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# Lattice walk area combinatorics, some remarkable trigonometric sums and Apéry-like numbers

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## Abstract

Explicit algebraic area enumeration formulae are derived for various lattice walks generalizing the canonical square lattice walks, and in particular for the triangular lattice chiral walks recently introduced by the authors. A key element in the enumeration is the derivation of some identities involving some remarkable trigonometric sums –which are also important building blocks of non trivial quantum models such as the Hofstadter model– and their explicit rewriting in terms of multiple binomial sums. An intriguing connection is also made with number theory and some classes of Apéry-like numbers, the cousins of the Apéry numbers which play a central role in irrationality considerations for  $\zeta(2)$  and  $\zeta(3)$ .

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# 1 Introduction

Random walks on lattices emerge in the study of various problems of physical interest. The dynamics of electrons (or quasiparticles) on an atomic lattice can be well approximated by their hopping to the ground state levels of different atoms in the lattice. Hopping to excited states would introduce extra effective discrete degrees of freedom but such transitions are generally energetically suppressed. Likewise, hopping to atoms beyond the few near neighbors of the atom presently binding the electron are also suppressed. As a consequence, the entire dynamical process can be described by a random lattice walk. Percolation processes and various other statistical processes can also be modeled as random walks.

The canonical example of a square lattice random walk is the one mapping to the Hofstadter model, describing a charged particle hopping on nearest neighbor sites on a square lattice pierced by a constant magnetic field. The dynamics of this process is described in terms of the length of random walks, quantified by the number of jumps, as well as their area, due to the interaction with the magnetic field. In the sum over lattice walks the magnetic term appears as a chemical potential dual to the algebraic area of the walk (defined precisely in the next paragraph) measured in unit cells, in the form  $e^{i\Phi(\text{algebraic area})}$ , with  $\Phi$  the magnetic flux per unit cell. The parameter  $Q = e^{i\Phi}$  plays the role of “fugacity” for the algebraic area. For a flux per unit cell equal to a rational multiple of the flux quantum  $2\pi$  (setting the particle charge  $e = 1$ ),  $\Phi = 2\pi p/q$ , the model develops an intricate discrete spectrum, leading to the famous “Hofstadter butterfly” of the energy spectrum as a function of the magnetic field. The Hofstadter problem reduces effectively to enumerating paths of given length and algebraic area.

The exact enumeration of random walks of given algebraic area on a two-dimensional lattice is a hard and challenging problem. The algebraic area is defined as the oriented area spanned by the walk as it traces the lattice. A unit lattice cell enclosed in the counterclockwise (positive) way has an area  $+1$ , whereas when enclosed in the clockwise (negative) way it has an area  $-1$ . The total algebraic area is the area enclosed by the walk weighted by the winding number: if the walk winds around more than once, the area is counted with multiplicity. The combinatorics of such walks depend on the exact rule generating them and on the lattice geometry.

An exact formula for the number of square lattice walks of given length and algebraic area for a rational flux  $\Phi = 2\pi p/q$ , with  $p$  and  $q$  coprime positive integers, was only recently obtained [1]. The analysis revealed some trigonometric sums to be key ingredients for the algebraic area enumeration. They are defined as

$$\frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) \quad (1)$$

where  $b_{p/q}(k)$  is a trigonometric function called spectral function which depends on the rational number  $p/q$ , and  $l_1, l_2, \dots, l_j$  is a set of positive or null integers. In view of

the algebraic area enumeration of square lattice walks these integers will be parts in the compositions of the integer  $n$ , i.e.,  $n = l_1 + \dots + l_j$  and all  $l_i$  positive, with  $n$  fixing the length of the walks. But in the sequel we will consider that some of the  $l_i$ 's can be null, in a way to be specified below, for more general lattice walks relevant to other physical processes.

In [1] the focus was on the spectral function

$$b_{p/q}(k) = (2 \sin(\pi k p/q))^2 \quad (2)$$

which encodes the Hofstadter dynamics. The algebraic area enumeration was obtained in part thanks to an explicit rewriting of the trigonometric sum (1), when evaluated for the Hofstadter spectral function (2), in terms of the binomial multiple sums

$$\frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) = \sum_{\substack{A=-\infty \\ A \text{ even}}}^{\infty} e^{i\pi A p/q} \quad (3)$$

$$\sum_{k_3=-l_3}^{l_3} \dots \sum_{k_j=-l_j}^{l_j} \binom{l_1 + A/2 + \sum_{i=3}^j (i-2)k_i}{l_1 + A/2 + \sum_{i=3}^j (i-2)k_i} \binom{l_2 - A/2 - \sum_{i=3}^j (i-1)k_i}{l_2 - A/2 - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{2l_i}{l_i + k_i}$$

The summation variable  $A$  in the right hand side of (3) will later on play the role of the algebraic area of the square lattice walks, thus making contact with the enumeration problem of interest,  $\exp(i\pi A p/q)$  being the magnetic flux factor. Eq. (3) is valid for any set of positive or null integers  $l_i$  with an  $A$ -summation range finite due to the first two binomials, where  $A$  appears. In the specific case where the  $l_i$ 's are all positive –as is the case for the square lattice walks algebraic area enumeration–  $A$  is restricted in the interval  $[-2\lfloor(l_1 + \dots + l_j)^2/4\rfloor, 2\lfloor(l_1 + \dots + l_j)^2/4\rfloor]$ . When some of the  $l_i$ 's are null these bounds can be generalized (see, e.g., the bounds in eq. (11)).

We note that when we replace  $e^{i\pi A p/q}$  by 1 in (3), which corresponds to vanishing magnetic field in the Hofstadter model and leads to a simple counting of random walks of all areas for given length  $l$ , we get the binomial identity

$$\binom{2(l_1 + \dots + l_j)}{l_1 + \dots + l_j} = \sum_{\substack{A=-\infty \\ A \text{ even}}}^{\infty} \quad (4)$$

$$\sum_{k_3=-l_3}^{l_3} \dots \sum_{k_j=-l_j}^{l_j} \binom{l_1 + A/2 + \sum_{i=3}^j (i-2)k_i}{l_1 + A/2 + \sum_{i=3}^j (i-2)k_i} \binom{l_2 - A/2 - \sum_{i=3}^j (i-1)k_i}{l_2 - A/2 - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{2l_i}{l_i + k_i}$$

where the resulting binomial products in the LHS<sup>1</sup> will be interpreted later on as factors contributing to the counting of lattice walks. Again, formula (4) is valid for any set of positive or null integers  $l_i$ ; if the  $l_i$ 's are all positive the bounds on  $A$  are as specified above.

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<sup>1</sup>This binomial counting can be easily checked by first summing over  $A$  and subsequently over the  $k_i$ 's, redefining them appropriately; see [1].

We remark here that the trigonometric sum (1) reduces to the binomial multiple sum given in (3) in the case  $b_{p/q}(k) = (2 \sin(\pi k p/q))^2$  only when  $l_1 + \dots + l_j < q$ , i.e., for large enough values of  $q$  (or walks of length less than  $q$ ). This constraint on  $q$  eliminates from the counting open walks with endpoint coordinates coinciding modulo  $q$ , that is, closed walks on a doubly periodic lattice of periods  $q$  in each direction, which the formula counts along with closed walks on the plane<sup>2</sup>.

In [3] we revisited the algebraic area enumeration of [1] and noted that it admits a statistical mechanical interpretation in terms of particles obeying generalized *exclusion statistics* [4] with exclusion parameter  $g = 2$  ( $g = 0$  for bosons,  $g = 1$  for fermions, higher  $g$  means a stronger exclusion beyond Fermi). Other lattice walks admit a similar interpretation with higher integer values of  $g$ . We also introduced the notion of  $g$ -compositions where some zeros can be inserted at will inside the set of the  $l_i$ 's with the restriction that no more than  $g - 2$  zeros lay in succession. The integer  $n$  admits  $g^{n-1}$  such compositions. In particular,  $g = 1$ -exclusion refers to the unique composition  $n = n$ , whereas  $g = 2$ -exclusion corresponds to the standard compositions with no zeros at all. We also constructed triangular lattice chiral walks realizing  $g = 3$ -exclusion with spectral function

$$b_{p/q}(k) = (2 \sin(2\pi k p/q))(2 \sin(2\pi(k+1)p/q)) \quad (5)$$

We finally hinted at other walks corresponding to statistics with higher values of the exclusion parameter  $g$  and to other spectral functions. However, for the triangular lattice chiral walks, as well as for other cases, an explicit algebraic area enumeration formula was missing due to the lack of binomial expressions analogous to (3) for the triangular spectral function (5).

In the present work we focus on filling this gap by uncovering such expressions for entire classes of trigonometric spectral functions generalizing (2) and (5). Namely, we consider, on the one hand

$$b_{p/q}(k) = (2 \sin(\pi k p/q))^r \quad (6)$$

and on the other hand

$$b_{p/q}(k) = (2 \sin(\pi k p/q))(2 \sin(\pi(k+1)p/q)) \dots (2 \sin(\pi(k+r-1)p/q)) \quad (7)$$

where in both instances  $r$  can be even or odd. The case  $r = 2$  reproduces<sup>3</sup> (2) and (5) respectively, thus deriving the area counting and statistics of triangular chiral walks as a special case. We will see that the basic structure of the binomial multiple sum (3) naturally generalizes to these cases. In the Appendix we will also derive the relevant generalization for the spectral function

$$b_{p/q}(k) = (2 \sin(\pi k p/q))^{r/2} (2 \sin(\pi(k+1)p/q))^{r/2} \quad (8)$$

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<sup>2</sup>Extrapolating (3) as such to any value of  $q \geq 1$  would amount to enforcing, for any given integer  $l$ , the identity  $\sum_{k=1}^q e^{2ik\pi l/q} = 0$  even though this is valid only when  $l$  is not a multiple of  $q$  (when  $l$  is a multiple of  $q$  the sum is actually equal to  $q$ ).

<sup>3</sup>The actual spectral function (5) for triangular lattice chiral walks has a factor 2 in front of the  $\pi$ 's which we omit here to stay in line with (6); it anyway amounts to a trivial redefinition of  $p/q \rightarrow 2p/q$ .

where  $r$  is even, yet another possible generalization of (5) that is relevant to other types of random walks.

Turning to the algebraic area combinatorics *per se*, these expressions, as already mentioned, will allow for explicit enumeration formulae analogous to the square lattice walks formula obtained in [1] for  $g = 2$  and the Hofstadter spectral function (2). Achieving this requires the additional step of introducing an appropriate weighting coefficient depending on the composition and summing over all compositions of the integer  $n$ . We refer to [1] for detailed explanations of how this procedure unfolds and to [3] for the connection to  $g$ -exclusion statistics and the resulting generalizations. With the  $g$ -exclusion statistics weighting coefficients [3]

$$\begin{aligned} c_g(l_1, l_2, \dots, l_j) &= \frac{(l_1 + \dots + l_{g-1} - 1)!}{l_1! \dots l_{g-1}!} \prod_{i=1}^{j-g+1} \binom{l_i + \dots + l_{i+g-1} - 1}{l_{i+g-1}} \\ &= \frac{\prod_{i=1}^{j-g+1} (l_i + \dots + l_{i+g-1} - 1)!}{\prod_{i=1}^{j-g} (l_{i+1} + \dots + l_{i+g-1} - 1)!} \prod_{i=1}^j \frac{1}{l_i!} \end{aligned}$$

we can express the lattice walks algebraic area enumeration for  $g \geq 2$ -exclusion and a general periodic spectral function  $b_{p/q}(k)$  by means of the  $g$ -cluster coefficient<sup>4</sup>

$$b(n) = gn \sum_{\substack{l_1, l_2, \dots, l_j \\ g\text{-composition of } n}} c_g(l_1, l_2, \dots, l_j) \frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k-1) \dots b_{p/q}^{l_j}(k-j+1) \quad (9)$$

As already stressed, (9) yields the algebraic area combinatorics provided that an expression analogous to (3) is known for the trigonometric sum involving the specific  $b_{p/q}(k)$ . Indeed, the summation index  $A$  in (3) will be interpreted in (9) as the algebraic area for square lattice walks, and the coefficient multiplying the exponential factor  $e^{i\pi A p/q}$  will be the desired algebraic area counting number. Eq. (9) will also yield the triangular lattice chiral walk counting described by  $g = 3$ -exclusion and spectral function (5).

Finally, we will discuss the unexpected occurrence of Apéry-like numbers in the cluster coefficient (9) evaluated at particular values of  $p/q$  for certain  $g$ -exclusions and spectral functions. Apéry-like numbers are interesting *per se* since they are cousins of the celebrated Apéry numbers which allow for a proof of the irrationality of  $\zeta(2)$  and  $\zeta(3)$ . One key characteristic of these numbers is that they are integer solutions of second order recursion relations. As we will see, some of the  $\zeta(2)$  Apéry-like numbers fascinatingly emerge in the algebraic enumeration formula (9).

## 2 Trigonometric sums $\sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1)$

We aim at uncovering explicit binomial multiple sums analogous to (3) for the spectral functions (6) and (7). In fact, the form of (3) is quite robust and suggestive, and allows

<sup>4</sup>For statistical mechanics considerations the  $g$ -cluster coefficient introduced in [3] is the expression in (9) multiplied by  $(-1)^{n-1} q/(gn)$ .

deducing such generalizations by simple deformations while preserving its overall structure. We stress that, from now on, some  $l_i$ 's can be null according to the  $g$ -composition structure discussed previously, i.e., no more than  $g - 2$  zeros in succession inside the set. The  $A$ -summation bounds, when specified, will explicitly depend on the parameter  $g$ .

## 2.1 Square lattice walks generalization: $b_{p/q}(k) = (2 \sin(\pi kp/q))^r$

We first point out two basic facts:

- When  $q \rightarrow \infty$  the sum over  $k$  in (3) goes over to an integral and one obtains the overall counting

$$\int_0^1 (2 \sin(\pi s))^{r l_1 + l_2 + \dots + r l_j} ds = \binom{r(l_1 + l_2 + \dots + l_j)}{r(l_1 + l_2 + \dots + l_j)/2} \quad (10)$$

so we focus on  $(l_1 + l_2 + \dots + l_j)$  such that  $r(l_1 + l_2 + \dots + l_j)$  be even. It means that for  $r$  even any set  $l_1, l_2, \dots, l_j$  is admissible, whereas for  $r$  odd the  $l_i$ 's have to be such that their sum be even.

- It is obvious that for a given  $r$

$$\frac{1}{q} \sum_{k=1}^q ((2 \sin(\pi kp/q))^r)^{l_1} ((2 \sin(\pi(k+1)p/q))^r)^{l_2} \dots ((2 \sin(\pi(k+j-1)p/q))^r)^{l_j}$$

amounts to

$$\frac{1}{q} \sum_{k=1}^q ((2 \sin(\pi kp/q))^2)^{r l_1/2} ((2 \sin(\pi(k+1)p/q))^2)^{r l_2/2} \dots ((2 \sin(\pi(k+j-1)p/q))^2)^{r l_j/2}$$

which is essentially the Hofstadter case  $r = 2$ , i.e., for the spectral function  $(2 \sin(\pi kp/q))^2$ , but now with  $l_i \rightarrow r l_i/2$ .

Based on the above observations, the binomial multiple sum in (3) for the  $r = 2$

Hofstadter case becomes, for  $b_{p/q}(k) = (2 \sin(\pi kp/q))^r$  with  $r$  even<sup>5</sup>,

$$\frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) = \sum_{\substack{A=-(g-1)r \lfloor (l_1+\dots+l_j)^2/4 \rfloor \\ A \text{ even}}}^{(g-1)r \lfloor (l_1+\dots+l_j)^2/4 \rfloor} e^{i\pi Ap/q} \quad (11)$$

$$\sum_{k_3=-rl_3/2}^{rl_3/2} \dots \sum_{k_j=-rl_j/2}^{rl_j/2} \binom{rl_1}{rl_1/2 + A/2 + \sum_{i=3}^j (i-2)k_i} \binom{rl_2}{rl_2/2 - A/2 - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{rl_i}{rl_i/2 + k_i}$$

which is valid when  $r(l_1 + \dots + l_j)/2 < q$  holds, and where we have specified the range  $[-(g-1)r \lfloor (l_1 + \dots + l_j)^2/4 \rfloor, (g-1)r \lfloor (l_1 + \dots + l_j)^2/4 \rfloor]$  in which  $A$  needs to be restricted.

In the  $r$  odd case we expect a binomial multiple sum analogous to (11). To see this in full generality, and to give a full proof of the original formula with even  $r$ , let us first recall the Poisson resummation formula for any  $q$ -periodic function  $f(x) = f(x+q)$

$$\sum_{k=1}^q f(k) = \sum_{n=-\infty}^{\infty} \tilde{f}(nq) \quad (12)$$

where  $\tilde{f}$  is the Fourier transform of  $f$  defined as

$$\tilde{f}(k) = \int_0^q f(x) e^{-2i\pi kx/q} dx, \quad f(x) = \frac{1}{q} \sum_{k=-\infty}^{\infty} \tilde{f}(k) e^{2i\pi kx/q}$$

Let us consider the function  $f(x) = \frac{1}{q} b_{p/q}^{l_1}(x) b_{p/q}^{l_2}(x+1) \dots b_{p/q}^{l_j}(x+j-1)$  which is

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<sup>5</sup>The overall counting, found by replacing  $e^{i\pi Ap/q}$  by 1 is

$$\binom{r(l_1 + l_2 + \dots + l_j)}{r(l_1 + l_2 + \dots + l_j)/2} = \sum_{\substack{A=-(g-1)r \lfloor (l_1+\dots+l_j)^2/4 \rfloor \\ A \text{ even}}}^{(g-1)r \lfloor (l_1+\dots+l_j)^2/4 \rfloor}$$

$$\sum_{k_3=-rl_3/2}^{rl_3/2} \dots \sum_{k_j=-rl_j/2}^{rl_j/2} \binom{rl_1}{rl_1/2 + A/2 + \sum_{i=3}^j (i-2)k_i} \binom{rl_2}{rl_2/2 - A/2 - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{rl_i}{rl_i/2 + k_i}$$



indeed  $q$ -periodic due to  $r(l_1 + l_2 + \dots + l_j)$  being assumed even. We have

$$\begin{aligned}
\tilde{f}(nq) &= \int_0^q f(k) e^{-2i\pi kn} dk \\
&= \frac{1}{q} \int_0^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) e^{-2i\pi kn} dk \\
&= \frac{1}{q} \int_0^q \prod_{i=1}^j \frac{1}{i^{rl_i}} \left( e^{i\pi(k+i-1)p/q} - e^{-i\pi(k+i-1)p/q} \right)^{rl_i} e^{-2i\pi kn} dk \\
&= \frac{1}{q} \int_0^q \prod_{i=1}^j \frac{1}{i^{rl_i}} \sum_{k_i=-rl_i/2}^{rl_i/2} \binom{rl_i}{rl_i/2 + k_i} e^{2i\pi(k+i-1)k_i p/q} (-1)^{rl_i/2 - k_i} e^{-2i\pi kn} dk \\
&= \sum_{k_1=-rl_1/2}^{rl_1/2} \dots \sum_{k_j=-rl_j/2}^{rl_j/2} \prod_{i=1}^j \binom{rl_i}{rl_i/2 + k_i} \frac{(-1)^{rl_i/2 - k_i}}{i^{rl_i}} \int_0^1 e^{2i\pi \sum_{i=1}^j k_i s p + 2i\pi \sum_{i=1}^j (i-1)k_i p/q} e^{-2i\pi s q n} ds \\
&= \sum_{k_1=-rl_1/2}^{rl_1/2} \dots \sum_{k_j=-rl_j/2}^{rl_j/2} \prod_{i=1}^j \binom{rl_i}{rl_i/2 + k_i} \frac{(-1)^{rl_i/2 - k_i}}{i^{rl_i}} e^{2i\pi \sum_{i=1}^j (i-1)k_i p/q} \delta \left( \sum_{i=1}^j k_i p - nq \right)
\end{aligned} \tag{13}$$

As stressed above,  $r(l_1 \dots + l_j)$  is even and thus the sum of the  $k_i$  is an integer. Further,  $p$  and  $q$  are coprime. These facts imply that enforcing the Kronecker  $\delta$  in (13) yields

$$p \sum_{i=1}^j k_i = qn \quad \text{and thus} \quad \sum_{i=1}^j k_i = tq \quad \text{and} \quad n = tp$$

for some integer  $t$ . Now  $|\sum_{i=1}^j k_i| \leq r(l_1 + \dots + l_j)/2$  and thus, under the condition  $r(l_1 + \dots + l_j)/2 < q$ ,  $t$  is necessarily equal to 0, implying that  $\sum_{i=1}^j k_i = 0$  and  $n = 0$ . From the Poisson resummation formula (12) then we infer  $\sum_{k=1}^q f(k) = \tilde{f}(0) = \int_0^q f(x) dx$ ; that is, for  $b_{p/q}(k) = (2 \sin(\pi k p/q))^r$ ,

$$\frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) = \frac{1}{q} \int_0^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) dk \tag{14}$$

What has been achieved in (14) is the trading of the original sum over  $k$  from 1 to  $q$  in the LHS for the integral over  $k$  from 0 to  $q$  in the RHS, which is valid provided that  $r(l_1 + \dots + l_j)/2 < q$ .

We can easily check that the trigonometric integral yields the binomial multiple sum (3) in the  $r = 2$  case, or more generally (11) in the  $r$  even case. The integral is essentially

evaluated in the last line of (13) for  $n = 0$ , which implies  $\sum_{i=1}^j k_i = 0$ . We obtain

$$\begin{aligned} \frac{1}{q} \int_0^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) dk &= \int_0^1 dt \prod_{i=1}^j \left( 2 \sin(\pi t + \pi(i-1)p/q) \right)^{rl_i} \\ &= \sum_{k_1=-rl_1/2}^{rl_1/2} \dots \sum_{k_j=-rl_j/2}^{rl_j/2} \delta\left(\sum_{i=1}^j k_i\right) \prod_{i=1}^j \binom{rl_i}{rl_i/2 + k_i} e^{2i\pi \sum_{i=1}^j (i-1)k_i p/q} \end{aligned} \quad (15)$$

The change of integration from  $(1/q) \int_0^q dk$  to  $\int_0^1 dt$  in the variable  $t = kp/q$  in the first line is justified since  $r(l_1 + \dots + l_j)$  is even and the integrand has period 1 in  $t$ . To reproduce the  $A$ -expansion with exponential factors  $e^{i\pi Ap/q}$  in the binomial multiple sums (3) and (11), we denote by  $A$  the coefficient  $2 \sum_{i=1}^j (i-1)k_i$  of  $i\pi p/q$  appearing in the exponential of the last line in (15). We also need to enforce the Kronecker  $\delta$  constraint in the summation variables  $k_i$ . The resulting system of two equations,  $\sum_{i=1}^j k_i = 0$  and  $A = 2 \sum_{i=1}^j (i-1)k_i$ , can be readily solved for, e.g.,  $k_1$  and  $k_2$ , to yield

$$k_1 = -A/2 + \sum_{i=3}^j (i-2)k_i \quad , \quad k_2 = A/2 - \sum_{i=3}^j (i-1)k_i$$

Finally, changing summation variables from  $k_i$  to  $-k_i$  and noting that each binomial is invariant under changing the sign of  $k_i$ , we obtain

$$\begin{aligned} \frac{1}{q} \sum_{k=0}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) \\ &= \sum_{\substack{A=-(g-1)r[(l_1+\dots+l_j)^2/4] \\ \text{in steps of 2}}}^{(g-1)r[(l_1+\dots+l_j)^2/4]} e^{i\pi Ap/q} \sum_{k_3=-rl_3/2}^{rl_3/2} \dots \sum_{k_j=-rl_j/2}^{rl_j/2} \\ &\quad \binom{rl_1}{rl_1/2 + A/2 + \sum_{i=3}^j (i-2)k_i} \binom{rl_2}{rl_2/2 - A/2 - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{rl_i}{rl_i/2 + k_i} \end{aligned} \quad (16)$$

i.e., precisely (11) but now valid for  $r$  even and  $r$  odd, with a specific  $A$ -summation dictated by the condition that in (16) the first two binomial entries  $rl_1/2 + A/2 + \sum_{i=3}^j (i-2)k_i$  and  $rl_2/2 - A/2 - \sum_{i=3}^j (i-1)k_i$  take integer values for all  $k_i \in [-rl_i/2, rl_i/2]$ ,  $i = 3, \dots, j$ , as was the case in (15) for the first two binomial entries  $rl_1/2 + k_1$  and  $rl_2/2 + k_2$  for all  $k_1 \in [-rl_1/2, rl_1/2]$  and  $k_2 \in [-rl_2/2, rl_2/2]$ . It follows that in the case  $r$  even, where the  $k_i$ 's are all integers,  $A$  has to be even, and in the case  $r$  odd, where the  $k_i$ 's are either integers or half integers,  $l_1 + l_2 + \dots + l_j$  has to be even and  $A$  of the same parity as  $l_1 + l_3 + \dots$  (or  $l_2 + l_4 + \dots$ ). In both cases this boils down to  $A \in [-(g-1)r[(l_1 + \dots + l_j)^2/4], (g-1)r[(l_1 + \dots + l_j)^2/4]]$  in steps of 2.

We can express the  $A$ -binomial block in (16) in an integral form by augmenting the LHS to the double integral  $\frac{1}{2} \int_0^1 dt \int_0^2 dt' \prod_{i=1}^j \left( 2 \sin(\pi t + \pi(i-1)t') \right)^{rl_i} \delta(p/q - t')$  and

using  $2 \sum_{n=-\infty}^{\infty} \delta(p/q - t' - 2n) = \sum_{A=-\infty}^{\infty} e^{i\pi A(p/q - t')}$  to get

$$\begin{aligned} & \frac{1}{2} \int_0^2 dt' \int_0^1 dt \prod_{i=1}^j \left( 2 \sin(\pi t + \pi(i-1)t') \right)^{rl_i} e^{i\pi A t'} \quad (17) \\ &= \sum_{k_3=-rl_3/2}^{rl_3/2} \cdots \sum_{k_j=-rl_j/2}^{rl_j/2} \binom{rl_1}{rl_1/2 + A/2 + \sum_{i=3}^j (i-2)k_i} \binom{rl_2}{rl_2/2 - A/2 - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{rl_i}{rl_i/2 + k_i} \end{aligned}$$

In the multiple sum of the RHS  $A$  is constrained as above, depending on  $r$  being even or odd. However, the integral in the LHS is valid for all integer values of  $A$ , yielding zero for the values that do not appear in the RHS.

## 2.2 Triangular generalization: $b_{p/q}(k) = (2 \sin(\pi k p/q)) (2 \sin(\pi(k+1)p/q)) \dots (2 \sin(\pi(k+r-1)p/q))$

We can proceed in exactly the same way for triangular-like spectral functions of the type  $b_{p/q}(k) = (2 \sin(\pi k p/q)) (2 \sin(\pi(k+1)p/q)) \dots (2 \sin(\pi(k+r-1)p/q))$ . Again

- $q \rightarrow \infty$  recovers the overall counting

$$\int_0^1 (2 \sin(\pi s))^{rl_1 + rl_2 + \dots + rl_j} ds = \binom{r(l_1 + l_2 + \dots + l_j)}{r(l_1 + l_2 + \dots + l_j)/2}$$

as in (10), so we still focus on sets of  $l_i$ 's such that  $r(l_1 + l_2 + \dots + l_j)$  is even, again ensuring the  $q$ -periodicity of the functions at hand

- The rewriting of the trigonometric sum as a trigonometric integral proceeds along the same lines as in (13) under the same condition  $r(l_1 + \dots + l_j)/2 < q$  since the sole input in this condition is the highest power of  $e^{i\pi k p/q}$  that appears in  $b_{p/q}(k)$  given by (7), which happens to be again  $r$

### 2.2.1 Triangular chiral walks $r = 2$ : $b_{p/q}(k) = (2 \sin(\pi k p/q)) (2 \sin(\pi(k+1)p/q))$

Following the same steps as in 2.1, we can rewrite the trigonometric sum corresponding to  $b_{p/q}(k) = (2 \sin(\pi k p/q)) (2 \sin(\pi(k+1)p/q))$  as the simple integral

$$\begin{aligned} & \frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) = \\ & \int_0^1 dt \left( 2 \sin(\pi t) \right)^{l_1} \prod_{i=2}^j \left( 2 \sin(\pi t + \pi(i-1)p/q) \right)^{l_{i-1} + l_i} \left( 2 \sin(\pi t + \pi j p/q) \right)^{l_j} \quad (18) \end{aligned}$$

provided that  $l_1 + \dots + l_j < q$ .

Integrating (18) leads to the appropriate deformation of the binomial multiple sum (3) for the spectral function  $b_{p/q}(k) = (2 \sin(\pi k p/q))(2 \sin(\pi(k+1)p/q))$ , a deformation which could also have been directly guessed by simple manipulations: in (1) the integer  $l_1$  is associated with the index  $k$ ,  $l_1 + l_2$  with  $k+1$ ,  $l_2 + l_3$  with  $k+2$ , etc. This leads to<sup>6</sup>

$$\begin{aligned}
& \frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) \\
&= \int_0^1 dt \left( 2 \sin(\pi t) \right)^{l_1} \prod_{i=2}^j \left( 2 \sin(\pi t + \pi(i-1)p/q) \right)^{l_{i-1}+l_i} \left( 2 \sin(\pi t + \pi j p/q) \right)^{l_j} \\
&= \sum_{\substack{A = -\lceil (l_1 + \dots + l_j)^2/2 \rceil - (g-2)\lfloor (l_1 + \dots + l_j)^2/2 \rfloor \\ A \text{ same parity as } l_1 + l_2 + \dots + l_j}} e^{i\pi A p/q} \sum_{k_3 = -(l_2 + l_3)/2}^{(l_2 + l_3)/2} \dots \sum_{k_j = -(l_{j-1} + l_j)/2}^{(l_{j-1} + l_j)/2} \sum_{k_{j+1} = -l_j/2}^{l_j/2} \\
& \left( \binom{l_1}{l_1/2 + A/2 + \sum_{i=3}^{j+1} (i-2)k_i} \right) \left( \binom{l_1 + l_2}{(l_1 + l_2)/2 - A/2 - \sum_{i=3}^{j+1} (i-1)k_i} \right) \\
& \times \prod_{i=3}^j \left( \binom{l_{i-1} + l_i}{(l_{i-1} + l_i)/2 + k_i} \right) \left( \binom{l_j}{l_j/2 + k_{j+1}} \right) \tag{19}
\end{aligned}$$

We note that  $A$  in the summation (19) spans the interval  $[-\lceil (l_1 + \dots + l_j)^2/2 \rceil - (g-2)\lfloor (l_1 + \dots + l_j)^2/2 \rfloor, \lceil (l_1 + \dots + l_j)^2/2 \rceil + (g-2)\lfloor (l_1 + \dots + l_j)^2/2 \rfloor]$  increasing by steps of 2, which in particular implies that  $A$  is of the same parity as  $l_1 + l_2 + \dots + l_j$ .

### 2.2.2 $r = 3$ : $b_{p/q}(k) = (2 \sin(\pi k p/q))(2 \sin(\pi(k+1)p/q))(2 \sin(\pi(k+2)p/q))$ with $l_1 + \dots + l_j$ even

Similarly to the previous cases one can rewrite the  $r = 3$  triangular trigonometric sum as the simple integral

$$\begin{aligned}
& \frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) = \\
& \int_0^1 dt \left( 2 \sin(\pi t) \right)^{l_1} \left( 2 \sin(\pi t + \pi p/q) \right)^{l_1 + l_2} \prod_{i=3}^j \left( 2 \sin(\pi t + \pi(i-1)p/q) \right)^{l_{i-2} + l_{i-1} + l_i} \\
& \times \left( 2 \sin(\pi t + \pi j p/q) \right)^{l_{j-1} + l_j} \left( 2 \sin(\pi t + \pi(j+1)p/q) \right)^{l_j}
\end{aligned}$$

<sup>6</sup>With overall counting, obtained in the  $q \rightarrow \infty$  limit by replacing  $e^{iA p/q}$  by 1:

$$\binom{2(l_1 + l_2 + \dots + l_j)}{l_1 + l_2 + \dots + l_j}$$

provided that  $3(l_1 + \dots + l_j)/2 < q$ .

Likewise one obtains the binomial multiple sum<sup>7</sup>

$$\begin{aligned}
\frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) &= \sum_{\substack{A=-\infty \\ \text{A same parity as } l_1+l_3+\dots \text{ or } l_2+l_4+\dots}}^{\infty} e^{i\pi Ap/q} \\
&\sum_{k_3=-(l_1+l_2+l_3)/2}^{(l_1+l_2+l_3)/2} \dots \sum_{k_j=-(l_{j-2}+l_{j-1}+l_j)/2}^{(l_{j-2}+l_{j-1}+l_j)/2} \sum_{k_{j+1}=-(l_{j-1}+l_j)/2}^{(l_{j-1}+l_j)/2} \sum_{k_{j+2}=-l_j/2}^{l_j/2} \\
&\binom{l_1}{l_1/2 + A/2 + \sum_{i=3}^{j+2} (i-2)k_i} \binom{l_1+l_2}{(l_1+l_2)/2 - A/2 - \sum_{i=3}^{j+2} (i-1)k_i} \\
&\times \prod_{i=3}^j \binom{l_{i-2}+l_{i-1}+l_i}{(l_{i-2}+l_{i-1}+l_i)/2 + k_i} \binom{l_{j-1}+l_j}{(l_{j-1}+l_j)/2 + k_{j+1}} \binom{l_j}{l_j/2 + k_{j+2}} \quad (20)
\end{aligned}$$

where  $A$  has to be of the same parity as  $l_1 + l_3 + \dots$  (or  $l_2 + l_4 + \dots$ ) and obviously a finite range. The cases  $r = 4$  and beyond are treated in the Appendix.

### 3 Algebraic area enumeration

From the obtained trigonometric identities and the cluster coefficient (9) we can retrieve algebraic area enumeration formulae for various random lattice walks. For example, from (16) for  $b_{p/q}(k) = (2 \sin(\pi k p/q))^r$  with  $r$  even and  $g$ -exclusion, (9) becomes

$$\begin{aligned}
b(n) &= gn \sum_{\substack{A=-(g-1)r\lfloor n^2/4\rfloor \\ A \text{ even}}}^{(g-1)r\lfloor n^2/4\rfloor} e^{i\pi Ap/q} \sum_{\substack{l_1, l_2, \dots, l_j \\ g\text{-composition of } n}} c_g(l_1, l_2, \dots, l_j) \quad (21) \\
&\sum_{k_3=-rl_3/2}^{rl_3/2} \dots \sum_{k_j=-rl_j/2}^{rl_j/2} \binom{rl_1}{rl_1/2 + A/2 + \sum_{i=3}^j (i-2)k_i} \binom{rl_2}{rl_2/2 - A/2 - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{rl_i}{rl_i/2 + k_i}
\end{aligned}$$

with overall counting, given by replacing  $e^{i\pi Ap/q}$  by 1

$$\binom{gn}{n} \binom{rn}{rn/2} \quad (22)$$

This reproduces the area counting of various types of square walks. Note that the second binomial in (22), as initially discussed in (4) and displayed in the various overall counting cases of subsection (2.1), results from the trigonometric sums replacing  $e^{i\pi Ap/q}$  by 1 in the limit  $q \rightarrow \infty$ , whereas the first one results from the summation of the exclusion weight coefficients  $c_g$  over all  $g$ -compositions of the integer  $n$ .

<sup>7</sup>With overall counting, obtained by replacing  $e^{i\pi Ap/q}$  by 1:

$$\binom{3(l_1 + l_2 + \dots + l_j)}{3(l_1 + l_2 + \dots + l_j)/2}$$

### 3.1 Square lattice walks: $b_{p/q}(k) = (2 \sin(\pi kp/q))^2$

As already stated, the standard square lattice walks are specifically  $g = 2$  and  $r = 2$  and are defined in terms of the Hamiltonian [3]

$$H = (1 - u)v + v^{-1}(1 - u^{-1})$$

where  $u$  and  $v$  respectively stand for the *right* and *up* hopping operators on the lattice, with commutation  $vu = Q uv$ , where  $Q = e^{i\Phi} = e^{i2\pi p/q}$  is the noncommutativity parameter encoding the presence of the magnetic field perpendicular to the lattice, with  $\Phi$  the magnetic flux per plaquette. We recover the Hofstadter spectral function as

$$b_{p/q}(k) = (1 - Q^{-k})(1 - Q^k) = (2 \sin(\pi kp/q))^2$$

The Hamiltonian describes a random walk with elementary steps *up*, *right* followed by *up*, *down*, and *down* followed by *left*. It means that starting from the origin  $(0, 0)$  it reaches after one step the lattice points  $(0, 1)$ ,  $(1, 1)$ ,  $(0, -1)$  or  $(-1, -1)$  with equal probability. This generates deformed walks on the square lattice (see Fig.1) which are equivalent through a modular transformation to the usual square lattice walks. (This modular transformation amounts to the transformation  $u \rightarrow -uv$ , which leaves the  $u, v$  commutation relation unchanged and turns  $H$  into  $u + v + u^{-1} + v^{-1}$ .)  $b(n)$  in (21) then yields the desired algebraic area counting [1]

$$b(n) = \sum_{\substack{A=-2\lfloor n^2/4\rfloor \\ A \text{ even}}}^{2\lfloor n^2/4\rfloor} e^{i\pi Ap/q} C_{2n}(A)$$

where

$$C_{2n}(A) = 2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{2-composition of } n}} c_2(l_1, l_2, \dots, l_j) \quad (23)$$

$$\sum_{k_3=-l_3}^{l_3} \cdots \sum_{k_j=-l_j}^{l_j} \binom{2l_1}{l_1 + A/2 + \sum_{i=3}^j (i-2)k_i} \binom{2l_2}{l_2 - A/2 - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{2l_i}{l_i + k_i}$$

with  $A$  even in the interval  $[-2\lfloor n^2/4\rfloor, 2\lfloor n^2/4\rfloor]$ .  $C_{2n}(A)$  counts the number of closed square lattice walks of length  $2n$  –there are overall  $\binom{2n}{n}^2$  of them, see (22)– enclosing an algebraic area  $A/2$  in the interval<sup>8</sup>  $[-\lfloor n^2/4\rfloor, \lfloor n^2/4\rfloor]$ : indeed the mapping of random walk algebraic area to the Hofstadter model [1] is via the weighting factor  $Q^{\text{algebraic area}}$ , where  $Q = e^{2i\pi p/q}$ , so here, with  $e^{i\pi Ap/q}$  appearing in (21), the algebraic area is  $A/2$ .

<sup>8</sup>This can be easily seen geometrically for lattice walks of length  $2n$  with  $n$  even, which have largest possible area  $\pm(n/2)^2$ : this is the walk circling a square of side  $n/2$  anti-clockwise or clockwise.

### 3.2 Square lattice walks: $b_{p/q}(k) = (2 \sin(\pi kp/q))^4$

Let us now look at square lattice walks with  $g = 2$  and  $r = 4$  which are defined in terms of the Hamiltonian

$$H = (u + u^{-1})^2 v + v^{-1}(u + u^{-1})^2 \quad (24)$$

The corresponding spectral function

$$b_{p/q}(k) = (Q^k + Q^{-k})^4 = (2 \cos(2\pi kp/q))^4$$

can be put in the standard form (6) for  $r = 4$  by redefining  $u \rightarrow iu$  and  $Q \rightarrow \sqrt{Q}$ , which does not affect the counting of walks nor the area weighting.

The Hamiltonian (24) describes a random walk with elementary steps in groups of one random step *up* or *down* and two independent random steps *right* or *left*. It means that starting from the origin  $(0, 0)$  it reaches after one step the lattice points  $(2, 1)$ ,  $(-2, 1)$ ,  $(2, -1)$  or  $(-2, -1)$  with probability  $1/8$ , or the lattice points  $(1, 0)$  or  $(-1, 0)$  with probability  $1/4$ . The same walk can be described as a particle hopping on an *even* or *odd* square sublattice, where even points are those with  $x$  and  $y$  coordinates adding to an even integer, the remaining being odd. The walk proceeds randomly on one of the sublattices but at each step it has the option to move to the nearest up or down point of the opposite sublattice, with each such jump contributing a factor of two in the weight of the walk. The Hamiltonian (24) counts the weighted number of such closed walks of a given total area.

There are  $\binom{2n}{n} \binom{4n}{2n}$  such closed walks of length  $2n$ , as in (22). The enumeration of such walks enclosing a given algebraic area, with the proper weight, is given by (21):

$$b(n) = \sum_{\substack{A=-4\lfloor n^2/4\rfloor \\ A \text{ even}}}^{4\lfloor n^2/4\rfloor} e^{i\pi Ap/q} C'_{2n}(A)$$

where

$$C'_{2n}(A) = 2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{2-composition of } n}} c_2(l_1, l_2, \dots, l_j) \\ \sum_{k_3=-2l_3}^{2l_3} \cdots \sum_{k_j=-2l_j}^{2l_j} \binom{4l_1}{2l_1 + A/2 + \sum_{i=3}^j (i-2)k_i} \binom{4l_2}{2l_2 - A/2 - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{4l_i}{2l_i + k_i}$$

with  $A$  even in the interval  $[-4\lfloor n^2/4\rfloor, 4\lfloor n^2/4\rfloor]$ .  $C'_{2n}(A)$  counts the number of closed square lattice walks described above of length  $2n$  and enclosing an algebraic area  $A/2$ .

### 3.3 Square lattice walks: $b_{p/q}(k) = \left( (2 \sin(\pi kp/q)) (2 \sin(\pi(k+1)p/q)) \right)^2$

Now consider square lattice walks with  $g = 2$  and  $r = 4$  defined by the Hamiltonian

$$H = (u + u^{-1})v(u + u^{-1}) + (u + u^{-1})v^{-1}(u + u^{-1}) \quad (25)$$

The spectral function can be brought to the standard form (8) for  $r = 4$  by an appropriate redefinition of  $u \rightarrow -iu$

$$b_{p/q}(k) = (2 \sin(\pi kp/q))^2 (2 \sin(\pi(k+1)p/q))^2$$

Its treatment is given in the subsection 6.2 of the Appendix.

This walk proceeds with sets of one step left or right, one step up or down and another step left or right. With an appropriate redefinition of  $u$  and  $v$  (modular transformation) this walk can also be mapped to a walk proceeding on odd or even square sublattices, as in the last subsection, but now the weight of jumping on the opposite sublattice is not 2, as before, but rather  $Q + Q^{-1}$ . So in this description the weight of the walks depends explicitly on  $Q$ , unlike any other walk we encountered before.

There are again  $\binom{2n}{n} \binom{4n}{2n}$  such closed walks of length  $2n$ . The enumeration of such walks enclosing a given algebraic area, with the proper weight, is given by

$$b(n) = \sum_{\substack{A=-\infty \\ A \text{ even}}}^{\infty} e^{i\pi Ap/q} C''_{2n}(A)$$

where

$$\begin{aligned} C''_{2n}(A) &= 2n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{2-composition of } n}} c_2(l_1, l_2, \dots, l_j) \\ &\sum_{k_3=-(l_2+l_3)}^{l_2+l_3} \cdots \sum_{k_j=-(l_{j-1}+l_j)}^{l_{j-1}+l_j} \sum_{k_{j+1}=-l_j}^{l_j} \binom{2l_1}{l_1+A/2 + \sum_{i=3}^{j+1} (i-2)k_i} \binom{2(l_1+l_2)}{l_1+l_2-A/2 - \sum_{i=3}^{j+1} (i-1)k_i} \\ &\times \prod_{i=3}^j \binom{2(l_{i-1}+l_i)}{l_{i-1}+l_i+k_i} \binom{2l_j}{l_j+k_{j+1}} \end{aligned} \quad (26)$$

$C''_{2n}(A)$  counts again the weighted number of closed square lattice walks described above of length  $2n$  enclosing an algebraic area  $A/2$ . It differs from the corresponding number (25) only in the weighting factor when jumping sublattices.



### 3.4 Triangular lattice chiral walks: $b_{p/q}(k) = (2 \sin(\pi kp/q))(2 \sin(\pi(k+1)p/q))$

From (19) for the triangular spectral function (7) with  $r = 2$  and  $g$ -exclusion we obtain

$$\begin{aligned}
b(n) = gn & \sum_{\substack{A = -\lceil n^2/2 \rceil - (g-2)\lfloor n^2/2 \rfloor \\ A \text{ same parity as } n}}^{\lceil n^2/2 \rceil + (g-2)\lfloor n^2/2 \rfloor} e^{i\pi Ap/q} \sum_{\substack{l_1, l_2, \dots, l_j \\ g\text{-composition of } n}} c_g(l_1, l_2, \dots, l_j) \\
& \sum_{k_3 = -(l_2+l_3)/2}^{(l_2+l_3)/2} \dots \sum_{k_j = -(l_{j-1}+l_j)/2}^{(l_{j-1}+l_j)/2} \sum_{k_{j+1} = -l_j/2}^{l_j/2} \\
& \binom{l_1}{l_1/2 + A/2 + \sum_{i=3}^{j+1} (i-2)k_i} \binom{l_1 + l_2}{(l_1 + l_2)/2 - A/2 - \sum_{i=3}^{j+1} (i-1)k_i} \\
& \times \prod_{i=3}^j \binom{l_{i-1} + l_i}{(l_{i-1} + l_i)/2 + k_i} \binom{l_j}{l_j/2 + k_{j+1}} \tag{27}
\end{aligned}$$

with overall counting given by replacing  $e^{i\pi Ap/q}$  by 1

$$\binom{gn}{n} \binom{2n}{n}$$

Triangular  $g = 3$  lattice chiral walks correspond to the quantum Hamiltonian

$$H = i(-u + u^{-1})v + v^{-2}$$

with spectral function

$$b_{p/q}(k) = (2 \sin(2\pi kp/q))(2 \sin(2\pi(k+1)p/q))$$

as already given in (5). They are depicted in Figs.2–4 (see [3] for more details; these walks are the generalization to four quadrants of the Kreweras walks [5]). Since the exclusion parameter is  $g = 3$  the counting above reduces to

$$\binom{3n}{n, n, n}$$

which is the number of closed triangular lattice chiral walks of length  $3n$ . The cluster coefficient (27) then yields the triangular lattice chiral walks algebraic area counting

$$b(n) = \sum_{\substack{A = -n^2 \\ A \text{ in steps of } 2}}^{n^2} e^{i\pi Ap/q} C_{3n}(A)$$

where

$$\begin{aligned}
C_{3n}(A) = & 3n \sum_{\substack{l_1, l_2, \dots, l_j \\ \text{3-compositions of } n}} c_3(l_1, l_2, \dots, l_j) \sum_{k_3 = -(l_2+l_3)/2}^{(l_2+l_3)/2} \dots \sum_{k_j = -(l_{j-1}+l_j)/2}^{(l_{j-1}+l_j)/2} \sum_{k_{j+1} = -l_j/2}^{l_j/2} \\
& \binom{l_1}{l_1/2 + A/2 + \sum_{i=3}^{j+1} (i-2)k_i} \binom{l_1 + l_2}{(l_1 + l_2)/2 - A/2 - \sum_{i=3}^{j+1} (i-1)k_i} \\
& \times \prod_{i=3}^j \binom{l_{i-1} + l_i}{(l_{i-1} + l_i)/2 + k_i} \binom{l_j}{l_j/2 + k_{j+1}} \quad (28)
\end{aligned}$$

with  $A$  in the interval  $[-n^2, n^2]$  with same parity as  $n$ .

$C_{3n}(A)$  counts the number of closed triangular lattice chiral walks of length  $3n$  enclosing an algebraic area  $A$ . Indeed, the mapping of triangular algebraic area-quantum triangular Hamiltonian discussed in [3] is via  $\mathcal{Q}^{\text{algebraic area}}$  where  $\mathcal{Q} = e^{2i\pi p/q}$ . Since in  $b_{p/q}(k)$  of (7) the building block  $2 \sin(\pi k p/q)$  is used, rather than  $2 \sin(2\pi k p/q)$  as in (5), we end up with  $e^{i\pi A p/q}$  in (27) in place of  $e^{2i\pi A p/q}$ , so that the algebraic area is  $A$ . One can directly check by explicit enumeration that when  $n$  is odd  $A$  is also odd (see, e.g.,  $n = 1$  with 3 walks of algebraic area 1 and 3 walks of algebraic area  $-1$ ) and when  $n$  is even  $A$  is also even (as in  $n = 2$ , with algebraic areas  $0, \pm 2$  and  $\pm 4$ ).

We conclude our discussion of algebraic area counting by remarking that it was possible to extract explicit expressions in terms of binomial sums for  $C_{2n}(A)$  in (23),  $C'_{2n}(A)$  in (25) and  $C_{3n}(A)$  in (28) from the cluster coefficients (21) or (27) because the summation constraints over  $A$  in the relevant binomial multiple sums (16) with  $r = 2, 4$  ( $A$  even) or (19) with  $r = 2$  ( $A$  same parity as  $l_1 + l_2 + \dots + l_j$ ), as well as the summation ranges, depend only on  $l_1 + l_2 + \dots + l_j = n$  and not on the individual  $l_i$ 's. Similar expressions would apply for walks deriving from odd  $r$  binomial sums, like (16) or (20), provided that the binomials appearing in the expressions are understood to vanish for values of  $A$  leading to noninteger entries, as discussed after (16).

It is a curious fact that if, in the binomial multiple sums or the cluster coefficients, we sum over *all* integer values of  $A$  without restrictions, and analytically continue the binomials to fractional values using gamma functions, the resulting infinite sums are closely related to the finite ones over the allowed values of  $A$ . This point is detailed and explained in the subsection 6.3 of the Appendix. Considering, for example, the binomial multiple sum (16), this means in particular that for even  $r$  and any set of  $l_i$ 's, the cumulative sum of the infinite sequence of coefficients of *odd*  $A$ , which are rational numbers times  $1/\pi^2$ , converges to the standard binomial counting  $\binom{r(l_1+l_2+\dots+l_j)}{r(l_1+l_2+\dots+l_j)/2}$ .

## 4 Apéry-like numbers

We finally turn to the occurrence of Apéry-like numbers in cluster coefficients (9) when evaluated at certain values of  $p/q$ . We consider  $b(n)$  as a stand-alone mathematical entity

that happens to lead to such occurrences. The interpretation of  $b(n)$  as the generating function of algebraic area enumerations of actual lattice walks, however, leads to the emergence of these Apéry-like numbers as specific weighted sums of random walks.

#### 4.1 Apéry-like numbers $g = 2$ and $r = 2$ : $b_{p/q}(k) = \left(2 \sin(\pi kp/q)\right)^2$

Let us consider<sup>9</sup>  $b(n)$  in (21). For  $g = 2$  and  $r = 2$  it gives, for  $n = 1, 2, 3, \dots$

$$\begin{aligned} p/q = 1 &\Rightarrow b(n) = \binom{2n}{n}^2 \Leftrightarrow \text{closed square lattice walks counting} \\ p/q = 1/2 &\Rightarrow b(n) = 4, 20, 112, 676, 4304, 28496, \dots \end{aligned}$$

These are the Apéry-like numbers  $\zeta(2)$  sequence OEIS A081085

$$\sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} 4^{n-2k} \binom{n}{2k} \binom{2k}{k}^2$$

with recurrence relation

$$(n+1)^2 b(n+1) - (12n(n+1) + 4)b(n) + 32n^2 b(n-1) = 0$$

The above Apéry-like numbers appear as *alternating* sums of square lattice random walks, weighted by the parity of their area.

#### 4.2 Apéry-like numbers $g = 2$ and $r = 1$ : $b_{p/q}(k) = 2 \sin(\pi kp/q)$

Let us still focus on (21) but now for  $g = 2$  and  $r = 1$ , with  $n$  necessarily even<sup>10</sup>. We find, for  $n = 2, 4, 6, \dots$

$$\begin{aligned} p/q = 1 &\Rightarrow b(n) = (-1)^{n/2} \binom{n}{n/2}^2 \\ p/q = 1/2 &\Rightarrow b(n) = 4, 20, 112, 676, 4304, 28496, \dots \end{aligned}$$

These are the same Apéry-like numbers as above

$$\sum_{k=0}^{n/2} \binom{n/2}{k} \binom{2k}{k} \binom{n-2k}{n/2-k}$$

---

<sup>9</sup>Or equivalently, using (14)

$$b(n) = gn \sum_{\substack{l_1, l_2, \dots, l_j \\ g\text{-composition of } n}} c_g(l_1, l_2, \dots, l_j) \int_0^1 dt \prod_{i=1}^j \left(2 \sin(\pi t + \pi(i-1)p/q)\right)^{r l_i}$$

<sup>10</sup> $n$  is necessarily even because  $l_1 + l_2 + \dots + l_j$  (which is equal to  $n$ ) has to be even.

now occurring for even  $n$ 's. Indeed, cases  $r = 2$  and ( $r = 1$ ,  $n$  even) are essentially equivalent: calling  $n = 2n'$  for  $r = 1$ , then  $(2 \sin(\pi k p/q))^{n=l_1+l_2+\dots+l_j}$  with  $l_1, l_2, \dots, l_j$  a composition of  $n$ , is in fact  $((2 \sin(\pi k p/q))^2)^{l'_1+l'_2+\dots+l'_j=n'}$  with  $l'_1, l'_2, \dots, l'_j$  a composition of  $n'$ , which is the  $r = 2$  result.

### 4.3 Apéry-like numbers $g = 2$ and $r = 4$ : $b_{p/q}(k) = (2 \sin(\pi k p/q))^4$

Let us again focus on  $b(n)$  in (21) but now for  $g = 2$  and  $r = 4$ : we find, for  $n = 1, 2, 3, \dots$

$$\begin{aligned} p/q = 1 &\Rightarrow b(n) = \binom{2n}{n} \binom{4n}{2n} \\ p/q = 1/2 &\Rightarrow b(n) = 12, 164, 2352, 34596, 516912, 7806224, \dots \end{aligned}$$

These are the Apéry-like numbers  $\zeta(2)$  sequence OEIS A143583

$$\sum_{k=0}^n \binom{2k}{k} \binom{4k}{2k} \binom{2n-2k}{n-k} \binom{4n-4k}{2n-2k} / \binom{2n}{n} = \sum_{k=0}^n 4^{n-k} \binom{2n-2k}{n-k} \binom{2k}{k}^2$$

with recurrence relation

$$(n+1)^2 b(n+1) - (32n(n+1) + 12)b(n) + 256n^2 b(n-1) = 0$$

These are, again, related to alternating sums of square random walks of the modified type defined in section 3.2.

### 4.4 Apéry-like numbers $g = 3$ and $r = 2$ : $b_{p/q}(k) = (2 \sin(\pi k p/q))(2 \sin(\pi(k+1)p/q))$

Finally we focus<sup>11</sup> on  $b(n)$  in (27). We find, for  $g = 3$  and  $n = 1, 2, 3, \dots$

$$\begin{aligned} p/q = 1 &\Rightarrow b(n) = (-1)^n \binom{3n}{n} \binom{2n}{n} \Leftrightarrow \text{triangular lattice chiral walks counting} \\ p/q = 1/2 &\Rightarrow b(n) = \binom{3n/2}{n/2} \binom{n}{n/2} \text{ if } n \text{ multiple of } 2 \text{ and } 0 \text{ otherwise} \\ p/q = 1/3 &\Rightarrow b(n) = 3, 9, 21, 9, -297, -2421, \dots \end{aligned}$$

---

<sup>11</sup>Or equivalently, using (18), on

$$\begin{aligned} b(n) = gn &\sum_{\substack{l_1, l_2, \dots, l_j \\ \text{g-composition of } n}} c_g(l_1, l_2, \dots, l_j) \\ &\int_0^1 dt \left(2 \sin(\pi t)\right)^{l_1} \prod_{i=2}^j \left(2 \sin(\pi t + \pi(i-1)p/q)\right)^{l_{i-1}+l_i} \left(2 \sin(\pi t + \pi j p/q)\right)^{l_j} \end{aligned}$$

These are Apéry-like numbers  $\zeta(2)$  sequence OEIS A006077

$$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \binom{2k}{k} \binom{3k}{k} = \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{n-3k, k, k, k}$$

with recurrence relation

$$(n+1)^2 b(n+1) + (9n(n+1) + 3)b(n) + 27n^2 b(n-1) = 0$$

We see that now Apéry-like numbers emerge as sums of random walks weighted by a *phase*  $e^{i2\pi A/3}$  with  $A$  the area of the walk in fundamental triangles, which reflects the triangular, rather than square, nature of the walk.

## 5 Conclusions

The trigonometric identities analyzed in this work, as well as their generalizations to other spectral functions that can be derived along the lines presented here, allow us to obtain expressions for the algebraic area counting of a broad set of random walks on two-dimensional lattices. The only requirement is that these walks be described by a Hamiltonian of the general form introduced in [3], admitting an interpretation as systems of generalized exclusion statistics with specific spectral functions. A wide class of lattice walk models can be embedded into this framework, and we gave a few examples in the present work, most notably the triangular chiral walk introduced originally in [3]. Using our present results we were able to derive the algebraic area counting formula for this chiral triangular walk.

The most obvious and interesting extension of our results would be in obtaining the area counting of other, more general types of walks. From the algebraic point of view, an immediate choice presents itself: the Hamiltonian

$$H_m = (u + u^{-1})^m v + v^{-1}(u + u^{-1})^m, \quad m = 1, 2, \dots$$

describes a class of Hofstadter-like models representing generalized random walks on the square lattice, with  $m = 1$  the standard (Hofstadter) random walk and  $m = 2$  the walk studied in subsection **3.2**. The model for general  $m$  represents a walk that proceeds in groups of one random step up or down and then  $m$  independent random steps left or right, but other representations are possible by performing modular transformations to the lattice (or redefinitions of the  $u, v$  operators in the Hamiltonian). All these walks belong to the class of  $g = 2$  exclusion statistics and their area counting is readily given by the relevant  $g = 2$  cluster coefficients and generalized trigonometric sums.

Clearly this is just the tip of a large iceberg as far as lattice walk models are concerned. For instance, another class of walks at  $g = 2$  would be described by the Hamiltonian

$$\tilde{H}_m = (u^m + u^{m-1} + \dots + u^{-m})v + v^{-1}(u^m + u^{m-1} + \dots + u^{-m})$$

This represents walks proceeding with a random step up or down to one of the  $2m + 1$  neighboring points in the left-right direction of distance up to  $m$  from the original horizontal position with equal probability. Again, the combinatorics of these walks are readily obtained with our methods. Yet other walks can be constructed, with asymmetrical propagation rules and belonging to higher  $g$  statistics. The only limitation, or criterion, is the potential relevance and physical significance of these walks, and this remains an open field of investigation. In particular, an adaptation of the methods of [3] and of the present work for walks on a hexagonal lattice, with or without chirality, would be of considerable physical interest, as such models would provide a description of the motion of electrons on graphene in the tight-binding electron-atom regime in the presence of a magnetic field, perhaps with additional chiral dynamics. This is a line of investigation with substantial potential physical payoff.

The emergence of Apéry-like numbers within the mathematical structure of these walks is another intriguing but obscure issue. At the present level of our understanding this is something of a mystery, or curiosity. It would be satisfying to have a better understanding of the relation between random walks and Apéry numbers, with an eye to possible applications in the mathematics of  $\zeta$ -functions and/or statistical models.

Finally, the Hamiltonians  $H_m$  and  $\tilde{H}_m$  presented above are all Hermitian and thus have a real spectrum, generalizing the corresponding spectrum of the Hofstadter model that leads to the celebrated “butterfly” fractal structure. It is expected that the spectrum of all the above models will have a similarly fractal structure, and preliminary numerical investigations confirm this fact. The shape and eigenvalue statistics of the spectrum of these generalized models are nontrivial deformations of the basic Hofstadter butterfly and their properties are an intriguing topic for further research.

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## 6 Appendix

### 6.1 Triangular $r = 4$ : $b_{p/q}(k) = (2 \sin(\pi k p/q)) (2 \sin(\pi(k+1)p/q)) (2 \sin(\pi(k+2)p/q)) (2 \sin(\pi(k+3)p/q))$

Likewise

$$\begin{aligned} \frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) &= \sum_{\substack{A=-\infty \\ A \text{ even}}}^{\infty} e^{i\pi A p/q} \\ &\sum_{k_3=-(l_1+l_2+l_3)/2}^{(l_1+l_2+l_3)/2} \sum_{k_4=-(l_1+l_2+l_3+l_4)/2}^{(l_1+l_2+l_3+l_4)/2} \dots \sum_{k_j=-(l_{j-3}+l_{j-2}+l_{j-1}+l_j)/2}^{(l_{j-3}+l_{j-2}+l_{j-1}+l_j)/2} \sum_{k_{j+1}=-(l_{j-2}+l_{j-1}+l_j)/2}^{(l_{j-2}+l_{j-1}+l_j)/2} \sum_{k_{j+2}=-(l_{j-1}+l_j)/2}^{(l_{j-1}+l_j)/2} \sum_{k_{j+3}=-l_j/2}^{l_j/2} \\ &\binom{l_1}{l_1/2 + A/2 + \sum_{i=3}^{j+3} (i-2)k_i} \binom{l_1+l_2}{(l_1+l_2)/2 - A/2 - \sum_{i=3}^{j+3} (i-1)k_i} \binom{l_1+l_2+l_3}{(l_1+l_2+l_3)/2 + k_3} \\ &\prod_{i=4}^j \binom{l_{i-3}+l_{i-2}+l_{i-1}+l_i}{(l_{i-3}+l_{i-2}+l_{i-1}+l_i)/2 + k_i} \binom{l_{j-2}+l_{j-1}+l_j}{(l_{j-2}+l_{j-1}+l_j)/2 + k_{j+1}} \binom{l_{j-1}+l_j}{(l_{j-1}+l_j)/2 + k_{j+2}} \binom{l_j}{l_j/2 + k_{j+3}} \end{aligned} \quad (29)$$

with overall counting

$$\begin{pmatrix} 4(l_1 + l_2 + \dots + l_j) \\ 2(l_1 + l_2 + \dots + l_j) \end{pmatrix}$$

One notes that as in previous cases the binomial multiple sum (29) is nothing but the trigonometric integral

$$\begin{aligned} \frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) &= \\ \int_0^1 dt \left( 2 \sin(\pi t) \right)^{l_1} \left( 2 \sin(\pi t + \pi p/q) \right)^{l_1+l_2} \left( 2 \sin(\pi t + \pi 2p/q) \right)^{l_1+l_2+l_3} \\ \prod_{i=4}^j \left( 2 \sin(\pi t + \pi(i-1)p/q) \right)^{l_{i-3}+l_{i-2}+l_{i-1}+l_i} \left( 2 \sin(\pi t + \pi j p/q) \right)^{l_{j-2}+l_{j-1}+l_j} \\ \left( 2 \sin(\pi t + \pi(j+1)p/q) \right)^{l_{j-1}+l_j} \left( 2 \sin(\pi t + \pi(j+2)p/q) \right)^{l_j} \end{aligned}$$

under the provision that  $2(l_1 + \dots + l_j) < q$ .

Clearly for a general  $r$  the spectral function  $b_{p/q}(k) = (2 \sin(\pi k p/q)) (2 \sin(\pi(k+1)p/q)) \dots (2 \sin(\pi(k+r-1)p/q))$  can be treated along the same lines as in subsections (2.2.1) and (2.2.2) and above.

## 6.2 Another triangular chiral walks generalization:

$$b_{p/q}(k) = (2 \sin(\pi k p/q))^{r/2} (2 \sin(\pi(k+1)p/q))^{r/2} \quad \text{with } r \text{ even}$$

When  $b_{p/q}(k) = (2 \sin(\pi k p/q))^2$  we have seen that (3), rewritten as

$$\frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) = \sum_{\substack{A=-\infty \\ A \text{ even}}}^{\infty} e^{i\pi A p/q}$$

$$\sum_{k_3=-l_3}^{l_3} \dots \sum_{k_j=-l_j}^{l_j} \binom{2l_1}{l_1 + A/2 + \sum_{i=3}^j (i-2)k_i} \binom{2l_2}{l_2 - A/2 - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{2l_i}{l_i + k_i}$$

generalizes for  $b_{p/q}(k) = (2 \sin(\pi k p/q))^r$  and  $r$  is even to

$$\frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) = \sum_{\substack{A=-\infty \\ A \text{ even}}}^{\infty} e^{i\pi A p/q}$$

$$\sum_{k_3=-rl_3/2}^{rl_3/2} \dots \sum_{k_j=-rl_j/2}^{rl_j/2} \binom{rl_1}{rl_1/2 + A/2 + \sum_{i=3}^j (i-2)k_i} \binom{rl_2}{rl_2/2 - A/2 - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{rl_i}{rl_i/2 + k_i}$$

Likewise, when  $b_{p/q}(k) = (2 \sin(\pi k p/q))(2 \sin(\pi(k+1)p/q))$ , equation (19),

$$\frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1)$$

$$= \sum_{\substack{A=-\infty \\ A \text{ same parity } l_1+l_2+\dots+l_j}}^{\infty} e^{i\pi A p/q} \sum_{k_3=-(l_2+l_3)/2}^{(l_2+l_3)/2} \dots \sum_{k_j=-(l_{j-1}+l_j)/2}^{(l_{j-1}+l_j)/2} \sum_{k_{j+1}=-l_j/2}^{l_j/2}$$

$$\binom{l_1}{l_1/2 + A/2 + \sum_{i=3}^{j+1} (i-2)k_i} \binom{l_1 + l_2}{(l_1 + l_2)/2 - A/2 - \sum_{i=3}^{j+1} (i-1)k_i}$$

$$\prod_{i=3}^j \binom{l_{i-1} + l_i}{(l_{i-1} + l_i)/2 + k_i} \binom{l_j}{l_j/2 + k_{j+1}}$$

generalizes for  $b_{p/q}(k) = ((2 \sin(\pi k p/q))^{r/2} (2 \sin(\pi(k+1)p/q))^{r/2})$  and  $r$  even to



$$\begin{aligned}
& \frac{1}{q} \sum_{k=1}^q b_{p/q}^{l_1}(k) b_{p/q}^{l_2}(k+1) \dots b_{p/q}^{l_j}(k+j-1) \\
&= \sum_{\substack{A=-\infty \\ \text{A same parity } r(l_1+l_2+\dots+l_j)/2}}^{\infty} e^{i\pi Ap/q} \sum_{k_3=-r(l_2+l_3)/4}^{r(l_2+l_3)/4} \dots \sum_{k_j=-r(l_{j-1}+l_j)/4}^{r(l_{j-1}+l_j)/4} \sum_{k_{j+1}=-rl_j/4}^{rl_j/4} \\
& \left( \binom{rl_1/2}{rl_1/4 + A/2 + \sum_{i=3}^{j+1} (i-2)k_i} \right) \left( \binom{r(l_1+l_2)/2}{r(l_1+l_2)/4 - A/2 - \sum_{i=3}^{j+1} (i-1)k_i} \right) \\
& \prod_{i=3}^j \left( \binom{r(l_{i-1}+l_i)/2}{r(l_{i-1}+l_i)/4 + k_i} \right) \left( \binom{rl_j/2}{rl_j/4 + k_{j+1}} \right)
\end{aligned}$$

### 6.3 Regarding (16): summing over $A$ odd when $r$ is even

So far one has considered the  $r(l_1 + l_2 + \dots + l_j)$  even cases so that the  $q \rightarrow \infty$  limit in the trigonometric sum (1) yields an overall binomial counting which is an integer and contributes as such to the overall counting of closed lattice walks. We have seen that this trigonometric sum can be rewritten as a multiple binomial sum of the type (16) or (19) with some constraints on the evenness or oddness of the  $A$ 's (and additionally of  $l_1 + l_2 + \dots + l_j$  in the case  $r$  odd). In the  $r(l_1 + l_2 + \dots + l_j)$  odd cases, on the other hand, (1) would not rewrite anymore as a multiple binomial sum.

Still, and quite generally, one could take the binomial multiple sums (16) (and likewise (19)) at face value for all possible entries  $A$  even or odd and  $l_1 + l_2 + \dots + l_j$  even or odd. In the  $r$  even case we already know that the  $A$  even summation in (16) has a finite range and yields the overall integer counting binomial. The  $A$  odd summation happens to yield again the same overall binomial but with each term in the sum a rational number times  $1/\pi^2$  and an infinite summation range. The  $1/\pi^2$  factor comes from the first two binomials in (16) due the relaxation of the constraint that their entries be integers (since  $A$  is now odd). Likewise in the  $r$  odd case, when  $l_1 + l_2 + \dots + l_j$  is even, we already know that  $A$  even or odd summations, depending on the parity of  $l_1 + l_3 + \dots$ , have a finite range and yield the usual overall integer counting binomial; it is still true that summing over  $A$  even with  $l_1 + l_3 + \dots$  odd or on  $A$  odd with  $l_1 + l_3 + \dots$  even would yield the same overall counting binomial with again terms  $1/\pi^2$  times rational numbers and an infinite summation range. Finally when both  $r$  and  $l_1 + l_2 + \dots + l_j$  are odd,  $A$  even and odd summations have finite range to yield the overall binomial which is in this case  $1/\pi$  times a rational number. In all these instances the coefficients sum up to  $\binom{r(l_1+l_2+\dots+l_j)}{r(l_1+l_2+\dots+l_j)/2}$  for both  $A$  even or odd summations, with finite or infinite ranges depending on the situation.

To better understand these weird  $A$ -summations, let us first focus on the regular  $A$ -summations and consider the RHS of (17) i.e., the binomial multiple sum

$$\sum_{k_3=-rl_3/2}^{rl_3/2} \cdots \sum_{k_j=-rl_j/2}^{rl_j/2} \binom{rl_1}{rl_1/2 + A/2 + \sum_{i=3}^j (i-2)k_i} \binom{rl_2}{rl_2/2 - A/2 - \sum_{i=3}^j (i-1)k_i} \prod_{i=3}^j \binom{rl_i}{rl_i/2 + k_i}$$

One wishes to go backward and get the double integral in the LHS of (17), which, when summed over  $A$ , directly yields the overall counting binomial

$$\binom{r(l_1 + l_2 + \dots + l_j)}{r(l_1 + l_2 + \dots + l_j)/2}$$

For simplicity let us consider the case  $r$  even: since  $r$  is even, all the  $k_i$ 's  $i = 3, \dots, j$  are integers, and since we know that  $A$  has then to be even (see below (16)), in the first two binomials both  $rl_1/2 + A/2 + \sum_{i=3}^j (i-2)k_i$  and  $rl_2/2 - A/2 - \sum_{i=3}^j (i-1)k_i$  are integers. Using that for an integer  $n$  and integer  $k$

$$\int_0^1 dt e^{2i\pi(k-n)t}$$

is the Kronecker  $\delta(k, n)$  meaning<sup>12</sup>

$$\sum_{\substack{k=-\infty \\ k \text{ integer}}}^{\infty} \delta(k, n) f(k) = f(n)$$

we can rewrite these binomials as

$$\binom{rl_1}{rl_1/2 + A/2 + \sum_{i=3}^j (i-2)k_i} = \sum_{\substack{k_1=-rl_1/2 \\ k_1 \text{ integer}}}^{rl_1/2} \int_0^1 dt e^{2i\pi(k_1 - (A/2 + \sum_{i=3}^j (i-2)k_i))t} \binom{rl_1}{rl_1/2 + k_1}$$

$$\binom{rl_2}{rl_2/2 - A/2 - \sum_{i=3}^j (i-1)k_i} = \sum_{\substack{k_2=-rl_2/2 \\ k_2 \text{ integer}}}^{rl_2/2} \int_0^1 dt' e^{2i\pi(k_2 + A/2 + \sum_{i=3}^j (i-1)k_i)t'} \binom{rl_2}{rl_2/2 + k_2}$$

where the summations are restricted to  $[-rl_1/2, rl_1/2]$  and  $[-rl_2/2, rl_2/2]$  since there is no point to sum outside these intervals where the binomials trivially vanish. So the LHS of (17) becomes

$$\sum_{\substack{k_1=-rl_1/2 \\ k_1 \text{ integer}}}^{rl_1/2} \cdots \sum_{\substack{k_j=-rl_j/2 \\ k_j \text{ integer}}}^{rl_j/2} \int_0^1 dt e^{2i\pi(k_1 - (A/2 + \sum_{i=3}^j (i-2)k_i))t} \int_0^1 dt' e^{2i\pi(k_2 + A/2 + \sum_{i=3}^j (i-1)k_i)t'} \prod_{i=1}^j \binom{rl_i}{rl_i/2 + k_i}$$

<sup>12</sup>From now on for clarity we explicitly specify summation indices to be integers or half integers.

which is

$$\sum_{\substack{k_1=-r l_1/2 \\ k_1 \text{ integer}}}^{r l_1/2} \cdots \sum_{\substack{k_j=-r l_j/2 \\ k_j \text{ integer}}}^{r l_j/2} \int_0^1 dt \int_0^1 dt' e^{i\pi A(t'-t)} \prod_{i=1}^j \binom{r l_i}{r l_i/2 + k_i} e^{2i\pi k_i((i-1)t' - (i-2)t)}$$

i.e., since obviously

$$\sum_{\substack{k_i=-r l_i/2 \\ k_i \text{ integer}}}^{r l_i/2} \binom{r l_i}{r l_i/2 + k_i} e^{2i\pi k_i((i-1)t' - (i-2)t)} = \left( 2 \cos(\pi((i-1)t' - (i-2)t)) \right)^{r l_i}$$

and calling  $t' - t = t''$ , we obtain<sup>13</sup>

$$\int_0^1 dt \int_0^1 dt'' e^{i\pi A t''} \prod_{i=1}^j \left( 2 \cos(\pi((i-1)t'' + t)) \right)^{r l_i}$$

We have to sum over  $A$  even: since  $\sum_{A \text{ even}} e^{i\pi A t''} = \sum_{n=-\infty}^{\infty} \delta(t'', n)$

$$\begin{aligned} \sum_{A \text{ even}} \int_0^1 dt \int_0^1 dt'' e^{i\pi A t''} \prod_{i=1}^j \left( 2 \cos(\pi((i-1)t'' + t)) \right)^{r l_i} &= \int_0^1 dt (2 \cos(\pi t))^{r(l_1 + l_2 + \dots + l_j)} \\ &= \binom{r(l_1 + l_2 + \dots + l_j)}{r(l_1 + l_2 + \dots + l_j)/2} \end{aligned} \quad (30)$$

where the overall binomial counting has been obtained as expected.

Now still assuming  $r$  being even, so that all the  $k_i$ 's  $i = 3, \dots, j$  are integers, let us insist that the summation over  $A$  be on  $A$  odd so that both  $r l_1/2 + A/2 + \sum_{i=3}^j (i-2)k_i$  and  $r l_2/2 - A/2 - \sum_{i=3}^j (i-1)k_i$  are half-integers. Using that for an half-integer  $n/2$  and half integer  $k$

$$\int_0^1 dt e^{2i\pi(k-n/2)t}$$

is the Kronecker  $\delta(k, n/2)$  meaning

$$\sum_{\substack{k=-\infty \\ k \text{ half integer}}}^{\infty} \delta(k, n/2) f(k) = f(n/2)$$

---

<sup>13</sup>Or equivalently as in the RHS of (17)

$$\frac{1}{2} \int_0^2 dt' \int_0^1 dt \prod_{i=1}^j \left( 2 \sin(\pi t + \pi(i-1)t') \right)^{r l_i} e^{i\pi A t'}$$

we rewrite the same two binomials as

$$\binom{rl_1}{rl_1/2 + A/2 + \sum_{i=3}^j (i-2)k_i} = \sum_{\substack{k_1=-\infty \\ k_1 \text{ half integer}}}^{\infty} \int_0^1 dt e^{2i\pi(k_1 - (A/2 + \sum_{i=3}^j (i-2)k_i))t} \binom{rl_1}{rl_1/2 + k_1}$$

$$\binom{rl_2}{rl_2/2 - A/2 - \sum_{i=3}^j (i-1)k_i} = \sum_{\substack{k_2=-\infty \\ k_2 \text{ half integer}}}^{\infty} \int_0^1 dt' e^{2i\pi(k_2 + A/2 + \sum_{i=3}^j (i-1)k_i)t'} \binom{rl_2}{rl_2/2 + k_2}$$

Doing the same manipulations as above except for the first two binomials the LHS of (17) then becomes

$$\int_0^1 dt \int_0^1 dt'' \sum_{\substack{k_1=-\infty \\ k_1 \text{ half integer}}}^{\infty} \sum_{\substack{k_2=-\infty \\ k_2 \text{ half integer}}}^{\infty} \binom{rl_1}{rl_1/2 + k_1} \binom{rl_2}{rl_2/2 + k_2} e^{i\pi A t''} e^{2i\pi(k_1+k_2)t} e^{2i\pi k_2 t''} \prod_{i=3}^j \left( 2 \cos(\pi((i-1)t'' + t)) \right)^{rl_i}$$

Summing over all  $A$  odd i.e., over  $A + 2k_2$  even –since  $k_2$  is an half integer– yields again a Kronecker enforcing  $t'' = 0$  so that after summation one obtains

$$\int_0^1 dt \sum_{\substack{k_1=-\infty \\ k_1 \text{ half integer}}}^{\infty} \sum_{\substack{k_2=-\infty \\ k_2 \text{ half integer}}}^{\infty} \binom{rl_1}{rl_1/2 + k_1} \binom{rl_2}{rl_2/2 + k_2} e^{2i\pi(k_1+k_2)t} (2 \cos(\pi t))^{r(l_3+\dots+l_j)}$$

Comparing with (30) we see that in order to get the same overall binomial counting everything boils down to showing that in the same way that obviously

$$\sum_{\substack{k_1=-rl_1/2 \\ k_1 \text{ integer}}}^{rl_1/2} \sum_{\substack{k_2=-rl_2/2 \\ k_2 \text{ integer}}}^{rl_2/2} \binom{rl_1}{rl_1/2 + k_1} \binom{rl_2}{rl_2/2 + k_2} e^{2i\pi(k_1+k_2)t} = (2 \cos(\pi t))^{r(l_1+l_2)} \quad (31)$$

holds,

$$\sum_{\substack{k_1=-\infty \\ k_1 \text{ half integer}}}^{\infty} \sum_{\substack{k_2=-\infty \\ k_2 \text{ half integer}}}^{\infty} \binom{rl_1}{rl_1/2 + k_1} \binom{rl_2}{rl_2/2 + k_2} e^{2i\pi(k_1+k_2)t} = (2 \cos(\pi t))^{r(l_1+l_2)}$$

should also hold.

To show this let us focus on the trivial identity (31) which is nothing but

$$\sum_{\substack{k_1=-rl_1/2 \\ k_1 \text{ integer}}}^{rl_1/2} \sum_{\substack{k_2=-rl_2/2 \\ k_2 \text{ integer}}}^{rl_2/2} \binom{rl_1}{rl_1/2 + k_1} \binom{rl_2}{rl_2/2 + k_2} e^{2i\pi(k_1+k_2)t} = \sum_{\substack{k=-r(l_1+l_2)/2 \\ k \text{ integer}}}^{r(l_1+l_2)/2} \binom{r(l_1+l_2)}{r(l_1+l_2)/2 + k} e^{2i\pi kt}$$

or equivalently, harmlessly relaxing the range of  $k_1, k_2$  and  $k$  summations,

$$\sum_{\substack{k_1=-\infty \\ k_1 \text{ integer}}}^{\infty} \sum_{\substack{k_2=-\infty \\ k_2 \text{ integer}}}^{\infty} \binom{rl_1}{rl_1/2 + k_1} \binom{rl_2}{rl_2/2 + k_2} e^{2i\pi(k_1+k_2)t} = \sum_{\substack{k=-\infty \\ k \text{ integer}}}^{\infty} \binom{r(l_1+l_2)}{r(l_1+l_2)/2 + k} e^{2i\pi kt} \quad (32)$$

Let us to rederive it in an other way : defining  $k = k_1 + k_2$  we can rewrite

$$\begin{aligned} & \sum_{\substack{k_1=-\infty \\ k_1 \text{ integer}}}^{\infty} \sum_{\substack{k_2=-\infty \\ k_2 \text{ integer}}}^{\infty} \binom{rl_1}{rl_1/2 + k_1} \binom{rl_2}{rl_2/2 + k_2} e^{2i\pi(k_1+k_2)t} \\ &= \sum_{\substack{k=-\infty \\ k \text{ integer}}}^{\infty} \sum_{\substack{k_1=-\infty \\ k_1 \text{ integer}}}^{\infty} \binom{rl_1}{rl_1/2 + k_1} \binom{rl_2}{rl_2/2 + k - k_1} e^{2i\pi kt} \end{aligned}$$

Thanks to the Chu-Vandermonde identity

$$\binom{l_1+l_2}{l'_1+l'_2} = \sum_{\substack{k_1=-\infty \\ k_1 \text{ integer}}}^{\infty} \binom{l_1}{l'_1+k_1} \binom{l_2}{l'_2-k_1}$$

we conclude that we indeed recover (32).

It is clear that the same conclusion can be reached when  $k_1$  and  $k_2$  are now both half integers namely

$$\begin{aligned} & \sum_{\substack{k_1=-\infty \\ k_1 \text{ half integer}}}^{\infty} \sum_{\substack{k_2=-\infty \\ k_2 \text{ half integer}}}^{\infty} \binom{rl_1}{rl_1/2 + k_1} \binom{rl_2}{rl_2/2 + k_2} e^{2i\pi(k_1+k_2)t} \\ &= \sum_{\substack{k=-\infty \\ k \text{ integer}}}^{\infty} \binom{r(l_1+l_2)}{r(l_1+l_2)/2 + k} e^{2i\pi kt} \end{aligned} \quad (33)$$

Indeed  $k_1$  and  $k_2$  being both half integers then  $k = k_1 + k_2$  is again an integer so we can write

$$\begin{aligned} & \sum_{\substack{k_1=-\infty \\ k_1 \text{ half integer}}}^{\infty} \sum_{\substack{k_2=-\infty \\ k_2 \text{ half integer}}}^{\infty} \binom{rl_1}{rl_1/2 + k_1} \binom{rl_2}{rl_2/2 + k_2} e^{2i\pi(k_1+k_2)t} \\ &= \sum_{\substack{k=-\infty \\ k \text{ integer}}}^{\infty} \sum_{\substack{k_1=-\infty \\ k_1 \text{ half integer}}}^{\infty} \binom{rl_1}{rl_1/2 + k_1} \binom{rl_2}{rl_2/2 + k - k_1} e^{2i\pi kt} \end{aligned}$$

Thanks to the generalized Chu-Vandermonde identity

$$\binom{l_1+l_2}{l'_1+l'_2} = \sum_{\substack{k_1=-\infty \\ k_1 \text{ half integer}}}^{\infty} \binom{l_1}{l'_1+k_1} \binom{l_2}{l'_2-k_1}$$

we reach indeed the identity (33) for the half integers summations. From which it directly follows that in the presence of the additional  $(2 \cos(\pi t))^{r(l_3+\dots+l_j)}$  term integrating over  $t$  from 0 to 1 one ends up getting again the same overall binomial counting, as desired.

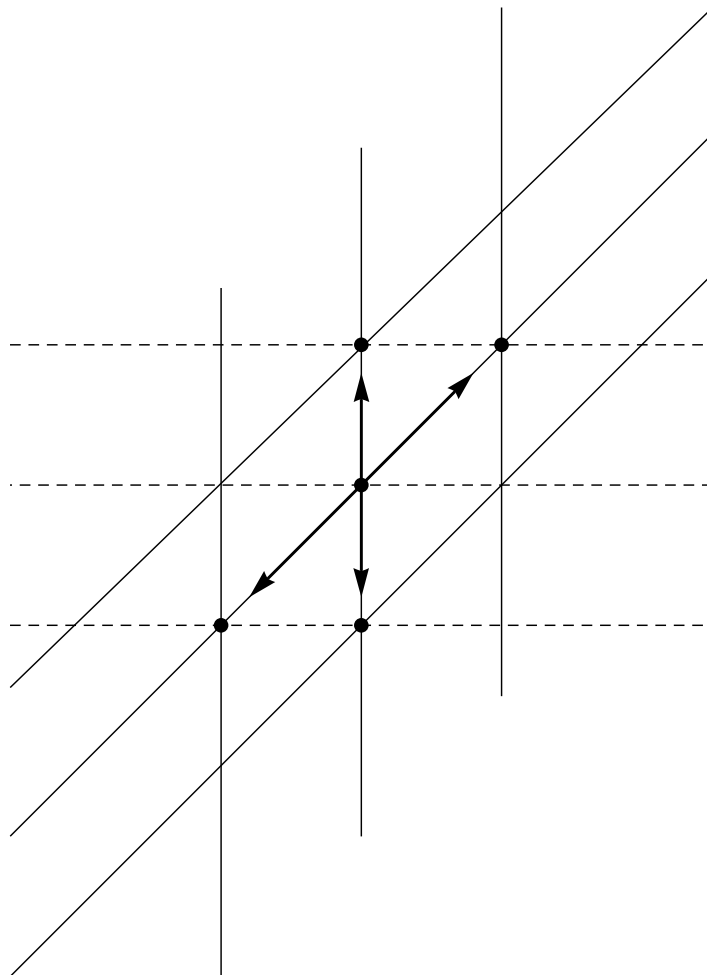


Figure 1: The lattice in (3.1).

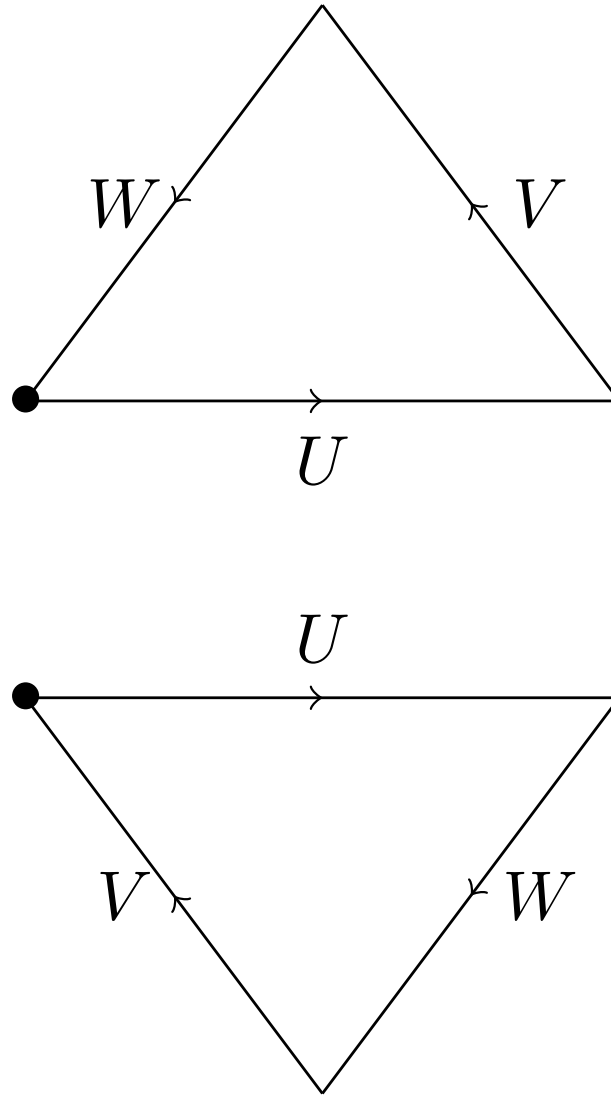


Figure 2:  $U$ ,  $V$  and  $W$  are the three possible hoppings on the triangular lattice. As an illustration two chiral walks going around *up-vertex* and *down-vertex* triangular cells starting from the black bullet lattice site.

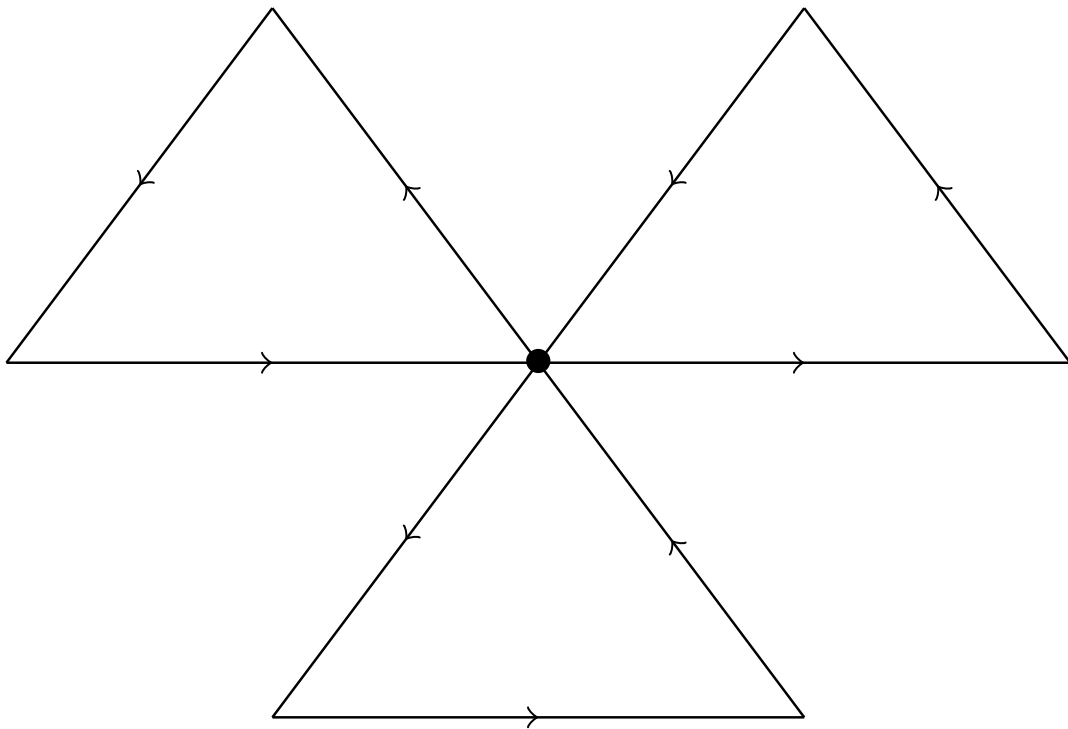
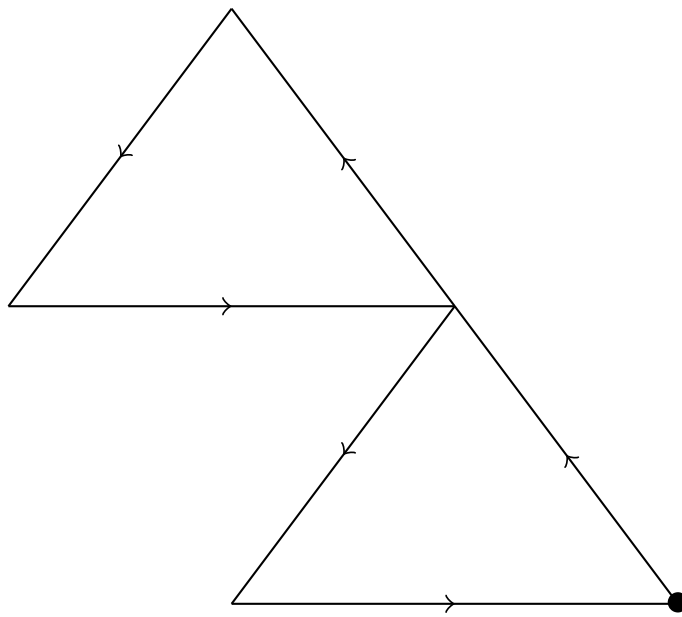
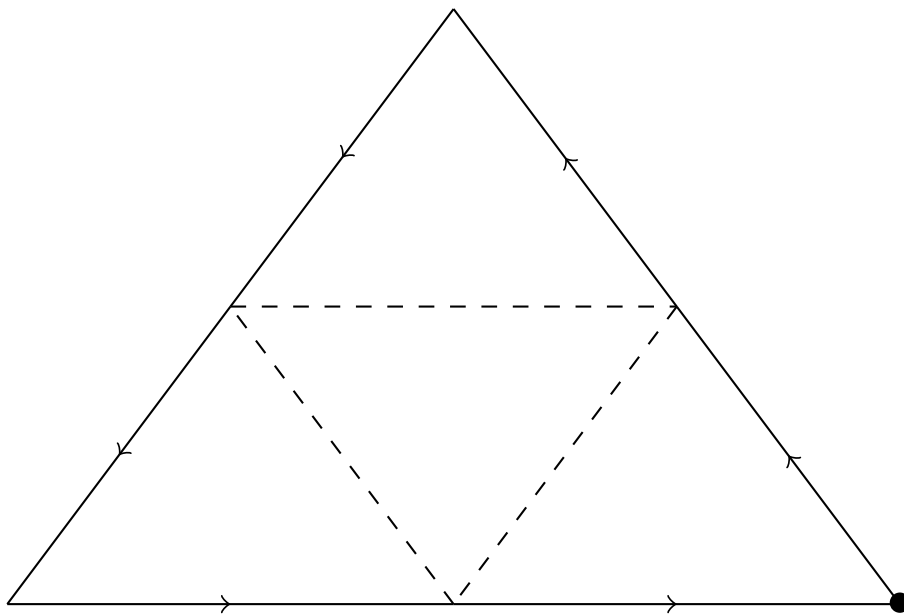


Figure 3: Three of the 6 possible chiral walks starting from the same black bullet lattice site. Only the 3 outgoing arrows represent possible motions from the original site.





$$UWUWV^2$$



$$U^2W^2V^2$$

Figure 4:  $UWUWV^2$  and  $U^2W^2V^2$  walks.

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