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To cite this version:

Leonid Vainerman, Jean-Michel Vallin. Yetter-Drinfel'd algebras and coideals of Weak Hopf C * -Algebras. Theory and Applications of Categories, 2021, 37 (3), pp.57-94. 10.48550/arXiv.2006.16330 . hal-02886793v2

HAL Id: hal-02886793

https://hal.science/hal-02886793v2

Submitted on 1 Apr 2021

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YETTER-DRINFEL'D ALGEBRAS AND COIDEALS OF WEAK HOPF C^*-ALGEBRAS

LEONID VAINERMAN, JEAN-MICHEL VALLIN

Abstract. We characterize braided commutative Yetter-Drinfel'd C^*-algebras over weak Hopf C^*-algebras in categorical terms. Using this, we then study quotient type coideal subalgebras of a given weak Hopf C^*-algebra $\mathfrak{G}$ and coideal subalgebras invariant with respect to the adjoint action of $\mathfrak{G}$. Finally, as an example, we explicitly describe quotient type coideal subalgebras of the weak Hopf C^*-algebras associated with Tambara-Yamagami categories.

1. Introduction

This paper continues the study of coactions of weak Hopf C*-algebras on C*-algebras and their applications which was initiated in two articles [Vainerman-Vallin,2017] and [Vainerman-Vallin,2020]. Let us first recall our motivation.

It is known that any finite tensor category equipped with a fiber functor to the category of finite dimensional vector spaces is equivalent to the representation category of some Hopf algebra - see, for example, [Etingof et all.,2015], Theorem 5.3.12. But many tensor categories do not admit a fiber functor, so they cannot be presented as representation categories of Hopf algebras. On the other hand, T. Hayashi [Hayashi,1999] showed that any fusion category admits a tensor functor to the category of bimodules over some semisimple (even commutative) algebra. Then it was proved in [Hayashi,1999], [Szlachanyi,2001], [Ostrik,2003] that any fusion category is equivalent to the representation category of some algebraic structure generalizing Hopf algebras called a *weak Hopf algebra* [Bohm-Nill-Szlachanyi,1999] or a *finite quantum groupoid* [Nikshych-Vainerman,2002].

The main difference between weak and usual Hopf algebra is that in the former the coproduct Δ is not necessarily unital. In addition, a representation category of a weak Hopf algebra is, in general, multitensor, i.e., its unit object is not necessarily simple (see, for example, [Etingof et all.,2015], 4.1). By this reason, in the present paper we work mainly in the context of multitensor categories.

Apart from (multi)tensor categories, weak Hopf algebras have interesting applications to the subfactor theory. In particular, for any finite index and finite depth II_1-subfactor $N \subset M$, there exists a weak Hopf C^*-algebra $\mathfrak{G}$ such that the corresponding Jones tower can be expressed in terms of crossed products of N and M with $\mathfrak{G}$ and its dual. Moreover,

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2020 Mathematics Subject Classification: Primary 18D10, Secondary 16T05, Tertiary 46L05.

Key words and phrases: Coactions and corepresentations of quantum groupoids, C^*-categories, reconstruction theorem..

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there is *a Galois correspondence* between intermediate subfactors in this Jones tower and coideal C^*-subalgebras of $\mathfrak{G}$ - see [Nikshych-Vainerman 2,2000]. This motivates the study of coideal C^*-subalgebras of weak Hopf C^*-algebras (in what follows - WHAs).

A unital C^*-algebra A equipped with a *coaction* $\mathfrak{a}$ of a WHA $\mathfrak{G} = (B, \Delta, S, \varepsilon)$ is called a $\mathfrak{G}$-C^*-algebra. When A is a unital C^*-subalgebra of B and $\mathfrak{a} = \Delta$, we call it a *coideal C^*-subalgebra* or briefly a coideal of B.

The structure of the paper is as follows. Sections 2 to 4 contain basic definitions and facts needed for the comprehension of the main results of the paper. In particular, in Section 2 we describe three C^*-multitensor categories associated with any weak Hopf C^*-algebra and in Section 3 we explain how to reconstruct a weak Hopf C^*-algebra if one of these categories is given. Various results of this kind are known, for example [Szlachanyi,2001], [Hayashi,1999], [Calaque-Etingof,2008], [Pfeiffer,2009], [Ostrik,2003], and we present them in the form convenient for our goals.

It was shown in [Vainerman-Vallin,2017] that any $\mathfrak{G}$-C^*-algebra $(A, \mathfrak{a})$ corresponds to a pair $(\mathcal{M}, M)$, where $\mathcal{M}$ is a module C^*-category with a generator M over the category of unitary corepresentations of $\mathfrak{G}$. Here, in section 5, we study an important special class of $\mathfrak{G}$-C^*-algebras - braided-commutative Yetter-Drinfel'd C^*-algebras and characterize the corresponding C^*-module categories:

1.1. Theorem. *Given a WHA $\mathfrak{G}$, the following two categories are equivalent:*

(i) Category $YD_{brc}(\mathfrak{G})$ of unital braded-commutative Yetter-Drinfel'd $\mathfrak{G}$-C^*-algebras with unital $\mathfrak{G}$- and $\hat{\mathfrak{G}}$-equivariant $*$-homomorphisms as morphisms.

(ii) Category $Tens(UCorep(\mathfrak{G}))$ of pairs $(\mathcal{C}, \mathcal{E})$, where $\mathcal{C}$ is a C^*-multitensor category whose associativities reduce to the changing of brackets and $\mathcal{E} : UCorep(\mathfrak{G}) \longrightarrow \mathcal{C}$ is a unitary tensor functor such that $\mathcal{C}$ is generated by the image of $\mathcal{E}$. Morphisms $(\mathcal{C}, \mathcal{E}) \longrightarrow (\mathcal{C}', \mathcal{E}')$ of this category are equivalence classes of pairs $(\mathcal{F}, \eta)$, where $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{C}'$ is a unitary tensor functor and $\eta : \mathcal{F}\mathcal{E} \longrightarrow \mathcal{E}'$ is a natural unitary monoidal functor isomorphism.

Moreover, given a morphism $[(\mathcal{F}, \eta)] : (\mathcal{C}, \mathcal{E}) \longrightarrow (\mathcal{C}', \mathcal{E}')$, the corresponding homomorphism of YD $\mathfrak{G}$-C^*-algebras is injective if and only if $\mathcal{F}$ is faithful, and it is surjective if and only if $\mathcal{F}$ is full.

A similar result for compact quantum group coactions on C*-algebras was obtained earlier in [Neshveyev-Yamashita,2014]. When it is possible, we follow the same strategy. However, instead of tensor products over $\mathbb{C}$ we have to deal with tensor products over, in general, non commutative algebras which makes many reasonings and calculations much more complicated.

In Section 6, we study, as an application of Theorem 1.1, coideals C^*-subalgebras which belong to the category $YD_{brc}(\mathfrak{G})$: quotient type and invariant with respect to the adjoint action of a WHA and the relationship between them. We prove

1.2. Theorem. *Any quotient type coideal C^*-subalgebra is invariant. Conversely, for any invariant coideal C^*-subalgebra I of $\mathfrak{G}$ there exists a unique, up to isomorphism, quantum subgroupoid (i.e., a WHA $\mathfrak{H}$ equipped with an epimorphism $\pi : \mathfrak{G} \longrightarrow \mathfrak{H}$) such that I is isomorphic as a $\mathfrak{G}$-C^*-algebra to the quotient type coideal C^*-subalgebra $I(\mathfrak{H}\backslash\mathfrak{G})$.*

In the Hopf algebraic setting, similar result was obtained in [Takeuchi,1994]

Let us note that the coideal C^*-subalgebra (B, Δ) is invariant (quotient type) if and only if $\mathfrak{G}$ is a usual Hopf algebra and that invariant (quotient type) coideal C^*-subalgebras form a sublattice of the lattice of all coideal C^*-subalgebras of a WHA introduced in [Nikshych-Vainerman 2,2000].

A concrete example illustrating the results of the paper is considered in Section 7. Namely, we describe invariant and quotient type coideal C^*-subalgebras of WHAs constructed using the Tambara-Yamagami categories [Tambara-Yamagami,1998] whose simple objects are elements of a finite abelian group G and one separate element m. In particular, it is shown that a coideal C^*-subalgebra is invariant if and only if it is of quotient type and that the lattice of invariant (quotient type) coideal C^*-subalgebras is isomorphic to the lattice of subgroups of G completed by the new maximal element $G \sqcup \{m\}$.

Notation: for any category $\mathcal{C}$ we denote by $\Omega = Irr(\mathcal{C})$ an exhaustive set of representatives of the equivalence classes of its simple objects.

Our references are: [Etingof et all.,2015] for the multitensor categories, [Neshveyev-Tuset,2013] for C^*-tensor categories, [Nikshych-Vainerman,2002] for WHAs.

2. Weak Hopf C^*-algebras

2.1. Weak Hopf C^*-algebras. A *weak bialgebra* $\mathfrak{G} = (B, \Delta, \varepsilon)$ is a finite dimensional algebra B with the comultiplication $\Delta : B \longrightarrow B \otimes B$ and counit $\varepsilon : B \longrightarrow \mathbb{C}$ such that (B, Δ, ε) is a coalgebra and the following axioms hold for all $b, c, d \in B$:

(1) Δ is a (not necessarily unital) homomorphism : $\Delta(bc) = \Delta(b)\Delta(c)$.

(2) The unit and counit satisfy the identities (we use the Sweedler leg notation $\Delta(c) = c_{(1)} \otimes c_{(2)}$, $(\Delta \otimes id_B)\Delta(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$ etc.):

$$\begin{aligned} \varepsilon(bc_{(1)})\varepsilon(c_{(2)}d) &= \varepsilon(bcd), \\ (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) &= (\Delta \otimes id_B)\Delta(1). \end{aligned}$$

A *weak Hopf algebra* is a weak bialgebra equipped with an antipode $S : B \longrightarrow B$ which is an anti-algebra and anti-coalgebra homomorphism such that

$$\begin{aligned} m(id_B \otimes S)\Delta(b) &= (\varepsilon \otimes id_B)(\Delta(1)(b \otimes 1)), \\ m(S \otimes id_B)\Delta(b) &= (id_B \otimes \varepsilon)((1 \otimes b)\Delta(1)), \end{aligned}$$

where m denotes the multiplication.

The right hand sides of two last formulas are called *target* and *source counital maps* ε_t and ε_s, respectively. Their images are unital subalgebras of B called *target* and *source counital subalgebras* B_t and B_s, respectively. They commute elementwise, we have $S \circ \varepsilon_s = \varepsilon_t \circ S$ and $S(B_t) = B_s$. We say that B is *connected* if $B_t \cap Z(B) = \mathbb{C}$ (where $Z(B)$ is the center of B), coconnected if $B_t \cap B_s = \mathbb{C}$, and *biconnected* if both conditions are satisfied.

Finally, if B is a C^*-algebra and $\Delta(b^*) = \Delta(b)^*$, the collection $\mathfrak{G} = (B, \Delta, S, \varepsilon)$ is called a *weak Hopf C^*-algebra* (WHA). Then B_t and B_s are also C^*-subalgebras.

The dual vector space $\hat{B}$ has a natural structure of a WHA, namely $\hat{\mathfrak{G}} = (\hat{B}, \hat{\Delta}, \hat{S}, \hat{\varepsilon})$ given by dualizing the structure operations of B:

$$\begin{aligned} < \varphi\psi,\, b > &= < \varphi \otimes \psi,\, \Delta(b) >, \\ < \hat{\Delta}(\varphi),\, b \otimes c > &= < \varphi,\, bc >, \\ < \hat{S}(\varphi),\, b > &= < \varphi,\, S(b) >, \\ < \phi^*, b > &= \overline{< \varphi,\, S(b)^* >}, \end{aligned}$$

for all $b, c \in B$ and $\varphi, \psi \in \hat{B}$. The unit of $\hat{B}$ is ε and the counit is 1.

The antipode S is unique, invertible, and satisfies $(S \circ *)^2 = id_B$. Since it was mentioned in [Nikshych,2003], Remark 3.7 that problems regarding general WHAs can be translated to problems regarding those with the property $S^2|_{B_t} = id$ which are called *regular*, we will only consider such WHAs (see also [Vallin,2003]). In this case, there exists a canonical positive element H in the center of B_t such that S^2 is an inner automorphism implemented by $G = HS(H)^{-1}$, i.e., $S^2(b) = GbG^{-1}$ for all $b \in B$. The element G is called the canonical group-like element of B, it satisfies the relation $\Delta(G) = (G \otimes G)\Delta(1) = \Delta(1)(G \otimes G)$.

An element $\hat{l} \in \hat{B}$ is called a left integral (or a left invariant measure on B) if $(id_B \otimes \hat{l})\Delta = (\varepsilon_t \otimes \hat{l})\Delta$. Similarly one gives the definition of a right integral (or a right invariant measure on B). In any WHA there is a unique positive left and right integral h on B such that $(id_B \otimes h)\Delta(1) = 1$, called a *normalized Haar measure*.

We will dehote by H_h the GNS Hilbert space generated by B and h and by $\Lambda_h : B \longrightarrow H_h$ the corresponding GNS map.

2.2. Three categories associated with a WHA.

. **1. Unitary representations.** Let $\mathfrak{G} = (B, \Delta, S, \varepsilon)$ be a weak bialgebra. Objects of the category $Rep(\mathfrak{G})$ of representations of $\mathfrak{G}$ are finite rank left B-modules, simple objects are irreducible B-modules and morphisms are B-linear maps. The tensor product of two objects $H_1, H_2 \in Rep(\mathfrak{G})$ is the subspace $\Delta(1_B) \cdot (H_1 \otimes H_2)$ of the usual tensor product together with the action of B given by Δ. Tensor product of morphisms is the restriction of the usual tensor product of B-module morphisms. Any $H \in Rep(\mathfrak{G})$ is automatically a B_t-bimodule via $z \cdot v \cdot t := zS(t) \cdot v,\ \forall z, t \in B_t, v \in E$, and the above tensor product is in fact $\otimes_{B_t}$, moreover the B_t-bimodule structure on $H_1 \otimes_{B_t} H_2$ is given by $z \cdot \xi \cdot t = (z \otimes S(t)) \cdot \xi,\ \forall z, t \in B_t, \xi \in H_1 \otimes_{B_t} H_2$. This tensor product is associative, so the associativity isomorphisms are trivial. The unit object of $URep(\mathfrak{G})$ is B_t with the action of B given by $b \cdot z := \varepsilon_t(bz),\ \forall b \in B, z \in B_t$. When $\mathfrak{G}$ is a WHA, it is natural to consider the category $URep(\mathfrak{G})$ of its *unitary* representations formed by finite rank left B-modules whose underlying vector spaces are Hilbert spaces H with scalar product $< \cdot, \cdot >$ satisfying $< b \cdot v, w >=< v, b^* \cdot w >$, for all $v, w \in H,\ b \in B$. Then the above tensor product is also a Hilbert space because $\Delta(1_B)$ is an orthogonal projection. The scalar product on B_t is defined by $< z, t >= h(t^* z)$.

For any morphism $f : H_1 \longrightarrow H_2$, let $f^* : H_2 \longrightarrow H_1$ be the adjoint linear map: $< f(v), w >=< v, f^*(w) >$, $\forall v \in H_1, w \in H_2$. Clearly, f^* is B-linear, $f^{**} = f$, $(f \otimes_{B_t} g)^* = f^* \otimes_{B_t} g^*$, and $End(H)$ is a C^*-algebra, for any object H. So $URep(\mathfrak{G})$ is a finite C^*-multitensor category ($\mathbf{1}$ can be decomposable).

The conjugate object for any $H \in URep(\mathfrak{G})$ is the dual vector space $\hat{H}$ naturally identified $(v \mapsto \overline{v})$ with the conjugate Hilbert space $\overline{H}$ with the action of B defined by $b \cdot \overline{v} = \overline{G^{1/2}S(b)^*G^{-1/2} \cdot v}$, where G is the canonical group-like element of $\mathfrak{G}$. Then the rigidity morphisms defined by

$$R_H(1_B) = \Sigma_i(G^{1/2} \cdot \overline{e}_i \otimes_{B_t} \cdot e_i),\ \overline{R}_H(1_B) = \Sigma_i(e_i \otimes_{B_t} G^{-1/2} \cdot \overline{e}_i), \tag{1}$$

where $\{e_i\}_i$ is any orthogonal basis in H, satisfy all the needed properties - see [Bohm-Nill-Szlachanyi,2000], 3.6. Also, it is known that the B-module B_t is irreducible if and only if $B_t \cap Z(B) = \mathbb{C}1_B$, i.e., if $\mathfrak{G}$ is *connected*. So that, we have

2.3. Proposition. *$URep(\mathfrak{G})$ is a rigid finite C^*-multitensor category with trivial associativity constraints. It is C^*-tensor if and only if $\mathfrak{G}$ is connected.*

2.4. Remark. *If $\{z_\alpha\}_{\alpha\in\Gamma}$ is the set of minimal orthoprojectors of $B_t \cap Z(B)$, then the trivial representation denoted by $\mathbf{1}$ admits a decomposition $\mathbf{1} = \underset{\alpha\in\Gamma}{\oplus} \mathbf{1}_\alpha$ with $\mathbf{1}_\alpha$ irreducibles and according to [Etingof et all.,2015], Remark 4.3.4 we have:*

$$URep(\mathfrak{G}) = \underset{\alpha,\beta\in\Gamma}{\oplus} \mathcal{C}_{\alpha\beta}, \tag{2}$$

where $\mathcal{C}_{\alpha\beta}$ are called the component subcategories of $\mathcal{C}$. Moreover:

(1) Every irreducible of $\mathcal{C}$ belongs to one of $\mathcal{C}_{\alpha\beta}$.

(ii) The tensor product maps $\mathcal{C}_{\alpha\beta} \times \mathcal{C}_{\gamma\delta}$ to $\mathcal{C}_{\alpha\delta}$ and equals to 0 unless $\beta = \gamma$.

(ii) Every $\mathcal{C}_{\alpha\alpha}$ is a rigid finite C^*-tensor category with unit object $\mathbf{1}_\alpha$.

(iv) The conjugate of any $X \in \mathcal{C}_{\alpha\beta}$ belongs to $\mathcal{C}_{\beta\alpha}$.

. 2. Unitary comodules

2.5. Definition. *A right unitary $\mathfrak{G}$-comodule is a pair $(H, \mathfrak{a})$, where H is a Hilbert space with scalar product $< \cdot, \cdot >$, $\mathfrak{a} : H \longrightarrow H \otimes B$ is a bounded linear map between Hilbert spaces H and $H \otimes H_h = H \otimes \Lambda_h(B)$, and such that:*

(i) $(\mathfrak{a} \otimes id_B)\mathfrak{a} = (id_H \otimes \Delta)\mathfrak{a}$;

(ii) $(id_H \otimes \varepsilon)\mathfrak{a} = id_H$;

(iii) $< v^{(1)}, w > v^{(2)} =< v, w^{(1)} > S(w^{(2)})^*$, $\forall v, w \in H$, where we used the leg notation $\mathfrak{a}(v) = v^{(1)} \otimes v^{(1)}$.

A morphism of unitary $\mathfrak{G}$-comodules H_1 and H_2 is a linear map $T : H_1 \longrightarrow H_2$ such that $\mathfrak{a}_{H_2} \circ T = (T \otimes id_B)\mathfrak{a}_{H_1}$ (i.e., a B-colinear map).

Right unitary $\mathfrak{G}$-comodules with **finite dimensional** *underlying Hilbert spaces and their morphisms form a category which we denote by $UComod(\mathfrak{G})$.*

We say that two unitary $\mathfrak{G}$-comodules are equivalent (resp., unitarily equivalent) if the space of morphisms between them contains an invertible (resp., unitary) operator.

2.6. Example. *Let us equip a right coideal C^*-subalgebra $I \subset B$ with the scalar product $< v, w >:= h(w^*v)$. Then the strong invariance of h gives:*

$$< v^{(1)}, w > v^{(2)} = (h \otimes id_B)((w^* \otimes 1_B)\Delta(v)) =$$

$$= (h \otimes S^{-1})(\Delta(w^*)(v \otimes 1_B)) =< v, w^{(1)} > S(w^{(2)})^*.$$

If $(H, \mathfrak{a})$ is a right unitary $\mathfrak{G}$-comodule, then H is naturally a unitary left $\hat{\mathfrak{G}}$-module via

$$\hat{b} \cdot v := v^{(1)} < \hat{b}, v^{(2)} >, \quad \forall \hat{b} \in \hat{B}, \ v \in H. \tag{3}$$

Due to the canonical identifications $B_t \cong \hat{B}_s$ and $B_s \cong \hat{B}_t$ given by the maps $z \mapsto \hat{z} = \varepsilon(\cdot z)$ and $t \mapsto \hat{t} = \varepsilon(t\cdot)$, H is also a B_s-bimodule via $z \cdot v \cdot t = v^{(1)}\varepsilon(zv^{(2)}t)$, for all $z, t \in B_s, \ v \in V$. The maps $\alpha, \beta : B_s \longrightarrow B(H)$ defined by $\alpha(z)v := z \cdot v$ and $\beta(z)v := v \cdot z$, for all $z \in B_s, v \in H$ are a $*$-algebra homomorphism and antihomomorphism, respectively, with commuting images. Indeed, for instance, for all $v, w \in H, z \in B_s$, one has:

$$< \alpha(z)v, w >:=< v^{(1)}\varepsilon(zv^{(2)}), w >= \varepsilon(< v^{(1)}, w > zv^{(2)}) =$$

$$= \varepsilon(< v, w^{(1)} > zS(w^{(2)})^*) =< v, w^{(1)} > \overline{\varepsilon(S(w^{(2)})z^*)} =$$

$$=< v, w^{(1)}\varepsilon(S(z^*)w^{(2)}) >=< v, \alpha(z^*)w^{(1)}\varepsilon(w^{(2)}) >=< v, \alpha(z^*)w > .$$

So that, $\alpha(z)^* = \alpha(z^*)$, and similarly for the map β.

The correspondence (3) is bijective since one has the inverse formula: if $(b_i)_i$ is a basis for B and $(\hat{b}_i)$ is its dual basis in $\hat{B}$, then set:

$$\mathfrak{a}(v) = \sum_i (\hat{b}_i \cdot v) \otimes b_i \quad \forall v \in H. \tag{4}$$

Moreover, formulas (3) and (4) also lead to a bijection of morphisms, and we have two functors, $\mathcal{F}_1 : UComod(\mathfrak{G}) \longrightarrow URep(\hat{\mathfrak{G}})$ and $\mathcal{G}_1 : URep(\hat{\mathfrak{G}}) \longrightarrow UComod(\mathfrak{G})$, which are mutually inverse to each other. Hence, these categories are isomorphic and we can transport various additional structures from $URep(\hat{\mathfrak{G}})$ to $UComod(\mathfrak{G})$ and vice versa.

For instance, let us define tensor product of two unitary $\mathfrak{G}$-comodules, $(H_1, \mathfrak{a}_{H_1})$ and $(H_2, \mathfrak{a}_{H_2})$. As a vector space, it is

$$H_1 \otimes_{\hat{B}_t} H_2 := \hat{\Delta}(\hat{1})(H_1 \otimes H_2) = \hat{1}_{(1)} \cdot H_1 \otimes \hat{1}_{(2)} \cdot H_2$$

and can be identified with $H_1 \otimes_{B_s} H_2$ (see [Pfeiffer 2,2009], 2.2 or [Nill,1998], Chapter 4). The unitary comodule structure on $H_1 \otimes_{B_s} H_2$ is given by

$$v \otimes_{B_s} w \mapsto v^{(1)} \otimes_{B_s} w^{(1)} \otimes v^{(2)}w^{(2)}, \ \forall v \in H_1, w \in H_2.$$

Thus, $UComod(\mathfrak{G})$ is a multitensor category with trivial associativity isomorphisms whose unit object $(B_s, \Delta|_{B_s})$ is simple if and only if $\mathfrak{G}$ is coconnected. The conjugate object for $(H, \mathfrak{a}) \in UComod(\mathfrak{G})$ is $(\overline{H}, \tilde{\mathfrak{a}})$ with

$$\tilde{\mathfrak{a}}(\overline{v}) = \overline{v^{(1)}} \otimes [\hat{G}^{-1/2} \rightharpoonup (v^{(2)})^* \leftharpoonup \hat{G}^{1/2}], \tag{5}$$

where $\hat{b} \rightharpoonup b :=< \hat{b}, b_{(2)}b_{(1)} >,\ b \leftharpoonup \hat{b} :=< \hat{b}, b_{(1)} > b_{(2)}$ $(\forall b \in B, \hat{b} \in \hat{B})$ are the Sweedler arrows and $\hat{G}$ is the canonical group-like element of $\hat{\mathfrak{G}}$.

The rigidity morphisms are given by (1) with B_t replaced by B_s. For any morphism f, f^* is the conjugate linear map of the corresponding Hilbert spaces, the colinearity of f implies that f^* is colinear. So that, we have

2.7. Proposition. *$UComod(\mathfrak{G})$ is a strict rigid finite C^*-multitensor category isomorphic to $URep(\hat{\mathfrak{G}})$. It is C^*-tensor if and only if $\mathfrak{G}$ is coconnected (i.e., $B_t \cap B_s = \mathbb{C}1_B$).*

. **3. Unitary corepresentations.**

2.8. Definition. *A* **right unitary corepresentation** *U of $\mathfrak{G}$ on a Hilbert space H_U is a partial isometry $U \in B(H_U) \otimes B$ such that:*

(i) $(id \otimes \Delta)(U) = U_{12}U_{13}$.

(ii) $(id \otimes \varepsilon)(U) = id$.

A morphism between two right corepresentations U and V is a bounded linear map $T \in B(H_U, H_V)$ such that $(T \otimes 1_B)U = V(T \otimes 1_B)$. We denote by $UCorep(\mathfrak{G})$ the category whose objects are unitary corepresentations on **finite dimensional** *Hilbert spaces and above mentioned morphisms.*

Any H_U is a unitary right B-comodule via $v \mapsto U(v \otimes 1_B)$. Conversely, given $(H, \mathfrak{a}) \in UComod(\mathfrak{G})$, one can construct $V \in UCorep(\mathfrak{G})$ as follows:

$$V(x \otimes \Lambda_h y) := x^{(1)} \otimes \Lambda_h(x^{(2)}y)), \text{ for all } x \in H,\ y \in B.$$

Hence, the categories $UComod(\mathfrak{G})$ and $UCorep(\mathfrak{G})$ are isomorphic. The tensor product $U \otimes V$ equals $U_{13}V_{23}$ and acts on $H_U \otimes_{B_s} H_V$, the conjugate object $\overline{U}$ is the unitary corepresentation acting on $\overline{H}_U$ via $\overline{U}(\overline{x} \otimes \Lambda_h(y)) = \overline{x}^{[1)} \otimes \Lambda_h((\overline{x}^{[2)})^* y)$, where $\tilde{\mathfrak{a}}(\overline{x})$ is given by (5), the unit object $U_\varepsilon \in B(B_s) \otimes B$ is defined by $z \otimes b \mapsto \Delta(1_B)(1_B \otimes zb)$, $\forall z \in B_s, b \in B$, and the rigidity morphisms are given by (1) with B_t replaced by B_s. For any morphism T, T^* is the conjugate linear map of the corresponding Hilbert spaces. Thus, we have

2.9. Proposition. *$UCorep(\mathfrak{G})$ is a strict rigid finite C^*-multitensor category isomorphic to $UComod(\mathfrak{G})$ and to $URep(\hat{\mathfrak{G}})$. It is C^*-tensor if and only if $\mathfrak{G}$ is coconnected.*

2.10. Remark. *1. Using the leg notation $U = U^{(1)} \otimes U^{(2)}$, we define, for any $\eta, \zeta \in H_U$, the* matrix coefficient *$U_{\eta,\zeta} :=< U^{(1)}\zeta, \eta > U^{(2)} \in B$ of U. If $\{\zeta_i\}$ is an orthonormal basis in H_U, denote $U_{i,j} := U_{\zeta_i,\zeta_j}$. Then the formula*

$$U = \oplus_{i,j} m_{i,j} \otimes U_{i,j}, \text{ where } m_{i,j} \text{ are the matrix units of } B(H_U) \text{ in basis } \{\zeta_i\},$$

defines a corepresentation of $\mathfrak{G}$ if and only if for all $i, j = 1, ..., dim(H_U)$:

$$\Delta(U_{i,j}) = \Sigma_{k=1}^{dim(H_U)} U_{i,k} \otimes U_{k,j}, \quad \varepsilon(U_{i,j}) = \delta_{i,j}, \quad U_{i,j} = S(U_{j,i})^*. \tag{6}$$

2. We also have $(U \otimes V)_{i,j,k,l} = U_{i,j}V_{k,l}$ for all $i, j = 1, ., dim(H_U), k, l = 1, ., dim(H_V)$ and for all $U, V \in UCorep(\mathfrak{G})$.

3. For $U \in UCorep(\mathfrak{G})$, denote $B_U := Span\{U_{i,j}|i,j = 1,...,dim(H_U)\}$. Then (6) implies: $\Delta(B_U) \subset \Delta(1_B)(B_U \otimes B_U)$, $B_U = S(B_U)^*$, $B_{\overline{U}} = (B_U)^*$.

4. a) $B_{\oplus_{k=1}^p U_k} = span\{B_{U_1},...,B_{U_p}\}$ for any finite direct sum of unitary corepresentations. In particular, $B = \oplus_{x\in\Omega} B_{U^x}$.

b) Decomposition $U \otimes V = \oplus_z d_z U^z$ with multiplicities d_z implies $B_U B_V \subset \oplus_z B_{U^z}$, where z parameterizes the irreducibles of the above decomposition.

3. Reconstruction theorems.

1. Let $\mathcal{C}$ be a rigid finite C^*-multitensor category with unit object $\mathbf{1}$ and let $\mathcal{J}$ be a unitary tensor functor (see [Neshveyev-Tuset,2013], Definition 2.1.3) from $\mathcal{C}$ to the C^*-multitensor category $Corr_f(R)$ of finite dimensional Hilbert R-bimodules (R-correspondences), where $R = \mathcal{J}(\mathbf{1})$ is a finite dimensional C^*-algebra. A discussion of the category $Corr_f(R)$ can be found in [Vainerman-Vallin,2017], pp. 86,87.

Put $H_U := \mathcal{J}(U)$, for all $U \in \mathcal{C}$, in particular, $H^x := \mathcal{J}(x)$, for all $x \in \Omega = Irr(\mathcal{C})$. Let $J_{U,V} : H_U \underset{R}{\otimes} H_V \longrightarrow H_{U\otimes V}$ be the natural isomorphisms defining the tensor structure of $\mathcal{J}$ and choose an orthonormal basis $\{v_y^x | y \in \Omega_x := \{1,...,dim(H^x)\}\}$ in each H^x.

Let U^* be the conjugate of $U \in \mathcal{C}$, $R_U : \mathbf{1} \longrightarrow U^* \otimes U$ and $\overline{R}_U : \mathbf{1} \longrightarrow U \otimes U^*$ be the corresponding rigidity morphisms.

Then the conjugate of H_U is H_{U^*} with the rigidity morphisms $J^{-1}_{U^*,U} \circ \mathcal{J}(R_U)$ and $J^{-1}_{U,U^*} \circ \mathcal{J}(\overline{R}_U)$. The properties of the rigidity morphisms imply that the duality $< v, w >:= tr_R \circ \mathcal{J}(R_x^*)(J^{-1}_{x^*,x})^*(v \underset{R}{\otimes} w)$, where $v \in H^{x^*}, w \in H^x$ and tr_R is the trace of the left regular representation of R, is non degenerate. Hence, there exist isomorphisms $\Psi_x : \overline{H}^x \cong (H^x)^* \longrightarrow H^{x^*}$ and $\Phi_x : H^x \longrightarrow \overline{H}^{x^*} \cong (H^{x^*})^*$.

Next is a combined C^*-version of several reconstruction theorems scattered in various papers - see [Szlachanyi,2001], [Hayashi,1999], [Calaque-Etingof,2008], [Pfeiffer,2009], [Ostrik,2003].

3.1. Theorem. *A couple $(\mathcal{C}, \mathcal{J})$ defines on the vector spaces*

$$B = \bigoplus_{x\in\Omega} H^x \otimes \overline{H}^x \quad \text{and} \quad \hat{B} = \bigoplus_{x\in\Omega} B(H^x) \tag{7}$$

two WHA structures, $\mathfrak{G}$ and $\hat{\mathfrak{G}}$, respectively, dual to each other with respect to the bracket

$$< A, w \otimes \overline{v} >=< Av, w >_x \quad \text{where} \quad x \in \Omega, A \in B(H^x), v, w \in H^x,$$

such that $\mathcal{C} \cong UComod(\mathfrak{G}) \cong UCorep(\mathfrak{G}) \cong URep(\hat{\mathfrak{G}})$.

A sketch of the proof. Clearly, $\hat{B}$ is a C^*-algebra with usual matrix multiplication, conjugation and unit. Since all of H^x are Hilbert R-bimodules, there are homomorphisms $t : R \longrightarrow \hat{B}$ and $s : R^{op} \longrightarrow \hat{B}$ defined by $t(r)v = r \cdot v$ and $s(r)v = v \cdot r$, respectively (here $r \in R, v \in \underset{x\in\Omega}{\oplus} H^x$).

The coalgebra structure in B dual to the algebra structure in $\hat{B}$, is:

$$\Delta(w \otimes \overline{v}) = \bigoplus_{y \in \Omega_x} (w \otimes \overline{v}_y^x)_x \otimes (v_y^x \otimes \overline{v})_x, \tag{8}$$

$$\varepsilon(w \otimes \overline{v}) =< w, v >_x, \quad \text{where} \quad v, w \in H^x. \tag{9}$$

Then, as in [Calaque-Etingof,2008], 2.3.2, define the coproduct $\hat{\Delta} : \hat{B} \longrightarrow \hat{B} \otimes \hat{B}$:

$$\hat{\Delta}(b) := \eta \circ (J^{-1} \cdot b \cdot J) \quad (b \in \hat{B}, x, y \in \Omega), \tag{10}$$

where $J := \underset{x,y \in \Omega}{\oplus} J_{x,y}$, $\eta : B(H^x) \underset{R}{\otimes} B(H^y) \longrightarrow B(H^x) \otimes B(H^y)$ is the canonical map defined by $\eta(a_x \underset{R}{\otimes} c_y) = \sum_{i \in I} (s(e_i)a_x \otimes t(e^i)c_y)$ (here $a, c \in \hat{B}, \{e_i\}$ and $\{e^i\}$ are dual bases of R with respect to the duality $< a, b >= tr_R(L(a)L(b))$, where $a, b \in R, L(a), L(b)$ are the corresponding left multiplication operators.

The multiplication in B dual to $\hat{\Delta}$, is as follows:

$$(w \otimes \overline{v})_x \cdot (g \otimes \overline{h})_z = (J_{x,z}(w \underset{R}{\otimes} g) \otimes \overline{J_{x,z}(v \underset{R}{\otimes} h)})_{x \otimes z} \in H^{(x \otimes z)} \otimes \overline{H^{(x \otimes z)}}, \tag{11}$$

where $v, w \in H^x, g, h \in H^z$, for all $x, z \in \Omega$. For any $b = \underset{x \in \Omega}{\oplus} b_x$, the component $b_{\mathbf{1}} \in B(H^{\mathbf{1}}) \cong B(R)$, so $b_{\mathbf{1}}(1_R)$ can be viewed as an element of R. Then the triple $(\hat{B}, \hat{\Delta}, \hat{\varepsilon})$ with $\hat{\varepsilon}(b) = tr_R(L(b_{\mathbf{1}}(1_R)))$ is a weak bialgebra.

The antipode in $\hat{B}$ is defined by $\hat{S}(b)_x := (\Psi_{x^*} \circ i_{x^*})(b_{x^*})^*(i_{x^*}^{-1} \circ \Phi_x)$, where $b \in \hat{B}, x \in \Omega$, $i_x : H^x \longrightarrow \overline{H}^x$ is a canonical antilinear isomorphism. Dually:

$$S(w \otimes \overline{v}) = v^\natural \otimes \overline{w}^\flat, (w \otimes \overline{v})^* = w^\natural \otimes \overline{v}^\flat \ (\forall v, w \in H^x), \tag{12}$$

where $w^\natural = \Psi_x(\overline{w}), \overline{v}^\flat = \Phi_x(v)$. Any H^x is a unitary right B-comodule via

$$\mathfrak{a}_x(v) = \sum_{y \in \Omega_x} v \otimes v_y^x \otimes \overline{v_y^x}, \quad \text{where} \quad v \in H_x.$$

2. Let $(\mathcal{C}, \mathcal{J})$ be as in Theorem 3.1, let $\mathcal{C}'$ be a rigid finite C^*-multitensor category and let $\mathcal{P} : \mathcal{C}' \longrightarrow \mathcal{C}$ be a unitary tensor functor. Then Theorem 3.1 shows the existence of two WHAs, $\mathfrak{G} = (B, \Delta, \varepsilon, S)$ and $\mathfrak{G}' = (B', \Delta', \varepsilon', S')$, generated by the couples $(\mathcal{C}, \mathcal{J})$ and $(\mathcal{C}', \mathcal{J}' = \mathcal{J} \circ \mathcal{P})$, respectively, such that $UCorep(\mathfrak{G}) \cong UComod(\mathfrak{G}) \cong \mathcal{C}$ and $UCorep(\mathfrak{G}') \cong UComod(\mathfrak{G}') \cong \mathcal{C}'$. The following theorem reveals the structure of the functor $\mathcal{P}$.

3.2. Theorem. *In the above conditions, there is a WHA morphism $p : \mathfrak{G}' \longrightarrow \mathfrak{G}$ such that $\mathcal{P}$ viewed as a functor $UComod(\mathfrak{G}') \longrightarrow UComod(\mathfrak{G})$ is of the form $(V, \rho'_V) \mapsto (V, (id \otimes p)\rho'_V)$, where $(V, \rho'_V) \in UComod(\mathfrak{G}')$. $\mathcal{P}$ is faithful (resp., full) if and only if p is is injective (resp., surjective).*

Let us comment on the proof. For any $(V, \rho'_V) \in UComod(\mathfrak{G}')$ the condition $\mathcal{J}' = \mathcal{J} \circ \mathcal{P})$ implies that $\mathcal{P}(V, \rho'_V) = (V, \rho_V) \in UComod(\mathfrak{G})$, where $\rho_V : V \longrightarrow V \otimes B$ is a right coaction. In particular, $(B', \Delta') \in UComod(\mathfrak{G}')$, so $\mathcal{P}(B', \Delta') = (B', \rho_{B'}) \in UComod(\mathfrak{G})$, where $\rho_{B'} : B' \longrightarrow B' \otimes B$ is a right coaction. Then the composition $p := (\varepsilon' \otimes id_B) \circ \rho_{B'} : B' \longrightarrow B$ is a linear map. Theorem 3.5 of [Wakui,2020] proves that p is a weak bialgebra morphism and that $\mathcal{P}(V, \rho'_V) = (V, (id \otimes p)\rho'_V)$ for any $(V, \rho'_V) \in UComod(\mathfrak{G}')$. Corollary 3.6 of [Wakui,2020] shows that $\mathcal{P}$ is an equivalence if and only if p is an isomorphism.

In our context the comodules (B', Δ') and $\mathcal{P}(B', \Delta') = (B', (id_{B'} \otimes p)\Delta')$ are unitary which gives for all $b, c \in B'$:

$$< b_{(1)}, c > (b_{(2)}) =< b, c_{(1)} > S'(c_{(2)})^*,$$

$$< b_{(1)}, c > p(b_{(2)}) ==< b, c_{(1)} > p(S(c_{(2)})^*).$$

For $b = 1_{B'}$ this implies $S(p(c))^* = p(S'(c)^*)$, for all $c \in B'$.

Then, the conjugate object for (B', Δ') in $UComod(\mathfrak{G}')$ is $(\overline{B}', \tilde{\Delta}')$, where

$$\tilde{\Delta}'(\overline{b}) = \overline{b_{(1)}} \otimes [\hat{G}^{-1/2} \rightharpoonup (b_{(2)})^* \leftharpoonup \hat{G}^{1/2}],$$

and $\hat{G}$ is the canonical group-like element of the dual WHA $\hat{\mathfrak{G}}'$. Let us note that G and $\hat{G}$ belong to $B_t B_s$, so p just sends them respectively to the canonical group-like elements of $\mathfrak{G}$ and its dual. The conjugate object for $\mathcal{P}(B', \Delta) = (B', (id_{B'} \otimes p)\Delta')$ in $UComod(\mathfrak{G})$ is described by a similar formula. As $\mathcal{P}$ respects the rigidity of the categories in question, we have:

$$\overline{b_{(1)}} \otimes p[\hat{G}^{-1/2} \rightharpoonup (b_{(2)})^* \leftharpoonup \hat{G}^{1/2}] = \overline{b_{(1)}} \otimes [(p(\hat{G})^{-1/2}) \rightharpoonup (p(b_{(2)}))^* \leftharpoonup p(\hat{G})^{1/2}]$$

which gives by the invertibility of $\hat{G}$: $\overline{b_{(1)}} \otimes p(b^*_{(2)}) = \overline{b_{(1)}} \otimes (p(b_{(2)}))^*$. Applying ε' to the first leg, we get $p(b)^* = p(b^*)$, then also $S(p(b)) = p(S'(b))(\forall b \in B')$.

For the WHA $p(\mathfrak{G}')$ we have $UComod(p(\mathfrak{G}')) = \mathcal{P}(UComod(\mathfrak{G}'))$. The functor $\mathcal{P}$ splits into the composition of the full functor $UComod(\mathfrak{G}') \longrightarrow \mathcal{P}(UComod(\mathfrak{G}'))$ and the inclusion $\mathcal{P}(UComod(\mathfrak{G}') \longrightarrow UComod(\mathfrak{G})$. Respectively, p splits into the composition of the surjective WHA homomorphism $\mathfrak{G}' \longrightarrow p(\mathfrak{G}')$ and the inclusion $p(\mathfrak{G}') \longrightarrow \mathfrak{G}$. Then $\mathcal{P}$ is full if and only if $UComod(p(\mathfrak{G}')) = UComod(\mathfrak{G})$ or if and only if $p(\mathfrak{G}') = \mathfrak{G}$.

Also, $\mathcal{P}$ is faithful if and only if the first functor in the above decomposition is an equivalence which happens if and only if $p : \mathfrak{G}' \longrightarrow p(\mathfrak{G}')$ is an isomorphism of WHAs or if only if the map $p : \mathfrak{G}' \longrightarrow \mathfrak{G}$ is injective. ∎

4. Coactions.

4.1. Definition. *A right coaction of a WHA $\mathfrak{G}$ on a unital $*$-algebra A, is any $*$-homomorphism $\mathfrak{a} : A \longrightarrow A \otimes B$ such that:*

1) $(\mathfrak{a} \otimes i)\mathfrak{a} = (id_A \otimes \Delta)\mathfrak{a}$.

2) $(id_A \otimes \varepsilon)\mathfrak{a} = id_A$.
3) $\mathfrak{a}(1_A) \in A \otimes B_t$.
One also says that $(A, \mathfrak{a})$ *is a* $\mathfrak{G}$-$*$-*algebra.*

If A is a C^*-algebra, then $\mathfrak{a}$ is automatically continuous, even an isometry.

There are $*$-homomorphism $\alpha : B_s \longrightarrow A$ and $*$-antihomomorphism $\beta : B_s \longrightarrow A$ with commuting images defined by $\alpha(x)\beta(y) := (id_A \otimes \varepsilon)[(1_A \otimes x)\mathfrak{a}(1_A)(1_A \otimes y)]$, for all $x, y \in B_s$. We also have $\mathfrak{a}(1_A) = (\alpha \otimes id_B)\Delta(1_B)$,

$$\mathfrak{a}(\alpha(x)a\beta(y)) = (1_A \otimes x)\mathfrak{a}(a)(1_A \otimes y), \tag{13}$$

and

$$(\alpha(x) \otimes 1_B)\mathfrak{a}(a)(\beta(y) \otimes 1_B) = (1_A \otimes S(x))\mathfrak{a}(a)(1_A \otimes S(y)). \tag{14}$$

The set $A^{\mathfrak{a}} = \{a \in A | \mathfrak{a}(a) = \mathfrak{a}(1_A)(a \otimes 1_B)\}$ is a unital $*$-subalgebra of A (it is a unital C^*-subalgebra of A when A is a C^*-algebra) commuting pointwise with $\alpha(B_s)$. A coaction $\mathfrak{a}$ is called *ergodic* if $A^{\mathfrak{a}} = \mathbb{C}1_A$.

4.2. Definition. *A* $\mathfrak{G} - C^*$*-algebra* $(A, \mathfrak{a})$ *is said to be indecomposable if it cannot be presented as a direct sum of two* $\mathfrak{G} - C^*$*-algebras.*

It is easy to see that $(A, \mathfrak{a})$ is indecomposable if and only if $Z(A) \cap A^{\mathfrak{a}} = \mathbb{C}1_A$. Clearly, any ergodic $\mathfrak{G} - C^*$-algebra is indecomposable.

For any $(U, H_U) \in UCorep(\mathfrak{G})$, we define the *spectral subspace* of A corresponding to (U, H_U) by

$$A_U := \{a \in A | \mathfrak{a}(a) \in \mathfrak{a}(1_A)(A \otimes B_U)\}.$$

Let us recall the properties of the spectral subspaces:

(i) All A_U are closed.

(ii) $A = \oplus_{x \in \Omega} A_{U^x}$.

(iii) $A_{U^x} A_{U^y} \subset \oplus_z A_{U^z}$, where z runs over the set of all irreducible direct summands of $U^x \otimes U^y$.

(iv) $\mathfrak{a}(A_U) \subset \mathfrak{a}(1_A)(A_U \otimes B_U)$ and $A_{\overline{U}} = (A_U)^*$.

(v) A_ε is a unital C^*-algebra.

4.2.1. Let us note that the usage of C^*-multitensor categories allows to get without much effort the following slight generalization of the main result of [Vainerman-Vallin,2017]:

4.3. Theorem. *Given a WHA* $\mathfrak{G}$*, the following categories are equivalent:*

(i) The category of unital $\mathfrak{G}$-C^**-algebras with unital* $\mathfrak{G}$*-equivariant* $*$*-homomorphisms as morphisms.*

(ii) The category of pairs $(\mathcal{M}, M)$*, where* $\mathcal{M}$ *is a left module* C^**-category with trivial module associativities over* $UCorep(\mathfrak{G})$ *and* M *is a generator in* $\mathcal{M}$*, with equivalence classes of unitary module functors respecting the prescribed generators as morphisms.*

In particular, given a unital $\mathfrak{G}$-C^*-algebra A, one constructs the C^*-category $\mathcal{M} = \mathcal{D}_A$ of finitely generated right Hilbert A-modules which are equivariant, that is, equipped

with a compatible right coaction [Baaj-Skandalis,1989]. Any its object is automatically a (B_s, A)-bimodule, and the bifunctor $U \boxtimes X := H_U \otimes_{B_s} X \in \mathcal{D}_A$, for all $U \in UCorep(\mathfrak{G})$ and $X \in \mathcal{D}_A$, turns $\mathcal{D}_A$ into a left module C^*-category over $UCorep(\mathfrak{G})$ with generator A and trivial associativities.

Vice versa, if a pair $(\mathcal{M}, M)$ is given, the construction of a $\mathfrak{G}$-C^*-algebra $(A, \mathfrak{a})$ contains the following steps. First, denote by R the unital C^*-algebra $End_{\mathcal{M}}(M)$ and consider the functor $F : \mathcal{C} \longrightarrow Corr(R)$ defined on the objects by $F(U) = Hom_{\mathcal{M}}(M, U \boxtimes M)\ \forall U \in \mathcal{C}$. Here $X = F(U)$ is a right R-module via the composition of morphisms, a left R-module via $rX = (id \otimes r)X$, the R-valued inner product is given by $< X, Y >= X^*Y$, the action of F on morphisms is defined by $F(T)X = (T \otimes id)X$. The weak tensor structure of F (in the sense of [Neshveyev,2014]) is given by $J_{X,Y}(X \otimes Y) = (id \otimes Y)X$, for all $X \in F(U), Y \in F(V), U, V \in UCorep(\mathfrak{G})$.

Then consider two vector spaces:

$$A = \bigoplus_{x \in \Omega} A_{U^x} := \bigoplus_{x \in \Omega} (F(U^x) \otimes \overline{H^x}) \tag{15}$$

and

$$\tilde{A} = \bigoplus_{U \in \|UCorep(\mathfrak{G})\|} A_U := \bigoplus_{U \in \|UCorep(\mathfrak{G})\|} (F(U) \otimes \overline{H_U}), \tag{16}$$

where $F(U) = \bigoplus_i F(U_i)$ corresponds to the decomposition $U = \bigoplus U_i$ into irreducibles, and $\|UCorep(\mathfrak{G})\|$ is an exhaustive set of representatives of the equivalence classes of objects in $UCorep(\mathfrak{G})$ (these classes constitute a countable set). $\tilde{A}$ is a unital associative algebra with the product

$$(X \otimes \overline{\xi})(Y \otimes \overline{\eta}) = (id \otimes Y)X \otimes (\overline{\xi} \otimes_{B_s} \overline{\eta}),\ \forall (X \otimes \overline{\xi}) \in A_U, (Y \otimes \overline{\eta}) \in A_V,$$

and the unit

$$1_{\tilde{A}} = id_M \otimes \overline{1_B}.$$

Note that $(id \otimes Y)X = J_{X,Y}(X \otimes Y) \in F(U \otimes V)$. Then, for any $U \in UCorep(G)$, choose isometries $w_i : H_i \longrightarrow H_U$ defining the decomposition of U into irreducibles, and construct the projection $p_A : \tilde{A} \longrightarrow A$ by

$$p_A(X \otimes \overline{\xi}) = \sum_i (F(w_i^*)X \otimes \overline{w_i^* \xi}),\ \forall (X \otimes \overline{\xi}) \in A_U, \tag{17}$$

which does not depend on the choice of w_i. Then A is a unital $*$-algebra with the product $x \cdot y := p(xy)$, for all $x, y \in A$ and the involution $x^* := p(x^\bullet)$, where $(X \otimes \overline{\xi})^\bullet := (id \otimes X^*)F(\overline{R}_U) \otimes \overline{\hat{G}^{1/2}\xi}$, for all $\xi \in H_U, X \in F(U), U \in UCorep(\mathfrak{G})$. Here $\overline{R}_U$ is the rigidity morphism from (1). Finally, the map

$$\mathfrak{a}(X \otimes \overline{\xi}_i) = \sum_j (X \otimes \overline{\xi}_j) \otimes U^x_{j,i}, \tag{18}$$

where $\{\xi_i\}$ is an orthogonal basis in H^x and $(U^x_{i,j})$ are the matrix elements of U^x in this basis, is a right coaction of $\mathfrak{G}$ on A. Moreover, A admits a unique C^*-completion $\overline{A}$ such that $\mathfrak{a}$ extends to a continuous coaction of $\mathfrak{G}$ on it.

4.4. Remark. *We say that a $UCorep(\mathfrak{G})$-module category is indecomposable if it is not equivalent to a direct sum of two nontrivial $UCorep(\mathfrak{G})$-module subcategories. Theorem 4.3 implies that a $\mathfrak{G}-C^*$-algebra $(A,\mathfrak{a})$ is indecomposable if and only if the $UCorep(\mathfrak{G})$-module category $\mathcal{M}$ is indecomposable.*

4.5. Remark. *Equivalence between $\mathcal{M}$ and $\mathcal{D}_A$ maps all morphism $f : Hom_{\mathcal{M}}(U\otimes_{B_s} M, V\otimes_{B_s} M)$ to a morphism $\tilde{f} : H_U\otimes_{B_s} A\longrightarrow H_V\otimes_{B_s} A$ $(U,V\in UCorep(\mathfrak{G}))$, $\tilde{f}$ is an A-linear map on the right intertwining $\delta_{H_U\otimes_{B_s}A} = U_{13}(id\otimes_{B_s}\alpha)$ and $\delta_{H_V\otimes_{B_s}A} = V_{13}(id\otimes_{B_s}\alpha)$, so it can be written as*

$$\tilde{f} = \sum_i s_i\otimes_{B_s} a_i \in B(H_U,H_V)\otimes_{B_s} A$$

acting by $\tilde{f}(\xi\otimes_{B_s} a) = \Sigma s_i(\xi)\otimes_{B_s} a_i a$, where $\xi\in H_U, a\in A$, and such that $V_{13}(id\otimes\alpha)\tilde{f} = (\tilde{f}\otimes id)U_{13}(id\otimes\alpha)$.

5. Yetter-Drinfel'd C^*-algebras over WHA

5.1. Basic definitions and results. Let $\mathfrak{G}$ be a WHA, $\hat{\mathfrak{G}}$ be its dual and $(A,\mathfrak{a})$ be a right unital $\mathfrak{G}$-C^*-algebra which is also a left unital $\hat{\mathfrak{G}}$-C^*-algebra via a left coaction $\mathfrak{b} : A\longrightarrow \hat{B}\otimes A$. The coaction $\mathfrak{b}$ defines a right B-module algebra structure $\triangleleft : A\otimes B\longrightarrow A$ by

$$a\triangleleft b := (b\otimes id_A)\mathfrak{b}(a), \quad \text{for all} \quad a\in A,\ b\in B.$$

One can check that the following relations hold:

$$a\triangleleft 1_B = a, \quad (ac)\triangleleft b = (a\triangleleft b_{(1)})(c\triangleleft b_{(2)})\ \forall a,c\in A,\ b\in B,$$

$$a^*\triangleleft b = (a\triangleleft S(b)^*)^* \quad \text{and} \quad 1_A\triangleleft b = 1_A\triangleleft \varepsilon_s(b). \tag{19}$$

Below we will use the leg notations for coactions and write 1 instead of 1_B.

5.2. Lemma. *The following two conditions are equivalent:*

(i) the identity

$$\mathfrak{a}(a\triangleleft b) = (a^{(1)}\triangleleft b_{(2)})\otimes S(b_{(1)})a^{(2)}b_{(3)}, \tag{20}$$

holds for all $a\in A,\ b\in B$.

(ii) the identity

$$(id_{\hat{B}}\otimes\mathfrak{a})\mathfrak{b}(a) = W_{13}^*(\mathfrak{b}\otimes id_B)\mathfrak{a}(a)W_{13}, \tag{21}$$

holds for all $a\in A$, where the operator $W\in\mathcal{L}(\lambda_{h\otimes h}(B\otimes B))$ is defined by

$$W(\lambda_{h\otimes h}(b\otimes c)) := \lambda_{h\otimes h}(\Delta(c)(b\otimes 1)),$$

for all $b,c\in B$ (it is the adjoint of the regular multiplicative partial isometrty I of B - see [Vallin,2001]), and W_{13} is the usual leg notation.

Proof. As $A = \oplus_{x\in\Omega} A_{U^x}$, where A_{U^x} is the spectral subspace of A corresponding to an irreducible corepresentation U^x of A, it suffices to prove the statement for $a \in A_{U^x}$ only. The matrix units $\{m^x_{i,j}\}$ of $B(H^x)$ with respect to some orthogonal basis $\{e_i\}$ in H^x and the corresponding matrix coefficients $U^x_{i,j}$ of U^x with all possible i, j, x form dual bases in $\hat{B}$ and B, respectively, so that $\mathfrak{b}$ can be restored from $\triangleright$ by

$$\mathfrak{b}(a) = \Sigma_{i,j}\ m^x_{i,j} \otimes (a \triangleleft U^x_{i,j}), \quad \text{for all } a \in A_{U^x}. \tag{22}$$

Since $W = \oplus_{x\in\Omega} dim(H^x)U^x$ implements Δ and $\Delta(U^x_{i,j}) = \Sigma_k U^x_{i,k} \otimes U^x_{k,j}$, the right hand side of (21) can be written for any $a \in A_{U^x}$ as

$$(U^x_{13})^*(\mathfrak{b} \otimes id_B))\mathfrak{a}(a)U^x_{13} =$$
$$= \Sigma_{i,j,p,q}(m^x_{i,j} \otimes 1_A \otimes (U^x_{j,i})^*)(\mathfrak{b}(a^{(1)} \otimes a^{(2)}))(m^x_{q,p} \otimes 1_A \otimes U^x_{q,p}) =$$
$$= \Sigma_{i,j,p,q,r,s}(m^x_{i,j}m^x_{r,s}m^x_{q,p} \otimes (a^{(1)} \triangleleft U^x_{r,s}) \otimes (U^x_{j,i})^* a^{(2)} U^x_{q,p}) =$$
$$= \Sigma_{i,j,p,q}(m^x_{i,p} \otimes (a^{(1)} \triangleleft U^x_{j,q}) \otimes (U^x_{j,i})^* a^{(2)} U^x_{q,p}).$$

On the other hand, if (20) holds, the left hand side of (21) can be written as

$$(id_{\hat{B}} \otimes \mathfrak{a})\mathfrak{b}(a) = \Sigma_{i,p}(m^x_{i,p} \otimes \mathfrak{a}(a \triangleleft U^x_{i,p})) =$$
$$= \Sigma_{i,p}(m^x_{i,p} \otimes (a^{(1)} \triangleleft (U^x_{i,p})_{(2)}) \otimes S((U^x_{i,p})_{(1)})a^{(2)}(U^x_{i,p})_{(3)}) =$$
$$\Sigma_{i,p,j,q}(m^x_{i,p} \otimes (a^{(1)} \triangleleft U^x_{j,q}) \otimes S(U^x_{i,j})a^{(2)}U^x_{q,p}) =$$
$$\Sigma_{i,p,j,q}(m^x_{i,p} \otimes (a^{(1)} \triangleleft U^x_{j,q}) \otimes (U^x_{j,i})^* a^{(2)} U^x_{q,p}).$$

So (20) implies (21). Conversely, writing in (21) $\mathfrak{b}$ as above, we get (20) for any $b = U^x_{i,j}$ $(x \in \Omega, i, j = 1, ..., dimH^x)$, $a \in A_{U^x}$ which gives the result. ∎

5.3. Definition. *(cf. [Neshveyev-Yamashita,2014]) A is a right-right Yetter-Drinfel'd (YD) $\mathfrak{G}$-C^*-algebra if one of the above equivalent conditions is satisfied.*

We say that a Yetter-Drinfel'd $\hat{\mathfrak{G}}$-C^*-algebra A is braided-commutative if

$$ab = b^{(1)}(a \triangleleft b^{(2)}), \quad \textit{for all} \quad a, b \in A. \tag{23}$$

In particular, if $b \in A^{\mathfrak{a}}$, then $b^{(1)} \otimes b^{(2)} = 1^{(1)}b \otimes 1^{(2)}$, and since b commutes with $1^{(1)} \in \alpha(B_s)$, the right hand side of (23) can be written as $b1^{(1)}(a \triangleleft 1^{(2)})$. But (23) implies that $1^{(1)}(a \triangleleft 1^{(2)}) = a$, so $ab = ba$. Hence, $A^{\mathfrak{a}} \in Z(A)$.

Given a WHA $\mathfrak{G}$, let us construct a new WHA $D(\mathfrak{G})$ called the **Drinfel'd double** of $\mathfrak{G}$ as follows. The C^*-algebra of $D(\mathfrak{G})$ is $B \otimes \hat{B}$, where $\hat{\mathfrak{G}} = (\hat{B}, \hat{\Delta}, \hat{S}, \hat{\varepsilon})$ is the dual of $\mathfrak{G}$. The coproduct Δ_D on $B \otimes \hat{B}$ is defined by

$$\Delta_D = Ad(1 \otimes \sigma \circ W \otimes 1_{\hat{B}})(\Delta \otimes \hat{\Delta}),$$

where $W \in H_h \otimes H_{\hat{h}}$ is the multiplicative partial isometry canonically associated with $\mathfrak{G}$ - see [Vallin 1,2003], and σ is the flip. The antipode S_D and the counit ε_D on $B \otimes \hat{B}$ are defined, respectively, by

$$S_D = Ad(W^*)(S \otimes \hat{S}) \quad \text{and} \quad \varepsilon_D = m(\varepsilon \otimes \hat{\varepsilon}).$$

5.4. Lemma. *The collection $D(\mathfrak{G}) = (B \otimes \hat{B}, \Delta_D, S_D, \varepsilon_D)$ is a WHA.*

Proof. It suffices to note that the WHA called there the Drinfel'd double, given in [Nikshych-Turaev-Vainerman,2003], is dual to $D(\mathfrak{G})$. ∎

Theorem. *A* **YD $\mathfrak{G}$-C^*-algebra is the same as a $D(\mathfrak{G})$-C^*-algebra.**

The proof is similar to the one of [Nenciu,2002], Theorem 3.4.

5.5. Categorical duality for Yetter-Drinfel'd algebras over WHAs. Let us give the proof of Theorem 1.1. The condition that $\mathcal{C}$ is generated by $\mathcal{E}(\mathcal{C})$ means that any object of $\mathcal{C}$ is isomorphic to a subobject of $\mathcal{E}(U)$ for some $U \in UCorep(\mathfrak{G})$. Assume without loss of generality that $\mathcal{C}$ is closed with respect to subobjects, but its unit object is not necessarily simple.

Let us precise the equivalence relation on the set of pairs $(\mathcal{F}, \eta)$ in (ii). Given such a pair, we can consider, for all $U, V \in UCorep(\mathfrak{G})$, linear maps

$$\mathcal{C}(\mathcal{E}(U), \mathcal{E}(V)) \longrightarrow \mathcal{C}'(\mathcal{E}'(U), \mathcal{E}'(V)) : \quad T \mapsto \eta_V \mathcal{F}(T) \eta_U^{-1}.$$

We say that two pairs, $(\mathcal{F}, \eta)$ and $(\mathcal{F}', \eta')$, are equivalent if the above maps are equal for all $U, V \in UCorep(\mathfrak{G})$.

The proof of Theorem 1.1 will be done in several steps.

a) From YD $\mathfrak{G}$-C^*-algebras to C^*-multitensor categories.

Given a braided commutative YD $\mathfrak{G}$-C^*-algebra A, let us show that the C^*-category $\mathcal{D}_A$ is in fact a C^*-multitensor category. We start with

5.6. Remark. *Recall the following relations:*

1) $\delta_{H_U}(\zeta) := \zeta^{(1)} \otimes \zeta^{(2)} := U(\zeta \otimes 1)$, where $\zeta \in H_U, U \in UCorep(\mathfrak{G})$.

2) $\delta_{H_{U\otimes V}}(\zeta \otimes_{B_s} \eta) := U_{13}V_{23}(\zeta \otimes_{B_s} \eta \otimes 1)$ or $(\zeta \otimes_{B_s} \eta)^{(1)} = \zeta^{(1)} \otimes_{B_s} \eta^{(1)}$, $(\zeta \otimes_{B_s} \eta)^{(2)} = \zeta^{(2)}\eta^{(2)}$, where $\zeta \in H_U, \eta \in H_V, U, V \in UCorep(\mathfrak{G})$.

3) $\delta_{H_U \otimes_{B_s} A}(\zeta \otimes_{B_s} a) := U_{13}(\zeta \otimes_{B_s} \mathfrak{a}(a))$ or $(\zeta \otimes_{B_s} a)^{(1)} = \zeta^{(1)} \otimes_{B_s} a^{(1)}$, $(\zeta \otimes_{B_s} a)^{(2)} = \zeta^{(2)}a^{(2)}$. Then $\delta_{H_{U\otimes V} \otimes_{B_s} A}(\zeta \otimes_{B_s} \eta \otimes_{B_s} a) = \zeta^{(1)} \otimes_{B_s} \eta^{(1)} \otimes_{B_s} a^{(1)} \otimes \zeta^{(2)}\eta^{(2)}a^{(2)}$, where $\zeta \in H_U, \eta \in H_V$ $U, V \in UCorep(\mathfrak{G}), a \in A$.

4) It follows from the equality $\mathfrak{a}(1_A) = (\alpha \otimes id)\Delta(1)$ that $(id_A \otimes \varepsilon_t)\mathfrak{a}(b) = \mathfrak{a}(1_A)(b \otimes 1)$ (see [Vainerman-Vallin,2017]) . One can deduce from here, using 3) and the relations $(id \otimes \varepsilon_t)U = (id \otimes \varepsilon_s)U = 1$ that $(id \otimes \varepsilon_t)\delta_{H_U \otimes_A A}(\zeta \otimes_A a) = \zeta \otimes_A a^{(1)} \otimes \varepsilon_t(a^{(2)})$.

5.7. Lemma. *For any $X \in \mathcal{D}_A$, there exists a unique unital $*$-homomorphism $\pi_X : A \longrightarrow \mathcal{L}_A(X)$ such that $\pi_X(a)(\zeta) = \zeta^{(1)}(a \triangleleft \zeta^{(2)})$ and $\delta_X(\pi_X(a)\zeta) = (\pi_X \otimes id)\mathfrak{a}(a)\delta_X(\zeta)$, for all $a \in A$ and $\zeta \in X$.*

Proof. It suffices to consider $X = H^x \otimes_{B_s} A$ because A is a generator of $\mathcal{D}_A$. If $\{v_i^x\}$ is an orthonormal basis in H^x, Remark 5.6, 2) is equivalent to

$$\delta_X(v_i^x \otimes_{B_s} b) = \sum_j [v_j^x \otimes_{B_s} b^{(1)} \otimes U_{j,i}^x b^{(2)}], \tag{24}$$

here $U^x_{i,j}$ are the matrix coefficients of U^x. The braided commutativity gives:

$$(v^x_i \otimes_{B_s} b)^{(1)}(a \triangleleft (v^x_i \otimes_{B_s} b)^{(2)}) = \sum_j (v^x_j \otimes_{B_s} b^{(1)})(a \triangleleft (U^x_{j,i} b^{(2)})) =$$

$$= \sum_j (v^x_j \otimes_{B_s} b^{(1)}(a \triangleleft U^x_{j,i}) \triangleleft b^{(2)}) = \sum_j (v^x_j \otimes_{B_s} (a \triangleleft U^x_{j,i}))b.$$

Now it is clear how to define π_X explicitly:

$$\pi_X(a) = \Sigma_{i,j} m^x_{i,j} \otimes (a \triangleleft U^x_{j,i}),$$

where $m^x_{i,j}$ are the corresponding matrix units of $B(H_x), a \in A$. This gives the first statement of the lemma. In order to prove the second statement, take an arbitrary $X \in \mathcal{D}_A$, then for any $a \in A$ and $\zeta \in X$ we have:

$$\delta_X(\pi_X(a)\zeta) = \delta_X(\zeta^{(1)}(a \triangleleft \zeta^{(2)})) = (\zeta^{(1)}(a \triangleleft \zeta^{(2)}))^{(1)} \otimes (\zeta^{(1)}(a \triangleleft \zeta^{(2)}))^{(2)}.$$

The Yetter-Drinfel'd condition (20) shows that the last expression equals to

$$\zeta^{(1)}(a \triangleleft \zeta^{(3)})^{(1)} \otimes \zeta^{(2)}(a \triangleleft \zeta^{(3)})^{(2)} = \zeta^{(1)}(a^{(1)} \triangleleft \zeta^{(4)}) \otimes \zeta^{(2)} S(\zeta^{(3)}) a^{(2)} \zeta^{(5)} =$$

$$= \zeta^{(1)}(a^{(1)} \triangleleft \zeta^{(3)}) \otimes \varepsilon_t(\zeta^2) a^{(2)} \zeta^{(4)}.$$

If again $X = H^x \otimes_{B_s} A$ and $\zeta = v^x_i \otimes_{B_s} b$, Remark 5.6, 4) shows that $(id \otimes \varepsilon_t)\delta_X(\zeta) = \zeta \otimes 1$ for the above $\zeta \in X$. This gives $\delta_X(\pi_X(a)\zeta) = \zeta^{(1)}(a^{(1)} \triangleleft \zeta^{(2)}) \otimes a^{(2)}\zeta^{(3)}$. On the other hand,

$$(\pi_X \otimes id)\mathfrak{a}(a)\delta_X(\zeta) = \pi_X(a^{(1)})\zeta^{(1)} \otimes a^{(2)}\zeta^{(2)} = \zeta^{(1)}(a^{(1)} \triangleleft \zeta^{(2)}) \otimes a^{(2)}\zeta^{(3)},$$

and we are done. ∎

This lemma implies that any $X \in \mathcal{D}_A$ is a $\mathfrak{G}$-equivariant (A, A)-correspondence and any $\mathfrak{G}$-equivariant endomorphism of the right Hilbert A-module X is automatically an (A, A)-bimodule map. Therefore, $\mathcal{D}_A$ is a full subcategory of the C^*-multitensor category of $\mathfrak{G}$-equivariant (A, A)-correspondences. In order to show that $\mathcal{D}_A$ is invariant with respect to $\otimes_A$, take $X, Y \in \mathcal{D}_A$ and prove two statements:

(i) $(X \otimes_A Y) \in \mathcal{D}_A$;

(ii) the left A-module structure on $X \otimes_A Y$ induced by that of X is the same as the left A-module structure given by Lemma 5.7 using the coaction of $\mathfrak{G}$ and the right A-module structure on $X \otimes_A Y$.

The statement (ii) is proved by direct computations similar to those in the proof of Lemma 5.7. In order to prove (i), it suffices to prove

5.8. Lemma. *The map $T_{U,V} : X \otimes_A Y \longrightarrow H_{U\otimes V} \otimes_{B_s} A$, where $X = H_U \otimes_{B_s} A$, $Y = (H_V \otimes_{B_s} A)$ defined for all $\zeta \in H_U, \eta \in H_V, a, b \in A$ by*

$$T_{U,V} : (\zeta \otimes_{B_s} a) \otimes_A (\eta \otimes_{B_s} b) \mapsto \zeta \otimes_{B_s} \pi_Y(a)(\eta \otimes_{B_s} b), \tag{25}$$

is a $\mathfrak{G}$-equivariant unitary isomorphism of right Hilbert A-modules and

$$T_{U\otimes V,W}(T_{U,V} \otimes_A id) = T_{U,V\otimes W}(id \otimes_A T_{V,W}) \ (\forall\ U, V, W \in UCorep(\mathfrak{G})) \tag{26}$$

Proof. Clearly, (25) defines $T_{U,V}$ as a right A-module isomorphism. Let us note that the vectors of the form $(\zeta \otimes_{B_s} 1_A) \otimes_A (\eta \otimes_{B_s} 1_A)$ generate $X \otimes_A Y$ as a right A-module and that $T_{U,V}$ is isometric on these vectors. This implies that $T_{U,V}$ is a unitary isomorphism of right Hilbert A-modules.

Let us check the $\mathfrak{G}$-equivariance of $T_{U,V}$, i.e., we must have the equality $(T_{U,V} \otimes id_B)\delta_{X\otimes_A Y} = \delta_{H_{U\otimes V}\otimes_{B_s} A} \circ T_{U,V}$. Since $\pi_Y(a)(\eta \otimes_{B_s} b) = \eta^{(1)} \otimes_{B_s} (a \triangleleft \eta^{(2)})b$, we have:

$$\delta_{X\otimes_A Y}((\zeta \otimes_{B_s} a) \otimes_A (\eta \otimes_{B_s} b)) = (\zeta \otimes_{B_s} a)^{(1)} \otimes_A (\eta \otimes_{B_s} b)^{(1)} \otimes$$

$$\otimes (\zeta \otimes_{B_s} a)^{(2)} (\eta \otimes_{B_s} b)^{(2)} = (\zeta^{(1)} \otimes_{B_s} b^{(1)}) \otimes_A (\eta^{(1)} \otimes_{B_s} a^{(1)}) \otimes \zeta^{(2)} a^{(2)} \eta^{(2)} b^{(2)}$$

Applying $T_{U,V} \otimes id_B$, we get $\zeta^{(1)} \otimes_{B_s} \eta^{(1)} \otimes_{B_s} (a^{(1)} \triangleleft \eta^{(2)}) b^{(1)} \otimes_{B_s} \zeta^{(2)} a^{(1)} \eta^{(3)} b^{(2)}$.

On the other hand,

$$\delta_{H_{U\otimes V}\otimes_{B_s} A} \circ T_{U,V}[(\zeta \otimes_{B_s} a) \otimes_A (\eta \otimes_{B_s} b)] = \delta_{H_{U\otimes V}\otimes_{B_s} A}(\zeta \otimes_{B_s} \eta^{(1)} \otimes_{B_s} (a \triangleleft \eta^{(2)})b) =$$

$$= \zeta^{(1)} \otimes_{B_s} \eta^{(1)} \otimes_{B_s} (a \triangleleft \eta^{(3)})^{(1)} b^{(1)} \otimes \zeta^{(2)} \eta^{(2)} (a \triangleleft \eta^{(3)})^{(2)} b^{(2)}$$

Applying (20), we see that the last expression equals to

$$\zeta^{(1)} \otimes_{B_s} \eta^{(1)} \otimes_{B_s} (a \triangleleft \eta^{(4)}) b^{(1)} \otimes \zeta^{(2)} \eta^{(2)} S(\eta^{(3)}) a^{(2)} \eta^{(5)} b^{(2)}$$

As $\eta^{(2)} S(\eta^{(3)}) = \varepsilon_t(\eta^{(2)})$ and $\eta^{(1)} \otimes \varepsilon_t(\eta^{(2)}) = 1^{(1)}\eta^{(1)}) \otimes 1^{(2)}$, the last expression also equals to $\zeta^{(1)} \otimes_{B_s} \eta^{(1)} \otimes_{B_s} (a^{(1)} \triangleleft \eta^{(2)}) b^{(1)} \otimes_{B_s} \zeta^{(2)} a^{(1)} \eta^{(3)} b^{(2)}$.

Finally, the relation (26) can be justified by direct computations. ∎

5.9. Corollary. *If $V = \sum_{i,j} m_{i,j} \otimes V_{i,j}$, then for all $\zeta \in H_U, \eta \in H_V, a, b \in A$*

$$T_{U,V}(\zeta \otimes_{B_s} a) \otimes_A (\eta \otimes_{B_s} b) = \zeta \otimes_{B_s} \sum_{i,j} m_{i,j} \eta \otimes_{B_s} (a \triangleleft V_{j,i}) b.$$

Let us summarize the above mentioned results.

5.10. Theorem. *Let A be a unital braided commutative YD $\mathfrak{G} - C^*$-algebra. Then $\mathcal{D}_A$ is a C^*-multitensor category with tensor product $\otimes_A$ and trivial associativities equipped with a unitary tensor functor $\mathcal{E}_A : UCorep(\mathfrak{G}) \longrightarrow \mathcal{D}_A$ sending U to $H_U \otimes_{B_s} A$ whose structural unitary isomorphisms $T_{U,V} : \mathcal{E}_A(U) \otimes_A \mathcal{E}_A(V) \longrightarrow \mathcal{E}_A(U \otimes V)$ are given by (25). Clearly, $\mathcal{E}_A(U_\varepsilon) = A = \mathbf{1}_{D_A}$.*

b) From C^*-multitensor categories to YD- $\mathfrak{G}$-C^*-algebras.

Consider a pair $(\mathcal{C}, \mathcal{E}) \in Tens(UCorep(\mathfrak{G}))$. The category $\mathcal{C}$ is a left $UCorep(\mathfrak{G})$-module category: $U \boxtimes X := \mathcal{E}(U) \otimes X$, $\forall X \in \mathcal{C}$, with generator $\mathcal{E}(\mathcal{U}_\varepsilon) = \mathbf{1}_{\mathcal{C}}$. So $R = End_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}})$ and weak tensor functor F sends any U to $Hom_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, \mathcal{E}(U))$. By Theorem 4.3 we can construct a $\mathfrak{G} - C^*$-algebra $\overline{A}$ with right coaction $\mathfrak{a} : \overline{A} \longrightarrow \overline{A} \otimes B$. Now our goal is to prove

5.11. Theorem. *The above $\mathfrak{G}-C^*$-algebra $\overline{A}$ has a natural structure of a unital braided commutative YD $\mathfrak{G}-C^*$-algebra.*

First, define a right B-module algebra structure on A given by (15). Let $\tilde{A}$ be an algebra (16) with the projection $p_A:\tilde{A}\longrightarrow A$ (17) and $\tilde{B}=\bigoplus_{U\in\|UCorep(\mathfrak{G})\|}(H_U\otimes\overline{H}_U)$, $B=\bigoplus_{x\in\hat{G}}(H_x\otimes\overline{H}_x)$ be the algebras with the similar projection $p_B:\tilde{B}\longrightarrow B$ (see [Vainerman-Vallin,2017], Example 6.7).s Then define a linear map $\tilde{\triangleleft}:\tilde{A}\otimes B\longrightarrow\tilde{A}$ for all $X\otimes\overline{\eta}\in A_U$, $\zeta\otimes\overline{\xi}\in H_V\otimes\overline{H}_V$ and $U,V\in UCorep(\mathfrak{G})$:

$$(X\otimes\overline{\eta})\tilde{\triangleleft}(\zeta\otimes\overline{\xi})=(id\otimes X\otimes id)F(R_V)\otimes\overline{(\hat{G}^{-1/2}\cdot\overline{\zeta}\otimes_{B_s}\eta\otimes_{B_s}\xi)},\tag{27}$$

where R_V comes from (1). Both sides of (27) are in $A_{\overline{V}\otimes U\otimes V}$. Identifying B with the subspace of $\tilde{B}$, define a linear map $\triangleleft:A\otimes B\longrightarrow A$ putting $a\triangleleft b:=p_A(a\tilde{\triangleleft}b)$, for all $a\in A,b\in B$.

5.12. Lemma. *The map $\triangleleft$ defines a right B-module algebra structure on A such that $p_A(a\tilde{\triangleleft}b)=p_A(a)\triangleleft p_B(b)$, for all $a\in\tilde{A},b\in\tilde{B}$.*

Proof. Put $a=X\otimes\overline{\eta}\in F(U)\otimes\overline{H}_U$, $b=\zeta\otimes_{B_s}\overline{\xi}\in H_V\otimes_{B_s}\overline{H}_V$ and choose isometries $u_i:H_{x_i}\longrightarrow H_U$ and $v_j:H_{x_j}\longrightarrow H_V$ defining the decompositions of U and V into irreducibles. Then:

$$p_A(a)\triangleleft p_B(b)=p_A(\sum_{i,j}(F(u_i^*)X\otimes\overline{u_i^*\eta})\tilde{\triangleleft}(v_j^*\zeta\otimes_{B_s}\overline{v_j^*\xi}))=$$

$$=p_A(\sum_{i,j}(id\otimes F(u_i^*)X\otimes id)F(R_{V_{x_j}})\otimes\overline{(\hat{G}^{-1/2}\cdot\overline{v_j^*\zeta}\otimes_{B_s}u_i^*\eta\otimes_{B_s}v_j^*\xi)}).$$

On the other hand,

$$p_A(a\tilde{\triangleleft}b)=p_A((id\otimes X\otimes id)F(R_V)\otimes\overline{(\hat{G}^{-1/2}\cdot\overline{\zeta}\otimes_{B_s}\eta\otimes_{B_s}\xi)})=$$

$$=p_A(\sum_{i,j,k}(F(\overline{v}_j^*)\otimes F(u_i^*)X\otimes F(v_k^*))F(R_V)\otimes\overline{(\overline{v_j^*\hat{G}^{-1/2}\cdot\zeta}\otimes_{B_s}u_i^*\eta\otimes_{B_s}v_k^*\xi)}),$$

where the morphism $\overline{v}_j:H_{\overline{V}_{x_j}}=\overline{H}_{V_{x_j}}\longrightarrow\overline{H}_V=H_{\overline{V}}$ is defined by $\overline{v}_j\overline{\zeta}=\overline{v_j\zeta}$. Since $v_j^*(\hat{G}^{-1/2}\cdot\xi)=\hat{G}^{-1/2}\cdot(v_j^*\xi)$, $\forall j$, $R_V=\sum_j(\overline{v}_j\otimes v_j)R_{V_j}$ and the partial isometries v_j have mutually orthogonal images, the two expressions are equal.

In order to show that $\triangleleft$ defines a right B-module algebra on A, take a,b as above and $c=\mu\otimes_{B_s}\overline{\nu}\in H_W\otimes_{B_s}\overline{H}_W$, where $W\in UCorep(\mathfrak{G})$. Then:

$$(a\tilde{\triangleleft}b)\tilde{\triangleleft}c=(id\otimes X\otimes id)F(R_V)\otimes\overline{(\hat{G}^{-1/2}\cdot\overline{\zeta}\otimes_{B_s}\eta\otimes_{B_s}\xi)}\tilde{\triangleleft}(\mu\otimes_{B_s}\overline{\nu})=$$

$$(id\otimes id\otimes X\otimes id\otimes id)(id\otimes F(R_V)\otimes id)F(R_W)\otimes$$

$$\otimes\overline{(\hat{G}^{-1/2}\cdot\overline{\mu}\otimes_{B_s}\hat{G}^{-1/2}\cdot\overline{\zeta}\otimes_{B_s}\eta\otimes_{B_s}\xi\otimes_{B_s}\nu)}.$$

The result belongs to $\tilde{A}_{\overline{W}\otimes(\overline{V}\otimes U\otimes V)\otimes W}$. On the other hand,

$$a\tilde{\triangleleft}(bc) = (X\otimes\overline{\eta})\tilde{\triangleleft}(\zeta\otimes_{B_s}\mu\otimes_{B_s}\overline{\xi\otimes_{B_s}\nu}) =$$

$$= (id\otimes X\otimes id)F(R_{V\otimes W})\otimes\overline{\hat{G}^{-1/2}\cdot\overline{(\zeta\otimes_{B_s}\mu)}\otimes_{B_s}\eta\otimes_{B_s}(\xi\otimes_{B_s}\nu)}.$$

This result belongs to $\tilde{A}_{\overline{V\otimes W}\otimes U\otimes V\otimes W}$ and is different from the previous one because $\overline{W}\otimes\overline{V}\neq\overline{V\otimes W}$. But the map $\sigma:\overline{H}_W\otimes_{B_s}\overline{H}_V\longrightarrow\overline{H_V\otimes_{B_s}H_W}$ defined by $\sigma(\overline{\mu}\otimes_{B_s}\overline{\zeta})=\overline{\zeta\otimes_{B_s}\mu}$ gives the equivalence of these corepresentations, so $R_{V\otimes W}=(\sigma\otimes id\otimes id)(id\otimes R_V\otimes id)R_W$. Then, applying p_A to the above elements, we have an exact equality $p_A((a\tilde{\triangleleft}b)\tilde{\triangleleft}c)=p_A(a\tilde{\triangleleft}(bc))$.

In order to check the relation $(ad)\triangleleft b=(a\triangleleft b_{(1)})(d\triangleleft b_{(2)})$, take $a=X\otimes\overline{\eta}\in\tilde{A}_U, d=Y\otimes\overline{\mu}\in\tilde{A}_V, b=\zeta_i\otimes_{B_s}\overline{\xi}_j$, where $\{\zeta_i\otimes_{B_s}\overline{\xi}_j\}_{i,j}$ is an orthonormal basis in $H_W\otimes_{B_s}\overline{H}_W$ and $U,V,W\in UCorep(\mathfrak{G})$. Since $p_B(\zeta_i\otimes_{B_s}\overline{\xi}_j)=W_{i,j}$ and $\Delta(W_{i,j})=\sum_k(W_{i,k}\otimes W_{k,j})$, we have to show that

$$p_A((ad)\tilde{\triangleleft}(\zeta_i\otimes_{B_s}\overline{\xi}_j))=\sum_k p_A((a\tilde{\triangleleft}(\zeta_i\otimes_{B_s}\overline{\xi}_k))(d\tilde{\triangleleft}(\zeta_i\otimes_{B_s}\overline{\xi}_j))).$$

The formula for the product in $\tilde{A}$ and (27) give:

$$p_A((ad)\tilde{\triangleleft}(\zeta_i\otimes_{B_s}\overline{\xi}_j)) =$$

$$= p_A((id\otimes X\otimes Y\otimes id)F(R_W)\otimes\overline{\hat{G}^{-1/2}\cdot\overline{\zeta}_i\otimes_{B_s}\eta\otimes_{B_s}\mu\otimes_{B_s}\xi_j}).$$

On the other hand,

$$\sum_k(a\tilde{\triangleleft}(\zeta_i\otimes_{B_s}\overline{\xi}_k))(d\tilde{\triangleleft}(\zeta_k\otimes_{B_s}\overline{\xi}_j))) =$$

$$= \sum_k((id\otimes X\otimes id)F(R_W)\otimes\overline{\hat{G}^{-1/2}\cdot\overline{\zeta_i}\otimes_{B_s}\eta\otimes_{B_s}\xi_k})$$

$$((id\otimes Y\otimes id)F(R_W)\otimes\overline{\hat{G}^{-1/2}\cdot\overline{\zeta_k}\otimes_{B_s}\mu\otimes_{B_s}\xi_j}) = \sum_k(id\otimes X\otimes id\otimes id\otimes Y\otimes id)$$

$$F(R_W\otimes R_W)\otimes\overline{(\hat{G}^{-1/2}\cdot\overline{\zeta_i}\otimes_{B_s}\eta\otimes_{B_s}\xi_k\otimes_{B_s}\hat{G}^{-1/2}\cdot\overline{\zeta_k}\otimes_{B_s}\mu\otimes_{B_s}\xi_j)}$$

Since $\overline{R}_W(\mathbf{1})=\sum_k(\xi_k\otimes_{B_s}\hat{G}^{-1/2}\cdot\overline{\zeta}_k)$, and $\overline{R}_W$ is, up to a scalar factor, an isometric embedding of $\mathbf{1}$ to $W\otimes\overline{W}$, by applying p_A to this element, we get

$$(id\otimes X\otimes F(\overline{R}_W^*)\otimes Y\otimes id)F(R_W\otimes R_W)\otimes\overline{(\hat{G}^{-1/2}\cdot\overline{\zeta_i}\otimes_{B_s}\eta\otimes_{B_s}\otimes_{B_s}\mu\otimes_{B_s}\xi_j)}$$

Since $(\overline{R}_W^*\otimes id)(id\otimes R_W)=id_W$, this is equal to $p_A((ad)\tilde{\triangleleft}(\zeta_i\otimes_{B_s}\overline{\xi}_j))$. ∎

Let us check now the compatibility of $\triangleleft$ with the involution.

5.13. Lemma. *We have $a^* \triangleleft b = (a \triangleleft S(b)^*)^*$ for all $a \in A, b \in B$.*

Proof. Recall that if $a = X \otimes \overline{\eta} \in \tilde{A}_U$, then $a^\bullet = (id \otimes X^*)F(R_U) \otimes \overline{\overline{\hat{G}^{1/2} \cdot \eta}} \in \tilde{A}_{\overline{U}}$. If $b = \zeta \otimes_{B_s} \overline{\xi} \in H_V \otimes_{B_s} \overline{H}_V$, put $b^\bullet = \xi \otimes_{B_s} \overline{\zeta}$. Since $p_B(b^\bullet) = p_B(\xi \otimes_{B_s} \overline{\zeta}) = V_{\xi,\zeta} = (S(V_{\zeta,\xi}))^* = (S(p_B(b))^*$, we have to prove that $p_A(a^* \tilde{\triangleleft} b) = p_A((a \tilde{\triangleleft} b^\bullet)^*)$. Let us compute:

$$(X \otimes \overline{\eta})^\bullet \tilde{\triangleleft} (\zeta \otimes_{B_s} \overline{\xi}) =$$

$$= (id \otimes id \otimes X^* \otimes id)(id \otimes F(R_U) \otimes id)F(R_V) \otimes \overline{(\hat{G}^{-1/2} \cdot \overline{\zeta} \otimes_{B_s} \overline{\hat{G}^{1/2} \cdot \eta} \otimes_{B_s} \xi)}.$$

On the other hand,

$$(a \tilde{\triangleleft} (\zeta \otimes_{B_s} \overline{\xi})^\bullet)^\bullet = ((id \otimes X \otimes id)F(R_V) \otimes \overline{(\hat{G}^{-1/2} \cdot \overline{\xi} \otimes_{B_s} \eta \otimes_{B_s} \zeta)})^\bullet =$$

$$= (id \otimes (id \otimes X \otimes id)F(R_V))^* F(R_{\overline{V} \otimes U \otimes V}) \otimes \overline{(\xi \otimes_{B_s} \overline{\hat{G}^{1/2} \cdot \eta} \otimes_{B_s} \overline{\hat{G}^{1/2} \cdot \zeta})}.$$

Comparing these expressions and using the fact that $\hat{G}^{-1/2} \cdot \overline{\xi} = \overline{\hat{G}^{1/2} \cdot \xi}$, we see that they are not equal only by the reason that the corepresentations $\overline{V} \otimes U \otimes V$ and $\overline{V \otimes \overline{U} \otimes \overline{V}}$ are not equal. But they are equivalent via the map $\sigma(\overline{\zeta} \otimes_{B_s} \eta \otimes_{B_s} \xi) = \overline{\overline{\xi} \otimes_{B_s} \overline{\eta} \otimes_{B_s} \zeta}$ which gives the relation

$$R_{\overline{V} \otimes U \otimes V} = (\sigma \otimes id \otimes id \otimes id)(id \otimes \overline{R}_V \otimes id \otimes id)(id \otimes R_U \otimes id)R_V)$$

Since $(R_V^* \otimes id)(id \otimes \overline{R}_V) = id_{\overline{V}}$, we have

$$(id \otimes (id \otimes X \otimes id)F(R_V))^* F(R_{\overline{V} \otimes U \otimes V}) =$$

$$= \sigma(id \otimes id \otimes X^* \otimes id)(id \otimes F(R_U) \otimes id)F(R_V).$$

Hence, the images of these expressions after applying p_A are equal ∎

Now let us check the Yetter-Drinfel'd relation (20).

5.14. Lemma. *For all $a \in A$ and $b \in B$ we have*

$$\mathfrak{a}(a \triangleleft b) = (a^{(1)} \triangleleft b_{[2)}) \otimes S(b_{[1)})a^{(2)}b_{(3)}$$

Proof. Let $U, V \in UCorep`(\mathfrak{G})$ and $\{\eta_i\} \in H_U$, $\{\zeta_j\} \in H_V$ be two orthonormal bases. For the simplicity, consider ζ_j as eigenvectors of the strictly positive operator $\zeta \mapsto \hat{G} \cdot \zeta$ in $H_V : \hat{G} \cdot \zeta_j = \lambda_j(V)\zeta_j$. Then one has the following relations between the matrix coefficients of V with respect to $\{\zeta_j\}$ and of $\overline{V}$ with respect to $\{\overline{\zeta}_j\}$: $\lambda_j^{-1/2}(V)\lambda_k^{1/2}(V)\overline{V}_{j,k} = V_{j,k}^* = S(V_{k,j})$.

Now take $a = X \otimes \overline{\eta}_{k_0} \in \tilde{A}_U$ and $b = V_{i_0,j_0}$, then we have, using (18):

$$(a^{(1)} \triangleleft b_{(2)}) \otimes S(b_{(1)})a^{(2)}b_{(3)} = \sum_{i,j,k} [p_A(X \otimes \overline{\eta}_k) \triangleleft V_{i,j}] \otimes S(V_{i_0,i})U_{k_0,k}V_{j_0,j} =$$

$$= \sum_{i,j,k} [p_A((id \otimes X \otimes id)F(R_V) \otimes \overline{(\overline{\zeta}_i \otimes_{B_s} \eta_k \otimes_{B_s} \zeta_j)} \otimes \lambda_i^{1/2}(V)S(V_{i_0,i})U_{k,k_0}V_{j,j_0}.$$

On the other hand,

$$\mathfrak{a}(a \triangleleft b) = \mathfrak{a}(p_A((id \otimes X \otimes id)F(R_V) \otimes \overline{(\overline{\zeta}_{i_0} \otimes_{B_s} \eta_{k_0} \otimes_{B_s} \zeta_{j_0})})) =$$

$$= \lambda_{i_0}^{1/2}(V) \sum_{i,j,k} [p_A((id \otimes X \otimes id)F(R_V) \otimes \overline{(\overline{\zeta}_i \otimes_{B_s} \eta_k \otimes_{B_s} \zeta_j)}) \otimes \overline{V}_{i,i_0} U_{k,k_0} V_{j,j_0}.$$

As $\lambda_i^{-1/2}(V)\lambda_{i_0}^{1/2}(V)\overline{V}_{i,i_0} = S(V_{i_0,i})$, the two expressions are equal. ∎

Finally, let us check the braided commutativity relation (23)

5.15. Lemma. *For all $a, b \in A$ we have $ab = b^{(1)}(a \triangleleft b^{(2)})$.*

Proof. Let $a = p_A(X \otimes \overline{\eta}), b = p_A(Y \otimes \overline{\zeta}_i)$, where $X \in F(U), Y \in F(V), \eta \in H_U$, also U, V are in $UCorep(\mathfrak{G})$ and bases $\{\eta_i\} \in H_U$ and $\{\zeta_j\} \in H_V$ as above. Then we compute:

$$b^{(1)}(a \triangleleft b^{(2)}) = \sum_j p_A((Y \otimes \overline{\zeta}_j)((X \otimes \overline{\eta}) \tilde{\triangleleft} V_{j,i})) =$$

$$= \sum_j p_A((Y \otimes \overline{\zeta}_j)((id \otimes X \otimes id)F(R_V) \otimes \lambda_j^{1/2} \overline{(\overline{\zeta}_j \otimes_{B_s} \eta \otimes_{B_s} \zeta_i)}) =$$

$$= \sum_j p_A((id \otimes (id \otimes X \otimes id)F(R_V))Y \otimes \lambda_j^{1/2}(\overline{\zeta}_j \otimes \zeta_j \otimes_{B_s} \overline{\eta} \otimes_{B_s} \overline{\zeta}_i)) =$$

$$= p_A((id \otimes id \otimes X \otimes id)(id \otimes F(R_V))Y \otimes \overline{(\overline{R}_V(1_B) \otimes_{B_s} \eta \otimes_{B_s} \zeta_i)}$$

Since $\overline{R}_V$ is, up to a scalar factor, an isometric embedding of $\mathbf{1}$ into $V \otimes \overline{V}$, the last expression equals

$$p_A((F(\overline{R}_V^*) \otimes X \otimes id)(id \otimes F(R_V))Y \otimes \overline{(\eta \otimes_{B_s} \zeta_i)} = p_A((id \otimes Y)X \otimes \overline{(\eta \otimes_{B_s} \zeta_i)}),$$

which is exactly ab. ∎

Passing to the C^*-completion of A, we finish the proof of Theorem 5.11.

c) Functoriality. Given a morphism $\overline{A}_0 \longrightarrow \overline{A}_1$ in $YD_{brc}(\mathfrak{G})$, the map $X \longrightarrow X \otimes_{\overline{A}_0} \overline{A}_1$ defines a unitary functor $\mathcal{D}_{\overline{A}_0} \longrightarrow \mathcal{D}_{\overline{A}_1}$ (see [Vainerman-Vallin,2017], Theorem 4.12). By Theorem 5.10, both $\mathcal{D}_{\overline{A}_0}$ and $\mathcal{D}_{\overline{A}_1}$ are C^*-multitensor categories, and similarly to the proof of Lemma 5.8 one shows that the isomorphisms

$$(X \otimes_{\overline{A}_0} \overline{A}_1) \otimes_{\overline{A}_1} (Y \otimes_{\overline{A}_0} \overline{A}_1) \cong (X \otimes_{\overline{A}_0} Y) \otimes_{\overline{A}_0} \overline{A}_1$$

defined by $(x \otimes_{\overline{A}_0} a) \otimes_{\overline{A}_1} (y \otimes_{\overline{A}_0} b) \mapsto x \otimes_{\overline{A}_0} y^{(1)} \otimes_{\overline{A}_1} (a \triangleleft (y^{(2)})b$, for all $x \in X \in \mathcal{D}_{\overline{A}_0}$ and $y \in Y \in \mathcal{D}_{\overline{A}_1}$, define a tensor structure on this functor. This functor together with obvious isomorphisms $\eta_U : (H_U \otimes_{B_s} \overline{A}_0) \otimes_{\overline{A}_0} \overline{A}_1 \longrightarrow H_U \otimes_{B_s} \overline{A}_1$ define a morphism $(\mathcal{D}_{\overline{A}_0}, \mathcal{E}_{\overline{A}_0}) \longrightarrow (\mathcal{D}_{\overline{A}_1}, \mathcal{E}_{\overline{A}_1})$. Thus, we have a functor $\mathcal{T} : YD_{brc}(\mathfrak{G}) \longrightarrow Tens(UCorep(\mathfrak{G}))$.

Let now $[(\mathcal{F}, \eta)] : (\mathcal{C}_0, \mathcal{E}_0) \longrightarrow (\mathcal{C}_1, \mathcal{E}_1)$ be a morphism in $Tens(UCorep(\mathfrak{G}))$, and let $\overline{A}_0$ and $\overline{A}_1$ be the corresponding braided-commutative YD $\mathfrak{G} - C^*$-algebras - see Theorem

5.11. It follows from the construction of the $*$-algebras A_0 and A_1 that the maps $(\mathcal{E}_0(U^x)\otimes \overline{H}^x) \longrightarrow (\mathcal{E}_1(U^x)\otimes \overline{H}^x)$ given by $(X\otimes\overline{\xi})\mapsto(\mathcal{F}(X)\otimes\overline{\xi})$ define a unital $*$-homomorphism $A_0\longrightarrow A_1$ that respects their B-comodule and B-module structures. It then extends to a homomorphism of unital braided-commutative YD $\mathfrak{G}-C^*$-algebras $f:\overline{A}_0\longrightarrow\overline{A}_1$ which depends only on the equivalence class of $(\mathcal{F},\eta)]$. Thus, we have constructed a functor $\mathcal{S}: Tens(UCorep(\mathfrak{G}))\longrightarrow YD_{brc}(\mathfrak{G})$.

The homomorphism $f:\overline{A}_0\longrightarrow\overline{A}_1$ is injective (resp., surjective) if and only if the maps $Hom_{\mathcal{C}_0}(\mathbf{1},\mathcal{E}_0(U^x))\longrightarrow Hom_{\mathcal{C}_1}(\mathbf{1},\mathcal{E}_1(U^x))$ are injective (resp., surjective), for all $x\in\Omega$. But thanks to the equalities of the type $Hom_{\mathcal{C}}(\mathbf{1},V\otimes\overline{U})=Hom_{\mathcal{C}}(U,V)$, this holds if and only if, for all $U,V\in UCorep(\mathfrak{G})$, the maps $Hom_{\mathcal{C}_0}(\mathcal{E}_0(U),\mathcal{E}_0(V))\longrightarrow Hom_{\mathcal{C}_1}(\mathcal{E}_1(U),\mathcal{E}_1(V))$ are injective (resp., surjective). Since $\mathcal{C}_i$ is generated by $\mathcal{E}_i(UCorep(\mathfrak{G}))$ for $i\in\{0,1\}$, it follows that f is injective (resp., surjective) if and only if the functor $\mathcal{F}$ is faithful (resp., full).

d) Equivalence of categories. In order to show that the above functors $\mathcal{T}$ and $\mathcal{S}$ are inverse to each other up to an isomorphism, let us start with a pair $(\mathcal{C},\mathcal{E})$ as above, the corresponding braided-commutative YD $\mathfrak{G}-C^*$-algebra $\overline{A}$ and describe explicitly the image of any morphism $T\in Hom_{\mathcal{C}}(\mathcal{E}(U),\mathcal{E}(V))$, where $U,V\in UCorep(\mathfrak{G})$, under the unitary equivalence $\mathcal{F}:\mathcal{C}\longrightarrow\mathcal{D}_{\overline{A}}$ as left $UCorep(\mathfrak{G})$-module C^*-categories given by Theorem 4.3.

In particular, $\mathcal{F}$ maps a morphism $T\in Hom_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}},\mathcal{E}(V))=F(V)$ to the morphism $\mathcal{F}(T):H^{\varepsilon}\otimes_{B_s}A_{\varepsilon}\longrightarrow H_V\otimes_{B_s}A_V$ sending 1_B to $\sum_j[\zeta_j\otimes_{B_s}p_A(T\otimes\overline{\zeta}_j)]$, where $\{\zeta_j\}\in H_V$ is an orthonormal basis (see the proof of [Vainerman-Vallin,2017],Theorem 6.3). Now write any $T\in Hom_{\mathcal{C}}(\mathcal{E}(U),\mathcal{E}(V))$ as $T=(\mathcal{E}(\overline{R}_U^*)\otimes id)(id\otimes S)$, where $S=(id\otimes T)\mathcal{E}(R_U)\in Hom_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}},\mathcal{E}(\overline{U}\otimes V))$. Choose an orthonormal basis $\{\xi_i\}$ in H_U (as in the proof of Lemma 5.14, it is convenient to choose $\{\xi_i\}$ such that $\hat{G}\cdot\xi_i=\lambda_i(U)\xi_i$ for all i). Then the image of the morphism S is:

$$1\mapsto\sum_{i,j}[\overline{\xi}_i\otimes_{B_s}\zeta_j\otimes_{B_s}p_A(S\otimes\overline{(\overline{\xi}_i\otimes_{B_s}\zeta_j)})].$$

It follows that $T=(\mathcal{E}(\overline{R}_U^*)\otimes id)(id\otimes S)$ is mapped into

$$\sum_{i,j}[\mathcal{E}(\overline{R}_U^*)(\cdot\otimes_{B_s}\overline{\xi}_i)\zeta_j\otimes_{B_s}p_A(S\otimes\overline{(\overline{\xi}_i\otimes_{B_s}\zeta_j)})].$$

Using the second of formulas (1), we conclude that the image of T is:

$$\sum_{i,j}\theta_{\zeta_j,\xi_i}\otimes_{B_s}p_A((id\otimes T)\mathcal{E}(R_U)\otimes\overline{(\hat{G}^{-1/2}\cdot\overline{\xi}_i\otimes_{B_s}\zeta_j)},\tag{28}$$

where $\theta_{\zeta_j,\xi_i}\in B(H_U,H_V)$ is defined by $\theta_{\zeta_j,\xi_i}(\eta)=<\eta,\xi_i>\zeta_j$ for all $\eta\in H_U$.

In order to show that $\mathcal{F}$ is a strict tensor functor on $\mathcal{E}(UCorep(\mathfrak{G}))$ and hence on $\mathcal{C}$, we have to show that $\mathcal{F}(S\otimes T)=\mathcal{F}(S)\otimes\mathcal{F}(T)$ on morphisms in $\mathcal{E}(UCorep(\mathfrak{G}))$. Since $\mathcal{F}$ is an equivalence of left $UCorep(\mathfrak{G})$-module categories, we already know that $\mathcal{F}(id\otimes T)=id\otimes\mathcal{F}(T)$, so it remains to show that $\mathcal{F}(S\otimes id)=\mathcal{F}(S)\otimes id$.

If $S : \mathcal{E}(U) \longrightarrow \mathcal{E}(V)$ and $\{\eta_k\}$ is an orthonormal basis in H_W ($W \in UCorep(\mathfrak{G})$), then according to (28) $\mathcal{F}(S \otimes id_W)$ equals

$$\sum_{i,j,k,l} \theta_{\zeta_j \otimes_{B_s} \eta_k, \xi_i \otimes_{B_s} \eta_l} \otimes_{B_s} p_A((id \otimes (S \otimes id))\mathcal{E}(R_{U\otimes W}) \otimes$$

$$\otimes \overline{\hat{G}^{-1/2} \cdot \overline{(\xi_i \otimes_{B_s} \eta_l)} \otimes_{B_s} \zeta_j \otimes_{B_s} \eta_k}).$$

As in the proof of Lemma 5.12, $R_{U\otimes W}$ coincides, modulo the equivalence $\overline{U \otimes W} \equiv \overline{W} \otimes \overline{U}$, with $(id \otimes \overline{R}_U \otimes id)\overline{R}_W)$, so the above expression equals

$$\sum_{i,j,k,l} \theta_{\zeta_j \otimes_{B_s} \eta_k, \xi_i \otimes_{B_s} \eta_l} \otimes_{B_s} p_A((id \otimes (id \otimes S)\mathcal{E}(R_U) \otimes id)\mathcal{E}(R_W) \otimes$$

$$\otimes \overline{\hat{G}^{-1/2} \cdot \overline{\eta}_l \otimes_{B_s} \hat{G}^{-1/2} \cdot \overline{\xi}_i \otimes_{B_s} \zeta_j \otimes_{B_s} \eta_k}).$$

The operators θ_{η_k, η_l} are the matrix units $m_{k,l}$ in $B(H_W)$. Recalling the definition of $\triangleleft$, we can rewrite the above expression as

$$\sum_{i,j,k,l} \theta_{\zeta_j, \xi_i} \otimes_{B_s} m_{k,l} \otimes_{B_s} [p_A((id \otimes S)\mathcal{E}(R_U) \otimes \overline{(\hat{G}^{-1/2} \cdot \overline{\xi}_i \otimes_{B_s} \zeta_j)}) \triangleleft W_{k,l}]$$

On the other hand, $\mathcal{F}(S) : H_U \otimes_{B_s} A \longrightarrow H_V \otimes_{B_s} A$ can be presented as $\mathcal{F}(S) = \sum_i s_i \otimes_{B_s} a_i$, where $s_i \in B(H_U, H_V), a_i \in A$, with the action $\mathcal{F}(S)(\xi \otimes_{B_s} a) = \sum_i s_i(\xi) \otimes_{B_s} a_i a$, for all $\xi \in H_U, a \in A$. Considering $\mathcal{F}(S) \otimes id$ as a morphism from $H_{U\otimes W} \otimes_{B_s} A$ to $H_{V\otimes W} \otimes_{B_s} A$, we have for all $\zeta \in H_U, \eta \in H_W$:

$$(\mathcal{F}(S) \otimes id)(\zeta \otimes_{B_s} \eta \otimes_{B_s} 1_{\overline{A}}) = T_{V,W}(\sum_i s_i(\zeta) \otimes_{B_s} \eta \otimes_{B_s} a_i) =$$

$$= T_{V,W}(\sum_i s_i(\zeta) \otimes_{B_s} a_i) \otimes_{\overline{A}} (\eta \otimes_{B_s} 1_{\overline{A}}) =$$

$$\sum_i s_i(\zeta) \otimes_{B_s} \sum_{k,l} m_{k,l}\eta \otimes_{B_s} (a_i \triangleleft W_{k,l}).$$

Hence, the actions of $\mathcal{F}(S \otimes id)$ and $\mathcal{F}(S) \otimes id$ on generating vectors $\zeta \otimes_{B_s} \eta \otimes_{B_s} 1_{\overline{A}}$ coincide.

5.16. Remark. *Similar calculation and the fact that* $1_{\overline{A}} \triangleleft U_{k,l} = \delta_{k,l} 1_{\overline{A}}$ *give*

$$(id \otimes \mathcal{F}(S))(\eta \otimes_{B_s} \zeta \otimes_{B_s} 1_{\overline{A}}) = \eta \otimes_{B_s} \sum_i s_i(\zeta) \otimes_{B_s} a_i.$$

Conversely, consider a unital braided-commutative YD $\mathfrak{G} - C^*$-algebra $\overline{A}$ and the corresponding pair $(\mathcal{C}_{\overline{A}}, \mathcal{E}_{\overline{A}})$, and let $\overline{A}_{\mathcal{C}}$ be the braided-commutative YD $\mathfrak{G} - C^*$-algebra constructed from this pair. By Theorem 4.3, there is an isomorphism $\lambda : \overline{A}_{\mathcal{C}} \longrightarrow \overline{A}$ intertwining the coactions of $\mathfrak{G}$ and defined by $\lambda(p_{\overline{A}}(T \otimes \overline{\zeta})) = (\overline{\zeta} \otimes id)T$, for all $\zeta \in H_V, T \in \mathcal{C}_{\overline{A}}(\mathbf{1}, V) \subset \mathcal{L}(B_s, H_V) \otimes \overline{A} = H_V \otimes \overline{A}$. So it only remains to show that λ is a right B-module map.

As above, fix $U, V \in UCorep(\mathfrak{G})$ and orthonormal bases $\{\xi_i\} \in H_U$ and $\{\zeta_k\} \in H_V$, and let U_{kl} be the matrix coefficients of U. Take $T = \sum_k (\zeta_k \otimes_{B_s} a_k) \in H_V \otimes_{B_s} A$, then $\lambda(p_{\overline{A}}(T \otimes \overline{\zeta}_{k_0})) = a_{k_0}$, and check that $\lambda(p_{\overline{A}}(T \otimes \overline{\zeta}_{k_0}) \triangleleft U_{i_0,j_0}) = a_{k_0} \triangleleft U_{i_0,j_0}$. By (27) we have

$$p_{\overline{A}}(T \otimes \overline{\zeta}_{k_0}) \triangleleft U_{i_0,j_0} = (id \otimes T \otimes id)F(R_U) \otimes \overline{(\hat{G}^{-1/2}\overline{\xi}_{i_0} \otimes_{B_s} \zeta_{k_0} \otimes_{B_s} \xi_{j_0})}.$$

In order to compute the image of this element under λ, we need an explicit formula for $(id \otimes T \otimes id)F(R_U) \ : \ \mathbf{1} \longrightarrow H_{\overline{U} \otimes V \otimes U} \otimes_{B_s} A$. Remark 5.16 and the computation before it show that the element $id_{\overline{U}} \otimes T \otimes id_U \ : \ H_{\overline{U}} \otimes_{B_s} H_U \longrightarrow H_{\overline{U}} \otimes_{B_s} H_V \otimes_{B_s} H_U$ equals $1 \otimes_{B_s} \sum_{i,j,k} \zeta_k \otimes_{B_s} m_{i,j} \otimes_{B_s} (a_k \triangleleft U_{j,i})$. Then

$$(id \otimes T \otimes id)F(R_U) = \sum_{i,j,k} \hat{G}^{1/2} \cdot \overline{\xi}_j \otimes_{B_s} \zeta_k \otimes_{B_s} \xi_i \otimes_{B_s} (a_k \triangleleft U_{j,i}).$$

Therefore, $p_{\overline{A}}(T \otimes \overline{\zeta}_{k_0}) \triangleleft U_{i_0,j_0}$ equals

$$p_{\overline{A}}(\sum_{i,j,k} \hat{G}^{1/2} \cdot \overline{\xi}_j \otimes_{B_s} \zeta_k \otimes_{B_s} \xi_i \otimes_{B_s} (a_k \triangleleft U_{j,i}) \otimes \overline{(\hat{G}^{-1/2}\overline{\xi}_{i_0} \otimes_{B_s} \zeta_{k_0} \otimes_{B_s} \xi_{j_0})})$$

Applying λ, we get the required equality $\lambda(p_{\overline{A}}(T \otimes \overline{\zeta}_{k_0}) = a_{k_O} \triangleleft U_{i_0,j_0}$. As the algebra A is spanned by elements a_{k_0} for various $V \in UCorep(\mathfrak{G})$, it follows that λ is a right B-module map. This completes the proof of Theorem 1.1.

6. Quotient type and invariant coideal C^* subalgebras

6.1. Quotient type coideal C^*-subalgebras. The notion of a quotient type coideal C^*-subalgebra of a WHA $\mathfrak{G}$ is closely related to the notion of a quantum subgroupoid which is just another WHA $\mathfrak{H}$ equipped with an epimorphism $\pi : \mathfrak{G} \longrightarrow \mathfrak{H}$. We start with basic definitions and results.

6.2. Definition. *A morphism between two WHAs, $\mathfrak{G} = (B, \Delta^B, S^B, \varepsilon^B)$ and $\mathfrak{H} = (C, \Delta^C, S^C, \varepsilon^C)$, is a unital morphism $\pi : B \longrightarrow C$ of their C^*-algebras such that $\Delta^C \circ \pi = (\pi \otimes \pi)\Delta^B$, $S^C \circ \pi = \pi \circ S^B$ and $\varepsilon^C \circ \pi = \varepsilon^B$.*

6.3. Remark. *1. One checks that this definition implies: $\pi \circ \varepsilon^B_t = \varepsilon^C_t \circ \pi$ and $\pi \circ \varepsilon^B_s = \varepsilon^C_s \circ \pi$, so $\pi(B_t) = C_t$ and $\pi(B_s) = C_s$.*

2. If $\mathfrak{G}$ and $\mathfrak{H}$ are usual Hopf C^*-algebras, Definition 6.2 coincides with the usual definition of a morphism of Hopf C^*-algebras..

6.4. Lemma. *If $\pi : \mathfrak{G} \longrightarrow \mathfrak{H}$ is surjective, the map $\mathcal{E}_\pi : U \mapsto (id \otimes \pi)U$ is a unitary tensor functor $UCorep(\mathfrak{G}) \longrightarrow UCorep(\mathfrak{H})$. Moreover, $(id \otimes \pi)U \in B(H^\pi_U) \otimes C$, where $(H^\pi_U)^\perp = \{\zeta \in H_U | (id \otimes \pi)U(\zeta \otimes 1) = 0\}$.*

PROOF. Considering H_U as a left $\hat{B}$-module in the following way:

$$\hat{b}\cdot\zeta =< U^{(2)}, \hat{b} > U^{(1)}\zeta \quad (\forall \hat{b}\in\hat{B}, \zeta\in H_U),$$

and the $*$-algebra inclusion $\pi_* : \hat{C} \longrightarrow \hat{B}$ dual to $\pi : B \longrightarrow C$, one has:

$$< \pi(U_{\eta,\zeta}), \hat{c} >=< (id\otimes\pi)U(\zeta\otimes 1_B), \eta\otimes\hat{c} >=< U^{(1)}\zeta, \eta >< U^{(2)}, \pi_*(\hat{c}) >=$$

$$=< \pi_*(\hat{c})\zeta, \eta >=< \zeta, \pi_*(\hat{c})^*\eta > \quad (\forall \eta, \zeta\in H_U, \hat{c}\in\hat{C}),$$

from where $H_U^\pi = \pi_*(\hat{C})H_U$. In particular, $\pi(U_{\eta,\zeta}) = 0$ for all $\zeta\in H_U^\pi, \eta\in (H_U^\pi)^\perp$ which gives the result. ∎

6.5. COROLLARY. *The functor $\mathcal{E}_\pi$ transforms H_U into H_U^π and intertwiners $H_U \longrightarrow H_V$ into intertwiners $H_U^\pi \longrightarrow H_V^\pi$ for all $U, V \in UCorep(\mathfrak{G})$.*

6.6. LEMMA. *Let $\hat{l}\in\hat{C}$. The matrix coefficient $\pi(U_{\pi_*(\hat{l})\zeta,\eta})\in C_s$ for all $U\in UCorep(\mathfrak{G})$ and $\eta,\zeta\in H_U$ if and only if $\hat{l}$ is a left integral.*

PROOF. Combining the above mentioned relations, first one has:

$$\pi(U_{\eta,\pi_*(\hat{l})\zeta}) =< U^{(1)}\pi_*(\hat{l})\zeta, \eta > \pi(U^{(2)}) =$$

$$=< U^{(1)}U'^{(1)}\zeta, \eta > \pi(U^{(2)}) < U'^{(2)}, \pi_*(\hat{l}) >=$$
$$< U^{(1)}\zeta, \eta > (id\otimes\hat{l})\Delta_C(\pi(U^{(2)})).$$

Since C is spanned by the π-images of matrix elements of all $U\in UCorep(\mathfrak{G})$, this element is in C_t if and only if $(id\otimes\hat{l})\Delta_C = (\varepsilon_t^C\otimes\hat{l})\Delta_C$, i.e., if and only if $\hat{l}$ is a left integral. Finally, $\pi(U_{\pi_*(\hat{l})\zeta,\eta}) = S_C(\pi(U_{\eta,\pi_*(\hat{l})\zeta}))^*$. ∎

Consider now a surjective $*$-homomorphism $\pi : \mathfrak{G} \twoheadrightarrow \mathfrak{H}$ of WHAs $\mathfrak{G} = (B, \Delta_B, S_B, \varepsilon_B)$ and $\mathfrak{H} = (C, \Delta_C, S_C, \varepsilon_C)$. Then the map $\mathfrak{a} = (\pi\otimes id_B)\Delta$ is a *left* coaction of $\mathfrak{H}$ on B.

6.7. DEFINITION. *The fixed point unital $*$-subalgebra $I(\mathfrak{H}\backslash\mathfrak{G})$ of B with respect to the coaction $(\pi\otimes id_B)\Delta$ of $\mathfrak{H}$ is called a quotient type coideal C^*-subalgebra (briefly, quotient type coideal) of B. Equivalently,*

$$I(\mathfrak{H}\backslash\mathfrak{G}) = \{b\in B | (\pi\otimes id)\Delta_B(b) = (\pi\otimes id)((1\otimes b)\Delta_B(1))\}$$

Obviously, $\pi(I)\subset C_s$ and $B_s\subset I$, so $\pi(I) = C_s$.

Clearly, the smallest quotient type coideal of B is B_s. It corresponds to $\mathfrak{H} = \mathfrak{G}$, $\pi = id$. Since $I(\mathfrak{H}\backslash\mathfrak{G})$ is the fixed point $*$-subalgebra with respect to the coaction $(\pi\otimes id)\Delta$, it is included into B_t' $(= \alpha(B_t)')$.

6.8. LEMMA. *B_t' is the greatest quotient type coideal C^*-subalgebra.*

Proof. Let $\hat{\mathfrak{G}}$ be the dual of a WHA $\mathfrak{G} = (B, \Delta, S, \varepsilon)$. Let $\hat{\mathfrak{G}}_{min} = \hat{B}_t\hat{B}_s$ be the minimal WHA contained in $\hat{\mathfrak{G}}$ and $i_{min} : \hat{\mathfrak{G}}_{min} \longrightarrow \hat{\mathfrak{G}}$ the corresponding inclusion of WHAs (see [Bohm-Nill-Szlachanyi,1999], [Nikshych,2003]). Then the adjoint map $\pi_{min} : \mathfrak{G} \longrightarrow \hat{\mathfrak{G}}^*_{min}$ given by $< i_{min}(\hat{B}_{min}), B >=< \hat{B}_{min}, \pi_{min}(B) >$, is an epimorphism of WHAs. The corresponding quotient type coideal I_{max} is the set of such $b \in B$ that

$$< \hat{z} \otimes \hat{b}, (\pi_{min} \otimes id)\Delta(b) >=< \hat{z} \otimes \hat{b}, (\pi_{min} \otimes id)(1_B \otimes b)\Delta({}_B1) >,$$

which is equivalent to

$$< \hat{b}, b \leftharpoonup \hat{z} >=< \hat{b}, b(1_B \leftharpoonup \hat{z}) >, \quad \text{for all} \quad \hat{b} \in \hat{B},\ \hat{z} \in \hat{B}_{min}, \tag{29}$$

where, by definition, $c \leftharpoonup \hat{z} := c_2 < \hat{z}, c_1 >$, for any $c \in B$. As $\hat{B}_{min} = \hat{B}_s\hat{B}_t$, it suffices to consider $\hat{z} = \hat{u}\hat{v}$, where $\hat{u} \in \hat{B}_t$ and $\hat{v} \in \hat{B}_s$. So, $b \in I_{max}$ if and only if $b \in B$ and

$$b \leftharpoonup (\hat{u}\hat{v}) = b(1_B \leftharpoonup (\hat{u}\hat{v}))b \quad \text{or} \quad (b \leftharpoonup \hat{u}) \leftharpoonup \hat{v} = b((1_B \leftharpoonup \hat{u}) \leftharpoonup \hat{v}).$$

By [Bohm-Nill-Szlachanyi,1999], (2.21a), one can rewrite the last equality as $(ub) \leftharpoonup (\hat{v}) = b(u \leftharpoonup \hat{v})$, where we denoted $1_B \leftharpoonup \hat{u} \in B_t$ by u. Now, by [Bohm-Nill-Szlachanyi,1999], (2.20b), this can be rewritten as $(ub)v = b(uv)$, where we denoted $1_B \leftharpoonup \hat{v} \in B_s$ by v. But this is true exactly for $b \in B'_t$, and we are done. ∎

6.9. Lemma. *Let $\zeta, \eta \in H_U$, then $U_{\zeta,\eta} \in I$ if and only if $\pi(U_{\zeta,\theta}) \in C_s$, for all $\theta \in H_U$.*

Proof. If $\{\theta_k\}$ is an orthonormal basis is H_U and $U_{\zeta,\eta} \in I$, then $\Delta(U_{\zeta,\eta}) = \sum_k (U_{\zeta,\theta_k} \otimes U_{\theta_k,\eta}) \in I \otimes B$, so all $U_{\zeta,\theta_k} \in I$ and $\pi(U_{\zeta,\theta_k}) \in C_s$ for all k. So, $\pi(U_{\zeta,\theta}) \in C_s$, for all $\theta \in H_U$.

Conversely, if $\pi(U_{\zeta,\theta}) \in C_s$, then

$$(\pi \otimes id)\Delta_B(U_{\zeta,\theta}) = (\varepsilon^C_s \otimes id)(\pi \otimes id)\Delta_B(U_{\zeta,\theta}) =$$

$$= (\pi \otimes id)(\varepsilon^B_s \otimes id)\Delta_B(U_{\zeta,\theta}) = (\pi \otimes id)(1 \otimes U_{\zeta,\theta})\Delta_B(1).$$

∎

The following lemma generalizes [Vainerman-Vallin,2017], Example 6.7 and describes module categories associated with quotient type coideals.

6.10. Lemma. *If $(\mathfrak{H}, \pi)$ is a quantum subgroupoid of $\mathfrak{G}$ and Λ is the set of left integrals of $\hat{\mathfrak{H}}$, then $I = I(\mathfrak{H}\backslash\mathfrak{G})$ admits the decomposition*

$$I(\mathfrak{H}\backslash\mathfrak{G}) = \bigoplus_{x\in\Omega} \pi_*(\Lambda)H^x \otimes H^x$$

and the corresponding $UCorep(\mathfrak{G})$-module C^*-category $\mathcal{M}$ is equivalent to $UCorep(\mathfrak{H})$ viewed as a $UCorep(\mathfrak{G})$-module C^*-category via the functor $\mathcal{E}_\pi$.

Proof. It suffices to prove that I is equivariantly isomorphic to the $\mathfrak{G}-C^*$-algebra A corresponding to the couple $(UCorep(\mathfrak{H}),\mathbf{1})$. Following the categorical duality, we first construct an algebra $\tilde{A}$ of the form (16), where $F(U)=Hom_{UCorep(\mathfrak{H})}(\mathbf{1},\mathcal{E}_\pi(H_U))=Hom_{URep(\hat{\mathfrak{H}})}(C_s,\pi_*(\hat{C})(H_U))$, where C_s is a left $\hat{C}$-module via $\hat{c}\cdot z:=\hat{c}\rightharpoonup z$ $(\hat{c}\in\hat{C},z\in B_s)$. For any such morphism f the vector $f(1)$ is cyclic for $Im(f)$, so in fact we have to describe $Hom_{Rep(\hat{\mathfrak{H}})}(C_s,\hat{C})$. But [Bohm-Nill-Szlachanyi,1999], Lemma 3.3 shows that this is exactly the set Λ of left integrals in $\hat{C}$. Thus, we can identify $Hom_{URep(\hat{\mathfrak{H}})}(C_s,\pi_*(\hat{C})(H_U))$ with the subspace $\pi_*(\Lambda)H_U\subset H_U$.

So $\tilde{A}$ is a subalgebra of the algebra $\tilde{B}=\bigoplus_U(H_U\otimes\overline{H}_U)$, and the map $p_B:\tilde{B}\longrightarrow B$ sending $\zeta\otimes\overline{\eta}\in H_U\otimes\overline{H}_U$ onto the matrix coefficient $U_{\zeta,\eta}$ induces an $\mathfrak{G}$-equivariant isomorphism of A onto the coideal C^*-subalgebra $I=Vec\{U_{\zeta,\eta}|\zeta\in\pi_*(\Lambda)H_U,\eta\in H_U,U\in UCorep(\mathfrak{G})\}$. Finally, Corollary 6.6 and Lemma 6.9 show that $I=I(\mathfrak{H}\backslash\mathfrak{G})$. ∎

6.11. Invariant coideal C^*-subalgebras. Consider the **right adjoint action** of B on itself defined by

$$b\triangleleft x:=S(x_{(1)})bx_{(2)},\ \text{for all } x,b\in B. \tag{30}$$

It follows from [Nikshych-Turaev-Vainerman,2003], Lemma 2.2 that the map $P_\triangleleft:b\mapsto b\triangleleft 1_B$ is a projection from B onto B'_t, from where $B\triangleleft B=B'_t$.

6.12. Definition. *A right coideal C^*-subalgebra I is called invariant if $I\triangleleft B=I$.*

6.13. Remark. *1. I is invariant if and only if $I\subset B'_t$ and $I\triangleleft B\subset I$. Indeed, Definition 6.12 implies $I=I\triangleleft B\subset B\triangleleft B=B'_t$ and $I\triangleleft B=I\subset I$. Conversely, if $I\subset B'_t$ and $I\triangleleft B\subset I$, then $I=P_\triangleleft(I)=I\triangleleft 1_B\subset I\triangleleft B\subset I$.*

2. B is invariant if and only if $\mathfrak{G}$ is a $*$-Hopf algebra.

3. One can check that B'_t is the greatest invariant coideal C^*-subalgebra.

It is known [Nikshych-Vainerman,2000] that all coideal C^*-subalgebras of B form a lattice $l(B)$ with minimal element B_s and maximal element B under the usual operations: $I_1\wedge I_2=I_1\cap I_2,\ I_1\vee I_2=(I_1\cup I_2)''$.

6.14. Lemma. *Invariant coideal C^*-subalgebras form its sublattice $invl(B)$ with minimal (resp., maximal) element B_s (resp., B'_t).*

Proof. If I,I' are two invariant coideal C^*-subalgebras, then for any natural number $k\geq 2$ and all $j_1,..,j_k\in I\cup I',b\in B$ one has:

$$j_1..j_k\triangleleft b=(j_1...j_{k-1}\triangleleft b_{(1)})(j_k\triangleleft b_{(2)}).$$

So by obvious iteration $j_1..j_k\triangleleft b\in I\vee I'$, but I,I' are $*$-invariant so $I\vee I'$ is spanned by sums of type $j_1..j_k$. Moreover, the map: $j\mapsto j\triangleleft b$ is linear which gives that $(I\vee I')\triangleleft B\subset(I\vee I')$. But $(I\vee I')\subset B'_t$, so by Remark 6.13, 1 $I\vee I'$ is invariant. Also, $(I\cap I')\triangleleft B\subset(I\cap I')$ and $(I\cap I')\subset B'_t$, so $I\cap I'$ is invariant and the result follows. ∎

6.15. Lemma. *Any invariant coideal C^*-subalgebra I belongs to the category $YD_{brc}(\mathfrak{G})$.*

Proof. All the relations (19) are obvious. Let us check the Yetter-Drinfel'd and the braided commutativity relations:

$$\Delta(S(b_{(1)})ab_{(2)}) = S(b_{(2)})a_{(1)}b_{(3)} \otimes S(b_{(1)})a_{(2)}b_{(4)} = (a_{(1)} \triangleleft b_{(2)}) \otimes S(b_{(1)})a_{(2)}b_{(3)},$$

where $a \in I, b \in B$. Finally, using the fact that $I \in B'_t$:

$$b_{(1)}(a \triangleleft b_{(2)}) = b_{(1)}S(b_{(2)})ab_{(3)} = \varepsilon_t(b_{(1)})ab_{(2)} = a\varepsilon_t(b_{(1)})b_{(2)} = ab, \ \forall a, b \in I.$$

■

Now discuss the relationship between quotient type and invariant coideals.

6.16. Lemma. *Any quotient type coideal C^*-subalgebra I is invariant.*

Proof. Let us show that $(I \triangleleft x) \in I$ for all $x \in B$. Indeed, using Proposition 2.2.1 of [Nikshych-Vainerman,2002], we have for all $b \in I$:

$$\begin{aligned}
&(\pi \otimes id)\Delta(S(x_{(1)})bx_{(2)})) = \\
&\quad = (\pi(S(x_{(2)}) \otimes S(x_{(1)}))((\pi \otimes id)((1 \otimes b)\Delta(1))(\pi(x_{(3)}) \otimes x_{(4)}) = \\
&\quad = (\pi \otimes id)(S(x_{(2)})1_{(1)}x_{(3)} \otimes S(x_{(1)})b1_{(2)}x_{(4)}) = \\
&\quad = (id \otimes \pi)(S(x_{(2)})x_{(3)} \otimes S(x_{(1)})bS(x_{(4)}))) = \\
&\quad = (\pi \otimes id)(\varepsilon_s(x_{(2)}) \otimes S(x_{(1)})bx_{(3)}) = (\pi \otimes id)(1_{(1)} \otimes S(x_{(1)})bx_{(2)}1_{(2)}).
\end{aligned}$$

■

The inverse statement - Theorem 1.2 is proved as follows:

Proof. Lemma 6.15 shows that I is a braided-commutative YD $\mathfrak{G}$-C^*-algebra. Then Theorem 5.10 shows that the corresponding $UCorep(\mathfrak{G})$-module category $\mathcal{D}_I$ is a C^*-multitensor category with tensor product $\otimes_I$ and trivial associativities equipped with a unitary tensor functor

$$\mathcal{E}_I : UCorep(\mathfrak{G}) \longrightarrow \mathcal{D}_I \quad , \quad U \mapsto H_U \otimes_{B_s} I.$$

Let us equip now the category $\mathcal{D}_I$ with the tensor functor $\mathcal{F}$ to $Corr_f(B_s)$ sending $H_U \otimes_{B_s} I$ to H_U. Then the reconstruction theorems for WHA's and their morphisms allow to construct a WHA $\mathfrak{H}_I$ such that $UCorep(\mathfrak{H}_I) \cong \mathcal{C}_I$ together with an epimorphism $\pi : \mathfrak{G} \longrightarrow \mathfrak{H}_I$. In its turn, this allows to construct the quotient type coideal C^*-subalgebra $J = I(\mathfrak{H}_I \backslash \mathfrak{G})$. Now, Lemma 6.10 shows that $\mathcal{D}_J \cong UCorep(\mathfrak{H}_I)$, therefore, $\mathcal{D}_J \cong \mathcal{D}_I$. But due to the categorical duality this implies a covariant isomorphism $J \cong I$.

In order to prove the uniqueness of $\mathfrak{H}$, we prove that J determines $Ker(\pi)$. Indeed, as $J = Vec\{U_{\eta,\zeta}\}$, where $U \in UCorep(\mathfrak{G})\}, \eta \in H_U$ and $\zeta \in \pi_*(\hat{L})H_U \subset H_U$, the last subspace is determined by J. This means that for any $U \in UCorep(\mathfrak{G}$, the space $Hom_{UCorep(\mathfrak{H})}(\mathbf{1}, (H_U^\pi))$ is determined by J. The duality morphisms $Hom_{UCorep(\mathfrak{G})}(\mathbf{1}, U \otimes$

$\overline{V}) = Hom_{UCorep(\mathfrak{G})}(V,U)$ $(U,V \in UCorep(\mathfrak{G}))$ show that the same is true for all subspaces
$Hom_{UCorep(\mathfrak{H}}(H_V^\pi), H_U^\pi) \subset B(H_V, H_U)$. Finally, an operator from $B(H_U)$ belongs to $Ker(\pi)$ if and only if all matrix coefficients corresponding to the commutant of the set $End_{UCorep(\mathfrak{H}}(H_U^\pi)$ in $B(H_U)$ equal to 0. ∎

6.17. Remark. *In [Vainerman-Vallin,2020] we introduced the notion of a* weak *coideal I of B, the difference of which from a coideal C^*-subalgebra is that 1_I is not necessarily equal to 1_B. One can show that if I is invariant and $1_I \in Z(B)$, then $(I,\Delta) \in YD_{brc}(\mathfrak{G})$. Then the same reasoning as in the proof of Theorem 1.2 shows that I is isomorphic, as a $\mathfrak{G}$-C^*-algebra, to a unique quotient type coideal C^*-subalgebra.*

7. Example: the Tambara - Yamagami case

7.1. Remark. *In this example $\mathcal{C}$ is not only C^*-multitensor, but a rigid finite C^*-tensor category. Let $\mathcal{F} : \mathcal{C} \longrightarrow Corr_f(R)$ be a unitary tensor functor, where R is a finite dimensional unital C^*-algebra. Then it was shown in [Szlachanyi,2001] that the WHA reconstructed from the pair $(\mathcal{C},\mathcal{F})$ as in Theorem 3.1 is biconnected: $B_t \cap B_s = \mathbb{C} = B_t \cap Z(B)$. Moreover, Hayashi [Hayashi,1999] proved that for any given $\mathcal{C}$, the class of C^*-algebras R for which such a functor $\mathcal{F}$ exists, contains at least $R = \mathbb{C}^{|\Omega|}$ (where $\Omega = Irr(\mathcal{C})$). He also constructed the corresponding particular functor $\mathcal{H}$. In general, this class of C^*-algebras R contains several elements, and the corresponding WHAs are called Morita equivalent. In particular, if this class contains $R = \mathbb{C}$, the corresponding WHAs are Morita equivalent to a usual C^*-Hopf algebra.*

7.2. Reconstruction for Tambara-Yamagami categories. The description of the Hayashi's functor for Tambara-Yamagami categories and the corresponding WHA's was first obtained in [Mevel,2010]. Below we follow [Vainerman-Vallin,2020], 2.3 and 4.1, where one can find more details.

Given a finite abelian group G, a non degenerate symmetric bicharacter χ on it and a number $\tau = \pm|G|^{-1/2}$, one can define a fusion category denoted by $\mathcal{TY}(G,\chi,\tau)$ [Tambara-Yamagami,1998]. Its set of simple objects is $\Omega = G \sqcup \{m\}$ (m is a separate element), its Grothendieck ring is isomorphic to the $\mathbb{Z}_2$-graded fusion ring $\mathcal{TY}_G = \mathbb{Z}G \oplus \mathbb{Z}\{m\}$ such that $g \cdot m = m \cdot g = m$, $m^2 = \sum\limits_{g\in G} g$, $g^* = g^{-1}$, $m = m^*$. The associativities $a_{U,V,W} : (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)$ are

$$a_{g,h,k} = id_{g+h+k}, \quad a_{g,h,m} = id_m, \quad a_{m,g,h} = id_m,$$

$$a_{g,m,g} = \chi(g,h) id_m, \quad a_{g,m,m} = \bigoplus_{h\in G} id_h, \quad a_{m,m,g} = \bigoplus_{h\in G} id_h,$$

$$a_{m,g,m} = \bigoplus_{h\in G} \chi(g,h) id_h, \quad a_{m,m,m} = (\tau\chi(g,h)^{-1} id_m)_{g,h},$$

where $g,h,k \in G$. The unit isomorphisms are trivial. $\mathcal{TY}(G,\chi,\tau)$ becomes a C^*-tensor category when $\chi : G \times G \longrightarrow T = \{z \in \mathbb{C} | |z| = 1\}$, from now on we assume that this is the

case. The dual objects are: $g^* = -g$, for all $g \in G$, and $m^* = m$. The rigidity morphisms are defined by $R_g : 0 \xrightarrow{id_0} g^* \otimes g$, $\overline{R_g} : 0 \xrightarrow{id_0} g \otimes g^*$, $R_m = \tau |G|^{1/2} \iota$, and $\overline{R_m} = |G|^{1/2} \iota$, where $\iota : 0 \longrightarrow m \otimes m$ is the inclusion. Then $dim_q(g) = 1$, for all $g \in G$, and $dim_q(m) = \sqrt{|G|}$.

Using now the Hayashi's functor $\mathcal{H} : \mathcal{TY}(G, \chi, \tau) \longrightarrow Corr_f(R)$, where $R \cong \mathbb{C}^{|G|+1}$ (see [Mevel,2010], [Vainerman-Vallin,2020]), one can apply Theorem 3.1 in order to construct a biconnected regular WHA $\mathfrak{G}_{\mathcal{TY}} = (B, \Delta, S, \varepsilon)$ with $UCorep(\mathfrak{G}_{\mathcal{TY}}) \cong \mathcal{TY}(G, \chi, \tau)$ as C^*-tensor categories. It happens that $\mathfrak{G}_{\mathcal{TY}}$ is selfdual.

Denoting $\Omega_g = \Omega := G \sqcup \{m\}$ and $\Omega_m := G \sqcup \overline{G}$, where $g \in G$ and $\overline{G}$ is the second copy of G, one computes that $H^g \cong \mathbb{C}^{|G|+1}$, for all $g \in G$, and $H^m :\cong \mathbb{C}^{2|G|}$. Let us fix a basis $\{v^x_y\}(y \in \Omega_x)$ in each H^x $(x \in \Omega)$ choosing a norm one vector in every 1-dimensional vector subspace: $v^g_h \in Hom(h, (h-g) \otimes g)$, $v^g_m \in Hom(m, m \otimes g)$, $v^m_g \in Hom(m, g \otimes m)$, and $v^m_{\overline{g}} \in Hom(g, m \otimes m)$, where $g \in G$. Now the whole WHA structure of $\mathfrak{G}_{\mathcal{TY}} = (B, \Delta, S, \varepsilon)$ is given by formulas (7), (8), (9), (11) and (12). In particular, the C^*-algebra $B = \bigoplus_{x \in \Omega} H^x \otimes \overline{H^x}$ has a canonical basis $\{f^x_{\alpha,\beta} = v^x_\alpha \otimes \overline{v^x_\beta}\}_{x \in \Omega, \alpha, \beta \in \Omega_x}$.

For all $x, y \in \Omega$ and all $v \in H^x$, $w \in H^y$, denote $v \circ w = \mathcal{J}_{x,y}(v \otimes_R w)$. Then for all $\alpha, \beta \in \Omega_x, \gamma, \delta \in \Omega_y$, one has:

$$f^x_{\alpha,\beta} f^y_{\gamma,\delta} = (v^x_\alpha \circ v^y_\gamma) \otimes \overline{(v^x_\beta \circ v^y_\delta)},$$

where computations made in [Mevel,2010] 2.1.5, give, for all $g, h, k \in G$:

$$v^g_k \circ v^h_x = \delta_{x,h+k} v^{g+h}_{h+k},\ v^g_m \circ v^h_x = \delta_{x,m} v^{g+h}_m,$$

$$v^m_k \circ v^g_x = \delta_{x,m} \chi(g,k) v^m_k,\ v^m_{\overline{k}} \circ v^g_x = \delta_{x,g+k} v^m_{\overline{g+k}},$$

$$v^g_x \circ v^m_{\overline{k}} = \delta_{x,m} \chi(g,k) v^m_{\overline{k}},\ v^g_x \circ v^m_k = \delta_{x,k} v^m_{k-g},$$

$$v^m_h \circ v^m_{\overline{k}} = v^{k-h}_k,\ v^m_{\overline{h}} \circ v^m_k = \delta_{h,k} \tau \sum_{g \in G} \chi(g,h)^{-1} v^g_m.$$

The coproduct and the counit are defined, respectively, by

$$\Delta(f^x_{\alpha,\beta}) = \sum_{\alpha', \beta' \in \Omega_x} f^x_{\alpha,\alpha'} \otimes f^x_{\beta',\beta}$$

and $\varepsilon(f^x_{\alpha,\beta}) = \delta_{\alpha,\beta}$. The antipode and the involution are as follows:

$$S(f^g_{h,k}) = f^{-g}_{k-g,h-g},\ S(f^g_{h,m}) = f^{-g}_{m,h-g},\ S(f^g_{m,h}) = f^{-g}_{h-g,m}, \tag{31}$$

$$S(f^g_{m,m}) = f^{-g}_{m,m},\ S(f^m_{g,h}) = f^m_{\overline{h},\overline{g}},\ S(f^m_{g,\overline{h}}) = \tau^{-1} f^m_{h,\overline{g}}, \tag{32}$$
$$S(f^m_{\overline{g},h}) = \tau f^m_{\overline{h},g},\ S(f^m_{\overline{g},\overline{h}}) = f^m_{h,g}$$

and:

$$(f^g_{h,k})^* = f^{-g}_{h-g,k-g},\ (f^g_{h,m})^* = f^{-g}_{h-g,m},\ (f^g_{m,h})^* = f^{-g}_{m,h-g},$$

$$(f^g_{m,m})^* = f^{-g}_{m,m},\ (f^m_{g,h})^* = f^m_{\overline{g},\overline{h}},\ (f^m_{g,\overline{h}})^* = \tau f^m_{\overline{g},h},$$
$$(f^m_{\overline{g},h})^* = \tau^{-1} f^m_{g,\overline{h}},\ (f^m_{\overline{g},\overline{h}})^* = f^m_{g,h}.$$

Recall that H^0 is a commutative C^*-algebra isomorphic to $R \sim \mathbb{C}^\Omega$.

7.3. Remark. *Since $\mathfrak{G}_{\mathcal{TY}}$ is selfdual, we also have $B = \bigoplus_{g\in G} B^g \oplus B^m$, where $B^g \cong M_{|G|+1}(\mathbb{C})$, $\forall g \in G$, $B^m \cong M_{2|G|}(\mathbb{C})$ (see [Mevel,2010], 2.1). Using the basis $\{f^x_{y,z}\}$ and the matrix units $\{e^x_{y,z}\}$ of B with respect to the basis $\{v^x_t\}$ of H^x, any irreducible corepresentation U^x $(x \in \Omega)$ of $\mathfrak{G}_{\mathcal{TY}}$ can be written as*

$$U^x = \sum_{y,z\in\Omega_x} e^x_{y,z} \otimes f^x_{y,z}.$$

7.4. Quantum subgroupoids and quotient type coideal C^*-subalgebras.

7.5. Remark. *The lattice $\underline{Subgrp(G)}$ of subgroups of G with operations $\wedge = \cap$ and $\vee = +$ can be extended to $\overline{Subgrp(G)} := Subgrp(G) \sqcup \{\Omega\}$, where $\Omega = G \sqcup \{m\}$, by putting $L \cap \Omega = \Omega \cap L = L$ and $L \vee \Omega = \Omega \vee L = \Omega \cap \Omega = \Omega \vee \Omega = \Omega$, for any subgroup L of G. Any rigid tensor C^*-subcategory of $\mathcal{TY}(G,\chi,\tau)$ is equivalent either to Vec_L - the category of finite dimensional L-graded vector spaces $(L < G)$ or to $\mathcal{TY}(G,\chi,\tau)$. Let $\mathcal{C}^x$ $(x \in \overline{Subgrp(G)})$ be a representative in such equivalence class of subcategories, in particular, $\mathcal{C}^\Omega = \mathcal{TY}(G,\chi,\tau)$.*

In order to construct all quotient type coideal C^*-subalgebras of $\mathfrak{G}_{\mathcal{TY}}$, first construct all its quantum subgroupoids (up to isomorphism). Theorems 3.1 and 3.2 imply that any quantum subgroupoid of $\mathfrak{G}_{\mathcal{TY}}$ is isomorphic to one of the quantum subgroupoids $(\mathfrak{G}^x, \pi^x)$ such that $URep(\mathfrak{G}^x) \cong \mathcal{C}^x$, where $x \in \overline{Subgrp(G)}$. Define $(\mathfrak{G}^\Omega, \pi^\Omega) = (\mathfrak{G}_{\mathcal{TY}}, id)$ and, for any $L < G$, $(\mathfrak{G}^L, \pi^L)$ as follows:

7.6. Lemma. *If $e^x_{y,z}$ $(x \in \Omega, y, z \in \Omega_x)$ are the matrix units of B (see Remark 7.3), $l \in L, g, h \in G$, the collection $(B_L, \Delta_L, S_L, \varepsilon_L)$, where $B_L = \bigoplus_{l\in L} B(H^l)$,*

$$\Delta_L(e^l_{g,g'}) = \sum_{\substack{l_1,l_2\in L\\ l_1+l_2=l}} e^{l_1}_{g-l_2,g'-l_2} \otimes e^{l_2}_{g,g'},\ \Delta_L(e^l_{g,m}) = \sum_{\substack{l_1,l_2\in L\\ l_1+l_2=l}} e^{l_1}_{g-l_2,m} \otimes e^{l_2}_{g,m},$$

$$\Delta_L(e^l_{m,g'}) = \sum_{\substack{l_1,l_2\in L\\ l_1+l_2=l}} e^{l_1}_{m,g'-l_2} \otimes e^{l_2}_{m,g'},\ \Delta_L(e^l_{m,m}) = \sum_{\substack{l_1,l_2\in L\\ l_1+l_2=l}} e^{l_1}_{m,m} \otimes e^{l_2}_{m,m},$$

$$S_L(e^l_{g,g'}) = e^{-l}_{g'-l,g-l}, \qquad S_L(e^l_{g,m}) = e^{-l}_{m,g-l},$$
$$S_L(e^l_{m,g'}) = e^{-l}_{g'-l,m}, \qquad S_L(e^l_{m,m}) = e^{-l}_{m,m},$$
$$\varepsilon_L(e^l_{g,g'}) = \varepsilon_L(e^l_{m,g'}) = \varepsilon_L(e^l_{g,m'}) = \varepsilon_L(e^l_{m,m}) = \delta_{l,0},$$

defines a WHA $\mathfrak{G}^L$. The canonical projection $\pi_L : B \longrightarrow B_L$ defined, for all $x \in \Omega, \alpha, \beta \in \Omega_x$, by $\pi_L(e^x_{\alpha,\beta}) = \delta_{x,L} e^x_{\alpha,\beta}$, where $\delta_{x,L} = 1$ if $x \in L$ and $= 0$ otherwise, gives to $\mathfrak{G}^L$ the structure of a quantum subgroupoid of $\mathfrak{G}_{TY}$.

Proof. Straightforward computations. ∎

7.7. Corollary. *A linear basis for $(B_L)_t$ (resp., $(B_L)_s$) is given by $(e(L)^\alpha)_{\alpha\in\Omega}$ (resp., $(e(L)_\alpha)_{\alpha\in\Omega}$), where, for all $g\in G$, one has: $e(L)^g=\sum_{l\in L}e^l_{g+l,g+l}$, $e(L)_g=\sum_{l\in L}e^l_{g,g}$ and $e(L)^m=e(L)_m=\sum_{l\in L}e^l_{m,m}$.*

The counital maps are given by:

$$\epsilon_t^{B_L}(e^l_{x,y})=\delta_{l,0}e(L)^x,\quad \epsilon_s^{B_L}(e^l_{x,y})=\delta_{l,0}e(L)_y.$$

A linear basis for $(B_L)_s\cap(B_L)_t$ is given by $(z_\beta)_{\beta\in G/L\sqcup\{m\}}$, where, for all $\beta\in G/L$, one has: $z_\beta=\sum_{l\in L,g,\in\beta}e^l_{g,g}$ and $z_m=e(L)^m=e(L)_m=\sum_{l\in L}e^l_{m,m}$. Moreover, $(B_L)_s\cap Z(B_L)=\mathbb{C}$, so $\mathfrak{G}^L$ is connected and not coconnected.

7.8. Remark. *Any $\mathfrak{G}^L$ is Morita equivalent to a commutative and cocommutative Hopf C^*-algebra generated by the group L.*

7.9. Proposition. *Denote $I^x:=I(\mathfrak{G}^x\backslash\mathfrak{G})$. Then $I^\Omega=B_s$ and, for any subgroup L of G, setting $v^0_Y:=\sum_{y\in Y}v^0_y$, where $Y\in G/L$, one has:*

$$I^L=Vec<v^0_Y,Y\in G/L>\otimes\overline{H^0}\bigoplus_{l\in L^\perp}v^l_m\otimes\overline{H^l}.$$

Proof. We will use Lemma 6.10. For all $\hat{c}\in\hat{\mathfrak{G}}^L,\beta\in\Omega_\alpha$, one has:

$$(\pi_L)_*(\hat{c})v^\alpha_\beta=\sum_{i,j\in\Omega_\alpha}<f^\alpha_{i,j},(\pi_L)_*(\hat{c})>e^\alpha_{i,j}v^\alpha_\beta=\sum_{i\in\Omega_\alpha}<\pi_L(f^\alpha_{i,\beta}),\hat{c}>v^\alpha_i \tag{33}$$

A linear form ϕ on B_L is a left integral if and only if $(i\otimes\phi)\Delta_L(e^l_{x,y})$ is in $(B_L)_t$, for all $l\in L,x,y\in\Omega$. Then Lemma 7.6 implies that $\Lambda=Vec<\lambda_x,x\in\Omega>$.

For all $\alpha,\gamma\in\Omega,\beta\in\Omega_\alpha$, using (33), one has:

$$(\pi_L)_*(\lambda_\gamma)v^\alpha_\beta=\sum_{i\in\Omega_\alpha,l\in L}<\pi_L(f^\alpha_{i,\beta}),(e^l_{\gamma,\gamma})^*>v^\alpha_i.$$

This gives, in particular:

$$(\pi_L)_*(\lambda_\gamma)v^p_k=\delta_{p,0}\delta_{\gamma,-k}\sum_{l\in L}v^0_{k+l},$$

$$(\pi_L)_*(\lambda_\gamma)v^p_m=\delta_{\gamma,m}\delta_{p,L^\perp}|L|v^p_m$$

And also:

$$(\pi_L)_*(\lambda_\gamma)v^m_k=\sum_{h\in G,l\in L}<\pi_L(e^{h-k}_{-k,m}),(e^l_{\gamma,\gamma})^*>v^p_h=0,$$

$$(\pi_L)_*(\lambda_\gamma)v^m_{\overline{k}} = \sum_{h\in G, l\in L} < \pi_L(e^{h-k}_{m,-k}), (e^l_{\gamma,\gamma})^* > v^m_{\overline{h}} = 0.$$

So if one sets: $v^0_X = \sum_{x\in X} v^0_x$, for all $X \subset \Omega$, then:

$$(\pi_L)_*(\Lambda)H^p = \delta_{p,0}(Vec < v^0_Y, v^0_m/Y \in G/L >) + \delta_{p,L^\perp}\mathbb{C}v^p_m \text{ for all } p \in G,$$

$$(\pi_L)_*(\Lambda)H^m = \{0\}.$$

These calculations and Lemma 6.10 give the result. ∎

7.10. The lattice of invariant coideal C^*-subalgebras. In order to precise the relationship between quotient type and invariant coideal C^*-subalgebras and to characterize the lattice of these coideals in the Tambara-Yamagami case, rewrite the definition (30) of $\triangleleft$ using (31) and (32) as follows:

$$(\eta^y \otimes \overline{\xi^y}) \triangleleft (\eta^x \otimes \overline{\xi^x}) = (\sum_{z\in\Omega_x} (v^x_z)^\natural \circ \eta^y \circ v^x_z) \otimes (\eta^x)^\flat \circ \overline{\xi^y} \circ \overline{\xi^x},$$

where $x, y \in \Omega, \eta^x, \xi^x \in H^x, \eta^y, \xi^y \in H^y$. This expression allows to define the map $P^x : \bigoplus_{y\in\Omega} H^y \longrightarrow \bigoplus_{y\in\Omega} H^y$ by putting for any fixed $x, y \in \Omega, \eta^y \in H^x$:

$$P^x(\eta^y) = \sum_{z\in\Omega_x} (v^x_z)^\natural \circ \eta^y \circ v^x_z,$$

and we have:

7.11. Lemma. *A coideal C^*-subalgebra $I = \sum_{y\in\Omega} X^y \otimes \overline{H^y}$ is invariant if and only if $\bigoplus_{y\in\Omega} X^y = P^x(\bigoplus_{y\in\Omega} X^y)$ for all $x \in \Omega$.*

A straightforward calculation of P^x on the basic elements v^x_z proves

7.12. Lemma. *For all $g, k, h \in G$, one has:*

$$P^h(v^g_k) = \delta_{g,0}v^0_{k+h} \ , \ P^m(v^g_k) = \delta_{g,0}sign(\tau)\sum_{p\in G}\chi(p,k)v^p_m$$

$$P^h(v^g_m) = v^g_m \ , \ P^m(v^g_m) = \tau^{-1}|G|^{1/2}\sum_{p\in G}\chi(g,p)v^0_p$$

$$P^h_{|H^m} = P^m_{|H^m} = 0$$

In the Tambara-Yamagami case any invariant coideal C^*-subalgebra is not only isomorphic but is itself of quotient type:

7.13. Proposition. *For any invariant coideal C^*-subalgebra $I = \bigoplus_{y\in\Omega} X^y \otimes \overline{H^y}$ there is a unique quotient type coideal C^*-subalgebra $I^x = I(\mathfrak{G}^x\backslash\mathfrak{G})$ such that $I = I^x$.*

Proof. Due to [Vainerman-Vallin,2020], Lemma 3.3, b) there is a partition $(\Gamma_i)_{i\in I^0}$ of Ω such that $X^0 = Vec < v^0_{\Gamma_i}/i \in I^0 >$. Moreover, putting $K := \{g \in G/Dim(X^g) \neq 0\}$, one has by Lemmas 7.11 and 7.12: $X^m = \{0\}$, and $X^g = \mathbb{C}v^g_m$ for all $g \in K$, $g \neq 0$. With the convention $m + g = m$ for all $g \in G$:, one has by Lemma 7.12: $v^0_{\Gamma_i+g} \in X^0$ for all $i \in I^0$. As a consequence, for all $g \in G$ there is $j \in J$ such that $\Gamma_i + g = \Gamma_j$; this allows only two possibilities:

1) there is a single class $\Gamma_i = \Omega$, so $I = B_s + \bigoplus_{g\in K\setminus\{0\}} \mathbb{C}v^g_m$. But for all $g \in K \setminus \{0\}$ one has $P^m(v^g_m) = \tau^{-1}|G|^{1/2}\sum_{p\in G}\chi(g,p)v^0_p$ which must be collinear to $v^0_\Omega = \sum_{y\in\Omega} v^0_y$. Hence, $K = \{0\}$ and $I = B_s = I^\Omega$.

2) The partition (Γ_i) is $\{m\}$ together with a partition $(\Gamma^G_p)_{p\in P}$ of G. Moreover, for any $p, q \in P$ there is $g \in G$ such that $\Gamma^G_p + g = \Gamma^G_q$. For any $p \in P$ denote $L_p = \{g \in G/\Gamma^G_p + g = \Gamma^G_p\}$, then L_p is a subgroup of G. But since for all $q \in P$, there is $h \in G$ such that $\Gamma_q = \Gamma_p + h$, the group L_p does not depend on $p \in P$, denote it by L. Let us show that $\Gamma^G_p \in G/L$ for any $p \in P$.

If $h \in \Gamma^G_p - \Gamma^G_p$, then $(\Gamma^G_p + h) \cap \Gamma^G_p \neq \emptyset$, so $\Gamma^G_p + h = \Gamma^G_p$. Hence, $K = \Gamma^G_p - \Gamma^G_p$. For all $p \in P$, let z be in Γ^G_p, obviously one has $z + L \subset \Gamma^G_p$, let $t \in \Gamma^G_p$, then $t - z \in L = \Gamma^G_p - \Gamma^G_p$ hence $t \in z + L$, as a consequence $z + L = \Gamma^G_p$ so $\Gamma^G_p \in G/L$, as we deal with a partition of G: $\{\Gamma^G_p/p \in P\} = G/L$. So we have: $X^0 = Vec < (v^0_Y)_{Y\in G/L}, v^0_m >$.

If now $g \in K$, then due to Lemma 7.12 one must have $\sum_{h\in G}\chi(g,h)v^0_h \in X^0$, but: $\sum_{h\in G}\chi(g,h)v^0_h = \sum_{p\in G/L}\sum_{h\in p}\chi(g,h)v^0_h$, so this element has to belong to X^0 and must be of the form $\sum_{p\in G/K}\mu_p v^0_p$, i.e., for all $p \in G/K$ and all $h \in p$, one has $\mu_p = \chi(g,h)$, which means that $K \subset L^\perp$. Conversely, by Lemma 7.12, one must have $\sum_{k\in K,p\in G}\chi(p,k)v^p_m \in \bigoplus_{k\in G} A^k$, but on the other hand:

$$\sum_{k\in K}\sum_{p\in G}\chi(p,k)v^p_m = \sum_{p\in G}(\sum_{k\in K}\chi(k,p))v^p_m = |K|\sum_{p\in K^\perp} v^p_m$$

So $L^\perp \subset K$ and in case 2) we have $I = I^L$. ∎

7.14. Corollary. *In $\mathfrak{G}_{\mathcal{TY}}(G,\chi,\tau)$, the sets of quotient type and invariant coideal C^*-subalgebras coincide and are in bijection with $\overline{Subgrp(G)}$.*

Proof. By 7.13 any invariant coideal is quotient type, conversely any quotient type coideal is invariant by lemma 6.16, moreover the set $\{I^x, x \in \overline{Subgrp(G)}\}$ contains all invariant coideals and is included in the set of quotient type coideals. ∎

Finally, we describe the lattice of invariant (or quotient type) coideal C^*-subalgebras.

7.15. Proposition. *The map: $x \mapsto I^x$ is an anti-isomorphism of the lattices $\overline{Subgr(G)}$ and $Invl(B)$.*

Proof. For any subgroup K of G, since $I^K = Vec < v_Y^0, Y \in G/K > \otimes \overline{H^0} \underset{k\in K^\perp}{\oplus} v_m^k \otimes \overline{H^k}$, one sees that the map $x \mapsto I^x$ is decreasing. Hence, for all $L, K < G$, one has $I^{L+K} \subset I^L \cap I^K$. Conversely, for all $z \in I^L \cap I^K$, there exist some families of complex numbers $(\lambda_u), (\mu_v)$ and non zero vectors $(\xi_w^0), (\eta_m^g)$ such that:

$$z = \sum_{Y\in G/L} \lambda_Y v_Y^0 \otimes \overline{\xi_Y^0} + \sum_{l\in L^\perp} \lambda_l v_m^l \otimes \overline{\eta_m^l} = \sum_{Z\in G/K} \mu_Z v_Z^0 \otimes \overline{\xi_Z^0} + \sum_{k\in K^\perp} \mu_k v_m^k \otimes \overline{\eta_m^k}.$$

This gives:

$$\sum_{g\in G} \lambda_{\dot g} v_g^0 \otimes \overline{\xi_{\dot g}^0} + \sum_{l\in L^\perp} \lambda_l v_m^l \otimes \overline{\eta_m^l} = \sum_{g\in G} \lambda_{\ddot g} v_g^0 \otimes \overline{\xi_{\ddot g}^0} + \sum_{k\in K^\perp} \lambda_k v_m^k \otimes \overline{\eta_m^k},$$

where $\dot g$ (resp., $\ddot g$) is the class of g in G/L (resp., in G/K). As a consequence, one has:

1) if $l \in L^\perp$ and $l \notin K^\perp$, then $\lambda_l = 0$.

2) for any $g \in G$ the following equality holds: $\lambda_{\dot g}\overline{\xi_{\dot g}^0} = \lambda_{\ddot g}\overline{\xi_{\ddot g}^0}$.

Condition 2) implies that for all $g \in G, p \in L + K$, one has $\lambda_{\dot g}\overline{\xi_{\dot g}^0} = \lambda_{\underline{\cdot}_{g+p}}\overline{\xi^0_{\underline{\cdot}_{g+p}}}$. Hence, for any $Y \in G/(L + K)$ we can define $\overline{\xi_Y^0}$ such that for all $g \in Y$ one has: $\lambda_{\dot g}\overline{\xi_{\dot g}^0} = \overline{\xi_Y^0}$. Then, using the fact that $L^\perp \cap K^\perp = (L + K)^\perp$ (see [Hewitt-Ross,1963], (23),(29)(b) on p.369), one has:

$$\begin{aligned} z &= \sum_{g\in G} \lambda_{\dot g} v_g^0 \otimes \overline{\xi_{\dot g}^0} + \sum_{l\in L^\perp\cap K^\perp} \lambda_l v_m^l \otimes \overline{\eta_m^l} \\ &= \sum_{Y\in G/(L+K)} \sum_{g\in Y} v_g^0 \otimes \overline{\xi_Y^0} + \sum_{l\in (L+K)^\perp} \lambda_l v_m^l \otimes \overline{\eta_m^l} \\ &= \sum_{Y\in G/(L+K)} v_Y^0 \otimes \overline{\xi_Y^0} + \sum_{l\in (L+K)^\perp} \lambda_l v_m^l \otimes \overline{\eta_m^l} \end{aligned}$$

Hence, $z \in I^{L+K}$, which proves that $I^{L+K} = I^L \cap I^K$.

Obviously, $I^L \vee I^K \subset I^{L\cap K}$. Conversely, since $(L \cap K)^\perp = L^\perp + K^\perp$, for any $p \in (L \cap K)^\perp$ there exist $l \in L^\perp$ and $k \in K^\perp$ such that $p = k + l$. Then $v_m^p \otimes \overline{H^p} = (v_m^l \otimes \overline{H^l})(v_m^k \otimes \overline{H^k})$, so it belongs to $I^L \vee I^K$. For all $g \in G$ one has:

$$v_{g+L}^0 \circ v_{g+K}^0 = v_{g+L\cap K}^0,$$

hence, $v_{g+L\cap K}^0 \otimes \overline{H^0}$ belongs to $I^L \vee I^K$. All basic elements of $I^{L\cap K}$ are in $I^L \vee I^K$ which gives the converse inclusion $I^{L\cap K} \subset I^L \vee I^K$, and the result follows. ∎

References

Saad Baaj and Georges Skandalis. C^*-algèbres de Hopf et théorie de Kasparov équivariante. *K-Theory*, 2(6):683–721, 1989.

Gabriella Böhm, Florian Nill, and Kornél Szlachányi. Weak Hopf algebras. I. Integral theory and C^*-structure. *J. Algebra*, 221(2):385–438, 1999.

Gabriella Böhm and Kornél Szlachányi. Weak Hopf algebras. II. Representation theory, dimensions, and the Markov trace. *J. Algebra*, 233(1):156–212, 2000.

Damien Calaque and Pavel Etingof. Lectures on tensor categories. In *Quantum groups*, volume 12 of *IRMA Lect. Math. Theor. Phys.*, pages 1–38. Eur. Math. Soc., Zürich, 2008.

Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

T. Hayashi. A canonical Tannaka duality for semi finite tensor categories. Preprint, math.QA/9904073, 1999.

E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory. Group representations.*, volume 115. Springer, Berlin, 1963.

C.Mevel. Exemples et applications des groupoides quantiques finis, https://tel.archives-ouvertes.fr/tel-00498884/document. Thèse, Université de Caen, 2010.

Adriana Nenciu. The center construction for weak Hopf algebras. *Tsukuba J. Math.*, 26(1):189–204, 2002.

Sergey Neshveyev. Duality theory for nonergodic actions. *Münster J. Math.*, 7(2):413–437, 2014.

Sergey Neshveyev and Lars Tuset. Hopf algebra equivariant cyclic cohomology, K-theory and index formulas. *K-Theory*, 31(4):357–378, 2004.

Sergey Neshveyev and Lars Tuset. *Compact quantum groups and their representation categories*, volume 20 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2013.

Sergey Neshveyev and Makoto Yamashita. Categorical duality for Yetter-Drinfeld algebras. *Doc. Math.*, 19:1105–1139, 2014.

Dmitri Nikshych. On the structure of weak Hopf algebras. *Adv. Math.*, 170(2):257–286, 2002.

Dmitri Nikshych, Vladimir Turaev, and Leonid Vainerman. Invariants of knots and 3-manifolds from quantum groupoids. In *Proceedings of the Pacific Institute for the Mathematical Sciences Workshop "Invariants of Three-Manifolds" (Calgary, AB, 1999)*, volume 127 1-2, pages 91–123, 2003.

Dmitri Nikshych and Leonid Vainerman. Algebraic versions of a finite-dimensional quantum groupoid. In *Hopf algebras and quantum groups (Brussels, 1998)*, volume 209 of *Lecture Notes in Pure and Appl. Math.*, pages 189–220. Dekker, New York, 2000.

Dmitri Nikshych and Leonid Vainerman. A Galois correspondence for II_1 factors and quantum groupoids. *J. Funct. Anal.*, 178(1):113–142, 2000.

Dmitri Nikshych and Leonid Vainerman. Finite quantum groupoids and their applications. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 211–262. Cambridge Univ. Press, Cambridge, 2002.

Florian Nill. Axioms for weak bialgebras. Preprint,arXiv: math/9805104 [math.QA], 1998.

Victor Ostrik. Module categories, weak Hopf algebras and modular invariants. *Transform. Groups*, 8(2):177–206, 2003.

Hendryk Pfeiffer. Finitely semisimple spherical categories and modular categories are self-dual. *Adv. Math.*, 221(5):1608–1652, 2009.

Hendryk Pfeiffer. Tannaka-Kreĭn reconstruction and a characterization of modular tensor categories. *J. Algebra*, 321(12):3714–3763, 2009.

Kornel Szlachányi. Finite quantum groupoids and inclusions of finite type. *Fields Inst.Commun.*, 30:314–343. AMS Providence, RI, 2001.

Mitsuhiro Takeuchi. Quotient spaces for Hopf algebras. *Comm. Algebra*, 22(7):2503–2523, 1994.

Daisuke Tambara and Shigeru Yamagami. Tensor categories with fusion rules of self-duality for finite abelian groups. *J. Algebra*, 209(2):692–707, 1998.

Leonid Vainerman and Jean-Michel Vallin. Tannaka-Krein reconstruction for coactions of finite quantum groupoids. *Methods Funct. Anal. Topology*, 23(1):76–107, 2017.

Leonid Vainerman and Jean-Michel Vallin. Classifying (weak) coideal subalgebras of weak Hopf C*-algebras. *J. Algebra*, 550:333–357, 2020.

Jean-Michel Vallin. Groupoïdes quantiques finis. *J. Algebra*, 239(1):215–261, 2001.

J.-M. Vallin. Deformation of finite dimensional C^*-quantum groupoids. Preprint,arXiv: math/0310265 [math.QA], 2003.

Jean-Michel Vallin. Multiplicative partial isometries and finite quantum groupoids. In *Locally compact quantum groups and groupoids (Strasbourg, 2002)*, volume 2 of *IRMA Lect. Math. Theor. Phys.*, pages 189–227. de Gruyter, Berlin, 2003.

Mishihisa Wakui. Reconstruction of weak bialgebra maps and its applications. Preprint, arXiv:2002.12568v1 [mathRA], 2020.

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