \( \Omega^*_{O_X/f^* O_S} \in \text{C}^b(X) \) is then a bounded complex with \( \Omega^*_{O_X/f^* O_S} \in \text{PSh}_{O_X}(X) \) a locally free \( O_X \) module of finite rank.

(ii): Follows from proposition 9 (ii) since \( \Omega^*_{O_X/f^* O_S} \in \text{C}^b(X) \) is then a bounded complex with \( \Omega^*_{O_X/f^* O_S} \in \text{PSh}_{O_X}(X) \) a locally free \( O_X \) module of finite rank. □
4.1.3 The support section functor for $D$ module on ringed spaces

Let $(S, O_S) \in \text{RTop}$ with $O_S$ commutative. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \hookrightarrow S$ the open complementary embedding,

- For $G \in C_{D(O_S)}(S)$, $\Gamma_Z G := \text{Cone}(\text{ad}(j^*, j_*)(G) : F \to j_* j^* G)[-1]$ has a (unique) structure of $D(O_S)$ module such that $\gamma_Z(G) : \Gamma_Z G \to G$ is a map in $C_{D(O_S)}(S)$. This gives the functor
  \[ \Gamma_Z : C_{D(O_S)}(S) \to C_{D(O_S)}(S), (G, F) \mapsto \Gamma_Z(G, F) \]
  together with the canonical map $\gamma_Z(G, F) : \Gamma_Z G \to (G, F)$. Let $Z_2 \subset Z$ a closed subset, then for $G \in C_{D(O_S)}(S)$, $T(Z_2/Z, \gamma)(G) : \Gamma_{Z_2} G \to \Gamma_Z G$ is a map in $C_{D(O_S)}(S)$.

- For $G \in C_{O_S}(S)$, $\Gamma_Z^\vee G := \text{Cone}(\text{ad}(j_!, j^*)(G) : j_! j^* G \to G)$ has a unique structure of $D(O_S)$ module, such that $\gamma_Z^\vee(G) : G \to \Gamma_Z^\vee G$ is a map in $C_{D(O_S)}(S)$. This gives the functor
  \[ \Gamma_Z^\vee : C_{D(O_S)}(S) \to C_{D(O_S)}(S), (G, F) \mapsto \Gamma_Z^\vee(G, F), \]
  together with the canonical map $\gamma_Z^\vee(G, F) : (G, F) \to \Gamma_Z^\vee(G, F)$. Let $Z_2 \subset Z$ a closed subset, then for $G \in C_{D(O_S)}(S)$, $T(Z_2/Z, \gamma^\vee)(G) : \Gamma_Z^\vee G \to \Gamma_{Z_2}^\vee G$ is a map in $C_{D(O_S)}(S)$.

- For $G \in C_{D(O_S)}(S)$,
  \[ \Gamma_Z^{\vee, h} G : = \mathcal{D}_S^O L_O \gamma_Z(\mathcal{D}_S^O G) : = \text{Cone}(\mathcal{D}_S^O L_O \gamma_Z(\mathcal{D}_S^O G)) : \mathcal{D}_S^O L_O j_* j^* \mathcal{D}_S^O G \to \mathcal{D}_S^O L_O \mathcal{D}_S^O G \]
  has also canonical $D(O_S)$-module structure, and $\gamma_Z^{\vee, h}(G) : G \to \Gamma_Z^{\vee, h} G$ is a map in $C_{D(O_S)}$. This gives the functor
  \[ \Gamma_Z^{\vee, h} : C_{D(O_S)}(S) \to C_{D(O_S)}(S), (G, F) \mapsto \Gamma_Z^{\vee, h}(G, F), \]
  together with the canonical map $\gamma_Z^{\vee, h}(G, F) : (G, F) \to \Gamma_Z^{\vee, h}(G, F)$.

- Consider $\mathcal{I}^Z_2 \subset O_S$ the ideal of vanishing function on $Z$ and $\mathcal{I}_Z \subset D_S$ the right ideal of $D_S$ generated by $\mathcal{I}^Z_2$. We have then $\mathcal{I}^Z_2 \subset \mathcal{I}_Z$, where $\mathcal{I}^Z_2 \subset D_S$ is the left and right ideal consisting of sections which vanish on $Z$. For $G \in \text{PSh}_{D(O_S)}(S)$, we consider, $S^o \subset S$ being an open subset,
  \[ \mathcal{I}_Z G(S^o) = \{ f, m \in G(S^o), f \in \mathcal{I}_Z(S^o) \} \subset G(S^o) \]
  the $D(O_S)$-submodule generated by the functions which vanish on $Z$ ($\mathcal{I}_Z$ is a right $D(O_S)$ ideal). This gives the functor,
  \[ \Gamma_Z^{O, \mathcal{I}^Z_2} := \Gamma_Z^{O, \mathcal{I}^Z_2} : C_{D(O_S)}(S) \to C_{D(O_S)}(S), (G, F) \mapsto \Gamma_Z^{O, \mathcal{I}^Z_2}(G, F) := \text{Cone}(b_2(G, F) : \mathcal{I}_Z(G, F) \to (G, F)), b_2(-) := b_{\mathcal{I}^Z_2}(-) \]
  together with the canonical map $\gamma_Z^{O, \mathcal{I}^Z_2}(G, F) : (G, F) \to \Gamma_Z^{O, \mathcal{I}^Z_2}(G, F)$, which factors through
  \[ \gamma_Z^{O, \mathcal{I}^Z_2}(G) : G \xrightarrow{\gamma_Z^{\vee}(G)} \Gamma_Z^\vee G \xrightarrow{b_{\mathcal{I}^Z_2}(G)} \Gamma_Z^{O, \mathcal{I}^Z_2} G. \]
  with $b_{\mathcal{I}^Z_2}(-) = b_{\mathcal{I}^Z_2}$ and we have an homotopy equivalence $c_Z(G) : c_{\mathcal{I}^Z_2}(G) : \Gamma_Z^{O, \mathcal{I}^Z_2} G \to G/\mathcal{I}_Z G$.

**Lemma 3.** Let $(Y, O_Y) \in \text{RTop}$ and $i : X \hookrightarrow Y$ a closed embedding.

(i) For $(M, F) \in C_{D(O_Y)}(Y)$ and $(N, F) \in \text{PSh}_{D(O_Y)}(Y)$ such that $a_* N$ is a locally free $D(O_Y)$ module of finite rank, the canonical map
  \[ T(\gamma, \otimes)(E(M, F), (N, F)) := (I, T(j, \otimes)(E(M, F), (N, F))) : \]
  \[ (\Gamma_Y E(M, F)) \otimes_{D(O_Y)} (N, F) \to \Gamma_Y E((M, F) \otimes_{D(O_Y)} (N, F)) \]
  is an equivalence top local.
(ii) For \((M, F) \in C_{D(O_Y)^{op}} ffil(Y)\) and \((N, F) \in PSh_{D(O_Y)^{op}} ffil(Y)\) such that \(a_Y N\) is a locally free \(O_Y\) module of finite rank, the canonical map

\[
T(\gamma, \otimes)(E(M, F), (N, F)) := (I, T(j, \otimes)(E(M, F), (N, F)))
\]

\[
\Gamma X E(M, F) \otimes_{O_Y} (N, F) \rightarrow \Gamma X ((M) \otimes_{O_Y} (N, F))
\]

is a filtered top local equivalence.

**Proof.** Follows from proposition 9. Also note that \(T(j, \otimes)(-,-) = T^{mod}(j, \otimes)(-,-)\).

We now look at the pullback map and the transformation map of De Rahm complexes together with the support section functor. The following is a generalization of definition-proposition 3:

**Definition-Proposition 17.** Consider a commutative diagram in RTop

\[
D_0 = f : (X, O_X) \rightarrow (Y, O_Y) \rightarrow (S, O_S)
\]

\[
g \downarrow \quad g' \downarrow \quad g
\]

\[
f' : (X', O_{X'}) \rightarrow (Y', O_{Y'}) \rightarrow (T, O_T)
\]

with \(i, i'\) being closed embeddings. Denote by \(D\) the right square of \(D\). We have a factorization \(i' : X' \rightarrow Y'\), where \(i_0, i_1\) are closed embedding.

(i) For \((M, F) \in C_{D(O_Y)^{op}} ffil(Y)\), the canonical map,

\[
E(\Omega_{O_Y/g}\times\circ O_Y \otimes_{/g O_S} (M, F)) \circ T(g'', E)(- \otimes T(g'', \gamma) \rightarrow g'^{''mod}(M, F))
\]

unique up to homotopy such that the following diagram in \(C_{g'^{''mod}O_S ffil(Y')} = C_{g'^{''mod}O_S ffil(Y')}\) commutes

\[
g'^{''mod}(M, F) \otimes_{O_Y} (M, F) \rightarrow \Gamma X Y E(\Omega_{O_Y/g}(M, F)) \otimes \Gamma X Y E(\Omega_{O_Y/g}(M, F))
\]

(ii) For \(M \in C_D(Y)\), there is a canonical map

\[
T'_D(M, F) : g^{''mod}LOp_* \Gamma X E((\Omega_{O_Y/p} \otimes_{O_S} F_b) \otimes_{O_Y} (M, F)) \rightarrow p_* \Gamma X Y E((\Omega_{O_Y/p} \otimes_{O_S} F_b) \otimes_{O_Y} (M, F))
\]

unique up to homotopy such that the following diagram in \(C_{O_T ffil(T)}\) commutes

\[
g^{''mod}LOp_* \Gamma X E((\Omega_{O_Y/p} \otimes_{O_S} F_b) \otimes_{O_Y} (M, F)) \rightarrow p_* \Gamma X Y E((\Omega_{O_Y/p} \otimes_{O_S} F_b) \otimes_{O_Y} (M, F))
\]

\[
\rightarrow \Gamma X Y E((\Omega_{O_Y/p} \otimes_{O_S} F_b) \otimes_{O_Y} (M, F))
\]

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(iii) For $N \in C_D(Y \times T)$, the canonical map in $C_{k^\ast,\mathcal{O}_T,fil}(Y')$

$$T(X' / X \times Y', \gamma)(-) : \Gamma_X E((\Omega^\gamma_{y/T}, F_b) \otimes_{\mathcal{O}_Y} (N, F)) \to \Gamma_X \gamma Y' E((\Omega^\gamma_{y/T}, F_b) \otimes_{\mathcal{O}_Y} (N, F))$$

is unique up to homotopy such that $\gamma_{X \times Y'}(-) \circ T(X' / X \times Y', \gamma)(-) = \gamma_X(-)$.

(iv) For $M = O_Y$, we have $T^O_\omega(D)(O_Y \times S)^\gamma = T^O_\omega(D)^\gamma$ and $T^O_\omega(x \times Y' / Y')(O_Y)^\gamma = T^O_\omega(x \times Y' / Y')(S, F)$ (see definition-proposition 3).

Proof. Immediate from definition. We take for the map of point (ii) the composite

$$T^O_\omega(D)(M, F)^\gamma : g^\ast \text{mod } L_D p_* \Gamma X E((\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} (M, F)) \xrightarrow{\gamma}$$

$$g^\ast p_* \Gamma X E((\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} (M, F)) \otimes_{g^\ast \mathcal{O}_S} \mathcal{O}_T \xrightarrow{T(g'', E)(-)-T(g''', (\gamma)(-))T_\omega(D)(\Omega^\gamma_{O_Y / p^\ast O_S}, F_b))$$

$$(p_4 \Gamma X \times Y'E((\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} (M, F))) \otimes_{g^\ast \mathcal{O}_S} \mathcal{O}_T \xrightarrow{E((\Omega^\gamma_{O_Y / p^\ast O_S}, F_b))}$$

$$p_4 \Gamma X \times Y'E((\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} (M, F)) \otimes_{g^\ast \mathcal{O}_S} \mathcal{O}_T \xrightarrow{m}$$

with $m(n \otimes s) = s.n.$

Let $p : (Y, O_Y) \to (S, O_S)$ a morphism with $(Y, O_Y), (S, O_S) \in \mathcal{R}T_{op}$. Let $i : X \hookrightarrow Y$ a closed embedding. Denote by $j : Y \setminus X \hookrightarrow Y$ the complementary open embedding. Consider, for $(M, F) \in C_{D(O_Y), fil}(Y)$ (see definition 60):

$$k \circ T^O_\omega(j, \otimes)(E(M, F)) : (\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} j_* j^\ast E(M, F))$$

$$D(R(O_Y / p^\ast O_S), \text{ad}(j^\ast j_\ast)(-))$$

$$j_* j^\ast((\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} j_* j^\ast E(M, F)) = j_* j^\ast((\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} j_* j^\ast E(M, F))$$

$$\xrightarrow{k \circ D(R(O_Y / p^\ast O_S), \text{ad}(j^\ast j_\ast)(j^\ast E(M))))}$$

$$j_* E(j^\ast((\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} j_* E(M, F))) = j_* E(j^\ast((\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} E(M, F)))$$

Definition 61. Let $p : (Y, O_Y) \to (S, O_S)$ a morphism with $(Y, O_Y), (S, O_S) \in \mathcal{R}T_{op}$. Let $i : X \hookrightarrow Y$ a closed embedding. Denote by $j : Y \setminus X \hookrightarrow Y$ the complementary open embedding. We consider, for $(M, F) \in C_{D(O_Y), fil}(Y)$ the canonical map in $C_{p^\ast O_S, fil}(Y)$

$$T^O_\omega(\gamma, \otimes)(M, F) := (I, k \circ T^O_\omega(j, \otimes)(E(M, F))) :$$

$$(\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} \Gamma_X E(M, F) \to \Gamma_X E((\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} E(M, F)).$$

Proposition 48. Let $p : (Y, O_Y) \to (S, O_S)$ a morphism with $(Y, O_Y), (S, O_S) \in \mathcal{R}T_{op}$. Let $i : X \hookrightarrow Y$ a closed embedding. Then, if $\Omega^\gamma_{O_Y / p^\ast O_S}$ is a locally free $O_Y$ module, for $(M, F) \in C_{D(O_Y), fil}(Y)$

(i) the map

$$T^O_\omega(\gamma, \otimes)(M, F) : (\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} \Gamma_X E(M, F) \to \Gamma_X E((\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} E(M, F))$$

is a 1-filtered top local equivalence,

(ii) the map in $D_{p^\ast O_S, fil}(Y)$

$$T^O_\omega(\gamma, \otimes) := D(R(O_Y / p^\ast O_S), k)^{-1} \circ T^O_\omega(\gamma, \otimes)(M, F) :$$

$$(\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} \Gamma_X E(M, F) \to \Gamma_X E((\Omega^\gamma_{O_Y / p^\ast O_S}, F_b) \otimes_{\mathcal{O}_Y} (M, F))$$

is an isomorphism.
4.2 The D-modules on smooth complex algebraic varieties and on complex analytic manifolds and their functorialities in the filtered case

For convenience, we will work with and state the results for presheaves of D-modules. In this section, it is possible to assume that all the presheaves are sheaves and that the sheafification functor after the pullback functor $f^*$ for a morphism $f : X \to S$, $X, S \in \text{Var}(\mathbb{C})$ or $X, S \in \text{AnSp}(\mathbb{C})$, and after the internal hom functors and tensor products of presheaves of modules on $S \in \text{Var}(\mathbb{C})$ or $S \in \text{AnSp}(\mathbb{C})$.

For $S = (S, O_S) \in \text{SmVar}(\mathbb{C})$, resp. $S = (S, O_S) \in \text{AnSp}(\mathbb{C})$, we denote by

- $D_S := D(O_S) \subset \text{Hom}_{\mathcal{C}_S}(O_S, O_S)$ the subsheaf consisting of differential operators. By a $D_S$ module, we mean a left $D_S$ module.

- we denote by

  - $\text{PSh}_{\mathcal{D}}(S)$ the abelian category of Zariski (resp. usu) presheaves on $S$ with a structure of left $D_S$ module, and by $\text{PSh}_{\mathcal{D}, h}(S) \subset \text{PSh}_{\mathcal{D}, c}(S) \subset \text{PSh}_{\mathcal{D}}(S)$ the full subcategories whose objects are coherent, resp. holonomic, sheaves of left $D_S$ modules, and by $\text{PSh}_{\mathcal{D}, rh}(S) \subset \text{PSh}_{\mathcal{D}, h}(S)$ the full subcategory of regular holonomic sheaves of left $D_S$ modules,

  - $\text{PSh}_{\mathcal{D}^{op}}(S)$ the abelian category of Zariski (resp. usu) presheaves on $S$ with a structure of right $D_S$ module, and by $\text{PSh}_{\mathcal{D}^{op}, h}(S) \subset \text{PSh}_{\mathcal{D}^{op}, c}(S) \subset \text{PSh}_{\mathcal{D}^{op}}(S)$ the full subcategories whose objects are coherent, resp. holonomic, sheaves of right $D_S$ modules, and by $\text{PSh}_{\mathcal{D}^{op}, rh}(S) \subset \text{PSh}_{\mathcal{D}^{op}, h}(S)$ the full subcategory of regular holonomic sheaves of right $D_S$ modules,

- we denote by

  - $C_{\mathcal{D}}(S) = C(\text{PSh}_{\mathcal{D}}(S))$ the category of complexes of Zariski presheaves on $S$ with a structure of left $D_S$ module,

    $$C_{\mathcal{D}, h}(S) \subset C_{\mathcal{D}, c}(S) \subset C_{\mathcal{D}}(S)$$

    the full subcategories consisting of complexes of presheaves $M$ such that $a_\tau H^n(M)$ are coherent, resp. holonomic, resp. regular holonomic, sheaves of $D_S$ modules, $a_\tau$ being the sheafification functor for the Zariski, resp. usual, topology,

  - $C_{\mathcal{D}^{op}}(S) = C(\text{PSh}_{\mathcal{D}^{op}}(S))$ the category of complexes of Zariski presheaves on $S$ with a structure of right $D_S$ module,

    $$C_{\mathcal{D}^{op}, h}(S) \subset C_{\mathcal{D}^{op}, c}(S) \subset C_{\mathcal{D}^{op}}(S)$$

    the full subcategories consisting of complexes of presheaves $M$ such that $a_\tau H^n(M)$ are coherent, resp. holonomic, resp. regular holonomic, sheaves of right $D_S$ modules,

- in the filtered case we have

  - $C_{\mathcal{D}^{(2)}fil}(S) \subset C(\text{PSh}_{\mathcal{D}}(S), F, W) := C(\text{PSh}_{D(O_S)}(S), F, W)$ the category of (bi)filtered complexes of algebraic (resp. analytic) $D_S$ modules such that the filtration is biregular (see definition 54),

    $$C_{\mathcal{D}^{(2)}fil, h}(S) \subset C_{\mathcal{D}^{(2)}fil, c}(S) \subset C_{\mathcal{D}^{(2)}fil}(S),$$

Proof. By proposition 47,

- $\text{Gr}^p_{f^*}(k \circ T^O_w(j, \otimes)(E(M, F))) : \Omega^*_D \otimes_{O_Y} j_* j^* F^{p-\bullet} E(M) \to j_* E(j^*(\Omega^*_D \otimes_{O_Y} F^{p-\bullet} E(M)))$ is a top local equivalence and

- $DR(O_Y/p^* O_S)(k) : \Omega^*_D \otimes_{O_Y} (M, F) \to \Omega^*_D \otimes_{O_Y} E(M, F)$ is a filtered top local equivalence. \hfill \Box
the full subcategories consisting of filtered complexes of presheaves \((M, F)\) such that \(a_\tau H^n(M, F)\) are filtered coherent, resp. filtered holonomic, resp. filtered regular holonomic, sheaves of \(D_S\) modules, that is \(a_\tau H^n(M)\) are coherent, resp. holonomic, resp. regular holonomic, sheaves of \(D_S\) modules and \(F\) induces a good filtration on \(a_\tau H^n(M)\) (in particular \(F^p a_\tau H^n(M) \subset a_\tau H^n(M)\) are coherent sub \(O_S\) modules),

- \(C_{D^f} \subset C_{D^f}(S)\) the full subcategory such that the filtration is a filtration by \(D_S\) submodule (which is stronger then Griffiths transversality), \(C_{D^{(1,0)}f} \subset C_{D^{2f}}(S)\) the full subcategory such that \(W\) is a filtration by \(D_S\) submodules (see definition 54),

\[
C_{D^{(1,0)}f,h}(S) = C_{D^{2f},h}(S) \cap C_{D^{(1,0)}f}(S) \subset C_{D^{2f},h}(S),
\]

the full subcategory consisting of filtered complexes of presheaves \((M, F, W)\) such that \(a_\tau H^n(M, F)\) are filtered holonomic sheaves of \(D_S\) modules and such that \(W^p M \subset M\) are \(D_S\) submodules (recall that the \(O_S\) submodules \(F^p M \subset M\) are NOT \(D_S\) submodules but satisfy by definition \(md: F^p D_S \subset F^p M \subset F^{p+r} M\)),

\[
C_{D^{(1,0)}f,h,\tau}(S) = C_{D^{2f},\tau}(S) \cap C_{D^{(1,0)}f}(S) \subset C_{D^{2f},\tau}(S),
\]

the full subcategory consisting of filtered complexes of presheaves \((M, F, W)\) such that \(a_\tau H^n(M, F)\) are filtered regular holonomic sheaves of \(D_S\) modules and such that \(W^p M \subset M\) are \(D_S\) submodules

- \(C_{D^{op}}(S) \subset C_{PSh_{D^{op}}(S), F, W) := C_{PSh_{D^{op}}(S), F, W)\} the category of (bi)filtered complexes of algebraic (resp. analytic) right \(D_S\) modules such that the filtration is biregular, as in the left case we consider the subcategories

\[
C_{D^{op}(2, f),r}(S) \subset C_{D^{op}(2, f),h}(S) \cap C_{D^{op}(2, f),r}(S) \subset C_{D^{op}(2, f),r}(S),
\]

the full subcategories consisting of filtered complexes of presheaves \((M, F)\) such that \(a_\tau H^n(M, F)\) are filtered coherent, resp. filtered holonomic, resp. filtered regular holonomic, sheaves of right \(D_S\) modules.

For \(S = (S, O_S) \in AnSm(\mathbb{C})\), we have the natural extension \(D_S \subset D_S^{\infty} \subset \text{Hom}_{\mathcal{O}_S} (O_S, O_S)\) where \(D_S^{\infty} \subset \text{Hom}_{\mathcal{O}_S} (O_S, O_S)\) is the subsheaf of differential operators of possibly infinite order (see [19]) for the definition of the action of a differential operator of infinite order on \(O_S\) Similarly, we have

- \(C_{D^{\infty}(2, f),h}(S) \subset C_{D^{\infty}(2, f),h}(S) \cap C_{D^{\infty}(2, f),r}(S) \subset C_{D^{\infty}(2, f),r}(S)\) the category of (bi)filtered complexes of \(D_S^{\infty}\) modules such that the filtration is biregular,

\[
C_{D^{\infty}(2, f),h}(S) \subset C_{D^{\infty}(2, f),r}(S) \subset C_{D^{\infty}(2, f),r}(S),
\]

the full subcategories consisting of filtered complexes of presheaves \((M, F)\) such that \(a_\tau H^n(M, F)\) are filtered coherent (resp. holonomic) sheaves of \(D_S^{\infty}\) modules that is \(a_\tau H^n(M)\) are coherent (resp. holonomic) sheaves of \(D_S^{\infty}\) modules and \(F\) induces a good filtration on \(a_\tau H^n(M)\).

- \(C_{D^{op}}(S) \subset C_{D^{\infty}(2, f),h}(S)\) the full subcategory such that the filtration is a filtration by \(D_S^{\infty}\) submodule, \(C_{D^{(1,0)}f}(S) \subset C_{D^{2f}}(S)\) the full subcategory such that \(W\) is a filtration by \(D_S^{\infty}\) submodules,

\[
C_{D^{(1,0)}f,\tau}(S) = C_{D^{2f},\tau}(S) \cap C_{D^{(1,0)}f}(S) \subset C_{D^{2f},\tau}(S),
\]

the full subcategory consisting of filtered complexes of presheaves \((M, F, W)\) such that \(a_\tau H^n(M, F)\) are filtered holonomic sheaves of \(D_S^{\infty}\) modules and such that \(W^p M \subset M\) are \(D_S\) submodules

- \(C_{D^{op}}(S) \subset C_{D^{op}(2, f),h}(S), F, W) := C_{PSh_{D^{op}}(S), F, W)\} the category of (bi)filtered complexes of right \(D_S^{\infty}\) modules such that the filtration is biregular,

\[
C_{D^{op}(2, f),h}(S) \subset C_{D^{op}(2, f),h}(S) \cap C_{D^{op}(2, f),r}(S) \subset C_{D^{op}(2, f),r}(S),
\]

the full subcategories consisting of filtered complexes of presheaves \((M, F)\) such that \(a_\tau H^n(M, F)\) are filtered coherent (resp. holonomic) sheaves of \(D_S\) modules.
For \( f : X \to S \) a morphism with \( X, S \in \SmVar(\mathbb{C}) \) or with \( (X, S) \in \AnSm(\mathbb{C}) \),

- we denote by
  
  \[
  \begin{align*}
  \text{PSh}_{f^* \mathcal{D}}(X) & \text{ the abelian category of Zariski (resp. usu) presheaves on } S \text{ with a structure of left } f^* D_S \text{ module, and } C_{f^* \mathcal{D}}(X) = C(\text{PSh}_{f^* \mathcal{D}}(X)), \\
  \text{PSh}_{D^\mathcal{D} f} (X) & \text{ the abelian category of Zariski (resp. usu) presheaves on } S \text{ with a structure of left } f^* D_S \text{ module and right } D_X \text{ module, and } C_{D^\mathcal{D} f}(X) = C(\text{PSh}_{D^\mathcal{D} f}(X)), \\
  \text{PSh}_{D^\mathcal{D} f} (X) & \text{ the abelian category of Zariski (resp. usu) presheaves on } S \text{ with a structure of left } f^* D_S \text{ module and right } D_X \text{ module and } C_{D^\mathcal{D} f}(X) = C(\text{PSh}_{D^\mathcal{D} f}(X)), \\
  \end{align*}
  \]

- we denote by
  
  \[
  \begin{align*}
  C_{f^* \mathcal{D}^\mathcal{D} f}(X) & \subseteq C(\text{PSh}_{f^* \mathcal{D}^\mathcal{D} f}(X), F) := C(\text{PSh}_{f^* \mathcal{D}^\mathcal{D} f}(X), F) \text{ the category of filtered complexes of algebraic (resp. analytic) left } f^* D_S \text{ modules such that the filtration is biregular}, \\
  C_{D^\mathcal{D} f}(X) & \subseteq C(\text{PSh}_{D^\mathcal{D} f}(X), F) \text{ the category of filtered complexes of algebraic (resp. analytic) left } f^* D_S \text{ modules such that the filtration is biregular}, \\
  C_{D^\mathcal{D} f}(X) & \subseteq C(\text{PSh}_{D^\mathcal{D} f}(X), F) \text{ the category of filtered complexes of algebraic (resp. analytic) left } f^* D_S \text{ modules such that the filtration is biregular}. \\
  \end{align*}
  \]

For \( f : X \to S \) a morphism with \( X, S \in \AnSm(\mathbb{C}) \), we denote by

- \( C_{f^* \mathcal{D}^\mathcal{D} f}(S) \subseteq C(\text{PSh}_{f^* \mathcal{D}^\mathcal{D} f}(S), F) := C(\text{PSh}_{f^* \mathcal{D}^\mathcal{D} f}(S), F) \text{ the category of filtered complexes of left } f^* D_S \text{ modules such that the filtration is biregular}, \)

- \( C_{D^\mathcal{D} f}(S) \subseteq C(\text{PSh}_{D^\mathcal{D} f}(S), F) \text{ the category of filtered complexes of algebraic (resp. analytic) left } f^* D_S \text{ modules such that the filtration is biregular,} \)

- \( C_{D^\mathcal{D} f}(S) \subseteq C(\text{PSh}_{D^\mathcal{D} f}(S), F) \text{ the category of filtered complexes of algebraic (resp. analytic) left } f^* D_S \text{ modules such that the filtration is biregular.} \)

For \( S \in \AnSm(\mathbb{C}) \), we denote by

\[ J_S : C_{D^\mathcal{D} f}(S) \to C_{D^\mathcal{D} f}(S), (M, F) \mapsto J_S(M, F) := (M, F) \otimes_{D_S} (D_S^\mathcal{D}, F) \]

the natural functor. For \( (M, F) \in C_{D^\mathcal{D} f}(S) \), we will consider the map

\[ J_S(M, F) : J_S(M, F) := (M, F) \otimes_{D_S} (D_S^\mathcal{D}, F) \to (M, F), m \otimes P \mapsto Pm \]

Of course \( J_S(C_{D^\mathcal{D} f}(S)) \subseteq C_{D^\mathcal{D} f}(S) \). More generally, for \( f : X \to S \) a morphism with \( X, S \in \AnSm(\mathbb{C}) \), we denote by

\[ J_{X/S} : C_{f^* \mathcal{D}^\mathcal{D} f}(X) \to C_{f^* \mathcal{D}^\mathcal{D} f}(X), (M, F) \mapsto J_{X/S}(M, F) := (M, F) \otimes_{f^*(D_S, F)} f^*(D_S^\mathcal{D}, F) \]

the natural functor, together with, for \( (M, F) \in C_{f^* \mathcal{D}^\mathcal{D} f}(X) \), the map \( J_S(M, F) : J_S(M, F) \to (M, F) \).

**Definition 62.** Let \( S \in \SmVar(\mathbb{C}) \), resp. \( S \in \AnSm(\mathbb{C}) \). Let \( Z \subseteq S \) a closed subset and denote by \( j : S \setminus Z \to S \) the open embedding.

(i) We denote by

- \( \text{PSh}_{D, Z}(S) \subseteq \text{PSh}_{D}(S) \), the full subcategory consisting of presheaves \( M \in \text{PSh}_{D}(S) \), such that \( j^* M = 0 \),

- \( \text{C}_{D, Z}(S) \subseteq \text{C}_{D}(S) \), the full subcategory consisting of complexes presheaves \( M \in \text{C}_{D}(S) \) such that \( \alpha^* j^* H^n M = 0 \) for all \( n \in \mathbb{Z} \),
- $C_{D,Z,h}(S) := C_{D,Z}(S) \cap C_{D,h}(S) \subset C_{D}(S)$ the full subcategory consisting of $M \in C_{D}(S)$ such that $a_{r}H^{n}(M)$ are holonomic and $a_{r}j^{*}H^{n}M = 0$ for all $n \in \mathbb{Z}$.

- $C_{D,Z,c}(S) := C_{D,Z}(S) \cap C_{D,c}(S) \subset C_{D}(S)$ the full subcategory consisting of $M \in C_{D}(S)$ such that $a_{r}H^{n}(M)$ are coherent and $a_{r}j^{*}H^{n}M = 0$ for all $n \in \mathbb{Z}$.

(ii) We denote by

- $C_{D(2)}fil,Z(S) \subset C_{D(2)}fil(S)$, the full subcategory consisting of $(M, F) \in C_{D(2)}fil(S)$ such that there exists $r \in \mathbb{N}$ and an $r$-filtered homotopy equivalence $\phi : (M, F) \to (N, F)$ with $(N, F) \in C_{D(2)}fil(S)$ such that $a_{r}j^{*}H^{n}Gr^{p}_{F}(M, F) = 0$ for all $n, p \in \mathbb{Z}$, note that by definition this $r$ does NOT depend on $n$ and $p$,

- $C_{D(2)}fil,Z,rh(S) := C_{D(2)}fil,Z(S) \cap C_{D(2)}fil,rh(S) \subset C_{D(2)}fil(S)$ the full subcategory consisting of $(M, F)$ such that $a_{r}H^{n}(M, F)$ are filtered regular holonomic for all $n \in \mathbb{Z}$ and such that there exists $r \in \mathbb{N}$ and an $r$-filtered homotopy equivalence $\phi : (M, F) \to (N, F)$ with $(N, F) \in C_{D(2)}fil(S)$ such that $a_{r}j^{*}H^{n}Gr^{p}_{F}(M, F) = 0$ for all $n, p \in \mathbb{Z}$,

- $C_{D(2)}fil,Z,h(S) := C_{D(2)}fil,Z(S) \cap C_{D(2)}fil,h(S) \subset C_{D(2)}fil(S)$ the full subcategory consisting of $(M, F)$ such that $a_{r}H^{n}(M, F)$ are filtered coherent for all $n \in \mathbb{Z}$ and such that there exists $r \in \mathbb{N}$ and an $r$-filtered homotopy equivalence $\phi : (M, F) \to (N, F)$ with $(N, F) \in C_{D(2)}fil(S)$ such that $a_{r}j^{*}H^{n}Gr^{p}_{F}(M, F) = 0$ for all $n, p \in \mathbb{Z}$.

(iii) We have then the full subcategories

- $C_{D(1)}fil,Z(S) = C_{D(1)}fil(S) \cap C_{D2fil,Z}(S) \subset C_{D2fil}(S)$,

- $C_{D(1)}fil,Z,rh(S) = C_{D(1)}fil(S) \cap C_{D2fil,Z,rh}(S) \subset C_{D2fil}(S)$.

- $C_{D(1)}fil,Z,h(S) = C_{D(1)}fil(S) \cap C_{D2fil,Z,h}(S) \subset C_{D2fil}(S)$.

Similarly:

**Definition 63.** Let $S \in \text{AnSm}(\mathbb{C})$. Let $Z \subset S$ a closed subset and denote by $j : S \setminus Z \to S$ the open embedding.

(i) We denote by

- $C_{D\infty(2)}fil,Z(S) \subset C_{D\infty(2)}fil(S)$. the full subcategory consisting of $(M, F) \in C_{D\infty}(S)$ such that $j^{*}M$ is acyclic

- $C_{D\infty(2)}fil,Z,h(S) := C_{D\infty(2)}fil,Z(S) \cap C_{D\infty(2)}fil,h(S) \subset C_{D\infty(2)}fil(S)$ the full subcategory consisting of $(M, F)$ such that $a_{r}H^{n}(M)$ are holonomic and such that there exist $r \in \mathbb{Z}$ and an $r$-filtered homotopy equivalence $\phi : (M, F) \to (N, F)$ with $(N, F) \in C_{D\infty(2)}fil(S)$ such that $a_{r}j^{*}H^{n}Gr^{p}_{F}(M, F) = 0$.

(ii) We have then the full subcategories

- $C_{D\infty(1)}fil,Z(S) = C_{D\infty(1)}fil(S) \cap C_{D\infty(2)fil,Z}(S) \subset C_{D\infty(2)fil}(S)$,

- $C_{D\infty(1)}fil,Z,h(S) := C_{D\infty(1)}fil(S) \cap C_{D\infty(2)fil,Z,h}(S) \subset C_{D\infty(2)fil}(S)$.

Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. We recall (see section 2) that a morphism $m : (M, F) \to (N, F)$ with $(M, F), (N, F) \in C_{Dfil}(S)$ is said to be an $r$-filtered zariski, resp. usu. local equivalence if there exist morphisms $m_{i} : (C_{i}, F) \to (C_{i+1}, F)$ with $(C_{i}, F), (C_{i+1}, F) \in C_{Dfil}(S)$, $0 \leq i \leq s$, $(C_{0}, F) = (M, F)$, $(C_{s}, F) = (N, F)$, such that

$$m = m_{s} \circ \cdots \circ m_{i} \circ \cdots m_{0} : (M, F) \to (N, F)$$
and \( m_i : (C_i, F) \rightarrow (C_{i+1}, F) \) is either a filtered zariski, resp. usu, local equivalence or an \( r \)-filtered homotopy.

**Definition 64.**
(i) Let \( f : X \rightarrow S \) a morphism with \( X, S \in \text{SmVar}(\mathbb{C}) \), or with \( X, S \in \text{AnSm}(\mathbb{C}) \). We have, for \( r = 1, \ldots, \infty \), the homotopy categories

\[
K_{D,f}^{(2)}(S) := \text{Ho}_{r,0}C_{D,f}^{(2)}(S), \quad K_{D^p,f}^{(2)}(S) := \text{Ho}_{r,0}C_{D^p,f}^{(2)}(S),
\]

whose objects are those of \( C_{D,f}^{(2)}(S) \), resp. those of \( C_{D^p,f}^{(2)}(S) \), and whose morphisms are \( r \)-filtered for the first filtration (filtered for the second) homotopy classes of morphisms (see section 2.1). We have then

\[
D_{D,f}^{(2)}(S) := K_{D,f}^{(2)}(S)[[E_1]^{-1}], \quad D_{D^p,f}^{(2)}(S) := K_{D^p,f}^{(2)}(S)[[E_1]^{-1}],
\]

the localizations with respect to the classes of filtered Zariski, resp. usu, local equivalences (see section 2). Note that the classes of filtered \( \tau \) local equivalence constitute a right multiplicative system. If \( m : (M,F) \rightarrow (N,F) \) with \( (M,F), (N,F) \in D_{D,f}^{(2)}(S) \) is an \( r \)-filtered zariski, resp. usu, local equivalence, then \( m = D(\text{top}) : (M,F) \rightarrow (N,F) \) is an isomorphism in \( D_{D,f}^{(2)}(S) \).

(ii) Let \( S \in \text{SmVar}(\mathbb{C}) \) or \( S \in \text{AnSm}(\mathbb{C}) \). We denote by

\[
D_{D,f}^{(2),\infty,h}(S) \subset D_{D(fu,h)}^{(2),\infty}(S) \subset D_{D(fu,\infty)}^{(2),\infty}(S),
\]

the full subcategories consisting of the image of \( C_{D,f}^{(2),h}(S) \), resp. \( C_{D(fu,h)}^{(2),h}(S) \), by the localization functor

\[
D(\text{top}) : C_{D(fu,h)}^{(2)}(S) \rightarrow D_{D(fu,\infty)}^{(2),\infty}(S)
\]

that is consisting of \( (M,F) \in C_{D(fu)}^{(2)}(S) \) such that \( a_rH^n(M,F) \) are filtered holonomic, resp. filtered regular holonomic for all \( n \in \mathbb{Z} \).

(iii) Let \( S \in \text{SmVar}(\mathbb{C}) \) or \( S \in \text{AnSm}(\mathbb{C}) \). We denote by

\[
D_{D(1),f}^{(2),\infty,h}(S) \subset D_{D(1),f}^{(2),\infty,h}(S) \subset D_{D(2),f}^{(2),\infty}(S),
\]

the full subcategories consisting of the image of \( C_{D(1),f}^{(2),h}(S) \), resp. \( C_{D(1),f}^{(2),h}(S) \), by the localization functor

\[
D(\text{top}) : C_{D(1),f}^{(2)}(S) \rightarrow D_{D(2),f}^{(2),\infty}(S)
\]

that is consisting of \( (M,F,W) \in C_{D(2)}^{(2)}(S) \) such that \( a_rH^n(M,F) \) are filtered holonomic, resp. filtered regular holonomic, and \( W^pM^n \subset M^n \) are \( D_S \) submodules for all \( n \in \mathbb{Z} \).

Let \( S \in \text{SmVar}(\mathbb{C}) \) or \( S \in \text{AnSm}(\mathbb{C}) \). By definition (see section 2), we have sequences of functors

\[
C_{D(fu)}^{(2)}(S) \rightarrow K_{D(fu)}^{(2)}(S) \rightarrow D_{D(fu)}^{(2)}(S) \rightarrow D_{D(fu,2)}^{(2)}(S) \rightarrow \cdots \rightarrow D_{D(fu,\infty)}^{(2),\infty}(S).
\]

and commutative diagrams of functors

\[
\begin{array}{ccc}
K_{D(fu)}^{(2)}(S) & \rightarrow & D_{fu}^{(2)}(S) \quad , \quad K_{D(fu,r)}^{(2)}(S) & \rightarrow & D_{D(fu,r)}^{(2),\infty}(S) \\
\text{Ver} & & \text{Ver} & & \text{Ver} \\
K_{D(fu,2)}^{(2)}(S) & \rightarrow & D_{D(fu,2)}^{(2),\infty}(S) & & K_{D(fu,r+1)}^{(2)}(S) & \rightarrow & D_{D(fu,r+1)}^{(2),\infty}(S)
\end{array}
\]

Then, for \( r = 1 \), \( K_{D(fu)}^{(2)}(S) \) and \( D_{D(fu)}^{(2),\infty}(S) \) are in the canonical way triangulated categories. However, for \( r > 1 \), the categories \( K_{D(fu,r)}^{(2)}(S) \) and \( D_{D(fu,r)}^{(2),\infty}(S) \) together with the canonical triangles does NOT satisfy the 2 of 3 axiom of triangulated categories.

Similarly,
Definition 65. (i) Let \( f : X \to S \) a morphism with \( X, S \in \text{AnSm}(\mathbb{C}) \). We have, for \( r = 1, \ldots, \infty \), the categories

\[
K_{D^\infty,f^*D^\infty(2)fil,\tau}(S) := \text{Ho}_{r,0} C_{D^\infty,f^*D^\infty(2)fil}(S), \quad K_{D^{\infty,op,f^*D^\infty(2)fil,\tau}}(S) := \text{Ho}_{r,0} C_{D^{\infty,op,f^*D^\infty(2)fil}}(S),
\]

whose objects are those of \( C_{D^\infty,f^*D^\infty(2)fil}(S) \), resp. those of \( C_{D^{\infty,op,f^*D^\infty(2)fil}}(S) \), and whose morphisms are \( r \)-filtered for the first filtration (filtered for the second) homotopy classes of morphisms (see section 2.1). We have the categories

\[
D_{D^\infty,f^*D^\infty(2)fil,\tau}(S) := K_{D^\infty,f^*D^\infty(2)fil,\tau}(S)([E_1]^{-1}), \quad D_{D^{\infty,op,f^*D^\infty(2)fil,\tau}}(S) := K_{D^{\infty,op,f^*D^\infty(2)fil}}(S)([E_1]^{-1}),
\]

the localizations with respect to the classes of filtered usu local equivalence (see section 2.1). Note that the classes of filtered usu local equivalence constitute a right multiplicative system.

(ii) Let \( S \in \text{AnSm}(\mathbb{C}) \). We denote by

\[
D_{D^\infty(2)fil,\tau}(S) \subset D_{D^\infty(2)fil,\infty}(S), \quad D_{D^{\infty,1}(0)fil,\tau}(S) \subset D_{D^{\infty,1}(0)fil,\infty}(S)
\]

the full subcategories consisting of the image of \( C_{D^\infty(2)fil,h}(S) \), resp. \( C_{D^{\infty,1}(0)fil,h}(S) \), by the localization functor

\[
D(\text{top}) : C_{D^\infty(2)fil}(S) \to D_{D^\infty(2)fil,\tau}(S).
\]

We begin this subsection by recalling the following well known facts.

Proposition 49. Let \( S \in \text{SmVar}(\mathbb{C}) \) or \( S \in \text{AnSm}(\mathbb{C}) \).

(i) The sheaf of differential operators \( D_S \) is a locally free sheaf of \( O_S \) module. Hence, a coherent \( D_S \) module \( M \in \text{Coh}_D(S) \) is a quasi-coherent sheaf of \( O_S \) modules.

(ii) A coherent sheaf \( M \in \text{Coh}_{O_S}(S) \) of \( O_S \) module admits a \( D_S \) module structure if and only if it is locally free (of finite rank by coherency) and admits an integrable connexion. In particular if \( i : Z \to S \) is a closed embedding for the Zariski topology, then \( i_* O_Z \) does NOT admit a \( D_S \) module structure since it is a coherent but not locally free \( O_S \) module.

Proof. Standard.

In order to prove a version of the first GAGA theorem for coherent \( D \) modules, we will need to following. We start by a definition (cf. [17] definition 1.4.2):

Definition 66. An \( X \in \text{SmVar}(\mathbb{C}) \) is said to be \( D \)-affine if the following two condition hold:

(i) The global section functor \( \Gamma(X, \cdot) : \text{QCoh}_D(X) \to \text{Mod}(\Gamma(X, D_X)) \) is exact.

(ii) If \( \Gamma(X, M) = 0 \) for \( M \in \text{QCoh}_D(X) \), then \( M = 0 \).

Proposition 50. If \( X \in \text{SmVar}(\mathbb{C}) \) is \( D \)-affine, then :

(i) Any \( M \in \text{QCoh}_D(X) \) is generated by its global sections.

(ii) The functor \( \Gamma(X, \cdot) : \text{QCoh}_D(X) \to \text{Mod}(\Gamma(X, D_X)) \) is an equivalence of category whose inverse is \( L \in \text{Mod}(\Gamma(X, D_X)) \mapsto D_X \otimes_{\Gamma(X, D_X)} L \in \text{QCoh}_D(X) \).

(iii) We have \( \Gamma(X, \cdot) (\text{Coh}_D(X)) = \text{Mod}(\Gamma(X, D_X))_f \), that is the global sections of a coherent \( D_X \) module is a finite module over the differential operators on \( X \).

Proof. See [17].

The following proposition is from Kashiwara.

Proposition 51. Let \( S \in \text{AnSm}(\mathbb{C}) \).
(i) For $K \in C_c(S)$ a complex of presheaves with constructible cohomology sheaves, we have $\mathsf{Hom}(L(K), E(O_S)) \in C_{\mathcal{D}^{\infty, h}}(S)$.

(ii) The functor $J_S : \mathcal{D}_{(2)}^\bullet \to \mathcal{D}_{(2)}^\bullet$ satisfy $J_S(C_{\mathcal{D}_{(2)}^\bullet}(S)) \subset C_{\mathcal{D}^{\infty, h}}(S)$, derive trivially, and induce an equivalence of category

$$J_S : \mathcal{D}_{(2)}^\bullet \to \mathcal{D}_{(2)}^\bullet.$$ 

whose inverse satify, for $(M, F) \in \mathsf{Hol}_{\mathcal{D}_{(2)}^\bullet}(S)$ a (filtered) holonomic $D_S^\infty$ module, that $J_S^{-1}(M, F) = (M_{reg}, F) \subset (M, F)$ is the $D_S$ sub-module of $M$ which is the regular part.

(iii) We have $J_S(C_{\mathcal{D}_{(2)}^\bullet}(S)) \subset C_{\mathcal{D}^{\infty, h}}(S)$ and $J_S(D_{\mathcal{D}_{(2)}^\bullet}(S)) = D_{\mathcal{D}^{\infty, h}}(S)$.

Proof. Follows from [19].

Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$, and let $i : Z \hookrightarrow S$ a closed embedding and denote by $j : S\backslash Z \hookrightarrow S$ the open complementary. For $M \in \text{PSh}_S(S)$, we denote $\mathcal{I}_Z M$ the (left) $D_S$ submodule given by, for $S^o \subset S$ an open subset, $\mathcal{I}_Z M(S^o) \subset M(S^o)$ is the (left) $D_S(S^o)$ submodule

$$\mathcal{I}_Z M(S^o) := \{ f \in \mathcal{I}_Z(S^o), m \in M(S^o) \} \supset M(S^o)$$

generated by the elements of the form $fm$. We denote by $b_Z(M) : \mathcal{I}_Z M \to M$ the inclusion map and $c_Z(M) : M \to M/\mathcal{I}_Z M$ the quotient map of (left) $D_S$ modules. For $M \in \text{PSh}_S(S)$, we denote $M_Z \subset M$ the right $D_S$ submodule given by, for $S^o \subset S$ an open subset, $\mathcal{I}_Z M(S^o) \subset M(S^o)$ is the right $D_S(S^o)$ submodule

$$\mathcal{I}_Z M(S^o) := \{ mf, f \in \mathcal{I}_Z(S^o), m \in M(S^o) \} \supset M(S^o)$$

generated by the elements of the form $mf$. We denote by $b_Z(M) : \mathcal{I}_Z M \to M$ the inclusion map and $c_Z(M) : M \to M/\mathcal{I}_Z M$ the quotient map of right $D_S$ modules.

4.2.1 Functorialities

Let $f : X \to S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, we recall from section 4.1, the transfers modules

- $(D_X \to S, F^{C}) := F^{C \leftarrow} \otimes f_* O_S (O_X, F_b)$ which is a left $D_X$ module and a left and right $f^* D_S$ module
- $(D_X \to S, F^{C}) := (K_X, F_b) \otimes f_* (D_X \to S, F^{C}) \otimes f_* O_S f^*(K_X, F_b)$ which is a right $D_X$ module and a left and right $f^* D_S$ module.

Let $f : X \to S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the transfers modules of infinite order are

- $(D_X^{\infty} \to S, F^{C}) := F^{C \leftarrow} \otimes f_* O_S (D_X^{\infty} \to S, F^{C}) \otimes f_* O_S f^*(K_X, F_b)$ which is a left $D_X^{\infty}$ module and a left and right $f^* D_S^{\infty}$ module
- $(D_X^{\infty} \to S, F^{C}) := (K_X, F_b) \otimes f_* (D_X^{\infty} \to S, F^{C}) \otimes f_* O_S f^*(K_X, F_b)$ which is a right $D_X^{\infty}$ module and a left and right $f^* D_S^{\infty}$ module.

We have the following:

Lemma 4. Let $f_1 : X \to Y, f_2 : Y \to S$ be two morphism with $X, S, Y \in \text{SmVar}(\mathbb{C})$, or let $f_1 : X \to Y, f_2 : Y \to S$ be two morphism with $X, Y \in \text{AnSm}(\mathbb{C})$.

(i) We have $(D_X \to S, F^{C}) = f_1^*(D_Y \to S, F^{C}) \otimes f_1^* D_Y (D_X \to Y, F^{C})$ in $C_{\mathcal{D}_{(f_2 \circ f_1)^*} \mathcal{D}^\infty}(X)$ and

$$(D_X \to S, F^{C}) = f_1^*(D_Y \to S, F^{C}) \otimes f_1^* D_Y (D_X \to Y, F^{C}) = f_1^*(D_Y \to S, F^{C}) \otimes f_1^* D_Y (D_X \to Y, F^{C})$$

in $D_{\mathcal{D}_{(f_2 \circ f_1)^*} \mathcal{D}^\infty}(X)$. 177
(ii) We have \((D_{X \to S}, F_{\text{ord}}) = f_1^*(D_{Y \to S}, F_{\text{ord}}) \otimes f_1^*D_Y (D_{X \to Y}, F_{\text{ord}}) \) in \(\mathcal{C}_{D_{\text{for}},(f_2 \circ f_1)^*}D_{\text{fil}}(X)\) and
\[(D_{X \to S}, F_{\text{ord}}) = f_1^*(D_{Y \to S}, F_{\text{ord}}) \otimes f_1^*D_Y (D_{X \to Y}, F_{\text{ord}}) = f_1^*(D_{Y \to S}, F_{\text{ord}}) \otimes f_1^*D_Y (D_{X \to Y}, F_{\text{ord}}),\]
in \(\mathcal{C}_{D_{\text{for}},(f_2 \circ f_1)^*}D_{\text{fil},r}(X)\).

Proof. Follows immediately from definition. The first assertions of (i) and (ii) are particular cases of lemma 2. See [17] for example.

In the analytical case we also have

Lemma 5. Let \(f_1 : X \to Y, f_2 : Y \to S\) be two morphism with \(X,Y \in \text{AnSm}(\mathbb{C})\).

(i) We have \((D_{X \to S}, F_{\text{ord}}) = f_1^*(D_{Y \to S}, F_{\text{ord}}) \otimes f_1^*D_Y (D_{X \to Y}, F_{\text{ord}})\) in \(\mathcal{C}_{D_{\text{for}},(f_2 \circ f_1)^*}D_{\text{fil}}(X)\) and
\[(D_{X \to S}, F_{\text{ord}}) = f_1^*(D_{Y \to S}, F_{\text{ord}}) \otimes f_1^*D_Y (D_{X \to Y}, F_{\text{ord}}) = f_1^*(D_{Y \to S}, F_{\text{ord}}) \otimes f_1^*D_Y (D_{X \to Y}, F_{\text{ord}}),\]
in \(\mathcal{C}_{D_{\text{for}},(f_2 \circ f_1)^*}D_{\text{fil},r}(X)\).

(ii) We have \((D_{X \to S}, F_{\text{ord}}) = f_1^*(D_{Y \to S}, F_{\text{ord}}) \otimes f_1^*D_Y (D_{X \to Y}, F_{\text{ord}})\) in \(\mathcal{C}_{D_{\text{for}},(f_2 \circ f_1)^*}D_{\text{fil}}(X)\) and
\[(D_{X \to S}, F_{\text{ord}}) = f_1^*(D_{Y \to S}, F_{\text{ord}}) \otimes f_1^*D_Y (D_{X \to Y}, F_{\text{ord}}) = f_1^*(D_{Y \to S}, F_{\text{ord}}) \otimes f_1^*D_Y (D_{X \to Y}, F_{\text{ord}}),\]
in \(\mathcal{C}_{D_{\text{for}},(f_2 \circ f_1)^*}D_{\text{fil},r}(X)\).

Proof. Similar to the proof of lemma 4

For closed embeddings, we have:

Proposition 5.2. (i) Let \(i : Z \to S\) be a closed embedding with \(Z,S \in \text{SmVar}(\mathbb{C})\). Then, \(D_{Z \to S} = i^*D_S/D_SI_Z\) and it is a locally free (left) \(D_Z\) module. Similarly, \(D_{Z \to S} = i^*D_S/I_ZD_S\) and it is a locally free right \(D_Z\) module.

(ii) Let \(i : Z \to S\) be a closed embedding with \(Z,S \in \text{AnSm}(\mathbb{C})\). Then, \(D_{Z \to S} = i^*D_S/D_SI_Z\) and it is a locally free (left) \(D_Z\) module. Similarly, \(D_{Z \to S} = i^*D_S/I_PD_S\) and it is a locally free right \(D_Z\) module.

(iii) Let \(i : Z \to S\) be a closed embedding with \(Z,S \in \text{AnSm}(\mathbb{C})\). Then, \(D_{\text{ord}}^{\infty}_{Z \to S} = i^*D_S^{\infty}/D_SI_Z\) and it is a locally free (left) \(D_Z^{\infty}\) module. Similarly, \(D_{\text{ord}}^{\infty}_{Z \to S} = i^*D_S^{\infty}/I_PD_S^{\infty}\) and it is a locally free right \(D_Z^{\infty}\) module.

Proof. (i): See [17].
(ii): See [26].
(iii): Similar to (ii).

We now enumerate some functorialities we will use, all of them are particular case of the functoriality given in subsection 2.3 for any ringed spaces:

- Let \(f : X \to S\) be a morphism with \(X,S \in \text{Var}(\mathbb{C})\), or let \(f : X \to S\) be a morphism with \(X,S \in \text{AnSp}(\mathbb{C})\). Then, the inverse image functor
  \[f^{\ast\text{mod}} : \text{PSh}_{O_S}(S) \to \text{PSh}_{O_X}(X), M \mapsto f^{\ast\text{mod}}M := O_X \otimes f_{\ast}O_S f^{\ast}M\]
is a Quillen adjunction which induces in the derived category the functor
  \[Lf^{\ast\text{mod}} : D_{O_S}(S) \to D_{O_X}(X), M \mapsto Lf^{\ast\text{mod}}M := O_X \otimes f_{\ast}O_S f^{\ast}L_{O_M} = O_X \otimes f_{\ast}O_S f^{\ast}L_{O_M},\]
The adjunction \((f^{\ast\text{mod}}, f_{\ast}) : \text{PSh}_{O_S}(S) \leftrightarrows \text{PSh}_{O_X}(X)\) is a Quillen adjunction, the adjunction map are the maps
- for $M \in C_{O_S}(S)$, $\text{ad}(f^{*\text{mod}}, f_*)(M) : M \overset{\text{ad}(f^*, f_*)(M)}{\rightarrow} f_*(f^*M) \overset{f_*}{\rightarrow} f_*(f^*M \otimes f^*O_S O_X) = f_*f^{*\text{mod}}M$ where $m(m) = m \otimes 1$,

- for $M \in C_{O_X}(X)$, $\text{ad}(f^{*\text{mod}}, f_*)(M) : f^{*\text{mod}}f_*M = f^*f_*M \otimes f^*O_S O_X \overset{\text{ad}(f^*, f_*)(M) \otimes f^*O_S O_X}{\rightarrow} M \otimes f^*O_S O_X \overset{n}{\rightarrow} M$, where $n(m \otimes h) = h.m$ is the multiplication map.

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$.

- For $M \in C_D(S)$, we have the canonical projective resolution $q : L_D(M) \rightarrow M$ of complexes of $D_S$ modules.

- For $M \in C_{\mathcal{D}}(S)$, there exist a unique structure of $D_S$ module on the flasque presheaves $E^i(M)$ such that $E(M) \in C_D(S)$ (i.e. a complex of $D_S$ modules) and that the map $k : M \rightarrow E(M)$ is a morphism of complexes of $D_S$ modules.

Let $S \in \text{AnSm}(\mathbb{C})$.

- For $M \in C_{\mathcal{D}\mathcal{D}}(S)$, we have the canonical projective resolution $q : L_{D\mathcal{D}}(M) \rightarrow M$ of complexes of $D_S^\infty$ modules.

- For $M \in C_{\mathcal{D}\mathcal{D}}(S)$, there exist a unique structure of $D_S^\infty$ module on the flasque presheaves $E^i(M)$ such that $E(M) \in C_{\mathcal{D}\mathcal{D}}(S)$ (i.e. a complex of $D_S^\infty$ modules) and that the map $k : M \rightarrow E(M)$ is a morphism of complexes of $D_S^\infty$ modules.

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathcal{D}(op)}(S)$, $N \in C(S)$, we will consider the induced $D$ module structure (right $D_S$ module in the case one is a left $D_S$ module and the other one is a right one) on the presheaf $M \otimes N := M \otimes_{O_S} N$ (see section 2). We get the bifunctor

$$C(S) \times C_{\mathcal{D}}(S) \rightarrow C_{\mathcal{D}}(S), (M, N) \mapsto M \otimes N$$

For $S \in \text{AnSm}(\mathbb{C})$, we also have the bifunctor $C(S) \times C_{\mathcal{D}\mathcal{D}}(S) \rightarrow C_{\mathcal{D}\mathcal{D}}(S), (M, N) \mapsto M \otimes N$.

- Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. For $M, N \in C_{\mathcal{D}(op)}(S)$, $M \otimes_{O_S} N$ (see section 2), has a canonical structure of $D_S$ modules (right $D_S$ module in the case one is a left $D_S$ module and the other one is a right one) given by (in the left case) for $S^o \subset S$ an open subset,

$$m \otimes n \in \Gamma(S^o, M \otimes_{O_S} N), \gamma \in \Gamma(S^o, D_S), \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes \gamma.n$$

This gives the bifunctor

$$C_{\mathcal{D}(op)}(S)^2 \rightarrow C_{\mathcal{D}(op)}(S), (M, N) \mapsto M \otimes_{O_S} N$$

More generally, let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$. Assume $S$ smooth. For $M, N \in C_{f^{\ast}, \mathcal{D}(op)}(X)$, $M \otimes_{f^{\ast}O_S} N$ (see section 2), has a canonical structure of $f^{\ast}D_S$ modules (right $f^{\ast}D_S$ module in the case one is a left $f^{\ast}D_S$ module and the other one is a right one) given by (in the left case) for $X^o \subset X$ an open subset,

$$m \otimes n \in \Gamma(X^o, M \otimes_{f^{\ast}O_S} N), \gamma \in \Gamma(X^o, f^{\ast}D_S), \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes \gamma.n$$

This gives the bifunctor

$$C_{f^{\ast}, \mathcal{D}(op)}(X)^2 \rightarrow C_{f^{\ast}, \mathcal{D}(op)}(X), (M, N) \mapsto M \otimes_{f^{\ast}O_S} N$$

For $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSp}(\mathbb{C})$ and $S$ smooth, we also have the bifunctor $C_{f^{\ast}, \mathcal{D}\mathcal{D}(op)}(X)^2 \rightarrow C_{f^{\ast}, \mathcal{D}\mathcal{D}(op)}(X), (M, N) \mapsto M \otimes_{f^{\ast}O_S} N$.  

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• Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{D^{op}}(S)$ and $N \in C_D(S)$, we have $M \otimes_{D_S} N \in C(S)$ (see section 2). This gives the bifunctor
\[ C_{D^{op}}(S) \times C_D(S) \to C(S), (M, N) \mapsto M \otimes_{D_S} N \]
For $S \in \text{AnSm}(\mathbb{C})$, we also have the bifunctor $C_{D^{op}}(S) \times C_D(S) \to C(S), (M, N) \mapsto M \otimes_{D_S^\op} N$.

• Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. The internal hom bifunctor
\[ \mathcal{H}om(\cdot, \cdot) := \mathcal{H}om_{D_S}(\cdot, \cdot) : C(S)^2 \to C(S) \]
induces a bifunctor
\[ \mathcal{H}om(\cdot, \cdot) := \mathcal{H}om_{D_S}(\cdot, \cdot) : C(S) \times C_D(S) \to C_D(S) \]
such that, for $F \in C(S)$ and $G \in C_D(S)$, the $D_S$ structure on $\mathcal{H}om^*(F,G)$ is given by
\[ \gamma \in \Gamma(S^\circ, D_S) \mapsto (\phi) \in \mathcal{H}om^*(F_{|S^\circ}, G_{|S^\circ}) \mapsto (\gamma \cdot \phi : \alpha \in F^*(S^\circ) \mapsto \gamma \cdot \phi^*(S^\circ)(\alpha)) \]
where $\phi^*(S^\circ)(\alpha) \in \Gamma(S^\circ, G)$. For $S \in \text{AnSm}(\mathbb{C})$, it also induces the bifunctor
\[ \mathcal{H}om(\cdot, \cdot) := \mathcal{H}om_{D_S}(\cdot, \cdot) : C(S) \times C_{D^\op}(S) \to C_{D^\op}(S) \]

• Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$. For $M, N \in C_D(S)$, $\mathcal{H}om_{O_S}(M, N)$, has a canonical structure of $D_S$ modules given by $S^\circ \subset S$ an open subset and $\phi \in \Gamma(S^\circ, \mathcal{H}om(M, O_S))$, $\gamma \in \Gamma(S^\circ, D_S)$, $(\gamma, \phi)(m) := \gamma(\phi(m)) - \phi(\gamma, m)$ This gives the bifunctor
\[ \mathcal{H}om^*_{O_S}(-, -) : C_D(S)^2 \to C(D(S)^{op}, (M, N) \mapsto \mathcal{H}om^*_{O_S}(M, N) \]
In particular, for $M \in C_D(S)$, we get the dual
\[ D^O_S(M) := \mathcal{H}om^*_{O_S}(M, O_S) \in C_D(S) \]
with respect to $O_S$, together with the canonical map $d(M) : M \to D^O_S(M)$. Let $f : X \to S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. We have, for $M \in C_D(S)$, the canonical transformation map
\[ T(f, D^o)(M) : f^* \mathcal{H}om^*_{O_S}(M, O_S) \otimes_{f^* O_S} O_X \]
\[ T_{\mathcal{H}om}(f, O_S)(M) \to \mathcal{H}om_{O_X}(f^* M \otimes_{O_S} O_X, O_X) =: D^O_X(f^* M) \]

• Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSm}(\mathbb{C})$, we have the bifunctors
\[ \mathcal{H}om_{D_S}(\cdot, \cdot) : C_D(S)^2 \to C(S), (M, N) \mapsto \mathcal{H}om_{D_S}(M, N) \]
if $N$ is a bimodule (i.e. has a right $D_S$ module structure whose opposite coincide with the left one), $\mathcal{H}om_{D_S}(M, N) \in C_{D^{op}}(S)$ given by $S^\circ \subset S$ an open subset and $\phi \in \Gamma(S^\circ, \mathcal{H}om(M, N))$, $\gamma \in \Gamma(S^\circ, D_S)$, $(\phi, \gamma)(m) := (\phi(m)) \gamma$

\[ \mathcal{H}om_{D_S}(\cdot, \cdot) : C_D(S)^2 \to C(S), (M, N) \mapsto \mathcal{H}om_{D_S}(M, N) \]
and if $N$ is a bimodule, $\mathcal{H}om_{D_S}(M, N) \in C_D(S)$

For $M \in C_D(S)$, we get in particular the dual with respect $D_S$,
\[ D_S := \mathcal{H}om_{D_S}(M, D_S) \in C_D(S) ; D^S_S(M) := \mathcal{H}om_{D_S}(M, D_S) \otimes_{O_S} D^O_S w(K_S)[d_S] \in C_D(S) \]
and we have canonical map $d : M \to D^O_S M$. This functor induces in the derived category, for $M \in D_D(S)$,
\[ \mathcal{L}^O_S M := R \mathcal{H}om_{D_S}(L_D M, D_S) \otimes_{O_S} D^O_S w(K_S)[d_S] = D^O_S L_D M \in D_D(S) \]
where $D^O_S w(S) : D^O_S w(K_S) \to D^O_S K_S = K_S^{-1}$ is the dual of the Koczul resolution of the canonical bundle (proposition 72), and the canonical map $d : M \to \mathcal{L}^O_S M$. For $S \in \text{AnSm}(\mathbb{C})$, we also have the bifunctors
For $M \in C_D(S)$, we get in particular the dual with respect $D_S^∞$, 

$$D_S^∞ M := \mathcal{H}om_{D_S}^∞ (M, D_S^∞) \in C_D(S), \text{ and } D_S^∞ K := \mathcal{H}om_{D_S}^∞ (M, D_S^∞) \otimes_{O_S} D_S^∞ w(K_S)[d_S] \in C_D(S)$$

and we have canonical maps $d : M \rightarrow D_S^{∞,2}M$, $d : M \rightarrow D_S^{∞,2}M$. This functor induces in the derived category, for $M \in D_D^∞(S)$,

$$L D_S^∞ M := R \mathcal{H}om_{D_S}^∞ (M, D_S^∞) \otimes_{O_S} D_S^∞ w(K_S)[d_S] = D_S^{∞,K} L D_D^∞ M \in D_D^∞(S).$$

and the canonical map $d : M \rightarrow L D_S^{∞,2}M$.

- **Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$ or with $X, S \in \text{AnSm}(\mathbb{C})$. For $N \in C_D^∞(X)$ and $M \in C_D(X)$, $N \otimes_{DX} M$ has the canonical $f^* D_S$ module structure given by, for $X^o \subset X$ an open subset, 

$$\gamma \in \Gamma(X^o, f^* D_S), m \in \Gamma(X^o, M), n \in \Gamma(X^o, N), \gamma.(n \otimes m) = (\gamma.n) \otimes m.$$ 

This gives the functor 

$$C_D, f^* D(X) \times C_D(X) \rightarrow C_f, D(X), (M, N) \mapsto M \otimes_{DX} N$$

- **Let $f : X \rightarrow S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $N \in C_D^∞, f^* D(S)$ and $M \in C_D(S)$, $N \otimes_{DX} M$ has the canonical $f^* D_S^∞$ module structure given by, for $X^o \subset X$ an open subset, 

$$\gamma \in \Gamma(X^o, f^* D_S), m \in \Gamma(X^o, M), n \in \Gamma(X^o, N), \gamma.(n \otimes m) = (\gamma.n) \otimes m.$$ 

This gives the functor 

$$C_D^∞, f^* D^∞(X) \times C_D^∞(X) \rightarrow C_f^∞, D^∞(X), (M, N) \mapsto M \otimes_{DX} N$$

- **Let $f : X \rightarrow S$ a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \rightarrow S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, for $M \in C_D(S)$, $O_X \otimes_{f^* O_S} f^* M$ has a canonical $D_X$ module structure given by, for $X^o \subset X$ an open subset, 

$$m \otimes n \in \Gamma(X^o, O_X \otimes_{f^* O_S} f^* M), \gamma \in \Gamma(X^o, D_X), \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes df(\gamma).n.$$ 

This gives the inverse image functor  

$$f^{*\text{mod}} : \text{PSh}_D(S) \rightarrow \text{PSh}_D(X), \text{ and } M \mapsto f^{*\text{mod}} M := O_X \otimes_{f^* O_S} f^* M = D_X \rightarrow S \otimes_{f^* D_S} f^* M$$

which induces in the derived category the functor 

$$L f^{*\text{mod}} : D_D(S) \rightarrow D_D(X), \text{ and } M \mapsto L f^{*\text{mod}} M := O_X \otimes_{f^* O_S} f^* M = O_X \otimes_{f^* O_S} f^* L_D M,$$

We will also consider the shifted inverse image functor 

$$L f^{*\text{mod},[-]} := L f^{*\text{mod}}[d_S - d_X] : D_D(S) \rightarrow D_D(X).$$
Let $f : X \to S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, for $M \in C_{D^\infty}(S)$, $O_X \otimes_{f^* O_S} f^* M$ has a canonical $D_X^\infty$ module structure induced by the finite order case. This gives the inverse image functor

$$f^{\ast \text{mod}} : \text{PSh}_{D^\infty}(S) \to \text{PSh}_{D^\infty}(X), \quad M \mapsto f^{\ast \text{mod}} M := O_X \otimes_{f^* O_S} f^* M = D_{X \to S} \otimes_{f^* D_S^\infty} f^* M$$

which induces in the derived category the functor

$$L f^{\ast \text{mod}} : D_{D^\infty}(S) \to D_{D^\infty}(X), \quad M \mapsto L f^{\ast \text{mod}} M := O_X \otimes_{f^* O_S} f^* M = O_X \otimes_{f^* O_S} f^* L D^\infty_M,$$

We will also consider the shifted inverse image functor

$$L f^{\ast \text{mod}}[-] := L f^{\ast \text{mod}}[d_S - d_X] : D_{D^\infty}(S) \to D_{D^\infty}(X).$$

Let $f : X \to S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{D^\infty}(X)$, $D_{X \leftarrow S} \otimes_{D_X} M$ has the canonical $f^* D_S$ module structure given above. Then, the direct image functor

$$f^\ast_{\text{mod}} : \text{PSh}_{D^\infty}(X) \to \text{PSh}_{D^\infty}(S), \quad M \mapsto f^\ast_{\text{mod}} M := f_*(D_{X \leftarrow S} \otimes_{D_X} M)$$

induces in the derived category the functor

$$\int_f = R f^\ast_{\text{mod}} : D_{D}(X) \to D_{D}(S), \quad M \mapsto \int_f M = R f_*(D_{X \leftarrow S} \otimes_{D_X}^L M).$$

Let $f : X \to S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{D^\infty}(X)$, $D_{X \leftarrow S} \otimes_{D_X} M$ has the canonical $f^* D_S$ module structure given above. Then, the direct image functor

$$f^0_{\ast \text{mod}} : \text{PSh}_{D^\infty}(X) \to \text{PSh}_{D^\infty}(S), \quad M \mapsto f^0_{\ast \text{mod}} M := f_*(D_{X \leftarrow S}^\infty \otimes_{D_X}^\infty M)$$

induces in the derived category the functor

$$\int_{f^0} = R f^0_{\ast \text{mod}} : D_{D}(X) \to D_{D}(S), \quad M \mapsto \int_{f^0} M = R f_*(D_{X \leftarrow S}^\infty \otimes_{D_X}^L M).$$

Let $f : X \to S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. The direct image functor with compact support

$$f^0_{\ast \text{mod}} : \text{PSh}_{D^\infty}(X) \to \text{PSh}_{D^\infty}(S), \quad M \mapsto f^0_{\ast \text{mod}} M := f_!(D_{S \leftarrow X} \otimes_{D_X} M)$$

induces in the derived category the functor

$$\int_{f^0} = R f^0_{\ast \text{mod}} : D_{D}(X) \to D_{D}(S), \quad M \mapsto \int_{f^0} M = R f_!(D_{S \leftarrow X} \otimes_{D_X} M).$$

Let $S \in \text{SmVar}(\mathbb{C})$. The analytical functor of a $D_S$ modules has a canonical structure of $D_{S^{an}}$ module:

$$(-)^{an} : C_{D}(S) \to C_{D}(S^{an}), \quad M \mapsto M^{an} := \text{an}_{S}^{\ast \text{mod}} M := M \otimes_{\text{an}_{S}^{\ast}} O_S O_{S^{an}}$$

which induces in the derived category

$$(-)^{an} : D_{D}(S) \to D_{D}(S^{an}), \quad M \mapsto M^{an} := \text{an}_{S}^{\ast \text{mod}} M$$

since $\text{an}_{S}^{\ast \text{mod}}$ derive trivially.
The functorialities given above induce:

- Let $f : X \to S$ be a morphism with $X, S \in \text{Var}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \text{AnSp}(\mathbb{C})$. The adjonction map induces:
  - for $(M, F) \in C_{O_S \text{fil}}(S)$, the map in $D_{O_S \text{fil}}(S)$
    \[ \text{ad}(Lf^\text{smod}, Rf_*)(M, F) : (M, F) \xrightarrow{\text{ad}(f^\text{smod}, Rf_*)(M, F)} f_*E(f^*(M, F)) = Rf_*f^*(M, F) \]
    \[ f^m \mapsto Rf_*(f^*(M, F) \otimes^{L}_{f^*O_S} O_X) = Rf_*f^*(M, F), \]
    where $m(m) = m \otimes 1$,
  - for $(M, F) \in C_{O_X \text{fil}}(X)$, the map in $D_{O_X \text{fil}}(X)$
    \[ \text{ad}(Lf^\text{smod}, Rf_*)(M, F) : Lf^\text{smod} Rf_*(M, F) = f^*f_*E(M, F) \otimes^{L}_{f^*O_X} O_X \]
    \[ \xrightarrow{\text{ad}(f^\text{smod}, Rf_*)(E(M, F)) \otimes^{L}_{f^*O_X} O_X} (M, F) \otimes^{L}_{f^*O_X} O_X \xrightarrow{n} (M, F), \]
    where $n(m \otimes h) = h \cdot m$ is the multiplication map.

- For a commutative diagram in $\text{Var}(\mathbb{C})$ or in $\text{AnSp}(\mathbb{C})$:
  \[
  D = \begin{array}{c}
  Y \\
  \xrightarrow{g_2} X \\
  \xrightarrow{f_2} f_1
  \end{array}
  \]
  \[
  T \xrightarrow{g_1} S
  \]
  we have, for $(M, F) \in C_{O_X \text{fil}}(X)$, the canonical map in $D_{O_X \text{fil}}(T)$
  \[
  T^\text{mod}(D)(M, F) : Lg_1^\text{smod} f_1*(M, F) \xrightarrow{\text{ad}(Lf_2^\text{smod}, Rf_2)(Lg_1^\text{smod} f_1(E(M, F))} Rf_2*Lf_2^\text{smod} Lg_1^\text{smod} Rf_1*(M, F) = Rf_2*Lf_2^\text{smod} Lg_1^\text{smod} Rf_1*(M, F)
  \]
  \[ \xrightarrow{\text{ad}(Lf_2^\text{smod}, Rf_2)(M, F)} Rf_2*Lf_2^\text{smod}(M, F) \]
  the canonical transformation map given by the adjonction maps.

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\text{fil}}(S)$ and $(N, F) \in C_{\text{fil}}(S)$, recall that (see section 2)
  \[ F^p((M, F) \otimes (N, F)) := \text{Im}(\oplus_q F^q M \otimes F^{p-q} N \to M \otimes N) \]

  This gives the functor
  \[ (\cdot, \cdot) : C_{\text{fil}}(S) \times C_{\text{Dfil}}(S) \to C_{\text{Dfil}}(S), ((M, F), (N, F)) \mapsto (M, F) \otimes (N, F). \]

  It induces in the derived categories by taking r-projective resolutions the bifunctors, for $r = 1, \ldots, \infty$,
  \[ (\cdot, \cdot) : D_{\text{Dfil}, r}(S) \times D_{\text{Dfil}, r}(S) \to D_{\text{Dfil}, r}(S), ((M, F), (N, F)) \mapsto (M, F) \otimes^R(N, F) = L_D(M, F) \otimes(N, F). \]

  For $S \in \text{AnSm}(\mathbb{C})$, it gives the bifunctor
  \[ (\cdot, \cdot) : C_{\text{fil}}(S) \times C_{\text{Dfil}}(S) \to C_{\text{Dfil}}(S), ((M, F), (N, F)) \mapsto (M, F) \otimes (N, F), \]
  and its derived functor.
Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSp}(\mathbb{C})$ and $\mathcal{O}_S \in \text{PSh}(\mathbb{C})_c$ or let $\mathcal{O}_S \in \text{PSh}(\mathbb{C})_c$ sheaf of commutative ring. For $(M, F) \in C_{\mathcal{O}_S, \text{fil}}(S)$ and $(N, F) \in C_{\mathcal{O}_S, \text{fil}}(S)$, recall that (see section 2)

$$F^p((M, F) \otimes \mathcal{O}_S(N, F)) := \text{Im}(\otimes_q F^q M \otimes \mathcal{O}_S^{F^q} N \rightarrow M \otimes \mathcal{O}_S N)$$

It induces in the derived categories by taking r-projective resolutions the bifunctors, for $r = 1, \ldots, \infty$,

$$\langle \cdot, \cdot \rangle : D_{\mathcal{D}_{\text{fil}, r}}(S)^2 \rightarrow D_{\mathcal{D}_{\text{fil}, r}}(S), (\langle M, F \rangle, \langle N, F \rangle) \mapsto (M, F) \otimes_{\mathcal{D}_{\text{fil}}(S)}^L (N, F).$$

More generally, let $f : X \rightarrow S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$. Assume $S$ smooth. We have the bifunctors

$$\langle \cdot, \cdot \rangle : D_{f_{\mathcal{D}_{\text{fil}, r}}(X)^2} \rightarrow D_{f_{\mathcal{D}_{\text{fil}, r}}(X)}, (\langle M, F \rangle, \langle N, F \rangle) \mapsto (M, F) \otimes_{f_{\mathcal{D}_{\text{fil}}}(S)}^L (N, F)).$$

Note that the filtration given by $W$ satisfy that the $W^p$ are $D_S$ submodule which is stronger than Griffiths transversality.

Let $S \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSp}(\mathbb{C})$. The hom functor induces the bifunctor

$$\text{Hom}(\langle \cdot, \cdot \rangle) : C_{\mathcal{D}_{\text{fil}}}(S) \times C_{\mathcal{D}_{\text{fil}}}(S) \rightarrow C_{D_{\text{fil}, (1, 0)}}(S), ((M, W), (N, F)) \mapsto \text{Hom}(\langle M, W \rangle, \langle N, F \rangle).$$

For $S \in \text{AnSp}(\mathbb{C})$, the hom functor also induces the bifunctor

$$\text{Hom}(\langle \cdot, \cdot \rangle) : C_{D^{(1, 0)}}(S) \times C_{D^{(1, 0)}}(S) \rightarrow C_{D^{(1, 0)}}(S), ((M, W), (N, F)) \mapsto \text{Hom}(\langle M, W \rangle, \langle N, F \rangle).$$

We get the filtered dual

$$\mathbb{D}_S^K(\cdot) : C_{D_{(2, 1)}}(S) \rightarrow C_{D_{(2, 1)}}(S)^{op}, (M, F) \mapsto \mathbb{D}_S^K(M, F) := \text{Hom}(M, F, D_S) \otimes \mathcal{O}_S \mathbb{D}_S^K w(K_S)[d_S]$$

together with the canonical map $d(M, F) : (M, F) \rightarrow \mathbb{D}_S^K(M, F)$. Of course $\mathbb{D}_S^K(\cdot)(C_{D_{(1, 0)}}(S)) \subset C_{D_{(1, 0)}}(S)$. It induces in the derived categories $D_{\mathcal{D}_{\text{fil}, r}}(S)$, for $r = 1, \ldots, \infty$, the functors

$$L\mathbb{D}_S(\cdot) : D_{D_{(2, 1), 0}}(S) \rightarrow D_{D_{(2, 1), 0}}(S)^{op}, (M, F) \mapsto L\mathbb{D}_S(M, F) := \mathbb{D}_S^K L_D(M, F).$$

together with the canonical map $d(M, F) : L_D(M, F) \rightarrow \mathbb{D}_S^K L_D(M, F)$.

Let $S \in \text{AnSp}(\mathbb{C})$. The hom functor also induces the bifunctors

$$\text{Hom}(\langle \cdot, \cdot \rangle) : C_{D^{(1, 0)}}(S) \times C_{D^{(1, 0)}}(S) \rightarrow C_{D^{(1, 0)}}(S), ((M, W), (N, F)) \mapsto \text{Hom}(\langle M, W \rangle, \langle N, F \rangle).$$

We get the filtered dual

$$\mathbb{D}_S^{(1, 0), K}(\cdot) : C_{D^{(2, 1)}}(S) \rightarrow C_{D^{(2, 1)}}(S)^{op}, (M, F) \mapsto \mathbb{D}_S^{(1, 0), K}(M, F) := \text{Hom}(M, F, D_S^{(1, 0)}) \otimes \mathcal{O}_S \mathbb{D}_S^{(1, 0)} w(K_S)[d_S]$$

together with the canonical map $d(M, F) : (M, F) \rightarrow \mathbb{D}_S^{(1, 0), K}(M, F)$. Of course $\mathbb{D}_S^{(1, 0), K}(\cdot)(C_{D^{(1, 0)}}(S)) \subset C_{D^{(1, 0)}}(S)$. It induces in the derived categories $D_{\mathcal{D}_{\text{fil}, r}}(S)$, for $r = 1, \ldots, \infty$, the functors

$$L\mathbb{D}_S^{(1, 0)}(\cdot) : D_{D^{(2, 1), 0}}(S) \rightarrow D_{D^{(2, 1), 0}}(S)^{op}, (M, F) \mapsto L\mathbb{D}_S^{(1, 0)}(M, F) = \mathbb{D}_S^{(1, 0), K} L_D^{(1, 0)}(M, F).$$

together with the canonical map $d(M, F) : (M, F) \rightarrow L\mathbb{D}_S^{(1, 0), K}(M, F)$.
• Let \( f : X \to S \) be a morphism with \( X, S \in \text{SmVar}(\mathbb{C}) \), or let \( f : X \to S \) be a morphism with \( X, S \in \text{AnSm}(\mathbb{C}) \). Then, the inverse image functor
\[
\begin{align*}
f^{*}\text{mod} : & \ C_{D(2)fil}(S) \to C_{D(2)fil}(X), \\
(M, F) \mapsto & f^{*}\text{mod}(M, F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M, F) = (D_X \to S, F^{ord}_D) \otimes_{f^*D_S} f^*(M, F),
\end{align*}
\]
induces in the derived categories the functors, for \( r = 1, \ldots, \infty \) (resp. \( r \in (1, \ldots, \infty)^2 \)),
\[
\begin{align*}
L(f)^{*}\text{mod} : & \ D_{D(2)fil,r}(S) \to D_{D(2)fil,r}(X), \\
(M, F) \mapsto & Lf^{*}\text{mod}M := (O_X, F_b) \otimes_{f^*O_S} f^*(M, F) = (O_X, F_b) \otimes_{f^*O_S} f^*L_D(M, F).
\end{align*}
\]
Of course \( f^{*}\text{mod}(C_{D(1,0)fil}(S)) \subseteq C_{D(1,0)fil}(X) \). Note that
- If the \( M \) is a complex of locally free \( O_S \) modules, then \( Lf^{*}\text{mod}(M, F) = f^{*}\text{mod}(M, F) \) in \( D_{D(2)fil,\infty}(S) \).
- If the \( Gr^p_f M \) are complexes of locally free \( O_S \) modules, then \( Lf^{*}\text{mod}(M, F) = f^{*}\text{mod}(M, F) \) in \( D_{D(2)fil}(S) \).

We will consider also the shifted inverse image functors
\[
L f^{*}\text{mod}|_{[-]} := L f^{*}\text{mod}[d_S - d_X] : D_{D(2)fil,r}(S) \to D_{D(2)fil,r}(X).
\]

• Let \( f : X \to S \) be a morphism with \( X, S \in \text{SmVar}(\mathbb{C}) \). Then, the inverse image functor
\[
\begin{align*}
f^{*}\text{mod} : & \ C_{D(\infty)fil}(S) \to C_{D(\infty)fil}(X), \\
(M, F) \mapsto & f^{*}\text{mod}(M, F) := (O_X, F_b) \otimes_{f^*O_S} f^*(M, F),
\end{align*}
\]
induces in the derived categories the functors, for \( r = 1, \ldots, \infty \) (resp. \( r \in (1, \ldots, \infty)^2 \)),
\[
\begin{align*}
L(f)^{*}\text{mod} : & \ D_{D(\infty)fil,r}(S) \to D_{D(\infty)fil,r}(X), \\
(M, F) \mapsto & Lf^{*}\text{mod}M := (O_X, F_b) \otimes_{f^*O_S} f^*(M, F) = (O_X, F_b) \otimes_{f^*O_S} f^*L_D(M, F).
\end{align*}
\]
Of course \( f^{*}\text{mod}(C_{D(\infty,1,0)fil}(S)) \subseteq C_{D(\infty,1,0)fil}(X) \). Note that We will consider also the shifted inverse image functors
\[
L f^{*}\text{mod}|_{[-]} := L f^{*}\text{mod}[d_S - d_X] : D_{D(\infty,2)fil,r}(S) \to D_{D(\infty,2)fil,r}(X).
\]

• Let \( f : X \to S \) be a morphism with \( X, S \in \text{SmVar}(\mathbb{C}) \), or let \( f : X \to S \) be a morphism with \( X, S \in \text{AnSm}(\mathbb{C}) \). Then, the direct image functor
\[
\begin{align*}
f_{00}^{*}\text{mod} : & \ (PSh_{D}(X), F) \to (PSh_{D}(S), F), \\
(M, F) \mapsto & f_{00}^{*}\text{mod}(M, F) := f_*((D_{S-X}, F^{ord}_D) \otimes_{D_X} (M, F))
\end{align*}
\]

induces in the derived categories by taking \( r \)-injective resolutions the functors, for \( r = 1, \ldots, \infty \),
\[
\begin{align*}
\int f & = Rf^{*}\text{mod} : D_{D(2)fil,r}(X) \to D_{D(2)fil,r}(S), \\
(M, F) \mapsto & \int f(M, F) = Rf_*((D_{S-X}, F^{ord}_D) \otimes_{D_X} (M, F)).
\end{align*}
\]
Let \( f_1 : X \to Y \) and \( f_2 : Y \to S \) two morphism with \( X, Y, S \in \text{SmVar}(\mathbb{C}) \) or with \( X, Y, S \in \text{AnSm}(\mathbb{C}) \). We have, for \( (M, F) \in C_{Dfil}(X) \), the canonical transformation map in \( D_{D(2)fil,r}(S) \)
\[
T(\int f_2 \circ f_1, \int f_2 \circ f_1)(M, F) :
\]
\[
\begin{align*}
\int f_2 & \int f_1(M, F) := Rf_2_*((D_{Y-S}, F^{ord}_D) \otimes_{D_Y} Rf_1_*((D_{X-Y}, F^{ord}_D) \otimes_{D_X} (M, F))) \\
& \overset{T(f_1, \otimes)(-,-)}{\approx} Rf_2_*Rf_1_*((f_1^*(D_{Y-S}, F^{ord}_D) \otimes_{D_Y} (D_{X-Y}, F^{ord}_D) \otimes_{D_X} (M, F))) \\
& \overset{\approx}{\approx} Rf_2_*Rf_1_*((D_{X-S}, F^{ord}_D) \otimes_{D_X} (M, F)) := \int f_2 \circ f_1(M, F)
\end{align*}
\]
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• Let $f : X \to S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the direct image functor

$$f_{*mod}^{00} : (\text{PSh}_D^\infty(X), F) \to (\text{PSh}_D^\infty(S), F), \ (M, F) \mapsto f_{*mod}(M, F) := f_*((D^\infty_{S^r-X}, F^{ord}) \otimes_{D^\infty_X} (M, F))$$

induces in the derived categories by taking $r$-injective resolutions the functors, for $r = 1, \ldots, \infty$,

$$\int_f = Rf_{*mod} : D^\infty_{D^\infty fil}(X) \to D^\infty_{D^\infty fil}(S), (M, F) \mapsto \int_f (M, F) = Rf_*((D^\infty_{S^r-X}, F^{ord}) \otimes_{D^\infty_X} (M, F)).$$

We have, similarly, for $(M, F) \in C^\infty_{D^\infty fil}(X)$, the canonical transformation map in $D^\infty_{D^\infty fil}(S)$

$$T((\int_{f_2} \circ \int_{f_1} \cdot \int_{f_2 \circ f_1})(M, F)) : \int_{f_2} \int_{f_1} (M, F) \to \int_{f_2 \circ f_1} (M, F)$$

• Let $f : X \to S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, the direct image functor with compact support

$$f_{!mod}^{00} : (\text{PSh}_D^\infty(X), F) \to (\text{PSh}_D^\infty(S), F), \ (M, F) \mapsto f_{!mod}(M, F) := f_!(((D^\infty_{S^r-X}, F^{ord}) \otimes_{D^\infty_X} (M, F))$$

induces in the derived categories by taking $r$-injective resolutions the functors, for $r = 1, \ldots, \infty$,

$$\int_f = Rf_{!mod} : D^\infty_{D^\infty fil}(X) \to D^\infty_{D^\infty fil}(S), (M, F) \mapsto \int_f (M, F) = Rf_!(((D^\infty_{S^r-X}, F^{ord}) \otimes_{D^\infty_X} (M, F)).$$

We have, similarly, for $(M, F) \in C^\infty_{D^\infty fil}(X)$, the canonical transformation map in $D^\infty_{D^\infty fil}(S)$

$$T((\int_{f_2} \circ \int_{f_1} \cdot \int_{(f_2 \circ f_1)})(M, F)) : \int_{f_2} \int_{f_1} (M, F) \to \int_{f_2 \circ f_1} (M, F)$$

• Let $S \in \text{SmVar}(\mathbb{C})$. The analytical functor for filtered $D_S$-modules is

$$(\cdot)^{an} : C^\infty_{D^\infty fil}(S) \to C^\infty_{D^\infty fil}(S^{an}), (M, F) \mapsto (M, F)^{an} := (\otimes_{\text{an}}^S)_{O_S^{an}} (O_S^{an}, F_b).$$

It induces in the derived categories the functors, for $r = 1, \ldots, \infty$,

$$(\cdot)^{an} : D^\infty_{D^\infty fil}(S) \to D^\infty_{D^\infty fil}(S^{an}), (M, F) \mapsto (M, F)^{an} := (\otimes_{\text{an}}^S)_{O_S^{an}} (O_S^{an}, F_b).$$

• Let $f : X \to S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then the functor

$$f^{*mod} : C^\infty_{D^\infty fil}(S) \to C^\infty_{D^\infty fil}(X), (M, F) \mapsto f^{*mod}(M, F) := \mathbb{D}_X^K L_D f^{*mod} L_D \mathbb{D}_S^K (M, F)$$
induces in the derived categories the exceptional inverse image functors, for \( r = 1, \ldots, \infty \) (resp. \( r \in (1, \ldots \infty)^2 \)),
\[
Lf^{\ast, \text{mod}} : D_{D(2)}(S) \to D_{D(2)}(X),
\]
\[
(M, F) \mapsto Lf^{\ast, \text{mod}}(M, F) := L\mathcal{D}_X Lf^{\ast, \text{mod}} L\mathcal{D}_S(M, F) := f^{\ast, \text{mod}} L_D(M, F).
\]

Of course \( f^{\ast, \text{mod}}(C_{D(1,0)}(S)) \subset C_{D(1,0)}(X) \). We will also consider the shifted exceptional inverse image functors
\[
Lf^{\ast, \text{mod}}[-] := Lf^{\ast, \text{mod}}[d_S - d_X] : D_{D(2)}(S) \to D_{D(2)}(X).
\]

• Let \( f : X \to S \) be a morphism with \( X, S \in \text{AnSm}(\mathbb{C}) \). Then the functor
\[
f^{\ast, \text{mod}} : C_{D^{\infty}(2)}(S) \to C_{D^{\infty}(2)}(X), (M, F) \mapsto f^{\ast, \text{mod}}(M, F) := D^{\infty}_X Lf^{\ast, \text{mod}} L_D^{\infty}(M, F)
\]

induces in the derived categories the exceptional inverse image functors, for \( r = 1, \ldots, \infty \) (resp. \( r \in (1, \ldots \infty)^2 \)),
\[
L_f^{\ast, \text{mod}}(S) : D_{D^{\infty}(2)}(S) \to D_{D^{\infty}(2)}(X),
\]
\[
(M, F) \mapsto Lf^{\ast, \text{mod}}(M, F) := L\mathcal{D}_X Lf^{\ast, \text{mod}} L\mathcal{D}_S^{\infty}(M, F) := f^{\ast, \text{mod}}(M, F).
\]

Of course \( f^{\ast, \text{mod}}(C_{D^{\infty}(1,0)}(S)) \subset C_{D^{\infty}(1,0)}(X) \). We will also consider the shifted exceptional inverse image functors
\[
Lf^{\ast, \text{mod}}[-] := Lf^{\ast, \text{mod}}[d_S - d_X] : D_{D^{\infty}(2)}(S) \to D_{D^{\infty}(2)}(X).
\]

• Let \( S_1, S_2 \in \text{SmVar}(\mathbb{C}) \) or \( S_1, S_2 \in \text{AnSm}(\mathbb{C}) \). Consider \( p : S_1 \times S_2 \to S_1 \) the projection. Since \( p \) is a projection, we have a canonical embedding \( p^* D_S \hookrightarrow D_{S_1 \times S_2} \). For \( (M, F) \in C_{D(2)}(S_1 \times S_2) \), \( (M, F) \) has a canonical \( p^* D_S \) module structure. Moreover, with this structure, for \( (M_1, F) \in C_{D(2)}(S_1) \)
\[
\text{ad}(p^{\ast, \text{mod}}, p)(M_1, F) : (M_1, F) \to p_* p^{\ast, \text{mod}}(M_1, F)
\]
is a map of complexes of \( D_{S_1} \) modules, and for \( (M_1, F) \in C_{D(2)}(S_1 \times S_2) \)
\[
\text{ad}(p^{\ast, \text{mod}}, p)(M_1, F) : p^{\ast, \text{mod}} p_* (M_1, F) \to (M_1, F)
\]
is a map of complexes of \( D_{S_1 \times S_2} \) modules. Indeed, for the first adjunction map, we note that \( p^{\ast, \text{mod}}(M_1, F) \) has a structure of \( p^* D_S \) module, hence \( p^* D_S \) has a structure of \( p^* D_S \) module, hence a structure of \( D_S \) module using the adjunction map \( \text{ad}(p^*, p_*)(D_S) : D_S \to p_* p^* D_S \). For the second adjunction map, we note that \( (M_1, F) \) has a structure of \( p^* D_S \) module, hence \( p_*(M_1, F) \) has a structure of \( p_* p^* D_S \) module, hence a structure of \( D_S \) module using the adjunction map \( \text{ad}(p^*, p_*)(D_S) : D_S \to p_* p^* D_S \).

• Let \( S_1, S_2 \in \text{AnSm}(\mathbb{C}) \). Consider \( p : S_1 \times S_2 \to S_1 \) the projection. Since \( p \) is a projection, we have a canonical embedding \( p^* D_S^{\infty} \hookrightarrow D_{S_1 \times S_2}^{\infty} \). For \( (M, F) \in C_{D^{\infty}(2)}(S_1 \times S_2) \), \( (M, F) \) has a canonical \( p^* D_S^{\infty} \) module structure. Moreover, with this structure, for \( (M_1, F) \in C_{D^{\infty}(2)}(S_1) \)
\[
\text{ad}(p^{\ast, \text{mod}}, p)(M_1, F) : (M_1, F) \to p_* p^{\ast, \text{mod}}(M_1, F)
\]
is a map of complexes of \( D_{S_1}^{\infty} \) modules, and for \( (M_1, F) \in C_{D^{\infty}(2)}(S_1 \times S_2) \)
\[
\text{ad}(p^{\ast, \text{mod}}, p)(M_1, F) : p^{\ast, \text{mod}} p_* (M_1, F) \to (M_1, F)
\]
is a map of complexes of \( D_{S_1 \times S_2}^{\infty} \) modules, similarly to the finite order case.
We following proposition concern the commutativity of the inverse images functors and the commutativity of the direct image functors.

**Proposition 53.** 
(i) Let \( f_1 : X \to Y \) and \( f_2 : Y \to S \) two morphism with \( X, Y, S \in \text{SmVar}(\mathbb{C}) \).

- Let \( (M, F) \in C_{D(2)fil,r}(S) \). Then \( (f_2 \circ f_1)^{smod}(M, F) = f_2^{smod}f_1^{smod}(M, F) \).
- Let \( (M, F) \in D_{D(2)fil,r}(S) \). Then \( L(f_2 \circ f_1)^{smod}(M, F) = Lf_1^{smod}(Lf_2^{smod}(M, F)) \).

(ii) Let \( f_1 : X \to Y \) and \( f_2 : Y \to S \) two morphism with \( X, Y, S \in \text{SmVar}(\mathbb{C}) \). Let \( M \in D_P(X) \). Then, 
\[
T(\int_{f_2 \circ f_1} \int_{f_2} f_1^*(M)) : \int_{f_2} \int_{f_1} (M) \xrightarrow{\sim} \int_{f_2 \circ f_1} (M)
\]
is an isomorphism in \( D_P(S) \) (i.e. if we forget filtration).

(iii) Denote \( D \) (possibly infinite) direct sums (see [17]).

**Remark 8.** Let \( f_1 : X \to Y \) and \( f_2 : Y \to S \) two morphism with \( X, Y, S \in \text{SmVar}(\mathbb{C}) \). Then, for \((M, F) \in D_{D(2)fil,\infty}(X), \int_{f_2} \int_{f_1} (M, F) \) is NOT isomorphic to \( \int_{f_2 \circ f_1} (M, F) \) in general, the filtrations on the isomorphic cohomology sheaves may be different.

**Proposition 54.** Let \( f : X \to S \) a morphism with \( X, S \in \text{SmVar}(\mathbb{C}) \). Then,

(i) For \((M, F) \in C_{D(2)fil,h}(S)\), we have \( Lf^{smod}(M, F) \in D_{D(2)fil,\infty,h}(X) \).
(ii) For $M \in C_{D,h}(X)$, we have $\int_f M \in D_{D,h}(S)$.

(iii) If $f$ is proper, for $(M, F) \in C_{D(2)fil,h}(X)$, we have $\int_f (M, F) \in D_{D(2)fil,\infty,h}(S)$.

**Proof.** See [17] for the non filtered case. The filtered case follows immediately from the non filtered case and the fact the pullback of a good filtration is a good filtration (since the pullback of a coherent $O_S$ module is coherent) and the direct image of a good filtration by a proper morphism is a good filtration (since the pushforward of a coherent $O_X$ module by a proper morphism is coherent).

The following easy proposition says that the analytical functor commutes the pullback of $D$ modules and the tensor product. Again it is well known in the non filtered case. Note that for $S \in \text{SmVar}(\mathbb{C})$, $D_S^an = D_{S^an}$.

**Proposition 55.** (i) Let $f : T \to S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$.

- Let $(M, F) \in C_{Dfil,r}(S)$. Then $(f^*\text{mod}(M, F))^an = f^*\text{mod}(M, F)^an$.
- Let $(M, F) \in D_{Dfil,r}(S)$, for $r = 1, \ldots, \infty$. Then, $(Lf^*\text{mod}(M, F))^an = Lf^*\text{mod}(M, F)^an$.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$.

- Let $(M, F), (N, F) \in C_{Dfil}(S)$. Then, $((M, F) \otimes_{O_S} (N, F))^an = (M, F)^an \otimes_{O_{S^an}} (N, F)^an$.
- Let $(M, F), (N, F) \in D_{Dfil,r}(S)$, for $r = 1, \ldots, \infty$. Then, $((M, F) \otimes_{L_r} (N, F))^an = (M, F)^an \otimes_{O_{S^an}} (N, F)^an$.

**Proof.** (i): For $(M, F) \in C_{Dfil}(S)$, we have, since $f^* a^n_X = a^n_X f^an$,\[
\begin{align*}
(f^*\text{mod}(M, F))^an &= \text{an}(X)^* (f^*\text{mod}(M, F) \otimes f^*O_S O_X) \otimes_{\text{an}(X)^*O_X} O_{X^an} \\
&= f^an^\ast a^n_X (M, F) \otimes f^an^\ast O_{X^an} \otimes O_{X^an} =: f^an*\text{mod}(M^an, F)
\end{align*}
\]
For $(M, F) \in D_{Dfil,r}(S)$, we take $(M, F) \in C_{Dfil}(S)$ an r-projective $f^*O_S$ module such that $D_{top,r}(M, F) = (M, F)$ so that
\[
(Lf^*\text{mod}(M, F))^an = (f^*\text{mod}(M, F))^an = f^an*\text{mod}(M^an, F) = Lf^an*\text{mod}(M^an, F)
\]

(ii): For $(M, F), (N, F) \in C_{Dfil}(S)$, we have
\[
((M, F) \otimes_{O_S} (N, F))^an : = a^n_X ((M, F) \otimes_{O_S} (N, F)) \otimes_{a^n_X O_S} O_{S^an} = a^n_X (M, F) \otimes a^n_X O_S \otimes a^n_X (N, F) \otimes a^n_X O_S \otimes a^n_X O_{S^an} \otimes a^n_X (N, F) \otimes a^n_X O_S \otimes a^n_X O_{S^an} = : (M^an, F) \otimes O_{S^an} (N^an, F)
\]
It implies the isomorphism in the derived category by taking an r-projective resolution of $(M, F)$ (e.g $(L_D(M), F) = L_D(M, F))$.

**Proposition 56.** (i) Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \text{AnSm}(\mathbb{C})$.

- Let $(M, F) \in C_{D(2)fil}(S)$ or let $(M, F) \in C_{D(\infty)fil}(S)$. Then $(f_2 \circ f_1)^*\text{mod}(M, F) = f_1^*\text{mod} f_2^*\text{mod}(M, F)$.
- Let $(M, F) \in D_{Dfil}(S)$ or let $(M, F) \in D_{D(\infty)fil}(S)$. Then $L(f_2 \circ f_1)^*\text{mod}(M, F) = Lf_1^*\text{mod} Lf_2^*\text{mod}(M, F)$.

(ii) Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphisms with $X, Y, S \in \text{AnSm}(\mathbb{C})$. Let $M \in D_D(X)$. If $f_1$ is proper, we have $\int_{f_2 \circ f_1} M = \int_{f_2} (\int_{f_1} M)$.

(iii) Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphisms with $X, Y, S \in \text{AnSm}(\mathbb{C})$. Let $M \in D_{D\infty}(X)$. If $f_1$ is proper, we have $\int_{f_2 \circ f_1} M = \int_{f_2} (\int_{f_1} M)$. 189
(iii) Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphisms with $X, Y, S \in \text{AnSm}(\mathbb{C})$. Let $(M, F) \in D_\mathcal{D}(X)$. We have $f_{1(f_2f_1)}^* M = f_1^*(f_2^* M)$.

(iv) Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphisms with $X, Y, S \in \text{AnSm}(\mathbb{C})$. Let $M \in D_{\mathcal{D}^\infty}(X)$. We have $f_{1(f_2f_1)}^* M = f_2^*(f_1^* M)$.

Proof. (i): Similar to the proof of proposition 53(i).

(ii): Similar to the proof of proposition 53(ii) : we use lemma 5 and the fact that for $N \in C_{f_1^* \mathcal{D}}(X)$ and $N' \in C_{\mathcal{D}}(Y)$, the canonical morphism

$$T(f_1, \otimes)(N', N) : N' \otimes_{\mathcal{D}_Y} Rf_1* N \to Rf_1*(f_1^* N' \otimes_{f_1^* \mathcal{D}_Y} N)$$

is an isomorphism if $f_1$ is proper (in this case $f_1 = f_1$).

(iii): Similar to the proof of proposition 53(ii) : we use lemma 5 and the fact that for $N \in C_{f_1^* \mathcal{D}^\infty}(X)$ and $N' \in C_{\mathcal{D}^\infty}(Y)$, the canonical morphism

$$T(f_1, \otimes)(N', N) : N' \otimes_{\mathcal{D}^\infty_Y} Rf_1* N \to Rf_1*(f_1^* N' \otimes_{f_1^* \mathcal{D}^\infty_Y} N)$$

is an isomorphism if $f_1$ is proper (in this case $f_1 = f_1$).

(iii)’ : Similar to the proof of proposition 53(ii) : we use lemma 4 and the fact that for $N \in C_{f_1^* \mathcal{D}}(X)$ and $N' \in C_{\mathcal{D}}(Y)$, the canonical morphism

$$T(f_1!, \otimes)(N', N) : N' \otimes_{\mathcal{D}_Y} Rf_{1!} N \to Rf_{1!}(f_1^* N' \otimes_{f_1^* \mathcal{D}_Y} N)$$

is an isomorphism.

(iii) ’: Similar to the proof of proposition 53(ii) : we use lemma 5 and the fact that for $N \in C_{f_1^* \mathcal{D}^\infty}(X)$ and $N' \in C_{\mathcal{D}^\infty}(Y)$, the canonical morphism

$$T(f_1!, \otimes)(N', N) : N' \otimes_{\mathcal{D}^\infty_Y} Rf_{1!} N \to Rf_{1!}(f_1^* N' \otimes_{f_1^* \mathcal{D}^\infty_Y} N)$$

is an isomorphism.

(iv): Similar to the proof of proposition 53(iii) : we have

$$i_{2*}\text{mod}(M, F) = i_2\ast((M, F) \otimes_{D_{Z_2}} (D_{Z_2} \to S), F^{\text{ord}}) \xrightarrow{\sim} i_1\ast i_0\ast((M, F) \otimes_{D_{Z_2}} (D_{Z_2} \to Z_1), F^{\text{ord}}) \otimes_{i_1^* D_{Z_1}} i_0^*(D_{Z_1} \to S), F^{\text{ord}}) \xrightarrow{i_1\ast T(i_0, \otimes)(-)^{-1}} i_1\ast i_0\ast \text{mod}(M, F))$$

using lemma 4 and proposition 10.

(iv) ’: Similar to (iv): we have

$$i_{2*}\text{mod}(M, F) = i_2\ast((M, F) \otimes_{D_{Z_2}^\infty} (D_{Z_2}^\infty \to S), F^{\text{ord}}) \xrightarrow{\sim} i_1\ast i_0\ast((M, F) \otimes_{D_{Z_2}^\infty} (D_{Z_2}^\infty \to Z_1), F^{\text{ord}}) \otimes_{i_1^* D_{Z_1}^\infty} i_0^*(D_{Z_1}^\infty \to S), F^{\text{ord}}) \xrightarrow{i_1\ast T(i_0, \otimes)(-)^{-1}} i_1\ast i_0\ast \text{mod}(M, F))$$

using lemma 5 and proposition 10.

\[\square\]

**Proposition 57.** (i) Let $f : X \to S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{D(2)fil,h}(S)$, we have $L_f^*\text{mod}(M, F) \in D_{D(2)fil,\infty,h}(X)$. For $(M, F) \in C_{D(2)fil,h}(S)$, we have $L_f^*\text{mod}(M, F) \in D_{D(2)fil,\infty,h}(X)$. 

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(ii) Let $f : X \to S$ a proper morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, for $(M, F) \in C_{\mathcal{D}(2)fil,h}(X)$, we have $\int_f(M, F) \in D_{\mathcal{D}(2)fil,\infty,h}(S)$.

(iii) Let $f : X \to S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Then, for $(M, F) \in C_{\mathcal{D}(\infty)fil,h}(X)$, we have $\int_f(M, F) \in D_{\mathcal{D}(\infty)fil,\infty,h}(S)$.

**Proof.** (i) and (ii): Follows immediately from the non filtered case since we look at the complex in the derived category with respect to $\infty$ usu local equivalence. It says that the pullback and the proper pushforward of an holonomic $D$ module is still holonomic. See [17] for the non filtered case.

(iii): In the case the morphism is proper, it follows from the finite order case (ii). In the case of an open embedding, it follows from proposition 51(i): we have for $j : S^o \to S$ an open embedding,

$$j_*E(O_{S^o}) = j_*\text{Hom}(\mathbb{Z}_{S^o}, E(O_{S^o})) = \text{Hom}(j_*\mathbb{Z}_{S^o}, E(O_S)) \in C_{\mathcal{D}(\infty)h}(S).$$

and on the other hand

$$T(j, \otimes)(-,-) = T^\text{mod}(j, \otimes)(-,-) : \int_j(M, F) = j_*E(M, F) = j_*E(j^*O_S \otimes_{O_S} (M, F))$$

$$\overset{\sim}{\to} j_*E(O_{S^o}) \otimes_{O_S} (M, F)$$

is an isomorphism by proposition 9. \qed

For $X, Y \in \text{SmVar}(\mathbb{C})$ or $X, Y \in \text{AnSm}(\mathbb{C})$, we denote by

- $C_{O_X}(X) \times C_{O_Y}(Y) \to C_{O_{X \times Y}}(X \times Y), (M, N) \mapsto M \cdot N := O_{X \times Y} \otimes_{p_X^*O_X \otimes p_Y^*O_Y} p_X^*M \otimes p_Y^*N,$

- $C_{\mathcal{D}}(X) \times C_{\mathcal{D}}(Y) \to C_{\mathcal{D}_{X \times Y}}(X \times Y), (M, N) \mapsto M \cdot N := O_{X \times Y} \otimes_{p_X^*O_X \otimes p_Y^*O_Y} p_X^*M \otimes p_Y^*N$

the natural functors which induces functors in the filtered cases and the derived categories, $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ the projections.

We have then the following easy proposition :

**Proposition 58.** For $X \in \text{SmVar}(\mathbb{C})$ or $X \in \text{AnSm}(\mathbb{C})$, we have for $(M, F), (N, F) \in C_{O_X,fil}(X)$ or $(M, F), (N, F) \in C_{\mathcal{D},fil}(X)$,

$$(M, F) \otimes_{O_X} (N, F) = \Delta^*_{X,fil}(M, F) \cdot (N, F)$$

**Proof.** Standard. \qed

**Definition 67.** Let $f : X \to S$ a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. We have the canonical map in $C_{\mathcal{D}(\infty)fil}(X)$ modules:

$$T(f, \infty) : (D_{X \to S}, F^{ord}) \otimes_{D_X} (D_X^\infty, F^{ord}) \to (D_X^{\infty}_{X \to S}, F^{ord}), (h_X \otimes P_S) \otimes P_X \mapsto (P_X \cdot h_X \otimes P_S + h_X \otimes df(P_X)P_S$$

where $h_X \in \Gamma(X^o, O_X)$, $P_S \in \Gamma(X^o, f^*D_S)$ and $P_X \in \Gamma(X^o, D_X^\infty)$. This gives, for $(M, F) \in C_{\mathcal{D}(\infty)fil}(S)$, the following transformation map in $C_{\mathcal{D}(\infty)fil}(X)$

$$T(f, \infty)(M, F) : J_X(f^{*\text{mod}}(M, F)) := f^*(M, F) \otimes_{f^*\mathcal{D}_S} (D_X^\infty_{X \to S}, F^{ord}) \otimes_{D_X} (D_X^{\infty}, F^{ord}) \overset{\overset{\otimes T(f, \infty)}{\sim}}{\to} f^*(M, F) \otimes_{f^*\mathcal{D}_S} f^*D_S^\infty \otimes_{f^*D_S^\infty} (D_X^{\infty}_{X \to S}, F^{ord}) = f^{*\text{mod}}J_S(M, F)$$

where we recall that $J_S(M, F) = (M, F) \otimes_{\mathcal{D}_S} (D_S^\infty, F^{ord})$.

We now look at some properies of the dual functor for $D$ modules : For complex of $D$ module with coherent cohomology we have the following:

**Proposition 59.** (i) Let $S \in \text{SmVar}(\mathbb{C})$. For $M \in C_{\mathcal{D}(\infty)}(S)$, the canonical map $d(M) : M \to D_S^2 L_DM$ is an equivalence Zariski local.
Proposition 60.  

(ii) Let $S \in \text{AnSm}(\mathbb{C})$. For $M \in C_{\mathbb{D}}(S)$, the canonical map $d(M) : M \to \mathbb{D}_S^2L_D(M)$ is an equivalence usu local.

(iii) Let $S \in \text{AnSm}(\mathbb{C})$. For $(M, F) \in C_{\mathbb{D}_L}(S)$, the canonical map $d(M) : M \to \mathbb{D}_S^2L_D(M)$ is an equivalence usu local.

Proof. Standard ; follows from the definition of coherent sheaves. See [17] for example.

Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \to S_1$ the projection. In this case we have a canonical embedding $D_{S_1} \hookrightarrow D_{S_{12}}$. This gives, for $(M, F) \in C_{\mathbb{D}_{fil}}(S_1 \times S_2)$, the following transformation map in $C_{\mathbb{D}_{fil}}(S_1)$

$$T_*(p, D)(M, F) := p_* \mathcal{H}om_{DS_{12}}((M, F), D_{S_{12}}) \otimes_{O_{S_{12}}} \mathcal{D}_{S_{12}}^Q w(K_{S_{12}})[d_{S_{12}}]$$

$$\cong \mathcal{H}om_{DS_{12}}(p_*(M, F), D_{S_{12}}) \otimes_{O_{S_{12}}} \mathcal{D}_{S_{12}}^Q w(p_* K_{S_{12}})[d_{S_{12}}]$$

We have the canonical map

$$p(D) : p^* D_{S_1} = p^* D_{S_1} \otimes_{O_{S_1}} O_{S_{12}} \to D_{S_{12}}, \gamma \otimes f \mapsto f \gamma$$

induced by the embedding $p^* D_{S_1} \hookrightarrow D_{S_{12}}$. This gives, for $(M, F) \in C_{\mathbb{D}_{fil}}(S_1)$, the following transformation map in $C_{\mathbb{D}_{fil}}(S_1 \times S_2)$

$$T(p, D)(M, F) := p_* \mathcal{H}om_{DS_{12}}((M, F), D_{S_{12}}) \otimes_{O_{S_{12}}} p^* D_{S_1} \mathcal{D}_{S_1}^Q w(K_{S_{12}})[d_{S_{12}}]$$

$$\cong \mathcal{H}om_{DS_{12}}(p^*(M, F), D_{S_{12}}) \otimes_{O_{S_{12}}} p^* D_{S_1} \mathcal{D}_{S_1}^Q w(K_{S_{12}})[d_{S_{12}}]$$

where $K^{-1}(S_1/S_{12})$ is given by the wedge product with a generator of $\wedge^{d_{S_1}} D_{S_{12}}^\infty \cong K_{S_{12}}^{-1}$.

In the case $S_1, S_2 \in \text{AnSm}(\mathbb{C})$, we also have the embedding $p^* D_{S_1}^\infty \hookrightarrow D_{S_{12}}^\infty$. This gives in the same way, for $(M, F) \in C_{\mathbb{D}_{fil}}(S_1 \times S_2)$, the following transformation map in $C_{\mathbb{D}_{fil}}(S_1)$

$$T_*(p, D^\infty)(M, F) := p_* \mathcal{D}_{S_{12}}^{K, K}(M, F) \to \mathbb{D}_S^{\infty, K} p_*(M, F).$$

The map

$$p(D^\infty) : p^* D_{S_1}^\infty = p^* D_{S_1}^\infty \otimes_{O_{S_1}} O_{S_{12}} \to D_{S_{12}}^\infty, \gamma \otimes f \mapsto f \gamma$$

induced by the embedding $p^* D_{S_1}^\infty \hookrightarrow D_{S_{12}}^\infty$, gives in the same way, for $(M, F) \in C_{\mathbb{D}_L}(S_1)$, the transformation map in $C_{\mathbb{D}_{fil}}(S_1 \times S_2)$

$$T(p, D^\infty)(M, F) := p^* \mathcal{D}_{S_1}^{K, K}(M, F) \cong p^* \mathcal{D}_{S_1}^{\infty}(M, F) \otimes_{O_{S_1}} \mathcal{D}_{S_{12}}^Q w(K_{S_{12}})[d_{S_{12}}]$$

$$\cong \mathcal{H}om_{DS_{12}}(p^*(M, F), D_{S_{12}}^\infty) \otimes_{O_{S_{12}}} \mathcal{D}_{S_{12}}^Q w(K_{S_{12}})[d_{S_{12}}]$$

given in the same way then $T(p, D)(M, F)$.

**Proposition 60.** (i) Let $g : T \to S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. We have, for $M \in D_{\mathbb{D}}(S)$ canonical maps
\(T'(g, D)(M) : \mathbb{L}D_S Lg^{\text{mod}} M \rightarrow Lg^{\text{mod}} \mathbb{L}D_S M\)
\(T'(g, D)(M) : Lg^{\text{mod}} \mathbb{L}D_S M \rightarrow \mathbb{L}D_S Lg^{\text{mod}} M\)

Moreover, in the case where \(g\) is non characteristic with respect to \(M\) (e.g if \(g\) is smooth), these maps are isomorphism.

(ii) Let \(S_1, S_2 \in \text{SmVar}(\mathbb{C})\), \(p : S_1 \times S_2 \rightarrow S_1\) the projection. For \(M \in D_{\mathcal{D}}(S_1)\), we have \(T(p, D)(L_D(M)) = T'(p, D)(M)\) in \(D_{\mathcal{D}fhl}(S_1 \times S_2)\) (c.f.(i)).

\begin{proof}
(i): See the proof of (i) in [17].

(ii): See the proof of (i) in [17] □
\end{proof}

We have the followings :

**Proposition 61.** Let \(f_1 : X \rightarrow Y\) and \(f_2 : Y \rightarrow S\) two morphism with \(X, Y, S \in \text{SmVar}(\mathbb{C})\). Let \(M \in C_{D,h}(S)\). Then, we have \(L(f_2 \circ f_1)^{\text{mod}} M = (Lf_1^{\text{mod}}(Lf_2^{\text{mod}} M))\) in \(D_{D,h}(X)\).

\begin{proof}
Follows from proposition 53 (i), proposition 54 and proposition 59. □
\end{proof}

**Proposition 62.** Let \(f_1 : X \rightarrow Y\) and \(f_2 : Y \rightarrow S\) two morphism with \(X, Y, S \in \text{AnSm}(\mathbb{C})\).

(i) Let \((M) \in C_{D,h}(S)\). Then, we have \(L(f_2 \circ f_1)^{\text{mod}} M = (Lf_1^{\text{mod}}(Lf_2^{\text{mod}} M))\) in \(D_{D,h}(X)\).

(ii) Let \(M \in C_{D,h}(S)\). Then, we have \((f_2 \circ f_1)^{\text{mod}} M = (Lf_1^{\text{mod}}(Lf_2^{\text{mod}} M))\) in \(D_{D,h}(X)\).

\begin{proof}
Follows from proposition 56 (i), proposition 57 and proposition 59. □
\end{proof}

In the analytic case, we have the following transformation map which we will use in subsection 5.3:

**Definition 68.** Let \(S \in \text{AnSm}(\mathbb{C})\). We have for \((M, F) \in C_{Dfhl}(S)\) the canonical transformation map in \(C_{D\infty fhl}(S)\):

\[T(D, \infty)(M, F) : J_S(D_f^F(M, F)) := \text{Hom}_{D_S}((M, F), D_S) \otimes_{D_S} (D_S^\infty, F^{ord}) \otimes_{D_S} \mathbb{D}^O w(K_S)[d_S] \rightarrow \text{Hom}_{D_S}(L_D(M, F), D_S^\infty) \otimes_{D_S} \mathbb{D}^O w(K_S)[d_S] \rightarrow \text{Hom}_{D_S}((M, F) \otimes_{D_S} (D_S^\infty, F^{ord}), D_S^\infty) \otimes_{D_S} \mathbb{D}^O w(K_S)[d_S]
\]

which is an isomorphism.

4.2.2 The (relative) De Rahm of a (filtered) complex of a D-module and the filtered De Rahm direct image

Recall that for \(f : X \rightarrow S\) a morphism with \(X, S \in \text{Var}(\mathbb{C})\) or with \(X, S \in \text{AnSp}(\mathbb{C})\),

\[DR(X/S) := \Omega^\bullet_{X/S} \in C_f \cdot O_S(X)\]

denotes (see section 2) the relative De Rahm complex of the morphism of ringed spaces \(f : (X, O_X) \rightarrow (S, O_S)\), with \(\Omega^\bullet_{X/S} := \wedge^\bullet \Omega_{X/S} \in \text{PSh}_{O_X}(X)\) and \(\Omega_{X/S} := \text{coker}(f^* \Omega_S \rightarrow \Omega_X) \in \text{PSh}_{O_X}(X)\). Recall that \(\Omega_{X/S} \in C_f \cdot O_S(X)\) is a complex of \(f^* O_S\) modules, but is NOT a complex of \(O_X\) module since the differential is a derivation hence NOT \(O_X\) linear. Recall that (see section 4.1), for \((M, F) \in C_{D(O_X)fhl}(X)\), we have the relative (filtered) De rham complex of \((M, F)\)

\[DR(X/S)(M, F) := (\Omega^\bullet_{X/S}, F_0) \otimes_{O_X} (M, F) \in C_f \cdot O_S fhl(X),\]

and that if \(\phi : (M_1, F) \rightarrow (M_2, F)\) a morphism with \((M_1, F), (M_2, F) \in C_{D(O_X)fhl}(X)\),

\[(I \otimes \phi) : DR(X/S)(M_1, F) := (\Omega^\bullet_{X/S}, F_0) \otimes_{O_X} (M_1, F) \rightarrow DR(X/S)(M_2, F) := (\Omega^\bullet_{X/S}, F_0) \otimes_{O_X} (M_2, F)\]

for the maps of ringed spaces \((M_1, F), (M_2, F) \in C_{D(O_X)fhl}(X)\) and \(\phi) : (M_1, F) \rightarrow (M_2, F)\) a morphism with \((M_1, F), (M_2, F) \in C_{D(O_X)fhl}(X)\),

\[(I \otimes \phi) : DR(X/S)(M_1, F) := (\Omega^\bullet_{X/S}, F_0) \otimes_{O_X} (M_1, F) \rightarrow DR(X/S)(M_2, F) := (\Omega^\bullet_{X/S}, F_0) \otimes_{O_X} (M_2, F)\]
is by definition a morphism of complexes, that is a morphism in \( C_{f^*O_Sfil}(X) \). For \((N,F) \in C_{D(O_X)^{op}fil}(X)\), we have the relative (filtered) Spencer complex of \((N,F)\)

\[
SP(X/S)(N,F) := (T^\bullet_{X/S}, F_0) \otimes_{O_X} (N,F) \in C_{f^*O_Sfil}(X),
\]

and that if \(\phi : (N_1,F) \rightarrow (N_2,F)\) a morphism with \((N_1,F),(N_2,F) \in C_{D(O_X)^{op}fil}(X)\),

\[
(I \otimes \phi) : SP(X/S)(N_1,F) := (T^\bullet_{X/S}, F_0) \otimes_{O_X} (N_1,F) \rightarrow SP(X/S)(N_2,F) := (T^\bullet_{X/S}, F_0) \otimes_{O_X} (N_2,F)
\]
is by definition a morphism of complexes, that is a morphism in \( C_{f^*O_Sfil}(X) \).

**Proposition 63.** Let \(f : X \rightarrow S\) a smooth morphism with \(X,S \in \text{Var}(\mathbb{C})\) or with \(X,S \in \text{AnSp}(\mathbb{C})\), denote \(d = dx - ds\). The inner product gives, for \((M,F) \in C_{D(O_X)fil}(X)\), an isomorphism in \( C_{f^*O_Sfil}(X) \) and termwise \(O_X\) linear

\[
T(DR,SP)(M,F) : T^\bullet_{X/S} \otimes_{O_X} (M,F) \otimes_{O_X} K_{X/S} \xrightarrow{\sim} \Omega^d_{X/S} \otimes_{O_X} (M,F), \partial \otimes m \otimes \kappa \mapsto \iota(\partial)\kappa \otimes m
\]

**Proof.** Standard. \(\blacksquare\)

For a commutative diagram in \(\text{Var}(\mathbb{C})\) or in \(\text{AnSp}(\mathbb{C})\):

\[
\begin{array}{ccc}
D &=& X \\
\xrightarrow{f} & & \xrightarrow{g} S \\
\xrightarrow{g'} & & \\
X' & & T
\end{array}
\]

we have (see section 2) the relative differential map of \(g'\) given by the pullback of differential forms:

\[
\Omega_{(X'/X)/T}(M,F) : \Omega^\bullet_{X/S} \rightarrow \Omega^\bullet_{X'/T}, \text{ given by for } X'^o \subset X', \ X'^o \supset (g'^{-1}(X^o)) (i.e. \ g'^{-1}(X^o) \supset X'^o), \\
\omega \in \Gamma(X^o, O^p_{X/S}) \mapsto \Omega_{(X'/X)/T}(M,F)(\omega) := [g'\omega] \in \Gamma(X'^o, O^p_{X'/T}).
\]

Moreover, by definition-proposition 16 (section 4.1), for \((M,F) \in C_{D(O_X)fil}(X)\) the map

\[
\Omega_{(X'/X)/T}(M,F) : g'^\ast(\Omega^\bullet_{X/S} \otimes_{O_X} (M,F)) \rightarrow \Omega^\bullet_{X'/T} \otimes_{O_{X'}} g'^{mod}(M,F)
\]
given in degree \((p,i)\) by, for \(X'^o \subset X'\) an open subset and \(X^o \subset X\) an open subset such that \(g'^{-1}(X^o) \supset X'^o\) (i.e. \( X'^o \supset (g'(X^o)) \), \( \omega \in \Gamma(X^o, O^p_{X/S}) \) and \( m \in \Gamma(X^o, M') \),

\[
\Omega_{(X'/X)/T}(M,F)(\omega \otimes m) = g'^\ast \omega \otimes (m \otimes 1)
\]
is a map of complexes, that is a map in \( C_{f^*O_Sfil}(X') \). This give, for \((M,F) \in C_{D(O_X)fil}(X)\), the following transformation map in \( C_{O^{(X')}_fil}(T) \)

\[
T^\omega_{o}(D)(M,F) : g^{mod}L_o(f, E(\Omega^\bullet_{X/S} \otimes_{O_X} (M,F))) \xrightarrow{T(g,L_o)(-)} (g^*f, E(\Omega^\bullet_{X/S} \otimes_{O_X} (M,F))) \otimes g^*O_{S} O_T \xrightarrow{T(g',E)(-)*T(D)(E(\Omega^\bullet_{X/S} \otimes_{O_X} M))} f'^*E(\Omega^\bullet_{X'/T} \otimes_{O_{X'}} g^{mod}(M,F))
\]

with \(m'(m) = m \otimes 1\). Under the canonical isomorphism \(\Omega^\bullet_{X/S} \xrightarrow{\sim} \Omega^\bullet_{X/S} \otimes_{O_X} O_X\) given by \(\omega \mapsto \omega \otimes 1\), we have (see remark 7)

\[
T^\omega_{o}(D)(O_X) = T^\omega_{o}(D) : g^{mod}L_o(f, E(\Omega^\bullet_{X/S})) \rightarrow f'^*E(\Omega^\bullet_{X'/T}).
\]

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Let $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Again by definition-proposition 16 (section 4.1), for $(M, F) \in C_{D(f)}(X)$ the map
\[
\Omega_{(X^n/X)/(S^n/S)}(M, F) : \text{an}_X^*\left(\Omega_{X/S}^* \otimes o_X (M, F)\right) \to \Omega_{X^n/S^n}^* \otimes o_{X^n}(M^n)
\]
given in degree $(p, i)$ by, for $X^o \subset X$ and $X^o \subset X^{oo}$ an open subsets of $X$ for the usual, resp. Zariski topology, $\omega \in \Gamma(X^o, \Omega_{X^o/S/S}^p)$ and $m \in \Gamma(X^o, M^i)$,
\[
\Omega_{(X^n/X)/(S^n/S)}(M, F) (\omega \otimes m) = \omega \otimes (m \otimes 1)
\]
is a map of complexes, that is a map in $C_{D(f)}(X)$. This gives, for $(M, F) \in C_{D(f)}(X)$, we have the following transformation map in $C_{D(f)}(S^n)$. Let $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$ or with $X, S \in \text{AnSp}(\mathbb{C})$. In the case where $X$ is smooth, for $(M, F) = (M^*, F) \in C_{D(f)}(X)$, the differential of the relative De Rham complex of $(M, F)$
\[
\text{DR}(X/S)(M, F) := (\Omega_{X/S}^* \otimes o_X (M, F) = \text{Tot}(\Omega_{X/S}^* \otimes o_X (M, F)) \in C_{D(f)}(X)
\]
are given by
\[
\begin{align*}
\bullet & \ d_{p,p+1} : \Omega_{X/S}^p \otimes o_X M^i \to \Omega_{X/S}^{p+1} \otimes o_X M^i, \text{ with for } X^o \subset X \text{ an open affine subset with } (x_1, \ldots, x_n) \text{ local coordinate (since } X \text{ is smooth, } T_X \text{ is locally free}, m \in \Gamma(X^o, M^i) \text{ and } \omega \in \Gamma(X^o, O_{X/S}^p), \\
\ & \ d_{p,p+1}(\omega \otimes m) := (d\omega) \otimes m + \sum_{i=1}^n (dx_i \wedge \omega) \otimes (\partial_i)m
\end{align*}
\]
\[
\begin{align*}
\bullet & \ d_{i,i+1} : \Omega_{X/S}^p \otimes o_X M^i \to \Omega_{X/S}^{p+1} \otimes o_X M^{i+1}, \text{ with for } X^o \subset X \text{ an open subset, } m \in \Gamma(X^o, M^i) \text{ and } \\
\ & \ d_{i,i+1}(\omega \otimes m) := (\omega \otimes dm).
\end{align*}
\]
For $D_X$ only, the differentials of its De Rham complex $(\Omega_{X/S}^*, F) \otimes o_X D_X$ are right linear, so that
\[
(\Omega_{X/S}^*, F_b) \otimes o_X (D_X, F^{ord}) \in C_{D(f)}(X)
\]
In the particular case of a projection $p : Y \times S \to S$ with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSp}(\mathbb{C})$ we have:

**Proposition 64.** Let $Y, S \in \text{SmVar}(\mathbb{C})$ or $Y, S \in \text{AnSp}(\mathbb{C})$. Let $p : Y \times S \to S$ the projection. For $(M, F) \in C_{D(f)}(Y \times S)$,
\[
\text{DR}(Y \times S/S)(M, F) := (\Omega_{Y \times S/S}^* \otimes o_{Y \times S} (M, F) \in C_{p^{*}D_{f}}(Y \times S)
\]
is a naturally a complex of filtered $p^{*}D_{S}$ modules, that is
\[
\text{DR}(Y \times S/S)(M, F) := (\Omega_{Y \times S/S}^* \otimes o_{Y \times S} (M, F) \in C_{p^{*}D_{f}}(Y \times S),
\]

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where the $p^* D_S$ module structure on $\Omega^p_{Y \times S/S} \otimes_{\mathcal{O}_{Y \times S}} M^n$ is given by for $(Y \times S)^o \subset Y \times S$ an open subset,

$$(\gamma \in \Gamma((Y \times S)^o, \mathcal{T}_{Y \times S}), \hat{\omega} \otimes m \in \Gamma((Y \times S)^o, \Omega^p_{Y \times S/S} \otimes_{\mathcal{O}_{Y \times S}} M^n)) \mapsto \gamma.(\hat{\omega} \otimes m) := (\hat{\omega} \otimes (\gamma.m)).$$

Moreover, if $\phi : (M_1, F) \to (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in C_{D_{fil}}(Y \times S)$,

$$DR(Y \times S/S)(\phi) := (I \otimes \phi) : (\Omega^*_{Y \times S/S}, F_b) \otimes_{\mathcal{O}_{Y \times S}} (M_1, F) \to (\Omega^*_{Y \times S/S}, F_b) \otimes_{\mathcal{O}_{Y \times S}} (M_2, F)$$

is a morphism in $C_{p^*D_{fil}}(Y \times S)$.

Proof. Standard. □

In the analytic case, we also have

**Proposition 65.** Let $Y, S \in \text{AnSm}((\mathbb{C})$. Let $p : Y \times S \to S$ the projection. For $(M, F) \in C_{D_{fil}}(Y \times S)$,

$$DR(Y \times S/S)(M, F) := (\Omega^*_{Y \times S/S}, F_b) \otimes_{\mathcal{O}_{Y \times S}} (M, F) \in C_{p^*O_Sfil}(Y \times S)$$

is naturally a complex of filtered $p^* D_S^\infty$ modules, that is

$$DR(Y \times S/S)(M, F) := (\Omega^*_{Y \times S/S}, F_b) \otimes_{\mathcal{O}_{Y \times S}} (M, F) \in C_{p^*D_{fil}}(Y \times S),$$

where the $p^* D_S^\infty$ module structure on $\Omega^p_{Y \times S/S} \otimes_{\mathcal{O}_{Y \times S}} M^n$ is given by for $(Y \times S)^o \subset Y \times S$ an open subset,

$$(\gamma \in \Gamma((Y \times S)^o, \mathcal{T}_{Y \times S}), \hat{\omega} \otimes m \in \Gamma((Y \times S)^o, \Omega^p_{Y \times S/S} \otimes_{\mathcal{O}_{Y \times S}} M^n)) \mapsto \gamma.(\hat{\omega} \otimes m) := (\hat{\omega} \otimes (\gamma.m)).$$

Moreover, if $\phi : (M_1, F) \to (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in C_{D_{fil}}(Y \times S)$,

$$DR(Y \times S/S)(\phi) := (I \otimes \phi) : (\Omega^*_{Y \times S/S}, F_b) \otimes_{\mathcal{O}_{Y \times S}} (M_1, F) \to (\Omega^*_{Y \times S/S}, F_b) \otimes_{\mathcal{O}_{Y \times S}} (M_2, F)$$

is a morphism in $C_{p^*D_{fil}}(Y \times S)$.

Proof. Standard : follows from the finite order case (proposition 64). □

We state on the one hand the commutativity of the tensor product with respect to $D_S$ and with respect to $O_S$, for $S \in \text{SmVar}((\mathbb{C})$ or $S \in \text{AnSm}((\mathbb{C})$ in the filtered case, and on the other hand the commutativity between the tensor product with respect to $D_S$ by $D_S$ and the De Rahm complex:

**Proposition 66.** (i) Let $f : X \to S$ a morphism with $X, S \in \text{SmVar}((\mathbb{C})$ or with $X, S \in \text{AnSm}((\mathbb{C})$. For $(M', F) \in C_{D_{fil}^{p^*fil}D_S}(X)$ and $(M, F), (N, F) \in C_{D_{fil}}(X)$, we have canonical isomorphisms of filtered $f^* D_S$ modules, i.e. isomorphisms in $C_{f^* D_S}(X)$,

$$(M', F) \otimes_{\mathcal{O}_X} (N, F) \otimes_{D_S} (M, F) = (M', F) \otimes_{D_S} ((M, F) \otimes_{\mathcal{O}_X} (N, F)) = ((M', F) \otimes_{\mathcal{O}_X} (M, F)) \otimes_{D_S} (N, F)$$

(ii) Let $f : X \to S$ a morphism with $X, S \in \text{Var}((\mathbb{C})$ or with $X, S \in \text{AnSp}((\mathbb{C})$. For $(M, F) \in C_{D_{fil}(O_S)}(X)$, we have a canonical isomorphisms of filtered $f^* O_S$ modules, i.e. isomorphisms in $C_{f^* O_Sfil}(X)$,

$$(\Omega^*_{X/S}, F_b) \otimes_{\mathcal{O}_X} (M, F) = (\Omega^*_{X/S}, F_b) \otimes_{\mathcal{O}_X} (D(O_X), F_b)) \otimes_{D_{fil}(O_X)} (M, F)$$

(iii) Let $p : Y \times S \to S$ a projection with $Y, S \in \text{SmVar}((\mathbb{C})$ or with $X, S \in \text{AnSm}((\mathbb{C})$. For $(M, F) \in C_{D_{fil}}(Y \times S)$, the isomorphisms of filtered $p^* O_S$ modules of (ii)

$$(\Omega^*_{Y \times S/S}, F_b) \otimes_{\mathcal{O}_{Y \times S}} (M, F) = (\Omega^*_{Y \times S/S}, F_b) \otimes_{\mathcal{O}_{Y \times S}} (D_{Y \times S}, F_b) \otimes_{D_{Y \times S}} (M, F)$$

are isomorphisms of filtered $p^* D_S$ modules, that is isomorphism in $C_{p^*D_{fil}}(Y \times S)$.
Proof. (i) and (ii) are particular case of proposition 46. (iii) follows immediately by definition of the \( p^*D_S \) module structure.

We now look at the functorialities of the relative De Rham complex of a smooth morphisms of smooth complex algebraic varieties:

**Proposition 67.** Consider a commutative diagram in \( \text{SmVar}(\mathbb{C}) \) or in \( \text{AnSm}(\mathbb{C}) \):

\[
D = \begin{array}{ccc}
Y \times S & \overset{p}{\longrightarrow} & S \\
g' = (g'' \times g) & \downarrow g & \downarrow \\
Y' \times T & \overset{p'}{\longrightarrow} & T
\end{array}
\]

with \( p \) and \( p' \) the projections. For \( (M, F) \in C_{D_{fil}}(Y \times S) \) the map in \( C_{g'' \cdot p^*O_{Sfil}}(Y' \times T) \)

\[
\Omega_{(Y' \times T)/(Y \times S)}(M, F) : g''^*(\Omega_{X \times Y}/S \circ F_{b}) \otimes_{O_{X \times Y}} (M, F) \rightarrow (\Omega_{Y' \times T}/T \circ F_{b}) \otimes_{O_{Y' \times T}} g''^*(M, F)
\]

is a map in \( C_{g'' \cdot p^*D_{fil}}(Y' \times T) \). Hence, for \( (M, F) \in C_{D_{fil}}(Y \times S) \), the map in \( C_{O_T}T_1(T) \) (with \( L_D \) instead of \( L_O \))

\[
T_{\omega}^D(D)(M) : g''^*L_{D}(p_{*}E((\Omega_{Y' \times T}/S, F_{b}) \otimes_{O_{Y' \times T}} (M, F))) \rightarrow p_{*}E((\Omega_{Y' \times T}/T \circ F_{b}) \otimes_{O_{Y' \times T}} g''^*(M, F)),
\]

is a map in \( C_{D_{fil}}(T) \).

Proof. Follows immediately by definition.

In the analytic case, we also have

**Proposition 68.** Consider a commutative diagram in \( \text{AnSm}(\mathbb{C}) \):

\[
D = \begin{array}{ccc}
Y \times S & \overset{p}{\longrightarrow} & S \\
g' = (g'' \times g) & \downarrow g & \downarrow \\
Y' \times T & \overset{p'}{\longrightarrow} & T
\end{array}
\]

with \( p \) and \( p' \) the projections. For \( (M, F) \in C_{D_{fil}}(Y \times S) \) the map in \( C_{g'' \cdot p^*O_{Sfil}}(Y' \times T) \)

\[
\Omega_{(Y' \times T)/(Y \times S)}(M, F) : g''^*(\Omega_{X \times Y}/S \circ F_{b}) \otimes_{O_{X \times Y}} (M, F) \rightarrow (\Omega_{Y' \times T}/T \circ F_{b}) \otimes_{O_{Y' \times T}} g''^*(M, F)
\]

is a map in \( C_{g'' \cdot p^*D_{fil}}(Y' \times T) \). Hence, for \( (M, F) \in C_{D_{fil}}(Y \times S) \), the map in \( C_{O_T}T_1(T) \) (with \( L_D \) instead of \( L_O \))

\[
T_{\omega}^D(D)(M) : g''^*L_{D}(p_{*}E((\Omega_{Y' \times T}/S, F_{b}) \otimes_{O_{Y' \times T}} (M, F))) \rightarrow p_{*}E((\Omega_{Y' \times T}/T \circ F_{b}) \otimes_{O_{Y' \times T}} g''^*(M, F)),
\]

is a map in \( C_{D_{fil}}(T) \).

Proof. Follows immediately by definition.

Similarly, we have

**Proposition 69.** Let \( p : Y \times S \rightarrow S \) a projection with \( Y, S \in \text{SmVar}(\mathbb{C}) \). For \( (M, F) \in C_{D_{fil}}(Y \times S) \) the map in \( C_{p^*O_{Sfil}}(Y^an \times S^an) \)

\[
\Omega_{(Y^an \times S^an)/(Y \times S)}(M, F) : an(Y \times S)^*(\Omega_{Y \times S} \circ F_{b}) \otimes_{O_{Y \times S}} (M, F) \rightarrow (\Omega_{Y^an \times S^an}/S^an \circ F_{b}) \otimes_{O_{Y \times S}} (M^an, F)
\]

is a map in \( C_{p^*D_{fil}}(Y^an \times S^an) \). For \( (M, F) \in C_{D_{fil}}(Y \times S) \), the map in \( C_{O_{Sfil}}(S^an) \)

\[
T_{\omega}^D(an, h)(M, F) : p_{*}E((\Omega_{Y \times S}/S, F_{b}) \otimes_{O_{Y \times S}} (M, F))^an \rightarrow p_{*}E((\Omega_{Y \times S}/S, F_{b}) \otimes_{O_{Y \times S}} (M, F)^an)
\]

is a map in \( C_{D_{fil}}(S^an) \).
Proof. Similar to the proof of proposition 67.

Proposition 70. Let $p : Y \times S \to S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSm}(\mathbb{C})$.

(i) If $\phi : (M, F) \to (N, F)$ is an $r$-filtered Zariski, resp. usu, local equivalence with $(M_1, F), (M_2, F) \in C_{\mathcal{D}_{\text{fil}}} (Y \times S)$, then

$$DR(Y \times S/S)(\phi) : (\Omega^r_{Y \times S/S}, F_b) \otimes_{O_{Y \times S}} (M, F) \to \Omega^r_{Y \times S/S} \otimes_{O_{Y \times S}} (N, F)$$

is an $r$-filtered equivalence Zariski, resp. usu, local in $C_{p^* \mathcal{D}_{\text{fil}}} (Y \times S)$.

(ii) Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ or in $\text{AnSm}(\mathbb{C})$

$$D = Y \times S \begin{array}{c}
p \\downarrow \\downarrow \\downarrow \\
\downarrow \\downarrow \\downarrow \\
V \rightarrow S
d = Y \times S \begin{array}{c}
p \\downarrow \\downarrow \\downarrow \\
\downarrow \\downarrow \\downarrow \\
V \rightarrow S
\end{array}$$

with $p$ the projection. For $(N, F) \in C_{\mathcal{D}_{\text{fil}}} (Y \times S)$, the map in $C_{p^* O_S} (Y \times S)$ (see definition 60)

$$k \circ T^O_\omega (l, \otimes) (E(N, F)) : (\Omega^r_{Y \times S/S}, F_b) \otimes_{O_{Y \times S}} l_* E(N, F) \to l_* ((\Omega^r_{Y/S}, F_b) \otimes_{O_S} E(N, F))$$

$$\to l_* E((\Omega^r_{Y/S}, F_b) \otimes_{O_S} E(N, F))$$

is a filtered equivalence Zariski, resp. usu, local in $C_{p^* \mathcal{D}_{\text{fil}}} (Y \times S)$.

Proof. (i): Follows from proposition 64 that it is a morphism of $p^* D_S$ module. The fact that it is an equivalence Zariski, resp usu, local is a particular case of proposition 47(i).

(ii): Follows from proposition 64 and the first part of proposition 67 that it is a morphism of $h^* D_S$ module. The fact that it is an equivalence Zariski, resp usu, local is a particular case of proposition 47(ii). □

In the analytical case, we also have

Proposition 71. Let $p : Y \times S \to S$ a projection with $Y, S \in \text{SmVar}(\mathbb{C})$ or with $Y, S \in \text{AnSm}(\mathbb{C})$.

(i) If $\phi : (M, F) \to (N, F)$ is an $r$-filtered usu local equivalence with $(M_1, F), (M_2, F) \in C_{\mathcal{D}^\infty_{\text{fil}}} (Y \times S)$, then

$$DR(Y \times S/S)(\phi) : (\Omega^r_{Y \times S/S}, F_b) \otimes_{O_{Y \times S}} (M, F) \to \Omega^r_{Y \times S/S} \otimes_{O_{Y \times S}} (N, F)$$

is an $r$-filtered equivalence usu local in $C_{p^* \mathcal{D}^\infty_{\text{fil}}} (Y \times S)$.

(ii) Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$

$$D = Y \times S \begin{array}{c}
p \\downarrow \\downarrow \\downarrow \\
\downarrow \\downarrow \\downarrow \\
V \rightarrow S
\end{array}$$

with $p$ the projection. For $(N, F) \in C_{\mathcal{D}^\infty_{\text{fil}}} (Y \times S)$, the map in $C_{p^* O_S} (Y \times S)$ (see definition 60)

$$k \circ T^O_\omega (l, \otimes) (E(N, F)) : (\Omega^r_{Y \times S/S}, F_b) \otimes_{O_{Y \times S}} l_* E(N, F) \to l_* ((\Omega^r_{Y/S}, F_b) \otimes_{O_S} E(N, F))$$

$$\to l_* E((\Omega^r_{Y/S}, F_b) \otimes_{O_S} E(N, F))$$

is a filtered equivalence usu local in $C_{p^* \mathcal{D}^\infty_{\text{fil}}} (Y \times S)$.

Proof. Follows from the finite order case : proposition 70. □

Dually of the De Rahm complex of a $D_S$ module $M$, we have the Spencer complex of $M$. In the particular case of $D_S$, we have the following:
Proposition 72. Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$.

- We have the filtered resolutions of $K_S$ by the following complex of locally free right $D_S$ modules:
  \[ \omega(S) : \omega(K_S) := (\Omega^*_S, F_b)[d_S] \otimes_{O_S} (D_S, F_b) \rightarrow (K_S, F_b) \text{ and } \omega(S) : \omega(K_S, F^{ord}) := (\Omega^*_S, F_b)[d_S] \otimes_{O_S} (D_S, F^{ord}) \rightarrow (K_S, F^{ord}) \]

- Dually, we have the filtered resolution of $O_S$ by the following complex of locally free (left) $D_S$ modules:
  \[ \omega^\vee(S) : \omega(O_S) := (\wedge^* T_S, F_b)[d_S] \otimes_{O_S} (D_S, F_b) \rightarrow (O_S, F_b) \text{ and } \omega^\vee(S) : \omega(O_S, F^{ord}) := (\wedge^* T_S, F_b)[d_S] \otimes_{O_S} (D_S, F^{ord}) \rightarrow (O_S, F^{ord}) \]

Let $S_1, S_2 \in \text{SmVar}(\mathbb{C})$ or $S_1, S_2 \in \text{AnSm}(\mathbb{C})$. Consider the projection $p = p_1 : S_1 \times S_2 \rightarrow S_1$.

- We have the filtered resolution of $D_{S_1 \times S_2 \rightarrow S_1}$ by the following complexes of (left) $(p^* D_{S_1}$ and right $D_{S_1 \times S_2})$ modules:
  \[ \omega(S_1 \times S_2 / S_1) : (\Omega^*_{S_1 \times S_2 / S_1}[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \rightarrow (D_{S_1 \times S_2 \rightarrow S_1}, F^{ord}) \]

- Dually, we have the filtered resolution of $D_{S_1 \times S_2 \rightarrow S_1}$ by the following complexes of (left) $(p^* D_{S_1}, D_{S_1 \times S_2})$ modules:
  \[ \omega^\vee(S_1 \times S_2 / S_1) : (\wedge^* T_{S_1 \times S_2 / S_1}[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \rightarrow (D_{S_1 \times S_2 \rightarrow S_1}, F^{ord}) \]

Proof. See [17]. \(\square\)

In the analytical case, we also have

Proposition 73. Let $S \in \text{AnSm}(\mathbb{C})$.

- We have the filtered resolutions of $K_S$ by the following complex of locally free right $D_S$ modules:
  \[ \omega(S) : \omega(K_S) := (\Omega^*_S, F_b)[d_S] \otimes_{O_S} (D_S^\infty, F^{ord}) \rightarrow (K_S, F_b) \]

- Dually, we have the filtered resolution of $O_S$ by the following complex of locally free (left) $D_S$ modules:
  \[ \omega^\vee(S) : \omega(O_S) := (\wedge^* T_S, F_b)[d_S] \otimes_{O_S} (D_S^\infty, F^{ord}) \rightarrow (O_S, F_b) \]

Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$. Consider the projection $p = p_1 : S_1 \times S_2 \rightarrow S_1$.

- We have the filtered resolution of $D_{S_1 \times S_2 \rightarrow S_1}$ by the following complexes of (left) $(p^* D_{S_1}^\infty$ and right $D_{S_1 \times S_2}^\infty)$ modules:
  \[ \omega(S_1 \times S_2 / S_1) : (\Omega^*_{S_1 \times S_2 / S_1}[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \rightarrow (D_{S_1 \times S_2 \rightarrow S_1}, F^{ord}) \]

- Dually, we have the filtered resolution of $D_{S_1 \times S_2 \rightarrow S_1}$ by the following complexes of (left) $(p^* D_{S_1}^\infty, D_{S_1 \times S_2}^\infty)$ modules:
  \[ \omega^\vee(S_1 \times S_2 / S_1) : (\wedge^* T_{S_1 \times S_2 / S_1}[d_{S_2}], F_b) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}, F^{ord}) \rightarrow (D_{S_1 \times S_2 \rightarrow S_1}, F^{ord}) \]

Proof. Similar to the finite order case: the first map on the right is a surjection and the kernel are obtained by tensoring $D_{S_1}^\infty$ with the kernel of the kozcul resolution of $K_S$ (note that $D_{S_1}^\infty$ is a locally free hence flat $O_S$ module). \(\square\)

Motivated by these resolutions, we make the following definition

Definition 69. (i) Let $i : Z \rightarrow S$ be a closed embedding, with $Z, S \in \text{SmVar}(\mathbb{C})$ or with $Z, S \in \text{AnSm}(\mathbb{C})$. Then, for $(M, F) \in C_{Dfil}(Z)$, we set

\[ i_{*mod}(M, F) := i_0^{\bullet}_{*mod}(M, F) := i_*(M, F) \otimes_{D_{Z \rightarrow S}} (D_{Z \rightarrow S}, F^{ord}) \in C_{Dfil}(S) \]

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Definition 70. (i) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p} S$, where $i$ is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection. Then, for $(M, F) \in C_{D_{fil}}(X)$ we set

$$- f_{*}\text{mod}(M, F) := p_*(DR(S_1 \times S_2/S_1)(M, F)) := p_*(\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)|_{[S_2]} \in C_{D_{fil}}(S_1),$$

$$- p_{*}\text{mod}(M, F) := p_*E(DR(S_1 \times S_2/S_1)(M, F)) := p_*E(\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)|_{[S_2]} \in C_{D_{fil}}(S_1).$$

(ii) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p} S$, where $i$ is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection. Then, for $(M, F) \in C_{D_{fil}}(X)$ we set

$$- f_{*}\text{mod}(M, F) := p_*(DR(S_1 \times S_2/S_1)(M, F)) := p_*(\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)|_{[S_2]} \in C_{D_{fil}}(S_1),$$

$$- p_{*}\text{mod}(M, F) := p_*E(DR(S_1 \times S_2/S_1)(M, F)) := p_*E(\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)|_{[S_2]} \in C_{D_{fil}}(S_1).$$

(iii) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p} S$, where $i$ is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection. Then, for $(M, F) \in C_{D_{fil}}(X)$ we set

$$- f_{*}\text{mod}(M, F) := p_*(DR(S_1 \times S_2/S_1)(M, F)) := p_*(\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)|_{[S_2]} \in C_{D_{fil}}(S_1),$$

$$- p_{*}\text{mod}(M, F) := p_*E(DR(S_1 \times S_2/S_1)(M, F)) := p_*E(\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)|_{[S_2]} \in C_{D_{fil}}(S_1).$$

(iv) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p} S$, where $i$ is the graph embedding and $p_S : X \times S \rightarrow S$ is the projection. Then, for $(M, F) \in C_{D_{fil}}(X)$ we set

$$- f_{*}\text{mod}(M, F) := p_*(DR(S_1 \times S_2/S_1)(M, F)) := p_*(\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)|_{[S_2]} \in C_{D_{fil}}(S_1),$$

$$- p_{*}\text{mod}(M, F) := p_*E(DR(S_1 \times S_2/S_1)(M, F)) := p_*E(\Omega_{S_1 \times S_2/S_1}^\bullet, F_b) \otimes_{O_{S_1 \times S_2}} (M, F)|_{[S_2]} \in C_{D_{fil}}(S_1).$$
Proposition 75. (i) Let \( i : Z \to S \) a closed embedding with \( S,Z \in \text{SmVar}(\mathbb{C}) \) or with \( S,Z \in \text{AnSm}(\mathbb{C}) \). Then for \((M,F) \in C_{D_{fil}}(Z)\), we have

\[
\int_i (M,F) := R\pi_* ((M,F) \otimes_{D_\varnothing} (D_{\varnothing \leftarrow S}, F^{ord})) = i_* (M,F) \otimes_{D_\varnothing} (D_{\varnothing \leftarrow S}, F^{ord}) = i_* \text{mod}(M,F).
\]

(ii) Let \( S_1,S_2 \in \text{SmVar}(\mathbb{C}) \) or \( S_1,S_2 \in \text{AnSm}(\mathbb{C}) \) and \( p : S_{12} := S_1 \times S_2 \to S_1 \) be the projection. Then, for \((M,F) \in C_{D_{fil}}(S_{1 \times S_2})\) we have

\[
\int_p (M,F) := R\pi_* ((M,F) \otimes_{D_{S_{1 \times S_2}}} (D_{S_{1 \times S_2} \leftarrow S_1}, F^{ord})) = p_* E((\Omega^\bullet_{S_{1 \times S_2}/S_1}, F_0) \otimes_{O_{S_{1 \times S_2}}} (D_{S_{1 \times S_2}}, F^{ord})) \otimes_{D_{\varnothing \leftarrow S_1 \times S_2}} (M,F) [d_{S_2}]
\]

where the second equality follows from Griffitz transversality (the canonical isomorphism map respect by definition the filtration).

(iii) Let \( f : X \to S \) be a morphism with \( X,S \in \text{SmVar}(\mathbb{C}) \) or with \( X,S \in \text{AnSm}(\mathbb{C}) \). Then for \( M \in C_D(X)\), we have \( \int_f^{FDR} M = \int_f M \).

Proof. (i): Follows from the fact that \( D_{\varnothing \leftarrow S} \) is a locally free \( D_{\varnothing} \) module and that \( i_* \) is an exact functor. (ii): Since \( \Omega^\bullet_{S_{12}/S_1} d_{S_2}, F_0) \otimes_{O_{S_{12}}} D_{S_{12}} \) is a complex of locally free \( D_{S_1 \times S_2} \) modules, we have in \( D_{fil}(S_1 \times S_2)\), using proposition 72,

\[
(D_{S_1 \times S_2 \leftarrow S_1}, F^{ord}) \otimes_{D_{S_{1 \times S_2}}} (M,F) = (\Omega^\bullet_{S_{12}/S_1} d_{S_2}, F_0) \otimes_{O_{S_{12}}} (D_{S_{12}}, F^{ord}) \otimes_{D_{\varnothing \leftarrow S_{12}}} (M,F).
\]

(iii): Follows from (i) and (ii) by proposition 53 (ii) in the algebraic case and by proposition 56(ii) in the analytic case since a closed embedding is proper.

\[\square\]

Proposition 74. (i) Let \( i : Z \to S \) a morphism with \( Z \) an analytic space and \( S \) a closed embedding with \( S \in \text{AnSm}(\mathbb{C}) \). Then for \((M,F) \in C_{D_{fil}}(Z)\), we have \( \int_i (M,F) = i_* \text{mod}(M,F) \).
(ii) Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \rightarrow S_1$ be the projection. Then, for $(M, F) \in C_{D^{\infty}}(S_1 \times S_2)$ we have

\[
\int_p (M, F) := \text{Rp}_*(M, F) \otimes_{D_{S_1 \times S_2}^{\infty}} (D_{S_1 \times S_2}^{\infty}, F_{\text{ord}})
\]

\[
= p_* E((\Omega_{S_1 \times S_2}^{\infty}, F_{b}) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}^{\infty}, F_{\text{ord}}) \otimes_{D_{S_1 \times S_2}^{\infty}} (M, F)[d_{S_2}])
\]

\[
= p_* E((\Omega_{S_1 \times S_2}^{\infty}, F_{b}) \otimes_{O_{S_1 \times S_2}} (M, F)[d_{S_2}]) := p_{\ast \text{mod}}(M, F).
\]

(iii) Let $S_1, S_2 \in \text{AnSm}(\mathbb{C})$ and $p : S_{12} := S_1 \times S_2 \rightarrow S_1$ be the projection. Then, for $(M, F) \in C_{D^l}(S_1 \times S_2)$ or $(M, F) \in C_{D^{l_h}}(S_1 \times S_2)$, we have

\[
\int^{F_{\text{ord}}} (M, F) := \text{Rp}_{l}(M, F) \otimes_{D_{S_1 \times S_2}} (D_{S_1 \times S_2}, F_{\text{ord}})
\]

\[
= p_* E((\Omega_{S_1 \times S_2}^{\infty}, F_{b}) \otimes_{O_{S_1 \times S_2}} (D_{S_1 \times S_2}^{\infty}, F_{\text{ord}}) \otimes_{D_{S_1 \times S_2}^{\infty}} (M, F)[d_{S_2}])
\]

\[
= p_* E((\Omega_{S_1 \times S_2}^{\infty}, F_{b}) \otimes_{O_{S_1 \times S_2}} (M, F)[d_{S_2}]) := p_{\ast \text{mod}}(M, F).
\]

Proof. (i): Follows from the fact that $D_{S_1 \times S_2}^{\infty}$ is a locally free $D_{S_1}^{\infty}$ module and that $i_*$ is an exact functor.

(ii): Similar to the proof of proposition 74(ii): follows from proposition 73.

(iii): Similar to the proof of proposition 74(ii) follows from proposition 73.

Proposition 76. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{SmVar}(\mathbb{C})$.

(i) Let $(M, F) \in C_{D^{(2)f_1}}(X)$. Then

\[
\int^{F_{\text{ord}}}_{f_2 \circ f_1} (M, F) = \int^{F_{\text{ord}}}_{f_1} \int^{F_{\text{ord}}}_{f_2} (M, F) \in D_{D^{(2)f_1 \circ f_2}}(S).
\]

(ii) Let $(M, F) \in C_{D^{(2)f_1 \circ f_2}}(X)$. Then

\[
\int^{F_{\text{ord}}}_{f_2 \circ f_1} (M, F) = \int^{F_{\text{ord}}}_{f_2} \int^{F_{\text{ord}}}_{f_1} (M, F) \in D_{D^{(2)f_1 \circ f_2}}(S).
\]

Proof. See [22].

Proposition 77. Let $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow S$ two morphism with $X, Y, S \in \text{AnSm}(\mathbb{C})$.

(i) Let $(M, F) \in C_{D^{(2)f_1 \circ f_2}}(X)$. Then

\[
\int^{F_{\text{ord}}}_{f_2 \circ f_1} (M, F) = \int^{F_{\text{ord}}}_{f_1} \int^{F_{\text{ord}}}_{f_2} (M, F).
\]

(ii) Let $(M, F) \in C_{D^{(2)f_1 \circ f_2}}(X)$. Then

\[
\int^{F_{\text{ord}}}_{f_2 \circ f_1} (M, F) = \int^{F_{\text{ord}}}_{f_1} \int^{F_{\text{ord}}}_{f_2} (M, F).
\]

Proof. Similar to proposition 76.

Definition 71. (i) Let $f : X \rightarrow S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$.

Consider the graph factorization $f : X \xrightarrow{l} X \times S \xrightarrow{p} S$, with $l$ the graph embedding and $p$ the projection. We have the transformation map given by, for $(M, F) \in C_D(X),

\[
T(\int^{F_{\text{ord}}}_f, \int^{F_{\text{ord}}}_f) (M, F) := \int_p \int_l (M, F) T((p \circ f_1 \circ f_2)^{\text{ord}}(M, F)) \int_f (M, F)
\]

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(ii) Let \( j : S^o \hookrightarrow S \) an open embedding with \( S \in \text{Var}(\mathbb{C}) \). Consider the graph factorization \( j : S^o \overset{l}{\to} S^o \times S \overset{\gamma}{\to} S \), with \( l \) the graph embedding and \( p \) the projection. We have, for \( (M,F) \in C_{D}fui(S^o) \), the canonical map in \( C_{D}fui(S) \),

\[
T(j_{mod}^{FDR}, j)(M,F) : j_{mod}^{FDR}(M,F) := p_*E((\Omega^{•}_{S^o \times S/S}, F_b) \otimes_{O_{S^o \times S}} l_{mod}(M,F)) \xrightarrow{k_{\omega}(S^o \times S/S)}
\]

\[
p_*E((D_{S^o \times S \to S}, F^{ord}) \otimes_{D_{S^o \times S}} l(D_{S^o \times S} \otimes D_{S^o} E(M,F)) \xrightarrow{T(l, \otimes)(-,-)} j_*E(M,F)
\]

We have, for \( (M,F) \in C_{D}fui(S) \), the canonical map in \( C_{D}fui(S) \),

\[
ad(j^*, j_{mod}^{FDR})(M,F) : (M,F) \xrightarrow{ad(p^{mod} \circ j)(M,F)} p_*E((\Omega^{•}_{S^o \times S/S}, F_b) \otimes_{O_{S^o \times S}} p^{mod}(M,F))
\]

### 4.2.3 The support section functors for D modules and the graph inverse image

Let \( S \in \text{SmVar}(\mathbb{C}) \) or \( S \in \text{AnSm}(\mathbb{C}) \). Let \( i : Z \to S \) a closed embedding and denote \( j : S \setminus Z \to S \) the complementary open embedding. More generally, let \( h : Y \to S \) a morphism with \( Y, S \in \text{Var}(\mathbb{C}) \) or \( Y, S \in \text{AnSp}(\mathbb{C}) \), \( S \) smooth, and let \( i : X \to Y \) a closed embedding and denote by \( j : Y \setminus X \to Y \) the open complementary. We then get from section 2 the following functors:

- **We get the functor**
  \[
  \Gamma_Z : C_{D(2)}fui(S) \to C_{D(2)}fui(S),
  \]
  \[
  (M,F) \mapsto \Gamma_Z(M,F) := \text{Cone}(ad(j^*, j_*)((M,F)) : (M,F) \to j_*j^*(M,F))[1],
  \]
  together we the canonical map \( \gamma_Z(M,F) : \Gamma_Z(M,F) \to (M,F) \), and more generally the functor
  \[
  \Gamma_X : C_{h^*D(2)}fui(Y) \to C_{h^*D(2)}fui(Y),
  \]
  \[
  (M,F) \mapsto \Gamma_X(M,F) := \text{Cone}(ad(j^*, j_*)((M,F)) : (M,F) \to j_*j^*(M,F))[1],
  \]
  together we the canonical map \( \gamma_X(M,F) : \Gamma_X(M,F) \to (M,F) \).

- **We get the functor**
  \[
  \Gamma_Z^{\vee} : C_{D(2)}fui(S) \to C_{D(2)}fui(S),
  \]
  \[
  (M,F) \mapsto \Gamma_Z^{\vee}(M,F) := \text{Cone}(ad(j^*, j_*)((M,F)) : (M,F) \to j_*j^*(M,F)),
  \]
  together we the canonical map \( \gamma_Z^{\vee}(M,F) : \Gamma_Z^{\vee}(M,F) \to (M,F) \), and more generally the functor
  \[
  \Gamma_X^{\vee} : C_{h^*D(2)}fui(Y) \to C_{h^*D(2)}fui(Y),
  \]
  \[
  (M,F) \mapsto \Gamma_X^{\vee}(M,F) := \text{Cone}(ad(j^*, j_*)((M,F)) : (M,F) \to j_*j^*(M,F)),
  \]
  together we the canonical map \( \gamma_X^{\vee}(M,F) : \Gamma_X^{\vee}(M,F) \to (M,F) \).

- **We get the functor**
  \[
  \Gamma_Z^{\vee,h} : C_{D(2)}fui(S) \to C_{D(2)}fui(S),
  \]
  \[
  (M,F) \mapsto \Gamma_Z^{\vee,h}(M,F) := \mathbb{D}_S^K L_D \Gamma_Z E(\mathbb{D}_S^K(M,F)),
  \]
  together with the factorization
  \[
  \gamma_Z^{\vee,h}(L_D(M,F)) : L_D(M,F) \xrightarrow{\gamma_Z^{\vee,h}(L_D(M,F))} \Gamma_Z^{\vee,h}(L_D(M,F)) \xrightarrow{k_{\omega}(L_D(-,-)) \otimes(-)} \Gamma_Z^{\vee,h}(L_D(M,F)),
  \]
  and more generally the functor
  \[
  \Gamma_X^{\vee,h} : C_{h^*D(2)}fui(Y) \to C_{h^*D(2)}fui(Y),
  \]
  \[
  (M,F) \mapsto \Gamma_X^{\vee,h}(M,F) := \mathbb{D}_Y^{h^*D,K} L_{h^*D} \Gamma_X E(\mathbb{D}_Y^{h^*D,K}(M,F)),
  \]
  together with the factorization
  \[
  \gamma_X^{\vee,h}(L_{h^*D}(M,F)) : L_{h^*D}(M,F) \xrightarrow{\gamma_X^{\vee,h}(L_{h^*D}(M,F))} \Gamma_X^{\vee,h}(L_{h^*D}(M,F)) \xrightarrow{k_{\omega}(h^*D,K)(L_{h^*D}(-,-)) \otimes(-)} \Gamma_X^{\vee,h}(L_{h^*D}(M,F)).
  \]

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• We get the functor
\[ \Gamma^\vee_O : C_{D^{(2)}_{fil}}(S) \to C_{D^{(2)}_{fil}}(S), \]
\[ (M, F) \mapsto \Gamma^\vee_O(M, F) := \text{Cone}(b_Z((M, F)) : \mathcal{I}_Z(M, F) \to (M, F)), \]
together we the factorization
\[ \gamma^\vee_O(M, F) : (M, F) \xrightarrow{\beta_{Z}(M, F)} \Gamma^\vee_O(M, F) \xrightarrow{\gamma_{Z}(M, F)} \Gamma^\vee_O(M, F). \]

Since \( M \mapsto M/\mathcal{I}_Z M \) is a right exact functor, \( M \mapsto \Gamma^\vee_O M \) send Zariski, resp. usu, local equivalence between projective complexes of presheaves to Zariski, resp. usu local equivalence, and thus induces in the derived category
\[ L\Gamma^\vee_O : D_{D^{(2)}_{fil}, \infty}(S) \to D_{D^{(2)}_{fil}, \infty}(S), \]
\[ (M, F) \mapsto \Gamma^\vee_O L_D(M, F) := \text{Cone}(b_Z(L_D(M, F)) : \mathcal{I}_Z L_D(M, F) \to L_D(M, F)). \]

• We get the functor
\[ \Gamma^O_Z : C_{D^{(2)}_{fil}}(S) \to C_{D^{(2)}_{fil}}(S), \]
\[ (M, F) \mapsto \Gamma^O_Z(M, F) := \text{Cone}(b_Z((M, F)) : (M, F) \to (M, F) \otimes_{O_S} D^b_S(\mathcal{I}_Z D_S)), \]
together we the factorization
\[ \gamma^O_Z(M, F) : (M, F) \xrightarrow{\beta_{Z}(M, F)} \Gamma^O_Z(M, F) \xrightarrow{\gamma_{Z}(M, F)} (M, F). \]

• We have, for \((M, F) \in C_{D_{fil}}(S)\), a canonical isomorphism
\[ I(D, \gamma^O) : D^K_{S}\Gamma^\vee_O(M, F) \cong \Gamma^O_Z D^K_S(M, F) \]
which gives the transformation map in \( C_{D_{fil}}(S) \)
\[ T(D, \gamma^O) : \Gamma^\vee_O D^K_S(M, F) \to D^K_S \Gamma^O_Z(M, F) \]

Let \( S \in \text{AnSm}(\mathbb{C}) \). Let \( i : Z \hookrightarrow S \) a closed embedding and denote \( j : S \backslash Z \hookrightarrow S \) the complementary open embedding. More generally, let \( h : Y \to S \) a morphism with \( Y, S \in \text{AnSp}(\mathbb{C}) \), \( S \) smooth, and let \( i : X \hookrightarrow Y \) a closed embedding and denote by \( j : Y \backslash X \to Y \) the open complementary.

• We get the functor
\[ \Gamma_Z : C_{D^{(2)}_{fil}}(Y) \to C_{D^{(2)}_{fil}}(Y), \]
\[ (M, F) \mapsto \Gamma_Z(M, F) := \text{Cone}(\text{ad}(j^* j_* ((M, F))) : (M, F) \to j_* j^*(M, F))[-1], \]
together we the canonical map \( \gamma_Z(M, F) : \Gamma_Z(M, F) \to (M, F) \), and more generally the functor
\[ \Gamma_X : C_{h^* D^{(2)}_{fil}}(Y) \to C_{h^* D^{(2)}_{fil}}(Y), \]
\[ (M, F) \mapsto \Gamma_X(M, F) := \text{Cone}(\text{ad}(j^* j_* ((M, F))) : (M, F) \to j_* j^*(M, F))[-1], \]
together we the canonical map \( \gamma_X(M, F) : \Gamma_X(M, F) \to (M, F) \).

• We get the functor
\[ \Gamma^\vee_Z : C_{D^{(2)}_{fil}}(Y) \to C_{D^{(2)}_{fil}}(Y), \]
\[ (M, F) \mapsto \Gamma^\vee_Z(M, F) := \text{Cone}(\text{ad}(j^* j_* ((M, F))) : j_* j^*(M, F) \to (M, F)), \]
together we the canonical map \( \gamma^\vee_Z(M, F) : (M, F) \to \Gamma^\vee_Z(M, F) \), and more generally the functor
\[ \Gamma^\vee_X : C_{h^* D^{(2)}_{fil}}(Y) \to C_{h^* D^{(2)}_{fil}}(Y), \]
\[ (M, F) \mapsto \Gamma^\vee_X(M, F) := \text{Cone}(\text{ad}(j^* j_* ((M, F))) : j_* j^*(M, F) \to (M, F)), \]
together we the canonical map \( \gamma^\vee_X(M, F) : (M, F) \to \Gamma^\vee_X(M, F) \).

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We get the functor
\[
\Gamma^\vee,h_{Z,S}(C_D^{\infty,fil}(S)) \to C_D^{\infty,fil}(S), \quad (M, F) \mapsto \Gamma^\vee,h_{Z,S}(M, F) := D_{S}^\infty K L_D^\infty \Gamma Z E(D_S^{\infty, K}(M, F)),
\]

together with the factorization
\[
\gamma^\vee,h_{Z,S}(L_D^\infty (M, F)) : L_D^\infty (M, F) \xrightarrow{\Delta} \Gamma Z L_D^\infty (M, F) \xrightarrow{\ker D^\infty (j, j^*) \od (\cdot)} \Gamma Z L_D^\infty (M, F),
\]

and more generally the functor
\[
\Gamma^\vee,h_{X,S}(C_h^{\infty,fil}(Y)) \to C_h^{\infty,fil}(Y), \quad (M, F) \mapsto \Gamma^\vee,h_{X,S}(M, F) := D_Y^h \Gamma X E(D_Y^h (M, F)),
\]

together with the factorization
\[
\gamma^\vee,h_{X,S}(L_{h^*} (M, F)) : L_{h^*} (M, F) \xrightarrow{\Delta} \Gamma X L_{h^*} (M, F) \xrightarrow{\ker h^* (j, j^*) \od (\cdot)} \Gamma X L_{h^*} (M, F).
\]

We get the functor
\[
\Gamma^\vee,O_{Z,S}(C_D^{\infty,fil}(S)) \to C_D^{\infty,fil}(S), \quad (M, F) \mapsto \Gamma^\vee,O_{Z,S}(M, F) := \text{Cone}(b_Z((M, F)) : \mathcal{I}_Z(M, F) \to (M, F)),
\]

together we the factorization
\[
\gamma^\vee,O_{Z,S}(M, F) : (M, F) \xrightarrow{\Delta} \Gamma^\vee,O_{Z,S}(M, F) \xrightarrow{b_Z((M, F))} \Gamma^\vee,O_{Z,S}(M, F).
\]

We get the functor
\[
\Gamma^\vee,O_{X,S}(C_h^{\infty,fil}(Y)) \to C_h^{\infty,fil}(Y), \quad (M, F) \mapsto \Gamma^\vee,O_{X,S}(M, F) := \text{Cone}(b_Z((M, F)) : (M, F) \to (M, F) \otimes K D_S^K (\mathcal{I}_Z D_S)),
\]

together we the factorization
\[
\gamma^\vee,O_{X,S}(M, F) : (M, F) \xrightarrow{\Delta} \Gamma^\vee,O_{X,S}(M, F) \xrightarrow{b_Z((M, F))} \Gamma^\vee,O_{X,S}(M, F).
\]

We have, for \((M, F) \in C_D^{\infty,fil}(S)\), a canonical isomorphism
\[
I(D, \gamma^O)(M, F) : D_S^{K, \infty} \Gamma^\vee,O_{Z,S}(M, F) \xrightarrow{\sim} \Gamma^\vee,O_{Z,S}(D_S^{K, \infty})(M, F)
\]

which gives the transformation map in \(C_D^{\infty,fil}(S)\)
\[
T(D, \gamma^O)(M, F) : \Gamma^\vee,O_{Z,S} D_S^{K, \infty}(M, F) \to D_S^{K, \infty} \Gamma^\vee,O_{Z,S}(M, F)
\]

In the analytic case, we have

**Definition 72.** Let \(S \in \text{AnSm}(\mathbb{C})\). For \((M, F) \in C_D^{\infty,fil}(S)\), we have the map in \(C_D^{\infty,fil}(S)\)
\[
T(\infty, \gamma)(M, F) := (I, T(j, \otimes (\cdot), -)) : J_S(\Gamma_Z(M, F)) := \Gamma_Z(M, F) \otimes D_S^\infty (D_S^\infty, \text{ord}) \to \Gamma_Z((M, F) \otimes D_S^\infty (D_S^\infty, \text{ord})) = \Gamma Z J_S(M, F)
\]

Let \(i : Z \hookrightarrow S\) a closed embedding, with \(Z, S \in \text{SmVar}(\mathbb{C})\) or \(Z, S \in \text{AnSm}(\mathbb{C})\). We have the functor
\[
i^\sharp : C_D^{\infty}(S) \to C_D^{\infty}(Z), (M, F) \mapsto i^\sharp(M, F) := \text{Hom}_{D_S}((D_S^{\infty, T}(D_S, \text{ord}), i^\sharp(M, F))
\]

where the (left) \(D_Z\) module structure on \(i^\sharp M\) comes from the right module structure on \(D_S^{\infty, T}\), resp. \(O_Z\). We denote by
• for \((M, F) \in C_{Dfil}(S)\), the canonical map in \(C_{Dfil}(S)\)

\[
\text{ad}(i_{*}\text{mod}, i^\sharp)(M, F) : i_{*}\text{mod}i^\sharp(M, F) := i_*(\text{Hom}_{i^*D_S}(D_S^\tau, F), i^*(M, F)) \otimes_{D_S} (D_S^\tau, F) \\
\rightarrow (M, F), \phi \otimes P \mapsto \phi(P)
\]

• for \((N, F) \in C_{Dfil}(Z)\), the canonical map in \(C_{Dfil}(Z)\)

\[
\text{ad}(i_{*}\text{mod}, i^\sharp)(N, F) : (N, F) \rightarrow i^\sharp i_{*}\text{mod}(N, F) := \text{Hom}_{i^*D_S}(D_S^\tau, i^*i_*(N, F)) \otimes_{D_S} (D_S^\tau, F) \\
n \mapsto (P \mapsto n \otimes P)
\]

The functor \(i^\sharp\) induces in the derived category the functor :

\[
Ri^\sharp : D_{Dfil}(S) \rightarrow D_{Dfil}(Z),
\]

\((M, F) \mapsto R\text{Hom}_{i^*D_S}(D_Z^\tau, F), i^*(M, F)) = \text{Hom}_{i^*D_S}(D_Z^\tau, F), E(i^*(M, F))).

**Proposition 78.** Let \(i : Z \hookrightarrow S\) a closed embedding, with \(Z, S \in \text{SmVar}(\mathbb{C})\) or \(Z, S \in \text{AnSm}(\mathbb{C})\). The functor \(i_{*}\text{mod} : C_{D}(Z) \rightarrow C_{D}(S)\) admit a right adjoint which is the functor \(i^\sharp : C_{D}(S) \rightarrow C_{D}(Z)\) and

\[
\text{ad}(i_{*}\text{mod}, i^\sharp)(N) : N \rightarrow i^\sharp i_{*}\text{mod}N \quad \text{and} \quad \text{ad}(i_{*}\text{mod}, i^\sharp)(M) : i_{*}\text{mod}i^\sharp M \rightarrow M
\]

are the adjonction maps.

**Proof.** See [17] for the algebraic case. The analytic case is completely analogue. \(\square\)

One of the main results in D modules is Kashiwara equivalence :

**Theorem 20.**

(i) Let \(i : Z \hookrightarrow S\) a closed embedding with \(Z, S \in \text{SmVar}(\mathbb{C})\).

- The functor \(i_{*}\text{mod} : \text{QCoh}_D(Z) \rightarrow \text{QCoh}_D(S)\) is an equivalence of category whose inverse is

\[
i^\sharp := a_r i^\sharp : \text{QCoh}_D(S) \rightarrow \text{QCoh}_D(Z).
\]

That is, for \(M \in \text{QCoh}_D(S)\) and \(N \in \text{QCoh}_D(Z)\), the adjonction maps

\[
\text{ad}(i_{*}\text{mod}, i^\sharp)(M) : i_{*}\text{mod}i^\sharp M \xrightarrow{\sim} M, \text{ad}(i_{*}\text{mod}, i^\sharp)(N) : i^\sharp i_{*}\text{mod}N \xrightarrow{\sim} N
\]

are isomorphisms.

- The functor \(\int_i = i_{*}\text{mod} : D_{D}(Z) \rightarrow D_{D}(S)\) is an equivalence of category whose inverse is

\[
Ri^\sharp : D_{D}(S) \rightarrow D_{D}(Z).
\]

That is, for \(M \in D_{D}(S)\) and \(N \in D_{D}(Z)\), the adjonction maps

\[
\text{ad}(\int_i, Ri^\sharp)(M) : \int_i Ri^\sharp M \xrightarrow{\sim} M, \text{ad}(\int_i, Ri^\sharp)(N) : Ri^\sharp \int_i N \xrightarrow{\sim} N
\]

are isomorphisms.

(ii) Let \(i : Z \hookrightarrow S\) a closed embedding with \(Z, S \in \text{AnSm}(\mathbb{C})\).

- The functor \(i_{*}\text{mod} : \text{Coh}_D(Z) \rightarrow \text{Coh}_D(S)\) is an equivalence of category whose inverse is

\[
i^\sharp := a_r i^\sharp : \text{Coh}_D(S) \rightarrow \text{Coh}_D(Z).
\]

That is, for \(M \in \text{Coh}_D(S)\) and \(N \in \text{Coh}_D(Z)\), the adjonction maps

\[
\text{ad}(i_{*}\text{mod}, i^\sharp)(M) : i_{*}\text{mod}i^\sharp M \xrightarrow{\sim} M, \text{ad}(i_{*}\text{mod}, i^\sharp)(N) : i^\sharp i_{*}\text{mod}N \xrightarrow{\sim} N
\]

are isomorphisms.

- The functor \(\int_i = i_{*}\text{mod} : D_{D,c}(Z) \rightarrow D_{D,c}(S)\) is an equivalence of category whose inverse is

\[
Ri^\sharp : D_{D,c}(S) \rightarrow D_{D,c}(Z).
\]

That is, for \(M \in D_{D,c}(S)\) and \(N \in D_{D,c}(Z)\), the adjonction maps

\[
\text{ad}(\int_i, Ri^\sharp)(M) : \int_i Ri^\sharp M \xrightarrow{\sim} M, \text{ad}(\int_i, Ri^\sharp)(N) : Ri^\sharp \int_i N \xrightarrow{\sim} N
\]

are isomorphisms.
Proof. (i): Standard. Note that the second point follows from the first.

(ii): Standard. Note that the second point follows from the first.

We have a canonical embedding of rings \( D_Z \hookrightarrow D_{Z \to S} := i^* D_S \otimes_{i^* O_S} O_Z \). We denote by \( C_{T, D}(Z) \) the category whose objects are complexes of presheaves \( M \) of \( i^* D_S \) modules on \( Z \) such that the cohomology presheaves \( H^n M \) have an induced structure of \( D_Z \) modules. We denote by

\[
q_K : K_{O_S}(i_* O_Z) \to i_* O_Z
\]

the Koszul complex which is a resolution of the \( O_S \) module \( i_* O_Z \) of length \( c = \text{codim}(Z, S) \) by locally free sheaves of finite rank. The fact that it is a locally free resolution of finite rank comes from the fact that \( Z \) is a locally complete intersection in \( S \) since both \( Z \) and \( S \) are smooth. We denote again

\[
q_K = i^* q_K : K_{i^* O_S}(O_Z) := i^* K_{O_S}(i_* O_Z) \to i^* i_* O_Z = O_Z
\]

We denote by \( K_{i^* O_S}^\vee \)(\( O_Z \)) := \( \text{Hom}_{i^* O_S}(K_{i^* O_S}(O_Z), i^* O_S) \) its dual, so that we have a canonical map

\[
q_K^\vee : K_{i^* O_S}(O_Z) \to O_Z[-c].
\]

Let \( M \in C_D(S) \). The \( i^* D_S \) module structure on \( \text{Hom}_{i^* O_S}(K_{i^* O_S}(O_Z), i^* M) \) and \( K_{i^* O_S}(O_Z) \otimes_{i^* O_S} i^* M \) induce a canonical \( D_Z \) module structure on the cohomology groups \( H^n \text{Hom}_{i^* O_S}(K_{i^* O_S}(O_Z), i^* M) \) and \( H^n(K_{i^* O_S}(O_Z) \otimes_{i^* O_S} i^* M) \) for all \( n \in \mathbb{Z} \).

The projection formula for ringed spaces (proposition 9) implies the following lemma:

**Lemma 6.** Let \( i : Z \hookrightarrow S \) a closed embedding with \( Z, S \in \text{Var}(\mathbb{C}) \) or with \( Z, S \in \text{AnSp}(\mathbb{C}) \). Denote by \( j : U := S \setminus Z \to Z \) the open complementary embedding. Then, if \( i \) is a locally complete intersection embedding (e.g. if \( Z, S \) are smooth), we have for \( M \in C_{O_U}(U) \), \( Li^{\text{smod}} R j_* M = 0 \).

**Proof.** We have

\[
i_* Li^{\text{smod}} R j_* M := i_* (i^* L_O(j_* E(M)) \otimes_{i^* O_Z} O_Z) \xrightarrow{T(i, \otimes)(L_O(j_* E(M)), O_Z)^{-1}} L_O(j_* E(M)) \otimes_{i^* O_S} i_* O_Z
\]

\[
\xrightarrow{q_K^{-1}} (j_* E(M)) \otimes_{O_S} i_* K_{i^* O_S}(O_Z) \xrightarrow{T(j, \otimes)(E(M), K_{O_Z}(i_* O_Z))} j_* (E(M) \otimes_{O_U} j^* K_{O_S}(i_* O_Z)),
\]

\( T(i, \otimes)(L_O(j_* E(M)), O_Z) \) being an equivalence Zariski, resp. usu, local by proposition 10 and follows from the fact that \( j^* K_{O_S}(i_* O_Z) \) is acyclic. But

\[
T(j, \otimes)(E(M), K_{O_S}(i_* O_Z)) : (j_* E(M)) \otimes_{O_S} K_{O_S}(i_* O_Z) \to j_* (E(M) \otimes_{O_U} j^* K_{i^* O_S}(O_Z))
\]

is an equivalence Zariski, resp. usu, local by proposition 9 since \( K_{O_S}(i_* O_Z) \) is a finite complex of locally free \( O_S \) modules of finite rank.

We deduce from theorem 20(i) and lemma 6 the localization for \( D \)-modules for a closed embedding of smooth algebraic varieties:

**Theorem 21.** Let \( i : Z \hookrightarrow S \) a closed embedding with \( Z, S \in \text{SmVar}(\mathbb{C}) \). Denote by \( c = \text{codim}(Z, S) \). Then, for \( M \in C_D(S) \), we have by Kashiwara equivalence the following map in \( C_D(S) \):

\[
K_{Z/S}(M) : \Gamma_Z E(M) \xrightarrow{\text{ad}(i^{\text{smod}}, i^*)^{-1}} i^{\text{smod}} L \Gamma_Z E(M)
\]

\[
\xrightarrow{q_Z^{-1}} i^{\text{smod}} L \Gamma_Z E(M) \xrightarrow{\text{Hom}(K_E, i^{*} M) \otimes_{\text{Hom}(O_Z, T(i, \otimes)(E(M)))} \text{Hom}(K_{i^* O_S}(O_Z) \otimes_{i^* O_S} M, i^{\text{smod}} L i^{\text{smod}} M[c]}
\]

**Proof.** Follows from theorem 20(i) and lemma 6 : see [17] for example.
Definition 73. Let \( f : X \to S \) be a morphism, with \( X, S \in \text{SmVar}(\mathbb{C}) \) or \( X, S \in \text{AnSm}(\mathbb{C}) \). Consider the factorization \( f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S \), where \( i \) is the graph embedding and \( p_S : X \times S \to S \) is the projection.

(i) Then, for \((M, F) \in C_{\mathcal{D}(2)fil}(S)\) we set
\[
 f^{*\text{mod}[\cdot]},\Gamma(M, F) := \Gamma_X E(p_S^{*\text{mod}[\cdot]}(M, F)) \in C_{\mathcal{D}(2)fil,\infty}(X \times S),
\]
It induces in the derived category
\[
 Rf^{*\text{mod}[\cdot]},\Gamma(M, F) := f^{*\text{mod}[\cdot]},\Gamma(M, F) := \Gamma_X E(p_S^{*\text{mod}[\cdot]}(M, F)) \in D_{\mathcal{D}(2)fil,\infty}(X \times S),
\]
By definition-proposition 21, we have in the algebraic case \( \mathcal{L}i^{*\text{mod}}f^{*\text{mod}},\Gamma M = Lf^{*\text{mod}}M \in D_{\mathcal{D}}(X) \).

(ii) Then, for \((M, F) \in C_{\mathcal{D}(2)fil}(S)\) we set
\[
 Lf^{*\text{mod}[\cdot]},\Gamma(M, F) := \Gamma_X^{v,h} LDP_S^{*\text{mod}[\cdot]}(M, F) := D_S^K L\Gamma_X E(D_S^K LDP_S^{*\text{mod}[\cdot]}(M, F)) \in D_{\mathcal{D}(2)fil,\infty}(X \times S).
\]
In the analytical case we also have

Definition 74. Let \( f : X \to S \) be a morphism, with \( X, S \in \text{AnSm}(\mathbb{C}) \). Consider the factorization \( f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S \), where \( i \) is the graph embedding and \( p_S : X \times S \to S \) is the projection.

(i) Then, for \((M, F) \in C_{\mathcal{D}(2)\infty}(S)\) we set
\[
 f^{*\text{mod}[\cdot]},\Gamma(M, F) := \Gamma_X E(p_S^{*\text{mod}[\cdot]}(M, F)) \in C_{\mathcal{D}(2)\infty,\infty}(X \times S),
\]
It induces in the derived category
\[
 Rf^{*\text{mod}[\cdot]},\Gamma(M, F) := f^{*\text{mod}[\cdot]},\Gamma(M, F) := \Gamma_X E(p_S^{*\text{mod}[\cdot]}(M, F)) \in D_{\mathcal{D}(2)\infty,\infty}(X \times S),
\]
(ii) Then, for \((M, F) \in C_{\mathcal{D}(2)\infty}(S)\) we set
\[
 Lf^{*\text{mod}[\cdot]},\Gamma(M, F) := \Gamma_X^{v,h} LDP_S^{*\text{mod}[\cdot]}(M, F) := D_S^K L\Gamma_X E(D_S^K LDP_S^{*\text{mod}[\cdot]}(M, F)) \in D_{\mathcal{D}(2)\infty,\infty}(X \times S).
\]

4.2.4 The 2 functors and transformations maps for D modules on the smooth complex algebraic varieties and the complex analytic manifolds

By the definitions and the propositions 53, 54, 76, for the algebraic case, and the propositions 56, 57, 77, for the analytical case,

- we have the 2 functors on \( \text{SmVar}(\mathbb{C}) \):
  - \( C_{\mathcal{D}(2)\text{fil}}(\cdot) : \text{SmVar}(\mathbb{C}) \to C_{\mathcal{D}(2)\text{fil}}(\cdot), \quad S \mapsto C_{\mathcal{D}(2)\text{fil}}(S), \quad (f : T \to S) \mapsto f^{*\text{mod}}, \quad (f : T \to S) \mapsto f^{*\text{mod}[\cdot]}, \)
  - \( D_{\mathcal{D}(2)\text{fil},r}(\cdot) : \text{SmVar}(\mathbb{C}) \to D_{\mathcal{D}(2)\text{fil},r}(\cdot), \quad S \mapsto D_{\mathcal{D}(2)\text{fil},r}(S), \quad (f : T \to S) \mapsto Lf^{*\text{mod}}, \quad (f : T \to S) \mapsto Lf^{*\text{mod}[\cdot]}, \)
  - \( D_{\mathcal{D}(2)\text{fil},\infty}(\cdot) : \text{SmVar}(\mathbb{C}) \to D_{\mathcal{D}(2)\text{fil},\infty}(\cdot), \quad S \mapsto D_{\mathcal{D}(2)\text{fil},\infty}(S), \quad (f : T \to S) \mapsto f^{FDR}, \)
- we have the 2 functors on \( \text{AnSm}(\mathbb{C}) \):
  - \( C_{\mathcal{D}(2)\text{fil}}(\cdot) : \text{AnSm}(\mathbb{C}) \to C_{\mathcal{D}(2)\text{fil}}(\cdot), \quad S \mapsto C_{\mathcal{D}(2)\text{fil}}(S), \quad (f : T \to S) \mapsto f^{*\text{mod}}, \quad (f : T \to S) \mapsto f^{*\text{mod}[\cdot]}, \)
- \( D_{D(2)fil,r}(\cdot) : \text{AnSm}(\mathbb{C}) \to D_{D(2)fil,r}(\cdot) \), \( S \mapsto D_{D(2)fil,r}(S), (f : T \to S) \mapsto Lf^{\text{mod}}, (f : T \to S) \mapsto Lf^{\text{mod}[-]} \),
- \( D_{D(2)fil,\infty}(\cdot) : \text{AnSm}(\mathbb{C}) \to D_{D(2)fil,\infty}(\cdot) \), \( S \mapsto D_{D(2)fil,\infty}(S), (f : T \to S) \mapsto f^{\text{FDR}} \),

we have also the 2 functors on \( \text{AnSm}(\mathbb{C}) \):

- \( C_{D_{D}^{\infty}(2)fil}(\cdot) : \text{AnSm}(\mathbb{C}) \to C_{D_{D}^{\infty}(2)fil}(\cdot) \), \( S \mapsto C_{D_{D}^{\infty}(2)fil}(S), (f : T \to S) \mapsto f^{\text{mod}}, (f : T \to S) \mapsto f^{\text{mod}[-]} \),
- \( D_{D_{D}^{\infty}(2)fil,r}(\cdot) : \text{AnSm}(\mathbb{C}) \to D_{D_{D}^{\infty}(2)fil,r}(\cdot) \), \( S \mapsto D_{D_{D}^{\infty}(2)fil,r}(S), (f : T \to S) \mapsto Lf^{\text{mod}}, (f : T \to S) \mapsto Lf^{\text{mod}[-]} \),
- \( D_{D_{D}^{\infty}(2)fil,r}(\cdot) : \text{AnSm}(\mathbb{C}) \to D_{D_{D}^{\infty}(2)fil,r}(\cdot) \), \( S \mapsto D_{D_{D}^{\infty}(2)fil,r}(S), (f : T \to S) \mapsto f^{\text{FDR}} \),

inducing the following commutative diagrams of functors:

\[
\begin{array}{ccc}
\text{SmVar}(\mathbb{C}) & \xrightarrow{f \mapsto f^{\text{mod}}} & C_{D(2)fil}(\cdot) \\
\downarrow \text{An} & & \downarrow \text{An} \\
\text{AnSm}(\mathbb{C}) & \xrightarrow{f \mapsto f^{\text{mod}}} & C_{D(2)fil}(\cdot)
\end{array}
\quad \text{,} \quad
\begin{array}{ccc}
\text{SmVar}(\mathbb{C}) & \xrightarrow{f \mapsto Lf^{\text{mod}}} & D_{D(2)fil,r}(\cdot) \\
\downarrow \text{An} & & \downarrow \text{An} \\
\text{AnSm}(\mathbb{C}) & \xrightarrow{f \mapsto Lf^{\text{mod}}} & D_{D(2)fil,r}(\cdot)
\end{array}
\]

where, for \( S \in \text{AnSm}(\mathbb{C}) \),

- \( D_{D(2)fil,\infty,h}(S) \subset D_{D(2)fil,\infty,h}(S) \) is the full subcategory consisting of filtered complexes of \( D_S \) module whose cohomology sheaves are regular holonomic,
- \( J : C_{D(2)fil}(S) \to C_{D_{D}^{\infty}(2)fil}(S) \) is the functor \( (M, F) \mapsto J(M, F) := (M, F) \otimes_{D_S} D_S^{\infty} \), which derive trivially.

We first look at the pullback map and the transformation map of De Rahm complexes (see definition 16 and definition-proposition 17) together with the support section functor:

**Proposition 79.** Consider a commutative diagram and a factorization

\[
D_0 = X \xrightarrow{f} S \xrightarrow{g} X = X' \xrightarrow{f'} T \xrightarrow{g'} T
\]

with \( X, X', Y, S, T \in \text{Var}(\mathbb{C}) \) or \( X, X', Y, S, T \in \text{AnSp}(\mathbb{C}) \), \( i, i' \) being closed embeddings, and \( p, p' \) the projections. Denote by \( D \) the right square of \( D_0 \). We have a factorization \( i' : X' \xrightarrow{i'_0} X_T = X \times_Y S \xrightarrow{i'_1} Y \times T \xrightarrow{\gamma} Y \times T \), where \( i'_0, i'_1 \) are closed embedding. Assume \( S, T, Y, Y' \) are smooth.

(i) For \( (M, F) \in C_{Dfil}(Y \times S) \), the canonical map in \( C_{p' \cdot O_Tfil}(Y \times T) \) (c.f. definition-proposition 17),

\[
\gamma^{\ast} \Gamma_X E((\Omega_{Y \times S}^\bullet, \mathcal{F}) \otimes_{O_Y \times S} (M, F)) \to \Gamma_{X_T} E((\Omega_{Y \times T/T}^\bullet, \mathcal{F}) \otimes_{O_{Y \times T}} g^{\ast \text{mod}}(M, F))
\]

is a map in \( C_{p' \cdot Dfil}(Y \times T) \).

(ii) For \( (M, F) \in C_{Dfil}(Y \times S) \), the canonical map in \( C_{O_Tfil}(T) \) (c.f. definition-proposition 17 with \( L_D \) instead of \( L_O \))

\[
\gamma^{\ast} \Gamma_X E((\Omega_{Y \times S}^\bullet \otimes_{O_Y \times S} (M, F)) \to p'_\ast \Gamma_{X_T} E((\Omega_{Y \times T/T}^\bullet \otimes_{O_{Y \times T}} g^{\ast \text{mod}}(M, F))
\]

is a map in \( C_{Dfil}(T) \).

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(iii) For $\mathbf{(N,F)} \in C_{D\fil}(Y \times T)$, the canonical map in $C_{p'\cdot O_T\fil}(Y \times T)$

$$T(X'/X_T, \gamma)(-): \Gamma_X E((\Omega_Y^*_{\cdot X/T}, F_b) \otimes_{O_Y \times T} (N,F)) \rightarrow \Gamma_{X_T} E((\Omega_Y^*_{\cdot X/T}, F_b) \otimes_{O_Y \times T} (N,F))$$

is a map in $C_{p'\cdot D\fil}(Y \times T)$.

(iv) For $M = O_Y$, we have $T^O_\omega(D_{(O_Y \times S)})^\gamma = T^O_\omega(D)^\gamma$ as complexes of $D_T$ modules and $T^O_\omega(X_T/Y \times T)(O_Y \times T)^\gamma = T^O_\omega(X_T/Y \times T)^\gamma$ as complexes of $p' \cdot D_T$ modules.

**Proof.** Follows by definition from proposition 67. □

In the analytical case, we also have

**Proposition 80.** Consider a commutative diagram and a factorization

```
\begin{equation*}
\begin{array}{ccc}
D_0 = & X & \rightarrow S \\
\downarrow g' & \downarrow f & \downarrow g \\
X' & \rightarrow T \\
\end{array}
\end{equation*}
```

with $X, X', Y, S, T \in \text{AnSp}(\mathbb{C})$, $i, i'$ being closed embeddings, and $p, p'$ the projections. Denote by $D$ the right square of $D_0$. We have a factorization $i' : X' \rightarrow X_T = X \times Y \times S Y \times T \rightarrow X \times T$, where $i_0, i'_1$ are closed embeddings. Assume $S, T, Y, Y'$ are smooth.

(i) For $(M, F) \in C_{D\fil}(Y \times S)$, the canonical map in $C_{p'\cdot O_T\fil}(Y \times T)$ (c.f. definition-proposition 17),

$$E(\Omega_{(Y'/X_T)}((X \times S)/(T/S))(M,F)) \circ T(g'', E)(-) \circ T(g'', \gamma)(-) : \gamma^{\ast} \Gamma_X E((\Omega_Y^*_{\cdot X/S}, F_b) \otimes_{O_Y \times S} (M,F)) \rightarrow \Gamma_{X_T} E((\Omega_Y^*_{\cdot X/T}, F_b) \otimes_{O_Y \times T} g''^{\ast \mod}(M,F))$$

is a map in $C_{p'\cdot D\fil}(Y \times T)$.

(ii) For $(M, F) \in C_{D\fil}(Y \times S)$, the canonical map in $C_{O_T\fil}(T)$ (c.f. definition-proposition 17 with $L_{D\fil}$ instead of $L_D$)

$$T^O_\omega(D)(M,F)^\gamma : g^{\ast \mod} L_{D\fil} p_\ast \Gamma_X E((\Omega_Y^*_{\cdot X/S}, F_b) \otimes_{O_Y \times S} (M,F)) \rightarrow p'_\ast \Gamma_{X_T} E((\Omega_Y^*_{\cdot X/T}, F_b) \otimes_{O_Y \times T} g''^{\ast \mod}(M,F))$$

is a map in $C_{D\fil}(T)$.

(iii) For $(N, F) \in C_{D\fil}(Y \times T)$, the canonical map in $C_{p'\cdot O_T\fil}(Y \times T)$

$$T(X'/X_T, \gamma)(-): \Gamma_X E((\Omega_Y^*_{\cdot X/T} \otimes_{O_Y \times T} (N,F)) \rightarrow \Gamma_{X_T} E((\Omega_Y^*_{\cdot X/T} \otimes_{O_Y \times T} (N,F))$$

is a map in $C_{p'\cdot D\fil}(Y \times T)$.

(iv) For $M = O_Y$, we have $T^O_\omega(D_{(O_Y \times S)})^\gamma = T^O_\omega(D)^\gamma$ as complexes of $D_T$ modules and $T^O_\omega(X_T/Y \times T)(O_Y \times T)^\gamma = T^O_\omega(X_T/Y \times T)^\gamma$. as complexes of $p' \cdot D_T$ modules.

**Proof.** Follows from proposition 79. □

Similarly, we have:

**Proposition 81.** Let $p : Y \times S \rightarrow S$ a projection and $i : X \hookrightarrow Y \times S$ a closed embedding with $S, Y \in \text{SmVar}(\mathbb{C})$. 210
(i) For \((M,F) \in C_{\mathcal{D}fil}(Y \times S)\) the canonical map in \(C_{p^*O}fil(Y^{an} \times S^{an})\) (see definition-proposition 17)

\[
E(\Omega_{(Y^{an} \times S^{an})/Y \times S}^{an}/(S^{an}/S)(M,F)) \circ T(an,\gamma)(-): (\Gamma_X E((\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F)))^{an} \rightarrow \Gamma_X E((\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F)))^{an}
\]

is a map in \(C_{\mathcal{D}fil}(Y^{an} \times S)\).

(ii) For \((M,F) \in C_{\mathcal{D}fil}(Y \times S)\) the canonical map in \(C_{O}fil(S^{an})\) (see definition-proposition 17)

\[
T^O_w(an,p)(M,F) : (\Gamma_X E((\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F)))^{an} \rightarrow \Gamma_X E((\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F)))^{an}
\]

is a map in \(C_{\mathcal{D}fil}(S^{an})\).

(iii) For \(M = O_Y\), we have \(T^O_w(an,h)(O_Y) = T^O_w(an,h)^\gamma\) as complexes of \(D_S\) modules.

Proof. Follows by definition from proposition 69.

Let \(p : Y \times S \rightarrow S\) a projection with \(Y, S \in \text{SmVar}(\mathbb{C})\) or with \(Y, S \in \text{AnSm}(\mathbb{C})\). Let \(j : V \hookrightarrow Y \times S\) an open embedding. Consider (see proposition 70), for \((M,F) \in C_{\mathcal{D}fil}(Y \times S)\), the canonical transformation map in \(C_{p^*\mathcal{D}fil}(Y \times S)\)

\[
k \circ T^O_w(j,\otimes)(E(M,F)) : (\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F))^{an} \rightarrow \Gamma_X E((\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F)))^{an}
\]

We have then :

**Proposition 82.** Let \(p : Y \times S \rightarrow S\) a projection with \(Y, S \in \text{SmVar}(\mathbb{C})\) or with \(Y, S \in \text{AnSm}(\mathbb{C})\). Let \(i : X \hookrightarrow Y \times S\) a closed embedding. Then, for \((M,F) \in C_{\mathcal{D}fil}(Y \times S)\)

(i) the canonical map in \(C_{p^*\mathcal{D}fil}(Y)\) (definition 61)

\[
T^O_w(\gamma,\otimes)(M,F) : (\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F))^{an} \rightarrow \Gamma_X E((\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F)))^{an}
\]

is a \((1,1)\)-filtered Zariski, resp usu, local equivalence.

(ii) the map of point (i) gives the following canonical isomorphism in \(D_{p^*\mathcal{D}fil}(Y)\)

\[
T^O_w(\gamma,\otimes)(M,F) : (\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F))^{an} \rightarrow \Gamma_X E((\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F)))^{an}
\]

Proof. By proposition 70.

- \(T^O_w(j,\otimes)(M,F) : (\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F))^{an} \rightarrow \Gamma_X E((\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F)))^{an}\) is a filtered Zariski, resp usu, local equivalence in \(C_{p^*\mathcal{D}fil}(Y \times S)\) and

- \(DR(Y \times S)(k) : (\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F))^{an} \rightarrow (\Omega_{Y^{an} \times S^{an}}^{an}/(S^{an}/S)(M,F))^{an}\) is a filtered Zariski, resp usu, local equivalence in \(C_{p^*\mathcal{D}fil}(Y \times S)\).
(ii): Follows from (i). □

In the analytic case, we also have

**Proposition 83.** Let \( p : Y \times S \to S \) a projection with \( Y, S \in \text{AnSm}(\mathbb{C}) \). Let \( i : X \to Y \) a closed embedding. Then, for \((M,F) \in C_{\mathcal{D}fild}(Y \times S)\)

(i) the canonical map in \( C_{p^*\mathcal{D}fild}(Y) \)

\[
T_w^O(\gamma, \otimes)(M, F) := (I, T_w^O(j, \otimes)(E(M, F))) :
(\Omega_{Y \times S/S}^\bullet, F_0) \otimes_{O_{Y \times S}} \Gamma_X E(M, F) \to \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_0) \otimes_{O_{Y \times S}} E(M, F))
\]

is a map in \( C_{p^*\mathcal{D}fild}(Y \times S) \). Proposition 82 says that it is a filtered equivalence usu local,

(ii) the map of point (i) gives the following canonical isomorphism in \( D_{p^*\mathcal{D}fild}(Y \times S) \)

\[
T_w^O(\gamma, \otimes)(M, F) : (\Omega_{Y \times S/S}^\bullet, F_0) \otimes_{O_{Y \times S}} \Gamma_X E(M, F) \xrightarrow{\Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_0) \otimes_{O_{Y \times S}} (M, F))}
\]

\[
\xrightarrow{\text{DR}(Y \times S/k)^{-1}} \Gamma_X E((\Omega_{Y \times S/S}^\bullet, F_0) \otimes_{O_{Y \times S}} (M, F)).
\]

**Proof.** (i): By Proposition 71

- \( T_w^O(j, \otimes)(M) : \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} j_* j^* E(M) \to j_* E(j^*(\Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} E(M))) \) is an equivalence usu local in \( C_{p^*\mathcal{D}fild}(Y \times S) \) and

- \( \text{DR}(Y \times S/S)(k) : \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} M \to \Omega_{Y \times S/S}^\bullet \otimes_{O_{Y \times S}} E(M) \) is an equivalence usu local in \( C_{p^*\mathcal{D}fild}(Y \times S) \).

(ii): Follows from (i). □

In the projection case, we consider the following canonical maps: Let \( S_1, S_2 \in \text{SmVar}(\mathbb{C}) \) or let \( S_1, S_2 \in \text{AnSm}(\mathbb{C}) \). Denote by \( p = p_1 : S_1 \times S_2 \to S_1 \) and \( p_2 : S_1 \times S_2 \to S_1 \) the projection. We consider

- \( p(M_1, F) : (M_1, F) \to p_s \circ p_{s_2} \circ \text{pm}^{-1}(M_1, F) \) in \( C_{\mathcal{D}fild}(S_1) \), for \((M_1, F) \in C_{\mathcal{D}fild}(S_1) \), which is the composite

\[
p(M_1, F) : (M_1, F) \xrightarrow{\text{ad}(p^*, p_2)(M_1)} p_2^* p_1^*(M_1, F) \xrightarrow{\text{ad}(p_2^*, p_1)(M_1)} p_2^* p_1^*(\Omega_{S_1/S_2}^\bullet, F_0) \otimes_{p_s^* O_{S_1}} p_1^*(M_1, F))
\]

\[
= p_2^* p_1^*(\Omega_{S_1/S_2}^\bullet, F_0) \otimes_{p_s^* O_{S_1}} p_1^*(M_1, F) = p_{s_2} \circ p_{s_1} \circ \text{pm}^{-1}(M_1, F)
\]

where \( m_1 : p_s^* M_1 \to p_2^* M_1 \otimes_{p_s^* \Omega_{S_1}} \Omega_{S_1/S_2}^\bullet \) is given by \( m_1(m) = m \otimes 1 \),

- \( p(M_1, F) : p_s \circ \text{pm}^{-1}(M_1, F) \to (M_1, F) \) in \( C_{\mathcal{D}fild}(S_1 \times S_2) \), for \((M_1, F) \in C_{\mathcal{D}}(S_1 \times S_2) \), which is the composite

\[
p(M_1, F) : p_{s_2} \circ \text{pm}^{-1}(M_1, F) \to (M_1, F) \xrightarrow{\text{ad}(p^*, p_2)(- \otimes_{p_s^* O_{S_1}})(M_1, F)} O_{S_1/S_2} O_{S_1/S_2} (M_1, F) \otimes_{p_s^* O_{S_1}} \Omega_{S_1/S_2}^\bullet \otimes_{p_s^* O_{S_1}} \Omega_{S_1/S_2}^\bullet \xrightarrow{m_1} (M_1, F)
\]

We have then \( p(p_{s_2} \circ \text{pm}^{-1}(M_1, F)) \circ p_{s_2} \circ \text{pm}^{-1}(M_1, F) = I_{p_s \circ \text{pm}^{-1}(M_1, F)} \). It gives the following maps
Definition 75. (i) Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ which is cartesian, together with its factorization

$$D = (f, q) = X \times T \xrightarrow{f'} T \times S$$

$$D = (f, q) = f'' : X \times T \xrightarrow{i''} X \times T \times S \xrightarrow{p''} T \times S,$$

where the squares are cartesian, $f = p \circ i$ being the graph factorization and $q, q'$ the projections. We have, for $(M, F) \in C_{D(2),\text{fil},\infty}(X)$, the following transformation map in $C_{D(2),\text{fil},\infty}(T \times S)$:

$$T^{D_{\text{mod}}} (f, q)(M, F) : q''^{\text{mod}}p_{s}E((\Omega_{X \times T \times S}, F_{b}) \otimes_{O_{X \times T \times S}} \mathbb{i}_{\text{mod}}(M, F))$$

$$\xrightarrow{T^{D_{\text{mod}}} (f, q)(i_{\text{mod}}(M, F))} p''_{s}E((\Omega_{X \times T \times S}, F_{b}) \otimes_{O_{X \times T \times S}} \mathbb{q}''^{\text{mod}}\mathbb{i}_{\text{mod}}(M, F))$$

$$\xrightarrow{p_{s}E(T^{D_{\text{mod}}} (i, q'')(M, F)) \otimes i} p'_{s}E((\Omega_{X \times T \times S}, F_{b}) \otimes_{O_{X \times T \times S}} \mathbb{q}'^{\text{mod}}\mathbb{i}_{\text{mod}}(M, F))$$

where

$$T^{D_{\text{mod}}} (i, q'')(M, F) : q''^{\text{mod}}\mathbb{i}_{\text{mod}}(M, F) := \mathbb{i}_{\text{mod}}(D^{X_{\text{mod}}}(D_{X \times S}, F^{\text{ord}}))$$

$$\xrightarrow{T^{\text{mod}}(q'', i)(-)} \mathbb{i}_{\text{mod}}(D_{X \times S}, F^{\text{ord}})$$

$$\xrightarrow{\mathbb{i}_{\text{mod}}(q''^{\text{mod}}(M, F) \otimes_{q'} D_{X} \mathbb{q}'^{\text{mod}}(D_{X \times S}, F^{\text{ord}}))} \mathbb{i}_{\text{mod}}(D_{X \times S}, F^{\text{ord}})$$

(ii) Consider a commutative diagram in $\text{SmVar}(\mathbb{C})$ which is cartesian, together with its factorization

$$D = (f, g) = X_{T} \xrightarrow{f'} T \xrightarrow{g} X\ ,$$

$$D = (f, g) = f' : X_{T} \xrightarrow{i'} X \times T \xrightarrow{p'} T \times S$$

$$f'' : X \times T \xrightarrow{i''} X \times T \times S \xrightarrow{p''} T \times S$$

$$f : X \xrightarrow{i} X \times S \xrightarrow{p} S\ .$$
where the squares are cartesian, $f = p \circ i$, $g = q \circ l$ being the graph factorizations. We have, for $(M, F) \in D_{D(2)fil,\infty}(X)$, the following transformation map in $D_{D(2)fil,\infty}(T \times S)$:

$$T^{D^{\text{mod}}(f,g)}((M, F)) : Rg^{*^{\text{mod}}}(M, F) \xrightarrow{f} \int^{\text{FDR}} (M, F) := \Gamma_{T}E(q^{*^{\text{mod}}}, E(((\Omega_{X S/S}, F_{b}) \otimes_{\Omega_{X S/S}} i^{*^{\text{mod}}}(M, F))$$

$$(ii')$$ We have, for $(M, F) \in D_{D}(X)$, the following transformation map in $D_{D}(T)$:

$$T^{D^{\text{mod}}(f,g)}((M, F)) : g^{*^{\text{mod}}}(M, F) \xrightarrow{i^{*^{\text{mod}}}} \int^{\text{FDR}} i^{*^{\text{mod}}}(M, F)$$

$$(ii)$$ We have, for $(M, F) \in D_{D(2)fil,\infty}(X)$, the following transformation map in $D_{D(2)fil,\infty}(T \times S)$:

$$T^{D^{\text{mod}}(f,g)}((M, F)) : Rg^{*^{\text{mod}}}(M, F) \xrightarrow{f} \int^{\text{FDR}} (M, F) := \Gamma_{T}E(q^{*^{\text{mod}}}, E(((\Omega_{X S/S}, F_{b}) \otimes_{\Omega_{X S/S}} i^{*^{\text{mod}}}(M, F))$$

$$(ii')$$ We have, for $(M, F) \in D_{D}(X)$, the following transformation map in $D_{D}(T)$:

$$T^{D^{\text{mod}}(f,g)}((M, F)) : g^{*^{\text{mod}}}(M, F) \xrightarrow{i^{*^{\text{mod}}}} \int^{\text{FDR}} i^{*^{\text{mod}}}(M, F)$$

$$(ii)$$ We have, for $(M, F) \in D_{D(2)fil,\infty}(X)$, the following transformation map in $D_{D(2)fil,\infty}(T \times S)$:

$$T^{D^{\text{mod}}(f,g)}((M, F)) : Rg^{*^{\text{mod}}}(M, F) \xrightarrow{f} \int^{\text{FDR}} (M, F) := \Gamma_{T}E(q^{*^{\text{mod}}}, E(((\Omega_{X S/S}, F_{b}) \otimes_{\Omega_{X S/S}} i^{*^{\text{mod}}}(M, F))$$

$$(ii')$$ We have, for $(M, F) \in D_{D}(X)$, the following transformation map in $D_{D}(T)$:

$$T^{D^{\text{mod}}(f,g)}((M, F)) : g^{*^{\text{mod}}}(M, F) \xrightarrow{i^{*^{\text{mod}}}} \int^{\text{FDR}} i^{*^{\text{mod}}}(M, F)$$

In the analytic case, we have:

**Definition 76.** Consider a commutative diagram in $\text{AnSm}(\mathbb{C})$ which is cartesian together with a factorization

$$D = (f, g) = X \xrightarrow{i'} T \xrightarrow{p'} Y \times T \xrightarrow{p} S,$$

where $Y \in \text{AnSm}(\mathbb{C})$, $i$, $i'$ are closed embeddings and $p$, $p'$ the projections.

(i) We have, for $(M, F) \in D_{D(2)fil,\infty,h}(X)$, the following transformation map in $D_{D(2)fil,\infty}(T \times S)$:

$$T^{D^{\text{mod}}(f,g)}((M, F)) : Rg^{*^{\text{mod}}}(M, F) \xrightarrow{f} \int^{\text{FDR}} (M, F) := \Gamma_{T}E(q^{*^{\text{mod}}}, E(((\Omega_{X S/S}, F_{b}) \otimes_{\Omega_{X S/S}} i^{*^{\text{mod}}}(M, F))$$

define in the same way as in definition 75

(ii) For $(M, F) \in D_{D(2)fil,\infty}(X)$, the following transformation map in $D_{D(2)fil,\infty}(T \times S)$:

$$T^{D^{\text{mod}}(f,g)}((M, F)) : Rg^{*^{\text{mod}}}(M, F) \xrightarrow{f} \int^{\text{FDR}} (M, F) := \Gamma_{T}E(q^{*^{\text{mod}}}, E(((\Omega_{X S/S}, F_{b}) \otimes_{\Omega_{X S/S}} i^{*^{\text{mod}}}(M, F))$$

is defined in the same way as in (ii) : see definition 75.
In the algebraic case, we have the following proposition:

**Proposition 84.** Consider a cartesian square in $\text{SmVar}(\mathbb{C})$

\[
\begin{array}{ccc}
D = X_T & \xrightarrow{g'} & X \\
\downarrow f & & \downarrow f \\
T & \xrightarrow{g} & S
\end{array}
\]

(i) For $(M,F) \in D_{D(2)fil,\infty,c}(X)$,

\[
T^{D_{mod}}(f,g)((M,F)) : Rg^{\ast_{mod}} \int_f^{FDR} (M,F) \xrightarrow{\sim} \int_{f'}^{FDR} Rg'_{\ast_{mod}}(M,F)
\]

is an isomorphism in $D_{D(2)fil,\infty}(T)$.

(ii) For $M \in D_{D,c}(X)$,

\[
T^{D_{mod}}(f,g)(M) : g^{\ast_{mod}} \int_f M \xrightarrow{\sim} \int_{f'} g'_{\ast_{mod}} M
\]

is an isomorphism in $D_{D}(T)$.

**Proof.** Follows from the projection case and the closed embedding case.

In the analytic case, we have similarly:

**Proposition 85.** Consider a cartesian square in $\text{AnSm}(\mathbb{C})$

\[
\begin{array}{ccc}
D = X_T & \xrightarrow{g'} & X \\
\downarrow f & & \downarrow f \\
T & \xrightarrow{g} & S
\end{array}
\]

(i) Assume that $f$, hence $f'$ is proper. For $(M,F) \in D_{D(2)fil,\infty,h}(X)$,

\[
T^{D_{mod}}(f,g)((M,F)) : Rg^{\ast_{mod}} \int_f^{FDR} (M,F) \xrightarrow{\sim} \int_{f'}^{FDR} Rg'_{\ast_{mod}}(M,F)
\]

is an isomorphism in $D_{D(2)fil,\infty}(T)$.

(ii) For $(M,F) \in D_{D,\infty}(2)fil,\infty,h(X)$,

\[
T^{D_{mod}}(f,g)((M,F)) : Rg^{\ast_{mod}} \int_f^{FDR} (M,F) \xrightarrow{\sim} \int_{f'}^{FDR} Rg'_{\ast_{mod}}(M,F)
\]

is an isomorphism in $D_{D,\infty}(2)fil,\infty(T)$.

**Proof.** (i):Similar to the proof of proposition 84.

(ii):Similar to the proof of proposition 84.

**Definition 77.** Let $f : X \rightarrow S$ a morphism with $X,S \in \text{SmVar}(\mathbb{C})$.

(i) We have, for $(M,F) \in C_{Dfil}(S)$ and $(N,F) \in C_{Dfil}(X)$, we have the map in $C_{Dfil}(S)$

\[
T^{D_{mod,0}}(\otimes,f)((M,F),(N,F)) : (M,F) \otimes_{O_S} f^{0}_{\ast_{mod}}(N,F) := (M,F) \otimes_{O_S} f_*(N,F) \otimes_{D_X} (D_{X-S,F_{ord}})
\]

\[
\overset{\lor}{\underset{\lor}{\longrightarrow}} f_*(f^{\ast_{mod}}(M,F) \otimes_{f^{\ast_{mod}}(N,F)} \otimes_{D_X} (D_{X-S,F_{ord}})) \xrightarrow{\lor} f_*(f^{\ast_{mod}}(M,F) \otimes_{O_X} (N,F) \otimes_{D_X} (D_{X-S,F_{ord}})) =: f^{0}_{\ast_{mod}}(f^{\ast_{mod}}(M,F) \otimes_{O_X} (N,F))
\]

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(ii) Consider the cartesian square
\[
    \begin{array}{ccc}
    D & \xrightarrow{i} & X \times S \\
    \downarrow f & & \downarrow f \times I_S \\
    S & \xrightarrow{\Delta_S} & S \times S
    \end{array}
\]
where \(i_f = (f \times I_S) \circ \Delta_X : X \hookrightarrow X \times S\) is the graph embedding. Then, for \((M, F) \in C_{D(2)}(S)\) and \((N, F) \in C_{Dfil}(X)\), we have the map in \(D_{D(2)}(f, r, S)\)
\[
    T^{D,0}_{\text{mod}}(\otimes, f)((M, F), (N, F)) : \left(\int_f \left((N, F) \otimes_{O_X} f^*_\text{mod} FDR(M, F)\right)\right) \xrightarrow{\int_f i^*_\text{mod} FDR(p^*_X N \otimes p^*_S M)} \left(\int_f \left((N, F) \otimes_{O_X} f^*_\text{mod} FDR(M, F)\right)\right)
\]
Clearly if \(i : Z \hookrightarrow S\) is a closed embedding with \(Z, S \in \text{SmVar}(\mathbb{C})\) or with \(Z, S \in \text{AnSm}(\mathbb{C})\), then \(T^{D,0}_{\text{mod}}(\otimes, i)(M, N) = T^D(\otimes, i)(M, N)\) in \(D_{D(2)}(f, r, S)\).

We have then the following :

**Proposition 86.**

(i) Let \(i : Z \hookrightarrow S\) is a closed embedding with \(Z, S \in \text{SmVar}(\mathbb{C})\), then for \((M, F) \in C_{Dfil}(S)\) and \((N, F) \in C_{Dfil}(Z)\)
\[
    T^{D,0}_{\text{mod}}(\otimes, i)((M, F), (N, F)) : (M, F) \otimes_{O_S} i^*_\text{mod} (N, F) \xrightarrow{\sim} i^*_\text{mod} (i^*_\text{mod} (M, F) \otimes_{O_Z} (N, F))
\]
is an isomorphism in \(C_{Dfil}(S)\).

(ii) Let \(f : X \to S\) a morphism with \(X, S \in \text{SmVar}(\mathbb{C})\). Then, for \((M, F) \in C_{D(2)}(S)\) and \((N, F) \in C_{D(2)}(S)\),
\[
    T^{D,0}_{\text{mod}}(\otimes, f)((M, F), (N, F)) : \left(\int_f \left((M, F) \otimes_{O_X} f^*_\text{mod} FDR(N, F)\right)\right) \xrightarrow{\sim} \left(\int_f (M, F) \otimes_{O_Y} (N, F)\right)
\]
is an isomorphism in \(D_{D(2)}(f, r, S)\).

**Proof.** (i): Follows from proposition 10. (ii): Follows from proposition 84(i). \(\square\)

Let \(f : X \to S\) a morphism with \(X, S \in \text{SmVar}(\mathbb{C})\). Consider the graph embedding \(f : X \xrightarrow{\Delta_X} X \times S \xrightarrow{\sim} S\), with \(X, Y, S \in \text{SmVar}(\mathbb{C})\). We have, for \((M, F) \in C_{Dfil}(X)\), the canonical isomorphism in \(C_{D(2)}(S)\)
\[
    \text{an}^*_{\text{mod}} i^*_\text{mod} L_D(p^*_\text{mod} (M, F) \otimes_{O_X} \mathcal{V}_X)) \xrightarrow{\sim} i^*_\text{mod} L_Dp^*_\text{mod} ((M, F) \otimes_{O_X} \mathcal{V}_X)
\]
We then define and study the transformation map between the direct image functor and the analytical functor for \(D\)-modules :

**Definition 78.** Let \(f : X \to S\) a morphism with \(X, S \in \text{SmVar}(\mathbb{C})\).

(i) We have for \((M, F) \in C_{D(2)}(X)\) the canonical map in \(C_{D(2)}(S)\)
\[
    T^{D,0}_{\text{mod}}(\text{an}, f)(M, F) : \text{an}^*_{\text{mod}} f_* E(D_{X \to S} F_{\text{ord}}) \otimes_{D_X} L_D(M, F)) \xrightarrow{T^{\text{mod}}(\text{an}, f)(-)} f_* E((D_{X \to S} F_{\text{ord}}) \otimes_{D_X} L_D(M, F))\]
(i) Consider the graph embedding \( f : X \to X \times S \xrightarrow{p} S \), with \( X, Y, S \in \text{SmVar}(\mathbb{C}) \). We have, for \((M, F) \in C_{D\text{fil}}(X)\), the canonical map in \( C_{D(2)\text{fil}}(S^{an})\)

\[
T^{\text{Dmod}}_{\text{an}, f}(an, f)(M, F) : an_{S}^{\text{mod}}(p_{*}E((\Omega^{\bullet}_{Y \times S/S}, F_{b}) \otimes_{O_{Y \times S}} i_{\text{mod}}(M, F)))
\]

\[
\xrightarrow{T^{\text{Dmod}}_{\text{an}, f}(an, f)(i_{\text{mod}}(M, F))} p_{*}E((\Omega^{\bullet}_{Y \times S/S}, F_{b}) \otimes_{O_{Y \times S}} i_{\text{mod}}(M, F))^{an}
\]

\[
p_{*}E((\Omega^{\bullet}_{Y \times S/S}, F_{b}) \otimes_{O_{Y \times S}} i_{\text{mod}}(M, F))^{an}
\]

In order to prove that this map gives an isomorphism in the derived category in the non filtered case if \( f \) is proper and \( M \) coherent, we will need the following (c.f. [17]):

**Theorem 22.** A product \( X \times S \) of a smooth projective variety \( X \) and a smooth affine variety \( S \) is \( D \)-affine.

**Proof.** See [17] theorem 1.6.5. \( \square \)

A main result is that we have the following version of the first GAGA theorem for coherent \( D \)-modules:

**Theorem 23.** Let \( f : X \to S \) a morphism with \( X, S \in \text{SmVar}(\mathbb{C}) \). Let \( M \in D_{D(2)\text{fil},c}(X) \), for \( r = 1, \ldots, \infty \). If \( f \) is proper,

\[
T^{\text{Dmod}}_{\text{an}, f}(M, F) : (\int_{r}^{f} M)^{an} \xrightarrow{\sim} \int_{r}^{f} (M^{an})
\]

is an isomorphism.

**Proof.** We may assume that \( f \) is projective, so that we have a factorization \( f : X \xrightarrow{i} \mathbb{P}^{N} \times S \xrightarrow{p} S \) where \( i \) is a closed embedding and \( p \) the projection. The question being local on \( S \), we may assume that \( S \) is affine. Since \( \mathbb{P}^{N} \times S \) is \( D \)-affine by theorem 22, we have by proposition 50(iii) a complex \( F \in C_{D}(\mathbb{P}^{N} \times S) \) such that \( i_{\text{mod}} M = F \simeq F \in D_{D,r}(\mathbb{P}^{N} \times S) \) and each \( F^{n} \) is a direct summand of a free \( D_{\mathbb{P}^{N} \times S}^{an} \) module of finite rank. The theorem now follows from the fact that \( \int_{r}^{f} D_{\mathbb{P}^{N} \times S}^{an} \simeq D_{S}[-N] \) and the fact that \( (D_{S})^{an} = D_{S}^{an} \). \( \square \)

We also have

**Definition 79.** (i) Let \( f : X \to S \) a morphism with \( X, S \in \text{SmVar}(\mathbb{C}) \) or with \( X, S \in \text{AnSm}(\mathbb{C}) \). We have, for \( M, N \in C_{D\text{fil}}(X) \), the canonical transformation map in \( D_{D\text{fil}, \infty}(S) \)

\[
T^{0}(f, \text{hom})(M, F), (N, F)) : Rf_{*}R\text{Hom}_{D_{X}}((M, F), (N, F)) \rightarrow
\]

\[
Rf_{*}R\text{Hom}_{D_{X}}((M, F) \otimes_{D_{X}} L_{D}(D_{X}^{\text{r}, S}, F^{\text{ord}})), (N, F) \otimes_{D_{X}} L_{D}(D_{X}^{\text{r}, S}, F^{\text{ord}}))
\]

\[
= R\text{Hom}_{D_{X}}((f, f_{*}M), f_{*}(N, F))
\]

(ii) Let \( f : X \to S \) a morphism with \( X, S \in \text{AnSm}(\mathbb{C}) \). We have, for \( (M, F), (N, F) \in C_{D\text{fil}}(X) \), the canonical transformation map in \( D_{D\text{fil}, \infty}(S) \)

\[
T^{0}(f_{*}, \text{hom})(M, F), (N, F)) : Rf_{*}R\text{Hom}_{D_{S}}((M, F), (N, F)) \rightarrow
\]

\[
Rf_{*}R\text{Hom}_{D_{X}}((M, F) \otimes_{D_{X}} L_{D}(D_{X}^{\text{r}, S}, F^{\text{ord}})), (N, F) \otimes_{D_{X}} L_{D}(D_{X}^{\text{r}, S}, F^{\text{ord}}))
\]

\[
= R\text{Hom}_{D_{X}}((f_{*}, f_{*}M), f_{*}(N, F))
\]

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\textbf{Definition 80.} Let \( f : X \to S \) a morphism with \( X,S \in \text{SmVar}(\mathbb{C}) \) or with \( X,S \in \text{AnSm}(\mathbb{C}) \). We have, for \((M,F),(N,F) \in C_{Dfil}(S)\), the canonical transformation map in \( C_{Dfil}(X) \)
\[
T^D(f,\text{hom})((M,F),(N,F)) : f^*\text{Hom}_{D_S}((M,F),(N,F)) \to \text{Hom}_{D_S}(f^*(M,F),(f^*(N,F)) \to \text{Hom}_{D_X}(f^*(M,F) \otimes _{f^*D_S} Lf_*(D_X \to S,F^{ord}),f^*(N,F) \otimes _{f^*D_S} Lf_*(D_X \to S,F^{ord})) \to \text{Hom}_{D_X}(f^{\text{mod}}(M,F),f^{\text{mod}}(N,F))
\]
which is the one given by Kashiwara (see [20]).

In the algebraic case, we have, in the non filtered case, the six functor formalism for holonomic D-modules:

\textbf{Theorem 24.} Let \( f : X \to S \) a morphism with \( X,S \in \text{SmVar}(\mathbb{C}) \).

(i) We have, for \( M \in D_{D,h}(X) \) and \( N \in D_{D,h}(S) \) a canonical isomorphism in \( D_D(S) \)
\[
I^{\text{mod}}(L^f_{f^{\text{mod}}[-]},\int_f)(M,N) : Rf_*R\text{Hom}_{D_X}(L^f_{f^{\text{mod}}[-]}N,M) \cong R\text{Hom}_{D_S}(N,\int_f M).
\]

(ii) We have, for \( M \in D_{D,h}(X) \) and \( N \in D_{D,h}(S) \) a canonical isomorphism in \( D_D(X) \)
\[
I^{\text{mod}}(\int_{f!} L^f_{f^{\text{mod}}[-]}(M,N) : R\text{Hom}_{D_X}(\int_{f!} M,N) \cong Rf_*R\text{Hom}_{D_S}(M,L^f_{f^{\text{mod}}[-]}N).
\]

\textbf{Proof.} Follows from the projection case and the closed embedding case. \qed

\textbf{Corollary 2.} Let \( f : X \to S \) a morphism with \( X,S \in \text{SmVar}(\mathbb{C}) \). Then,

- \((L^f_{f^{\text{mod}}[-]},\int_f) : D_{D,h}(S) \to D_{D,h}(X) \) is a pair of adjoint functors.

- \((\int_{f!} L^f_{f^{\text{mod}}[-]}) : D_{D,h}(S) \to D_{D,h}(X) \) is a pair of adjoint functors.

\textbf{Proof.} Follows immediately from theorem 24 by taking global sections. \qed

Consider a commutative diagram in \( \text{SmVar}(\mathbb{C}) \),
\[
\begin{array}{ccc}
D & \xrightarrow{f'} & T \\
\downarrow{g} & & \downarrow{g} \\
X & \xrightarrow{f} & S
\end{array}
\]

We have, for \( M \in C_{D,h}(X) \), the following transformation maps
\[
T_1^{D^{\text{mod}}}(D)(M) : Lg_{f^{\text{mod}}[-]} \int_f M \xrightarrow{\text{ad}(Lg_{f^{\text{mod}}[-]}f_f)(-)} \int_{f'} Lg'_{f^{\text{mod}}[-]}Lg_{f^{\text{mod}}[-]} \int_f M \cong \int_{f'} Lg'_{f^{\text{mod}}[-]}Lg_{f^{\text{mod}}[-]} \int_{f!} M
\]
and
\[
T_2^{D^{\text{mod}}}(D)(M,F) : \int_{f!} Lg'_{f^{\text{mod}}[-]}M \xrightarrow{\text{ad}(f_{f!}Lg'_{f^{\text{mod}}[-]}(-))} \int_{f!} Lg'_{f^{\text{mod}}[-]}Lg_{f^{\text{mod}}[-]} \int_{f!} M \cong \int_{f!} Lg'_{f^{\text{mod}}[-]}Lg_{f^{\text{mod}}[-]} \int_{f!} F^{DR} M
\]

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Proposition 87. Consider a cartesian square in $\text{SmVar}(\mathbb{C})$

\[
D = \begin{array}{c}
X_f \\
\downarrow g' \\
T \\
\downarrow f
\end{array} \begin{array}{c}
X \\
\downarrow f \\
S
\end{array}
\]

Assume that $f$ (and hence $f'$) is proper. Then, for $(M, F) \in D_{\text{P}(2)_{\text{fit, \infty, h}}}(X)$,

- $T^\text{D}_{1}^\text{mod}(f, g)(M) : Lg^\text{smod[\cdot]} \overset{\sim}{\to} \int_{f'} Lg^\text{smod[\cdot]} M$ and
- $T^\text{D}_{2}^\text{mod}(f, g)(M) : \int_{f'} Lg^\text{smod[\cdot]} M \overset{\sim}{\to} Lg^\text{smod[\cdot]} \int_{f'} M$

are isomorphisms in $D_{\text{P}}(T)$.

Proof. Follows from proposition 84 and the fact that the map $T^\text{D}_{1}^\text{mod}(f, g)(M)$ is given by the composite

\[
T^\text{D}_{1}^\text{mod}(f, g)(M)[d_T - d_S] : Lg^\text{smod} \int_{f} (M, F) = L\text{D}_{T} Lg^\text{smod} L\text{D}_{S} \int_{f} M \overset{T(f, f)(\cdot)}{\longrightarrow} L\text{D}_{T} Lg^\text{smod} \int_{f} M \overset{\text{D} T(f, f)(\cdot)}{\longrightarrow} L\text{D}_{T} Lg^\text{smod} \int_{f} Lg^\text{smod[\cdot]} \int_{f'} M
\]

and the map $T^\text{D}_{2}^\text{mod}(f, g)(M, F)$ is given by the composite

\[
T^\text{D}_{2}^\text{mod}(f, g)(M)[d_T - d_S] : \int_{f'} Lg^\text{smod} M = L\text{D}_{T} \int_{f'} Lg^\text{smod} M \overset{\text{D} d(\cdot) \cdot T(f, f)(\cdot)}{\longrightarrow} L\text{D}_{T} Lg^\text{smod} \int_{f} Lg^\text{smod}(M, F) \int_{f'} M
\]

\[
\square
\]

4.3 The D modules on singular algebraic varieties and singular complex analytic spaces

In this subsection by defining the category of complexes of filtered D-modules in the singular case and there functionalities.

4.3.1 Definition

In all this subsection, we fix the notations:

- For $S \in \text{Var}(\mathbb{C})$, we denote by $\tilde{S} = \cup_i S_i$ an open cover such that there exits closed embeddings $i_i : S_i \to \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have then closed embeddings $i_I : S_I := \cap_{i \in I} S_i \hookrightarrow \tilde{S}_I := \Pi_{i \in I} \tilde{S}_i$. Then for $I \subset J$, we denote by $j_{IJ} : S_I \to S_I$ the open embedding and $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ the projection, so that $p_{IJ} \circ i_J = i_I \circ j_{IJ}$. This gives the diagram of algebraic varieties $(\tilde{S}_I) \in \text{Func}(\mathcal{P}(\mathbb{N}), \text{Var}(\mathbb{C}))$ which gives the diagram of sites $(\tilde{S}_I) : \text{Ouv}(\tilde{S}_I) \in \text{Func}(\mathcal{P}(\mathbb{N}), \text{Cat})$. It also gives the diagram of sites $(\tilde{S}_I)^{\text{op}} : \text{Ouv}(\tilde{S}_I)^{\text{op}} \in \text{Func}(\mathcal{P}(\mathbb{N}), \text{Cat})$. For $I \subset J$, we denote by $m : \tilde{S}_I \setminus (S_I \setminus S_J) \hookrightarrow \tilde{S}_I$ the open embedding.

- For $S \in \text{AnSp}(\mathbb{C})$ we denote by $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have then closed embeddings $i_I : S_I := \cap_{i \in I} S_i \hookrightarrow \tilde{S}_I := \Pi_{i \in I} \tilde{S}_i$. 219
Then for $I \subset J$, we denote by $j_{IJ} : S_J \hookrightarrow S_I$ the open embedding and $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ the projection, so that $p_{IJ} \circ i_J = i_I \circ j_{IJ}$. This gives the diagram of analytic spaces $(\tilde{S}_I) \in \Fun(P(N), \AnSp(C))$ which gives the diagram of sites $(\tilde{S}_I) := \Ouv(\tilde{S}_I) \in \Fun(P(N), \Cat)$. It also gives the diagram of sites $(\tilde{S}_I)^{op} := \Ouv(\tilde{S}_I)^{op} \in \Fun(P(N), \Cat)$. For $I \subset J$, we denote by $m : \tilde{S}_I \setminus (\tilde{S}_I \setminus S_I) \to \tilde{S}_I$ the open embedding.

The first definition is from [28] remark 2.1.20, where we give a shifted version to have compatibility with perverse sheaves.

**Definition 81.** Let $S \in \Var(C)$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \rightarrow \tilde{S}_i$ with $\tilde{S}_i \in \SmVar(C)$; or let $S \in \AnSp(C)$ and let $S = \cup_i S_i$ an open cover such that there exit closed embeddings $i_i : S_i \rightarrow \tilde{S}_i$ with $\tilde{S}_i \in \AnSp(C)$. Then, $\PSh_{D(2)fil}(S/(\tilde{S}_I)) \subset \PSh_{D(2)fil}(\tilde{S}_I)$ is the full subcategory

- whose objects are $(M, F) = ((M_I, F)_{I \subset [1, \ldots, l]}, s_{IJ})$, with

  - $(M_I, F) \in \PSh_{D(2)fil}(\tilde{S}_I)$ such that $I_S, M_I = 0$, in particular $(M_I, F) \in \PSh_{D(2)fil,S}(\tilde{S}_I)$

  - $s_{IJ} : m^*(M_I, F) \cong m^* p_{IJ*}(M_I, F)[d_{\tilde{S}_I} - d_{\tilde{S}_J}]$ for $I \subset J$, are isomorphisms, $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{JI} : \tilde{S}_K \to \tilde{S}_I$;

- the morphisms $m : (M, F) \rightarrow (N, F)$ between $(M_I, F)_{I \subset [1, \ldots, l]}, s_{IJ})$ and $(N, F) = ((N_I, F)_{I \subset [1, \ldots, l]}, r_{IJ})$ are by definition a family of morphisms of complexes,

  $m = (m_I : (M_I, F) \to (N_I, F))_{I \subset [1, \ldots, l]}$

such that $r_{IJ} \circ m_J = p_{IJ*} m_I \circ s_{IJ}$ in $C_{D_{S_I}}(\tilde{S}_J)$.

We denote by

$$\PSh_{D(2)fil, rh}(S/(\tilde{S}_I)) \subset \PSh_{D(2)fil,h}(S/(\tilde{S}_I)) \subset \PSh_{D(2)fil}(S/(\tilde{S}_I))$$

the full subcategory consisting of $(M_I, s_{IJ})$ such that $(M_I, F)$ is filtered coherent, resp. filtered holonomic, resp. filtered regular holonomic, i.e. $M_I$ are coherent, resp. holonomic, resp. filtered regular holonomic, sheaves of $D_{\tilde{S}_I}$ modules and $F$ is a good filtration. We have the full subcategories

$$\PSh_{D(1,0)fil, rh}(S/(\tilde{S}_I)) \subset \PSh_{D(2)fil, rh}(S/(\tilde{S}_I)), \quad \PSh_{D(1,0)fil,h}(S/(\tilde{S}_I)) \subset \PSh_{D(2)fil,h}(S/(\tilde{S}_I)), \quad \PSh_{D(1,0)fil,h}(S/(\tilde{S}_I)) \subset \PSh_{D(2)fil,h}(S/(\tilde{S}_I)),$$

consisting of $(M_I, F, W, s_{IJ})$ such that $W^p M_I$ are $D_{S_I}$ submodules.

We recall from section 2 the following

- A morphism $m = (m_I) : ((M_I), s_{IJ}) \to ((N_I), r_{IJ})$ in $C(\PSh_{D}(S/(\tilde{S}_I)))$ is a Zariski, resp. usu, local equivalence if and only if all the $m_I$ are Zariski, resp. usu, local equivalences.

- A morphism $m = (m_I) : ((M_I), s_{IJ}) \to ((N_I), r_{IJ})$ in $C(\PSh_{D(2)fil}(S/(\tilde{S}_I)))$ is a filtered Zariski, resp. usu, local equivalence if and only if all the $m_I$ are filtered Zariski, resp. usu, local equivalence.

- By definition, a morphism $m = (m_I) : ((M_I), s_{IJ}) \to ((N_I), r_{IJ})$ in $C(\PSh_{D(2)fil}(S/(\tilde{S}_I)))$ is an $r$-filtered Zariski, resp. usu, local equivalence if there exist $m_0 : (C_{iI}, F), s_{iIJ}) \to (C_{i+1I}, F), s_{i+1IJ})$, $0 \leq i \leq s$, with $(C_{iI}, F), s_{iIJ}) \in C(\PSh_{D(2)fil}(S/(\tilde{S}_I)))$, $(C_{iI}, F), s_{iIJ}) = (M_I, F), s_{IJ}) = (N_I, F), r_{IJ})$ such that

  $$m = m_s \circ \cdots m_0 \circ m_1 : ((M_I), s_{IJ}) \to ((N_I), r_{IJ})$$

with $m_0 : (C_{iI}, F), s_{iIJ}) \to (C_{i+1I}, F), s_{i+1IJ})$ either filtered Zariski, resp. usu, local equivalence or $r$-filtered homotopy equivalence.

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Let $S \in \text{Var}(\mathbb{C})$ or $S \in \text{AnSp}(\mathbb{C})$.

- If $S \in \text{Var}(\mathbb{C})$, let $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \hat{S}_i$ with $\hat{S}_i \in \text{SmVar}(\mathbb{C})$, and let $S = \bigcup_{I_i=1}^{I} S_{I_i}$ another open cover such that there exist closed embeddings $i_{I_i} : S_{I_i} \hookrightarrow \hat{S}_{I_i}$ with $\hat{S}_{I_i} \in \text{SmVar}(\mathbb{C})$.

- If $S \in \text{AnSp}(\mathbb{C})$, let $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \hat{S}_i$ with $\hat{S}_i \in \text{AnSm}(\mathbb{C})$, and let $S = \bigcup_{I_i=1}^{I} S_{I_i}$ another open cover such that there exist closed embeddings $i_{I_i} : S_{I_i} \hookrightarrow \hat{S}_{I_i}$ with $\hat{S}_{I_i} \in \text{AnSm}(\mathbb{C})$.

Denote $L = [1, \ldots, l]$, $L' = [1, \ldots, l']$ and $L'' := [1, \ldots, l] \cup [1, \ldots, l']$. We have then the refined open cover $S = \cup_{k \in L} S_k$ and we denote for $I, I' \subseteq L''$, $S_{I \cup I'} := \cap_{k \in I \cup I'} S_k$ and $\hat{S}_{I \cup I'} := \Pi_{k \in I \cup I'} \hat{S}_k$, so that we have a closed embedding $i_{I \cup I'} : S_{I \cup I'} \hookrightarrow \hat{S}_{I \cup I'}$. For $I \subseteq I' \subseteq J$, denote by $p_{I \cup I', J} : S_{I \cup I'} \rightarrow \hat{S}_{I \cup I'}$ the projection. We then have a natural transfer map

$$T_{S}^{L / L'} : \text{PSh}_{D^{fil}}(S/(S_{I})) \rightarrow \text{PSh}_{D^{fil}}(S/(S_{I}')),$$

$$(s_{I}, F) \mapsto (\lim_{I \subseteq I'} \text{PSh}_{D^{fil}}(p_{I \cup I', I}^{*}(F)))/(S_{I \cup I'}),$$

with, in the homotopy limit, the natural transition morphisms

$$p_{I \cup I' *} \ast (p_{I \cup I'}^{*}(F)) = p_{I \cup I'}^{*} \ast (p_{I \cup I'}^{*}(F))$$

for $J \subseteq I$, and

$$s_{I \cup I'} : \lim_{I \subseteq I'} \text{PSh}_{D^{fil}}(p_{I \cup I', I}^{*}(F)) \rightarrow \lim_{I \subseteq I'} \text{PSh}_{D^{fil}}(p_{I \cup I', I}^{*}(F))$$

$$\rightarrow \text{PSh}_{D^{fil}}(p_{I \cup I', I}^{*}(F))$$

**Definition-Proposition 18.** Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \hat{S}_i$ with $\hat{S}_i \in \text{SmVar}(\mathbb{C})$, or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \hat{S}_i$ with $\hat{S}_i \in \text{AnSm}(\mathbb{C})$. Then $\text{PSh}_{D^{(2)}^{fil}}(S/(\hat{S}_i))$ does not depend on the open covering of $S$ and the closed embeddings and we set

$$\text{PSh}_{D^{(2)}^{fil}}(S) := \text{PSh}_{D^{(2)}^{fil}}(S/(\hat{S}_i))$$

We denote by $C_{D^{(2)}^{fil}}^{0}(S) := C(\text{PSh}_{D^{(2)}^{fil}}(S/(\hat{S}_i)))$ and by $D_{D^{(2)}^{fil}}^{0}(S) := K_{D^{(2)}^{fil,r}}^{0}(S) \text{mod}^{-1}$ the localization of the $r$-filtered homotopy category with respect to the classes of filtered Zariski, resp. usu, local equivalences.

**Proof.** It is obvious that $T_{S}^{L / L'} : \text{PSh}_{D^{fil}}(S/(S_{I})) \rightarrow \text{PSh}_{D^{fil}}(S/(S_{I}'))$ is an equivalence of category with inverse $T_{S}^{L' / L} : \text{PSh}_{D^{fil}}(S/(S_{I}')) \rightarrow \text{PSh}_{D^{fil}}(S/(S_{I}))$.

We now give the definition of our category:

**Definition 82.** Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \hat{S}_i$ with $\hat{S}_i \in \text{SmVar}(\mathbb{C})$, or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \hat{S}_i$ with $\hat{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $C_{D^{(2)}^{fil}}(S/(\hat{S}_i)) \subset C_{D^{(2)}^{fil}}((\hat{S}_i))$ is the full subcategory

- whose objects are $(M, F) \in C_{D^{(2)}^{fil}}(S/(\hat{S}_i))$ (see definition 62),

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we denote by $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_i)) \subset C_{\mathcal{D}(2)fil}(S/(\tilde{S}_1))$ the full subcategory consisting of objects $((M_i, F), u_{iJ})$ such that the $u_{ij}$ are Zariski, resp. usu, local equivalences.

Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$ or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, we denote by

$$C_{\mathcal{D}(2)fil,rh}(S/(\tilde{S}_1)) \subset C_{\mathcal{D}(2)fil}(S/(\tilde{S}_i)) \subset C_{\mathcal{D}(2)fil,c}(S/(\tilde{S}_1)) \subset C_{\mathcal{D}(2)fil}(S/(\tilde{S}_i))$$

the full subcategories consisting of those $((M_i, F), u_{iJ}) \in C^\sim_{\mathcal{D}(2)fil}(S/(\tilde{S}_i))$ such that $(M_i, F) \in C_{\mathcal{D}(2)fil,S_i,c}(\tilde{S}_i)$, that is such that $a_r H^n(M_i, F)$ are filtered coherent for all $n \in \mathbb{Z}$ and all $I \subset \{1, \ldots, l\}$ (i.e. $a_r H^n(M_i)$) are coherent sheaves of $D_{\tilde{S}_i}$ modules and $F$ induces a good filtration on $a_r H^n(M_i)$, resp. such that $(M_i, F) \in C_{\mathcal{D}(2)fil,S_i,rh}(\tilde{S}_i)$, that is such that $a_r H^n(M_i, F)$ are filtered holonomic for all $n \in \mathbb{Z}$ and all $I \subset \{1, \ldots, l\}$ (i.e. $a_r H^n(M_i)$) are regular holonomic sheaves of $D_{\tilde{S}_i}$ modules and $F$ induces a good filtration on $a_r H^n(M_i)$.

We denote by

$$C_{\mathcal{D}(1,0)fil,h}(S/(\tilde{S}_1)) \subset C_{\mathcal{D}(2)fil,h}(S/(\tilde{S}_i)) \subset C_{\mathcal{D}(2)fil,rh}(S/(\tilde{S}_1)) \subset C_{\mathcal{D}(2)fil,h}(S/(\tilde{S}_i))$$

the full subcategories consisting of those $((M_i, F, W), u_{iJ}) \in C^\sim_{\mathcal{D}(2)fil}(S/(\tilde{S}_i))$ such that $W^r M_i$ are $D_{\tilde{S}_i}$ submodules (resp. and $a_r H^n(M_i, F)$ are filtered holonomic).

We recall from section 2 the following

• A morphism $m = (m_i) : ((M_i), u_{iJ}) \to ((N_i), v_{ij})$ in $C_{\mathcal{D}}(S/(\tilde{S}_1))$ is a Zariski, resp. usu, local equivalence if and only if all the $m_i$ are Zariski, resp. usu, local equivalences.

• A morphism $m = (m_i) : ((M_i, F), u_{iJ}) \to ((N_i, F), v_{ij})$ in $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_1))$ is a filtered Zariski, resp. usu, local equivalence if and only if all the $m_i$ are filtered Zariski, resp. usu, local equivalence.

• Let $r \in \mathbb{N}$. By definition, a morphism $m = (m_i) : ((M_i, F), u_{iJ}) \to ((N_i, F), v_{ij})$ in $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_1))$ is an $r$-filtered Zariski, resp. usu, local equivalence if there exist $m_i : (C_{i,l}, F, u_{iJ}) \to (C_{(i+1)l}, F, u_{(i+1)J})$, $0 \leq i \leq s$, with $(C_{i,l}, F, u_{iJ}) \in C_{\mathcal{D}(2)fil}(S/(\tilde{S}_1))$, $(C_{0l}, F, u_{iJ}) = (M_i, F, u_{iJ})$, $(C_{sl}, F, u_{sJ}) = (N_i, F, v_{ij})$ such that

$$m = m_s \circ \cdots \circ m_0 : ((M_i, F), u_{iJ}) \to ((N_i, F), v_{ij})$$

with $m_i : (C_{i,l}, F, u_{iJ}) \to (C_{(i+1)l}, F, u_{(i+1)J})$ either filtered Zariski, resp. usu, local equivalence or $r$-filtered homotopy equivalence (i.e. $r$-filtered for the first filtration and filtered for the second filtration).

In the analytic case, we also define in the same way:
Definition 83. Let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \bigcup_{i} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $C_{D^\infty(2)fil}(S/(\tilde{S}_i)) \subset C_{D^\infty(2)fil}(S/(\tilde{S}_i))$ is the full subcategory

- whose objects are $(M, F) = ((M_i, F)_{i \in [1, \cdots, l]}, u_{II})$, with
  
  $ (M_i, F) \in C_{D^\infty(2)fil}(S/(\tilde{S}_i)) $  \ (see definition 63),
  
  $ u_{II} : m_i^* p_{IJ*}(M_i, F)[d_{\tilde{S}_i} - d_{\tilde{S}_j}], $  \ for $J \subset I$, are morphisms, $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ} \circ u_{IK} = u_{IK}$ in $C_{D^\infty(2)fil}(S/(\tilde{S}_i))$ ;

- the morphisms $m : ((M_1, F), u_{II}) \to ((N_1, F), v_{II})$ between $(M, F) = ((M_i, F)_{i \in [1, \cdots, l]}, u_{II})$ and $(N, F) = ((N_i, F)_{i \in [1, \cdots, l]}, v_{II})$ being a family of morphisms of complexes,
  
  $ m = (m_I : (M_i, F) \to (N_i, F))_{I \in [1, \cdots, l]} $  

such that $v_{II} \circ m_I = p_{IJ} \circ m_{II} \circ u_{IJ}$ in $C_{D^\infty(2)fil}(S/(\tilde{S}_i))$.

We denote by $C_{D^\infty(2)fil}(S/(\tilde{S}_i))$ the full subcategory consisting of objects $(M, F), u_{II})$ such that the $u_{II}$ are $\infty$-filtered usu local equivalence.

Let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \bigcup_{i} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We denote by

$ C_{D^\infty(2)fil,h}(S/(\tilde{S}_i)) \subset C_{D^\infty(2)fil}(S/(\tilde{S}_i)) $  \ $ C_{D^\infty(2)fil,c}(S/(\tilde{S}_i)) \subset C_{D^\infty(2)fil}(S/(\tilde{S}_i)) $

the full subcategories consisting of $(M_i, F), u_{II}) \in C_{D^\infty(2)fil}(S/(\tilde{S}_i))$ such that $(M_i, F) \in C_{D^\infty(2)fil,S_i,c}(\tilde{S}_i)$, that is such that $a_n H^n(M_i, F)$ are filtered coherent for all $n \in \mathbb{Z}$ and all $I \subset [1, \cdots, l]$, resp. such that $(M_i, F) \in C_{D^\infty(2)fil,S_i,h}(\tilde{S}_i)$, that is such that $a_n H^n(M_i, F)$ are filtered holonomic for all $n \in \mathbb{Z}$ and all $I \subset [1, \cdots, l]$. We denote by

$ C_{D^\infty(1,0)fil}(S/(\tilde{S}_i)) \subset C_{D^\infty(2)fil}(S/(\tilde{S}_i)) $  \ $ C_{D^\infty(1,0)fil,h}(S/(\tilde{S}_i)) \subset C_{D^\infty(2)fil,h}(S/(\tilde{S}_i)) $

the full subcategories consisting of those $(M_i, F, W), u_{II}) \in C_{D^\infty(2)fil}(S/(\tilde{S}_i))$ such that $W^\infty M_I$ are $D_{\tilde{S}_i}$ submodules (resp. and $a_n H^n(M_i, F)$ filtered holonomic).

We recall from section 2 the following

- A morphism $m = (m_I : ((M_i, F), u_{II}) \to ((N_i, F), v_{II})$ in $C_{D^\infty}(S/(\tilde{S}_i))$ is a usu local equivalence if and only if all the $m_I$ are usu local equivalences.

- A morphism $m = (m_I : ((M_i, F), u_{II}) \to ((N_i, F), v_{II})$ in $C_{D^\infty(2)fil}(S/(\tilde{S}_i))$ is a filtered usu local equivalence if and only if all the $m_I$ are filtered usu local equivalence.

- Let $r \in \mathbb{N}$. By definition, a morphism $m = (m_I : ((M_i, F), u_{II}) \to ((N_i, F), v_{II})$ in $C_{D^\infty(2)fil}(S/(\tilde{S}_i))$ is an $r$-filtered usu local equivalence if there exist $m_i : (C_i, F), u_{II}) \to (C_{(i+1)}, F), u_{II}$ such that $0 \leq i \leq s$, with $(C_i, F), u_{II}) \in C_{D^\infty(2)fil}(S/(\tilde{S}_i))$, $(C_{(i)}, F), u_{II}) = (M_i, F), u_{II})$, $(C_{(i+1)}, F), u_{II}) = (N_i, F), v_{II})$ such that

  $ m = m_s \circ \cdots \circ m_i \cdots \circ m_0 : ((M_i, F), u_{II}) \to ((N_i, F), v_{II})) $  

with $m_i : (C_i, F), u_{II}) \to (C_{(i+1)}, F), u_{II})$ either filtered usu local equivalence or $r$-filtered homotopy equivalence (i.e. $r$-filtered for the first filtration and filtered for the second filtration).

Definition 84. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \bigcup_{i} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \bigcup_{i} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. 223
(i) We have, for $r = 1, \ldots, \infty$, (see section 2.1) the $\infty$-filtered homotopy category

$$K_{D(2)fil,r}(S/(\bar{S}_I)) := \text{Ho}_r(C_{D(2)fil}(S/(\bar{S}_I)))$$

whose objects are those of $C_{D(2)fil}(S/(\bar{S}_I))$ and whose morphism are $r$-filtered homotopy classes of morphisms ($r$-filtered for the first filtration and filtered for the second), and

$$D_{D(2)fil,r}(S/(\bar{S}_I)) := K_{D(2)fil,r}(S/(\bar{S}_I))([E_1]^{-1})$$

the localization with respect to the classes of filtered Zariski, resp. usu, local equivalences, together with the localization functor

$$D(\text{top}) : C_{D(2)fil}(S/(\bar{S}_I)) \to K_{D(2)fil,r}(S/(\bar{S}_I)) \to D_{D(2)fil,r}(S/(\bar{S}_I))$$

Note that the classes of filtered $\tau$ local equivalence constitute a right multiplicative system. By definition if $m : ((M_I, F), u_{ij}) \to ((N_I, F), v_{ij})$ in $C_{D(2)fil}(S/(\bar{S}_I))$ is an $r$-filtered Zariski, resp. usu, local equivalence, then $m = D(\tau)(m) : ((M_I, F), u_{ij}) \to ((N_I, F), v_{ij})$ is an isomorphism in $D_{D(2)fil,r}(S/(\bar{S}_I))$

(ii) We have

$$D_{D(1,0)fil,\infty,h}(S/(\bar{S}_I)) \subset D_{D(2)fil,\infty,h}(S/(\bar{S}_I)) \subset D_{D(2)fil,\infty}(S/(\bar{S}_I))$$

the full subcategories which are the image of $C_{D(2)fil,h}(S/(\bar{S}_I))$, resp. of $C_{D(1,0)fil,h}(S/(\bar{S}_I))$, by the localization functor $D(\text{top}) : C_{D(2)fil}(S/(\bar{S}_I)) \to D_{D(2)fil,\infty}(S/(\bar{S}_I))$.

In the analytic case, we also have

Definition 85. Let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_1 : S_i \hookrightarrow \bar{S}_i$ with $S_i \in \text{AnSm}(\mathbb{C})$.

(i) We have, for $r = 1, \ldots, \infty$, (see section 2.1) the $\infty$-filtered homotopy category

$$K_{D(\infty)fil,r}(S/(\bar{S}_I)) := \text{Ho}_r(C_{D(\infty)fil}(S/(\bar{S}_I)))$$

whose objects are those of $C_{D(\infty)fil}(S/(\bar{S}_I))$ and whose morphism are $r$-filtered homotopy classes of morphisms ($r$-filtered for the first filtration and filtered for the second), and

$$D_{D(\infty)fil,r}(S/(\bar{S}_I)) := K_{D(\infty)fil,r}(S/(\bar{S}_I))([E_1]^{-1})$$

the localization with respect to the classes of filtered usu local equivalences, together with the localization functor

$$D(\text{usu}) : C_{D(\infty)fil}(S/(\bar{S}_I)) \to K_{D(\infty)fil,r}(S/(\bar{S}_I)) \to D_{D(\infty)fil,r}(S/(\bar{S}_I))$$

Note that the classes of filtered usu local equivalence constitute a right multiplicative system. By definition if $m : ((M_I, F), u_{ij}) \to ((N_I, F), v_{ij})$ in $C_{D(\infty)fil}(S/(\bar{S}_I))$ is an $r$-filtered Zariski, resp. usu, local equivalence, then $m = D(\tau)(m) : ((M_I, F), u_{ij}) \to ((N_I, F), v_{ij})$ is an isomorphism in $D_{D(\infty)fil,r}(S/(\bar{S}_I))$

(ii) We have then

$$D_{D(\infty)fil,(1,0),\infty,h}(S/(\bar{S}_I)) \subset D_{D(\infty)fil,\infty,h}(S/(\bar{S}_I)) \subset D_{D(\infty)fil,\infty}(S/(\bar{S}_I))$$

the full subcategories which are the image of $C_{D(\infty)fil,h}(S/(\bar{S}_I))$, resp. of $C_{D(\infty)fil,(1,0),h}(S/(\bar{S}_I))$, by the localization functor $D(\text{usu}) : C_{D(\infty)fil}(S/(\bar{S}_I)) \to D_{D(\infty)fil,\infty}(S/(\bar{S}_I))$. 

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Definition 86. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \bigcup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \bigcup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$.

(i) We denote by

$$C_{D(2)}(S/\tilde{S}_i)^0 \subset C_{D(2)}(S/\tilde{S}_i)$$

the full subcategory consisting of $((M_1, F), u_{1,J}) \in C_{D(2)}(S/\tilde{S}_i)$ such that

$$H^n((M_1, F), u_{1,J}) = (H^n(M_1, F), H^n u_{1,J}) \in \text{PSh}_{D(2)}^0(S/\tilde{S}_i)$$

that is such that the $H^n u_{1,J}$ are isomorphism. We denote by $D_{D(2)}(S/\tilde{S}_i)^0 := D(\text{top})(C_{D(2)}(S/\tilde{S}_i)^0)$ its image by the localization functor.

(ii) We have the full embedding functor

$$i_{S/\tilde{S}_i}^0 : C_{D(2)}(S) := C_{D(2)}(S/\tilde{S}_i) \hookrightarrow C_{D(2)}(S/\tilde{S}_i),$$

$$((M_1, F), s_{1,J}) \mapsto ((M_1, F), s_{1,J})$$

By definition, $i_{S/\tilde{S}_i}^0(C_{D(1,0)}(S/\tilde{S}_i)) \subset C_{D(1,0)}(S/\tilde{S}_i)$. This full embedding induces in the derived category the functor

$$i_{S/\tilde{S}_i}^0 : D_{D(2)}^{\text{fil}, \infty}(S) := D_{D(2)}^{\text{fil}, \infty}(S/\tilde{S}_i) \to D_{D(2)}^{\text{fil}, \infty}(S/\tilde{S}_i),$$

$$((M_1, F), s_{1,J}) \mapsto ((M_1, F), s_{1,J}).$$

Proposition 88. Let $S \in \text{Var}(\mathbb{C})$ and let $S = \bigcup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \bigcup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then,

$$i_{S/\tilde{S}_i}^0 : D_{D(2)}^{\text{fil}, \infty}(S) \to D_{D(2)}^{\text{fil}, \infty}(S/\tilde{S}_i)$$

is a full embedding whose image is $D_{D(2)}^{\text{fil}, \infty}(S/\tilde{S}_i)^0$, that is consists of $((M_1, F), s_{1,J}) \in C_{D(2)}^{\text{fil}}(S/\tilde{S}_i)$ such that

$$H^n((M_1, F), s_{1,J}) := (H^n(M_1, F), H^n(s_{1,J})) \in \text{PSh}_{D}^0(S/\tilde{S}_i).$$

and

$$i_{S}^0 := i_{S/\tilde{S}_i}^0 : D_{D(2)}^{\text{fil}, \infty}(S) \equiv D_{D(2)}^{\text{fil}, \infty}(S/\tilde{S}_i)^0$$

the induced equivalence of categories.

Proof. Follows immediately from the fact that for $((M_1, F), s_{1,J}), ((N_1, F), r_{1,J}) \in C_{D(2)}^{\text{fil}}(S)$

$$\text{Hom}_{D_{D(2)}^{\text{fil}, \infty}(S)}(((M_1, F), s_{1,J}), ((N_1, F), r_{1,J})) = \text{Hom}_{D_{D(2)}^{\text{fil}, \infty}(S/\tilde{S}_i)}(((L(M_1, F), s_{1,J}^J), (E(N_1, F), r_{1,J}^J))$$

$$= \text{Hom}_{D_{D(2)}^{\text{fil}, \infty}(S/\tilde{S}_i)}(((M_1, F), s_{1,J}), ((N_1, F), r_{1,J})).$$

We finish this subsection by the statement a result of kashiwara in the singular case.

Definition 87. Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \bigcup_{i=1}^n S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We will consider the functor

$$J_S : C_{D(2)}(S) \to C_{D(2)}(S/\tilde{S}_i),$$

$$((M_1, F), u_{1,J}) \mapsto J_S((M_1, F), u_{1,J}) := (J_{S_i}(M_1, F), J(u_{1,J})) := ((M_1 \otimes D_{\mathbb{S}}, F), J(u_{1,J}))$$

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with, denoting for short $d_{IJ} := d_{S_j} - d_{S_i}$,

$$J(u_{IJ}) : J(M_I, F) \xrightarrow{J(u_{IJ})} J(p_{I,J*}(M_J, F)[d_{IJ}]) \xrightarrow{T_s(p_{I,J*})} p_{I,J*}J(M_J, F)[d_{IJ}].$$

Of course $J_S(C_D(1,0)_{fil}(S/(\tilde{S}_I))) \subset C_D^{\infty}(1,0)_{fil}(S/(\tilde{S}_I))$.

**Proposition 89.** Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \bigcup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$. Then the functor

$$J_S : C_D(2)_{fil}(S/(\tilde{S}_I)) \to C_D^{\infty}(2)_{fil}(S/(\tilde{S}_I)),$$

satisfy $J_S : C^\infty_{D(2)}(S/(\tilde{S}_I)) \subset C_D^{\infty}(2)_{fil}(S/(\tilde{S}_I))$ and induces an equivalence of category

$$J_S : D_{D(2)}_{fil,\infty,rh}(S/(\tilde{S}_I)) \to D_{D^{\infty}(2)}_{fil,\infty,h}(S/(\tilde{S}_I)).$$

Proof. Follows immediately from the smooth case (proposition 51).

### 4.3.2 Duality in the singular case

The definition of Saito’s category comes with a dual functor :

**Definition 88.** Let $S \in \text{Var}(\mathbb{C})$ and let $S = \bigcup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and $S = \bigcup_{i=1}^l S_i$ an open cover such that there exist closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$. We have the dual functor :

$$\mathbb{D}_S^K : C_{D(1,0)}^{\infty}(S/(\tilde{S}_I)) \to C_{D(1,0)}^{\infty}(S/(\tilde{S}_I)), \quad ((M_I, F), s_{1J}) \mapsto \left(\mathbb{D}_S^K(M_I, F), s_{1J}^d\right),$$

with, denoting for short $d_{IJ} := d_{S_j} - d_{S_i}$,

$$u_{IJ}^d : \mathbb{D}_S^K(M_I, F) \xrightarrow{\mathbb{D}_S^K(s_{IJ}^d)} \mathbb{D}_S^K(p_{I,J*}(M_J, F)[d_{IJ}]) \xrightarrow{T_s(p_{I,J*})} p_{I,J*}\mathbb{D}_S^K(M_J, F)[d_{IJ}].$$

It induces in the derived category the functor

$$L\mathbb{D}_S^K : D_{D(1,0)}^{\infty}(S/(\tilde{S}_I)) \to D_{D(1,0)}^{\infty}(S/(\tilde{S}_I)), \quad ((M_I, F), s_{1J}) \mapsto \left(L\mathbb{D}_S^K(M_I, F), s_{1J}\right),$$

with $q : Q((M_I, F), s_{1J}) \to ((M_I, F), s_{1J})$ a projective resolution.

In the analytic case we also define

**Definition 89.** Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \bigcup_{i=1}^l S_i$ an open cover such that there exist closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSp}(\mathbb{C})$. We have the dual functor :

$$\mathbb{D}_S^{K,\infty} : C_{D^{\infty}}(S/(\tilde{S}_I)) \to C_{D^{\infty}}(S/(\tilde{S}_I)), \quad ((M_I, F), u_{1J}) \mapsto \left(\mathbb{D}_S^{K,\infty}(M_I, F), u_{1J}^d\right),$$

with $u_{1J}^d$ defined similarly as in definition 88. It induces in the derived category the functor

$$L\mathbb{D}_S^{K,\infty} : D_{D^{\infty}}(S/(\tilde{S}_I)) \to D_{D^{\infty}}(S/(\tilde{S}_I)), \quad ((M_I, F), u_{1J}) \mapsto \left(L\mathbb{D}_S^{K,\infty}(M_I, F), u_{1J}^d\right),$$

with $q : Q((M_I, F), s_{1J}) \to ((M_I, F), s_{1J})$ a projective resolution.
4.3.3 Inverse image in the singular case

We give in this subsection the inverse image functors between our categories.

Let \( n : S^o \to S \) be an open embedding with \( S \in \text{Var}(\mathbb{C}) \) and let \( S = \bigcup_i S_i \) be an open cover such that there exist closed embeddings \( i_i : S_i \to \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \); or let \( n : S^o \to S \) be an open embedding with \( S \in \text{AnSp}(\mathbb{C}) \) and let \( S = \bigcup_i S_i \) be an open cover such that there exist closed embeddings \( i_i : S_i \to \tilde{S}_i \) with \( \tilde{S}_i \in \text{AnSm}(\mathbb{C}) \). Denote \( S_i^o := n^{-1}(S_i) = S_i \cap S^o \) and \( n_I := n|_{S_i^o} : S_i^o \to S^o \) the open embeddings. Consider open embeddings \( \tilde{n}_I : \tilde{S}_i^o \to \tilde{S}_i \) such that \( \tilde{S}_i^o \cap \tilde{S}_i = S_i^o \), that is, which are lift of \( n_I \). We have the functor

\[
 n^* : C_{D_{\mathbb{D}}}(S/(\tilde{S}_i)) \to C_{D_{\mathbb{D}}}(S^o/(\tilde{S}_i^o)),
\]

which derive trivially.

Let \( f : X \to S \) be a morphism, with \( X, S \in \text{Var}(\mathbb{C}) \), such that there exist a factorization \( f : X \xrightarrow{f} Y \times S \xrightarrow{p_S} S \) with \( Y \in \text{SmVar}(\mathbb{C}), l \) a closed embedding and \( p_S \) the projection, and consider \( S = \bigcup_{I=1}^l S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \to \tilde{S}_i \), with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \); or let \( f : X \to S \) be a morphism, with \( X, S \in \text{AnSp}(\mathbb{C}) \), such that there exist a factorization \( f : X \xrightarrow{f} Y \times S \xrightarrow{p_S} S \) with \( Y \in \text{AnSm}(\mathbb{C}), l \) a closed embedding and \( p_S \) the projection and consider \( S = \bigcup_{I=1}^l S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \to \tilde{S}_i \), with \( \tilde{S}_i \in \text{AnSm}(\mathbb{C}) \). Then, \( X = \bigcup_{I=1}^l X_i \) with \( X_i := f^{-1}(S_i) \). Denote by \( p_{IJ} : J \to I \) and \( p^I_{IJ} : Y \times \tilde{S}_J \to Y \times \tilde{S}_I \) the projections and by

\[
\begin{align*}
E_{IJ} &= \tilde{S}_J \xrightarrow{m_{IJ}} \tilde{S}_J \quad E'_{IJ} = Y \times \tilde{S}_J \xrightarrow{m_{IJ}} Y \times \tilde{S}_J , \quad E_{IJ} \xrightarrow{\tilde{l}_J} \tilde{S}_J \quad E'_{IJ} \xrightarrow{\tilde{l}_J} Y \times \tilde{S}_J , \\
&= Y \times \tilde{S}_J \xrightarrow{m_{IJ}} Y \times \tilde{S}_J \quad E_{IJ} \xrightarrow{\tilde{l}_J} \tilde{S}_J \quad E'_{IJ} \xrightarrow{\tilde{l}_J} Y \times \tilde{S}_J
\end{align*}
\]

the commutative diagrams. The (graph) inverse image functors are:

\[
 f^* \Gamma : C_{D_{\mathbb{D}}}(S/(\tilde{S}_i)) \to C_{D_{\mathbb{D}}}(X/(\tilde{S}_i)),
\]

\[
 (M, F) = ((M_I, F), u_{IJ}) \to f^* \Gamma (M, F) := (\Gamma_{X_i} E(p_{S_i}^{*\Gamma}(M_I, F)), f^* u_{IJ})
\]

with, denoting for short \( d_{IJ} := d_{\tilde{S}_i} - d_{\tilde{S}_j} \),

\[
 f^* \Gamma u_{IJ} : \Gamma_{X_i} E(p_{S_i}^{*\Gamma}(M_I, F)) \xrightarrow{\Gamma_{X_i} E(p_{S_i}^{*\Gamma}(u_{IJ}))} \Gamma_{X_i} E(p_{S_i}^{*\Gamma}(M_I, F)[d_{IJ}])
\]

\[
 = \Gamma_{X_i} E(p_{S_i}^{*\Gamma}(M_I, F)[d_{IJ} + d_{IJ}])
\]

It induces in the derived categories the functor

\[
 R^* f^* \Gamma : D_{\mathbb{D}}(S/(\tilde{S}_i)) \to D_{\mathbb{D}}(X/(\tilde{S}_i)),
\]

\[
 (M, F) = ((M_I, F), u_{IJ}) \to R^* f^* \Gamma (M, F) := (\Gamma_{X_i} E(p_{S_i}^{*\Gamma}(M_I, F)), f^* \Gamma u_{IJ}).
\]

It gives by duality the functor

\[
 Lf^* \Gamma : D_{\mathbb{D}}(S/(\tilde{S}_i))^0 \to D_{\mathbb{D}}(X/(\tilde{S}_i))^0,
\]

\[
 (M, F) = ((M_I, F), u_{IJ}) \to Lf^* \Gamma (M, F) := LD_{S_i} K R f^* \Gamma (M, F).
\]

where \( i_S^0 : D_{\mathbb{D}}(S/(\tilde{S}_i))^0 \to D_{\mathbb{D}}(S/(\tilde{S}_i))^0 \) is the isomorphism of definition 86.
Let $f : X \to S$ be a morphism, with $X, S \in \text{AnSp}(\mathbb{C})$, such that there exist a factorization $f; X \xrightarrow{p} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, $p$ a closed embedding and $p_S$ the projection and consider $S = \bigcup_{i=1}^{n} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow S_i$, with $S_i \in \text{AnSm}(\mathbb{C})$. Then, $X = \bigcup_{i=1}^{n} X_i$ with $X_i := f^{-1}(S_i)$. We have also the functors,

$$f_{*}^{\text{mod}[-1],\Gamma} : C_{D_{\infty}}(\mathcal{F})_{\text{int}}(S/(\tilde{S}_i)) \to C_{D_{\infty}}(\mathcal{F})_{\text{int}}(X/(Y \times \tilde{S}_i)),
$$

$$(M, F) = ((M_1, F), u_{1J}) \mapsto f_{*}^{\text{mod}[-1],\Gamma}(M, F) := (\Gamma X_i(p_{S_i}^{*}M, F), f_{*}^{\text{mod}[-1],u_{1J}}(M, F))$$

with, denoting for short $d_{1J} := d_{S_j} - d_{S_i}$,

$$f_{*}^{\text{mod}[-1],u_{1J}} : X_i(p_{S_i}^{*}M, F) \to \Gamma X_i(p_{S_i}^{*}M, F)[d_{1J}]$$

$$\Gamma X_i(p_{S_i}^{*}M, F)[d_{1J}] \to \Gamma X_i(p_{S_i}^{*}M, F)[d_{1J}]$$

It induces in the derived categories, the functor

$$(M, F) = ((M_1, F), u_{1J}) \mapsto Rf_{*}^{\text{mod}[-1],\Gamma}(M, F) := (\Gamma X_i(p_{S_i}^{*}M, F), f_{*}^{\text{mod}[-1],u_{1J}}(M, F))$$

It gives by duality the functor

$$L_{*}^{\text{mod}[-1],\Gamma} : D_{D_{\infty}}(\mathcal{F})_{\text{int}}(S/(\tilde{S}_i)) \to D_{D_{\infty}}(\mathcal{F})_{\text{int}}(X/(Y \times \tilde{S}_i)),
$$

$$(M, F) = ((M_1, F), u_{1J}) \mapsto Lf_{*}^{\text{mod}[-1],\Gamma}(M, F) := L_{D_{\infty}}(\mathcal{F})_{\text{int}}(S/(\tilde{S}_i))^{0} = D_{D_{\infty}}(\mathcal{F})_{\text{int}}(X/(Y \times \tilde{S}_i))^{0},
$$

where $L_{*}^{\text{mod}[-1],\Gamma} : D_{D_{\infty}}(\mathcal{F})_{\text{int}}(S/(\tilde{S}_i))^{0} \sim D_{D_{\infty}}(\mathcal{F})_{\text{int}}(S/(\tilde{S}_i))^{0}$ is the isomorphism of definition 86.

The following proposition are then easy :

**Proposition 90.** Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \text{Var}(\mathbb{C})$. Assume there exist factorizations $f_1 : X \xrightarrow{p_{12}} Y' \times Y \xrightarrow{p_{12}} Y$ and $f_2 : Y \xrightarrow{p_{12}} Y'' \times S \xrightarrow{p_{12}} S$ with $Y', Y'' \in \text{SmVar}(\mathbb{C})$, $l_{12}$ closed embeddings and $p_{12}$, $p_{12}$ the projections. We have then the factorization

$$f_2 \circ f_1 : X \xrightarrow{(l_{12}f_{12})_{\text{int}}} Y' \times Y'' \times S \xrightarrow{p_{12}} S.$$

We have, for $(M, F) \in C_{D_{\infty}}(\mathcal{F})_{\text{int}}(S/(\tilde{S}_i))$, $R(f_2 \circ f_1)^{*}M, F) = Rf_{2}^{*}f_{1}^{*}M, F) = Rf_{1}^{*}M, F)$.\]

**Proof.** Follows from the fact that for $(M, F) = ((M_1, F), u_{1J}) \in C_{D_{\infty}}(\mathcal{F})_{\text{int}}(S/(\tilde{S}_i))$,

$$(\Gamma X_i(p_{S_i}^{*}M, F), f_{*}^{\text{mod}[-1],u_{1J}}(M, F)) \mapsto (\Gamma X_i(p_{S_i}^{*}M, F), (f_{*}^{\text{mod}[-1],u_{1J}}(M, F))$$

by proposition 53(i) and the fact that $X_i \subset f_{12}^{-1}(Y_1)$.\]

**Proposition 91.** Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \text{Var}(\mathbb{C})$. Assume there exist factorizations $f_1 : X \xrightarrow{p_{12}} Y' \times Y \xrightarrow{p_{12}} Y$ and $f_2 : Y \xrightarrow{p_{12}} Y'' \times S \xrightarrow{p_{12}} S$ with $Y', Y'' \in \text{SmVar}(\mathbb{C})$, $l_{12}$ closed embeddings and $p_{12}$, $p_{12}$ the projections. We have then the factorization

$$f_2 \circ f_1 : X \xrightarrow{(l_{12}f_{12})_{\text{int}}} Y' \times Y'' \times S \xrightarrow{p_{12}} S.$$

We have, for $(M, F) \in C_{D_{\infty}}(\mathcal{F})_{\text{int}}(S/(\tilde{S}_i))$, $R(f_2 \circ f_1)^{*}M, F) = Rf_{2}^{*}f_{1}^{*}M, F) = Rf_{1}^{*}M, F)$.\]

**Proof.** Similar to the proof of proposition 90.\]
4.3.4 Direct image functor in the singular case

We define the direct image functors between our category.

Let $f : X \to S$ be a morphism with $X, S \in \text{Var}(\mathbb{C})$, and assume there exist a factorization $f : X \xrightarrow{\rho} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ a the projection; or let $f : X \to S$ be a morphism with $X, S \in \text{AnSp}(\mathbb{C})$, and assume there exist a factorization $f : X \xrightarrow{\rho} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, $l$ a closed embedding and $p_S$ a the projection. Let $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; resp. let $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then $X = \bigcup_{i=1}^{l} X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \ldots, l]$, $S_I = \bigcap_{i \in I} S_i$ and $X_I = \bigcap_{i \in I} X_i$. For $I \subset [1, \ldots, l]$, denote by $\tilde{S}_I = \Pi_{i \in I} \tilde{S}_i$, We then have, for $I \subset [1, \ldots, l]$, closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_I$ and the following commutative diagrams which are cartesian (we take $Y = \mathbb{P}^N, \mathbb{A}$ in the algebraic case)

$$f_I = f_{|X_I} : X_I \xrightarrow{I_i} Y \times S_I \xrightarrow{p_S} S_I \xrightarrow{i_i} S_I' \xrightarrow{p_S'} S_I' \xrightarrow{p_{I,J}} S_I \xrightarrow{i_i} S_I,$$

with $l_I : l_{X,I} = I \times i_i$, $p_{S_I} = p_{S_{I,J}}$ and $p_{S_I}$ are the projections and $p_{I,J} = I \times p_{I,J}$. Then $\tilde{f}_I := p_{S_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $f_I = f_{|X_I}$. We define the direct image functor on our category by

$$f_{\text{mod}}^\text{FDR} : C_{D(2)fiu}(X/(Y \times \tilde{S}_I)) \to C_{D(2)fiu}(S/(\tilde{S}_I)),$$

$$((M, F), u_{IJ}) \mapsto (\tilde{f}_{\text{mod}}^\text{FDR}(M_I, F), f^k(u_{IJ})) := (p_{S_i}E((\Omega_{Y \times \tilde{S}_I/S_i}^i, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F)[dy]), f^k(u_{IJ}))$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_I} - d_{S_I}$,

$$f^k(u_{IJ})[dy] : p_{S_I}E((\Omega_{Y \times \tilde{S}_I/S_i}^i, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F))$$

$$\xrightarrow{p_{S_I}E(DR(Y \times \tilde{S}_I/S_i)(u_{IJ}))} p_{S_I}E((\Omega_{Y \times \tilde{S}_I/S_i}^i, F_b) \otimes_{O_{Y \times \tilde{S}_I}} p_{I,J}^*E((M_I, F)[dy]), f^k(u_{IJ}))$$

with denoting for short $d_{IJ} := d_{\tilde{S}_I} - d_{S_I}$,

$$\int_f^\text{FDR} : D_{D(2)fiu, \infty}(X) \to D_{D(2)fiu, \infty}(S), ((M, F), u_{IJ}) \mapsto (\tilde{f}_{\text{mod}}^\text{FDR}(M_I, F), f^k(u_{IJ}))$$

Let $f : X \to S$ be a morphism with $X, S \in \text{AnSp}(\mathbb{C})$, and assume there exist a factorization $f : X \xrightarrow{\rho} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(\mathbb{C})$, $l$ a closed embedding and $p_S$ a the projection. Let $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then $X = \bigcup_{i=1}^{l} X_i$ with $X_i := f^{-1}(S_i)$. We also have the functors

$$f_{\text{mod}}^\text{FDR} : C_{D^{\infty}(2)fiu}(X/(Y \times \tilde{S}_I)) \to C_{D^{\infty}(2)fiu}(S/(\tilde{S}_I)),$$

$$((M, F), u_{IJ}) \mapsto (\tilde{f}_{\text{mod}}^\text{FDR}(M_I, F), f^k(u_{IJ})) := (p_{S_i}E((\Omega_{Y \times \tilde{S}_I/S_i}^i, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F)[dy]), f^k(u_{IJ}))$$

where $f^k(u_{IJ})[dy]$ is given as above,

$$\int_f^\text{FDR} : D_{D^{\infty}(2)fiu, \infty}(X) \to D_{D^{\infty}(2)fiu, \infty}(S), ((M, F), u_{IJ}) \mapsto (\tilde{f}_{\text{mod}}^\text{FDR}(M_I, F), f^k(u_{IJ})) := (p_{S_i}E((\Omega_{Y \times \tilde{S}_I/S_i}^i, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F)[dy]), f^k(u_{IJ}))$$
where \( f^k(u_{IJ})[dy] \) is given as above.

In the algebraic case, we have the followings:

**Proposition 92.** Let \( f_1 : X \to Y \) and \( f_2 : Y \to S \) two morphism with \( X,Y,S \in \text{QPVar}(\mathbb{C}) \) quasi-projective. Then there exist factorizations \( f_1 : X \xrightarrow{l_1} Y' \to Y \xrightarrow{pr} Y \) and \( f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{pr_2} S \) with \( Y' = \mathbb{P}^N,o \subset Y'' = \mathbb{P}^N,o \subset \mathbb{P}^N \) open subsets, \( l_1,l_2 \) closed embeddings and \( p_S,pr \) the projections. We have then the factorization \( f_2 \circ f_1 : X \xrightarrow{(l_2\circ l_1)\circ f_1} Y' \times Y'' \times S \xrightarrow{pr_S} S \). Let \( i : S \hookrightarrow \tilde{S} \) a closed embedding with \( \tilde{S} = \mathbb{P}^n,o \subset \mathbb{P}^n \) an open subset.

(i) Let \( (M,F) \in C_{D(2)}(X/(Y' \times Y'' \times \tilde{S})) \). Then, we have \( \int_{f_2 \circ f_1}^{\text{FDR}}(M,F) = \int_{f_1}^{\text{FDR}}(\int_{f_2}^{\text{FDR}}(M,F)) \) in \( D_{D(2)}(\tilde{S}) ) \).

(ii) Let \( (M,F) \in C_{D(2)}(X/(Y' \times Y'' \times \tilde{S})) \). Then, we have \( \int_{f_2 \circ f_1}^{\text{FDR}}(M,F) = \int_{f_2}^{\text{FDR}}(\int_{f_1}^{\text{FDR}}(M,F)) \) in \( D_{D(2)}(\tilde{S}) ) \).

**Proof.** (i): By the smooth case : proposition 76, we have an isomorphism
\[
\int_{f_2}^{\text{FDR}} \int_{f_1}^{\text{FDR}} (M,F) = \int_{f_1}^{\text{FDR}} \int_{pr_1 \times \tilde{S}}^{\text{FDR}} (M,F) \sim \int_{f_2 \circ f_1}^{\text{FDR}} (M,F).
\]

(ii): Follows from (i). \( \square \)

In the analytic case, we have the followings:

**Proposition 93.** Let \( f_1 : X \to Y \) and \( f_2 : Y \to S \) two morphism with \( X,Y,S \in \text{AnSp}(\mathbb{C}) \) quasi-projective. Then there exist factorizations \( f_1 : X \xrightarrow{l_1} Y' \times Y \xrightarrow{pr} Y \) and \( f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{pr_2} S \) with \( Y' = \mathbb{P}^N,o \subset Y'' = \mathbb{P}^N,o \subset \mathbb{P}^N \) open subsets, \( l_1,l_2 \) closed embeddings and \( p_S,pr \) the projections. We have then the factorization \( f_2 \circ f_1 : X \xrightarrow{(l_2\circ l_1)\circ f_1} Y' \times Y'' \times S \xrightarrow{pr_S} S \). Let \( i : S \hookrightarrow \tilde{S} \) a closed embedding with \( \tilde{S} = \mathbb{P}^n,o \subset \mathbb{P}^n \) an open subset.

(i) Let \( (M,F) \in C_{D(2)}(X/(Y' \times Y'' \times \tilde{S})) \). Then, we have \( \int_{f_2 \circ f_1}^{\text{FDR}}(M,F) = \int_{f_2}^{\text{FDR}}(\int_{f_1}^{\text{FDR}}(M,F)) \) in \( D_{D(2)}(\tilde{S}) ) \).

(ii) Let \( (M,F) \in C_{D(2)}(X/(Y' \times Y'' \times \tilde{S})) \). Then, we have \( \int_{f_2 \circ f_1}^{\text{FDR}}(M,F) = \int_{f_2}^{\text{FDR}}(\int_{f_1}^{\text{FDR}}(M,F)) \) in \( D_{D(2)}(\tilde{S}) ) \).

**Proof.** (i): By the smooth case : proposition 77, we have an isomorphism
\[
\int_{f_2}^{\text{FDR}} \int_{f_1}^{\text{FDR}} (M,F) = \int_{f_1}^{\text{FDR}} \int_{pr_1 \times \tilde{S}}^{\text{FDR}} (M,F) \sim \int_{f_2 \circ f_1}^{\text{FDR}} (M,F).
\]

(ii): Follows from (i). \( \square \)

### 4.3.5 Tensor product in the singular case

Let \( S \in \text{Var}(\mathbb{C}) \) and let \( S = \cup S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \) or let \( S \in \text{AnSp}(\mathbb{C}) \) and \( S = \cup S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{AnSp}(\mathbb{C}) \). We have the tensor product functors
\[
(-) \otimes_{\tilde{S}_i} (-) : C_{D(2)}(S/(\tilde{S}_i)) \to C_{D(2)}(S/(\tilde{S}_i)),
((M_1,F),(u_{IJ})) \mapsto ((M_1,F) \otimes_{\tilde{S}_i} (N_1,F),u_{IJ} \otimes v_{IJ}),
\]

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with, denoting for short \(d_{IJ} := d_{\tilde{S}_j} - d_{\tilde{S}_i}\) and \(d_I := d_{\tilde{S}_i},\)

\[
\begin{align*}
u_{IJ} \otimes v_{IJ} : (M_I, F) \otimes_{O_{\tilde{S}_i}} (N_I, F)[d_I] &\xrightarrow{T[p_{I,J}^{\text{mod}},p_{IJ}](-)[d_I]} p_{IJ,*}p_{IJ}^{\text{mod}}((M_I, F) \otimes_{O_{\tilde{S}_j}} (N_I, F))[d_I] \\
&\xrightarrow{\cong} p_{IJ,*}(p_{IJ}^{\text{mod}}(M_I, F) \otimes_{O_{\tilde{S}_j}} p_{IJ}^{\text{mod}}(N_I, F))[d_I] \\
I(p_{I,J}^{\text{mod}},p_{IJ})(-,-)(u_{IJ} \otimes I(p_{I,J}^{\text{mod}},p_{IJ})(-,-)(v_{IJ}))[d_I] &\xrightarrow{p_{IJ,*}} p_{IJ,*}((M_J, F) \otimes_{O_{\tilde{S}_j}} (N_J, F))[d_J + d_{IJ}].
\end{align*}
\]

Let \(S \in \text{AnSp}(\mathbb{C})\) and \(S = \cup S_i\) an open cover such that there exist closed embeddings \(i_i : S_i \rightarrow \tilde{S}_i\) with \(\tilde{S}_i \in \text{SmVar}(\mathbb{C})\). We have the tensor product functors

\[
(-) \otimes_{O_{\tilde{S}_i}}^! (-) : C_{D^\text{int}}(S/(\tilde{S}_i)) \rightarrow C_{D^\text{int}}(S/(\tilde{S}_i)),
\]

\[
((M_I, F), u_{IJ}), ((N_I, F), v_{IJ})) \mapsto ((M_I, F) \otimes_{O_{\tilde{S}_i}} (N_I, F), u_{IJ} \otimes v_{IJ}),
\]

with \(u_{IJ} \otimes v_{IJ}\) as above.

**Proposition 94.** Let \(S \in \text{Var}(\mathbb{C}).\) Denote \(\Delta_S : S \rightarrow S \times S\) the diagonal embedding. Let \(S = \cup S_i\) an open cover such that there exist closed embeddings \(i_i : S_i \rightarrow \tilde{S}_i\) closed embedding with \(\tilde{S}_i \in \text{SmVar}(\mathbb{C})\); or let \(S \in \text{AnSp}(\mathbb{C})\) and \(S = \cup S_i\) an open cover such that there exist closed embedding \(i_i : S_i \rightarrow \tilde{S}_i\) with \(\tilde{S}_i \in \text{SmVar}(\mathbb{C}).\) We have, for \((M_I, F), u_{IJ}), ((N_I, F), v_{IJ})) \in C_{D^\text{int}}(S/(\tilde{S}_i)),

\[
((M_I, F), u_{IJ}) \otimes_{O_{\tilde{S}_i}}^! ((N_I, F), v_{IJ})) = \Delta_S^\text{mod}(((M_I, F), u_{IJ}).((N_I, F), v_{IJ})).
\]

**Proof.** Follows from proposition 58. \(\square\)

### 4.3.6 The 2 functors of D modules on the category of complex algebraic varieties and on the category of complex analytic spaces, and the transformation maps

**Definition 90.** Consider a commutative diagram in \(\text{Var}(\mathbb{C})\) which is cartesian:

\[
\begin{array}{ccc}
D = X_T \xrightarrow{\ell'} T. & \Downarrow{g'} & \Downarrow{g} \\
\downarrow{g} & \downarrow{g} & \downarrow{g} \\
X \xrightarrow{f} S
\end{array}
\]

Assume there exist factorizations \(f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S,\) \(g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S,\) with \(Y_1, Y_2 \in \text{SmVar}(\mathbb{C}),\)

\(l_1, l_2\) closed embeddings and \(p_S, p_S\) the projections. Then, the above commutative diagram factors through

\[
\begin{array}{ccc}
D = f' : X_T \xrightarrow{\ell'_1} Y_1 \times T \xrightarrow{p_T} T. & \Downarrow{\ell'_{1,2}} & \Downarrow{\ell_{1,2}} \\
\downarrow{\ell_{1,2}} & \downarrow{\ell_{1,2}} & \downarrow{\ell_{1,2}} \\
f'' : X \times Y_2 \xrightarrow{p_Y \times S} Y_1 \times Y_2 \xrightarrow{p_Y \times S} Y_2 \times S & \Downarrow{p_Y \times S} & \Downarrow{p_Y \times S} \\
\downarrow{p_X} & \downarrow{p_X} & \downarrow{p_X} \\
f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_S} S
\end{array}
\]

whose squares are cartesian. Let \(S = \cup_i S_i\) be an open cover such that there exist closed embeddings \(i_i : S_i \rightarrow \tilde{S}_i\) with \(\tilde{S}_i \in \text{SmVar}(\mathbb{C}).\) Then \(X = \cup_i X_i\) and \(T = \cup_i T_i\) with \(X_i := f^{-1}(S_i)\) and \(T_i := f^{-1}(S_i).\) Moreover, \(f_i = f|_{X_i} : X_i \rightarrow S_i\) lift to \(f_i := p_{S_i} : Y_1 \times S_i \rightarrow S_i\) and \(g_i = g_{T_i} : T_i \rightarrow S_i\) lift to

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\[ \tilde{g}_i := p_{S_i} : Y_2 \times \tilde{S}_i \to \tilde{S}_i. \]

We then have the following commutative diagram whose squares are cartesian

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y_1 \times T_1 \\
\downarrow{l_2''} & & \downarrow{l_2} \\
X_1 \times Y_2 & \xrightarrow{i_1 \circ i_2} & Y_1 \times Y_2 \times \tilde{S}_1 \\
\downarrow{p_X} & & \downarrow{p_{Y_1 \times \tilde{S}_1}} \\
X_1 & \xrightarrow{i_1} & Y_1 \times \tilde{S}_1 & \xrightarrow{i_2} & \tilde{S}_1
\end{array}
\]

We then define, for \((M,F) = ((M_1,F),u_{1,1}) \in C_{D(2)}(X/(Y_1 \times \tilde{S}_1))\), the following canonical transformation map in \(D_{D(2)}(T/(Y_2 \times \tilde{S}_1))\), using proposition 82,

\[
T^{D\text{-mod}}(f,g)(M,F) := \int_{f'} FDR Rg_{\text{mod},T} (M,F) = (\Gamma_{T_1} E(g_{\text{mod}}^* p_{S_1}^* E((\Omega_{Y_1 \times Y_2 / Y_1 \times Y_2})_{\mathcal{O}_{Y_1 \times \tilde{S}_1}} (M_1,F)), (\tilde{g}_{\text{mod}}^* f_k(u_{1,1}))}
\]

\[
\begin{array}{ccc}
\Gamma_{T_1} E(p_{Y_1 \times \tilde{S}_1}^* E((\Omega_{Y_1 \times Y_2 / Y_1 \times Y_2})_{\mathcal{O}_{Y_1 \times \tilde{S}_1}} p_{Y_1 \times \tilde{S}_1}^* (M_1,F)), f_k(u_{1,1}))
\end{array}
\]

\[
\begin{array}{ccc}
\Gamma_{Y_1 \times T_1} E(p_{Y_1 \times \tilde{S}_1}^* (M_1,F)), f_k(u_{1,1}))
\end{array}
\]

In the analytic case, we have

**Definition 91.** Consider a commutative diagram in \(\text{AnSp}(\mathbb{C})\) which is cartesian:

\[
D = (f,g) = \begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow{g} & & \downarrow{g'} \\
S & \xrightarrow{l_1} & Y_1 \times S & \xrightarrow{p_2} & S
\end{array}
\]

Assume there exist factorizations \(f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_2} S\), \(g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_2} S\), with \(Y_1, Y_2 \in \text{AnSp}(\mathbb{C})\), \(l_1, l_2\) closed embeddings and \(p_2, p_2\) the projections.

(i) We have, for \((M,F) \in D_{D(2)}(X/(Y_1 \times \tilde{S}_1))\), the following transformation map in \(D_{D(2)}(T/(Y_2 \times \tilde{S}_1))\)

\[
T^{D\text{-mod}}(f,g)((M,F)) = Rg_{\text{mod},T} \int_{f'} FDR (M,F) \to \int_{f'} FDR Rg_{\text{mod},T}^*(M,F)
\]

define in the same way as in definition 90.

(ii) For \((M,F) \in D_{D(2)}(X/(Y_1 \times \tilde{S}_1))\), the following transformation map in \(D_{D(2)}(T/(Y_2 \times \tilde{S}_1))\)

\[
T^{D\text{-mod}}(f,g)((M,F)) = Rg_{\text{mod},T} \int_{f'} FDR (M,F) \to \int_{f'} FDR Rg_{\text{mod},T}^*(M,F)
\]

is defined in the same way as in (ii) : see definition 90.
In the algebraic case, we have the following:

**Proposition 95.** Consider a commutative diagram in \( \text{Var}(\mathbb{C}) \)

\[
D = (f, g) = \xymatrix{ X \ar[r]^f & T \ar[d]^g \\
X \ar[r]^f & S }
\]

which is cartesian. Assume there exist factorizations \( f : X \overset{1}{\to} Y_1 \times S \overset{p_2}{\to} S, \ g : T \overset{2}{\to} Y_2 \times S \overset{p_2}{\to} S, \) with \( Y_1, Y_2 \in \text{SmVar}(\mathbb{C}), \ l_1, l_2 \) closed embeddings and \( p_S, p_S \) the projections. For \((M, F) = ((M_1, F), u_{1,2}) \in C_{D(2)fil,c}(X/(Y \times \tilde{S}_I)), \)

\[
T^{D\text{mod}}(f, g) : Rg^{*\text{mod}, \Gamma} \int_{f}^{FDR} (M, F) \to \int_{f'}^{FDR} Rg'^{*\text{mod}, \Gamma}(M, F)
\]

is an isomorphism in \( D_{D(2)fil,\infty}(T/(Y_2 \times \tilde{S}_I)). \)

**Proof.** Similar to the proof of proposition 84: the maps

\[
T^{O}(p_{S_I}, \tilde{g}_I)(M_1, F) : \tilde{g}_I^{*\text{mod}} p_{S_{I,J}}^* E((\Omega^\bullet_{Y \times \tilde{S}_I/S_I}, F_b) \otimes_{O_{Y \times S_I}} (M_1, F)) \to
p_{T_1}^* E((\Omega^\bullet_{Y \times \tilde{T}_I/T_I}, F_b) \otimes_{O_{Y \times \tilde{T}_I}} \tilde{g}_I''^{*\text{mod}}(M_1, F))
\]

are \( \infty \)-filtered Zariski local equivalences since \( \tilde{g}_I : Y_2 \times \tilde{S}_I \to \tilde{S}_I \) are projections. \( \square \)

**Proposition 96.** Consider a commutative diagram in \( \text{AnSp}(\mathbb{C}) \)

\[
D = (f, g) = \xymatrix{ X \ar[r]^f & T \ar[d]^g \\
X \ar[r]^f & S }
\]

which is cartesian. Assume that \( f \) (hence \( f' \)) is proper and that there exist factorizations \( f : X \overset{1}{\to} Y_1 \times S \overset{p_2}{\to} S, \ g : T \overset{2}{\to} Y_2 \times S \overset{p_2}{\to} S, \) with \( Y_1, Y_2 \in \text{AnSm}(\mathbb{C}), \ l_1, l_2 \) closed embeddings and \( p_S, p_S \) the projections.

(i) For \((M, F) = ((M_1, F), u_{1,2}) \in C_{D(2)fil,h}(X/(Y_1 \times \tilde{S}_I)) \)

\[
T^{D\text{mod}}(f, g) : Rg^{*\text{mod}, \Gamma} \int_{f}^{FDR} (M, F) \to \int_{f'}^{FDR} Rg'^{*\text{mod}, \Gamma}(M, F)
\]

is an isomorphism in \( D_{D(2)fil,\infty}(T/(Y_2 \times \tilde{S}_I)). \)

(ii) For \((M, F) = ((M_1, F), u_{1,2}) \in C_{D^{\infty}(2)fil,h}(X/(Y_1 \times \tilde{S}_I)) \)

\[
T^{D\text{mod}}(f, g) : Rg^{*\text{mod}, \Gamma} \int_{f}^{FDR} (M, F) \to \int_{f'}^{FDR} Rg'^{*\text{mod}, \Gamma}(M, F)
\]

is an isomorphism in \( D_{D^{\infty}(2)fil,\infty}(T/(Y_2 \times \tilde{S}_I)). \)

**Proof.** (i): Similar to the proof of proposition 95.
(ii): Similar to the proof of proposition 95. \( \square \)
Definition 92. Let $f : X \to S$ be a morphism, with $X, S \in \text{Var}(\mathbb{C})$, such that there exist a factorization $f : X \xrightarrow{p_S} Y \times S \xrightarrow{p_Y} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection, and consider $S = \bigcup_{i=1}^d S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow S_i$, with $S_i \in \text{SmVar}(\mathbb{C})$; then, $X = \bigcup_{i=1}^d X_i$ with $X_i := f^{-1}(S_i)$. We have, for $(M, F) = ((M_1, F), u_{J_1}) \in C_{D(ful)}(S/(S_i))$, the canonical transformation map in $D_{D(ful)}(T^{an}/(T^{an}_i))$

$$
T^{mod}(an, \gamma_T)(M, F) :
$$

$$
f^{\ast \text{mod}[-]}(\gamma_T(M, F))^{an} := ((\Gamma_{T_i}E(p_{S_i}^{\ast \text{mod}[-]}(M_1, F)))^{an}, (f^{\ast \text{mod}[-]}u_{J_i})^{an})
$$

$$
\xrightarrow{\big((\Gamma_{T_i}E(p_{S_i}^{\ast \text{mod}[-]}(M_1, F)))^{an}, f^{\ast \text{mod}[-]}u_{J_i})^{an}\big)_{\gamma_T}} ((\Gamma_{T_i}E(p_{S_i}^{\ast \text{mod}[-]}(M_1, F)))^{an}, f^{\ast \text{mod}[-]}u_{J_i})^{an}) =: f^{\ast \text{mod}[-]}(\gamma_T(M, F))^{an}
$$

where the equality is obvious (see proposition 55).

Definition 93. Let $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{p_S} Y \times S \xrightarrow{p_Y} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \bigcup_{i=1}^d S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow S_i$ closed embeddings with $S_i \in \text{SmVar}(\mathbb{C})$. We have, for $(M, F) = ((M_1, F), u_{J_1}) \in C_{D(ful)}(X/Y \times S_i)$, the following transformation map in $D_{D(ful)}(X^{an}/(X/Y \times S_i))$

$$
T^{D\text{mod}}(an, f)(M, F) : \int_f (M, F)^{an} = (p_{S_i}^{\ast}E((\Omega_{Y \times S_i}^{\ast}/\mathcal{I}_i/F_0) \otimes_{O_Y} L_D(M_1, F))^{an}, (f^{k}(u_{J_i}))^{an})
$$

$$
\xrightarrow{(\Gamma_{p_{S_i}^{\ast}(an)}(M_1, F))_{\gamma_T}} (p_{F_i}^{\ast}E((\Omega_{Y \times S_i}^{\ast}/\mathcal{I}_i/F_0) \otimes_{O_Y} L_D(M_1, F))^{an}, f^{k}(u_{J_i}))^{an}) =: \int_{f}(M, F)^{an}
$$

Theorem 25. Let $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{p_S} Y \times S \xrightarrow{p_Y} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \bigcup_{i=1}^d S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow S_i$ closed embeddings with $S_i \in \text{SmVar}(\mathbb{C})$. Let $M \in D_{D(ful)}(X/Y \times S_i)$. If $f$ is proper,

$$
T^{D}(an, f)(M) : \int_f M^{an} \xrightarrow{\sim} \int_{f}(M)^{an}
$$

is an isomorphism.

Proof. By theorem 23, $T^{\ast}(p_{S_i}, an)(M_1)$ are usu local equivalences. \qed

In the analytic case, we have the following canonical transformation maps

Definition 94. Let $f : X \to S$ be a morphism, with $X, S \in \text{AnSp}(\mathbb{C})$, such that there exist a factorization $f : X \xrightarrow{p_S} Y \times S \xrightarrow{p_Y} S$ with $Y \in \text{AnSm}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection, and consider $S = \bigcup_{i=1}^d S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow S_i$, with $S_i \in \text{AnSm}(\mathbb{C})$; then, $X = \bigcup_{i=1}^d X_i$ with $X_i := f^{-1}(S_i)$. We have, for $(M, F) = ((M_1, F), u_{J_1}) \in C_{D(2)(ful)}(S/(S_i))$, the canonical transformation map in $D_{D(2)(ful)}(T/(T_i))$ obtained by the canonical maps given in definition 67 and definition 72:

$$
T(f, \infty)(M, F) : J_T(f^{\ast \text{mod}[-]}(\gamma_T(M, F))) := (J_{T_i}(\Gamma_{T_i}E(p_{S_i}^{\ast \text{mod}[-]}(M_1, F))), J(f^{\ast \text{mod}[-]}u_{J_i}))
$$

$$
\xrightarrow{(\Gamma_{T_i}E(p_{S_i}^{\ast \text{mod}[-]}(M_1, F)), N_{J_i})} (\Gamma_{T_i}E(p_{S_i}^{\ast \text{mod}[-]}J_{S_i}(M_1, F), f^{\ast \text{mod}[-]}(u_{J_i})) =: f^{\ast \text{mod}[-]}(J_S(M, F))
$$

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4.4 The category of complexes of quasi-coherent sheaves whose cohomology sheaves has a structure of D-modules

4.4.1 Definition on a smooth complex algebraic variety or smooth complex analytic space and the functorialities

Let \( X \in \text{SmVar}(\mathbb{C}) \) or let \( X \in \text{AnSm}(\mathbb{C}) \). Recall that (see definition 57 section 4.1) \( \mathcal{C}_{O_{X}^{fil},D}(X) \) is the category

- whose objects \((M, F) \in \mathcal{C}_{O_{X}^{fil},D}(X)\) are filtered complexes of presheaves of \( O_X \) modules \((M, F) \in \mathcal{C}_{O_{X}^{fil},D}(X)\) whose cohomology presheaves \( H^n(M, F) \in \text{PSh}_{O_X^{fil}}(X) \) are endowed with a structure of filtered \( D_X \) modules for all \( n \in \mathbb{Z} \).

- whose set of morphisms \( \text{Hom}_{\mathcal{C}_{O_{X}^{fil},D}(X)}((M, F), (N, F)) \subset \text{Hom}_{\mathcal{C}_{O_{X}^{fil},D}(X)}((M, F), (N, F)) \) between \((M, F), (N, F) \in \mathcal{C}_{O_{X}^{fil},D}(X)\) are the morphisms of filtered complexes of \( O_X \) modules \( m : (M, F) \rightarrow (N, F) \) such that \( H^n m : H^n(M, F) \rightarrow H^n(N, F) \) is \( D_X \) linear, i.e. is a morphism of (filtered) \( D_X \) modules, for all \( n \in \mathbb{Z} \).

More generally, let \( h : X \rightarrow S \) be a morphism with \( X, S \in \text{SmVar}(\mathbb{C}) \) or with \( X, S \in \text{AnSm}(\mathbb{C}) \). Then, \( \mathcal{C}_{h^{*}O_{X}^{fil},h^{*}D}(X) \) the category

- whose objects \((M, F) \in \mathcal{C}_{h^{*}O_{X}^{fil},h^{*}D}(X)\) are filtered complexes of presheaves of \( h^{*}O_S \) modules \((M, F) \in \mathcal{C}_{h^{*}O_{X}^{fil},h^{*}D}(X)\) whose cohomology presheaves \( H^n(M, F) \in \text{PSh}_{h^{*}O_S^{fil}}(X) \) are endowed with a structure of filtered \( h^{*}D_S \) modules for all \( n \in \mathbb{Z} \).

- whose set of morphisms \( \text{Hom}_{\mathcal{C}_{h^{*}O_{X}^{fil},h^{*}D}(X)}((M, F), (N, F)) \subset \text{Hom}_{\mathcal{C}_{h^{*}O_{X}^{fil},h^{*}D}(X)}((M, F), (N, F)) \) between \((M, F), (N, F) \in \mathcal{C}_{h^{*}O_{X}^{fil},h^{*}D}(X)\) are the morphisms of filtered complexes of \( h^{*}O_S \) modules \( m : (M, F) \rightarrow (N, F) \) such that \( H^n m : H^n(M, F) \rightarrow H^n(N, F) \) is \( h^{*}D_S \) linear, i.e. is a morphism of (filtered) \( h^{*}D_S \) modules, for all \( n \in \mathbb{Z} \).

**Definition 95.** Let \( S \in \text{SmVar}(\mathbb{C}) \) or \( S \in \text{AnSm}(\mathbb{C}) \). Let \( Z \subset S \) a closed subset. Denote by \( j : S \setminus Z \hookrightarrow S \) the open complementary embedding.

(i) We denote by \( \mathcal{C}_{O_S,h^{*}D,Z}(S) \subset \mathcal{C}_{O_S,h^{*}D}(S) \) the full subcategory consisting of \( M \in \mathcal{C}_{O_S,h^{*}D}(S) \) such that such that \( j^{*}H^n M = 0 \) for all \( n \in \mathbb{Z} \).

(ii) We denote by \( \mathcal{C}_{O_{X}^{fil},h^{*}D,Z}(S) \subset \mathcal{C}_{O_{X}^{fil},h^{*}D}(S) \) the full subcategory consisting of \((M, F) \in \mathcal{C}_{O_{X}^{fil},h^{*}D}(S)\) such that there exist \( r \in \mathbb{N} \) and an \( r \)-filtered homotopy equivalence \( m : (M, F) \rightarrow (M', F) \) with \((M', F) \in \mathcal{C}_{O_{X}^{fil},h^{*}D}(S) \) such that \( j^{*}H^n Gr^p_p(M', F) = 0 \) for all \( n, p \in \mathbb{Z} \).

**Definition 96.** Let \( S \in \text{SmVar}(\mathbb{C}) \) or let \( S \in \text{AnSm}(\mathbb{C}) \). We have then (see section 2), for \( r = 1, \cdots, \infty \), the homotopy category \( \mathcal{K}_{O_{X}^{fil},h^{*}D, r}(S) = \text{Ho}_r(\mathcal{C}_{O_{X}^{fil},h^{*}D}(S)) \) whose objects are those of \( \mathcal{C}_{O_{X}^{fil},h^{*}D}(S) \) and whose morphisms are \( r \)-filtered homotopy classes of morphism, and its localization \( D_{O_{X}^{fil},h^{*}D, r}(S) = K_{O_{X}^{fil},h^{*}D}(S)([E_1]^{-1}) \) with respect to filtered zariski, resp. usu local equivalence. Note that the classes of filtered \( \tau \) local equivalence constitute a right multiplicative system.

We look at functoriality

- Let \( S \in \text{SmVar}(\mathbb{C}) \) or \( S \in \text{AnSm}(\mathbb{C}) \). Let \((M, F) \in \mathcal{C}_{O_{X}^{fil},h^{*}D}(S) \). Then, the canonical morphism \( q : L_0(M, F) \rightarrow (M, F) \in \mathcal{C}_{O_{X}^{fil},h^{*}D}(S) \) being a quasi-isomorphism of \( O_S \) modules, we get in a unique way \( L_0(M, F) \in \mathcal{C}_{O_{X}^{fil},h^{*}D}(S) \) such that \( q : L_0(M, F) \rightarrow (M, F) \) is a morphism in \( \mathcal{C}_{O_{X}^{fil},h^{*}D}(S) \).

- Let \( f : X \rightarrow S \) be a morphism with \( X, S \in \text{SmVar}(\mathbb{C}) \), or let \( f : X \rightarrow S \) be a morphism with \( X, S \in \text{AnSm}(\mathbb{C}) \). Let \((M, F) \in \mathcal{C}_{O_{X}^{fil},h^{*}D}(S) \). Then, \( f^{\text{sm}} H^n(M, F) := (O_X, F_0) \otimes f^{*}O_S f^{*}H^n(M, F) \) is canonical a filtered \( D_X \) module (see section 4.1 or 4.2). Consider the canonical surjective map \( q(f) : H^n f^{\text{sm}}(M, F) \rightarrow f^{\text{sm}}H^n(M, F) \). Then, \( q(f) \) is an isomorphism if \( f \) is smooth. Let
$h : U \to S$ be a smooth morphism with $U, S \in \text{SmVar}(\mathbb{C})$, or let $h : U \to S$ be a smooth morphism with $U, S \in \text{AnSm}(\mathbb{C})$. We get the functor

$$h^{\ast \text{mod}} : C_{O_S} \leftarrow (S) \to C_{O_U} \leftarrow (U), (M, F) \mapsto h^{\ast \text{mod}}(M, F),$$

- Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$, and let $i : Z \to S$ a closed embedding and denote by $j : S \setminus Z \to S$ the open complementary. For $M \in C_{O_S}(S)$, the cohomology presheaves of

  $$\Gamma_Z M := \text{Cone}(\text{ad}(j^*, j_*)(M) : M \to j_* j^* M)[-1]$$

has a canonical $D_S$-module structure (as $j^* H^m M$ is a $j^* D_S$ module, $H^m j_* j^* M = j_* j^* H^m M$ has an induced structure of $D_S$ module), and $\gamma_Z(M) : \Gamma_Z M \to M$ is a map in $C_{O_S}(S)$. For $Z_X \subset Z$ a closed subset and $M \in C_{O_S}(S)$, $T(Z_X/Z, \gamma)(M) : \Gamma Z_X \to \Gamma Z_M$ is a map in $C_{O_S}(S)$. We get the functor

$$\Gamma_Z : C_{O_S} \leftarrow (S) \to C_{O_S}(S),$$

$$(M, F) \mapsto \Gamma_Z(M, F) := \text{Cone}(\text{ad}(j^*, j_*)((M, F)) : (M, F) \to j_* j^* (M, F))[-1],$$

together we the canonical map $\gamma_Z(M, F) : \Gamma Z(M, F) \to (M, F)$

More generally, let $h : Y \to S$ a morphism with $Y, S \in \text{Var}(\mathbb{C})$ or $Y, S \in \text{AnSp}(\mathbb{C})$, $S$ smooth, and let $i : X \to Y$ a closed embedding and denote by $j : Y \setminus X \to Y$ the open complementary. For $M \in C_{h \ast O_S,h \ast D}(Y)$,

$$\Gamma_X := \text{Cone}(\text{ad}(j^*, j_*)(M) : M \to j_* j^* M)[-1]$$

has a canonical $h^* D_Y$-module structure, (as $j^* H^n M$ is a $j^* D_Y$ module, $H^n j_* j^* M = j_* j^* H^n M$ has an induced structure of $j^* h^* D_Y$ module), and $\gamma_X(M) : \Gamma_X M \to M$ is a map in $C_{h \ast O_S,h \ast D}(Y)$. For $X_Y \subset X$ a closed subset and $M \in C_{h \ast O_S,h \ast D}(Y)$, $T(Z_Y/Z, \gamma)(M) : \Gamma X_Y \to \Gamma X_M$ is a map in $C_{h \ast O_S,h \ast D}(Y)$. We get the functor

$$\Gamma_X : C_{h \ast O_S,h \ast D}(Y) \to C_{h \ast O_S,h \ast D}(Y),$$

$$(M, F) \mapsto \Gamma_X(M, F) := \text{Cone}(\text{ad}(j^*, j_*)((M, F)) : (M, F) \to j_* j^* (M, F))[-1],$$

together we the canonical map $\gamma_X(M, F) : \Gamma_X(M, F) \to (M, F)$

- Let $f : X \to S$ be a morphism with $X, S \in \text{SmVar}(\mathbb{C})$, or let $f : X \to S$ be a morphism with $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \xrightarrow{i} X \times S \xrightarrow{p} S$, where $l$ is the graph embedding and $p$ the projection. We get from the two preceding points the functor

$$f^{\ast \text{mod}, \Gamma} : C_{O_X} \leftarrow (X \times S), (M, F) \mapsto f^{\ast \text{mod}, \Gamma}(M, F) := \Gamma_X p^{\ast \text{mod}}(M, F),$$

and

$$f^{\ast \text{mod}[-1], \Gamma} : C_{O_X} \leftarrow (X \times S), (M, F) \mapsto f^{\ast \text{mod}[-1], \Gamma}(M, F) := \Gamma_X E(p^{\ast \text{mod}}(M, F))[-d_X],$$

which induces in the derived categories the functor

$$R f^{\ast \text{mod}[-1], \Gamma} : D_{O_X} \leftarrow (X \times S), (M, F) \to R f^{\ast \text{mod}[-1], \Gamma}(M, F) := \Gamma_X E(p^{\ast \text{mod}[-1]}(M, F)).$$

For $(M, F) \in C_{O_S}(S)$ or $(M, F) \in C_{O_S}(S)$, the canonical map in $C_{O_S}(S)$

$$\text{ad}(i^{\ast \text{mod}}, i_*)(-) : L \Gamma X E(p^{\ast \text{mod}}(M, F)) \to i_* i^{\ast \text{mod}} L \Gamma X E(p^{\ast \text{mod}}(M, F))$$

gives in the derived category, the canonical map in $D_{O_X}(X \times S)$

$$I (f^{\ast \text{mod}, \Gamma}(M, F), R f^{\ast \text{mod}, \Gamma}(M, F) = L \Gamma X E(p^{\ast \text{mod}}(M, F)) \xrightarrow{\text{ad}(i^{\ast \text{mod}}, i_*)(-)} i_* i^{\ast \text{mod}} L \Gamma X E(p^{\ast \text{mod}}(M, F)) \xrightarrow{\sim} i_* i^{\ast \text{mod}} L p^{\ast \text{mod}}(M, F) = L f^{\ast \text{mod}}(M, F)$$

where the isomorphism is given by lemma 6.
• Let $S \in \text{SmVar}(\mathbb{C})$. We have the analytical functor:

$(-)^{an}: C_{O_{fil,D}}(S) \to C_{O_{fil,D}}(S^{an})$, $(M,F) \mapsto (M,F)^{an} := \text{an}_{\mathbb{Z}}(M,F) := (M,F) \otimes_{\mathbb{Z}} O_{S^{an}}$

which induces in the derived category

$(\cdot)^{an}_{D}: D_{O_{fil,D,r}}(S) \to D_{O_{fil,D,r}}(S^{an})$, $((M,F) \mapsto (M,F)^{an} := \text{an}_{\mathbb{Z}}(M,F))$

since $\text{an}_{\mathbb{Z}}$ is an exact functor.

We have, for $f: T \to S$ with $T, S \in \text{SmVar}(\mathbb{C})$ or with $T, S \in \text{AnSm}(\mathbb{C})$, the commutative diagrams of functors

\[
\begin{array}{ccc}
C_{D_{fil}}(S) & \overset{\alpha_S}{\longrightarrow} & C_{O_{fil,D}}(S) \\
\downarrow_{f^*\mod[-],F} & & \downarrow_{f^*\mod[-],F} \\
C_{D_{fil}}(T) & \overset{\alpha_T}{\longrightarrow} & C_{O_{fil,D}}(T)
\end{array}
\quad
\begin{array}{ccc}
D_{D_{fil,r}}(S) & \overset{\alpha_S}{\longrightarrow} & D_{O_{fil,D,r}}(S) \\
\downarrow_{Rf^*\mod[-],F} & & \downarrow_{Rf^*\mod[-],F} \\
D_{D_{fil,r}}(T) & \overset{\alpha_T}{\longrightarrow} & D_{O_{fil,D,r}}(T)
\end{array}
\]

where $\alpha_S$ and $\alpha_T$ are the forgetful functors.

4.4.2 Definition on a singular complex algebraic variety or singular complex analytic space and the functorialities

**Definition 97.** Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i: S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i: S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. Then, $C_{O_{fil,D}}(S/(\tilde{S}_I))$ is the category

- whose objects are $(M,F) = ((M_I,F)_I \cup [1,...,l], u_{IJ})$, with
  - $(M_I,F) \in C_{O_{fil,D}}(\tilde{S}_I)$,
  - $u_{IJ}: m^* (M_I,F) \to m^* p_{IJ,*}(M_J,F)[d_{\tilde{S}_J} - d_{\tilde{S}_I}]$ for $J \subset I$, are morphisms, $p_{IJ}: \tilde{S}_I \to \tilde{S}_J$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ,J,K} \circ u_{IJ} = u_{IK}$ in $C_{O_{fil,D}}(\tilde{S}_I)$;

- whose morphisms $m: ((M_I,F), u_{IJ}) \to ((N_I,F), v_{IJ})$ between $(M,F) = ((M_I,F)_I \cup [1,...,l], u_{IJ})$ and $(N,F) = ((N_I,F)_I \cup [1,...,l], v_{IJ})$ are a family of morphisms of complexes,

$m = (m_I: (M_I,F) \to (N_I,F))_I \cup [1,...,l]$

such that $v_{IJ} \circ m_I = p_{IJ,*} m_J$ in $C_{O_{fil,D}}(\tilde{S}_I)$.

We denote by $C_{\tilde{O}_{fil,D}}(S/(\tilde{S}_I)) \subset C_{O_{fil,D}}(S/(\tilde{S}_I))$ the full subcategory consisting of objects $(M_I,F,u_{IJ})$ such that the $u_{IJ}$ are $\infty$-filtered Zariski, resp. usd, local equivalences.

**Definition 98.** Let $S \in \text{Var}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i: S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$; or let $S \in \text{AnSp}(\mathbb{C})$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i: S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. We have then (see section 2), for $r = 1, \ldots, \infty$, the homotopy category

$K_{O_{fil,D,r}}(S/(\tilde{S}_I)) := \text{Ho}_r(C_{O_{fil,D}}(S/(\tilde{S}_I)))$

whose objects are those of $C_{O_{fil,D}}(S/(\tilde{S}_I))$ and whose morphisms are $r$-filtered homotopy classes of morphism, and its localization

$D_{fil,D,r}(S/(\tilde{S}_I)) := K_{O_{fil,D,r}}(S/(\tilde{S}_I))[E_1^{-1}]$

with respect to the classes of filtered zariski, resp. usd local equivalence. Note that the classes of filtered $\tau$ local equivalence constitute a right multiplicative system.
Let $f : X \to S$ be a morphism, with $X, S \in \text{Var}(C)$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(C)$, $l$ a closed embedding and $p_S$ the projection, and consider $S = \cup_{i=1}^d S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow S_i$, with $S_i \in \text{SmVar}(C)$; or let $f : X \to S$ be a morphism, with $X, S \in \text{AnSp}(C)$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{AnSm}(C)$, $l$ a closed embedding and $p_S$ the projection and consider $S = \cup_{i=1}^d S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow S_i$, with $S_i \in \text{AnSm}(C)$. Then, $X = \cup_{i=1}^d X_i$ with $X_i := f^{-1}(S_i)$. Denote by $p_{IJ} : S_J \rightarrow S_I$ and $p_{IJ}^\prime : Y \times S_J \rightarrow Y \times S_I$ the projections and by

$$E_{IJ} = \xymatrix{ \tilde{S}_J \setminus S_J \ar[r]^{m_J} & \tilde{S}_J \ar[d]^{p_{IJ}} \ar[r]^{m_J} & Y \times \tilde{S}_J \setminus X_J \ar[r]^{m_J} & Y \times \tilde{S}_J \ar[d]^{p_{IJ}^\prime} \ar[r]^{m_J} & Y \times \tilde{S}_J \ar[d]^{p_{IJ}^\prime} \ar[r]^{m_J} & Y \times \tilde{S}_J \ar[d]^{p_{IJ}^\prime} \ar[r]^{m_J} & Y \times \tilde{S}_J \ar[d]^{p_{IJ}^\prime} \ar[r]^{m_J} & Y \times \tilde{S}_J \ar[d]^{p_{IJ}^\prime} \ar[r]^{m_J} & Y \times \tilde{S}_J \ar[d]^{p_{IJ}^\prime} \ar[r]^{m_J} & Y \times \tilde{S}_J }$$

the commutative diagrams. We then have the filtered De Rham the inverse image functor:

$$f^{*\text{mod}[-], \Gamma} : C_{\text{Ofil},D}(S/(\tilde{S}_I)) \rightarrow C_{\text{Ofil},D}(X/(Y \times \tilde{S}_I)), \quad (M, F) = ((M_I, F), u_{IJ}) \mapsto f^{*\text{mod}[-], \Gamma}(M, F) := (\Gamma X_I E(p_{IJ}^{*\text{mod}[-]}(M_I, F)), f_J^{*\text{mod}[-]} u_{IJ})$$

with, denoting for short $d_{IJ} := d_{S_J} - d_{S_I}$

$$f_J^{*\text{mod}[-]} u_{IJ} : \Gamma X_I E(p_{IJ}^{*\text{mod}[-]}(M_I, F)) \xrightarrow{\Gamma X_I E(p_{IJ}^{*\text{mod}[-]}(u_{IJ}))} \Gamma X_I E(p_{IJ}^{*\text{mod}[-]} p_{IJ}* (M_I, F)[d_{IJ}])$$

$$\xrightarrow{\Gamma X_I E(T(p_{IJ}^{*\text{mod}[-]} p_{IJ})(-)^{-1})[d_Y + d_{IJ}]} \Gamma X_I E(p_{IJ}* p_{IJ}^{*\text{mod}[-]} (M, F)[dY + d_{IJ}])$$

$$\xrightarrow{\sim} p_{IJ}^\prime \Gamma X_I E(p_{IJ}^{*\text{mod}[-]}(M, F)[d_{IJ}]).$$

It induces in the derived categories, the functor

$$Rf^{*\text{mod}[-], \Gamma} : D_{\text{Ofil},D,\infty}(S/(\tilde{S}_I)) \rightarrow D_{\text{Ofil},D,\infty}(X/(Y \times \tilde{S}_I)), \quad (M, F) = ((M_I, F), u_{IJ}) \mapsto$$

$$Rf^{*\text{mod}[-], \Gamma}((M, F)) := (\Gamma X_I E(p_{IJ}^{*\text{mod}[-]}(M_I, F)), f_J^{*\text{mod}[-]} u_{IJ}).$$

By definition, for $f : T \to S$ with $T, S \in \text{QPVar}(C)$ or with $T, S \in \text{AnSp}(C)^{QP}$, after considering a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(C)$, $l$ a closed embedding and $p_S$ the projection, the commutative diagrams of functors

$$\xymatrix{ C_{\text{Ofil}}(S/(\tilde{S}_I)) \ar[r]^{o_S} & C_{\text{Ofil},D}(S/(\tilde{S}_I)) \ar[d]^{f^{*\text{mod}[-], \Gamma}} \ar[r]^{o_S} & D_{\text{Ofil},D,\infty}(S/(\tilde{S}_I)) \ar[d]^{Rf^{*\text{mod}[-], \Gamma}} \ar[r]^{o_S} & D_{\text{Ofil},D,\infty}(S/(\tilde{S}_I)) \ar[d]^{Rf^{*\text{mod}[-], \Gamma}} \ar[r]^{o_S} & D_{\text{Ofil},D,\infty}(S/(\tilde{S}_I)) \ar[d]^{Rf^{*\text{mod}[-], \Gamma}} \ar[r]^{o_S} & D_{\text{Ofil},D,\infty}(S/(\tilde{S}_I)) }$$

where $o_S$ and $o_T$ are the forgetful functors.

Let $f : X \to S$ be a morphism, with $X, S \in \text{Var}(C)$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(C)$, $l$ a closed embedding and $p_S$ the projection, and consider $S = \cup_{i=1}^d S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow S_i$, with $S_i \in \text{SmVar}(C)$; Then, $X = \cup_{i=1}^d X_i$ with $X_i := f^{-1}(S_i)$. We have, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\text{Ofil},D}(S/(\tilde{S}_I))^{\Gamma}$, the
canonical transformation map in $D_{Ofil,D}(T^{an}/(T^{an}_I))$: 
\[ T^{an}(\gamma_T)(M, F) : f^{an}(\Gamma(M, F), F)^{an} \] 
\[ \frac{(T^{an}(\gamma_T))^{-1}}{\Gamma_T} = (\Gamma_T E((p^{an}_{S_I})^*(M, F), F)^{an}), f^{an}(\Gamma(M, F), F)^{an}) \]

\[ \cong (\Gamma_T E((p^{an}_{S_I})^*(M, F), F)^{an}), f^{an}(\Gamma(M, F), F)^{an}) =: f^{an}(\Gamma(M, F), F)^{an} \] 
where the equality is obvious.

5 The category of mixed Hodge modules on complex algebraic varieties and complex analytic spaces and the functorialities

For $S \in \text{Top}$ a topological space endowed with a stratification $S = \bigcup_{k=1}^d S_k$ by locally closed subsets $S_k$ together with the perversity $p(S_k)$, we denote by $\mathcal{P}(S, W) \subset D_{fil}(S)$ the category of filtered perverse sheaves of abelian groups. For a locally compact (hence Hausdorff) topological space, we denote by $D_c(S) \subset D(S)$ the full subcategory of complexes of presheaves whose cohomology sheaves are constructible.

5.1 The De Rahm functor for D modules on a complex analytic space

Let $S \in \text{AnSm}(\mathbb{C})$. Recall we have the dual functor
\[ \mathbb{D}_S : C(S) \to C(S), K \mapsto \mathbb{D}_S(K) := \mathcal{H}om(K, E(\mathbb{Z}_S)) \]

which induces the functor
\[ L\mathbb{D}_S : D(S) \to D(S), K \mapsto L\mathbb{D}_S(K) := \mathbb{D}_S(LK) := \mathcal{H}om(LK, E_{et}(\mathbb{Z}_S)). \]

Let $S \in \text{AnSm}(\mathbb{C})$.

- The functor
\[ M \in \text{PSh}_{D}(S) \mapsto DR(S)(M) := \Omega^\bullet_S \otimes_{O_S} M \in C_{cs}(S) \]

which sends a $D_S$ module to its De Rham complex (see section 4) induces, after shifting by $d_S$ in order to send holonomic module (degree zero) to perverse sheaves, in the derived category the functor
\[ DR(S)^{-1} : D_{P}(S) \to D_{cs}(S), M \mapsto DR(S)^{-1}(M) := DR(S)(M)[d_S] := \Omega^\bullet_S \otimes_{O_S} M[d_S] \simeq K_S \otimes_{D_S} M \simeq \mathcal{H}om_{D_S}(\mathbb{D}_S(LD_M, E(O_S))[d_S] \]

and, by functoriality, the functor
\[ DR(S)^{-1} : D_{D_{P},\infty}(S) \to D_{cs,fil,\infty}(S), (M, W) \mapsto DR(S)^{-1}(M, W) := (\Omega^\bullet_S, F_b) \otimes_{O_S} (M, W)[d_S] = K_S \otimes_{D_S} (M, W) \]

- On the other hand, we have the functor
\[ C_{cs}(S) \to C_{D^{\infty}}(S), K \mapsto \mathcal{H}om_{cs}(L_C \mathbb{D}_S(LK), E(O_S)) = \mathcal{H}om_{cs}(L_C \mathbb{D}_S(LK), E(O_S))[d_S] \]

together with, for $K \in C_{cs}(S)$, the canonical map
\[ s(K) : K \to DR(S)^{-1}(J^{-1}_S \mathcal{H}om_{cs}(L_C \mathbb{D}_S(LK), E(O_S))[d_S]) \]
\[ \cong \mathcal{H}om_{D_S}(\mathbb{D}_S(LD_M, E(O_S)), E(O_S)), c \in \Gamma(S^o, L(K)) \mapsto s(K)(c) = (\phi \in \Gamma(S^{oo}, LD_{\mathcal{H}om}(L_C(K), E(O_S)))) \mapsto \phi(c) \]

where $S^{oo} \subset S^o \subset S$ are open subsets.
Theorem 26. Let $S \in \text{AnSm}(\mathbb{C})$.

(i) The functor $\mathcal{J}_S : D_{D,\text{rh}}(S) \to D_{\mathcal{D}^-,h}(S)$ is an equivalence of category. Moreover, for $K \in \mathcal{C}(S)$, we have $\mathcal{H}\text{om}(L(K), E(O_S)) \in C_{D^-,h}(S)$.

(ii) The restriction of the De Rham functor to the full subcategory $D_{D,\text{rh}}(S) \subset D_S$ is an equivalence of category

$$DR(S)^{-1} : D_{D,\text{rh}}(S) \sim \to D_{\mathcal{C}_S, c}(S)$$

whose inverse is the functor

$$K \in C_{\mathcal{C}_S, c}(S) \mapsto J^{-1} \mathcal{H}\text{om}_{\mathcal{C}_S}(\mathbb{D}_S L(K), E(O_S))[-d_S],$$

the map $s(K) : K \mapsto DR(S)^{-1}(J^{-1} \mathcal{H}\text{om}_{\mathcal{C}_S}(L_{\mathbb{D}_S} L(K), E(O_S)))$ being an isomorphism.

(iii) The De Rham functor $DR(S)^{-1}$ sends regular holonomic modules to perverse sheaves.

Proof. See [19].

Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \cup_{i=1}^I S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow S_i$.

- The De Rham functor is in this case

$$DR(S)^{-1} : D_{D^{\text{fil}}, \infty}(S) \to D_{\mathcal{C}^{\text{fil}}, \infty}(S), M = ((M_i, W), u_{ij}) \mapsto DR(S)^{-1}(M, W) := (DR(S_i)^{-1}(M_i, W), DR^{[-1]}(u_{ij})) := (\mathcal{O}_S^* \otimes O_{S_i}(M_i, W), DR^{[-1]}(u_{ij}))$$

with, denoting for short $d_I = d_{S_i}$

$$DR^{[-1]}(u_{ij}) : \mathcal{O}_{S_j} \otimes O_{S_j}(M_j, W)[d_I] \xrightarrow{\text{ad}(p_{ij}, p_{ij}^*)(-)} p_{ij}^* \mathcal{O}_{S_i}(M_i, W)[d_I]$$

- Considering the diagrams

$$D_{ij} = \begin{array}{ccc}
S_j & \xrightarrow{p_{ij}} & S_i \\
\downarrow i_j & & \downarrow i_i \\
\tilde{S}_j & \xrightarrow{\tilde{p}_{ij}} & \tilde{S}_i
\end{array}$$

we get the functor

$$C_{\mathcal{C}^{\text{fil}}}(S) \xrightarrow{T(S/(S_i))} C_{\mathcal{C}^{\text{fil}}}(S/(S_i)) \to C_{D^{\text{fil}}}(S/(S_i)),$$

$$(K, W) \mapsto (\mathcal{H}\text{om}_{\mathcal{C}_S}(L_{\mathbb{D}_S} L(i_i, j^*_i(K, W)), E(O_{S_i}))[-d_{S_i}], u_{ij}(K, W))$$

where

$$u_{ij}(K, W) : \mathcal{H}\text{om}_{\mathcal{C}_{S_j}}(L_{\mathbb{D}_{S_j}} L(i_i, j^*_i(K, W)), E(O_{S_i}))[-d_{S_i}]$$

$$\xrightarrow{\text{ad}(p_{ij}^* \circ p_{ij})(-)} p_{ij}^* \mathcal{H}\text{om}_{\mathcal{C}_{S_j}}(L_{\mathbb{D}_{S_j}} L(i_i, j^*_i(K, W)), E(O_{S_j}))[-d_{S_j}]$$

$$\xrightarrow{\mathcal{H}\text{om}(\cdot, E_o(p_{ij})) \circ T(p_{ij}, \text{hom})(-)} p_{ij}^* \mathcal{H}\text{om}_{\mathcal{C}_{S_j}}(L_{\mathbb{D}_{S_j}} L(i_i, j^*_i(K, W)), E(O_{S_j}))[-d_{S_j}]$$

$$\xrightarrow{\mathcal{H}\text{om}(T(p_{ij}, \mathbb{D})(-)^{-1}, -)} p_{ij}^* \mathcal{H}\text{om}_{\mathcal{C}_{S_j}}(L_{\mathbb{D}_{S_j}} p_{ij}^* L(i_i, j^*_i(K, W)), E(O_{S_j}))[-d_{S_j}]$$

$$\xrightarrow{\mathcal{H}\text{om}(L_{\mathbb{D}_{S_j}} T^{-1}(D_{ij}), (j^*_i(K, W)), E(O_{S_j}))} p_{ij}^* \mathcal{H}\text{om}_{\mathcal{C}_{S_j}}(L_{\mathbb{D}_{S_j}} L(i_i, j^*_i(K, W)), E(O_{S_j}))[-d_{S_j}].$$
Moreover, for \((K, W) \in C_{fil}(S)\), we have
\[
(\text{Hom}_{C_{\bar{S}}} (L_c \mathbb{D}_{\bar{S}} L(i_! j_*^! (K, W)), E(O_{\bar{S}}_i))[-d_{\bar{S}_i}], u_{IJ}(K, W)) \in C_{D^{\infty}_{0,fil,h}(S)}^0
\]
and a canonical map in \(D_{fil}(S) = D_{fil}(S/(\bar{S}_i))\)
\[
s(K) : T(S/(S_i))(K, W) := (L(i_! j_*^! (K, W)), I) \to DR(S)[-1](J^{-1}_{S} \text{Hom}_{C_{\bar{S}}} (L_c \mathbb{D}_{\bar{S}} L(i_! j_*^! (K, W)), E(O_{\bar{S}}_i))[-d_{\bar{S}_i}], u_{IJ}(K, W))
\]

**Corollary 3.** Let \(S \in \text{AnSp}(\mathbb{C})\). Let \(S = \bigcup_{i=1}^d S_i\) an open cover such that there exists closed embeddings \(i_i : S_i \hookrightarrow S_i\). The restriction of the De Rahm functor to the full subcategory \(D^0_{D, rh}(S) \subset D^0_{D}(S)\) is an equivalence of category
\[
DR(S)[-1] : D^0_{D, rh}(S) \xrightarrow{\sim} D_{C_{S,c}}(S)
\]
whose inverse is the functor
\[
K \mapsto J_{S}^{-1}(\text{Hom}_{C_{\bar{S}}} (L_c \mathbb{D}_{\bar{S}} L(i_! j_*^! K), E(O_{\bar{S}}_i))[-d_{\bar{S}_i}], u_{IJ}(K))
\]
the map
\[
s(K) : T(S/(S_i))(K, W) := (L(i_! j_*^! (K, W)), I) \to DR(S)[-1](J^{-1}_{S} \text{Hom}_{C_{\bar{S}}} (L_c \mathbb{D}_{\bar{S}} L(i_! j_*^! (K, W)), E(O_{\bar{S}}_i))[-d_{\bar{S}_i}], u_{IJ}(K, W))
\]
being an isomorphism.

**Proof.** Follows from theorem 26(ii), see [28].

**Proposition 97.** (i) Let \(S \in \text{AnSm}(\mathbb{C})\). Then, for \(M \in C_{D,c}(S)\), there is a canonical isomorphism
\[
T(D, DR)(M) : \mathbb{D}^c_S DR(S)[-1](M) \xrightarrow{\sim} DR(S)[-1](\mathbb{D}^c_S L_DM)
\]
(ii) Let \(S \in \text{AnSp}(\mathbb{C})\). Let \(S = \bigcup_{i=1}^d S_i\) an open cover such that there exists closed embeddings \(i_i : S_i \hookrightarrow S_i\). Then, for \(M = (M_I, u_{IJ}) \in C_{D,c}(S/(\bar{S}_i))\), there is a canonical isomorphism
\[
T(D, DR)(M) : \mathbb{D}^c_S DR(S)[-1](M) \xrightarrow{\sim} DR(S)[-1](\mathbb{D}^c_S M)
\]

**Proof.** (i):See [17]. (ii):Follows from (i), see [28].

We have the following transformation maps :

- Let \(g : T \to S\) a morphism with \(T, S \in \text{AnSm}(\mathbb{C})\). We have, for \((M, W) \in C_{D_{0,fil}}(S)\), the canonical transformation map in \(D_{fil}(T)\)
\[
T(g, DR)(M, W) : g^* DR(S)[-1](M, W) := g^*(\Omega^*_T \otimes_{O_T} L_D(M, W))[d_s]
\]
\[
\xrightarrow{\Omega^*_{T,S}(L_D(M, W))[d_s]} \Omega^*_{T} \otimes_{O_T} g^{* \text{mod}} L_D(M, W)[d_s]
\]
\[
\xrightarrow{\sim} \Omega^*_{T} \otimes_{O_T} g^{* \text{mod}}[-1] L_D(M, W)[d_s] =: DR(T)(-)(Lg^{* \text{mod}}[-1](M, W))
\]

Note that this transformation map is NOT an isomorphism in general. It is an isomorphism if \(g\) is a smooth morphism. If \(g\) is a closed embedding, it is an isomorphism for \(M\) non caracteristic with respect to \(g\).
• Let \( j : S^o \rightarrow S \) an open embedding with \( S \in \text{AnSm}(\mathbb{C}) \). We have, for \((M,W) \in C_{D_{0fil}}(S^o)\), the canonical transformation map in \( D_{fil}(S) \):

\[
T_c(j, DR)(M,W) : DR(S)[-1](j_!(M,W)) := \Omega^*_S \otimes_{O_S} j_!(M,W)[d_S]
\]

which is an isomorphism (see proposition 82).

• Let \( g : T \rightarrow S \) a morphism with \( T,S \in \text{AnSp}(\mathbb{C}) \). Assume there exist a factorization \( g : T \rightarrow Y \times S \xrightarrow{p_S} S \) with \( Y \in \text{AnSm}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. Let \( S = \cup_i S_i \) an open covers such that there exist closed embeddings \( i : S_i \rightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{AnSm}(\mathbb{C}) \). We have, for \( M = (M_I, u_{lj}) \in C_D(S/(\tilde{S}_I)) \), the canonical transformation map in \( D_{fil}(T/(Y \times \tilde{S}_I)) \):

\[
T_f(g, DR)(M) := T(T/(Y \times \tilde{S}_I))(g_!DR(S)[-1](M,W))
\]

which is an isomorphism.

• Let \( f : T \rightarrow S \) a morphism with \( T,S \in \text{Var}(\mathbb{C}) \). Assume there exist a factorization \( f : T \rightarrow Y \times S \xrightarrow{p_S} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. Let \( S = \cup_i S_i \) an open covers such that there exist closed embeddings \( i : S_i \rightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). We have, for \( M = (M_I, u_{lj}) \in C_D(S/(\tilde{S}_I)) \), the canonical transformation map in \( D_{fil}(T/(Y \times \tilde{S}_I)) \):

\[
DR(T)[-1](T^{\text{mod}}(an, \gamma_T)(M)) := DR(T)[-1]((\Gamma^* T, E(f^{\text{mod}}[-1] \Gamma^* M)^{an})); (f^{\text{mod}}[-1]u_{lj}^an)
\]

\[
\rightarrow DR(T)[-1](Rf^{\text{mod}}[-1] \Gamma^* M^{an}) := DR(T)[-1]((\Gamma^* T, E(f^{\text{mod}}[-1] \Gamma^* M)^{an})); (f^{\text{mod}}[-1]u_{lj}^an).
\]

Proposition 98. Let \( f : T \rightarrow S \) a morphism with \( T,S \in \text{Var}(\mathbb{C}) \). Assume there exist a factorization \( f : T \rightarrow Y \times S \xrightarrow{p_S} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. Let \( S = \cup_i S_i \) an open covers such that there exist closed embeddings \( i : S_i \rightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). Then, for \( M = (M_I, u_{lj}) \in C_{Drh}(S/(\tilde{S}_I)) \), the map in \( D_{fil}(T/(Y \times \tilde{S}_I)) \):

\[
DR(T)[-1](T^{\text{mod}}(an, \gamma_T)(M)) := DR(T)[-1]((\Gamma^* T, E(f^{\text{mod}}[-1] \Gamma^* M)^{an})); (f^{\text{mod}}[-1]u_{lj}^an)
\]

\[
\rightarrow DR(T)[-1](Rf^{\text{mod}}[-1] \Gamma^* M^{an}) := DR(T)[-1]((\Gamma^* T, E(f^{\text{mod}}[-1] \Gamma^* M)^{an})); (f^{\text{mod}}[-1]u_{lj}^an).
\]

given above is an isomorphism.

Proof. See [17].

In the algebraic case, we have, by proposition 98, for complexes of \( D \)-modules whose cohomology sheaves are regular holonomic the following canonical isomorphisms:
Definition 99. (i) Let $f : T \to S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{L} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. We have, for $M = (M_1, u_{ij}) \in C_{\mathbb{D}, rh}(S/\tilde{S})^0$, the canonical map

$$
T(f, DR)(M) : f^* DR(S)[-](M^{an}) \xrightarrow{T(f, DR)(M^{an})} DR(T)[-](Rf^* \text{mod}[-], \Gamma(M)) \\
\xrightarrow{DR(T)[-](\text{mod}(\alpha, \gamma)(M))} DR(T)[-](Rf^* \text{mod}[-], \Gamma M^{an}) =: DR(T)[-](Rf^* \text{mod}[-], \Gamma M^{an}),
$$

which is an isomorphism by proposition 98.

(ii) Let $f : T \to S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : T \xrightarrow{k} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding, and $p_S$ the projection. We have, for $M = (M_1, u_{ij}) \in C_{\mathbb{D}, rh}(S/\tilde{S})^0$, the canonical transformation map

$$
T(f, DR)(M) : DR(T)[-](\text{mod}[-], \Gamma M^{an}) := DR(T)[-](\text{mod}[-], \Gamma L\mathbb{D}_{S}^{K} M^{an}) \\
\xrightarrow{T(D, DR)(-)} L\mathbb{D}_{S}^{K} DR(T)[-](\text{mod}[-], \Gamma L\mathbb{D}_{S}^{K} M^{an}) \\
\xrightarrow{L\mathbb{D}_{S}^{K} f^* T(D, DR)(-)} L\mathbb{D}_{S}^{K} f^* DR(S)[-](\text{mod}[-], \Gamma L\mathbb{D}_{S}^{K} M^{an})
$$

which is an isomorphism by (i) and proposition 97.

(iii) Let $f : T \to S$ a morphism with $T, S \in \text{QPVar}(\mathbb{C})$. Consider a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y = \mathbb{P}^{N, o} \subset \mathbb{P}^N$ an open subset, $l$ a closed embedding, and $p_S$ the projection. We have, for $M = (M_1, u_{ij}) \in C_{\mathbb{D}, rh}(T/Y \times S)^0$, the canonical transformation map

$$
T_{\ast}(f, DR)(M) : DR(S)[-]((\int_{f}^{} M)^{an}) \xrightarrow{\text{ad}(f^*, Rf_{\ast})(-)} Rf_{\ast} f^* DR(S)[-]((\int_{f}^{} M)^{an}) \\
\xrightarrow{Rf_{\ast} DR(T)[-](\text{mod}[-], \Gamma \int_{f}^{} M))} Rf_{\ast} DR(T)[-](\text{mod}[-], \Gamma \int_{f}^{} M))
$$

which is an isomorphism by GAGA in the proper case and by the open embedding case (cf. proposition 98).

(iv) Let $f : T \to S$ a morphism with $T, S \in \text{QPVar}(\mathbb{C})$. Consider a factorization $f : T \xrightarrow{k} Y \times S \xrightarrow{p_S} S$ with $Y = \mathbb{P}^{N, o} \subset \mathbb{P}^N$ an open subset, $l$ a closed embedding, and $p_S$ the projection. We have, for $M = (M_1, u_{ij}) \in C_{\mathbb{D}, rh}(T)$, the canonical transformation map

$$
T_{\ast}(f, DR)(M) : Rf_{\ast} DR(T)[-](\text{mod}[-], \Gamma \int_{f_{\ast}}^{} M)) \\
\xrightarrow{Rf_{\ast} DR(T)[-](\text{mod}[-], \Gamma \int_{f_{\ast}}^{} M))} Rf_{\ast} DR(T)[-](\text{mod}[-], \Gamma \int_{f_{\ast}}^{} M))
$$

which is an isomorphism by (iii) and proposition 97.

In the filtered case, we will consider the weight monodromy filtration for open embeddings:

Definition 100. Let $k \subset \mathbb{C}$ a subfield.

(i) Let $S \in \text{Var}(\mathbb{C})$ and $j : S^o \to S$ an open embedding such that $D := S \setminus S^o = V(s) \subset S$ is a Cartier divisor.
For $(K, W) \in P_{fil}(S^{o, an})$, we consider as in \cite{[28]}

\[ j_{sw}(K, W) := (R_j, K, W) \in P_{fil}(S^{an}), W_k R_j, K := R_j, W_k, K, W(N) \ni K \supset R_j, K \]
so that $j^* j_{sw}(K, W) = (K, W)$, where $W_k R_j, K \subset R_j, K$ is given by $W$ and the weight
monodromy filtration $W(N)$ of the universal cover $\pi : \tilde{S}^{o, an} \to S^{o, an}$. Note that a stratification
of $W_k R_j, K$ is given by the closure of a stratification of $W_k K$ and $D := S \setminus S^o$.

For $(K', W) \in P_{fil}(S^{an})$, there is, by construction,

- a canonical map $\text{ad}((j^*, j_{sw})(K', W)) = \text{ad}(j^*, j_{sw})(K') : (K', W) \to j_{sw} j^*(K', W)$ in $P_{fil}(S^{an})$,
- a canonical map $\text{ad}(j_{sw}, j^*)(K', W) = \text{ad}(j_{sw}, j')(K') : j_{sw} j^*(K', W) \to (K', W)$ in $P_{fil}(S^{an})$.

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $j : S^{o} := S \setminus Z \to S$ an open embedding with $Z = V(\mathcal{I}) \subset S$ an arbitrary
closed subset, $\mathcal{I} \subset O_S$ being an ideal subsheaf. Taking generators $\mathcal{I} = (s_1, \ldots, s_r)$, we get $Z = V(s_1, \ldots, s_r) = \cap_{i=1}^r Z_i \subset S$ with $Z_i = V(s_i) \subset S$, $s_i \in \Gamma(S, \mathcal{L}_i)$ and $\mathcal{L}_i$ a line bundle. Note that $Z$ is an arbitrary closed subset, $d_Z \geq d_X - r$ needing not be a complete intersection. Denote by $j^!: S^{o, \text{fil}} := \cap_{i \in I}(S, Z_i) = S \setminus (\cup_{i \in I} Z_i) \supset S \supset S^o$ the open complementary
embeddings, where $I \subset \{1, \cdots, r\}$. Denote

\[ D(Z/S) := \{(Z_i)_{i \in \{1, \cdots, r\}}, Z_i \subset S, \cap Z_i = Z\}, Z'_i \subset Z_i \]

the flag category. For $(K, W) \in C(P_{fil}(S^{o, an}))$, we define by (i)

- the (bi-)filtered complex of $D_S$-modules

\[ j_{sw}(K, W) := \lim_{D(Z/S)} \text{Tot}_{\text{card} = 3}(j_{sw} j_{sw}^!(K, W)) \in C(P_{fil}(S^{an})) \]

where the horizontal differential are given by, if $I \subset J, d_{IJ} := \text{ad}(j^*, j_{sw})(j^! j_{sw}(K, W))$,

\[ j_{IJ} : S^{o, \text{fil}} \to S^{o, \text{fil}} \text{ being the open embedding, and } d_{IJ} = 0 \text{ if } I \notin J, \]

- the (bi-)filtered complex of $D_S$-modules

\[ j_{sw}(K, W) := \lim_{D(Z/S)} \text{Tot}_{\text{card} = 0}(j_{sw} j_{sw}^!(K, W)) = D_S j_{sw} D_S^!(K, W) \in C(P_{fil}(S^{an})), \]

where the horizontal differential are given by, if $I \subset J, d_{IJ} := \text{ad}(j^*, j_{sw})(j^! j_{sw}(K, W))$,

\[ j_{IJ} : S^{o, \text{fil}} \to S^{o, \text{fil}} \text{ being the open embedding, and } d_{IJ} = 0 \text{ if } I \notin J. \]

By definition, we have for $(K, W) \in C(P_{fil}(S^{o, an}))$, $j^* j_{sw}(K, W) = (K, W)$ and $j^* j_{sw}(K, W) = (K, W)$. For $(K', W) \in C(P_{fil}(S^{an}))$, there is, by (i),

- a canonical map $\text{ad}(j^*, j_{sw})(K', W) : (K', W) \to j_{sw} j^!(K', W)$ in $C(P_{fil}(S^{an}))$,
- a canonical map $\text{ad}(j_{sw}, j^*)(K', W) : j_{sw} j^!(K', W) \to (K', W)$ in $C(P_{fil}(S^{an}))$.

**Definition 101.** Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Denote by $j : S \setminus Z \to S$ the complementary
open embedding.

(i) We define using definition 100, the filtered Hodge support section functor

\[ \Gamma^\infty_Z : C(P_{fil}(S^{an})) \to C(P_{fil}(S^{an})), \]

\[ (K, W) \mapsto \Gamma^\infty_Z(K, W) := \text{Cone}(\text{ad}(j^*, j_{sw})(K, W) : (K, W) \to j_{sw} j^!(K, W))[1], \]

together we the canonical map $\gamma_Z^\infty(K, W) : \Gamma_Z^\infty(K, W) \to (K, W)$.

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(i) Since $j_{*,w}: C(P_{fil}(S^{o,an})) \to C(P_{fil}(S^{an}))$ is an exact functor, $\Gamma_{Z}^{w}$ induces the functor

$$\Gamma_{Z}^{w}: D_{fil,c}(S^{an}) \to D_{fil,c}(S^{an}), \ (K,W) \mapsto \Gamma_{Z}^{w}(K,W)$$

(ii) We define using definition 100, the dual filtered Hodge support section functor

$$\Gamma_{Z}^{\vee,w}: C(P_{fil}(S^{an})) \to C(P_{fil}(S^{an})), \ (K,W) \mapsto \Gamma_{Z}^{\vee,w}(K,W) := \text{Cone}(\text{ad}(j_{w},j^{*})(K,W) : j_{w},j^{*}(K,W) \to (K,W)),$$

together we the canonical map $\gamma_{Z}^{Hdg}(K,W) : (K,W) \to \Gamma_{Z}^{\vee,w}(K,W)$.

(iii) Since $j_{w}: C(P_{fil}(S^{o,an})) \to C(P_{fil}(S^{an}))$ is an exact functor, $\Gamma_{Z}^{\vee,w}$ induces the functor

$$\Gamma_{Z}^{\vee,w}: D_{fil,c}(S^{an}) \to D_{fil,c}(S^{an}), \ (K,W) \mapsto \Gamma_{Z}^{\vee,w}(K,W)$$

Let $S \in \text{Var}(\mathbb{C})$ and $D = V(s) \subset S$ a Cartier divisor. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^{o} := S\backslash D \hookrightarrow S$ the open embedding. Let $\pi : S^{o,an} \to S^{o,an}$ the universal covering. We then consider, for $(K,W) \in D_{fil,c}(S^{o,an}) = \text{Ho}(C(P_{fil}(S^{o,an})))$,

- the filtered nearby cycle functor
  $$\psi_{D}(K,W) := (\psi_{D}K,W) \in D_{fil,c}(D^{an}), \ W_{k}(\psi_{D}(K,W)) := <W_{k}\psi_{D}K, W(N)_{k}\psi_{D}K > \supset \psi_{D}K,$$

- the vanishing cycle functor
  $$\phi_{D}(K,W) := \text{Cone}(\text{ad}(j \circ \pi^{*}, j \circ \pi_{*})(K) : i^{*}(K,W) \to \psi_{D}(K,W)) \in D_{fil,c}(D^{an}),$$

- the canonical morphisms in $D_{fil,c}(D^{an})$
  $$\text{can}(K,W) := c(\phi_{D}(K,W)) : \psi_{D}(K,W) \to \phi_{D}(K,W),$$
  $$\text{var}(K,W) := D^{\text{an}}(\phi_{D}(D^{an}))(K,W) : \phi_{D}(K,W) \to \psi_{D}(K,W).$$

Definition 102. Let $k \subset \mathbb{C}$ a subfield.

(i) Let $f : X \to S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Consider the graph factorization $f : X \xrightarrow{l} X \times S \xrightarrow{p} S$ of $f$ where $l$ the the graph closed embedding and $p$ is the projection. We have, using definition 101,

- the inverse image functor
  $$f^{*w} : D_{fil,c}(S^{an}) \to D_{fil,c}(X^{an}), \ (K,W) \mapsto f^{*w}(K,W) := l^{*}\Gamma_{X}^{\vee,w}p^{*}(K,W)$$

- the exceptional inverse image functor
  $$f^{*w} : D_{fil,c}(S^{an}) \to D_{fil,c}(X^{an}), \ (K,W) \mapsto f^{*w}(K,W) := l^{*}\Gamma_{X}^{w}p^{*}(K,W).$$

(ii) Let $f : X \to S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Consider a compactification $f : X \hookrightarrow j\overline{X} \xrightarrow{\bar{f}} S$ of $f$ with $\overline{X} \in \text{Var}(\mathbb{C})$, $j$ an open embedding and $\bar{f}$ a proper morphism. We have, using definition 100,

- the direct image functor
  $$Rf_{*,w} : D_{fil,c}(X^{an}) \to D_{fil,c}(S^{an}), \ (K,W) \mapsto Rf_{*,w}(K,W) := R\bar{f}_{*}j_{*,w}(K,W)$$

- the proper direct image functor
  $$Rf_{!,w} : D_{fil,c}(X^{an}) \to D_{fil,c}(S^{an}), \ (K,W) \mapsto Rf_{!,w}(K,W) := R\bar{f}_{!}j_{!,w}(K,W).$$
5.2 The filtered Hodge direct image, the filtered Hodge inverse image, and the hodge support section functors for mixed hodge modules

- Let $S \in \text{SmVar}(\mathbb{C})$. The category $C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S^{an})$ is the category
  - whose set of objects is the set of triples $\{((M,F,W), (K,W), \alpha)\}$ with
    $$(M,F,W) \in C_{\mathcal{D}(1,0)fil,rh}(S), \ (K,W) \in D_{fil}(S^{an}), \ \alpha : (K,W) \otimes \mathbb{C}_{S^{an}} \rightarrow DR(S)[^{-1}]((M,W)^{an})$$
  - and whose set of morphisms are
    $$\phi = (\phi_D, \phi_C, [\theta]) : ((M_1,F,W), (K_1,W), \alpha_1) \rightarrow ((M_2,F,W), (K_2,W), \alpha_2)$$
where $\phi_D : (M_1,F,W) \rightarrow (M_2,F,W)$ and $\phi_C : (K_1,W) \rightarrow (K_2,W)$ are morphisms, and

$$\theta = (\theta^*, I(DR(S)(\phi_D^{an})) \circ I(\alpha_1), I(\alpha_2) \circ I(\phi_C \otimes I)) : I(K_1,W) \otimes \mathbb{C}_{S^{an}}[1] \rightarrow I(DR(S)(M_2^{an},W))$$
is an homotopy, i.e. for all $i \in \mathbb{Z}$,

$$\theta^i \circ \partial^i - \partial^{i+1} \circ \theta^i = (I(DR(S)(\phi_D^{an})) \circ I(\alpha_1))^i - (I(\alpha_2) \circ I(\phi_C))^i,$$

$I : C_{fil}(S^{an}) \rightarrow K_{fil}(S^{an})$ being the injective resolution functor: for $(K,W) \in C_{fil}(S^{an})$, we take an injective resolution $k : (K,W) \rightarrow I(K,W)$ with $I(K,W) \in C_{fil}(S^{an})$ which is unique modulo homotopy, and the class $[\theta]$ of $\theta$ does NOT depend of the injective resolution; in particular, we have

$$DR(S)[^{-1}](\phi_D^{an}) \circ \alpha_1 = \alpha_2 \circ (\phi_C \otimes I)$$
in $D_{fil}(S^{an})$; and for

* $\phi = (\phi_D, \phi_C, [\theta]) : ((M_1,F,W), (K_1,W), \alpha_1) \rightarrow ((M_2,F,W), (K_2,W), \alpha_2)$
* $\phi' = (\phi_D', \phi_C', [\theta']) : ((M_2,F,W), (K_2,W), \alpha_2) \rightarrow ((M_3,F,W), (K_3,W), \alpha_3)$

the composition law is given by

$$\phi' \circ \phi := (\phi'' \circ \phi_D, \phi_C' \circ \phi_C, I(DR(S)(\phi_D'^{an})) \circ [\theta] + [\theta'] \circ I(\phi_C \otimes I)[1]) : ((M_1,F,W), (K_1,W), \alpha_1) \rightarrow ((M_3,F,W), (K_3,W), \alpha_3),$$
in particular for $((M,F,W), (K,W), \alpha) \in C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S^{an})$,

$$I_{((M,F,W), (K,W), \alpha)} = (I_M, I_K, 0).$$

We have then the full embedding

$$\text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil}(S^{an})$$

where $\text{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil}(S^{an})$ is the category

- whose set of objects is the set of triples $\{((M,F,W), (K,W), \alpha)\}$ with
  $$(M,F,W) \in \text{PSh}_{\mathcal{D}(1,0)fil,rh}(S), \ (K,W) \in P_{fil}(S^{an}), \ \alpha : (K,W) \otimes \mathbb{C}_{S^{an}} \rightarrow DR(S)[^{-1}]((M,W)^{an})$$
  where $DR(S)[^{-1}]$ is the De Rahm functor and $\alpha$ is an isomorphism in $D_{fil}(S^{an})$,

- and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) = (\phi_D, \phi_C, 0) : ((M_1,F,W), (K_1,W), \alpha_1) \rightarrow ((M_2,F,W), (K_2,W), \alpha_2)$$
where $\phi_D : (M_1,F,W) \rightarrow (M_2,F,W)$ and $\phi_C : (K_1,W) \rightarrow (K_2,W)$ are morphisms (of filtered sheaves) and $DR(S)[^{-1}](\phi_D^{an}) \circ \alpha_1 = \alpha_2 \circ (\phi_C \otimes I)$ in $P_{fil}(S^{an})$.
Let $S \in \text{Var}(C)$. Let $S = \bigcup_{i \in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(C)$. The category $C_{D(1,0)fil,(\tilde{S}_i)} \times I D_{fil}(S^{an})$ is the category

- whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with

  \[
  (((M_I, F, W), u_{IJ}) \in C_{D(1,0)fil,(\tilde{S}_i)})(K, W) \in D_{fil}(S^{an}),
  \]

  \[
  \alpha : T(S/(\tilde{S}_i))(K, W) \otimes \mathbb{C}_{S^{an}} \to DR(S)^{[-1]}(((M_I, W), u_{IJ})^{an})
  \]

  where

  \[
  DR(S)^{[-1]} := DR(S)^{[-1]} : C_{D(1,0)fil,(\tilde{S}_i)}(S^{an}/(\tilde{S}_i)) \to C_{fil}(S^{an}/(\tilde{S}_i))
  \]

  is the De Rahm functor and $\alpha$ is a morphism in $D_{fil}(S^{an}/(\tilde{S}_i))$,

- and whose set of morphisms consists of

  \[
  \phi = (\phi_D, \phi_C, [\theta]) : (((M_I, F, W), u_{IJ}), (K, W), \alpha) \to (((M_I, F, W), u_{IJ}), (K_2, W), \alpha_2)
  \]

  where $\phi_D : ((M_I, F, W), u_{IJ}) \to ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \to (K_2, W)$ are morphisms, and

  \[
  \theta = (\theta^\ast, I(DR(S))(\phi_D^{an})) \circ I(\alpha_1), I(\alpha_2) \circ I(\phi_C \otimes I) : 
  \]

  \[
  I(T(S/(\tilde{S}_i))(K, W) \otimes \mathbb{C}_{S^{an}}[1] \to I(DR(S)(((M_2, F, W), u_{IJ})^{an}))
  \]

  is an homotopy, $I : C_{fil}(S^{an}/(\tilde{S}_I)) \to K_{fil}(S^{an}/(\tilde{S}_I))$ being the injective resolution functor

  : for $((K_1, W), t_{IJ}) \in C_{fil}(S^{an}/(\tilde{S}_i))$, we take an injective resolution

  \[
  k : ((K_1, W), t_{IJ}) \to I((K_1, W), t_{IJ})
  \]

  with $I((K_1, W), t_{IJ}) \in C_{fil}(S^{an}/(\tilde{S}_I))$ which is unique modulo homotopy, and the class $[\theta]$ of $\theta$ does NOT depend of the injective resolution ; in particular we have

  \[
  DR(S)^{[-1]}((\phi_D^{an}) \circ \alpha_1 = \alpha_2 \circ (\phi_C \otimes I)
  \]

  in $D_{fil}(S^{an}/(\tilde{S}_I))$ ; and for

  * $\phi = (\phi_D, \phi_C, [\theta]) : (((M_I, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_2, F, W), u_{IJ}), (K_2, W), \alpha_2)$

  * $\phi' = (\phi_D', \phi_C', [\theta']) : (((M_2', F, W), u_{IJ}), (K_2', W), \alpha_2) \to (((M_3, F, W), u_{IJ}), (K_3, W), \alpha_3)$

  the composition law is given by

  \[
  \phi' \circ \phi := (\phi_D' \circ \phi_D, \phi_C' \circ \phi_C, [\theta'] \circ [\theta] \circ I(\phi_C \otimes I)[1]) : 
  \]

  \[
  (((M_I, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_3, F, W), u_{IJ}), (K_3, W), \alpha_3)
  \]

  in particular for $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C_{D(1,0)fil,(\tilde{S}_i)} \times I D_{fil}(S^{an})$,

  \[
  I(((M_I, F, W), u_{IJ}), (K, W), \alpha) = ((I_{M_I}, I_K), 0).
  \]

We have then full embeddings

\[
\text{PSh}_{D(1,0)fil,(\tilde{S}_i)}^0(S/(\tilde{S}_i)) \times I P_{fil}(S^{an}) \hookrightarrow C_{D(1,0)fil,(\tilde{S}_i)}^0 \times I D_{fil}(S^{an})
\]

\[
\iota_{/\tilde{S}_i} : C_{D(1,0)fil,(\tilde{S}_i)}^0 \times I D_{fil}(S^{an}) \hookrightarrow C_{D(1,0)fil,(\tilde{S}_i)} \times I D_{fil}(S^{an})
\]

where $\text{PSh}_{D(1,0)fil,(\tilde{S}_i)}(S/(\tilde{S}_i)) \times I P_{fil}(S^{an})$ is the category
Proposition 99.

Definition 103. Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$.

(i) Let $D = V(s) \subset S$ be a smooth (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to $D$. Let $M \in \text{PSh}(S)$. A $V_D$-filtration $V$ for $M$ (see definition 56) is called a Kashiwara-Malgrange $V_D$-filtration for $M$ if

- $V_k M$ are coherent $O_S$ modules for all $k \in \mathbb{Z}$, that is $V$ is a good filtration,
- $sV_k M = V_{k-1} M$ for $k << 0$,
- all eigenvalues of $s\partial_s : \text{Gr}_{V,k} := V_k M/V_{k-1} M \rightarrow \text{Gr}_{V,k} := V_k M/V_{k-1} M$ have real part between $k - 1$ and $k$.

Almost by definition, a Kashiwara-Malgrange $V_D$-filtration for $M$ if it exists is unique (see [29]) so that we denote it by $(M, V_D) \in \text{PSh}_{O_S}^{\text{fil}}(S)$ and $(M, V_D)$ is strict. In particular if $m : (M_1, F) \rightarrow (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in \text{PSh}_{\mathcal{D}(2)}^{\text{fil}}(S)$ such that $M_1$ and $M_2$ admit the Kashiwara-Malgrange filtration for $D \subset S$, we have $m(V_{D,q} F^p M_1) \subset V_{D,q} F^p M_2$, that is we get $m : (M_1, F, V_D) \rightarrow (M_2, F, V_D)$ a filtered morphism, and if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence, $0 \rightarrow (M', V_D) \rightarrow (M, V_D) \rightarrow (M'', V_D) \rightarrow 0$ is an exact sequence (strictness).

(ii) More generally, let $Z = V(s_1, \ldots, s_r) = D_1 \cap \cdots \cap D_r \subset S$ be a smooth Zariski closed subset, where $s_i \in \Gamma(S, L_i)$ is a section of the line bundle $L = L_{D_i}$ associated to $D_i$. Let $M \in \text{PSh}(S)$. A $V_Z$-filtration $V$ for $M$ (see definition 56) is called a Kashiwara-Malgrange $V_Z$-filtration for $M$ if

- $V_k M$ are coherent $O_S$ modules for all $k \in \mathbb{Z}$,
- $\sum_{i=1}^r s_i \partial_{s_i} : \text{Gr}_{V,k} := V_k M/V_{k-1} M \rightarrow \text{Gr}_{V,k} := V_k M/V_{k-1} M$ have real part between $k - 1$ and $k$.

Almost by definition, a Kashiwara-Malgrange $V_Z$-filtration for $M$ if it exists is unique (see [29]) so that we denote it by $(M, V_Z) \in \text{PSh}_{O_S}^{\text{fil}}(S)$ and $(M, V_Z)$ is strict. In particular if $m : (M_1, F) \rightarrow (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in \text{PSh}_{\mathcal{D}(2)}^{\text{fil}}(S)$ such that $M_1$ and $M_2$ admit the Kashiwara-Malgrange filtration for $D \subset S$, we have $m(V_{\mathcal{Z},q} F^p M_1) \subset V_{\mathcal{Z},q} F^p M_2$, that is we get $m : (M_1, F, V_Z) \rightarrow (M_2, F, V_Z)$ a filtered morphism, and if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence, $0 \rightarrow (M', V_Z) \rightarrow (M, V_Z) \rightarrow (M'', V_Z) \rightarrow 0$ is an exact sequence (strictness).

Proposition 99. (i) Let $S \in \text{AnSm}(\mathbb{C})$.

- Let $D = V(s) \subset S$ a smooth (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to $D$. If $M \in \text{PSh}_{\mathcal{D}}^{\text{fil}}(S)$, the Kashiwara-Malgrange $V_D$-filtration for $M$ (see definition 103) exist so that we denote it by $(M, V_D) \in \text{PSh}_{O_S}^{\text{fil}}(S)$.

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More generally, let \( Z = V(s_1, \ldots, s_r) = D_1 \cap \cdots \cap D_r \subset S \) be a smooth Zariski closed subset, where \( s_i \in \Gamma(S, L_i) \) is a section of the line bundle \( L = L_{D_i} \) associated to \( D_i \). If \( M \in \text{PSh}_{D,rh}^\varnothing(S) \), the Kashiwara-Malgrange \( V_Z \)-filtration for \( M \) (see definition 103) exist so that we denote it by \((M, V_Z) \in \text{PSh}_{O_{Z,fil}}(S)\).

(ii) Let \( S \in \text{SmVar}(\mathbb{C}) \).

- Let \( D = V(s) \subset S \) a smooth (Cartier) divisor, where \( s \in \Gamma(S, L) \) is a section of the line bundle \( L = L_D \) associated to \( D \). If \( M \in \text{PSh}_{D,rh}^\varnothing(S) \), the Kashiwara-Malgrange \( V_D \)-filtration for \( M \) (see definition 103) exist so that we denote it by \((M, V_D) \in \text{PSh}_{O_{fil}}(S)\).

- More generally, let \( Z = V(s_1, \ldots, s_r) = D_1 \cap \cdots \cap D_r \subset S \) be a smooth Zariski closed subset, where \( s_i \in \Gamma(S, L_i) \) is a section of the line bundle \( L = L_{D_i} \) associated to \( D_i \). If \( M \in \text{PSh}_{D,rh}^\varnothing(S) \), the Kashiwara-Malgrange \( V_Z \)-filtration for \( M \) (see definition 103) exist so that we denote it by \((M, V_Z) \in \text{PSh}_{O_{Z,fil}}(S)\).

Proof. (i): Follows from the work of Kashiwara. Note that the second point is a particular case of the first by induction. (ii): Take a compactification \( \bar{S} \in \text{PSh}_{\text{Var}}(\mathbb{C}) \) of \( S \) and denote by \( \bar{D} \subset \bar{S} \) the closure of \( D \). Using the closed embedding \( i : S \hookrightarrow L_{\bar{D}} \) given by the zero section, we may assume that \( \bar{D} \) is smooth. Denote by \( j : S \setminus D \hookrightarrow \bar{S} \) the open complement. Then, \( j_*M \in \text{PSh}_{D,rh}^\varnothing(\bar{S}) \) is regular holonomic. The result then follows by (i) and GAGA for \( j_*M \) and we get \((j_*M, V_{\bar{D}}) \in \text{PSh}_{O_{fil}}(\bar{S})\) and \((M, V_D) = (j^*j_*M, j^*V_{\bar{D}}) \in \text{PSh}_{O_{fil}}(S)\). We can also prove the algebraic case directly using the theory of meromorphic connections since a simple holonomic \( D_S \)-module with support \( Z \subset S \) is an integrable connexion on \( Z^o = Z \cap S^o \), \( S^o \subset S \) being an open subset.

We have from Kashiwara or Malgrange the following which relates the graded piece of the Kashiwara-Malgrange \( V \)-filtration \( V_D \) of a \( D_S \) module \( M \in \text{PSh}_{D,rh}^\varnothing(S) \) along a smooth divisor \( D \) with the nearby and vanishing cycle functors of \( DR(S)(M) \) with respect to \( D : \)

\[ \text{Theorem 27.} \quad \text{Let } S \in \text{AnSm}(\mathbb{C}). \text{ Let } D = V(s) \subset S \text{ be a smooth (Cartier) divisor, where } s \in \Gamma(S, L) \text{ is a section of the line bundle } L = L_D \text{ associated to } D. \text{ Denote by } j : S \setminus D \hookrightarrow \bar{S} \text{ the open complementary embedding and by } k : \bar{S} \rightarrow S \text{ with } k \text{ the universal covering of } S^o. \text{ For } M \in \text{PSh}_{D,rh}^\varnothing(S) \text{ a regular holonomic } D_S \text{ module, consider } (M, V_D) \in \text{PSh}_{O_{fil}}(S) \text{ it together with its } V_D \text{ filtration. Then,} \]

- there is canonical isomorphism

\[ T(\psi_D, DR)(M) : DR(S)(\oplus_{-1 \leq \alpha < 0} \text{Gr}_{V_D, \alpha} M) \xrightarrow{\sim} \psi_D(DR(S)(M)) := Rk_*k^*DR(S)(M) \]

- there is canonical isomorphism

\[ T(\phi_D, DR)(M) : DR(S)(\oplus_{-1 < \alpha \leq 0} \text{Gr}_{V_D, \alpha} M) \xrightarrow{\sim} \phi_D(DR(S)(M)) := \text{Cone}(DR(S)(M) \xrightarrow{\text{add}(k^*, Rk_*)(-)} \psi_D(DR(S)(M))) \]

- \( DR(S)(\partial_s) = \text{can}(DR(S)(M)) \circ T(\psi_D, DR)(M), \) with \( \text{can}(DR(S)(M)) : \psi_D DR(S)(M) \rightarrow \phi_D(DR(S)(M)) \) the structural embedding of complexes of the cone,

- \( DR(S)(s\partial_s) = T \circ T(\psi_D, DR)(M), \) with \( T : \psi_D(DR(S)(M)) \rightarrow \psi_D(DR(S)(M)) \) the monodromy morphism.

- \( T(\phi_D, DR)(M) : DR(S)(s) \simeq \text{var}(DR(S)(M)) \) with \( \text{var}(DR(S)(M)) := D^v \text{can}(D^v DR(S)(M)) : \phi_D(DR(S)(M)) \rightarrow \psi_D(DR(S)(M)). \)

Proof. See [29].\[ \]
The main tool is the nearby and vanishing cycle functors for Cartier divisors. We need for the definition of Hodge modules on a smooth complex algebraic variety $S$ to extend the V-filtration associated to a smooth Cartier divisor $D \subset S$ of regular holonomic $D_S$ module $M$ such that the monodromy morphism $T : \psi_D(DR(S)(M)) \to \psi_D(DR(S))$ is quasi-unipotent by a rational V-filtration (i.e. indexed by rational numbers).

**Definition 104.** Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $D = V(s) \subset S$ a (Cartier) divisor, where $s \in \Gamma(S,L)$ is a section of the line bundle $L = L_D$ associated to $D$. We then have the zero section embedding $i : S \to L$. We denote $D_0 = i(S)$ and $D' := L \setminus D_0$ the open complementary subset. We denote $\text{PSh}_{D(2)}D_{fil,rh}(S)^{sp,0} \subset \text{PSh}_{D(2)}D_{fil,rh}(S)$ the full subcategory consiting of objects such that the monodromy operator $T : \psi_D(DR(S)(M^{an})) \to \psi_D(DR(S)(M^{an}))$ is quasi-unipotent.

(i) Let $(M,F) \in \text{PSh}_{D(2)}D_{fil,rh}(S)^{sp,0}$. By proposition 99, we have the Kashiwara-Malgrange $V_S$-filtration on $i_{*mod}M$. We refine it to all rational numbers as follows: for $\alpha = k + r/q \in \mathbb{Q}$, $k,q,r \in \mathbb{Z}$, $q \leq 0$, $0 \leq r \leq q - 1$, we set

$$V_{S,\alpha}M := q_{V,k}^{-1}(\oplus_{k - 1 < \beta \leq \alpha} \text{Gr}_{k,\beta}V_M) \subset V_{S,k}M$$

with $\text{Gr}_{k,\beta}V_M := \ker(\partial_k s - \beta I) \subset \text{Gr}_kV_M$ and $q_{V,k} : V_{S,k}M \to \text{Gr}_kV_M$ is the projection. We set similarly

$$V_{S,\leq \alpha}M := q_{V,k}^{-1}(\oplus_{k - 1 < \beta \leq \alpha} \text{Gr}_{k,\beta}V_M) \subset V_{S,k}M$$

The Hodge filtration induced on $\text{Gr}_V^\alpha M$ is

$$F^p \text{Gr}_V^\alpha M := (F^pM \cap V_{S,\alpha}M) / (F^pM \cap V_{S,\leq \alpha}M)$$

(ii) we have using (i) the nearby cycle functor

$$\psi_D : \text{PSh}_{D(2)}D_{fil,rh}(S)^{sp,0} \to \text{PSh}_{D(2)}D_{fil,rh}(D/(S)), (M,F) \mapsto \psi_D(M,F) := \oplus_{1 \leq \alpha < 0} \text{Gr}_{V_S,\alpha}i_{*mod}(M,F),$$

and vanishing cycle functors

$$\phi_D : \text{PSh}_{D(2)}D_{fil,rh}(S)^{sp,0} \to \text{PSh}_{D(2)}D_{fil,rh}(D/(S)), (M,F) \mapsto \phi_D(M,F) := \oplus_{1 < \alpha \leq 0} \text{Gr}_{V_S,\alpha}i_{*mod}(M,F),$$

$$\phi_{D1} : \text{PSh}_{D(2)}D_{fil,rh}(S)^{sp,0} \to \text{PSh}_{D(2)}D_{fil,rh}(D/(S)), (M,F) \mapsto \phi_{D1}(M,F) := \text{Gr}_{V_S,0}i_{*mod}(M,F).$$

(iii) This induces, by theorem 27, the nearby cycle functor

$$\psi_D : \text{PSh}_{D(2)}D_{fil,rh}(S)^{sp,0} \times_1 P_{fil,D}(S^{an}) \to \text{PSh}_{D(2)}D_{fil,rh}(D^{an} \times_1 P_{fil,D}(S^{an})),$$


and the vanishing cycle functor

$$\phi_{D1} : \text{PSh}_{D(2)}D_{fil,rh}(S)^{sp,0} \times_1 D_{fil}(S^{an}) \to \text{PSh}_{D(2)}D_{fil,rh}(D^{an} \times_1 P_{fil,D}(S^{an})),$$

$$((M,F,W),(K,W),\alpha) \mapsto \phi_{D1}((M,F,W),(K,W),\alpha) := (\phi_{D1}(M,F,W),\phi_D(K,W),\phi_D(\alpha))$$

We have the category of mixed Hodge modules over a complex algebraic variety or a complex analytic space $S$ defined by, for $S$ smooth, by induction on dimension of $S$, and for $S$ singular using embeddings into smooth complex algebraic varieties, resp. smooth complex analytic spaces:

**Definition 105.** [28]

(i) Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Denote $\text{PSh}_{Dfil,rh}(S)^{sp,ass} \subset \text{PSh}_{Dfil,rh}(S)$ the full subcategory consisting of objects $(M,F)$
such that for all Cartier divisor $D = V(s) \subset S$, $s \in \Gamma(S, L)$, denoting $i : S \hookrightarrow L$ the closed embedding the monodromy morphism $T : \psi_D(DR(S)(\mathcal{M}^{(\alpha)})) \to \psi_D(DR(S)(\mathcal{M}^{(\alpha)}))$ is quasi-unipotent, $FPV_{S,\alpha} \iota_{\text{mod}} M = FPV_{S,\alpha-1} \iota_{\text{mod}} M$ for $\alpha < 0, \partial F\mathcal{G}_{S,\alpha} \iota_{\text{mod}} M = \mathcal{G}_{\alpha+1} \iota_{\text{mod}} M$ for $\alpha > -1$, the filtration induced by $F$ on $\mathcal{G}_{\alpha} \iota_{\text{mod}} M$ is good,

which admits a decomposition with $D_S$ module with strict support on closed irreducible subvarieties.

The category of Hodge modules over $S$ of weight $w$ is the full subcategory

$$i_S : \mathcal{H}M(S, w) = \bigoplus_{d \in \mathbb{N}} \mathcal{H}M_{\leq d}(S, w) \hookrightarrow \mathcal{PSh}_{\mathcal{D}_{\text{fil,rh}}} S \times I P(S^{(\alpha)}), \hookrightarrow \mathcal{PSh}_{\mathcal{D}_{\text{fil,rh}}} S \times I P(S^{(\alpha)})$$
given inductively by, $d$ being the dimension of the support of the $D_S$ modules,

- for $i_0 : S_0 \hookrightarrow S$ a closed point, $i_0 \times \text{pt} : \mathcal{H}M_{s_0}(S, w) = HS \hookrightarrow \mathcal{PSh}_{\mathcal{D}_{\text{fil,rh}},s_0}(S) \times I P(S^{(\alpha)})$ consists of Hodge structures of weight $w$, this gives $\mathcal{H}M_{s_0}(S, w)$

- for $Z \subset S$ an irreducible closed subvariety of dimension $d$, $((M, F), K, \alpha) \in \mathcal{PSh}_{\mathcal{D}_{\text{fil,rh}}} S \times I P(S^{(\alpha)})$ belongs to $\mathcal{H}M_Z(S, w)$ if and only if $M$ has strict support $Z$ (i.e. $\text{supp}(M) = Z$ and for all non trivial subobject $N$ or quotient of $M$ $\text{supp}(N) = Z$), and for all proper maps $f : S' \to S$ such that $f_{\mathbb{Z} \cap S'} \neq 0$, $j : S' \hookrightarrow S$ being an open subset,

$$\mathcal{G}_{k}^{W}(N) \psi_{f-1(0)}(j^*(M, F), j^*K, j^*\alpha) \in \mathcal{H}M_{\leq d-1}(S'', w-1+k) \hookrightarrow \mathcal{PSh}_{\mathcal{D}_{\text{fil,rh}},f^{-1}(0)}(S''\times IP_{f^{-1}(0)}(S^{(\alpha)}))$$

for all $k \in \mathbb{Z}$, see definition 104, $W(N)$ being the weight filtration associated to the monodromy morphism $T : \psi_{f^{-1}(0)}(DR(S)(\mathcal{M}^{(\alpha)})) \to \psi_{f^{-1}(0)}(DR(S)(\mathcal{M}^{(\alpha)}))$, we then set $\mathcal{H}M_{\leq d}(S, w) := \bigoplus_{Z \subset S, \text{dim}(Z) = d} \mathcal{H}M_Z(S, w)$.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. The category of mixed Hodge modules over $S$ is the full subcategory

$$i_S : \mathcal{M}HM(S) \hookrightarrow \mathcal{M}HW(S) \hookrightarrow \mathcal{PSh}_{\mathcal{D}_{\text{1,0} \text{fil,rh}}} S \times I P(S^{(\alpha)}), \hookrightarrow \mathcal{PSh}_{\mathcal{D}_{\text{1,0} \text{fil,rh}}} S \times I P(S^{(\alpha)})$$

where the full subcategory $\mathcal{M}HW(S)$ consists of objects $((M, F, W), (K, W), \alpha) \in \mathcal{PSh}_{\mathcal{D}_{\text{1,0} \text{fil,rh}}} S \times I P(S^{(\alpha)})$ satisfy

$$(\mathcal{G}_{i}^{W}((M, F, W), \mathcal{G}_{i}^{W}(K, W), \mathcal{G}_{i}^{W}(\alpha) \in \mathcal{H}M(S))$$

and the objects of $\mathcal{M}HM(S)$ satisfy in addition an admissibility condition (in particular the three filtration $F, W, V_S$ are compatible). As usual, for $Z \subset S$ a closed subset and $j : S' \hookrightarrow S$ the open complementary subset, we denote $\mathcal{M}HM_Z(S) \subset \mathcal{M}HM(S)$ the full subcategory consisting of $((M, F, W), (K, W), \alpha) \in \mathcal{M}HM(S)$ such that

$$j^*((M, F, W), (K, W), \alpha) := (j^*(M, F, W), j^*(K, W), j^*\alpha) = 0.$$

(iii) Let $S \in \text{Var}(\mathbb{C})$ or $S \in \text{AnSp}(\mathbb{C})$ non smooth. Take an open cover $S = \bigcup S_i$ so that there are closed embedding $S_i \hookrightarrow \tilde{S}_i$, with $S_i \in \text{SmVar}(\mathbb{C})$, resp $S_i \in \text{AnSm}(\mathbb{C})$. The category of mixed Hodge modules over $S$ is the full subcategory

$$i_S : \mathcal{M}HM(S) \hookrightarrow \mathcal{M}HW(S) \hookrightarrow \mathcal{PSh}_{\mathcal{D}_{\text{1,0} \text{fil,rh}}} (S/\tilde{S}_i) \times I P_{fil}(S^{(\alpha)})$$

consisting of objects

$$((M_i, F, W), u_{iJ}, (K, W), \alpha) \in \mathcal{PSh}_{\mathcal{D}_{\text{1,0} \text{fil,rh}}} (S/\tilde{S}_i) \times I P_{fil}(S^{(\alpha)})$$

such that $((M_i, F, W), T(S/\tilde{S}_i)(K, W), \alpha) \in (\mathcal{M}HM_{S_i}(\tilde{S}_i))$ (see (ii)). The category $\mathcal{M}HM(S)$ does NOT depend on the open cover an the closed embedding by proposition 100.
(iv) Let $S \in \text{Var}(\mathbb{C})$. We get from (iii) $D(MHM(S)) := (\text{Ho}_{\text{zar}}, I)(C(MHM(S)))$. By induction, using the result for mixed hodge structure and the strictness of the Kashiwara-Malgrange $V$-filtration for morphism of $D$-module, the morphism of $HM(S)$ are strict for $F$ and $W$ (see [28]).

(iv)' Let $S \in \text{AnSp}(\mathbb{C})$. We get from (iii) $D(MHM(S)) := (\text{Ho}_{\text{usn}}, I)(C(MHM(S)))$. By induction, using the result for mixed hodge structure and the strictness of the Kashiwara-Malgrange $V$-filtration for morphism of $D$-module, the morphism of $HM(S)$ are strict for $F$ and $W$ (see [28]).

- Let $S \in \text{SmVar}(\mathbb{C})$. We consider the canonical functor

$$\pi_S : C(MHW(S)) \xrightarrow{\pi_S} C_{D(1,0)fil}(S) \times_I D_{fil}(S^an) \xrightarrow{p_S} C_{D(1,0)fil}(S),$$

$$((M, F, W), (K, W), \alpha) \mapsto (M, F, W)$$

where $p_S$ is the projection functor. Then $\pi_S(MHW(S)) \subset \text{PSh}_{D(1,0)fil}(S)$ is the subcategory consisting of $(M, F, W) \in \text{PSh}_{D(1,0)fil}(S)$ such that $((M, F, W), (K, W), \alpha) \in \text{HM}(S)$ is a $W$ filtered Hodge module for some $(K, W) \in C_{fil}(S)$. It induces in the derived category the functor

$$\pi_S : D(MHW(S)) \xrightarrow{\pi_S} D_{D(1,0)fil}(S) \times_I D_{fil}(S^an) \xrightarrow{p_S} D_{D(1,0)fil}(S),$$

$$((M, F, W), (K, W), \alpha) \mapsto (M, F, W)$$

after localization with respect to filtered Zariski and usu local equivalence.

- Let $S \in \text{Var}(\mathbb{C})$ non smooth. Take an open cover $S = \bigcup_i S_i$ such that there are closed embedding $S_I \hookrightarrow \tilde{S}_I$, with $S_I \in \text{SmVar}(\mathbb{C})$. We consider the canonical functor

$$\pi_S : C(MHW(S)) \rightarrow C_{D(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil}(S^an) \xrightarrow{p_S} C_{D(1,0)fil}(S/(\tilde{S}_I)),

(((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto ((M_I, F, W), u_{IJ})$$

where $p_S$ is the projection functor. Then $\pi_S(MHW(S)) \subset \text{PSh}_{D(1,0)fil}(S/(\tilde{S}_I))$ is the subcategory consisting of $((M_I, F, W), u_{IJ}) \in \text{PSh}_{D(1,0)fil}(S/(\tilde{S}_I))$ such that $((M_I, F, W), u_{IJ}, (K, W), \alpha) \in \text{HM}(S)$ is a $W$ filtered Hodge module for some $(K, W) \in C_{fil}(S)$. It induces in the derived category the functor

$$\pi_S : D(MHW(S)) \rightarrow D_{D(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil}(S^an) \xrightarrow{p_S} D_{D(1,0)fil}(S/(\tilde{S}_I)),$$

$$(((M_I, F, W), u_{IJ}, (K, W), \alpha) \mapsto ((M, F, W), u_{IJ})$$

after localization with respect to filtered Zariski and usu local equivalence.

Let $S \in \text{Var}(\mathbb{C})$ or $S \in \text{AnSp}(\mathbb{C})$.

- If $S \in \text{Var}(\mathbb{C})$, let $S = \bigcup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$, and let $S = \bigcup_{i=1}^l S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{SmVar}(\mathbb{C})$.

- If $S \in \text{AnSp}(\mathbb{C})$, let $S = \bigcup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$, and let $S = \bigcup_{i=1}^l S_{i'}$ an other open cover such that there exist closed embeddings $i_{i'} : S_{i'} \hookrightarrow \tilde{S}_{i'}$ with $\tilde{S}_{i'} \in \text{AnSm}(\mathbb{C})$.

Denote $L = [1, \ldots, l]$, $L' = [1, \ldots, l']$ and $L'' = [1, \ldots, l] \cup [1, \ldots, l']$. We have then the refined open cover $S = \bigcup_{k \in L} S_k$ and we denote for $I \cup I' \subset L''$, $S_{IJ} := \cap_{k \in I \cup I'} S_k$ and $\tilde{S}_{IJ} := \cap_{k \in I \cup I'} \tilde{S}_k$, so that we have a closed embedding $i_{IJ} : S_{IJ} \hookrightarrow \tilde{S}_{IJ}$. Consider $\pi_S(MHM(S)) \subset \text{PSh}_{Dfil}(S/(\tilde{S}_I))$ and $\pi_S(MHM(S)) \subset \text{PSh}_{Dfil}(S/(\tilde{S}_I))$. For $I \cap I' \subset J \cup J'$, denote by $p_{IJ \cap I'J'} : \tilde{S}_{IJ \cap I'J'} \rightarrow \tilde{S}_{I'J'}$ the projection. We then have a natural transfer map

$$T_{S/L''}^{S/L'} : \pi_S^L(MHM(S)) \rightarrow \pi_S^{L'}(MHM(S)),

((M_I, F, W), s_{IJ}) \mapsto (\text{ho lim}_{I \in L} p_{IJ \cap I'J'}^* \text{Gr}_{IJ \cap I'J'} \pi^{\text{smod}}_{IJ \cap I'J'}(M_I, F), (s_{IJ})),

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with, in the homotopy limit, the natural transition morphisms
\[ p_{I'}(I,J') \mapsto \text{ad}(p_{I,J}, p_{I,J'})(p_{I,J'}^*(M_I, F)) : \]
\[ p_{I'}(J,J') \mapsto (\text{Gr}_\text{v}, p_{I,J'}^*(\text{Gr}_\text{v}, \text{p}_{I,J')}^*(M_I, F)) \]
for \( J \subset I \), and
\[ s_{I', J'} : \text{holim}_{I \in L} m_* p_{I'(I,J')} \mapsto (\text{Gr}_\text{v}, p_{I,J'}^*(\text{Gr}_\text{v}, \text{p}_{I,J')}^*(M_I, F)) \]
\[ \text{holim}_{I \in L} p_{I', J'}^*(\text{Gr}_\text{v}, p_{I,J'}^*(\text{Gr}_\text{v}, \text{p}_{I,J')}^*(M_I, F)) \]

**Proposition 100.** (i) Let \( S \in \text{Var}(\mathbb{C}) \) and let \( S = \cup_i S_i \) an open cover such that there exist closed embeddings \( i_i S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). Then \( \pi_S((\text{MHM}(S) \subset \text{PSh}_{\mathbb{D}(\mathbb{C})}(S/\tilde{S}_i)) \) does not depend on the open covering of \( S \) and the closed embeddings. More precisely, let \( S = \cup_i S_i \) an other open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( S_i \in \text{SmVar}(\mathbb{C}) \). Then,
\[ T_{S}^{L/L'} : \pi_S^L((\text{MHM}(S)) \rightarrow \pi_S^L((\text{MHM}(S)), \]
is an equivalence of category with inverse is \( T_{S}^{L'/L} : \pi_S^L((\text{MHM}(S)) \rightarrow \pi_S^L((\text{MHM}(S)). \]

(ii) Let \( S \in \text{AnSm}(\mathbb{C}) \) and let \( S = \cup_i S_i \) an open cover such that there exist closed embeddings \( i_i S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{AnSm}(\mathbb{C}) \). Then \( \pi_S((\text{MHM}(S)) \subset \text{PSh}_{\mathbb{D}(\mathbb{C})}(S/\tilde{S}_i)) \) does not depend on the open covering of \( S \) and the closed embeddings. More precisely, let \( S = \cup_i S_i \) an other open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow S_i \) with \( \tilde{S}_i \in \text{AnSm}(\mathbb{C}) \). Then,
\[ T_{S}^{L/L'} : \pi_S^L((\text{MHM}(S)) \rightarrow \pi_S^L((\text{MHM}(S)), \]
is an equivalence of category with inverse is \( T_{S}^{L'/L} : \pi_S^L((\text{MHM}(S)) \rightarrow \pi_S^L((\text{MHM}(S)). \]

**Proof.** Follows from the definition of the Hodge filtration which use the V-filtration : see [28].

The main results of Saito, which implies in the algebraic case the six functor formalism on \( \text{DMHM}(-) \) are the followings

**Definition 106.** Let \( S \in \text{SmVar}(\mathbb{C}) \) or \( S \in \text{AnSm}(\mathbb{C}) \). We denote by \( \text{VMHS}(S) \subset \text{PSh}_{\mathbb{D}(\mathbb{C})}(S, \tilde{S}_i) \times I \) \( P_{\text{fil}}(S^{an}) \) the full subcategory consisting of variation of mixed Hodge structure, whose objects consist of
\[ ((L_S, W) \otimes O_S, F), (L_S, W, \alpha) \subset \text{PSh}_{\mathbb{D}(\mathbb{C})}(S, \tilde{S}_i) \times I P_{\text{fil}}(S^{an}) \]
with
- \( L_S \in \text{PSh}(S^{an}) \) a local system,
- the \( D_S \) module structure on \( (L_S, W) \otimes O_S \) is given by the flat connection associated to the local system \( L_S \),
- \( F^p(W^p L_S \otimes O_S) \subset (W^p L_S \otimes O_S) \) are locally free \( O_S \) subbundle satisfying Griffiths transversality for the \( D_S \) module structure (i.e. for the flat connection).
- \( \alpha : (L_S, W) \rightarrow DR(S^{-1})((L_S, W) \otimes O_S) \) is the isomorphism given by theorem 26.

**Theorem 28.** Let \( S \in \text{SmVar}(\mathbb{C}) \) or \( S \in \text{AnSm}(\mathbb{C}) \).

(i) A variation of mixed Hodge structure \(((L_S, W) \otimes O_S, F), (L_S, W, \alpha) \in \text{VMHS}(S) \) (see definition 106) is a mixed module. That is \( \text{VMHS}(S) \subset \text{MHM}(S) \).
Theorem 29. (i) Let $((M,F,W),(K,W),\alpha) \in MFM$ a mixed Hodge module with support $\text{supp} M = Z$, there exist an open subset $J : S^0 \to S$, such that $j_*((M,F,W),(K,W),\alpha) := (j_*(M,F,W),j^*(K,W),j^*\alpha) \in VMHS(Z \cap S^0)$. That is a mixed Hodge module is generically a variation of mixed Hodge structure.

**Proof.** See [28].

(ii) For $((M,F,W),(K,W),\alpha) \in MFM(S)$ a mixed Hodge module with support $\text{supp} M = Z$, there exist an open subset $J : S^0 \to S$, such that $j_*((M,F,W),(K,W),\alpha) := (j_*(M,F,W),j^*(K,W),j^*\alpha) \in VMHS(Z \cap S^0)$. That is a mixed Hodge module is generically a variation of mixed Hodge structure.

**Theorem 29.** (i) Let $f : X \to S$ a projective morphism with $X,S \in \text{AnSp}(\mathbb{C})$, where projective means that there exist a factorization $f : X \xrightarrow{\iota} \mathbb{P}^N \times S \xrightarrow{p_S} S$ with $l$ a closed embedding and $p_S$ the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exits closed embeddings $i_1 : S_i \to S_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$. For $I \subset [1,\ldots,s]$, recall that we denote $S_I := \cap_{i \in I} S_i$ and $X_I := f^{-1}(S_I)$. We have then the following commutative diagram

$$
\begin{array}{ccc}
X_I & \xrightarrow{i_{1\circ l_I}} & \tilde{S}_I \\
\downarrow j_{1J} & & \downarrow p_{IJ} \\
X_J & \xrightarrow{i_{1\circ l_J}} & \tilde{S}_J
\end{array}
$$

whose right square is cartesian (see section 5). Then, for

$$
((M,F,W),(K,W),\alpha) = (((M_1,F,W),u_{1J}),(K,W),\alpha) \in MFM(X),
$$

where $((M_1,F,W),u_{1J}) \in C_{D2fil}(X_I/(\mathbb{P}^N \times \tilde{S}_I)), (K,W) \in C_{fil}(X)$, we have for all $n \in \mathbb{Z}$,

$$
(H^n \int_f^{\text{FDR}} ((M_1,F,W),u_{1J}), R^n f_*(K,W), H^n f_*(\alpha)) \in MFM(S)
$$

and for all $p \in \mathbb{Z}$, the differentials of $\text{Gr}_F^p \int_f^{\text{FDR}} ((M_1,F,W),u_{1J})$ are strict for the Hodge filtration $F$.

(ii) Let $f : X \to S$ a projective morphism with $X,S \in \text{Var}(\mathbb{C})$, where projective means that there exist a factorization $f : X \xrightarrow{\iota} \mathbb{P}^N \times S \xrightarrow{p_S} S$ with $l$ a closed embedding and $p_S$ the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exits closed embeddings $i_1 : S_i \to S_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1,\ldots,s]$, recall that we denote $S_I := \cap_{i \in I} S_i$ and $X_I := f^{-1}(S_I)$. We have then the following commutative diagram

$$
\begin{array}{ccc}
X_I & \xrightarrow{i_{1\circ l_I}} & \tilde{S}_I \\
\downarrow j_{1J} & & \downarrow p_{IJ} \\
X_J & \xrightarrow{i_{1\circ l_J}} & \tilde{S}_J
\end{array}
$$

whose right square is cartesian (see section 5). Then, for

$$
((M,F,W),(K,W),\alpha) = (((M_1,F,W),u_{1J}),(K,W),\alpha) \in D(MFM(X)),
$$

where $((M_1,F,W),u_{1J}) \in C_{D2fil}(X_I/(\mathbb{P}^N \times \tilde{S}_I)), (K,W) \in C_{fil}(X^{an})$, we have

$$
H^n \int_f^{\text{FDR}} ((M_1,F,W),u_{1J}), Rf_*(K,W), f_*(\alpha)) \in MFM(S)
$$

for all $n \in \mathbb{Z}$, and for all $p \in \mathbb{Z}$, the differentials of $\text{Gr}_F^p \int_f^{\text{FDR}} ((M_1,F,W),u_{1J})$ are strict for the Hodge filtration $F$.
Proof. (i): See [28].

(ii): By (i) \( (H^n \int f_j((M,F,W)^{an}), R^n f_* (K,W), H^n f_*(\alpha)) \in MHM(S^{an}) \) for all \( n \in \mathbb{Z} \). On the other hand, \( T^D(\alpha, f)(M,F,W) : (\int f_j((M,F,W)^{an}) \simeq \int f_j((M,F,W)^{an}) \) is an isomorphism since \( f \) is proper by theorem GAGA for mixed hodge modules: see [28].

\[ \square \]

**Theorem 30.** (i) Let \( S \in \text{AnSp}(\mathbb{C}) \). Let \( Y \in \text{AnSm}(\mathbb{C}) \) and \( p_S : Y \times S \to S \) the projection. Let \( S = \cup_{i=1}^{s} S_i \) an open cover such that there exits closed embeddings \( i_f : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{AnSp}(\mathbb{C}) \). For \( I \subset [1, \ldots, s] \), recall that we denote \( S_I := \cap_{i \in I} S_i \). We have then the following commutative diagram

\[
\begin{array}{ccc}
Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \\
\downarrow \quad p_{I_J} & & \downarrow \\
Y \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J
\end{array}
\]

which is cartesian (see section 5). Then, for

\[
((M,F,W),(K,W),\alpha) = (((M_I,F,W),u_{I_J}),(K,W),\alpha) \in MHM(S),
\]

where \( ((M_I,F,W),u_{I_J}) \in C_{\text{D2fil}}(S_I/(\tilde{S}_I)), (K,W) \in C_{\text{fil}}(S) \),

\[
\begin{align*}
- \ (p_{S_I}^{\text{mod}[-]}(M,F,W),p_{S_I}^*(K,W),p_{S_I}^*(\alpha)) := ((p_{S_I}^{\text{mod}[-]}u_{I_J},p_{S_I}^{*(\text{mod}[-]}u_{I_J},p_{S_I}^*(K,W),p_{S_I}^*(\alpha)) \in MHM(S) \\
- \ (p_{S_J}^{\text{mod}[-]}(M,F,W),p_{S_J}^*(K,W),p_{S_J}^*(\alpha)) := ((p_{S_J}^{\text{mod}[-]}u_{I_J},p_{S_J}^{*(\text{mod}[-]}u_{I_J},p_{S_J}^*(K,W),p_{S_J}^*(\alpha)) \in MHM(S)
\end{align*}
\]

(ii) Let \( S \in \text{Var}(\mathbb{C}) \). Let \( Y \in \text{SmVar}(\mathbb{C}) \) and \( p_S : Y \times S \to S \) the projection. Let \( S = \cup_{i=1}^{s} S_i \) an open cover such that there exits closed embeddings \( i_I : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). For \( I \subset [1, \ldots, s] \), recall that we denote \( S_I := \cap_{i \in I} S_i \). We have then the following commutative diagram

\[
\begin{array}{ccc}
Y \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \\
\downarrow \quad p_{I_J} & & \downarrow \\
Y \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J
\end{array}
\]

which is cartesian (see section 5). Then, for

\[
((M,F,W),(K,W),\alpha) = (((M_I,F,W),u_{I_J}),(K,W),\alpha) \in D(MHM(S)),
\]

where \( ((M_I,F,W),u_{I_J}) \in C_{\text{D2fil}}(S_I/(\tilde{S}_I)), (K,W) \in C_{\text{fil}}(S^{an}) \), we have

\[
\begin{align*}
- \ (p_{S_I}^{\text{mod}[-]}(M,F,W),(K,W),\alpha) := ((p_{S_I}^{\text{mod}[-]}u_{I_J},p_{S_I}^{*(\text{mod}[-]}u_{I_J},p_{S_I}^*(K,W),p_{S_I}^*(\alpha)) \in D(MHM(S)) \\
- \ (p_{S_J}^{\text{mod}[-]}(M,F,W),(K,W),\alpha) := ((p_{S_J}^{\text{mod}[-]}u_{I_J},p_{S_J}^{*(\text{mod}[-]}u_{I_J},p_{S_J}^*(K,W),p_{S_J}^*(\alpha)) \in D(MHM(S)).
\end{align*}
\]

Proof. (i): See [28].

(ii): Follows immediately from (i) since \( (p_{S_I}^{\text{mod}[-]}(M_I,F,W))^{an} = p_{S_I}^{\text{mod}[-]}((M_I,F,W)^{an}). \)

We have, by the results of Saito, the following key definition.
Definition 107. (i) Let $S \in \text{SmVar}(\mathbb{C})$ or $S \in \text{AnSm}(\mathbb{C})$. Let $D = V(s) \subset S$ a divisor with $s \in \Gamma(S, L)$ and $L$ a line bundle ($S$ being smooth, $D$ is Cartier). Denote by $j^o : S^o := S \setminus D \hookrightarrow S$ the open complementary embedding. Let $(M, F, W) \in \pi_{S^o}(MHW(S^o))$. Consider the $V_S$-filtration on $i_*\text{mod} M$ (see proposition 99). If $(M, F, W)$ is extendable (which is always the case in the algebraic case), then,

- we have

$$j_*\text{Hdg}(M, F, W) := (j_*, M, F, W) \in \pi_S(MHM(S)),$$

with $F^p j_* M := \sum_{k \in \mathbb{N}} \partial_S^{F^p+k} V_S j_* M \subset j_* M$, $W_k j_* M := \left< j_* W_k M, W(N) \right>_k \supset j_* M$

which is unique such that $j^* j_* \text{Hdg}(M, F, W) = (M, F, W)$ and $\text{DR}(S)(j_* \text{Hdg}(M, F, W)) = j_* \text{DR}(S^o)(M, W)$,

- there exist

$$j_\text{Hdg}(M, F, W) := \mathbb{D}_S^{\text{Hdg}} j_* \text{Hdg} \mathbb{D}_S^{\text{Hdg}} (M, F, W) \in \pi_S(MHM(S))$$

unique such that $j^* j_\text{Hdg}(M, F, W) = (M, F, W)$ and $\text{DR}(S)(j_\text{Hdg}(M, F, W)) = j_\text{DR}(S^o)(M, W)$.

Moreover for $(M', F, W) \in \pi_S(MHM(S))$, by proposition 102

- there is a canonical map $\text{ad}(j^*, j_* \text{Hdg})(M', F, W) : (M', F, W) \to j_* \text{Hdg} j^*(M', F, W)$ in $\pi_S(MHM(S))$,

- there is a canonical map $\text{ad}(j_{\text{Hdg}} j^*)(M', F, W) : j_{\text{Hdg}} j^*(M', F, W) \to (M', F, W)$ in $\pi_S(MHM(S))$.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$. Let $Z = V(\mathcal{I}) \subset S$ an arbitrary closed subset, $\mathcal{I} \subset O_S$ being an ideal subsheaf. Taking generators $\mathcal{I} = (s_1, \ldots, s_r)$, we get $Z = V(s_1, \ldots, s_r) = \cap_{i=1}^r \mathcal{Z}_i \subset S$ with $\mathcal{Z}_i = V(s_i) \subset S$, $s_i \in \Gamma(S, \mathcal{L}_i)$ and $\mathcal{L}_i$ a line bundle. Note that $Z$ is an arbitrary closed subset, $d_Z \supset d_X - r$ needing not be a complete intersection. Denote by $j : S^o := S \setminus Z \hookrightarrow S$, $j_I : S^{o,I} := \cap_{i \in I}(S \setminus \mathcal{Z}_i) = S \setminus (\cup_{i \in I} \mathcal{Z}_i) \hookrightarrow S^o \hookrightarrow S$ the open complementary embeddings, where $I \subset \{1, \ldots, r\}$. For $(M, F, W) \in \pi_{S^o}(C(MHM(S^o)))$, we define by (i)

- the (bi)-filtered complex of $D_S$-modules

$$j_\text{Hdg}(M, F, W) := \lim_{\text{Tot}_{\text{card}I}=-} \bullet \left( j_{\text{Hdg}}^I j^*_I(M, F, W) \right) \in \pi_S(C(MHM(S)))$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \text{ad}(j^I_{IJ}, j_{IJ}^{\text{Hdg}})(j^*_I(M, F, W))$, $j_{IJ} : S^{o,I} \hookrightarrow S^{o,J}$ being the open embedding, and $d_{IJ} = 0$ if $I \notin J$,

- the (bi)-filtered complex of $D_S$-modules

$$j_{\text{Hdg}}(M, F, W) := \lim_{\text{Tot}_{\text{card}I}=-} \bullet \left( j_{\text{Hdg}} j^*_I(M, F, W) \right)$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \text{ad}(j_{\text{Hdg}}^I, j^I_{IJ})(j^*_I(M, F, W))$, $j_{IJ} : S^{o,I} \hookrightarrow S^{o,J}$ being the open embedding, and $d_{IJ} = 0$ if $I \notin J$.

By definition, we have for $(M, F, W) \in \pi_{S^o}(C(MHM(S^o)))$, $j^* j_* \text{Hdg}(M, F, W) = (M, F, W)$ and $j^* j_\text{Hdg}(M, F, W) = (M', F, W)$. For $(M', F, W) \in \pi_S(C(MHM(S)))$, there is, by construction,

- a canonical map $\text{ad}(j^*, j_* \text{Hdg})(M', F, W) : (M', F, W) \to j_* \text{Hdg} j^*(M', F, W),$

- a canonical map $\text{ad}(j_{\text{Hdg}} j^*)(M', F, W) : j_{\text{Hdg}} j^*(M', F, W) \to (M', F, W).$
For \((M, F, W) \in \pi_S^*(C(MHM(S^o)))\),
- we have the canonical map in \(C_{D(1)fil}(S)\)
  \[ T(j_{\ast Hdg}, j_\ast)(M, F, W) := k \circ \text{ad}(j^*, j_\ast)(j_{\ast Hdg}(M, F, W)) : j_{\ast Hdg}(M, F, W) \to j_\ast E(M, F, W), \]
- we have the canonical map in \(C_{D(1)fil}(S)\)
  \[ T(j_{\ast Hdg}, j_\ast)(M, F, W) := \mathbb{D}^S_\ast L_D(k \circ \text{ad}(j^*, j_\ast)(-)) : j_\ast(M, F, W) := \mathbb{D}^K_\ast L_Dj_{\ast Hdg}\mathbb{D}^K_\ast (M, F, W) = j_{\ast Hdg}(M, F, W) \]
the canonical maps.

**Remark 9.** Let \(j : S^o \hookrightarrow S\) an open embedding, with \(S \in \text{SmVar}(\mathbb{C})\). Then, for \(((M, F, W), (K, W), \alpha) \in MHM(S^o)\),
- the map \(T(j_!, j_{! Hdg})(M, W) : j_{! w}(M, W) \to j_{! Hdg}(M, W)\) in \(C_{D(1)fil}(S)\) is a filtered quasi-isomorphism (apply the functor \(DR^{[\cdot]}(S^o)\) and use theorem 26 and theorem 99).
- the map \(T(j_{\ast Hdg}, j_\ast)(M, W) : j_{\ast Hdg}(M, W) \to j_{\ast w}(M, W)\) in \(C_{D(1)fil}(S)\) is a filtered quasi-isomorphism (apply the functor \(DR^{[\cdot]}(S^o)\) and use theorem 26 and theorem 99).

Hence, for \(((M, F, W), (K, W), \alpha) \in MHM(S^o)\),
- we get, for all \(p, n \in \mathbb{N}\), monomorphisms
  \[ F^pH^nT(j_!, j_{! Hdg})(M, F, W) : F^pH^n j_{! w}(M, F, W) \hookrightarrow F^pH^n j_{! Hdg}(M, F, W) \]
in \(PSh_{D_\ast}(S)\), but \(F^pH^n j_{! w}(M, F, W) \neq F^pH^n j_{! Hdg}(M, F, W)\) (it leads to different \(F\)-filtrations), since \(F^pH^n j_{! w}(M, F) \subset H^n j_{! M}\) are sub \(D_S\) module while the \(F\)-filtration on \(H^n j_{! Hdg}(M, F)\) is given by Kashiwara-Malgrange \(V\)-filtrations, hence satisfy a non trivial Griffith transversality property, thus \(H^n j_{!}(M, F)\) and \(H^n j_{! Hdg}(M, F)\) are isomorphic as \(D_S\)-modules but NOT isomorphic as filtered \(D_S\)-modules.
- we get, for all \(p, n \in \mathbb{N}\), monomorphisms
  \[ T(j_{\ast Hdg}, j_\ast)(M, F, W) : F^pH^n j_{\ast Hdg}(M, F, W) \hookrightarrow F^pH^n j_\ast E(M, F, W) \]
in \(PSh_{D_\ast}(S)\), but \(F^pH^n j_{\ast Hdg}(M, F, W) \neq F^pH^n j_{\ast w}(M, F, W)\) (it leads to different \(F\)-filtrations), since \(F^pH^n j_{\ast E}(M, F) \subset H^n j_{\ast E}(M)\) are sub \(D_S\) module while the \(F\)-filtration on \(H^n j_{\ast Hdg}(M, F)\) is given by Kashiwara-Malgrange \(V\)-filtrations, hence satisfy a non trivial Griffith transversality property, thus \(H^n j_{\ast E}(M, F)\) and \(H^n j_{\ast Hdg}(M, F)\) are isomorphic as \(D_S\)-modules but NOT isomorphic as filtered \(D_S\)-modules.

We make the following key definition

**Definition 108.** Let \(S \in \text{SmVar}(\mathbb{C})\). Let \(Z \subset S\) a closed subset. Denote by \(j : S \setminus Z \hookrightarrow S\) the complementary open embedding.

(i) We define using definition 107, the filtered Hodge support section functor

\[ \Gamma^H_{Z} : \pi_S(C(MHM(S))) \to \pi_S(C(MHM(S))), \]
\[ (M, F, W) \mapsto \Gamma^H_{Z}(M, F, W) := \text{Cone}(\text{ad}(j^*, j_{\ast Hdg})(M, F) : (M, F) \to j_{\ast Hdg}j_{\ast}(M, F))[\mathbb{L}], \]

together we have the canonical map \(\gamma^H_{Z}(M, F, W) : \Gamma^H_{Z}(M, F, W) \to (M, F, W)\). We then have the canonical map in \(C_{D(2)fil}(S)\)

\[ T(\Gamma^H_{Z}, \gamma_{Z})(M, F, W) := (I, T(j_{\ast Hdg}, j_\ast)(M, F, W)) : \Gamma^H_{Z}(M, F, W) \to \Gamma_{Z}E(M, F, W) \]
unique up to homotopy such that \(\gamma^H_{Z}(M, F, W) = \gamma_{Z}(E(M, F, W)) \circ T(\Gamma^H_{Z}, \gamma_{Z})(M, F, W)\).
(i) Since $j_*^{Hdg} : \pi_S(C(MHM(S^0)) \to \pi_S(C(MHM(S))$ is an exact functor, $\Gamma_Z^{Hdg}$ induces the functor

\[ \Gamma_Z^{Hdg} : \pi_S(D(MHM(S)) \to \pi_S(D(MHM(S))), (M, F, W) \mapsto \Gamma_Z^{Hdg}(M, F, W) \]

(ii) We define using definition 107, the dual filtered Hodge support section functor

\[ \Gamma_Z^{\vee, Hdg} : \pi_S(C(MHM(S)) \to \pi_S(C(MHM(S))), (M, F, W) \mapsto \text{Cone}(\text{ad}(j_*^{Hdg}, j^*)(M, F, W) : j_*^{Hdg}, j^*(M, F, W) \to (M, F, W)), \]

together we the canonical map $\gamma_Z^{\vee, Hdg}(M, F, W) : (M, F, W) \to \Gamma_Z^{\vee, Hdg}(M, F)$. We then have the canonical map in $C_{D(2)}$ limit

\[ T(\Gamma_Z^{\vee, h}, \Gamma_Z^{\vee, Hdg})(M, F, W) := (I, T(j_*, j_*^{Hdg})(M, F, W)) : \Gamma_Z^{\vee, h}(M, F, W) \to \Gamma_Z^{\vee, Hdg}(M, F, W) \]

unique up to homotopy such that $\gamma_Z^{\vee, Hdg}(M, F) = T(\Gamma_Z^{\vee, h}, \Gamma_Z^{\vee, Hdg})(M, F, W) \circ \gamma_Z^{\vee, h}(M, F, W).

(iii) Since $j^{Hdg} : \pi_S(C(MHM(S^0)) \to \pi_S(C(MHM(S))$ is an exact functor, $\Gamma_Z^{Hdg, \vee}$ induces the functor

\[ \Gamma_Z^{\vee, Hdg} : \pi_S(D(MHM(S)) \to \pi_S(D(MHM(S))), (M, F, W) \mapsto \Gamma_Z^{\vee, Hdg}(M, F, W) \]

We now give the definition of the filtered Hodge inverse image functor:

**Definition 109.**

(i) Let $i : Z \hookrightarrow S$ be a closed embedding, with $Z, S \in \text{SmVar}(\mathbb{C})$. Then, for $(M, F, W) \in \pi_S(C(MHM(S)))$, we set

\[ i_*^{mod}(M, F, W) := i_*S_i^{-1}\Gamma_Z^{Hdg}(M, F, W) \in \pi_Z(D(MHM(Z)) \]

and

\[ i_*^{mod}(M, F, W) := i_*S_i^{-1}\Gamma_Z^{\vee, Hdg}(M, F, W) \in \pi_Z(D(MHM(Z)) \]

using the fact that $S : \pi_Z(D(MHM(Z)) \to \pi_S(D(MHMZ(S)))$ is an equivalence of category since $S : D(MHMZ(S)) \to D(MHMZ(S))$ is an equivalence of category by [28].

(ii) Let $f : X \to S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$. Consider the factorization $f : X \overset{i}{\to} X \times S \overset{p_S}{\to} S$, where $i$ is the graph embedding and $p_S : X \times S \to S$ is the projection.

- For $(M, F, W) \in \pi_S(C(MHM(S)))$ we set

\[ f_*^{mod}p_S^{mod^{-1}}(M, F, W)(d_X)[2d_X] \in \pi_X(D(MHM(X))) \]

- For $(M, F, W) \in \pi_S(C(MHM(S)))$ we set

\[ f_*^{mod}p_S^{mod^{-1}}(M, F, W) \in \pi_S(D(MHM(S))) \]

If $j : S^0 \to S$ is a closed embedding, we have (see [28]), for $(M, F, W) \in \pi_S(C(MHM(S)))$,

\[ j_*^{mod}(M, F, W) = j_*^{mod}(M, F, W) \]

(iii) Let $f : X \to S$ be a morphism, with $X, S \in \text{SmVar}(\mathbb{C})$ or $X, S \in \text{AnSm}(\mathbb{C})$. Consider the factorization $f : X \overset{i}{\to} X \times S \overset{p_S}{\to} S$, where $i$ is the graph embedding and $p_S : X \times S \to S$ is the projection.
For \((M,F,W) \in \pi_S(C(MHM(S)))\) we set
\[
f_{Hdg}^{\text{mod}}(M,F,W) := \Gamma_X^{Hdg} p_S^{\text{mod}[-]}(M,F,W) =: \Gamma_X^{Hdg} p_S^{\text{mod}[-]}(M,F,W)
\]
We have for \((M,F,W) \in \pi_S(C(MHM(S)))\), the canonical map in \(C_{\mathcal{D}(1,0)}(X \times S)\)
\[
T(f_{Hdg}^{\text{mod},\Gamma}, f_{Hdg}^{\text{mod},\Gamma})(M,F,W) : f_{Hdg}^{\text{mod},\Gamma}(M,F,W) := \Gamma_X^{Hdg} p_S^{\text{mod}[-]}(M,F,W)
\]
For \((M,F,W) \in \pi_S(C(MHM(S)))\) we set
\[
f_{Hdg}^{\text{mod}}(M,F,W) := \Gamma_X^{Hdg} p_S^{\text{mod}[-]}(M,F,W) \in \pi_{X \times S}(C(MHM(X \times S))),
\]
We have for \((M,F,W) \in \pi_S(C(MHM(S)))\), the canonical map in \(C_{\mathcal{D}(1,0)}(X \times S)\)
\[
T(f_{Hdg}^{\text{mod},\Gamma}, f_{Hdg}^{\text{mod},\Gamma})(M,F,W) : f_{Hdg}^{\text{mod},\Gamma}(M,F,W) := \Gamma_X^{Hdg} p_S^{\text{mod}[-]}(M,F,W) =: f_{Hdg}^{\text{mod}}(M,F,W)
\]

**Definition-Proposition 19.**

(i) Let \(g : S' \rightarrow S\) a morphism with \(S', S \in \text{SmVar}(\mathbb{C})\) and \(i : Z \hookrightarrow S\) a closed subset. Then, for \((M,F,W) \in \pi_S(C(MHM(S)))\), there is a canonical map in \(\pi_S(C(MHM(S') \times S))\)
\[
T^{Hdg}(g, \gamma)(M,F,W) : g^{\text{mod}^{\Gamma},\Gamma}_Z^{Hdg}(M,F,W) \rightarrow \Gamma^{Hdg}_{Z \times S' S} g^{\text{mod}^{\Gamma},\Gamma}_Z^{Hdg}(M,F,W)
\]
unique up to homotopy such that
\[
\gamma^{Hdg}_{Z \times S'}(g^{\text{mod}^{\Gamma},\Gamma}_Z^{Hdg}(M,F,W)) \circ T^{Hdg}(g, \gamma)(M,F,W) = g^{\text{mod}^{\Gamma},\Gamma}_Z^{Hdg}(M,F,W).
\]

(i)' Let \(g : S' \rightarrow S\) a morphism with \(S', S \in \text{SmVar}(\mathbb{C})\) and \(i : Z \hookrightarrow S\) a closed subset. Then, for \((M,F,W) \in \pi_S(C(MHM(S)))\), there is a canonical isomorphism in \(\pi_S(C(MHM(S') \times S))\)
\[
T^{Hdg}(g, \gamma^{\vee})(M,F,W) : \Gamma^{Hdg}_{Z \times S' S} g^{\text{mod}^{\Gamma},\Gamma}_Z^{Hdg}(M,F,W) \cong g^{\text{mod}^{\Gamma},\Gamma}_Z^{Hdg}(M,F,W)
\]
unique up to homotopy such that
\[
\gamma^{\vee,Hdg}_{Z \times S'}(g^{\text{mod}^{\Gamma},\Gamma}_Z^{Hdg}(M,F,W)) \circ g^{\text{mod}^{\Gamma},\Gamma}_Z^{Hdg}(M,F,W) = T^{Hdg}(g, \gamma)(M,F,W).
\]

(ii) Let \(S \in \text{SmVar}(\mathbb{C})\) and \(i_1 : Z_1 \hookrightarrow S, i_2 : Z_2 \hookrightarrow Z_1\) be closed embeddings. Then, for \((M,F,W) \in \pi_S(C(MHM(S)))\),

- there is a canonical map \(T(Z_2/Z_1, \gamma^{Hdg})(M,F,W) : \Gamma^{Hdg}_{Z_1}(M,F,W) \rightarrow \Gamma^{Hdg}_{Z_2}(M,F,W)\) in \(\pi_S(C(MHM(S)))\) unique up to homotopy such that
\[
\gamma^{Hdg}_{Z_2}(G,F) \circ T(Z_2/Z_1, \gamma^{Hdg})(G,F) = \gamma^{Hdg}_{Z_2}(G,F)
\]

Together with a distinguish triangle in \(K(\pi_S(MHM(S)))\)
\[
\begin{array}{c}
\Gamma^{Hdg}_{Z_2}(M,F,W) \xrightarrow{T(Z_2/Z_1, \gamma^{Hdg})(M,F,W)} \Gamma^{Hdg}_{Z_1}(M,F,W) \\
\xrightarrow{\text{adj}(\gamma^{Hdg}_{Z_2}, \gamma^{Hdg}_{Z_1})} \Gamma^{Hdg}_{Z_2}(G,F) \rightarrow \Gamma^{Hdg}_{Z_2}(G,F)[1]
\end{array}
\]
there is a canonical map $T(Z_2/Z_1, \gamma_{Z_2}^{\nu,Hdg})(M, F, W) : \Gamma_{Z_2}^{\nu,Hdg}(M, F, W) \to \Gamma_{Z_2}^{\nu,Hdg}(M, F, W)$ in $\pi_S(C(MHM(S)))$ unique up to homotopy such that

\[
\gamma_{Z_2}^{\nu,Hdg}(M, F, W) = T(Z_2/Z_1, \gamma_{Z_1}^{\nu,Hdg})(M, F, W) \circ \gamma_{Z_1}^{\nu,Hdg}(M, F, W).
\]

Proof. Follows from the projection case and the closed embedding case using the adjunction maps. \qed

The definitions 108 and 109 immediately extends to the non smooth case:

**Definition 110.** Let $S \in \mathcal{V}(C)$. Let $Z \subset S$ a closed subset. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow S$ with $S_i \in \mathcal{S}(C)$. Denote $Z_I := Z \cap S_I$. Denote by $j : S/I \to S$ and $j_I : S/I \to S/I$ the complementary open embeddings.

(i) We define using definition 107, the filtered Hodge support section functor

\[
\Gamma_{Z_I}^{\nu,Hdg} : \pi(C(MHM(S))) \to \pi(C(MHM(S))), ((M_I, F, W), u_I) \mapsto \Gamma_{Z_I}^{\nu,Hdg}((M_I, F, W), u_I) := \text{Cone(ad}(j^*, j_{*Hdg}))((M_I, F, W), u_I) : ((M_I, F, W), u_I) \to (j_{*Hdg}j^*(M_I, F, W), j_{*}(u_I))[-1],
\]

unique up to homotopy such that

\[
\gamma_{Z_I}^{\nu,Hdg}((M_I, F, W), u_I) = (\gamma_{Z_I}(E(M_I, F, W))) \circ (\Gamma_{Z_I}^{\nu,Hdg}, \Gamma_Z)((M_I, F, W), u_I).
\]

(ii) Since $j_{*Hdg} : \pi_S(C(MHM(S/I))) \to \pi_S(C(MHM(S/I)))$ are exact functors, $\Gamma_{Z_I}^{\nu,Hdg}$ induces the functor

\[
\Gamma_{Z_I}^{\nu,Hdg} : \pi_S(D(MHM(S/I))) \to \pi_S(D(MHM(S/I)), ((M_I, F, W), u_I) \mapsto \Gamma_{Z_I}^{\nu,Hdg}((M_I, F, W), u_I)
\]

We then have the canonical map $\gamma_{Z_I}^{\nu,Hdg}((M_I, F, W), u_I) : \Gamma_{Z_I}^{\nu,Hdg}((M_I, F, W), u_I) \to (\Gamma_{Z_I}^{\nu,Hdg}, \Gamma_Z)((M_I, F, W), u_I)$.

(ii) We define using definition 107, the dual filtered Hodge support section functor

\[
\Gamma_{Z_I}^{\nu,Hdg} : \pi(C(MHM(S))) \to \pi(C(MHM(S))), ((M_I, F, W), u_I) \mapsto \Gamma_{Z_I}^{\nu,Hdg}((M_I, F, W), u_I) := \text{Cone(ad}(j_{*Hdg}, j^*))((M_I, F, W), u_I) : ((M_I, F, W), u_I) \to (j_{*Hdg}j^*(M_I, F, W), j_{*}(u_I))[-1],
\]

together we the canonical map $\gamma_{Z_I}^{\nu,Hdg}((M_I, F, W), u_I) : ((M_I, F, W), u_I) \mapsto \Gamma_{Z_I}^{\nu,Hdg}((M_I, F, W), u_I)$.

We then have the canonical map in $\mathcal{C}(D(\mathcal{P}(S/I)))$

\[
T(\Gamma_{Z_I}^{\nu,Hdg}, \Gamma_{Z_I}^{\nu,Hdg})((M_I, F, W), u_I) := (I, T(j_{*Hdg}, j_{*}((M_I, F, W), u_I)) : (\Gamma_{Z_I}^{\nu,Hdg}(M_I, F, W), \Gamma_{Z_I}^{\nu,Hdg}(u_I)) \to \Gamma_{Z_I}^{\nu,Hdg}((M_I, F, W), u_I)
\]

unique up to homotopy such that

\[
\gamma_{Z_I}^{\nu,Hdg}((M_I, F, W), u_I) = T(\Gamma_{Z_I}^{\nu,Hdg}, \Gamma_{Z_I}^{\nu,Hdg})((M_I, F, W), u_I) \circ (\gamma_{Z_I}^{\nu,Hdg}(M_I, F, W)).
\]
(ii) Since $j^Hdg_{\mathcal{I}t} : \pi_{\mathcal{I}t}^Hdg(C(MHM(\tilde{S}_t \setminus S_t))) \to \pi_{\mathcal{I}t}(C(MHM(\tilde{S}_t)))$ are exact functors, $\Gamma^Hdg_{\mathcal{I}t}$ induces the functor

$$\Gamma^Hdg_{\mathcal{I}t} : \pi_S(D(MHM(S))) \to \pi_S(D(MHM(S))), ((M_I, F, W), u_{IJ}) \mapsto \Gamma^Hdg_{\mathcal{I}t}((M_I, F, W), u_{IJ})$$

**Definition 111.** Let $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \sqcup_{i \in I} S_I$ an open cover such that there exist closed embeddings $i : S_I \to \tilde{S}_I$ with $S_I \in \text{SmVar}(\mathbb{C})$. Denote $X_I := f^{-1}(S_I)$. We have then $X = \sqcup_{i \in I} X_I$, and the commutative diagrams

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\xrightarrow{i_I} & \xrightarrow{p_S} & \xrightarrow{i_I} \tilde{S}_I \\
Y & \xrightarrow{p_{S_I}} & \tilde{S}_I
\end{array}$$

(i) For $((M_I, F, W), u_{IJ}) \in \pi_S(C(MHM(S)))$ we set (see definition 110 for $l$)

$$f^*_{\text{mod}}((M_I, F, W), u_{IJ}) := \Gamma^Hdg_{X}((p_{S_I}^*{\text{mod}}[\cdot])(M_I, F, W), u_{IJ})(d_Y[2d_Y]) \in \pi_X(C(MHM(X)))$$

We have for $((M_I, F, W), u_{IJ}) \in \pi_S(C(MHM(S)))$, the canonical map in $C_{D(1,0)fii}(X/(Y \times \tilde{S}_1))$

$$T(f^*_{\text{mod}}, f^*_{\text{mod}, \Gamma})((M_I, F, W), u_{IJ}) : f^*_{\text{mod}}((M_I, F, W), u_{IJ}) := \Gamma^Hdg_{X}((p_{S_I}^*{\text{mod}}[\cdot])(M_I, F, W), p_{S_I}^*{\text{mod}}[\cdot]u_{IJ})$$

$$\xrightarrow{T(l^*_{\text{mod}, \Gamma})} \Gamma_X E(p_{S_I}^*{\text{mod}}[\cdot])(M_I, F, W), f^*_{\text{mod}}[\cdot]u_{IJ}) := f^*_{\text{mod}}[\cdot] \Gamma(M, F, W)$$

(ii) For $((M_I, F, W), u_{IJ}) \in \pi_S(C(MHM(S)))$ we set (see definition 110 for $l$)

$$f^*_{\text{mod}}(M, F, W) := \Gamma^Hdg_{X}((p_{S_I}^*{\text{mod}}[\cdot])(M_I, F, W), p_{S_I}^*{\text{mod}}[\cdot]u_{IJ}) \in \pi_X(C(MHM(X)))$$

We have for $(M_I, F, W) \in \pi_S(C(MHM(S)))$, the canonical map in $C_{D(1,0)fii}(X/(Y \times \tilde{S}_1))$

$$T(f^*_{\text{mod}, \Gamma}, f^*_{\text{mod}})\Gamma((M_I, F, W), u_{IJ}) : f^*_{\text{mod}}[\cdot], \Gamma(M, F, W) := D_{X}^K f^*_{\text{mod}}[\cdot], \Gamma\times_{X}^K ((M_I, F, W), u_{IJ})$$

$$\xrightarrow{D_{X}^K T(l^*_{\text{mod}, \Gamma})} \Gamma^Hdg_{X}((p_{S_I}^*{\text{mod}}[\cdot])(M_I, F, W), p_{S_I}^*{\text{mod}}[\cdot]u_{IJ}) =: f^*_{\text{mod}}[\cdot](M, F, W)$$

**Proposition 101.** Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \text{QPVar}(\mathbb{C})$ or with $X, Y, S \in \text{SmVar}(\mathbb{C})$.

(i) Let $(M, F, W) \in \pi_S(C(MHM(S)))$. Then,

$$(f_2 \circ f_1)^*_{\text{mod}}(M, F) = f^*_{\text{mod}} f^*_{\text{mod}}(M, F) \in \pi_X(D(MHM(X))).$$

(ii) Let $(M, F, W) \in \pi_S(C(MHM(S)))$. Then,

$$(f_2 \circ f_1)^*_{\text{mod}}(M, F) = f^*_{\text{mod}} f^*_{\text{mod}}(M, F) \in \pi_X(D(MHM(X)))$$

**Proof.** (i):Follows from the unicity of the functor $j^Hdg_{\mathcal{I}t}$.

(ii):Follows from the unicity of the functor $j^Hdg_{\mathcal{I}t}$. 

$\square$

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• Let $f : X \to S$ a morphism with $S, X \in \text{SmVar}(\mathbb{C})$. Let $\bar{f} : \bar{X} \to \bar{S}$ a compactification of $f$ with $\bar{X} \in \text{SmVar}(\mathbb{C})$ and $j$ the open embedding. We set, for $(M, F, W) \in \pi_X(D(MHM(X)))$, using definition 107
\[
\int^\text{Hdg}_f (M, F, W) := \int_f^{\text{FDR}} j_*^{\text{Hdg}}(M, F, W) \in D_{D(1,0)fil}(S)
\]
and
\[
\int^\text{Hdg}_{\bar{f}} (M, F, W) := \int_{\bar{f}}^{\text{FDR}} j_{\text{Hdg}}(M, F, W) \in D_{D(1,0)fil}(S)
\]

• Let $f : X \to S$ a morphism with $S, X \in \text{QPVar}(\mathbb{C})$. Consider a factorization $f : X \xrightarrow{j} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $\bar{Y} \in \text{PSmVar}(\mathbb{C})$ a smooth compactification of $Y$ with $j : Y \hookrightarrow \bar{Y}$ the open embedding. Then $\bar{f} : \bar{X} \xrightarrow{j} \bar{Y} \times S \xrightarrow{\bar{p}_S} S$ is a compactification of $f$, with $\bar{X} \subset \bar{Y} \times S$ the closure of $X$ and $\bar{l}$ the closed embedding. Let $S = \cup_i S_i$ an open affine cover and $i : S_i \to S$ closed embedding with $\bar{S}_i \subset \text{SmVar}(\mathbb{C})$. For $((M_l, F, W), u_{l,i}) \in \pi_X(C(MHM(X))) \subset C_{D(1,0)fil,rh}(X/(Y \times \bar{S}_i))$, we set using definition 107
\[
\int^\text{Hdg}_f ((M_l, F, W), u_{l,i}) := \int_f^{\text{FDR}} ((I \times j)_*^{\text{Hdg}}((M_l, F, W), u_{l,i})) \in D_{D(1,0)fil}(S/(\bar{S}_i))
\]
and
\[
\int^\text{Hdg}_{\bar{f}} ((M_l, F, W), u_{l,i}) := \int_{\bar{f}}^{\text{FDR}} ((I \times j)^{\text{Hdg}}((M_l, F, W), u_{l,i})) \in D_{D(1,0)fil}(S/(\bar{S}_i))
\]

From the D-module case on algebraic varieties and the constructible sheaves case on CW complexes, we get:

**Definition 112.** (i) Let $f : X \to S$ a morphism with $S, X \in \text{SmVar}(\mathbb{C})$. Let $\bar{f} : \bar{X} \to \bar{S}$ a compactification of $f$ with $\bar{X} \in \text{SmVar}(\mathbb{C})$ and $j$ the open embedding. Let
\[
\alpha : (K, W) \otimes \mathbb{C}_{X^{an}} \to DR(X)((M, W)^{an})
\]
a morphism in $D_{fil}(X^{an})$, with $(M, F, W) \in \pi_X(C(MHM(X)))$ and $(K, W) \in D_{fil}(X^{an})$. We then consider the maps in $D_{fil}(S^{an})$
\[
f_*\alpha : Rf_*w(K, W) \otimes \mathbb{C}_{S^{an}} := R\bar{f}_*Rj_*w(K, W) \otimes \mathbb{C}_{S^{an}}
\]
\[
\xrightarrow{R\bar{f}_* j_* w, \alpha, \alpha} R\bar{f}_* Rj_* w DR(X)((M, W)^{an}) \xrightarrow{T^{\alpha}(\cdot, \otimes)^{-1}} R\bar{f}_* Rj_{\text{Hdg}}(\bar{X})(j_{\text{Hdg}}(M, W)^{an})
\]
\[
\xrightarrow{T(\bar{f}, DR)^{-1}} DR(S)((\int_f^{\text{Hdg}} (M, W))^{an}) = DR(S)((\int_{\bar{f}}^{\text{Hdg}} (M, W))^{an})
\]
and
\[
f_\alpha : Rf_*w(K, W) \otimes \mathbb{C}_{S^{an}} := R\bar{f}_* Rj_*w(K, W) \otimes \mathbb{C}_{S^{an}}
\]
\[
\xrightarrow{R\bar{f}_* j_* \alpha, \alpha} R\bar{f}_* Rj_*w DR(X)((M, W)^{an}) \xrightarrow{BT^{\alpha}(\cdot, \otimes)^{-1}} R\bar{f}_* DR(\bar{X})(j_{\text{Hdg}}(M, W)^{an})
\]
\[
\xrightarrow{T(\bar{f}, DR)^{-1}} DR(S)((\int_{\bar{f}}^{\text{Hdg}} (M, W))^{an}) = DR(S)((\int_f^{\text{Hdg}} (M, W))^{an}),
\]
see definition 102 and definition 107.
(ii) Let $f : X \to S$ a morphism with $S, X \in \text{QPVar}(\mathbb{C})$. Consider a factorization $f : X \xrightarrow{i} Y \xrightarrow{p} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p$ the projection. Let $\tilde{Y} \in \text{PSmVar}(\mathbb{C})$ a smooth compactification of $Y$ with $j : Y \to \tilde{Y}$ the open embedding. Then $\tilde{f} : \tilde{X} \xrightarrow{i} \tilde{Y} \xrightarrow{p} S$ is a compactification of $f$, with $\tilde{X} \subset \tilde{Y} \times S$ the closure of $X$ and $\tilde{i}$ the closed embedding. Let $S = \bigcup_i S_i$ an open affine cover and $i : S_i \to \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let

$$\alpha : T(X/(Y \times \tilde{S}_i))((K,W) \otimes \mathbb{C}_{\text{an}}) \to DR(S)((M_I,W)^{an},u_{\text{an}}^{an})$$

a morphism in $D_{fil}(X^{an}/(Y \times \tilde{S}_i^{an}))$, with $((M_I,W),u_{II}) \in \pi_X(C(MHM(X))) \subset C_{D0fil,\text{rk}}(X/(Y \times \tilde{S}_i))$ and $(K,W) \in D_{fil}(X^{an})$. We then consider the maps in $D_{fil}(S^{an}/(\tilde{S}_i^{an}))$

$$f_* \alpha = f_*(\alpha) : T(S/\tilde{S}_i)(Rf_{*w}(K,W)) \otimes \mathbb{C}_{\text{an}}$$

and

$$f_1 \alpha = f_1(\alpha) : T(S/\tilde{S}_i)(Rf_{1w}(K,W)) \otimes \mathbb{C}_{\text{an}}$$

see definition 102 and definition 107.

(iii) Let $l : S^{0} \hookrightarrow S$ an open embedding with $S \in \text{Var}(\mathbb{C})$ and denote $Z = S \setminus S^{0}$. Let $S = \bigcup S_i$ an open affine cover and $i : S_i \to \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $l : \tilde{S}_i \to \tilde{S}_I$ open embeddings such that $\tilde{S}_i \cap S = S^{0} \cap S_i$. Let

$$\alpha : T(S/(\tilde{S}_I))((K,W) \otimes \mathbb{C}_{\text{an}}) \to DR(S)((M_I,W),u_{II})^{an}$$

a morphism in $D_{fil}(S^{0}/(\tilde{S}_I^{an}))$, with $((M_I,W),u_{II}) \in \pi_S(C(\text{DRM}(S))) \subset C_{D0fil,\text{rk}}(S/(\tilde{S}_I))$ and $(K,W) \in D_{fil}(S^{0})$. We then consider the maps in $D_{fil}(S^{an}/(\tilde{S}_I^{an}))$

$$\Gamma_Z(\alpha) : T(S/(\tilde{S}_I))((\gamma_Z^{w}(K,W) \otimes \mathbb{C}_{\text{an}}) \xrightarrow{Rf_{\gamma_Z}^{an}} \Gamma_Z^{w}T(S/(\tilde{S}_I))((K,W) \otimes \mathbb{C}_{\text{an}}))$$

and

$$\Gamma_Z(\alpha) : T(S/(\tilde{S}_I))((\gamma_Z^{w}(K,W) \otimes \mathbb{C}_{\text{an}}) \xrightarrow{Rf_{\gamma_Z}^{an}} \Gamma_Z^{w}T(S/(\tilde{S}_I))((K,W) \otimes \mathbb{C}_{\text{an}}))$$

see definition 101 and definition 110.
(iv) Let $f : X \to S$ a morphism with $S, X \in \text{QPVar}(\mathbb{C})$. Consider a factorization $f : X \to Y \times S \xrightarrow{\pi} S$ with $Y \in \text{SmVar}(\mathbb{C})$. Let $S = \bigcup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let

$$\alpha : T(S/(\tilde{S}_i))((K, W) \otimes \mathbb{C}_{S^{an}}) \to DR(S)(((M_i, W), u_{ij})^{an}),$$

a morphism in $D_{fil}(S^{an}/(\tilde{S}_i^{an}))$, with $((M_i, W), u_{ij}) \in C(\text{ARM}(S)) \subset C_{\text{Dolfil,rh}}(S/(\tilde{S}_i))$ and $(K, W) \in D_{fil}(S^{an})$. We then consider, see (iii), the maps in $D_{fil}(X^{an}/(Y \times S)^{(\alpha)})$

$$f^\prime \alpha = f^\prime(\alpha) : T(X/(Y \times \tilde{S}_i))((f^w(K, W) \otimes \mathbb{C}_{X^{an}}) \cong T(X/(Y \times \tilde{S}_i))(\Gamma_X^w p^* (K, W) \otimes \mathbb{C}_{X^{an}})$$

$$\Rightarrow (\Gamma_X^w p^\prime_{\tilde{S}_i} T(S/\tilde{S}_i)((K, W) \otimes \mathbb{C}_{S^{an}}), \Gamma_X^w p^* T(D_{ij})(-))$$

$$\Rightarrow (\Gamma_X^w p^\prime_{\tilde{S}_i} DR(S)((M_i, W), u_{ij})_I, \Gamma_X^w p^* DR(u_{ij}))$$

$$T^\prime(f, DR)(-) := T((\gamma_X^w, DR)(-) \cdot T^\prime(p, DR)(-)) \to DR(X)(f^{\text{mod}}_{Hdg}((M_i, W), u_{ij})^{an})$$

and

$$f^* \alpha = f^*(\alpha) : T(X/(Y \times \tilde{S}_i))(f^{w*}(K, W) \otimes \mathbb{C}_{X^{an}}) \cong T(X/(Y \times \tilde{S}_i))((\Gamma_X^w p^* (K, W) \otimes \mathbb{C}_{X^{an}})$$

$$\Rightarrow (\Gamma_X^w p^\prime_{\tilde{S}_i} T(S/\tilde{S}_i)((K, W) \otimes \mathbb{C}_{S^{an}})_I, \Gamma_X^w p^* T(D_{ij})(-))$$

$$\Rightarrow (\Gamma_X^w p^\prime_{\tilde{S}_i} DR(S)((M_i, W), u_{ij})_I, \Gamma_X^w p^* DR(u_{ij}))$$

$$T^*(f, DR)(-) := T((\gamma_X^w, DR)(-) \cdot T^*(p, DR)(-)) \to DR(X)(f^{\text{mod}}_{Hdg}((M_i, W), u_{ij})^{an})$$

(v) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let

$$\alpha : T(S/(\tilde{S}_i))((K, W) \otimes \mathbb{C}_{S^{an}}) \to DR(S)(((M_i, W), u_{ij})^{an}),$$

$$\alpha^j : T(S/(\tilde{S}_i))((K^j, W) \otimes \mathbb{C}_{S^{an}}) \to DR(S)(((M_i^j, W), u_{ij})^{an})$$

two morphism in $D_{fil}(S^{an}/(\tilde{S}_i^{an}))$, with $((M_i, W), u_{ij}), ((M_i^j, W), u_{ij}) \in \pi_S(C(MHM(S)) \subset C_{\text{Dolfil,rh}}(S/(\tilde{S}_i))$ and $(K, W), (K^j, W) \in D_{fil}(S^{an})$. We then consider the map

$$\alpha \otimes \alpha^j : T(S/(\tilde{S}_i))((K, W) \otimes \mathbb{C}_{S^{an}}) \to T(S/\tilde{S}_i)_J((K, W) \otimes \mathbb{C}_{S^{an}})$$

$$\Rightarrow DR(S)(((M_i, W), u_{ij})^{an}) \otimes DR(S)(((M_i^j, W), u_{ij})^{an})$$

$$\Rightarrow DR(S)(((M_i, W), u_{ij}) \otimes_{O_{\text{S}}} ((M_i^j, W), u_{ij})^{an})$$

in $D_{fil}(S^{an}/(\tilde{S}_i^{an}))$.

**Definition 113.** Let $k \subset \mathbb{C}$ a subfield. Let $S \in \text{SmVar}(\mathbb{C})$. Let $j : S^o \hookrightarrow S$ an open embedding. Let $Z := S \setminus S^o = V(\mathcal{I}) \subset S$ the closed complementary subset, $\mathcal{I} \subset O_S$ being an ideal sheaf. Taking generators $\mathcal{I} = (s_1, \ldots, s_r)$, we get $Z = V(s_1, \ldots, s_r) = \cap_{i=1}^r Z_i \subset S$ with $Z_i = V(s_i) \subset S$, $s_i \in \Gamma(S, \mathcal{I}_i)$ and $L_i$ a line bundle. Note that $Z$ is an arbitrary closed subset, $d_Z \geq d_X - r$ needing not be a complete intersection. Denote by $j_1 : S^o \cap \cap_{i \in \mathcal{I}} Z_i = S \setminus \cup_{i \in \mathcal{I}} Z_i \xrightarrow{j_1} \to S^o \xrightarrow{j} S$ the open embeddings. Let $(M, F, W) \in MHM(S^o)$. We then define, using definition 107 and definition 100

- the canonical extension

$$j_*^{\text{Hdg}}((M, F, W), (K, W), \alpha) := (j_*^{\text{Hdg}}(M, F, W), j_*^{w}(K, W), j_*^{\alpha})$$

$$:= \text{Tot}((j_*^{\text{Hdg}}j_1^*(M, F, W), j_1^{w*}j_1^*(K, W), j_1^{\alpha})) \in \text{MHM}(S),$$

so that $j^*(j_*^{\text{Hdg}}((M, F, W), (K, W), \alpha)) = ((M, F, W), (K, W), \alpha)$,
• the canonical extension

\[ j_{\text{Hdg}}((M, F, W), (K, W), \alpha) := (j_{\text{Hdg}}(M, F, W), j_{\text{w}}(K, W), j_{\alpha}) := \text{Tot}((j_{\text{Hdg}}j_{\text{f}}(M, F, W), j_{\text{w}}j_{\text{f}}(K, W), j_{\alpha})) \in \text{MHM}(S), \]

so that \( j^* (j_{\text{Hdg}}((M, F, W), (K, W), \alpha)) = ((M, F, W), (K, W), \alpha) \).

Moreover for \( ((M', F, W), (K', W), \alpha') \in \text{MHM}(S) \),

• there is a canonical map in \( \text{MHM}(S) \)

\[ \text{ad}(j^*, j_{\text{Hdg}})((M', F, W), (K', W), \alpha') : ((M', F, W), (K', W), \alpha') \rightarrow j_{\text{Hdg}}j^* ((M', F, W), (K', W), \alpha'), \]

• there is a canonical map in \( \text{MHM}(S) \)

\[ \text{ad}(j_{\text{Hdg}}, j^*)((M', F, W), (K', W), \alpha') : j_{\text{Hdg}}j^* ((M', F, W), (K', W), \alpha') \rightarrow ((M', F, W), (K', W), \alpha'). \]

**Definition 114.** Let \( S \in \text{SmVar}(\mathbb{C}) \). Let \( Z \subset S \) a closed subset. Denote by \( j : S \setminus Z \rightarrow S \) the complementary open embedding.

(i) We define using definition 108, definition 101 and definition 112(iii), the filtered Hodge support section functor

\[ \Gamma^\text{Hdg}_Z : C(\text{MHM}(S)) \rightarrow C(\text{MHM}(S)), \quad ((M, F, W), (K, W), \alpha) \mapsto (\Gamma^\text{Hdg}_Z(M, F, W), \Gamma^\text{Hdg}_Z(K, W), \Gamma_Z(\alpha)) \]

\[ = \text{Cone}(\text{ad}(j^*, j_{\text{Hdg}})(-)) : j_{\text{Hdg}}j^* ((M, F, W), (K, W), \alpha) \rightarrow ((M, F, W), (K, W), \alpha)[-1] \]

see definition 113 for the last equality, together we the canonical map

\[ \gamma^\text{Hdg}_Z((M, F, W), (K, W), \alpha) : \Gamma^\text{Hdg}_Z((M, F, W), (K, W), \alpha) \rightarrow ((M, F, W), (K, W), \alpha). \]

(i') Since \( j_{\text{Hdg}} : C(\text{MHM}(S^o)) \rightarrow C(\text{MHM}(S)) \) is an exact functor, \( \Gamma^\text{Hdg}_Z \) induces the functor

\[ \Gamma^\text{Hdg}_Z : D(\text{MHM}(S)) \rightarrow D(\text{MHM}(S)), \]

\[ ((M, F, W), (K, W), \alpha) \mapsto \Gamma^\text{Hdg}_Z((M, F, W), (K, W), \alpha) \]

(ii) We define using definition 108, definition 101 and definition 112(iii) the dual filtered Hodge support section functor

\[ \Gamma'^\text{Hdg}_Z : C(\text{MHM}(S)) \rightarrow C(\text{MHM}(S)), \quad ((M, F, W), (K, W), \alpha) \mapsto \Gamma'^\text{Hdg}_Z(M, F, W), \Gamma'^\text{Hdg}_Z(K, W), \Gamma'_Z(\alpha)) \]

\[ = \text{Cone}(\text{ad}(j_{\text{Hdg}}, j^*)(-)) : j_{\text{Hdg}}, j^* ((M, F, W), (K, W), \alpha) \rightarrow ((M, F, W), (K, W), \alpha) \]

see definition 113 for the last equality, together we the canonical map

\[ \gamma'^\text{Hdg}_Z((M, F, W), (K, W), \alpha) : ((M, F, W), (K, W), \alpha) \mapsto \Gamma'^\text{Hdg}_Z((M, F, W), (K, W), \alpha). \]

(ii') Since \( j_{\text{Hdg}} : C(\text{MHM}(S^o)) \rightarrow C(\text{MHM}(S)) \) is an exact functor, \( \Gamma'^{\text{Hdg}}_Z \) induces the functor

\[ \Gamma'^\text{Hdg}_Z : D(\text{MHM}(S)) \rightarrow D(\text{MHM}(S)), \]

\[ ((M, F, W), (K, W), \alpha) \mapsto \Gamma'^\text{Hdg}_Z((M, F, W), (K, W), \alpha) \]

In the singular case it gives :
Definition 115. Let \( S \subseteq \text{Var}(\mathbb{C}) \). Let \( Z \subseteq S \) a closed subset. Let \( S = \bigcup_{i=1}^n S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \subseteq \text{SmVar}(\mathbb{C}) \). Denote \( \tilde{Z}_i := S \cap S_i \). Denote by \( n : S \setminus Z \to S \) and \( n_i : \tilde{S}_i \setminus \tilde{Z}_i \to \tilde{S}_i \) the complementary open embeddings.

(i) We define using definition 110, definition 101 and definition 112(iii) the filtered Hodge support section functor

\[
\Gamma^{\text{Hdg}}_Z : C(\text{MHM}(S)) \to C(\text{MHM}(S)),
\]

\[
(((M_I, F, W), u_{1J}), (K, W), \alpha) \mapsto \Gamma^{\text{Hdg}}_Z(((M_I, F, W), u_{1J}), (K, W), \alpha) := (\Gamma^{\text{Hdg}}_Z((M_I, F, W), u_{1J}), \Gamma^w_Z(K, W), \Gamma_Z(\alpha))
\]

together with the canonical map

\[
\gamma^{\text{Hdg}}_Z(((M_I, F, W), u_{1J}), (K, W), \alpha) : \Gamma^{\text{Hdg}}_Z(((M_I, F, W), u_{1J}), (K, W), \alpha) \to (((M_I, F, W), u_{1J}), (K, W), \alpha).
\]

(ii) By exactness of \( \Gamma^{\text{Hdg}}_Z \) and \( \Gamma^w_Z \) it induces the functor

\[
\Gamma^{\text{Hdg}}_Z : D(\text{MHM}(S)) \to D(\text{MHM}(S)),
\]

\[
(((M_I, F, W), u_{1J}), (K, W), \alpha) \mapsto \Gamma^{\text{Hdg}}_Z(((M_I, F, W), u_{1J}), (K, W), \alpha)
\]

(ii) We define using definition 110, definition 101 and definition 112(iii) the dual filtered Hodge support section functor

\[
\Gamma^{\vee, \text{Hdg}}_Z : C(\text{MHM}(S)) \to C(\text{MHM}(S)),
\]

\[
(((M_I, F, W), u_{1J}), (K, W), \alpha) \mapsto \Gamma^{\vee, \text{Hdg}}_Z(((M_I, F, W), u_{1J}), (K, W), \alpha) := (\Gamma^{\vee, \text{Hdg}}_Z((M_I, F, W), u_{1J}), \Gamma^{\vee, w}_Z(K, W), \Gamma^{\vee}_Z(\alpha)),
\]

together with the canonical map

\[
\gamma^{\vee, \text{Hdg}}_Z(((M_I, F, W), u_{1J}), (K, W), \alpha) : \Gamma^{\vee, \text{Hdg}}_Z(((M_I, F, W), u_{1J}), (K, W), \alpha) \to (((M_I, F, W), u_{1J}), (K, W), \alpha).
\]

(ii) By exactness of \( \Gamma^{\vee, \text{Hdg}}_Z \) and \( \Gamma^{\vee, w}_Z \), it induces the functor

\[
\Gamma^{\vee, \text{Hdg}}_Z : D(\text{MHM}(S)) \to D(\text{MHM}(S)),
\]

\[
(((M_I, F, W), u_{1J}), (K, W), \alpha) \mapsto \Gamma^{\vee, \text{Hdg}}_Z(((M_I, F, W), u_{1J}), (K, W), \alpha) := (\Gamma^{\vee, \text{Hdg}}_Z((M_I, F, W), u_{1J}), \Gamma^{\vee, w}_Z(K, W), \Gamma^{\vee}_Z(\alpha))
\]

For \( X \subseteq \text{SmVar}(\mathbb{C}) \), we have, by definition

\[
z^{\text{Hdg}}_X := a^{\text{Hdg}}_X z^{\text{Hdg}}_{\text{pt}} := ((O_X, F_b)[d_X], Z_X, \alpha(X)) \in D(\text{MHM}(X)),
\]

with \( \alpha(X) : C_X \hookrightarrow (0 \to O_X \to \Omega_X \to \cdots K_X) \). If \( X \subseteq \text{SmVar}(\mathbb{C}) \),

\[
z^{\text{Hdg}}_{\text{pt}} := a^{\text{Hdg}}_{\text{pt}} z^{\text{Hdg}}_{\text{pt}} := ((O_X, F_b)[d_X], Z_X^{an}, \alpha(X^{an})) \in D(\text{MHM}(X)).
\]

Let \( X \subseteq \text{Var}(\mathbb{C}) \) non smooth. Take an open cover \( X = \bigcup_{i=1}^l X_i \) such that there exists closed embeddings \( i_i : X_i \hookrightarrow \tilde{X}_i \) with \( \tilde{X}_i \subseteq \text{SmVar}(\mathbb{C}) \). Then, by definition

\[
z^{\text{Hdg}}_X := a^{\text{Hdg}}_X z^{\text{Hdg}}_{\text{pt}} := (((\Gamma^{\vee, \text{Hdg}}_{X_i}(O_{\tilde{X}_i}, F_b)[d_{\tilde{X}_i}], o_{\tilde{S}_j/\tilde{S}_i}), (Z_X^{an}, W), \alpha(X/\tilde{X}_i)) \in D(\text{MHM}(X)).
\]

with \( \alpha(X/\tilde{X}_i) : (\Gamma^{\vee, w}_{X_i} \alpha(\tilde{X}_i)) : T(X/(\tilde{X}_i))(\mathbb{Z}^{an}) : (i_i, Z_X^{an}, I) \to DR(X)^{-1}((\Gamma^{\vee, \text{Hdg}}_{X_i}(O_{\tilde{X}_i}, [d_{\tilde{X}_i}], o_{\tilde{S}_j/\tilde{S}_i})
\]

We have from [28] the following proposition which shows us how to construct inductively mixed Hodge modules, as we do for perverse sheaves:
Proposition 102. (i) Let $S \in \text{AnSm}(\mathbb{C})$. Let $D = V(s) \subset S$ a (Cartier) divisor, where $s \in \Gamma(S,L)$ is a section of the line bundle $L = L_D$ associated to $D$. We then have the zero section embedding $i : S \hookrightarrow L$. We denote $L_0 = i(S)$ and $j : L^o := L \setminus L_0 \hookrightarrow L$ the open complementary subset. We denote by $\text{MHW}(S\setminus D)^{\text{ex}} \times_J \text{MHW}(D)$ the category whose set of objects consists of

$$\{(\mathcal{M},\mathcal{N},a,b), \mathcal{M} \in \text{MHW}(S\setminus D)^{\text{ex}}, \mathcal{N} \in \text{MHW}(D), a : \psi_{D1} \mathcal{M} \to \mathcal{N}, b : \mathcal{N} \to \psi_{D1} \mathcal{M}\}$$

where $\text{MHW}(S\setminus D)^{\text{ex}} \subset \text{MHW}(S\setminus D)$ is the full subcategory of extendable objects. The functor (see definition 104)

$$j^*, \phi_{D1}, c, v : \text{MHW}(S) \to \text{MHW}(S\setminus D)^{\text{ex}} \times_J \text{MHW}(D),$$

$$(\mathcal{M}, \mathcal{N}, (K,W), \alpha) \mapsto ((j^*(\mathcal{M}, F, W), j^*(K,W), j^*\alpha), \phi_{D1}((\mathcal{M}, F, W), (K,W), \alpha), \text{can}, \text{var})$$

is an equivalence of category.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$. Let $D = V(s) \subset S$ a (Cartier) divisor, where $s \in \Gamma(S,L)$ is a section of the line bundle $L = L_D$ associated to $D$. We then have the zero section embedding $i : S \hookrightarrow L$. We denote $L_0 = i(S)$ and $j : L^o := L \setminus L_0 \hookrightarrow L$ the open complementary subset. We denote by $\text{MHW}(S\setminus D)^{\text{ex}} \times_J \text{MHW}(D)$ the category whose set of objects consists of

$$\{(\mathcal{M},\mathcal{N},a,b), \mathcal{M} \in \text{MHW}(S\setminus D), \mathcal{N} \in \text{MHW}(D), a : \psi_{D1} \mathcal{M} \to \mathcal{N}, b : \mathcal{N} \to \psi_{D1} \mathcal{M}\}$$

The functor (see definition 104)

$$(j^*, \phi_{D1}, c, v) : \text{MHW}(S) \to \text{MHW}(S\setminus D)^{\text{ex}} \times_J \text{MHW}(D),$$

$$(\mathcal{M}, \mathcal{N}, (K,W), \alpha) \mapsto ((j^*(\mathcal{M}, F, W), j^*(K,W), j^*\alpha), \phi_{D1}((\mathcal{M}, F, W), (K,W), \alpha), \text{can}, \text{var})$$

is an equivalence of category.

Proof. See [28].

Theorem 31. Let $S \in \text{Var}(\mathbb{C})$. The category of mixed Hodge modules is the full subcategory

$$\iota_S : \text{MHM}(S) \hookrightarrow \text{MHW}(S) \hookrightarrow \text{PSh}_{\text{Dr}(1,0)\text{fil},\text{rh}}(S) \times_I \text{Pfil}(S^{an})$$

consisting of objects

$$((\mathcal{M}, F,W), (K,W), \alpha) = (((\mathcal{M}_I, F,W), u_{IJ}), (K,W), \alpha) \in \text{PSh}_{\text{Dr}(1,0)\text{fil},\text{rh}}(S) \times_I \text{Pfil}(S^{an})$$

such that $((\mathcal{M}, F,W)^{an}, (K,W), \alpha) : (((\mathcal{M}_I)^{an}, F,W), u_{IJ}), (K,W), \alpha) \in \text{MHM}(S^{an})$.

Proof. Follows from GAGA and the extendableness in the algebraic case (proposition 102).
Moreover, and whose set of morphisms consists of
\[ \phi = (\phi_D, \phi_C, [\theta]) : (((M_1, F, W), u_{I,J}), (K_1, W), \alpha_1) \to (((M_2, F, W), u_{I,J}), (K_2, W), \alpha_2) \]
where \( \phi_D : ((M_1, F, W), u_{I,J}) \to ((M_2, F, W), u_{I,J}) \) and \( \phi_C : (K_1, W) \to (K_2, W) \) are morphisms and
\[
\theta = (\theta^*, I(DR(S)(\phi_D^{an})) \circ I(\alpha_1), I(\alpha_2) \circ I(\phi_C \otimes I)) : I(T(S/(\tilde{S}_I))(K_1, W) \otimes \mathbb{C}_{S=\{1\}} \to I(DR(S(((M_2, F, W), u_{I,J})^{an})))
\]
is an homotopy, \( I : D_{fil}(S^{an}/(\tilde{S}_I^{an})) \to K_{fil}(S^{an}/(\tilde{S}_I^{an})) \) being the injective resolution functor, and for
\[
\phi = (\phi_D, \phi_C, [\theta]) : (((M_1, F, W), u_{I,J}), (K_1, W), \alpha_1) \to (((M_2, F, W), u_{I,J}), (K_2, W), \alpha_2)
\]
and\( \phi' : (\phi_D', \phi_C', [\theta']) : (((M_2, F, W), u_{I,J}), (K_2, W), \alpha_2) \to (((M_3, F, W), u_{I,J}), (K_3, W), \alpha_3) \)
the composition law is given by
\[
\phi' \circ \phi := (\phi_D' \circ \phi_D, \phi_C' \circ \phi_C, I(DR(S)(\phi_D^{an})) \circ \theta + [\theta'] \circ I(\phi_C \otimes I)[1]) : (((M_1, F, W), u_{I,J}), (K_1, W), \alpha_1) \to (((M_3, F, W), u_{I,J}), (K_3, W), \alpha_3),
\]
in particular for \( ((M_I, F, W), u_{I,J}), (K, W), \alpha) \in D_{D_{fil}(0,fil,rel}, (S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}),
\]
\[
I_{((((M_I, F, W), u_{I,J}), (K, W), \alpha))} = ((I_{M_I}), I_{K}, 0),
\]
and also the category \( D_{D_{fil}(0,fil,rel}, (S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \) defined in the same way, together with the localization functor
\[
(D_{zar}, I) : C_{D_{fil}(0,fil,rel}, (S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}) \to D_{D_{fil}(0,fil,rel}, (S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})
\]
\[
\to D_{D_{fil}(0,fil,rel}, (S/(\tilde{S}_I)) \times_I D_{fil}(S^{an})
\]
Moreover,
- For \( (((M_I, F, W), u_{I,J}), (K, W), \alpha) \in D_{D_{fil}(0,fil,rel}, (S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}), \)
we set
\[
(((M_I, F, W), u_{I,J}), (K, W), \alpha)[1] := (((M_I, F, W), u_{I,J})[1], (K, W)[1], \alpha[1]).
\]
- For
\[
\phi = (\phi_D, \phi_C, [\theta]) : (((M_1, F, W), u_{I,J}), (K_1, W), \alpha_1) \to (((M_2, F, W), u_{I,J}), (K_2, W), \alpha_2)
\]
a morphism in \( D_{D_{fil}(0,fil,rel}, (S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}), \)
we set (see [9] definition 3.12)
\[
\text{Cone}(\phi) := (\text{Cone}(\phi_D), \text{Cone}(\phi_C), ((\alpha_1, \theta), (\alpha_2, 0))) \in D_{D_{fil}(0,fil,rel}, (S/(\tilde{S}_I)) \times_I D_{fil}(S^{an}).
\]
together with the canonical maps
\[
- c_1(-) = (c_1(\phi_D), c_1(\phi_C), 0) : (((M_2, F, W), u_{I,J}), (K_2, W), \alpha_2) \to \text{Cone}(\phi)
\]
\[
- c_2(-) = (c_2(\phi_D), c_2(\phi_C), 0) : \text{Cone}(\phi) \to (((M_I, F, W), u_{I,J}), (K_1, W), \alpha_1)[1].
\]
We now state and prove the following key theorem :
Theorem 32. (i) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then the full embedding

$$t_S : MHM(S) \hookrightarrow \text{PSh}_{D(1,0)}^{\mathbb{C}}(S/(\tilde{S}_1)) \times_1 P_{\mathbb{F}}(S^{an}) \hookrightarrow C_{D(1,0)}^{\mathbb{C}}(S/(\tilde{S}_1)) \times_1 D_{\mathbb{F}}(S^{an})$$

induces a full embedding

$$t_S : D(MHM(S)) \hookrightarrow D_{D(1,0)}^{\mathbb{C}}(S/(\tilde{S}_1)) \times_1 D_{\mathbb{F}}(S^{an})$$

whose image consists of $(((M_1,F,W),u_{11}), (K,W),\alpha) \in D_{D(1,0)}^{\mathbb{C}}(S/(\tilde{S}_1)) \times_1 D_{\mathbb{F}}(S^{an})$ such that

$$ (((H^n(M_1,F,W),H^n(u_{11})), H^n(K,W), H^n\alpha) \in MHM(S)$$

for all $n \in \mathbb{Z}$ and such that for all $p \in \mathbb{Z}$, the differentials of $Gr^p_W(M_1,F)$ are strict for the filtrations $F$.

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then the full embedding

$$t_S : MHM(S) \hookrightarrow \text{PSh}_{D(1,0)}^{\mathbb{C}}(S/(\tilde{S}_1)) \times_1 P_{\mathbb{F}}(S^{an}) \hookrightarrow C_{D(1,0)}^{\mathbb{C}}(S/(\tilde{S}_1)) \times_1 D_{\mathbb{F}}(S^{an})$$

induces a full embedding

$$t_S : D(MHM(S)) \hookrightarrow D_{D(1,0)}^{\mathbb{C}}(S/(\tilde{S}_1)) \times_1 D_{\mathbb{F}}(S^{an})$$

whose image consists of $(((M_1,F,W),u_{11}), (K,W),\alpha) \in D_{D(1,0)}^{\mathbb{C}}(S/(\tilde{S}_1)) \times_1 D_{\mathbb{F}}(S^{an})$ such that

$$ (((H^n(M_1,F,W),H^n(u_{11})), H^n(K,W), H^n\alpha) \in MHM(S)$$

for all $n \in \mathbb{Z}$ and such that there exist $r \in \mathbb{Z}$ and an $r$-filtered homotopy equivalence $((M_1,F,W),u_{11}) \rightarrow ((M'_1,F,W),u_{11})$ such that for all $p \in \mathbb{Z}$ the differentials of $Gr^p_W(M'_1,F)$ are strict for the filtrations $F$.

Proof. (i): We first show that $t_S$ is fully faithful, that is for all $\mathcal{M} = (((M_1,F,W),u_{11}), (K,W),\alpha), \mathcal{M}' = (((M'_1,F,W),u_{11}), (K',W),\alpha') \in MHM(S)$ and all $n \in \mathbb{Z}$,

$$t_S : \text{Ext}^n_{D(MHM(S))}(\mathcal{M}, \mathcal{M}') := \text{Hom}_{D(MHM(S))}(\mathcal{M}, \mathcal{M}'[n]) \rightarrow \text{Ext}^n_{D(S)}(\mathcal{M}, \mathcal{M}') := \text{Hom}_{D(S)}(\mathcal{M}, \mathcal{M}'[n])$$

For this it is enough to assume $S$ smooth. We then proceed by induction on $\max(\dim \text{supp}(M), \dim \text{supp}(M'))$.

- For $\text{supp}(M) = \text{supp}(M') = \{s\}$, it is the theorem for mixed hodge complexes or absolute Hodge complexes, see [9]. If $\text{supp}(M) = \{s\}$ and $\text{supp}(M') = \{s'\}$, then by the localization exact sequence

$$\text{Ext}^n_{D(MHM(S))}(\mathcal{M}, \mathcal{M}') = 0 = \text{Ext}^n_{D(S)}(\mathcal{M}, \mathcal{M}')$$

- Denote $\text{supp}(M) = Z \subset S$ and $\text{supp}(M') = Z' \subset S$. There exist an open subset $S'' \subset S$ such that $Z'' := Z \cap S''$ and $Z' := Z' \cap S''$ are smooth, and $M_{i,j} := ((i^*j^*\text{Gr}_{Z''\neq 0} M_{i,j}, F,W), i^*j^*(K,W), \alpha^*(i)) \in MHM(Z'')$ and $M'_{i,j} := ((i'^*j'^*\text{Gr}_{Z''\neq 0} M'_{i,j}, F,W), i'^*j'^*(K,W), \alpha^*(i')) \in MHM(Z'')$ are variation of mixed Hodge structure, where $j : S'' \rightarrow S$ is the open embedding and $i : Z'' \hookrightarrow S''$, $i : Z'' \hookrightarrow S''$ the closed embeddings. Considering the connected components of $Z''$ and $Z'\subset Z''$ we may assume that $Z''$ and $Z'$ are connected. Shrinking $S''$ if necessary, we may assume that either $Z'' = Z'$ or $Z'' \cap Z' = \emptyset$. We denote $D = S/S''$. Shrinking $S''$ if necessary, we may assume that $D$ is a divisor and denote by $l : S \hookrightarrow L_D$ the zero section embedding.
If $Z^o = Z'^o$, denote $i : Z^o \hookrightarrow S^o$ the closed embedding. We have then the following commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^n_{D(MHM(S^o))}(\mathcal{M}_{|S^o}, \mathcal{M}'_{|S^o}) & \xrightarrow{i_{S^o}} & \text{Ext}^n_{D(S^o)}(\mathcal{M}_{|S^o}, \mathcal{M}'_{|S^o}) \\
(i^* \text{Gr}_{S^o, o,i^*}, \alpha^*(i)) \downarrow & & \downarrow (i^* \text{Gr}_{S^o, o,i^*}, \alpha^*(i)) \\
\text{Ext}^n_{D(MHM(Z^o))}(\mathcal{M}_{|Z^o}, \mathcal{M}'_{|Z^o}) & \xrightarrow{i_{Z^o}} & \text{Ext}^n_{D(Z^o)}(\mathcal{M}_{|Z^o}, \mathcal{M}'_{|Z^o})
\end{array}
\]

Now we prove that $i_{Z^o}$ is an isomorphism similarly to the proof the the generic case of 32. On the other hand the left and right column are isomorphisms. Hence $i_{S^o}$ is an isomorphism by the diagram.

If $Z^o \cap Z'^o = \emptyset$, we consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^n_{D(MHM(S^o))}(\mathcal{M}_{|S^o}, \mathcal{M}'_{|S^o}) & \xrightarrow{i_{S^o}} & \text{Ext}^n_{D(S^o)}(\mathcal{M}_{|S^o}, \mathcal{M}'_{|S^o}) \\
(i^* \text{Gr}_{S^o, o,i^*}, \alpha^*(i)) \downarrow & & \downarrow (i^* \text{Gr}_{S^o, o,i^*}, \alpha^*(i)) \\
\text{Ext}^n_{D(MHM(Z^o))}(\mathcal{M}_{|Z^o}, 0) = 0 & \xrightarrow{i_{Z^o}} & \text{Ext}^n_{D(Z^o)}(\mathcal{M}_{|Z^o}, 0) = 0
\end{array}
\]

where the left and right column are isomorphism by strictness of the $V_{Z^o}$ filtration (use a bi-filtered injective resolution with respect to $F$ and $V_{Z^o}$ for the right column).

- We consider now the following commutative diagram in $C(Z)$ where we denote for short $H := D(MHM(S))$

\[
\begin{array}{ccc}
0 & \xrightarrow{H} & \text{Hom}^*(H_{D} \mathcal{M}, \Gamma_{D} H_{dg} \mathcal{M}'_{\delta}(\mathcal{M}')) \\
\downarrow i_{S} & & \downarrow i_{S} \\
0 & \xrightarrow{H} & \text{Hom}^*(H_{D} \mathcal{M}, \Gamma_{D} H_{dg} \mathcal{M}'_{\delta}(\mathcal{M}'))
\end{array}
\]

whose lines are exact sequence. We have the one hand,

\[
\text{Hom}^*_{D(MHM(S))}(\Gamma_{D} H_{dg} \mathcal{M}, j_{*} H_{dg} j_{*} \mathcal{M}') = \text{Hom}^*_{D(S)}(\Gamma_{D} H_{dg} \mathcal{M}, j_{*} H_{dg} j_{*} \mathcal{M}')
\]

On the other hand by induction hypothesis

\[
i_{S} : \text{Hom}^*_{D(MHM(S))}(\Gamma_{D} H_{dg} \mathcal{M}, \Gamma_{D} H_{dg} \mathcal{M}') \rightarrow \text{Hom}^*_{D(S)}(\Gamma_{D} H_{dg} \mathcal{M}, \Gamma_{D} H_{dg} \mathcal{M}')
\]

is a quasi-isomorphism. Hence, by the diagram

\[
i_{S} : \text{Hom}^*_{D(MHM(S))}(\Gamma_{D} H_{dg} \mathcal{M}, \mathcal{M}') \rightarrow \text{Hom}^*_{D(S)}(\Gamma_{D} H_{dg} \mathcal{M}, \mathcal{M}')
\]

is a quasi-isomorphism.

- We consider now the following commutative diagram in $C(Z)$ where we denote for short $H := D(MHM(S))$

\[
\begin{array}{ccc}
0 & \xrightarrow{H} & \text{Hom}^*(H_{D} \mathcal{M}, \mathcal{M}') \\
\downarrow i_{S} & & \downarrow i_{S} \\
0 & \xrightarrow{H} & \text{Hom}^*(H_{D} \mathcal{M}, \mathcal{M}')
\end{array}
\]

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whose lines are exact sequence. On the one hand, the commutative diagram

\[
\begin{array}{ccc}
\Hom_{D(MHM(S))}(j_Hdg^{*}M, M') & \rightarrow & \Hom_{D(MHM(S))}(j^{*}M, j^{*}M') \\
\downarrow \iota_{S} & & \downarrow \iota_{S} \\
\Hom_{D(S)}(j_Hdg^{*}M, M') & \rightarrow & \Hom_{D(S)}(j^{*}M, j^{*}M')
\end{array}
\]

together with the fact that the horizontal arrows \(j^{*}\) are quasi-isomorphism by the functoriality given the uniqueness of the \(V_{S}\) filtration for the embedding \(l : S \rightarrow L_{D}\), (use a bi-filtered injective resolution with respect to \(F\) and \(V_{S}\) for the lower arrow) and the fact that \(\iota_{S}\) is a quasi-isomorphism by the first two point, show that

\[\iota_{S} : \Hom_{D(MHM(S))}(j_Hdg^{*}M, M') \rightarrow \Hom_{D(S)}(j_Hdg^{*}M, M')\]

is a quasi-isomorphism. On the other hand, by the third point

\[\iota_{S} : \Hom_{D(MHM(S))}(\Gamma_{D}^{V_{Hdg}}M, M') \rightarrow \Hom_{D(S)}(\Gamma_{D}^{V_{Hdg}}M, M')\]

is a quasi-isomorphism. Hence, by the diagram

\[\iota_{S} : \Hom_{D(MHM(S))}(\Gamma_{D}^{V_{Hdg}}M, M') \rightarrow \Hom_{D(S)}(\Gamma_{D}^{V_{Hdg}}M, M')\]

is a quasi-isomorphism.

This shows the fully faithfulness. We now prove the essential surjectivity: let

\[(((M_{I}, F, W), u_{IJ}), (K, W), \alpha) \in C_{D_{(1, 0)}fil, \rho h}(S/(\tilde{S}_{I})) \times \Gamma D_{fil}(S^{an})\]

such that the cohomology are mixed hodge modules and such that the differential are strict. We proceed by induction on \(\text{card }n \in \mathbb{Z}\), s.t. \(H^{n}(M_{I}, F, W) \neq 0\) by taking for the cohomological truncation

\[\tau_{\leq n}(((M_{I}, F, W), u_{IJ}), (K, W), \alpha) := ((\tau_{\leq n}(M_{I}, F, W), \tau_{\leq n}u_{IJ}), \tau_{\leq n}(K, W), \tau_{\leq n}\alpha)\]

and using the fact that the differential are strict for the filtration \(F\) and the fully faithfulness.

(ii): Follows from (i) and the fact that the image of the embedding given by (i) consists of classes of complexes whose differential are strict for \(F\).

\[\square\]

**Definition 116.** Let \(f : X \rightarrow S\) a morphism with \(X, S \in \text{Var}(\mathbb{C})\). Assume there exist a factorization \(f : X \xleftarrow{l} Y \times S \xrightarrow{p_{S}} S\) with \(Y \in \text{SmVar}(\mathbb{C})\), \(l\) a closed embedding and \(p_{S}\) the projection. Let \(\bar{Y} \in \text{PSmVar}(\mathbb{C})\) a smooth compactification of \(Y\) with \(n : Y \rightarrow \bar{Y}\) the open embedding. Then \(\bar{f} : \bar{X} \xleftarrow{\tilde{l}} \bar{Y} \times S \xrightarrow{\bar{p}_{S}} S\) is a compactification of \(f\), with \(\bar{X} \subset \bar{Y} \times S\) the closure of \(X\) and \(\tilde{l}\) the closed embedding, and we denote by \(n' : X \rightarrow \bar{X}\) the open embedding so that \(f = f \circ n'\).

(i) For \(((M_{I}, F, W), u_{IJ}), (K, W), \alpha) \in C(MHM(X))\), we define, using definition 113 and theorem 32

\[R_{f_{*}Hdg}(((M_{I}, F, W), u_{IJ}), (K, W), \alpha) := \iota_{S}^{-1}(\int_{f}^{FDR} (n \times I)_{*}Hdg(((M_{I}, F, W), u_{IJ}), Rf_{*w}(K, W), f_{*}(\alpha))\]

\[\in D(MHM(S))\]

where \(f_{*}(\alpha)\) is given in definition 112, and since

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there exist a factorization $f$.

Let $\iota : (i \times n) \to (M_I, F, W, u_{IJ})$, $Rf_* \mathcal{G} \mathcal{W}^{k'}(K, W), Rf_* \mathcal{G} \mathcal{W}^{k}(K, W, f_* \alpha) \in H M(S)$

for all $i, k \in \mathbb{Z}$, hence by the spectral sequence for the filtered complexes $\int_{f}^{FDR}(I \times n)_{Hdg}((M_I, F, W), u_{IJ})$
and $Rf_*((I \times n)_{sw}(K, W))$

Gr$^{k}_W H^i(\int_{f}^{Hdg} (M_I, F, W, u_{IJ}), Rf_*\mathcal{W}^{k'}(K, W, f_* \alpha)) :=$

$(\text{Gr}^{k}_W H^i \int_{f}^{FDR} (I \times n)_{Hdg}((M_I, F, W), u_{IJ}), \text{Gr}^{k}_W H^i Rf_*\mathcal{W}^{k'}(K, W, f_* \alpha) \in H M(S)$

this gives by definition $H^i(\int_{f}^{Hdg}((M_I, F, W), u_{IJ}), Rf_*\mathcal{W}^{k}(K, W, f_* \alpha)) \in M H M(S)$ for all $i \in \mathbb{Z}$.

- $\int_{f}^{Hdg}((M_I, F, W), u_{IJ})$ is the class of a complex such that the differential are strict for $F$ by theorem 29.

(ii) For $((M_I, F, W), u_{IJ}), (K, W, \alpha) \in C(M H M(X))$, we define, using definition 113 and theorem 32,

$Rf_{Hdg}(((M_I, F, W), u_{IJ}), (K, W, f_{\alpha}) := i_{\mathcal{S}}^{-1}(\int_{f}^{FDR} (n \times I)_{Hdg}((M_I, F, W), u_{IJ}), Rf_*\mathcal{W}(K, W, f_{\alpha}))$

$\in D(M H M(S))$

where $f_{\alpha}$ is given in definition 112, and since

- by definition

$H^i(\int_{f}^{FDR} \text{Gr}^{k}_W (n \times I)_{Hdg}((M_I, F, W), u_{IJ}), \text{Gr}^{k}_W f_* \mathcal{W}^{k'}(K, W, f_{\alpha}) \in H M(S)$

for all $i, k \in \mathbb{Z}$, hence by the spectral sequence for the filtered complexes $\int_{f}^{FDR}(n \times I)_{Hdg}((M_I, F, W), u_{IJ})$
and $Rf_*((n \times I)_{sw}(K, W))$

Gr$^{k}_W H^i(\int_{f}^{Hdg}((M_I, F, W), u_{IJ}), \text{Gr}^{k}_W H^i Rf_*\mathcal{W}^{k'}(K, W, f_{\alpha} \alpha) \in H M_{gm,k,C}(S)$

this gives by definition $H^i(\int_{f}^{Hdg}((M_I, F, W), u_{IJ}), Rf_*\mathcal{W}(K, W, f_{\alpha}) \in H M(S)$ for all $i \in \mathbb{Z}$.

- $\int_{f}^{Hdg}((M_I, F, W), u_{IJ})$ is the class of a complex such that the differential are strict for $F$ by theorem 29.

We have the six functors formalism for mixed Hodge modules on quasi-projective varieties:

**Definition 117.** Let $f : X \to S$ a morphism with $X, S \in QPVar(C)$. Then, since $X$ is quasi-projective, there exist a factorization $f : X \xrightarrow{l} \mathbb{P}^{N,0} \times S \xrightarrow{p_{S}} S$ with $u_0 : \mathbb{P}^{N,0} \hookrightarrow \mathbb{P}^{N}$ an open subset, $l$ a closed
embedding and $p_S$ the projection. Since $S$ is quasi-projective, there exist a closed embedding $i : S \hookrightarrow \tilde{S}$ with $\tilde{S} \in \text{SuVar}(\mathbb{C})$. We have then the commutative diagram

\[
\begin{array}{ccc}
  f : X & \xrightarrow{i} & \mathbb{P}^{N,o} \times S \xrightarrow{n} S \\
  & \xrightarrow{\ell' := (1 \times 1)} & \\
  & & \mathbb{P}^{N,o} \times \tilde{S} \xrightarrow{\ell} S \\
\end{array}
\]

(i) For $((M,F,W),(K,W),\alpha) \in D(\text{MHM}(X))$, where $(M,F,W) \in C_{\mathcal{D}(1.0)}(X/\mathbb{P}^{N,o} \times \tilde{S})$ and $(K,W) \in C_{\mathcal{D}}(X^\alpha)$, we define, using definition 116,

\[
f^*_{Hdg}((M,F,W),(K,W),\alpha) : = (R^*_{f^*_{Hdg}}(M,F,W), R_{f^*_{Hdg}}(K,W), f^*(\alpha))
\]

\[
: = \iota_S^{-1}( \int_{p_S} n^*_{Hdg}(M,F,W), R_{f^*_{Hdg}}(K,W), f^*(\alpha)) \in D(\text{MHM}(S))
\]

with $f^*(\alpha)$ given in definition 112.

(ii) For $((M,F,W),(K,W),\alpha) \in D(\text{MHM}(X))$, where $(M,F,W) \in C_{\mathcal{D}(1.0)}(X/\mathbb{P}^{N,o} \times \tilde{S})$ and $(K,W) \in C_{\mathcal{D}}(X^\alpha)$, we define, using definition 116

\[
f^1_{Hdg}((M,F,W),(K,W),\alpha) : = (R^1_{f^1_{Hdg}}(M,F,W), R_{f^1_{Hdg}}(K,W), f^1(\alpha))
\]

\[
: = \iota_S^{-1}( \int_{p_S} n^1_{Hdg}(M,F,W), R_{f^1_{Hdg}}(K,W), f^1(\alpha)) \in D(\text{MHM}(S))
\]

with $f^1(\alpha)$ given in definition 112.

(iii) For $((M,F,W),(K,W),\alpha) \in D(\text{MHM}(S))$, where $(M,F,W) \in C_{\mathcal{D}(1.0)}(S/\tilde{S})$, $(K,W) \in C_{\mathcal{D}}(S^\alpha)$, we define, using definition 111 (see theorem 30(ii) for $p_S$ and definition 108 for $i \circ l$),

\[
f^*_{Hdg}((M,F,W),(K,W),\alpha) : = (f^*_{Hdg}^\text{mod}(M,F,W), f^*_{Hdg}^{\text{w}}(K,W), f^*\alpha)
\]

\[
: = (\Gamma^\text{X}_{Hdg}^\text{mod}[\tilde{X}](M,F,W), \Gamma^\text{X}_{Hdg}^{\text{w}}[\tilde{X}](K,W), f^*\alpha) \in D(\text{MHM}(X))
\]

with $f^*\alpha$ given in definition 112. For $j : S^\alpha \hookrightarrow S$ an open embedding and $((M,F,W),(K,W),\alpha) \in D(\text{MHM}(S))$, we have (see [28])

\[
j^*_{Hdg}((M,F,W),(K,W),\alpha) = (j^*(M,F,W), j^*(K,W), j^*\alpha) \in D(\text{MHM}(S^\alpha)).
\]

(iv) For $((M,F,W),(K,W),\alpha) \in D(\text{MHM}(S))$, where $(M,F,W) \in C_{\mathcal{D}(1.0)}(S/\tilde{S})$, $(K,W) \in C_{\mathcal{D}}(S^\alpha)$, we define, using definition 111 (see theorem 30(ii) for $p_S$ and definition 108 for $i \circ l$),

\[
f^1_{Hdg}((M,F,W),(K,W),\alpha) : = (f^1_{Hdg}^\text{mod}(M,F,W), f^1_{Hdg}^{\text{w}}(K,W), f^1(\alpha))
\]

\[
: = (\Gamma^\text{X}_{Hdg}^\text{mod}[\tilde{X}](M,F,W)(d_X)[2\bar{d}X], \Gamma^\text{X}_{Hdg}^{\text{w}}[\tilde{X}](K,W), f^1(\alpha)) \in D(\text{MHM}(X))
\]

with $f^1\alpha$ given in definition 112. For $j : S^\alpha \hookrightarrow S$ an open embedding and $((M,F,W),(K,W),\alpha) \in D(\text{MHM}(S))$, we have (see [28])

\[
j^1_{Hdg}((M,F,W),(K,W),\alpha) = (j^*(M,F,W), j^*(K,W), j^*\alpha) \in D(\text{MHM}(S^\alpha)).
\]
Using the unicity of proposition 102, we see that these definitions does NOT depend on the choice of the factorization \( f : X \overset{1}{\to} Y \times S \overset{p_2}{\to} S \) of \( f \). Moreover, using the unicity of proposition 102 and proposition 53, we see that they are 2 functors on the category of quasi-projective complex algebraic varieties \( \text{Var}(\mathbb{C})^{\text{QP}} \).

(v) Let \( S \in \text{Var}(\mathbb{C}) \). Take an open cover \( S = \bigcup_{i=1}^{t} S_i \) such that there exists closed embeddings \( i_i : S_i \to \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). We have the functor

\[
((-) \otimes^H_{\text{St}} (-)) : C(MHM(S))^2 \to C(MHM(S)),
\]

\[
(((M_I, F, W), u_{IJ}), (K, W, \alpha), ((N_I, F, W), v_{IJ}), (K', W, \alpha')) \mapsto
(((M_I, F, W), u_{IJ}) \otimes^H_{\text{St}} (N_I, F, W), v_{IJ}), (K, W) \otimes (K', W), \alpha \otimes \alpha')
\]

where

\[
(((M, F, W), u_{IJ}), ((N, F, W), v_{IJ})) \mapsto ((M, F, W), u_{IJ}) \otimes^H_{\text{St}} (N, F, W), v_{IJ}) :=
\]

\[
\Delta^\text{mod}_{\text{St}}(p^\text{mod}_{SI} (M_I, F, W) \otimes \mathcal{O}_{\tilde{S}_i} \left|_{\tilde{S}_j} \right.) p^\text{mod}_{\text{St}}(N_I, F, W), p^\text{mod}_{\text{St}} u_{IJ} \otimes p^\text{mod}_{\text{St}} v_{IJ} :=
\]

\[
\Delta^\text{St}_{\text{mod}}(\Gamma^\text{mod}_{\text{St}} (p^\text{mod}_{SI} (M_I, F, W) \otimes \mathcal{O}_{\tilde{S}_i} \left|_{\tilde{S}_j} \right.) p^\text{mod}_{\text{St}}(N_I, F, W), p^\text{mod}_{\text{St}} u_{IJ} \otimes p^\text{mod}_{\text{St}} v_{IJ})
\]

and the map \( \alpha \otimes \alpha' \) is given in definition 112.

We have then the following

**Theorem 33.** Let \( f : X \to S \) be a morphism with \( X, S \in \text{Var}(\mathbb{C}) \), \( X \) quasi-projective. Then,

(i) \( (f^*_{\text{Hdg}}, f_*^{\text{Hdg}}) : D(MHM(S)) \to D(MHM(X)) \) is a pair of adjoint functors,

(ii) \( (f^*_{\text{Hdg}}, f_*^{\text{Hdg}}) : D(MHM(S)) \to D(MHM(X)) \) is a pair of adjoint functors.

**Proof.** For the projection case see section 4. For the open embedding see definition 107. \( \square \)

Definition 117 gives the following 2 functors:

- We have the following 2 functor on the category of complex algebraic varieties

\[
D(MHW(\cdot)) : \text{Var}(\mathbb{C}) \to \text{TriCat}, \ S \mapsto D(MHW(S)),
\]

\[
(f : T \to S) \mapsto (f^*_{\text{Hdg}} : ((M, F, W), (K, W), \alpha) \mapsto
f^*_{\text{Hdg}}((M, F, W), (K, W), \alpha) := (f^*_{\text{mod}}(M, F, W), f^{*w}(K, W), f^*(\alpha))).
\]

- We have the following 2 functor on the category of complex quasi-projective algebraic varieties

\[
D(MHW(\cdot)) : \text{QPVar}(\mathbb{C}) \to \text{TriCat}, \ S \mapsto D(MHW(S)),
\]

\[
(f : T \to S) \mapsto (f^*_{\text{Hdg}} : ((M, F, W), (K, W), \alpha) \mapsto
f^*_{\text{Hdg}}((M, F, W), (K, W), \alpha) := (Rf^*_{\text{Hdg}}(M, F, W), Rf^{*w}(K, W), f^*(\alpha))).
\]

- We have the following 2 functor on the category of complex quasi-projective algebraic varieties

\[
D(MHW(\cdot)) : \text{QPVar}(\mathbb{C}) \to \text{TriCat}, \ S \mapsto D(MHW(S)),
\]

\[
(f : T \to S) \mapsto (f^*_{\text{Hdg}} : ((M, F, W), (K, W), \alpha) \mapsto
f^*_{\text{Hdg}}((M, F, W), (K, W), \alpha) := (Rf^*_{\text{Hdg}}(M, F, W), Rf^{*w}(K, W), f^*(\alpha))).
\]

- We have the following 2 functor on the category of complex quasi-projective algebraic varieties

\[
D(MHW(\cdot)) : \text{Var}(\mathbb{C}) \to \text{TriCat}, \ S \mapsto D(MHW(S)),
\]

\[
(f : T \to S) \mapsto (f^*_{\text{Hdg}} : ((M, F, W), (K, W), \alpha) \mapsto
f^*_{\text{Hdg}}((M, F, W), (K, W), \alpha) := (f^*_{\text{mod}}(M, F, W), f_{\text{Hdg}}^w(K, W), f^*(\alpha))).
\]

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For a commutative diagram in \(\var{C}\)

\[
\begin{array}{ccc}
\ & X & \\
\text{g} & f & \text{S} \\
\text{g'} & \text{g} & \\
X' & \text{S} & \\
\end{array}
\]

with \(S, T, X', X\) quasi-projective, we have, for \(((M, F, W), (K, W), \alpha) \in \var{MHM}(X)\) using theorem 33, the following transformations maps

\[
\begin{align*}
T_1^{Hdg}(D)((M, F, W), (K, W), \alpha) : & \\
g^*h^df^*_h^dg^*((M, F, W), (K, W), \alpha) & \xrightarrow{\text{ad}(f^*h^df^*_h^dg^*)} f^*_h^dg^*(M, F, W), (K, W), \alpha) \\
& \xrightarrow{\text{ad}(f^*h^df^*_h^dg^*)} f^*_h^dg^*(M, F, W), (K, W), \alpha)
\end{align*}
\]

and

\[
\begin{align*}
T_2^{Hdg}(D)((M, F, W), (K, W), \alpha) : & \\
f^*_h^dg^*(M, F, W), (K, W), \alpha) & \xrightarrow{\text{ad}(f^*h^df^*_h^dg^*)} f^*_h^dg^*(M, F, W), (K, W), \alpha) \\
& \xrightarrow{\text{ad}(f^*h^df^*_h^dg^*)} g^*(M, F, W), (K, W), \alpha)
\end{align*}
\]

One consequence of the unicity of proposition 102 is the following:

**Proposition 103.** For a commutative diagram in \(\var{C}\)

\[
\begin{array}{ccc}
\ & X & \\
\text{g} & f & \text{S} \\
\text{g'} & \text{g} & \\
X_T & \text{S} & \\
\end{array}
\]

which is cartesian, with \(S, T, X', X\) quasi-projective and \(f\) (hence \(f'\) proper), and \(((M, F, W), (K, W), \alpha) \in \var{MHM}(X)\)

\[
T_1^{Hdg}(f, g) : ((M, F, W), (K, W), \alpha) : \\
g^*h^df^*_h^dg^*((M, F, W), (K, W), \alpha) \xrightarrow{\sim} f^*_h^dg^*(M, F, W), (K, W), \alpha)
\]

is an isomorphism.

**Proof.** See [28].

We have the following proposition

**Proposition 104.** Let \(Y \in \var{PSmVar}(\var{C})\) and \(i : Z \hookrightarrow S\) a closed embedding with \(Z\) smooth. Denote by \(j : U := S\backslash Z \hookrightarrow Y\) the complementary open subset.

(i) We have

\[
\begin{align*}
\text{ad}_{U^{Hdg}}(Z_U^{Hdg}) : & = a_{U^{Hdg}}((O_U, F_b), Z_U^{\text{un}}, \alpha(U)) \xrightarrow{\sim} (\int_{\partial Y}^{FDR} j_{U^{Hdg}}(O_U, F_b), (R_{U^{Hdg}}Z_U^{\text{un}}, W), a_U, \alpha(U)) \\
& \xrightarrow{\sim} (\int_{\partial Y}^{FDR} \text{ad}(i_{\text{mod}}, i^*)\alpha((O_Y, F_{b}), (R_{U^{Hdg}}Z_U^{\text{un}}, W), a_{U^{Hdg}}(U))) \\
& \xrightarrow{\sim} (\text{Cone}(E(\Omega_{D/Y}))(D) : \Gamma(Y, E(\Omega_{D/Y}^*, F_{b})) \rightarrow \Gamma(Z, E(\Omega_{D/Y}^*, F_{b})), W), (R_{U^{Hdg}}Z_U^{\text{un}}, W), a_{U^{Hdg}}(U))
\end{align*}
\]

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(ii) We have

\[ a_{U,Hdg}Z_U^{Hdg} := a_{U,Hdg,\ast}((O_U,F),Z_{U^\ast,n},\alpha(U)) \xrightarrow{\sim} (\int_{av}^{FDR} j_{\ast,Hdg}(O_U,F,W), (Ra_U,Z_{U^\ast,n},W), a_U,\alpha(U)) \]

Moreover we have (see [12]) canonical products

\[ a_{Hdg,U}Z_U^{Hdg} \xrightarrow{\sim} (\Gamma(Y,E(\Omega^\ast_Y,\log D),F,W)), (Ra_U,Z_{U^\ast,n},W), a_U,\alpha(U)) \]

and

\[ a_{Hdg,U}Z_U^{Hdg} := (\Gamma(Y,E(\Omega^\ast_Y(\text{null } D),F,W)), (Ra_U,Z_{U^\ast,n},W), a_U,\alpha(U)) \]

We recall the definition of the Deligne complex of a complex manifold and the Deligne cohomology class of an algebraic cycle of a complex algebraic variety.

**Definition 118.** (i) Let \( X \in \text{AnSm}(\mathbb{C}) \). We have for \( d \in \mathbb{Z} \) the Deligne complex

\[ Z_{D,X}(d) := (Z_X(d) \to \tau^{\leq d}DR(X)) = (Z(d) \to (O_X \to \cdots \to \Omega_X^{d-1}) \in C(X) \]

Let \( D \subset X \) a normal crossing divisor. We have for \( d \in \mathbb{Z} \) the Deligne complexes

\[ Z_{D,(X,D)}(d) := (Z_X(d) \to (O_X \to \cdots \to \Omega_X^{d-1}(\log D))) \in C(X) \]

Moreover we have (see [12]) canonical products

\[ (-) \cdot (-) : Z_{D,(X,D)}(d) \otimes Z_{D,(X,D)}(d') \to Z_{D,(X,D)}(d+d') \]

\[ (-) \cdot (-) : Z_{D,(X,D)}(d) \ast \otimes Z_{D,(X,D)}(d') \ast \to Z_{D,(X,D)}(d+d') \ast \]

(ii) Let \( X \in \text{AnSm}(\mathbb{C}) \). We have for \( d \in \mathbb{Z} \) the Deligne complex

\[ C^\bullet(X,Z(d)) := \text{Cone}(Z \text{Hom}_{Diff}(\mathbb{R})(\Delta^\bullet, X) \oplus \Gamma(X,F^dD_X) \to \Gamma(X,D_X)) \in C(\mathbb{C}) \]

Let \( D \subset X \) a normal crossing divisor. Denote \( U := X \setminus D \). We have for \( d \in \mathbb{Z} \) the Deligne complexes

\[ C^\bullet((X,D),Z(d)) := \text{Cone}(Z \text{Hom}_{Diff}(\mathbb{R})(\Delta^\bullet, U) \oplus \Gamma(X,F^dD_X(\log D)) \to \Gamma(X,D_X(\log D))) \in C(\mathbb{C}) \]

and

\[ C^\bullet(X,D,Z(d)) := \text{Cone}(Z \text{Hom}_{Diff}(\mathbb{R})(\Delta^\bullet, (X,D)) \oplus \Gamma(X,F^dD_X(D)) \to \Gamma(X,D_X(D))) \in C(\mathbb{C}) \]

(iii) Let \( X \in \text{PSmVar}(\mathbb{C}) \). We have, for \( k \in \mathbb{Z} \) and \( d \in \mathbb{Z} \), the Deligne cohomology

\[ H^k_D(X^{an},Z(d)) := H^k(X^{an},Z_{X,D}(d)) = H^kC^\bullet_D(X^{an},D,Z(d)) \]

Let \( U \in \text{SmVar}(\mathbb{C}) \). Let \( X \in \text{PSmVar}(\mathbb{C}) \) a compactification of \( U \) with \( D := X \setminus U \) a normal crossing divisor. We have, for \( k \in \mathbb{Z} \) and \( d \in \mathbb{Z} \), the Deligne cohomology

\[ H^k_D(U^{an},Z(d)) := H^k(X,Z(X^{an},D^{an}),D(d)) = H^kC^\bullet_D((X^{an},D^{an}),Z(d)) \]

and

\[ H^k_D(X,D,Z(d)) := H^k(X^{an},Z_{(X^{an},D^{an}),D}(d)) = H^kC^\bullet_D(X^{an},D^{an},Z(d)) \]

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(iv) Let $U \in \text{SmVar}(\mathbb{C})$. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of $U$ with $D := X \setminus U$ a normal crossing divisor. We define the Deligne cohomology of a (higher) cycle $Z \in Z^d(U, n)^{\partial = 0}$ by

$$[Z]_{\mathcal{D}} := \text{Im}(H^{2d-n}(\gamma_{\text{supp}(Z)})([Z])), \quad H^k(\gamma_{\text{supp}(Z)}): \mathbb{H}^{2d-n}_{\text{supp}(Z)}(X^\text{an}, \mathbb{Z}_{X^\text{an}, D^\text{an}}(d)) \to \mathbb{H}^{2d-n}(X^\text{an}, \mathbb{Z}_{X^\text{an}, D^\text{an}}(d))$$

with $\text{supp}(Z) := p_X(\text{supp}(Z)) \subset X$, where $\text{supp}(Z) \subset X \times \Box^n$ is the support of $Z$.

(v) Let $U \in \text{SmVar}(\mathbb{C})$. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of $U$ with $D := X \setminus U$ a normal crossing divisor. We have for $d \in Z$ the morphism of complexes

$$\mathcal{R}_d^U : Z^d(U, \bullet) \to C^d_U(X^\text{an}, D^{an}, \mathbb{Z}(d)), \quad Z \mapsto \mathcal{R}_U(d)(Z) := (T_Z, \Omega_Z, R_Z)$$

which gives for $Z \in Z^d(U, n)^{\partial = 0}$,

$$[\mathcal{R}_U^d(Z)] = [Z]_\mathcal{D} \in H^{2d-n}(U^\text{an}, \mathbb{Z}(d))$$

\textbf{Theorem 34.} \quad (i) Let $U \in \text{SmVar}(\mathbb{C})$. Denote by $a_U : U \to \text{pt}$ the terminal map. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of $U$ with $D := X \setminus U$ a normal crossing divisor. The embedding (see theorem 32)

$$\iota : D(MHM(\{\text{pt}\})) \to D_{fU}(\mathbb{C}) \times_i D(Z)$$

induces for $k \in Z$ and $d \in Z$, canonical isomorphisms

$$\iota(a_U \star H_{d} : Z) : H^k(a_U \star H_{d}, \mathbb{Z}(d)) \cong H^k(X^\text{an}, D^{an}, \mathbb{Z}(d)), \quad \text{and} \quad \iota(a_U \star H_{d} : Z) : H^k(a_U \star H_{d}, \mathbb{Z}(d)) \cong H^k(Z^d(U, n), \mathbb{Z}(d)).$$

(ii) Let $h : U \to S$ and $h' : U' \to S$ two morphism with $S, U, U' \in \text{SmVar}(\mathbb{C})$. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of $U$ with $D := X \setminus U$ a normal crossing divisor such that $h : U \to S$ extend to $f : X \to \bar{S}$. Let $X' \in \text{PSmVar}(\mathbb{C})$ a compactification of $U'$ with $D' := X' \setminus U'$ a normal crossing divisor such that $h' : U' \to S$ extend to $f' : X' \to \bar{S}$. The embedding $\iota : D(MHM(\{\text{pt}\})) \to D_{fU}(k) \times_i D(Z)$ (see theorem 32) induces for $k \in Z$ and $d \in Z$ a canonical isomorphism

$$\iota(a_U \times_s H_{d} \mathbb{Z}(d)) : \text{Hom}_{D(MHM'(S))}(h_U \star H_{d}, \mathbb{Z}(d)), \quad \text{and} \quad \iota(a_U \times_s H_{d} \mathbb{Z}(d)) : \text{Hom}_{D(MHM_{s, U}(\text{pt}))}(Z^d, (h_U \star H_{d}, \mathbb{Z}(d))(\mathbb{Z}(d)))$$

$$(Z \times_{S} X)^{\text{an}}, \quad ((X' \times_{S} X) \cup (U' \times_{S} X))^{\text{an}}, \quad \mathbb{Z}(d)).$$

(iii) Let $U \in \text{SmVar}(\mathbb{C})$. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of $U$ with $D := X \setminus U$ a normal crossing divisor. For $[Z] \in CH^d(U, n)$ and $[Z'] \in CH^d(U, n')$, we have

$$([Z] \cdot [Z'])_{\mathcal{D}} = [Z]_{\mathcal{D}} \cdot [Z']_{\mathcal{D}} \in H^{2d+2d'-n-n'}(U^{an}, \mathbb{Z}(d + d'))$$

where the product on the left is the intersection of higher Chow cycle which is well defined modulo boundary (they intersect properly modulo boundary) while the right product of Deligne cohomology classes is induced by the product of Deligne complexes $(-) \cdot (-) : Z_{D, (X, D)}(d) \otimes Z_{D, (X, D)}(d') \to Z_{D, (X, D)}(d + d')$.

(iv) Let $h : U \to S, h' : U' \to S, h'' : U'' \to S$ three morphism with $S, U, U', U'' \in \text{SmVar}(\mathbb{C})$. Let $X \in \text{PSmVar}(\mathbb{C})$ a compactification of $U$ with $D := X \setminus U$ a normal crossing divisor such that $h : U \to S$ extend to $f : X \to \bar{S}$. Let $X' \in \text{PSmVar}(\mathbb{C})$ a compactification of $U'$ with $D' := X' \setminus U'$ a normal crossing divisor such that $h' : U' \to S$ extend to $f' : X' \to \bar{S}$. Let $X' \in \text{PSmVar}(\mathbb{C})$ a
compactification of \( U' \) with \( D' := X'\backslash U' \) a normal crossing divisor such that \( h' : U' \to S \) extend to \( f' : X' \to \tilde{S} \). For \( [Z] \in \text{CH}^d(U \times_S U', n) \) and \( [Z'] \in \text{CH}^d(U' \times_S U'', n') \), we have

\[
([Z] \circ [Z'])_P = [Z]_P \circ [Z']_P \in H^{d' - n''}(U \times_S U'', [Z]_P \circ [Z']_P)
\]

where the composition on the left is the composition of higher correspondence modulo boundary while the composition on the right is given by (ii).

**Proof.** (i): Standard.
(ii): Follows on the one hand from (i) and on the other hand the six functor formalism on the 2-functor \( D\text{-MHM}(-) : \text{SmVar}(\mathbb{C}) \to \text{TriCat} \) gives the isomorphism \( R\text{I}(-,-) \).
(iii): Standard.
(iv): Follows from (iii).

\( \square \)

6 The algebraic and analytic filtered De Rham realizations for Voevodsky relative motives

6.1 The algebraic filtered De Rham realization functor

6.1.1 The algebraic Gauss-Manin filtered De Rham realization functor and its transformation map with pullbacks

Consider, for \( S \in \text{Var}(\mathbb{C}) \), the following composition of morphism in RCat (see section 2)

\[
\tilde{e}(S) : (\text{Var}(\mathbb{C})/S, O_{\text{Var}(\mathbb{C})/S}) \xrightarrow{\rho_S} (\text{Var}(\mathbb{C})^{sm}/S, O_{\text{Var}(\mathbb{C})^{sm}/S}) \xrightarrow{\tilde{e}(S)} (S, O_S)
\]

with, for \( X/S = (X, h) \in \text{Var}(\mathbb{C})/S \),

- \( O_{\text{Var}(\mathbb{C})/S}(X/S) := O_X(X) \),
- \( (\tilde{e}(S))^* O_S(X/S) \to O_{\text{Var}(\mathbb{C})/S}(X/S)) := (h^* O_S \to O_X) \).

and \( O_{\text{Var}(\mathbb{C})^{sm}/S} := \rho_S, O_{\text{Var}(\mathbb{C})/S} \), that is, for \( U/S = (U, h) \in \text{Var}(\mathbb{C})^{sm}/S \), \( O_{\text{Var}(\mathbb{C})^{sm}/S}(U/S) := O_{\text{Var}(\mathbb{C})/S}(U/S) := O_U(U) \)

**Definition 119.** (i) For \( S \in \text{Var}(\mathbb{C}) \), we consider the complexes of presheaves

\[
\Omega^s_{/S} := \text{coker}(\Omega^*_{O_{\text{Var}(\mathbb{C})^{sm}/S}} \circ O_S : \Omega^*_{\tilde{e}(S)^* O_S} \to \Omega^*_{O_{\text{Var}(\mathbb{C})/S}}) \in \mathcal{C}_{O_S}(\text{Var}(\mathbb{C})/S)
\]

which is by definition given by

- for \( X/S \) a morphism \( \Omega^s_{/S}(X/S) = \Omega^s_{X/S}(X) \)
- for \( g : X'/S \to X/S \) a morphism,

\[
\Omega^s_{/S}(g) := \Omega^s_{X'/S/(S/S)}(X') : \Omega^*_{X/S}(X) \to g^* \Omega^*_{X/S}(X) \to \Omega^s_{X/S}(X')
\]

\( \omega \mapsto \Omega^s_{X'/S/(S/S)}(X') (\omega) := g^*(\omega) : (\alpha \in \wedge^k T_X(X') \mapsto \omega(dg(\alpha))) \)

(ii) For \( S \in \text{Var}(\mathbb{C}) \), we consider the complexes of presheaves

\[
\Omega^s_{/S} := \rho_S, \Omega^s_{/S} = \text{coker}(\Omega^*_{O_{\text{Var}(\mathbb{C})^{sm}/S}} \circ O_S : \Omega^*_{\tilde{e}(S)^* O_S} \to \Omega^*_{O_{\text{Var}(\mathbb{C})^{sm}/S}}) \in \mathcal{C}_{O_S}(\text{Var}(\mathbb{C})^{sm}/S)
\]

which is by definition given by

- for \( U/S \) a smooth morphism \( \Omega^s_{/S}(U/S) = \Omega^s_{U/S}(U) \)
– for \( g : U' / S \to U / S \) a morphism,

\[
\Omega^*_S(g) := \Omega_{(U' / U) / (S / S)}(U') : \Omega^*_{U / S}(U) \to g^* \Omega_{U / S}(U') \to \Omega^*_{U' / S}(U')
\]

\[
\omega \mapsto \Omega_{(U' / U) / (S / S)}(U') (\omega) := g^* (\omega) : (\alpha, h) \to \omega (dg(\alpha))
\]

**Remark 10.** For \( S \in \text{Var}(\mathbb{C}) \), \( \Omega^*_S \in C(\text{Var}(\mathbb{C}) / S) \) is by definition a natural extension of \( \Omega^*_S \in C(\text{Var}(\mathbb{C})^m / S) \). However \( \Omega^*_S \) does NOT satisfy cdh descent.

For a smooth morphism \( h : U \to S \) with \( S, U \in \text{SmVar}(\mathbb{C}) \), the cohomology presheaves \( H^n \Omega^*_U \) of the relative De Rham complex

\[
DR(U / S) := \Omega^*_U := \text{coker}(h^* \Omega_S \to \Omega_U) \in C_{h^* \mathcal{O}_S}(U)
\]

for all \( n \in \mathbb{Z} \), have a canonical structure of a complex of \( h^* \mathcal{D}_S \) modules given by the Gauss Manin connexion : for \( \gamma \in \Gamma(S^0, T_S) \) a vector field and \( \tilde{\omega} \in \Omega_U^p(U^o)^c \) a closed form, the action is given by

\[
\gamma \cdot [\omega] = [\tilde{\gamma} \tilde{\omega}],
\]

\( \omega \in \Omega_U^p(U^o) \) being a representative of \( \omega \) and \( \tilde{\gamma} \in \Gamma(U^o, T_U) \) a relevement of \( \gamma \) (\( h \) is a smooth morphism), so that

\[
DR(U / S) := \Omega^*_U := \text{coker}(h^* \Omega_S \to \Omega_U) \in C_{h^* \mathcal{O}_S, h^* \mathcal{D}}(U)
\]

with this \( h^* \mathcal{D}_S \) structure. Hence we get \( h^* \Omega^*_U \in C_{\mathcal{O}_u, \mathcal{D}}(S) \) considering this structure. Since \( h \) is a smooth morphism, \( \Omega^*_U / \mathcal{D} \) are locally free \( \mathcal{O}_U \) modules.

The point (ii) of the definition 129 above gives the object in \( \mathcal{D} \text{A}(S) \) which will, for \( S \) smooth, represent the algebraic Gauss-Manin De Rham realisation. It is the class of an explicit complex of presheaves on \( \text{Var}(\mathbb{C})^m / S \).

**Proposition 105.** Let \( S \in \text{Var}(\mathbb{C}) \).

(i) For \( U / S = (U, h) \in \text{Var}(\mathbb{C})^m / S \), we have \( e(U) h^* \Omega^*_S = \Omega^*_U / \mathcal{D} \).

(ii) The complex of presheaves \( \Omega^*_S \in C_{\mathcal{O}_u}(\text{Var}(\mathbb{C})^m / S) \) is \( \mathbb{A}^1 \) homotopic, in particular \( \mathbb{A}^1 \) invariant. Note that however, for \( p > 0 \), the complexes of presheaves \( \Omega^*_S^p \) are NOT \( \mathbb{A}^1 \) local. On the other hand, \( (\Omega^*_S, F_b) \) admits transfers (recall that means \( \text{Tr}(S), \text{Tr}(S)^* \Omega^*_S = \Omega^*_S^p \)).

(iii) If \( S \) is smooth, we get \( (\Omega^*_S, F_b) \in C_{\mathcal{O}_u, \mathcal{D}_S}(\text{Var}(\mathbb{C})^m / S) \) with the structure given by the Gauss Manin connexion. Note that however the \( \mathcal{D}_S \) structure on the cohomology groups given by Gauss Main connexion does NOT comes from a structure of \( \mathcal{D}_S \) module structure on the filtered complex of \( \mathcal{O}_S \) module. The \( \mathcal{D}_S \) structure on the cohomology groups satisfy a non trivial Griffiths transversality (in the non projection cases), whereas the filtration on the complex is the trivial one.

*Proof. (i):* Let \( h' : V \to U \) a smooth morphism with \( V \in \text{Var}(\mathbb{C}) \). We have then

\[
h^* \Omega^*_U (V \xrightarrow{h'} U) = \Omega^*_V (V \xrightarrow{h'} U \xrightarrow{h} S).
\]

Hence, if \( h' : V \to U \) is in particular an open embedding, \( h^* \Omega^*_U (V \xrightarrow{h'} U) = \Omega^*_U / \mathcal{D} \). This proves the equality.

(ii): We prove that \( E_{et}(\Omega^*_S, F_b) \in C_{\mathcal{O}_u, \mathcal{D}_S}(\text{Var}(\mathbb{C})^m / S) \) is 2-filtered \( \mathbb{A}^1 \) invariant. We follow [21]. Consider the map in \( C(\text{Var}(\mathbb{C})^m / S) \)

\[
\phi := \text{ad}(p_a, p_{a^*})(-) : \Omega^*_S \to p_a^* p_{a^*} \Omega^*_S
\]

which is given, for \( U / S \in \text{Var}(\mathbb{C})^m / S \) by

\[
\text{ad}(p_a, p_{a^*})(-) (U / S) = \Omega_{(U \times \mathbb{A}^1 / U) / (S / S)}(U \times \mathbb{A}^1) : \Omega^*_U (U) \to \Omega^*_U / \mathbb{A}^1 (U \times \mathbb{A}^1), \omega \mapsto p^* \omega
\]

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where \( p : U \times \mathbb{A}^1 \to U \) is the projection. On the other hand consider the map in \( C(\text{Var}(\mathbb{C})^{sm}/S) \)

\[
\psi := I_0^* : p_{a*}p_a^* \Omega^*_{U/S} \to \Omega^*_{U/S}
\]
given, for \( U/S \in \text{Var}(\mathbb{C})^{sm}/S \) by

\[
I_0^*(U/S) : \Omega^*_{U \times \mathbb{A}^1/S}(U \times \mathbb{A}^1) \to \Omega^*_{U/S}(U), \ \omega \mapsto i_0^* \omega
\]

where \( i_0 : U \hookrightarrow U \times \mathbb{A}^1 \) is closed embedding given by \( i_0(x) := (x, 0) \). Then,

- we have \( \phi \circ \psi = I \)
- considering the map in \( \text{PSh}( \mathbb{N} \times \text{Var}(\mathbb{C})^{sm}/S) \)

\[
H : p_{a*}p_a^* \Omega^*_{U/S}[1] \to p_{a*}p_a^* \Omega^*_{U/S}
\]
given for \( U/S \in \text{Var}(\mathbb{C})^{sm}/S \) by

\[
H(U/S)\Omega^p_{U \times \mathbb{A}^1/S}(U \times \mathbb{A}^1) \to \Omega^{p-1}_{U \times \mathbb{A}^1/S}(U \times \mathbb{A}^1),
\]

\[
H(U/S)(p^* \omega \wedge q^*(f(s)ds)) = (\int_0^t f(s)ds)p^* \omega, \ H(U/S)(p^* \omega \wedge q^* f) = 0,
\]

note that \( g(t) = \int_0^t f(s)ds \) is algebraic since \( f \in O_{\mathbb{A}^1}(\mathbb{A}^1) \) is a polynomial, we have \( \psi \circ \phi - I = \partial H + H \partial \).

This shows that

\[
\text{ad}(p_{a*}, p_{a*})(-) : \Omega^*_{U/S} \to p_{a*}p_a^* \Omega^*_{U/S}
\]
is an homotopy equivalence whose inverse is \( I_0^* \). Hence,

\[
\text{ad}(p_{a*}, p_{a*})(-) : (\Omega^*_{U/S}, F_b) \to p_{a*}p_a^*(\Omega^*_{U/S}, F_b)
\]
is a 2-filtered homotopy equivalence whose inverse is

\[
I_0^* : p_{a*}p_a^*(\Omega^*_{U/S}, F_b) \to (\Omega^*_{U/S}, F_b).
\]

(iii): For \( h : U \to S \) a smooth morphism with \( U, S \in \text{SmVar}(\mathbb{C}) \), recall that the \( h^* D_S(U) = D_S(h(U)) \) structure on \( H^p\Omega^*_{U/S}(U/S) := H^p\Omega^*_{U/S}(U) \) is given by, for \( \tilde{\omega} \in \Omega^p_{U/S}(U)^e \), \( \gamma \cdot [\tilde{\omega}] = [\hat{\gamma} \bar{\gamma} \partial \omega], \ \omega \in \Omega^p_{U/S}(U^o) \) being a representative of \( \tilde{\omega} \) and \( \hat{\gamma} \in \Gamma(U^o, T_{U^o}) \) a relevevant of \( \bar{\gamma} \) (\( h \) is a smooth morphism). Now, if \( g : V/S \to U/S \) is a morphism, where \( h' : V \to S \) is a smooth morphism with \( V \in \text{SmVar}(\mathbb{C}) \), we have

\[
g^*(\gamma \cdot \tilde{\omega}) = g^*(\hat{\gamma} \bar{\gamma} \partial \omega) = \hat{\gamma} \bar{\gamma} g^* \omega = \gamma \cdot (g^* \tilde{\omega})
\]

that is \( H^p\Omega^*_{U/S}(g) : H^p\Omega^*_{U/S} \to H^p\Omega^*_{V/S} \) is a map of \( D_S(h(U)) \) modules. \( \square \)

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let \( g : T \to S \) a morphism with \( T, S \in \text{Var}(\mathbb{C}) \). Consider the following commutative diagram in \( \text{RCat} \):

\[
\begin{array}{ccc}
D(g, e) : (\text{Var}(\mathbb{C})^{sm}/T, O_{\text{Var}(\mathbb{C})^{sm}/T}) & \xrightarrow{P(g)} & (\text{Var}(\mathbb{C})^{sm}/S, O_{\text{Var}(\mathbb{C})^{sm}/S}) \\
| & & | \\
\varepsilon(T) & \xrightarrow{\varepsilon(T)} & \varepsilon(S) \\
(T, O_T) & \xrightarrow{P(g)} & (S, O_S)
\end{array}
\]

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It gives (see section 2) the canonical morphism in \( C_{g^*O_S}f_{U}(\text{Var}(\mathbb{C})_{sm}/T) \)

\[
\Omega_\text{/(T/S)} := \Omega_{(\text{Var}(\mathbb{C})_{sm}/T/g^*\text{Var}(\mathbb{C})_{sm}/S)}(O_T/g^*O_S)
\]

\[
g^*(\Omega^*_S,F_b) = \Omega^*_0\text{Var}(\mathbb{C})_{sm}/S/g^*eS \rightarrow (\Omega^*_T,F_b) = \Omega^*_0\text{Var}(\mathbb{C})_{sm}/T/e(T)\cdot O_T
\]

which is by definition given by the pullback on differential forms: for \((V/T) = (V,h) \in \text{Var}(\mathbb{C})_{sm}/T,\)

\[
\Omega_\text{/(T/S)}(V/T) := g^*(\Omega^*_S)(V/T) := \lim_{(b':U\rightarrow S,sm,g':V\rightarrow U,h,g)} \Omega^*_U(U) \xrightarrow{\Omega(V/U)/(T/S)(V/T)} \Omega^*_V(V) =: \Omega^*_T(V/T)
\]

\[
\dot{\omega} \rightarrow \Omega(V/U)/(T/S)(V/T)(\omega) := g^*\dot{\omega}.
\]

If \(S\) and \(T\) are smooth, \(\Omega_\text{/(T/S)} : g^*(\Omega^*_S,F_b) \rightarrow (\Omega^*_T,F_b)\) is a map in \( C_{g^*O_S}f_{U,g^*D_S}(\text{Var}(\mathbb{C})_{sm}/T) \) It induces the canonical morphisms in \( C_{g^*O_S}f_{U,g^*D_S}(\text{Var}(\mathbb{C})_{sm}/T) \):

\[
E\Omega_\text{/(T/S)} : g^*E_{ct}(\Omega^*_S,F_b) \xrightarrow{T(g,E_{ct})(\Omega^*_S,F_b)} E_{ct}(g^*(\Omega^*_S,F_b)) \xrightarrow{E_{ct}(\Omega_\text{/(T/S)})} E_{ct}(\Omega^*_T,F_b).
\]

and

\[
E\Omega_\text{/(T/S)} : g^*E_{zar}(\Omega^*_S,F_b) \xrightarrow{T(g,E_{zar})(\Omega^*_S,F_b)} E_{zar}(g^*(\Omega^*_S,F_b)) \xrightarrow{E_{zar}(\Omega_\text{/(T/S)})} E_{zar}(\Omega^*_T,F_b).
\]

**Definition 120.** (i) Let \( g : T \rightarrow S \) a morphism with \( T,S \in \text{Var}(\mathbb{C}) \). We have, for \( F \in C(\text{Var}(\mathbb{C})_{sm}/S) \), the canonical transformation in \( C_{O_Tf_{U}}(T) \):

\[
T^O(g,\Omega_\text{/(T/S)})(F) : g^{*\text{mod}}\text{LO}\text{e}(S),\text{Hom}^*(F,E_{ct}(\Omega^*_S,F_b)) \xrightarrow{\text{mod}} g^{*\text{Loe}(S),\text{Hom}^*(F,E_{ct}(\Omega^*_S,F_b))} \otimes g^*O_S OT
\]

\[
\xrightarrow{T(e,g)(\cdot)\circ T(g,\text{Lo}(\cdot))} \xrightarrow{L_0(e(T),g^*\text{Hom}^*(F,E_{ct}(\Omega^*_S,F))} \otimes g^*O_S OT
\]

\[
\xrightarrow{T(e,\text{hom})(F,E_{ct}(\Omega^*_S,F))} \text{lo}(e(T),\text{Hom}^*(g^*F,g^*E_{ct}(\Omega^*_S,F_b))} \otimes g^*O_S OT
\]

\[
\xrightarrow{\text{ev}(\cdot)\circ m} \text{lo}(e(T),\text{Hom}^*(g^*F,E_{ct}(\Omega^*_T,F_b))} \otimes g^*O_S OT
\]

\[
\xrightarrow{m} \text{lo}(e(T),\text{Hom}^*(g^*F,E_{ct}(\Omega^*_T,F_b))} \otimes g^*O_S OT
\]

where \( m(\alpha \otimes h) := h.\alpha \) is the multiplication map.

(ii) Let \( g : T \rightarrow S \) a morphism with \( T,S \in \text{Var}(\mathbb{C}) \), \( S \) smooth. Assume there is a factorization \( g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. We have, for \( F \in C(\text{Var}(\mathbb{C})_{sm}/S) \), the canonical transformation in \( C_{O_Tf_{U}}(Y \times S) \):

\[
T(g,\Omega_\text{/(T/S)})(F) : g^{*\text{mod}}\text{e}(S),\text{Hom}^*(F,E_{ct}(\Omega^*_S,F_b)) \xrightarrow{\text{mod}} \Gamma_T E_{zar}(p_S^{*\text{mod}}\text{e}(S),\text{Hom}^*(F,E_{ct}(\Omega^*_S,F_b)))
\]

\[
\xrightarrow{T^O(p_S,\Omega_\text{/(T/S)})(F)} \Gamma_T E_{zar}(e(T \times S),\text{Hom}^*(p_S^{*}F,E_{ct}(\Omega^*_Y \times S,F_b)))
\]

\[
\xrightarrow{\text{mod}} e(T \times S) \cdot \Gamma_T (\text{Hom}^*(p_S^{*}F,E_{ct}(\Omega^*_Y \times S,F_b)))
\]

\[
\xrightarrow{L(\gamma,\text{hom})(\cdot,\cdot)} e(T \times S) \cdot \text{Hom}^*(T^0p_S^{*}F,E_{ct}(\Omega^*_Y \times S,F_b)).
\]

For \( Q \in \text{Proj PSh}(\text{Var}(\mathbb{C})_{sm}/S) \),

\[
T(g,\Omega_\text{/(T/S)})(Q) : g^{*\text{mod}}\text{e}(S),\text{Hom}^*(Q,E_{ct}(\Omega^*_S,F_b)) \rightarrow e(T \times S),\text{Hom}^*(T^0p_S^{*}Q,E_{ct}(\Omega^*_Y \times S,F_b))
\]

is a map in \( C_{O_Tf_{U}}(Y \times S) \).
The following easy lemma describe these transformation map on representable presheaves:

**Lemma 7.** Let \( g : T \to S \) a morphism with \( T, S \in \text{Var}(\mathbb{C}) \) and \( h : U \to S \) is a smooth morphism with \( U \in \text{Var}(\mathbb{C}) \). Consider a commutative diagram whose square are cartesian:

\[
g : T \xrightarrow{l} S \times Y \xrightarrow{p_S} S \xleftarrow{h} \space \xrightarrow{h''=h \times I} \space U
\]

\[
g' : U_T \xrightarrow{l'} U \times Y \xrightarrow{p_U} U
\]

with \( l, l' \) the graph embeddings and \( p_S, p_U \) the projections. Then \( g^*Z(U/S) = Z(U_T/T) \) and

(i) we have the following commutative diagram in \( C_{\text{Orf}^d}(T) \) (see definition 1 and definition 120(i)) :

\[
g^*\mathcal{L}oc\mathcal{H}om^*(\mathbb{Z}(U/S), E_\text{et}(\Omega^*_U/F_b)) \xrightarrow{T(g,\Omega_J)(\mathbb{Z}(U/S))} e(T) \mathcal{H}om^*(\mathbb{Z}(U_T/T), E_\text{et}(\Omega^*_T/F_b))
\]

(ii) if \( Y, S \in \text{SmVar}(\mathbb{C}) \), we have the following commutative diagram in \( C_{\text{Orf}^d,D}(Y \times S) \) (see definition 1 and definition 120(ii)) :

\[
g^*\mathcal{L}oc\mathcal{H}om^*(\mathbb{Z}(U/S), E_\text{et}(\Omega^*_U/F_b)) \xrightarrow{T(g,\Omega_J)(\mathbb{Z}(U/S))} e(Y \times S) \mathcal{H}om^*(\Gamma^\vee_{U_T} \mathbb{Z}(U \times Y/S \times Y), E_{\text{et}}(\Omega^*_U\times Y/S\times Y))
\]

where \( j : T \setminus T \times S \to T \times S \) is the open complementary embedding.

with

\[
k : E_\text{et}(h^*\Omega^*_J/F_b) \to E_{\text{et}}(E_\text{et}(h^*\Omega^*_J/F_b)) = E_{\text{et}}(h^*\Omega^*_J/F_b).
\]

which is a filtered Zariski local equivalence.

**Proof.** The commutative diagram follows from Yoneda lemma and proposition 105(i). On the other hand, \( k : E_\text{et}(\Omega^*_J/F_b) \to E_{\text{et}}(\Omega^*_J/F_b) \) is a (1-)filtered Zariski local equivalence by theorem 10 and proposition 105(ii)

\[
T(p,\Omega_J)(Q) : p^*\mathcal{L}oc\mathcal{H}om^*(Q, E_\text{et}(\Omega^*_S/F_b)) \to e(S_1) \mathcal{H}om^*(p^*Q, E_{\text{et}}(\Omega^*_S/F_b))
\]

is an isomorphism.

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Proof. Follows from lemma 7 and base change by smooth morphisms of quasi-coherent sheaves. \qed

Let $S \in \text{Var}(\mathbb{C})$ and $h : U \to S$ a morphism with $U \in \text{Var}(\mathbb{C})$. We then have the canonical map given by the wedge product

$$w_{U/S} : \Omega_{U/S}^* \otimes_{O_S} \Omega_{U/S}^2 \to \Omega_{U/S}^2; \alpha \otimes \beta \mapsto \alpha \wedge \beta.$$ 

Let $S \in \text{Var}(\mathbb{C})$ and $h_1 : U_1 \to S$, $h_2 : U_2 \to S$ two morphisms with $U_1, U_2 \in \text{Var}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \to S$ and $p_{112} : U_1 \times_S U_2 \to U_1$, $p_{212} : U_1 \times_S U_2 \to U_2$ the projections. We then have the canonical map given by the wedge product

$$w_{(U_1,U_2)/S} : p_{112}^*\Omega^*_{U_1/S} \otimes_{O_S} p_{212}^*\Omega^*_{U_2/S} \to \Omega^*_{U_{12}/S}; \alpha \otimes \beta \mapsto p_{112}^*\alpha \wedge p_{212}^*\beta$$

which gives the map

$$Ew_{(U_1,U_2)/S} : h_{12}^*E_{zar}(\Omega^*_{U_{12}/S}) \otimes_{O_S} h_{22}^*E_{zar}(\Omega^*_{U_2/S}) \to E_{zar}(\Omega^*_{U_{12}/S})$$

Let $S \in \text{Var}(\mathbb{C})$. We have the canonical map in $C_{O_S,fil}(\text{Var}(\mathbb{C})_{et}/S)$

$$w_S : (\Omega^*_{S/F_b} \otimes_{O_S} \Omega^*_{S,F_b}) \to (\Omega^*_{S,F_b})$$

given by for $h : U \to S \in \text{Var}(\mathbb{C})_{et}/S$.

$$w_S(U/S) : (\Omega^*_{U/S,F_b} \otimes_{O_{S,U}} (\Omega^*_{U/S,F_b})(U) \xrightarrow{w_{U/S}(U)} (\Omega^*_{U/S,F_b})(U)$$

It gives the map

$$Ew_S : E_{et}(\Omega^*_{S,F_b} \otimes_{O_S} \Omega^*_{S,F_b}) \xrightarrow{\sim} E_{et}((\Omega^*_{S,F_b} \otimes_{O_S} (\Omega^*_{S,F_b})) \xrightarrow{E_{et}(w_S)} E_{et}(\Omega^*_{S,F_b})$$

If $S \in \text{SmVar}(\mathbb{C})$,

$$w_S : (\Omega^*_{S,F_b} \otimes_{O_S} \Omega^*_{S,F_b}) \to (\Omega^*_{S,F_b})$$

is a map in $C_{O_S,fil,D_S}(\text{Var}(\mathbb{C})_{et}/S)$.}

**Definition 121.** Let $S \in \text{Var}(\mathbb{C})$. We have, for $F,G \in C(\text{Var}(\mathbb{C})_{et}/S)$, the canonical transformation in $C_{O_S,fil}(S)$:

$$T(\otimes,\cdot)(F,G) : e(S)_*\text{Hom}(F,E_{et}(\Omega^*_{S,F_b})) \otimes_{O_S} e(S)_*\text{Hom}(G,E_{et}(\Omega^*_{S,F_b})) \xrightarrow{\sim} e(S)_*\text{Hom}(F \otimes G,E_{et}(\Omega^*_{S,F_b})) \xrightarrow{\text{Hom}(F \otimes G,\text{Ew}_S)} e(S)_*\text{Hom}(F \otimes G,E_{et}(\Omega^*_{S,F_b}))$$

If $S \in \text{SmVar}(\mathbb{C})$, $T(\otimes,\cdot)(F,G)$ is a map in $C_{O_S,fil,D_S}(S)$.

**Lemma 8.** Let $S \in \text{Var}(\mathbb{C})$ and $h_1 : U_1 \to S$, $h_2 : U_2 \to S$ two smooth morphisms with $U_1, U_2 \in \text{Var}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \to S$ and $p_{112} : U_1 \times_S U_2 \to U_1$, $p_{212} : U_1 \times_S U_2 \to U_2$ the projections. We then have the following commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
\text{e}(S)_*\text{Hom}(F,E_{et}(\Omega^*_{S,F_b})) & \otimes_{O_S} & \text{e}(S)_*\text{Hom}(G,E_{et}(\Omega^*_{S,F_b})) \\
\downarrow k & & \downarrow k \\
\text{h}_{12}^*E_{zar}(\Omega^*_{U_{12}/S,F_b}) & \otimes_{O_S} & \text{h}_{22}^*E_{zar}(\Omega^*_{U_{22}/S,F_b}) \\
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\xrightarrow{Ew_{(U_1,U_2)/S}} & & \xrightarrow{Ew_{(U_1,U_2)/S}} \\
\end{array}
\]

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Proof. Follows from Yoneda lemma.

Let \( S \in \text{Var}(\mathbb{C}) \) and \( S = \bigcup_{i=1}^{l} S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). For \( I \subset [1, \cdots, l] \), denote by \( S_I := \cap_{i \in I} S_i \) and \( j_I : S_I \hookrightarrow S \) the open embedding. We then have closed embeddings \( i_{I J} : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i \). Consider, for \( I \subset J \), the following commutative diagram

\[
D_{I J} = \begin{array}{ccc}
S_I & \xrightarrow{i_{I J}} & \tilde{S}_I \\
\downarrow{j_{I J}} & & \downarrow{p_{I J}} \\
S_J & \xrightarrow{i_{J}} & \tilde{S}_J
\end{array}
\]

and \( j_{I J} : S_J \hookrightarrow S_I \) is the open embedding so that \( j_J o j_{I J} = j_I \). Considering the factorization of the diagram \( D_{I J} \) by the fiber product:

\[
D_{I J} = \tilde{S}_J = \tilde{S}_I \times \tilde{S}_{J \setminus I} \xrightarrow{p_{I J}} \tilde{S}_I \]

the square of this factorization being cartesian, we have for \( F \in C(\text{Var}(\mathbb{C})^{sm}/S) \) the canonical map in \( C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I) \)

\[
S(D_{I J})(F) : L_i j_* j_I^* F \xrightarrow{q} i_{I J}^* j_I^* F = (i_I \times I) \ast l_{J J} j_J^* F \xrightarrow{(i_I \times I), ad(p_{I J}^* j_I^*)} j_I^* F = p_{IJ}^* j_I^* F
\]

which factors through

\[
S(D_{I J})(F) : L_i j_* j_I^* F \xrightarrow{S(D_{I J})(F)} p_{IJ}^* L_i j_I^* F \xrightarrow{q} p_{IJ}^* j_I^* F
\]

Definition 122. (i) Let \( S \in \text{SmVar}(\mathbb{C}) \). We have the functor

\[
C(\text{Var}(\mathbb{C})^{sm}/S)^{op} \to C_{O_{\text{fil}}(\mathbb{C})}(S), \quad F \mapsto e(S), \text{Hom}^* (L(i_{I J} j_I^* F), e_{\text{et}}(\Omega^*_S, F_b))\]

(ii) Let \( S \in \text{Var}(\mathbb{C}) \) and \( S = \bigcup_{i=1}^{l} S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). For \( I \subset [1, \cdots, l] \), denote by \( S_I := \cap_{i \in I} S_i \) and \( j_I : S_I \hookrightarrow S \) the open embedding. We then have closed embeddings \( i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i \). We have the functor

\[
C(\text{Var}(\mathbb{C})^{sm}/S)^{op} \to C_{O_{\text{fil}}(\mathbb{C})}(S/\tilde{S}_I), \quad F \mapsto e(\tilde{S}_I), \text{Hom}^* (L(i_{I J} j_I^* F), e_{\text{et}}(\Omega^*_S, F_b))\]

where

\[
u_{I J}^* F[d_{S_I}] : e(\tilde{S}_I), \text{Hom}^* (L(i_{I J} j_I^* F), e_{\text{et}}(\Omega^*_S, F_b))
\]

...
For $I \subset J \subset K$, we have obviously $p_{IJ \ast} u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$.

We then have the following key proposition

**Proposition 107.**

(i) Let $S \in \text{Var}(\mathbb{C})$. Let $m : Q_1 \to Q_2$ be an equivalence $(\mathbb{A}^1, \text{et})$ local in $C(\text{Var}(\mathbb{C})^{sm}/S)$ with $Q_1, Q_2$ complexes of projective presheaves. Then,

$$e(S) \hom(m, E_{et}(\Omega^*_{/S}, F_b)) : e(S) \hom^\bullet(Q_2, E_{et}(\Omega^*_{/S}, F_b)) \to e(S) \hom^\bullet(Q_1, E_{et}(\Omega^*_{/S}, F_b))$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{O_{fil, \infty}}(S)$ if $S$ is smooth.

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^n S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $m = (m_1) : (Q_{11}, s_{11}) \to (Q_{21}, s_{21})$ be an equivalence $(\mathbb{A}^1, \text{et})$ local in $C(\text{Var}(\mathbb{C})^{sm}/(\tilde{S}_i)_{et})$ with $Q_1, Q_2$ complexes of projective presheaves. Then,

$$(e(\tilde{S}_i))_\ast \hom(m_1, E_{et}(\Omega^*_{/\tilde{S}_i}, F_b)) :$$

$$(e(\tilde{S}_i))_\ast \hom^\bullet(Q_{21}, E_{et}(\Omega^*_{/\tilde{S}_i}, F_b)), u_{11}(Q_{21}, s_{21}) \to (e(\tilde{S}_i))_\ast \hom^\bullet(Q_{11}, E_{et}(\Omega^*_{/\tilde{S}_i}, F_b)), u_{11}(Q_{11}, s_{11})$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{O_{fil, \infty}}(\tilde{S}_i)$.

**Proof.** (i): By definition of an $(\mathbb{A}^1, \text{et})$ local equivalence (see proposition 17), there exist

$$\{X_{1, \alpha}/S, \alpha \in \Lambda_1\}, \ldots \ldots, \{X_{s, \alpha}/S, \alpha \in \Lambda_s\} \subset \text{Var}(\mathbb{C})^{sm}/S$$

such that we have in $H_{et}(C(\text{Var}(\mathbb{C})^{sm}/S))$

$$\text{Cone}(m) \Rightarrow \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(Z(X_{1, \alpha} \times \mathbb{A}^1/S) \rightarrow Z(X_{1, \alpha}/S))$$

$$\rightarrow \cdots \rightarrow \oplus_{\alpha \in \Lambda_s} \text{Cone}(Z(X_{s, \alpha} \times \mathbb{A}^1/S) \rightarrow Z(X_{s, \alpha}/S))$$

This gives in $D_{fil}(\mathbb{Z}) := H_{fil}(\mathbb{Z})$,

$$\text{Cone}(\hom(m, E_{et}(\Omega^*_{/S}, F_b))) \Rightarrow \text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(E_{et}(\Omega^*_{/S}, F_b)(X_{1, \alpha}/S) \rightarrow E_{et}(\Omega^*_{/S}, F_b)(X_{1, \alpha} \times \mathbb{A}^1/S))$$

$$\rightarrow \cdots \rightarrow \oplus_{\alpha \in \Lambda_s} \text{Cone}(E_{et}(\Omega^*_{/S}, F_b)(X_{s, \alpha}/S) \rightarrow E_{et}(\Omega^*_{/S}, F_b)(X_{s, \alpha} \times \mathbb{A}^1/S)))$$

Since $\Omega^*_{/S} \subset C(\text{Var}(\mathbb{C})^{sm}/S)$ is $\mathbb{A}^1$ homotopic, for all $1 \leq i \leq s$ and all $\alpha \in \Lambda_i$,

$$\text{Cone}(E_{et}(\Omega^*_{/S})(X_{i, \alpha}/S) \rightarrow E_{et}(\Omega^*_{/S})(X_{i, \alpha} \times \mathbb{A}^1/S)) \rightarrow 0$$

are homotopy equivalent. Hence $\text{Cone}(\hom(m, E_{et}(G,F))) \rightarrow 0$ is a 2-filtered quasi-isomorphism.

(ii): Similar to (i): see proposition 20

**Definition 123.**

(i) We define, using definition 122, by proposition 107, the filtered algebraic Gauss-Manin realization functor defined as

$$\mathcal{F}^{GM}_S : \text{DA}_c(S)_{op} \to D_{O_{fil, \infty}}(S), \ M \mapsto \mathcal{F}^{GM}_S(M) := e(S)_\ast \hom^\bullet(L(F), E_{et}(\Omega^*_{/S}, F_b))[-d_S]$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, \text{et})(F)$,

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \bigcup_{i=1}^n S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset \{1, \ldots, \ell\}$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $j_i : S_I \hookrightarrow \tilde{S}_I := \bigcap_{i \in I} \tilde{S}_i$. We define, using definition 122 and proposition 107 the filtered algebraic Gauss-Manin realization functor defined as

$$\mathcal{F}^{GM}_S : \text{DA}_c(S)_{op} \to D_{O_{fil, \infty}}(S/(\tilde{S}_I)) \mapsto$$

$$\mathcal{F}^{GM}_S(M) := ((e(\tilde{S}_i), \hom^\bullet(L(j_{I_1} \ast F), E_{et}(\Omega^*_{/\tilde{S}_i}, F_b))[-d_{\tilde{S}_i}], u_{I_1}(F))$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(K, \text{et})(F)$.
Proposition 108. For $S \in \text{Var}(\mathbb{C})$, the functor $\mathcal{F}^{\text{GM}}_S$ is well defined.

Proof. Let $S \in \text{Var}(\mathbb{C})$ and $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Denote, for $I \subseteq \{1, \cdots, l\}$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i \in I} \tilde{S}_i$. Let $M \in D(\mathbb{A}_1, et)$. Let $F, F' \in C(\text{Var}(\mathbb{C})^{gm}/S)$ such that $M = D(A_1, et)(F) = D(A_1, et)(F')$. Then there exist by definition a sequence of morphisms in $C(\text{Var}(\mathbb{C})^{gm}/S)$:

$$ F = F_1 \xrightarrow{s_1} F_2 \xrightarrow{s_2} \cdots \xrightarrow{s_k} F_s = F $$

where, for $1 \leq k \leq s$, and $s_k$ are $(\mathbb{A}^1, et)$ local equivalence. But if $s : F_1 \to F_2$ is an equivalence $(\mathbb{A}^1, et)$ local,

$$ L(i_{I*}j_I^*s) : L(i_{I*}j_I^*F_1) \to L(i_{I*}j_I^*F_2) $$

is an equivalence $(\mathbb{A}^1, et)$ local, hence

$$ \hom(L(i_{I*}j_I^*s), E_{et}(\Omega^*_j, F_b)) : (e(\tilde{S}_I), \hom(L(i_{I*}j_I^*F_2), E_{et}(\Omega^*_j, F_b), u_{I*}(F_2)) $$

$$ \to (e(\tilde{S}_I), \hom(L(i_{I*}j_I^*F_1), E_{et}(\Omega^*_j, F_b), u_{I*}(F_1)) $$

is an $\infty$-filtered quasi-isomorphism by proposition 107.

Let $f : X \to S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Assume that there is a factorization

$$ f : X \xrightarrow{i} Y \times S \xrightarrow{p_S} S $$

of $f$, with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have $X = \bigcup_{i=1}^{l} X_i$ with $X_i = f^{-1}(S_i)$. Denote, for $I \subseteq \{1, \cdots, l\}$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. For $I \subseteq \{1, \cdots, l\}$, denote by $\tilde{S}_I = \Pi_{i \in I} \tilde{S}_i$. We then have, for $I \subseteq \{1, \cdots, l\}$, closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and the following commutative diagrams which are cartesian

$$
\begin{array}{ccc}
X_I & \xrightarrow{i_I} & Y \times S_I \xrightarrow{p_{S_I}} S_I \\
\downarrow^{j_I} & & \downarrow^{p_{I*}} \\
\tilde{S}_I & \xrightarrow{p_{I*}} & S_I \\
\end{array}
$$

with $l_I : l_{|X_I}, i_I = I \times i_I, p_{S_I}$ and $p_{I*}$ are the projections and $p_{I*} = I \times p_{I*}$. We then recall that we denote by $j_I : \tilde{S}_I \setminus S_I \hookrightarrow \tilde{S}_I$ and $j_I^* : Y \times \tilde{S}_I \setminus X_I \hookrightarrow Y \times S_I$ the open complementary embeddings. We then have the following commutative diagrams

$$
\begin{array}{ccc}
D_{IJ} = S_I & \xrightarrow{i_{IJ}} & \tilde{S}_I \\
\downarrow^{j_{IJ}} & & \downarrow^{p_{IJ}} \\
S_I & \xrightarrow{p_{IJ}} & S_I \\
\end{array}
, \quad
\begin{array}{ccc}
D'_{IJ} = X_I & \xrightarrow{i'_{IJ}} & Y \times \tilde{S}_I \\
\downarrow^{j'_{IJ}} & & \downarrow^{p'_{IJ}} \\
X_I & \xrightarrow{p'_{IJ}} & X_I \\
\end{array}
$$

and the factorization of $D'_{IJ}$ by the fiber product:

$$
\begin{array}{ccc}
D'_{IJ} = X_I \xrightarrow{i'_{IJ}} Y \times \tilde{S}_I, \quad D_{IJ} = X_I \xrightarrow{i_{IJ}} Y \times S_I \\
\downarrow^{j'_{IJ}} & & \downarrow^{j_{IJ}} \\
X_I \xrightarrow{i'_{IJ}} Y \times \tilde{S}_I & & \tilde{S}_I \times Y \xrightarrow{\pi_{IJ}} X_I \times S_I = X_I \times \tilde{S}_{IJ} \\
\downarrow^{p'_{IJ}} & & \downarrow^{p_{IJ}} \\
Y \times \tilde{S}_I & & Y \times \tilde{S}_I \\
\end{array}
\quad (55)
$$

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where \( j_{i,j} : X_j \hookrightarrow X_i \) is the open embedding. Consider
\[
F(X/S) := p_{X,i}^! \Gamma_X^\vee Z(Y \times S/Y \times S) \in \text{C}(\text{Var}(\mathbb{C})^{sm}/S)
\]
so that \( D(\mathbb{A}^1,\text{et})(F(X/S)) = M(X/S) \). Then, by definition,
\[
\mathcal{F}_{S}^{GM}(M^{BM}(X/S)) := (e(\bar{S}_I), \text{Hom}(L_{\Omega^{*}_{S_j}} F(X/S)), E_{ct}(\Omega^{*}_{S_j}, F_b))[-d_{S_j}, u^q_{I,j}(F(X/S))]
\]
On the other hand, let
\[
Q(X_i/\bar{S}_I) := p_{S_i,F}^! \Gamma_{X_i}^\vee Z(Y \times \bar{S}_I/Y \times \bar{S}_I) \in \text{C}(\text{Var}(\mathbb{C})^{sm}/\bar{S}_I),
\]
see definition 10. We have then for \( I \subset [1, l] \) the following map in \( \text{C}(\text{Var}(\mathbb{C})^{sm}/\bar{S}_I) \):
\[
\begin{align*}
N_I(X/S) : Q(X_i/\bar{S}_I) & = p_{S_i,F}^! \Gamma_{X_i}^\vee Z(Y \times \bar{S}_I/Y \times \bar{S}_I) \xrightarrow{p_{S_i,F}^! \text{ad}(i^*_I, p_{S_i,F}^!)} \frac{p_{S_i,F}^!}{\text{ad}(i^*_I, p_{S_i,F}^!)}(\gamma) \\
p_{S_i,F}^! i_I^* \Gamma_{X_i}^\vee Z(Y \times \bar{S}_I/Y \times \bar{S}_I)[d_I] & \xrightarrow{p_{S_i,F}^! (T(i^*_I, \gamma^\vee))} p_{S_i,F}^! i_I^* \Gamma_{X_i}^\vee Z(Y \times \bar{S}_I/Y \times \bar{S}_I)[d_I] \\
 & \xrightarrow{\bar{i}_{I,F}} i_{I,F} p_{S_i,F}^! \Gamma_{X_i}^\vee Z(Y \times \bar{S}_I/Y \times \bar{S}_I)[d_I]
\end{align*}
\]
We have then for \( I \subset J \) the following commutative diagram in \( \text{C}(\text{Var}(\mathbb{C})^{sm}/\bar{S}_J) \):
\[
\begin{align*}
p_{I,F} p_{S_i,F}^! \Gamma_{X_i}^\vee Z(Y \times \bar{S}_I/Y \times \bar{S}_I) & \xrightarrow{p_{I,F} p_{S_i,F}^! \text{ad}(i^*_I, p_{I,F})} p_{I,F} (i^*_{I,F}) \Gamma_{X_i}^\vee Z(Y \times \bar{S}_I/Y \times \bar{S}_I) \\
 & \xrightarrow{p_{I,F} \text{ad}(i^*_I, p_{I,F})} p_{I,F} \Gamma_{X_i}^\vee Z(Y \times \bar{S}_I/Y \times \bar{S}_I)
\end{align*}
\]
with
\[
H_{I,J} : p_{S_i,F}^! \Gamma_{X_i}^\vee Z(Y \times \bar{S}_I/Y \times \bar{S}_I)
\]
This say that the maps \( N_I(X/S) \) induces a map in \( \text{C}(\text{Var}(\mathbb{C})^{sm}/(S/\bar{S}_I)) \)
\[
(N_I(X/S)) : (Q(X_i/\bar{S}_I), H_{I,J}) \rightarrow (i_{I,F} \Gamma_{X_i}^\vee Z(Y \times \bar{S}_I/Y \times \bar{S}_I), S(D_{I,J})(F(X/S))).
\]
We denote by \( v_{I,J}^q(F(X/S)) \) the composite
\[
v_{I,J}^q(F(X/S))[d_{S_j}] : e(\bar{S}_I), \text{Hom}(Q(X_i/\bar{S}_I), E_{ct}(\Omega^{*}_{S_j}, F_b))
\]
\[
\xrightarrow{\text{ad}(p_{I,F}^{*}\text{mod}(p_{I,F}))} p_{I,F} p_{S_i}^{*}\text{mod}(e(\bar{S}_I), \text{Hom}(Q(X_i/\bar{S}_I), E_{ct}(\Omega^{*}_{S_j}, F_b)))
\]
\[
\xrightarrow{p_{I,F} T(p_{I,F}, \Omega^{*})(Q(X_i/\bar{S}_I))} p_{I,F} p_{S_i}^{*}\text{mod}(Q(X_i/\bar{S}_I), E_{ct}(\Omega^{*}_{S_j}, F_b))
\]
\[
\xrightarrow{\text{Hom}(H_{I,J}, E_{ct}(\Omega^{*}_{S_j}, F_b))} p_{I,F} p_{S_i}^{*}\text{mod}(Q(X_i/\bar{S}_I), E_{ct}(\Omega^{*}_{S_j}, F_b)).
\]
On the other hand, we have the following map in \( \text{C}(\text{Var}(\mathbb{C})^{sm}/\bar{S}_J) \)
\[
\begin{align*}
\omega_{I,J}(X/S)[d_{S_j}] : p_{S_i,F}^! \Gamma_{X_i} E_{zar}(\Omega^{*}_{Y \times S_j/\bar{S}_I}, F_b) & \xrightarrow{\text{ad}(p_{I,F}^{*}\text{mod}(p_{I,F}))} p_{I,F} p_{S_i}^{*}\text{mod}(\Gamma_{X_i} E_{zar}(\Omega^{*}_{Y \times S_j/\bar{S}_I}, F_b))
\end{align*}
\]
\[
\xrightarrow{T(\omega_{I,J})(p_{I,F}^{*}\text{mod}(p_{I,F}))} p_{I,F} p_{S_i}^{*}\text{mod}(\Gamma_{X_i} E_{zar}(\Omega^{*}_{Y \times S_j/\bar{S}_I}, F_b))
\]
\[
\xrightarrow{\text{Hom}(H_{I,J}, E_{zar}(\Omega^{*}_{Y \times S_j/\bar{S}_I}, F_b))} p_{I,F} p_{S_i}^{*}\text{mod}(\Gamma_{X_i} E_{zar}(\Omega^{*}_{Y \times S_j/\bar{S}_I}, F_b)).
\]
Lemma 9.  
(i) The map in $C(\text{Var}(\mathbb{C}))_{/S}/(S/\tilde{S}_I)$

$$(N_I(X/S)) : (Q(X_I/\tilde{S}_I), H_{I,J}) \to (L(i_1, j_1^* F(X/S)), S^q(D_{I,J})(F(X/S))).$$

is an equivalence ($\mathbb{A}^1, et$) local.

(ii) The maps $(N_I(X/S))$ induces an $\infty$-filtered quasi-isomorphism in $\text{COfd, D}(S/(\tilde{S}_I))$

$$(\text{Hom}(N_I(X/S), E_{et}(\Omega^*_{S_I}, F_b))) :$$

$$(e(\tilde{S}_I), \text{Hom}(L(i_1, j_1^* F(X/S)), E_{et}(\Omega^*_{S_I}, F_b))[-d_{\tilde{S}_I}], u^q_{I,J}(F(X/S))) \to$$

$$(e(\tilde{S}_I), \text{Hom}(Q(X_I/\tilde{S}_I), E_{et}(\Omega^*_{S_I}, F_b))[-d_{S_I}], v^q_{I,J}(F(X/S)))$$

(iii) The maps $(I(\gamma, \text{hom})(-,-))$ and $(k : E_{zar}(p_{S_I}^* \Omega^*_{S_I}, F_b) \to E_{et}(p_{S_I}^* \Omega^*_{S_I}, F_b))$ induce an $(1)$-filtered Zariski local equivalence in $\text{COfd, D}(S/(\tilde{S}_I))$

$$(k \circ I(\gamma, \text{hom})(-,-)) : (p_{S_I}^* \Gamma_X, E_{zar}(\Omega^*_{Y \times S_I/\tilde{S}_I}, F_b)[-d_{\tilde{S}_I}], w_{I,J}(X/S))$$

$$\to (e(\tilde{S}_I), \text{Hom}(Q(X_I/\tilde{S}_I), E_{et}(\Omega^*_{S_I}, F_b))[-d_{S_I}], v^q_{I,J}(F(X/S))).$$

Proof. (i): Follows from theorem 16.

(ii): These maps induce a morphism in $\text{COfd, D}(S/(\tilde{S}_I))$ by construction. The fact that it is an $\infty$-filtered quasi-isomorphism follows from (i) and proposition 107.

(iii): These maps induce a morphism in $\text{COfd, D}(S/(\tilde{S}_I))$ by construction. □

Proposition 109. Let $f : X \to S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \to \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \cdots, l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. Assume there exist a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of $f$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. We then have, for $I \subset [1, \cdots, l]$, the following commutative diagrams which are cartesian

$$f_I = f|_{X_I} : X_I \xrightarrow{l_I} Y \times S_I \xrightarrow{p_{S_I}} S_I \xrightarrow{i_I} Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I$$

Let $F(X/S) := p_{S_I}^* \Gamma_X^* \mathbb{Z}(Y \times S/Y \times S)$. The transformations maps $(N_I(X/S)) : Q(X_I/\tilde{S}_I) \to i_1, j_1^* F(X/S))$ and $(k \circ I(\gamma, \text{hom})(-,-))$, for $I \subset [1, \cdots, l]$, induce an isomorphism in $\text{DOfd, D}_{\infty}(S/(\tilde{S}_I))$

$$(F_{GM}^S(M(X/S)) := (e(\tilde{S}_I), \text{Hom}(L(i_1, j_1^* F(X/S)), E_{et}(\Omega^*_{S_I}, F_b))[-d_{\tilde{S}_I}], u^q_{I,J}(F(X/S)))$$

$$(e(\tilde{S}_I), \text{Hom}(L_{N_I}(X/S), E_{et}(\Omega^*_{S_I}, F_b))[-d_{\tilde{S}_I}], v^q_{I,J}(F(X/S)))$$

$$(k \circ I(\gamma, \text{hom})(-,-))^{-1} : (p_{S_I}^* \Gamma_X, E_{zar}(\Omega^*_{Y \times S_I/\tilde{S}_I}, F_b)[-d_{S_I}], w_{I,J}(X/S)).$$

Proof. Follows from lemma 9. □

We now define the functorialities of $F_{GM}^S$ with respect to $S$ which makes $F_{GM}^S$ a morphism of 2-functor.
Definition 124. Let \( g : T \rightarrow S \) a morphism with \( T, S \in \text{SmVar}(\mathbb{C}) \). Consider the factorization \( g : T \xrightarrow{T \rightarrow T \times S} P_{S/T} S \) where \( l \) is the graph embedding and \( p_S \) the projection. Let \( M \in \text{DA}_c(S) \) and \( F \in C(\text{Var}(\mathbb{C}))^{\text{sm}}/S \) such that \( M = D(\Delta^1_S, \text{et})(F) \). Then, \( D(\Delta^1_T, \text{et})(g^*F) = g^*M \).

(i) We have then the canonical transformation in \( D_{O,T,S,\infty}(T \times S) \) (see definition 120):

\[
T(g, \mathcal{F}^{GM})(M) : Rg^{\text{mod}-1}[1] \mathcal{F}^{GM}_S(M) := g^{\text{mod}-1}e(S)_* \text{Hom}^*(L^F, E_{et}(\Omega^*_S, F))[-d_T]
\]

\[
e(T \times S)_* \text{Hom}^*(T^*_{g\mathcal{T}} L^F, E_{et}(\Omega^*_{T \times S}, F))[-d_T] =: \mathcal{F}^{GM}_T(I, g^*M, W).
\]

(ii) We have then the canonical transformation in \( D_{O,T,S,\infty}(T) \) (see definition 120):

\[
T^O(g, \mathcal{F}^{GM})(M, W) : Lg^{\text{mod}-1}[1] \mathcal{F}^{GM}_S(M) := g^{\text{mod}-1}e(S)_* \text{Hom}^*(L^F, E_{et}(\Omega^*_S, F))[-d_T]
\]

\[
e(T)_* \text{Hom}^*(g^*L^F, E_{et}(\Omega^*_T, F))[-d_T] =: \mathcal{F}^{GM}_T(g^*M).
\]

We give now the definition in the non smooth case Let \( g : T \rightarrow S \) a morphism with \( T, S \in \text{Var}(\mathbb{C}) \). Assume we have a factorization \( g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. Let \( S = \bigcup_{i=1}^l S_i \) be an open cover such that there exists closed embeddings \( i_i : S_i \hookrightarrow S_i \) with \( S_i \in \text{SmVar}(\mathbb{C}) \) Then, \( T = \bigcup_{i=1}^l T_i \) with \( T_i := g^{-1}(S_i) \) and we have closed embeddings \( i_i^* := i_i \circ l : T_i \hookrightarrow Y \times S_i \). Moreover \( \tilde{g}_i := p_{S_i} : Y \times S_i \rightarrow S_i \) is a lift of \( g_i := g|_{T_i} : T_i \rightarrow S_i \). We recall the commutative diagram :

\[
E_{i,J} = (Y \times S_i)\setminus T_i \xrightarrow{\tilde{g}_i} Y \times S_i, \quad E_{i,J} = \tilde{S}_j \setminus S_j \xrightarrow{\tilde{g}_i} \tilde{S}_j \quad E_{i,J} = (Y \times S_i)\setminus T_i \xrightarrow{\tilde{g}_i} Y \times S_i
\]

For \( I \subset J \), denote by \( \tilde{p}_{i,J} : \tilde{S}_j \rightarrow \tilde{S}_i \) and \( \tilde{p}_{i,J}^* := I_{Y \times \tilde{S}_j} : Y \times \tilde{S}_j \rightarrow Y \times \tilde{S}_i \) the projections, so that \( \tilde{g}_i \circ \tilde{p}_{i,J} = \tilde{p}_{i,J} \circ \tilde{g}_i \). Consider, for \( I \subset J \subset [1, \ldots, l] \), resp. for each \( I \subset [1, \ldots, l] \), the following commutative diagrams in \( \text{Var}(\mathbb{C}) \)

\[
D_{i,J} = S_i \xrightarrow{i_i} \tilde{S}_j ; \quad D_{i,J} = T_i \xrightarrow{j_i} Y \times \tilde{S}_j ; \quad D_{i,J} = T_i \xrightarrow{i_i} Y \times \tilde{S}_j
\]

and \( j_{i,J} : S_j \hookrightarrow S_i \) is the open embedding so that \( j_i \circ j_{i,J} = j_{i,J} \). Let \( F \in C(\text{Var}(\mathbb{C}))^{\text{sm}}/S \). Recall (see section 2) that since \( j_i^* j_{i,J}^* g^* F = 0 \), the morphism \( T(D_{i,J})(j_i^* F) : \tilde{g}_i^* i_i^* j_{i,J}^* g^* F \) factors through

\[
T(D_{i,J})(j_i^* F) : \tilde{g}_i^* i_i^* j_{i,J}^* g^* F \xrightarrow{\gamma_{X_{S_i}}} \Gamma_{X_{S_i}} \tilde{g}_i^* i_i^* j_{i,J}^* g^* F \xrightarrow{T^\gamma(D_{i,J})(j_i^* F)} i_i^* j_{i,J}^* g^* F
\]

We then have, for each \( I \subset [1, \ldots, l] \), the morphism

\[
T^{\gamma \cdot}(D_{i,J})(j_i^* F) : \Gamma^\gamma_{X_{S_i}} \tilde{g}_i^* i_i^* j_{i,J}^* F \xrightarrow{T^{\gamma \cdot}(S_{i,J})(j_i^* F)} \Gamma^\gamma_{T_i} L(\tilde{g}_i^* i_i^* j_{i,J}^* F) \xrightarrow{L(T^\gamma(D_{i,J})(j_i^* F))} L(i_i^* j_{i,J}^* g^* F)
\]
and the following diagram in $\text{C}(\text{Var}(\mathbb{C})^m/Y \times \tilde{S}_J)$ commutes

$$\Gamma^\vee_{T_J} \tilde{g}_I^* L(i, j^*_I F) \xrightarrow{\Gamma^\vee_{T_J} \tilde{g}_I^* q_I(F)} \Gamma^\vee_{T_J} \tilde{g}_I^* (i, j^*_I F)$$

$T^{q, \gamma}(D_{g_I})(j^*_I F)$

$L(i, j^*_I g^* F) \xrightarrow{q_I(g^* F)} \tilde{g}_I^* L(i, j^*_I g^* F) = i, j^*_I g^* F$

We have the following commutative diagram in $\text{C}(\text{Var}(\mathbb{C})^m/Y \times \tilde{S}_J)$

$$p_{I, j}^* \tilde{g}_I^* L(i, j^*_I F) = \tilde{g}_I^* p_{I, j}^* i, j^*_I F = p_{I, j}^* \tilde{g}_I^* L(i, j^*_I F)$$

$T^{q, \gamma}(D_{g_I})(j^*_I F)$

$\tilde{g}_I^* L(i, j^*_I g^* F) = \tilde{g}_I^* L(i, j^*_I g^* F)$

This gives, after taking the functor $L$, the following commutative diagram in $\text{C}(\text{Var}(\mathbb{C})^m/Y \times \tilde{S}_J)$

$$\Gamma^\vee_{T_J} p_{I, j}^* \Gamma^\vee_{T_J} \tilde{g}_I^* L(i, j^*_I F) = \Gamma^\vee_{T_J} \tilde{g}_I^* p_{I, j}^* L(i, j^*_I F)$$

$T^{q, \gamma}(D_{g_I})(j^*_I F)$

$L(i, j^*_I g^* F)$

The fact that the diagrams (60) commutes says that the maps $T^{q, \gamma}(D_{g_I})(j^*_I F)$ define a morphism in $\text{C}(\text{Var}(\mathbb{C})^m/(T/Y \times \tilde{S}_J))$)

$(T^{q, \gamma}(D_{g_I})(j^*_I F)) : (\Gamma^\vee_{T_J} \tilde{g}_I^* L(i, j^*_I F), \tilde{g}_J S^q(D_{1, j})(F)) \rightarrow (L(i, j^*_I g^* F), S^q(D_{1, j})(g^* F))$

We denote by $\tilde{g}_J u_{I, j}^* (F)$ the composite

$$\tilde{g}_J u_{I, j}^* (F) \xrightarrow{\text{ad}(p_{I, j}^* p_{I, j}^* \gamma)^{-1}} p_{I, j}^* \tilde{g}_J u_{I, j}^* (F) \xrightarrow{T^{q, \gamma}(D_{g_I})(j^*_I F)} p_{I, j}^* \tilde{g}_J u_{I, j}^* (F) \xrightarrow{\text{Hom}(\tilde{g}_J L(i, j^*_I F), E_c(t^*_{\tilde{Y} \times \tilde{S}_J}, F))}$$

$$\Gamma^\vee_{T_J} \tilde{g}_I^* L(i, j^*_I g^* F)$$

$$T^{q, \gamma}(D_{g_I})(j^*_I F)$$

$L(i, j^*_I g^* F)$

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We denote by $\tilde{g}_I^j u_{I,j}^q(F)$2 the composite

$$\tilde{g}_I^j u_{I,j}^q(F) \to \text{ad}(p_{I,j}^* \circ p_{I,j}^* \circ e(T_{\tilde{g}_I}^j)) \Hom(\Gamma_T^j, \tilde{g}_I^j L (i_1, j_1^* F), Ect(\Omega^*_{Y \times \tilde{S}_I}, F_b))$$

$$\xrightarrow{\text{T}(p_{I,j, \tilde{S}_I})} \Hom(\Gamma_T^j, \tilde{g}_I^j L (i_1, j_1^* F), Ect(\Omega^*_{Y \times \tilde{S}_I}, F_b))$$

$$\xrightarrow{\text{Cone}(\text{ad}(p_{I,j}^* \circ p_{I,j}^* \circ e(T_{\tilde{g}_I}^j)))} \Hom(\Gamma_T^j, \tilde{g}_I^j L (i_1, j_1^* F), Ect(\Omega^*_{Y \times \tilde{S}_I}, F_b))$$

We then have the following lemma:

**Lemma 10.** (i) The morphism in $C(\text{Var}(\mathbb{C})^\text{sm}/(T/(Y \times \tilde{S}_I)))$

$$\Gamma_T^j \Lambda \tilde{g}_I^j i_1, j_1^* F, \tilde{g}_I^j S^q(D_I, j_1^* F) \to (i_1, j_1^* F, S^q(D_I, j_1^* F))$$

is an equivalence $(\mathbb{A}^1, \text{et})$ local.

(ii) Denote for short $d_{Y_1} = -d_Y - d_{\tilde{S}_I}$. The maps $\Hom((\Gamma_T^j \Lambda \tilde{g}_I^j i_1, j_1^* F), Ect(\Omega^*_{Y \times \tilde{S}_I}, F_b))$ induce an infinite filtered quasi-isomorphism in $C_{O_{\tilde{S}_I}, T}(T/(Y \times \tilde{S}_I))$

$$(\Gam\text{T}(\Gamma_{\tilde{g}_I}^j \Lambda \tilde{g}_I^j i_1, j_1^* F), Ect(\Omega^*_{Y \times \tilde{S}_I}, F_b))$$

$$(e(Y \times \tilde{T}_I), \text{Hom}(\Lambda(i_1, j_1^* F), Ect(\Omega^*_{Y \times \tilde{S}_I}, F_b))[d_{Y_1}], u_{I,j}^q(g^* F) \to$$

$$(e(Y \times \tilde{T}_I), \text{Hom}(\Gamma_T^j \Lambda \tilde{g}_I^j i_1, j_1^* F), Ect(\Omega^*_{Y \times \tilde{S}_I}, F_b))[d_{Y_1}], \tilde{g}_I^j u_{I,j}^q(F_1))$$

(iii) The maps $T(\tilde{g}_I, \Omega_1)(L(i_1, j_1^* F))$ (see definition 120) induce a morphism in $C_{O_{\tilde{S}_I}, T}(T/(Y \times \tilde{S}_I))$

$$T(\tilde{g}_I, \Omega_1)(L(i_1, j_1^* F))$$

$$(\Gamma_T, \text{e}_{\text{zar}}(\tilde{g}_I^j \Lambda \tilde{g}_I^j i_1, j_1^* F), Ect(\Omega^*_{\tilde{S}_I}, F_b))[d_{Y_1}], \tilde{g}_I^j u_{I,j}^q(F_1)) \to$$

$$(\Gamma_{T_1}(e(Y \times \tilde{S}_I), \text{Hom}(\tilde{g}_I^j L(i_1, j_1^* F), Ect(\Omega^*_{Y \times \tilde{S}_I}, F_b))[d_{Y_1}], \tilde{g}_I^j u_{I,j}^q(F_1))$$

Proof. (i): Follows from theorem 16.

(ii): These maps induce a morphism in $C_{O_{\tilde{S}_I}, T}(T/(Y \times \tilde{S}_I))$ by construction. The fact that this morphism is an infinite filtered equivalence Zariski local follows from (i) and proposition 107.

(iii): These maps induce a morphism in $C_{O_{\tilde{S}_I}, T}(T/(Y \times \tilde{S}_I))$ by construction.

**Definition 125.** Let $g : T \to S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{i} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \bigcup_{i=1}^{n} S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_I$ with $\tilde{S}_I \in \text{SmVar}(\mathbb{C})$. Then $T = \bigcup_{i=1}^{n} T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_I$. Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_1 := g|_{T_i} : T_i \to S_i$. Denote for short $d_{Y_1} = d_Y + d_{\tilde{S}_I}$. Let $M \in D_{A_{\mathfrak{c}}}(S)$ and $F \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$ such that $M = D(\mathbb{A}^1, \text{et})(F)$. Then $D(\mathbb{A}^1, \text{et})(g^* F) = g^* M$. We
have, by lemma 10, the canonical transformation in $D_{Oj,k,l}(T/(Y \times \tilde{S}_1))$

\[
T(g, \mathcal{F}_{GM}^S(M)) \cdot Rg^{* \text{mod}[\cdot]} \mathcal{F}_{GM}^S(M) :=
\]

\[
(\Gamma_{T,1}E_{\text{zar}}(\tilde{g}_I^{* \text{mod}}c(\tilde{S}_1), \text{Hom}^\bullet(L(i_1, j_1^* F, E_{et}(\Omega_{/\tilde{S}_1}', F_0)))[{-d_Y} - d_{\tilde{S}_1}], \tilde{g}_J^{* \text{mod}}u_{I,J}^*(F))
\]

\[
(\Gamma_{T,1}E(T(\tilde{g}_I, \Omega_{/\tilde{S}_1}))(L(I_1, j_1^* F))\mathcal{F}_{GM}(M))
\]

\[
(\Gamma_{T,1}E(\tilde{g}_I, \Omega_{/\tilde{S}_1}))(L(I_1, j_1^* F))\mathcal{F}_{GM}(M))
\]

\[
(\Gamma_T, e(Y \times \tilde{S}_1), \text{Hom}^\bullet(\tilde{g}_I^{* \text{mod}}L(i_1, j_1^* F, E_{et}(\Omega_{/\tilde{S}_i}', F_0)))[{-d_Y} - d_{\tilde{S}_1}], \tilde{g}_J^{* \text{mod}}u_{I,J}^*(F))
\]

\[
(\tilde{g}_I^{* \text{mod}}L(i_1, j_1^* F, E_{et}(\Omega_{/\tilde{S}_i}', F_0)))[{-d_Y} - d_{\tilde{S}_1}], \tilde{g}_J^{* \text{mod}}u_{I,J}^*(F))
\]

\[
(\tilde{g}_I^{* \text{mod}}L(i_1, j_1^* F, E_{et}(\Omega_{/\tilde{S}_i}', F_0)))[{-d_Y} - d_{\tilde{S}_1}], u_{I,J}^*(F))
\]

\[
(\text{Hom}(\tilde{T}_2, \tilde{S}_1), \text{Hom}(\tilde{T}_2, \tilde{S}_1), \text{Hom}(\tilde{T}_2, \tilde{S}_1), \text{Hom}(\tilde{T}_2, \tilde{S}_1))\mathcal{F}_{GM}(g^* M).
\]

**Proposition 110.** (i) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(C)$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{p_2} S$ with $Y_2 \in \text{SmVar}(C)$, $l$ a closed embedding and $p_2$ the projection. Let $S = \bigcup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow S_i$ with $S_i \in \text{SmVar}(C)$ Then, $T = \bigcup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i_i' : i_i \circ l : T_i \rightarrow Y_2 \times S_i$. Moreover $g_I := p_{S_i} : Y \times \tilde{S}_i \rightarrow \tilde{S}_i$ is a lift of $g_I : T_i \rightarrow S_i$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(C)$. Assume that there is a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_2} S$, with $Y_1 \in \text{SmVar}(C)$, $l$ a closed embedding and $p_2$ the projection. We have then the following commutative diagram whose squares are cartesiens

\[
\begin{array}{ccc}
  f' : X_T & \rightarrow & Y_1 \times T \\
  \downarrow & & \downarrow \\
  f'' = f \times I : Y_2 \times X & \rightarrow & Y_1 \times Y_2 \times S \\
  \downarrow & & \downarrow \\
  f : X & \rightarrow & Y_2 \times S \\
  \downarrow & & \downarrow \\
  & & S
\end{array}
\]

Consider $F(X/S) := p_{S_2} \Gamma_X Z(Y_1 \times S/Y_1 \times S)$ and the isomorphism in $C(\text{Var}(C)^{sm}/S)$

\[
T(f, g, F(X/S)) : g^* F(X/S) := g^* p_{S_2} \Gamma_X^\vee Z(Y_1 \times S/Y_1 \times S) \xrightarrow{\sim} p_{T_2} \Gamma_X^\vee Z(Y_1 \times T/Y_1 \times T) =: F(X_T/T).
\]

which gives in $\text{DA}(S)$ the isomorphism $T(f, g, F(X/S)) : g^* M(X/S) \xrightarrow{\sim} M(X_T/T)$. Then, the following diagram in $D_{Oj,k,l}(T/(Y_2 \times S_i))$ commutes

\[
Rg^{* \text{mod}[\cdot]} \mathcal{F}_{GM}^S(M(X/S)) \xrightarrow{T(g, \mathcal{F}_{GM}^S(M(X/S)))} \mathcal{F}_{GM}^S(M(X_T/T))
\]

\[
\downarrow g^{* \text{mod}[\cdot]} \mathcal{F}_{GM}^S(M(X/S)) \xrightarrow{T(g, \mathcal{F}_{GM}^S(M(X/S)))} \mathcal{F}_{GM}^S(M(X_T/T))
\]

\[
\downarrow g^{* \text{mod}[\cdot]} \mathcal{F}_{GM}^S(M(X/S)) \xrightarrow{T(g, \mathcal{F}_{GM}^S(M(X/S)))} \mathcal{F}_{GM}^S(M(X_T/T))
\]

\[
(g^{* \text{mod}[\cdot]} \Gamma_{X,i} E_{zar}(\Omega_{Y_1 \times S_i/Y_1 \times S_i}, F_0)[{-d_{\tilde{S}_i}}, (\tilde{g}_I, \gamma)(-) \mathcal{F}_{GM}^S(\tilde{g}_I, p_{S_i})]) \mathcal{F}_{GM}^S(M(X_T/T))
\]

\[
(g^{* \text{mod}[\cdot]} \Gamma_{X,i} E_{zar}(\Omega_{Y_1 \times S_i/Y_1 \times S_i}, F_0)[{-d_{\tilde{S}_i}}, (\tilde{g}_I, \gamma)(-) \mathcal{F}_{GM}^S(\tilde{g}_I, p_{S_i})]) \mathcal{F}_{GM}^S(M(X_T/T))
\]

\[
(g^{* \text{mod}[\cdot]} \Gamma_{X,i} E_{zar}(\Omega_{Y_1 \times S_i/Y_1 \times S_i}, F_0)[{-d_{\tilde{S}_i}}, (\tilde{g}_I, \gamma)(-) \mathcal{F}_{GM}^S(\tilde{g}_I, p_{S_i})]) \mathcal{F}_{GM}^S(M(X_T/T))
\]

(ii) Let $g : T \rightarrow S$ a morphism with $T, S \in \text{SmVar}(C)$. Let $f : X \rightarrow S$ a morphism with $X \in \text{Var}(C)$. Assume that there is a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_2} S$, with $Y \in \text{SmVar}(C)$, $l$ a closed embedding and $p_2$ the projection. Consider $F(X/S) := p_{S_2} \Gamma_X^\vee Z(Y \times S/Y \times S)$ and the isomorphism
in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T(f, g, F(X/S)) : g^* F(X/S) := g^* p_{S,T} \Gamma_X^\infty \mathbb{Z}(Y \times S/Y \times S) \rar \text{pr}_2^* \Gamma_X^\infty \mathbb{Z}(Y \times T/Y \times T) =: F(X_T/T).$$

which gives in $DA(S)$ the isomorphism $T(f, g, F(X/S)) : g^* M(X/S) \rar M(X_T/T)$. Then, the following diagram in $D_{O_{fil, \infty}}(T)$ commutes

\[
\begin{array}{ccc}
Lg^{* \text{mod}[-]} F_S^{GM}(M(X/S)) & \xrightarrow{T^O(g, F^{GM})(M(X/S))} & F_T^{GM}(M(X_T/T)) \\
\text{I}^{GM}(X/S) & \downarrow & \text{I}^{GM}(X_T/T) \\
g^{* \text{mod}L_O(p_{S,T} \Gamma_X \mathbb{E}_{zar}(\Omega_{Y \times S/X}, F_b)[-d_T]} \rightarrow p_{Y \times T} \Gamma_{X_T} \mathbb{E}_{zar}(\Omega_{Y \times T/X}, F_b)[-d_T] & \xrightarrow{(T(g \times I, \gamma)(-)) \circ T_S^O(g, p_S)} & p_{Y \times T} \Gamma_{X_T} \mathbb{E}_{zar}(\Omega_{Y \times T/X}, F_b)[-d_T] \\
Lg^{* \text{mod}FDR_{p_S} \Gamma_X \mathbb{E}(\mathbb{O} \times S, F_b)[-d_T]} \rightarrow \Gamma_{X_T} \mathbb{E}(\mathbb{O} \times T, F_b)[-d_T] & \xrightarrow{T^{\text{mod}FDR}(g, f)(\mathbb{E}(\mathbb{O} \times S, F_b))} & \Gamma_{X_T} \mathbb{E}(\mathbb{O} \times T, F_b)[-d_T] \\
\end{array}
\]

Proof. Follows immediately from definition.

We have the following theorem:

**Theorem 35.** (i) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \rar Y \times S \rar S$, with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \hat{S}_i$ with $\hat{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M \in DA_c(S)$

$$T(g, F^{GM})(M) : Rg^{* \text{mod}[-]} F_S^{GM}(M) \rightrightarrows F_T^{GM}(g^* M)$$

is an isomorphism in $D_{O_{fil, d, \infty}}(T/(Y \times \hat{S}_T))$.

(ii) Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Then, for $M \in DA_c(S)$

$$T^O(g, F^{GM})(M) : Lg^{* \text{mod}[-]} F_S^{GM}(M) \rightrightarrows F_T^{GM}(g^* M)$$

is an isomorphism in $D_{O_T}(T)$.

(iii) A base change theorem for algebraic De Rham cohomology: Let $g : T \rightarrow S$ is a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Let $h : U \rightarrow S$ a smooth morphism with $U \in \text{Var}(\mathbb{C})$. Then the map (see definition 1)

$$T^O_w(g, h) : Lg^{* \text{mod}Rh_* (\Omega^\infty_{U/S}, F_b)} \rightrightarrows Rh'_* (\Omega^\infty_{U_T/T}, F_b)$$

is an isomorphism in $D_{O_T}(T)$.

Proof. (i):Follows from proposition 106.
(ii): Follows from proposition 110(ii) and the base change for algebraic D modules (proposition 84).
(iii): Follows from (ii) and lemma 7.

We finish this subsection by some remarks on the absolute case and on a particular case of the relative case:

**Proposition 111.** (i) Let $X \in \text{PSmVar}(\mathbb{C})$ and $D = \cup D_i \subset X$ a normal crossing divisor. Consider the open embedding $j : U := X \setminus D \hookrightarrow X$. Then,
The map in $D_{fil, \infty}(\mathbb{C})$

$$\text{Hom}(\mathbb{D}(\mathbb{Z}(U)), k)^{-1} \circ \text{Hom}((0, \text{ad}(j^*, j_*)(\mathbb{Z}(X/X))), \mathbb{E}_c(\Omega^*, F_b)) :$$

$$\mathcal{F}^{GM}(\mathbb{D}(\mathbb{Z}(U))) := \text{Hom}(\mathbb{D}(\mathbb{Z}(U)), \mathbb{E}_c(\Omega^*, F_b))$$

$$\xrightarrow{\sim} \text{Hom}((\text{Cone}(Z(D) \to \mathbb{Z}(X)), \mathbb{E}_zar(\Omega^*, F_b)) = \Gamma(X, \mathbb{E}_zar(\Omega^*_X(\text{mul} D), F_b)).$$

is an isomorphism, where we recall $\mathbb{D}(\mathbb{Z}(U)) := a_{X, *}, E_c(\mathbb{Z}(U/U)) = a_{U, *}, E_c(\mathbb{Z}(U/U))$.

$$\mathcal{F}^{GM}(\mathbb{Z}(U)) = \Gamma(U, \mathbb{E}_zar(\Omega^*_U, F_b)) \subset D_{fil, \infty}(\mathbb{C})$$

is NOT isomorphic to $\Gamma(X, \mathbb{E}_zar(\Omega^*_X(\text{log} D), F_b))$ in $D_{fil, \infty}(\mathbb{C})$ in general. For example $U$ is affine, then $H^n(U, \Omega^*_U) = 0$ for all $p \in \mathbb{N}$, $p \neq 0$, so that the $E_p^2(\Gamma(U, \mathbb{E}_zar(\Omega^*_U, F_b)))$ are NOT isomorphic to $E_p^2(\Gamma(X, \mathbb{E}_zar(\Omega^*_X(\text{log} D), F_b)))$ in this case. In particular, the map,

$$j^* := \text{ad}(j^*, j_*)(-) : H^n \Gamma(X, \mathbb{E}_zar(\Omega^*_X(\text{log} D))) \xrightarrow{\sim} H^n \Gamma(U, \mathbb{E}_zar(\Omega^*_U))$$

which is an isomorphism in $D(\mathbb{C})$ (i.e. if we forgot filtrations), gives embeddings

$$j^* := \text{ad}(j^*, j_*)(-) : F^p H^n(\mathbb{C}, U) := F^p H^n \Gamma(X, \mathbb{E}_zar(\Omega^*_X(\text{log} D), F_b)) \hookrightarrow F^p H^n \Gamma(U, \mathbb{E}_zar(\Omega^*_U, F_b))$$

which are NOT an isomorphism in general for $n, p \in \mathbb{Z}$. Note that, since $a_U : U \to \{\text{pt}\}$ is not proper,

$$[\Delta_U] : \mathbb{Z}(U) \to a_{U, *}, E_c(\mathbb{Z}(U/U))[2dv]$$

is NOT an equivalence $(\mathbb{A}^1, \text{et})$ local.

Let $Z \subset X$ a smooth subvariety and denote $U := X \setminus Z$ the open complementary. Denote

$$M_Z(X) := \text{Cone}(M(U) \to M(X)) \in D(\mathbb{C})$$. The map in $D_{fil, \infty}(\mathbb{C})$

$$\text{Hom}(\mathcal{G}(X, Z), \mathbb{E}_c(\Omega^*, F_b))^{-1} \circ \text{Hom}(a_{X, *}, \mathcal{G}(Z(X/X)), \mathbb{E}_c(\Omega^*, F_b))^{-1} :$$

$$\mathcal{F}^{GM}(M_Z(X)) := \text{Hom}(a_{X, *}, \mathcal{G}(Z(X/X)), \mathbb{E}_c(\Omega^*, F_b)) \xrightarrow{\sim}$$

$$\Gamma(X, \mathcal{G}(Z(\mathbb{Z}(X/X)), \mathbb{E}_c(\Omega^*, F_b))) \xrightarrow{\sim} \mathcal{F}^{GM}(M(Z)(c)[2e]) = \Gamma(Z, \mathbb{E}_zar(\Omega^*_Z, F_b)(-c)[-2e])$$

is an isomorphism, where $c = \text{codim}(Z, X)$ and $\mathcal{G}(X, Z) : a_{X, *}, \mathcal{G}(Z(X/X) \to \mathbb{Z}(Z)(c)[2e]$ is the Gysin morphism.

Let $D \subset X$ a smooth divisor and denote $U := X \setminus Z$ the open complementary. Note that the canonical distinguish triangle in $D(\mathbb{C})$

$$M(U) \xrightarrow{\text{ad}(j^*, j_*)} M(X) \xrightarrow{\gamma_Z^*(\mathbb{Z}(X/X))} M(D) \to M(U)[1]$$

give a canonical triangle in $D_{fil, \infty}(\mathbb{C})$

$$\mathcal{F}^{GM}(M(D)(X)) \xrightarrow{\mathcal{F}^{GM}(\gamma_Z^*(\mathbb{Z}(X/X)))} \mathcal{F}^{GM}(M(X)) \xrightarrow{\mathcal{F}^{GM}(\text{ad}(j^*, j_*))} \mathcal{F}^{GM}(M(U)) \to \mathcal{F}^{GM}(M(D)(X))[1],$$

which is NOT the image of a distinguish triangle in $\pi(D(\mathbb{M}HM(\mathbb{C})))$, as $\mathcal{F}^{GM}(M(U)) \notin \pi(D(\mathbb{M}HM(\mathbb{C})))$ since the morphism

$$\text{ad}(j^*, j_*) : H^n(X, \mathbb{E}_zar(\Omega^*_X(\text{log} D), F_b)) \to H^n(U, \mathbb{E}_zar(\Omega^*_U, F_b))$$

is not strict. Note that if $U := S, D$ is affine, then by the exact sequence in $C(Z)$

$$0 \to \Gamma(Z(X, \mathbb{E}_zar(\Omega^*_Z)) \to \Gamma(X, \mathbb{E}_zar(\Omega^*_X)) \to \Gamma(U, \mathbb{E}_zar(\Omega^*_U)) \to 0$$

we have $H^q \Gamma(Z(X, \mathbb{E}_zar(\Omega^*_Z))) = H^q(\Gamma(X, \mathbb{E}_zar(\Omega^*_X)))$. In particular, the map,

$$j^* := \text{ad}(j^*, j_*)(-): \Gamma(X, \mathbb{E}_zar(\Omega^*_X(\text{log} D), F_b)) \to \Gamma(U, \mathbb{E}_zar(\Omega^*_U, F_b))$$

is strict in $D_{fil, \infty}(\mathbb{C})$.
and hence the map
\[
\begin{aligned}
j^* &:= \text{ad}(j^*, j_*)(-): \text{Cone}(\Gamma(X, \Omega_X^\bullet, F_b)) \to \Gamma(X, E_{zar}(\Omega_X^\bullet(\log D), F_b)) \\
&\quad \to \text{Cone}(\Gamma(X, \Omega_X^\bullet, F_b)) \to \Gamma(U, E_{zar}(\Omega_X^\bullet, F_b)) =: \Gamma(X, \Gamma_Z E_{zar}(\Omega_X^\bullet, F_b))
\end{aligned}
\]
are quasi-isomorphisms (i.e. if we forgot filtrations), but the first one is NOT an \(\infty\)-filtered quasi-isomorphism whereas the second one is an \(\infty\)-filtered quasi-isomorphism (recall that for \(r > 1\) the \(r\)-filtered quasi-isomorphisms does NOT satisfy the 2 of 3 property for morphism of canonical triangles: see section 2.1).

(ii) More generally, let \(f: X \to S\) a smooth projective morphism with \(S, X \in \text{SmVar}(\mathbb{C})\). Let \(D = \cup D_i \subset X\) a normal crossing divisor such that \(f|_{D_i} := f \circ i_i : D_i \to S\) are \textsc{smooth} morphisms (note that it is a very special case), with \(i_i : D_i \hookrightarrow X\) the closed embeddings. Consider the open embedding \(j : U := X \setminus D \hookrightarrow X\) and \(h := f \circ j : U \to S\).

- The map in \(D_{\bullet\text{filt}, \infty}(S)\)
\[
\begin{aligned}
\text{Hom}(L\mathbb{D}(\mathbb{Z}(U)), k)^{-1} \circ \text{Hom}(\text{ad}(j^*, j_*)(\mathbb{Z}(X/X)), E_{et}(\Omega^\bullet_{/S}, F_b)) &:= F_S^{GM}(\mathbb{Z}(U/S)) := \text{Hom}(L\mathbb{D}(\mathbb{Z}(U)), E_{et}(\Omega^\bullet_{/S}, F_b)) \\
\quad \to \text{Hom}(\text{Cone}(\mathbb{Z}(D) \to \mathbb{Z}(X)), E_{zar}(\Omega^\bullet_{/S}, F_b)) = f_* E_{zar}(\Omega^\bullet_{/S}^{(\text{null} D), F_b}).
\end{aligned}
\]
is an isomorphism, where we recall \(D(U) := f_* j_* E_{et}(\mathbb{Z}(U/U)) = h_* E_{et}(\mathbb{Z}(U/U))\).

- \(F_S^{GM}(\mathbb{Z}(U/S)) = h_* E_{zar}^{\Omega^\bullet_{/S}, F_b} \in D_{\bullet\text{filt}, \infty}(S)\) is NOT isomorphic to \(f_* E_{zar}(\Omega^\bullet_{/S}(\log D), F_b)\) in \(D_{\bullet\text{filt}, \infty}(S)\) in general. In particular, the map,
\[
j^* := \text{ad}(j^*, j_*)(-): H^n f_* E_{zar}(\Omega^\bullet_{/S}(\log D)) \to H^n h_* E_{zar}(\Omega^\bullet_{/S})
\]
which is an isomorphism in \(D_{\bullet}(S)\) (i.e. if we forgot filtrations), gives embeddings
\[
j^* := \text{ad}(j^*, j_*)(-): F^p H^n h_* \mathbb{C}_U := F^p H^n f_* E_{zar}(\Omega^\bullet_{/S}(\log D), F_b) \to F^p H^n h_* E_{zar}(\Omega^\bullet_{/S}, F_b)
\]
which are NOT an isomorphism in general for \(n, p \in \mathbb{Z}\). Note that, since \(\mathbb{C} \to U \to \{\text{pt}\}\) is not proper,
\[
[\Delta_U]: \mathbb{Z}(U/S) \to h_* E_{et}(\mathbb{Z}(U/U))[2d_U]
\]
is NOT an equivalence \((\mathbb{A}^1, \text{et})\) local.

- Let \(Z \subset X\) a subvariety and denote \(U := X \setminus Z\) the open complementary. Denote \(M_Z(X/S) := \text{Cone}(M(U/S) \to M(X/S)) \in \text{DA}(S)\). If \(j_Z := f \circ i_Z : Z \to S\) is a \textsc{smooth} morphism, the map in \(D_{\bullet\text{filt}, \infty}(S)\)
\[
\begin{aligned}
\text{Hom}(\text{Cone}(\mathbb{Z}(X/Z), E_{et}(\Omega^\bullet_{/S}, F_b)) \circ \text{Hom}(\Gamma_Z^\bullet \mathbb{Z}(X/X)), k)^{-1} \\
\quad \to \text{Hom}(j_Z \Gamma_Z^\bullet \mathbb{Z}(X/X), E_{et}(\Omega^\bullet_{/S}, F_b)) \to \text{Hom}(\text{Cone}(\mathbb{Z}(X/Z), E_{et}(\Omega^\bullet_{/S}, F_b)) \\
\quad \to \text{Hom}(\text{Cone}(\mathbb{Z}(X/Z), E_{et}(\Omega^\bullet_{/S}, F_b)) \circ \text{Hom}(\Gamma_Z^\bullet \mathbb{Z}(X/X), k)^{-1})
\end{aligned}
\]
is an isomorphism, where \(c = \text{codim}(Z, X)\) and \(G(Z/X) := f_Z \Gamma_Z^\bullet \mathbb{Z}(X/X) \to \mathbb{Z}(Z/S)(c)[2c]\) is the \textsc{gysin} morphism.

- Let \(D \subset X\) a smooth divisor and denote \(U := X \setminus Z\) the open complementary. Note that the canonical distinguish triangle in \(\text{DA}(S)\)
\[
M(U/S) \xrightarrow{\text{ad}(j_1 j^*)} M(X/S) \xrightarrow{\gamma_Z^\bullet(\mathbb{Z}(X/X))} M_D(X/S) \to (M(U/S)[1]
\]
give a canonical triangle in \(D_{\bullet\text{filt}, \infty}(S)\)
\[
\begin{aligned}
F_S^{GM}(M_D(X/S)) &:= F_S^{GM}(\mathbb{Z}(X/S)) \to F_S^{GM}(M(X/S)) \xrightarrow{F_S^{GM}(\text{ad}(j_1 j^*))} F_S^{GM}(M(U/S)) \\
&\to F_S^{GM}(M_D(X/S))[1]
\end{aligned}
\]
which is NOT the image of a distinguish triangle in \(D(M\mathbb{H}M(S))\).
Proof. (i): For simplicity, we may assume that $i : D \rightarrow X$ is a smooth divisor. Then, by theorem 16, the map

$$(0, \text{ad}(j_1, j_+)(Z(X/X))) : Z(D) \rightarrow Z(X) \rightarrow \mathbb{D}(Z(U/U))$$

is an equivalence $(\mathbb{A}^1, et)$ local in $C(\text{SmVar}(\mathbb{C}))$. The result then follows from proposition 105. By theorem 16, we have an equivalence $(\mathbb{A}^1, et)$ local in $C(\text{SmVar}(\mathbb{C}))$

$$G(X, Z) : a_{X_2} \Gamma^*Z(X/X) \rightarrow Z(Z)(c)[2c]$$

The result then follows from proposition 105.

(ii): For simplicity, we may assume that $i : D \rightarrow X$ is a smooth divisor. Then, by theorem 16, the map

$$(0, \text{ad}(j_1, j_+)(Z(X/X))) : Z(D/S) \rightarrow Z(X/S) \rightarrow \mathbb{D}(Z(U/U))$$

is an equivalence $(\mathbb{A}^1, et)$ local in $C(\text{Var}(\mathbb{C})^{sm}/S)$. The result then follows from proposition 105. By theorem 16, we have an equivalence $(\mathbb{A}^1, et)$ local in $C(\text{SmVar}(\mathbb{C}))$

$$G(X, Z) : f_2 \Gamma^*Z(X/X) \rightarrow Z(Z)(c)[2c]$$

The result then follows from proposition 105.

Definition 126. Let $S \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in DA(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ projective such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{O_{\text{f}, D}(S)}$

$$T(F_S^{GM}, \otimes)(M, N) : F_S^{GM}(M) \otimes_{O_S}^{L[-]} F_S^{GM}(N) := (e(S), \text{Hom}(F, E_{ct}(\Omega^*_{S}, F_b))) \otimes_{O_S} (e(S), \text{Hom}(F, E_{ct}(\Omega^*_{S}, F_b)))[-d_S]$$

$$\xrightarrow{T(\otimes, \Omega_{S}(F, G))} e(S), \text{Hom}(F \otimes G, E_{ct}(\Omega^*_{S}, F_b))[-d_S]$$

Definition 127. Let $S \in \text{Var}(\mathbb{C})$ and $S = \bigcup_{i=1}^{l} S_i$ an open affine covering and denote, for $I \subset [1, \cdots, l]$, $S_I = \cap_{i \in I} S_i$ and $j_i : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_I \hookrightarrow S_i$ closed embeddings, with $S_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in DA(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{O_{\text{f}, D}(S/\{S_I\})}$

$$T(F_S^{-GM}, \otimes)(M, N) : F_S^{GM}(M) \otimes_{O_S}^{L[-]} F_S^{GM}(N) := (e(S_I), \text{Hom}(L(i_{I*}, j_I^*F), E_{ct}(\Omega^*_I, F_b))[-d_{S_I}], u_{I*}(F)) \otimes_{O_S}$$

$$(e(S_I), \text{Hom}(L(i_{I*}, j_I^*F), E_{ct}(\Omega^*_I, F_b))[-d_{S_I}], u_{I*}(G))$$

$$\xrightarrow{T(\otimes, \Omega_{S_I}(L(i_{I*}, j_I^*F), L(i_{I*}, j_I^*G)))} e(S_I), \text{Hom}(L(i_{I*}, j_I^*F) \otimes L(i_{I*}, j_I^*G), E_{ct}(\Omega^*_{S_I}, F_{DR}))[-d_{S_I}], u_{I*}(F \otimes G))$$

$$\xrightarrow{(e(S_I), \text{Hom}(L(i_{I*}, j_I^*F) \otimes L(i_{I*}, j_I^*G), E_{ct}(\Omega^*_{S_I}, F_{DR}))[-d_{S_I}], u_{I*}(F \otimes G))}$$

Proposition 112. Let $f_1 : X_1 \rightarrow S$, $f_2 : X_2 \rightarrow S$ two morphism with $X_1, X_2, S \in \text{Var}(\mathbb{C})$. Assume that there exist factorizations $f_1 : X_1 \xrightarrow{l_1} Y_1 \xrightarrow{p_S} S$, $f_2 : X_2 \xrightarrow{l_2} Y_2 \xrightarrow{p_S} S$ with $Y_1, Y_2 \in \text{SmVar}(\mathbb{C})$, $l_1, l_2$ closed embeddings and $p_S$ the projections. We have then the factorization

$$f_1 \times f_2 : X_{12} := X_1 \times_S X_2 \xrightarrow{l_1 \times l_2} Y_1 \times Y_2 \xrightarrow{p_S} S$$
Let $S = \bigcup_{i=1}^{l} S_i$ an open affine covering and denote, for $I \subset [1, \cdots, l]$, $S_I = \bigcap_{i \in I} S_i$ and $j_I : S_I \rightarrow S$ the open embedding. Let $i_t : S_t \rightarrow S_i$ closed embeddings, with $S_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following commutative diagram in $D_{O^f, PM}(S/(S_I))$

\[
\begin{array}{ccc}
\mathcal{F}^G_M(M(X_1/S) \otimes M(X_2/S)) & \xrightarrow{I^{GM}(X_1/S) \otimes I^{GM}(X_2/S)} & \mathcal{F}^G_M(M(X_1/S) \otimes M(X_2/S)) \\
(p_{S_i} \Gamma_{X_1} E_{zar}(\Omega^*_{Y_1 \times X_1/S_1}, F_b)[-d_{S_i}], w_{fJ}(X_1/S)) & \xrightarrow{O_{p}(p_{S_i} \Gamma_{X_1} E_{zar}(\Omega^*_{Y_1 \times X_1/S_1}, F_b)[-d_{S_i}], w_{fJ}(X_1/S))} & (p_{S_i} \Gamma_{X_2} E_{zar}(\Omega^*_{Y_2 \times X_2/S_1}, F_b)[-d_{S_i}], w_{fJ}(X_2/S))
\end{array}
\]

Proof. Immediate from definition. \[\square\]

6.1.2 The algebraic filtered De Rham realization functor and the commutativity with the six operations

We recall (see section 2), for $f : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$, the commutative diagrams of sites (29) and (30)

\[
\begin{array}{ccc}
\text{Var}(\mathbb{C})^2/T & \xrightarrow{\rho_T} & \text{Var}(\mathbb{C})^{2,pr}/T \\
P(f) & & P(f) \\
\text{Var}(\mathbb{C})^2/S & \xrightarrow{\rho_S} & \text{Var}(\mathbb{C})^{2,sm}/S \\
P(f) & & P(f) \\
\text{Var}(\mathbb{C})^{2,sm}/T & \xrightarrow{\rho_T} & \text{Var}(\mathbb{C})^2/T \\
P(f) & & P(f) \\
\text{Var}(\mathbb{C})^2/pr/S & \xrightarrow{\rho_S} & \text{Var}(\mathbb{C})^{2,sm}/S \\
P(f) & & P(f) \\
\text{Var}(\mathbb{C})^{2,sm}/T & \xrightarrow{\rho_T} & \text{Var}(\mathbb{C})^2/T \\
P(f) & & P(f) \\
\text{Var}(\mathbb{C})^{2,sm}/S & \xrightarrow{\rho_S} & \text{Var}(\mathbb{C})^2/pr/S
\end{array}
\]

Let $S \in \text{Var}(\mathbb{C})$. We have for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ the canonical map in $C(\text{Var}(\mathbb{C})^{sm}/S)$

\[
\text{Gr}(F) : \text{Gr}^{2, pr}_{\rho_S, \mu_S, F^T} \rightarrow F,
\]

\[
\text{Gr}(F)(U/S) : \Gamma_U^{\rho_S, F^T}(U \times S/U \times S) \xrightarrow{\text{ad}(l, l_*) \rho^* F(U \times S/U \times S)} h^* F(U/S) = F(U/S)
\]

where $h : U \rightarrow S$ is a smooth morphism with $U \in \text{Var}(\mathbb{C})$ and $h : U \xrightarrow{l} U \times S \xrightarrow{\rho} S$ is the graph factorization with $l$ the graph embedding and $p$ the projection.
For \( s : I \rightarrow J \) a functor, with \( I, J \in \text{Cat} \), and \( f_\bullet : T_\bullet \rightarrow S_\bullet \) a morphism with \( T_\bullet \in \text{Fun}(J, \text{Var}(C)) \) and \( S_\bullet \in \text{Fun}(I, \text{Var}(C)) \), we have then the commutative diagrams of sites (31) and (32)

We will use the following map from the property of mixed Hodge module (see section 5) together with the specialization map of a filtered D module for a closed embedding (see definition 56):

**Definition-Proposition 20.**

(i) Let \( l : Z \hookrightarrow S \) a closed embedding with \( S, Z \in \text{SmVar}(C) \). Consider an open embedding \( j : S^o \hookrightarrow S \). We then have the cartesian square

\[
\begin{array}{ccc}
S^o & \xrightarrow{j} & S \\
\downarrow{l'} & & \downarrow{l} \\
Z^o := Z \times_S S^o & \rightarrow & Z
\end{array}
\]

where \( j' \) is the open embedding given by base change. Using proposition 99, the morphisms \( Q^0_{V_Z,V_D}(O_S,F_b) \) for \( D \subset S \) a closed subset of definition-proposition 15 induces a canonical morphism in \( C^*_{\text{O}_{\text{fil}}}(Z) \)

\[
Q(Z,j')(O_S,F_b) : l^*Q^0_{V_Z,V_D}(O_S,F_b) \rightarrow j'_!Hdg(O_{Z^o},F_b),
\]

where \( V_Z \) is the Kashiwara-Malgrange \( V_Z \)-filtration and \( V_D \) is the Kashiwara-Malgrange \( V_D \)-filtration, which commutes with the action of \( T_Z \).

(ii) Let \( l : Z \hookrightarrow S \) and \( k : Z' \hookrightarrow Z \) be closed embeddings with \( S, Z, Z' \in \text{SmVar}(C) \). Consider an open
embedding \( j : S^o \to S \). We then have the commutative diagram whose squares are cartesian.

\[
\begin{array}{ccc}
S^o & \xrightarrow{j} & S \\
\downarrow{i'} & & \downarrow{i} \\
Z^o := Z \times_S S^o & \xrightarrow{j'} \cong & Z \\
\uparrow{k} & & \uparrow{k}
\end{array}
\]

where \( j' \) is the open embedding given by base change. Then,

\[
Q(Z',j_1)(O_S,F_0) = Q(Z',j_1')(O_Z,F_0) \circ (k^*Q_{Vz',0}Q(Z,j_1)(O_S,F_0)) : k^*Q_{Vz,0}l^*Q_{Vz,0}Hdg(O_S,F_b) \xrightarrow{Q(Z,j_1)(O_S,F_0)} k^*Q_{Vz,0}j_1'Hdg(O_Z,F_b) \xrightarrow{Q(Z',j_1')(O_Z,F_0)} j_1''Hdg(O_{Z^o},F_b)
\]

in \( C_{k^*l^*O_SfHil}(Z') \) which commutes with the action of \( T_{Z^o} \).

(iii) Consider a commutative diagram whose squares are cartesian

\[
\begin{array}{ccc}
S^o & \xrightarrow{j_2} & S^o \\
\downarrow{l'} & & \downarrow{l} \\
Z^o := Z \times_S S^o & \xrightarrow{j_2'} \cong & Z \\
\uparrow{j_1} & & \uparrow{j_1}
\end{array}
\]

where \( j_1, j_2 \), and hence \( j_1', j_2' \) are open embeddings. We have then the following commutative diagram

\[
l^*Q_{Vz,0}j_1'Hdg(O_{Z^o},F_b) \xrightarrow{\ad(j_2'Hdg,j_2')}{j_1'Hdg}(O_{S^o},F_b) \xrightarrow{Q(Z,j_1)(O_S,F_0)} l^*Q_{Vz,0}(j_1 \circ j_2)!Hdg(O_{S^o},F_b) \xrightarrow{Q(Z,(j_1 \circ j_2))}{j_1'Hdg}(O_{Z^o},F_b)
\]

in \( C_{l^*O_SfHil}(Z) \) which commutes with the action of \( T_Z \).

Proof. (i): By definition of \( j!Hdg : \pi_{S^o}(MHM(S^o)) \to \pi_S(C(MHM(S))) \), we have to construct the isomorphism for each complement of a (Cartier) divisor \( j = j_D : S^o = S\backslash D \to S \). In this case, we have the closed embedding \( i : S \hookrightarrow L \) given by the zero section of the line bundle \( L = L_D \) associated to \( D \). We have then, using definition-proposition 15, the canonical morphism in \( PSh_{l^*O_SfHil}(Z) \) which commutes with the action of \( T_Z \)

\[
Q(Z,j_1)(O_S,F_b) := \lim_{(D_1)\subseteq L} \left((Q(Z,j_D)(O_S,F_b)) : l^*Q_{Vz,0}j_1'Hdg(O_{S^o},F_b) \xrightarrow{T(i,l)(j_1')}{j_1'Hdg}(O_{Z^o},F_b) = j_1''Hdg(O_{Z^o},F_b)\right)
\]

and \( V_{Z^o}^pT_{(l,j)}(-1) = Q_{Vz,Vz}(i_{*mod}(O_S,F_b)) \). Now for \( j : S^o = S\backslash R \to S \) an arbitrary open embedding, we set

\[
Q(Z,j_1)(O_S,F_b) := \lim_{(D_1)\subseteq L} \left((Q(Z,j_D)(O_S,F_b)) : l^*Q_{Vz,0}j_1'Hdg(O_{S^o},F_b) \xrightarrow{T(i,l)(j_1')}{j_1'Hdg}(O_{Z^o},F_b) \right)
\]

(ii): Follows from definition-proposition 15.

(iii): Follows from definition-proposition 15.

\[\square\]
Lemma 11. (i) For \( g : T \to S \) and \( g' : T' \to T \) two morphism with \( S,T,T' \in \text{SmVar}(\mathbb{C}) \), considering the commutative diagram whose squares are cartesian

Definition 128. Consider a commutative diagram in \( \text{SmVar}(\mathbb{C}) \) whose square are cartesian

\[
\begin{array}{cccc}
Z_T & \xrightarrow{i'} & T & \xrightarrow{j'} \ T \setminus Z_T \\
\downarrow & & \downarrow & \downarrow \\
T \setminus Z & \xrightarrow{j} & T \times Z & \\
\downarrow & & \downarrow & \downarrow \\
Z & \xrightarrow{j} & S \setminus Z & \\
\end{array}
\]

where \( i \) and hence \( I \times i \) and \( i' \), are closed embeddings, \( j \), \( I \times j \), \( j' \) are the complementary open embeddings and \( g : T \xrightarrow{i} T \times S \xrightarrow{p_S} S \) is the graph factorization, where \( l \) is the graph embedding and \( p_S \) the projection. Then, the map in \( C_{T,O_T \times S,f} (T) \)

\[
s p_{V_T} (\Gamma^\vee, Hdg (O_{T \times S}, F_b)) : l^* \Gamma^\vee, Hdg (O_{T \times S}, F_b) \xrightarrow{\text{Q}(T,(I \times j))} l^* Q_{V_T,0} (\Gamma^\vee, Hdg (O_{T \times S}, F_b)) \xrightarrow{\text{Q}(T,(I \times j), (O_{T \times S},F_b))} \Gamma^\vee, Hdg (O_T,F_b)
\]

which commutes with the action of \( T_T \), where the first map is given in definition 56 and the last map is studied definition-proposition 20, factors through

\[
s p_{V_T} (\Gamma^\vee, Hdg (O_{T \times S}, F_b)) : l^* \Gamma^\vee, Hdg (O_{T \times S}, F_b) \xrightarrow{\text{Q}(T,(I \times j))} l^* Q_{V_T,0} (\Gamma^\vee, Hdg (O_{T \times S}, F_b)) \xrightarrow{\text{Q}(T,(I \times j), (O_{T \times S},F_b))} \Gamma^\vee, Hdg (O_T,F_b),
\]

with for \( \text{U} \subset T \times S \) an open subset, \( m \in \Gamma (\text{U}, O_{T \times S}) \) and \( h \in \Gamma (\text{U}, O_T) \), \( n(m) := n \otimes 1 \) and \( s p_{V_T} (-)(m \otimes h) = h \cdot s p_{V_T} (m) ; \) see definition-proposition 19, proposition 99 and theorem 28. Then,

\[
s p_{V_T} (\Gamma^\vee, Hdg (O_{T \times S}, F_b)) : l^* \Gamma^\vee, Hdg (O_{T \times S}, F_b) \to \Gamma^\vee, Hdg (O_T,F_b),
\]

is a map in \( C_{D,(1,0)} (T) \), i.e. is \( D_T \) linear. We then consider the canonical map in \( C_{D,(1,0)} (T) \)

\[
a(g,Z)(O_S,F_b) : g^* \Gamma^\vee, Hdg (O_S,F_b) \xrightarrow{\text{I}(\text{mod})} p^*_S \Gamma^\vee, Hdg (O_S,F_b) \xrightarrow{\text{I}(\text{mod})} \Gamma^\vee, Hdg (O_T,F_b).
\]
we have then
\[ a(g \circ g', Z)(O_S, F_b) = a(g', Z_T)(O_T, F_b) \circ (g'*mod a(g, Z)(O_S, F_b)) : \]
\[ (g \circ g')*mod \Gamma^* dg_{Z}(O_S, F_b) = \frac{\Gamma^* dg_{Z}(O_S, F_b) \circ (g'*mod a(g, Z)(O_S, F_b))}{\Gamma^* dg_{Z_T}(O_T, F_b)} \]

(ii) For \( g : T \rightarrow S \) a morphism with \( S, T \in \text{SmVar}(\mathbb{C}) \), considering the commutative diagram whose squares are cartesian

we have then the following commutative diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{k'} & Z_T \\
\downarrow{g} & & \downarrow{g} \\
Z & \xrightarrow{k} & S
\end{array}
\]

Proof. (i):Follows from definition-proposition 20 (ii)
(ii):Follows from definition-proposition 20 (iii)

We can now define the main object:

**Definition 129.** (i) For \( S \in \text{SmVar}(\mathbb{C}) \), we consider the filtered complexes of presheaves

\[ (\Omega^*_{/S}, F_{D_R}) \in C_{D_R}(\text{Var}(\mathbb{C})^{2,smpr}, S) \]

given by,

- for \( (Y \times S, Z)/S = ((Y \times S, Z), p) \in \text{Var}(\mathbb{C})^{2,smpr}, S \),

\[ (\Omega^*_{/S}((Y \times S, Z)/S), F_{D_R}) := ((\Omega^*_{Y \times S/S}, F_b) \otimes_{O_Y \times S} \Gamma^* dg_{Z}(O_{Y \times S}, F_b))(Y \times S) \]

with the structure of \( p^* D_S \) module given by proposition 64,

- for \( g : (Y_1 \times S, Z_1)/S = ((Y_1 \times S, Z_1), p_1) \rightarrow (Y \times S, Z)/S = ((Y \times S, Z), p) \) a morphism in \( \text{Var}(\mathbb{C})^{2,smpr}, S \), denoting for short \( Z := Z \times_{Y \times S} (Y_1 \times S) \),

\[ \Omega^*_{/S} : (\Omega^*_{Y \times S/S}, F_b) \otimes_{O_Y \times S} \Gamma^* dg_{Z}(O_{Y \times S}, F_b))(Y_1 \times S) \]

\[ DR(Y_1 \times S/S)(a(g, Z)(O_{Y \times S}, F_b))(Y_1 \times S) \]

\[ DR(Y_1 \times S/S)(T(Z_1/Z_T, \gamma' \cdot Hdg_{T'})(O_{Y_1 \times S}, F_b))(Y_1 \times S) \]

where

* \( i_- \) is the arrow of the inductive limit,
* we recall that
\[ \Omega(Y \times S/Y \times S)/S) \Gamma_Z^{\nu,H^d}(O_{Y \times S}, F_b) : g^*(\Omega(Y \times S/S, F_b) \otimes O_{Y \times S}) \Gamma_Z^{\nu,H^d}(O_{Y \times S}, F_b) \]
\[ \rightarrow (\Omega(Y \times S/S, F_b) \otimes O_{Y \times S}) g^{*\text{mod}} \Gamma_Z^{\nu,H^d}(O_{Y \times S}, F_b) \]
is the map given in definition-proposition 16, which is \( p_1^{\ast DS} \) linear by proposition 67,
* the map
\[ a(g, Z)(O_{Y \times S}, F_b) : g^{*\text{mod}} \Gamma_Z^{\nu,H^d}(O_{Y \times S}, F_b) \rightarrow \Gamma_Z^{\nu,H^d}(O_{Y \times S}, F_b) \]
is the map given in definition 128
* the map
\[ T(Z_1/\hat{Z}, \gamma^{\nu,H^d})(O_{Y \times S}, F_b) : \Gamma_Z^{\nu,H^d}(O_{Y \times S}, F_b) \rightarrow \Gamma_Z^{\nu,H^d}(O_{Y \times S}, F_b) \]
is given in definition-proposition 19.

For \( g : ((Y_1 \times S, Z_1), p_1) \rightarrow ((Y \times S, Z), p) \) and \( g' : ((Y'_1 \times S, Z'_1), p_1) \rightarrow ((Y_1 \times S, Z_1), p) \) two morphisms in \( \text{Var}(\mathbb{C})^{\ast \text{ampr}}/S \), we have
\[ \Omega_{/S}^{\Gamma,\text{pr}}(g \circ g') = \Omega_{/S}^{\Gamma,\text{pr}}(g) \circ \Omega_{/S}^{\Gamma,\text{pr}}(g') : ((\Omega_{Y \times S/S} \otimes O_{Y \times S}) \Gamma^{\nu,H^d}(O_{Y \times S}, F_b)(Y \times S) \]
\[ \xrightarrow{\Omega_{/S}^{\Gamma,\text{pr}}(g)} \Omega_{Y \times S/S}^{\Gamma,\text{pr}}(F_b) \otimes O_{Y \times S} \Gamma^{\nu,H^d}(O_{Y \times S}, F_b)(Y_1 \times S) \]
\[ \xrightarrow{\Omega_{/S}^{\Gamma,\text{pr}}(g')} \Omega_{Y \times S/S}^{\Gamma,\text{pr}}(F_b) \otimes O_{Y \times S} \Gamma^{\nu,H^d}(O_{Y \times S}, F_b)(Y'_1 \times S) \]
since, denoting for short \( \hat{Z} := Z \times Y \times S \) (\( Y_1 \times S \)) and \( \hat{Z}' := Z \times Y \times S \) (\( Y'_1 \times S \)),
- we have by lemma 11(i)
\[ a(g \circ g', \hat{Z}')(O_{Y \times S}, F_b) = a(g', \hat{Z})(O_{Y_1 \times S}, F_b) \circ g^{*\text{mod}} a(g, Z)(O_{Y \times S}, F_b) \]
- we have by lemma 11(ii)
\[ T(Z_1/\hat{Z}', \gamma^{\nu,H^d})(O_{Y_1 \times S}, F_b) \circ a(g', \hat{Z})(O_{Y \times S}, F_b) \]
\[ = a(g', Z_1)(O_{Y_1 \times S}, F_b) \circ g^{*\text{mod}} T(Z_1/\hat{Z}, \gamma^{\nu,H^d})(O_{Y \times S}, F_b). \]

(ii) For \( S \in \text{SnVar}(\mathbb{C}) \), we have the canonical map \( C_{O_S^{\ast \text{fil}}.DS}(\text{Var}(\mathbb{C})^{\ast \text{am}})/S) \)
\[ \text{Gr}(\Omega_{/S}) : \text{Gr}_{S}^{12}(\Omega_{/S}^{\Gamma,\text{pr}}, F_{DR}) \rightarrow (\Omega_{/S}^{\Gamma,\text{pr}}, F_b) \]
given by, for \( U/S = (U, h) \in \text{Var}(\mathbb{C})^{\ast \text{am}}/S \)
\[ \text{Gr}(\Omega_{/S})(U/S) : \text{Gr}_{S}^{12}(\Omega_{/S}^{\Gamma,\text{pr}}, F_{DR})(U/S) := ((\Omega_{U \times S/S} \otimes O_{U \times S}) \Gamma^{\nu,H^d}(O_{U \times S}, F_b)(U \times S) \]
\[ \xrightarrow{\text{ad}(u)} i^* ((\Omega_{U \times S/S} \otimes O_{U \times S}) \Gamma^{\nu,H^d}(O_{U \times S}, F_b))(U) \]
\[ \xrightarrow{DR(U/S)(a(i_{U}^{U}))} (\Omega_{U \times S/S}(U/S) \otimes O_{U \times S})^{i^{\ast \text{mod}} \Gamma^{\nu,H^d}(O_{U \times S}, F_b)}(U) \]
where \( h : U \xrightarrow{i_U} U \times S \xrightarrow{p_S} S \) is the graph factorization with \( i_U \) the graph embedding and \( p_S \) the projection, note that \( a(i_U, U) \) is an isomorphism since for \( j_U : U \times S \setminus U \rightarrow U \times S \) the open complementary \( i_U^{i\ast \text{mod}} \Gamma^{\nu,H^d}(M, F, W) = 0 \).
Definition 130. For $S \in \text{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_S \text{fil},D_S}(\text{Var}(\mathbb{C})^{2,sm}/S)$

$$T(\Omega^\Gamma_S) : \mu_{S*}(\Omega^\bullet_{/S}^\Gamma, F_b) \to (\Omega^\bullet_{/S}^\Gamma, F_{DR})$$

given by, for $(Y \times S, X)/S = ((Y \times S, Z), p) \in \text{Var}(\mathbb{C})^{2,sm}/S$

$$T(\Omega^\Gamma_S)((Y \times S, Z)/S) :$$

$$(\Omega^\bullet_{/S}^\Gamma, F_b)((Y \times S, Z)/S) := \mathbb{D}_{p^*OS}L_p^*\Omega_{\text{zar}}E_{\text{zar}}(\mathbb{D}_{p^*OS}L_p^*(\Omega^\bullet_{Y \times S/S}^\Gamma, F_b))(Y \times S)$$

$$\xrightarrow{DR(Y \times S/S)(\gamma_{Z}^\text{Hdg}(O_{Y \times S}))(Y \times S)}$$

$$\mathbb{D}_{p^*OS}L_p^*\Omega_{\text{zar}}E_{\text{zar}}((\Omega^\bullet_{Y \times S/S}^\Gamma, F_b) \otimes_{O_{Y \times S}}\Gamma_{Z}^\text{Hdg}(O_{Y \times S}, F_b))(Y \times S) \xrightarrow{=}$$

$$((\Omega^\bullet_{Y \times S/S}^\Gamma, F_b) \otimes_{O_{Y \times S}}\Gamma_{Z}^\text{Hdg}(O_{Y \times S}, F_b))(Y \times S) := (\Omega^\bullet_{/S}^\Gamma, F_{DR})(Y \times S)/S).$$

By definition $\text{Gr}(\Omega/S) \circ \text{Gr}_s, T(\Omega^\Gamma_S) = \text{Gr}^O(\Omega/S)$.

Remark 11. (i) Let $S \in \text{Var}(\mathbb{C})$. We have by definition $\alpha_{2*}(\Omega^\bullet_{/S}^\Gamma, F_b) = (\Omega^\bullet_{/S}^\Gamma, F_b) \in C_{O_S \text{fil},D_S}(\text{Var}(\mathbb{C})^{sm}/S)$. Moreover, if $S \in \text{SmVar}(\mathbb{C})$, $\alpha_{2*}(\Omega^\bullet_{/S}^\Gamma, F_b) = (\Omega^\bullet_{/S}^\Gamma, F_b) \in C_{O_S \text{fil},D_S}(\text{Var}(\mathbb{C})^{sm}/S)$.

(ii) Let $S \in \text{Var}(\mathbb{C})$. Then, $(\Omega^\bullet_{/S}^\Gamma, F_b) \in C_{O_S \text{fil},D_S}(\text{Var}(\mathbb{C})^{2}/S)$ is a natural extension of $(\Omega^\bullet_{/S}^\Gamma, F_b) := \rho_{S*}(\Omega^\bullet_{/S}^\Gamma, F_b) \in C_{O_S \text{fil},D_S}(\text{Var}(\mathbb{C})^{2,sm}/S)$,

but does NOT satisfy cdh descent.

We have the following canonical transformation map given by the pullback of (relative) differential forms:

- Let $g : T \to S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. We have the canonical morphism in $C_{g^*O_S \text{fil},g^*D_S}(\text{Var}(\mathbb{C})^{2,sm}/T)$

$$\Omega^\Gamma_{/(T/S)} : g^*(\Omega^\bullet_{/S}^\Gamma, F_b) \to (\Omega^\bullet_{/T}^\Gamma, F_b)$$

induced by the pullback of differential forms: for $((V, Z_1)/T) = ((V, Z_1), h) \in \text{Var}(\mathbb{C})^{2,sm}/T$,

$$g^*\Omega^\Gamma_{/S}((V, Z_1)/T) := \lim_{(h; (U, Z) \to \text{SSm}, g; (V, Z_1) \to (U, Z_2), h, g)} \Omega^\Gamma_{/(U, Z)/S}((V, Z)/S)$$

$$\xrightarrow{\Omega^\Gamma_{/(U, Z)/S}(g^*\alpha_{g^{sm}})} \Omega^\Gamma_{/(U, Z)/S}((V, Z_1)/S) \xrightarrow{\Gamma_Z^\text{Hdg} h q_{(Y, T)}} \Omega^\Gamma_{/(U, Z_1)/T},$$

where $g^T : U_T := U \times S T \to U$ is the base change map and $q : \Omega^\bullet_{Y_1 \times T/S} \to \Omega^\bullet_{Y_1 \times T/T}$ is the quotient map. If $T, S \in \text{SmVar}(\mathbb{C})$,

$$\Omega^\Gamma_{/(T/S)} : g^*(\Omega^\bullet_{/S}^\Gamma, F_b) \to (\Omega^\bullet_{/T}^\Gamma, F_b)$$

is a morphism in $C_{g^*O_S \text{fil},g^*D_S}(\text{Var}(\mathbb{C})^{2,sm}/T)$ It induces the canonical morphisms in $C_{g^*O_S \text{fil},g^*D_S}(\text{Var}(\mathbb{C})^{2,sm}/T)$:

$$E_{\text{et}}^\Gamma_{/(T/S)} : g^*E_{\text{et}}(\Omega^\bullet_{/S}^\Gamma, F_b) \xrightarrow{T(g,E_{\text{et}})(\Omega^\bullet_{/S}^\Gamma, F_b)} E_{\text{et}}(g^*(\Omega^\bullet_{/S}^\Gamma, F_b)) \xrightarrow{E_{\text{et}}(\Omega^\Gamma_{/(T/S)})} E_{\text{et}}(\Omega^\bullet_{/T}^\Gamma, F_b)$$

and

$$E_{\text{zar}}^\Gamma_{/(T/S)} : g^*E_{\text{zar}}(\Omega^\bullet_{/S}^\Gamma, F_b) \xrightarrow{T(g,E_{\text{zar}})(\Omega^\bullet_{/S}^\Gamma, F_b)} E_{\text{zar}}(g^*(\Omega^\bullet_{/S}^\Gamma, F_b)) \xrightarrow{E_{\text{zar}}(\Omega^\Gamma_{/(T/S)})} E_{\text{zar}}(\Omega^\bullet_{/T}^\Gamma, F_b).$$

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• Let \( g : T \to S \) a morphism with \( T, S \in \text{SmVar}(\mathbb{C}) \). We have the canonical morphism in \( C_{g^*D_{S,fib}}(\text{Var}(\mathbb{C}) \Sigma^{2,\text{smpr}}) / T \)

\[
\Omega^*_T / (T / S) : g^*(\Omega^*_S \otimes, F_{DR}) \to (\Omega^*_T)^{\Gamma,pr}, F_{DR})
\]

induced by the pullback of differential forms: for \( ((Y_1 \times T, Z_1)) / T = ((Y_1 \times T, Z_1), p) \in \text{Var}(\mathbb{C}) \Sigma^{2,\text{smpr}}) / T \)

\[
g^*\Omega^*_S (Y_1 \times T, Z_1) / T := \lim_{h: (Y \times S, Z) \to S, g_1 : (Y_1 \times T, Z_1) \to (Y \times T, Z_T), h, g} \Omega^*_S (Y \times T, Z) / S
\]

where \( g' = (I_Y \times g) : Y \times T \to Y \times S \) is the base change map and \( q(M) : \Omega_{Y_1 \times T / S} \otimes \Omega_{Y_1 \times T} \to \Omega_{Y_1 \times T / S} \) is the quotient map. It induces the canonical morphisms in \( C_{g^*D_{S,fib}}(\text{Var}(\mathbb{C}) \Sigma^{2,\text{smpr}}) / T \):

\[
E_{\Omega^*_T / (T / S)} : g^*E_{et}(\Omega^*_S \otimes, F_{DR}) \xrightarrow{T(g,E)(-)} E_{et}(g^*(\Omega^*_S \otimes, F_{DR})) \xrightarrow{E_{et}(\Omega^*_T / (T / S))} E_{et}(\Omega^*_T \otimes, F_{DR})
\]

and

\[
E_{\Omega^*_T / (T / S)} : g^*E_{zar}(\Omega^*_S \otimes, F_{DR}) \xrightarrow{T(g,E)(-)} E_{zar}(g^*(\Omega^*_S \otimes, F_{DR})) \xrightarrow{E_{zar}(\Omega^*_T / (T / S))} E_{zar}(\Omega^*_T \otimes, F_{DR}).
\]

Definition 131. Let \( g : T \to S \) a morphism with \( T, S \in \text{SmVar}(\mathbb{C}) \). We have, for \( F \in C(\text{Var}(\mathbb{C}) \Sigma^{2,\text{smpr}}) / S \), the canonical transformation in \( C_{D_{fj}}(T) \):

\[
T(g, \Gamma(S)(F)) : g^*\text{mod } L_{De}(S), \text{Gr}^2 \text{Hom}^*(F, E_{et}(\Omega^*_S \otimes, F_{DR})) \xrightarrow{T(g,\Gamma(S))(-) \circ T(e,g)(-)} e(T), \text{Gr}^2 \text{Hom}^*(F, E_{et}(\Omega^*_S \otimes, F_{DR})) \otimes g^*O_S OT
\]

which gives the map

\[
w_{U / S} : \Omega^*_U / S \otimes O_S \to \Omega^*_U / S \otimes O_S \to \Omega^*_U / S \otimes O_S \otimes \alpha \otimes \beta \mapsto \alpha \wedge \beta.
\]

Let \( S \in \text{Var}(\mathbb{C}) \) and \( h_1 : U_1 \to S, h_2 : U_2 \to S \) two morphisms with \( U_1, U_2 \in \text{Var}(\mathbb{C}) \). Denote \( h_{12} : U_{12} := U_1 \times S U_2 \to S \) and \( p_{112} : U_1 \times S U_2 \to U_1, p_{212} : U_1 \times S U_2 \to U_2 \) the projections. Recall we have the canonical map given by the wedge product

\[
w_{(U_1, U_2) / S} : \Omega^*_{U_1 / S} \otimes O_S \otimes p^*_{212} \Omega^*_{U_2 / S} \to \Omega^*_{U_{12} / S} \otimes \alpha \otimes \beta \mapsto p_{112}^* \alpha \wedge p_{212}^* \beta
\]

which gives the map

\[
Ew_{(U_1, U_2) / S} : h_{12} \circ E_{zar}(\Omega^*_1 / S) \otimes O_S \otimes h_2 \circ E_{zar}(\Omega^*_2 / S) \to h_{12} \circ E_{zar}(p_{112}^* \Omega^*_{U_1 / S} \otimes O_S \otimes p_{212}^* \Omega^*_{U_2 / S})
\]

Let \( S \in \text{SmVar}(\mathbb{C}) \).
The complex of presheaves \((\Omega^\bullet, F_b) \in C_{O_S} fil, D_S(Var(\mathbb{C})^{2, sm}/S)\) have a monoidal structure given by the wedge product of differential forms: for \(h : (U, Z) \to S \in Var(\mathbb{C})^2/S\), the map

\[ DR(-)(\gamma^\vee, h(-)) \circ w_U/S : (\Omega^\bullet/S, F_b) \otimes_{pr} (\Omega^\bullet/S, F_b) \to \Gamma^\vee_S Lh^* O_S (\Omega^\bullet/S, F_b) \]

factors through

\[ DR(-)(\gamma^\vee, Hdg(-)) \circ w_U/S : (\Omega^\bullet/S, F_b) \otimes_{pr} (\Omega^\bullet/S, F_b) \to \Gamma^\vee_S Lh^* O_S (\Omega^\bullet/S, F_b) \]

unique up to homotopy, giving the map in \(C_{O_S} fil, D_S(Var(\mathbb{C})^{2, smpr}/S)\):

\[ w_S : (\Omega^\bullet, F_b) \otimes_{os} (\Omega^\bullet, F_b) \to (\Omega^\bullet, F_b) \]

given by for \(h : (U, Z) \to S \in Var(\mathbb{C})^{2, sm}/S\),

\[ w_S((U, Z)/S) : (\Gamma^\vee_S Lh^* O_S (\Omega^\bullet/S, F_b) \otimes_{pr} \Gamma^\vee_S Lh^* O_S (\Omega^\bullet/S, F_b))(U) \]

which induces the map in \(C_{O_S} fil, D_S(Var(\mathbb{C})^{2, sm}/S)\)

\[ Ew_S : E_{et}(\Omega^\bullet/S, F_b) \otimes_{os} E_{et}(\Omega^\bullet/S, F_b) \to E_{et}(\Omega^\bullet/S, F_b) \to E_{et}(\Omega^\bullet/S, F_b) \]

given by the functoriality of the Godement resolution (see section 2).

The complex of presheaves \((\Omega^\bullet, F_{DR}) \in C_{D_S} fil(Var(\mathbb{C})^{2, smpr}/S)\) have a monoidal structure given by the wedge product of differential forms: for \(p : (Y \times S, Z) \to S \in Var(\mathbb{C})^{2, smpr}/S\), the map

\[ DR(-)(\gamma^\vee, Hdg(-)) \circ w_{Y \times S/S} : (\Omega^\bullet/Y \times S/S \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \otimes_{pr} (\Omega^\bullet/Y \times S/S \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \to \Omega^\bullet/Y \times S/S \otimes_{O_{Y \times S}} \Gamma^\vee_S Lh^* O_S (O_{Y \times S}, F_b) \]

factors through

\[ DR(-)(\gamma^\vee, Hdg(-)) \circ w_{Y \times S/S} : (\Omega^\bullet/Y \times S/S \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \otimes_{pr} (\Omega^\bullet/Y \times S/S \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \to \Omega^\bullet/Y \times S/S \otimes_{O_{Y \times S}} \Gamma^\vee_S Lh^* O_S (O_{Y \times S}, F_b) \]

unique up to homotopy, giving the map in \(C_{D_S} fil(Var(\mathbb{C})^{2, smpr}/S)\):

\[ w_S : (\Omega^\bullet, F_{DR}) \otimes_{os} (\Omega^\bullet, F_{DR}) \to (\Omega^\bullet, F_{DR}) \]

given by for \(p : (Y \times S, Z) \to S \in Var(\mathbb{C})^{2, smpr}/S\),

\[ (((\Omega^\bullet/Y \times S/S \otimes_{O_{Y \times S}} \Gamma^\vee_S Lh^* O_S (O_{Y \times S}, F_b)) \otimes_{pr} (\Omega^\bullet/Y \times S/S \otimes_{O_{Y \times S}} \Gamma^\vee_S Lh^* O_S (O_{Y \times S}, F_b))(Y \times S) \]

unique up to homotopy, giving the map in \(C_{D_S} fil(Var(\mathbb{C})^{2, smpr}/S)\):

\[ w_S((Y \times S, Z)/S) : \]

\[ (((\Omega^\bullet/Y \times S/S \otimes_{O_{Y \times S}} \Gamma^\vee_S Lh^* O_S (O_{Y \times S}, F_b)) \otimes_{pr} (\Omega^\bullet/Y \times S/S \otimes_{O_{Y \times S}} \Gamma^\vee_S Lh^* O_S (O_{Y \times S}, F_b))(Y \times S) \]

unique up to homotopy, giving the map in \(C_{D_S} fil(Var(\mathbb{C})^{2, smpr}/S)\):

\[ w_S((Y \times S, Z)/S) : \]
which induces the map in $C_{D_{fil}}(\text{Var}(\mathbb{C})^{2, smpr}/S)$

$$Ew_S : E_{et}(\Omega^\bullet_{/S}^{, pr}, F_{DR}) \otimes_{O_S} E_{et}(\Omega^\bullet_{/S}^{, pr}, F_{DR}) \xrightarrow{\phi}$$

by the functoriality of the Godement resolution (see section 2).

**Definition 132.** Let $S \in \text{SmVar}(\mathbb{C})$. We have, for $F, G \in C(\text{Var}(\mathbb{C})^{2, smpr}/S)$, the canonical transformation in $C_{D_{fil}}(S)$:

$$T(\otimes, \Omega)(F, G) : e(S)_* \text{Gr}_{S_*}^{12} \text{Hom}(F, E_{et}(\Omega^\bullet_{/S}^{, pr}, F_{DR})) \otimes_{O_S} e(S)_* \text{Gr}_{S_*}^{12} \text{Hom}(G, E_{et}(\Omega^\bullet_{/S}^{, pr}, F_{DR}))$$

This gives the diagram of algebraic varieties $T(\otimes, \Omega)(F, G)$.

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^{l} S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow S_j$ closed embeddings, with $S_i \subset \text{Var}(\mathbb{C})$. For $I \subset [1, \ldots, l]$, denote by $S_I = \Pi_{i \in I} S_i$. We then have closed embeddings $i_I : S_I \hookrightarrow S_I$ and for $J \subset I$ the following commutative diagram

$$D_{IJ} = S_I \xrightarrow{i_I} S_J \xrightarrow{p_{IJ}} S_I$$

where $p_{IJ} : S_J \rightarrow S_I$ is the projection and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_I$. This gives the diagram of algebraic varieties $(S_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Var}(\mathbb{C}))$ which the diagram of sites $\text{Var}(\mathbb{C})^{2, smpr}/(S_I) \in \text{Fun}(\mathcal{P}(\mathbb{N}), \text{Cat})$. This gives also the diagram of algebraic varieties $(S_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N})^{op}, \text{Var}(\mathbb{C}))$ which the diagram of sites $\text{Var}(\mathbb{C})^{2, smpr}/(S_I)^{op} \in \text{Fun}(\mathcal{P}(\mathbb{N})^{op}, \text{Cat})$. We then get

$$(\Omega^\bullet_{/S_I}(\Omega^\bullet_{/S_I})(F_{DR})[-d_{S_I}], T_{IJ}) \in C_{D_{IK}}(\text{Var}(\mathbb{C})^{2, smpr}/(S_I))$$

with

$$T_{IJ} : (\Omega^\bullet_{/S_I}(\Omega^\bullet_{/S_I})(F_{DR})[-d_{S_I}] \xrightarrow{\text{ad}_{p_{IJ}, j_{IJ}}(-)} p_{IJ}^* p_{IJ}^* (\Omega^\bullet_{/S_I}(\Omega^\bullet_{/S_I})(F_{DR})[-d_{S_I}] \otimes_{O_{S_I}} O_{S_I} \xrightarrow{m_{p_{IJ}, j_{IJ}}} p_{IJ}^* (\Omega^\bullet_{/S_J}(\Omega^\bullet_{/S_J})(F_{DR})[-d_{S_J}]).$$

For $(G_I, K_{IJ}) \in C(\text{Var}(\mathbb{C})^{2, smpr}/(S_I)^{op})$, we denote (see section 2)

$$e'((S_I)), \text{Hom}((G_I, K_{IJ}), (E_{zar}(\Omega^\bullet_{/S_I}(\Omega^\bullet_{/S_I})(F_{DR})[-d_{S_I}], T_{IJ}) := (e'((S_I)), \text{Hom}(G_I, E_{zar}(\Omega^\bullet_{/S_I}(\Omega^\bullet_{/S_I})(F_{DR})[-d_{S_I}], u_{IJ}((G_I, K_{IJ})) ) \in C_{D_{fil}}((S_I))$$

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with

\[
\begin{align*}
    u_{IJ}((G_I, K_{IJ})) : e'(\tilde{S}_I), \mathcal{H}om(G_I, E_{zar}(\Omega^{\bullet, pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}] \\
    \text{ad}(\rho_{IJ}^{-1}, \rho_{IJ}, (-) \circ T(p_{IJ}, e)(-) \circ \mathcal{T}(p_{IJ}, e)) & \rightarrow p_{IJ}e'(\tilde{S}_J, p_{IJ}^* \mathcal{H}om(G_I, E_{zar}(\Omega^{\bullet, pr}_{/\tilde{S}_I}, F_{DR})) \otimes p_{IJ}O_{\tilde{S}_J}[-d_{\tilde{S}_J}] \\
    \mathcal{T}(p_{IJ}, \mathcal{H}om)(-,-) & \rightarrow p_{IJ}e'(\tilde{S}_J, \mathcal{H}om(p_{IJ}^* G_I, p_{IJ}^* E_{zar}(\Omega^{\bullet, pr}_{/\tilde{S}_I}, F_{DR})) \otimes p_{IJ}O_{\tilde{S}_J}[-d_{\tilde{S}_J}] \\
    \mathcal{H}om(K_{IJ}, E_{zar}(\Omega^{\bullet, pr}_{/\tilde{S}_J}, F_{DR})) & \rightarrow p_{IJ}e'(\tilde{S}_J), \mathcal{H}om(p_{IJ}^* G_I, E_{zar}(\Omega^{\bullet, pr}_{/\tilde{S}_J}, F_{DR}))[-d_{\tilde{S}_J}].
\end{align*}
\]

This gives in particular

\[
(\Omega^{\bullet, pr}_{/\tilde{S}_J}, F_{DR})[-d_{\tilde{S}_J}], T_{IJ}) \in C_{D_{(\tilde{S}_J), fil}(\text{Var}(\mathbb{C})^{2,(sm)pr}/(\tilde{S}_J)^{op}).
\]

We now define the filtered De Rahm realization functor.

**Definition 133.** (i) Let \( S \in \text{SmVar}(\mathbb{C}) \). We have, using definition 129 and definition 41, the functor

\[
\mathcal{F}^{_{F_{DR}}} : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{D_{fil}}(S),\ F \mapsto
\mathcal{F}^{_{F_{DR}}}(F) := e(S)_* \mathcal{G}^{12}_{S} \mathcal{H}om^{\bullet}(\mathcal{R}^{\mathcal{C}H}(\rho_{S}^* L(F)), E_{zar}(\Omega^{\bullet, pr}_{/S}, F_{DR}))[-d_{S}]
\]

(ii) Let \( S \in \text{Var}(\mathbb{C}) \) and \( S = \bigcup_{i=1}^{l} S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). For \( I \subset \{1, \cdots, l\} \), denote by \( S_I := \bigcap_{i \in I} S_i \) and \( j_I : S_I \hookrightarrow S \) the open embedding. We then have closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i := \Pi_{\tilde{S}_i \in I} \). Consider, for \( I \subset J \), the following commutative diagram

\[
\begin{array}{ccc}
    D_{IJ} & \xrightarrow{i_i} & \tilde{S}_I \\
    \downarrow{j_{IJ}} & & \downarrow{p_{IJ}} \\
    S_I & \xrightarrow{i_j} & \tilde{S}_J
\end{array}
\]

and \( j_{IJ} : S_I \rightarrow S_J \) is the open embedding so that \( j_I \circ j_{IJ} = j_J \). We have, using definition 129 and definition 41, the functor

\[
\mathcal{F}^{_{F_{DR}}} : C(\text{Var}(\mathbb{C})^{sm}/S) \rightarrow C_{D_{fil}}(S/\tilde{S}_I),\ F \mapsto
\mathcal{F}^{_{F_{DR}}}(F) := e'(\tilde{S}_I), \mathcal{H}om^{\bullet}(\mathcal{R}^{\mathcal{C}H}(\rho_{\tilde{S}_I}^* L(i_i j_I F)), \mathcal{R}^{\mathcal{C}H}(\rho_{S_I}^* T^{q}(D_{IJ})(j_I^* F))),
\]

\[
\mathcal{H}om^{\bullet}(\mathcal{R}^{\mathcal{C}H}(\rho_{S_I}^* L(i_i j_I F)), E_{zar}(\Omega^{\bullet, pr}_{/S_I}, F_{DR}))[-d_{S_I}], T_{IJ})
\]

\[
:= (e'(\tilde{S}_I), \mathcal{H}om^{\bullet}(\mathcal{R}^{\mathcal{C}H}(\rho_{\tilde{S}_I}^* L(i_i j_I F)), E_{zar}(\Omega^{\bullet, pr}_{/S_I}, F_{DR}))[-d_{S_I}], u_{IJ}^q(F))
\]

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where we have denoted for short $e'(\tilde{S}_i) = e(\tilde{S}_i) \circ \text{Gr}_{\tilde{S}_i}^{12}$, and

\[
\begin{align*}
&\text{ad}^{(p_{ij}^{\text{smpr}}, p_{ij})} \circ p_{ij} \circ e' \circ (\tilde{S}_i)_* \text{Hom}^*(\hat{R}^CH(p_{\tilde{S}_i}^* L(1_{i,J}^* F)), E_{2ar}(\Omega^*_{/\tilde{S}_i}, F_{DR})) \\
&= p_{ij} \circ e' \circ (\tilde{S}_i)_* \text{Hom}^*(\hat{R}^CH(p_{\tilde{S}_i}^* L(1_{i,J}^* F)), E_{2ar}(\Omega^*_{/\tilde{S}_i}, F_{DR})) \\
&= p_{ij} \circ e' \circ (\tilde{S}_i)_* \text{Hom}^*(\hat{R}^CH(p_{\tilde{S}_i}^* L(1_{i,J}^* F)), E_{2ar}(\Omega^*_{/\tilde{S}_i}, F_{DR})).
\end{align*}
\]

For $I \subset J \subset K$, we have obviously $p_{ij} \circ u_{jk}(F) = u_{ik}(F)$.

Recall, see section 2, that we have the projection morphisms of sites $p_a : \text{Var}(\mathbb{C})^{2,\text{smpr}}/(\tilde{S}_i)^{op} \rightarrow \text{Var}(\mathbb{C})^{2,\text{smpr}}/(\tilde{S}_i)^{op}$ given by the functor

\[
p_a : \text{Var}(\mathbb{C})^{2,\text{smpr}}/(\tilde{S}_i)^{op} \rightarrow \text{Var}(\mathbb{C})^{2,\text{smpr}}/(\tilde{S}_i)^{op},
\]

\[
p_a((Y \times \tilde{S}_i, Z_{ij})/\tilde{S}_i, s_{ij}) := ((Y_i \times \mathbb{A}^1 \times \tilde{S}_i, Z_{ij} \times \mathbb{A}^1)/\tilde{S}_i, s_{ij} \times I),
\]

\[
p_a((g_i) : ((Y_i' \times \tilde{S}_i, Z_{ij}')/\tilde{S}_i, s_{ij}) \rightarrow ((Y_i \times \tilde{S}_i, Z_{ij})/\tilde{S}_i, s_{ij})) = (g_i \times I) : ((Y_i' \times \mathbb{A}^1 \times \tilde{S}_i, Z_{ij}' \times \mathbb{A}^1)/\tilde{S}_i, s_{ij}' \times I) \rightarrow ((Y_i \times \mathbb{A}^1 \times \tilde{S}_i, Z_i \times \mathbb{A}^1)/\tilde{S}_i, s_{ij} \times I)).
\]

We have the following key proposition :

**Proposition 113.** (i) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^3 S_i$ an open cover such that there exist closed embeddings $i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. The complex of presheaves $(\Omega^*_{/\tilde{S}_i}, F_{DR}) \in C_{D_{\text{fil}}}(\text{Var}(\mathbb{C})^{2,\text{smpr}}/(\tilde{S}_i)^{op})$ is 2-filtered $\mathbb{A}^1$ homotopic, that is

\[
\text{ad}(p_a, p_{is}) \circ (\Omega^*_{/\tilde{S}_i}, F_{DR}) : (\Omega^*_{/S}, F_{DR}) \rightarrow p_{is} \circ p_a \circ (\Omega^*_{/\tilde{S}_i}, F_{DR})
\]

is a 2-filtered homotopy.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$. The complex of presheaves $(\Omega^*_{/S}, F_{DR}) \in C_{D_{\text{fil}}}(\text{Var}(\mathbb{C})^{2,\text{smpr}}/S)$ admits transfers, i.e.

\[
\text{Tr}(S), \text{Tr}(S)^* \circ (\Omega^*_{/S}, F_{DR}) = (\Omega^*_{/S}, F_{DR}).
\]

(iii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^3 S_i$ an open cover such that there exist closed embeddings $i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $m = (m_1) : (Q_{11}, K_{11}) \rightarrow (Q_{21}, K_{21})$ be an equivalence (cf., et.) local with $(Q_{11}, K_{11}) \rightarrow (Q_{21}, K_{21}) \in C(\text{Var}(\mathbb{C})^{smpr}/(\tilde{S}_i)^{op})$ complexes of representable presheaves. Then, the map in $C_{D_{\text{fil}}}(\tilde{S}_i))$

\[
M := (e(\tilde{S}_i), H^* (m_1, E_{2ar}(\Omega^*_{/\tilde{S}_i}, F_{DR})[-d_{\tilde{S}_i}])) :
\]

\[
e'(\tilde{S}_i), H^* ((Q_{21}, K_{21}), (E_{2ar}(\Omega^*_{/\tilde{S}_i}, F_{DR})[-d_{\tilde{S}_i}], T_{II}))
\]

\[
e'(\tilde{S}_i), H^* ((Q_{11}, K_{11}), (E_{2ar}(\Omega^*_{/\tilde{S}_i}, F_{DR})[-d_{\tilde{S}_i}], T_{II}))
\]

is a 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{D_{\text{fil}}}(\tilde{S}_i))$.  

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Proof. (i1): Let $(Y \times S, Z)/S = ((Y \times S, Z), p) \in \operatorname{Var}(\mathbb{C})^{2, smpr}/S$ so that $p_a : (Y \times \mathbb{A}^1 \times S, Z \times \mathbb{A}^1) \to (Y \times S, Z)$ is the projection, and $i_0 : (Y \times S, Z) \to (Y \times \mathbb{A}^1 \times S, Z \times \mathbb{A}^1)$ the closed embedding. Then,

$$a(p_a, Z) : p_a^* \cdot \Gamma^\vee_{Z,s, Hdg}(O_{Y \times S}, F_b) \to \Gamma^\vee_{Z,s, Hdg}(O_{Y \times S}, F_b).$$

is a quasi-isomorphism in $\pi_{Y \times \mathbb{A}^1 \times S}(C(MHM(Y \times \mathbb{A}^1 \times S))).$ Since a morphism of mixed Hodge module is strict for the $F$-filtration,

$$a(p_a, Z) : p_a^* \cdot \Gamma^\vee_{Z,s, Hdg}(O_{Y \times S}, F_b) \to \Gamma^\vee_{Z,s, Hdg}(O_{Y \times S}, F_b).$$

is a filtered quasi-isomorphism in $C_{D, jil}(Y \times \mathbb{A}^1 \times S).$ Hence, as

$$I(p_a^*, p_a^*)(-,-)(\Omega_{(Y \times \mathbb{A}^1 \times S)/S}(\mathcal{M}_Z, F_b) : (\Omega_{Y \times S, F_b} \otimes O_{Y \times S} \Gamma^\vee_{Z,s, Hdg}(O_{Y \times S}, F_b)) : (\Omega_{Y \times S, F_b} \otimes O_{Y \times S} \Gamma^\vee_{Z,s, Hdg}(O_{Y \times S}, F_b)) \to (\Omega_{Y \times S, F_b} \otimes O_{Y \times S} (\Gamma^\vee_{Z,s, Hdg}(O_{Y \times S}, F_b)))$$

is a 2-filtered homotopy equivalence whose inverse is

$$p_a^* \cdot I(i_0^*, i_0^*)(-,-)(\Omega_{(Y \times \mathbb{A}^1 \times S)/S}(\mathcal{M}_Z, F_b) : (\Omega_{Y \times S, F_b} \otimes O_{Y \times S} \Gamma^\vee_{Z,s, Hdg}(O_{Y \times S}, F_b)) : (\Omega_{Y \times S, F_b} \otimes O_{Y \times S} \Gamma^\vee_{Z,s, Hdg}(O_{Y \times S}, F_b)) \to (\Omega_{Y \times S, F_b} \otimes O_{Y \times S} (\Gamma^\vee_{Z,s, Hdg}(O_{Y \times S}, F_b)))$$

(see the proof of proposition 105), the map

$$\operatorname{ad}(p_a^*, p_a^*) : (\Omega^\bullet_{S, F}^{\mathcal{M}_pr}, F_{DR}) \to (\Omega^\bullet_{S, F}^{\mathcal{M}_pr}, F_{DR})$$

is an homotopy equivalence whose inverse is

$$\operatorname{ad}(i_0^*, i_0^*) : (\Omega^\bullet_{S, F}^{\mathcal{M}_pr}, F_{DR}) \to (\Omega^\bullet_{S, F}^{\mathcal{M}_pr}, F_{DR}).$$

(iii): Let us show that $\Omega^\bullet_{S, F}^{\mathcal{M}_pr} \in C_{D_{jil}}(\operatorname{Var}(\mathbb{C})^{2, smpr}/S)$ admits transferts. Let $\alpha \in \operatorname{Cor}(\operatorname{Var}(\mathbb{C})^{2, smpr}/S)((Y_1 \times S, Z_1)/S, (Y_2 \times S, Z_2)/S)$ irreducible. Denote by $i : \alpha \to Y_1 \times Y_2 \times S$ the closed embedding, and $p_1 : Y_1 \times Y_2 \times S \to Y_1 \times S, p_2 : Y_1 \times Y_2 \times S \to Y_2 \times S$ the projections. The morphism $p_1 \circ i : \alpha \to Y_1 \times S$ is then finite surjective and $(Z_1 \times Y_2) \cap \alpha \subset Y_1 \times Z_2$ (i.e. $p_2(p_1^{-1}(Z_2) \cap \alpha) \subset Z_2$). Then, the transfert map is given by

$$\begin{align*}
\Omega_{Y_1 \times Y_2 \times S, F_b}^{\mathcal{M}_pr} : (\Omega_{Y_1 \times Y_2 \times S, F_b} \otimes O_{Y_1 \times Y_2 \times S}) & \epsilon (\Omega_{Z_1 \times Y_2, F_b} \otimes O_{Z_1 \times Y_2}) \Gamma^\vee_{Z_1 \times Y_2, Hdg}(O_{Y_1 \times Y_2 \times S}, F_b) \\
\Delta_{DR}(-)(\Delta((\Omega_{Z_1 \times Y_2, F_b} \otimes O_{Z_1 \times Y_2}) \epsilon (\Omega_{Z_1 \times Y_2, F_b} \otimes O_{Z_1 \times Y_2}) \Gamma^\vee_{Z_1 \times Y_2, Hdg}(O_{Y_1 \times Y_2 \times S}, F_b)) & \epsilon (\Omega_{Z_1 \times Y_2, F_b} \otimes O_{Z_1 \times Y_2}) \Gamma^\vee_{Z_1 \times Y_2, Hdg}(O_{Y_1 \times Y_2 \times S}, F_b) \\
\Omega_{Y_1 \times Y_2 \times S, F_b}^{\mathcal{M}_pr} & \epsilon (\Omega_{Z_1 \times Y_2, F_b} \otimes O_{Z_1 \times Y_2}) \Gamma^\vee_{Z_1 \times Y_2, Hdg}(O_{Y_1 \times Y_2 \times S}, F_b)
\end{align*}$$

$$\Omega_{Y_1 \times Y_2 \times S, F_b}^{\mathcal{M}_pr} \epsilon (\Omega_{Z_1 \times Y_2, F_b} \otimes O_{Z_1 \times Y_2}) \Gamma^\vee_{Z_1 \times Y_2, Hdg}(O_{Y_1 \times Y_2 \times S}, F_b)$$

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(ii): Two morphisms of complexes of representable presheaves in $C(\text{Var}(\mathbb{C})^{smpr}/(\tilde{S}_1)^{op})$

$m = (m^i_a), m^i = (m^i_a) : \oplus_{\alpha \in \Lambda_i}(\mathbb{Z}((Y^{1,\ast}_{\alpha} \times \tilde{S}_1, Z^{2,\ast}_{\alpha})/\tilde{S}_1), s_{I \ast J}^1) \rightarrow \oplus_{\alpha \in \Lambda_i}(\mathbb{Z}((Y^{2,\ast}_{\alpha} \times \tilde{S}_1, Z^{3,\ast}_{\alpha})/\tilde{S}_1), s_{I \ast J}^2)$

are said to induce a twisted homotopy if there exist morphisms in $C_{D(S_1)^{pr}}(N \times \mathbb{Z})$

$$h = (h^{p, \ast}_{a, \alpha}) : (Q^2_{Y^{2,\ast}_{\alpha} \times \tilde{S}_1/\tilde{S}_1} \otimes_{Q^1_{Y^{1,\ast}_{\alpha} \times \tilde{S}_1/\tilde{S}_1}} \Gamma^2_{Z^{3,\ast}_{\alpha}}(Q_{Y^{2,\ast}_{\alpha} \times \tilde{S}_1, F_b}(Y^{2,\ast}_{\alpha} \times \tilde{S}_1, u_{IJ}(-)^p))$$

$$\rightarrow (Q^1_{Y^{1,\ast}_{\alpha} \times \tilde{S}_1/\tilde{S}_1} \otimes_{Q^1_{Y^{1,\ast}_{\alpha} \times \tilde{S}_1/\tilde{S}_1}} \Gamma^1_{Z^{3,\ast}_{\alpha}}(Q_{Y^{1,\ast}_{\alpha} \times \tilde{S}_1, F_b}(Y^{1,\ast}_{\alpha} \times \tilde{S}_1, u_{IJ}(-)^{p-1}))$$

where $p \in \mathbb{N}$, such that

$$\Omega_{(\mathbb{Z})}^{\Gamma, pr}(m) - \Omega_{(\mathbb{Z})}^{\Gamma, pr}(m') = d_1 \circ h + h \circ d_2.$$

As for homotopy, twisted homotopy satisfy the 2 of 3 property for morphism of canonical triangles. Moreover, by definition, if two morphisms of complexes of representable presheaves in $C(\text{Var}(\mathbb{C})^{smpr}/(\tilde{S}_1)^{op})$

$m, m' : Q_{1I}, K^1_{I \ast J} \rightarrow (Q_{2I}, K^2_{I \ast J})$ induce a twisted homotopy, then

$$e((\tilde{S}_1)), \text{Hom}^\bullet((h, m, m'), (E_{zar}(\Omega^\Gamma_{/\tilde{S}_1}^{\Gamma, pr}, F_{DR})[-d_{\tilde{S}_1}], T_{IJ})) :$$

$$e'((\tilde{S}_1)), \text{Hom}^\bullet((Q_{2I}, K^2_{I \ast J}),(E_{zar}(\Omega^\Gamma_{/\tilde{S}_1}^{\Gamma, pr}, F_{DR})[-d_{\tilde{S}_1}], T_{IJ}))[1]$$

$$\rightarrow e'((\tilde{S}_1)), \text{Hom}^\bullet((Q_{1I}, K^1_{I \ast J}),(E_{zar}(\Omega^\Gamma_{/\tilde{S}_1}^{\Gamma, pr}, F_{DR})[-d_{\tilde{S}_1}], T_{IJ}))$$

is a 2-filtered homotopy. Now, by proposition 27, there exists

$$\left\{((Y_{1I,\alpha} \times \tilde{S}_1, Z_{1I,\alpha})/\tilde{S}_1, s_{I \ast J}^{1, \alpha}), \alpha \in \Lambda_1\right\}, \ldots, \left\{((Y_{sI,\alpha} \times \tilde{S}_1, Z_{sI,\alpha})/\tilde{S}_1, s_{I \ast J}^{s, \alpha}), \alpha \in \Lambda_s\right\}$$

such that we have in $\text{Ho}_{zar}(C(\text{Var}(\mathbb{C})^{2(sm)pr}/(\tilde{S}_1)^{op})$

$$\text{Cone}(m) \rightarrow$$

$$\text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}((\mathbb{Z}((Y_{1I,\alpha} \times A^1 \times \tilde{S}_1, Z_{1I,\alpha} \times A^1)/\tilde{S}_1), s_{I \ast J}^{1, \alpha} \times I) \rightarrow (\mathbb{Z}((Y_{1I,\alpha} \times S, Z_{1I,\alpha})/\tilde{S}_1), s_{I \ast J}^{1, \alpha})))$$

$$\rightarrow \cdots \rightarrow \oplus_{\alpha \in \Lambda_1} \text{Cone}((\mathbb{Z}((Y_{sI,\alpha} \times A^1 \times \tilde{S}_1, Z_{sI,\alpha} \times A^1)/\tilde{S}_1), s_{I \ast J}^{s, \alpha} \times I) \rightarrow (\mathbb{Z}((Y_{sI,\alpha} \times S, Z_{sI,\alpha})/\tilde{S}_1), s_{I \ast J}^{s, \alpha})))$$

This gives in $D_{fil}(\tilde{S}_1)) := \text{Ho}_{zar,fil}(\tilde{S}_1),$

$$\text{Cone}(M) \rightarrow$$

$$\text{Cone}(\oplus_{\alpha \in \Lambda_1} \text{Cone}(e'((\tilde{S}_1)), \text{Hom}^\bullet((\mathbb{Z}((Y_{1I,\alpha} \times A^1 \times \tilde{S}_1, Z_{1I,\alpha} \times A^1)/\tilde{S}_1), s_{I \ast J}^{1, \alpha} \times I) \rightarrow (\mathbb{Z}((Y_{1I,\alpha} \times S, Z_{1I,\alpha})/\tilde{S}_1), s_{I \ast J}^{1, \alpha}))), (E_{zar}(\Omega^\Gamma_{/\tilde{S}_1}^{\Gamma, pr}, F_{DR})[-d_{\tilde{S}_1}], T_{IJ}))))$$

$$\rightarrow \cdots \rightarrow \oplus_{\alpha \in \Lambda_1} \text{Cone}(e'((\tilde{S}_1)), \text{Hom}^\bullet((\mathbb{Z}((Y_{sI,\alpha} \times A^1 \times \tilde{S}_1, Z_{sI,\alpha} \times A^1)/\tilde{S}_1), s_{I \ast J}^{s, \alpha} \times I) \rightarrow (\mathbb{Z}((Y_{sI,\alpha} \times S, Z_{sI,\alpha})/\tilde{S}_1), s_{I \ast J}^{s, \alpha}))), (E_{zar}(\Omega^\Gamma_{/\tilde{S}_1}^{\Gamma, pr}, F_{DR})[-d_{\tilde{S}_1}], T_{IJ}))))$$

Then by (ii),(ii) and theorem 12, for all $1 \leq i \leq s$ and all $\alpha \in \Lambda_i$

$$e'((\tilde{S}_1)), \text{Hom}^\bullet((\mathbb{Z}((Y_{iI,\alpha} \times A^1 \times \tilde{S}_1, Z_{iI,\alpha} \times A^1)/\tilde{S}_1), s_{I \ast J}^{1, \alpha} \times I) \rightarrow (\mathbb{Z}((Y_{iI,\alpha} \times S, Z_{iI,\alpha})/\tilde{S}_1), s_{I \ast J}^{1, \alpha}))), (E_{zar}(\Omega^\Gamma_{/\tilde{S}_1}^{\Gamma, pr}, F_{DR})[-d_{\tilde{S}_1}], T_{IJ})))) \rightarrow 0$$

are twisted homotopy equivalence. Hence, we have in $C_{D_{fil}}(\tilde{S}_1))$

$$M = m_1 \circ \cdots \circ m_i : e'((\tilde{S}_1)), \text{Hom}^\bullet((Q_{2I}, K^2_{I \ast J}),(E_{zar}(\Omega^\Gamma_{/\tilde{S}_1}^{\Gamma, pr}, F_{DR})[-d_{\tilde{S}_1}], T_{IJ}))))$$

$$\rightarrow e'((\tilde{S}_1)), \text{Hom}^\bullet((Q_{1I}, K^1_{I \ast J}),(E_{zar}(\Omega^\Gamma_{/\tilde{S}_1}^{\Gamma, pr}, F_{DR})[-d_{\tilde{S}_1}], T_{IJ}))))$$

with $m_i$ either a filtered quasi-isomorphism or a twisted homotopy equivalence. This proves (ii). \qed
We deduce the following:

**Proposition 114.** Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

(i) Let $m = (m_i : (Q_{11}, K^1_{1j}) \to (Q_{21}, K^2_{1j}))$ be an etale local equivalence local with $(Q_{11}, K^1_{1j}), (Q_{21}, K^2_{1j}) \in C(\text{Var}(\mathbb{C})^m/(\tilde{S}_i))$ complexes of projective presheaves. Then,

$$e'(\tilde{S}_i), \text{Hom}^*((\tilde{R}^S_{CH}(\rho^S_{\Omega}Q_{11}), \tilde{R}^CH(K^1_{1j}))|E_{\text{zar}}(\Omega^*_{/\tilde{S}_i}|W, F_{D,t})(-d_{\tilde{S}_i}], T_{1j}))$$

is a filtered quasi-isomorphism. It is thus an isomorphism in $D_{\text{fil}}((\tilde{S}_i))$.

(ii) Let $m = (m_i : (Q_{11}, K^1_{1j}) \to (Q_{21}, K^2_{1j}))$ be an equivalence $(\mathbb{A}^1, et)$ local equivalence local with $(Q_{11}, K^1_{1j}), (Q_{21}, K^2_{1j}) \in C(\text{Var}(\mathbb{C})^m/(\tilde{S}_i))$ complexes of projective presheaves. Then,

$$e'(\tilde{S}_i), \text{Hom}^*((\tilde{R}^CH(\rho^S_{\Omega}Q_{11}), \tilde{R}^CH(K^1_{1j}))|E_{\text{zar}}(\Omega^*_{/\tilde{S}_i}|W, F_{D,t})(-d_{\tilde{S}_i}], T_{1j}))$$

is a filtered quasi-isomorphism. It is thus an isomorphism in $D_{\text{fil}}((\tilde{S}_i))$.

**Proof.** Follows from proposition 113 and the fact that the differentials of the complexes are stricts for the filtration by theorem 29.

**Definition 134.** (i) Let $S \in \text{SmVar}(\mathbb{C})$. We define using definition 133(i) and proposition 114(ii) the filtered algebraic De Rahm realization functor defined as

$$F^{FDR}_S : \text{DA}_c(S) \to D_{\text{fil}}(S), \ M \mapsto$$

$$F^{FDR}_S(M) := e(S), \text{Gr}^{I}_{/S}, \text{Hom}^*(\tilde{R}^CH(\rho^S_{\Omega}L(F)), E_{\text{zar}}(\Omega^*_{/S}|F_{D,t})(-d_{S}])$$

where $F \in C(\text{Var}(\mathbb{C})^m/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(ii) For the Corti-Hanamura weight structure $W$ on $\text{DA}_c(S)^-$, we define using definition 133(ii) and proposition 114(ii)

$$F^{FDR}_S : \text{DA}^-_c(S) \to D_{\text{fil}}(S), \ M \mapsto$$

$$F^{FDR}_S(M) := e(S), \text{Gr}^{I}_{/S}, \text{Hom}^*(\tilde{R}^CH(\rho^S_{\Omega}L(F,W)), E_{\text{zar}}(\Omega^*_{/S}|F_{D,t})(-d_{S}])$$

where $(F,W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^m/S)$ is such that $M = D(\mathbb{A}^1, et)((F,W))$ using corollary 1. Note that the filtration induced by $W$ is a filtration by sub $D_{S}$ module, which is a stronger property then Griffitz transversality. Of course, the filtration induced by $F$ satisfy only Griffitz transversality in general.

(iii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \ldots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_{II} : S_I \hookrightarrow \tilde{S}_I := S_{I \cap I}$. We define, using definition 133(ii) and proposition 114(ii), the filtered algebraic De Rahm realization functor defined as

$$F^{FDR}_S : \text{DA}_c(S) \to D_{\text{fil}}(S(\tilde{S}_I)), \ M \mapsto$$

$$F^{FDR}_S(M) := (e(\tilde{S}_I), \text{Hom}^*(\tilde{R}^CH(\rho^S_{\Omega}L(I)), E_{\text{zar}}(\Omega^*_{/S}|F_{D,t})(-d_{\tilde{S}_I}], u_{I}(F))$$

where $F \in C(\text{Var}(\mathbb{C})^m/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$, see definition 133.
(ii)’ For the Corti-Hanamura weight structure \( W \) on \( DA^-(S) \), using definition 133(ii) and proposition 114(ii),

\[
\mathcal{F}^{FDR}_S : DA_c(S) \to D_{D(1,0)fil}(S/(\tilde{S}_I)), \quad M \mapsto \mathcal{F}^{FDR}_S((M, W)) := (e'(\tilde{S}_I), \text{Hom}(\mathcal{R}^{CH}(\rho^I_{\tilde{S}_I} L(i_1, j_1^I(F, W))), E_{zar}(\Omega^{•,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^I_{j_1}(F, W))
\]

where \((F, W) \in C_{fil}(\text{Var}(\mathbb{C}))^m/S\) is such that \((M, W) = D(\mathbb{A}^1, et)(F, W)\) using corollary 1. Note that the filtration induced by \( W \) is a filtration by sub \( D_{\tilde{S}_I} \)-modules, which is a stronger property then Griffitz transversality. Of course, the filtration induced by \( F \) satisfy only Griffitz transversality in general.

**Proposition 115.** For \( S \in \text{Var}(\mathbb{C}) \) and \( i = \bigcup_{i=1}^l S_i \) an open cover such that there exist closed embeddings \( i_1 : S_i \hookrightarrow S_i \) with \( S_i \in \text{SmVar}(\mathbb{C}) \), the functor \( F^{FDR}_S \) is well defined.

**Proof.** Let \( S \in \text{Var}(\mathbb{C}) \) and \( i = \bigcup_{i=1}^l S_i \) an open cover such that there exist closed embeddings \( i_1 : S_i \hookrightarrow S_i \) with \( S_i \in \text{SmVar}(\mathbb{C}) \). Denote, for \( I \subset [1, \cdots, l], S_I = \cap_{i \in I} S_i \) and \( j_1 : S_I \hookrightarrow S \) the open embedding. We then have closed embeddings \( i_1 : S_I \hookrightarrow S_I := \Pi_{i \in I} S_i \). Let \( M \in DA(S) \). Let \( F, F' \in C(\text{Var}(\mathbb{C}))^m/S \) such that \( M = D(\mathbb{A}^1, et)(F) = D(\mathbb{A}^1, et)(F') \). Then there exist by definition a sequence of morphisms in \( C(\text{Var}(\mathbb{C}))^m/S \):

\[
F = F_1 \xrightarrow{s_1} F_2 \xrightarrow{s_2} F_3 \xrightarrow{s_3} F_4 \to \cdots \xrightarrow{s_i} F' = F_s
\]

where, for \( 1 \leq k \leq s \), and \( s_k \) are \((\mathbb{A}^1, et)\) local equivalence. But if \( s : F_1 \to F_2 \) is an equivalence \((\mathbb{A}^1, et)\) local,

\[
L(i_1, j_1^I s) : L(i_1, j_1^I F_1) \to L(i_1, j_1^I F_2)
\]

is an equivalence \((\mathbb{A}^1, et)\) local, hence

\[
\text{Hom}((\mathcal{R}^{CH}(\rho^I_{\tilde{S}_I} L(i_1, j_1^I s)), E_{zar}(\Omega^{•,pr}_{/\tilde{S}_I}, F_{DR}))) : (e'(\tilde{S}_I), \text{Hom}(\mathcal{R}^{CH}(\rho^I_{\tilde{S}_I} L(i_1, j_1^I F_1)), E_{zar}(\Omega^{•,pr}_{/\tilde{S}_I}, F_{DR})), u^I_{j_1}(F_1)) \to (e'(\tilde{S}_I), \text{Hom}(\mathcal{R}^{CH}(\rho^I_{\tilde{S}_I} L(i_1, j_1^I F_2)), E_{zar}(\Omega^{•,pr}_{/\tilde{S}_I}, F_{DR})), u^I_{j_1}(F_2))
\]

is a filtered quasi-isomorphism by proposition 114.

Let \( f : X \to S \) a morphism with \( S, X \in \text{Var}(\mathbb{C}) \). Assume there exists a factorization

\[
f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S
\]

of \( f \), with \( l \) a closed embedding, \( Y \in \text{SmVar}(\mathbb{C}) \) and \( p_S \) the projection. Let \( \bar{Y} \in \text{PSmVar}(\mathbb{C}) \) a smooth compactification of \( Y \) with \( \bar{Y}\backslash Y = D \) a normal crossing divisor, denote \( k : D \hookrightarrow \bar{Y} \) the closed embedding and \( n : Y \hookrightarrow \bar{Y} \) the open embedding. Denote \( \bar{X} \subset \bar{Y} \times S \) the closure of \( X \subset \bar{Y} \times S \). We have then the following commutative diagram in \( \text{Var}(\mathbb{C}) \)

\[
\begin{array}{ccc}
X & \xrightarrow{l} & Y \times S \\
\downarrow \quad \quad \quad & \quad & \downarrow \quad \quad \quad F_S \\
\bar{X} & \xrightarrow{l} & \bar{Y} \times S \\
\quad \downarrow \quad \quad \quad (k \times l) & \quad \quad & \quad \quad \\
Z := \bar{X}\backslash X & \xrightarrow{l_Z} & D \times S
\end{array}
\]

Let \( S = \bigcup_{i=1}^l S_i \) an open cover such that there exist closed embeddings \( i_1 : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). We have \( \bar{X} = \bigcup_{i=1}^l X_i \) with \( X_i := f^{-1}(S_i) \). Denote, for \( I \subset [1, \cdots, l], S_I = \cap_{i \in I} S_i \) and \( X_I = \cap_{i \in I} X_i \). For
We have then for $I \subset [1, \ldots, l]$, closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and for $I \subset J$, the following commutative diagrams which are cartesian

\[
\begin{array}{ccc}
f_I = f_{I|X_I} : X_I \xrightarrow{i_I} Y \times S_I \xrightarrow{p_{S_I}} S_I, & Y \times \tilde{S}_I \xrightarrow{p_{\tilde{S}_I}} \tilde{S}_I, \\
\end{array}
\]

with $l_I : l_{I|X_I}$, $i_I = I \times i_I$, $p_{S_I}$ and $p_{\tilde{S}_I}$ are the projections and $p'_{IJ} = I \times p_{IJ}$, and we recall that we denote by $j_I : \tilde{S}_I \hookrightarrow \tilde{S}_I$ and $j^I_I : Y \times \tilde{S}_I \hookrightarrow Y \times S_I$ the open complementary embeddings. We then have the commutative diagrams

\[
D_{IJ} = S_I \xrightarrow{i_J} \tilde{S}_I, \quad D'_{IJ} = X_J \xrightarrow{i'_{IJ} = i_J \circ l_{IJ}} Y \times \tilde{S}_I.
\]

and the factorization of $D'_{IJ}$ by the fiber product:

\[
D'_{IJ} = X_J \xrightarrow{i'_{IJ}} Y \times \tilde{S}_I, \quad D'_{IJ} = X_J \xrightarrow{i_J} Y \times \tilde{S}_I.
\]

where $j'_{IJ} : X_J \hookrightarrow X_I$ is the open embedding. Consider

\[
F(X/S) := p_{S_I,2} \Gamma_X^\lor \mathbb{Z}(Y \times S/Y \times S) \in C(\text{Var}(\mathbb{C})^{sm}/S)
\]

so that $D(\mathbb{A}^1, et)(F(X/S)) = M(X/S)$ since $Y$ is smooth. Then, by definition,

\[
\mathcal{F}^{DR}_S(M(X/S)) := (e'_{(\tilde{S}_I)})_* \mathcal{Hom}(\tilde{R}^{CH}(p_{\tilde{S}_I}^* L(i_{IJ}, j^I_I)^! F(X/S)), E_{zar}(\Omega^{\bullet, \Gamma}_{\tilde{S}_I}, F_{DR})[-d_{\tilde{S}_I}, u_{IJ}^0(F(X/S))])
\]

On the other hand, let

\[
Q(X_I/\tilde{S}_I) := p_{\tilde{S}_I,2} \Gamma_X^\lor \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I), \in C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I),
\]

We have then for $I \subset [1, l]$ the map (56) in $C(\text{Var}(\mathbb{C})^{sm}/\tilde{S}_I)$:

\[
N_I(X/S) : Q(X_I/\tilde{S}_I) = p_{\tilde{S}_I,2} \Gamma_X^\lor \mathbb{Z}(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \xrightarrow{\text{ad}(i'_{IJ}, i_{IJ})} p_{S_I,2} \Gamma_X^\lor \mathbb{Z}(Y \times S_I/Y \times S_I) \xrightarrow{T_{(i'_{IJ}, i_{IJ})}(-)} p_{S_I,2} \Gamma_X^\lor \mathbb{Z}(Y \times S_I/Y \times S_I) \xrightarrow{\hat{T}_{(\tilde{S}_I, i_{IJ})}(-)} \hat{i}^* \Gamma_X^\lor \mathbb{Z}(Y \times S_I/Y \times S_I) = i^* j_{IJ}^! F(X/S).
\]

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We then have the commutative diagram in $C(\text{Var}(\mathbb{C})^m/\tilde{S}_J)$

$$p^*_{I,J}Q(X_I/\tilde{S}_I) \xrightarrow{p^*_{I,J}N_I(X/S)} p^*_{I,J}L_{I,J}F(X/S)$$

with

$$H_{IJ} : p^*_{I,J}p_{\tilde{S}_J}\Gamma^r_{X,I}Z(Y \times \tilde{S}_I/Y \times \tilde{S}_I) \xrightarrow{T^q(p_{I,J},p_{\tilde{S}_J})(\cdot)^{-1}} p^*_{I,J}p^*_{I,J}\Gamma^r_{X,I}Z(Y \times \tilde{S}_I/Y \times \tilde{S}_I)$$

The diagram 63 say that the maps $N_I(X/S)$ induces a map in $C(\text{Var}(\mathbb{C})^m/(S/\tilde{S}_I))$

$$(N_I(X/S)) : (Q(X_I/\tilde{S}_I), I(p^*_{I,J},p_{\tilde{S}_J})(\cdot,-)(H_{IJ}))$$

$$(L_{I,J}F(X/S), I(p^*_{I,J},p_{\tilde{S}_J})(\cdot,\cdot)(T^q(D_{IJ})(j_I^*F(X/S))))$$

Denote $\tilde{X}_I := X \cap (Y \times \tilde{S}_I) \subset \tilde{Y} \times \tilde{S}_I$ the closure of $X_I \subset \tilde{Y} \times \tilde{S}_I$, and $Z_I := Z \cap (Y \times \tilde{S}_I) = \tilde{X}_I \setminus X_I$. Consider for $I \subset [1, \cdots l]$ and $I \subset J$ the following commutative diagrams in $\text{Var}(\mathbb{C})$

Let $\epsilon_1 : ((\tilde{Y} \times \tilde{S}_I)_1, E_1) \to (\tilde{Y} \times \tilde{S}_I, Z_I)$ a strict desingularization of the pair $(\tilde{Y} \times \tilde{S}_I, Z_I)$, $\epsilon_2 : ((\tilde{Y} \times \tilde{S}_I)_2, E_2) \to (\tilde{Y} \times \tilde{S}_I, \tilde{X}_I)$ a strict desingularization of the pair $(\tilde{Y} \times \tilde{S}_I, \tilde{X}_I)$ and a morphism $\epsilon_{12} : (\tilde{Y} \times \tilde{S}_I)_2 \to (\tilde{Y} \times \tilde{S}_I)_1$ such that the following diagram commutes (see definition-proposition 12):

$$((\tilde{Y} \times \tilde{S}_I)_2)_{\epsilon_{12}} \xrightarrow{\epsilon_{12}} ((\tilde{Y} \times \tilde{S}_I)_1)_{\epsilon_1} \quad \tilde{Y} \times \tilde{S}_I$$

and we denote by $l'_Z_I : E_1 \hookrightarrow (\tilde{Y} \times \tilde{S}_I)_2$, $l'_Z : E_2 \hookrightarrow (\tilde{Y} \times \tilde{S}_I)_2$ the closed embeddings. We have then the canonical map in $C(\text{Var}(\mathbb{C})^{2,\text{smpr}}/(\tilde{S}_I)^{\text{op}})$

$$(I_\epsilon((\tilde{X}_I, Z_I)/\tilde{S}_I)) : (\hat{R}^{CH}(\rho_{\tilde{S}_I}^*Q(X_I/\tilde{S}_I)), \hat{R}^{CH}(H_{IJ}))$$

$$(\text{Cone}(\text{Cone}((Z^{tr}((E_1, E_1, E_1) \times \tilde{S}_I, u_{I,J})) \to Z^{tr}(((\tilde{Y} \times \tilde{S}_I)_1, (\tilde{Y} \times \tilde{S}_I)_1), (\tilde{Y} \times \tilde{S}_I)_1, (\tilde{Y} \times \tilde{S}_I)_1)))) \to \text{Cone}((Z^{tr}((E_2, E_2) \times \tilde{S}_I, u_{I,J}) \to Z^{tr}(((\tilde{Y} \times \tilde{S}_I)_2, (\tilde{Y} \times \tilde{S}_I)_2), (\tilde{Y} \times \tilde{S}_I)_2, (\tilde{Y} \times \tilde{S}_I)_2))))$$

$$(\text{Cone}(\tilde{Z}^{tr}((\tilde{Y} \times \tilde{S}_I, Z_I)/\tilde{S}_I)) \to \tilde{Z}^{tr}((\tilde{Y} \times \tilde{S}_I, \tilde{X}_I)/\tilde{S}_I)))/(\tilde{y}_Y \cdot d_{\tilde{S}_I} - 2d_{\tilde{S}_I} - 2d_{\tilde{S}_I}, \tilde{Z}^{tr}(I \times p_{I,J}))$$
We denote by $v_{I,J}^q(F(X/S))$ the composite

$$v_{I,J}^q(F(X/S))[d_{ZJ}]: e'((S_I), \mathcal{H}\text{om}(R_{\{Y \times S_I\}^\prime, E^\prime})(\rho_{S_I}^*Q(X_I/\tilde{S}_I)), E_{zar}(\Omega^{*,pr}_{S_I}, F_{DR}))$$

$$p_{I,J_*}T(p_{I,J^*}(\Omega^{*,pr}_{S_I}), (\rho_{S_I}^*Q(X_I/\tilde{S}_I)), E_{zar}(\Omega^{*,pr}_{S_I}, F_{DR})))$$

$$\mathcal{H}\text{om}(T(p_{I,J^*}, \mathcal{R}^{CH})(Q(X_I/\tilde{S}_I)), (\rho_{S_I}^*Q(X_I/\tilde{S}_I)), E_{zar}(\Omega^{*,pr}_{S_I}, F_{DR}))$$

$$p_{I,J_*}e'((S_I), \mathcal{H}\text{om}(R_{\{Y \times S_I\}^\prime, E^\prime})(\rho_{S_I}^*Q(X_I/\tilde{S}_I)), E_{zar}(\Omega^{*,pr}_{S_I}, F_{DR})))$$

$$p_{I,J_*}e'((S_I), \mathcal{H}\text{om}(R_{\{Y \times S_I\}^\prime, E^\prime})(\rho_{S_I}^*Q(X_I/\tilde{S}_I)), E_{zar}(\Omega^{*,pr}_{S_I}, F_{DR}))).$$

On the other hand, we have in $\pi_X(C(MHM(\tilde{X}))) \subset C_{Dfi}(\tilde{X}/(Y \times \tilde{S}_I))$

$$(\mathcal{C}(T(Z_I/\tilde{X}_I, \gamma^{*,Hdg}(\gamma)))/(\gamma))) : (\Gamma_{X_I}^{*,Hdg}(O_{Y \times S_I}, F_b), x_{I,J}(\tilde{X}/S)) \xrightarrow{(n \times I)^{Hdg}b} (\Gamma_{X_I}^{*,Hdg}(O_{Y \times S_I}, F_b), x_{I,J}(X/S))$$

with

- for the closed embedding $\tilde{X} \subset \tilde{Y} \times S_I$ we consider the map in $\pi_{Y \times \tilde{S}_I}(C(MHM(\tilde{Y} \times \tilde{S}_I)))$

$$x_{I,J}(\tilde{X}/S) : \Gamma_{X_I}^{*,Hdg}(O_{Y \times S_I}, F_b) \xrightarrow{\mathcal{O}(p_{I,J^*}, p_{I,J^*})} p_{I,J^*}^{*,mod}p_{I,J^*}^{*,mod}\Gamma_{X_I}^{*,Hdg}(O_{Y \times S_I}, F_b)$$

- for the closed embedding $Z \subset \tilde{Y} \times S_I$ we consider the map in $\pi_{Y \times \tilde{S}_I}(C(MHM(\tilde{Y} \times \tilde{S}_I)))$

$$x_{I,J}(Z/S) : \Gamma_{Z_I}^{*,Hdg}(O_{Y \times S_I}, F_b) \xrightarrow{\mathcal{O}(p_{I,J^*}, p_{I,J^*})} p_{I,J^*}^{*,mod}p_{I,J^*}^{*,mod}\Gamma_{Z_I}^{*,Hdg}(O_{Y \times S_I}, F_b)$$

- for the closed embedding $X \subset Y \times S_I$ we consider the map in $\pi_{Y \times \tilde{S}_I}(C(MHM(\tilde{Y} \times \tilde{S}_I)))$

$$x_{I,J}(X/S)(-d_Y)[-2d_Y] : \Gamma_{X_I}^{*,Hdg}(O_{Y \times S_I}, F_b) \xrightarrow{\mathcal{O}(p_{I,J^*}, p_{I,J^*})} p_{I,J^*}^{*,mod}p_{I,J^*}^{*,mod}\Gamma_{X_I}^{*,Hdg}(O_{Y \times S_I}, F_b)$$

The maps $x_{I,J}(X/S)$ gives the following maps in $C_{Dfi,S_I}(\tilde{S}_I)$

$$\mathcal{O}(p_{I,J^*}, p_{I,J^*}) \xrightarrow{\mathcal{O}(p_{I,J^*}, p_{I,J^*})} p_{I,J^*}^{*,mod}p_{I,J^*}^{*,mod}\Gamma_{X_I}^{*,Hdg}(O_{Y \times S_I}, F_b)$$

We have then the following lemma
Lemma 12.  
(i) The map in \(C(\mathcal{V}(\mathbb{C})^{sm}/(S/\hat{S}_I))\)

\[(N_I(X/S)) : (Q(X_I/\hat{S}_I), H_{I,J}) \rightarrow (L(i_+) p F(X/S), T^{a}(D_{I,J})(F(X/S))).\]

is an equivalence \((A^1, et)\) local.

(ii) The maps \((N_I(X/S))\) induces a filtered quasi-isomorphism in \(C_{Dfil}(S/\hat{S}_I))\)

\[\begin{align*}
(e'_{(i)} & , \Hom(\hat{R}^{GH}_{/}\hat{S}_I}(N_I(X/S)), \operatorname{Ezar}(\Omega^{\Gamma, pr}_{/\hat{S}_I}, F_0)) : \\
& \rightarrow (e'_{(i)} , \Hom(\hat{R}^{GH}(\rho_{\hat{S}_I} X_I/\hat{S}_I)), \operatorname{Ezar}(\Omega^{\Gamma, pr}_{/\hat{S}_I}, F_{DR})[-d_{\hat{S}_I}], v_{i_I} F(X/S)))
\end{align*}\]

(iii) The map \((I_{I_I}(X_I, Z_I)/\hat{S}_I))\) induces a filtered Zariski local equivalence in \(C_{Dfil}(S/\hat{S}_I))\)

\[(I((X_I, Z_I)/\hat{S}_I)) := \Hom((I_{I_I}(X_I, Z_I)/\hat{S}_I)), -) \circ (DR(Y \times \hat{S}_I/\hat{S}_I)(\operatorname{ad}(n \times I)_! H_{dg}(O_Y \times \hat{S}_I, F_0)), 0) : \]

\[\begin{align*}
=p_{I_I}^\ast \operatorname{Ezar}(\Omega^{\Gamma}_{X/Y \times \hat{S}_I}, F_0) \otimes \operatorname{O}_{X \times \hat{S}_I} (n \times I)_! H_{dg}(O_Y \times \hat{S}_I, F_0) d_Y | [2d_Y + d_{\hat{S}_I}], u_{I_I} F(X/S)) \\
& \rightarrow (e'_{(i)}, \Hom(\hat{R}^{GH}(\rho_{\hat{S}_I} X_I/\hat{S}_I)), \operatorname{Ezar}(\Omega^{\Gamma, pr}_{/\hat{S}_I}, F_{DR})[-d_{\hat{S}_I}], v_{i_I} F(X/S)))
\end{align*}\]

Proof. (i): See lemma 9(i)

(ii): These maps induce a morphism in \(C_{D}(S/\hat{S}_I))\) by construction. It is a filtered quasi-isomorphism by (i) and proposition 113.

(iii): See section 5.2. \(\Box\)

Proposition 116. Let \(f : X \rightarrow S\) a morphism with \(S, X \in \mathcal{V}(\mathbb{C})\). Assume there exist a factorization

\[f : X \xrightarrow{i} Y \times S \xrightarrow{p_S} S\]

of \(f\) with \(Y \in \mathcal{E}_{\mathcal{V}(\mathbb{C})}\) compactification of \(Y\) with \(\bar{Y} \setminus Y = D\) a normal crossing divisor, denote \(k : D \hookrightarrow \bar{Y}\) the closed embedding and \(n : Y \hookrightarrow \bar{Y}\) the open embedding. Denote \(\bar{X} \subset \bar{Y} \times S\) the closure of \(X \subset \bar{Y} \times S\). We have then the following commutative diagram in \(\mathcal{V}(\mathbb{C})\)

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \times S \\
\downarrow & & \downarrow \\
\bar{X} & \xrightarrow{i} & \bar{Y} \times S \\
\downarrow & \downarrow \uparrow & \downarrow \uparrow \\
Z := X \setminus X & \xrightarrow{i} & D \times S \\
\end{array}
\]

Let \(S = \bigcup_{i=1}^{l} S_i\) an open cover such that there exist closed embeddings \(i_i : S_i \hookrightarrow \bar{S}_i\) with \(\bar{S}_i \in \mathcal{E}(\mathbb{C})\). 

Then \(X = \bigcup_{i=1}^{l} X_i\) with \(X_i := f^{-1}(S_i)\). Denote, for \(I \subset \{1, \cdots, l\}\), \(S_I = \bigcap_{i \in I} S_i\) and \(X_I = \bigcap_{i \in I} X_i\). Denote \(X_I := X \cap (Y \times S_I) \subset \bar{Y} \times \bar{S}_I\) the closure of \(X_I \subset \bar{Y} \times S_I\), and \(Z_I := Z \cap (Y \times S_I) = \bar{X}_I \setminus X_I \subset \bar{Y} \times \bar{S}_I\)

We have then for \(I \subset \{1, \cdots, l\}\), the following commutative diagram in \(\mathcal{V}(\mathbb{C})\)

\[
\begin{array}{ccc}
X_I & \xrightarrow{i_I} & \bar{Y} \times \bar{S}_I \\
\downarrow & & \downarrow \\
\bar{X}_I & \xrightarrow{i_I} & \bar{Y} \times \bar{S}_I \\
\downarrow & \downarrow \uparrow & \downarrow \uparrow \\
Z_I = \bar{X}_I \setminus X_I & \xrightarrow{i_I} & D \times \bar{S}_I \\
\end{array}
\]

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Let \( F(X/S) := p_{S*} \Gamma_X^\infty \mathcal{Z}(Y \times S/Y \times S) \subset C(\text{Var}(C)^{\text{sm}}/S). \) We have then the following isomorphism in \( D_{\text{fil}}(S/(\tilde{S}_1)) \)

\[
I(X/S) : \mathcal{F}_S^{FDR}(M(X/S)) \cong (e'(\tilde{S}_1)_*, \text{Hom}(\bar{R}^{\text{CH}}(p_{S_1}^*L(i_1*J_1^!(F(X/S))), E_{\text{zar}}(\Omega^\bullet_{/S_1}^{\Gamma, pr}, F_{\text{D}}))[d_{S_1}], \nu_{ij}^q(F(X/S)))
\]

\[
(\text{Hom}(\bar{R}^{\text{CH}}(N_1(X/S)), E_{\text{zar}}(\Omega^\bullet_{/S_1}^{\Gamma, pr}, F_{\text{D}})) \rightarrow (e'(\tilde{S}_1)_*, \text{Hom}(\bar{R}^{\text{CH}}(p_{S_1}^*Q(X_1/S_1)), E_{\text{zar}}(\Omega^\bullet_{/S_1}^{\Gamma, pr}, F_{\text{D}}))[d_{S_1}], \nu_{ij}^q(F(X/S)))
\]

\[
i((X_i, Z_i)/\tilde{S}_1) \rightarrow (p_{S_1*}E_{\text{zar}}((\Omega^\bullet_{/S_1}^{\Gamma, pr}, F_{\text{D}})) \otimes_{O_{\tilde{S}_1}} (n \times I)_! Hdg^{\Gamma, Hdg}(O_{Y \times S_1/F_1}))(dy + d_{S_1})[2d_{Y_S} + d_{S_1}], u_{ij}(X/S))
\]

\[
\cong \iota_S R f_1^{Hdg} Hdg^1 \mathcal{Z}(U'/S, F_1)(dy)[2d_{Y_{S_1}}], x_{ij}(X/S)) \cong \iota_S R f_1^{Hdg} f_{\text{smooth}}^{Hdg} \in \iota_S S(MHM(S).
\]

Proof. Follows from lemma 121, proposition 30.

**Corollary 4.** Let \( S \in \text{Var}(C) \) and \( S = \bigcup_{i \in I} S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \rightarrow S \) with \( S_i \in \text{SmVar}(\mathcal{C}) \). Then, for \( F \in C(\text{Var}(C)^{\text{sm}}/S) \) such that \( M = D(h^1, e_1)(F) \in D_{\mathcal{C}}(S), \)

\[
H^i \mathcal{F}_{S}^{FDR}(M, W) := (a_{zar} H^i e'(\tilde{S}_1)_*, \text{Hom}(L_{\rho_{S_1*}} \mu_{S_1*} \bar{R}^{\text{CH}}(p_{S_1}^*L(i_1*J_1^!(F, W))), E_{\text{zar}}(\Omega^\bullet_{/S_1}^{\Gamma, pr}, F_{\text{D}}))[d_{S_1}], H^i u_{ij}^q(F, W) \in \pi_S(MHM(S)
\]

for all \( i \in \mathbb{Z}, \) and for all \( p \in \mathbb{Z}, \)

\[
\mathcal{F}_S^{FDR}(M, W) \in D_{D(1,0)\text{fil}}(S/(\tilde{S}_1))
\]

is the class of a complex \( \mathcal{F}_S^{FDR}(F, W) \in C_{D(1,0)\text{fil}}(S/(\tilde{S}_1)) \) such that for all \( k \in \mathbb{Z}, \) the differentials of \( \text{Gr}_k \mathcal{F}_S^{FDR}(F, W) \) are strict for the filtration \( F. \)

Proof. Follows from theorem 29: Indeed, for \( S \) smooth and \( g : U'/S \rightarrow U/S \) a morphism with \( U/S = (U, h), U'/S = (U', h') \) in \( \text{Var}(C)^{\text{sm}}/S, U, U' \) connected, hence irreducible by smoothness, the complex

\[
\mathcal{F}_S^{FDR}(g) := e'(S)_* \text{Hom}((\bar{R}^{\text{CH}}(\mathbb{Z}(U'/S)), E_{\text{zar}}(\Omega^\bullet_{/S}^{\Gamma, pr}, F_{\text{D}}))[d_{S_1}], H^i u_{ij}^q(F, W) \in \pi_S(MHM(S)
\]

satisfy \( H^i \mathcal{F}_S^{FDR}(g) \in \pi_S(MHM(S)) \) and is the class of a complex such that the differentials are strict for \( F. \) Let \( U \subset X \) a compactification of \( U \) and \( U' \subset X' \) a compactification of \( U', S \subset \mathbb{C} \) a compactification of \( S \) with \( X, X' \in \text{PSmVar}(\mathcal{C}), i : D = \bigcup_{i \in I} D_i := X \setminus U \rightarrow X, i : D' = \bigcup_{i \in I} D'_i := X' \setminus U' \rightarrow X' \) normal crossing divisors, such that \( g : U'/S \rightarrow U/S \) extend to \( \tilde{g} : X'/\tilde{S} \rightarrow X/\tilde{S} \) see section 2. Denote \( n : U \rightarrow X, n' : U' \rightarrow X' \) and \( n'' : U' \rightarrow X' \) the open embeddings. We get the following map in \( C_{D\text{fil}}(S) \)

\[
\Omega_{/S}^{\Gamma, pr}(\Gamma_g^t) := (p_{S*} E(\Omega_{X', S/S}^{\Gamma, pr} \otimes_{O_{X', S}} (n' \times I)_! Hdg^{\Gamma, Hdg}(O_{U'/S, F_1})(d') [2d'])
\]

\[
\text{Hom}((\mathcal{F}_S^{FDR}(g), -)) \rightarrow e'(S)_* \text{Hom}((\bar{R}^{\text{CH}}(\mathbb{Z}(U'/S)), E_{\text{zar}}(\Omega^\bullet_{/S}^{\Gamma, pr}, F_{\text{D}}))[d_{S_1}], H^i u_{ij}^q(F, W) \in \pi_S(MHM(S)
\]

\[
\cong \iota_S R f_1^{Hdg} Hdg^1 \mathcal{Z}(U'/S, F_1)(dy)[2d_{Y_{S_1}}], x_{ij}(X/S) \cong \iota_S R f_1^{Hdg} f_{\text{smooth}}^{Hdg} \in \iota_S S(MHM(S).
\]

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which is a filtered quasi-isomorphism by proposition 113. Note that in the particular case where \( \tilde{g} = i : Z \hookrightarrow X \) is a closed embedding of codimension \( c = \dim(X) - \dim(Z) \),

\[
p_{S*}E(\Omega_{X/S}^\bullet \otimes_{O_Z(S)} \Gamma_Z^{\vee, Hdg}(O_{Z \times S}, F_b)(-c)[-2c]) \xrightarrow{\Omega^{pr}(\Gamma_g^t)} p_{S*}E(\Omega_{X/S}^\bullet \otimes_{O_Z(S)} \Gamma_X^{\vee, Hdg}(O_{X \times S}, F_b))
\]

is the Gysin morphism, where \( \gamma \in H^0(\Gamma(X, N^\bullet_{Z/X}[2c])) \) is the De Rham fundamental class of the conormal bundle, also note that denoting by \( j : X \setminus Z \hookrightarrow X \) the open complementary the composite in \( C_{D, fil}(S) \)

\[
p_{S*}E(\Omega_{X/S}^\bullet \otimes_{O_Z(S)} \Gamma_Z^{\vee, Hdg}(O_{Z \times S}, F_b)(-c)[-2c]) \xrightarrow{\Omega^{pr}(\Gamma_g^t)} p_{S*}E(\Omega_{X/S}^\bullet \otimes_{O_Z(S)} \Gamma_X^{\vee, Hdg}(O_{X \times S}, F_b))
\]

is NOT equal to zero, and the composite in \( C(\Var(\mathbb{C})^{2, sm pr}/S) \)

\[
Z^{tr}((X\setminus Z) \times S, X\setminus Z)/S \xrightarrow{Z^{tr}(j \times I)} Z^{tr}((X \times S, X)/S) \xrightarrow{[\Gamma_g^t]} Z^{tr}((Z \times S, Z)/S)(c)[2c]
\]

is NOT equal to zero. Now, for \( g : U'/\tilde{S}_I \hookrightarrow U/\tilde{S}_I \) the commutative diagram

\[
p_{I j^t}^{mod} F^{DR}_{\tilde{S}_j}(g) \xrightarrow{T(p_{I j}^{\triangleright}, \Omega^{pr})(-)} F^{DR}_{\tilde{S}_j}(g \times I) \xrightarrow{\Omega^{pr}(\Gamma_g^t)} \Omega^{pr}(\Gamma_g^t) \xrightarrow{A_I} \Omega^{pr}(\Gamma_g^t)
\]

with \( g \times I : U' \times \tilde{S}_{I,1}/\tilde{S}_j \rightarrow U \times \tilde{S}_{I,1}/\tilde{S}_j \) and

\[
A_I := (\langle DR(\tilde{X} \times \tilde{S}_I/\tilde{S}_J), (n' \times I)^{Hdg}(O_{\tilde{X} \times \tilde{S}_I}, F_b)), 0 \rangle, \langle DR(\tilde{X} \times \tilde{S}_I/\tilde{S}_J), (n \times I)^{Hdg}(O_{\tilde{X} \times \tilde{S}_I}, F_b)), 0 \rangle).
\]

\[\Box\]

**Proposition 117.** For \( S \in \Var(\mathbb{C}) \) not smooth, the functor (see corollary 4)

\[
\iota^{-1}_S F^{DR}_{S} : DA^-_{c}(S)^{op} \rightarrow \pi_S(D(MHM(S))
\]

does not depend on the choice of the open cover \( S = \cup_i S_i \) and the closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \).

**Proof.** Let \( S = \cup_{i=n+1}^{n'} S_i \) is an other open cover together with closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with
\( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \) for \( l + 1 \leq i \leq l' \). Then, for \( J' \subset I' \subset [l + 1, \ldots, l'] = L' \) and \( J \subset I \subset L = [1, \ldots, l] \),

\[
T_S^{L/L'}(\iota_{S_1}^{-1}(e'(\tilde{S}_I)), \text{Hom}(\hat{R}CH(\rho_{S_1}^p L(i_1, j_1^* F)), E_{zar}(\Omega^*_S \cdot \pi_{pr}, F_{DR}))[−d_{S_1}], u_{IJ}^* F))
\]

\[
:= (\lim_{i \in I} \rho_{S_i}^{\chi_{Hdg}^* \text{mod}[-]}(\iota_{S_1}^{-1}(e'(\tilde{S}_I)), \text{Hom}(\hat{R}CH(\rho_{S_1}^p L(i_1, j_1^* F)),
E_{zar}(\Omega^*_S \cdot \pi_{pr}, F_{DR}))[−d_{S_1}], u_{IJ}^* F))
\]

is a filtered Zariski local equivalence, since all the morphisms are filtered Zariski local equivalences by proposition 116 and proposition 100.

We have the canonical transformation map between the filtered De Rham realization functor and the Gauss-Manin realization functor:

**Definition 135.** Let \( S \in \text{Var}(\mathbb{C}) \) and \( S = \bigcup_{i=1}^l S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). Let \( M \in \text{DA}_c(S) \) and \( F \in C(\text{Var}(\mathbb{C}))^m/S \) such that \( M = D(\mathbb{A}^1, \text{et})(F) \). We have, using definition 129(ii), the canonical map in \( D_{\text{O}S_0} \text{Fil}_*, \mathbb{D}_\infty(S/\{\tilde{S}_i\}) \)

\[
T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : \mathcal{F}_S^{GM}(L_{S} M) := (e(\tilde{S}_I), \text{Hom}^*(L(i_1, j_1^* S L F), E_{zar}(\Omega^*_{S_1} F, F))[−d_{S_1}], u_{IJ}^* F))
\]

\[
\xrightarrow{(\text{Hom}(−, \text{Gr}(\Omega_{S_1}^*)))^{-1}} (e(\tilde{S}_I), \text{Hom}^*(L^0_{S_1} L(i_1, j_1^* F), E_{zar}(\Omega^*_{S_1} F, F))[−d_{S_1}], u_{IJ}^* F))
\]

\[
\xrightarrow{(\text{Hom}(−, \text{Gr}(\Omega_{S_1}^*)))^{-1}} (e(\tilde{S}_I), \text{Hom}^*(\hat{R}CH(\rho_{S_1}^p L(i_1, j_1^* F), E_{zar}(\Omega^*_{S_1} F, F))[−d_{S_1}], u_{IJ}^* F))
\]

\[
\xrightarrow{(\text{Gr}_{S_1}^* \cdot \text{Gr}_{S_1}^*)(−, −)} (e(\tilde{S}_I), \text{Hom}^*(\hat{R}CH(\rho_{S_1}^p L(i_1, j_1^* F), E_{zar}(\Omega^*_{S_1} F, F))[−d_{S_1}], u_{IJ}^* F)) = \mathcal{F}_S^{FDR}(M)
\]

**Proposition 118.** Let \( S \in \text{Var}(\mathbb{C}) \) and \( S = \bigcup_{i=1}^l S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \).

(i) For \( M \in \text{DA}_c(S) \) the map in \( D_{\text{O}S_0, \mathbb{D}_*}(S/\{\tilde{S}_I\}) = D_{\text{O}S_0, \mathbb{D}_*}(S) \)

\[
o_{\mathbb{D}_*}T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : o_{\mathbb{D}_*}\mathcal{F}_S^{GM}(L_{S} M) \xrightarrow{\sim} o_{\mathbb{D}_*}\mathcal{F}_S^{FDR}(M)
\]

given in definition 135 is an isomorphism if we forgot the Hodge filtration \( F \).

(ii) For \( M \in \text{DA}_c(S) \) and all \( n, p \in \mathbb{Z} \), the map in \( \text{PSh}_{\text{O}S_0, \mathbb{D}_*}(S/\{\tilde{S}_I\}) \)

\[
F^n H^p T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : F^n H^p \mathcal{F}_S^{GM}(L_{S} M) \hookrightarrow F^n H^p \mathcal{F}_S^{FDR}(M)
\]
given in definition 135 is a monomorphism. Note that $F^pH^nT(F_S^{GM}, F_S^{FDR})(M)$ is NOT an isomorphism in general: take for example $M(S^o/S) = D(\mathbb{A}, et)(j, E_{zar}(\mathbb{Z}(S^o/S)))$ for an open embedding $j : S^o \hookrightarrow S,$ then

$$H^n F_S^{GM}(L\mathbb{D}_S M(S^o/S)') = F_S^{GM}(\mathbb{Z}(S^o/S)) = j_* E(O_{S^o}, F_0) \notin \pi_S(MHM(S))$$

and hence is NOT isomorphic to $H^n F_S^{FDR}(L\mathbb{D}_S M(S^o/S))' \in \pi_S(MHM(S))$ as filtered $D_S$-modules (see remark 9). It is an isomorphism in the very particular cases where $M = D(\mathbb{A}, et)(\mathbb{Z}(X/S))$ or $M = D(\mathbb{A}, et)(\mathbb{Z}(X^o/S))$ for $f : X \to S$ a smooth proper morphism and $n : X^o \to X$ an open subset such that $X \setminus X^o = \cup D_i$ is a normal crossing divisor and such that $f_i|D_i = f \circ i_i : D_i \to X$ are SMOOTH morphism with $i_i : D_i \hookrightarrow X$ the closed embedding and considering $f|_{X^o} = f \circ n : X^o \to S$ (see proposition 111).

Proof. (i):Follows from the computation for a Borel-Moore motive.

(ii):Follows from (i).

We now define the functorialities of $F_{S}^{FDR}$ with respect to $S$ which makes $F_{S}^{FDR}$ a morphism of 2 functor.

**Definition 136.** Let $S \in \text{Var}(\mathbb{C})$. Let $Z \subset S$ a closed subset. Let $S = \cup_{i=1}^{t} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow S$ with $S_i \in \text{SmVar}(\mathbb{C})$. Denote $Z_i := Z \cap S_i$. We then have closed embeddings $Z_i \hookrightarrow S_i \hookrightarrow S$.

(i) For $F \in C(\text{Var}(\mathbb{C})^{em}/S)$, we will consider the following canonical map in $\pi_S(D(MHM(S))) \subset D_{\mathcal{D}(1, 0)\text{fun}}(S/(S_i))$

$$T(\Gamma_Z^{V, Hdg}, \Omega^\bullet_{S}) (F, W) : \Gamma_Z^{V, Hdg} t_S^{-1}(e'_* \text{Hom}^\bullet(\hat{R}^{CH} (p_{S_i}^* \Gamma_{S} L(i_1, j_i^* F)), E_{zar}(\Omega^\bullet_{S}, F_{DR}))[-d_S], u_{i j}^{Z}(F, W))$$

$$\xrightarrow{\text{Hom}^\bullet(\hat{R}^{CH} (\gamma_{V, Z} (L(i_1, j_i^* F))), E_{zar}(\Omega^\bullet_{S}, F_{DR}))[-d_S]} u_{i j}^{Z}(F, W)$$

with

$$u_{i j}^{Z}(F, W) : e'_*(\hat{S}_i), \text{Hom}^\bullet(\hat{R}^{CH} (p_{S_i}^* Z L(i_1, j_i^* F)), E_{zar}(\Omega^\bullet_{S}, F_{DR}))$$

$$p_{i j} \cdot T(p_{i j}, \gamma_{V, S}) \cdot \text{load}(p_{i j}, \gamma_{V, S})^{-1}$$

(ii) For $F \in C(\text{Var}(\mathbb{C})^{em}/S)$, we have also the following canonical map in $\pi_S(D(MHM(S))) \subset$
\[ D_{\mathcal{D}(1,0)}(S/\langle \tilde{S} \rangle) \]

\[ T(\Gamma_{Z}^{Hdg}, \Omega_{S}^{pr})(F, W) : \]

\[ \iota_{S}^{-1}(e_{s}^{*}\text{Hom}^{*}(R^{CH}(\rho_{S}^{*} L \Gamma_{Z} E(i_{1}, j_{1}^{*} D_{S}(F, W))), E_{zar}^{*}(\Omega^{\Gamma_{S}^{pr}, F_{DR}^{*}})[d_{S}]), u_{I_{J}}^{*}(d_{S}, F, W)) \]

\[ \Gamma_{Z}^{Hdg} \iota_{S}^{-1}(e_{s}^{*}\text{Hom}^{*}(R^{CH}(\rho_{S}^{*} L \Gamma_{Z} E(i_{1}, j_{1}^{*} D_{S}(F, W))), E_{zar}^{*}(\Omega^{\Gamma_{S}^{pr}, F_{DR}^{*}})[d_{S}]), u_{I_{J}}^{*}(d_{S}, F, W)) \]

\[ \text{with} \]

\[ u_{I_{J}}^{*}(F)[d_{S}] : e_{s}(\tilde{S})_{*}, \text{Hom}^{*}(L_{\rho_{S}^{*} H_{S}^{*}}, R^{CH}(\rho_{S}^{*} L \Gamma_{Z} E(i_{1}, j_{1}^{*} D_{S}(L F))), E_{zar}^{*}(\Omega^{\Gamma_{S}^{pr}, F_{DR}^{*}})) \]

\[ p_{I_{J}} \Gamma_{I_{J}}^{*} e_{s}(\tilde{S})_{*}, \text{Hom}^{*}(p_{I_{J}}^{*} R^{CH}(\rho_{S}^{*} L \Gamma_{Z} E(i_{1}, j_{1}^{*} D_{S}(F))), E_{zar}^{*}(\Omega^{\Gamma_{S}^{pr}, F_{DR}^{*}})) \]

\[ \text{This transformation map will, with the projection case, gives the transformation between the pullback functor :} \]

**Definition 137.** Let \( g : T \to S \) a morphism with \( T, S \in \text{SmVar}(\mathbb{C}) \). Consider the factorization \( g : T \to T \times S \xrightarrow{p_{S}} S \) where \( l \) is the graph embedding and \( p_{S} \) the projection. Let \( M \in \text{DA}_{\mathcal{S}}(S)^{-} \) and \((F, W) \in C_{\text{fud}}(\text{Var}(\mathbb{C})^{ex})^{S} / S \) such that \((M, W) = D(\hat{\mathbb{A}}^{1}, et)(F, W) \). Then, \( D(\hat{\mathbb{A}}^{1}, et)(g^{*} M) \) and there exist \( (F', W) \in C_{\text{fud}}(\text{Var}(\mathbb{C})^{ex})^{S} / T \times S \) and an equivalence \( (\mathbb{A}^{1}, et) \) local \( e : \Gamma_{S}^{*} p_{S}^{*} F, W \to (F', W) \) such that \( D(\hat{\mathbb{A}}^{1}, et)(F', W) = \Gamma_{S}^{*} p_{S}^{*} M, W \). We have then the canonical transformation in \( \pi_{T}(D(MHM(T)) \) using definition 131 and definition 136(i) :}

\[ T(g, F_{DR}^{F})(M) : g^{*} \text{mod, Hdg} \varphi_{S} F_{DR}^{F}(M) := \]

\[ \Gamma_{T}^{*} \varphi_{S}^{mod}[d_{S}] : e_{T}(\tilde{S})_{*}, \text{Hom}^{*}(\rho_{S}^{*} L E(T \times S), E_{zar}^{*}(\Omega^{\Gamma_{S}^{pr}, F_{DR}^{*}})[d_{S}]) \]

\[ \text{with} \]

\[ T(\Gamma_{T}^{*} \varphi_{S}^{mod}, \Omega_{S}^{pr})(F, W) : \]

\[ \Gamma_{T}^{*} \varphi_{S}^{mod}[d_{S}] : e_{T}(\tilde{S})_{*}, \text{Hom}^{*}(\rho_{S}^{*} L E(T \times S), E_{zar}^{*}(\Omega^{\Gamma_{S}^{pr}, F_{DR}^{*}})[d_{S}]) \]

\[ =: \varphi_{T} F_{DR}^{F}(l_{*} g^{*} M) = \varphi_{T} F_{DR}^{F}(g^{*} M) \]
where the last equality follows from proposition 117.

We give now the definition in the non smooth case. Let $g : T \to S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{p} Y \times S \xrightarrow{\rho} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \bigcup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_1 : S_i \hookrightarrow S_i$ with $S_i \in \text{SmVar}(\mathbb{C})$ Then, $T = \bigcup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i \circ l : T_i \hookrightarrow Y \times S_i$. Moreover $\tilde{g}_I := p_{S_i} : Y \times S_i \to S_i$ is a lift of $g_I := g|_{T_i} : T_i \to S_i$. We recall the commutative diagram :

$$E_{I,J,g} = (Y \times S_i)|_{T_i} \xrightarrow{m_{I,J}} Y \times S_J, E_{I,J} = \tilde{S}_I \setminus S_J \xrightarrow{m_{I,J}} \tilde{S}_I \quad E_{I,J} = (Y \times S_i)|_{T_i} \xrightarrow{m_{I,J}} Y \times S_J$$

For $I \subset J$, denote by $p_{I,J} : \tilde{S}_I \to \tilde{S}_J$ and $p'_{I,J} := I_Y \times p_{I,J} : Y \times \tilde{S}_J \to Y \times \tilde{S}_I$ the projections, so that $\tilde{g}_I \circ p'_{I,J} = p_{I,J} \circ \tilde{g}_J$. Consider, for $I \subset J \subset \{1, \ldots, l\}$, resp. for each $I \subset \{1, \ldots, l\}$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$D_{I,J} = \begin{array}{ccc}
S_I & \xrightarrow{i_I} & \tilde{S}_I \\
\downarrow j_I & & \downarrow j_I \\
S_J & \xrightarrow{i_J} & \tilde{S}_J 
\end{array}, D'_{I,J} = \begin{array}{ccc}
T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_J \\
\downarrow p_{I,J} & & \downarrow p_{I,J} \\
T_J & \xrightarrow{i'_J} & Y \times \tilde{S}_J 
\end{array}, D_{g_I} = \begin{array}{ccc}
S_I & \xrightarrow{i_I} & \tilde{S}_I \\
\downarrow g_I & & \downarrow g_I \\
T_I & \xrightarrow{i'_I} & Y \times \tilde{S}_J 
\end{array}$$

and $j_I : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{I,J} = j_I$. Let $F \in C(\text{Var}(\mathbb{C})^{em}/S)$. Recall (see section 2) that since $j_I'^* i'_{I,J} g^* F = 0$, the morphism $T(D_{g_I})(j_I^* F) : \tilde{g}_I'^* i'_{I,J} g^* F \to i_I'^* g^* F$ factors trough

$$T(D_{g_I})(j_I^* F) : \tilde{g}_I'^* i'_{I,J} g^* F \xrightarrow{T^{-1}(D_{g_I})(j_I^* F)} \Gamma_{\tilde{S}_I} i_I'^* g^* F \xrightarrow{T^\vee(D_{g_I})(j_I^* F)} i_I'^* g^* F.$$

and that the fact that the diagrams (61) commutes says that the maps $T^\vee(D_{g_I})(j_I^* F)$ define a morphism in $C(\text{Var}(\mathbb{C})^{em}/(T/(Y \times S_I)))$

$$(T^\vee(D_{g_I})(j_I^* F)) : (\Gamma_{\tilde{S}_I} \tilde{g}_I'^* L(i_{I,J}^* g^* F), T^\vee(D_{I,J})(j_I^* F) \circ T(T_{I,J}/T_I \times \tilde{S}_{J,I}, \gamma^{-})(-)) \xrightarrow{T^\vee(D_{I,J})(j_I^* g^* F)} (\Gamma_{\tilde{S}_I} i_I'^* g^* F, T^\vee(D_{I,J})(j_I^* g^* F))$$

Denote for short $d_{Y^I} := -d_Y - d_{S_J}$. We denote by $\tilde{g}_I'^* u_{I,J}^* F$ the composite

$$\tilde{g}_I'^* u_{I,J}^* F[-d_{Y^I}] : e'(Y \times \tilde{S}_I), \text{Hom}((\hat{R}_Y^{\mathbb{C}}(\rho_{Y \times S_J}^{*} \tilde{g}_I'^* L(i_{I,J}^* g^* F)), E_{zar}(\Omega_{Y \times S_J}^{* \Gamma, pr}, F_{DR}))$$

$$p_{I,J}^*, T(p_{I,J}^* \Omega_{Y \times S_J}^{* \Gamma, pr})(-) \circ (\rho_{Y \times S_J}^{* \tau, \text{mod}} \tilde{g}_I'^*)(-)$$

$$p_{I,J}^*, e'(Y \times \tilde{S}_I), \text{Hom}(p_{I,J}^* \hat{R}_Y^{\mathbb{C}}(\rho_{Y \times S_J}^{*} \tilde{g}_I'^* L(i_{I,J}^* g^* F)), E_{zar}(\Omega_{Y \times S_J}^{* \Gamma, pr}, F_{DR}))$$

$$\text{Hom}(T(p_{I,J}^* \hat{R}_Y^{\mathbb{C}})(-1), E_{zar}(\Omega_{Y \times S_J}^{* \Gamma, pr}, F_{DR}))$$

$$p_{I,J}^*, e'(Y \times \tilde{S}_I), \text{Hom}(\hat{R}_Y^{\mathbb{C}}(\rho_{Y \times S_J}^{*} p_{I,J}^* \tilde{g}_I'^* L(i_{I,J}^* g^* F)), E_{zar}(\Omega_{Y \times S_J}^{* \Gamma, pr}, F_{DR}))$$

$$\text{Hom}(\hat{R}_Y^{\mathbb{C}}(T^\vee(D_{I,J})(j_I^* g^* F)), E_{zar}(\Omega_{Y \times S_J}^{* \Gamma, pr}, F_{DR}))$$

$$e'(Y \times \tilde{S}_I), \text{Hom}(\hat{R}_Y^{\mathbb{C}}(\rho_{Y \times S_J}^{*} \tilde{g}_I'^* L(i_{I,J}^* g^* F)), E_{zar}(\Omega_{Y \times S_J}^{* \Gamma, pr}, F_{DR})).$$

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We denote by \( \tilde{g}_j^* \gamma u_{ijJ}^q(F) \) the composite
\[ e'(Y \times \tilde{S}_1), \text{Hom}(\tilde{R}^{CH}(\rho_{Y \times \tilde{S}_1} \Gamma^\vee g_1^* L(i_1,j_1^* F)), E_{zar}(\Omega^\bullet_{Y \times \tilde{S}_1}^F, F_{DR})) \]
\[ p_{1J,J}' e'(Y \times \tilde{S}_1), \text{Hom}(\tilde{p}_{1J,J}' \tilde{R}^{CH}(\rho_{Y \times \tilde{S}_1} \Gamma^\vee g_1^* L(i_1,j_1^* F)), E_{zar}(\Omega^\bullet_{Y \times \tilde{S}_1}^F, F_{DR})) \]
\[ \text{Hom}(\tilde{p}_{1J,J}' R^{CH}(\rho_{Y \times \tilde{S}_1} \Gamma^\vee g_1^* L(i_1,j_1^* F)), E_{zar}(\Omega^\bullet_{Y \times \tilde{S}_1}^F, F_{DR})) \]
\[ e'(Y \times \tilde{S}_1), \text{Hom}(\tilde{R}^{CH}(\rho_{Y \times \tilde{S}_1} \Gamma^\vee g_1^* L(i_1,j_1^* F)), E_{zar}(\Omega^\bullet_{Y \times \tilde{S}_1}^F, F_{DR})) \]
\[ e'(Y \times \tilde{S}_1), \text{Hom}(\tilde{R}^{CH}(\rho_{Y \times \tilde{S}_1} \Gamma^\vee g_1^* L(i_1,j_1^* F)), E_{zar}(\Omega^\bullet_{Y \times \tilde{S}_1}^F, F_{DR})) \]

We then have the following lemma:

**Lemma 13.** (i) The morphism in \( (\text{C}(\text{Var}(\mathbb{C}))^\text{sm} / (T / (Y \times \tilde{S}_1))) \)
\[ (T^q \gamma (D_{gt})(j_1^* F)) : (\Gamma_{Y, 1} L g_1^* i_1, j_1^* F, T^q(D_{gt})(j_1^* F)) \circ T(T_1 / T_1 \times \tilde{S}_1, Y, Y, Y, (-) \circ T(p_{1J,J}', 
\[ g^* F) \]
\[ \text{is an equivalence (\mathbb{A}^1, et) local.} \]

(ii) The maps \( \text{Hom}((T^q \gamma (D_{gt})(j_1^* F)), E_{zar}(\Omega^\bullet_{Y \times \tilde{S}_1}^F, F_{DR})) \) induce a filtered quasi-isomorphism in \( C_{\text{fil}}(T / (Y \times \tilde{S}_1)) \)
\[ (\text{Hom}(\tilde{R}^{CH}(\rho_{Y \times \tilde{S}_1} T^q \gamma (D_{gt})(j_1^* F)), E_{zar}(\Omega^\bullet_{Y \times \tilde{S}_1}^F, F_{DR})) [dy_1] : \]
\[ (\text{Hom}(\tilde{R}^{CH}(\rho_{Y \times \tilde{S}_1} T^q \gamma (D_{gt})(j_1^* F)), E_{zar}(\Omega^\bullet_{Y \times \tilde{S}_1}^F, F_{DR})) [dy_1], \tilde{g}_j^* \gamma u_{ijJ}^q(F) \]
\[ \to (\text{Hom}(\tilde{R}^{CH}(\rho_{Y \times \tilde{S}_1} T^q \gamma (D_{gt})(j_1^* F)), E_{zar}(\Omega^\bullet_{Y \times \tilde{S}_1}^F, F_{DR})) [dy_1], u_{ijJ}^q(g^* F) \]

(iii) The maps \( T(\tilde{g}_1, \Omega^\cdot F_{DR})(-)(\text{see definition 131), induce a morphism in } C_{\text{fil}}(T / (Y \times \tilde{S}_1)) \)
\[ T(\tilde{g}_1, \Omega^\cdot F_{DR})(-)[dy_1] : \]
\[ (\text{Hom}(\tilde{R}^{CH}(\rho_{Y \times \tilde{S}_1} T^q \gamma (D_{gt})(j_1^* F)), E_{zar}(\Omega^\bullet_{Y \times \tilde{S}_1}^F, F_{DR})) [dy_1], \tilde{g}_j^* \gamma u_{ijJ}^q(F) \]
\[ \to (\text{Hom}(\tilde{R}^{CH}(\rho_{Y \times \tilde{S}_1} T^q \gamma (D_{gt})(j_1^* F)), E_{zar}(\Omega^\bullet_{Y \times \tilde{S}_1}^F, F_{DR})) [dy_1], \tilde{g}_j^* \gamma u_{ijJ}^q(F) \]

**Proof.** (i):Follows from theorem 16.
(ii):These morphism induce a morphism in \( C_{\text{fil}}(T / (Y \times \tilde{S}_1)) \) by construction. The fact that this morphism is an filtered equivalence Zariski local follows from (i) and proposition 114.
(iii):These morphism induce a morphism in \( C_{\text{fil}}(T / (Y \times \tilde{S}_1)) \) by construction. \( \square \)

**Definition 138.** Let \( g : T \to S \) a morphism with \( T, S \in \text{Var}(\mathbb{C}) \). Assume we have a factorization \( g : T \to \to Y \times S \xrightarrow{p_S} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. Let \( S = \bigcup_{i=1}^l S_i \) be an open cover such that there exists closed embeddings \( i : S_i \to \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). Then, \( T = \bigcup_{i=1}^l T_i \) with \( T_i := g^{-1}(S_i) \) and we have closed embeddings \( i_i : i_i : l : T_i \to Y \times \tilde{S}_i \). Moreover \( \tilde{g}_1 := p_{\tilde{S}_1} : Y \times \tilde{S}_1 \to \tilde{S}_1 \) is a lift of \( g_1 := g_{\tilde{S}_1} : T_1 \to \tilde{S}_1 \). Let \( M \in \text{DA}_*(S)^- \) and \( (F, W) \in C_{fil}(\text{Var}(\mathbb{C})^\cdot / S) \) such that \( (M, W) = D(A^\cdot, et)(F, W) \). Then, \( D(A^\cdot, et)(g^* M) = g^* M \) and there exist \( (F', W) \in C_{fil}(\text{Var}(\mathbb{C})^\cdot / S) \)
and an equivalence \((\mathcal{A}, \mathcal{B})\) local e. g. \(g^*(F, W) \to (F', W)\) such that \(D(\mathcal{A}, \mathcal{B}) = (g^*M, W)\). Denote for short \(d_Y := -d_Y - d_S\). We have, using definition 131 and definition 136(i), by lemma 13, the canonical map in \(\pi_T(D(MHM(T))) \subset D_{D(1,0)}(T/(Y \times S))\)

\[
T(\mathcal{F}^{FDR}(M)) : g_{Hdg}^{\star} F^*_{\mathcal{S}}(M) := \Gamma^{\mathcal{F},Hdg}_{T} t^{-1}((\mathcal{F})^{\star \text{mod}}(e'_{\mathcal{F}} \text{Hom}^*(\mathcal{R}^{CH}(\rho_{S_i} (L(i_1, j_1^r (F, W)))), \pi_{zar}(\Omega^*_J, F_{\mathcal{DR}})))|_{dY1}, g_{J^r}^{\star \text{mod}}(u_{J^r J} (F, W)))
\]

\[
\Gamma^{\mathcal{F},Hdg}_{T} t^{-1}((\mathcal{F})^{\star \text{mod}}(e'_{\mathcal{F}} \text{Hom}^*(\mathcal{R}^{CH}(\rho_{S_i} (L(i_1, j_1^r (F, W)))), \pi_{zar}(\Omega^*_J, F_{\mathcal{DR}})))|_{dY1}, g_{J^r}^{\star \text{mod}}(u_{J^r J} (F, W)))
\]

\[
\Gamma^{\mathcal{F},Hdg}_{T} t^{-1}((\mathcal{F})^{\star \text{mod}}(e'_{\mathcal{F}} \text{Hom}^*(\mathcal{R}^{CH}(\rho_{S_i} (L(i_1, j_1^r (F, W)))), \pi_{zar}(\Omega^*_J, F_{\mathcal{DR}})))|_{dY1}, g_{J^r}^{\star \text{mod}}(u_{J^r J} (F, W)))
\]

\[
\Gamma^{\mathcal{F},Hdg}_{T} t^{-1}((\mathcal{F})^{\star \text{mod}}(e'_{\mathcal{F}} \text{Hom}^*(\mathcal{R}^{CH}(\rho_{S_i} (L(i_1, j_1^r (F, W)))), \pi_{zar}(\Omega^*_J, F_{\mathcal{DR}})))|_{dY1}, g_{J^r}^{\star \text{mod}}(u_{J^r J} (F, W)))
\]

*Definition 139.* Let \(f : X \to S\) be a morphism with \(X, S \in \text{Var}(\mathcal{C})\). Assume there exist a factorization \(f : X \xrightarrow{p} Y \to S\) with \(Y \in \text{SmVar}(\mathcal{C})\), \(l\) a closed embedding and \(p_S\) the projection. We have, for \(M \in \text{DA}_c(X)\), the following transformation map in \(\pi_S(D(MHM(S)))\)

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]

\[
T_{\mathcal{F}}(f, F_{\mathcal{DR}})(M) : F^*_{\mathcal{S}}(M) \to F^*_{\mathcal{S}}(M) := \text{ad}(\mathcal{F})^{\mathcal{F}_{\mathcal{D}}}(f) \to F^*_{\mathcal{S}}(M)
\]
Let $f : X \to S$ a morphism with $X, S \in \Var(C)$. Assume there exist a factorization $f : X \xrightarrow{\sim} Y \times S \xrightarrow{p_S} S$ with $Y \in \SmVar(C)$, $l$ a closed embedding and $p_S$ the projection. We have, using the third point, for $M \in \DA(S)$, the following transformation map in $\pi_X(D(MHM(X)))$

\[
T(f, T_F^\text{DR})(M) : T_S^\text{DR}(f^! M) \xrightarrow{\text{ad}(Rf^! Hdg)} T_S^\text{DR}(f^! Rf^! M) \xrightarrow{\text{Fr}^\text{mod}_S Hdg} T_S^\text{DR}(f^! M)
\]

Let $S \in \Var(C)$. Let $S = \bigcup_{i=1}^n S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \SmVar(C)$. We have, using the preceding point, denoting $\Delta_S : S \hookrightarrow S$ the diagonal closed embedding and $p_1 : S \times S \to S$, $p_2 : S \times S \to S$ the projections, for $M, N \in \DA(S)$ and $(F, W), (G, W) \in C_{fil}(\Var(C)^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$ and $(N, W) = D(\mathbb{A}^1, et)(G, W)$, the following transformation map in $\pi_S(D(MHM(S)))$

\[
T(S^\text{DR}, \otimes)(M, N) : S^\text{DR}(M) \otimes_{\OS} S^\text{DR}(N) := \Delta_S^Hdg(p_1^\text{mod}_S S^\text{DR}(M) \otimes_{\OS \times S} p_2^\text{mod}_S S^\text{DR}(N) \xrightarrow{\text{ad}(\OS \times S)} T(S^\text{DR}(M) \otimes_{\OS} S^\text{DR}(N) \xrightarrow{\text{Fr}^\text{mod}_S} S^\text{DR}(p_1^\text{mod}_S \otimes p_2^\text{mod}_S N)
\]

where the last equality follows from the equality in $\DA(S)$

\[
\Delta_S(p_1^\text{mod}_S \otimes p_2^\text{mod}_S N) = \Delta_S^\text{DR}(p_1^\text{mod}_S \otimes p_2^\text{mod}_S N) = \Delta_S^\text{DR}(M \otimes N)
\]

**Proposition 119.** Let $g : T \to S$ a morphism with $T, S \in \Var(C)$. Assume we have a factorization $g : T \xrightarrow{\sim} Y_2 \times S \xrightarrow{p_S} S$ with $Y_2 \in \SmVar(C)$, $l$ a closed embedding and $p_S$ the projection. Let $S = \bigcup_{i=1}^n S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \SmVar(C)$ Then, $T = \bigcup_{i=1}^n T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$, Moreover $\tilde{g}_i := p_{S_i} : Y \times \tilde{S}_i \to S_i$ is a lift of $g_i := g_{T_i} : T_i \to S_i$. Let $f : X \to S$ a morphism with $X \in \Var(C)$ such that there exists a factorization $f : X \xrightarrow{\sim} Y_1 \times S \xrightarrow{p_S} S$, with $Y_1 \in \SmVar(C)$, $l$ a closed embedding and $p_S$ the projection. We have then the following commutative diagram whose squares are cartesians

\[
f : X \to \begin{array}{ccc} Y_1 \times T & \xrightarrow{f'} & Y_1 \times T \\ \downarrow & & \downarrow \\ Y_1 \times S & \xrightarrow{g'} & Y_1 \times \tilde{S}_i \\ \downarrow & & \downarrow \\ \tilde{Y}_1 \times \tilde{S}_i & \xrightarrow{S} & \tilde{Y}_1 \times \tilde{S}_i \end{array}
\]

Take a smooth compactification $\tilde{Y}_1 \in \PSmVar(C)$ of $Y_1$, denote $\tilde{X}_I \subset \tilde{Y}_1 \times \tilde{S}_i$ the closure of $X_I$, and $Z_I := \tilde{X}_I \setminus X_I$. Consider $F(X/S) := p_{S_2} \Gamma_X^Y Z(Y_1 \times S/Y_1 \times S) \in C(\Var(C)^{sm}/S)$ and the isomorphism in $C(\Var(C)^{sm}/T)$

\[
T(f, g, F(X/S)) : g^* F(X/S) := g^* p_{S_2} \Gamma_X^Y Z(Y_1 \times S/Y_1 \times S) \xrightarrow{\sim} p_{T_2} \Gamma_{X_T}^Y Z(Y_1 \times T/Y_1 \times T) := F(X_T/T).
\]

which gives in $\DA(T)$ the isomorphism $T(f, g, F(X/S)) : g^* M(X/S) \xrightarrow{\sim} (X_T/T)$. Then the following diagram in $\pi_T(D(MHM(T))) \subset D_{\text{D}(1.0,f)(T/(Y_2 \times S_i))}$, where the horizontal maps are given by proposition

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Proof. Follows immediately from definition.

Proposition 120. Let $S \in \text{Var}(\mathbb{C})$. Let $Y \in \text{SmVar}(\mathbb{C})$ and $p : Y \times S \to S$ the projection. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i^j : S_i \to \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset \{1, \cdots, l\}$, we denote by $S_I = \cap_{i \in I} S_i$, $j_i^j : S_I \to S$ and $j_I : Y \times S_I \to Y \times S$ the open embeddings. We then have closed embeddings $i'_I : Y \times S_I \to Y \times \tilde{S}_I$, and we denote by $p_{S_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ the projections. Let $f' : X' \to Y \times S$ a morphism, with $X' \in \text{Var}(\mathbb{C})$ such that there exists a factorization $f' : X' \xrightarrow{j'} Y' \times S \xrightarrow{f} Y \times S$ with $Y' \in \text{SmVar}(\mathbb{C})$, $l'$ a closed embedding and $p'$ the projection. Denoting $X'_I := f'^{-1}(Y \times S_I)$, we have closed embeddings $i'_I : X'_I \to Y' \times \tilde{S}_I$. Consider

$$F(X'/Y \times S) := p_{Y \times S} \Gamma_{X'/Z}(Y' \times X \times Y' \times X \times Y),$$

and $F(X'/S) := p_S F(X'/Y \times S) \in C(\text{Var}(\mathbb{C})^m / S)$, so that $Lp_{S} M(X'/S) [-2d_Y] =: M(X'/S)$. Then, the following diagram in $\pi_S(D(MHM(S))) \subset D^{(1,0)}_{\text{fil}}(S/Y \times S_I)$, where the vertical maps are given by proposition 116, commutes

$$\begin{array}{c}
\begin{array}{c}
\xymatrix{ R_{p_H^{S_H}} F_{S}^{\text{HDR}}(M(X'/Y \times S)) \ar[r]^{T(p_H^{S_H})} & F_{S}^{\text{HDR}}(M(X'/S)) \ar[d]^I(X'/S)} \\
R_{p_H^{S_H}} R_{j'_I}^{\text{HDR}}(i'^{-1}(Y \times S_I)) \ar[r]^{\mathbf{f}_{Y \times S}^{\text{HDR}}} & R_{j'_I}^{\text{HDR}}(i'^{-1}(Y \times S_I)) \ar[l]_{f'_{Y \times S}^{\text{HDR}}}
\end{array}
\end{array}$$

Proof. Immediate from definition.

Proposition 121. Let $f_1 : X_1 \to S$, $f_2 : X_2 \to S$ two morphism with $X_1, X_2, S \in \text{Var}(\mathbb{C})$. Assume that there exist factorizations $f_1 : X_1 \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $f_2 : X_2 \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$ with $Y_1, Y_2 \in \text{SmVar}(\mathbb{C})$, $l_1, l_2$ closed embeddings and $p_S$ the projections. We have then the factorization

$$f_{12} := f_1 \times f_2 : X_{12} := X_1 \times X_2 \xrightarrow{l_{12}} Y_1 \times Y_2 \times S \xrightarrow{p_S} S$$

Let $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset \{1, \cdots, l\}$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \to S$ the open embedding. Let $i_i : S_I \to \tilde{S}_I$ closed embeddings, with $\tilde{S}_I \in \text{SmVar}(\mathbb{C})$. We have then the following commutative diagram in $\pi_S(D(MHM(S))) \subset D^{(1,0)}_{\text{fil}}(S/\tilde{S}_I)$ where the vertical maps are given by proposition 116

$$\begin{array}{c}
\begin{array}{c}
\xymatrix{ \mathcal{F}_{S}^{\text{HDR}}(M(X_1/S)) \otimes_{O_S} \mathcal{F}_{S}^{\text{HDR}}(M(X_2/S)) \ar[r]^{I(X_1/S) \otimes I(X_2/S)} & \mathcal{F}_{S}^{\text{HDR}}(M(X_1/S) \otimes M(X_2/S) = M(X_1 \times X_2/S) \ar[d]^{T(\mathcal{F}_{S}^{\text{HDR}} \otimes M(X_1/S), M(X_2/S))}}} \\
R_{j_1}^{\text{HDR}}(i_{X_1}^{S_H}(O_{Y_1 \times S_1}, F_b)(d_1)[2d_1], x_{IJ}(X_1/S) \otimes O_S) \ar[r] & R_{j_2}^{\text{HDR}}(i_{X_2}^{S_H}(O_{Y_2 \times S_1}, F_b)(d_1)[2d_2], x_{IJ}(X_2/S)) \ar[l]_{R_{j_{12}}^{\text{HDR}}(i_{X_1 \times X_2}^{S_H}(O_{Y_1 \times Y_2 \times S_1}, F_b)(d_12)[2d_12], x_{IJ}(X_1/S))}
\end{array}
\end{array}$$

with $d_1 = d_{Y_1}$, $d_2 = d_{Y_2}$ and $d_{12} = d_{Y_1} + d_{Y_2}$.
Theorem 36. (i) Let $g : T \to S$ a morphism, with $S,T \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $M \in \text{DA}_c(S)$. Then map in $\pi_T(D(MHM(T)))$

$$T(g, \mathcal{F}^{FDR})(M) : g^{*\text{mod}}_{Hdg} \mathcal{F}^{FDR}_S(M) \xrightarrow{\sim} \mathcal{F}^{FDR}_T(g^{*}M)$$

given in definition 138 is an isomorphism.

(ii) Let $f : X \to S$ a morphism with $X,S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Then, for $M \in \text{DA}_c(X)$, the map given in definition 139

$$T_l(f, \mathcal{F}^{FDR})(M) : Rf^*_l Rf^*_t \mathcal{F}^{FDR}_X(M) \xrightarrow{\sim} \mathcal{F}^{FDR}_S(Rf_l M)$$

is an isomorphism in $\pi_S(D(MHM(S)))$.

(iii) Let $f : X \to S$ a morphism with $X,S \in \text{Var}(\mathbb{C})$, $S$ quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. We have, for $M \in \text{DA}_c(X)$, the map given in definition 139

$$T_s(f, \mathcal{F}^{FDR})(M) : \mathcal{F}^{FDR}_S(Rf_l M) \xrightarrow{\sim} Rf^*_s Rf^*_t \mathcal{F}^{FDR}_X(M)$$

is an isomorphism in $\pi_X(D(MHM(X)))$.

(iv) Let $f : X \to S$ a morphism with $X,S \in \text{Var}(\mathbb{C})$, $S$ quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Then, for $M \in \text{DA}_c(S)$, the map in $\pi_S(D(MHM(S)))$

$$T(f, \mathcal{F}^{FDR})(M) : \mathcal{F}^{FDR}_X(f^1 M) \xrightarrow{\sim} f^*_l Rf^*_t \mathcal{F}^{FDR}_S(M)$$

is an isomorphism in $\pi_X(D(MHM(X)))$.

(v) Let $S \in \text{Var}(\mathbb{C})$ and $S = \bigcup_{i=1}^{l} S_i$ an open affine covering and denote, for $I \subset [1,\cdots,l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M,N \in \text{DA}_c(S)$, the map in $\pi_S(D(MHM(S)))$

$$T(\mathcal{F}^{FDR}_S, \otimes)(M,N) : \mathcal{F}^{FDR}_S(M) \otimes_{\mathcal{O}_S} \mathcal{F}^{FDR}_S(N) \xrightarrow{\sim} \mathcal{F}^{FDR}_S(M \otimes N)$$

given in definition 139 is an isomorphism.

Proof. (i): Follows from proposition 119 and proposition 116.

(ii): By proposition 120 and proposition 116, for $S \in \text{Var}(\mathbb{C})$, $Y \in \text{SmVar}(\mathbb{C})$, $p : Y \times S \to S$ the projection and $M \in \text{DA}_c(Y \times S)$,

$$T_l(p, \mathcal{F}^{FDR})(M) : Rp^*_l Rf^*_t \mathcal{F}^{FDR}_{Y \times S}(M) \to \mathcal{F}^{FDR}_S(Rp_l M)$$

is an isomorphism.

(iii): Consider first an open embedding $n : S^o \hookrightarrow S$ with $S \in \text{Var}(\mathbb{C})$ so that there exist a closed embedding $i : S \hookrightarrow \tilde{S}$ with $\tilde{S} \in \text{SmVar}(\mathbb{C})$. Then, since

$$n^* : C(\text{Var}(\mathbb{C})^\text{sm}/S) \to C(\text{Var}(\mathbb{C})^\text{sm}/S^o)$$

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is surjective, $n^*: \text{DA}(S) \to \text{DA}(S^n)$ is surjective. Denote by $i: Z = S\setminus S^o \hookrightarrow S$ the complementary closed embedding. By [1], $\text{DA}(S)$ is generated by motives of the form

$$
\text{DA}(S) = \langle M(X'/S) = f'_* E(\mathbb{Z}_{X'}) \rangle, \quad f': X' \to S \text{ proper with } X' \in \text{SmVar}(\mathbb{C}),
$$

s.t. $f^{-1}(Z) = X'$ or $f^{-1}(Z) = \cup D_i = D \subset X' > .$

If $f^{-1}(Z) = X'$, $n^* M(X'/S) = 0$. So let consider the case $f^{-1}(Z) = \cup D_i$. Then $D_i \cap X'$ is a normal crossing divisor. Denote $f^o_D: f^o_D: D \to Z$, $D_i = \cap_{i \in I} D_i$ and $i^o_f: D_i \hookrightarrow X'$, $n^o: X^o := X'\setminus D \to X'$ the complementary open embedding and $f^o: f'[X'^o : X^o \to S^o$ Denote $L = [1, \ldots, l]$. We have then a generalized distinguished triangle in $\text{DA}(X')$

$$
\begin{align*}
a(n^o, i^o) & : n^o_* n^o^* \text{et}_{Z}(\mathbb{Z}_{X'}) \xrightarrow{\sim} \text{Cone}(\gamma_D(-): \Gamma_D \text{et}_{Z}(\mathbb{Z}_{X'}) \to \text{et}_{Z}(\mathbb{Z}_{X'})) \\
& \to \text{Cone}(\Gamma_D \text{et}_{Z}(\mathbb{Z}_{X'}) \to \sum_{i=1}^{l} \text{et}_{Z}(\mathbb{Z}_{X'}) \\
& \to \text{Cone}(i^o_* n^o_* \text{et}_{Z}(\mathbb{Z}_{X'}) \to \bigoplus_{i=1}^{l} i^o_* \text{et}_{Z}(\mathbb{Z}_{X'})) \xrightarrow{\text{add}(i^o_* n^o^* \text{et}_{Z}(\mathbb{Z}_{X'}))} \text{et}_{Z}(\mathbb{Z}_{X'})) \\
& \to \text{Cone}(i^o_* \text{et}_{ZD_1}[-1] \to \cdots \bigoplus_{i=1}^{l} i^o_* \text{et}_{ZD_i}[-1] \to \text{et}_{Z}(\mathbb{Z}_{X'}))
\end{align*}
$$

where the first isomorphism is the image of an homotopy equivalence by definition, the second one is the image of an homotopy equivalence by definition-proposition 4(ii), the third one follows by the localization property of mixed Hodge modules (see section 5). Similarly, we have a generalized distinguished triangle in $D(MHM(S'))$

$$
a^\text{mod}(n^o, i^o) : n^o_* \text{Hdg} n^o^*(O_{X'}, F_b) \xrightarrow{\sim} \text{Cone}(\gamma_D^{\text{Hdg}}(O_{X'}, F_b) : \Gamma_D^{\text{Hdg}}(O_{X'}, F_b) \to (O_{X'}, F_b)) \\
\to \text{Cone}(\Gamma_D^{\text{Hdg}}(O_{X'}, F_b) \to \sum_{i=1}^{l} \Gamma_D^{\text{Hdg}}(O_{X'}, F_b) \xrightarrow{\oplus \gamma_D^{\text{Hdg}}(-)} (O_{X'}, F_b)) \\
\to \text{Cone}(i^o_* \text{mod}(O_{D_1}, F_b)[-1] \to \cdots \bigoplus_{i=1}^{l} i^o_* \text{mod}(O_{D_i}, F_b)[-1] \\
\xrightarrow{\text{add}(i^o_* \text{mod}(n^o, i^o)(O_{X'}, F_b))} \to (O_{X'}, F_b))
$$

where the first isomorphism is the image of an homotopy equivalence by definition, the second one is the image of an homotopy equivalence by definition-proposition 19, and the third one follows by the localization property of mixed Hodge modules (see section 5). Consider $n^* M(X'/S) = M(X^o/S^o)$. We have then the following commutative diagram in $\pi_S(MHM(S))$

$$
\begin{align*}
\text{F}^\text{FDR}_S(R_n^* n^* M(X'/S)) &= \text{F}^\text{FDR}_S(R_n^* n^* R^f_n \text{et}_{Z}(\mathbb{Z}_{X'})) \\
\text{F}^\text{FDR}_S(R_n^* T_f(n^* M(X'/S)) &= \text{F}^\text{FDR}_S(R_f^* R_n^* n^* \text{et}_{Z}(\mathbb{Z}_{X'})) \\
\text{F}^\text{FDR}_S(R_n^* n^* \text{et}_{Z}(\mathbb{Z}_{X'})) &= \text{F}^\text{FDR}_S(R_f^* R_n^* n^* \text{et}_{Z}(\mathbb{Z}_{X'})) \\
\text{F}^\text{FDR}_S(R_n^* n^* \text{et}_{Z}(\mathbb{Z}_{X'})) &= \text{F}^\text{FDR}_S(R_f^* R_n^* n^* \text{et}_{Z}(\mathbb{Z}_{X'})) \\
\text{F}^\text{FDR}_S(R_n^* n^* \text{et}_{Z}(\mathbb{Z}_{X'})) &= \text{F}^\text{FDR}_S(R_f^* R_n^* n^* \text{et}_{Z}(\mathbb{Z}_{X'})) \\
\text{F}^\text{FDR}_S(R_n^* n^* \text{et}_{Z}(\mathbb{Z}_{X'})) &= \text{F}^\text{FDR}_S(R_f^* R_n^* n^* \text{et}_{Z}(\mathbb{Z}_{X'}))
\end{align*}
$$

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Since all the morphism involved are isomorphisms, $T_*(n, F_{\text{FDR}}(n^* M(X'/S)))$ is an isomorphism. Hence, $T_*(n, F_{\text{FDR}})(M)$ is an isomorphism for all $M \in \text{DA}(S^n)$. Consider now the case of a general morphism $f : X \to S$, $X, S \in \text{Var}(\mathbb{C})$ quasi-projective, which factors trough $g : X \xrightarrow{i} Y \times S \xrightarrow{p_S} S$ with some $Y \in \text{SmVar}(\mathbb{C})$. By definition, for $M \in \text{DA}_c(X)$

$$T_*(f, F_{\text{FDR}})(M) : F_{\text{FDR}}^S(Rf_* M) = F_{\text{FDR}}^S(Rp_{S*} l_* M)$$

$$T_*(p_S, F_{\text{FDR}})(l_* M) \to Rp_{S*}^Hdg F_{\text{FDR}}^S(l_* M) =: Rf_{*}^Hdg F_{\text{FDR}}^S(M).$$

Hence, we have to show that for $S \in \text{Var}(\mathbb{C})$, $Y \in \text{SmVar}(\mathbb{C})$, $p : Y \times S \to S$ the projection, and $M \in \text{DA}_c(Y \times S)$,

$$T_*(p, F_{\text{FDR}})(M) : F_{\text{FDR}}^S(Rp_* M) \to R_{\text{Hdg}}^p F_{\text{FDR}}^S(M)$$

is an isomorphism. Take a smooth compactification $\bar{Y} \in \text{PSmVar}(\mathbb{C})$ of $Y$. Denote by $n_0 : Y \hookrightarrow \bar{Y}$ and $n := n_0 \times l_{\bar{S}} : Y \times S \hookrightarrow \bar{Y} \times S$ the open embeddings and by $\bar{p} : \bar{Y} \times S \to S$ the projection. We have $p = \bar{p} \circ n : Y \times S \to S$, which gives the factorization

$$T_*(p, F_{\text{FDR}})(M) : F_{\text{FDR}}^S(Rp_* M) = F_{\text{FDR}}^S(R\bar{p}_{\text{S*}} Rn_* M) \xrightarrow{T_*(p, F_{\text{FDR}})(Rn_* M)} R_{\text{Hdg}}^p F_{\text{FDR}}^S(Rn_* M)$$

By the open embedding case $T_*(n, F_{\text{FDR}})(M)$ is an isomorphism. On the other hand, since $\bar{p}$ is proper, $T_*(\bar{p}, F_{\text{FDR}})(Rn_* M) = T_! (\bar{p}, F_{\text{FDR}})(Rn_* M)$ is an isomorphism by (i).

(iv): Denote by $n : Y \times S \setminus X \hookrightarrow Y \times S$ the complementary open embedding. We have, for $M \in \text{DA}_c(S)$, the factorization

$$T^i(f, F_{\text{FDR}})(M) : F_{\text{FDR}}^X(t^i M)$$

$$\xrightarrow{T^i_!(f, F_{\text{FDR}})(t^i M)} F_{\text{FDR}}^S(Cone(p_S^1 M \to Rn_* n^* p_S^1 M)[-1])$$

$$\xrightarrow{\sim} Cone(F_{\text{FDR}}^S(p^1_S M) \to F_{\text{FDR}}^S(Rn_* n^* p_S^1 M)[-1])$$

$$\to Cone(p_{S*}^1 M \to R_{\text{Hdg}}^p n^* p_{S*}^1 M)[-1] \to f_{\text{Hdg}}^* F_{\text{FDR}}^S(M).$$

By (ii), $T(n, F_{\text{FDR}})(p_S^1 M)$ is an isomorphism. On the other hand, since $p_S$ is a smooth morphism, $T^i(p_S, F_{\text{FDR}})(M)$ is an isomorphism by (i).

**Lemma 14.** Let $g : T \to S$ be a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{i} Y \times S \xrightarrow{p_S \circ i} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \bigcup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow S_i$, $\bar{S}_i \in \text{SmVar}(\mathbb{C})$ then, $T = \bigcup_{i=1}^l T_i$, with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i_1^*: i_i : T_i \hookrightarrow Y \times S_i$. Moreover $\bar{g}_i := p_{S_i} : Y \times S_i \to \bar{S}_i$ is a lift of $g_i := g|_{T_i} : T_i \to \bar{S}_i$. Let $M \in \text{DA}_c(S^-)$ and $(F, W) \in C_{\text{fms}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\Delta^1, e)t(F, W)$. Then, $g^* M = L \Delta S g^* L \Delta S M$, $D(\Delta^1, e)(g^* L \Delta S L F) = g^* L \Delta S M$ and there exist $(F', W) \in C_{\text{fms}}(\text{Var}(\mathbb{C})^{sm}/S)$ and an equivalence $(\Delta^1, e)$ local $e : (F', W) \to (g^* L \Delta S L(F, W))$ such that $D(\Delta^1, e)(F', W) = g^* L \Delta S (M, W)$ and, using definition 131 and definition 136(ii) and lemma 13, the map in $\pi_T(D(MHM(T))) \subset D_{(1, 0)}(T/(Y \times S_i))$

$$T^i(g, F_{\text{FDR}})(M) : F_{\text{FDR}}^T(g^1 M) \to g_{\text{Hdg}}^* F_{\text{FDR}}^S(M)$$

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given in definition 139 is the inverse of the following map

\[ T_{g}^{f,FDR}(M) : i_{Hdg}^{mod[-1]} F_{S}(M) \rightarrow (\Gamma_{T}^{Hdg})^{-1}(g_{i}^{\otimes} \otimes \bigwedge^{n} g_{T}^{\otimes} (L_{i} \otimes L_{j})_{L(D,S)L}(F,W)) \]

\[ (\Gamma_{T}^{Hdg})^{-1}(g_{i}^{\otimes} \otimes \bigwedge^{n} g_{T}^{\otimes} (L_{i} \otimes L_{j})_{L(D,S)L}(F,W)) \]

Proof. Follows from lemma 14.

We have the following easy proposition

**Proposition 123.** Let \( S \in \text{Var}(\mathbb{C}) \) and \( S = \bigcup_{i=1}^{l} S_{i} \) an open affine covering and denote, for \( I \subseteq [1, \ldots, l] \), \( \bigcap_{i \in I} S_{i} \) and \( j_{I} : S_{I} \rightarrow S \) the open embedding. Let \( i_{i} : S_{i} \rightarrow S_{i} \) closed embeddings, with \( S_{i} \in \text{SmVar}(\mathbb{C}). \) We have, for \( M, N \in \text{DA}(\mathbb{C})S \) and \( F, G \in (\text{Var}(\mathbb{C})^{sm}/S) \) such that \( M = D(A_{1}, et)(F) \) and \( N = D(A_{1}, et)(G) \), the following commutative diagram in \( D_{OSfil,\text{fil},\text{zar}}(T/(Y \times \hat{S}_{I})) \)

\[ \begin{array}{ccc}
F_{T}^{G}(g^{*}L_{S}M) & \xrightarrow{T(\text{F}_{S}^{GM}, \text{F}_{S}^{FDR})(g^{*}M)} & F_{T}^{G}(g^{*}L_{S}M) \\
T(g_{T}, F_{S}^{GM})(L_{D,S}M) & \xrightarrow{T(g_{T}, F_{S}^{GM})(L_{D,S}N)} & T(g_{T}, F_{S}^{GM})(L_{D,S}N) \\
\end{array} \]

\[ \begin{array}{ccc}
F_{S}^{GM}(L_{D,S}M) & \xrightarrow{T(\text{F}_{S}^{GM}, \text{F}_{S}^{FDR})(M \otimes N)} & F_{S}^{GM}(L_{D,S}N) \\
T(\text{F}_{S}^{GM}, \text{F}_{S}^{FDR})(M \otimes N) & \xrightarrow{T(\text{F}_{S}^{GM}, \text{F}_{S}^{FDR})(M \otimes N)} & T(\text{F}_{S}^{GM}, \text{F}_{S}^{FDR})(M \otimes N) \\
\end{array} \]
Remark 12.

For $C$ Definition 140.

6.2 The analytic filtered De Rahm realization functor

Proof. Immediate from definition. \hfill \Box

6.2.1 The analytic Gauss-Manin filtered De Rham realization functor and its transformation map with pullbacks

Consider, for $S \in \text{AnSp}(\mathbb{C})$, the following composition of morphism in RCat (see section 2)

$$
\tilde{\epsilon}(S) : (\text{AnSp}(\mathbb{C})/S, O_{\text{AnSp}(\mathbb{C})}/S) \xrightarrow{\rho_S} (\text{AnSp}(\mathbb{C})^{\text{sm}}/S, O_{\text{AnSp}(\mathbb{C})^{\text{sm}}}/S) \xrightarrow{\epsilon(S)} (S, O_S)
$$

with, for $X/S = (X, h) \in \text{AnSp}(\mathbb{C})/S$,

- $O_{\text{AnSp}(\mathbb{C})/S}(X/S) := O_X(X)$,
- $(\tilde{\epsilon}(S)^* O_S(X/S) \to O_{\text{AnSp}(\mathbb{C})/S}(X/S)) := (h^* O_S \to O_X)$.

and $O_{\text{AnSp}(\mathbb{C})^{\text{sm}}}/S := \rho_S^* O_{\text{AnSp}(\mathbb{C})}/S$, that is, for $U/S = (U, h) \in \text{AnSp}(\mathbb{C})^{\text{sm}}/S$, $O_{\text{AnSp}(\mathbb{C})^{\text{sm}}}/S(U/S) := O_{\text{AnSp}(\mathbb{C})/S}(U/S) := O_{U}(U)$

Definition 140. (i) For $S \in \text{Var}(\mathbb{C})$, we consider the complexes of presheaves

$$
\Omega^*_S := \text{coker}(\Omega_{\text{AnSp}(\mathbb{C})/S} \tilde{\epsilon}(S)^* O_S : \Omega^*_{\text{AnSp}(\mathbb{C})/S} \to \Omega^*_{\text{AnSp}(\mathbb{C})/S}) \in C_{O_S}(\text{AnSp}(\mathbb{C})/S)
$$

which is by definition given by

- for $X/S$ a morphism $\Omega^*_S(X/S) = \Omega^*_X(X)$
- for $g : X'/S \to X/S$ a morphism,

$$
\Omega^*_S(g) := \Omega_{(X'/S)/(S/S)}(X') : \Omega^*_X(X) \to g^* \Omega^*_X(X') \to \Omega^*_{X'}(X')
$$

$$
\omega \mapsto \Omega_{(X'/S)/(S/S)}(X')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{X'}(X') \mapsto \omega(\alpha))
$$

(ii) For $S \in \text{AnSp}(\mathbb{C})$, we consider the complexes of presheaves

$$
\Omega^*_S := \rho_S^* \Omega^*_S := \text{coker}(\Omega_{\text{AnSp}(\mathbb{C})^{\text{sm}}/S} \epsilon(S)^* O_S : \Omega^*_{\text{AnSp}(\mathbb{C})^{\text{sm}}/S} \to \Omega^*_{\text{AnSp}(\mathbb{C})^{\text{sm}}/S}) \in C_{O_S}(\text{AnSp}(\mathbb{C})^{\text{sm}}/S)
$$

which is by definition given by

- for $U/S$ a smooth morphism $\Omega^*_S(U/S) = \Omega^*_U(U)$
- for $g : U'/S \to U/S$ a morphism,

$$
\Omega^*_S(g) := \Omega_{(U'/S)/(S/S)}(U') : \Omega^*_U(U) \to g^* \Omega^*_U(U') \to \Omega^*_{U'}(U')
$$

$$
\omega \mapsto \Omega_{(U'/S)/(S/S)}(U')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{U'}(U') \mapsto \omega(\alpha))
$$

Remark 12. For $S \in \text{AnSp}(\mathbb{C})$, $\Omega^*_S \in \text{C}(\text{AnSp}(\mathbb{C})/S)$ is by definition a natural extension of $\Omega^*_S \in \text{C}(\text{AnSp}(\mathbb{C})^{\text{sm}}/S)$. However $\Omega^*_S \in \text{C}(\text{AnSp}(\mathbb{C})/S)$ does NOT satisfy cdl descent.
For a smooth morphism $h : U \to S$ with $S, U \in \text{AnSm}(\mathbb{C})$, the cohomology presheaves $H^n\Omega^\bullet_{U/S}$ of the relative De Rham complex

$$DR(U/S) := \Omega^\bullet_{U/S} := \text{coker}(h^*\Omega_S \to \Omega_U) \in C_{h^*O_S}(U)$$

for all $n \in \mathbb{Z}$, have a canonical structure of a complex of $h^*D^\infty_S$ modules given by the Gauss Manin connexion : for $S^o \subset S$ an open subset, $U^o = h^{-1}(S^o)$, $\gamma \in \Gamma(S^o, T_S)$ a vector field and $\hat{\omega} \in \Omega^p_{U/S}(U^o)$ a closed form, the action is given by

$$\gamma \cdot [\hat{\omega}] = [\iota(\hat{\gamma})\partial \omega],$$

$\omega \in \Omega^p_U(U^o)$ being a representative of $\hat{\omega}$ and $\hat{\gamma} \in \Gamma(U^o, T_U)$ a releveement of $\gamma$ ($h$ is a smooth morphism), so that

$$DR(U/S) := \Omega^\bullet_{U/S} := \text{coker}(h^*\Omega_S \to \Omega_U) \in C_{h^*O_S,h^*D^\infty_S}(U)$$

with this $h^*D^\infty_S$ structure. Hence we get $h_*\Omega^\bullet_{U/S} \subset C_{O_S,D^\infty_S}(S)$ considering this structure. Since $h$ is a smooth morphism, $\Omega^p_{U/S}$ are locally free $O_U$ modules.

The point (ii) of the definition 148 above gives the object in $DA(S)$ which will, for $S$ smooth, represent the analytic Gauss-Manin De Rham realisation. It is the class of an explicit complex of presheaves on $\text{AnSp}(\mathbb{C})^{sm}/S$.

**Proposition 124.** Let $S \in \text{Var}(\mathbb{C})$.

(i) For $U/S = (U, h) \in \text{AnSp}(\mathbb{C})^{sm}/S$, we have $e(U)_h^*\Omega^\bullet_{U/S} = \Omega^\bullet_{U/S}$.

(ii) The complex of presheaves $\Omega^\bullet_{h:S} \in C_{O_S,\text{AnSp}(\mathbb{C})^{sm}/S}$ is $\mathbb{D}^1$ homotopic. Note that however, for $p > 0$, the complexes of presheaves $\Omega^\bullet_{h:S}^{\geq p}$ are NOT $\mathbb{D}^1$ local. On the other hand, $\Omega^\bullet_{f/S}$ admits transfers (recall that means $\text{Tr}(S)_* \text{Tr}(S)^\bullet \Omega^p_{f/S} = \Omega^p_{f/S}$).

(iii) If $S$ is smooth, we get $(\Omega^\bullet_{h:S}, F_h) \in C_{O_S,\text{fil},D^\infty_S}(\text{Var}(\mathbb{C})^{sm}/S)$ with the structure given by the Gauss Manin connexion. Note that however the $D^\infty_S$ structure on the cohomology groups given by Gauss Manin connexion does NOT comes from a structure of $D^\infty_S$ module structure on the filtered complex of $O_S$ module. The $D_S$ structure on the cohomology groups satisfy a non trivial Griffitz transversality (in the non projection cases), whereas the filtration on the complex is the trivial one.

*Proof.* Similar to the proof of proposition 105. \hfill \square

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let $g : T \to S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$. Consider the following commutative diagram in $\text{RCat}$:

$$
\begin{array}{ccc}
D(g, e) : (\text{AnSp}(\mathbb{C})^{sm}/T, O_{\text{AnSp}(\mathbb{C})^{sm}/T}) & \xrightarrow{P(g)} & (\text{AnSp}(\mathbb{C})^{sm}/S, O_{\text{AnSp}(\mathbb{C})^{sm}/S}) \\
\downarrow{e(T)} & & \downarrow{e(S)} \\
(T, O_T) & \xrightarrow{P(g)} & (S, O_S)
\end{array}
$$

It gives (see section 2) the canonical morphism in $C_{g^*O_S,\text{fil}}(\text{Var}(\mathbb{C})^{sm}/T)$

$$
\Omega^{(T/S)} := \Omega^{(O_{\text{AnSp}(\mathbb{C})^{sm}/T}/g^*O_{\text{AnSp}(\mathbb{C})^{sm}/S})}((O_T/g^*O_S) : g^*(\Omega^\bullet_{h:S}, F_h) = \Omega^\bullet_{g^*O_{\text{AnSp}(\mathbb{C})^{sm}/S}/g^*O_{\text{AnSp}(\mathbb{C})^{sm}/S}} \to (\Omega^\bullet_{T}, F_T) = \Omega^{\bullet}_{O_{\text{AnSp}(\mathbb{C})^{sm}/T}/e(T)^*O_T}
$$

which is by definition given by the pullback on differential forms : for $(V/T) = (V, h) \in \text{Var}(\mathbb{C})^{sm}/T$,

$$
\Omega^{(T/S)}(V/T) := \lim_{(h': U \to \text{Sm}, g': V \to U, h, g)} \Omega_{U/S}(U) \xrightarrow{\Omega(V/U)/(T/S)(V/T)} \Omega_{V/T}(V) =: \Omega^\bullet_{T}(V/T)
$$

$$\hat{\omega} \mapsto \Omega_{(V/U)/(T/S)}(V/T)(\omega) := g^*\hat{\omega}.$$
Lemma 15. If $S$ and $T$ are smooth, $\Omega_{/(T/S)}: g^*(\Omega^*_S, F_b) \to (\Omega^*_T, F_b)$ is a map in $C_{g^*O_Sfd, g^*D_S^*} (\text{AnSp}(\mathbb{C})^{sm}/T)$ It induces the canonical morphism in $C_{g^*O_Sfd, g^*D_S^*} (\text{AnSp}(\mathbb{C})^{sm}/T)$:

$$E\Omega_{/(T/S)}: g^*E_{usu}(\Omega^*_S, F_b) \xrightarrow{T(g, E_{usu}((\Omega^*_S, F_b))} E_{usu}(g^*(\Omega^*_S, F_b) \xrightarrow{E_{usu}(\Omega_{/(T/S)})} E_{usu}(\Omega^*_T, F_b).$$

Definition 141. (i) Let $g: T \to S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$. We have, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_Tfd}(T)$:

$$T^O(g, \Omega_{/T})(F): g^*L^1\alpha(S), \text{Hom}^*(F, E_{usu}(\Omega^*_S, F_b)) \xrightarrow{\sim} \Gamma_T E_{usu}(p^*_SF, \text{Hom}^*(F, E_{usu}(\Omega^*_S, F_b))) \otimes g^*O_S T$$

$$\xrightarrow{T(e, g)(-) \circ T(g, L^1)(-)} L^O(e(T), \text{Hom}^*(g^*F, g^*E_{usu}(\Omega^*_S, F_b)) \otimes g^*O_S T)$$

$$\xrightarrow{\text{ev}(\text{hom}, \otimes)(-,-,-)} L^O(e(T), \text{Hom}^*(g^*F, g^*E_{usu}(\Omega^*_S, F_b)) \otimes g^*O_S e(T)^*O_T)$$

$$\xrightarrow{m} L^O(e(T), \text{Hom}^*(g^*F, E_{usu}(\Omega^*_T, F_b)) \otimes g^*O_S e(T)^*O_T)$$

where $m(\alpha \otimes h) := h.\alpha$ is the multiplication map.

(ii) Let $g: T \to S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$, $S$ smooth. Assume there is a factorization $g: T \overset{1}{\to} Y \times S \overset{p_S}{\to} S$ with $1$ a closed embedding, $Y \in \text{AnSm}(\mathbb{C})$ and $p_S$ the projection. We have, for $F \in C(\text{AnSp}(\mathbb{C})^{sm}/S)$, the canonical transformation in $C_{O_Tfd, D^{\infty}}(Y \times S)$:

$$T(g, \Omega_{/T})(F): g^*L^1\alpha(S), \text{Hom}^*(F, E_{usu}(\Omega^*_S, F_b)) \xrightarrow{\sim} \Gamma_T E_{usu}(p^*_SF, \text{Hom}^*(F, E_{usu}(\Omega^*_S, F_b))) \otimes g^*O_S T$$

$$\xrightarrow{T^O(p_S, \Omega_{/T})(F)} \Gamma_T E_{usu}(e(T \times S), \text{Hom}^*(p^*_SF, E_{usu}(\Omega^*_T \times S, F_b))) \xrightarrow{\sim} e(T \times S)^*E_{usu}((\Omega^*_T \times S, F_b))) \xrightarrow{I(\gamma, \text{hom})(-,-,-)} e(T \times S)^*\text{Hom}^*(\Gamma_T^p p^*_S F, E_{usu}(\Omega^*_T \times S, F_b)).$$

For $Q \in \text{Proj PSh}(\text{AnSp}(\mathbb{C})^{sm}/S)$,

$$T(g, \Omega_{/T})(Q): g^*L^1\alpha(S), \text{Hom}^*(Q, E_{usu}(\Omega^*_S, F_b)) \to e(T \times S)^*\text{Hom}^*(\Gamma_T^p p^*_S Q, E_{usu}(\Omega^*_T \times S, F_b))$$

is a map in $C_{O_Tfd, D^{\infty}}(Y \times S)$.

The following easy lemma describe these transformation map on representable presheaves:

Lemma 15. Let $g: T \to S$ a morphism with $T, S \in \text{AnSp}(\mathbb{C})$ and $h: U \to S$ is a smooth morphism with $U \in \text{AnSp}(\mathbb{C})$. Consider a commutative diagram whose square are cartesian:

$$
g: T \xrightarrow{l} S \times Y \xrightarrow{p_S} S$$

$$h \uparrow \quad h' \uparrow \quad h'' := h \times 1 \uparrow$$

$$g': U_T \xrightarrow{l'} U \times Y \xrightarrow{p_U} U$$

with $l$, $l'$ the graph embeddings and $p_S$, $p_U$ the projections. Then $g^*Z(U/S) = Z(U_T/T)$ and
(i) we have the following commutative diagram in $C_{O_{\text{fil}}(T)}$ (see definition 1 and definition 141(i)):

$$
g^*\text{mod}_{\text{fil}}(S) \circ \text{Hom}^*(\mathbb{Z}(U/S), E_{\text{usu}}(\Omega^*_S, F_b)) \xrightarrow{T(g, \Omega_1)(\mathbb{Z}(U/S))} c(T) \circ \text{Hom}^*(\mathbb{Z}(U/T), E_{\text{usu}}(\Omega^*_T, F_b)) = g^*\text{mod}_{\text{fil}}h_+ E_{\text{usu}}(\Omega^*_U/T, F_b)
$$

(ii) if $Y, S \in \text{AnSm}(\mathbb{C})$, we have the following commutative diagram in $C_{\text{O_{\text{fil}}}, D_{\text{fil}}}(T)$ (see definition 1 and definition 141(ii)):

$$
g^*\text{mod}_{\text{fil}}(S) \circ \text{Hom}^*(\mathbb{Z}(U/S), E_{\text{usu}}(\Omega^*_S, F_b)) \xrightarrow{T(g, \Omega_1)(\mathbb{Z}(U/S))} c(T) \circ \text{Hom}^*(\mathbb{Z}(U/T), E_{\text{usu}}(\Omega^*_T, F_b)) = g^*\text{mod}_{\text{fil}}h_+ E_{\text{usu}}(\Omega^*_U/T, F_b)
$$

where $j : T \setminus T \times S \rightarrow T \times S$ is the open complementary embedding.

Proof. Obvious. □

**Proposition 125.** Let $p : S_{12} \rightarrow S_1$ is a smooth morphism with $S_1, S_{12} \in \text{AnSp}(\mathbb{C})$. Then if $Q \in C(\text{AnSp}(\mathbb{C})^{\text{sm}}/S_1)$ is projective,

$$T(p, \Omega_1)(Q) : p^*\text{mod}(S_1) \circ \text{Hom}^*(Q, E_{\text{usu}}(\Omega^*_S, F_b)) \rightarrow e(S_{12}) \circ \text{Hom}^*(p^*Q, E_{\text{usu}}(\Omega^*_{S_{12}}, F_b))$$

is an isomorphism.

Proof. Similar to the proof of proposition 106. □

Let $S \in \text{AnSp}(\mathbb{C})$ and $h : U \rightarrow S$ a morphism with $U \in \text{AnSp}(\mathbb{C})$. We then have the canonical map given by the wedge product

$$w_{U/S} : \Omega^*_U/S \otimes_{O_S} \Omega^*_U/S \rightarrow \Omega^*_U/S; \alpha \otimes \beta \mapsto \alpha \wedge \beta.$$  

Let $S \in \text{Var}(\mathbb{C})$ and $h_1 : U_1 \rightarrow S$, $h_2 : U_2 \rightarrow S$ two morphisms with $U_1, U_2 \in \text{AnSp}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \rightarrow S$ and $p_{121} : U_1 \times S U_2 \rightarrow U_1$, $p_{212} : U_1 \times S U_2 \rightarrow U_2$ the projections. We then have the canonical map given by the wedge product

$$w_{(U_1, U_2)/S} : p_{121}^* \Omega^*_U/S \otimes_{O_S} p_{212}^* \Omega^*_U/S \rightarrow \Omega^*_{U_{12}/S}; \alpha \otimes \beta \mapsto p_{112}^* \alpha \wedge p_{212}^* \beta$$

which gives the map

$$Ew_{(U_1, U_2)/S} : h_{12} \circ E_{\text{usu}}(\Omega^*_{U_{12}/S}) \otimes_{O_S} h_{12} \circ E_{\text{usu}}(\Omega^*_{U_{12}/S})$$

$$\xrightarrow{\text{ad}(p_{112}^*, p_{212}^*) \cdot (-) \circ \text{ad}(p_{112}^*, p_{212}^*) \cdot (-)} (h_1 \circ p_{112} \circ p_{112}^* \circ E_{\text{usu}}(\Omega^*_{U_{12}/S})) \otimes_{O_S} (h_2 \circ p_{212} \circ p_{212}^* \circ E_{\text{usu}}(\Omega^*_{U_{12}/S})) = h_{12} \circ p_{112} \circ p_{212} \circ E_{\text{usu}}(\Omega^*_{U_{12}/S})$$

$$T(\cdot, E)(\cdot) \circ (T(p_{112}, E)(\cdot) \circ T(p_{212}, E)(\cdot)) \rightarrow h_{12} \circ E_{zar}(p_{112}^* \Omega^*_{U_{12}/S} \otimes_{O_S} p_{212}^* \Omega^*_{U_{12}/S})$$

Let $S \in \text{AnSp}(\mathbb{C})$. We have the canonical map in $C_{O_{\text{fil}}}(\text{AnSp}(\mathbb{C})^{\text{sm}}/S)$

$$w_S : (\Omega^*_S, F_b) \otimes_{O_S} (\Omega^*_S, F_b) 

given by for $h : U \to S \in \text{AnSp}(\mathbb{C})^{sm}/S$

$$w_{\sigma}(U/S) : \left(\Omega_{U/S}^\bullet, F_h \right) \otimes_{\mathcal{O}_S} \left(\Omega_{U/S}^\bullet, F_h \right)(U) \xrightarrow{\text{w}_{U/S}(U)} \left(\Omega_{U/S}^\bullet, F_h \right)(U)$$

It gives the map

$$Ew_{\sigma} : E_{\text{usu}}(\Omega_{/S}^\bullet, F_h) \otimes_{\mathcal{O}_S} E_{\text{usu}}(\Omega_{/S}^\bullet, F_h) \xrightarrow{=} E_{\text{usu}}(\left(\Omega_{/S}^\bullet, F_h \right) \otimes_{\mathcal{O}_S} \left(\Omega_{/S}^\bullet, F_h \right)) \xrightarrow{E_{\text{usu}}(w_{\sigma})} E_{\text{usu}}(\Omega_{/S}^\bullet, F_h)$$

If $S \in \text{AnSm}(\mathbb{C})$,

$$w_{\sigma} : \left(\Omega_{/S}^\bullet, F_h \right) \otimes_{\mathcal{O}_S} \left(\Omega_{/S}^\bullet, F_h \right) \to \left(\Omega_{/S}^\bullet, F_h \right)$$

is a map in $\mathcal{C}_{\text{os fid}, D^2}(\text{Var}(\mathbb{C})^{sm}/S)$.

**Definition 142.** Let $S \in \text{AnSp}(\mathbb{C})$. We have, for $F, G \in \mathcal{C}(\text{AnSp}(\mathbb{C})^{sm}/S)$, the canonical transformation in $\mathcal{C}_{\text{os fid}}(S)$:

$$T(\otimes, \Omega)(F, G) : e(S)_* \mathcal{H}om(F, E_{\text{usu}}(\Omega_{/S}^\bullet, F_h)) \otimes_{\mathcal{O}_S} e(S)_* \mathcal{H}om(G, E_{\text{usu}}(\Omega_{/S}^\bullet, F_h))$$

\[ \xrightarrow{=} e(S)_* \mathcal{H}om(F \otimes G, E_{\text{usu}}(\Omega_{/S}^\bullet, F_h)) \otimes_{\mathcal{O}_S} e(S)_* \mathcal{H}om(G, E_{\text{usu}}(\Omega_{/S}^\bullet, F_h)) \]

If $S \in \text{AnSm}(\mathbb{C})$, $T(\otimes, \Omega)(F, G)$ is a map in $\mathcal{C}_{\text{os fid}, D^2}(S)$.

**Lemma 16.** Let $S \in \text{AnSp}(\mathbb{C})$ and $h_1 : U_1 \to S$, $h_2 : U_2 \to S$ two smooth morphisms with $U_1, U_2 \in \text{AnSp}(\mathbb{C})$. Denote $h_{12} : U_{12} := U_1 \times_S U_2 \to S$ and $p_{121} : U_1 \times_S U_2 \to U_1, p_{212} : U_1 \times_S U_2 \to U_2$ the projections. We then have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}om(F_{E_{\text{usu}}}(\Omega_{/S}^\bullet, F_h)) \otimes_{\mathcal{O}_S} \mathcal{H}om(G_{E_{\text{usu}}}(\Omega_{/S}^\bullet, F_h)) & \xrightarrow{T(\otimes, \Omega)(F, G)} & \mathcal{H}om(F \otimes G_{E_{\text{usu}}}(\Omega_{/S}^\bullet, F)) \\
\end{array}
\]

\[ e(S)_* \mathcal{H}om(F \otimes G_{E_{\text{usu}}}(\Omega_{/S}^\bullet, F)) \]

\[ h_{12}_* E_{\text{zar}}(\Omega_{U_{12}/S}^\bullet, F_h) \]

**Proof.** Follows from Yoneda lemma.

We now define the analytic Gauss Manin De Rahm realization functor.

Let $S \in \text{AnSp}(\mathbb{C})$ and $S = \bigcup_{i \in I} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{AnSm}(\mathbb{C})$ an affine space. For $I \subset \{1, \ldots, l\}$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ with $\tilde{S}_I$ an affine space. Consider, for $I \subset J$, the following commutative diagram

\[
\begin{array}{ccc}
D_{IJ} = \begin{array}{c}
S_J \\
S_I
\end{array} \\
\downarrow j_{IJ} \\
\tilde{S}_J \\
\downarrow i_{IJ} \\
\tilde{S}_I
\end{array}
\]

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. Considering the factorization of the diagram $D_{IJ}$ by the fiber product:

\[
D_{IJ} = \tilde{S}_J \times \tilde{S}_{J/I} \xrightarrow{p_{IJ}} \tilde{S}_I
\]
the square of this factorization being cartesian, we have for $F \in C(AnSp(C)^{sm}/S)$ the canonical map in $C(AnSp(C)^{sm}/S_j)$

$$S(D_{I,J})(F) : Li_{I,J}i^*_I j^*_J F \xrightarrow{\delta} i_J j^*_J F = (i_I \times I) \ast j^*_J F \mathrel{|} \begin{array}{c} (i_I \times I)_* p_{I,J}^* p_{I,J}^* F \ \ \\ \xrightarrow{T(p_{I,J,J})} \end{array} \begin{array}{c} p_{I,J}^* i_{I,J} F \\
\end{array} \xrightarrow{\varphi} p_{I,J}^* j^*_J F$$

which factors through

$$S(D_{I,J})(F) : Li_{I,J}i^*_I j^*_J F \xrightarrow{S(D_{I,J})} p_{I,J}^* Li_{I,J}i^*_I j^*_J F \xrightarrow{\varphi} p_{I,J}^* i_{I,J} F$$

**Definition 143.**

(i) Let $S \in AnSm(C)$. We have the functor

$$\mathcal{H}om^\bullet(\cdot, \mathcal{E}_{usu}(\Omega^*_{\bar{S}}, F_b)) : C(AnSp(C)^{sm}/S) \to C_{O_{fil,D}^\infty}(S), \ F \mapsto e(S)_* \mathcal{H}om^\bullet(LF, \mathcal{E}_{usu}(\Omega^*_{\bar{S}}, F_b)).$$

(ii) Let $S \in Var(C)$ and $S = \bigcup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \bar{S}_i$ with $\bar{S}_i \in SmVar(C)$. For $I \subset [1, \ldots, l]$, denote by $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \to S$ the open embedding. Then we have closed embeddings $i_i : S_I \to \bar{S}_i := \bigcap_{i \in I} \bar{S}_i$. We have the functor

$$C(Var(C)^{sm}/S) \to C_{O_{fil,D}^\infty}(S/\bar{S}_I), \ F \mapsto (e(\bar{S}_I)_* \mathcal{H}om^\bullet(\mathcal{A}_{\bar{S}_I}^\circ, L(i_{I,J}^* F), \mathcal{E}_{usu}(\Omega^*_{\bar{S}_I}, F_b))[-d_{\bar{S}_I}], u_{I,J}^*(F))$$

where

$$u_{I,J}^*(F)[d_{\bar{S}_I}] : e(\bar{S}_I)_* \mathcal{H}om^\bullet(\mathcal{A}_{\bar{S}_I}^\circ, L(i_{I,J}^* F), \mathcal{E}_{usu}(\Omega^*_{\bar{S}_I}, F_b))$$

$$\xrightarrow{\varphi} p_{I,J}^* i_{I,J}^* F$$

**Proposition 126.**

(i) Let $S \in AnSp(C)$. Let $m : Q_1 \to Q_2$ be an equivalence $(D^1, et)$ local in $C(AnSp(C)^{sm}/S)$ with $Q_1, Q_2$ complexes of presheaves. Then,

$$e(S)_* \mathcal{H}om^\bullet(m_* L\mathcal{E}_c(\Omega^*_{\bar{S}}, F_b)) : e(S)_* \mathcal{H}om^\bullet(Q_2, \mathcal{E}_c(\Omega^*_{\bar{S}}, F_b)) \to e(S)_* \mathcal{H}om^\bullet(Q_1, \mathcal{E}_c(\Omega^*_{\bar{S}}, F_b))$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{O_{fil,D}^\infty}(S)$ if $S$ is smooth.

(ii) Let $S \in AnSp(C)$. Let $S = \bigcup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \bar{S}_i$ with $\bar{S}_i \in AnSm(C)$. Let $m = (m_I) : (Q_{11}, s_{11}^1) \to (Q_{21}, s_{12}^1)$ be an equivalence $(D^1, usu)$ local in $C(AnSp(C)^{sm}/S)$ with $Q_{11}, Q_{21}$ complexes of projective presheaves. Then,

$$(e(\bar{S}_I)_* \mathcal{H}om^\bullet(m_* L\mathcal{E}_c(\Omega^*_{\bar{S}_I}, F_b))) : (e(\bar{S}_I)_* \mathcal{H}om^\bullet(Q_{21}, \mathcal{E}_c(\Omega^*_{\bar{S}_I}, F_b)), u_{I,J}^*(Q_{21}, s_{12}^1)) \to (e(\bar{S}_I)_* \mathcal{H}om^\bullet(Q_{11}, \mathcal{E}_c(\Omega^*_{\bar{S}_I}, F_b)), u_{I,J}^*(Q_{11}, s_{11}^1))$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{O_{fil,D}^\infty}(S/\bar{S}_I)$$.

**Proof.** Similar to the proof of proposition 107.

**Definition 144.**

(i) We define, according to proposition 126, the filtered analytic Gauss-Manin realization functor defined as

$$\mathcal{F}^{GM}_{S,an} : DA_*(S)^{op} \to D_{O_{fil,D}^\infty}(S), \ M \to$$

$$\mathcal{F}^{GM}_{S,an}(M) := e(S)_* \mathcal{H}om^\bullet(\mathcal{A}_{\bar{S}}^\circ, L(F), \mathcal{E}_{usu}(\Omega^*_{\bar{S}}, F_b))[-d_{\bar{S}}]$$

$$= e(S)_* \mathcal{H}om^\bullet(L(F), \mathcal{A}_{\bar{S}}^\circ, \mathcal{E}_{usu}(\Omega^*_{\bar{S}}, F_b))[-d_{\bar{S}}]$$

where $F \in C(Var(C)^{sm}/S)$ is such that $M = D(k^1, et)(F)$.
(ii) Let $S \in \text{Var} (\mathbb{C})$ and $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \cdots , l]$, denote by $S_I = \bigcap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \prod_{i \in I} \tilde{S}_i$. We define the filtered analytic Gauss-Manin realization functor defined as

$$\mathcal{F}^{GM}_{S, \text{an}} : \text{DA}_{c}(S)_{\text{op}} \to D_{\text{Ofl}, \text{fil}, \infty}(S/(\tilde{S}_I)), \quad M \mapsto \mathcal{F}^{GM}_{S, \text{an}} (M) := ((e(\tilde{S}_I), \text{Hom}^{\bullet}(\text{An}^{\bullet}_{\tilde{S}_I} L(i_{I*}, j_I^* F), E_{\text{usu}}(\Omega^*_{\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{IJ}^I (F)).$$

where $F \in C(\text{Var}(\mathbb{C})^m/S)$ is such that $M = D(\mathcal{A}^1, \text{et})(F)$, see definition 143.

**Proposition 127.** For $S \in \text{Var} (\mathbb{C})$, the functor $\mathcal{F}^{GM}_{S}$ is well defined.

**Proof.** Similar to the proof of proposition 108. \qed

**Proposition 128.** Let $f : X \to S$ a morphism with $S, X \in \text{Var} (\mathbb{C})$. Let $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then $X = \bigcup_{i=1}^{l} X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \cdots , l]$, $S_I = \bigcap_{i \in I} S_i$ and $X_I = \bigcap_{i \in I} X_i$. Assume there exist a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of $f$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. We then have, for $I \subset [1, \cdots , l]$, the following commutative diagrams which are cartesian

$$\begin{array}{ccc}
X_I & \xrightarrow{i} & Y \times S \xrightarrow{p_S} S_I \\
\downarrow & & \downarrow \\
S_I & \xrightarrow{i_I} & Y \times S \xrightarrow{p_S} S_I \\
\end{array}$$

Let $F(X/S) := p_{S*} \Gamma^{\chi}_X \mathcal{Z}(Y \times S/Y \times S)$. The transformations maps $(N_I (X/S) : Q(X_I/\tilde{S}_I) \to i_{I*} j_I^* F(X/S))$ and $(k \circ I(\gamma, \text{hom})(-, -))$, for $I \subset [1, \cdots , l]$, induce an isomorphism in $D_{\text{Ofl}, \text{fil}, \infty}(S/(\tilde{S}_I))$

$$I^{GM}_{S}(X/S) := (e(\tilde{S}_I), \text{Hom}^{\bullet}(\text{An}^{\bullet}_{\tilde{S}_I} L(i_{I*}, j_I^* F(X/S)), E_{\text{usu}}(\Omega^*_{\tilde{S}_I}, F_b)[-d_{\tilde{S}_I}], u_{IJ}^I (F(X/S))).$$

**Proof.** Similar to the proof of proposition 109. \qed

We now define the functorialities of $\mathcal{F}^{GM}_{S}$ with respect to $S$ which makes $\mathcal{F}^{GM}_{G}$ a morphism of 2-functor.

**Definition 145.** Let $g : T \to S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Consider the factorization $g : T \xrightarrow{l} T \times S \xrightarrow{p_S} S$ where $l$ is the graph embedding and $p_S$ the projection. Let $M \in \text{DA}_{c}(S)$ and $F \in C(\text{Var}(\mathbb{C})^m/S)$ such that $M = D(\mathcal{A}^1, \text{et})(F)$. Then, $D(\mathcal{A}^1, \text{et})(g^* F) = g^* M$.

(i) We have then the canonical transformation in $D_{\text{fil}, \infty}(T \times S)$ (see definition 141)

$$T(g, \mathcal{F}^{GM}) (M) := g^{* \text{mod}} e(S), \text{Hom}^{\bullet}(\text{An}^{\bullet}_{\tilde{S}_I} L(F), E_{\text{usu}}(\Omega^*_{\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}]$$

$$\xrightarrow{T(g, \Omega_{\gamma})(\text{An}^\gamma_{\tilde{S}} L(F))} e(T \times S), \text{Hom}^{\bullet}(\Gamma_T^\gamma \text{An}^\gamma_{\tilde{S}} L(F), E_{\text{usu}}(\Omega^*_{X \times S}, F_b))[-d_T]$$

$$\xrightarrow{\text{Hom}(T(\gamma, \text{hom})(-, -))} \mathcal{F}^{GM}_{T \times S, \text{an}} (l^* g^* M).$$

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(ii) We have then the canonical transformation in $D_{Ojul,\infty}(T)$ (see definition 141):

$$
T^G(g, F^GM)(M) : Lg^\text{sm mod} \circ F^GM \circ \ell(M) = g^\text{sm mod} L_{\text{OE}}(S), \text{Hom}^{\bullet}(\text{An}_S^* L(F), E_{\text{usu}}(\Omega_j^{\bullet} Y, F)) \rightarrow \ell \rightarrow -d_T
$$

We give now the definition in the non smooth case Let $g : T \rightarrow S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{i} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \bigcup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$ Then, $T = \bigcup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i_i' := i_i \circ l : T_i \hookrightarrow Y \times S_i$. Moreover $\tilde{g}_i := p_{S_i} : Y \times \tilde{S}_i \rightarrow \tilde{S}_i$ is a lift of $g_i := \tilde{g}_i : T_i \rightarrow \tilde{S}_i$. We recall the commutative diagram:

$$
E_{IJ} = (Y \times \tilde{S}_I) \xrightarrow{T_{ij}} Y \times \tilde{S}_J, E_{IJ} = \tilde{S}_J \ltimes \tilde{S}_J \xrightarrow{m_{ij}} \tilde{S}_I, E_{IJ} = (Y \times \tilde{S}_J) \xrightarrow{T_{ij}} Y \times \tilde{S}_J
$$

For $I \subset J$, denote by $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ and $p_{IJ}' := p_{IJ} \circ \tilde{g}_I$. Consider, for $I \subset J \subset [1, \ldots, l]$, resp. for each $I \subset [1, \ldots, l]$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$
D_{IJ} = S_I \xrightarrow{i_i} \tilde{S}_I, D_{IJ}' = T_I \xrightarrow{i_j} Y \times \tilde{S}_J, D_{gI} = S_I \xrightarrow{g_i} \tilde{S}_I
$$

and $j_{IJ} : S_I \hookrightarrow S_I$ is the open embedding so that $j_{IJ} \circ j_{IJ} = j_{IJ}$.

Let $F \in C(\text{Var}(\mathbb{C})^{\text{sm}}/S)$. The fact that the diagrams (61) commutes says that the maps $T^{\text{sm}}(\tilde{D}_{gI})(j_{ij}^* F)$ define a morphism in $C(\text{Var}(\mathbb{C})^{\text{sm}}/(T/(Y \times \tilde{S}_I)))$

$$
(T^{\text{sm}}(\tilde{D}_{gI})(j_{ij}^* F)) : (\Gamma^{\text{sm}_{ij}} L(i_i \times j_{ij}^* F), \tilde{g}_i^* S^q(D_{IJ})(F)) \rightarrow (L(i_i \times j_{ij}^* g_i^* F), S^q(D_{IJ}')(g_i^* F))
$$

We then have then the following lemma:

**Lemma 17.**

(i) The morphism in $C(\text{Var}(\mathbb{C})^{\text{sm}}/(T/(Y \times \tilde{S}_I)))$

$$
(T^{\text{sm}}(\tilde{D}_{gI})(j_{ij}^* F)) : (\Gamma^{\text{sm}_{ij}} L(i_i \times j_{ij}^* F), \tilde{g}_i^* S^q(D_{IJ})(F)) \rightarrow (L(i_i \times j_{ij}^* g_i^* F), S^q(D_{IJ}')(g_i^* F))
$$

is an equivalence ($\mathbb{A}^1$, etc.) local.

(ii) Denote for short $d_{gI} := -d_{Y \times S_I}$. The maps $\text{Hom}(\text{An}_Y \times S_I^* (T^{\text{sm}}(\tilde{D}_{gI})(j_{ij}^* F)), E_{\text{usu}}(\Omega_j^{\bullet} Y \times S_I^*, F))$

induce an $\infty$-filtered quasi-isomorphism in $C_{Ojul,\infty} (T/(Y \times \tilde{S}_I))$

$$
(\text{Hom}(\text{An}_Y \times S_I^* (T^{\text{sm}}(\tilde{D}_{gI})(j_{ij}^* F)), E_{\text{usu}}(\Omega_j^{\bullet} Y \times S_I^*, F)) : (e(Y \times \tilde{T}_I), \text{Hom}(\text{An}_Y \times S_I^* L(i_i \times j_{ij}^* g_i^* F), E_{\text{usu}}(\Omega_j^{\bullet} Y \times S_I^*, F)) [d_{Y \times S_I}], u_{ij}^q(F))
$$

$$
(e(Y \times \tilde{T}_I), \text{Hom}(\text{An}_Y \times S_I^* (\Gamma_{ij} L(i_i \times j_{ij}^* F), E_{\text{usu}}(\Omega_j^{\bullet} Y \times S_I^*, F)) [d_{Y \times S_I}], u_{ij}^q(F))
$$
Proposition 129. 

(iii) The maps $T(\tilde{g}_I, \Omega)(L(i_1^*j_1^*F))$ (see definition 141) induce a morphism in $C_{O_{fil,D}^\infty}(T/(Y \times \tilde{S}_I))$

$$(T(\tilde{g}_I, \Omega_I), (L(i_1^*j_1^*F))) :$$

$$(\Gamma_T, E_{zar}(\tilde{g}_I^{mod[-]}e(\tilde{S}_I), \Hom^*(A_{S_I}^* L(i_1^*j_1^*F), E_{usu}(\Omega^*_S/F_b))[dy_1], \tilde{g}_I^{mod}u_{1j}^*(F)) \rightarrow$$

$$(\Gamma_T, \Hom(\Delta_{Y \times \tilde{S}_I} \tilde{g}_I^* L(i_1^*j_1^*F), E_{usu}(\Omega^*_Y/\tilde{S}_I, F_b))[dy_1], \tilde{g}_I u_{1j}^*(F)).$$

Proof. (i): Follows from theorem 16
(ii): Similar to lemma 10(ii).
(iii): Similar to lemma 10(iii).

\[\square\]

Definition 146. Let $g : T \to S$ be a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{i} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \bigsqcup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$ Then, $T = \bigsqcup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \to Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{S_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{T_I} : T_I \to S_I$. Denote for short $d_{Y,T_I} := -d_Y - d_{S_I}$. Let $M \in D_{\mathcal{A}_c}(S)$ and $F \in C(\text{Var}(\mathbb{C})^{an}/S)$ such that $M = D(\mathcal{A}^*_S, \text{et})$. Then $D(\mathcal{A}^*_S, \text{et})(g^*F) = F^* M$. We have, by lemma 10, the canonical transformation in $D_{O_{fil,D}^\infty}(T/(Y \times \tilde{S}_I))$

$$T(g, \mathcal{F}_{GM})(M) = Rg^{mod[-]}_! \mathcal{F}_{GM}(M) :$$

$$(\Gamma_T, E_{zar}(\tilde{g}_I^{mod}e(\tilde{S}_I), \Hom^*(A_{S_I}^* L(i_1^*j_1^*F), E_{usu}(\Omega^*_S/F_b))[dy_1], \tilde{g}_I^{mod}u_{1j}^*(F))) \rightarrow$$

$$(\Gamma_T, \Hom(\Delta_{Y \times \tilde{S}_I} \tilde{g}_I^* L(i_1^*j_1^*F), E_{usu}(\Omega^*_Y/\tilde{S}_I, F_b))[dy_1], \tilde{g}_I u_{1j}^*(F)).$$

Proposition 129. (i) Let $g : T \to S$ be a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume we have a factorization $g : T \xrightarrow{i} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \bigsqcup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$ Then, $T = \bigsqcup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \to Y \times \tilde{S}_i$. Moreover $\tilde{g}_I := p_{S_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{T_I} : T_I \to S_I$. Denote for short $d_{Y,T_I} := -d_Y - d_{S_I}$. Let $f : X \to S$ be a morphism with $X \in \text{Var}(\mathbb{C})$. Assume that there is a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p_S} S$, with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. We have then the following commutative diagram whose squares are cartesiens
Proof. Theorem 37.

(i) Let $g : T \to S$ be a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Assume that there is a factorization $f : X \to Y \times S \overset{p_S} \to S$, with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Consider $F(X/S) := p_S \Gamma_X^* Z(Y \times S/Y \times S)$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T(f, g, F(X/S)) : g^* F(X/S) := g^* p_S \Gamma_X^* Z(Y \times S/Y \times S) \cong p_T \Gamma_T^* Z(Y \times T/Y \times T) =: F(X_T/T).$$

which gives in $\text{DA}(S)$ the isomorphism $T(f, g, F(X/S)) : g^* M(X/S) \cong M(X_T/T)$. Then, the following diagram in $D_{O(T)\times\infty}(T/(Y \times S))$

\[ \begin{array}{ccc}
Rg^{*\text{mod}, T} \mathcal{F}_{S,an}^G(M(X/S)) & \xrightarrow{T(g,F^G)(M(X/S))} & \mathcal{F}_{T,an}^G(M(X_T/T)) \\
\downarrow \mathcal{I}^{G_{\text{mod}}(X/S)} & & \downarrow \mathcal{I}^{G_{\text{mod}}(X_T/T)} \\
g^{*\text{mod}, [-]} \Gamma(p_{\tilde{S}_I}, \Gamma_{X, l}, \text{usu}^{\ast}_{Y_I \times \tilde{S}_I}, F_b)[-d_{\tilde{S}_I}] & \xrightarrow{(T(\gamma, l, \gamma)(-))_{\text{T}^{\text{us}}\text{s}}(\tilde{g}_{l, p_{\tilde{S}_I}})} & (p_{Y \times T, l}, \Gamma_{X_{\text{usu}}}^{\ast} \Omega_{Y_I \times \tilde{S}_I}^{\text{usu}} \otimes \Omega_{T}/T, F_b)[-d_{\tilde{S}_I}] & \xrightarrow{w_{1, l}(X/T)} \\
\end{array} \]

(ii) Let $g : T \to S$ be a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Assume that there is a factorization $f : X \to Y \times S \overset{p_S} \to S$, with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Consider $F(X/S) := p_S \Gamma_X^* Z(Y \times S/Y \times S)$ and the isomorphism in $C(\text{Var}(\mathbb{C})^{sm}/S)$

$$T(f, g, F(X/S)) : g^* F(X/S) := g^* p_S \Gamma_X^* Z(Y \times S/Y \times S) \cong p_T \Gamma_T^* Z(Y \times T/Y \times T)[d_Y] =: F(X_T/T).$$

which gives in $\text{DA}(S)$ the isomorphism $T(f, g, F(X/S)) : g^* M(X/S) \cong M(X_T/T)$. Then, the following diagram in $D_{O(T)\times\infty}(T)$

\[ \begin{array}{ccc}
Lg^{*\text{mod}, [-]} \mathcal{F}_{S,an}^G(M(X/S)) & \xrightarrow{T^{O}(g,F^G)(M(X/S))} & \mathcal{F}_{T,an}^G(M(X_T/T)) \\
\downarrow \mathcal{I}^{G_{\text{mod}}(X/S)} & & \downarrow \mathcal{I}^{G_{\text{mod}}(X_T/T)} \\
g^{*\text{mod}, L}(p_{S}, \Gamma_{X, l}, \text{usu}^{\ast}_{Y \times S}, F_b)[-d_T] & \xrightarrow{(T(\gamma, l, \gamma)(-))_{\text{T}^{\text{us}}\text{s}}(g_{l, p_{\tilde{S}_I}})} & (p_{T}, \Gamma_{X_{\text{usu}}}^{\ast} \Omega_{Y \times S}^{\text{usu}} \otimes \Omega_{T}/T, F_b)[-d_T] & \xrightarrow{w_{1, l}(X/T)} \\
\end{array} \]

Proof. Follows immediately from definition. □

We have the following theorem:

**Theorem 37.** (i) Let $g : T \to S$ be a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \overset{p_S} \to S$, with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \bigcup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow S_i$ with $S_i \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, F^G_{an})(M) : Rg^{*\text{mod}, [-]} \mathcal{F}^G_{S,an}(M) \to \mathcal{F}^G_{T,an}(g^* M)$$

is an isomorphism in $D_{O(T)\times\infty}(T/(Y \times S))$.

(ii) Let $g : T \to S$ be a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. Then, for $M \in \text{DA}_c(S)$

$$T(g, F^G_{an})(M) : Lg^{*\text{mod}, [-]} \mathcal{F}^G_{S,an}(M) \to \mathcal{F}^G_{T,an}(g^* M)$$

is an isomorphism in $D_{O(T)}(T)$.
Proof. (i): Follows from proposition 125.

(ii): First proof: Follows from proposition 129, proposition 135 and proposition 96.

Second proof: In the analytic case only, we can give a direct proof of this proposition: Indeed, let \( g : T \to S \) be a morphism with \( T, S \in \text{AnSp}(\mathbb{C}) \) and let \( h : U \to S \) a smooth morphism with \( U \in \text{AnSp}(\mathbb{C}) \), then,

\[
T^{\omega}(g, h) : g^{\text{mod}}L_{D}h_{\ast}E(\Omega_{U/S}^{\ast}, F) \to h^{\ast}E(\Omega_{U/T}^{\ast}, F)
\]
is an equivalence us local : consider the following commutative diagram

\[
\begin{array}{ccc}
g^{\text{mod}}L_{O}(h_{\ast}E(Z_{U}) \otimes O_{S}) & \xrightarrow{T(g,h)(E(Z_{U}))} & h'_{\ast}E(Z_{U/T}) \otimes O_{T} \\
\downarrow g^{\text{mod}}L_{O}T(h, \otimes)(-,-) & & \downarrow T(h', \otimes)(-,-) \\
g^{\text{mod}}L_{O}h_{\ast}E(h^{\ast}O_{S}) & \xrightarrow{T^{\text{mod}}(g,h)(h^{\ast}O_{S})} & h'_{\ast}E(h^{\ast}O_{T}) \\
\downarrow g^{\text{mod}}L_{O}h_{\ast}E(\Omega_{U/S}) & \xrightarrow{T^{\omega}(g,h)} & h'_{\ast}E(\Omega_{U/T})
\end{array}
\]

then,

- the maps \( T(h', \otimes)(-,-) \) and \( T(h, \otimes)(-,-) \) are us local equivalence by proposition 9,
- since \( h : U \to S \) is a smooth morphism, the inclusion \( i_{U/S} : h^{\ast}O_{S} \to \Omega_{U/S}^{\ast} \) is a quasi-isomorphism,
- since \( h' : U_{T} \to T \) is a smooth morphism, the inclusion \( i_{U_{T}/T} : h^{\ast}O_{T} \to \Omega_{U_{T}/T}^{\ast} \) is a quasi-isomorphism,
- since \( U, U_{T}, S, T \) are paracompact topological spaces (in particular Hausdorff), \( T(g,h)(E(Z_{U})) : g^{\ast}h_{\ast}E(Z_{U}) \to h'_{\ast}E(Z_{U/T}) \) is a quasi-isomorphism.

This fact, together with lemma 15, proves the proposition. \( \square \)

We finish this subsection by some remarks on the absolute case and on a particular case of the relative case:

**Proposition 130.** (i) Let \( X \in \text{PSmVar}(\mathbb{C}) \) and \( D = \cup D_{i} \subset X \) a normal crossing divisor. Consider the open embedding \( j : U := X \setminus D \hookrightarrow X \). Then,

- The map in \( D_{fil, \infty}(\mathbb{C}) \)

\[
\text{Hom}((0, \text{ad}(j^{\ast}, j_{\ast})(\mathbb{Z}(X/X)), E_{\text{usu}}(\Omega_{X}^{\ast}, F_{b}))) : \text{Hom}(\text{LD}(\mathbb{Z}(U)), E_{\text{usu}}(\Omega_{X}^{\ast}, F_{b}))
\]

\[
\xrightarrow{\sim} \text{Hom}(\text{Cone}(\mathbb{Z}(D) \to \mathbb{Z}(X)), E_{\text{zar}}(\Omega_{X}^{\ast}, F_{b})) = \Gamma(X, E_{\text{usu}}(\Omega_{X}^{\ast}(\text{null} D), F_{b})).
\]

is an isomorphism, where we recall \( \text{Hom}(\mathbb{Z}(U), E_{\text{usu}}(\Omega_{U}^{\ast}, F_{b})) \in D_{fil, \infty}(\mathbb{C}) \) is NOT isomorphic to \( \Gamma(X, E_{\text{usu}}(\Omega_{X}(\text{log} D), F_{b})) \) in \( D_{fil, \infty}(\mathbb{C}) \) in general. For example \( U \) is affine, then \( U_{an}^{\text{Fil}} \) is Stein so that \( H^{n}(U, \Omega_{U}^{p}) = 0 \) for all \( p \in \mathbb{N} \) if \( p \neq 0 \) so that the \( E_{\text{usu}}^{\infty}(\Gamma(U, E_{\text{usu}}(\Omega_{U}^{p}, F_{b}))) \) are NOT isomorphic to \( E_{\text{usu}}^{p, q}(\Gamma(U, E_{\text{usu}}(\Omega_{X}^{p}, (\text{log} D), F_{b}))) \) in this case. In particular, the map,

\[
j^{\ast} : = \text{ad}(j^{\ast}, j_{\ast})(- : H^{n}(X, E_{\text{usu}}(\Omega_{X}(\text{log} D))) \xrightarrow{\sim} H^{n}(U, E_{\text{zar}}(\Omega_{U}^{p})))
\]

which is an isomorphism in \( D(\mathbb{C}) \) (i.e. if we forgot filtrations), gives embeddings

\[
j^{\ast} : = \text{ad}(j^{\ast}, j_{\ast})(- : F^{p}H^{n}(U, \Omega_{U}^{p}) := F^{p}H^{n}(\Gamma(X, E_{\text{usu}}(\Omega_{X}(\text{log} D), F_{b}))) \hookrightarrow F^{p}H^{n}(U, E_{\text{usu}}(\Omega_{U}^{p}, F_{b}))
\]

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which are NOT an isomorphism in general for $n, p \in \mathbb{Z}$. Note that, since $a_U : U \to \{\text{pt}\}$ is not proper,
\[ [\Delta_U] : \mathbb{Z}(U) \to a_U, E_\mathbb{Z}(\mathbb{Z}(U/U))[2d_U] \]
is NOT an equivalence ($\mathbb{A}^1, \text{et}$) local.

- Let $Z \subset X$ a smooth subvariety and denote $U := X \setminus Z$ the open complementary. Denote $M_Z(X) := \text{Cone}(M(U) \to M(X)) \in \text{DA}(\mathbb{C})$. The map in $D_{fil, \infty}(\mathbb{C})$
\[ \text{Hom}(G(X, Z), E_{usu}(\Omega^\bullet, F_b))^{-1} : \mathcal{F}^{GM}_{an}(M_Z(X)) \cong \text{Hom}(a_X \Gamma_Z^\bullet \mathbb{Z}(X/X), E_{usu}(\Omega^\bullet, F_b)) \cong \Gamma(X, \Gamma_Z E_{usu}(\Omega^\bullet_X, F_b)) \cong \mathcal{F}^{GM}_{an}(M(Z)(c)[2c]) = \Gamma(Z, E_{usu}(\Omega^\bullet_Z, F_b))(-c)[-2c] \]
is an isomorphism, where $c = \text{codim}(Z, X)$ and $G(X, Z) : a_X \Gamma_Z^\bullet \mathbb{Z}(X/X) \to \mathbb{Z}(Z)(c)[2c]$ is the Gysin morphism.

- Let $D \subset X$ a smooth divisor and denote $U := X \setminus Z$ the open complementary. Note that the canonical distinguish triangle in $\text{DA}(\mathbb{C})$
\[ M(U) \xrightarrow{\text{ad}(j^*, j_*)} M_X \xrightarrow{\gamma_Z^\bullet (\mathbb{Z}(X/X))} M_D(X) \xrightarrow{M(U)[1]} \]
give a canonical triangle in $D_{fil, \infty}(\mathbb{C})$
\[ \mathcal{F}^{GM}(M_D(X)) \xrightarrow{\mathcal{F}^{GM}(\gamma_Z^\bullet (\mathbb{Z}(X/X)))} \mathcal{F}^{GM}(M_X) \xrightarrow{\mathcal{F}^{GM}(\text{ad}(j^*, j_*) \gamma_Z^\bullet (\mathbb{Z}(X/X)))} \mathcal{F}^{GM}(M(U)) \rightarrow \mathcal{F}^{GM}(M_D(X))[1], \]
which is NOT the image of a distinguish triangle in $\pi(D(\text{MMH}(\mathbb{C})))$, as $\mathcal{F}^{GM}(M(U)) \notin \pi(D(\text{MMH}(\mathbb{C})))$ since the morphism
\[ j^* := \text{ad}(j^*, j_*)(-) : H^n \Gamma(X, E_{usu}(\Omega^\bullet_X (\log D), F_b)) \to H^n \Gamma(U, E_{usu}(\Omega^\bullet_U, F_b)) \]
are not strict. Note that if $U := S \setminus D$ is affine, then by the exact sequence in $C(\mathbb{Z})$
\[ 0 \to \Gamma_Z(X, E_{usu}(\Omega^p_Z)) \to \Gamma(X, E_{usu}(\Omega^p_X)) \to \Gamma(U, E_{usu}(\Omega^p_U)) \to 0 \]
we have $H^q \Gamma_Z(X, E_{usu}(\Omega^p_Z)) = H^q(\Gamma(X, E_{usu}(\Omega^p_X))).$

In particular, the maps
\[ j^* := \text{ad}(j^*, j_*)(-) : \Gamma(X, E_{usu}(\Omega^\bullet_X (\log D), F_b)) \to \Gamma(U, E_{usu}(\Omega^\bullet_U, F_b)) \]
and
\[ j^* := \text{ad}(j^*, j_*)(-) : \text{Cone}(\Gamma(X, E_{usu}(\Omega^\bullet_X, F_b)) \to \Gamma(X, E_{usu}(\Omega^\bullet_X (\log D), F_b))) \to \]
\[ \text{Cone}(\Gamma(X, E_{usu}(\Omega^\bullet_X, F_b)) \to \Gamma(U, E_{usu}(\Omega^\bullet_U, F_b))) =: \Gamma(X, \Gamma_Z E_{usu}(\Omega^\bullet_X, F_b)) \]
are quasi-isomorphism (i.e. if we forgot filtrations), but the first one is NOT an $\infty$-filtered quasi-isomorphism whereas the second one is an $\infty$-filtered quasi-isomorphism (recall that for $r > 1$ the $r$-filtered quasi-isomorphism does NOT satisfy the 2 of 3 property for morphism of canonical triangles : see section 2.1).

(ii) More generally, let $f : X \to S$ a smooth projective morphism with $S, X \in \text{SmVar}(\mathbb{C})$. Let $D = \bigcup D_i \subset X$ a normal crossing divisor such that $f|_{D_i} := f \circ i_I : D_i \to S$ are SMOOTH morphisms (note that it is a very special case), with $i_I : D_i \hookrightarrow X$ the closed embeddings. Consider the open embedding $j : U := X \setminus D \hookrightarrow X$ and $h := f \circ j : U \to S$. 

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The map in $D_{\mathcal{D}fil,\infty}(S)$

\[
\text{Hom}( (0, \text{ad}(j^*, j_*)(\mathbb{Z}/X)), E_{et}(\Omega^*_{/S}, F_b)) : \\
\mathcal{F}^G_{\mathcal{S}, \text{an}}(\mathbb{D}(\mathbb{Z}/U:S)) := \text{Hom}(L\mathbb{D}(\mathbb{Z}/U:S), E_{usu}(\Omega^*_{/S}, F_b)) \\
\xrightarrow{\sim} \text{Hom}(\text{Cone}(\mathbb{Z}/D) \to \mathbb{Z}/(X)), E_{zar}(\Omega^*_{/S}, F_b)) = f_* E_{usu}(\Omega^*_{X/S}(\text{null} D), F_b).
\]

is an isomorphism, where we recall $\mathbb{D}(\mathbb{Z}/U) := f_*, j_* E_{et}(\mathbb{Z}/U:S) = h_* E_{usu}(\mathbb{Z}/U:S)$,

\[- \mathcal{F}^G_{\mathcal{S}, \text{an}}(\mathbb{Z}/U:S) = h_* E_{usu}(\Omega^*_{U:S}/S, F_b) \in D_{\mathcal{D}fil,\infty}(S) \text{ is NOT isomorphic to } f_* E_{usu}(\Omega^*_{X/S}(\text{log} D), F_b)
\]
in $D_{\mathcal{D}fil,\infty}(S)$ in general. In particular, the map,

\[ j^* := \text{ad}(j^*, j_*)(-): H^n f_* E_{usu}(\Omega^*_{X/S}(\text{log} D)) \xrightarrow{\sim} H^n h_* E_{usu}(\Omega^*_{U/S}) \]

which is an isomorphism in $D_{\mathcal{D}}(S)$ (i.e. if we forget filtrations), gives embeddings

\[ j^* := \text{ad}(j^*, j_*)(-) : \mathbb{F}^p H^n h_* \mathbb{C}_U := \mathbb{F}^p H^n f_* E_{usu}(\Omega^*_{X/S}(\text{log} D), F_b) \to \mathbb{F}^p H^n h_* E_{usu}(\Omega^*_{U/S}, F_b) \]

which are NOT an isomorphism in general for $n, p \in \mathbb{Z}$. Note that, since $a_U : U \to \{\text{pt}\}$ is not proper,

\[ [\Delta_U] : \mathbb{Z}/U:S \to h_* E_{usu}(\mathbb{Z}/U:S)[2d_U] \]

is NOT an equivalence ($\mathbb{A}^1$, et) local.

- Let $Z \subset X$ a subvariety and denote $U := X \setminus Z$ the open complementary. Denote $M_Z(X/S) := \text{Cone}(M(U/S) \to M(X/S)) \in \text{DA}(S)$. If $f|_Z := f \circ i_Z : Z \to S$ is a SMOOTH morphism, the map in $D_{\mathcal{D}fil,\infty}(S)$

\[
\text{Hom}(G(X, Z), E_{usu}(\Omega^*_{/S}, F_b)) : \\
\mathcal{F}^G_{\mathcal{S}, \text{an}}(M_Z(X/S)) := \text{Hom}(f|_Z^{\gamma^*_Z} \mathbb{Z}/X:Z), E_{usu}(\Omega^*_{/S}, F_b)) \\
\xrightarrow{\sim} \mathcal{F}^G_{\mathcal{S}}(M(Z/S) : \mathbb{C}(-c)[2c]) := f|_Z E_{usu}(\Omega^*_{Z/S}, F_b)(-c)[2c]
\]

is an isomorphism, where $c = \text{codim}(Z, X)$ and $G(X, Z) : f|_Z^{\gamma^*_Z} \mathbb{Z}/X:Z \to \mathbb{Z}/Z:S(c)[2c]$ is the Gysin morphism.

- Let $D \subset X$ a smooth divisor and denote $U := X \setminus Z$ the open complementary. Note that the canonical distinguish triangle in $\text{DA}(S)$

\[ M(U/S) \xrightarrow{\text{ad}(j_1 j^*)(\mathbb{Z}/X:X)} M(X/S) \xrightarrow{\gamma^*_Z(\mathbb{Z}/X:X)} M_D(X/S) \to M(U/S)[1] \]

give a canonical triangle in $D_{\mathcal{D}fil,\infty}(S)$

\[
\mathcal{F}^G_{\mathcal{S}, \text{an}}(M_D(X/S)) \xrightarrow{\mathcal{F}^G_{\mathcal{S}, \text{an}}(\gamma^*_Z(\mathbb{Z}/X:X))} \mathcal{F}^G_{\mathcal{S}, \text{an}}(M(X/S)) \xrightarrow{\mathcal{F}^G_{\mathcal{S}, \text{an}}(\text{ad}(j_1 j^*)(\mathbb{Z}/X:X))} \\
\mathcal{F}^G_{\mathcal{S}, \text{an}}(M(D(X/S)))[1],
\]

which is NOT the image of a distinguish triangle in $\pi_S(D_{\text{MHM}}(S))$. \hfill \Box

**Definition 147.** Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \ldots, l]$, $S_I := \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_I : S_i \hookrightarrow S_i$ closed embeddings, with $S_i \in \text{SmVar}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C(\text{Var}(\mathbb{C})^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and
Proposition 131. Let $S \in \text{Var}(\mathbb{C})$. Then, for $M, N \in \text{DA}_c(S)$

$$T(\otimes, \mathcal{F}^G_{S,an}(M, N)) : \mathcal{F}^G_{S,an}(M \otimes N) \rightarrow \mathcal{F}^G_{S,an}(M) \otimes_{O_S} \mathcal{F}^G_{S,an}(N)$$

is an isomorphism.

Proof. Assume first that $S$ is smooth. Let $h_1 : U_1 \rightarrow S$ and $h_2 : U_2 \rightarrow S$ smooth morphisms with $U_1, U_2 \in \text{Var}(\mathbb{C})$ and consider $h_{12} : U_1 \times_S U_2 \rightarrow S$. We then have by lemma 16 the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{F}^G_{S,an}(M) \otimes_{O_S} \mathcal{F}^G_{S,an}(N) := (e(\tilde{S}_I), \text{Hom}(\text{An}_{S_I}^n, L(i_{1*} j_I^* F), E_{usu}(\Omega_{S_I}^\bullet, F_b)), u_{1J}(F)) \otimes_{O_S} \\
\cong (e(\tilde{S}_I), \text{Hom}(\text{An}_{S_I}^n, L(i_{1*} j_I^* G), E_{usu}(\Omega_{S_I}^\bullet, F_b)), u_{1J}(G)) \\
\cong (e(\tilde{S}_I), \text{Hom}(\text{An}_{S_I}^n, L(i_{1*} j_I^* F), E_{usu}(\Omega_{S_I}^\bullet, F_b)), u_{1J}(F) \otimes u_{1J}(G)) \\
\longrightarrow (e(\tilde{S}_I), \text{Hom}(\text{An}_{S_I}^n, L(i_{1*} j_I^* F) \otimes \text{An}_{S_I}^n, L(i_{1*} j_I^* G), E_{usu}(\Omega_{S_I}^\bullet, F_b)), u_{1J}(F \otimes G)) \\
\cong (e(\tilde{S}_I), \text{Hom}(\text{An}_{S_I}^n, L(i_{1*} j_I^* (F \otimes G), E_{usu}(\Omega_{S_I}^\bullet, F_b)), u_{1J}(F \otimes G)) =: \mathcal{F}^G_{S,an}(M \otimes N)
\end{array}
$$

We have in the analytical case the following:

Since $U_1, U_2 \in \text{AnSp}(\mathbb{C})$ are locally contractible topological spaces, the lower row is an equivalence usu local by Kunneth formula for topological spaces (see section 2). This proves the proposition in the case $S$ is smooth. Let $S \in \text{Var}(\mathbb{C})$ and $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \rightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. By definition, for $F, G \in C(\text{Var}(\mathbb{C})^{an}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$,

$$T(\otimes, \mathcal{F}^G_{S,an}(M, N)) : e(\tilde{S}_I), \text{Hom}(\text{An}_{S_I}^n, L(i_{1*} j_I^* (F \otimes G), E_{usu}(\Omega_{S_I}^\bullet, F_b)), u_{1J}(F \otimes G))$$

$$\cong (e(\tilde{S}_I), \text{Hom}(\text{An}_{S_I}^n, L(i_{1*} j_I^* F), E_{usu}(\Omega_{S_I}^\bullet, F_b)), u_{1J}(F) \otimes u_{1J}(G))$$

Since $L(i_{1*} j_I^* F), L(i_{1*} j_I^* G) \in \text{DA}_c(\tilde{S}_I)$, by the smooth case applied to $\tilde{S}_I$ for each $I$, $T(\otimes, \mathcal{F}^{FDR}_{S,an}(M, N))$ is an equivalence usu local.\qed
6.2.2 The analytic filtered De Rham realization functor

Recall from section 2 that, for $S \in \text{Var}(\mathcal{C})$ we have the following commutative diagrams of sites

\[
\begin{array}{c}
\text{AnSp}(\mathcal{C})^2/S & \xrightarrow{\mu_S} & \text{AnSp}(\mathcal{C})^{2,pr}/S \\
\downarrow \rho_S & & \downarrow \rho_S \\
\text{AnSp}(\mathcal{C})^{2,sm}/S & \xrightarrow{\pi_1 \circ \pi_2} & \text{AnSp}(\mathcal{C})^{2,smpr}/S \\
\downarrow \rho_S & & \downarrow \rho_S \\
\text{Var}(\mathcal{C})^2/S & \xrightarrow{\pi_1 \circ \pi_2} & \text{Var}(\mathcal{C})^{2,smpr}/S \\
\downarrow \rho_S & & \downarrow \rho_S \\
\text{Var}(\mathcal{C})^2/S & \xrightarrow{\pi_1 \circ \pi_2} & \text{Var}(\mathcal{C})^{2,smpr}/S \\
\end{array}
\]

and

\[
\begin{array}{c}
\text{AnSp}(\mathcal{C})^{2,pr}/S & \xrightarrow{\text{Gr}^{12}} & \text{AnSp}(\mathcal{C})/S \\
\downarrow \rho_S & & \downarrow \rho_S \\
\text{AnSp}(\mathcal{C})^{2,smpr}/S & \xrightarrow{\text{Gr}^2} & \text{AnSp}(\mathcal{C})^{sm}/S \\
\downarrow \rho_S & & \downarrow \rho_S \\
\text{Var}(\mathcal{C})^{2,pr}/S & \xrightarrow{\text{Gr}^2} & \text{Var}(\mathcal{C})/S \\
\downarrow \rho_S & & \downarrow \rho_S \\
\text{Var}(\mathcal{C})^{2,sm}/S & \xrightarrow{\text{Gr}^{12}} & \text{Var}(\mathcal{C})^{sm}/S \\
\end{array}
\]

and that for $f : T \to S$ a morphism with $T, S \in \text{Var}(\mathcal{C})$ we have the following commutative diagrams of site,

\[
\begin{array}{c}
\text{AnSp}(\mathcal{C})^2/T^{\text{an}} & \xrightarrow{\text{An}_T} & \text{Var}(\mathcal{C})^2/T \\
\downarrow P(f) & & \downarrow P(f) \\
\text{AnSp}(\mathcal{C})^{2,sm}/T^{an} & \xrightarrow{\text{An}_f} & \text{Var}(\mathcal{C})^{2,sm}/T \\
\downarrow P(f) & & \downarrow P(f) \\
\text{AnSp}(\mathcal{C})^2/S^{an} & \xrightarrow{\text{An}_T} & \text{Var}(\mathcal{C})^2/S \\
\downarrow \rho_S & & \downarrow \rho_S \\
\text{AnSp}(\mathcal{C})^{2,sm}/S^{an} & \xrightarrow{\text{An}_f} & \text{Var}(\mathcal{C})^{2,sm}/S \\
\end{array}
\]

For $s : I \to J$ a functor, with $I, J \in \text{Cat}$, and $f_\bullet : T_\bullet \to S_{s(\bullet)}$ a morphism with $T_\bullet \in \text{Fun}(J, \text{Var}(\mathcal{C}))$
and $S_\bullet \in \text{Fun}(I, \text{Var}(\mathbb{C}))$, we have then the commutative diagram of sites 40

and the following commutative diagrams of site,

and the following commutative diagrams of site,

\[ \begin{align*}
\text{AnSp}(\mathbb{C})^2 / T_\bullet & \xrightarrow{\mu T_\bullet} \text{AnSp}(\mathbb{C})^2 / T_\bullet \\
\text{AnSp}(\mathbb{C})^2 / S_\bullet & \xrightarrow{\rho S_\bullet} \text{AnSp}(\mathbb{C})^2 / S_\bullet \\
\text{Var}(\mathbb{C})^2 / T_\bullet & \xrightarrow{\rho T_\bullet} \text{Var}(\mathbb{C})^2 / S_\bullet \\
\text{Var}(\mathbb{C})^2 / S_\bullet & \xrightarrow{\rho S_\bullet} \text{Var}(\mathbb{C})^2 / S_\bullet \\
\text{AnSp}(\mathbb{C})^2 / T_\bullet & \xrightarrow{\rho T_\bullet} \text{AnSp}(\mathbb{C})^2 / S_\bullet \\
\text{Var}(\mathbb{C})^2 / T_\bullet & \xrightarrow{\rho T_\bullet} \text{Var}(\mathbb{C})^2 / S_\bullet \\
\end{align*} \]

\[ \begin{align*}
\text{AnSp}(\mathbb{C})^2 / T^\text{an}_\bullet & \xrightarrow{\mu T^\text{an}_\bullet} \text{AnSp}(\mathbb{C})^2 / T^\text{an}_\bullet \\
\text{AnSp}(\mathbb{C})^2 / S^\text{an}_\bullet & \xrightarrow{\rho S^\text{an}_\bullet} \text{AnSp}(\mathbb{C})^2 / S^\text{an}_\bullet \\
\text{Var}(\mathbb{C})^2 / T^\text{an}_\bullet & \xrightarrow{\rho T^\text{an}_\bullet} \text{Var}(\mathbb{C})^2 / S^\text{an}_\bullet \\
\text{Var}(\mathbb{C})^2 / S^\text{an}_\bullet & \xrightarrow{\rho S^\text{an}_\bullet} \text{Var}(\mathbb{C})^2 / S^\text{an}_\bullet \\
\end{align*} \]

**Definition 148.** (i) For $S \in \text{SmVar}(\mathbb{C})$, we consider, using definition 129(i), the filtered complexes of presheaves

\[ (\Omega_{/S^\text{an}}^\text{I,pr}, F_{DR}) \in C_{D^\infty_{/S^\text{an}}}^{\text{fil}}(\text{Var}(\mathbb{C})^2 / S) \]

given by,

- for $(Y \times S, Z)/S = ((Y \times S, Z), p) \in \text{Var}(\mathbb{C})^2 / S$,

\[ (\Omega_{/S^\text{an}}^\text{I,pr})(Y \times S, Z)/S, F_{DR}) := (\Omega_{(Y \times S)_{/S^\text{an}}}, F_b) \otimes_{O_{(Y \times S)_{/S^\text{an}}}} (\Gamma_Z^{\text{I,pr}}(O_{(Y \times S, F_b)^{an}})(((Y \times S)^a)) \]

with the structure of $p^* D_S$ module given by proposition 64.

- for $g : (Y_1 \times S, Z_1)/S = ((Y_1 \times S, Z_1), p_1) \rightarrow (Y \times S, Z)/S = ((Y \times S, Z), p)$ a morphism in $\text{Var}(\mathbb{C})^2 / S$,

\[ \Omega_{/S^\text{an}}^\text{I,pr}(g) := (\Omega_{/S^\text{an}}^\text{I,pr}(g))^{an} : \Omega_{(Y_1 \times S)_{/S^\text{an}}}, F_b) \otimes_{O_{(Y_1 \times S)_{/S^\text{an}}}} (\Gamma_Z^{\text{I,pr}}(O_{(Y_1 \times S, F_b)^{an}})(((Y_1 \times S)^a)) \rightarrow \Omega_{(Y_1 \times S)_{/S^\text{an}}}, F_b) \otimes_{O_{(Y_1 \times S)_{/S^\text{an}}}} (\Gamma_Z^{\text{I,pr}}(O_{(Y_1 \times S, F_b)^{an}})(((Y_1 \times S)^a)). \]

For $S \in \text{SmVar}(\mathbb{C})$, we get the filtered complexes of presheaves

\[ (\Omega_{/S^\text{an}}^\text{I,pr,an}, F_{DR}) := F_{s_{/S^\text{an}}}^\text{mod}(\Omega_{/S^\text{an}}^\text{I,pr}, F_{DR}) := \text{An}_{/S^\text{an}}(\Omega_{/S^\text{an}}^\text{I,pr}, F_{DR}) \otimes_{O_{S^\text{an}}} \in C_{D^\infty_{/S^\text{an}}}^{\text{fil}}(\text{AnSp}(\mathbb{C})^2 / S). \]

(ii) For $S \in \text{SmVar}(\mathbb{C})$, we have the canonical map $C_{O_{/S^\text{an}}, D^\infty_{/S}} (\text{Var}(\mathbb{C})^m / S)$

\[ \text{Gr}(\Omega_{/S^\text{an}}) : \text{Gr}_{S^\text{an}}^{12}(\Omega_{/S^\text{an}}, F_b) \rightarrow \text{An}_{S^\text{an}}(\Omega_{/S}, F_b) \]

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given by
\[ \text{Gr}(\Omega_{/S^m})(U/S) := (\text{Gr}(\Omega_{/S})(U/S))^{an} \otimes m : \]
\[ J_{S^m}(\Omega_{(U \times S)^{an}/S^{an}}, F_b) \otimes \Omega_{(U \times S)^{an}}((\Gamma_U^\infty \Delta^g (\Omega_{(U \times S}, F_b))^{an})((U \times S)^{an}) \to (\Omega^\infty_{/S^{an}})^{an}, F_b), \]
where \( \text{Gr}(\Omega_{/S^m})(U/S)(\omega \otimes m \otimes P) := P(\text{Gr}(\Omega_{/S})(U/S)(\omega \otimes m)) \) with \( P \in \Gamma(S, D_S^m) \), see definition 129(ii), which gives by adjunction
\[ \text{Gr}(\Omega_{/S^m}) := I(\text{An}_{S^{an}}, \text{AnS})(\text{Gr}(\Omega_{/S^m})) : J_{S^m}(\text{Gr}_{S^{an}}^{12}(\Omega^\infty_{/S^{an}}^{an} \otimes F_b) \to (\Omega^\infty_{/S}, F_b) \]
in \( C_{O_S fil.D_S^m}^{an}(\text{AnSp}(C)^{an}/S). \)

Definition 149. For \( S \in \text{SmVar}(C) \), we have the canonical map in \( C_{O_S fil.D_S^m}^{an}(\text{Var}(C)^{2,anpr}/S) \)
\[ T(\Omega^\infty_{/S^{an}}) : \text{AnS}_* \mu_{S^*}(\Omega^\infty_{/S^{an}}, F_b) \to (\Omega^\infty_{/S^{an}} \otimes F_{DR}) \]
given by, for \((Y \times S, X) = ((Y \times S, Z), p) \in \text{Var}(C)^{2,an}/S \)
\[ T(\Omega^\infty_{/S^{an}})((Y \times S, Z)/S) := (T(\Omega^\infty_{/S})(((Y \times S, Z)/S)^{an}) : \]
\[ \Omega^\infty_{/S^m}((Y \times S, Z)/S) := (\Omega^\infty_{/S}(((Y \times S)^{an}, Z^{an})/S) := (\text{D}_{p^*} \text{D}_{p^*} \text{O} \text{G} \text{E} \text{M} \text{S}_{an} \text{D}_{p^*} \text{D}_{p^*} \text{O}(\Omega^\infty_{(Y \times S)^{an}/S^{an}}, F_b))((Y \times S)^{an}) \to \]
\[ ((\Omega^\infty_{(Y \times S)^{an}/S^{an}}, F_b) \otimes \Omega_{(Y \times S)^{an}}((\Gamma_U^\infty \Delta^g (\Omega_{Y \times S}, F_b))^{an})((Y \times S)^{an}) : (\Omega^\infty_{/S^{an}} \otimes F_{DR})((Y \times S, Z)/S), \]
see definition 130. By definition we have \( \text{Gr}^O(\Omega_{/S^m}) = \text{Gr}(\Omega_{/S^m}) \circ T(\Omega^\infty_{/S^{an}}). \)

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let \( g : T \to S \) a morphism with \( T, S \in \text{AnSm}(C). \)

- We have the canonical morphism in \( C_{g^*O_S fil.,g^*D_S^m}^{an}(\text{AnSp}(C)^{2,sm}/T) \)
\[ \Omega^\infty_{/(T/S)} : g^*(\Omega^\infty_{/S}, F_b) \to (\Omega^\infty_{/T}, F_b) \]
induced by the pullback of differential forms: for \((V, Z_1)/T = ((V, Z_1), h) \in \text{AnSp}(C)^{2,sm}/T, \)
\[ \Omega^\infty_{/(T/S)}((V, Z_1)/T) := (\text{h} : (U', Z_1) \to \text{sm}, g : (V, Z_1) \to (U_T, Z_1), h, g) \]
\[ \Omega^\infty_{/S}((U, Z)/S) \]
\[ g^*(\Omega^\infty_{/S})((V, Z_1)/T) := \lim_{\text{h} : (U', Z_1) \to \text{sm}, g : (V, Z_1) \to (U_T, Z_1), h, g} \Omega^\infty_{/S}((U, Z)/S) \]
\[ \Omega^\infty_{/(T/S)}((V, Z_1)/T), \]
where \( g' : U_T := U \times T \to U \) is the base change map and \( q : \Omega^\bullet_{(1 \times T)} \to \Omega^\bullet_{(1 \times T)} \) is the quotient map. It induces the canonical morphisms in \( C_{g^*O_S fil.,g^*D_S^m}^{an}(\text{AnSp}(C)^{2,sm}/T) : \)
\[ E\Omega^\infty_{/(T/S)} : g^*E_{ext}(\Omega^\infty_{/S}, F_b) \xrightarrow{T(g, E_{ext}(\Omega^\infty_{/S}, F_b))} E_{ext}(\Omega^\infty_{/(T/S)}), F_b) \]
\[ E_{ext}(\Omega^\infty_{/(T/S)}, F_b) \]

- We have the canonical morphism in \( C_{g^*D_S^m fil.,g^*D_S^m}^{an}(\text{Var}(C)^{2,anpr}/T) \)
\[ \Omega^\infty_{/(T/S)^{an}} : g^*(\Omega^\infty_{(S^{an}), F_{DR}}) \to (\Omega^\infty_{/(T/S)^{an}}, F_{DR}) \]
induced by the pullback of differential forms: for \((Y_1 \times T, Z_1)/T = ((Y_1 \times T, Z_1), p) \in \text{Var}(C)^{2,anpr}/T, \)
\[ \Omega^\infty_{/(T/S)^{an}}((Y_1 \times T, Z_1)/T) := (\text{h} : (U \times S, Z) \to \text{sm}, g : (Y_1 \times T, Z_1) \to (Y \times T, Z), h, g) \]
\[ \lim_{\text{h} : (U \times S, Z) \to \text{sm}, g : (Y_1 \times T, Z_1) \to (Y \times T, Z), h, g} \Omega^\infty_{/(T/S)^{an}}((Y \times T, Z)/S), \]
\[ \Omega^\infty_{/(T/S)^{an}}((Y_1 \times T, Z_1)/T), \]
where $g' = (I_Y \times g) : Y \times T \to Y \times S$ is the base change map and

$$q(M) : \Omega_{(Y_1 \times T)^{an}/S^{an}} \otimes_{\Omega_{(Y_1 \times T)^{an}/S^{an}}} (M, F) \to \Omega_{(Y_1 \times T)^{an}/T^{an}} \otimes_{\Omega_{(Y_1 \times T)^{an}}}(M, F)$$

is the quotient map. It induces the canonical morphisms in $C_{g'D_{2,fil}}(\text{Var}(\mathbb{C})^{2,smpr}/T)$:

$$E_{\Omega^\Gamma_{(T/S)}^\bullet}(\Omega^\bullet_{S/S}^\Gamma, F_{\text{DR}}) \xrightarrow{T(g, E)(-)} E_{\text{et}}(g^*(\Omega^\bullet_{S/S}^\Gamma, F_{\text{DR}})) \xrightarrow{E_{\text{et}}(\Omega^\bullet_{(T/S)^{an}}^\Gamma, F_{\text{DR}})} E_{\text{et}}(\Omega^\bullet_{T^{an}}^\Gamma, F_{\text{DR}})$$

and

$$E_{\Omega^\Gamma_{(T/S)^{an}}^\bullet}(\Omega^\bullet_{S/S}^\Gamma, F_{\text{DR}}) \xrightarrow{T(g, E)(-)} E_{\text{zar}}(g^*(\Omega^\bullet_{S/S}^\Gamma, F_{\text{DR}})) \xrightarrow{E_{\text{zar}}(\Omega^\bullet_{(T/S)^{an}}^\Gamma, F_{\text{DR}})} E_{\text{zar}}(\Omega^\bullet_{T^{an}}^\Gamma, F_{\text{DR}}).$$

**Definition 150.** Let $g : T \to S$ a morphism with $T, S \in \text{SmVar}(\mathbb{C})$. We have, for $F \in C(\text{Var}(\mathbb{C})^{2,smpr}/S)$, the canonical transformation in $C_{D = \text{fil}}(T)$:

$$T(g, \Omega_{T/S}^\Gamma(F)) : g^* \text{mod} L_D e(S) \ast \text{Gr}_S^{12} \ast \text{Hom}^\bullet(\text{An}_S^* F, E_{\text{et}}(\Omega_{T/S}^\bullet^\Gamma, F_{\text{DR}}))$$

$$\xrightarrow{\text{mod}} g^* L_D e(S) \ast \text{Hom}^\bullet(F, E_{\text{et}}(\Omega_{T/S}^\bullet^\Gamma, F_{\text{DR}})) \otimes_{g^* O_S} O_T$$

$$T(g, \text{Gr}_T^{12}(-) \circ T(e_g)(-)) \circ e(T) \ast \text{Gr}_T^{12} \ast \text{Hom}^\bullet(\text{An}_T^* g^* F, g^* E_{\text{et}}(\Omega_{T/S}^\bullet^\Gamma, F_{\text{DR}})) \otimes_{g^* O_S} O_T$$

$$\xrightarrow{(T(g, \text{hom})(-)) \otimes 1} e(T) \ast \text{Gr}_T^{12} \ast \text{Hom}^\bullet(\text{An}_T^* g^* F, g^* E_{\text{et}}(\Omega_{T/S}^\bullet^\Gamma, F_{\text{DR}})) \otimes_{g^* O_S} O_T$$

$$\xrightarrow{\text{ev}(\text{hom}, \otimes)(-,-,-,-)} e(T) \ast \text{Gr}_T^{12} \ast \text{Hom}^\bullet(\text{An}_T^* g^* F, g^* E_{\text{et}}(\Omega_{T/S}^\bullet^\Gamma, F_{\text{DR}})) \otimes_{g^* e(S) O_S} e(T) \ast O_T$$

$$\xrightarrow{\text{Hom}^\bullet(\text{An}_T^* g^* F, E_{\text{et}}(\Omega_{T/S}^\bullet^\Gamma, F_{\text{DR}}))} e(T) \ast \text{Gr}_T^{12} \ast \text{Hom}^\bullet(\text{An}_T^* g^* F, E_{\text{et}}(\Omega_{T/S}^\bullet^\Gamma, F_{\text{DR}})).$$

- **Let** $S \in \text{AnSm}(\mathbb{C})$. We have the map in $C_{O_{S^{an}}}^\bullet(\text{Var}(\mathbb{C})^{2,smpr}/S)$:

$$w_S : (\Omega_{T/S}^\Gamma, F_b) \otimes_{O_S} (\Omega_{T/S}^\Gamma, F_b) \to (\Omega_{T/S}^\Gamma, F_b)$$

given by for $h : (U, Z) \to S \in \text{Var}(\mathbb{C})^{2,sm}/S$,

$$w_S(U, Z) : \Gamma_Z^{h, \ast} L_h \ast O_S(\Omega_{U/S}^\bullet, F_b) \otimes_{O_S} \text{Gr}_Z^{v,h} L_h \ast O_S(\Omega_{U/S}^\bullet, F_b) \ast (U)$$

$$\xrightarrow{(D R(-)(\gamma_Z^{v,h}(-)) \otimes O_U/S)^7(U)} G^{v,h} L_h \ast O_S(\Omega_{U/S}^\bullet, F_b)(U)$$

which induces the map in $C_{O_{S^{an}}}^\bullet(\text{Var}(\mathbb{C})^{2,sm}/S)$

$$E_{w_S} : E_{\text{et}}(\Omega_{T/S}^\bullet, F_b) \otimes_{O_S} E_{\text{et}}(\Omega_{T/S}^\bullet, F_b) \xrightarrow{E_{\text{et}}(((\Omega_{T/S}^\bullet, F_b) \otimes_{O_S} (\Omega_{T/S}^\bullet, F_b)))} E_{\text{et}}(\Omega_{T/S}^\bullet, F_b).$$

- **Let** $S \in \text{SmVar}(\mathbb{C})$. We have the map in $C_{D_S^{an}}^\bullet(\text{Var}(\mathbb{C})^{2,smpr}/S)$:

$$w_S : (\Omega_{S^{an}}^\bullet, F_{\text{DR}}) \otimes_{O_S} (\Omega_{S^{an}}^\bullet, F_{\text{DR}}) \to (\Omega_{S^{an}}^\bullet, F_{\text{DR}})$$

given by for $p : (Y \times S, Z) \to S \in \text{Var}(\mathbb{C})^{2,smpr}/S$,

$$w_S((Y \times S, Z) / S) :$$

$$\xrightarrow{((\Omega_{Y \times S}^\bullet \otimes \text{O}_{Y \times S})(\Omega_{Y \times S}^\bullet, F_b)) \otimes_{O_{Y \times S}} (\Omega_{Y \times S}^\bullet \otimes \text{O}_{Y \times S})(\Omega_{Y \times S}^\bullet, F_b) \ast (Y \times S)}$$

$$\xrightarrow{(D R(-)(\gamma_{Z}^{v,Hd}(\Omega_{Y \times S}^\bullet, F_b)) \ast (Y \times S)} (\Omega_{Y \times S}^\bullet \otimes \text{O}_{Y \times S})(\Omega_{Y \times S}^\bullet, F_b)(Y \times S)$$

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which induces the map in $C_{D^\infty_{fil}}(\Var(\mathbb{C})^{2,\smpr}/S)$

$$Ew_S : E_{ct}(\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR}) \otimes_{O_S} E_{ct}(\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR}) \xrightarrow{\sim} E_{ct}(\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR}) \otimes_{O_S} (\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR})$$

by the functoriality of the Godement resolution (see section 2).

**Definition 151.** Let $S \in \SmVar(\mathbb{C})$. We have, for $F, G \in C(\Var(\mathbb{C})^{2,\smpr}/S)$, the canonical transformation in $C_{D^\infty_{fil}}(\SmVar(\mathbb{C}))$:

$$T(\otimes, \Omega)(F, G) :$$

$$e(S) \ast \Gr^{12}_S \Hom(\An^{*}_S F, E_{ct}(\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR})) \otimes_{O_S} e(S) \ast \Gr^{12}_S \Hom(\An^{*}_S G, E_{ct}(\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR}))$$

$$\xrightarrow{\sim} e(S) \ast \Gr^{12}_S \Hom(\An^{*}_S F, E_{ct}(\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR})) \otimes_{O_S} e(S) \ast \Gr^{12}_S \Hom(\An^{*}_S G, E_{ct}(\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR}))$$

$$\xrightarrow{T(\Hom, \otimes)(-)} e(S) \ast \Gr^{12}_S \Hom(\An^{*}_S F \otimes \An^{*}_S G, E_{ct}(\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR})) \otimes_{O_S} E_{usu}(\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR})$$

$$\xrightarrow{\sim} e(S) \ast \Gr^{12}_S \Hom(\An^{*}_S F \otimes G, E_{ct}(\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR})) \otimes_{O_S} E_{ct}(\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR})$$

$$\xrightarrow{\Hom(F \otimes G, \An_{/S_{an}}^{\smpr})} e(S) \ast \Gr^{12}_S \Hom(F \otimes G, E_{ct}(\Omega^{\Gamma,pr}_{/S_{an}}, F_{DR})).$$

Let $S \in \Var(\mathbb{C})$. Let $S = \bigcup_{i=1}^{l} S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_I : S_I \hookrightarrow \tilde{S}_I$ closed embeddings, with $\tilde{S}_I \in \Var(\mathbb{C})$. For $I \subset [1, \cdots, l]$, denote by $I = \Pi_{i \in I} S_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and for $J \subset I$ the following commutative diagram

$$D_{I,J} = \begin{array}{c}
S_I \\
\downarrow_{j_{IJ}}
\end{array} \xrightarrow{i_{IJ}} \begin{array}{c}
S_J \\
\uparrow_{p_{IJ}} \downarrow_{i_{IJ}}
\end{array} \begin{array}{c}
\tilde{S}_I \\
\downarrow_{j_{IJ}}
\end{array}$$

where $p_{IJ} : \tilde{S}_J \rightarrow \tilde{S}_I$ is the projection and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. This gives the diagram of algebraic varieties $(\tilde{S}_I) \in \Fun(\mathbb{P}(\mathbb{N}), \Var(\mathbb{C}))$ which the diagram of sites $\Var(\mathbb{C})^{2,\smpr}/(\tilde{S}_I) \in \Fun(\mathbb{P}(\mathbb{N}), \Cat)$. This gives also the diagram of algebraic varieties $(\tilde{S}_I)^{op} \in \Fun(\mathbb{P}(\mathbb{N})^{op}, \Var(\mathbb{C}))$ which the diagram of sites $\Var(\mathbb{C})^{2,\smpr}/(\tilde{S}_I)^{op} \in \Fun(\mathbb{P}(\mathbb{N})^{op}, \Cat)$. We then get

$$(\Omega^{\Gamma,pr}_{/\tilde{S}_I}, F_{DR})\lbrack-d_{\tilde{S}_I}\rbrack, T_{IJ}) \in C_{D^\infty_{fil}}(\AnSp(\mathbb{C})^{2,\smpr}/(\tilde{S}_I))$$

with

$$T_{IJ} : (\Omega^{\Gamma,pr}_{/\tilde{S}_I}, F_{DR})\lbrack-d_{\tilde{S}_I}\rbrack \xrightarrow{\ad(p^{\smpr}_{IJ})} p_{IJ}^\ast(\Omega^{\Gamma,pr}_{/\tilde{S}_I}, F_{DR}) \otimes_{p_{IJ}^\ast O_{\tilde{S}_I}} O_{\tilde{S}_J}\lbrack-d_{\tilde{S}_J}\rbrack$$

For $(G_I, K_{IJ}) \in C(\AnSp(\mathbb{C})^{2,\smpr}/(\tilde{S}_I)^{op})$, we denote (see section 2)

$$e'(\tilde{S}_I) \ast \Hom((G_I, K_{IJ}), (E_{usu}(\Omega^{\Gamma,pr}_{/\tilde{S}_I}, F_{DR})\lbrack-d_{\tilde{S}_I}\rbrack, T_{IJ})) := (e'(\tilde{S}_I) \ast \Hom(G_I, E_{ct}(\Omega^{\Gamma,pr}_{/\tilde{S}_I}, F_{DR})\lbrack-d_{\tilde{S}_I}\rbrack, u_{IJ}((G_I, K_{IJ}))) \in C_{D^\infty_{fil}}(\tilde{S}_I))$$
with

\[ u_{IJ}((G_I, K_{IJ})) : e'(\tilde{S}_I), \text{Hom}(G_I, E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \text{pr, an}}, F_{DR}))[-d_{\tilde{S}_I}] \]

\[
\frac{\text{ad}(p_{I,J}^{\bullet, \text{univ}}[-1], p_{IJ}^*) \circ T(p_{I,J}, e(-))}{p_{I,J}^* e'(\tilde{S}_I) \circ p_{IJ}^* \text{Hom}(G_I, E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \text{pr, an}}, F_{DR})) \otimes p_{I,J}^* O_{\tilde{S}_I}[-d_{\tilde{S}_I}]}
\]

\[
\frac{T(p_{IJ}, \text{hom})(-,-)}{p_{I,J}^* e'(\tilde{S}_I), \text{Hom}(p_{IJ}^* G_I, p_{IJ}^* E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \text{pr, an}}, F_{DR})) \otimes p_{I,J}^* O_{\tilde{S}_I}[-d_{\tilde{S}_I}]}
\]

\[
\frac{\text{moHom}(p_{IJ}^* G_I, T_{IJ})}{p_{I,J}^* e'(\tilde{S}_I), \text{Hom}(p_{IJ}^* G_I, E_{usu}(\Omega_{/\tilde{S}_I}^{\bullet, \text{pr, an}}, F_{DR}))[-d_{\tilde{S}_I}]}
\]

This gives in particular

\[
(\Omega_{/\tilde{S}_I}^{\bullet, \text{pr, an}}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}) \in C_{D^\infty_{\tilde{S}_I}, f_{\text{fil}}(\text{AnSp}(\mathbb{C})^{\text{sm}})^{\text{pr}}}/(\tilde{S}_I)^{\text{op}}.
\]

We now define the filtered analytic De Rahm realization functor.

**Definition 152.** (i) Let \( S \in \text{SmVar}(\mathbb{C}) \). We have, using definition 148 and definition 41, the functor

\[
F_{\text{S,an}}^{\text{FDR}} : C(\text{Var}(\mathbb{C})^{\text{sm}}/S) \to C_{D^\infty_{\tilde{S}_I}, f_{\text{fil}}(S^{\text{an}})}, F \mapsto
\]

\[
F_{\text{S,an}}^{\text{FDR}}(F) := e'(S), \text{Hom}^*(\text{An}_S^* \hat{R}^{\text{CH}}(\rho_S^* L(F)), E_{ct}(\Omega_{/\tilde{S}_I}^{\bullet, \text{pr, an}}, F_{DR}))[-d_S]
\]

\[
= e'(S), \text{Hom}^*(\hat{R}^{\text{CH}}(\rho_S^* L(F)), \text{An}_{S*} E_{ct}(\Omega_{/\tilde{S}_I}^{\bullet, \text{pr, an}}, F_{DR}))[-d_S]
\]

denoting for short \( e'(S) = e(S) \circ \text{Gr}^{12} \).

(ii) Let \( S \in \text{Var}(\mathbb{C}) \) and \( K = \bigcup_{i=1}^l S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). For \( I \subset [1, \cdots l] \), denote by \( \tilde{S}_I := \cap_{i \in I} S_i \) and \( j_I : S_I \hookrightarrow S \) the open embedding. We then have closed embeddings \( i_I : S_I \hookrightarrow \tilde{S}_I := \coprod_{i \in I} \tilde{S}_i \). Consider, for \( I \subset J \), the following commutative diagram

\[
D_{IJ} = S_I \xrightarrow{i_I} \tilde{S}_I \xrightarrow{p_{IJ}} S_J \xleftarrow{j_{IJ}} S_I
\]

and \( j_{IJ} : S_J \hookrightarrow S_I \) is the open embedding so that \( j_I \circ j_{IJ} = j_J \). We have, using definition 148 and definition 41, the functor

\[
F_{\text{S,an}}^{\text{FDR}} : C(\text{Var}(\mathbb{C})^{\text{sm}}/S) \to C_{D^\infty_{\tilde{S}_I}, f_{\text{fil}}(S^{\text{an}}/(\tilde{S}_I)^{\text{an}})}, F \mapsto
\]

\[
F_{\text{S,an}}^{\text{FDR}}(F) := e'(\tilde{S}_I), \text{Hom}^*(\text{An}_{S_I}^* \hat{R}^{\text{CH}}(\rho_{S_I}^* L(i_{I*} j_I^* F)), \hat{R}^{\text{CH}}(T^S(D_{IJ})(j_I^* F))),
\]

\[
(\text{E}_{\text{usu}}(\Omega_{/\tilde{S}_I}^{\bullet, \text{pr, an}}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))
\]

\[
= e'(\tilde{S}_I), \text{Hom}^*(\hat{R}^{\text{CH}}(\rho_{S_I}^* L(i_{I*} j_I^* F)), \text{An}_{S_I*} \text{E}_{\text{usu}}(\Omega_{/\tilde{S}_I}^{\bullet, \text{pr, an}}, F_{DR})[-d_{\tilde{S}_I}], u_{IJ}^*)(F)
\]

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where we have denoted for short $e'(\tilde{S}_I) = e(\tilde{S}_I) \circ \text{Gr}^{12}_{\tilde{S}_I}$, and

\[
\begin{align*}
    u^0_{IJ}(F)[d_{\tilde{S}_I}] : e'(\tilde{S}_I) &\ast \text{Hom}(\text{AnSp}(\tilde{S}_I) \text{RC}^H(\rho^*_{S_I}L(i_1,j_1^*F)), E_\text{at}(\Omega^\Gamma_{/\tilde{S}_I}^{\text{pr,an}}, F_{\text{DR}})) \\
    \text{ad}(p^*_{IJ}, p^*_{IJ}(\tilde{S}_I)) &\ast \text{Hom}(\text{AnSp}(\tilde{S}_I) \text{RC}^H(\rho^*_{S_I}L(i_1,j_1^*F)), E_\text{usu}(\Omega^\Gamma_{/\tilde{S}_I}^{\text{pr,an}}, F_{\text{DR}})) \\
    p_{IJ} \ast e'(\tilde{S}_I) &\ast \text{Hom}(\text{AnSp}(\tilde{S}_I) \text{RC}^H(\rho^*_{S_I}p_{IJ}^*L(i_1,j_1^*F)), E_\text{usu}(\Omega^\Gamma_{/\tilde{S}_I}^{\text{pr,an}}, F_{\text{DR}})) \\
    p_{IJ} \ast e'(\tilde{S}_I) &\ast \text{Hom}(\text{AnSp}(\tilde{S}_I) \text{RC}^H(\rho^*_{S_I}p_{IJ}^*L(i_1,j_1^*F)), E_\text{usu}(\Omega^\Gamma_{/\tilde{S}_I}^{\text{pr,an}}, F_{\text{DR}})).
\end{align*}
\]

For $I \subset J \subset K$, we have obviously $p_{IJ} \ast u_{JK}(F) = u_{IK}(F)$.

We have the following key proposition:

Recall, see section 2, that we have the projection morphisms of sites $p_a : \text{AnSp}(\mathbb{C})^{2,\text{smpr}}/(\tilde{S}_I)^{op} \to \text{AnSp}(\mathbb{C})^{2,\text{smpr}}/(\tilde{S}_I)^{op}$ given by the functor

\[
p_a : \text{AnSp}(\mathbb{C})^{2,\text{smpr}}/(\tilde{S}_I)^{op} \to \text{AnSp}(\mathbb{C})^{2,\text{smpr}}/(\tilde{S}_I)^{op},
\]

\[
p_a((Y \times \tilde{S}_I, Z_I)/\tilde{S}_I, s_{IJ}) := ((Y \times \mathbb{D}^1 \times \tilde{S}_I, Z_I \times \mathbb{D}^1)/\tilde{S}_I, s_{IJ} \times I),
\]

\[
p_a((g_I) : ((Y_I \times \tilde{S}_I, Z_I)/\tilde{S}_I, s_{IJ}) \to ((Y_I \times \tilde{S}_I, Z_I)/\tilde{S}_I, s_{IJ})) = (g_I \times I) : ((Y_I \times \mathbb{D}^1 \times \tilde{S}_I, Z_I \times \mathbb{D}^1)/\tilde{S}_I, s_{IJ} \times I) \to ((Y_I \times \mathbb{D}^1 \times \tilde{S}_I, Z_I \times \mathbb{D}^1)/\tilde{S}_I, s_{IJ} \times I)).
\]

We have the following key proposition:

**Proposition 132.** (i) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^r S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. The complex of presheaves $(\Omega^\Gamma_{/\tilde{S}_I}^{\text{pr,an}}, F_{\text{DR}}) \in C_{D^\infty_{\text{fil}}(\text{AnSp}(\mathbb{C})^{2,\text{smpr}}/(\tilde{S}_I)^{op})}$ is 2-filtered $\mathbb{D}^1$ homotopic, that is

\[
ad(p^*_{IJ}, p^*_{IJ})((\Omega^\Gamma_{/\tilde{S}_I}^{\text{pr,an}}, F_{\text{DR}}) : (\Omega^\Gamma_{/\tilde{S}_I}^{\text{pr,an}}, F_{\text{DR}}) \to p_a \ast p_a((\Omega^\Gamma_{/\tilde{S}_I}^{\text{pr,an}}, F_{\text{DR}})
\]

is a 2-filtered homotopy.

(ii) Let $S \in \text{SmVar}(\mathbb{C})$. The complex of presheaves $(\Omega^\Gamma_{/S}^{\text{pr,an}}, F_{\text{DR}}) \in C_{D^\infty_{\text{fil}}(\text{AnSp}(\mathbb{C})^{2,\text{smpr}}/S)}$ admits transfers, i.e.

\[
\text{Tr}(S), \text{Tr}(S)^* (\Omega^\Gamma_{/S}^{\text{pr,an}}, F_{\text{DR}}) = (\Omega^\Gamma_{/S}^{\text{pr,an}}, F_{\text{DR}}).
\]

(iii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_{i=1}^r S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $m = (m_1 ) : (Q_{11}, K_{11}) \to (Q_{21}, K_{21})$ be an equivalence $(\mathbb{D}^1, \text{usu})$ local with $(Q_{11}, K_{11}) \to (Q_{21}, K_{11}) \in C(\text{AnSp}(\mathbb{C})^{2,\text{smpr}}/(\tilde{S}_I)^{op})$ complexes of representable presheaves. Then, the map in $C_{D^{\infty}_{\text{fil}}(\tilde{S}_I)}$

\[
M := (e(\tilde{S}_I), \text{Hom}((\Omega^\Gamma_{/\tilde{S}_I}^{\text{pr,an}}, F_{\text{DR}})[-d_{\tilde{S}_I}])) : e'(\tilde{S}_I), \text{Hom}((Q_{21}, K_{11}), (\Omega^\Gamma_{/\tilde{S}_I}^{\text{pr,an}}, F_{\text{DR}})[-d_{\tilde{S}_I}, T_{1J}])
\]

\[
\to e'(\tilde{S}_I), \text{Hom}((Q_{11}, K_{11}), (\Omega^\Gamma_{/\tilde{S}_I}^{\text{pr,an}}, F_{\text{DR}})[-d_{\tilde{S}_I}, T_{1J}])
\]

is a 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{D^{\infty}_{\text{fil}}(\tilde{S}_I)}$. 

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Proof. Similar to proposition 113.

We deduce the following:

**Proposition 133.** Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

(i) Let $m = (m_I) : (Q_{1I}, K_{1I}^1) \rightarrow (Q_{2I}, K_{2I}^2)$ be an etale local equivalence local with $(Q_{1I}, K_{1I}^1), (Q_{2I}, K_{2I}^2) \in C(\text{Var}(\mathbb{C})^{an}/(\tilde{S}_I))$ complexes of projective sheaves. Then,
\[
\left( e'(\tilde{S}_I), \text{Hom}^* (\text{An}_S^* \check{R}^\text{CH}(\rho_S^*Q_{1I}), R^\text{CH}(K_{1I}^1)), (\text{Eusu}(\tilde{S}_I)_{/\tilde{S}_I}^* \Gamma, \text{pr}, \text{an}, \text{FDR})[-d_{\tilde{S}_I}]) \right) :
\]
\[
\left( e'(\tilde{S}_I), \text{Hom}^* (\text{An}_S^* \check{R}^\text{CH}(\rho_S^*Q_{2I}), R^\text{CH}(K_{2I}^2)), (\text{Eusu}(\tilde{S}_I)_{/\tilde{S}_I}^* \Gamma, \text{pr}, \text{an}, \text{FDR})[-d_{\tilde{S}_I}]), T_{IJ}) \right)
\]
is a filtered quasi-isomorphism. It is thus an isomorphism in $D_{\text{DR}^\text{an}*}(\tilde{S}_I)$.

(ii) Let $m = (m_I) : (Q_{1I}, K_{1I}^1) \rightarrow (Q_{2I}, K_{2I}^2)$ be an equivalence $(\mathbb{A}^1, \text{et})$ local equivalence local with $(Q_{1I}, K_{1I}^1), (Q_{2I}, K_{2I}^2) \in C(\text{Var}(\mathbb{C})^{an}/(\tilde{S}_I))$ complexes of projective sheaves. Then,
\[
\left( e'(\tilde{S}_I), \text{Hom}^* (\text{An}_S^* \check{R}^\text{CH}(\rho_S^*Q_{1I}), R^\text{CH}(K_{1I}^1)), (\text{Eusu}(\tilde{S}_I)_{/\tilde{S}_I}^* \Gamma, \text{pr}, \text{an}, \text{FDR})[-d_{\tilde{S}_I}]), T_{IJ}) \right)
\]
is a filtered quasi-isomorphism. It is thus an isomorphism in $D_{\text{DR}^\text{an}*}(\tilde{S}_I)$.

**Proof.** Similar to the proof of proposition 114.

**Definition 153.** (i) Let $S \in \text{SmVar}(\mathbb{C})$. We define using definition 152(i) and proposition 133(ii) the filtered algebraic De Rahm realization functor defined as
\[
\mathcal{F}_{S,\text{an}}^{\text{FDR}} : \text{DA}_{\text{c}}(S) \rightarrow D_{\text{DR}^\text{an}*}(\text{S}^{an}), M \mapsto \mathcal{F}_{S,\text{an}}^{\text{FDR}}(M) :=
\left( e'(S), \text{Hom}^* (\text{An}_S^* \check{R}^\text{CH}(\rho_S^*L(F)), (\text{Eusu}(\tilde{S}_I)_{/\tilde{S}_I}^* \Gamma, \text{pr}, \text{an}, \text{FDR})[-d_{\tilde{S}_I}]) \right)
\]
where $F \in C(\text{Var}(\mathbb{C})^{an}/S)$ is such that $M = D(\mathbb{A}^1, \text{et})(F)$.

(ii) For the Corti-Hanamura weight structure $W$ on $\text{DA}_{\text{c}}(S)^{-}$, we define using definition 152(ii) and proposition 133(ii)
\[
\mathcal{F}_{S,\text{an}}^{\text{FDR}} : \text{DA}_{\text{c}}^{-}(S) \rightarrow D_{\text{DR}^\text{an}*}(\text{S}^{an}), M \mapsto \mathcal{F}_{S,\text{an}}^{\text{FDR}}((M, W)) :=
\left( e'(S), \text{Hom}^* (\text{An}_S^* \check{R}^\text{CH}(\rho_S^*L(F, W)), (\text{Eusu}(\tilde{S}_I)_{/\tilde{S}_I}^* \Gamma, \text{pr}, \text{an}, \text{FDR})[-d_{\tilde{S}_I}]) \right)
\]
where $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{an}/S)$ is such that $M = D(\mathbb{A}^1, \text{et})(F, W)$ using corollary 1. Note that the filtration induced by $W$ is a filtration by sub $D_S$ module, which is a stronger property than Griffitz transversality. Of course, the filtration induced by $F$ satisfy only Griffitz transversality in general.

(ii) Let $S \in \text{Var}(\mathbb{C})$ and $S = \bigcup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset [1, \ldots, l]$, denote by $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \rightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i \in I} \tilde{S}_i$. We define, using definition 152(ii) and proposition 133(ii), the filtered algebraic De Rahm realization functor defined as
\[
\mathcal{F}_{S,\text{an}}^{\text{FDR}} : \text{DA}_{\text{c}}(S) \rightarrow D_{\text{DR}^\text{an}*}(\text{S}^{an}/(\tilde{S}_I^{an})), M \mapsto \mathcal{F}_{S,\text{an}}^{\text{FDR}}(M) :=
\left( e'(\tilde{S}_I), \text{Hom}^* (\text{An}_S^* \check{R}^\text{CH}(\rho_S^*L(i_I^*j_I^!F)), (\text{Eusu}(\tilde{S}_I)_{/\tilde{S}_I}^* \Gamma, \text{pr}, \text{an}, \text{FDR})[-d_{\tilde{S}_I}], u_{\tilde{S}_I}^{-1}F) \right)
\]
where $F \in C(\text{Var}(\mathbb{C})^{an}/S)$ is such that $M = D(\mathbb{A}^1, \text{et})(F)$, see definition 133.
(ii) For the Corti-Hanamura weight structure $W$ on $DA^-_c(S)$, using definition 133(ii) and proposition 114(ii),

\[ F_{S,an}^{FDR} \colon DA^-_c(S) \to D^{-\infty}_{(1,0)}(\mathbb{C}^n/\tilde{S}_I), M \mapsto F_{S,an}^{FDR}((M,W)) := (e'(\tilde{S}_I), \mathcal{H}om^\bullet(\text{An}^\bullet_{\tilde{S}_I} \tilde{R}CH(\rho_{S_I}^* L(i_* j_I^*(F,W))), E_{usu}(\Omega^{\bullet,pr,an}_{/S_I}, F_{DR})[-d_{\tilde{S}_I}], u_j^*(F,W)) \]

where $(F,W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $(M,W) = D(A^3, \text{et})(F,W)$ using corollary 1. Note that the filtration induced by $W$ is a filtration by sub $D_{\tilde{S}_I}$-modules, which is a stronger property than Griffiths transversality. Of course, the filtration induced by $F$ satisfy only Griffiths transversality in general.

**Proposition 134.** For $S \in \text{Var}(\mathbb{C})$ and $S = \bigcup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$, the functor $F_{S,an}^{FDR}$ is well defined.

**Proof.** Similar to the proof of proposition 115. \[ \square \]

**Proposition 135.** Let $f : X \to S$ a morphism with $S,X \in \text{Var}(\mathbb{C})$. Assume there exist a factorization

\[ f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S \]

of $f$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $\bar{Y} \in \text{PSmVar}(\mathbb{C})$ a compactification of $Y$ with $\bar{Y} \setminus Y = D$ a normal crossing divisor, denote $k : D \hookrightarrow \bar{Y}$ the closed embedding and $n : Y \hookrightarrow \bar{Y}$ the open embedding. Denote $\bar{X} \subset \bar{Y} \times S$ the closure of $X \subset \bar{Y} \times S$. We have then the following commutative diagram in $\text{Var}(\mathbb{C})$

\[
\begin{array}{ccc}
X & \xrightarrow{l} & Y \times S \\
\downarrow{l_1} & & \downarrow{(n \times I) \circ p_S} \\
\bar{X} & \xrightarrow{l_2} & \bar{Y} \times S \\
\downarrow{(k \times I)} & & \downarrow{p_S} \\
Z := \bar{X} \setminus X & \to & D \times S \\
\end{array}
\]

Let $S = \bigcup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.
Then $X = \bigcup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset \{1, \cdots, l\}$, $S_I = \bigcap_{i \in I} S_i$ and $X_I = \bigcap_{i \in I} X_i$. Denote $X_I := \bar{X} \setminus (\bar{Y} \times S_I)$ the closure of $X_I \subset \bar{Y} \times S_I$, and $Z_I := Z \cap (\bar{Y} \times S_I) = \bar{X}_I \setminus X_I \subset \bar{Y} \times S_I$.
We have then for $I \subset \{1, \cdots, l\}$, the following commutative diagram in $\text{Var}(\mathbb{C})$

\[
\begin{array}{ccc}
X_I & \xrightarrow{l_I} & Y \times S_I \\
\downarrow{l_{1I}} & & \downarrow{(n \times I) \circ p_{S_I}} \\
\bar{X}_I & \xrightarrow{l_{2I}} & \bar{Y} \times S_I \\
\downarrow{(k \times I)} & & \downarrow{p_{S_I}} \\
Z_I = \bar{X}_I \setminus X_I & \to & D \times S_I \\
\end{array}
\]
Let $F(X/S) := p_{S,i} \Gamma_{\tilde{S}_i}^{\omega}(X \times S/X \times S)$. We have then the following isomorphism in $D_{DFil}(S/(\tilde{S}_1))$

$$I(X/S) : F_{S,an}^{DR}(M(X/S)) \xrightarrow{\cong}$$

$$(e'_i \text{Hom}(\text{An}^*_{\tilde{S}_i} \tilde{R}^{CH}(\rho_{\tilde{S}_i}^* L(i_1)_i^j F(X/S))), E_{usu}(\Omega^{\bullet, pr.an}_{/\tilde{S}_i}, F_{DR}))[-d_{\tilde{S}_i}], u_{ij}^q(F(X/S)))$$

$$(\text{Hom}(\text{An}^*_{\tilde{S}_i} \tilde{R}^{CH}(N_1(X/S)), E_{usu}(\Omega^{\bullet, pr.an}_{/\tilde{S}_i}, F_{DR}))) \xrightarrow{\cong}$$

$$(e'_i \text{Hom}(\text{An}^*_{\tilde{S}_i} \tilde{R}^{CH}(\rho_{\tilde{S}_i}^* Q(X_1/\tilde{S}_i)), E_{usu}(\Omega^{\bullet, pr.an}_{/\tilde{S}_i}, F_{DR}))[-d_{\tilde{S}_i}], v_{ij}^q(F(X/S)))$$

$$(\tilde{p}_{\tilde{S}_i}, E_{usu}((\Omega^{\bullet}_{X_{\tilde{S}_i}/\tilde{S}_i}, F_b) \otimes O_{X_{\tilde{S}_i}}((n \times I)^{Hdg} \tilde{F}^{\rho, Hdg}_{X_{\tilde{S}_i}}(O_{\tilde{S}_i}^{an}, F_b))(dV + d_{\tilde{S}_i} | 2dV + d_{\tilde{S}_i}, w_{ij}(X/S)))$$

$\xrightarrow{=} \iota_S R^j_{Hdg} \Gamma^{Hdg}_{X_{\tilde{S}_i}}(O_{\tilde{S}_i}^{an}, F_b)(dV), x_{ij}(X/S)) \xrightarrow{=} \iota_S R^j_{Hdg} j_{remod,hdg}^{-1} \pi_{a,S}(MHM(S^{an}))$$

**Proof.** Similar to the proof of proposition 116. \qed

**Corollary 5.** Let $S \in \text{Var}(\mathbb{C})$ and $S = \bigcup_{i=1}^n S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $F \in C(\text{Var}(\mathbb{C})^{an}/S)$ such that $M = D(\mathbb{A}^1, et)(F) \in \text{DA}_c(S)$,

$$H^i F_{S,an}^{DR}(M, W) := (a_{usu} H^i e'_{\tilde{S}_i})_{*} \text{Hom}(\text{An}^*_{\tilde{S}_i} \tilde{R}^{CH}(\rho_{\tilde{S}_i}^* L(i_1)_i^j F(W))),$$

$$E_{usu}(\Omega^{\bullet, pr.an}_{/\tilde{S}_i}, F_{DR})[-d_{\tilde{S}_i}], H^i a_{ij}^q(F(W)) \in \pi_S(MHM(S^{an}))$$

for all $i \in \mathbb{Z}$, and for all $p \in \mathbb{Z},$

$$F_{S,an}^{DR}(M, W) \in D_{DFil}^{-1}(S/(\tilde{S}_1))$$

is the class of a complex $F_{S,an}^{DR}(F, W) \in C_{D^{+1}(1.0)fil}(S/(\tilde{S}_1))$ such that for all $k \in \mathbb{Z}$, the differentials of $\text{Gr}^W F_{S,an}^{DR}(F, W)$ are strict for the filtration $F$.

**Proof.** Similar to the proof of corollary 4. \qed

**Proposition 136.** For $S \in \text{Var}(\mathbb{C})$ not smooth, the functor (see corollary 4)

$$\iota_S^{-1} F_{S,an}^{DF} : \text{DA}_c^{-}(S)^{op} \rightarrow \pi_S(D(MHM(S^{an})))$$

does not depend on the choice of the open cover $S = \bigcup S_i$ and the closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$.

**Proof.** Similar to the proof of proposition 117. \qed

We have the canonical transformation map between the filtered analytic De Rham realization functor and the analytic Gauss-Manin realization functor:

**Definition 154.** Let $S \in \text{Var}(\mathbb{C})$ and $S = \bigcup_{i=1}^n S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{an}/S)$ such that
M = D(\mathbb{A}^1, et)(F). We have, using definition 148(ii), the canonical map in \(D_{\text{S,fil},D^{\infty,\infty}}(S^a/(\tilde{S}_j^n))\)

\[T(F_{S,an}, F_{\text{S,DR}})(M) : \]

\[\mathcal{F}^\text{GM}_{S,an}(L \otimes_{S} M) := (e(\tilde{S}_1), \text{Hom}^\bullet(\mathbb{A}_{\tilde{S}_1}^{\text{an}} \otimes L(i_1, j_1) \otimes S, LF), E_{\text{usu}}(\Omega^{\Gamma, \text{pr, an}}_{/\tilde{S}_1}(F), u_{IJ}^d(F)) \]

\[\xrightarrow{\sim} (e(\tilde{S}_1), \text{Hom}^\bullet(\mathbb{A}_{\tilde{S}_1}^{\text{an}} \otimes L(i_1, j_1)^L, E_{\text{usu}}(\Omega^{\Gamma, \text{pr, an}}_{/\tilde{S}_1}(F), u_{IJ}^d(F)) \]

\[\text{Hom}^{-1}(\text{Gr}(\Omega_{/\tilde{S}_1}^a)) \quad J_S(e(\tilde{S}_1), \text{Hom}^\bullet(\mathbb{A}_{\tilde{S}_1}^{\text{an}} \otimes L(\text{Gr}^\text{CH} \cdot \text{Gr}^\text{CH}(L(i_1, j_1)^F), F_{\text{DR}})), u_{IJ}^d(F)) \]

\[J_S(e(\tilde{S}_1), \text{Hom}^\bullet(\mathbb{A}_{\tilde{S}_1}^{\text{an}} \otimes L(\text{Gr}^\text{CH} \cdot \text{Gr}^\text{CH}(L(i_1, j_1)^F), F_{\text{DR}})), u_{IJ}^d(F)) \]

\[J_S(e(\tilde{S}_1), \text{Hom}^\bullet(\mathbb{A}_{\tilde{S}_1}^{\text{an}} \otimes L(\text{Gr}^\text{CH} \cdot \text{Gr}^\text{CH}(L(i_1, j_1)^F), F_{\text{DR}})), u_{IJ}^d(F)) \]

We now define the functorialities of \(F_{S,\text{DR}}\) with respect to \(S\) which makes \(F_{\text{DR}}\) a morphism of 2 functor.

**Definition 155.** Let \(S \in \text{Var}(\mathbb{C})\). Let \(Z \subset S\) a closed subset. Let \(S = \bigcup_{i=1}^d S_i\) an open cover such that there exist closed embeddings \(i_i : S_i \hookrightarrow S\) with \(S_i \in \text{SmVar}(\mathbb{C})\). Denote \(Z_i := Z \cap S_i\). We then have closed embeddings \(Z_i \hookrightarrow S_i \hookrightarrow S\).

(i) For \(F \in C(\text{Var}(\mathbb{C})^{an}/S), S\), we will consider the following canonical map in \(\pi_S(D(MHM(S^a))) \subset D_{\text{D}(1,0)fil}(S^a/(\tilde{S}_j^n))\)

\[T(\Gamma_{\tilde{S}_j}^{\Gamma, \text{pr, an}}(F, W) : \]

\[\Gamma_{\tilde{S}_j}^{\text{pr}^j}(e(\tilde{S}_i), \text{Hom}^\bullet(\mathbb{A}_{\tilde{S}_i}^{\text{an}} \otimes \hat{R}_{\tilde{S}_i} \cdot \text{Gr}^\text{CH}(L(i_1, j_1)^F, W), E_{\text{usu}}(\Omega^{\Gamma, \text{pr, an}}_{/\tilde{S}_i}(F, W)), u_{IJ}^d(F, W)) \]

\[\xrightarrow{\sim} (e(\tilde{S}_i), \text{Hom}^\bullet(\mathbb{A}_{\tilde{S}_i}^{\text{an}} \otimes \hat{R}_{\tilde{S}_i} \cdot \text{Gr}^\text{CH}(L(i_1, j_1)^F, W), E_{\text{usu}}(\Omega^{\Gamma, \text{pr, an}}_{/\tilde{S}_i}(F, W)), u_{IJ}^d(F, W)) \)

with \(u_{IJ}^Z(F)\) given as in definition 136(ii).

(ii) For \(F \in C(\text{Var}(\mathbb{C})^{an}/S), S\), we have also the following canonical map in \(\pi_S(D(MHM(S^a))) \subset D_{\text{D}(1,0)fil}(S^a/(\tilde{S}_j^n))\)

\[T(\Gamma_{\tilde{S}_j}^{\text{pr}^j}(F, W) : \]

\[\Gamma_{\tilde{S}_j}^{\text{pr}^j}(e(\tilde{S}_i), \text{Hom}^\bullet(\mathbb{A}_{\tilde{S}_i}^{\text{an}} \otimes \hat{R}_{\tilde{S}_i} \cdot \text{Gr}^\text{CH}(L(i_1, j_1)^F, W), E_{\text{usu}}(\Omega^{\Gamma, \text{pr, an}}_{/\tilde{S}_i}(F, W)), u_{IJ}^d(F, W)) \]

with \(u_{IJ}^Z(F)\) given as in definition 136(ii).

**Definition 156.** Let \(g : T \to S\) a morphism with \(T, S \in \text{Var}(\mathbb{C})\). Assume we have a factorization \(g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S\) with \(Y \in \text{SmVar}(\mathbb{C}), l \) a closed embedding and \(p_S\) the projection. Let
Let \( f : X \to S \) a morphism with \( X, S \in \var{C} \). Assume there exist a factorization map \( f : X \to Y \times S \) with \( Y \in \var{C} \), \( l \) a closed embedding and \( \pi_S \) the projection. We have, for \( M \in \DA_c(X) \), the following factorization map in \( \pi_S(D(MHM(S))) \)

\[
T_* \left( f, \mathcal{F} \right)(M) : \mathcal{F} \to \mathcal{F} \left( f^* \right)(M) \xrightarrow{\ad(f^* \text{mod} \ R f^*_H \text{Hdg})} R f^* \mathcal{F} \left( f^* \right)(M) = \mathcal{F} \left( f^* \right)(M) \xrightarrow{\Delta} \mathcal{F} \to \mathcal{F} \left( f^* \right)(M) = \mathcal{F} \left( f^* \right)(M)
\]

Clearly, for \( p : Y \to S \) a projection with \( Y \in \var{C} \), we have, for \( M \in \DA_c(Y) \times S \),

\[
T_* \left( p, \mathcal{F} \right)(M) = T_* \left( p, \mathcal{F} \right)(M)
\]

Let \( S \in \var{C} \). Let \( Y \in \var{C} \) and \( p : Y \to S \) the projection. We have then, for \( M \in \DA_c(Y) \times S \),

\[
T_* \left( p, \mathcal{F} \right)(M) : \mathcal{F} \left( f \right)(M) \xrightarrow{\ad(f^* \text{mod} \ R f^*_H \text{Hdg})} R f^* \mathcal{F} \left( f^* \right)(M) = \mathcal{F} \left( f^* \right)(M) \xrightarrow{\Delta} \mathcal{F} \to \mathcal{F} \left( f^* \right)(M) = \mathcal{F} \left( f^* \right)(M)
\]

Let \( f : X \to S \) a morphism with \( X, S \in \var{C} \). Assume there exist a factorization \( f : X \to Y \times S \) with \( Y \in \var{C} \), \( l \) a closed embedding and \( \pi_S \) the projection. We have then, using
Let \( f : X \to S \) a morphism with \( X, S \in \text{Var}(\mathbb{C}) \). Assume there exist a factorization \( f : X \overset{l}{\to} Y \times S \overset{p_S}{\to} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. We have, using the third point, for \( M \in \text{DA}(S) \), the following transformation map in \( \pi_X(D(MHM(X^{an}))) \)

\[
T_l(f, F^{\text{FDR}}_{\text{an}})(M) : R^{hdg} F^{\text{FDR}}_{X}(M) := R^{hdg} F^{\text{FDR}}_{Y \times S, an}(l_*M)
\]

\[
\xrightarrow{T(l, F^{\text{FDR}}_{\text{an}})(l_*M)} F^{\text{FDR}}_{S, an}(Lp_1 l_*M) \xrightarrow{=} F^{\text{FDR}}_{S, an}(Rf_*M)
\]

Let \( S \in \text{Var}(\mathbb{C}) \). Let \( S = \bigcup_{i=1}^k S_i \) an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). We have, using the preceding point, denoting \( \Delta_S : S \to S \) the diagonal closed embedding and \( p_1 : S \times S \to S \), \( p_2 : S \times S \to S \) the projections, for \( M, N \in \text{DA}(S) \) and \( (F, W), (G, W) \in C_{fd}(\mathbb{C}^{s_{\text{an}}}) \) such that \( (M, W) = D(A, \text{et})(F, W) \) and \( (N, W) = D(A, \text{et})(G, W) \), the following transformation map in \( \pi_S(D(MHM(S))) \)

\[
T(f, F^{\text{FDR}}_{\text{an}})(M, N) : F^{\text{FDR}}_{S, an}(f_1^*(M), f_2^*(N)) \xrightarrow{\Delta_S^{hdg}(p_1^{mod} F^{\text{FDR}}_{S, an}(M) \otimes O_S \otimes p_2^{mod} F^{\text{FDR}}_{S, an}(N))}
\]

\[
\xrightarrow{\Delta_S^{hdg}(F_{S, an}(p_1^{mod} \otimes p_2^{mod} N))} F^{\text{FDR}}_{S, an}(\Delta_S^{\text{mod}}(p_1^{mod} M \otimes p_2^{mod} N)) = F^{\text{FDR}}_{S, an}(M \otimes N)
\]

Proposition 137. Let \( g : T \to S \) a morphism with \( T, S \in \text{Var}(\mathbb{C}) \). Assume we have a factorization \( g : T \overset{l}{\to} Y_2 \times S \overset{p_S}{\to} S \) with \( Y_2 \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. Let \( S = \bigcup_{i=1}^k S_i \) be an open cover such that there exist closed embeddings \( i_i : S_i \to \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \). Then, \( T = \bigcup_{i=1}^k T_i \) with \( T_i := g^{-1}(S_i) \) and we have closed embeddings \( i'_i : i_i : S_i \to \tilde{S}_i \). Moreover \( \tilde{g}_i : X \times \tilde{S}_i \to \tilde{S}_i \) is a lift of \( g_i : T_i \to S_i \). Let \( f : X \to S \) a morphism with \( X \in \text{Var}(\mathbb{C}) \) such that there exists a factorization \( f : X \overset{l}{\to} Y_1 \times S \overset{p_S}{\to} S \), with \( Y_1 \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. We have then the following commutative diagram whose squares are cartesian

\[
\begin{array}{ccc}
\tilde{X}_i \subset \tilde{Y}_i \to \tilde{S}_i & \text{the closure of } X_i, & \text{and} \\
\end{array}
\]

\[
\begin{array}{ccc}
T(f, g, F(X/S)) : g^* F(X/S) := p_{S,0}^* \Gamma_X^* \mathcal{Z}(Y_1 \times S/Y_1 \times S) \xrightarrow{\cong} p_{T,0}^* \Gamma_X^* \mathcal{Z}(X_T \times T/Y_1 \times T) := F(X_T/T).
\end{array}
\]

Take a smooth compactification \( \bar{Y}_1 \in \text{PSmVar}(\mathbb{C}) \) of \( Y_1 \), denote \( \bar{X}_1 \subset \bar{Y}_1 \to \tilde{S}_i \) the closure of \( X_1 \), and \( \bar{Z}_1 := \bar{X}_1 \setminus X_1 \). Consider \( F(X/S) := p_{S,0}^* \Gamma_X^* \mathcal{Z}(Y_1 \times S/Y_1 \times S) \in C(\text{Var}(\mathbb{C})^{s_{\text{an}}}/S) \) and the isomorphism in \( C(\text{Var}(\mathbb{C})^{s_{\text{an}}}/T) \)

\[
\begin{array}{ccc}
T(f, g, F(X/S)) : g^* F(X/S) := p_{S,0}^* \Gamma_X^* \mathcal{Z}(Y_1 \times S/Y_1 \times S) \xrightarrow{\cong} p_{T,0}^* \Gamma_X^* \mathcal{Z}(X_T \times T/Y_1 \times T) := F(X_T/T).
\end{array}
\]
which gives in DA(T) the isomorphism $T(f, g, F(X/S)) = g^*M(X/S) \cong M(X/T)$. Then the following diagram in $\pi_T(D(MHM(T))) \subset D_{D(1,0)}(T/Y_2 \times \hat{S}_1)$, where the horizontal maps are given by proposition 135, commutes

$$\begin{array}{ccc}
g_{\hat{H}dg}^{mod, i} F_{S,an}^{FDR}(M(X/S)) & \overset{g_{\hat{H}dg}^{mod, i}(X/S)}{\longrightarrow} & g_{\hat{H}dg}^{mod, i}(X/S) \\
\downarrow T(g, f_{FDR})(M(X/S)) & & \downarrow T(p_{\hat{H}dg, \pi_{\hat{H}dg}^{mod, i}}) \\
 i_{\hat{H}dg, \pi_{\hat{H}dg}^{mod, i}} F_{T,an}^{FDR}(M(X_I/T)) & \overset{I(X_I/T)}{\longrightarrow} & F_{T,an}^{FDR}(M(X_I/T))
\end{array}$$

with $d_{Y_2} = d_{Y_1} + d_{Y_2}$.

Proof. Follows immediately from definition.

$\square$

**Proposition 138.** Let $S \in \text{Var}(\mathbb{C})$. Let $Y \in \text{SmVar}(\mathbb{C})$ and $p: Y \times S \rightarrow S$ the projection. Let $S = \bigcup_i S_i$ an open cover such that there exist closed embeddings $i_i: S_i \hookrightarrow S$ with $S_i \in \text{SmVar}(\mathbb{C})$. For $I \in [1, \ldots, l]$, we denote by $S_I = \bigcap_i S_i$, $j_I: S_I \hookrightarrow S$ and $j_I: Y \times S_I \rightarrow Y \times S$ the open embeddings.

Then, the following diagram in $\pi_S(D(MHM(S^an))) \subset D_{D(1,0)}(S^an/(Y^an \times \hat{S}_1^{an}))$, where the vertical maps are given by proposition 135, commutes

$$\begin{array}{ccc}
R^{\hat{H}dg}_{I,an} F_{S,an}^{FDR}(M(X'/Y \times S)) & \overset{T(p_{\hat{H}dg, \pi_{\hat{H}dg}^{mod, i}})(I(X'/Y \times S))}{\longrightarrow} & F_{S,an}^{FDR}(M(X'/Y \times S)) \\
\downarrow R^{\hat{H}dg}_{I'} F_{T,an}^{FDR}(M(X'/Y \times S)) & & \downarrow I(X'/Y \times S)
\end{array}$$

Proof. Immediate from definition.

$\square$

**Proposition 139.** Let $f_1: X_1 \rightarrow S$, $f_2: X_2 \rightarrow S$ two morphisms with $X_1, X_2, S \in \text{Var}(\mathbb{C})$. Assume that there exist factorizations $f_1: X_1 \overset{l_1}{\rightarrow} Y_1 \times S \overset{p_S}{\rightarrow} S$, $f_2: X_2 \overset{l_2}{\rightarrow} Y_2 \times S \overset{p_S}{\rightarrow} S$ with $Y_1, Y_2 \in \text{SmVar}(\mathbb{C})$, $l_1, l_2$ closed embeddings and $p_S$ the projections. We have then the factorization

$$f_{12} := f_1 \times f_2: X_{12} := X_1 \times S X_2 \overset{l_1 \times l_2}{\rightarrow} Y_1 \times Y_2 \times S \overset{p_S}{\rightarrow} S$$

Let $S = \bigcup_i S_i$ an open affine covering and denote, for $I \in [1, \ldots, l]$, $S_I = \bigcap_i S_i$ and $j_I: S_I \hookrightarrow S$ the open embedding. Let $i_i: S_i \hookrightarrow S_I$ closed embeddings, with $S_i \in \text{SmVar}(\mathbb{C})$. We have then the following commutative diagram in $\pi_S(D(MHM(S^an))) \subset D_{D(1,0)}(S^an/(\hat{S}_1^{an}))$ where the vertical maps are given by proposition 135

$$\begin{array}{ccc}
F_{S,an}^{FDR}(M(X_1/S)) \otimes_{O_{X_1/S}} F_{S,an}^{FDR}(M(X_2/S)) & \overset{I(X_1/S) \otimes I(X_2/S)}{\longrightarrow} & F_{S,an}^{FDR}(M(X_1/S) \otimes M(X_2/S) = M(X_1 \times S X_2/S)) \\
\downarrow T(F_{S,an}^{FDR}(M(X_1/S), M(X_2/S)) & & \downarrow I(X_1/S)
\end{array}$$

$$\begin{array}{ccc}
R^{\hat{H}dg}_{I_1} F_{S,an}^{FDR}(M(X_1/S)) & \overset{R^{\hat{H}dg}_{I_1} F_{S,an}^{FDR}(M(X_1/S))}{\longrightarrow} & F_{S,an}^{FDR}(M(X_1/S)) \\
\downarrow R^{\hat{H}dg}_{I_2} F_{S,an}^{FDR}(M(X_1/S)) & & \downarrow I(X_1/S)
\end{array}$$

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Proof. Immediate from definition. \hfill \Box

**Theorem 38.**  \( (i) \) Let \( g : T \to S \) a morphism, with \( S,T \in \text{Var}(\mathbb{C}) \). Assume we have a factorization \( g : T \xrightarrow{l} Y \times S \xrightarrow{pr_2} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( pr_2 \) the projection. Let \( M \in \text{DA}_c(S) \). Then map in \( \pi_T(D(MHM(T^{an}))) \)

\[
T(g_*^{\text{FDR}}) : g_{Hdg}^* \mathcal{F}_{S,an}^\text{FDR}(M) \xrightarrow{\sim} \mathcal{F}_{T,an}^\text{FDR}(g^*M)
\]
given in definition 156 is an isomorphism.

\( (ii) \) Let \( f : X \to S \) a morphism with \( X,S \in \text{Var}(\mathbb{C}) \). Assume there exist a factorization \( f : X \xrightarrow{l} Y \times S \xrightarrow{pr_2} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( pr_2 \) the projection. Then, for \( M \in \text{DA}_c(X) \),

\[
T_!(f_*^{\text{FDR}}) : Rf_1^Hdg \mathcal{F}_{X,an}^\text{FDR}(M) \xrightarrow{\sim} \mathcal{F}_{S,an}^\text{FDR}(Rf_1^*M)
\]
is an isomorphism in \( \pi_S(D(MHM(S^{an}))) \).

\( (iii) \) Let \( f : X \to S \) a morphism with \( X,S \in \text{Var}(\mathbb{C}) \), \( S \) quasi-projective. Assume there exist a factorization \( f : X \xrightarrow{l} Y \times S \xrightarrow{pr_2} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( pr_2 \) the projection. We have, for \( M \in \text{DA}_c(X) \),

\[
T_!(f_*^{\text{FDR}}) : \mathcal{F}_{S,an}^\text{FDR}(f_1^*M) \xrightarrow{\sim} Rf_1^Hdg \mathcal{F}_{X,an}^\text{FDR}(M)
\]
is an isomorphism in \( \pi_X(D(MHM(X^{an}))) \).

\( (iv) \) Let \( f : X \to S \) a morphism with \( X,S \in \text{Var}(\mathbb{C}) \), \( S \) quasi-projective. Assume there exist a factorization \( f : X \xrightarrow{l} Y \times S \xrightarrow{pr_2} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( pr_2 \) the projection. Then, for \( M \in \text{DA}_c(S) \)

\[
T_!(f_*^{\text{FDR}}) : \mathcal{F}_{S,an}^\text{FDR}(f_1^*M) \xrightarrow{\sim} f_{Hdg}^* \mathcal{F}_{X,an}^\text{FDR}(M)
\]
is an isomorphism in \( \pi_X(D(MHM(X^{an}))) \).

\( (v) \) Let \( S \in \text{Var}(\mathbb{C}) \) and \( S = \bigcup_{i=1}^k S_i \) an open affine covering and denote, for \( I \subset [1, \ldots, k] \), \( S_I = \cap_{i \in I} S_i \) and \( j_I : S_I \to S \) the open embedding. Let \( i_I : S_I \to \tilde{S}_I \) closed embeddings, with \( \tilde{S}_I \in \text{SmVar}(\mathbb{C}) \).

Then, for \( M,N \in \text{DA}_c(S) \), the map in \( \pi_S(D(MHM(S^{an}))) \)

\[
T(\mathcal{F}_{S,an}^\text{FDR}, \otimes)(M,N) : \mathcal{F}_{S,an}^\text{FDR}(M) \otimes_{Hdg}^\text{FDR} \mathcal{F}_{S,an}^\text{FDR}(N) \xrightarrow{\sim} \mathcal{F}_{S,an}^\text{FDR}(M \otimes N)
\]
given in definition is an isomorphism \( \pi_S(D(MHM(S^{an}))) \).

Proof. (i):Follows from proposition 137 and proposition 135.

(ii),(iii),(iv):Similar to the proof of theorem 36.

(v):Follows from proposition 139. \hfill \Box

**Proposition 140.** Let \( g : T \to S \) a morphism with \( T,S \in \text{Var}(\mathbb{C}) \). Assume we have a factorization \( g : T \xrightarrow{l} Y \times S \xrightarrow{pr_2} S \) with \( Y \in \text{SmVar}(\mathbb{C}) \), \( l \) a closed embedding and \( pr_2 \) the projection. Let \( S = \bigcup_{i=1}^k S_i \) be an open cover such that there exists closed embeddings \( i_i : S_i \to \tilde{S}_i \) with \( \tilde{S}_i \in \text{SmVar}(\mathbb{C}) \).

Then, \( T = \bigcup_{i=1}^k T_i \) with \( T_i := i_i^{-1}(S_i) \) and we have closed embeddings \( i_i' := i_i \circ l : T_i \to Y \times S_i \).

Moreover \( g_* := g_{|T_i} : T_i \to S_{T_i} \) is a lift of \( g_T := g_{|T} : T \to S \).

Let \( M \in \text{DA}_c(S) \) and \( F \in \mathcal{C}(\mathbb{C})^{an} / S \) such that \( M = D(\mathcal{A}_1^{an}, et)(F) \).

Then, \( D(\mathcal{A}_1^{an}, et)(g^*F) = g^*M \). Then the following diagram in \( D_{\text{Ofil}_T, \mathbb{C}^{an}}(T^{an} / (Y^{an} \times S_{T_i}^{an})) \) commutes

\[
\begin{array}{ccc}
Rg_!^{\text{FDR}}(\mathcal{F}_{S,an}^G \otimes \mathcal{F}_{S,an}^\text{FDR}(M)) & \xrightarrow{\phi_{\text{mod}}(\cdot)} & \mathcal{F}_{T,an}^{\text{FDR}}(g^*M) \\
T(\mathcal{F}_{S,an}^G, \mathcal{F}_{S,an}^\text{FDR}(M)) & \xrightarrow{\phi_{\text{mod}}(\cdot)} & \mathcal{F}_{T,an}^{\text{FDR}}(g^*M) \\
\end{array}
\]
Proof. Similar to the proof of proposition 122.

We have the following easy proposition

**Proposition 141.** Let \( S \in \Var(\mathbb{C}) \) and \( S = \bigcup_{i=1}^{l} S_i \) an open affine covering and denote, for \( I \subset [1, \ldots, l] \), \( S_I = \cap_{i \in I} S_i \) and \( j_I : S_I \hookrightarrow S \) the open embedding. Let \( i_I : S_I \hookrightarrow \tilde{S}_I \) closed embeddings, with \( \tilde{S}_I \in \Sm\Var(\mathbb{C}) \). We have, for \( M, N \in \DA(S) \) and \( F, G \in \C(\Var(\mathbb{C})^{sm}/S) \) such that \( M = D(h^1, et)(F) \) and \( N = D(h^1, et)(G) \), the following commutative diagram in \( \D_{\OS,fil, \D^{\infty}}(\S^{an}/(\S^{an}_I)) \):

\[
\begin{array}{cccccc}
\mathcal{F}_{S,an}(\LD_S M) \otimes_{\OS} \mathcal{F}_{S,an}(\LD_S N) & \xrightarrow{T(\mathcal{F}_{S,an}^{GM}(\D_S M) \otimes_{\OS} \mathcal{F}_{S,an}^{GM}(\D_S N))} & \mathcal{F}_{S,an}(\D_S^{GM}(M \otimes N)) \\
T(\mathcal{F}_{S,an}^{GM}(\D_S M) \otimes_{\OS} \mathcal{F}_{S,an}^{GM}(\D_S N)) & & T(\mathcal{F}_{S,an}^{GM}(\D_S^{GM}(M \otimes N))
\end{array}
\]

**Proof.** Immediate from definition.

### 6.3 The transformation map between the analytic De Rahm functor and the analytification of the algebraic De Rahm functor

#### 6.3.1 The transformation map between the analytic Gauss Manin realization functor and the analytification of the algebraic Gauss Manin realization functor

Recall from section 2 that, for \( f : T \to S \) a morphism with \( T, S \in \Var(\mathbb{C}) \), we have the following commutative diagram of sites (42):

\[
\begin{array}{cccccc}
\text{AnSp}(\mathbb{C})/T^{an} & \xrightarrow{\text{An}_T} & \Var(\mathbb{C})/T \\
\text{AnSp}(\mathbb{C})^{sm}/T^{an} & \xrightarrow{\text{An}_T^{sm}} & \Var(\mathbb{C})^{sm}/T \\
\text{AnSp}(\mathbb{C})/S^{an} & \xrightarrow{\text{An}_S} & \Var(\mathbb{C})/S \\
\text{AnSp}(\mathbb{C})^{sm}/S^{an} & \xrightarrow{\text{An}_S^{sm}} & \Var(\mathbb{C})^{sm}/S
\end{array}
\]

This gives for \( s : \mathcal{I} \to \mathcal{J} \) a functor with \( \mathcal{I}, \mathcal{J} \in \Cat \) and \( f : T \bullet \to S \bullet \) a morphism of diagram of algebraic varieties with \( T \bullet \in \Fun(\mathcal{I}, \Var(\mathbb{C})) \), \( S \bullet \in \Fun(\mathcal{J}, \Var(\mathbb{C})) \) the commutative diagram of sites (45):

\[
\begin{array}{cccccc}
\text{Dia}(S) := \text{AnSp}(\mathbb{C})/T^{an} & \xrightarrow{\text{An}_T} & \Var(\mathbb{C})/T \\
\text{AnSp}(\mathbb{C})^{sm}/T^{an} & \xrightarrow{\text{An}_T^{sm}} & \Var(\mathbb{C})^{sm}/T \\
\text{AnSp}(\mathbb{C})/S^{an} & \xrightarrow{\text{An}_S} & \Var(\mathbb{C})/S \\
\text{AnSp}(\mathbb{C})^{sm}/S^{an} & \xrightarrow{\text{An}_S^{sm}} & \Var(\mathbb{C})^{sm}/S
\end{array}
\]
Lemma 18. We have the following canonical transformation map given by the pullback of (relative) differential forms: Let \( S \in \text{Var}(\mathbb{C}) \). Consider the following commutative diagram in \( \text{RCat} \):

\[
\begin{array}{ccc}
D(an, e) : (\text{AnSp}(\mathbb{C})^{sm}/S^{an}, O_{\text{AnSp}(\mathbb{C})^{sm}/T}) & \xrightarrow{\text{An}_S} & (\text{Var}(\mathbb{C})^{sm}/S, O_{\text{Var}(\mathbb{C})^{sm}/S}) \\
\text{e}(T) & \downarrow & \text{e}(S) \\
(S^{an}, O_{S^{an}}) & \xrightarrow{\text{an}_S} & (S, O_S)
\end{array}
\]

It gives (see section 2) the canonical morphism in \( C_{an}_S O S (\text{AnSp}(\mathbb{C})^{sm}/S^{an}) \)

\[
\Omega/(S^{an}/S) := \Omega/O_{\text{AnSp}(\mathbb{C})^{sm}/S^{an}}/\text{An}_S e^*O_S, F_b) \rightarrow (\Omega^*_{S^{an}}, F_b) = (\Omega^*_{O_{\text{AnSp}(\mathbb{C})^{sm}/S^{an}}/e(S^{an})^*O_{S^{an}}, F_b})
\]

which is by definition given by the analytification on differential forms: for \((V/S^{an}) = (V, h) \in \text{AnSp}(\mathbb{C})^{sm}/S^{an},

\[
\Omega/(S^{an}/S)(V/S^{an}) : \dot{\omega} \in \text{An}_S^* (\Omega^*_S)(V/S^{an}) := \lim_{\nu} \Omega^*_S(U/S) \Rightarrow \Omega(V(U))/(S^{an}/S)(V/S^{an})(\omega) := \text{an}_S^* (\omega) \in \Omega_S^*(V/S^{an});
\]

with \( \omega \in \Gamma(U, \Omega^*_S) \) such that \( q(\omega) = \dot{\omega} \). If \( S \in \text{SmVar}(\mathbb{C}), \) the map \( \Omega/(T/S) : \text{An}_S^* \Omega^*_S \rightarrow \Omega^*_{S^{an}} \) is a map in \( C_{O_{S^{an}}} \text{Fil}_D(\text{AnSp}(\mathbb{C})^{sm}/S^{an}) \). It induces the canonical morphism in \( C_{O_{S^{an}}} \text{Fil}_D(\text{AnSp}(\mathbb{C})^{sm}/S^{an}) \):

\[
E \Omega/(S^{an}/S) : \text{An}_S^* E_{\text{et}}(\Omega^*_S, F_b) \xrightarrow{T(\text{An}, \Omega, \text{e})} E_{\text{et}}(\text{An}_S^* (\Omega^*_S, F_b)) \xrightarrow{E(\Omega/(S^{an}/S))} E_{\text{et}}(\Omega^*_{S^{an}}, F_b)
\]

We have the following canonical transformation map given by the analytical functor:

**Definition 158.** Let \( S \in \text{SmVar}(\mathbb{C}) \).

(i) For \( F \in C(\text{Var}(\mathbb{C})^{sm}/S) \), we have the canonical transformation map in \( C_{\text{Fil}_D}(S^{an}) \)

\[
((e(S), \text{Hom}^* (F, E_{\text{et}}(\Omega^*_S, F))))^{an} := O_{S^{an}} \otimes_{\text{an}_S^* O_S} \text{an}_S^* (e(S), \text{Hom}^* (F, E_{\text{et}}(\Omega^*_S, F)));
\]

(ii) We get from (i), for \( F \in C(\text{Var}(\mathbb{C})^{sm}/S) \), the canonical transformation map in \( \text{PSh}_{\text{Fil}^S} (S^{an}) \)

\[
T^n(an, \Omega, \text{e})(F) : J_S H^n((e(S), \text{Hom}^* (F, E_{\text{et}}(\Omega^*_S, F))))^{an}
\]

\[
\xrightarrow{J_S(H^n T(an, \Omega, \text{e})(F))} J_S H^n(e(S^{an}), \text{Hom}^* (\text{An}_S^* F, E_{\text{et}}(\Omega^*_{S^{an}}, F_b)));
\]

\[
\xrightarrow{J_S(-)} H^n(e(S^{an}), \text{Hom}^* (\text{An}_S^* F, E_{\text{et}}(\Omega^*_{S^{an}}, F_b))
\]

**Lemma 18.** Let \( S \in \text{SmVar}(\mathbb{C}) \).

(i) For \( h : U \rightarrow S \) a smooth morphism with \( U \in \text{SmVar}(\mathbb{C}) \), the following diagram commutes

\[
e(S), \text{Hom}^* (Z(U/S), E_{\text{et}}(\Omega^*_S, F_b))^{an} \xrightarrow{T(\Omega, \text{an})(\text{Z}(U/S))} e(S^{an}), \text{Hom}^* (Z(U^{an}/S^{an}), E_{\text{et}}(\Omega^*_{S^{an}}, F_b));
\]

\[
\xrightarrow{k} \rightarrow
\]

\[
(h_* E_{\text{zar}}(\Omega^*_U, F_b))^{an} \xrightarrow{T^\text{e}_w (an, h)} h_{an*} E_{\text{us}}(\Omega^*_{U^{an}/S^{an}}, F_b)
\]

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(ii) For \( h : U \to S \) a smooth morphism with \( U \in \text{SmVar}(\mathbb{C}) \), the following diagram commutes

\[
\begin{align*}
J_SH^n((e(S), \text{Hom}^\bullet(\mathbb{Z}(U/S), E_{et}(\Omega^\bullet_{/S}, F))))^{an} & \xrightarrow{T^n(\Omega_{/an}(\mathbb{Z}(U/S)))} H^ne(S^{an}, \text{Hom}^\bullet(\mathbb{Z}(U^{an}/S^{an}), E_{et}(\Omega^\bullet_{/S^{an}}, F_b))) \\
\end{align*}
\]

\[
\begin{align*}
J_SH^n((h_* E_{zar}(\Omega_{U/S}, F_b))^{an}) & \xrightarrow{\mathcal{J}_S(-) \circ J_N(\mathbf{an}, h)(\mathcal{O}_U, F)} H^nh_{an*}E_{usu}(\Omega_{U^{an}/S^{an}}, F_b)
\end{align*}
\]

Proof. Follows from Yoneda lemma.

By definition of the algebraic an analytic De Rahm realization functor, we have a natural transformation between them:

**Definition 159.** Let \( S \in \text{SmVar}(\mathbb{C}) \). Let \( M \in D_A(S) \) and \( Q \in C(\text{Var}(\mathbb{C})^{an}/S) \) projectively cofibrant such that \( M = D(A_S^{an}, et)(Q) \). We have the canonical transformation in \( D_{O_{fil, D}(S^{an})} \)

\[
T(\mathcal{A}_n, F_{GM}^\mathbb{C})(M) : (F_{GM}^\mathbb{C}(M))^{an} := (e(S), \text{Hom}^\bullet(Q, E_{et}(\Omega^\bullet_{/S}, F_b)))^{an}[-d_S]
\]

\[
\xrightarrow{T(\mathcal{A}_n, \Omega_{/S})(Q)} e(S), \text{Hom}^\bullet(F_{an}(Q), E_{usu}(\Omega^\bullet_{/S^{an}}, F_b))[-d_S]
\]

\[
\cong e(S), \text{Hom}^\bullet(F_{an}(Q), E_{usu}(\Omega^\bullet_{/S^{an}}, F_b))[-d_S] =: F_{GM}^\mathbb{C}^n(M)
\]

We give now the definition in the non smooth case: Let \( S \in \text{Var}(\mathbb{C}) \). Let \( S = \bigcup_{i=1}^l S_i \) be an open cover such that there exist closed embeddings \( i_i : S_i \to S \), with \( S_i \in \text{SmVar}(\mathbb{C}) \). For \( I \subset J \), denote by \( p_{IJ} : S_I \to S_J \) the projection. Consider, for \( I \subset J \subset [1, \ldots, l] \), resp. for each \( I \subset [1, \ldots, l] \), the following commutative diagrams in \( \text{Var}(\mathbb{C}) \)

\[
D_{IJ} = \begin{array}{ccc}
S_I & \xrightarrow{i_I} & \tilde{S}_I \\
\downarrow{i_{IJ}} & & \downarrow{p_{IJ}} \\
S_J & \xrightarrow{i_J} & \tilde{S}_J
\end{array}
\]

We then have the following lemma

**Lemma 19.** The maps \( T(\mathcal{A}_n, \Omega_{/S}(L(i_{1*}j^*_I F))) \) induce a morphism in \( C_{O_{fil, D}(S/(\tilde{S}_I))} \)

\[
(T(\mathcal{A}_n, \Omega_{/S})(L(i_{1*}j^*_I F)) : (e(\tilde{S}_I), \text{Hom}^\bullet(L(i_{1*}j^*_I F), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)))^{an}[-d_{\tilde{S}_I}], (u^q_{IJ}(F))^{an})
\]

\[
\to (e(\tilde{S}_I), \text{Hom}^\bullet(F_{an}(L(i_{1*}j^*_I F), E_{usu}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u^q_{IJ}(F))
\]

Proof. Obvious.

**Definition 160.** Let \( S \in \text{Var}(\mathbb{C}) \). Let \( S = \bigcup_{i=1}^l S_i \) be an open cover such that there exist closed embeddings \( i_i : S_i \to S_i \) with \( S_i \in \text{SmVar}(\mathbb{C}) \). For \( I \subset [1, \ldots, l] \), denote \( S_I = \cap_{i \in I} S_i \). We have then closed embeddings \( i_I : S_I \to S_I = \bigcap_{i \in I} S_i \). Let \( M \in D_A(S) \) and \( F \in C(\text{Var}(\mathbb{C})^{an}/S) \) such that \( M = D(A_S^{an}, et)(F) \). We have, by lemma 19, the canonical transformation in \( D_{O_{fil, D, \infty}(S^{an})} \)

\[
T(\mathcal{A}_n, F_{GM}^\mathbb{C})(M) : (F_{GM}^\mathbb{C}(M))^{an} := (e(S_I), \text{Hom}^\bullet(L(i_{1*}j^*_I F), E_{et}(\Omega^\bullet_{/S_I}, F_b)))^{an}[-d_{S_I}], (u^q_{IJ}(F))^{an})
\]

\[
\xrightarrow{T(\mathcal{A}_n, \Omega_{/S})(L(i_{1*}j^*_I F))} (e(S_I), \text{Hom}^\bullet(F_{an}(L(i_{1*}j^*_I F), E_{usu}(\Omega^\bullet_{/S_I}, F_b))[-d_{S_I}], u^q_{IJ}(F))
\]

\[
\cong (e(S_I), \text{Hom}^\bullet(F_{an}(L(i_{1*}j^*_I F), E_{usu}(\Omega^\bullet_{/S_I}, F_b))[-d_{S_I}], u^q_{IJ}(F)) =: F_{GM}^\mathbb{C}_{S, an}(M)
\]

The following proposition says this transformation map between \( F_{GM}^\mathbb{C} \) and \( (F_{GM}^{FDR})^{an} \) is functorial in \( S \in \text{Var}(\mathbb{C}) \), hence define a commutative diagram of morphism of 2-functor:

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Proposition 142. (i) Let \( g : T \to S \) a morphism with \( T,S \in \Var(\mathbb{C}) \). Assume there exist a factorization \( g : T \xrightarrow{p} Y \times S \xrightarrow{\pi_2} S \) with, \( Y \in \SmVar(\mathbb{C}) \), \( l \) a closed embedding and \( p_S \) the projection. Let \( S = \bigcup_{i=1}^t S_i \) be an open cover such that there exist closed embeddings \( i_i : S_i \hookrightarrow \bar{S}_i \) with \( \bar{S}_i \in \SmVar(\mathbb{C}) \). We then have closed embedding \( i_i : T_i \hookrightarrow Y \times \bar{S}_i \) and \( \bar{g}_i := p_{\bar{S}_i} : Y \times \bar{S}_i \to \bar{S}_i \) is a lift of \( g_i := g|_{T_i} : T_i \to S_i \). Then, for \( M \in \DA_c(S) \), the following diagram in \( D_{\OF,\mathcal{D},\infty}(T^{an}/(Y^{an} \times \bar{S}_i^{an})) \) commutes

\[
\begin{array}{ccc}
Rg^{* \mod[-],!}(\mathcal{F}^{\GM}(M))^\an & \xrightarrow{L_{g!}^{* \mod[-],!}} & Rg^{* \mod[-],!}(\mathcal{F}^{\GM}(M)) \\
\downarrow & & \downarrow \\
(F_T^{\GM}(g^*M))^\an & \xrightarrow{(T(M),\mathcal{F}^{\GM}(M))^{\an}} & (F_T^{\GM}(g^*M))
\end{array}
\]

(ii) Let \( S \in \Var(\mathbb{C}) \). Let \( S = \bigcup_{i=1}^t S_i \) be an open cover such that there exist closed embeddings \( i : S_i \hookrightarrow \bar{S}_i \) with \( \bar{S}_i \in \SmVar(\mathbb{C}) \). Then, for \( M,N \in \DA_c(S) \), the following diagram in \( D_{\OF,\mathcal{D},\infty}(S^{an}/(\bar{S}_i^{an})) \) commutes

\[
\begin{array}{ccc}
(F_S^{\GM}(M) \otimes_{O_S} F_S^{\GM}(N))^{\an} & \xrightarrow{(T(M),\mathcal{F}^{\GM}(M,N))^{\an}} & (F_S^{\GM}(M \otimes N))^{\an} \\
\downarrow & & \downarrow \\
(F_S^{\GM}(M))^{\an} \otimes_{O_S} F_S^{\GM}(N)^{\an} & \xrightarrow{T(M,\mathcal{F}^{\GM}(M)(N))} & T(M,\mathcal{F}^{\GM}(M \otimes N)) \\
\end{array}
\]

**Proof.** Immediate from definition.

Proposition 143. Let \( f : X \to S \) a morphism with \( S,X \in \Var(\mathbb{C}) \). Assume there exist a factorization

\[
f : X \xrightarrow{l} Y \times S \xrightarrow{p} S
\]

with \( Y \in \SmVar(\mathbb{C}) \), \( l \) a closed embedding and \( p \) the projection. Let \( S = \bigcup_{i} S_i \) an open cover such that there exists closed embeddings \( i : S_i \hookrightarrow \bar{S}_i \) with \( \bar{S}_i \in \SmVar(\mathbb{C}) \).

(i) We have then the following commutative diagram in \( D_{\OF,\mathcal{D},\infty}(S^{an}/(\bar{S}_i^{an})) \),

\[
\begin{array}{ccc}
(F_S^{\GM}(M(X/S)))^{\an} & \xrightarrow{I^{\GM}(X/S) \an} & F_S^{\GM}(M(X/S)) \\
\downarrow & & \downarrow \\
((p_{\bar{S}_i^*}\Gamma X, E_{zar}(\Omega^\bullet_{Y \times \bar{S}_i^*/S_i}))^{\an}_{-d_{\bar{S}_i}}, w_{IJ}(X/S))^{\an} & \xrightarrow{(T^\an_{(\an,p_{\bar{S}_i})})} & (p_{\bar{S}_i*}\Gamma X \mathcal{E}_{\mathcal{U}_S}(\Omega^\bullet_{(Y \times \bar{S}_i^*/S_i)})[-d_{\bar{S}_i}], w_f) \\
\downarrow & & \downarrow \\
((p, T^\an_{(\an,\gamma)(-)))^{\an}) & \xrightarrow{I^{\GM}(X/S)} & (p, T^\an_{(\an,\gamma)(-)))^{\an} \\
\end{array}
\]
(ii) We have then the following commutative diagram in \( \text{PSh}_D^{\infty} f_{\text{il}}(S^{an}/\tilde{S}_{1}^{an}) \),

\[
\begin{array}{ccc}
J_S H^n(F_S^{GM}(M_{BM}(X/S)))^{an} & \xrightarrow{J_S(-) \circ H^nT(\text{An}, F_{an}^{GM} (M(X/S)))} & H^n F^{GM}_{S,an}(M(X/S)) \\
J_S H^n((p_{S_1}, \Gamma_X E_{zar}(\Omega^*_{Y \times \tilde{S}_1/S_1}))^{an}[-d_{S_1}, w_{IJ}(X/S)^{an}] & \xrightarrow{H^n(T_{\gamma}^{an}(\text{An}, p_{\tilde{S}_1}))} & H^n(p_{S_1}, \Gamma_X E_{usu}(\Omega^*_{(Y \times \tilde{S}_1)^{an}/\tilde{S}_1}))^{an} \\
J_S H^n(f^{FDR}_{f} (\Gamma_X E_{zar}(O_{Y \times \tilde{S}_1}, F_b)[-d_Y - d_{S_1}, x_{IJ}(X/S)])^{an}) & \xrightarrow{H^n(T_{an,\gamma}^{an}(\text{An}, h) \circ f^{FDR}_{f}} & f^{FDR}_{f} (\Gamma_X E_{usu}(\Omega^*_{(Y \times \tilde{S}_1)^{an}/\tilde{S}_1}))^{an} \\
\end{array}
\]

Proof. (i): Immediate from definition. 

(ii): Follows from (i).

We deduce from proposition 143 and theorem 23 (GAGA for D-modules) the following:

**Theorem 39.**

(i) Let \( S \in \text{Var}(\mathbb{C}) \). Then, for \( M \in \text{DA}_{c}(S) \)

\[
J_S(-) \circ H^nT(\text{An}, F_{an}^{GM} (M)) : J_S(H^n(F_S^{GM}(M))^{an}) \xrightarrow{\sim} H^n F^{GM}_{S,an}(M)
\]

is an isomorphism in \( \text{PSh}_D(S^{an}/\tilde{S}_{1}^{an}) \).

(ii) A relative version of Grothendieck GAGA theorem for De Rham cohomology Let \( h : U \to S \) a smooth morphism with \( S, U \in \text{SmVar}(\mathbb{C}) \). Then,

\[
J_S(-) \circ J_S T_{\gamma}^{an}(h) : J_S((R^n h_* \Omega^*_{U/S})^{an}) \xrightarrow{\sim} R^n h_* \Omega^*_{U^{an}/S^{an}}
\]

is an isomorphism in \( \text{PSh}_D(S^{an}) \).

Proof. (i): Follows from proposition 143(ii) and theorem 23 using a resolution by Corti-Hanamura motives. (ii): Follows from (i) and lemma 18(ii). 

**6.3.2** The transformation map between the analytic filtered De Rham realization functor and the analytification of the filtered algebraic De Rham realization functor

Recall from section 2 that, for \( S \in \text{Var}(\mathbb{C}) \) we have the following commutative diagrams of sites
and

\[
\begin{align*}
\text{AnSp}(\mathbb{C})^{2,pr}/S & \xrightarrow{\text{Gr}^3} \text{AnSp}(\mathbb{C})/S, \\
\text{AnSp}(\mathbb{C})^{2,sm}/S & \xrightarrow{\text{AnS}} \text{AnSp}(\mathbb{C})^{sm}/S.
\end{align*}
\]

(65)

and that for \( f : T \to S \) a morphism with \( T, S \in \text{Var}(\mathbb{C}) \) we have the commutative diagram of site (43)

\[
\begin{align*}
\text{AnSp}(\mathbb{C})^{2}/T^{an} & \xrightarrow{\text{AnT}} \text{Var}(\mathbb{C})^{2}/T, \\
\text{AnSp}(\mathbb{C})^{2,sm}/T^{an} & \xrightarrow{\text{An}} \text{Var}(\mathbb{C})^{2,sm}/T. \\
\text{AnSp}(\mathbb{C})^{2}/S^{an} & \xrightarrow{\text{An}} \text{Var}(\mathbb{C})^{2}/S, \\
\text{AnSp}(\mathbb{C})^{2,sm}/S^{an} & \xrightarrow{\text{An}} \text{Var}(\mathbb{C})^{2,sm}/S.
\end{align*}
\]

This gives for \( s : \mathcal{I} \to \mathcal{J} \) a functor with \( \mathcal{I}, \mathcal{J} \in \text{Cat} \) and \( f : T_{\bullet} \to S_{\bullet} \) a morphism of diagram of algebraic varieties with \( T_{\bullet} \in \text{Fun}(\mathcal{I}, \text{Var}(\mathbb{C})), S_{\bullet} \in \text{Fun}(\mathcal{J}, \text{Var}(\mathbb{C})) \) the commutative diagram of sites (46)

\[
\begin{align*}
\text{Dia}^{12}(S) := \text{AnSp}(\mathbb{C})^{2}/T^{an}_{\bullet} & \xrightarrow{\text{AnT}} \text{Var}(\mathbb{C})^{2}/T_{\bullet}, \\
\text{AnSp}(\mathbb{C})^{2,sm}/T^{an}_{\bullet} & \xrightarrow{\text{An}} \text{Var}(\mathbb{C})^{2,sm}/T_{\bullet}. \\
\text{AnSp}(\mathbb{C})^{2}/S^{an}_{\bullet} & \xrightarrow{\text{An}} \text{Var}(\mathbb{C})^{2}/S_{\bullet}, \\
\text{AnSp}(\mathbb{C})^{2,sm}/S^{an}_{\bullet} & \xrightarrow{\text{An}} \text{Var}(\mathbb{C})^{2,sm}/S_{\bullet}.
\end{align*}
\]

Let \( S \in \text{SmVar}(\mathbb{C}) \). We have the canonical map in \( C(\text{Var}(\mathbb{C})^{2,sm}/S) \)

\[
\Omega^{\mathbf{F}_{pr}}_{/(S^{an}/S)} : (\Omega^{\mathbf{T}_{pr}}_{/S}, \mathcal{F}_{DR}) \to (\Omega^{\mathbf{T}_{pr}}_{/S^{an}}, \mathcal{F}_{DR})
\]

given by for \( p : (Y \times S, Z) \to S \) the projection with \( Y \in \text{SmVar}(\mathbb{C}) \),

\[
\Omega^{\mathbf{F}_{pr}}_{/(S^{an}/S)}((Y \times S, Z)/S) : \Omega_{Y \times S/S}^{\mathbf{pr}} \otimes O_{Y \times S} \Gamma_{Z, HDG}^{\mathbf{Y}_{Z, Hdg}}(O_{Y \times S}, F_{b})
\]

\[
\Omega_{(Y \times S)^{an}/(S^{an}/S)}^{\mathbf{pr}}((Y \times S)^{an}/S^{an}) \otimes O_{(Y \times S)^{an}} \Gamma_{Z, HDG}^{\mathbf{Y}_{Z, Hdg}}(O_{Y \times S}, F_{b})^{an}
\]

We have the following canonical transformation map given by the analytical functor:
**Definition 161.** Let $S \in \text{SmVar}(\mathbb{C})$. For $F \in C(\text{Var}(\mathbb{C})^{smpr}/S)$, we have the canonical transformation map in $C_{D^{\infty\text{fil}}(S^{an})}$

$$T(\alpha^\Gamma_{/S}(\Omega_{S}^I)(F):$$

$$(e(S),\text{Hom}^\bullet(F, E_{et}(\Omega^I_F, FDR)))^{an} \defeq O_{S^{an}} \otimes_{an^\sharp S} an^\sharp_S(e(S),\text{Hom}^\bullet(F, E_{et}(\Omega^I_F, FDR)))$$

$$\xrightarrow{T(\alpha, e)(-)} O_{S^{an}} \otimes_{an^\sharp S} O_S \xrightarrow{(e(S))_S \text{Hom}^\bullet(F, E_{et}(\Omega^I_F, FDR)))}$$

By definition of the algebraic analytic De Rham realization functor, we have a natural transformation between them:

**Definition 162.** Let $S \in \text{SmVar}(\mathbb{C})$. Let $M \in D\text{A}_{\mathbb{C}}(S)$ and $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^I_{\mathbb{C}}, et)(F, W)$. We have the canonical transformation map $\pi_S(D(\text{MHM}(S))) \subset D_{D(1,0)\text{fil}}(S^{an})$

$$T(\alpha_{an}^{FDR}(M): (\mathcal{F}^I_{FDR}(M))^{an} := (e(S),\text{Hom}^\bullet(\tilde{R}^C\text{H}(\rho_{S}^I L(F, W)), E_{zar}(\Omega^I_{/S}))^{an}$$

We give now the definition in the non smooth case: Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i: S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset J$, denote by $p_{IJ}: S_J \rightarrow \tilde{S}_J$ the projection. Consider, for $I \subset J \subset \{1, \ldots, l\}$, resp. for each $I \subset \{1, \ldots, l\}$, the following commutative diagrams in $\text{Var}(\mathbb{C})$

$$D_{IJ} = \xymatrix{ S_I \ar[r]^{i_i} \ar[d]_{j_I} & \tilde{S}_I \ar[d]^{p_{IJ}} \ar[l]_{i_j} \ar[r] \ar[d]_{j_J} & S_J \ar[d]^{j_J} \ar[l]_{i_j} }$$

We then have the following lemma

**Lemma 20.** The maps $T(\alpha, \Omega_{/S})(-)$ induce a morphism in $C_{D^{\infty\text{fil}}(S)/(\tilde{S}_I)}$

$$((\tilde{S}_I),\text{Hom}^\bullet(\tilde{R}^C\text{H}(\rho_{S}^I L(i_i \ast_j F)), E_{zar}(\Omega^I_{/S_I}))^{an}, (u^q_{IJ}(F))^{an})$$

$$\xrightarrow{T(\alpha_{\tilde{S}_I}^\Gamma_{/S}(\Omega_{/S_I}^I)(F))}$$

$$((\tilde{S}_I),\text{Hom}^\bullet(\tilde{R}^C\text{H}(\rho_{S}^I L(i_i \ast_j F)), E_{zar}(\Omega^I_{/S_I}))^{an}, (u^q_{IJ}(F))^{an})$$

**Proof.** Obvious. 

**Definition 163.** Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^l S_i$ be an open cover such that there exist closed embeddings $i_i: S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. For $I \subset \{1, \ldots, l\}$, denote $S_I = \bigcap_{i \in I} S_i$. We have then closed embeddings $i_I: S_I \hookrightarrow \tilde{S}_I = \bigcap_{i \in I} \tilde{S}_i$. Let $M \in D\text{A}_{\mathbb{C}}(S)$ and $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^I_{\mathbb{C}}, et)(F, W)$. We have, by lemma 20, the canonical transformation map in $\pi_S(D(\text{MHM}(S))) \subset D_{D(1,0)\text{fil}}(S^{an})$

$$T(\alpha_{an}^{FDR}(M): (\mathcal{F}_{S^{an}}^{FDR}(M))^{an} :=$$

$$((\tilde{S}_I),\text{Hom}^\bullet(\tilde{R}^C\text{H}(\rho_{S}^I L(i_i \ast_j F)), E_{zar}(\Omega^I_{/S_I})^{an}, (u^q_{IJ}(F))^{an})$$

$$\xrightarrow{T(\alpha_{\tilde{S}_I}^\Gamma_{/S}(\Omega^I_{/S_I})(F))}$$

$$(e'(\tilde{S}_I),\text{Hom}^\bullet(\tilde{R}^C\text{H}(\rho_{S}^I L(i_i \ast_j F)), E_{zar}(\Omega^I_{/S_I})^{an}, (u^q_{IJ}(F))^{an})$$

$$\xrightarrow{T(\alpha_{\tilde{S}_I}^\Gamma_{/S}(\Omega^I_{/S_I})(F))}$$

$$\mathcal{F}_{S^{an}}^{FDR}(M)$$

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The following proposition says this transformation map between $F^{S,an}$ and $(F^{S,FDR})^{an}$ is functorial in $S \in \text{Var}(\mathbb{C})$, hence define a commutative diagram of morphism of 2-functor:

**Proposition 144.**

(i) Let $g : T \to S$ a morphism with $T, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with, $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. Let $S = \bigcup_{i=1}^n S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \to S_i$ with $S_i \in \text{SmVar}(\mathbb{C})$. We then have closed embedding $i_i \circ l : T_i \hookrightarrow Y \times S_i$ and $\tilde{g}_i := p_{S_i} : Y \times S_i \rightarrow \tilde{S}_i$ is a lift of $g|_{T_i} : T_i \rightarrow S_i$.

(ii) Then, for $M \in DA_c(S)$, the following diagram in $\pi_T(D(MHM(T^{an})) \subset D_{\text{D}(1,0)}(\tilde{S}_i^{an}))$, see definition 138 and definition 156 commutes

$$g^{\text{mod}}((F^{S,FDR}(M))^{an}) = (g^{\text{mod}}(F^{S,FDR}(M)))^{\text{mod}}(g^{\text{mod}}(F^{\text{FDR}}(M)))^{\text{mod}}(F^{\text{FDR}}(M))^{an} \xrightarrow{T((g,F^{\text{FDR}})(M))^{an}} g^{\text{mod}}(F^{\text{FDR}}(M))^{an}$$

(ii) Then, for $M \in DA_c(T)$, the following diagram in $\pi_S(D(MHM(S^{an})) \subset D_{\text{D}(1,0)}(\tilde{S}_i^{an}))$ commutes

$$R^gH_{\text{dg}}((F^{S,FDR}(M))^{an}) = (R^gH_{\text{dg}}(F^{S,FDR}(M)))^{\text{FDR}}(R^gH_{\text{dg}})(F^{\text{FDR}}(M))^{an} \xrightarrow{T((g,F^{\text{FDR}})(M))^{an}} R^gH_{\text{dg}}(F^{\text{FDR}}(M))^{an}$$

(ii) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^n S_i$ be an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Then, for $M, N \in DA_c(S)$, the following diagram in $\pi_S(D(MHM(S))) \subset \pi_S(D(MHM(S)))$ commutes

$$R^gH_{\text{dg}}((F^{S,FDR}(M))^{an}) = (R^gH_{\text{dg}}(F^{S,FDR}(M)))^{\text{FDR}}(R^gH_{\text{dg}})(F^{\text{FDR}}(M))^{an} \xrightarrow{T((g,F^{\text{FDR}})(M))^{an}} R^gH_{\text{dg}}(F^{\text{FDR}}(M))^{an}$$
**Proof.** Immediate from definition.

**Proposition 145.** Let $f : X \to S$ a morphism with $S, X \in \text{Var}(\mathbb{C})$. Assume there exist a factorization

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p} S$$

with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p$ the projection. Let $S = \bigcup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. We have then the following commutative diagram in $D_{D^\infty \text{fil} \cap \infty}(S^{an})$:

$$
\begin{array}{c}
\text{(} \mathcal{F}_S^{FDR}(M) \otimes_{O_S} \mathcal{F}_S^{FDR}(N))^{an} \\
\downarrow \text{I}(X/S) \\
(R_{f!}^H \Gamma_{X_i}^!(\mathcal{O}_{Y \times S_i}, F_f)(d y)[2 d y], x_{ij}(X/S))^{an} \\
\downarrow \text{T}^{\text{mod}}(\text{an}, F_{S_i}(\cdot))(-) \\
R_{f!}^H \Gamma_{X_i}^!(\mathcal{O}_{Y \times \tilde{S}_i}^{an}, F_f)(d y)[2 d y], x_{ij}(X/S))^{an} \\
\end{array}
\text{commutes}
$$

Proof. Immediate from definition.

**Theorem 40.** Let $S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(S)$, the map in $\pi_S(D(MHM(S^{an}))) \subset D_{D(1,0)\text{fil} \cap \infty}(S^{an})$

$$T(\text{An}, \mathcal{F}_S^{FDR})(M) : (\mathcal{F}_S^{FDR}(M))^{an} \xrightarrow{\sim} \mathcal{F}_{S, an}^{FDR}(M)$$

given in definition 163 is an isomorphism.

Proof. Follows from proposition 145 and theorem 23.

We finish this subsection by the following easy proposition :

**Proposition 146.** Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Let $M \in \text{DA}_c(S)$ and $F \in C(\text{Var}(\mathbb{C})^{an}/S)$ such that $M = D(\otimes_S^1, \text{et})F$. Then the following diagram in $D_{\text{fil} \cap D(\infty)}(S/\tilde{S}_i)$ commutes

\[
\begin{array}{c}
\text{(} F_{S}^{GM}(L \otimes_S M) \otimes_{O_S} F_{S}^{GM}(F_{S}^{FDR}(M)))^{an} \\
\downarrow T(\text{An}, F_{S}^{GM})(L \otimes_S M) \\
\text{J}_S((\mathcal{F}_S^{FDR}(M))^{an}) \\
\end{array}
\text{commutes}
\]

Proof. Immediate from definition.
7 The Hodge realization functor for relative motives

7.1 The Betti realization functor

We have two definitions of the Betti realization functor which coincide at least for constructible motives, one given by [1] using the analytical functor and one given in [7] by composing the analytical functor with the forgetful functor to the topological space of a complex analytic space with a CW complex (see also [21] for the absolute case).

Definition 164. Let $S \in \text{Var}(\mathbb{C})$.

(i) The Ayoub's Betti realization functor is

$$\text{Bti}_S^*: \text{DA}(S) \to D^{an}(S), \; M \in \text{DA}(S) \mapsto \text{Bti}_S^* M = \text{Re}(S^{an}), \; \text{An}_S^* M = e(S^{an}), \; \text{An}_S^* F$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(ii) In [7], we define the Betti realization functor as

$$\widetilde{\text{Bti}}_S^*: \text{DA}(S) \to D^{cw}(S), \; M \mapsto \widetilde{\text{Bti}}_S^* M = \text{Re}(S^{cw}), \; \text{Cw}_S^* M = e(S^{cw}), \; \text{Cw}_S^* F$$

where $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(iii) For the Corti-Hanamura weight structure on $DA^-(S)$, we have by functoriality of (i) the functor

$$\text{Bti}_S^*: DA^-(S) \to D_{fil, \infty}(S^{an}), \; M \mapsto \text{Bti}_S^* M = e(S^{an}), \; \text{An}_S^* F, W$$

where $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ is such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$.

Note that by [7], $\text{An}_S^*$ and $\text{Cw}_S^*$ derive trivially.

Note that, by considering the explicit $D_S^1$ local model for presheaves on $\text{AnSp}(\mathbb{C})^{sm}/S^{an}$, $\text{Bti}_S^*(\text{DA}^-(S)) \subset D^{-(S^{an})}$; by considering the explicit $I_S^1$ local model for presheaves on $CW^{sm}/S^{cw}$, $\text{Bti}_S^*(\text{DA}^-(S)) \subset D^{-(S^{an})}$.

Let $f : T \to S$ a morphism, with $T, S \in \text{Var}(\mathbb{C})$. We have, for $M \in \text{DA}(S)$, $(F, W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$, and an equivalence $(\mathbb{A}^1, et)$ local $e : f^* (F, W) \to (F', W)$ with $(F', W) \in C_{fil}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $(f^* M, W) = D(\mathbb{A}^1, et)(F', W)$ the following canonical transformation map in $D_{fil}(T)$:

$$T^0(f, \text{Bti})(M) : f^* \text{Bti}_S^* M := f^* e(S^{an}), \; \text{An}_S^* (F, W) \xrightarrow{T(f,e)(-)} e(T^{an}), \; f^* \text{An}_T^* e \xrightarrow{T(f,e)(-)} e(T^{an}), \; \text{An}_T^* f^* (F, W)$$

Note 145. Let $f : T \to S$ a morphism, with $T, S \in \text{Var}(\mathbb{C})$. Consider the graph factorization $f : T \xrightarrow{l} T \times S \xrightarrow{p} S$ of $f$ with $l$ the graph closed embedding and $p$ the projection. We have, for $M \in \text{DA}_c(S)$, the following canonical transformation map in $D_{fil}(T^{an})$:

$$T(f, \text{Bti})(M, W) : f^* w \text{Bti}_S^* (M, W) := l^* \Gamma_T^{\vee, w} p^* \text{Bti}_S^* (F, W)$$

where we use definition 102.
Theorem 41.  
(i) Let $f : X \to S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. We have, for $M \in \text{DA}_c(X)$, the following transformation map in $D_{fil,c}(S^{an})$

$$T_*(f, \text{Bti})(M, W) : \text{Bti}^*_S(Rf_*(M, W)) \xrightarrow{\text{ad}(f^*, Rf_*(f_!(M, W)))} Rf_!w Rf^*_w \text{Bti}^*_S(Rf_*(M, W))$$

Clearly if $l : Z \to S$ is a closed embedding, then $T_*(l, \text{Bti})(M, W)$ is an isomorphism since $\text{ad}(l^*, l_*) : l^*l_*(M, W) \to (M, W)$ is an isomorphism (see section 3).

(ii) Let $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \text{SmVar}(\mathbb{C})$, $l$ a closed embedding and $p_S$ the projection. We have then, for $M \in \text{DA}_c(X)$, using theorem 41 for closed embeddings, the following transformation map in $D_{fil}((Y \times S)^{an})$

$$T_l(f, \text{Bti})(M) : Rf_!w \text{Bti}^*_X(M, W) = Rp_{Stw}!w \text{Bti}^*_X(M, W)$$

Clearly, for $f : X \to S$ a proper morphism, with $X, S \in \text{Var}(\mathbb{C})$ we have, for $M \in \text{DA}_c(Y \times S)$, $T_l(f, \text{Bti})(M, W) = T_*(f, \text{Bti})(M, W)$.

(iii) Let $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$. We have, using the second point, for $M \in \text{DA}(S)$, the following transformation map in $D_{fil}(X^{an})$

$$T^l(f, \text{Bti})(M, W) : \text{Bti}^*_X(f^!(M, W)) \xrightarrow{\text{ad}(f^!, f^!)(M, W))} f^!w Rf_!w \text{Bti}^*_X(f^!(M, W))$$

Let $S \in \text{Var}(\mathbb{C})$. We have, for $M, N \in \text{DA}(S)$ and $F, G \in C_! \text{Var}(\mathbb{C})^{an}(S)$ such that $M = D(A^1, et)(F)$ and $N = D(A^1, et)(G)$, the following transformation map in $D_{fil}(S^{an})$

$$\text{Bti}^*_S(M, W) \otimes \text{Bti}^*_S(N, W) := (e(S), \text{sing}_{\text{fil}}, \text{An}^*_S(F, W)) \otimes (e(S), \text{sing}_{\text{fil}}, \text{An}^*_S(G, F))$$

Theorem 41.  
(i) Let $f : X \to S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(S)$,

$$T(f, \text{Bti})(M, W) : f^*w \text{Bti}^*_S(M, W) \xrightarrow{\sim} \text{Bti}^*_X f^!(M, W)$$

is an isomorphism in $D_{fil}(X^{an})$.

(ii) Let $f : X \to S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(X)$,

$$T_l(f, \text{Bti})(M, W) : Rf_!w \text{Bti}^*_X(M, W) \xrightarrow{\sim} \text{Bti}^*_S Rf_!(M, W)$$

is an isomorphism.

(iii) Let $f : X \to S$ a morphism, with $X, S \in \text{Var}(\mathbb{C})$. For $M \in \text{DA}_c(X)$,

$$T_*(f, \text{Bti})(M, W) : Rf_!w \text{Bti}^*_X(M, W) \xrightarrow{\sim} \text{Bti}^*_S Rf_*(M, W)$$

is an isomorphism.
(iv) Let \( f : X \to S \) a morphism, with \( X, S \in \text{Var}(\mathbb{C}) \). For \( M \in \text{DA}_{c}(S) \),
\[
T^{i}(f, \text{Bti})(M, W) : f^{!}w \text{Bti}_{S}^{*}(M, W) \xrightarrow{\sim} \text{Bti}_{X}^{*} f^{i}(M, W)
\]
is an isomorphism.
(v) Let \( S \in \text{Var}(\mathbb{C}) \). For \( M, N \in \text{DA}_{c}(S) \),
\[
T(\otimes, \text{Bti})(M, W) : \text{Bti}_{S}^{*}(M, W) \otimes \text{Bti}_{S}^{*} N \xrightarrow{\sim} \text{Bti}_{X}^{*}((M, W) \otimes (N, W))
\]
is an isomorphism.

**Proof.** By functoriality it reduced to the case of Corti-Hanamura motives which then follows from [1].

The main result on the Betti realization functor is the following

**Theorem 42.**
(i) We have \( \text{Bti}_{S}^{*} = \text{Bti}_{S}^{*} \) on \( \text{DA}^{-}(S) \)

(ii) The canonical transformations \( T(f, \text{Bti}) \), for \( f : T \to S \) a morphism in \( \text{Var}(\mathbb{C}) \), define a morphism of 2 functor \( \text{Bti}_{S}^{*} : \text{DA}(\cdot) \to D(\text{an}) \), \( S \in \text{Var}(\mathbb{C}) \mapsto \text{Bti}_{S}^{*} : \text{DA}(S) \to D(\text{an}) \)

which is a morphism of homotopic 2 functor.

(ii)' The canonical transformations \( T(f, \text{Bti}) \), for \( f : T \to S \) a morphism in \( \text{Var}(\mathbb{C}) \), define a morphism of 2 functor \( \text{Bti}_{S}^{*} : \text{DA}(\cdot) \to D_{fil}(\text{an}) \), \( S \in \text{Var}(\mathbb{C}) \mapsto \text{Bti}_{S}^{*} : \text{DA}(S) \to D_{fil}(\text{an}) \)

which is a morphism of homotopic 2 functor.

**Proof.** (i): See [7].

(ii) and (ii)’: Follows from theorem 41.

**Remark 13.** For \( X \in \text{Var}(\mathbb{C}) \), the quasi-isomorphisms
\[
\mathbb{Z} \text{Hom}(\bar{\mathbb{D}}_{et}^{*}, X) \xrightarrow{\text{An}^{*}} \mathbb{Z} \text{Hom}(\bar{\mathbb{D}}^{n}(0, 1), X^{an}) \xrightarrow{\text{Hom}(i, X^{cw})} \mathbb{Z} \text{Hom}([0, 1]^{n}, X^{cw}),
\]
where,
\[
\bar{\mathbb{D}}^{n}_{e} := (e : U \to \mathbb{A}^{n}, \bar{\mathbb{D}}^{n}(0, 1) \subset e(U)) \in \text{Fun}(\mathbb{V}^{et}_{\mathbb{A}^{n}}(\bar{\mathbb{D}}^{n}(0, 1)), \text{Var}(\mathbb{C}))
\]
is the system of etale neighborhood of the closed ball \( \bar{\mathbb{D}}^{n}(0, 1) \subset \mathbb{A}^{n} \), and \( i : [0, 1]^{n} \to \bar{\mathbb{D}}^{n}(0, 1) \) is the closed embedding, shows that a closed singular chain \( \alpha \in \mathbb{Z} \text{Hom}^{n}([0, 1]^{n}, X^{cw}) \), is homologue to a closed singular chain
\[
\beta = \alpha + \partial \gamma = \bar{\beta}|_{[0, 1]^{n}} \in \mathbb{Z} \text{Hom}^{n}(\Delta^{n}, X^{cw})
\]
which is the restriction by the closed embedding \( [0, 1]^{n} \hookrightarrow U^{cw} \xrightarrow{e} \mathbb{A}^{n} \), where \( e : U \to \mathbb{A}^{n} \) an etale morphism with \( U \in \text{Var}(\mathbb{C}) \), of a complex algebraic morphism \( \bar{\beta} : U \to X \). Hence \( \beta([0, 1]^{n}) = \bar{\beta}([0, 1]^{n}) \subset X \) is the restriction of a real algebraic subset of dimension \( n \) in \( \text{Res}_{\mathbb{R}}(X) \) (after restriction a scalar that is under the identification \( \mathbb{C} \simeq \mathbb{R}^{2} \)).

**Definition 167.** Let \( S \in \text{Var}(\mathbb{C}) \) The cohomological Betti realization functor is
\[
\text{Bti}_{S}^{*} : \text{DA}(S) \to D(\text{S}^{cw}),
\]
\[
M \mapsto \text{Bti}_{S}^{*}(M) := R\text{Hom}(\text{Bti}_{S}^{*} M, Z_{S^{cw}}) = R\text{Hom}(M, \text{Bti}_{S}^{*} Z_{S^{cw}})
\]
where for \( \text{Bti}_{S}^{*} : K \in D(\text{S}^{cw}) \to R\text{An}_{S} e(\text{S}^{an})^{*} K \in \text{DA}(S) \) is the right ajoint to \( \text{Bti}_{S}^{*} \).
7.2 The Hodge realization functor for relative motives

Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^{n} S_i$ an open cover such that there exists closed embedding $i_i : S \rightarrow \tilde{S}_i$ with $\tilde{S}_i \in \text{SmVar}(\mathbb{C})$. Recall (see section 5.2) that $D_{D(1,0)fil, rh}(S/(\tilde{S}_i)) \times_I D_{fil}(S^{an})$ is the category

- whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with $((M_I, F, W), u_{IJ}) \in D_{D(1,0)fil, rh}(S/(\tilde{S}_i))$, $(K, W) \in D_{fil}(S^{an})$,
  $\alpha : T(S/(\tilde{S}_i))(K, W) \otimes C^{an} \rightarrow DR(S)[^i][((M_I, W), u_{IJ})^{an}]$

where $\alpha$ is a morphism in $D_{fil}(S^{an}/(\tilde{S}_i^{an}))$,

- and whose set of morphisms consists of

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_I, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{IJ}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

and

$$\theta = (\theta^*, I(DR(S)(\phi_D^{an})) \circ I(\alpha_1), I(\alpha_2) \circ I(\phi_C \otimes I) : I(T(S/(\tilde{S}_i))(K_1, W)) \otimes C^{an}[1] \rightarrow I(DR(S)((M_{IJ}, F, W), u_{IJ})^{an}))$$

is an homotopy, $I : D_{fil}(S^{an}/(\tilde{S}_i^{an})) \rightarrow K_{fil}(S^{an}/(\tilde{S}_i^{an}))$ being the injective resolution functor, and for

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_I, F, W), u_{IJ}), (K, W), \alpha_1) \rightarrow (((M_{IJ}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

the composition law is given by

$$\phi' \circ \phi := (\phi_D' \circ \phi_D, \phi_C' \circ \phi_C, I(DR(S)(\phi_D'^{an}) \circ [\theta] + [\theta'] \circ I(\phi_C \otimes I)[1]) :$$

$$(((M_{IJ}, F, W), u_{IJ}), (K_1, W), \alpha_1) \rightarrow (((M_{IJ}, F, W), u_{IJ}), (K_3, W), \alpha_3),$$

in particular for $((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{D(1,0)fil, rh}(S/(\tilde{S}_i)) \times_I D_{fil}(S^{an})$, $I(((M_I, F, W), u_{IJ}), (K, W), \alpha) = ((I_M), I_K, 0)$, together with the localization functor

$$(D(zar), I) : C_{D(1,0)fil, rh}(S/(\tilde{S}_i)) \times_I D_{fil}(S^{an}) \rightarrow D_{D(1,0)fil, rh}(S/(\tilde{S}_i)) \times_I D_{fil}(S^{an})$$

Moreover, $D_{D(1,0)fil, rh}(S/(\tilde{S}_i)) \times_I D_{fil}(S^{an})$, we set

$$(((M_I, F, W), u_{IJ}), (K, W), \alpha)[1] := (((M_I, F, W), u_{IJ})[1], (K, W)[1], \alpha[1]).$$

For a morphism in $D_{D(1,0)fil, rh}(S/(\tilde{S}_i)) \times_I D_{fil}(S^{an})$, we set (see [9] definition 3.12)

$$\text{Cone}(\phi) := (\text{Cone}(\phi_D), \text{Cone}(\phi_C), (((\alpha_1, \theta), (\alpha_2, 0)))) \in D_{D(1,0)fil, rh}(S/(\tilde{S}_i)) \times_I D_{fil}(S^{an})$$

$((\alpha_1, \theta), (\alpha_2, 0))$ being the matrix given by the composition law, together with the canonical maps

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\[-c_1(-) = (c_1(\phi_D), c_1(\phi_C), 0) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \rightarrow \text{Cone}(\phi)\]
\[-c_2(-) = (c_2(\phi_D), c_2(\phi_C), 0) : \text{Cone}(\phi) \rightarrow (((M_{1J}, F, W), u_{IJ}), (K_1, W), \alpha_1)[1].\]

Consider the category \[(D_{D(1,0)fil}(\hat{S}_I) \times_I D_{fil}(\hat{S}_{I}^{an})) \subset \text{Fun}(\Gamma(\hat{S}_I), \text{TriCat})\]

- whose objects are \(((M_I, F, W), (K_I, W), \alpha_I), u_{IJ}) \in \text{Fun}(\Gamma(\hat{S}_I), \text{TriCat})\) such that

\[((M_I, F, W), (K_I, W), \alpha_I) \in D_{D(1,0)fil}(\hat{S}_I) \times_I D_{fil}(\hat{S}_{I}^{an}) =: \mathcal{D}(\hat{S}_I)\]

and for \(I \subset J,\)

\[u_{IJ} : ((M_I, F, W), (K_I, W), \alpha_I) \rightarrow (M_J, F, W), (K_J, W), \alpha_J) := (p_{IJ_*}(M_I, F, W), p_{IJ_*}(K_I, W), p_{IJ_*}\alpha_I)\]

are morphisms in \(\mathcal{D}(\hat{S}_I),\)

- whose morphisms \(m = (m_I) : (((M_I, F, W), (K_I, W), \alpha_I), u_{IJ}) \rightarrow (((M_I', F, W), (K_I', W), \alpha_I'), v_{IJ})\) is a family of morphism such that \(u_{IJ} \circ m_I = p_{IJ_*} m_J \circ u_{IJ} \text{ in } \mathcal{D}(\hat{S}_I)\)

We have then the identity functor

\[I_S : D_{D(1,0)fil}(S/(\hat{S}_I))^0 \times_I D_{fil}(S^{an}) \rightarrow (D_{D(1,0)fil}(\hat{S}_I) \times_I D_{fil}(\hat{S}_{I}^{an})),\]

\[((M_I, F, W), u_{IJ}), (K_I, W), \alpha) \mapsto (((M_I, F, W), i_*\tilde{j}^*_I(K, W), \tilde{j}_I^*\alpha), (u_{IJ}, I, 0))\]

\[m = (m_I, n) \mapsto m = (m_I, i_*\tilde{j}_I^*n)\]

which is a full embedding since for \(((M_I, F, W), u_{IJ}) \in D_{D(1,0)fil}(S/(\hat{S}_I))^0,\)

\[u_{IJ} : (M_I, F, W) \rightarrow p_{IJ_*}(M_J, F, W)\]

are isomorphisms in \(D_{D(1,0)fil}(\hat{S}_I),\) and hence for \(((M_I, F, W), u_{IJ}), (K_I, W), \alpha) \in D_{D(1,0)fil}(S/(\hat{S}_I))^0 \times_I D_{fil}(S^{an}),\)

\[(u_{IJ}, I, 0) : ((M_I, F, W), i_*\tilde{j}_I^*(K, W), \tilde{j}_I^*\alpha) \rightarrow (p_{IJ_*}(M_J, F, W), i_*\tilde{j}_I^*(K, W), \tilde{j}_I^*\alpha)\]

are isomorphisms in \(\mathcal{D}(\hat{S}_I).\)

**Definition 168.** For \(h : U \rightarrow S\) a smooth morphism with \(S, U \in \text{SmVar}(\mathbb{C})\) and \(h : U \rightarrow X \xrightarrow{\phi} S\) a compactification of \(h\) with \(n\) an open embedding, \(X \in \text{SmVar}(\mathbb{C})\) such that \(D := X \backslash U = \cup_{i=1}^n D_i \subset X\) is a normal crossing divisor, we denote by, using definition 116 and definition 129

\[\text{I}(U/S) : h!_{Hdg} h!^{Hdg} \mathbb{Z}^{Hdg} \xrightarrow{\cong} \text{I}(U/S) : h!_{Hdg} h!^{Hdg} \mathbb{Z}^{Hdg} \xrightarrow{\cong}\]

\[\langle p_{S*} E_{zar}(\Omega^*_X \times _S \mathbb{Z}) \otimes O_{X \times S} \otimes (n \times I)_! Hdg_{\mathbb{Z}}(O_{U \times S}, F_b), D_{S h!} E_{usu} \mathbb{Z}^{Hdg}, h! \alpha(U) \rangle\]

\[\langle (DR(X \times S/S))_{\text{ad}(n \times I)_! Hdg_{\mathbb{Z}}}, h! (\alpha(U)) \rangle\]

\[\langle (\text{Cone}((\Omega^*_S \times_{D_I} X))_{\text{ad}(n \times I)_! Hdg_{\mathbb{Z}}}, h! (\alpha(U)) \rangle\]

\[\langle (\cdots \rightarrow p_{S*} E_{zar}(\Omega^*_X \times S \otimes O_{D \times S} \otimes \Gamma^V_{\mathbb{Z}}(O_{D \times S}, F_b) \rightarrow \cdots), D_{S h!} E_{usu} \mathbb{Z}^{Hdg}, h! \alpha(U) \rangle\]

\[\xrightarrow{\cong} \langle (F^D_{S F} \mathbb{Z}^{Hdg}(U/S), h! \mathbb{Z}^{Hdg}(U/S), \alpha(U)) \rangle\]

the canonical isomorphism in \(D_{D(1,0)fil}(S) \times_I D(S^{an}),\) with

\[h!^{Hdg} \mathbb{Z}^{Hdg} = (\Gamma^V_{\mathbb{Z}}(O_{U \times S}, F_b), \mathbb{Z}^{Hdg}, \alpha(U)) \in HM(U),\]

and \(i_{D_i} : D_i \hookrightarrow X\) are the closed embeddings.

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Lemma 21. Let $S \in \text{SmVar}(\mathbb{C})$. Let $g : U'/S \to U/S$ a morphism with $U/S := (U, h), U'/S := (U', h') \in \text{Var}(\mathbb{C})^{sm}/S$. Let $h : U \to X \xrightarrow{\Delta} S$ a compactification of $h$ with $n$ an open embedding, $X \in \text{SmVar}(\mathbb{C})$ such that $D := X \setminus U = \bigcup_{i=1}^r D_i \subset X$ is a normal crossing divisor, Let $h' : U' \xrightarrow{\nu'} X' \xrightarrow{\nu'} S$ a compactification of $h'$ with $n'$ an open embedding, $X' \in \text{SmVar}(\mathbb{C})$ such that $D' := X' \setminus U = U' \cup \bigcup_{i=1}^r D_i \subset X$ is a normal crossing divisor and such that $g : U' \to U$ extend to $\bar{g} : X' \to X$, see definition-proposition 12. Then, using definition 168, the following diagram in $D_{\text{fil}}(S) \times 1 D(S^{an})$ commutes

$$
\begin{array}{ccc}
\begin{array}{c}
h_1^{Hdg} h_1^{Hdg} \mathbb{Z}_S^{Hdg} \\
\text{ad}(\varphi_{Hdg}, g^{Hdg})(h_1^{Hdg} h_1^{Hdg} \mathbb{Z}_S^{Hdg})
\end{array}
& \xrightarrow{I(U'/S)} & \begin{array}{c}
(F_{S}^{FDR}(\mathbb{Z}(U'/S)), B\text{ti}_{S}^{*} \mathbb{Z}(U'/S), \alpha(\mathbb{Z}(U'/S)))
\end{array}
\\
\begin{array}{c}
h_1^{Hdg} h_1^{Hdg} \mathbb{Z}_S^{Hdg} \\
\text{ad}(\varphi_{Hdg}, g^{Hdg})(h_1^{Hdg} h_1^{Hdg} \mathbb{Z}_S^{Hdg})
\end{array}
& \xrightarrow{I(U/S)} & \begin{array}{c}
(F_{S}^{FDR}(\mathbb{Z}(U/S)), B\text{ti}_{S}^{*} \mathbb{Z}(U/S), \alpha(\mathbb{Z}(U/S)))
\end{array}
\end{array}
$$

where

$$
\theta(g) := R_{D}(\Gamma_{g}) : I(B\text{ti}_{S}^{*} \mathbb{Z}(U'/S) \otimes \mathbb{C})[1] \to I(DR(S)(\text{o}_{\text{fil}} F_{S}^{FDR}(\mathbb{Z}(U/S))^{an}))
$$

is the homotopy given by the third term of the Deligne homology class of the graph $\Gamma_{g} \subset U' \times_{S} U$ (see definition 118) and $\text{o}_{\text{fil}} : C_{\text{fil}}(S) \to C_{D}(S)$ is the forgetful functor and we recall (see section 5.2) that $I : C(S_{C}^{an} / S_{I}^{an}) \to K(S_{C}^{an} / S_{I}^{an})$ is the injective resolution functor.

Proof. Immediate from definition. \qed

We now define the Hodge realization functor.

Definition 169. Let $k \subset \mathbb{C}$ a subfield. Let $S \in \text{Var}(\mathbb{C})$. Let $S = \bigcup_{i=1}^r S_i$ an open cover such that there exists closed embedding $i_i : S \to S_i$ with $S_i \in \text{SmVar}(\mathbb{C})$. We define the Hodge realization functor, using definition 133, definition 164, and lemma 21

$$
\mathcal{F}_{S}^{Hdg} := (F_{S}^{FDR}, B\text{ti}_{S}^{*}) : C(\text{Var}(\mathbb{C})^{sm}/S) \to D_{D(1,0)\text{fil}}(S/\langle S_i \rangle) \times 1 D_{\text{fil}}(S^{an})
$$

first on objects and then on morphisms :

- for $F \in C(\text{Var}(\mathbb{C})^{sm}/S)$, taking $(F, W) \in C_{\text{fil}}(\text{Var}(\mathbb{C})^{sm}/S)$ such that $D(\mathbb{A}^{1}, \text{et})(F, W)$ gives the weight structure on $D(\mathbb{A}^{1}, \text{et})(F),$

$$
\mathcal{F}_{S}^{Hdg}(F) := (F_{S}^{FDR}(F), B\text{ti}_{S}^{*} F, \alpha(F)) := (e(S), \text{Hom}((\hat{R}_{S_i}^{CH}(\rho_{S_i}^{*} L_{i} \times j_{i}^{*} (F, W)), \hat{R}_{S_i}^{CH} (T^{q}_{D(1,1)}(-))), (E_{\text{zar}}(\Omega_{S_i}^{*} F, F_{DR}, T_{1})),

\begin{align*}
& e(S)_{\text{sing}, \alpha}, A_{S}^{*} L(F, W), \alpha(F) \in D_{D(1,0)\text{fil}}(S/\langle S_i \rangle) \times 1 D_{\text{fil}}(S^{an})
\end{align*}
$$

where $\alpha(F)$ is the map in $D_{\text{fil}}(S^{an}/\langle S_{I}^{an} \rangle)$, writing for short $DR(S) := DR(S)[\cdot] := (DR(S_{I})[-d_{S_i}])$}

$$
\alpha(F) : T(S/\langle S_{I} \rangle)((\text{Bti}_{S}^{*} M, W)) \otimes C_{S}) := (i_{I, J}^{*} (e(S)_{\text{sing}, \alpha}, A_{S}^{*} L(F, W)) \otimes C_{S}, I),
\begin{align*}
& \rightarrow (e(S)_{\text{sing}, \alpha}, A_{S}^{*} L_{i} \times j_{i}^{*} (F, W) \otimes C_{S}, T(p_{1,1}, An)(L_{i} \times j_{i}^{*} (F, W)))
\end{align*}
$$

$$
\rightarrow ((u_{i, \alpha, h_{i, a}})_{(i, h_{i, a})} \in \mathcal{V}_{I} h_{1a}^{1} Z_{S_{I}} \rightarrow \mathcal{H}_{i} h_{1a}^{1} Z_{S_{I}} \rightarrow \cdot), u_{1, J})
\rightarrow (\alpha(Z(U/\langle S_{I} \rangle), \alpha(\mathbb{Z}(U/\langle S_{I} \rangle)))
\rightarrow DR(S)(\text{o}_{\text{fil}}(e(S)_{\text{Hom}((\hat{R}_{S_i}^{CH}(\rho_{S_i}^{*} L_{i} \times j_{i}^{*} (F, W)), \hat{R}_{S_i}^{CH} (T^{q}_{D(1,1)}(-))), (E_{\text{zar}}(\Omega_{S_i}^{*} F, F_{DR}, T_{1}))))^{an})
\rightarrow DR(S)((o_{\text{fil}} F_{S}^{FDR}(M, W))^{an})
\rightarrow DR(S)((o_{\text{fil}} F_{S}^{FDR}(M, W))^{an})
using lemma 21, \((\alpha(Z(U_{1\alpha}/S)), \theta(g_{1,\alpha,\beta}^{\bullet}))\) being the matrix given inductively by the composition law in \(D_{D(1,0)FD}(\tilde{S}_1) \times_1 D_{FD}(\tilde{S}_1^{sm})\), that is we have the following isomorphism in \((D_{D(1,0)FD}(\tilde{S}_1) \times_1 D_{FD}(\tilde{S}_1^{sm}))\), denoting for short \(V_1 := \text{Var}(C)^{sm}/\tilde{S}_1\)

\[
\left(I^*(U_{1\alpha}/\tilde{S}_1)\right):
\]

\[
((\cdot \to \oplus_{(U_{1\alpha}, h_{1\alpha})} V_1 h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1) \xrightarrow{\text{ad}(g_{1,\alpha,\beta}^{\bullet} h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1)} \oplus_{(U_{1\alpha}, h_{1\alpha})} V_1 h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1) \xrightarrow{\cdot, u_{1\alpha}} I_1(F^{\text{FD}}(F)) := (F^{\text{FD}}(F), h_i, j_i(F, W, \alpha(F)))
\]

where we denote by \(g_{1,\alpha,\beta}^{\bullet}: U_{1\alpha} \to U_{1\beta}\) which satisfy \(h_{1\beta} \circ g_{1,\alpha,\beta}^{\bullet} = h_{1\alpha}\) the morphisms in the canonical projective resolution

\[
q: L i_{1\alpha, j_1}^* F := \cdots \oplus_{(U_{1\alpha}, h_{1\alpha})} V_1 h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1) \xrightarrow{\cdot, u_{1\alpha}} I_1(F^{\text{FD}}(F_1))
\]

\[
\text{for } m: F_1 \to F_2 \text{ a morphism in } C(\text{Var}(C)^{sm}/S), \text{ the morphism } F^{\text{Hdg}}_S(m) \text{ in } D_{D(1,0)FD}(S/\tilde{S}_1)^{\times 1} D_{FD}(S^{sm}) \text{ is given by}
\]

\[
F^{\text{Hdg}}_S(m) := I_1^{-1}((I^*(U_{1\alpha}/\tilde{S}_1)) \circ (\text{ad}(I_{1\alpha, \beta}^{\bullet} h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1)) \circ (I^*(U_{1\beta}/\tilde{S}_1)^{-1})
\]

\[
= (F^{\text{FD}}_S(m), h_i, j_i(m), \theta(m) := (\theta(I_{1\alpha, \beta}))) : F^{\text{Hdg}}_S(F_1) \to F^{\text{Hdg}}_S(F_2)
\]

using lemma 21, that is we have the following commutative diagram in \((D_{D(1,0)FD}(\tilde{S}_1)^{\times 1} D_{FD}(\tilde{S}_1^{sm}))\), denoting for short \(V_1 := \text{Var}(C)^{sm}/\tilde{S}_1\),

\[
((\cdot \to \oplus_{(U_{1\alpha}, h_{1\alpha})} V_1 h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1) \xrightarrow{\text{ad}(I_{1\alpha, \beta}^{\bullet} h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1)} \oplus_{(U_{1\alpha}, h_{1\alpha})} V_1 h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1) \xrightarrow{\cdot, u_{1\alpha}} I_1(F^{\text{FD}}(F_1))
\]

\[
\xrightarrow{\text{ad}(I_{2\alpha, \beta}^{\bullet} h_{2\alpha}! h_{2\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1)} (I^*(U_{1\alpha}/\tilde{S}_1)) \circ (I^*(U_{1\beta}/\tilde{S}_1))^{-1}
\]

\[
((\cdot \to \oplus_{(U_{1\alpha}, h_{1\alpha})} V_1 h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1) \xrightarrow{\cdot, u_{1\alpha}} I_1(F^{\text{FD}}(F_1)) \xrightarrow{\text{ad}(I_{2\alpha, \beta}^{\bullet} h_{2\alpha}! h_{2\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1)} (I^*(U_{1\alpha}/\tilde{S}_1)) \circ (I^*(U_{1\beta}/\tilde{S}_1))^{-1})
\]

\[
\xrightarrow{\text{ad}(I_{2\alpha, \beta}^{\bullet} h_{2\alpha}! h_{2\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1)} (I^*(U_{1\alpha}/\tilde{S}_1)) \circ (I^*(U_{1\beta}/\tilde{S}_1))^{-1}
\]

\[
\xrightarrow{\text{ad}(I_{2\alpha, \beta}^{\bullet} h_{2\alpha}! h_{2\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1)} (I^*(U_{1\alpha}/\tilde{S}_1)) \circ (I^*(U_{1\beta}/\tilde{S}_1))^{-1}
\]

\[
\]

\[
\]

where

- we denote for short \(A_{1\alpha, \beta}^{\text{Hdg}} h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1) := \text{ad}(g_{1,\alpha,\beta}^{\bullet} h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1)
\]

- we denote for short \(A_{2\alpha, \beta}^{\text{Hdg}} h_{2\alpha}! h_{2\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1) := \text{ad}(g_{2,\alpha,\beta}^{\bullet} h_{2\alpha}! h_{2\alpha}^{\text{Hdg}} h_{2\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1)
\]

- we denote by \(g_{1,\alpha,\beta}^{\bullet}: U_{1\alpha} \to U_{1\beta}\) which satisfy \(h_{1\beta} \circ g_{1,\alpha,\beta}^{\bullet} = h_{1\alpha}\) the morphisms in the canonical projective resolution

\[
q: L i_{1\alpha, j_1}^* F_1 := \cdots \oplus_{(U_{1\alpha}, h_{1\alpha})} V_1 h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1) \xrightarrow{\cdot, u_{1\alpha}} I_1(F^{\text{FD}}(F_1))
\]

\[
\oplus_{(U_{1\alpha}, h_{1\alpha})} V_1 h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1) \xrightarrow{\cdot, u_{1\alpha}} I_1(F^{\text{FD}}(F_1))
\]

\[
\]

- we denote by \(g_{2,\alpha,\beta}^{\bullet}: U_{1\alpha} \to U_{1\beta}\) which satisfy \(h_{1\beta} \circ g_{2,\alpha,\beta}^{\bullet} = h_{1\alpha}\) the morphisms in the canonical projective resolution

\[
q: L i_{1\alpha, j_1}^* F_2 := \cdots \oplus_{(U_{1\alpha}, h_{1\alpha})} V_1 h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1) \xrightarrow{\cdot, u_{1\alpha}} I_1(F^{\text{FD}}(F_2))
\]

\[
\oplus_{(U_{1\alpha}, h_{1\alpha})} V_1 h_{1\alpha}! h_{1\alpha}^{\text{Hdg}} h_{1\alpha}^{\text{Hdg}} \overset{\alpha}{\rightarrow} \tilde{S}_1) \xrightarrow{\cdot, u_{1\alpha}} I_1(F^{\text{FD}}(F_2))
\]

\[
\]

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we denote by \( p_{i_{\alpha,\beta}} : U_{i_{\alpha}} \to U_{i_{\beta}} \) which satisfy \( h_{i_{\beta}} \circ p_{i_{\alpha,\beta}} = h_{i_{\alpha}} \) and \( p_{i_{\alpha,\beta}} \circ g_{i_{\alpha,\beta}} = g_{i_{\alpha,\beta}} \circ p_{i_{\alpha,\beta}} \) the morphisms in the morphism of canonical projective resolutions.

\[
\begin{align*}
L_i \cdot j^*_i (m) : & \begin{array}{c}
\cdots \to \bigoplus_{(U_{i_{\alpha}}, h_{i_{\alpha}}) \in \text{Var}(C)^{\text{red}}} \mathbb{Z}(U_{i_{\alpha}}/\bar{S}_i) \to \cdots \end{array} \to (\mathbb{Z}(U_{i_{\alpha}}/\bar{S}_i)) \\
& \begin{array}{c}
\cdots \to \bigoplus_{(U_{i_{\alpha}}, h_{i_{\alpha}}) \in \text{Var}(C)^{\text{red}}} \mathbb{Z}(U_{i_{\alpha}}/\bar{S}_i) \to \cdots \end{array} =: L_i j^*_i \mathbb{F}_2,
\end{align*}
\]

the maps \( I^* (U_{i_{\alpha}}) \) are given by definition 168 and lemma 21.

Obviously \( \mathcal{F}_S^{Hdg} (F[1]) = \mathcal{F}_S^{Hdg}(F)[1] \) and \( \mathcal{F}_S^{Hdg} (\text{Cone}(m)) = \text{Cone}(\mathcal{F}_S^{Hdg} (m)) \). This functor induces by proposition 114 the functor

\[
\begin{align*}
\mathcal{F}_S^{Hdg} & := (\mathcal{F}_S^{FDR}, \text{Bti}_S^*): \text{DA}(S) \to D_{D(1,0)} (S/(\bar{S}_i)) \times I D_f (S^\alpha), \\
M & = D(h^1, \text{et})(F) \to \mathcal{F}_S^{Hdg} (M) := \mathcal{F}_S^{Hdg} (F) = (\mathcal{F}_S^{FDR}(M), \text{Bti}_S^* M, \alpha(M)),
\end{align*}
\]

with \( \alpha(M) = \alpha(F) \).

We now give the functoriality with respect to the five operation using the De Rahm realization case and the Betti realization case:

**Proposition 147.** (i) Let \( g : T \to S \) a morphism with \( T, S \in \text{Var}(C) \). Assume there exists a factorization \( g : T \to \bar{Y} \times S \to S \), with \( Y \in \text{SmVar}(C) \), \( l \) a closed embedding and \( p \) the projection. Let \( S = \cup_{i \in I} S_i \) an open cover and \( i_i : S_i \to \bar{S}_i \) closed embeddings with \( S_i \in \text{SmVar}(C) \). Then, \( \bar{y}_l : Y \times S_i \to \bar{S}_i \) is a lift of \( g_i = g_{|S_i} : T_i \to \bar{S}_i \) and we have closed embeddings \( i'_l := i_l \circ l \circ \bar{y}_l : T_l \to Y \times S_i \). Then, for \( M \in \text{DA}_c(S) \), the following diagram commutes:

\[
\begin{array}{ccc}
\text{Bti}_T^* M & \xrightarrow{\alpha(M)} & \text{DR}(T) \vert^{-1} ((\mathcal{F}_S^{FDR}(M))^\alpha) \\
T(g, \text{Bti})(M) \downarrow & & \downarrow \text{DR}(T) \vert^{-1} ((\mathcal{F}_T^* (\text{Bti}^* M))^\alpha) \\
\text{Bti}_T^* g^* M & \xrightarrow{\alpha(g^* M)} & \text{DR}(T) \vert^{-1} ((\mathcal{F}_T^* (g^* M))^\alpha)
\end{array}
\]

see section 5, definition 138 and definition 165

(ii) Let \( f : T \to S \) a morphism with \( T, S \in \text{QVPVar}(C) \). Then, for \( M \in \text{DA}_c(T) \), the following diagram commutes:

\[
\begin{array}{ccc}
\text{Bti}_S^* Rf_* M & \xrightarrow{\alpha(Rf_* M)} & \text{DR}(S) \vert^{-1} ((\mathcal{F}_S^{FDR}(Rf_* M))^\alpha) \\
T_c(f, \text{Bti})(M) \downarrow & & \downarrow \text{DR}(S) \vert^{-1} ((\mathcal{F}_S^{FDR}(Rf_* M))^\alpha) \\
\text{Bti}_S^* Rf_* M & \xrightarrow{\alpha(Rf_* M)} & \text{DR}(S) \vert^{-1} ((\mathcal{F}_S^{FDR}(Rf_* M))^\alpha)
\end{array}
\]

see section 5, definition 139 and definition 166

(iii) Let \( f : T \to S \) a morphism with \( T, S \in \text{QVPVar}(C) \). Then, for \( M \in \text{DA}_c(T) \), the following diagram commutes:

\[
\begin{array}{ccc}
\text{Bti}_T^* f_\ast M & \xrightarrow{\alpha(f_\ast M)} & \text{DR}(S) \vert^{-1} ((\mathcal{F}_S^{FDR}(f_\ast M))^\alpha) \\
T(f, \text{Bti})(M) \downarrow & & \downarrow \text{DR}(S) \vert^{-1} ((\mathcal{F}_S^{FDR}(f_\ast M))^\alpha) \\
\text{Bti}_T^* f_\ast M & \xrightarrow{\alpha(f_\ast M)} & \text{DR}(S) \vert^{-1} ((\mathcal{F}_S^{FDR}(f_\ast M))^\alpha)
\end{array}
\]

see section 5, definition 139 and definition 166.

}\]
(iv) Let \( f : T \to S \) a morphism with \( T, S \in \text{QPVar}(\mathbb{C}) \). Then, for \( M \in DA_c(S)\), the following diagram commutes:

\[
\begin{array}{ccc}
\text{Bti}_S^T M & \xrightarrow{f^!(M)} & DR(T) & \xrightarrow{(f^! \circ F^D)} (DR(S) \circ {F^D^T}_S(M))^n \\
T(f,M)(M,N) & & DR(T) & \xrightarrow{(f^! \circ F^D)} (DR(S) \circ {F^D^T}_S(M))^n \\
\text{Bti}_S^T f^! M & & DR(T) & \xrightarrow{(f^! \circ F^D)} (DR(S) \circ {F^D^T}_S(M))^n \\
\end{array}
\]

see section 5, definition 139 and definition 166.

(v) Let \( S \in \text{Var}(\mathbb{C}) \). Then, for \( M, N \in DA_c(S)\), the following diagram commutes:

\[
\begin{array}{ccc}
\text{Bti}_S^T M \otimes \text{Bti}_S^T N & \xrightarrow{(\alpha(M) \otimes \alpha(N))} & DR(S) & \xrightarrow{(\alpha \circ F^D)} (DR(S) \circ {F^D^T}_S(M))^n \\
T((\otimes, M))(M,N) & & DR(S) & \xrightarrow{(\alpha \circ F^D)} (DR(S) \circ {F^D^T}_S(M))^n \\
\text{Bti}_S^T (M \otimes N) & & DR(S) & \xrightarrow{(\alpha \circ F^D)} (DR(S) \circ {F^D^T}_S(M))^n \\
\end{array}
\]

see definition 139 and definition 166.

Proof. (i): Follows from the following commutative diagram in \((D_{(1,0)} m \times \hat{T}_I \times_{H \times \hat{T}_I})\):

\[
\begin{array}{ccc}
\text{Bti}_S^T M \otimes \text{Bti}_S^T N & \xrightarrow{(\alpha(M) \otimes \alpha(N))} & DR(S) & \xrightarrow{(\alpha \circ F^D)} (DR(S) \circ {F^D^T}_S(M))^n \\
T((\otimes, M))(M,N) & & DR(S) & \xrightarrow{(\alpha \circ F^D)} (DR(S) \circ {F^D^T}_S(M))^n \\
\text{Bti}_S^T (M \otimes N) & & DR(S) & \xrightarrow{(\alpha \circ F^D)} (DR(S) \circ {F^D^T}_S(M))^n \\
\end{array}
\]

where, we have denoted for short \( A_{g^H}^{Hdg} := \text{ad}(g^H_{I,\alpha,\beta} \circ g^H_{I,\alpha,\beta} \circ Hdg) \)

- we denoted for short \( A_{g^H}^{Hdg} := \text{ad}(g^H_{I,\alpha,\beta} \circ g^H_{I,\alpha,\beta} \circ Hdg) \)

- we denote by \( g^H_{I,\alpha,\beta} : U_{I,\alpha} \to U_{1,\beta} \), which satisfy \( h_{I,\beta} \circ g^H_{I,\alpha,\beta} = h_{I,\alpha} \), the morphisms in the canonical projective resolution

\[
q : L_{k_{I,\alpha}} j_{I,\beta}^* F := (\cdots \to \oplus(U_{I,\alpha,h_{I,\alpha}})_{\text{Var}(\mathbb{C})^m / \hat{T}_I} Z(U_{I,\alpha}/\hat{T}_I) \to \cdots) \to \text{Bti}_S^T \text{Bti}_S^T F
\]

- we denote by \( g^H_{I,\alpha,\beta} : U_{I,\alpha} \to U_{1,\beta} \), which satisfy \( h_{I,\beta} \circ g^H_{I,\alpha,\beta} = h_{I,\alpha} \), the morphisms in the canonical projective resolution

\[
q : L_{k_{I,\alpha}} j_{I,\beta}^* g^H = (\cdots \to \oplus(U_{I,\alpha,h_{I,\alpha}})_{\text{Var}(\mathbb{C})^m / \hat{T}_I} Z(U_{I,\alpha}/\hat{T}_I) \to \cdots) \to \text{Bti}_S^T \text{Bti}_S^T F
\]

(ii): Follows from (i) by adjunction.
(iii): The closed embedding case is given by (ii) and the smooth projection case follows from (i) by adjonction.

(iv): Follows from (iii) by adjonction.

(v): Obvious

We can now state the following key proposition and the main theorem:

**Proposition 148.** Let $k \subset \mathbb{C}$ a subfield.

(i) Let $S \in \text{Var}(\mathbb{C})$. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \bar{S}_i$ with $\bar{S}_i \in \text{SmVar}(\mathbb{C})$. Then we have the isomorphism in $D_{D(1,0)fil}(S/(\bar{S}_i)) \times I D_{fil}(S^{an})$

$$\mathcal{F}_S^{Hdg}(\mathbb{Z}_S) \to (\mathcal{F}_S^{FR}(\mathbb{Z}_S), B_{ti}\mathbb{Z}_S, \alpha(\mathbb{Z}_S))$$

$$(e(S), \text{Hom}((\bar{R}^{CH}(\Gamma_{S_1} \mathbb{Z}_{S_1}), \bar{R}^{CH}(x_{I,j})), (E_{zar}(\Omega^{*,pr}_{S_1}, F_{DR}), T_{I,J})), \mathbb{Z}^{S^{an}}, \alpha(S)) \to t_S((\Gamma_{S_1}^{\nu,Hdg}(O_{S_1}, F_b), x_{I,J}), \mathbb{Z}^{S^{an}}, \alpha(S) =: t_S(\mathbb{Z}_S^{Hdg})$$

with

$$\alpha(S) : T(S/(\bar{S}_i))(\mathbb{Z}^{S^{an}}) \otimes C_S := (i_{I,J}, C_{S^{an}}, I) \xrightarrow{(\text{ad}(i_{I,J}((C_{S^{an}})^n))^{-1}} \left(\Gamma_{S_1}^{\nu,C_{S_1}}, x_{I,J} \right) \xrightarrow{\left(\Gamma_{S_1}^{\nu,C_{S_1}}, x_{I,J} \right)} DR(S)(\alpha_{fil}(\Gamma_{S_1}^{\nu,Hdg}(O_{S_1}, F_b), x_{I,J}))$$

(ii) Let $f : X \to S$ a morphism with $X, S \in \text{Var}(\mathbb{C})$, $X$ quasi-projective. Consider a factorization $f : X \to Y \times S \to S$ with $Y = \mathbb{P}^{N,o} \subset \mathbb{P}^{N}$ an open subset, $l$ a closed embedding and $p_S$ the projection. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \bar{S}_i$ with $\bar{S}_i \in \text{SmVar}(\mathbb{C})$. Recall that $S_I := \cap_{i \in I} S_i$, $X_I = f^{-1}(S_I)$, and $\bar{S}_I := \Pi_{i \in I} \bar{S}_i$. Then, using proposition 147(iii), the maps of definition 139 and definition 166 gives an isomorphism in $D_{D(1,0)fil}(S/(\bar{S}_I)) \times I D_{fil}(S^{an})$

$$(T'(f, f^{FR})(\mathbb{Z}_X), T(f, B_{ti})(\mathbb{Z}_X), 0) : \mathcal{F}_S^{Hdg}(M^{BM}(X/S)) \to (\mathcal{F}_S^{FR}(Rf_{fil}^{BM}(X), B_{ti}^* Rf_{fil}^{BM}(X)), \alpha(Rf_{fil}^{BM}(X)))$$

$$\mathcal{F}_S^{Hdg}(M^{BM}(X/S)) \to t_S(Rf_{Hdg}^{BM}(O_{X \times S}, F_b), x_{I,J}(X/S)), Rf_{fil}^{BM}(X^{an}, f_{i}(\alpha(X))) =: t_S(Rf_{Hdg}^{BM}(X^{an}))$$

with

$$\mathbb{Z}_X^{Hdg} := ((\Gamma_{X_I}^{\nu,Hdg}(O_{X \times S_I}, F_b), x_{I,J}(X/Y \times S)), \mathbb{Z}^{X^{an}}, \alpha(X) \in C(MHM(X))$$

**Proof.** (i): Follows from proposition 114.

(ii): Follows from (i) by proposition 147(iii), theorem 36(i) and theorem 41(i).

The main theorem of this section is the following:

**Theorem 43.** Let $k \subset \mathbb{C}$ a subfield.

(i) For $S \in \text{Var}(\mathbb{C})$, we have $\mathcal{F}_S^{Hdg}((DA_{c}(S)) \subset D(MHM(S))$

$$t_S : \text{D}(MHM(S)) \to D_{D(1,0)fil}(S/(\bar{S}_I)) \times I D_{fil}(S^{an})$$

being a full embedding by theorem 32.

(ii) The Hodge realization functor $\mathcal{F}_S^{Hdg}(-)$ define a morphism of 2-functor on $\text{Var}(\mathbb{C})$

$$\mathcal{F}_S^{Hdg} : \text{Var}(\mathbb{C}) \to (DA_{c}(-) \to D(MHM(-)))$$

whose restriction to $\text{QPVar}(\mathbb{C})$ is an homotopic 2-functor in sense of Ayoub. More precisely,
(ii0) for \( g : T \to S \) a morphism, with \( T, S \in \text{QPVar}(\mathbb{C}) \), and \( M \in \text{DA}_e(S) \), the the maps of definition 138 and of definition 165 induce an isomorphism in \( D(MHM(T)) \)

\[
T(g, \mathcal{F}^{Hdg})(M) := (T(g, \mathcal{F}^{FDR})(M), T(g, \text{bti})(M), 0) : F^{Hdg}\to F^{FDR}(g^*M),
\]

\[
\mathcal{F}^{Hdg}(M) := \mathcal{F}_S^{Hdg}(M) := \iota_T^{-1}(g_{Hdg}^{\text{mod}}\mathcal{F}^{FDR}_S(M), g^w\text{Bti}_S(M), g^*(\alpha(M))) \cong \iota_T^{-1}(\mathcal{F}^{FDR}_T(g^*M), \text{Bti}_T^*(g^*M), \alpha(g^*M)) =: \mathcal{F}^{Hdg}_T(g^*M),
\]

(ii1) for \( f : T \to S \) a morphism, with \( T, S \in \text{QPVar}(\mathbb{C}) \), and \( M \in \text{DA}_e(T) \), the maps of definition 139 and of definition 166 induce an isomorphism in \( D(MHM(S)) \)

\[
T_*(f, \mathcal{F}^{Hdg})(M) := (T_*(f, \mathcal{F}^{FDR})(M), T_*(f, \text{bti})(M), 0) : F^{Hdg}\to F^{FDR},
\]

\[
\mathcal{F}^{Hdg}_S(M) := \mathcal{F}_S^{Hdg}_T(M) := \iota_s^{-1}(Rf^{Hdg}_*\mathcal{F}^{FDR}_T(M), Rf\text{Bti}_T(M), f_*(\alpha(M))) \cong \iota_s^{-1}(\mathcal{F}^{FDR}_S(Rf_*M), \text{Bti}_S^*(Rf_*M), \alpha(f_*(M))) =: \mathcal{F}^{Hdg}_S(Rf_*M),
\]

(ii2) for \( f : T \to S \) a morphism, with \( T, S \in \text{QPVar}(\mathbb{C}) \), and \( M \in \text{DA}_e(T) \), the maps of definition 139 and of definition 166 induce an isomorphism in \( D(MHM(S)) \)

\[
T^!(f, \mathcal{F}^{Hdg})(M) := (T^!(f, \mathcal{F}^{FDR})(M), T^!(f, \text{bti})(M), 0) : F^{Hdg}\to F^{FDR},
\]

\[
\mathcal{F}^{Hdg}_S(M) := \mathcal{F}_S^{Hdg}_T(M) := \iota_t^{-1}(f^{Hdg}^{\text{mod}}\mathcal{F}^{FDR}_S(M), f^w\text{Bti}_S(M), f^!(\alpha(M))) \cong \iota_t^{-1}(\mathcal{F}^{FDR}_T(f^*M), \text{Bti}_T^*(f^*M), \alpha(f^!(M))) =: \mathcal{F}^{Hdg}_T(f^*M),
\]

(ii3) for \( f : T \to S \) a morphism, with \( T, S \in \text{QPVar}(\mathbb{C}) \), and \( M \in \text{DA}_e(S) \), the maps of definition 139 and of definition 166 induce an isomorphism in \( D(MHM(T)) \)

\[
T^!(\otimes, \mathcal{F}^{Hdg})(M, N) := (T^!(\otimes, \mathcal{F}^{FDR})(M, N), T^!(\otimes, \text{bti})(M, N), 0) : F^{Hdg}\to F^{FDR},
\]

\[
\mathcal{F}^{Hdg}_S(M \otimes N) := \mathcal{F}_S^{Hdg}(M \otimes N) := \iota_s^{-1}(\mathcal{F}^{FDR}_S(M \otimes N), \text{Bti}_S(M \otimes N), \alpha(M \otimes N)) \cong \iota_s^{-1}(\mathcal{F}^{FDR}_S(\mathcal{F}^Hdg(M \otimes N)), \text{Bti}_S^*(\mathcal{F}^Hdg(M \otimes N)), \alpha(\mathcal{F}^Hdg(M \otimes N))),
\]

(iii) For \( S \in \text{Var}(\mathbb{C}) \), the following diagram commutes:

\[
\begin{array}{ccc}
\text{Var}(\mathbb{C})/S & \xrightarrow{M/(S)} & D(MHM(S)) \\
\downarrow{M/(S)} & & \downarrow{\iota_S} \\
\text{DA}(S) & \xrightarrow{\mathcal{F}^{Hdg}} & D_{D(1,0)fil}(S/(\tilde{S}_1)) \times_1 D_{fil}(S_{ann})
\end{array}
\]

Proof. (i): Let \( M \in \text{DA}_e(S) \). There exist by definition of constructible motives an isomorphism in \( \text{DA}(S) \)

\[
w(M) : M \cong \text{Cone}(M(X_0/S)[d_0] \to \cdots \to M(X_m/S)[d_m]),
\]

with \( f_n : X_n \to S \) morphisms and \( X_n \in \text{QPVar}(\mathbb{C}) \). This gives the isomorphism in \( D_{D(1,0)fil}(S/(\tilde{S}_1)) \times_1 D_{fil}(S_{ann}) \)

\[
\mathcal{F}^{Hdg}_S(w(M)) : \mathcal{F}^{Hdg}_S(M) \cong \text{Cone}(\mathcal{F}^{Hdg}_S(M(X_0/S))[d_0] \to \cdots \to \mathcal{F}^{Hdg}_S(M(X_m/S))[d_m]),
\]

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On the other hand, by proposition 148(i), we have
\[ \mathcal{F}_S^{Hdg}(M(X_n/S)) \to Rf_!Hdg\mathcal{Z}_X^{Hdg} \in D(MHM(S)). \]
This prove (i).

(ii): Follows from theorem 36(i), proposition 147(i) and theorem 41.

Theorem 43 gives immediately the following:

**Corollary 6.** Let \( f : U \to S, f' : U' \to S \) morphisms, with \( U, U', S \in \text{Var}(\mathbb{C}) \) irreducible, \( U' \) smooth. Let \( \overline{S} \in \text{PVar}(\mathbb{C}) \) a compactification of \( S \). Let \( \overline{X}, \overline{X}' \in \text{Var}(\mathbb{C}) \) compactification of \( U \) and \( U' \) respectively, such that \( f \) (resp. \( f' \)) extend to a morphism \( \overline{f} : \overline{X} \to \overline{S} \), resp. \( \overline{f}' : \overline{X}' \to \overline{S} \). Denote \( D = \overline{X} \setminus U \) and \( D' = \overline{X}' \setminus U' \) and \( \overline{E} = (\overline{X} \times_S \overline{X'}) \cup (\overline{X} \times_S D') \). Let \( i : D \hookrightarrow \overline{X} \), \( i' : D' \hookrightarrow \overline{X}' \) denote the closed embeddings and \( j : U \hookrightarrow \overline{X} \), \( j' : U' \rightarrow \overline{X}' \) the open embeddings. Denote \( d = \dim(U) \) and \( d' = \dim(U') \).

We have the following commutative diagram in \( D(\mathbb{Z}) \):

\[
\begin{array}{c}
\text{RHom}^\bullet_{DA(\overline{S})}(M(U'/\overline{S}), M((\overline{X}, \overline{D})/\overline{S})) \\
\text{RHom}^\bullet_{DA(\overline{S})}(M(pt), M(\overline{X} \times_{\overline{S}} \overline{X}, \overline{E})(d')[2d']) \\
Z_d(\overline{X} \times_{\overline{S}} \overline{E}, \bullet) \\
\end{array}
\]

where

\[
M((\overline{X}, \overline{D})/\overline{S}) := \text{Cone}(\text{ad}(i_* i'^* : M(\overline{D}/\overline{S}) \to M(\overline{X}/\overline{S})) = \overline{f}_j E_{ct}(\mathbb{Z}(U/U)) \in DA(\overline{S})
\]

and \( l \) the isomorphisms given by canonical embedding of complexes.

**Proof.** The upper square of this diagram follows from theorem 43(ii). On the other side, the lower square follows from the absolute case. \( \square \)

**References**


[22] G.Laumont, *Transformations canoniques et specialisation pour les V-modules filtrés*


LAGA UMR CNRS 7539
Université Paris 13, Sorbonne Paris Cité, 99 av Jean-Baptiste Clement,
93430 Villetaneuse, France,
bouali@math.univ-paris13.fr