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Antoine Benoit

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# Finite difference schemes approximation for hyperbolic boundary value problems on an interval: necessary and sufficient condition for lower exponential stability.

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## Abstract

In this article we are interested in the stability of finite difference schemes approximation for hyperbolic boundary value problems defined on the interval  $[0, 1]$ . The seminal work of [B. Gustafsson and Sundstrom., 1972], mainly devoted to the half-line, gives a necessary and sufficient invertibility condition ensuring the stability of the scheme, the so-called discrete uniform Kreiss-Lopatinskii condition. An interesting point is that this condition is a discretized version of the one imposed in the continuous setting to ensure the strong well-posedness of the hyperbolic boundary value problem. However as pointed in [B. Gustafsson and Sundstrom., 1972] and as soon as several boundary conditions are concerned the solution of the scheme may develop an exponential growth with respect to the discrete time variable. The question addressed here is to characterize the schemes having this growth or not. This is made under new invertibility conditions which are discretized versions of the ones preventing the exponential growth in time of the solution to continuous hyperbolic boundary value problems in the strip studied in [Benoit, a]. In some sense it shows that this continuous to discrete extension of the characterization occurs in the interval like in the half-line.

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# 1 Introduction

In this article we consider the approximation of the solution to first order linear hyperbolic boundary value problems by finite difference schemes. The continuous problem reads:

$$\begin{cases} L(\partial)u := \partial_t u + A\partial_x u = f & \text{for } (t, x) \in ]0, T] \times ]0, 1[, \\ B_0 u|_{x=0} = g_0, & \text{for } t \in ]0, T], \\ B_1 u|_{x=1} = g_1, & \text{for } t \in ]0, T], \\ u|_{t \leq 0} = u_0, & \text{for } x \in [0, 1], \end{cases} \quad (1) \quad \{\text{systeme\_intro}\}$$

where the unknown  $u \in \mathbb{R}^N$ ,  $N \geq 1$ , the interior coefficient  $A$  is a constant matrix in  $\mathbf{M}_{N \times N}(\mathbb{R})$  and where the boundary coefficient  $B_0$  (resp.  $B_1$ ) is a given matrix in  $\mathbf{M}_{p \times N}(\mathbb{R})$  (resp.  $\mathbf{M}_{(N-p) \times N}(\mathbb{R})$ ), for a fixed integer  $p$  encoding the good number of boundary conditions. More precisely if one considers the case where  $A = \text{diag}(\lambda_1, \dots, \lambda_N)$  then (1) decouples into  $N$  transport equations so that  $p$  must be equal to the number of strictly positive eigenvalues of  $A$  in order to make sure that the problem is not over or under determined.

The strong well-posedness (meaning existence and uniqueness of the solution satisfying an energy estimate in some exponentially weighted in time space based on  $L^2$ ) of this kind of problem is rather easy to establish if both problems in the half-lines  $\{x \leq 0\}$  and  $\{x \geq 1\}$  are strongly well-posed. With more details the desired energy estimate for (1) is the following straightforward generalization of the one in the half-line that is there exist  $C, \gamma_0 > 0$  such that for all  $\gamma \geq \gamma_0$ ,

$$\begin{aligned} & \gamma \int_{\mathbb{R}} \int_0^1 e^{-2\gamma t} |u(t, x)|^2 dx dt + \int_{\mathbb{R}} e^{-2\gamma t} |u|_{x=0}(t)|^2 dt + \int_{\mathbb{R}} e^{-2\gamma t} |u|_{x=1}(t)|^2 dt \\ & \leq C \left( \frac{1}{\gamma} \int_{\mathbb{R}} \int_0^1 e^{-2\gamma t} |f(t, x)|^2 dx dt + \int_{\mathbb{R}} e^{-2\gamma t} |g_0(t)|^2 dt + \int_{\mathbb{R}} e^{-2\gamma t} |g_1(t)|^2 dt \right). \end{aligned} \quad (2) \quad \{\text{eq\_est\_intro}\}$$

In particular let us point that (2) implies that the growth of  $u$  compared to  $t$  is bounded by the exponential factor  $e^{\gamma_0 t}$

Indeed localization arguments give the desired estimate near the boundaries of the interval while the desired estimate away from the boundaries follows from the well-posedness of the pure Cauchy problem.

However this proof gives *a priori* a solution admitting some exponential growth with respect to time, encoded by the threshold  $\gamma_0$ , growth which does not appear in the half-line geometry. A natural question is thus "Does this growth really occurs ?".

When one is dealing with finite difference schemes for boundary value problems then, for computational reasons, all the components of the approximated solution in the boundary conditions must be prescribed. However recall that in (1) only  $p$  or  $N - p$  components of the solution are given by the continuous problem in itself, so arbitrary choices of boundary conditions have to be made to complete the discrete problem.

A natural question is to ask "How can we choose good extra conditions ?". Indeed the issue is that typically a bad choice of the extra boundary conditions can lead to an exponentially growing solution of the scheme while the solution of the original continuous problem does not have this growth and *vice versa*. The aim of this paper is to try to give an answer to this question.

As far as the author knows little results can be found in the literature about this question. Finite difference scheme approximations in the interval are considered in [B. Gustafsson and Sundstrom., 1972] and the phenomenon of the exponential growth of the solution is studied on explicit examples and more precisely for some specific boundary conditions. We also have to mention [Trefethen, 1985] in which the author uses the notion of  $P$ -stability ( $P$  standing for practical) introduced in [Brown, 1984] that is to say stable schemes without exponential growth in time and like in [B. Gustafsson and Sundstrom., 1972] it is shown on examples that some choices of the discretized boundary conditions lead to this exponential growth of the solution.

Before to turn to a precise description of the characterization given in this article, let us recall some results concerning the stability of finite difference scheme approximations in the more studied geometry of the half-line.

The strong well-posedness of the continuous hyperbolic boundary value problem in the half-space:

$$\begin{cases} \partial_t u(t, x) + A\partial_x u(t, x) = f(t, x) & (t, x) \in ]-\infty, T] \times ]0, \infty[, \\ Bu|_{x=0}(t) = g(t) & t \in ]-\infty, T], \\ u|_{t \leq 0}(x) = 0 & x \in ]0, \infty[, \end{cases} \quad (3) \quad \{\text{ibvp\_intro}\}$$

has first been demonstrated in [Kreiss, 1970] in which the author establishes the energy estimate (that is an estimate analogous to (2) up to the control of only one boundary term) via the construction of a so-called Kreiss symmetrizer. The main tool used in this construction is the uniform Kreiss-Lopatinskii condition ensuring that in the normal mode analysis, no stable mode is solution to the homogeneous resolvent equation of (3) (that is the equation obtained after Laplace transform with respect to time).

Then in [Kreiss, 1968] and later in [B. Gustafsson and Sundstrom., 1972] the stability (without enter into the details by stability we mean that a discretized version of (2) holds) of numerical schemes obtained by finite difference approximations of (3) is established. We refer the interested reader to the full exposition of [Coulombel, ].

In the author's opinion one of the most significant point of the analysis is that the stability of the scheme is obtained via the construction of some discrete symmetrizer under some discrete version of the continuous uniform Kreiss-Lopatinskii condition.

Going back to the main question addressed in this article that is the possible exponential growth of the solution to the finite difference scheme approximation of (1), the analogous question has been considered for continuous problems in [Benoit, a] in which a characterization of the problems developing this growth is established. With more details this characterization asks the invertibility of some explicit matrices reading under the form  $I - \mathbf{T}$  and seems to be linked to the amplification coefficients of repetitive reflections of trapped wave packets (see [Benoit, b]).

Like in the half-space geometry the energy estimate is obtained via the construction of a symmetrizer (based on the one introduced in [Osher, 1973] to deal with the quarter space geometry).

The main result of this article is that, like in the half-line geometry and *mutatis mutandis*, the same discretized/continuous version of the same condition characterize the lower exponential stability of the scheme and the lower exponential strong stability of the continuous problem. More precisely that means that invertibility conditions on matrices reading  $I - \mathbb{T}$ , where  $\mathbb{T}$  is some kind of discrete version of  $\mathbf{T}$ , characterize the lower exponential stability of the finite difference scheme approximation.

The paper is organized as follows. In Section 2 we describe the scheme, introduce some notations and state the main assumptions. The main result of the article (namely Theorem 3.1) that is a characterization of lower exponential stability occupies Section 3. The proof of the main result is given in Sections 4 and 5.

More precisely Section 4 is divided in two: a preliminary required cancellation of the interior source term (see Paragraph 4.1) and a study of the reduction to the so-called resolvent problem, already used in the half-line geometry, where we recall the properties needed for the rest of the proof (see Paragraph 4.2).

Section 5 is the core of the proof of Theorem 3.1. Firstly we show that the invertibility conditions mentioned above are necessary for lower exponential stability and then we show that a little reinforcement of

one of these conditions is also sufficient for lower exponential stability via the construction of some discrete symmetrizer.

At last, Section 6 gives the explicit study of a simple example that is the scalar transport equation. However the study of this example permits in particular to recover some of the recent results about the finite difference scheme approximation of a transport equation in the interval with Neumann boundary condition of [Coulombel and Lagoutière, ].

## 2 Notations, description of the scheme and assumptions.

{part\_notati

### 2.1 Notations

In all this article, for  $a, b \in \mathbb{R}$ , the notation  $\llbracket a, b \rrbracket$  stands for the "interval" of integers between  $a$  and  $b$ . More precisely,  $\llbracket a, b \rrbracket := [a, b] \cap \mathbb{Z}$ .

For convenience, we define the following subsets of the complex plane which will be intensively used in the following:

$$\mathcal{U} := \{z \in \mathbb{C} \mid |z| > 1\}, \quad \overline{\mathcal{U}} := \{z \in \mathbb{C} \mid |z| \geq 1\} \quad \text{and} \quad \mathcal{D} := \{z \in \mathbb{C} \mid |z| < 1\}.$$

We also denote by  $\mathbb{S}^1$  the usual unit ball of  $\mathbb{C}$ .

Let  $K \in \mathbb{N}^*$  be fixed, the space step of discretization is defined by  $\Delta x := 1/K$  and we consider the regular space subdivision of  $[0, 1]$ ,  $(x_j)_j$  defined by  $x_j := j\Delta x$  for  $j \in \llbracket 0, K \rrbracket$ .

The time step of discretization is denoted by  $\Delta t > 0$  and the associated time subdivision  $(t^n)$  is defined by  $t^n = n\Delta t$ , for all  $n \in \mathbb{N}$ .

In order to make sure that the solution of the finite difference scheme in the full line  $\mathbb{R}$  converges to the solution of the continuous Cauchy problem a CFL (Courant-Friedrichs-Lewy) condition has to be assumed. More precisely we make the classical assumption that the ratio  $\lambda := \Delta t / \Delta x$  is kept constant when  $\Delta t, \Delta x \downarrow 0$ .

Let  $\mu \in \mathbb{Z}$  then for all sequence  $u = (u_j^n)_{n \in \mathbb{N}, j \in \mathbb{Z}}$  we introduce  $\mathbf{T}^\mu$  the space-shifting operator defined by  $(\mathbf{T}^\mu u)_{n,j} = u_{j+\mu}^n$ .

At last all the vectors in the following are assumed to be written in columns; for a fixed real matrix  $A$  the notation  $A^T$  stands for the transpose of  $A$  while when  $A$  is a complex matrix  $A^*$  stands for the usual composition of the transpose and complex conjugation of  $A$ .

### 2.2 Description of the scheme

We define the following finite difference scheme approximation of the interior equation of (1):

$$U_j^{n+1} = QU_j^n + \Delta t f_j^n \quad \text{for } n \geq 0 \text{ and } j \in \llbracket 1, K \rrbracket, \tag{4} \quad \{\text{discretizati}$$

where the matrix  $Q \in \mathbf{M}_{N \times N}(\mathbb{R})$  is defined by:

$$Q := \sum_{\mu=-\ell}^r A^\mu \mathbf{T}^\mu. \tag{5} \quad \{\text{def\_Q}\}$$

In (5),  $\ell, r \in \mathbb{N}$  are fixed (these integers correspond respectively to the stencil of the scheme in the left-hand side and in the right-hand side) and the matrices  $A^\mu \in \mathbf{M}_{N \times N}(\mathbb{R})$  are given.

As usual in the definition of finite difference schemes for boundary value problems we remark that the computation of  $U_1^{n+1}$  (resp.  $U_K^{n+1}$ ) from (4) requires the values of the boundary terms  $U_j^n$  for  $j \in \llbracket 1 - \ell, 0 \rrbracket$  (resp.  $j \in \llbracket K + 1, K + r \rrbracket$ ). In both cases these terms are not determined by the interior equation (that is (4)) so we need to add to (4) discretized boundary conditions in order to define properly these terms.

The discretized boundary conditions that we consider in this article are rather classical (see for example [B. Gustafsson and Sundstrom., 1972]) and are quite generic. More precisely we define:

$$U_j^{n+1} = \sum_{\sigma=0}^1 B_0^{\sigma,j} U_1^{n+\sigma} + g_{0,j}^{n+1}, \text{ for } n \geq 0 \text{ and } j \in \llbracket 1 - \ell, 0 \rrbracket, \quad (6) \quad \{\text{discretizati}\}$$

for the left-hand side boundary condition and

$$U_j^{n+1} = \sum_{\sigma=0}^1 B_1^{\sigma,j} U_K^{n+\sigma} + g_{1,j}^{n+1}, \text{ for } n \geq 0 \text{ and } j \in \llbracket K + 1, K + r \rrbracket, \quad (7) \quad \{\text{discretizati}\}$$

for the right one. In (6) and (7) the discretization coefficients are defined by:

$$B_0^{\sigma,j} := \sum_{\mu=0}^{b_0} B_0^{\sigma,j,\mu} \mathbf{T}^\mu \text{ and } B_1^{\sigma,j} := \sum_{\mu=0}^{b_1} B_1^{\sigma,j,\mu} \mathbf{T}^{-\mu}. \quad (8) \quad \{\text{def\_B}\}$$

where  $b_0$  and  $b_1 \in \mathbb{N}$  are fixed. Moreover to make sure that the two boundary conditions are uncorrelated we assume that  $b_0 + b_1 \ll K$  (another restriction on these parameters will be made in Assumption 2.3). The coefficients at the boundaries, that is the  $B_0^{\sigma,j,\mu}$  and the  $B_1^{\sigma,j,\mu}$ , are fixed matrices in  $\mathbf{M}_{N \times N}(\mathbb{R})$ .

At last to the finite difference approximation scheme we add a discretization of the initial condition and in this article we restrict our attention to homogeneous initial conditions:

$$U_j^0 = 0 \text{ for } j \in \llbracket 1 - \ell, K + r \rrbracket. \quad (9) \quad \{\text{discretizati}\}$$

To sum up the full finite difference scheme approximation considered in the following is given by:

$$\begin{cases} U_j^{n+1} = QU_j^n + \Delta t f_j^n & \text{for } n \geq 0 \text{ and } j \in \llbracket 1, K \rrbracket, \\ U_j^{n+1} = \sum_{\sigma=0}^1 B_0^{\sigma,j} U_1^{n+\sigma} + g_{0,j}^{n+1} & \text{for } n \geq 0 \text{ and } j \in \llbracket 1 - \ell, 0 \rrbracket, \\ U_j^{n+1} = \sum_{\sigma=0}^1 B_1^{\sigma,j} U_K^{n+\sigma} + g_{1,j}^{n+1} & \text{for } n \geq 0 \text{ and } j \in \llbracket K + 1, K + r \rrbracket, \\ U_j^0 = 0 & \text{for } j \in \llbracket 1 - \ell, K + r \rrbracket. \end{cases} \quad (10) \quad \{\text{approximatio}\}$$

Let us briefly give some comments about how the scheme (10) operates.

Assume that the values of the  $U_j^n$  for  $j \in \llbracket 1 - \ell, K + r \rrbracket$  are given at some time step  $n$ , we shall describe how the values of the  $U_j^{n+1}$  are computed. Firstly the interior equation of (10) gives the values of the  $U_j^{n+1}$  for  $j \in \llbracket 1, K \rrbracket$ , because the interior equation of (10) only depends on the  $U_j^n$  for  $j \in \llbracket 1 - \ell, K + r \rrbracket$ . So it remains to determine the value of the  $U_j^{n+1}$  for  $j \in \llbracket 1 - \ell, 0 \rrbracket \cup \llbracket K + 1, K + r \rrbracket$ .

From the definition of the discretization of the boundary condition on  $\{x = 0\}$  (resp.  $\{x = 1\}$ ), see (6) (resp. (7)) the  $U_j^{n+1}$  for  $j \in \llbracket 1 - \ell, 0 \rrbracket$  (resp.  $j \in \llbracket K + 1 - b_1, K + 1 \rrbracket$ ) depend on the  $U_j^k$  for  $k \in \{n, n + 1\}$  and  $j \in \llbracket 1, 1 + b_0 \rrbracket$  (resp.  $j \in \llbracket K - b_1, K \rrbracket$ ), terms that all have been determined at the previous step.

So that the determination of the  $U_j^{n+1}$  from the  $U_j^n$  for  $j \in \llbracket 1 - \ell, K + r \rrbracket$  is complete.

### 2.3 Assumptions on the scheme

In this paragraph we describe the several assumptions made upon the scheme (10) in order to show its stability.

The first one, which can be shown to be necessary for the stability of (10), is to ask that the Cauchy problem associated to (10) is stable. The aim of this paper is not to give a precise description of the stability of the Cauchy problem associated to (10) but it is a well-known result that this condition is equivalent to so-called Von Neumann condition. We refer, for example, to [[Coulombel, ] section 2.1] for more details

**Assumption 2.1 (Von Neumann condition)** For  $\kappa \in \mathbb{C} \setminus \{0\}$ , let  $\mathcal{A}$  be the amplification matrix associated to (10) defined by

$$\mathcal{A}(\kappa) := \sum_{\mu=-\ell}^r \kappa^\mu A^\mu$$

then there exists  $C > 0$  such that for all  $n \in \mathbb{N}$  and for all  $\eta \in \mathbb{R}$  we have  $|\mathcal{A}(e^{i\eta})^n| \leq C$ .

{hyp\_Neumann}

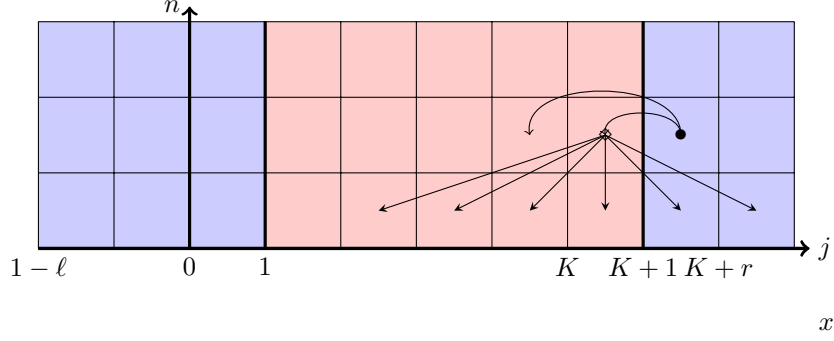


Figure 1: Time-space discretization of the full set of resolution for  $K = 5$ ,  $\ell = 3$  and  $r = 2$ . The red areas corresponds to the interior domain, the blue zones correspond to the boundaries

Then we define

$$\forall z \in \mathbb{C} \setminus \{0\}, \forall \mu \in \llbracket -\ell, r \rrbracket, \mathbb{A}_\mu(z) := \delta_{\mu,0} I - \frac{1}{z} A^{\sigma,\mu}, \quad (11) \quad \{\text{def\_A}\}$$

where the  $\delta_{\cdot,\cdot}$  stands for the Kronecker symbol.

The following assumption is rather classical in the study of boundary value problems and is referred as the non-characteristicity of the boundary condition; more precisely

**Assumption 2.2** [Non characteristic discrete condition] For all  $z \in \overline{\mathcal{U}}$ , the matrices  $\mathbb{A}_{-\ell}(z)$  and  $\mathbb{A}_r(z)$  are invertible. {\hyp\\_non\\_car}

The following assumption is possibly not necessary. It is a restriction on the value of the parameters defining the stencils in the scheme made to make sure that the so-called resolvent matrix  $\mathbb{M}(z)$  (see (24)) and the associated boundary matrices  $\mathbb{B}_0(z)$  and  $\mathbb{B}_1(z)$  (see (27) and (28)) have a suitable structure.

Indeed in the following these precise structures are used in a not tottaly trivial way. The generalization of our main result without Assumption 2.3 is possible in the author's opinion but it is left for future studies. {\hyp\\_restrict}

**Assumption 2.3** [On the parameters  $b_0$  and  $b_1$ ] We assume that the parameters  $b_0$  (resp.  $b_1$ ) defining the stencil of the boundary condition on the side  $\{x = 0\}$  (resp.  $\{x = 1\}$ ) (see (6) (resp. (7))) satisfies  $b_0 < r$  (resp.  $b_1 < \ell$ ).

Finally our last assumption on the finite difference approximation scheme is made to deal with possibly non-homogeneous source term in the interior. This assumption seems to be not harmless because firstly the construction of a symmetrizer requires a first reduction of the problem to a homogeneous forcing term in the interior and secondly because the proof of the stability estimate in itself requires some kind of discrete integration by part argument which uses Assumption 2.4. {\hyp\\_restrict}

**Assumption 2.4** The scheme (10) satisfies that for all  $u \in \ell^2(\mathbb{Z})$  the following inequality holds:

$$\|Qu\|_{\ell^2(\mathbb{Z})} \leq \|u\|_{\ell^2(\mathbb{Z})}.$$

### 3 Main result {\part\\_result}

Let us be more precise about the concept of stability used in this article. It is a natural generalization of the one used in the half-line geometry introduced in the seminal work of [B. Gustafsson and Sundstrom., 1972]. {\def\\_bien\\_pos}

**Definition 3.1** [B. Gustafsson and Sundstrom., 1972] The approximation scheme (10) is stable if for all source terms  $(f_j^n)_{n \in \mathbb{N}, j \in \llbracket 1, K \rrbracket}$ ,  $(g_{0,j}^n)_{n \in \mathbb{N}, j \in \llbracket 1 - \ell, 0 \rrbracket}$  and  $(g_{1,j}^n)_{n \in \mathbb{N}, j \in \llbracket K + 1, K + r \rrbracket}$  the unique solution of the scheme

(10) namely  $(U_j^n)_{n \in \mathbb{N}, j \in \llbracket 1-\ell, K+r \rrbracket}$  satisfies the following energy estimate: there exists  $C, \gamma_0 > 0$  such that for all  $\gamma \geq \gamma_0$  and  $\Delta t > 0$  we have

$$\begin{aligned} & \frac{\gamma - \gamma_0}{\gamma \Delta t + 1} \sum_{n \geq 0} \Delta t e^{-2\gamma n \Delta t} \sum_{j=1-\ell}^{K+r} \Delta x |U_j^n|^2 + \sum_{n \geq 0} \Delta t e^{-2\gamma n \Delta t} \sum_{j=1-\ell}^r |U_j^n|^2 + \sum_{n \geq 0} \Delta t e^{-2\gamma n \Delta t} \sum_{j=K+1-\ell}^{K+r} |U_j^n|^2 \quad (12) \quad \{\text{eq\_bien\_pose}\} \\ & \leq C \left( \frac{\gamma \Delta t + 1}{\gamma - \gamma_0} \sum_{n \geq 0} \Delta t e^{-2\gamma(n+1)\Delta t} \sum_{j=1}^K \Delta x |f_j^n|^2 + \sum_{n \geq 1} \Delta t e^{-2\gamma n \Delta t} \sum_{j=1-\ell}^0 |g_{0,j}^n|^2 + \sum_{n \geq 1} \Delta t e^{-2\gamma n \Delta t} \sum_{j=K+1}^{K+r} |g_{1,j}^n|^2 \right). \end{aligned}$$

ii) In particular when (12) holds with  $\gamma_0 = 0$  we say that the scheme (10) is Lower Exponentially (LE in the following) stable.

Let us give some comments about the energy estimate (10):

- Firstly note that (12) is just a discrete version of the the energy estimate for in the continuous setting (see (2)). Indeed let formally  $\Delta t \downarrow 0$  in (12) then we recover the energy estimate in the strip considered in [Benoit, a].
- Secondly recall that the parameters  $\Delta t$  and  $\Delta x$  are assumed to be linked by some CFL condition on  $\frac{\Delta t}{\Delta x}$  so that up to change the value of  $C$  in (12) there is no loss of generality by assuming that (12) does not depend of  $\Delta x$ . Moreover following [Coulombel, ], we can also assume that  $\Delta t = 1$  so that (12) is equivalent to the following energy estimate: there exists  $C, \gamma_0 > 0$  for all  $\gamma \geq \gamma_0$  we have the inequality

$$\begin{aligned} & \frac{\gamma - \gamma_0}{\gamma + 1} \sum_{n \geq 0} e^{-2\gamma n} \sum_{j=1-\ell}^{K+r} |U_j^n|^2 + \sum_{n \geq 0} \Delta t e^{-2\gamma n} \sum_{j=1-\ell}^r |U_j^n|^2 + \sum_{n \geq 0} e^{-2\gamma n \Delta t} \sum_{j=K-\ell}^{K+r} |U_j^n|^2 \quad (13) \quad \{\text{eq\_bien\_pose}\} \\ & \leq C \left( \frac{\gamma + 1}{\gamma - \gamma_0} \sum_{n \geq s+1} \Delta t e^{-2\gamma n} \sum_{j=1}^K |f_j^n|^2 + \sum_{n \geq s+1} e^{-2\gamma n} \sum_{j=1-\ell}^0 |g_{0,j}^n|^2 + \sum_{n \geq s+1} e^{-2\gamma n} \sum_{j=K+1}^{K+r} |g_{1,j}^n|^2 \right), \end{aligned}$$

equation that we will use up to now as the definition for the energy estimate.

- Then we describe the main difference between the strong stability and the LE-stability. The solution of a strong stable scheme may have a growth in the time variable with a exponential rate ruled by  $\gamma_0$  while in the case  $\gamma_0 = 0$  the solution of a LE-stable scheme have, as pointed in [[B. Gustafsson and Sundstrom., 1972]-Theorem 3.3], no exponential growth with respect to time. As mentioned in the introduction of this article we are here interested in LE-stable schemes. So that we are concern in (13) with  $\gamma - 0 = 0$ .
- Finally we point that the control of the traces values in the left-hand side of (12) are assumed to hold on the extended traces that is on  $\llbracket 1-\ell, r \rrbracket$  and on  $\llbracket K-\ell, K+r \rrbracket$  so the control differs a little from the one asked in [B. Gustafsson and Sundstrom., 1972] (in which the authors consider for the trace the values of the  $\llbracket 1-\ell, 0 \rrbracket$  only). This difference will be of real importance when we will define the traces operators (see equation (47)).

We now turn to the statement of the main result of this paper. More precisely this result characterizes the LE-stability of the scheme (10) in terms of an invertibility condition on some matrices referred as trace operators (that is matrices that take in input the value of the trace of the solution on one boundary and give in output the value of the trace on the other boundary (see (51) and (53) for a precise definition of these matrices).

This situation already occurs if one considers the continuous problem instead of the discrete one (see [Benoit, a] for more details). Moreover it is also already the case for the uniform Kreiss-Lopatinskii condition in the half-line geometry. We refer to [Kreiss, 1970]-[Benzoni-Gavage, 2007]-[Chazarain and Piriou, 1981] for a description of this condition in the continuous setting and [B. Gustafsson and Sundstrom., 1972]-[Coulombel, 2011] or Assumption 4.1 for a description in the discrete setting. In the author's opinion the generalization of this continuous/discrete requirement is interesting in its own because it seems to indicate that the same mechanisms occur in the strip and in the half-line.



**Theorem 3.1** Under Assumptions 2.1-2.4 if the scheme (10) is *LE-stable* in the sense of Definition 3.1 then for all  $z \in \mathcal{U}$  the matrix<sup>1</sup>  $I - \mathbb{T}_{0 \rightarrow 0}(z)$  is uniformly invertible from the stable subspace  $\mathbb{E}^s(z)$  into  $\mathbb{E}^s(z)$  while the matrix<sup>2</sup>  $I - \overline{\mathbb{T}}_{0 \rightarrow 0}(z)$  is invertible from  $\ker \mathbb{B}_0(z)$  into  $\ker \mathbb{B}_0(z)$ .

Conversely and under Assumptions 2.1-2.4 if  $I - \mathbb{T}_{0 \rightarrow 0}(z)$  is uniformly invertible from  $\mathbb{E}^s(z)$  into  $\mathbb{E}^s(z)$  and if  $I - \overline{\mathbb{T}}_{0 \rightarrow 0}(z)$  is **uniformly** invertible from  $\ker \mathbb{B}_0(z)$  into  $\ker \mathbb{B}_0(z)$  then the scheme (10) is *LE-stable* in the sense of Definition 3.1.

## 4 First analysis

### 4.1 Reduction to a homogeneous source term in the interior

In this paragraph we describe a method to restrict the study of the *LE-stability* of (10) to homogeneous source term in the interior. It is done because in Paragraph 5.2.2 the construction of the symmetrizer needs the scheme to be homogeneous in the interior.

This is also the case for the continuous problem (see [Benoit, a] for more details). However in the continuous setting we can, using some kind of integration by parts argument (which can easily be performed for symmetric interior matrices and strictly dissipative boundary conditions see [[Benzoni-Gavage, 2007], Section 3]) assume that the source term in the interior is zero.

The proof given here is a rather straightforward generalization of a proof establishing the so-called semi-group stability for finite difference schemes in the half-line given in [Coulombel and Gloria, 2011]. Consequently we will not give all the details of the proof but only describe the main points. Our aim is to show the following proposition:

**Proposition 4.1** [Reduction to a homogeneous source term in the interior] Under Assumptions 2.2 and 2.3 and Assumption 2.4. We consider the homogeneous scheme approximation with homogeneous source term in the interior:

$$\begin{cases} W_j^{n+1} = QW_j^n & n \geq 0, j \in \llbracket 1, K \rrbracket, \\ W_j^{n+1} = B_0^{0,j}W_1^n + B_0^{1,j}W_1^{n+1} + g_{0,j}^{n+1} & n \geq 0, j \in \llbracket 1 - \ell, 0 \rrbracket, \\ W_j^{n+1} = B_1^{0,j}W_K^n + B_1^{1,j}W_K^{n+1} + g_{1,j}^{n+1} & n \geq 0, j \in \llbracket K + 1, K + r \rrbracket, \\ W_j^0 = 0 & j \in \llbracket 1 - \ell, K + r \rrbracket. \end{cases} \quad (14)$$

Then (10) is *LE-stable* if and only if (14) is *LE-stable*.

The proof of Proposition 4.1 relies on the *LE-stability* of finite difference approximations with Dirichlet boundary conditions on each side of the boundary. Then the solution  $(U_j^n)_{n \in \mathbb{N}, j \in \llbracket 1 - \ell, K + r \rrbracket}$  to (10) is decomposed into the solution of this pure Dirichlet problem, namely  $(V_j^n)_{n \in \mathbb{N}, j \in \llbracket 1 - \ell, K + r \rrbracket}$  plus the solution to (14) with suitable source terms on the boundaries depending on the boundary value of  $(V_j^n)_{n \in \mathbb{N}, j \in \llbracket 1 - \ell, K + r \rrbracket}$ .

Finally, by error analysis, it remains to use the stability of (14) to estimate  $(U_j^n)$  in terms of  $(V_j^n)$  only. The stability of (10) then immediately follows from the one of the Dirichlet scheme.

#### 4.1.1 *LE-stability* of the pure Dirichlet problem.

In this paragraph we show that under Assumption 2.4 the finite difference scheme approximation with Dirichlet boundary conditions is automatically *LE-stable*<sup>3</sup>.

<sup>1</sup>We refer to (51) for a precise definition.

<sup>2</sup>See (??).

<sup>3</sup>In fact following all the proof exposed in [Coulombel and Gloria, 2011], that is by considering the case where the boundary terms and the initial condition are not trivial it is possible to show a stronger estimate than (16). Indeed, one can show that with Dirichlet boundary conditions the scheme in the half-line is semi-group stable in the sense that its solutions  $(V_j^n)$  satisfies the desired estimate with the extra term  $\sup_{n \geq 0} e^{-2\gamma n} \sum_{j=1-\ell}^{K+r} |V_j^n|^2$  in the left-hand side. This stronger estimate is of course interesting in its own, because little results are known about the semi-group stability of finite difference schemes with several boundaries but it will not be used in the following.

**Proposition 4.2** *Under Assumptions 2.2, 2.3 and Assumption 2.4, we consider  $(V_j^n)_{n \in \mathbb{N}, j \in \llbracket 1-\ell, K+r \rrbracket}$  the solution of*

$$\begin{cases} V_j^{n+1} = QV_j^n + f_j^n & n \geq 0, j \in \llbracket 1, K \rrbracket, \\ V_j^{n+1} = 0 & n \geq 0, j \in \llbracket 1-\ell, 0 \rrbracket, \\ V_j^{n+1} = 0 & n \geq 0, j \in \llbracket K+1, K+r \rrbracket, \\ V_j^0 = 0, & j \in \llbracket 1-\ell, K+r \rrbracket. \end{cases} \quad (15) \quad \{\text{problem\_diri}\}$$

Then  $(V_j^n)_{n \in \mathbb{N}, j \in \llbracket 1-\ell, K+r \rrbracket}$  satisfies the energy estimate: there exists  $C > 0$  such that for all  $\gamma > 0$  we have

$$\begin{aligned} \frac{\gamma}{\gamma+1} \sum_{n \geq 0} e^{-2\gamma n} \sum_{j=1-\ell}^{K+r} |V_j^n|^2 + \sum_{n \geq 0} e^{-2\gamma n} \sum_{j=1-\ell}^r |V_j^n|^2 + \sum_{n \geq 0} e^{-2\gamma n} \sum_{j=K+1-\ell}^{K+l} |V_j^n|^2 \\ \leq C \frac{\gamma+1}{\gamma} \sum_{n \geq 0} e^{-2\gamma(n+1)} \sum_{j=1}^K |f_j^n|^2. \end{aligned} \quad (16) \quad \{\text{eq\_est\_diric}\}$$

**Proof :** The proof exposed here follows the one given in the half-line geometry in [Coulombel and Gloria, 2011]. So that in the following we will only describe the main ideas of the proof for a sake of completeness. Let  $(V_j^n)_{n \in \mathbb{N}, j \in \llbracket 1-\ell, K+r \rrbracket}$  be the solution to (15), we introduce the decomposition  $Q = I + \tilde{Q}$  so that we have

$$\sum_{j=1}^K |V_j^{n+1}|^2 = \sum_{j=1}^K |V_j^n|^2 + 2 \sum_{j=1}^K (V_j^n)^T \tilde{Q} V_j^n + \sum_{j=1}^K |\tilde{Q} V_j^n|^2 + 2 \sum_{j=1}^K (QV_j^n)^T f_j^n + \sum_{j=1}^K |f_j^n|^2.$$

We define  $(W_j^n)_{n \in \mathbb{N}, j \in \mathbb{Z}}$  the extension of  $(V_j^n)$  by zero for  $j \leq 0$  and  $j \geq K+1$ . In terms of  $\tilde{Q}$ , Assumption 2.4 ensures that we have

$$\sum_{j=-\infty}^{\infty} (QW_j^n)^T W_j^n + \sum_{j=-\infty}^{\infty} |W_j^n|^2 \leq 0$$

that is

$$2 \sum_{j=1}^K (V_j^n)^T \tilde{Q} V_j^n + \sum_{j=1}^K |\tilde{Q} V_j^n|^2 + \sum_{j=-r}^0 |\tilde{Q} W_j^n|^2 + \sum_{j=K+1}^{K+l} |\tilde{Q} W_j^n|^2 \leq 0$$

From Lemma 2.2<sup>4</sup> of [Coulombel and Gloria, 2011] there exist two constants  $c_0, c_1 > 0$  such that

$$\sum_{j=-r}^0 |\tilde{Q} W_j^n|^2 \geq c_0 \sum_{j=1-\ell}^r |V_j^n|^2 \quad \text{and} \quad \sum_{j=K+1}^{K+l} |\tilde{Q} W_j^n|^2 \geq c_1 \sum_{j=K+1-\ell}^{K+r} |V_j^n|^2.$$

So we have:

$$\begin{aligned} \sum_{j=1}^K e^{-2\gamma(n+1)} |V_j^{n+1}|^2 - e^{-2\gamma} \sum_{j=1}^K e^{-2\gamma n} |V_j^n|^2 + e^{-2\gamma} \sum_{j=1-\ell}^r e^{-2\gamma n} |V_j^n|^2 + e^{-2\gamma} \sum_{j=K+1-\ell}^{K+l} e^{-2\gamma n} |V_j^n|^2 \\ \leq C \left( \sum_{j=1}^K e^{-2(n+1)\gamma} |f_j^n|^2 + e^{-2\gamma} \left( \sum_{j=1}^K e^{-2\gamma n} |V_j^n|^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^K e^{-2(n+1)\gamma} |f_j^n|^2 \right)^{\frac{1}{2}} \right) \end{aligned}$$

We sum from  $n = 0$  to  $N$  we have from Young inequality in the right-hand side with parameter  $(1 - e^{-2\gamma})$ :

$$\begin{aligned} e^{-2\gamma(N+1)} |V_j^{N+1}|^2 + \frac{1 - e^{-2\gamma}}{2} \sum_{n=0}^N \sum_{j=1}^K e^{-2\gamma n} |V_j^n|^2 + e^{-2\gamma} \sum_{n=0}^N \sum_{j=1-\ell}^r e^{-2\gamma n} |V_j^n|^2 \\ + e^{-2\gamma} \sum_{n=0}^N \sum_{j=K+1-\ell}^{K+r} e^{-2\gamma n} |V_j^n|^2 \leq C \frac{1}{1 - e^{-2\gamma}} \sum_{n=0}^N e^{-2\gamma(n+1)} |f_j^n|^2, \end{aligned}$$

<sup>4</sup>Note in particular that this lemma requires Assumption 2.2.

from which we derive

$$\begin{aligned} \frac{\gamma}{\gamma+1} \sum_{n=0}^N \sum_{j=1}^K e^{-2\gamma n} |V_j^n|^2 + e^{-2\gamma} \sum_{n=0}^N \sum_{j=1-\ell}^r e^{-2\gamma n} |V_j^n|^2 + e^{-2\gamma} \sum_{n=0}^N \sum_{j=K+1-\ell}^{K+r} e^{-2\gamma n} |V_j^n|^2 \\ \leq C \frac{\gamma+1}{\gamma} \sum_{n=0}^N e^{-2\gamma(n+1)} |f_j^n|^2, \end{aligned}$$

that is to say (16) up to the factor  $e^{-2\gamma}$  in front of the boundary term. So that (16) holds for small values of  $\gamma$  that is the values in which we are particularly interested in when we are dealing with  $LE$ -stability. The large values of  $\gamma$ , let us say  $\gamma \geq 1$ , are ruled exactly like in Lemma 2.4 of [Coulombel and Gloria, 2011].

□

#### 4.1.2 Proof of Proposition 4.1.

Clearly it is sufficient to show that the  $LE$ -stability of (14) implies the one of (10). Let  $(U_j^n)$  be the solution of (10) we decompose  $U_j^n = V_j^n + W_j^n$  where  $(V_j^n)$  satisfies the pure Dirichlet problem:

$$\begin{cases} V_j^{n+1} = QV_j^n + f_j^n & n \geq 0, j \in \llbracket 1, K \rrbracket, \\ V_j^{n+1} = 0 & n \geq 0, j \in \llbracket 1-\ell, 0 \rrbracket, \\ V_j^{n+1} = 0 & n \geq 0, j \in \llbracket K+1, K+r \rrbracket, \\ V_j^0 = 0 & j \in \llbracket 1-\ell, K+r \rrbracket, \end{cases} \quad (17) \quad \{\mathbf{eq\_V}\}$$

and where  $(W_j^n)$  satisfies

$$\begin{cases} W_j^{n+1} = QW_j^n & n \geq 0, j \in \llbracket 1, K \rrbracket, \\ W_j^{n+1} = B_0^{0,j} W_1^n + B_0^{1,j} W_1^{n+1} + \tilde{g}_{0,j}^{n+1} & n \geq 0, j \in \llbracket 1-\ell, 0 \rrbracket, \\ W_j^{n+1} = B_1^{0,j} W_K^n + B_1^{1,j} W_K^{n+1} + \tilde{g}_{1,j}^{n+1} & n \geq 0, j \in \llbracket K+1, K+r \rrbracket, \\ W_j^0 = 0 & j \in \llbracket 1-\ell, K+r \rrbracket. \end{cases} \quad (18) \quad \{\mathbf{eq\_W}\}$$

with for all  $n \geq 0$ ,

$$\begin{aligned} \forall j \in \llbracket 1-\ell, 0 \rrbracket, \tilde{g}_{0,j}^{n+1} &:= g_{0,j}^{n+1} + B_0^{0,j} V_1^n + B_0^{1,j} V_1^{n+1}, \\ \forall j \in \llbracket K+1, K+r \rrbracket, \tilde{g}_{1,j}^{n+1} &:= g_{1,j}^{n+1} + B_1^{0,j} V_K^n + B_1^{1,j} V_K^{n+1}. \end{aligned}$$

Equation (16) gives the desired estimate for the  $V_j^n$  in terms of the source term  $(f_j^n)$  only. Then the  $LE$ -stability of (14) gives

$$\begin{aligned} \frac{\gamma}{\gamma+1} \sum_{n \geq 0} e^{-2\gamma n} \sum_{j=1-\ell}^{K+r} |W_j^n|^2 + \sum_{n \geq 0} e^{-2\gamma n} \sum_{j=K+1-\ell}^{K+r} |W_j^n|^2 + \sum_{n \geq 0} e^{-2\gamma n} \sum_{j=1-\ell}^r |W_j^n|^2 \\ \leq C \left( \sum_{n \geq 0} e^{-2\gamma n} \sum_{j=1-\ell}^0 e^{-2\gamma n} |\tilde{g}_{0,j}^n|^2 + \sum_{n \geq 0} e^{-2\gamma n} \sum_{j=K+1}^{K+r} e^{-2\gamma n} |\tilde{g}_{0,j}^n|^2 \right), \end{aligned}$$

and from the definition of  $\tilde{g}_{0,j}^{n+1}$  and  $\tilde{g}_{1,j}^{n+1}$  it is sufficient to control the  $|V_j^n|^2$  and the  $|V_j^{n+1}|^2$  for  $j \in \llbracket 1, 1+b_0 \rrbracket$  and  $j \in \llbracket K-b_1, K \rrbracket$  which is possible from (16) and Assumption 2.3.

□

## 4.2 Study of the resolvent problem

{part\_resolv}

Like in the proof of the stability of the finite difference approximation scheme in the half-line the first step is to perform a normal mode analysis with respect to the time variable. More precisely we decompose for all  $z \in \mathbb{C} \setminus \{0\}$ ,  $U_j^n := z^n W_j$ , where  $(U_j^n)$  stands for the solution of (10). At the formal level the sequence  $(W_j)$  satisfies the scheme:

$$\begin{cases} W_j - z^{-1} Q W_j = 0 & j \in \llbracket 1, K \rrbracket, \\ W_j - z^{-1} B_0^{0,j} W_1 - B_0^{1,j} W_1 = G_{0,j} & j \in \llbracket 1 - \ell, 0 \rrbracket, \\ W_j - z^{-1} B_1^{0,j} W_K - B_1^{1,j} W_k = G_{1,j} & j \in \llbracket K + 1, K + r \rrbracket, \end{cases} \quad (19) \quad \{\text{schema\_resol}\}$$

where the precise expression of the new source terms  $(G_{0,j})_{j \in \llbracket 1 - \ell, 0 \rrbracket}$  and  $(G_{1,j})_{j \in \llbracket K + 1, K + r \rrbracket}$  is of little interest (because the new definition of the stability will be assumed to hold for all source terms). More precisely we now use the following definition of stability which from [B. Gustafsson and Sundstrom., 1972] (see also [Coulombel, ]) is equivalent to the stability of the scheme (10) in the sense of Definition 3.1.

{gateau}

**Definition 4.1 (LE-stability of the normal scheme)** *The finite difference approximation (19) is said to be LE-stable if there exists  $C > 0$  such that for all source terms  $(G_{0,j})_{j \in \llbracket 1 - \ell, 0 \rrbracket} \subset \mathbb{C}^N$  and  $(G_{1,j})_{j \in \llbracket K + 1, K + r \rrbracket} \subset \mathbb{C}^N$  the problem (19) admits a unique solution  $(W_j)_{j \in \llbracket 1 - \ell, K + r \rrbracket}$  satisfying that for all  $z \in \mathcal{U}$  we have the estimate*

$$\frac{|z| - 1}{|z|} \sum_{j=1-\ell}^{K+r} |W_j|^2 + \sum_{j=1-\ell}^r |W_j|^2 + \sum_{j=K+1-\ell}^{K+r} |W_j|^2 \leq C \left( \sum_{j=1-\ell}^0 |G_{0,j}|^2 + \sum_{j=K+1}^{K+r} |G_{1,j}|^2 \right). \quad (20) \quad \{\text{eq\_bien\_pose}\}$$

We conclude this paragraph with the following proposition which establishes that the control of the interior term in the left hand side of (20) comes for free in terms of the control of the boundary terms of the left hand side. More precisely

**Proposition 4.3** *Under Assumption 2.4, the resolvent finite difference approximation (19) is LE-stable in the sense of Definition 4.1 if and only if we have the following stability estimate: there exists  $C > 0$  such that for all source terms  $(G_{0,j})_{j \in \llbracket 1 - \ell, 0 \rrbracket} \subset \mathbb{C}^N$  and  $(G_{1,j})_{j \in \llbracket K + 1, K + r \rrbracket} \subset \mathbb{C}^N$  the problem (19) admits a unique solution  $(W_j)_{j \in \llbracket 1 - \ell, K + r \rrbracket}$  satisfying that for all  $z \in \mathcal{U}$  we have the estimate*

$$\sum_{j=1-\ell}^r |W_j|^2 + \sum_{j=K+1-\ell}^{K+r} |W_j|^2 \leq C \left( \sum_{j=1-\ell}^0 |G_{0,j}|^2 + \sum_{j=K+1}^{K+r} |G_{1,j}|^2 \right). \quad (21) \quad \{\text{eq\_bien\_pose}\}$$

{lapin}

**Proof :** Clearly it is sufficient to show that the energy estimate (21) implies the stronger estimate (20). Let  $(W_j)_{j \in \llbracket 1 - \ell, K + r \rrbracket}$  be the solution of (19) and let  $(\widetilde{W}_j)_{j \in \mathbb{Z}}$  be the extension of  $(W_j)_{j \in \llbracket 1 - \ell, K + r \rrbracket}$  by zero for  $j < 1 - \ell$  and  $j > K + r$ . Then  $(\widetilde{W}_j)_{j \in \mathbb{Z}}$  satisfies the equation  $\widetilde{W}_j = \frac{1}{z} Q \widetilde{W}_j + \widetilde{F}_j$  for all  $j \in \mathbb{Z}$  where the error source term is given by

$$\widetilde{F}_j := \begin{cases} 0 & \text{if } j < 1 - \ell - r \\ W_j - \frac{1}{z} \sum_{\sigma=1-\ell-j}^r A^\sigma W_\sigma & \text{if } j \in \llbracket 1 - \ell - r, 1 \rrbracket \\ 0 & \text{if } j \in \llbracket 1 - \ell, K \rrbracket \\ W_j - \frac{1}{z} \sum_{\sigma=-\ell}^{r+K-j} A^\sigma W_{j+\sigma} & \text{if } j \in \llbracket K + 1, K + r + \ell \rrbracket \\ 0 & \text{if } j > K + r + \ell \end{cases} \quad (22)$$

So that multiplying the evolution equation for  $(\widetilde{W}_j)_{j \in \mathbb{Z}}$  in the left hand side by  $\widetilde{W}_j^*$  and summing over  $j \in \mathbb{Z}$  gives by Cauchy-Scharzw inequality:

$$\sum_{j=1-\ell}^{K+r} |W_j|^2 \leq \frac{1}{|z|} \sum_{j=-\infty}^{\infty} |\widetilde{W}_j| |Q \widetilde{W}_j| + \sum_{j=1-\ell}^0 W_j^* \widetilde{F}_j + \sum_{j=K+1}^{K+r} W_j^* \widetilde{F}_j.$$

From Assumption 2.4 and because from the definition of  $\tilde{F}_j$  for  $j \in \llbracket 1 - \ell, 0 \rrbracket$  (resp.  $j \in \llbracket K + 1, K + r \rrbracket$ ) this term only involves the  $W_j$  for  $j \in \llbracket 1 - \ell, r \rrbracket$  (resp.  $j \in \llbracket K + 1 - \ell, K + r \rrbracket$ ) we obtain the estimate that for all  $z \in \mathcal{U}$ :

$$\frac{|z| - 1}{|z|} \sum_{j=1-\ell}^{K+r} \leq C \left( \sum_{j=1-\ell}^r |W_j|^2 + \sum_{j=K+1-\ell}^{K+r} |W_j|^2 \right), \quad (23) \quad \{\text{lapinou}\}$$

and the estimate (20) follows from (21) combined with (23)

□

#### 4.2.1 Reduction to the resolvent equation

To study (19) it is more convenient to change its expression in such a way that this scheme becomes an iterative process compared to  $j$ , for the higher dimensional vector  $\mathcal{W}_j := (W_{j+r-1}, \dots, W_{j-\ell})^T \in \mathbb{C}^{(\ell+r)N}$ .

To this aim, thanks to Assumption 2.2, we introduce the matrix:

$$\mathbb{M}(z) := \begin{bmatrix} -\mathbb{A}_r^{-1}(z)\mathbb{A}_{r-1}(z) & \cdots & \cdots & \mathbb{A}_r^{-1}(z)\mathbb{A}_\ell(z) \\ I & 0 & 0 & 0 \\ & \ddots & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \in \mathbf{M}_{(\ell+r)N \times (\ell+r)N}(\mathbb{C}) \quad (24) \quad \{\text{def\_M}\}$$

We also define the auxiliary boundary conditions:

$$\forall z \in \mathbb{C} \setminus \{0\}, \forall j \in \llbracket 1 - \ell, 0 \rrbracket, \forall \mu \in \llbracket 0, b_0 \rrbracket, \mathbb{B}_0^{j,\mu}(z) := \frac{1}{z} B_0^{0,j,\mu} + B_0^{1,j,\mu}, \quad (25) \quad \{\text{def\_B0\_aux}\}$$

$$\forall z \in \mathbb{C} \setminus \{0\}, \forall j \in \llbracket K + 1, K + r \rrbracket, \forall \mu \in \llbracket 0, b_1 \rrbracket, \mathbb{B}_1^{j,\mu}(z) := \frac{1}{z} B_1^{0,j,\mu} + B_1^{1,j,\mu}, \quad (26) \quad \{\text{def\_B1\_aux}\}$$

and then

$$\mathbb{B}_0(z) := \begin{bmatrix} 0 & \cdots & 0 & -\mathbb{B}_0^{0,b_0}(z) & \cdots & -\mathbb{B}_0^{0,0}(z) & I & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \ddots & \\ 0 & \cdots & 0 & -\mathbb{B}_0^{1-\ell,b_0}(z) & \cdots & -\mathbb{B}_0^{1-\ell,0}(z) & 0 & I \end{bmatrix} \in \mathbf{M}_{\ell N \times (\ell+r)N}(\mathbb{C}) \quad (27) \quad \{\text{def\_B0}\}$$

where the first block in (27) has  $(r - b_0 - 1)N$  columns, the second  $(b_0 + 1)N$  and the last one  $\ell N$ . Finally we define similarly

$$\mathbb{B}_1(z) := \begin{bmatrix} I & & -\mathbb{B}_1^{K+r,0} & \cdots & -\mathbb{B}_1^{K+r,b_1} & 0 & \cdots & 0 \\ & \ddots & \vdots & & \vdots & \vdots & \vdots & \\ & & I & -\mathbb{B}_1^{K+1,0} & \cdots & -\mathbb{B}_1^{K+1,b_1} & 0 & \cdots & 0 \end{bmatrix} \in \mathbf{M}_{rN \times (\ell+r)N}(\mathbb{C}), \quad (28) \quad \{\text{def\_B1}\}$$

where the first block in (28) has  $rN$  columns, the second  $(b_1 + 1)N$  and the last one  $(\ell - b - 1 - 1)N$ .

This change of unknown leads us to consider the following reformulation of (19)

$$\begin{cases} \mathcal{W}_{j+1} = \mathbb{M}(z)\mathcal{W}_j & j \in \llbracket 1, K \rrbracket, \\ \mathbb{B}_0(z)\mathcal{W}_1 = \mathcal{G}_0, \\ \mathbb{B}_1(z)\mathcal{W}_{K+1} = \mathcal{G}_1, \end{cases} \quad (29) \quad \{\text{eq\_total\_res}\}$$

where  $\mathcal{G}_0 := (G_{0,0}, \dots, G_{0,1-\ell}) \in \mathbb{C}^{\ell N}$  and  $\mathcal{G}_1 := (G_{1,K+r}, \dots, G_{1,K+1}) \in \mathbb{C}^{rN}$ .

We thus have the following proposition which is a slight adaptation [[Coulombel, ] Section 3.2] in order to deal with the extra boundary condition in the right hand-side.

**Proposition 4.4** *Under Assumption 2.2 the approximation (10) is LE-stable in the sense of Definition 4.1 if and only if there exists  $C > 0$  such that for all  $z \in \mathcal{U}$  and for all  $\mathcal{G}_0 \in \mathbb{C}^{\ell N}$ ,  $\mathcal{G}_1 \in \mathbb{C}^{rN}$  the resolvent equation (29) has a unique solution  $(\mathcal{W}_j)_{j \in \llbracket 1, K+1 \rrbracket}$  satisfying that*

$$\frac{|z| - 1}{|z|} \sum_{j=1}^K |\mathcal{W}_j|^2 + |\mathcal{W}_1|^2 + |\mathcal{W}_{K+1}|^2 \leq C (|\mathcal{G}_0|^2 + |\mathcal{G}_1|^2). \quad (30) \quad \{\text{est\_stable\_r}\}$$

So that with the results of Paragraph 4.1 and Proposition 4.2 in hand the question of the  $LE$ -stability of the finite difference scheme approximation (10) in the sense of Definition 3.1 can be reduced to the existence of  $C > 0$  such that for all  $z \in \mathcal{U}$  and for all  $\mathcal{G}_0 \in \mathbb{C}^{\ell N}$ ,  $\mathcal{G}_1 \in \mathbb{C}^{rN}$  the resolvent equation (29) has a unique solution  $(\mathcal{W}_j)_{j \in [1, K+1]}$  satisfying that

$$|\mathcal{W}_1|^2 + |\mathcal{W}_{K+1}|^2 \leq C (|\mathcal{G}_0|^2 + |\mathcal{G}_1|^2). \quad (31) \quad \{\text{est\_stable\_r}\}$$

Before to turn to the description of the necessary and sufficient conditions ensuring the  $LE$ -stability of (29), we shall give more informations about this scheme.

In the next paragraphs we firstly recall some classical properties about the resolvent matrix  $\mathbb{M}(z)$  which come from the study of finite difference schemes in the half-line. Then we construct a "dual" scheme for (29) because as we will see in Paragraph 5.2.2 the construction of the symmetrizer (that shows the stability estimate of the scheme) is intrinsically linked to this "dual" scheme. Moreover this "dual" scheme is an easy way to show the existence of a solution to (29).

#### 4.2.2 Properties of the matrix $\mathbb{M}(z)$

In this paragraph we recall the principal properties of the matrix  $\mathbb{M}(z)$  and more specifically the different behaviours of its spectrum when  $z \in \mathcal{U}$  and when  $z \in \overline{\mathcal{U}}$ .

We start by the following lemma describing the spectrum of the resolvent matrix  $\mathbb{M}(z)$  when  $z \in \mathcal{U}$ . More precisely this lemma indicates that when  $z \in \mathcal{U}$  the spectrum of  $\mathbb{M}(z)$  is well-separated in the sense that it only contains eigenvalues associated to exponentially decreasing modes and exponentially inscreasing modes. This lemma may be seen as the analogue of Hersh lemma for continuous problems (see [Hersh, 1963])

**Lemma 4.1** ([Kreiss, 1968]) *Under Assumptions 2.1 and 2.2 for all  $z \in \mathcal{U}$  the eigenvalues  $\kappa$  of the matrix  $\mathbb{M}(z)$  satisfy  $\kappa \neq 0$  and  $\kappa \notin \mathbb{S}^1$ . Moreover for all  $z \in \mathcal{U}$ ,  $\mathbb{M}(z)$  has  $\ell N$  eigenvalues in  $\mathcal{D}$  and  $rN$  in  $\mathcal{U}$ .*

We denote by  $\mathbb{E}^s(z)$  (resp.  $\mathbb{E}^u(z)$ ) the stable (resp. unstable) subspace of  $\mathbb{M}(z)$  that is the generalized eigenspace associated to eigenvalues in  $\mathcal{D}$  (resp.  $\mathcal{U}$ ).

In particular for all  $z \in \mathcal{U}$  the following decomposition holds:

$$\mathbb{C}^{(\ell+r)N} = \mathbb{E}^s(z) \oplus \mathbb{E}^u(z). \quad (32) \quad \{\text{decomp\_su}\}$$

An important corollary of Lemma 4.1 (see for example [Coulombel, ]) is the fact that the so-called discrete uniform Kreiss-Lopatinskii condition is necessary for the stability of the problem in the half-line. More precisely if we consider the schemes

$$\begin{cases} \mathcal{W}_{j+1} = \mathbb{M}(z)\mathcal{W}_j & j \geq 1 \\ \mathbb{B}_0(z)\mathcal{W}_1 = \mathcal{G}_0, \end{cases} \quad (33) \quad \{\text{approx\_demi1}\}$$

and

$$\begin{cases} \mathcal{W}_j = \mathbb{M}^{-1}(z)\mathcal{W}_{j+1} & j \leq K \\ \mathbb{B}_1(z)\mathcal{W}_{K+1} = \mathcal{G}_1, \end{cases} \quad (34) \quad \{\text{approx\_demi2}\}$$

then a necessary condition for these schemes to be stable is the following condition that we state as an assumption:

**Assumption 4.1 (Discrete uniform Kreiss-Lopatinskii condition)** *Under Assumptions 2.1 and 2.2 we assume that the discrete uniform Kreiss-Lopatinskii condition holds for the finite difference scheme (33) (resp. (34)) that is to say that there exists  $C_0 > 0$  (resp.  $C_1 > 0$ ) such that for all  $z \in \mathcal{U}$  we have that*

$$\forall \mathcal{W} \in \mathbb{E}^s(z), |\mathcal{W}| \leq C_0 |\mathbb{B}_0(z)\mathcal{W}| \quad (\text{resp. } \forall \mathcal{W} \in \mathbb{E}^u(z), |\mathcal{W}| \leq C_1 |\mathbb{B}_1(z)\mathcal{W}|),$$

where we stress that the constant  $C_0$  (resp.  $C_1$ ) does not depend on  $z$ .

In particular the restriction of  $\mathbb{B}_0(z)$  (resp.  $\mathbb{B}_1(z)$ ) to  $\mathbb{E}^s(z)$  (resp.  $\mathbb{E}^u(z)$ ) is uniformly invertible for all  $z \in \mathcal{U}$ , we denote this inverse by  $\Phi_0(z)$  (resp.  $\Phi_1(z)$ ).

The work of [B. Gustafsson and Sundstrom., 1972] then extended by [Coulombel, 2011] shows that in fact the discrete uniform Kreiss-Lopatinskii condition is also a sufficient condition for the stability of the finite difference scheme approximation in the half-line.

The sufficiency of this condition is shown from the construction of a so-called symmetrizer which brings back the proof of the energy estimate to a direct discrete integration by parts argument. However the construction of the symmetrizer in itself is rather technical and requires to understand what is the extension by continuity of the stable subspace  $\mathbb{E}^s(z)$  up to  $\overline{\mathcal{U}}$ . Without enter into technical details, we refer for example to [[Coulombel, ], Section 4], we admit here that  $\mathbb{E}^s(z)$  and  $\mathbb{E}^u(z)$  can be extended by continuity up to  $\overline{\mathcal{U}}$ . Thanks to this property we can give an equivalent formulation of the uniform Kreiss-Lopatinskii condition that will be used intensively in the following.

**Corollary 4.1** *Under Assumptions 2.1, 2.2 and 4.1, for all  $z \in \overline{\mathcal{U}}$  we have the following decompositions:* {cor\_ukl\_re}

$$\mathbb{C}^{(\ell+r)N} = \ker \mathbb{B}_0(z) \oplus \mathbb{E}^s(z), \quad (35) \quad \{\text{decomp}_0\}$$

and

$$\mathbb{C}^{(\ell+r)N} = \ker \mathbb{B}_1(z) \oplus \mathbb{E}^u(z). \quad (36) \quad \{\text{decomp}_1\}$$

To conclude this paragraph we introduce in the following definition the different projections that will be required later on. {def\_projo}

**Definition 4.2** *Under Assumptions 2.1 and 2.2:*

- for all  $z \in \mathcal{U}$  we denote by  $\Pi^s(z)$  (resp.  $\Pi^u(z)$ ) the projection upon  $\mathbb{E}^s(z)$  (resp.  $\mathbb{E}^u(z)$ ) with respect to the decomposition (32).
- Under Assumption 4.1, for all  $z \in \overline{\mathcal{U}}$  we denote by  $\overline{\Pi}^s(z)$  (resp.  $\overline{\Pi}_{\ker \mathbb{B}_0}(z)$ ) the projection upon  $\mathbb{E}^s(z)$  (resp.  $\ker \mathbb{B}_0$ ) with respect to the decomposition (35).  
Similarly we denote by  $\overline{\Pi}^u(z)$  (resp.  $\overline{\Pi}_{\ker \mathbb{B}_1}(z)$ ) the projection upon  $\mathbb{E}^u(z)$  (resp.  $\ker \mathbb{B}_1$ ) with respect to the decomposition (36).
- Finally, from the definition of the boundary matrices  $\mathbb{B}_0(z)$  and  $\mathbb{B}_1(z)$  (see (27) and (28)) we can define the projection  $\Pi_{\ker \mathbb{B}_0(z)}$  (resp.  $\Pi_{\ker \mathbb{B}_1(z)}$ ) with respect to the decomposition  $\mathbb{C}^{(\ell+r)N} = \ker \mathbb{B}_0(z) \oplus \ker \mathbb{B}_1(z)$  for all  $z \in \overline{\mathcal{U}}$ .

### 4.2.3 Construction of a dual problem for the resolvent equation {part\_dual}

In this paragraph we construct a dual problem for the scheme (29).

In the continuous setting dual problems are commonly used to show the existence of a solution of the partial differential equation. The argument is the following: in a first time an *a priori* energy estimate (the equivalent of the stability inequality (31), in the continuous setting) is established. Then we construct a dual problem to the continuous partial differential equation and show that this dual problem satisfies the *a priori* energy estimate. Consequently, by linearity, it shows the uniqueness of the solution of the dual problem and a duality argument gives the existence of a solution to the primal problem.

In the discrete setting this method is not used because the existence of a solution to (10) is clear. However, as first noticed in [Osher, 1973] for the corner geometry and then used in [Benoit, a] for the continuous problem in the strip, a simple way to construct a symmetrizer (used to show the *a priori* energy estimate (or equivalently the stability inequality (31))) is to consider the propagation operators, given by Duhamel formula for the dual problem.

The construction of the discrete symmetrizer for the discrete problem that we describe in Paragraph 5.2.2 also requires propagation operator for some dual scheme of approximation. That is why we introduce it and examine some of its properties.

Firstly we write (29) under the form:

$$\begin{cases} \mathbb{L}_{pri}(z)\mathcal{W}_j = \mathcal{F}_j & j \in [1, K], \\ \mathbb{B}_0(z)\mathcal{W}_1 = \mathcal{G}_0, \\ \mathbb{B}_1(z)\mathcal{W}_{K+1} = \mathcal{G}_1, \end{cases} \quad (37) \quad \{\text{eq\_total\_res}\}$$

where  $\mathbb{L}_{pri}(z)\mathcal{W}_j := \mathcal{W}_{j+1} - \mathbb{M}(z)\mathcal{W}_j$ .

**Definition 4.3** [Dual scheme] We say that the scheme

$$\begin{cases} \mathbb{L}_{dua}(z)\mathcal{V}_j = \widetilde{\mathcal{F}}_j & j \in \llbracket 1, K \rrbracket, \\ \mathbb{C}_0(z)\mathcal{V}_1 = \widetilde{\mathcal{G}}_0, \\ \mathbb{C}_1(z)\mathcal{V}_{K+1} = \widetilde{\mathcal{G}}_1, \end{cases} \quad (38) \quad \{\text{eq\_dual}\}$$

is a dual scheme for (37) if for all  $(\mathcal{V}_j)_{j \in \llbracket 1, K+1 \rrbracket} \subset \mathbb{C}^N$ ,  $(\mathcal{W}_j)_{j \in \llbracket 1, K+1 \rrbracket} \subset \mathbb{C}^N$  we have the duality formula:

$$\begin{aligned} \sum_{j=1}^K (\mathbb{L}_{pri}(z)\mathcal{W}_j)^* \mathcal{V}_{j+1} - \sum_{j=1}^K \mathcal{W}_j^* \mathbb{L}_{dua}(z)\mathcal{V}_j = & -(\mathcal{W}_1^* \mathbb{N}_0^*(z)\mathbb{C}_0(z)\mathcal{V}_1 + \mathcal{W}_1^* \mathbb{B}_0^*(z)\mathbb{M}_0(z)\mathcal{V}_1) \\ & + \mathcal{W}_{K+1}^* \mathbb{N}_1^*(z)\mathbb{C}_1(z)\mathcal{V}_{K+1} + \mathcal{W}_{K+1}^* \mathbb{B}_1^*(z)\mathbb{M}_1(z)\mathcal{V}_{K+1}, \end{aligned} \quad (39) \quad \{\text{eq\_def\_dual}\}$$

where the matrices  $\mathbb{C}_0(z), \mathbb{N}_0(z), \mathbb{M}_1(z) \in \mathbf{M}_{rN \times (\ell+r)N}(\mathbb{C})$  and  $\mathbb{C}_1(z), \mathbb{N}_1(z), \mathbb{M}_0(z) \in \mathbf{M}_{\ell N \times (\ell+r)N}(\mathbb{C})$

The following proposition, is an adaptation from the construction of [[Benzoni-Gavage, 2007]-Theorem 4.1 p116] in the continuous setting and shows the existence of a dual scheme.

**Proposition 4.5 (Existence of a dual problem)** Let  $\mathbb{L}_{dua}(z)\mathcal{V}_j := \mathcal{V}_j - \mathbb{M}^*(z)\mathcal{V}_{j+1}$ , then there exist matrices  $\mathbb{C}_0(z), \mathbb{N}_0(z), \mathbb{M}_1(z) \in \mathbf{M}_{rN \times (r+\ell)N}(\mathbb{C})$  and  $\mathbb{C}_1(z), \mathbb{N}_1(z), \mathbb{M}_1(z) \in \mathbf{M}_{\ell N \times (r+\ell)N}(\mathbb{C})$  such that (38) is a dual scheme for (37) in the sense of Definition 4.3.

Moreover the matrices  $\mathbb{C}_0(z), \mathbb{N}_0(z), \mathbb{M}_1(z), \mathbb{C}_1(z), \mathbb{N}_1(z)$  and  $\mathbb{M}_0(z)$  are characterized by the relations

$$I = \mathbb{C}_0^*(z)\mathbb{N}_0(z) + \mathbb{M}_0^*(z)\mathbb{B}_0(z) = \mathbb{C}_1^*(z)\mathbb{N}_1(z) + \mathbb{M}_1^*(z)\mathbb{B}_1(z), \quad (40) \quad \{\text{car\_dual1}\}$$

and

$$\ker \mathbb{C}_0(z) = \ker \mathbb{B}_0(z)^\perp, \quad \ker \mathbb{C}_1(z) = \ker \mathbb{B}_1(z)^\perp. \quad (41) \quad \{\text{car\_dual2}\}$$

**Proof :** We compute:

$$\begin{aligned} \sum_{j=1}^K (\mathbb{L}_{pri}(z)\mathcal{W}_j)^* \mathcal{V}_{j+1} &= \sum_{j=1}^K \mathcal{W}_{j+1}^* \mathcal{V}_{j+1} - \sum_{j=1}^K \mathcal{W}_j^* \mathbb{M}^*(z)\mathcal{V}_{j+1}, \\ &= \mathcal{W}_{K+1}^* \mathcal{V}_{K+1} - \mathcal{W}_1^* \mathcal{V}_1 + \sum_{j=1}^K \mathcal{W}_j^* (\mathcal{V}_j - \mathbb{M}^*(z)\mathcal{V}_{j+1}), \end{aligned}$$

so that

$$\sum_{j=1}^K (\mathbb{L}_{pri}(z)\mathcal{W}_j)^* \mathcal{V}_{j+1} - \sum_{j=1}^K \mathcal{W}_j^* \mathbb{L}_{dua}(z)\mathcal{V}_j = \mathcal{W}_{K+1}^* \mathcal{V}_{K+1} - \mathcal{W}_1^* \mathcal{V}_1,$$

and (39) follows if we can define matrices  $\mathbb{C}_0(z), \mathbb{N}_0(z), \mathbb{M}_1(z) \in \mathbf{M}_{rN \times (\ell+r)N}(\mathbb{C})$  and  $\mathbb{C}_1(z), \mathbb{N}_1(z), \mathbb{M}_0(z) \in \mathbf{M}_{\ell N \times (\ell+r)N}(\mathbb{C})$  such that (40) holds.

We describe here the construction of  $\mathbb{C}_0(z), \mathbb{N}_0(z) \in \mathbf{M}_{rN \times (\ell+r)N}(\mathbb{C})$  and  $\mathbb{M}_0(z) \in \mathbf{M}_{\ell N \times (\ell+r)N}(\mathbb{C})$  (the construction of  $\mathbb{C}_1(z), \mathbb{N}_1(z)$  and  $\mathbb{M}_1(z)$  follows the same lines).

The matrix  $\mathbb{B}_0(z)$  is clearly onto (because of the last block in its definition see (27)) so that we can find an onto matrix  $\mathbb{N}_0(z) \in \mathbf{M}_{rN \times (r+\ell)N}(\mathbb{C})$  such that the matrix  $\begin{bmatrix} \mathbb{B}_0(z) \\ \mathbb{N}_0(z) \end{bmatrix} \in \mathbf{GL}_{(\ell+r)N \times (\ell+r)N}(\mathbb{C})$  is invertible. We denote its inverse in the block form  $\begin{bmatrix} \mathbb{Y}_0(z) & \mathbb{D}_0(z) \end{bmatrix}$  where  $\mathbb{D}_0(z) \in \mathbf{M}_{(\ell+r)N \times rN}(\mathbb{C})$  and  $\mathbb{Y}_0(z) \in \mathbf{M}_{(\ell+r)N \times \ell N}(\mathbb{C})$ , so we have  $I = \mathbb{D}_0(z)\mathbb{N}_0(z) + \mathbb{Y}_0(z)\mathbb{B}_0(z)$  and the first equality in (40) holds if we define  $\mathbb{C}_0(z) = \mathbb{D}_0^*(z)$  and  $\mathbb{M}_0(z) = \mathbb{Y}_0^*(z)$ .

We now turn to the characterization (41). From its definition we have  $\ker \mathbb{C}_0(z) = \text{Ran } \mathbb{D}_0(z)^\perp = \ker \mathbb{B}_0(z)^\perp$ , because  $\mathbb{B}_0(z)\mathbb{D}_0(z) = 0$  in  $\mathbf{M}_{\ell N \times rN}(\mathbb{C})$ .



We denote by  $\tilde{\mathbb{E}}^s(z)$  (resp.  $\tilde{\mathbb{E}}^u(z)$ ) the stable (resp. unstable) subspace associated to the dual scheme (38). Let  $z \in \mathcal{U}$  because of the definition of  $\mathbb{L}_{dual}$  we have the decomposition  $\mathbb{C}^{(\ell+r)N} = \tilde{\mathbb{E}}^s(z) \oplus \tilde{\mathbb{E}}^u(z)$  for all  $z \in \mathcal{U}$ . Moreover the subspace  $\tilde{\mathbb{E}}^s(z)$  and  $\tilde{\mathbb{E}}^u(z)$  are characterized by the following proposition

**Proposition 4.6** *For all  $z \in \mathcal{U}$  we have the equalities:*

$$\tilde{\mathbb{E}}^s(z) = \mathbb{E}^s(z)^\perp, \quad \tilde{\mathbb{E}}^u(z) = \mathbb{E}^u(z)^\perp. \quad (42) \quad \{\text{eq\_prop\_dual}\}$$

**Proof :** Let  $\mathcal{V} \in \mathbb{E}^s(z)$  and  $\mathcal{W} \in \tilde{\mathbb{E}}^s(z)$ . Consider  $(\mathcal{V}_j)_{j \geq 1}$  (resp.  $(\mathcal{W}_j)_{j \geq 1}$ ) the solution of  $\mathbb{L}_{pri} \mathcal{V}_j = 0$ ,  $\mathcal{V}_1 = \mathcal{V}$ ,  $j \geq 1$  (resp.  $\mathbb{L}_{dual} \mathcal{W}_j = 0$ ,  $\mathcal{W}_1 = \mathcal{W}$ ). Then from the definition of the primal and the dual schemes we have that for all  $j \geq 1$ :

$$\mathcal{W}_j^* \mathcal{V}_j = \mathcal{W}_{j+1}^* \mathbb{M}(z) \mathbb{M}^{-1}(z) \mathcal{V}_{j+1} = \mathcal{W}_{j+1}^* \mathcal{V}_{j+1},$$

so the sequence  $(\mathcal{W}_j^* \mathcal{V}_j)_{j \geq 1}$  is constant and by definition of the stable subspaces this constant is zero. So that we obtain the first equality in (42). The proof of the second one is totally similar.  $\square$

The following proposition states that the fulfilment of the discrete uniform Kreiss-Lopatinskii condition for the primal problem is equivalent to the one for the dual problem. Once again the proof is adapted from the one of [[Benzoni-Gavage, 2007]-Theorem 4.2, p117] in the continuous setting.

**Proposition 4.7** *Let  $z \in \mathcal{U}$ , then we have the equivalence*

$$\begin{aligned} \mathbb{C}^{(\ell+r)N} = \ker \mathbb{B}_0(z) \oplus \mathbb{E}^s(z) &\Leftrightarrow \mathbb{C}^{(\ell+r)N} = \ker \mathbb{C}_0(z) \oplus \tilde{\mathbb{E}}^s(z). \\ \mathbb{C}^{(\ell+r)N} = \ker \mathbb{B}_1(z) \oplus \mathbb{E}^u(z) &\Leftrightarrow \mathbb{C}^{(\ell+r)N} = \ker \mathbb{C}_1(z) \oplus \tilde{\mathbb{E}}^u(z). \end{aligned}$$

Consequently for all  $z \in \mathcal{U}$  we define  $\tilde{\Phi}_0(z)$  (resp.  $\tilde{\Phi}_1(z)$ ) the inverse of the restriction of  $\mathbb{C}_0(z)$  (resp.  $\mathbb{C}_1(z)$ ) to  $\tilde{\mathbb{E}}^s(z)$  (resp.  $\tilde{\mathbb{E}}^u(z)$ ).

Moreover  $\Phi_0(z)$  (resp.  $\Phi_1(z)$ ) is uniformly bounded with respect to  $z \in \mathcal{U}$  if and only if  $\tilde{\Phi}_0(z)$  (resp.  $\tilde{\Phi}_1(z)$ ) is uniformly bounded with respect to  $z \in \mathcal{U}$ .

**Proof :** We only consider the boundary condition on the left-hand side, the proof follows the same lines for the right one. Let  $z \in \mathcal{U}$  we assume that  $\mathbb{C}^{(\ell+r)N} = \ker \mathbb{B}_0(z) \oplus \mathbb{E}^s(z)$ . Let  $\mathcal{W} \in \tilde{\mathbb{E}}^s(z) \cap \ker \mathbb{C}_0(z)$ , then for all  $\mathcal{V} \in \mathbb{E}^s(z)$  we have, from the preceding proposition  $\mathcal{W}^* \mathcal{V} = 0$ . So that from (40) it implies that  $\mathcal{W} \in \ker \mathbb{C}_0(z) \cap \ker \mathbb{M}_0(z) = \{0\}$  and the converse can be demonstrated in the same way.

We now turn to the proof of the uniform bound on  $\tilde{\Phi}_0(z)$ , it amounts to show that for all  $z \in \mathcal{U}$  and for all  $\mathcal{W} \in \tilde{\mathbb{E}}^s(z)$ ,  $|\mathcal{W}| \leq C |\mathbb{C}_0(z) \mathcal{W}|$ . From (40) we have  $|\mathcal{W}| \leq C (|\mathbb{C}_0(z) \mathcal{W}| + |\mathbb{M}_0(z) \mathcal{W}|)$ . Assuming the discrete uniform Kreiss-Lopatinskii condition for the primal scheme holds we can write (using again (40)):

$$|\mathbb{M}_0(z) \mathcal{W}| \leq C \sup_{\mathcal{V} \in \mathbb{E}^s(z)} \left\{ \frac{|\langle \mathbb{B}_0(z) \mathcal{V}, \mathbb{M}_0(z) \mathcal{W} \rangle|}{|\mathcal{V}|} \right\} \leq C \sup_{\mathcal{V} \in \mathbb{E}^s(z)} \left\{ \frac{|\langle \mathcal{V}, (I - \mathbb{N}_0(z)^* \mathbb{C}_0(z)) \mathcal{W} \rangle|}{|\mathcal{V}|} \right\}$$

so that the desired result follows from (42).  $\square$

For later purpose we also give the following proposition. Once again it is a discrete version of the analogous one in the continuous setting (see [Benoit, a]). \{\text{prop\\_dual2}\}

**Proposition 4.8** *For all  $z \in \mathcal{U}$  we have the equalities:*

$$(\tilde{\Phi}_0(z) \mathbb{C}_0(z))^* = I - \Phi_0(z) \mathbb{B}_0(z), \quad (\tilde{\Phi}_1(z) \mathbb{C}_1(z))^* = I - \Phi_1(z) \mathbb{B}_1(z). \quad (43) \quad \{\text{eq\_prop\_dual}\}$$

**Proof :** We only show the first equality in (43), the proof of the second is totally similar. Because  $\widetilde{\Phi}_0(z)\mathbb{C}_0(z)$  is a projection  $(\widetilde{\Phi}_0(z)\mathbb{C}_0(z))^*$  is characterized by its kernel and its range. However on the one hand we have:

$$\begin{aligned}\ker(\widetilde{\Phi}_0(z)\mathbb{C}_0(z))^* &= \text{Ran}\widetilde{\Phi}_0(z)\mathbb{C}_0(z)^\perp = \widetilde{\mathbb{E}}^s(z)^\perp = \mathbb{E}^s(z), \\ \text{Ran}(\widetilde{\Phi}_0(z)\mathbb{C}_0(z))^* &= \ker\widetilde{\Phi}_0(z)\mathbb{C}_0(z)^\perp = \ker\mathbb{C}_0(z)^\perp = \ker\mathbb{B}_0(z).\end{aligned}$$

On the other hand  $I - \Phi_0(z)\mathbb{B}_0(z)$  is also a projection satisfying that

$$\ker(I - \Phi_0(z)\mathbb{B}_0(z)) = \mathbb{E}^s(z), \text{ and } \text{Ran}(I - \Phi_0(z)\mathbb{B}_0(z)) = \ker\mathbb{B}_0(z),$$

so we deduce the desired equality. □

## 5 Necessary and sufficient conditions for $LE$ -stability : proof of the main result {part\_proof}

### 5.1 Necessary conditions for $LE$ -stability

In this subsection we want to exhibit necessary conditions for the  $LE$ -stability of (10). From Proposition 4.4, the  $LE$ -stability of (10) in the sense of Definition 3.1 is equivalent to the one of the resolvent scheme (29) consequently we assume that

$$\begin{cases} \mathcal{W}_{j+1} = \mathbb{M}(z)\mathcal{W}_j & j \in \llbracket 1, K \rrbracket, \\ \mathbb{B}_0(z)\mathcal{W}_1 = \mathcal{G}_0, \\ \mathbb{B}_1(z)\mathcal{W}_{K+1} = 0, \end{cases} \quad (44) \quad \{\text{scheme\_nec1}\}$$

is  $LE$ -stable in the sense described in Proposition 4.4. We recall that the matrices  $\mathbb{M}(z)$ ,  $\mathbb{B}_0(z)$  and  $\mathbb{B}_1(z)$  are defined in (24), (27) and (28) respectively and that  $\mathcal{W}_j := (W_{j+r-1}, \dots, W_{j-\ell})^T \in \mathbb{C}^{(\ell+r)N}$ .

We consider  $(\widetilde{\mathcal{W}}_j)_{j \geq 1}$  the extension of  $(\mathcal{W}_j)_{j \in \llbracket 1, K+1 \rrbracket}$  by zero for  $j > K+1$ . Clearly,  $(\widetilde{\mathcal{W}}_j)_{j \geq 1} \in \ell^2(\llbracket 1, \infty \rrbracket)$  satisfies:

$$\begin{cases} \widetilde{\mathcal{W}}_{j+1} = \mathbb{M}(z)\widetilde{\mathcal{W}}_j + \widetilde{\mathcal{F}}_j(z) & j \geq 1, \\ \mathbb{B}_0(z)\widetilde{\mathcal{W}}_1 = \mathcal{G}_0, \end{cases} \quad (45) \quad \{\text{scheme\_nec2}\}$$

where  $(\widetilde{\mathcal{F}}_j)_{j \geq 1}$  is an error term due to the extension by zero given by  $\widetilde{\mathcal{F}}_j(z) := -\delta_{j, K+1}\mathbb{M}(z)\mathcal{W}_j$ , where  $\delta_{\cdot, \cdot}$  stands for the Kronecker symbol.

We thus have

$$\forall j \geq 1, \widetilde{\mathcal{W}}_j = \mathbb{M}(z)^{j-1}\mathcal{W}_1 + \sum_{k=1}^{j-1} \mathbb{M}(z)^{j-1-k}\widetilde{\mathcal{F}}_k(z).$$

For  $z \in \mathcal{U}$  from the decomposition (32) (see Definition 4.2) we decompose the solution  $(\widetilde{\mathcal{W}}_j)_{j \geq 1}$  of (45) as:

$$\widetilde{\mathcal{W}}_j := \Pi^s(z)\widetilde{\mathcal{W}}_j + \Pi^u(z)\widetilde{\mathcal{W}}_j,$$

where each term in the right hand side is given by:

$$\begin{aligned}\Pi^s(z)\widetilde{\mathcal{W}}_j &= \mathbb{M}(z)^{j-1}\Pi^s(z)\mathcal{W}_1 + \sum_{k=1}^{j-1} \mathbb{M}(z)^{j-1-k}\Pi^s(z)\widetilde{\mathcal{F}}_k(z), \\ \Pi^u(z)\widetilde{\mathcal{W}}_j &= -\sum_{k=j}^{\infty} \mathbb{M}(z)^{j-1-k}\Pi^u(z)\widetilde{\mathcal{F}}_k(z).\end{aligned}$$

But from Assumption 4.1, the discrete uniform Kreiss-Lopatinskii condition holds for the boundary condition on  $\{x = 0\}$  so that we can write the stable part of  $\mathscr{W}_1$  as a function of the unstable part. More precisely we have

$$\Pi^s(z)\widetilde{\mathscr{W}}_1 = \Phi_0(z)\mathscr{G}_0 - \Phi_0(z)\mathbb{B}_0(z)\Pi^u(z)\widetilde{\mathscr{W}}_1,$$

from which we deduce that

$$\widetilde{\mathscr{W}}_1 = \Phi_0(z)\mathscr{G}_0 + (\Phi_0(z)\mathbb{B}_0(z) - I) \sum_{k=1}^{\infty} \mathbb{M}(z)^{-k} \Pi^u(z) \widetilde{\mathscr{F}}_k(z). \quad (46) \quad \{\text{necitracebru}$$

However recall that from the definition of  $(\widetilde{\mathscr{F}}_k)_{k \in \mathbb{Z}}$  all the terms (except the one associated to  $k = K + 1$ ) are zero so that the sum in the right-hand side of (46) reduces to:

$$\begin{aligned} \widetilde{\mathscr{W}}_1 &= \widetilde{\mathscr{W}}_1 = \Phi_0(z)\mathscr{G}_0 + (\Phi_0(z)\mathbb{B}_0(z) - I) \mathbb{M}(z)^{-(K+1)} \Pi^u(z) \widetilde{\mathscr{F}}_{K+1}(z), \\ &= \Phi_0(z)\mathscr{G}_0 + (I - \Phi_0(z)\mathbb{B}_0(z)) \mathbb{M}(z)^{-K} \Pi^u(z) \mathscr{W}_{K+1}, \end{aligned} \quad (47) \quad \{\text{relation_tra}$$

where we used the definition of  $\widetilde{\mathscr{F}}_{K+1}$  combined with the fact that  $\mathbb{M}(z)$  and  $\Pi^u(z)$  commute.

So that we obtain a relation between the boundary value of  $(\mathscr{W}_j)_{j \in \llbracket 1, K+1 \rrbracket}$  on the left-hand side of the interval in terms of the one on right-hand side of the interval.

**Remark** Note that  $\mathscr{W}_1$  depends on the  $(W_j)_{j \in \llbracket 1-\ell, r \rrbracket}$  and that  $\mathscr{W}_{K+1}$  depend on the  $(W_j)_{j \in \llbracket K+1-\ell, K+r \rrbracket}$ . This observation justifies the fact that in the *LE*-stability definition (see Definition 3.1) we have considered as the boundary values the extended ones like in [Coulombel, 2011] and not the ones used in [B. Gustafsson and Sundstrom., 1972].

Then the extension of  $(\mathscr{W}_j)_{j \in \llbracket 1, K+1 \rrbracket}$  by zero for  $j \leq 0$ , namely  $(\widetilde{W}_j)_{j \leq K+1} \in \ell^2(\llbracket -\infty, K+1 \rrbracket)$  satisfies the scheme:

$$\begin{cases} \widetilde{\mathscr{W}}_j = \mathbb{M}^{-1}(z)\widetilde{\mathscr{W}}_{j+1} + \widetilde{\mathscr{F}}_j(z) & j \leq K, \\ \mathbb{B}_1(z)\widetilde{\mathscr{W}}_{K+1} = 0, \end{cases}$$

where the source term  $(\widetilde{\mathscr{F}}_j)_{j \leq K}$  is now given by  $\widetilde{\mathscr{F}}_j(z) := -\delta_{j,0} \mathbb{M}^{-1}(z) \mathscr{W}_1$ . So we have

$$\forall j \geq 0, \widetilde{\mathscr{W}}_{K-j} = \mathbb{M}(z)^{-(j+1)} \widetilde{\mathscr{W}}_{K+1} + \sum_{k=0}^j \mathbb{M}(z)^{-(j-k)} \widetilde{\mathscr{F}}_{K-k},$$

so reiterating the same kind of computations as the ones exposed for the extension by zero for  $j > K + 1$  leads us to the following relation between  $\mathscr{W}_{K+1}$  and  $\mathscr{W}_1$ :

$$\mathscr{W}_{K+1} = (I - \Phi_1(z)\mathbb{B}_1(z)) \mathbb{M}(z)^K \Pi^s(z) \mathscr{W}_1. \quad (48) \quad \{\text{relation_tra}$$

Combining (47) with (48) immediately gives that the trace  $\mathscr{W}_1$  satisfies the compatibility condition:

$$(I - (I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z)\Pi^u(z)(I - \Phi_1(z)\mathbb{B}_1(z))\mathbb{M}^K(z)\Pi^s(z))\mathscr{W}_1 = \Phi_0(z)\mathscr{G}_0. \quad (49) \quad \{\text{eq_compatibi}$$

The end of this paragraph is devoted to a study of (49) to obtain two invertibility conditions from the compatibility condition (49).

Let  $z \in \mathscr{U}$ , identifying equation (49) on  $\mathbb{E}^s(z)$  and  $\mathbb{E}^u(z)$  with respect to the decomposition  $\mathbb{C}^{(\ell+r)N} = \mathbb{E}^s(z) \oplus \mathbb{E}^u(z)$  gives

$$\begin{cases} \Pi^s(z)\mathscr{W}_1 - \Pi^s(z)(I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z)\Pi^u(z)(I - \Phi_1(z)\mathbb{B}_1(z))\mathbb{M}^K(z)\Pi^s(z)\mathscr{W}_1 = \Phi_0(z)\mathscr{G}_0, \\ \Pi^u(z)\mathscr{W}_1 = \Pi^u(z)(I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z)\Pi^u(z)(I - \Phi_1(z)\mathbb{B}_1(z))\mathbb{M}^K(z)\Pi^s(z)\mathscr{W}_1. \end{cases} \quad (50) \quad \{\text{eq_premier_s}$$

Let us consider the first equation of (50). Indeed if this equation determines the stable component of the boundary value then the second equation of (50) determines in a unique way the unstable component in terms of the stable one.

Some computations gives:

$$\begin{aligned}
\Pi^s(z)\mathscr{W}_1 - \Pi^s(z)(I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z)\Pi^u(z)(I - \Phi_1(z)\mathbb{B}_1(z))\mathbb{M}^K(z)\Pi^s(z)\mathscr{W}_1 \\
= \Pi^s(z)\mathscr{W}_1 + \Pi^s(z)(I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z)\Pi^s(z)\mathscr{W}_1 \\
- \Pi^s(z)(I - \Phi_0(z)\mathbb{B}_0(z))\underbrace{\mathbb{M}^{-K}(z)\Pi^u(z)\mathbb{M}^K(z)\Pi^s(z)}_{=0}\mathscr{W}_1 \\
= \Pi^s(z)\mathscr{W}_1 - \Phi_0(z)\mathbb{B}_0(z)\mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z)\Pi^s(z)\mathscr{W}_1 \\
+ \underbrace{\Pi^s(z)\mathbb{M}^{-K}(z)\Pi^u(z)}_{=0}\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z)\Pi^s(z)\mathscr{W}_1.
\end{aligned}$$

Consequently we define the matrices  $\mathbb{T}_{0 \rightarrow 1}(z), \mathbb{T}_{1 \rightarrow 0}(z) \in \mathbf{M}_{(\ell+r)N \times (\ell+r)N}(\mathbb{C})$  by the relations:

$$\forall z \in \mathscr{U}, \mathbb{T}_{1 \rightarrow 0}(z) := \Phi_0(z)\mathbb{B}_0(z)\mathbb{M}^{-K}(z) \text{ and } \mathbb{T}_{0 \rightarrow 1}(z) := \Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z), \quad (51) \quad \{\text{def\_operateur}\}$$

and the first equation of (50) becomes:

$$(I - \mathbb{T}_{1 \rightarrow 0}(z)\mathbb{T}_{0 \rightarrow 1}(z))\Pi^s(z)\mathscr{W}_1 = \Phi_0(z)\mathscr{G}_0,$$

however recall that  $\Phi_0(z)\mathbb{C}^{\ell N} = \mathbb{E}^s(z)$ , so that it implies that if we define  $\mathbb{T}_{0 \rightarrow 0}(z) := \mathbb{T}_{1 \rightarrow 0}(z)\mathbb{T}_{0 \rightarrow 1}(z)$  then this operator is invertible from  $\mathbb{E}^s(z)$  to  $\mathbb{E}^s(z)$  and  $\Pi^s(z)\mathscr{W}_1$  is determined in a unique way in terms of the known source term  $\mathscr{G}_0$ . So that from the second equation of (50) we can also determine  $\Pi^u(z)\mathscr{W}_1$  and we deduce that:

$$\mathscr{W}_1 = (I - \mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z))(I - \mathbb{T}_{0 \rightarrow 0}(z))_{|\mathbb{E}^s(z)}^{-1}\Phi_0\mathscr{G}_0.$$

From the stability estimate (31) it turns out that there exists  $C > 0$  independent of  $z$  such that:

$$|(I - \mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z))(I - \mathbb{T}_{0 \rightarrow 0}(z))_{|\mathbb{E}^s(z)}^{-1}\Phi_0(z)\mathscr{G}_0| \leq C|\mathscr{G}_0|,$$

so that using the fact that the decomposition  $\mathbb{C}^{(\ell+r)N} = \mathbb{E}^s(z) \oplus \mathbb{E}^u(z)$  implies that the matrix  $(I - \mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z))$  is injective on  $\mathbb{E}^s(z)$  and the discrete uniform Kreiss-Lopatinskii condition for the left-hand side boundary condition we obtain the following proposition:

**Proposition 5.1 (First necessary condition for  $LE$ -stability)** *Assume that the resolvent problem (29) is  $LE$ -stable then for all  $z \in \mathscr{U}$  the matrix  $I - \mathbb{T}_{0 \rightarrow 0}(z)$  is uniformly invertible from  $\mathbb{E}^s(z)$  to  $\mathbb{E}^s(z)$ . More precisely there exists  $C > 0$  such that*

$$\forall z \in \mathscr{U}, \forall \mathscr{W} \in \mathbb{E}^s(z), |\mathscr{W}| \leq |(I - \mathbb{T}_{0 \rightarrow 0}(z))\mathscr{W}|. \quad \{\text{hyp\_inv1}\}$$

To obtain the second necessary condition for  $LE$ -stability we multiply (49) by  $\overline{\Pi}^s(z)$  and  $\overline{\Pi}_{\ker \mathbb{B}_0}(z)$  where we recall that these projections are the ones associated to the decomposition  $\mathbb{C}^{(\ell+r)N} = \ker \mathbb{B}_0(z) \oplus \mathbb{E}^s(z)$ ,  $z \in \overline{\mathscr{U}}$  given by the discrete uniform Kreiss-Lopatinskii condition on the left-hand side. Then we obtain the system:

$$\begin{cases} \overline{\Pi}^s(z)\mathscr{W}_1 = \Phi_0(z)\mathscr{G}_0, \\ \overline{\Pi}_{\ker \mathbb{B}_0}\mathscr{W}_1 = (I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z)\Pi^s(z)\mathscr{W}_1, \end{cases} \quad (52) \quad \{\text{eq\_second\_sy}\}$$

where we used the fact that  $\text{Ran}(I - \Phi_0(z)\mathbb{B}_0(z)) \subset \ker \mathbb{B}_0(z)$ .

The first equation of (52) determines in a unique way  $\overline{\Pi}^s(z)\mathscr{W}_1$  in terms of the source term  $\mathscr{G}_0$ . To obtain an invertibility condition like the one of Proposition 5.1 we thus make the right-hand side of the second equation more explicit in terms of  $\overline{\Pi}_{\ker \mathbb{B}_0}\mathscr{W}_1$ . Some computations give

$$\begin{aligned}
& (I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z)\Pi^s(z)\mathscr{W}_1 \\
&= (I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z)\underbrace{\Pi^s(z)\bar{\Pi}^s(z)}_{=\bar{\Pi}^s(z)}\mathscr{W}_1 \\
&\quad + (I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z)\Pi^s(z)\bar{\Pi}_{\ker \mathbb{B}_0(z)}\mathscr{W}_1, \\
&= (I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z)\Phi_0(z)\mathscr{G}_0 \\
&\quad - (I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z)(I - \Phi_1(z)\mathbb{B}_1(z))\mathbb{M}^K(z)\bar{\Pi}_{\ker \mathbb{B}_0(z)}\mathscr{W}_1 \\
&\quad - \underbrace{(I - \Phi_0(z)\mathbb{B}_0(z))\Pi^s(z)}_{=0}\bar{\Pi}_{\ker \mathbb{B}_0(z)}\mathscr{W}_1,
\end{aligned}$$

where we used the fact that for all  $\mathscr{X} \in \mathbb{C}^{(\ell+r)N}$  we have

$$(I - \Phi_1(z)\mathbb{B}_1(z))\mathbb{M}^K(z)\mathscr{X} = \underbrace{(I - \Phi_1\mathbb{B}_1(z))\mathbb{M}^K(z)\Pi^u(z)}_{=0}\mathscr{X} + \mathbb{M}^{-K}(z)(I - \Phi_1(z)\mathbb{B}_1(z))\mathbb{M}^K(z)\Pi^s(z)\mathscr{X}.$$

So that we obtain the following compatibility condition on  $\bar{\Pi}_{\ker \mathbb{B}_0}(z)\mathscr{W}_1$ :

$$(I - \bar{\mathbb{T}}_{1 \rightarrow 0}(z)\bar{\mathbb{T}}_{0 \rightarrow 1}(z))\bar{\Pi}_{\ker \mathbb{B}_0}(z)\mathscr{W}_1 = (I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z)\Phi_0(z)\mathscr{G}_0,$$

where the matrices  $\bar{\mathbb{T}}_{0 \rightarrow 1}, \bar{\mathbb{T}}_{1 \rightarrow 0} \in \mathbf{M}_{(\ell+r)N \times (\ell+r)N}(\mathbb{C})$  are defined by

$$\forall z \in \mathscr{U}, \bar{\mathbb{T}}_{0 \rightarrow 1}(z) := (I - \Phi_1(z)\mathbb{B}_1(z))\mathbb{M}^K(z) \text{ and } \bar{\mathbb{T}}_{1 \rightarrow 0}(z) := (I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{M}^{-K}(z). \quad (53) \quad \{\text{def\_operator}\}$$

We also define  $\bar{\mathbb{T}}_{0 \rightarrow 0} := \bar{\mathbb{T}}_{1 \rightarrow 0}(z)\bar{\mathbb{T}}_{0 \rightarrow 1}(z)$ .

Using the fact that  $(I - \Phi_0(z)\mathbb{B}_0(z))\mathbb{E}^u(z) = \ker \mathbb{B}_0(z)$  we thus obtain the following necessary condition for  $LE$ -stability:

**Proposition 5.2 (Second necessary condition for strong stability)** *If the resolvent problem (29) is  $LE$ -stable then for all  $z \in \mathscr{U}$  the operator  $I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z)$  is invertible from  $\ker \mathbb{B}_0(z)$  to  $\ker \mathbb{B}_0(z)$ .* {\hyp\\_inv2}

Note that compared to Proposition 5.1, Proposition 5.2 does not require any uniform bound of the inverse<sup>5</sup> of  $I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z)$ . In the following paragraph we will however show that assuming some uniform property on  $(I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z))|_{\ker \mathbb{B}_0(z)}^{-1}$  in terms of  $z$  is a sufficient condition for the  $LE$ -stability of the resolvent problem (29).

## 5.2 Sufficient conditions for $LE$ -stability via symmetrizer

Let us assume that the conclusion of holds Proposition 5.1 and that the following stronger version of the conclusion of Proposition 5.2 holds. {\hyp\\_inv\\_util}

**Assumption 5.1** *For all  $z \in \mathscr{U}$  we assume that:*

i) *the matrix  $I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z)$  is uniformly invertible from  $\mathbb{E}^s(z)$  to  $\mathbb{E}^s(z)$ , that is*

$$\exists C > 0 \text{ s.t. } \forall z \in \mathscr{U}, \forall \mathscr{V} \in \mathbb{E}^s(z), |\mathscr{V}| \leq C|(I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z))\mathscr{V}|.$$

ii) *The matrix  $I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z)$  is uniformly invertible from  $\ker \mathbb{B}_0(z)$  to  $\ker \mathbb{B}_0(z)$ , that is*

$$\exists \bar{C} > 0 \text{ s.t. } \forall z \in \mathscr{U}, \forall \mathscr{W} \in \ker \mathbb{B}_0(z), |\mathscr{W}| \leq \bar{C}|(I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z))\mathscr{W}|.$$

The aim of the following paragraphs is to show that Assumption 5.1 is a sufficient condition for the  $LE$ -stability of the resolvent problem (29) and consequently for the one of (10). The proof of this sufficiency holds on the construction of a discrete symmetrizer for the resolvent problem (29) and consequently follows

<sup>5</sup>In the author's opinion it is probably possible to use (49) to show that the uniform bound is necessary.

the same kinds of methods that the one used in the continuous framework in the strip exposed in [Benoit, a].

Let us recall that the  $LE$ -stability of the resolvent problem (29) amounts to show two points, the first one is the stability estimate (31) while the second one is the existence of a solution to (29). The fact that the existence of a discrete symmetrizer implies the stability estimate (31) is a direct consequence of its definition (see Paragraph 5.2.1).

Then the construction of the symmetrizer is described in Paragraph 5.2.2. An important point to keep in mind is that the construction of the discrete symmetrizer only requires  $ii$ ) of Assumption 5.1.

Finally the construction of a solution to (29) is made in Paragraph 5.2.3, requires the dual problem introduced in Paragraph 4.2.3 and uses  $i$ ) of Assumption 5.1. Indeed  $i$ ) of Assumption 5.1 for the primal scheme is nothing but  $ii$ ) of Assumption 5.1 so it gives the dual stability estimate from which we deduce the existence of a solution to the primal scheme.

It is thus interesting to remark that points  $i$ ) and  $ii$ ) are used at two totally different levels to establish the  $LE$ -stability of the resolvent problem (29).

### 5.2.1 Definition of the symmetrizer and the stability estimate

{part\_sym1}

In the following definition we introduce a tool, namely the discrete symmetrizer which gives the stability estimate for (29).

**Definition 5.1 (Discrete symmetrizer)** Under Assumption 2.2 we say that  $\mathbb{S}_j = \mathbb{S}_j(z)$ ,  $j \in \llbracket 1, K+1 \rrbracket$ ,  $z \in \mathcal{U}$  is a symmetrizer for (29) if  $(\mathbb{S}_j)_{j \in \llbracket 1, K+1 \rrbracket} \in \mathbf{M}_{(\ell+r)N \times (\ell+r)N}(\mathbb{C})$  satisfies the following properties:

{def\_sym\_1}

- $i$ ) for all  $z \in \mathcal{U}$ , for all  $j \in \llbracket 1, K+1 \rrbracket$ ,  $\mathbb{S}_j^*(z) = \mathbb{S}_j(z)$ .
- $ii$ ) There exist  $C_0, C_1 > 0$  such that for all  $z \in \mathcal{U}$ , for all  $\mathcal{V} \in \mathbb{E}^s(z)$  and for all  $\mathcal{W} \in \mathbb{E}^u(z)$  we have

$$|\mathbb{S}_1(z)\mathcal{V}| \leq C_0|\mathcal{V}| \text{ and } |\mathbb{S}_{K+1}(z)\mathcal{W}| \leq C_1|\mathcal{W}|,$$

where we stress that  $C_0$  and  $C_1$  do not depend on  $z$ .

- $iii$ ) For all  $z \in \mathcal{U}$ , for all  $j \in \llbracket 1, K \rrbracket$ ,  $\mathbb{S}_j(z)$  satisfies

$$\mathbb{S}_j(z) = \mathbb{M}^*(z)\mathbb{S}_{j+1}(z)\mathbb{M}(z).$$

- $iv$ ) There exists  $\varepsilon_0 > 0$  such that for all  $z \in \mathcal{U}$  and  $\mathcal{V} \in \ker \mathbb{B}_0(z)$  we have:

$$\mathcal{V}^*\mathbb{S}_1(z)\mathcal{V} \leq -\varepsilon_0|\mathcal{V}|^2.$$

- $v$ ) There exists  $\varepsilon_1 > 0$  such that for all  $z \in \mathcal{U}$  and  $\mathcal{W} \in \ker \mathbb{B}_1(z)$  we have:

$$\mathcal{W}^*\mathbb{S}_{K+1}(z)\mathcal{W} \geq \varepsilon_1|\mathcal{W}|^2.$$

The following proposition shows that the existence of a symmetrizer in the sense of Definition 5.1 is a sufficient condition for the stability estimate (31) to hold.

**Proposition 5.3** Under Assumptions 2.1, 2.2 and 4.1, assume that the scheme (29) admits a symmetrizer in the sense of Definition 5.1 then the solution of (29) satisfies the stability estimate (31).

**Proof :** Let  $\mathcal{W}_j$  be the solution of (29), multiply (in the left-hand side) the interior equation of (29) by  $\mathcal{W}_{j+1}^*\mathbb{S}_{j+1}(z)$  and sum for  $j \in \llbracket 1, K \rrbracket$  gives

$$\sum_{j=1}^K \mathcal{W}_{j+1}^*\mathbb{S}_{j+1}(z)\mathcal{W}_{j+1} - \sum_{j=1}^K \mathcal{W}_{j+1}^*\mathbb{S}_{j+1}(z)\mathbb{M}(z)\mathcal{W}_j = 0,$$

consequently using the relation  $\mathcal{W}_{j+1} = \mathbb{M}(z)\mathcal{W}_j$  we obtain:

$$\mathcal{W}_{K+1}^*\mathbb{S}_{K+1}(z)\mathcal{W}_{K+1} - \mathcal{W}_1^*\mathbb{S}_1(z)\mathcal{W}_1 + \sum_{j=1}^K \mathcal{W}_j^*\mathbb{S}_j(z)\mathcal{W}_j - \sum_{j=1}^K \mathcal{W}_j^*\mathbb{M}^*(z)\mathbb{S}_{j+1}(z)\mathbb{M}(z)\mathcal{W}_j = 0,$$

so that we have the equality:

$$\mathcal{W}_{K+1}^* \mathbb{S}_{K+1}(z) \mathcal{W}_{K+1} = \mathcal{W}_1^* \mathbb{S}_1(z) \mathcal{W}_1, \quad (54) \quad \{\text{chatonton}\}$$

by *iii*) of Definition 5.1.

From the discrete uniform Kreiss-Lopatinskii condition on the left-hand side boundary condition we decompose  $\mathcal{W}_1 = \mathcal{W}_1^b + \mathcal{W}_1^\sharp$  where  $\mathcal{W}_1^b \in \ker \mathbb{B}_0(z)$  and  $\mathcal{W}_1^\sharp \in \mathbb{E}^s(z)$  satisfies  $\mathbb{B}_0(z) \mathcal{W}_1^\sharp = \mathcal{G}_0$ . We thus have using *iv*) combined with Cauchy-Schwarz and Young inequality:

$$\mathcal{W}_1^* \mathbb{S}_1(z) \mathcal{W}_1 \leq -\frac{\varepsilon_0}{2} |\mathcal{W}_1^b|^2 + C \left( |\mathbb{S}_1(z) \mathcal{W}_1^\sharp|^2 + |\mathcal{W}_1|^2 \right)$$

so that we use the discrete uniform Kreiss-Lopatinskii condition for the left-hand side boundary condition combined with *ii*) of Definition 5.1 to obtain

$$\mathcal{W}_1^* \mathbb{S}_1(z) \mathcal{W}_1 \leq -\frac{\varepsilon_0}{2} |\mathcal{W}_1^b|^2 + C |\mathbb{B}_1(z) \mathcal{W}_1^\sharp|^2 \leq -\frac{\varepsilon_0}{2} |\mathcal{W}_1^b|^2 + C |\mathcal{G}_0|^2. \quad (55) \quad \{\text{chaton}\}$$

We then reiterate the same kind of computations for the term  $\mathcal{W}_{K+1}^* \mathbb{S}_{K+1}(z) \mathcal{W}_{K+1}$  to obtain the opposite estimate and (31) follows easily from (54) and (55).

□

## 5.2.2 Construction of the symmetrizer

{part\_sym}

The aim of this paragraph is to describe a precise construction of the symmetrizer. More precisely we aim to show the following proposition:

**Proposition 5.4** *Under Assumptions 2.1-4.1 and ii) of Assumption 5.1 then there exists a symmetrizer for (29) in the sense of Definition 5.1.*

We are looking for a discrete symmetrizer under the form:

$$\mathbb{S}_j(z) := \mathbb{J}_j(z) \mathbb{N}(z) \mathbb{J}_j^*(z), \quad (56) \quad \{\text{form_sym}\}$$

where  $\mathbb{J}_j(z), \mathbb{N}(z) \in \mathbf{M}_{(\ell+r)N \times (\ell+r)N}(\mathbb{C})$  have to be constructed and where  $\mathbb{N}(z)$  shall satisfy:

$$\mathbb{N}^*(z) = \mathbb{N}(z), \quad (57) \quad \{\text{cond_N}\}$$

in order that  $\mathbb{S}_j(z)$  reading (56) satisfies *i*) of Definition 5.1.

Let  $\mathbb{J}_j(z)$  be a solution to

$$\forall j \in \llbracket 1, K \rrbracket, \quad \mathbb{J}_j(z) = \mathbb{M}^*(z) \mathbb{J}_{j+1}(z). \quad (58) \quad \{\text{cond_J}\}$$

so that for  $\mathcal{V}_j \in \mathbb{C}^{(\ell+r)N}$ ,  $\mathbb{J}_j(z) \mathcal{V}_j$  is the solution to the so-called dual problem of Paragraph 4.2.3. Then it is clear that under this condition on  $\mathbb{J}_j$  a symmetrizer  $\mathbb{S}_j$  reading under the form (56) satisfies *iii*) of Definition 5.1. Indeed we have:

$$\mathbb{M}^*(z) \mathbb{S}_{j+1}(z) \mathbb{M}(z) = \mathbb{M}^*(z) \mathbb{J}_{j+1}(z) \mathbb{N}(z) (\mathbb{M}^*(z) \mathbb{J}_{j+1}(z))^* = \mathbb{J}_j(z) \mathbb{N}(z) \mathbb{J}_j^*(z) = \mathbb{S}_j(z).$$

We look for  $\mathbb{J}_j(z)$  in the decomposition form  $\mathbb{J}_{0,j}(z) + \mathbb{J}_{K,j}(z)$  where the matrices  $\mathbb{J}_{0,j}$  and  $\mathbb{J}_{1,j}$  are respectively defined by the following:

$$\begin{cases} \mathbb{J}_{0,j}(z) = \mathbb{M}^*(z) \mathbb{J}_{0,j+1}(z) & j \geq 1, \\ \mathbb{J}_{0,1}(z) = \tilde{\Phi}_0(z) \mathbb{C}_0(z), \end{cases} \quad (59) \quad \{\text{def_K_0}\}$$

and

$$\begin{cases} \mathbb{J}_{1,j}(z) = \mathbb{M}^*(z) \mathbb{J}_{1,j+1}(z) & j \leq K, \\ \mathbb{J}_{1,K+1}(z) = \tilde{\Phi}_1(z) \mathbb{C}_1(z), \end{cases} \quad (60) \quad \{\text{def_K_K}\}$$

where we recall that  $\mathbb{C}_0(z)$  and  $\mathbb{C}_1(z)$  are the boundary matrices for the dual problem and  $\tilde{\Phi}_0(z), \tilde{\Phi}_1(z)$  the associated discrete Kreiss-Lopatinskii inverse matrices (see Paragraph 4.2.3).

Equations (59) and (60) can be solved explicitly in terms of  $j$ . Indeed we have

$$\forall j \geq 1, \mathbb{J}_{0,j}(z) = \mathbb{M}(z)^{-(j-1)*} \tilde{\Phi}_0(z) \mathbb{C}_0(z) \text{ and } \forall j \leq K, \mathbb{J}_{1,j}(z) = \mathbb{M}(z)^{(K+1-j)*} \tilde{\Phi}_1(z) \mathbb{C}_1(z).$$

Now that *iii*) of Definition 5.1 is fulfilled we choose  $\mathbb{N}$  in (56) in a way ensuring that *iv*) and *v*) of Definition 5.1 hold.

These conditions read

$$\begin{cases} -((\mathbb{J}_{0,1}^*(z) + \mathbb{J}_{1,1}^*(z))\mathcal{V})^* \mathbb{N}(z) (\mathbb{J}_{0,1}^*(z) + \mathbb{J}_{1,1}^*(z))\mathcal{V} \leq -\varepsilon_0 |\mathcal{V}|^2 & \forall \mathcal{V} \in \ker \mathbb{B}_0(z), \\ ((\mathbb{J}_{0,K+1}^*(z) + \mathbb{J}_{1,K+1}^*(z))\mathcal{W})^* \mathbb{N}(z) (\mathbb{J}_{0,K+1}^*(z) + \mathbb{J}_{1,K+1}^*(z))\mathcal{W} \leq -\varepsilon_1 |\mathcal{W}|^2 & \forall \mathcal{W} \in \ker \mathbb{B}_1(z). \end{cases} \quad (61) \quad \{\text{confinement}\}$$

From Proposition 4.8 combined with the explicit expressions of  $\mathbb{J}_{0,j}$  and  $\mathbb{J}_{1,j}$  it turns out that

$$\mathbb{J}_{0,1}^*(z) + \mathbb{J}_{1,1}^*(z) = I - \Phi_0(z) \mathbb{B}_0(z) + \bar{\mathbb{T}}_{0 \rightarrow 1}(z) \text{ and } \mathbb{J}_{0,K+1}^*(z) + \mathbb{J}_{1,K+1}^*(z) = \bar{\mathbb{T}}_{1 \rightarrow 0}(z) + I - \Phi_1(z) \mathbb{B}_1(z). \quad (62) \quad \{\text{boules\_quies}\}$$

We write  $\mathbb{N}(z)$  under the form

$$\mathbb{N}(z) := \Pi_{\ker \mathbb{B}_0}^*(z) \mathbb{N}_0(z) \Pi_{\ker \mathbb{B}_0}(z) + \Pi_{\ker \mathbb{B}_1}^*(z) \mathbb{N}_1(z) \Pi_{\ker \mathbb{B}_1}(z) + 2\text{Re}(\Pi_{\ker \mathbb{B}_0}^*(z) \mathbb{N}_{01}(z) \Pi_{\ker \mathbb{B}_1}(z)) \quad (63) \quad \{\text{boules\_quies}\}$$

where  $\Pi_{\ker \mathbb{B}_0}(z)$  (resp.  $\Pi_{\ker \mathbb{B}_1}(z)$ ) is the projection on  $\ker \mathbb{B}_0(z)$  (resp.  $\ker \mathbb{B}_1(z)$ ) with respect to the decomposition  $\mathbb{C}^{(\ell+r)N} = \ker \mathbb{B}_0(z) \oplus \ker \mathbb{B}_1(z)$ . So that the boundary conditions (61) are equivalent to

$$\begin{cases} -\mathcal{V}^* \mathbb{N}_0(z) \mathcal{V} - \mathcal{V}^* \bar{\mathbb{T}}_{0 \rightarrow 1}^*(z) \mathbb{N}_1(z) \bar{\mathbb{T}}_{0 \rightarrow 1}(z) \mathcal{V} - 2\mathcal{V}^* \text{Re}(\mathbb{N}_{01}(z) \bar{\mathbb{T}}_{0 \rightarrow 1}(z)) \mathcal{V} \leq -\varepsilon_0 |\mathcal{V}|^2 & \forall \mathcal{V} \in \ker \mathbb{B}_0(z), \\ \mathcal{W}^* \bar{\mathbb{T}}_{1 \rightarrow 0}^*(z) \mathbb{N}_0(z) \bar{\mathbb{T}}_{1 \rightarrow 0}(z) \mathcal{W} + \mathcal{W}^* \mathbb{N}_1(z) \mathcal{W} + 2\mathcal{W}^* \text{Re}(\mathbb{N}_{01}^*(z) \bar{\mathbb{T}}_{1 \rightarrow 0}(z)) \mathcal{W} \leq -\varepsilon_1 |\mathcal{W}|^2 & \forall \mathcal{W} \in \ker \mathbb{B}_1(z). \end{cases}$$

In the following we solve these equations in terms of positive matrices more precisely we look for matrices  $\mathbb{N}_0$ ,  $\mathbb{N}_1$  and  $\mathbb{N}_{01}$  such that

$$\begin{cases} -\mathbb{N}_0(z) - \bar{\mathbb{T}}_{0 \rightarrow 1}^*(z) \mathbb{N}_1(z) \bar{\mathbb{T}}_{0 \rightarrow 1}(z) - 2\text{Re}(\mathbb{N}_{01}(z) \bar{\mathbb{T}}_{0 \rightarrow 1}(z)) \leq -\varepsilon_0 I, \\ \bar{\mathbb{T}}_{1 \rightarrow 0}^*(z) \mathbb{N}_0(z) \bar{\mathbb{T}}_{1 \rightarrow 0}(z) + \mathbb{N}_1(z) + 2\text{Re}(\mathbb{N}_{01}^*(z) \bar{\mathbb{T}}_{1 \rightarrow 0}(z)) \leq -\varepsilon_1 I, \end{cases} \quad (64) \quad \{\text{confinement2}\}$$

Without loss of generality we set  $\varepsilon_0 = \varepsilon_1 = 1$ . The ideas to find good candidates for  $\mathbb{N}_0$ ,  $\mathbb{N}_1$  and  $\mathbb{N}_{01}$  solving (64) follows the ones exposed in [Osher, 1973] for the construction of symmetrizers for quarter space continuous problems or in [Benoit, a] for the strip geometry. We reproduce here this analysis to explain the appearance of the operator  $(I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z))|_{\ker \mathbb{B}_0(z)}^{-1}$ .

We multiply the second equation of (64) by  $\bar{\mathbb{T}}_{0 \rightarrow 1}^*$  in the left-hand side and by  $\bar{\mathbb{T}}_{0 \rightarrow 1}$  in the right-hand side. Then we sum with the first equation of (64) to obtain

$$\mathbb{N}_0(z) - \bar{\mathbb{T}}_{0 \rightarrow 0}^*(z) \mathbb{N}_0(z) \bar{\mathbb{T}}_{0 \rightarrow 0}(z) + 2\text{Re}((I - \bar{\mathbb{T}}_{0 \rightarrow 0}^*(z)) \mathbb{N}_{01}(z) \bar{\mathbb{T}}_{0 \rightarrow 1}(z)) \leq -I - \bar{\mathbb{T}}_{0 \rightarrow 1}^*(z) \bar{\mathbb{T}}_{0 \rightarrow 1}(z). \quad (65) \quad \{\text{rond}\}$$

We choose the weights  $\mathbb{N}_0$  and  $\mathbb{N}_{01}$  such that the equality holds in (65). So that we define<sup>6</sup>  $\mathbb{N}_0(z) := -I$  and  $\mathbb{N}_{01}(z) := (I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z)) \mathbb{V}(z)$  so that (65) becomes:

$$\mathbb{V}(z) \bar{\mathbb{T}}_{0 \rightarrow 1}(z) + \bar{\mathbb{T}}_{0 \rightarrow 1}^*(z) \mathbb{V}^*(z) + \bar{\mathbb{T}}_{0 \rightarrow 0}^*(z) \bar{\mathbb{T}}_{0 \rightarrow 0}(z) = -\bar{\mathbb{T}}_{0 \rightarrow 1}^*(z) \bar{\mathbb{T}}_{0 \rightarrow 1}(z).$$

We can check that  $\mathbb{V}(z) := -\frac{1}{2}(I + \bar{\mathbb{T}}_{1 \rightarrow 0}^*(z) \bar{\mathbb{T}}_{1 \rightarrow 0}(z))$  is a suitable choice to obtain the desired equality.

We then define  $\mathbb{N}_1(z)$  such that the equality in (64) holds. So the weighted are defined by:

$$\begin{aligned} \mathbb{N}_0(z) &:= -I, \quad \mathbb{N}_{01}(z) = -\frac{1}{2}(I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z))|_{\ker \mathbb{B}_0(z)}^* \bar{\mathbb{T}}_{0 \rightarrow 1}^*(z) (\bar{\mathbb{T}}_{1 \rightarrow 0}^*(z) \bar{\mathbb{T}}_{1 \rightarrow 0}(z) + I), \\ \text{and } \mathbb{N}_1(z) &:= I - \bar{\mathbb{T}}_{1 \rightarrow 0}^*(z) \bar{\mathbb{T}}_{1 \rightarrow 0}(z) - 2\text{Re}(\bar{\mathbb{T}}_{1 \rightarrow 0}^*(z) \mathbb{N}_{01}(z)), \end{aligned} \quad (66) \quad \{\text{def\_poids1}\}$$

<sup>6</sup>The definition given here is possibly not the sharpest one, however it has the advantage of simplicity. But others definitions are probably possible because in the author's opinion we do not have uniqueness in the definitions of  $\mathbb{N}_0$ ,  $\mathbb{N}_1$  and  $\mathbb{N}_{01}$ . Indeed (64) is essentially equivalent to the resolution of a system of two equations with three degrees of freedom.



so that it remains to show that with such coefficients the first equation of (64) is satisfied. This equation can be developed under the form

$$-\bar{\mathbb{T}}_{0 \rightarrow 1}^*(z)\bar{\mathbb{T}}_{0 \rightarrow 1}(z) + \bar{\mathbb{T}}_{0 \rightarrow 0}^*(z)\bar{\mathbb{T}}_{0 \rightarrow 0}(z) + 2\operatorname{Re}((I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z))^* \mathbb{N}_{01}(z)\bar{\mathbb{T}}_{0 \rightarrow 1}(z)) \leq 0,$$

which by definition of  $\mathbb{N}_{01}(z)$  is equivalent to

$$-\bar{\mathbb{T}}_{0 \rightarrow 1}^*(z)\bar{\mathbb{T}}_{0 \rightarrow 1}(z) + \bar{\mathbb{T}}_{0 \rightarrow 0}^*(z)\bar{\mathbb{T}}_{0 \rightarrow 0}(z) + \mathbb{V}(z)\bar{\mathbb{T}}_{0 \rightarrow 1}(z) + \mathbb{V}^*(z)\bar{\mathbb{T}}_{0 \rightarrow 1}(z) \leq 0 \iff -2\bar{\mathbb{T}}_{0 \rightarrow 0}^*(z)\bar{\mathbb{T}}_{0 \rightarrow 0}(z) \leq 0,$$

that is clearly satisfied so the weights defined in (66) satisfy (64).

To conclude we have to justify that with the weights defined in (66) the uniform boundedness property *ii*) of Definition 5.1 holds. In the following we only consider the first bound that is  $|\mathbb{S}_1(z)\mathcal{V}| \leq C_0|\mathcal{V}|$  for all  $\mathcal{V} \in \mathbb{E}^s(z)$ , the proof of the second one follows exactly the same computations. From the construction above we have that for  $\mathcal{V} \in \mathbb{E}^s(z)$ ,

$$\begin{aligned} \mathbb{S}_1(z)\mathcal{V} &= (\mathbb{J}_{0,1}(z) + \mathbb{J}_{1,1}(z))\mathbb{N}(z)(\mathbb{J}_{0,1}(z) + \mathbb{J}_{1,1}(z))^*\mathcal{V} \\ &= (\tilde{\Phi}_0(z)\mathbb{C}_0(z) + \mathbb{M}^{K^*}(z)\tilde{\Phi}_1(z)\mathbb{C}_1(z))\mathbb{N}(z)\bar{\mathbb{T}}_{0 \rightarrow 1}(z)\mathcal{V}, \end{aligned}$$

because of equation (62). Next we combine the fact that  $\operatorname{Ran}(\bar{\mathbb{T}}_{0 \rightarrow 1}(z)) \subset \ker \mathbb{B}_1(z)$  with the definition of  $\mathbb{N}(z)$  (see (63)) to obtain the equality

$$\mathbb{S}_1(z)\mathcal{V} = \left( \tilde{\Phi}_0(z)\mathbb{C}_0(z) + \mathbb{M}(z)^{K^*}\tilde{\Phi}_1(z)\mathbb{C}_1(z) \right) \left( \Pi_{\ker \mathbb{B}_1(z)}^* \mathbb{N}_1(z) + \Pi_{\ker \mathbb{B}_0(z)} \mathbb{N}_{01}(z) \right) \bar{\mathbb{T}}_{0 \rightarrow 1}(z)\mathcal{V}$$

We then use the following lemma to simplify one last time the expression of  $\mathbb{S}_1(z)\mathcal{V}$ .

**Lemma 5.1** *For all  $z \in \mathcal{U}$  we have the decomposition  $\mathbb{C}^{(\ell+r)N} = \ker \mathbb{C}_0(z) \oplus \ker \mathbb{C}_1(z)$ . If we denote by  $\Pi_{\ker \mathbb{C}_0(z)}$  (resp.  $\Pi_{\ker \mathbb{C}_1(z)}$ ) the projection upon  $\ker \mathbb{C}_0(z)$  (resp.  $\ker \mathbb{C}_1(z)$ ) with respect to the previous decomposition we thus have the identities*

$$\Pi_{\ker \mathbb{C}_0(z)} = \Pi_{\ker \mathbb{B}_1(z)}^* \text{ and } \Pi_{\ker \mathbb{C}_1(z)} = \Pi_{\ker \mathbb{B}_0(z)}^*.$$

So that we shall study for  $\mathcal{V} \in \mathbb{E}^s(z)$ ,

$$\mathbb{S}_1(z)\mathcal{V} = \left( \tilde{\Phi}_0(z)\mathbb{C}_0(z)\Pi_{\ker \mathbb{C}_1(z)}\mathbb{N}_{01}(z) + \mathbb{M}^{K^*}(z)\tilde{\Phi}_1(z)\mathbb{C}_1(z)\Pi_{\ker \mathbb{C}_0(z)}\mathbb{N}_1(z) \right) \bar{\mathbb{T}}_{0 \rightarrow 1}(z)\mathcal{V}. \quad (67) \quad \{\text{boules\_quies}\}$$

In the following we consider the two terms in the right-hand side of (67) separately. About the first one, the uniform Kreiss-Lopatinskii condition for the dual problem implies that it is sufficient to show that  $|\Pi_{\ker \mathbb{C}_1(z)}\mathbb{N}_{01}(z)\bar{\mathbb{T}}_{0 \rightarrow 1}(z)\mathcal{V}| \leq C|\mathcal{V}|$  holds for all  $\mathcal{V} \in \mathbb{E}^s(z)$ . From the definition of  $\mathbb{N}_{01}(z)$  we write

$$\begin{aligned} \Pi_{\ker \mathbb{C}_1(z)}\mathbb{N}_{01}(z)\bar{\mathbb{T}}_{0 \rightarrow 1}(z)\mathcal{V} &= \\ &= -\frac{1}{2} \left[ (I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z))_{|\ker \mathbb{B}_0(z)}^{-1} \Pi_{\ker \mathbb{B}_0(z)} \right]^* \bar{\mathbb{T}}_{0 \rightarrow 1}^*(z) \left( \bar{\mathbb{T}}_{1 \rightarrow 0}(z)^* \bar{\mathbb{T}}_{1 \rightarrow 0}(z) + I \right) \bar{\mathbb{T}}_{0 \rightarrow 1}(z)\mathcal{V}, \end{aligned}$$

so that from *ii*) of Assumption 5.1 we have that

$$|\Pi_{\ker \mathbb{C}_1(z)}\mathbb{N}_{01}(z)\bar{\mathbb{T}}_{0 \rightarrow 1}(z)\mathcal{V}| \leq C \left( |\bar{\mathbb{T}}_{0 \rightarrow 1}^*(z)\bar{\mathbb{T}}_{0 \rightarrow 1}(z)\mathcal{V}| + |\bar{\mathbb{T}}_{0 \rightarrow 1}^*(z)\bar{\mathbb{T}}_{1 \rightarrow 0}^*(z)\bar{\mathbb{T}}_{0 \rightarrow 0}(z)\mathcal{V}| \right).$$

We then use the explicit formulas

$$\bar{\mathbb{T}}_{0 \rightarrow 1}^*(z) = \mathbb{M}^{K^*}(z)\tilde{\Phi}_1(z)\mathbb{C}_1(z), \quad \bar{\mathbb{T}}_{0 \rightarrow 1}(z) = \mathbb{M}^{-K^*}(z)\tilde{\Phi}_0(z)\mathbb{C}_0(z),$$

which implies, because  $\operatorname{Ran} \tilde{\Phi}_0(z) = \tilde{\mathbb{E}}^s(z)$  and  $\operatorname{Ran} \tilde{\Phi}_1(z) = \tilde{\mathbb{E}}^u(z)$  that we that the uniform bounds  $|\bar{\mathbb{T}}_{0 \rightarrow 1}^*(z)\mathcal{V}| \leq C|\mathcal{V}|$  and  $|\bar{\mathbb{T}}_{1 \rightarrow 0}^*(z)\mathcal{V}| \leq C|\mathcal{V}|$  for all  $\mathcal{V} \in \mathbb{C}^{(\ell+r)N}$ . So we only have to show that  $|\bar{\mathbb{T}}_{0 \rightarrow 1}(z)\mathcal{V}| \leq C|\mathcal{V}|$  and  $|\bar{\mathbb{T}}_{0 \rightarrow 0}(z)\mathcal{V}| \leq C|\mathcal{V}|$  for all  $\mathcal{V} \in \mathbb{E}^s(z)$ . The first inequality is clear from the definition of  $\bar{\mathbb{T}}_{0 \rightarrow 1}(z)$  and the fact that  $\mathcal{V} \in \mathbb{E}^s(z)$  while the second one is not so clear at first glance because

$\text{Ran } \overline{\mathbb{T}}_{0 \rightarrow 1}(z) \subset \ker \mathbb{B}_1(z)$ . However from the discrete uniform Kreiss-Lopatinskii condition on the left-hand side boundary condition we have

$$|\overline{\mathbb{T}}_{0 \rightarrow 0}(z)\mathcal{V}| \leq C|\mathbb{M}^{-K}(z)(I - \Phi_1(z)\mathbb{B}_1(z))\mathbb{M}^K(z)\mathcal{V}| \leq C(|\mathcal{V}| + |\mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z)\mathcal{V}|),$$

and the result follows because  $\text{Ran}(\Phi_1(z)) \subset \mathbb{E}^u(z)$  and  $\mathcal{V} \in \mathbb{E}^s(z)$ .

We then have to consider the second term in the right-hand side of (67). We firstly remark that it is sufficient to consider the term  $\mathbb{N}_1(z)\overline{\mathbb{T}}_{0 \rightarrow 1}\mathcal{V}$  for  $\mathcal{V} \in \mathbb{E}^s(z)$ . From the definition of  $\mathbb{N}_1(z)$  we have

$$\mathbb{N}_1(z)\overline{\mathbb{T}}_{0 \rightarrow 1}(z)\mathcal{V} = \overline{\mathbb{T}}_{0 \rightarrow 1}(z)\mathcal{V} - \overline{\mathbb{T}}_{1 \rightarrow 0}^*(z)\overline{\mathbb{T}}_{0 \rightarrow 0}(z)\mathcal{V} - 2\text{Re}(\overline{\mathbb{T}}_{0 \rightarrow 1}^*(z)\mathbb{N}_{01}(z))\overline{\mathbb{T}}_{0 \rightarrow 1}(z)\mathcal{V},$$

and we treat the terms separately. The first and the second one have already been considered in the proof of the uniform boundedness of the first term in the right-hand side of (67). The third one is treated exactly as the first term in the left-hand side of (67) with  $\overline{\mathbb{T}}_{0 \rightarrow 1}(z)$  instead of  $\Pi_{\ker \mathbb{B}_0}(z)$  so that we will omit the details here.

This concludes the proof of the fact that the constructed symmetrizer satisfies *i-v*) of Definition 5.1 and more generally the one of the stability estimate.

We sum up the construction of the previous paragraph in the following proposition

**Proposition 5.5** *Under Assumptions 2.1, 2.2, 2.3, 4.1 and ii) of Assumption 5.1 the resolvent scheme (29) admits a symmetrizer in the sense of Definition 5.1. Consequently under Assumption 2.4 the energy estimate (30) is satisfied.*

### 5.2.3 Construction of a solution to (29).

{part\_cons\_so

To conclude the proof of the sufficiency of Assumptions 5.1 for the *LE*-stability it remains to construct a solution  $\mathcal{W}$  of the resolvent equation (29). The proof exposed here follows the same lines as the construction of a weak solution by the duality method used in the continuous setting.

We introduce the following subspace

$$\mathfrak{X} := \left\{ \mathbb{L}_{dua}\mathcal{V} \text{ where } \mathcal{V}_j \in \mathbb{C}^{(\ell+r)N}, j \in \llbracket 1, K+1 \rrbracket \text{ satisfying } \mathcal{V}_1 \in \ker \mathbb{C}_0(z) \text{ and } \mathcal{V}_{K+1} \in \ker \mathbb{C}_1(z) \right\},$$

and for  $(\mathcal{F}_j)_{j \in \llbracket 1, K \rrbracket} \subset \mathbb{C}^{(\ell+r)N}$ ,  $\mathcal{G}_0, \mathcal{G}_1 \in \mathbb{C}^{(\ell+r)N}$  the following form  $f$  on  $\mathfrak{X}$  defined by

$$f(\mathbb{L}_{dua}\mathcal{V}) := \sum_{j=1}^K \mathcal{F}_j^* \mathcal{V}_{j+1} + \mathcal{G}_0^* \mathbb{M}_0(z) \mathcal{V}_1 - \mathcal{G}_1^* \mathbb{M}_1(z) \mathcal{V}_{K+1}.$$

We assume for a while that the dual scheme associated to (29), that is (38) is *LE*-stable so that we have the stability estimate:

$$\forall z \in \mathcal{U}, \frac{|z|-1}{|z|} \sum_{j=1}^K |\mathcal{V}_j|^2 + |\mathcal{V}_1|^2 + |\mathcal{V}_{K+1}|^2 \leq C \frac{|z|}{|z|-1} \sum_{j=1}^K |(\mathbb{L}_{dua}\mathcal{V})_j|^2,$$

then it is easy to show that  $f$  is a continuous form on  $\mathfrak{X}$ , consequently from Hahn-Banach and Riesz representation Theorem there exists  $(\underline{\mathcal{W}}_j)_{j \in \llbracket 1, K \rrbracket} \in \mathbb{C}^{(\ell+r)N}$  such that  $f(\mathbb{L}_{dua}\mathcal{V}) = \sum_{j=1}^K \underline{\mathcal{W}}_j^* \mathbb{L}_{dua}\mathcal{V}_j$ .

Then we use the equation defining the dual scheme that is (39) for  $\mathcal{W}_j = \underline{\mathcal{W}}_j$  and  $\mathcal{V}_j$  satisfying  $\mathcal{V}_1 = \mathcal{V}_{K+1} = 0$  so that we obtain  $\mathbb{L}_{pri}\mathcal{W}_j = \mathcal{F}_j$  for all  $j \in \llbracket 1, K \rrbracket$ . Next we test (39) with  $\mathcal{W}_j = \underline{\mathcal{W}}_j$  and  $\mathcal{V}_j$  satisfying  $\mathcal{V}_{K+1} = 0$  and  $\mathcal{V}_1 \in \ker \mathbb{C}_0(z)$  (resp.  $\mathcal{V}_1 = 0$  and  $\mathcal{V}_{K+1} \in \ker \mathbb{C}_1(z)$ ) to obtain  $(\mathcal{G}_0 - \mathbb{B}_0(z)\underline{\mathcal{W}}_1)^* \mathbb{M}_0(z) \mathcal{V}_1 = 0$  (resp.  $(\mathcal{G}_1 - \mathbb{B}_1(z)\underline{\mathcal{W}}_{K+1})^* \mathbb{M}_1(z) \mathcal{V}_{K+1} = 0$ ). To conclude it is sufficient to notice that  $\mathbb{M}_0(z)|_{\ker \mathbb{C}_0(z)}$  and  $\mathbb{M}_1(z)|_{\ker \mathbb{C}_1(z)}$  are onto. So  $\underline{\mathcal{W}}_j$  is a solution of (29) when the dual scheme (38) is *LE*-stable.

To conclude it is then sufficient to show that the dual scheme satisfy the stability inequality and from the results of Paragraphs 5.2.1 and 5.2.2 we shall show that the matrix  $(I - \overline{\mathbb{T}}_{0 \rightarrow 0}(z)) : \ker \mathbb{C}_0(z) \rightarrow \ker \mathbb{C}_0(z)$  is

uniformly invertible for all  $z \in \mathcal{U}$ , where  $\widetilde{\mathbb{T}}_{0 \rightarrow 0}(z)$  stands for the analogous of the matrix  $\mathbb{T}_{0 \rightarrow 0}(z)$  but for the dual scheme. More precisely this matrix is defined by  $\widetilde{\mathbb{T}}_{0 \rightarrow 0}(z) := \widetilde{\mathbb{T}}_{1 \rightarrow 0}(z)\widetilde{\mathbb{T}}_{0 \rightarrow 1}(z)$  with

$$\widetilde{\mathbb{T}}_{0 \rightarrow 1}(z) := (I - \Phi_1(z)\mathbb{C}_1(z))\mathbb{M}^{-K^*}(z) \text{ and } \widetilde{\mathbb{T}}_{1 \rightarrow 0}(z) := (I - \Phi_0(z)\mathbb{C}_0(z))\mathbb{M}^{K^*}(z). \quad (68) \quad \{\text{confinement1}\}$$

The main result of this paragraph is the following

**Proposition 5.6** *Under Assumptions 2.1, 2.2, 2.3 and 4.1 the primal scheme (29) satisfies i) of Assumption 5.1 if and only if the dual scheme (38) satisfies ii) of Assumption 5.1.*

**Proof :** Let us first remark that for all  $\mathcal{V} \in \ker \mathbb{C}_0(z)$  we have  $(I - \widetilde{\mathbb{T}}_{0 \rightarrow 0}(z))\mathcal{V} = (I - \widetilde{\mathbb{T}}_{0 \rightarrow 0}(z))(I - \widetilde{\Phi}_0(z)\mathbb{C}_0(z))\mathcal{V}$  consequently we show in the following that  $(I - \widetilde{\mathbb{T}}_{0 \rightarrow 0}(z))(I - \widetilde{\Phi}_0(z)\mathbb{C}_0(z))$  is uniformly invertible from  $\ker \mathbb{C}_0(z)$  to  $\ker \mathbb{C}_0(z)$ . In order to do so we use the decomposition  $\mathbb{C}^{(\ell+r)N} = \widetilde{\mathbb{E}}^s(z) \oplus \ker \mathbb{C}_0(z)$  for  $z \in \mathcal{U}$  and we show that  $\left[ (I - \widetilde{\mathbb{T}}_{0 \rightarrow 0}(z))(I - \widetilde{\Phi}_0(z)\mathbb{C}_0(z)) \right]^*$  is uniformly invertible from  $\widetilde{\mathbb{E}}^s(z)^\perp$  to  $\widetilde{\mathbb{E}}^s(z)^\perp$ . We recall that from (42) we have  $\widetilde{\mathbb{E}}^s(z)^\perp = \mathbb{E}^s(z)$ . From (43) we have  $(I - \widetilde{\Phi}_0(z)\mathbb{C}_0(z))^* = \Phi_0(z)\mathbb{B}_0(z)$  then (68) gives  $\widetilde{\mathbb{T}}_{0 \rightarrow 1}^*(z) = \mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)$  and  $\widetilde{\mathbb{T}}_{1 \rightarrow 0}^*(z) = \mathbb{M}^K(z)\Phi_0(z)\mathbb{B}_0(z)$ . So that for  $\mathcal{W} \in \mathbb{E}^s(z)$  we have

$$\begin{aligned} \left[ (I - \widetilde{\mathbb{T}}_{0 \rightarrow 0}(z))(I - \widetilde{\Phi}_0(z)\mathbb{C}_0(z)) \right]^* \mathcal{W} &= \Phi_0(z)\mathbb{B}_0(z) (I - \mathbb{M}^{-K}(z)\Phi_1(z)\mathbb{B}_1(z)\mathbb{M}^K(z)\Phi_0(z)\mathbb{B}_0(z)) \mathcal{W} \\ &= (I - \mathbb{T}_{0 \rightarrow 0}(z))\mathcal{W} \end{aligned}$$

because  $\mathcal{W} \in \mathbb{E}^s(z)$ . So that i) of Assumption 5.1 for the primal scheme (29) is equivalent to ii) of Assumption 5.1 for the dual scheme. □

We sum up the results of this paragraph in the following proposition which concludes the proof of Theorem 3.1.

**Proposition 5.7** *Under Assumptions 2.1, 2.2, 2.3 and 4.1 also assume that i) of Assumption 5.1 holds then the resolvent scheme (29) admits a solution  $(\mathcal{W}_j)_{j \in \llbracket 1, K+1 \rrbracket} \subset \mathbb{C}^{(\ell+r)N}$ .*

## 6 Application : the scalar transport equation \{part\\_examp\}

In this section we consider the discretization of a scalar transport equation and we recover some of the stability results of [Coulombel and Lagoutière, ].

More precisely we study the outgoing (for the side  $\{x = 0\}$ ) scalar transport equation defined in the interval  $[0, 1]$ . That is to say:

$$\begin{cases} \partial_t u + a \partial_x u = f & t > 0, x \in ]0, 1[, \\ u|_{x=0} = g & t > 0, \\ u|_{t=0} = 0 & x \in ]0, 1[, \end{cases} \quad (69) \quad \{\text{transport}\}$$

where  $f$  and  $g$  are given source terms and where  $a > 0$ .

Because  $a > 0$ , there is no boundary condition on the side  $\{x = 1\}$ . However if we want to approximate the solution of (69) by a finite difference scheme with a non trivial stencil in the right-hand side then a boundary condition on this side is required and in the following we focus our attention to Dirichlet or Neumann boundary conditions for this side.

The finite difference scheme used to approximate the transport equation (69) is the classical Lax-Friedrichs scheme. With more details we approximate the evolution equation of (69) by the finite difference relation:

$$u_j^{n+1} = \frac{1}{2} (u_{j+1}^n + u_{j-1}^n) - \frac{\lambda a}{2} (u_{j+1}^n - u_{j-1}^n) \Delta t f_j^{n+1}, \quad (70) \quad \{\text{eq\_LF\_aux}\}$$

where  $f_j^{n+1}$  stands for some approximation of  $f$ . In the notations used throughout the article we have  $r = \ell = 1$  so numerical boundary conditions on each side are required and should involve only one boundary value in order to satisfy Assumption 2.3. They thus read

$$u_0^{n+1} - \alpha u_1^{n+1} = g_0^{n+1}, \text{ and } u_{K+1}^{n+1} - \beta u_K^{n+1} = g_1^{n+1},$$

where  $\alpha, \beta > 0$  are given. In what follows we will mainly be interested in the cases  $\alpha, \beta \in \{0, 1\}$  corresponding to Dirichlet condition when  $\alpha$  or  $\beta$  is zero and Neumann condition when  $\alpha$  or  $\beta$  equals one<sup>7</sup>. So the full scheme reads:

$$\begin{cases} u_j^{n+1} = \frac{1}{2} (u_{j+1}^n + u_{j-1}^n) - \frac{\lambda a}{2} (u_{j+1}^n - u_{j-1}^n) + \Delta t f_j^{n+1} & n \geq 0, j \in \llbracket 1, K \rrbracket, \\ u_0^{n+1} - \alpha u_1^{n+1} = g_0^{n+1} & n \geq 0, \\ u_{K+1}^{n+1} - \beta u_K^{n+1} = g_1^{n+1} & n \geq 0, \\ u_j^0 = 0 & j \in \llbracket 0, K+1 \rrbracket, \end{cases} \quad (71) \quad \{\text{transport\_ap}\}$$

It is rather easy to show that this finite difference scheme satisfies Assumption 2.2, Assumption 2.4 and that it satisfies Assumption 2.1 if

$$\lambda a < 1,$$

which is nothing but the usual CFL condition for (70) to be stable in the full line.

In its resolvent form this approximation scheme reads:

$$\text{for } z \in \mathbb{C} \setminus \{0\} \quad \begin{cases} w_j = \frac{1}{2z} (1 - a\lambda) w_{j+1} + \frac{1}{2z} (1 + a\lambda) w_{j-1} + f_j & j \in \llbracket 1, K \rrbracket \\ w_0 - \alpha w_1 = g_0, \\ w_{K+1} - \beta w_K = g_1, \end{cases}$$

so in terms of the augmented vector  $\mathscr{W}_j := [w_j, w_{j-1}]^T$  we obtain:

$$\begin{cases} \mathscr{W}_j = \mathbb{M}(z) \mathscr{W}_j + \mathscr{F}_j, & j \in \llbracket 1, K \rrbracket \\ \mathbb{B}_0 \mathscr{W}_1 = \mathscr{G}_0, \\ \mathbb{B}_1 \mathscr{W}_{K+1} = \mathscr{G}_1, \end{cases}$$

where we defined

$$\mathbb{M}(z) := \begin{bmatrix} \frac{2z}{1-\lambda a} & -\frac{1+\lambda a}{1-\lambda a} \\ 1 & 0 \end{bmatrix}, \quad \mathbb{B}_0 := [-\alpha, 1] \text{ and } \mathbb{B}_1 := [1, -\beta].$$

As justified in [[Coulombel, ] Paragraph 4.5], for  $z \in \mathscr{U}$  the stable subspace  $\mathbb{E}^s(z)$  and the unstable subspace  $\mathbb{E}^u(z)$  of  $\mathbb{M}(z)$  can be parametrized as follows:

$$\mathbb{E}^s(z) = \text{vect} \left\{ [\kappa^s(z), 1]^T \right\} := \text{vect} \{v^s(z)\} \text{ and } \mathbb{E}^u(z) := \text{vect} \left\{ [\kappa^u(z), 1]^T \right\} = \text{vect} \{v^u(z)\}$$

where  $\kappa^s(z)$  (resp.  $\kappa^u(z)$ ) stands for the root in the  $\kappa$  variable of the equation

$$(1 - \lambda a) \kappa^2 - 2z\kappa + 1 + \lambda a = 0, \quad (72) \quad \{\text{eq\_dis\_LF\_tr}\}$$

satisfying that  $|\kappa^s(z)| < 1$  (resp.  $|\kappa^u(z)| > 1$ ) for all  $z \in \mathscr{U}$ . Formally we define the applications  $\Phi_0(z) : \mathbb{C} \rightarrow \mathbb{E}^s(z)$  and  $\Phi_1(z) : \mathbb{C} \rightarrow \mathbb{E}^u(z)$  by

$$\Phi_0(z)\zeta := \frac{\zeta}{1 - \alpha \kappa^s(z)} v^s(z) \text{ and } \Phi_1(z)\zeta := \frac{\zeta}{\kappa^u(z) - \beta} v^u(z), \quad (73) \quad \{\text{ukl\_transpor}\}$$

and it is clear that they are well-defined and that they respectively correspond to the inverses of  $\mathbb{B}_{0|\mathbb{E}^s(z)}$  and  $\mathbb{B}_{1|\mathbb{E}^u(z)}$  when  $\alpha, \beta < 1$ .

<sup>7</sup>In fact the analysis exposed bellow cover the range  $\alpha, \beta \in [0, 1]$ .

However, as we have mentioned before, one of our main case of interest is the Neumann boundary condition that corresponds to  $\alpha$  or  $\beta$  equals to one. From (73) it is equivalent to ask : Can  $\kappa^s(z)$  or  $\kappa^u(z)$  be equal to one ? Once again following the study made in [Coulombel, ] this situation can only occur if  $z = 1$  and in this case (72) becomes  $(1 - \lambda a)\kappa^2 - 2\kappa + 1 + \lambda a$ . It is clear that the roots of this new equation are

$$\kappa_1 = 1 \text{ and } \kappa_2 = \frac{1 + \lambda a}{1 - \lambda},$$

but recall that by assumption  $\lambda a < 1$  so  $\kappa_2 > 1$ . Consequently the only possible continuation of  $\kappa^s(z)$  up to 1 is 1. So we have the following result

- If  $\alpha = 1$  the discrete uniform Kreiss-Lopatinskii condition breaks down for the frequency<sup>8</sup>  $z = 1$ . Thus we restrict our attention to  $\alpha < 1$ .
- The discrete uniform Kreiss-Lopatinskii condition holds for all  $\beta \leq 1$ <sup>9</sup>.

In order to study the stability of (71) we first compute the restriction of  $\mathbb{T}_{0 \rightarrow 0}(z)$  to  $\mathbb{E}^s(z)$ . An easy computations gives:

$$(I - \mathbb{T}_{0 \rightarrow 0}(z))v^s(z) = \left(1 - (\kappa^s(z))^K (\kappa^u(z))^{-K} \frac{\kappa^s(z) - \beta}{\kappa^u(z) - \beta} \frac{1 - \alpha \kappa^u(z)}{1 - \alpha \kappa^s(z)}\right) v^s(z), \quad (74) \quad \{\text{equation\_T1.}\}$$

then we study the restriction of  $\bar{\mathbb{T}}_{0 \rightarrow 0}(z)$  to  $\text{vect} \left\{ [1, \alpha]^T \right\}$  and we also obtain

$$(I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z)) \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \left(1 - (\kappa^s(z))^K (\kappa^u(z))^{-K} \frac{\kappa^s(z) - \beta}{\kappa^u(z) - \beta} \frac{1 - \alpha \kappa^u(z)}{1 - \alpha \kappa^s(z)}\right) \begin{bmatrix} 1 \\ \alpha \end{bmatrix}.$$

Consequently it is sufficient to consider (74) and we want to have a look to the cases  $(\alpha, \beta) = (0, 0)$  (Dirichlet/Dirichlet) and  $(\alpha, \beta) = (0, 1)$  (Dirichlet/Neumann).

- **Dirichlet/Dirichlet:**

We remark that in such a setting  $\mathbb{T}_{0 \rightarrow 0}(z)v^s(z) = (\kappa^s(z))^{K+1} (\kappa^u(z))^{-(K+1)} v^s(z)$  which clearly implies that

$$|\mathbb{T}_{0 \rightarrow 0}(z)v^s(z)| < |v^s(z)|,$$

because  $|\kappa^s(z)| \leq 1$  and  $|\kappa^u(z)| > 1$ . So  $(I - \mathbb{T}_{0 \rightarrow 0}(z))|_{\mathbb{E}^s(z)}$  is clearly uniformly invertible and the same holds for  $(I - \bar{\mathbb{T}}_{0 \rightarrow 0}(z))|_{\ker \mathbb{B}_0(z)}$ . Assumptions 5.1 holds, Theorem 3.1 applies and we deduce the *LE*-stability of (71). Note that it is not surprising at all because we have already shown in Paragraph 4.1.1 that Dirichlet boundary conditions automatically lead to *LE*-stability.

However, let us point that imposing Dirichlet boundary condition on the side  $\{x = 1\}$  is a bad choice in terms of consistency of the scheme, that is why in the following we consider the case Dirichlet/Neumann.

- **Dirichlet/Neumann:**

Impose Neumann boundary condition  $\{x = 1\}$  on the side gives rise to a consistent scheme. 1 natural question is the stability of the scheme. We thus study the matrix  $\mathbb{T}_{0 \rightarrow 0}(z)$  (once again the computations are essentially the same for  $\bar{\mathbb{T}}_{0 \rightarrow 0}(z)$ ) as in the Dirichlet/Dirichlet framework we have:

$$|\mathbb{T}_{0 \rightarrow 0}(z)v^s(z)| < |\kappa^u(z)|^{-K} \frac{2}{|\kappa^u(z) - \beta|} |v^s(z)|,$$

where we used the fact that for all  $z \in \bar{\mathcal{W}}$ ,  $|\kappa^s(z)| \leq 1$ . And up to choose  $K$  large enough (recall that for all  $z \in \bar{\mathcal{W}}$ ,  $|\kappa^u(z)| > 1$ ) we can assume that  $\mathbb{T}_{0 \rightarrow 0}(z)$  is a contraction on  $\mathbb{E}^s(z)$  so that Assumption 5.1 is satisfied and Theorem 3.1 gives the *LE*-strong stability of (71).

<sup>8</sup>Note that because the transport phenomenon is outgoing for the side  $\{x = 0\}$  it is not surprising that imposing Neumann boundary condition leads to an unstable scheme.

<sup>9</sup>Consequently we can consider a Neumann boundary condition for the side  $\{x = 1\}$ , this is totally natural because the transform phenomemon is incoming for the side  $\{x = 1\}$

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