Quantum limits of products of Heisenberg manifolds

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Abstract

We study the spectral theory of a family of sub-Laplacians, defined on products of compact quotients of the Heisenberg group, which are examples of completely integrable sub-Riemannian manifolds. We classify all Quantum Limits of these sub-Laplacians, expressing them through a disintegration of measure result. This disintegration follows from a natural spectral decomposition of the sub-Laplacian in which harmonic oscillators appear.

Our results illustrate the fact that, because of the possibly high degeneracy of the spectrum, the spectral theory of general sub-Riemannian (or subelliptic) Laplacians can be very rich: the invariance properties of the Quantum Limits which we study are related to the classical dynamics of infinitely many vector fields on the cotangent bundle of the manifold. These phenomena contrast with what happens for Riemannian Laplacians, for which any Quantum Limit is simply invariant under the geodesic flow.

Contents

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1 Introduction and main results

1.1 Motivation

The main goal of this paper is to establish some properties of the eigenfunctions of a family of hypoelliptic operators in the high-frequency limit. A typical problem is the description of the Quantum Limits (QL) of the operator, i.e., the measures which are weak limits of a subsequence of squares of eigenfunctions.

The study of Quantum Limits for hypoelliptic operators started recently, with the paper [CdVHT18]. The authors proved Weyl laws (i.e., results "in average" on eigenfunctions), a result of decomposition of Quantum Limits, and also Quantum Ergodicity properties (i.e., equidistribution of Quantum Limits under an ergodicity assumption) for 3D contact sub-Laplacians.

We briefly recall the general definition of a sub-Laplacian. Let $n \in \mathbb{N}^*$ and let M be a smooth connected compact manifold of dimension n without boundary. We consider a smooth vector distribution $\mathcal{D}$ on M (possibly with non-constant rank), and a Riemannian metric g on $\mathcal{D}$. We also assume that $\mathcal{D}$ satisfies the Hörmander condition

$$\mathrm{Lie}(\mathcal{D}) = TM \tag{1}$$

(see [Mon02]). Let μ be a smooth volume form on M and let $\Delta_{g,\mu}$ be the selfadjoint sub-Laplacian associated with the metric g and with the volume form μ. If $\mathcal{D}$ is locally spanned by N vector fields $X_1, \dots, X_N$ that are g-orthonormal, then we set

$$\Delta_{g,\mu} = -\sum_{i=1}^{N} X_i^* X_i = \sum_{i=1}^{N} (X_i^2 + \mathrm{div}_\mu(X_i) X_i)$$

where the star designates the transpose in $L^2(M,\mu)$. This definition does not depend on the choice of the g-orthonormal frame $X_1, \dots, X_N$. We can also note that if $\mathcal{D} = TM$, g is a Riemannian metric on TM and μ is the canonical volume on (M,g), then $\Delta_{g,\mu}$ is the usual Laplace-Beltrami operator.

Under the assumption (1), $\Delta_{g,\mu}$ is hypoelliptic (see [Hör67]), has a compact resolvent, and there exists a sequence of (real-valued) eigenfunctions $(\varphi_k)_{k\in\mathbb{N}^*}$ of $-\Delta_{g,\mu}$ associated to the eigenvalues in increasing order $0 = \lambda_1 < \lambda_2 \leq ...$ (with $\lambda_k \to +\infty$ as $k \to +\infty$) which is orthonormal for the $L^2(M,\mu)$ scalar product. The main purpose of this paper is to understand the behaviour of the probability measure $|\varphi_k|^2 d\mu$ when $k \to +\infty$ for a particular family of sub-Laplacians, typically by describing its weak limits (in the sense of duality with continuous functions).

There is a phase-space extension of these weak limits whose behaviour is also of interest. Let us recall the following definition (see [Gér91b]):

Definition 1. *Let $(u_k)_{k\in\mathbb{N}^*}$ be a bounded sequence in $L^2(M)$ and weakly converging to 0. We call microlocal defect measure of $(u_k)_{k\in\mathbb{N}^*}$ any Radon measure ν on S^*M such that for any $a \in \mathscr{S}^0(M)$, there holds*

$$(Op(a)u_{\sigma(k)}, u_{\sigma(k)}) \underset{k\to+\infty}{\longrightarrow} \int_{S^*M} a d\nu$$

for some extraction σ. Here, $(\cdot,\cdot)$ denotes the $L^2(M,\mu)$ scalar product, $\mathscr{S}^0(M)$ is the space of classical symbols of order 0, and $Op(a)$ is the Weyl quantization of a (see Appendix A).

Microlocal defect measures are useful tools for studying the (asymptotic) concentration and oscillation properties of sequences, and they are necessarily non-negative.

Definition 2. *We call Quantum Limit (QL) associated with an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}^*}$ of eigenfunctions of $-\Delta$ any microlocal defect measure of $(\varphi_k)_{k\in\mathbb{N}^*}$.*

Remark 3. *Since $(\varphi_k)_{k\in\mathbb{N}^*}$ is orthonormal, any of its QLs is a probability measure on S^*M.*

For any Riemannian manifold (M, g), it is well known that any Quantum Limit ν of the Laplace-Beltrami operator Δ_g is invariant under the geodesic flow $\exp(t\vec{H})$: there holds $\exp(t\vec{H})\nu = 0$ for any $t \in \mathbb{R}$. To see it, we note that for any orthonormal basis $(\varphi_k)_{k\in\mathbb{N}^*}$ consisting of eigenfunctions of $-\Delta_g$, there holds

$$(\exp(-it\sqrt{-\Delta_g})\mathrm{Op}(a)\exp(it\sqrt{-\Delta_g})\varphi_k, \varphi_k)_{L^2} = (\mathrm{Op}(a)\varphi_k, \varphi_k)_{L^2}$$

for any $t \in \mathbb{R}$, any $k \in \mathbb{N}^*$ and any classical symbol $a \in \mathscr{S}^0(M)$. It follows from Egorov's theorem that $\exp(-it\sqrt{-\Delta_g})\mathrm{Op}(a)\exp(it\sqrt{-\Delta_g})$ is a pseudodifferential operator of order 0 with principal symbol $a \circ \exp(t\vec{H})$, which in turn implies $\exp(t\vec{H})\nu = 0$. As we will see, such a simple invariance property of Quantum Limits does not hold anymore for general sub-Laplacians $\Delta_{g,\mu}$. Indeed, the above computation does not work anymore since $\sqrt{-\Delta_{g,\mu}}$ is not a pseudodifferential operator near its characteristic manifold, and therefore Egorov's theorem does not apply.

1.2 The sub-Laplacian Δ

A first example of a sub-Laplacian which is not Riemannian can be defined on some appropriate quotient of the 3D Heisenberg group, and its Quantum Limits were studied in [CdVHT18]. Endow $\mathbb{R}^3$ with the product law

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' - xy').$$

With this law, $\tilde{\mathbf{H}} = (\mathbb{R}^3, \star)$ is a Lie group, which is isomorphic to the group of matrices

$$\left\{ \begin{pmatrix} 1 & x & -z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},\ x, y, z \in \mathbb{R} \right\}$$

endowed with the standard product law on matrices.

We consider the left quotient $\mathbf{H} = \Gamma\backslash\tilde{\mathbf{H}}$ where $\Gamma = (\sqrt{2\pi}\mathbb{Z})^2 \times 2\pi\mathbb{Z}$ is a cocompact subgroup of $\tilde{\mathbf{H}}$ (meaning that $\mathbf{H}$ is compact). The vector fields on $\mathbf{H}$

$$X = \partial_x \quad \text{and} \quad Y = \partial_y - x\partial_z$$

are left invariant, and we consider $\Delta_{\mathbf{H}} = X^2 + Y^2$ the associated sub-Laplacian (here μ is the Lebesgue measure $\mu = dxdydz$ and (X, Y) is orthonormal for g).

In this paper, we are interested in the product manifold $\mathbf{H}^m$ and the associated sub-Laplacian Δ for some integer $m \geq 2$, that is

$$\Delta = \Delta_{\mathbf{H}} \otimes (\mathrm{Id})^{\otimes m-1} + \mathrm{Id} \otimes \Delta_{\mathbf{H}} \otimes (\mathrm{Id})^{m-2} + \ldots + (\mathrm{Id})^{\otimes m-1} \otimes \Delta_{\mathbf{H}}, \tag{2}$$

which is a second-order pseudodifferential operator. Below, we give an expression (4) for Δ which is more tractable. In all the sequel, we fix once for all an integer $m \geq 2$. Also, in what follows, Δ denotes this sub-Laplacian, while $\Delta_{g,\mu}$ denotes any arbitrary sub-Laplacian.

Remark 4. *If $(\varphi_k)_{k\in\mathbb{N}^*}$ denotes an orthonormal Hilbert basis of $L^2(\mathbf{H})$ consisting of eigenfunctions of $-\Delta_{\mathbf{H}}$, then*

$$\{\varphi_{k_1} \otimes \ldots \otimes \varphi_{k_m} \ \mid \ k_1, \ldots, k_m \in \mathbb{N}^*\}$$

is an orthonormal Hilbert basis of $L^2(\mathbf{H}^m)$ consisting of eigenfunctions of $-\Delta$. However, there exist orthonormal Hilbert bases of $L^2(\mathbf{H}^m)$ which cannot be put in this tensorized form.

The structure and the invariance properties of the Quantum Limits of sub-Laplacians is more complicated than that of Riemannian Laplacians (recalled above), and it is important to note in particular that the Quantum Limits of sub-Laplacians are not necessarily invariant under the (sub-Riemannian) geodesic flow. In [CdVHT18, Theorem B], it was proved that for any sub-Laplacian $\Delta_{g,\mu}$, any of its Quantum Limit ν may be decomposed as a sum

$\nu = \nu_0 + \nu_\infty$ of mutually singular measures, where ν_0 is supported in the "elliptic part" of the principal symbol $g^* = \sigma_P(-\Delta_{g,\mu})$ and is invariant under the sub-Riemannian geodesic flow $\vec{g}^*$, and ν_∞ is supported in $(g^*)^{-1}(0)$ (and its invariance properties are far more difficult to establish, as will be seen below). It was also proved that for "most" QLs, $\nu_0 = 0$, and therefore most our efforts in this paper are devoted to understand ν_∞. The precise statement of [CdVHT18, Theorem B] is recalled in Proposition 1 below.

Let us introduce a few notations. If $\Delta_{g,\mu}$ is a sub-Laplacian, we set $g^* = \sigma_P(-\Delta_{g,\mu})$ where σ_P denotes the principal symbol (see Appendix A), and we denote by $\Sigma = (g^*)^{-1}(0) = \mathcal{D}^\perp \subset T^*M$ the characteristic cone (where $\perp$ is in the sense of duality). This is the region of the phase-space where $\Delta_{g,\mu}$ is not elliptic: in some sense, it is the region which is of most interest in the study of sub-Laplacians, in contrast with usual Riemannian Laplacians. We make the identification

$$S^*M = U^*M \cup S\Sigma \tag{3}$$

where S^*M is the cosphere bundle (i.e., the sphere bundle of T^*M), $U^*M = \{g^* = 1\}$ is a cylinder bundle and $S\Sigma$ is a sphere bundle consisting of the points at infinity of the compactification of U^*M.

In this introductory section, the sub-Laplacian we consider is either $\Delta_{\mathbf{H}}$, or Δ, or an arbitrary sub-Laplacian $\Delta_{g,\mu}$ on a general sub-Riemannian manifold $(M, \mathcal{D}, g)$. In all cases, we keep the same notations g^*, Σ and $S\Sigma$ to denote the objects we have just introduced, without any reference in the notation to the underlying manifold even for the particular sub-Laplacians $\Delta_{\mathbf{H}}$ and Δ. It should not lead to any confusion since the context is precisely stated when necessary.

We denote by ω the canonical symplectic form on the cotangent bundle T^*M of M. In local coordinates (q, p) of T^*M, we have $\omega = dq \wedge dp$. Given a smooth Hamiltonian function $p : T^*M \to \mathbb{R}$, we denote by $\vec{p}$ the corresponding Hamiltonian vector field on T^*M, defined by $\iota_{\vec{p}}\omega = dp$. Given any smooth vector field V on M, we denote by p_V the Hamiltonian function (momentum map) on T^*M associated with V, defined in local coordinates, by $p_V(q, p) = p(V(q))$. The Hamiltonian flow $\exp(t\vec{p}_V)$ of p_V projects onto the integral curves of V.

1.3 Main results

In order to give a precise statement of our main results, it is necessary to introduce a decomposition of the sub-Laplacian Δ defined by (2). Taking coordinates (x_j, y_j, z_j) on the j-th copy of $\mathbf{H}$, we may write

$$\Delta = \sum_{j=1}^{m} (X_j^2 + Y_j^2) \tag{4}$$

with $X_j = \partial_{x_j}$ and $Y_j = \partial_{y_j} - x_j\partial_{z_j}$.

Let us briefly describe Σ for the sub-Laplacian Δ. Denoting by (q, p) the canonical coordinates in $T^*\mathbf{H}^m$, i.e., $q = (x_1, y_1, z_1, \ldots, x_m, y_m, z_m)$ and $p = (p_{x_1}, p_{y_1}, p_{z_1}, \ldots, p_{x_m}, p_{y_m}, p_{z_m})$, we obtain that

$$\Sigma = \left\{(q, p) \in T^*\mathbf{H}^m \;\middle|\; p_{x_j} = p_{y_j} - x_j p_{z_j} = 0 \text{ for any } 1 \le j \le m\right\},$$

which is isomorphic to $\mathbf{H}^m \times \mathbb{R}^m$. Above any point $q \in \mathbf{H}^m$, the fiber of Σ is of dimension m, and therefore, above any point $q \in \mathbf{H}^m$, $S\Sigma$ consists of an $(m-1)$-dimensional sphere.

For $1 \le j \le m$, we consider the operator $R_j = \sqrt{\partial_{z_j}^* \partial_{z_j}}$ and we make a Fourier expansion with respect to the z_j-variable in the j-th copy of $\mathbf{H}$. On the eigenspaces corresponding to non-zero modes of this Fourier decomposition, we define the operator $\Omega_j = -R_j^{-1}(X_j^2 + Y_j^2) = -(X_j^2 + Y_j^2)R_j^{-1}$. For example, $-\Delta$ acts as

$$-\Delta = \sum_{j=1}^{m} R_j \Omega_j \tag{5}$$

on any eigenspace of $-\Delta$ on which $R_j \neq 0$ for any $1 \leq j \leq m$. Moreover, R_j and Ω_j are pseudodifferential operators of order 1 in any cone of $T^*\mathbf{H}^m$ whose intersection with some conic neighborhood of the set $\{p_{z_j} = 0\}$ is reduced to 0.

The operator Ω_j, seen as an operator on the j-th copy of $\mathbf{H}$, is an harmonic oscillator, having in particular eigenvalues $2n+1$, $n \in \mathbb{N}$ (see [CdVHT18, Section 3.1]). Moreover, the operators Ω_i (considered this time as operators on $\mathbf{H}^m$) commute with each other and with the operators R_j.

For our purpose, it is important to understand the precise structure of Σ. Indeed, it can be decomposed as a disjoint union

$$\Sigma = \bigcup_{\mathcal{J} \in \mathcal{P}} \Sigma_{\mathcal{J}} \tag{6}$$

where $\mathcal{P}$ is the set of all subsets of $\{1, \dots, m\}$, and, for $\mathcal{J} \in \mathcal{P}$, $\Sigma_{\mathcal{J}}$ is defined as the set of points $(q,p) \in \Sigma$ with $p = (p_{x_1}, p_{y_1}, p_{z_1}, \dots, p_{x_m}, p_{y_m}, p_{z_m})$ such that

$$\big(p_{z_j} \neq 0\big) \Leftrightarrow \big(j \in \mathcal{J}\big)\,.$$

For $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, we consider the simplex

$$\mathbf{S}_{\mathcal{J}} = \left\{ s = (s_j) \in \mathbb{R}_+^{\mathcal{J}},\ \sum_{j \in \mathcal{J}} s_j = 1 \right\}$$

and, for $s = (s_j) \in \mathbf{S}_{\mathcal{J}}$ and $(q,p) \in \Sigma_{\mathcal{J}}$, we set

$$\rho_s^{\mathcal{J}}(q,p) = \sum_{j \in \mathcal{J}} s_j |p_{z_j}|.$$

The Hamiltonian vector field $\vec{\rho}_s^{\mathcal{J}}$ is well-defined on $\Sigma_{\mathcal{J}}$ and smooth. Note that we have

$$\rho_s^{\mathcal{J}}(q,p) = (\sigma_P(R_s))_{|\Sigma_{\mathcal{J}}} \quad \text{where} \quad R_s = \sum_{j \in \mathcal{J}} s_j R_j \tag{7}$$

where σ_P denotes the principal symbol (see Appendix A).

Finally, denoting by $\mathscr{M}_+(E)$ (respectively $\mathscr{P}(E)$) the set of non-negative Radon measures (respectively Radon probability measures) on a given separated space E, we set[1]

$$\begin{aligned} \mathscr{P}_{S\Sigma} = \Bigg\{ \nu_\infty = \sum_{\mathcal{J} \in \mathcal{P}\setminus\{\emptyset\}} \nu^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m), \quad \nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \nu_s^{\mathcal{J}}\, dQ^{\mathcal{J}}(s), \\ \text{where } Q^{\mathcal{J}} \in \mathscr{M}_+(\mathbf{S}_{\mathcal{J}}),\ \nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m), \\ \nu_s^{\mathcal{J}}(S^*\mathbf{H}^m \setminus S\Sigma_{\mathcal{J}}) = 0 \text{ and, for } Q^{\mathcal{J}}\text{-almost any } s \in \mathbf{S}_{\mathcal{J}}\ ,\ \vec{\rho}_s^{\mathcal{J}} \nu_s^{\mathcal{J}} = 0 \Bigg\}. \end{aligned} \tag{8}$$

This last definition means that for any continuous function $a : S\Sigma \to \mathbb{R}$, there holds

$$\int_{S\Sigma} a d\nu_\infty = \sum_{\mathcal{J} \in \mathcal{P}\setminus\{\emptyset\}} \int_{\mathbf{S}_{\mathcal{J}}} \left(\int_{S\Sigma_{\mathcal{J}}} a d\nu_s^{\mathcal{J}} \right) dQ^{\mathcal{J}}(s).$$

In a few words, (8) means that any measure $\nu_\infty \in \mathscr{P}_{S\Sigma}$ is supported in $S\Sigma$, and that its invariance properties are given separately on each set $S\Sigma_{\mathcal{J}}$ (for $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$). Its restriction to any of these sets, denoted by $\nu^{\mathcal{J}}$, can be disintegrated with respect to $\mathbf{S}_{\mathcal{J}}$, and for any $s \in \mathbf{S}_{\mathcal{J}}$, there is a corresponding measure $\nu_s^{\mathcal{J}}$ which is invariant by the flow $e^{t\vec{\rho}_s^{\mathcal{J}}}$.

Our first main result is the following:

[1] The notation $S\Sigma_{\mathcal{J}}$ which appears for example in (8) designates in all the sequel the set of points (q,p) of $S\Sigma$ which have null (homogeneous) coordinate p_{z_i} for any $i \notin \mathcal{J}$ and non-null p_{z_j} for $j \in \mathcal{J}$. Note that this set is, in general, neither open nor closed.

Theorem 1. *Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be an orthonormal Hilbert basis of $L^2(\mathbf{H}^m)$ consisting of eigenfunctions of $-\Delta$ associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ labeled in increasing order. Let ν be a Quantum Limit associated to the sequence $(\varphi_k)_{k\in\mathbb{N}^*}$. Then, using the identification (3), we can write ν as the sum of two mutually singular measures $\nu = \beta\nu_0 + (1-\beta)\nu_\infty$, with $\nu_0, \nu_\infty \in \mathscr{P}(S^*\mathbf{H}^m)$, $\beta \in [0,1]$ and*

(1) $\nu_0(S\Sigma) = 0$ and ν_0 is invariant under the sub-Riemannian geodesic flow $e^{t\vec{g}^*}$;

(2) $\nu_\infty \in \mathscr{P}_{S\Sigma}$.

Moreover, there exists a density-one sequence $(k_\ell)_{\ell\in\mathbb{N}}$ of positive integers such that, if ν is a QL associated with a subsequence of $(k_\ell)_{\ell\in\mathbb{N}}$, then the support of ν is contained in $S\Sigma$, i.e., $\beta = 0$ in the previous decomposition.

Note that Theorem 1 holds for *any* orthonormal Hilbert basis of $L^2(\mathbf{H}^m)$ consisting of eigenfunctions of $-\Delta$, and not only for the bases described in Remark 4. In case $(\varphi_k)_{k\in\mathbb{N}^*}$ is of the form described in Remark 4, we can say much more about the associated QLs (see Proposition 13 in Appendix C): if an orthonormal Hilbert basis of eigenfunctions is in a "tensor form", then all associated QLs can be decomposed as tensorial products.

Note that the sub-Riemannian geodesic flow $e^{t\vec{g}^*}$ involved in Theorem 1 is completely integrable, see [ABB19, Chapter 18].

The converse of Theorem 1 also holds, in the following sense:

Theorem 2. *Let $\nu_\infty \in \mathscr{P}_{S\Sigma}$. Then ν_∞ is a Quantum Limit (associated to some orthonormal basis consisting of eigenfunctions of $-\Delta$).*

Together, Theorem 1 and Theorem 2 yield a classification of (nearly) all Quantum Limits of Δ.[2]

1.4 Comments on the main results

In order to explain the contents of Theorem 1 and Theorem 2, we recall the following result, which is valid for any sub-Laplacian $\Delta_{g,\mu}$.

Proposition 1. *[CdVHT18, Theorem B] Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be an orthonormal Hilbert basis of $L^2(M,\mu)$ consisting of eigenfunctions of $-\Delta_{g,\mu}$ associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ labeled in increasing order. Let ν be a QL associated with $(\varphi_k)_{k\in\mathbb{N}^*}$. Using the identification $S^*M = U^*M \cup S\Sigma$ (see (3)), the probability measure ν can be written as the sum $\nu = \beta\nu_0 + (1-\beta)\nu_\infty$ of two mutually singular measures with $\nu_0, \nu_\infty \in \mathscr{P}(S^*M)$, $\beta \in [0,1]$ and*

(1) $\nu_0(S\Sigma) = 0$ and ν_0 is invariant under the sub-Riemannian geodesic flow $\vec{g}^*$;

*(2) ν_∞ is supported on $S\Sigma$. Moreover, **in the 3D contact case**, ν_∞ is invariant under the lift to $S\Sigma$ of the Reeb flow.*[3]

Moreover, there exists a density-one sequence $(k_\ell)_{\ell\in\mathbb{N}}$ of positive integers such that, if ν is a QL associated with a subsequence of $(k_\ell)_{\ell\in\mathbb{N}}$, then the support of ν is contained in $S\Sigma$, i.e., $\beta = 0$ in the previous decomposition.[4]

The last part of Proposition 1 shows that ν_∞ is the "main part" of the QL, but, according to Point (2), its invariance properties were known only in the 3D contact case. Theorem 2

[2] The exact converse of Theorem 1 would guarantee that all measures $\nu \in \mathscr{P}(S^*\mathbf{H}^m)$ of the form $\nu = \beta\nu_0 + (1-\beta)\nu_\infty$ with the same assumptions on β, ν_0 and ν_∞ as in Theorem 1 are Quantum Limits. Our statement is weaker since it does not say anything about the measures ν for which $\beta \neq 0$ (which are rare, as stated in Theorem 1), but we do not think that a stronger converse statement for Theorem 1 holds.

[3] See [CdVHT18] for a definition of the Reeb flow, or Appendix D.

[4] The proof of this last fact follows from the results in [CdVHT18], although it is not explicitely stated there. Let us sketch the proof. By [CdVHT18, Proposition 4.3], we know that the microlocal Weyl measure of $\Delta_{g,\mu}$ is supported in $S\Sigma$. It then follows from [CdVHT18, Corollary 4.1] that for every $A \in \Psi^0(M)$ whose principal symbol vanishes on Σ, there holds $V(A) = 0$, where $V(A)$ is the variance introduced in [CdVHT18, Definition 4.1]. Finally, following the proof of Theorem B(2) in [CdVHT18], we get the result.

and Point (2) of Theorem 1 are the main novelties of this paper and they serve as substitutes to Point (2) of Proposition 1 for the sub-Laplacians Δ on $\mathbf{H}^m$.

Compared to the invariance properties of the QLs of 3D contact sub-Laplacians described in Proposition 1, the invariance property described by Point (2) of Theorem 1 involves an infinite number of different Hamiltonian vector fields $\vec{\rho}_s^{\mathcal{J}}$ on $S\Sigma$.

Spectrum of $-\Delta$. The particularly rich structure of the Quantum Limits of the sub-Laplacian $-\Delta$ described in Theorem 1 is due to the high degeneracy of its spectrum. To make an analogy with the Riemannian case, the QLs of the usual flat Riemannian torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ have a rich structure (see [Jak97]), whereas the QLs of irrational Riemannian tori are much simpler to describe.

Recall that the spectrum $\mathrm{sp}(-\Delta_{\mathbf{H}})$ is given by

$$\begin{aligned}\mathrm{sp}(-\Delta_{\mathbf{H}}) = \{\lambda_{\ell,\alpha} = (2\ell+1)|\alpha| \ \mid \ \ell \in \mathbb{N},\ \alpha \in \mathbb{Z}\setminus\{0\}\}\\ \cup\, \{\mu_{k_1,k_2} = 2\pi(k_1^2+k_2^2) \ \mid \ (k_1,k_2)\in\mathbb{Z}^2\}\end{aligned}$$

where $\lambda_{\ell,\alpha}$ is of multiplicity $|\alpha|$, multiplied by the number of decompositions of $\lambda_{\ell,\alpha}$ into the form $(2\ell'+1)|\alpha'|$ (see [CdVHT18, Proposition 3.1]). Therefore, using a tensorial orthonormal Hilbert basis of $L^2(\mathbf{H}^m)$ consisting of eigenfunctions of $-\Delta$, we get that

$$\mathrm{sp}(-\Delta) = \left\{ \sum_{j=1}^{J} (2n_j+1)\,|\alpha_j| + 2\pi \sum_{i=1}^{2(m-J)} k_i^2 \text{ with } 0\le J\le m,\ k_i\in\mathbb{Z},\ n_j\in\mathbb{N},\ \alpha_j\in(\mathbb{Z}\setminus\{0\}) \right\}$$

(see Section 2.1 for a detailed proof) and the multiplicities in $\mathrm{sp}(-\Delta)$ can be deduced from those in $\mathrm{sp}(-\Delta_{\mathbf{H}})$.

Note that the eigenvalues for which $J = m$ form a density-one subsequence of all eigenvalues labeled in increasing order.

Remark 5. *Contrarily to those of flat tori (see [Jak97]), the Quantum Limits of $\mathbf{H}^m$ (or, more precisely, their pushforward under the canonical projection onto $\mathbf{H}^m$) are not necessarily absolutely continuous. It was already remarked in the case $m = 1$ in [CdVHT18, Proposition 3.2(2)].*

A first illustration of Point (2) of Theorem 1. A way to get an intuition of Point (2) of Theorem 1 is to fix $(n_1,\dots,n_m)\in\mathbb{N}^m$, and to consider an orthonormal sequence of eigenfunctions $(\psi_k)_{k\in\mathbb{N}^*}$ of $-\Delta$ given in a tensor form as in Remark 4, such that, for any $k\in\mathbb{N}^*$, ψ_k is also, for any $1\le j\le m$, a sequence of eigenfunctions of R_j with eigenvalue tending to $+\infty$, and of Ω_j with eigenvalue $2n_j+1$. Such a sequence of eigenfunctions exists, and can be completed to an orthonormal basis of $L^2(\mathbf{H}^m)$ consisting of eigenfunctions of $-\Delta$. We notice that any associated Quantum Limit ν is supported in $S\Sigma$.[5]

Let $\mathcal{J} = \{1,\dots,m\}\in\mathcal{P}$. Then, ν is necessarily invariant under the Hamiltonian vector field $\vec{\rho}_s^{\mathcal{J}}$, where $s = (s_1,\dots,s_m)\in\mathbf{S}_{\mathcal{J}}$ is defined by $s_j = \frac{2n_j+1}{2n_1+1+\dots+2n_m+1}$ for $j = 1,\dots,m$. To see it, we set

$$R = \frac{\sum_{j=1}^m (2n_j+1)R_j}{\sum_{j=1}^m 2n_j+1}$$

and we note that for any $A\in\Psi^0(\mathbf{H}^m)$, we have

$$([A,R]\psi_k,\psi_k) = (AR\psi_k,\psi_k) - (A\psi_k,R\psi_k) = 0$$

[5]It follows from the classical Lemma 10 in Appendix A, using the fact that $P\psi_k = 0$ for sufficiently large $k\in\mathbb{N}^*$, with $P = -\Delta - \sum_{j=1}^m (2n_j+1)f(R_j)$ which is elliptic outside Σ. Here, $f\in C^\infty(\mathbb{R})$ vanishes in a neighborhood of 0 and is equal to 1 for sufficiently large $x\in\mathbb{R}$: since R_j is not a pseudodifferential operator in a neighborhood of $p_{z_j} = 0$, P is not necessarily a pseudodifferential operator for $f\equiv 1$, thus leading us to this non-constant choice for f.

since ψ_k is an eigenfunction of R. In the limit $k \to +\infty$, taking the principal symbol, we obtain $\int_{S\Sigma} \{a, \rho_s^{\mathcal{J}}\} d\nu = 0$ where $a = \sigma_P(A)$. Since it is true for any $a \in \mathscr{S}^0(\mathbf{H}^m)$, this implies $\vec{\rho}_s^{\mathcal{J}} \nu = 0$. Hence, for such sequences $(\psi_k)_{k\in\mathbb{N}^*}$, any QL verifies $\nu = \nu_s^{\mathcal{J}}$ (which is invariant under $\vec{\rho}_s^{\mathcal{J}}$), $Q^{\mathcal{J}}$ is a Dirac mass on s and $Q^{\mathcal{J}'} = 0$ for $\mathcal{P} \ni \mathcal{J}' \neq \mathcal{J}$.

In some sense, any QL supported on $S\Sigma$ is a linear combination of sequences as in the above example, for different $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$ and different $s \in \mathbf{S}_{\mathcal{J}}$.

Roles of R_j and Ω_j. The operators R_j and Ω_j play a key role in the proofs of Theorem 1 and Theorem 2. As illustrated in the previous paragraph, the operators Ω_j are linked with the parameters $s \in \mathbf{S}_{\mathcal{J}}$: in some sense, once the eigenfunctions have been orthogonally decomposed with respect to operators R_j and Ω_j (as explained in Section 2.1), the ratios between the Ω_j-s determines the invariance property of the associated Quantum Limits through the parameter s and the Hamiltonian vector field $\vec{\rho}_s^{\mathcal{J}}$. On the other side, the operators R_j 'determine' the microlocal support of the associated Quantum Limits, for example the element $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$ (such that the QL concentrates on $S\Sigma_{\mathcal{J}}$). The next paragraph, which is devoted to a sketch of proof of Theorem 1, will make these intuitions more precise.

Sketch of proof of Theorem 1. The last part of Theorem 1 is an immediate consequence of the last part of Proposition 1. Then, the proof of Points (1) and (2) in Theorem 1 is split into two steps which we now describe.

Step 1. First, in Lemma 2, extracting if necessary a subsequence, we split each eigenfunction φ_k into $\varphi_k = \varphi_k^0 + \varphi_k^\infty$, where φ_k^0 and φ_k^∞ are both eigenfunctions of $-\Delta$ with the same eigenvalue as φ_k, and with the property that, in the limit $k \to +\infty$, φ_k^∞ has ν_∞ for unique microlocal defect measure (and therefore microlocally concentrates on $S\Sigma$), while φ_k^0 admits ν_0 as unique microlocal defect measure. This proves that one can study independently Point (1) and Point (2) of Theorem 1. Since Point (1) is a consequence of Proposition 1, we focus on Point (2). In the next paragraphs, we omit the index ∞ in order to simplify notations.

In the proof of Lemma 2, we identify a decomposition of φ_k as a sum

$$\varphi_k = \sum_{\mathcal{J} \in \mathcal{P}\setminus\{\emptyset\}} \varphi_k^{\mathcal{J}}$$

with the property that for any $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, $\varphi_k^{\mathcal{J}}$ is a sequence of eigenfunctions of $-\Delta$ whose (unique) microlocal defect measure has all its mass contained in $S\Sigma_{\mathcal{J}}$. Using a "gluing lemma" for microlocal defect measures (Lemma 3), we obtain that it is sufficient to prove Point (2) of Theorem 1 for $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$, separately for each $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. Therefore, we focus on one of the sequences $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ for some $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. In other words, in Step 2, we prove that the (unique) microlocal defect measure $\nu = \nu^{\mathcal{J}}$ of the sequence $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ may be decomposed as

$$\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \nu_s^{\mathcal{J}} \, dQ^{\mathcal{J}}(s).$$

In order to simplify the presentation, in the next paragraphs, we assume that $\mathcal{J} = \{1, \dots, m\}$ and we omit this notation (writing for example $\mathbf{S}$ instead of $\mathbf{S}_{\mathcal{J}}$), but the proof is similar for any $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$.

Step 2. With this assumption, we can use the decomposition (5) to write each φ_k as a sum of eigenfunctions of operators of the form $\sum_{j=1}^m (2n_j + 1) R_j$ for some integers $n_1, \dots, n_m$:

$$\varphi_k = \sum_{(n_1,\dots,n_m)\in\mathbb{N}^m} \varphi_{k,n_1,\dots,n_m}. \tag{9}$$

We will see in Section 2.1 that the decomposition (9) is orthogonal, and therefore each eigenfunction $\varphi_{k,n_1,\dots,n_m}$ has the same eigenvalue λ_k as φ_k. Then, we do a careful analysis of this decomposition into modes, which, in the limit $k \to +\infty$, gives the disintegration

$\nu = \int_{\mathbf{S}} \nu_s dQ(s)$. This analysis builds upon a partition of the lattice $\mathbb{N}^m$ into positive cones, each of them gathering together the modes $\varphi_{k,n_1,\ldots,n_m}$ for which the m-tuples

$$\left(\frac{2n_1+1}{2n_1+1+\ldots+2n_m+1}, \ldots, \frac{2n_m+1}{2n_1+1+\ldots+2n_m+1} \right)$$

are approximately the same: each of these positive cones accounts for a small region of the simplex $\mathbf{S}$. If $\mathbb{N}^m$ is partitioned into 2^N positive cones C_ℓ^N (with $0 \leq \ell \leq 2^N - 1$), this gathering defines eigenfunctions

$$\varphi_{k,\ell}^N = \sum_{(n_1,\ldots,n_m)\in C_\ell^N} \varphi_{k,n_1,\ldots,n_m}$$

of $-\Delta$ such that

$$\varphi_k = \sum_{\ell=0}^{2^N-1} \varphi_{k,\ell}^N \tag{10}$$

for any $N \in \mathbb{N}^*$.

Taking a microlocal defect measure ν_ℓ^N in each sequence $(\varphi_{k,\ell}^N)_{k\in\mathbb{N}^*}$ and making $N \to +\infty$ (i.e., taking the limit where the positive cones degenerate to half-lines parametrized by $s \in \mathbf{S}$), we obtain from (10) the disintegration $\nu = \int_{\mathbf{S}} \nu_s dQ(s)$.

Given a certain $s = (s_1, \ldots, s_m) \in \mathbf{S}$, $dQ(s)$ accounts for the relative importance, in the limit $N \to +\infty$, of the eigenfunction $\varphi_{k,\ell(N)}^N$ in the sum (10), where $\ell(N)$ is chosen so that the positive cone $C_{\ell(N)}^N$ converges to the half-line with parameter s as $N \to +\infty$.

The invariance property $\vec{\rho}_s \nu_s = 0$ can be seen from the fact that, for any large N and any $0 \leq \ell \leq 2^N - 1$, each eigenfunction $\varphi_{k,n_1,\ldots,n_m}$ with $(n_1, \ldots, n_m) \in C_\ell^N$ is indeed an eigenfunction of the operator

$$\sum_{i=1}^m \left(\frac{2n_i+1}{2n_1+1+\ldots+2n_m+1} \right) R_i$$

which, by definition of $\varphi_{k,\ell}^N$, is approximately equal to $R_s = s_1 R_1 + \ldots + s_m R_m$ if $s = (s_1, \ldots, s_m) \in \mathbf{S}$ denotes the parameter of the limiting half-line of the positive cones C_ℓ^N as $N \to +\infty$. Hence, $\varphi_{k,\ell}^N$ is an approximate eigenfunction of R_s, from which it follows by a classical argument that ν_s is invariant under the Hamiltonian vector field $\vec{\rho}_s$ of $\rho_s = (\sigma_P(R_s))_{|\Sigma}$.

Remark 6. *There is no clear link of our result with the concept of "second microlocalization", although such a link may seem possible at first sight. Focusing on a Quantum Limit supported in $S\Sigma$, our study builds upon a* spectral *decomposition of it, and not upon a second direction of microlocalization as is usually done while studying fine properties of sequences of solutions of an operator (see for example [FK00]).*

Remark 7. *Two generalizations of Theorem 1 may be considered. The first one consists in adding a potential V to $-\Delta$ and to look for the invariance properties of the associated QLs in the spirit, for example, of [MR19]. A second generalization consists in studying the QLs of products of general 3D contact manifolds, thus replacing the quotient Heisenberg manifold $\mathbf{H}$ by an arbitrary 3D contact manifold. Both generalizations are open issues.*

Bibliographical comments. The study of Quantum Limits for Riemannian Laplacians is a long-standing question. Over the years, a particular attention has been drawn towards Riemannian manifolds whose geodesic flow is ergodic since in this case, up to extraction of a density-one subsequence, the set of Quantum Limits is reduced to the Liouville measure, a phenomenon which is called Quantum Ergodicity (see for example [Shn74], [CdV85], [Zel87]). For compact arithmetic surfaces, a detailed study of invariant measures lead to the resolution of the Quantum Unique Ergodicity conjecture for these manifolds, meaning that

the extraction of a density-one subsequence in the previous result is even not necessary for these particular manifolds ([Lin06]). In manifolds which have more integrability properties, the set of Quantum Limits is generally richer: see for example [Jak97] for the description of Quantum Limits on flat tori (which are product manifolds, as the ones we consider) or [ALM16] for the case of the disk.

An important literature is also devoted to the study of semiclassical measures, which are analog to Quantum Limits in a time-dependent and semiclassical setting, and which are a natural tool used for understanding the Schrödinger flow (see for example [AM12]). With this in mind, in [FKF19], the authors developed a notion of semiclassical measures adapted to "Heisenberg type" sub-Laplacians, with the aim of studying the Schrödinger equation with a sub-Laplacian replacing the usual Laplace-Beltrami operator (see also [FKL] for application to the study of controllability of the Schrödinger equation in Heisenberg type groups).

The study of Quantum Limits of general sub-Laplacians was undertaken in the work [CdVHT18], which was mainly devoted to the 3D contact case - encompassing for example the case of the manifold $\mathbf{H}$ - although some results are valid for any sub-Laplacian (see Proposition 1 of the present paper). The understanding of Quantum Limits of general sub-Laplacians remains a largely unexplored question.

Structure of the paper. In Section 2.1, we explain the spectral decomposition of $L^2(\mathbf{H}^m)$ according to the eigenspaces of the harmonic oscillators Ω_j which gives the possibility to write any eigenfunction φ_k of $-\Delta$ as a sum of the form (9). It replaces, in some sense, the Fourier decomposition of eigenfunctions usually done in Riemannian tori.

This decomposition plays a key role in the proof of Point (2) of Theorem 1, which is divided into two steps, as explained previously.

Section 2.2 is devoted to the first step: using a pseudodifferential cut-off procedure, we show how to deduce Point (1) of Theorem 1 from Proposition 1, and to reduce the proof of Theorem 1 to the case where $(\varphi_k)_{k\in\mathbb{N}^*}$ has a unique microlocal defect measure, with all its mass contained in $S\Sigma_{\mathcal{J}}$ for some $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$.

Building upon this reduction and the spectral decomposition of Section 2.1, we establish in Section 2.3 the second and final step of the proof of Point (2) of Theorem 1.

In Section 3, we prove Theorem 2 by constructing explicitely a sequence of eigenfunctions with prescribed Quantum Limit.

In Appendix A, we recall some basic facts of pseudodifferential calculus and two related elementary lemmas. In Appendix B, we provide another way for obtaining the measures $Q^{\mathcal{J}}$ and $\nu_s^{\mathcal{J}}$ (but without proving the invariance properties), which relies on pure functional analysis arguments and sheds a different light on Theorem 1. In Appendix C, we show that the Quantum Limits of a Hilbert basis of eigenfunctions of $\mathbf{H}^m$ given in a "tensor form" can themselves be expressed in a tensor form. Finally, in Appendix D, we prove a result concerning Quantum Limits of flat contact manifolds in any dimension: for such manifolds, the invariance properties of Quantum Limits are essentially the same as in the 3D case. This gives further motivation for our main results, since the Quantum Limits on the manifold $\mathbf{H}^m$ reveal a more interesting behaviour than those on flat contact manifolds.

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2 Proof of Theorem 1

2.1 Spectral decomposition of $-\Delta$

In this section, we study in detail the action of $-\Delta$ on $L^2(\mathbf{H}^m)$: the existence of orthonormal Hilbert bases of $L^2(\mathbf{H}^m)$ constituted of eigenfunctions of $-\Delta$ in a tensor form (see Remark 4) allows to write a decomposition of $-\Delta$ and also an orthogonal decomposition of $L^2(\mathbf{H}^m)$.

Let us recall that, for $1 \leq j \leq m$, we have set $R_j = \sqrt{\partial^*_{z_j}\partial_{z_j}}$ and we made a Fourier expansion with respect to the z_j-variable. On the eigenspaces corresponding to non-zero modes of this Fourier decomposition, we defined the operator $\Omega_j = -R_j^{-1}(X_j^2 + Y_j^2) = -(X_j^2 + Y_j^2)R_j^{-1}$. For example, $-\Delta$ acts as

$$-\Delta = \sum_{j=1}^{m} R_j \Omega_j$$

on any eigenspace of $-\Delta$ on which $R_j \neq 0$ for any $1 \leq j \leq m$. Moreover, R_j and Ω_j are pseudodifferential operators of order 1 in any cone of $T^*\mathbf{H}^m$ whose intersection with some conic neighborhood of the set $\{p_{z_j} = 0\}$ is reduced to 0 (for example in small conic neighborhoods of $\Sigma_{\mathcal{J}}$ for $\mathcal{J}$ containing j).

The operator Ω_j, seen as an operator on the j-th copy of $\mathbf{H}$, is an harmonic oscillator, having in particular eigenvalues $2n+1$, $n \in \mathbb{N}$ (see [CdVHT18, Section 3.1]). Moreover, the operators Ω_i (considered this time as operators on $\mathbf{H}^m$) commute with each other and with the operators R_j.

Recall that $\mathcal{P}$ stands for the set of all subsets of $\{1, \dots, m\}$. We fix $\mathcal{J} \in \mathcal{P}$. In the sequel, we think of $\mathcal{J}$ as the set of j for which $R_j \neq 0$. For $j \in \mathcal{J}$ and $n \in \mathbb{N}$, we denote by $E_n^j \subset L^2(\mathbf{H})$ the eigenspace of Ω_j corresponding to the eigenvalue $2n+1$. For $(n_j) \in \mathbb{N}^{\mathcal{J}}$, we set

$$\mathcal{H}^{\mathcal{J}}_{(n_j)} = F^1 \otimes \ldots \otimes F^m \subset L^2(\mathbf{H}^m)$$

where $F^j = E^j_{n_j}$ for $j \in \mathcal{J}$ and $F^j = L^2(\mathbf{H})$ otherwise.

We have the orthogonal decomposition

$$L^2(\mathbf{H}^m) = \bigoplus_{\mathcal{J}\in\mathcal{P}} \bigoplus_{(n_j)\in\mathbb{N}^{\mathcal{J}}} \mathcal{H}^{\mathcal{J}}_{(n_j)}. \tag{11}$$

We can also write the associated decomposition of $-\Delta$:

$$-\Delta = \bigoplus_{\mathcal{J}\in\mathcal{P}} \bigoplus_{(n_j)\in\mathbb{N}^{\mathcal{J}}} H^{\mathcal{J}}_{(n_j)}$$

$$\text{with} \quad H^{\mathcal{J}}_{(n_j)} = \sum_{j\in\mathcal{J}} (2n_j+1)\, R_j - \sum_{i\notin\mathcal{J}} (\partial^2_{x_i} + \partial^2_{y_i}).$$

From this, we deduce

$$\begin{aligned}
\mathrm{sp}(-\Delta) &= \bigcup_{\mathcal{J}\in\mathcal{P}} \bigcup_{(n_j)\in\mathbb{N}^{\mathcal{J}}} \mathrm{sp}(H^{\mathcal{J}}_{(n_j)}) \\
&= \Bigg\{ \sum_{j\in\mathcal{J}} (2n_j+1)\,|\alpha_j| + 2\pi \sum_{i\notin\mathcal{J}} (k_i^2 + \ell_i^2), \\
&\qquad\qquad \text{with } k_i, \ell_i \in \mathbb{Z},\ \mathcal{J} \in \mathcal{P},\ n_j \in \mathbb{N},\ \alpha_j \in (\mathbb{Z}\setminus\{0\}) \Bigg\}
\end{aligned}$$

where sp denotes the spectrum.

2.2 Step 1: Identification of ν_0 and ν_∞

In all the sequel, we fix $(\varphi_k)_{k\in\mathbb{N}^*}$ an orthonormal basis of eigenfunctions of $-\Delta$ associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ with $\lambda_k \to +\infty$, and we consider ν, a Quantum Limit associated to the sequence $(\varphi_k)_{k\in\mathbb{N}^*}$.

In this section, we identify different parts in the Quantum Limit ν: our goal is to provide an adequate decomposition which is preliminary to the detailed analysis performed in Section 2.3. This corresponds to Step 1 in the sketch of proof given in Section 1.4.

Lemma 2. *Let us assume that $(\varphi_k)_{k\in\mathbb{N}^*}$ is an orthonormal sequence of eigenfunctions[6] of $-\Delta$ with associated eigenvalues $\lambda_k \to +\infty$. Then, up to extraction of a subsequence, one can decompose*

$$\varphi_k = \varphi_k^0 + \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} \varphi_k^{\mathcal{J}},$$

with the following properties:

- *The sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ has a unique Quantum Limit ν;*
- *φ_k^0 and all $\varphi_k^{\mathcal{J}}$, for $k\in\mathbb{N}^*$ and $\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}$, are eigenfunctions of $-\Delta$ with eigenvalue λ_k;*
- *Using the identification $S^*\mathbf{H}^m = U^*\mathbf{H}^m \cup S\Sigma$ (see (3)), the sequence $(\varphi_k^0)_{k\in\mathbb{N}^*}$ admits a unique microlocal defect measure $\beta\nu_0$, where $\beta\in[0,1]$, $\nu_0 \in \mathscr{P}(S^*\mathbf{H}^m)$ and $\nu_0(S\Sigma) = 0$, and, for any $\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}$, the sequence $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ also admits a unique microlocal defect measure $\nu^{\mathcal{J}}$, having all its mass contained in $S\Sigma_{\mathcal{J}}$;*
- *There holds*

$$\nu = \beta\nu_0 + \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} \nu^{\mathcal{J}} \tag{12}$$

and the sum in (12) is supported in $S\Sigma$.

Let us first give an intuition of how the proof goes. Using the spectral decomposition of Section 2.1, for fixed $k\in\mathbb{N}$, we decompose φ_k as a sum of functions which are eigenfunctions of R_j for any $1\le j\le m$, and simultaneously eigenfunctions either of Ω_j (if the corresponding eigenvalue of R_j is non-null) or of ∂_{x_j} and ∂_{y_j} (if the corresponding eigenvalue of R_j is zero). Each of these functions is an eigenfunction of $-\Delta$ with same eigenvalue λ_k as φ_k, and also an eigenfunction of $-(X_j^2+Y_j^2)$ for any j. Then, roughly speaking, we gather some of these functions into φ_k^0 or into $\varphi_k^{\mathcal{J}}$ for some $\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}$, depending on their eigenvalues with respect to the operators $-\Delta$, R_j and $-(X_j^2+Y_j^2)$ for $1\le j\le m$. More precisely, the functions which we select (asymptotically as $k\to+\infty$) to be in $\varphi_k^{\mathcal{J}}$ are those such that:

- For any $j\notin\mathcal{J}$, the corresponding eigenvalue of $-(X_j^2+Y_j^2)$ is negligible in comparison to λ_k, as $k\to+\infty$;
- For any $j\in\mathcal{J}$, the corresponding eigenvalue of $-(X_j^2+Y_j^2)$ is *not* negligible in comparison to λ_k, but the corresponding eigenvalue of R_j is much larger than that of Ω_j as $k\to+\infty$.

It corresponds to the intuition that $\Sigma_{\mathcal{J}}$ is the set of points (q,p) of $T^*\mathbf{H}^m$ for which, for any $j\notin\mathcal{J}$, $p_{x_j} = p_{y_j} = p_{z_j} = 0$, and for any $j\in\mathcal{J}$, $|p_{z_j}|$ is much larger than $|p_{x_j}|$ and $|p_{y_j} - x_j p_{z_j}|$ (which are indeed equal to 0 on Σ).

Proof of Lemma 2. For $n\in\mathbb{N}^*$, let $\chi_n\in C_c^\infty(\mathbb{R},[0,1])$ such that $\chi_n(x) = 1$ for $|x|\le\frac{1}{2n}$ and $\chi_n(x) = 0$ for $|x|\ge\frac{1}{n}$. We set $\Delta_j = X_j^2+Y_j^2$ for $1\le j\le m$, and we have

$$\sigma_P(-\Delta_j) = p_{x_j}^2 + (p_{y_j} - x_j p_{z_j})^2.$$

We also introduce

$$E = -\Delta + \sum_{j=1}^m R_j^2 \in \Psi^2(\mathbf{H}^m),$$

[6]Note that $(\varphi_k)_{k\in\mathbb{N}^*}$ is not supposed to be a basis of $L^2(\mathbf{H}^m)$.

which is elliptic, with principal symbol

$$\sigma_P(E) = \sum_{j=1}^{m} \left(p_{x_j}^2 + (p_{y_j} - x_j p_{z_j})^2 \right) + p_{z_j}^2.$$

For $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, we consider the operator

$$P_n^{\mathcal{J}} = \prod_{i \notin \mathcal{J}} \chi_n \left(\frac{-\Delta_i + R_i^2}{E} \right) \prod_{j \in \mathcal{J}} \left((1 - \chi_n) \left(\frac{-\Delta_j + R_j^2}{E} \right) \chi_n \left(\frac{-\Delta_j}{-\Delta_j + R_j^2} \right) \right) \tag{13}$$

defined thanks to functional calculus. As we will see, $P_n^{\mathcal{J}} \in \Psi^0(\mathbf{H}^m)$ and, as $n \to +\infty$, its principal symbol tends to the characteristic function $\mathbf{1}_{\Sigma_{\mathcal{J}}} : T^*\mathbf{H}^m \to \mathbb{R}$, where $\Sigma_{\mathcal{J}}$ has been defined in (6).

For any $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, the following properties hold:

(1) $P_n^{\mathcal{J}} \in \Psi^0(\mathbf{H}^m)$;

(2) $[P_n^{\mathcal{J}}, \Delta] = 0$;

(3) $\sigma_P(P_n^{\mathcal{J}}) \to \mathbf{1}_{\Sigma_{\mathcal{J}}}$ pointwise as $n \to +\infty$.

Let us prove Point (1). Since E is elliptic, it is invertible, and since $-\Delta_i, R_i^2 \in \Psi^0(\mathbf{H}^m)$, by [HV00], $\chi_n \left(\frac{-\Delta_i + R_i^2}{E} \right) \in \Psi^0(\mathbf{H}^m)$ with principal symbol

$$\chi_n \left(\frac{\sigma_P(-\Delta_i) + p_{z_i}^2}{\sigma_P(E)} \right).$$

Similarly, the operator $(1 - \chi_n) \left(\frac{-\Delta_j + R_j^2}{E} \right)$ belongs to $\Psi^0(\mathbf{H}^m)$ and its principal symbol is supported in the cone of $T^*\mathbf{H}^m$ given by

$$\tilde{S}_n^j = \left\{ \frac{\sigma_P(-\Delta_j) + p_{z_j}^2}{\sigma_P(E)} \geq \frac{1}{2n} \right\}.$$

In this cone, $-\Delta_j + R_j^2$ is elliptic, hence invertible. Therefore, $\chi_n \left(\frac{-\Delta_j}{-\Delta_j + R_j^2} \right)$ is a 0-th order pseudodifferential operator in $\tilde{S}_n^j$, from which we conclude that

$$(1 - \chi_n) \left(\frac{-\Delta_j + R_j^2}{E} \right) \chi_n \left(\frac{-\Delta_j}{-\Delta_j + R_j^2} \right) \in \Psi^0(\mathbf{H}^m).$$

Finally, both products in the definition of $P_n^{\mathcal{J}}$ belong to $\Psi^0(\mathbf{H}^m)$. Hence, $P_n^{\mathcal{J}} \in \Psi^0(\mathbf{H}^m)$.

Point (2) is an immediate consequence of functional calculus, since all operators $-\Delta_i$ and R_j for $1 \leq i, j \leq m$ commute one with each other.

Let us prove Point (3). The support of $\sigma_P(P_n^{\mathcal{J}})$ is contained in the intersection of several conic subsets of $T^*\mathbf{H}^m$: it is contained in the cone

$$S_n^i := \left\{ \frac{\sigma_P(-\Delta_i) + p_{z_i}^2}{\sigma_P(E)} \leq \frac{1}{n} \right\}$$

for any $i \notin \mathcal{J}$, and, for any $j \in \mathcal{J}$, in the cone $\tilde{S}_n^j$ intersected with the cone

$$T_n^j := \left\{ \frac{\sigma_P(-\Delta_j)}{\sigma_P(-\Delta_j) + p_{z_j}^2} \leq \frac{1}{n} \right\} = \left\{ \sigma_P(-\Delta_j) \leq \frac{p_{z_j}^2}{n-1} \right\}.$$

It follows that, in the limit $n \to +\infty$, $\sigma_P(P_n^{\mathcal{J}})$ vanishes everywhere outside the set defined by the relations

$$\sigma_P(-\Delta_i) = p_{z_i} = 0, \quad \forall i \notin \mathcal{J}$$
$$\sigma_P(-\Delta_j) = 0 \quad \text{and} \quad p_{z_j} \neq 0, \quad \forall j \in \mathcal{J}.$$

We note that these relations exactly define the set $\Sigma_{\mathcal{J}}$.

Let $(q,p) \in \Sigma_{\mathcal{J}}$ with $p = (p_{x_1}, p_{y_1}, p_{z_1}, \dots, p_{x_m}, p_{y_m}, p_{z_m})$ and $p \neq 0$. Our goal is to show that $\sigma_P(P_n^{\mathcal{J}})(q,p) = 1$. It follows from a separate analysis of the principal symbol of each factor in the product (13):

- For $i \notin \mathcal{J}$, since $(q,p) \in \Sigma$ and $p_{z_i} = 0$, we have $p_{x_i} = p_{y_i} = 0$. Hence, at (q,p),
$$\chi_n\left(\frac{\sigma_P(-\Delta_i) + p_{z_i}^2}{\sigma_P(E)}\right) = 1.$$
- For $j \in \mathcal{J}$, we know that $p_{z_j} \neq 0$. Hence, for n sufficiently large, at (q,p),
$$(1-\chi_n)\left(\frac{\sigma_P(-\Delta_j) + p_{z_j}^2}{\sigma_P(E)}\right) = 1.$$
- For $j \in \mathcal{J}$, using that $p_{z_j} \neq 0$ and $\sigma_P(-\Delta_j) = 0$, we get, at (q,p),
$$\chi_n\left(\frac{\sigma_P(-\Delta_j)}{\sigma_P(-\Delta_j) + p_{z_j}^2}\right) = 1.$$

All in all, $\sigma_P(P_n^{\mathcal{J}})(q,p) = 1$ for sufficiently large n, which finally proves Point (3).

We now conclude the proof of Lemma 2. We consider, for fixed $n \in \mathbb{N}$ and $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, the sequence $(P_n^{\mathcal{J}}\varphi_k)_{k\in\mathbb{N}^*}$, which, thanks to Points (1) and (2), is also a sequence of eigenfunctions of $-\Delta$ with same eigenvalues as φ_k. We denote by $\nu_n^{\mathcal{J}}$ a microlocal defect measure of $(P_n^{\mathcal{J}}\varphi_k)_{k\in\mathbb{N}^*}$ and by ν_n^0 a microlocal defect measure of the sequence given by the eigenfunctions

$$\varphi_k - \sum_{\mathcal{J}\in\mathcal{P}\setminus\{0\}} P_n^{\mathcal{J}}\varphi_k.$$

Furthermore, we may assume thanks to the diagonal extraction process that the extraction used to obtain all these microlocal defect measures is the same for any $n \in \mathbb{N}^*$ and any $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$.

Finally, we take $\nu^{\mathcal{J}}$ a weak-star limit of $(\nu_n^{\mathcal{J}})_{n\in\mathbb{N}}$ and $\beta\nu_0$ a weak-star limit of $(\nu_n^0)_{n\in\mathbb{N}}$, with $\nu \in \mathscr{P}(S^*\mathbf{H}^m)$ and $\beta \in [0,1]$. Thanks to the analysis done while proving Point (3), we know that $\nu^{\mathcal{J}}$ gives no mass to the complementary of $S\Sigma_{\mathcal{J}}$ in $S^*\mathbf{H}^m$, and that $\nu_0(S\Sigma) = 0$. Again, thanks to the diagonal extraction process, up to extraction of a subsequence in $k \in \mathbb{N}^*$, we can write

$$\varphi_k = \varphi_k^0 + \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} \varphi_k^{\mathcal{J}} \tag{14}$$

where the unique microlocal defect measure of $(\varphi_k^0)_{k\in\mathbb{N}^*}$ is $\beta\nu_0$, and $\varphi_k^{\mathcal{J}} = P_{r(k)}^{\mathcal{J}}\varphi_k$ (for some function r tending (slowly) to $+\infty$ as $k \to +\infty$) has a unique microlocal defect measure as $k \to +\infty$, which is $\nu^{\mathcal{J}}$.

Let us prove that (14) implies (12). For that, we first recall a definition and an elementary lemma concerning joint microlocal defect measures.

Definition 8. *Let $(u_k)_{k\in\mathbb{N}^*}, (v_k)_{k\in\mathbb{N}^*}$ be bounded sequences in $L^2(M)$ such that u_k and v_k weakly converge to 0 as $k \to +\infty$. We call joint microlocal defect measure of $(u_k)_{k\in\mathbb{N}^*}$ and $(v_k)_{k\in\mathbb{N}^*}$ any Radon measure ν_{joint} on S^*M such that for any $a \in \mathscr{S}^0(M)$, there holds*

$$(Op(a)u_{\sigma(k)}, v_{\sigma(k)}) \underset{k\to+\infty}{\longrightarrow} \int_{S^*M} a d\nu_{joint}$$

for some extraction σ.

Note that joint microlocal defect measures are not necessarily non-negative, and that joint Quantum Limits (similarly defined) are not necessarily invariant by the geodesic flow, even in the Riemannian case. However, the following lemma, proved in Appendix A, shows a regularity property for these joint microlocal defect measures.

Lemma 3. *Let $(u_k), (v_k)$ be two sequences of functions weakly converging to 0, each with a unique microlocal defect measure, which we denote respectively by μ_{11} and μ_{22}. Then, any joint microlocal defect measures μ_{12} (resp. μ_{21}) of $(u_k)_{k\in\mathbb{N}^*}$ and $(v_k)_{k\in\mathbb{N}^*}$ (resp. of $(v_k)_{k\in\mathbb{N}^*}$ and $(u_k)_{k\in\mathbb{N}^*}$) is absolutely continuous with respect to both μ_{11} and μ_{22}.*

Using Lemma 3, we then notice that if $\mathcal{J}, \mathcal{J}' \in \mathcal{P} \setminus \{\emptyset\}$ are distinct, the joint microlocal defect measures of $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ and $(\varphi_k^{\mathcal{J}'})_{k\in\mathbb{N}^*}$ vanish. Similarly, the joint microlocal defect measure of $(\varphi_k^0)_{k\in\mathbb{N}^*}$ with the sequence $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ vanishes for any $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. Therefore, evaluating $(\mathrm{Op}(a)\varphi_k, \varphi_k)$ and using (14), we obtain (12), which finishes the proof of Lemma 2. □

Remark 9. *The above proof is inspired by the proof of a slightly different fact (see [Gér91a, Proposition 3.3]): if θ is the unique microlocal defect measure of a sequence $(\psi_k)_{k\in\mathbb{N}^*}$ of functions over a manifold M, A (resp. B) is a closed (resp. open) subset of S^*M, and A and B form a partition of S^*M, then we can write $\theta = \theta_A + \theta_B$, with θ_A (resp. θ_B) supported in A (resp. $\theta_B(A) = 0$) and $\psi_k = \psi_k^A + \psi_k^B$ such that θ_A (resp. θ_B) is a microlocal defect measure of $(\psi_k^A)_{k\in\mathbb{N}^*}$ (resp. of $(\psi_k^B)_{k\in\mathbb{N}^*}$). The proof just consists in choosing symbols $p_n \in \mathscr{S}^0(M)$ concentrating on A and taking $\psi_k^A = Op(p_n)\psi_k$ as in the proof above.*

In the proof of Lemma 2, we had to choose particular symbols p_n in order to ensure that $\varphi_k^{\mathcal{J}}$ and φ_k^0 are still eigenfunctions of $-\Delta$.

Restriction to a fixed $\mathcal{J} \in \mathcal{P}\setminus\{\emptyset\}$. Combining Lemma 2 with Point (1) of Proposition 1, we see that it is enough to prove Point (2) of Theorem 1, and that it is possible to assume that $(\varphi_k)_{k\in\mathbb{N}^*}$ is a sequence of eigenfunctions with eigenvalue tending to $+\infty$, and with a unique microlocal defect measure ν, which can be assumed to be supported in $S\Sigma$. Indeed, thanks to Lemma 2, we can even assume that all the mass of ν is contained in $S\Sigma_{\mathcal{J}}$ for some $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, i.e., $\nu = \nu^{\mathcal{J}}$: once we have established the decomposition

$$\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \nu_s^{\mathcal{J}} dQ^{\mathcal{J}}(s),$$

Point (2) of Theorem 1 follows by just gluing all pieces of ν together thanks to Lemma 2.

Therefore, in order to establish Point (2) of Theorem 1, we assume that the unique microlocal defect measure of $(\varphi_k)_{k\in\mathbb{N}^*}$ has no mass outside $S\Sigma_{\mathcal{J}}$ for some $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. By symmetry, we may even assume that $\mathcal{J} = \{1, \dots, J\}$ with $J = \mathrm{Card}(\mathcal{J})$.

To sum up, the sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ that we consider is no more an orthonormal Hilbert basis as stated in Theorem 1, but it satisfies the following assumption:

Assumption 1. *$(\varphi_k)_{k\in\mathbb{N}^*}$ is a bounded sequence of eigenfunctions of $-\Delta$ labeled with increasing eigenvalues tending to $+\infty$, and with unique microlocal defect measure ν. Moreover, there exist $J \leq m$ and $r(k) \to +\infty$ as $k \to +\infty$ such that*

$$\varphi_k = P_{r(k)}^{\mathcal{J}} \varphi_k \tag{15}$$

for $\mathcal{J} = \{1, \dots, J\}$ and for any $k \in \mathbb{N}^*$, where $P_n^{\mathcal{J}}$ is defined in (13). In particular, ν has no mass outside $S\Sigma_{\mathcal{J}}$.

2.3 Step 2: End of the proof of Point (2) of Theorem 1

In the sequel, the notation $(\cdot, \cdot)$ stands for the $L^2(\mathbf{H}^m)$ scalar product, and the associated norm is denoted by $\|\cdot\|_{L^2}$.

Positive cones. We set $V = \left(-\frac{1}{2}, \dots, -\frac{1}{2}\right) \in \mathbb{R}^J$ and we consider the quadrant

$$V + \mathbb{R}_+^J = \left\{ (x_1, \dots, x_J) \in \mathbb{R}^J \mid x_j \geq -\frac{1}{2} \text{ for any } 1 \leq j \leq J \right\}.$$

We now define a series of partitions of $V + \mathbb{R}_+^J$ into positive cones with vertex at V, each of these partitions (indexed by N) being composed of 2^N thin positive cones, with the property that each partition is a refinement of the preceding one.

More precisely, these positive cones $C_\ell^N \subset V + \mathbb{R}_+^J$, for $N \in \mathbb{N}^*$ and $0 \leq \ell \leq 2^N - 1$, satisfy the following properties, some of which are illustrated on Figure 1 below:

(1) For any $N \in \mathbb{N}^*$ and any $0 \leq \ell \leq 2^N - 1$, C_ℓ^N is a positive cone with vertex at V, i.e.,

$$V + \lambda(W - V) \in C_\ell^N, \quad \forall \lambda > 0, \ \forall W \in C_\ell^N;$$

(2) For any $N \in \mathbb{N}^*$, $(C_\ell^N)_{0 \leq \ell \leq 2^N - 1}$ is a partition of $V + \mathbb{R}_+^J$, i.e.,

$$\bigcup_{\ell=0}^{2^N-1} C_\ell^N = V + \mathbb{R}_+^J \quad \text{and} \quad C_\ell^N \cap C_{\ell'}^N = \emptyset, \ \forall \ell \neq \ell';$$

(3) Each partition is a refinement of the preceding one: for any $N \geq 2$ and any $0 \leq \ell \leq 2^N - 1$, there exists a unique $0 \leq \ell' \leq 2^{N-1} - 1$ such that $C_\ell^N \subset C_{\ell'}^{N-1}$.

Denote by $\mathscr{L}$ the set of half-lines issued from V and contained in $V + \mathbb{R}_+^J$. Note that $\mathscr{L}$ is parametrized by $s \in \mathbf{S}_\mathcal{J}$. We also assume the following property:

(4) For any $L \in \mathscr{L}$ parametrized by $s \in \mathbf{S}_\mathcal{J}$, there exists a subsequence $(C_{\ell(s,N)}^N)_{N \in \mathbb{N}^*}$ which converges to $\mathscr{L}$, in the following sense. There exists $d : \mathbb{N} \to \mathbb{R}^+$ with $d \to 0$ as $N \to +\infty$, such that, for any $s' \in \mathbf{S}_\mathcal{J}$ parametrizing a half-line $L' \in \mathscr{L}$ contained in $\mathbf{S}_{\ell(s,N)}^N$, we have

$$\|s' - s\|_1 \leq d(N). \tag{16}$$

This last property is equivalent to saying that the size of the positive cones tends uniformly to 0 as $N \to +\infty$.

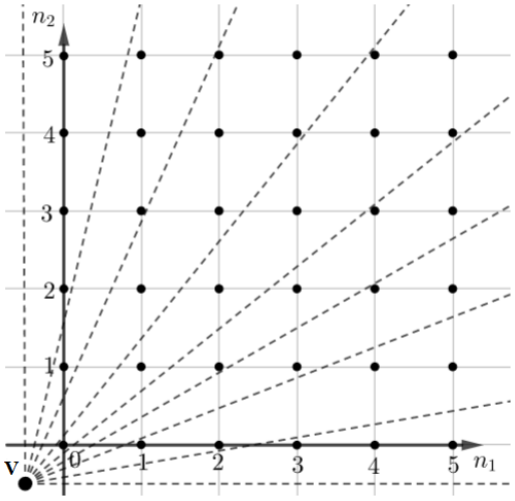

Figure 1: The positive cones C_ℓ^N, for $J = 2$, $N = 3$.

Spectral decomposition. Decomposing φ_k on the spaces $\mathcal{H}^{\mathcal{J}}_{(n_j)}$ defined in Section 2.1, we write

$$\varphi_k = \sum_{\ell=0}^{2^N-1} \varphi^N_{k,\ell} \tag{17}$$

where

$$\varphi^N_{k,\ell} = \sum_{(n_1,\dots,n_J)\in C^N_\ell} \varphi_{k,n_1,\dots,n_J}$$

and, for any $(n_j) \in \mathbb{N}^{\mathcal{J}}$, $k \in \mathbb{N}^*$ and $j \in \mathcal{J}$,

$$\Omega_j \varphi_{k,n_1,\dots,n_J} = (2n_j+1)\varphi_{k,n_1,\dots,n_J}.$$

For any $N \in \mathbb{N}^*$ and any $0 \leq \ell \leq 2^N - 1$, we take ν^N_ℓ to be a microlocal defect measure of the sequence $(\varphi^N_{k,\ell})_{k\in\mathbb{N}^*}$. By diagonal extraction in $k \in \mathbb{N}^*$ (which we omit in the notations), we can assume that any of these microlocal defect measures is obtained with respect to the same subsequence.

Lemma 4. *The following properties hold:*

(1) All the mass of ν^N_ℓ is contained in $S\Sigma_{\mathcal{J}}$ for any $N \in \mathbb{N}^*$ and any $0 \leq \ell \leq 2^N - 1$;

(2) For $N \in \mathbb{N}^*$ and $\ell \neq \ell'$ with $0 \leq \ell, \ell' \leq 2^N - 1$, the joint microlocal defect measure (see Definition 8) of $(\varphi^N_{k,\ell})_{k\in\mathbb{N}^*}$ and $(\varphi^N_{k,\ell'})_{k\in\mathbb{N}^*}$ vanishes. In particular, for any $N \in \mathbb{N}^*$,

$$\nu = \sum_{\ell=0}^{2^N-1} \nu^N_\ell. \tag{18}$$

Proof. We first prove Point (1). Using (15), (17) and the fact that $P^{\mathcal{J}}_n \in \Psi^0(\mathbf{H}^m)$ commutes with the operators Ω_j and R_j, we get that

$$\varphi^N_{k,\ell} = P^{\mathcal{J}}_{r(k)} \varphi^N_{k,\ell}.$$

Point (1) now follows from the fact that $\sigma_P(P^{\mathcal{J}}_{r(k)}) \to \mathbf{1}_{\Sigma_{\mathcal{J}}}$ as $k \to +\infty$ (see the proof of Lemma 2).

We now turn to the proof of Point (2). Let N, ℓ, ℓ' be as in the statement. By Point (1) and Lemma 3, we know that the joint microlocal defect measure of $(\varphi^N_{k,\ell})_{k\in\mathbb{N}^*}$ and $(\varphi^N_{k,\ell'})_{k\in\mathbb{N}^*}$ has no mass outside $S\Sigma_{\mathcal{J}}$.

Let $b \in \mathscr{S}^0(\mathbf{H}^m)$ which is microlocally supported in a conic set in which R_j, Ω_j act as first-order pseudodifferential operators for any $j \in \mathcal{J}$. A typical example of microlocal support for b is given by any conic subset of $T^*\mathbf{H}^m$ whose intersection with some conic neighborhood of the set $\{p_{z_j} = 0\}$ is reduced to 0, for any $j \in \mathcal{J}$. We set $U(t) = U(t_1, \dots, t_J) = e^{i(t_1\Omega_1 + \dots + t_J\Omega_J)}$ for $t = (t_1, \dots, t_J) \in (\mathbb{R}/2\pi\mathbb{Z})^J$.

The average of $\mathrm{Op}(b)$ is then defined by

$$A = \int_{(\mathbb{R}/2\pi\mathbb{Z})^J} U(-t)\mathrm{Op}(b)U(t)dt$$

(see [Wei77]). For $1 \leq j \leq J$, since

$$\frac{d}{dt_j} U(-t)\mathrm{Op}(b)U(t) = U(-t)[\mathrm{Op}(b), \Omega_j]U(t),$$

integrating in the t_j variable, using that all Ω_i commute together, and that $\exp(2i\pi\Omega_j) = \mathrm{Id}$ (since the eigenvalues of Ω_j belong to $\mathbb{N}$), we get that $[A, \Omega_j] = 0$ for any $1 \leq j \leq J$.

By a bracket computation, A has principal symbol

$$a := \sigma_P(A) = \int_{(\mathbb{R}/2\pi\mathbb{Z})^J} b \circ \theta_1(t_1) \circ \dots \circ \theta_J(t_J)\ dt.$$

Here, $\theta_j(\cdot)$ denotes, for $1 \leq j \leq J$, the 2π-periodic flow of the Hamiltonian vector field of $\sigma_P(\Omega_j)$ (see [CdVHT18, Lemma 6.1] for similar arguments).

Remark 10. *If D is a 0^{th}-order pseudodifferential operator on $\mathbf{H}^m$ which satisfies $[D,\Omega_j]=0$ for any $j\in\mathcal{J}$, then D leaves $\mathcal{H}^{\mathcal{J}}_{(n_j)}$ invariant for any $(n_j)=(n_1,\dots,n_J)\in\mathbb{N}$. It follows that for any $f\in\mathcal{H}^{\mathcal{J}}_{(n_j)}$ and any $g\in\mathcal{H}^{\mathcal{J}}_{(n'_j)}$ such that $(n_1,\dots,n_J)\neq(n'_1,\dots,n'_J)$, we have $(Df,g)=0$.*

We know that $\sigma_P(A)=b$ on $S\Sigma_{\mathcal{J}}$. Therefore,

$$(\mathrm{Op}(b)\varphi^N_{k,\ell},\varphi^N_{k,\ell'})-(A\varphi^N_{k,\ell},\varphi^N_{k,\ell'})\underset{k\to+\infty}{\longrightarrow}0.$$

Since A commutes with Ω_j for any $1\leq j\leq J$, by Remark 10, we know that $(A\varphi^N_{k,\ell},\varphi^N_{k,\ell'})=0$. Hence, $(\mathrm{Op}(b)\varphi^N_{k,\ell},\varphi^N_{k,\ell'})$ tends to 0 as $k\to+\infty$. Using this result for all possible b with microlocal support satisfying the property recalled at the beginning of the proof, we obtain that the joint microlocal defect measure of $(\varphi^N_{k,\ell})_{k\in\mathbb{N}^*}$ and of $(\varphi^N_{k,\ell'})_{k\in\mathbb{N}^*}$ vanishes. Evaluating $(\mathrm{Op}(b)\varphi_k,\varphi_k)$ in the limit $k\to+\infty$ and using (17), we conclude the proof of Point (2). □

Approximate invariance. We fix $N\in\mathbb{N}^*$ and $0\leq\ell\leq 2^N-1$ and we consider $s\in\mathbf{S}_{\mathcal{J}}$ such that the half-line issued from V and defined by the J equations $\frac{2x_j+1}{2x_1+1+\dots+2x_J+1}=s_j$ (and $x_j\geq-1/2$) lies in C^N_ℓ.

Let A be a 0-th order pseudodifferential operator microlocally supported in a conic set where R_j,Ω_j act as first-order pseudodifferential operators for any $j\in\mathcal{J}$. Assume moreover that A commutes with $\Omega_1,\dots,\Omega_J$ and $\Delta_{J+1},\dots,\Delta_m$. Recall that R_s was defined in (7). Using that $[A,R_s]$ commutes with $\Omega_1,\dots,\Omega_J$ in order to kill crossed terms (see Remark 10), we have

$$\begin{aligned}([A,R_s]\varphi^N_{k,\ell},\varphi^N_{k,\ell}) &= ([A,R_s]\sum_{(n_1,\dots,n_J)\in C^N_\ell}\varphi_{k,n_1,\dots,n_J},\sum_{(n_1,\dots,n_J)\in C^N_\ell}\varphi_{k,n_1,\dots,n_J})\\ &= \sum_{(n_1,\dots,n_J)\in C^N_\ell}([A,R_s]\varphi_{k,n_1,\dots,n_J},\varphi_{k,n_1,\dots,n_J}) \qquad (19)\end{aligned}$$

Let us fix $(n_1,\dots,n_J)\in C^N_\ell$ and prove that

$$\begin{aligned}&([A,R_s]\varphi_{k,n_1,\dots,n_J},\varphi_{k,n_1,\dots,n_J})\\ &\qquad=\sum_{j=1}^J\left(s_j-\frac{2n_j+1}{\sum_{i=1}^J2n_i+1}\right)([A,R_j]\varphi_{k,n_1,\dots,n_J},\varphi_{k,n_1,\dots,n_J}) \qquad (20)\end{aligned}$$

We set

$$R=\frac{\sum_{j=1}^J(2n_j+1)R_j-\sum_{i=J+1}^m\Delta_i}{\sum_{j=1}^J2n_j+1}.$$

and, for the sake of simplicity of notations, $\varphi=\varphi_{k,n_1,\dots,n_J}$. Using that R is selfadjoint (since R_j is selfadjoint for any j) and that φ is an eigenfunction of R, we get

$$([A,R]\varphi,\varphi)=(AR\varphi,\varphi)-(A\varphi,R\varphi)=0$$

and therefore, since A commutes with $\Delta_{J+1},\dots,\Delta_m$, we get

$$([A,R_s]\varphi,\varphi)=([A,R_s-R]\varphi,\varphi)=\sum_{j=1}^J\left(s_j-\frac{2n_j+1}{\sum_{i=1}^J2n_i+1}\right)([A,R_j]\varphi,\varphi)$$

which is exactly (20).

Thanks to our choice of microlocal support for A, we know that $[A,R_j]\in\Psi^0(\mathbf{H}^m)$ for $1\leq j\leq J$. Combining (19) and (20), we obtain

$$\begin{aligned}\left|([A,R_s]\varphi^N_{k,\ell},\varphi^N_{k,\ell})\right| &\leq C\sum_{(n_1,\dots,n_J)\in C^N_\ell}\sum_{j=1}^J\left|s_j-\frac{2n_j+1}{\sum_{i=1}^J2n_i+1}\right|\|\varphi_{k,n_1,\dots,n_J}\|^2_{L^2} \qquad (21)\\ &\leq Cd(N)\|\varphi^N_{k,\ell}\|^2_{L^2}\end{aligned}$$

where in the last line, we used (16) and the fact that the decomposition (11) is orthogonal.

In order to pass to the limit $k \to +\infty$ in these last inequalities, we note that

$$\sigma_P([A, R_s])_{|\Sigma_{\mathcal{J}}} = \{a_{|\Sigma_{\mathcal{J}}}, \rho_s\}_{\omega_{|\Sigma_{\mathcal{J}}}} \tag{22}$$

(see [CdVHT18, Lemma 6.2] for a similar identity). Here, the Poisson bracket $\{\cdot,\cdot\}_{\omega_{|\Sigma_{\mathcal{J}}}}$ is the Poisson bracket on the manifold $(\Sigma_{\mathcal{J}}, \omega_{|\Sigma_{\mathcal{J}}})$ which is symplectic as it is defined as a product of symplectic manifolds (recall that for $m = 1$, the 4-dimensional manifold Σ is symplectic, see for example [CdVHT18]).

Since all the mass of ν_ℓ^N is contained in $S\Sigma_{\mathcal{J}}$ by Lemma 4, we finally deduce from (21) the upper bound

$$\int_{S\Sigma_{\mathcal{J}}} \{a_{|\Sigma_{\mathcal{J}}}, \rho_s\}_{\omega_{|\Sigma_{\mathcal{J}}}} \, d\nu_\ell^N \leq Cd(N)\nu_\ell^N(S\Sigma_{\mathcal{J}}). \tag{23}$$

The upper bound (23) has been established only for $a_{|\Sigma_{\mathcal{J}}}$ the restriction to $\Sigma_{\mathcal{J}}$ of the symbol of an operator A of order 0 *which commutes with* $\Omega_1, \dots, \Omega_J$ *and* $\Delta_{J+1}, \dots, \Delta_m$, and we would like to remove this commutation assumption. Let $b \in \mathscr{S}^0(\mathbf{H})$ of the form

$$b(q,p) = b_{\mathcal{J}}(q_1, \dots, q_J, p_1, \dots, p_J)$$

where (q,p) denote the coordinates in $T^*\mathbf{H}^m$, (q_j, p_j) the coordinates in the cotangent bundle of the j-th copy of $\mathbf{H}$, and $b_{\mathcal{J}} \in \mathscr{S}^0(\mathbf{H}^{\mathcal{J}})$ is an arbitrary 0-th order symbol supported in a subset of $T^*\mathbf{H}^{\mathcal{J}}$ where R_j, Ω_j act as first-order pseudodifferential operators for any $j \in \mathcal{J}$. We consider the operator

$$A = \int_{(\mathbb{R}/2\pi\mathbb{Z})^J} U(-t)\mathrm{Op}(b)U(t)dt \ \in \Psi^0(\mathbf{H}^m)$$

where $U(t) = U(t_1, \dots, t_J) = e^{i(t_1\Omega_1 + \dots + t_J\Omega_J)}$ for $t = (t_1, \dots, t_J) \in (\mathbb{R}/2\pi\mathbb{Z})^J$. By an argument that we have already in the proof of Point (2) of Lemma (4), A commutes with Ω_j for any $1 \leq j \leq J$, and it also commutes with $\Delta_{J+1}, \dots, \Delta_m$. Moreover, the principal symbol of A on $S\Sigma_{\mathcal{J}}$ coincides with $b_{\mathcal{J}}$ by the Egorov theorem. Using (23) for A, this proves that (23) is valid for any symbol a of order 0 on $\mathbf{H}^m$ supported far from the sets $\{p_{z_j} = 0\}$ for $j \in \mathcal{J}$, without any assumption of commutation on A.

Disintegration of measures. From the equality (18) taken in the limit $N \to +\infty$, we will deduce that $\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \nu_s^{\mathcal{J}} dQ^{\mathcal{J}}(s)$. Note that a simple Fubini argument does not suffice since $Q^{\mathcal{J}}$ is not the Lebesgue measure in general (it may contain Dirac masses, see Section 1.4). Instead, we have to adapt the proof of the classical disintegration of measure theorem (see [Roh62]).

First of all, we define a measure $Q^{\mathcal{J}}$ over $\mathbf{S}_{\mathcal{J}}$ as follows. It was explained at the beginning of Section 2.3 that the set $\mathscr{L}$ of half-lines issued from V and contained in $V + \mathbb{R}_+^J$ is parametrized by $s \in \mathbf{S}_{\mathcal{J}}$. For $N \in \mathbb{N}^*$ and $0 \leq \ell \leq 2^N - 1$, we consider the subset of $\mathbf{S}_{\mathcal{J}}$ given by

$$\mathbf{S}_\ell^N = \left\{ s \in \mathbf{S}_{\mathcal{J}}, \ s \text{ parametrizes a half-line of } \mathscr{L} \text{ contained in } C_\ell^N \right\}. \tag{24}$$

Then we define

$$Q^{\mathcal{J}}(\mathbf{S}_\ell^N) = \nu_\ell^N(S\Sigma). \tag{25}$$

This definition is consistent thanks to the partition of $V + \mathbb{R}_+^J$ into nested positive cones: $Q^{\mathcal{J}}$ is well-defined on any $\mathbf{S}_\ell^N$ and it is also additive. By the properties of the positive cones C_ℓ^N, for any $s \in \mathbf{S}_{\mathcal{J}}$, there exists a sequence $(\ell(s,N))_{N\in\mathbb{N}^*}$ such that $\mathbf{S}_{\ell(s,N)}^N \subset \mathbf{S}_{\mathcal{J}}$ converges to s, in the sense that any sequence $(s^N)_{N\in\mathbb{N}^*}$ such that $s^N \in \mathbf{S}_{\ell(s,N)}^N$ for any $N \in \mathbb{N}^*$ converges to s as $N \to +\infty$. Therefore, by extension, (25) defines a (unique) non-negative Radon measure $Q^{\mathcal{J}}$ on $\mathbf{S}_{\mathcal{J}}$.

Given $N \geq 1$, $0 \leq \ell \leq 2^N - 1$ and a continuous function $f : S\Sigma_{\mathcal{J}} \to \mathbb{R}$, we set

$$f_\ell^N = \frac{1}{\nu_\ell^N(S\Sigma_{\mathcal{J}})} \int_{S\Sigma_{\mathcal{J}}} f d\nu_\ell^N \tag{26}$$

if $\nu_\ell^N(S\Sigma_{\mathcal{J}}) \neq 0$, and $f_\ell^N = 0$ otherwise.

Proposition 5. *Given any continuous function $f : S\Sigma \to \mathbb{R}$, for $Q^{\mathcal{J}}$-almost all $s \in \mathbf{S}_{\mathcal{J}}$, there exists a real number $e(f)(s)$ such that*

$$f^N_{\ell(s,N)} \underset{N\to+\infty}{\longrightarrow} e(f)(s),$$

where, for any $N \in \mathbb{N}^*$, $\ell(s,N)$ is the unique integer $0 \leq \ell(s,N) \leq 2^N - 1$ such that $s \in \mathbf{S}^N_{\ell(s,N)}$.
In the sequel, we call $\ell(s,N)$ the approximation at order N of s.

Proof. By linearity of formula (26), it is sufficient to prove the statement for $f \geq 0$. Therefore, in the sequel, we fix $f \geq 0$. For $N \geq 1$, we define the function $f^N : \mathbf{S}_{\mathcal{J}} \to \mathbb{R}$ by $f^N(s) = f^N_{\ell(s,N)}$, where $\ell(s,N)$ is the approximation at order N of s. Note that f^N is constant on $\mathbf{S}^N_\ell$ for $0 \leq \ell \leq 2^N - 1$.

For $0 \leq \alpha < \beta \leq 1$, we define $S(\alpha,\beta)$ as the set of $s \in \mathbf{S}_{\mathcal{J}}$ such that

$$\liminf_{N\to+\infty} f^N(s) < \alpha < \beta < \limsup_{N\to+\infty} f^N(s).$$

To prove Proposition 5, it is sufficient to prove that $S(\alpha,\beta)$ has $Q^{\mathcal{J}}$-measure 0 for any $0 \leq \alpha < \beta \leq 1$. Fix such α, β. For $s \in S(\alpha,\beta)$, take a sequence $1 \leq N_1^\alpha(s) < N_1^\beta(s) < N_2^\alpha(s) < N_2^\beta(s) < ... < N_k^\alpha(s) < N_k^\beta(s) < ...$ of integers such that $f^{N_k^\alpha(s)}(s) < \alpha$ and $f^{N_k^\beta(s)}(s) > \beta$ for any $k \geq 1$. We finally define the following sets:

$$A_k = \bigcup_{s\in S(\alpha,\beta)} \mathbf{S}^{N_k^\alpha(s)}_{\ell(s,N_k^\alpha(s))}$$

$$B_k = \bigcup_{s\in S(\alpha,\beta)} \mathbf{S}^{N_k^\beta(s)}_{\ell(s,N_k^\beta(s))}$$

We have $S(\alpha,\beta) \subset A_{k+1} \subset B_k \subset A_k$ for every $k \geq 1$. In particular,

$$S(\alpha,\beta) \subset \tilde{S}(\alpha,\beta) := \bigcap_{k\in\mathbb{N}^*} A_k = \bigcap_{k\in\mathbb{N}^*} B_k. \tag{27}$$

Given any two of the sets $\mathbf{S}^{N_k^\alpha(s)}_{\ell(s,N_k^\alpha(s))}$ that form A_k, either they are disjoint or one is contained in the other. Consequently, A_k may be written as a disjoint union of such sets, denoted by $A_k^{k'}$. Therefore,

$$\int_{A_k} f dQ^{\mathcal{J}} = \sum_{k'} \int_{A_k^{k'}} f dQ^{\mathcal{J}} < \sum_{k'} \alpha Q^{\mathcal{J}}(A_k^{k'}) = \alpha Q^{\mathcal{J}}(A_k)$$

and analogously, with similar notations,

$$\int_{B_k} f dQ^{\mathcal{J}} = \sum_{k'} \int_{B_k^{k'}} f dQ^{\mathcal{J}} > \sum_{k'} \beta Q^{\mathcal{J}}(B_k^{k'}) = \beta Q^{\mathcal{J}}(B_k).$$

Since $B_k \subset A_k$, we get $\alpha Q^{\mathcal{J}}(A_k) > \beta Q^{\mathcal{J}}(B_k)$. Taking the limit $k \to +\infty$, it yields $\alpha Q^{\mathcal{J}}(\tilde{S}(\alpha,\beta)) > \beta Q^{\mathcal{J}}(\tilde{S}(\alpha,\beta))$, which is possible only if $Q^{\mathcal{J}}(\tilde{S}) = 0$. Therefore, using (27), we get $Q^{\mathcal{J}}(S) = 0$, which concludes the proof of the proposition. □

From (18) and (26), we infer that for any $N \geq 1$,

$$\int_{S\Sigma_{\mathcal{J}}} f d\nu^{\mathcal{J}} = \sum_{\ell=0}^{2^N-1} \int_{S\Sigma_{\mathcal{J}}} f d\nu_\ell^N = \sum_{\ell=0}^{2^N-1} f_\ell^N \nu_\ell^N(S\Sigma_{\mathcal{J}}),$$

and the dominated convergence theorem together with the definition of $Q^{\mathcal{J}}$ and Proposition 5 yield

$$\int_{S\Sigma_{\mathcal{J}}} f d\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} e(f)(s) dQ^{\mathcal{J}}(s). \tag{28}$$

We see that for a fixed $s \in \mathbf{S}_{\mathcal{J}}$,

$$C^0(S\Sigma_{\mathcal{J}}, \mathbb{R}) \ni f \mapsto e(f)(s) \in \mathbb{R}$$

is a non-negative linear functional on $C^0(S\Sigma_{\mathcal{J}}, \mathbb{R})$. By the Riesz-Markov theorem, there exists a unique Radon probability measure $\nu_s^{\mathcal{J}}$ on $S\Sigma_{\mathcal{J}}$ such that

$$e(f)(s) = \int_{S\Sigma_{\mathcal{J}}} f d\nu_s^{\mathcal{J}}. \tag{29}$$

Putting (28) and (29) together, we get

$$\int_{S\Sigma_{\mathcal{J}}} f d\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \left(\int_{S\Sigma_{\mathcal{J}}} f d\nu_s^{\mathcal{J}} \right) dQ^{\mathcal{J}}(s)$$

which is the desired disintegration of measures formula.

Conclusion of the proof. There remains to show that $\nu_s^{\mathcal{J}}$ is invariant by $\vec{\rho}_s^{\mathcal{J}}$. Let $a \in \mathscr{S}^0(\mathbf{H}^m)$ be supported in cone of $T^*\mathbf{H}^m$ whose intersection with some conic neighborhood of the set $\{p_{z_j} = 0\}$ is reduced to 0, for any $j \in \mathcal{J}$. For $Q^{\mathcal{J}}$-almost every $s \in \mathbf{S}_{\mathcal{J}}$, we have

$$\begin{aligned}
\int_{S\Sigma_{\mathcal{J}}} \{a, \rho_s^{\mathcal{J}}\} d\nu_s^{\mathcal{J}} &= e(\{a, \rho_s^{\mathcal{J}}\})(s) \qquad \text{(by (29))} \\
&= \lim_{N \to +\infty} \frac{1}{\nu_{\ell(s,N)}^N(S\Sigma_{\mathcal{J}})} \int_{S\Sigma_{\mathcal{J}}} \{a, \rho_s^{\mathcal{J}}\} d\nu_{\ell(s,N)}^N \qquad (30) \\
&\leq \lim_{N \to +\infty} Cd(N) \qquad \text{(by (23))} \\
&= 0
\end{aligned}$$

with the convention that if the denominator in (30) is null, then the whole expression is null. For an arbitrary $a \in \mathscr{S}^0(\mathbf{H}^m)$, taking a sequence $a_n \in \mathscr{S}^0(\mathbf{H}^m)$ whose support has the above property and such that $a_n \to a$ in $S\Sigma_{\mathcal{J}}$ (in the space of symbols) as $n \to +\infty$, we see that the above quantity also vanishes since $\nu_s^{\mathcal{J}}$ has finite mass and $\{a_n, \rho_s^{\mathcal{J}}\} \to \{a, \rho_s^{\mathcal{J}}\}$ in $S\Sigma_{\mathcal{J}}$ as $n \to +\infty$. This implies that $\nu_s^{\mathcal{J}}$ is invariant by the flow $e^{t\vec{\rho}_s^{\mathcal{J}}}$, which concludes the proof of Theorem 1.

3 Proof of Theorem 2

In this section, we prove Theorem 2. The four steps are the following:

1. In Lemma 6 and Lemma 7, we prove the result for a fixed $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, $Q^{\mathcal{J}}$ the Dirac mass at some $s \in \mathbf{S}_{\mathcal{J}}$, and $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$
   (i) has no mass outside $S\Sigma_{\mathcal{J}}$,
   (ii) is invariant under the flow of $\vec{\rho}_s^{\mathcal{J}}$,
   (iii) and *is in a simple tensor form* that we make precise below.

In other words, if $\nu_\infty = \nu_s^{\mathcal{J}}$ with $\nu_s^{\mathcal{J}}$ satisfying (i), (ii) and (iii), then it is a QL.

2. In Lemma 8, we extend the result of Step 1 to the case where (iii) is not necessarily satisfied, i.e., $\nu_\infty = \nu_s^{\mathcal{J}}$ satisfies only (i) and (ii).
3. In Lemma 9, we extend the result of Steps 1 and 2 to the case where $\nu_\infty \in \mathscr{P}_{S\Sigma}$ has no mass outside $S\Sigma_{\mathcal{J}}$ for some $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, i.e., $\nu_\infty = \nu^{\mathcal{J}}$.
4. Finally, using the previous result for all $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, we prove Theorem 2 in full generality (i.e., for arbitrary $\nu_\infty \in \mathscr{P}_{S\Sigma}$).

The map $\Sigma \to \mathbf{H}^m \times \mathbb{R}^m$, $(q,p) \mapsto (q, p_{z_1}, \dots, p_{z_m})$ is an isomorphism, and thus, in the sequel, we consider the coordinates $(q, p_{z_1}, \dots, p_{z_m})$ on Σ and in the coordinates $(q, p_{z_1} : \cdots : p_{z_m})$ on $S\Sigma$, where the notation $p_{z_1} : \cdots : p_{z_m}$ stands for homogeneous coordinates.

Let us summarize the proof. We fix $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. Since any two of the operators R_j and $\Omega_{j'}$ for $j, j' \in \mathcal{J}$ commute, the orthogonal decomposition (11) can be refined: more precisely, given $(n_j) \in \mathbb{N}^{\mathcal{J}}$ and $(\alpha_j) \in (\mathbb{Z}\setminus\{0\})^{\mathcal{J}}$, we consider the joint eigenspace $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)} \subset L^2(\mathbf{H}^m)$ on which the operator $\frac{1}{i}\partial_{z_j}$ acts as α_j and Ω_j acts as $2n_j + 1$.

ν_∞ is obtained as a QL of an orthonormal sequence of eigenfunctions $(\varphi_k)_{k\in\mathbb{N}^*}$ which is described through its components in these eigenspaces. Moreover, each of the four steps is achieved by taking linear combinations of eigenfunctions (with same eigenvalues) used in the previous step. Therefore, the number of eigenspaces $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}$ used for building $(\varphi_k)_{k\in\mathbb{N}^*}$ increases at each step.

In order to achieve Step 1, we focus on the eigenspaces $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}$ corrreponding to

$$\frac{2n_j+1}{\sum_{i\in\mathcal{J}}(2n_i+1)} \approx s_j \quad \text{and} \quad \frac{\alpha_j}{\alpha_{j'}} \approx \frac{p_{z_j}}{p_{z_{j'}}}$$

for any $j, j' \in \mathcal{J}$.

For Step 2, we add the results of the previous step for different $p \in S\Sigma_{\mathcal{J}}$, and we take care that each term in the sum corresponds to the same value of $-\Delta$. Hence, $(n_j) \in \mathbb{N}^{\mathcal{J}}$ is the same as in Step 1, but we use various $(\alpha_j) \in (\mathbb{Z} \setminus \{0\})^{\mathcal{J}}$ to reach all p.

For Step 3, we add the results of Step 2 for different $s \in \mathbf{S}_{\mathcal{J}}$. Therefore, we use the eigenspaces $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}$ also for different $(n_j) \in \mathbb{N}^{\mathcal{J}}$. Finally, in Step 4, we sum the sequences obtained at Step 3 for $\mathcal{J}$ ranging over $\mathcal{P} \setminus \{0\}$.

In order to describe the measures in a "tensor form" which we consider for Step 1, we need to introduce a few notations.

Notations. For the first three steps, we fix $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. Any $s \in \mathbf{S}_{\mathcal{J}}$ can be identified to some homogeneous coordinate $p_{z_1} : \cdots : p_{z_m}$ (with $p_{z_i} = 0$ for $i \notin \mathcal{J}$), in a way which does not depend on $q \in \mathbf{H}^m$. Thus, for any $q \in \mathbf{H}^m$, $t \in \mathbb{R}$ and $s \in \mathbf{S}_{\mathcal{J}}$, it makes sense to consider the point $q + ts \in \mathbf{H}^m$, which has the same coordinates x_j and y_j as q for any $1 \le j \le m$ (only the coordinates z_j for $j \in \mathcal{J}$ change).

Let us consider the set

$$M_q^s = \overline{\{q + ts,\ t \in \mathbb{R}\}} \subset \mathbf{H}^m$$

where the bar denotes the closure in $\mathbf{H}^m$. The set M_q^s is a submanifold of $\mathbf{H}^m$ of dimension $d_q^s \le m$, and we denote by $\mathscr{H}_q^s$ the Hausdorff measure of dimension d_q^s on M_q^s.

For any $(q,p) \in S\Sigma$ and any $q' \in \mathbf{H}^m$, it makes sense to consider the point $(q', p) \in S\Sigma$, which is the point in the fiber of $S\Sigma$ over q that has the same homogeneous coordinates $p_{z_1} : \cdots : p_{z_m}$ as p.

Lemma 6. *Let $(q,p) \in S\Sigma_{\mathcal{J}}$ and $s \in \mathbf{S}_{\mathcal{J}}$ be such that there exists a J-tuple $(n_j) \in \mathbb{N}^{\mathcal{J}}$ with*

$$s_j = \frac{2n_j+1}{\sum_{i\in\mathcal{J}}(2n_i+1)} \tag{31}$$

for any $j \in \mathcal{J}$. Then, the measure $\mathscr{H}_q^s \otimes \delta_p$ is a Quantum Limit.[7]

Proof. Since the s_j are pairwise rationally related, the mapping $t \mapsto q + ts$ is periodic and $d_q^s = 1$. Without loss of generality, we assume that $\mathcal{J} = \{1, \dots, J\}$ for some $1 \leq J \leq m$.

We construct a sequence of eigenfunctions $(\varphi_k)_{k\in\mathbb{N}^*}$ of $-\Delta$ which admits $\mu_{q,p}^s$ as unique Quantum Limit. In our construction, for any $k \in \mathbb{N}^*$, φ_k belongs to the eigenspace $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}$ for some $(n_j) \in \mathbb{N}^{\mathcal{J}}$ and some $(\alpha_j) \in (\mathbb{Z} \setminus \{0\})^{\mathcal{J}}$, and it does not depend on the variables in the i-th copy of $\mathbf{H}$ for $i \notin J$. Our goal is to choose adequately the J-tuples (n_j) and (α_j). Note that a similar argument for $m = 1$ is done in the proof of Point 2 of Proposition 3.2 in [CdVHT18].

We fix a sequence of J-tuples $(\alpha_{1,k}, \dots, \alpha_{J,k}) \in (\mathbb{Z} \setminus \{0\})^J$, for $k \in \mathbb{N}^*$, such that:

- For any $1 \leq j \leq J$, $\alpha_{j,k} \to +\infty$ as $k \to +\infty$, so that for any $1 \leq j, j' \leq J$, there holds
$$\frac{n_{j'}}{\alpha_{j,k}} \underset{k\to+\infty}{\longrightarrow} 0; \tag{32}$$
- For any $1 \leq j, j' \leq J$,
$$\frac{\alpha_{j,k}}{\alpha_{j',k}} \underset{k\to+\infty}{\longrightarrow} \frac{p_{z_j}}{p_{z_{j'}}}, \tag{33}$$
where $p_{z_1} : \cdots : p_{z_m}$ are the homogeneous coordinates of p in $S\Sigma$.

Now, for any $k \in \mathbb{N}^*$, denoting by $\mathbf{1}$ the constant function equal to 1 (on some copy of $\mathbf{H}$), we define
$$\varphi_k = \Phi_k^1 \otimes \ldots \otimes \Phi_k^J \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{m-J \text{ times}}, \tag{34}$$
where, for $1 \leq j \leq J$,
$$\Phi_k^j(x_j, y_j, z_j) = \phi_{j,k}(x_j, y_j) e^{i\alpha_{j,k} z_j}$$
is an eigenfunction of $-\Delta_j$ (on the j-th copy of $\mathbf{H}$) with eigenvalue $(2n_j + 1)|\alpha_{j,k}|$. The precise form of $\phi_{j,k}$ will be given below.

Using (32) and the proof of Lemma 2, notably the pseudodifferential operators $P_n^{\mathcal{J}}$ introduced in (13), we obtain that the mass of any Quantum Limit of $(\varphi_k)_{k\in\mathbb{N}^*}$ is contained in $S\Sigma_{\mathcal{J}}$. Moreover, from the decomposition into cones done in Section 2.3 and the equality (31), we infer that any Quantum Limit of $(\varphi_k)_{k\in\mathbb{N}^*}$ is invariant under $\vec{\rho}_s^{\mathcal{J}}$.

In the next paragraphs, we explain how to choose $\phi_{j,k}$ with eigenvalue $2n_j + 1$ in order to ensure that $(\varphi_k)_{k\in\mathbb{N}^*}$ has a unique QL, which is $\mu_{0,p}^s$. For the sake of simplicity of notations, we set $\alpha = \alpha_{j,k}$. The eigenspace of $-\Delta_j$ corresponding to the eigenvalue $(2n_j + 1)|\alpha|$ is of the form $(A_\alpha^*)^{n_j}(\ker(A_\alpha))e^{i\alpha z}$, where $A_\alpha = \partial_{x_j} + i\partial_{y_j} + i\alpha x_j$ locally, and, accordingly, $A_\alpha^* = -\partial_{x_j} + i\partial_{y_j} + i\alpha x_j$ locally (see for example [CdV84, Section 2]). This follows from a Fourier expansion in the z_j variable, which gives
$$-\Delta_j = \bigoplus_{\gamma\in\mathbb{Z}} B_\gamma, \quad \text{where } B_\gamma = A_\gamma^* A_\gamma + \gamma \text{ for } \gamma \in \mathbb{Z}.$$

We note that the function $f_{j,k}(x_j, y_j) = c_k \exp(-\alpha\frac{x_j^2}{2} + \frac{\alpha}{4}(x_j + iy_j)^2)$ (normalized to 1 thanks to c_k) is a quasimode of A_α, as $\alpha \to +\infty$, for the eigenvalue 0. Moreover, a well-known computation on coherent states (see Example 1 of Chapter 5 in [Zwo12]) guarantees that for any $a \in \mathscr{S}^0(\mathbb{R}^{2m})$,
$$(\mathrm{Op}(a)(A_\alpha^*)^{n_j} f_{j,k}, (A_\alpha^*)^{n_j} f_{j,k}) \underset{k\to+\infty}{\longrightarrow} a(0,0).$$
In other words, $(A_\alpha^*)^{n_j} f_{j,k}$, seen as a sequence of functions of $\mathbb{R}^{2m}$, has a unique Quantum Limit, which is $\delta_{0,0}$.

[7] The associated orthonormal sequence of eigenfunctions is specified in the proof, see also Remark 11.

Now, using that the spectrum of B_α has gaps that are uniformly bounded below, this property is preserved when we consider eigenfunctions of $-\Delta_j$: when α varies, the projection Φ_k^j of $(A_\alpha^*)^{n_j} f_{j,k} e^{i\alpha z}$ onto the eigenspace of $-\Delta_j$ corresponding to the eigenvalue $(2n_j+1)|\alpha|$ has a unique QL, which is $\mathscr{H}_0^s \otimes \delta_p$. The Dirac mass at p comes from (33) and from Lemma 10 applied, for any $1 \leq i, j \leq J$, to the operator $\frac{R_i}{R_j} - \frac{p_i}{p_j}$. Note that the point $q = 0$ plays no specific role, and therefore any measure $\mathscr{H}_q^s \otimes \delta_p$ can be obtained as a QL, when $d_q^s = 1$ and under (31). □

Lemma 7. *Let $(q,p) \in S\Sigma_{\mathcal{J}}$ and $s \in \mathbf{S}_{\mathcal{J}}$ be arbitrary. Then, the measure $\mathscr{H}_q^s \otimes \delta_p$ is a Quantum Limit.*[8]

Proof. We still assume that $\mathcal{J} = \{1, \dots, J\}$. Using Lemma 6, we can assume that $q \in \mathbf{H}^m$ and $s \in \mathbf{S}_j$ verify either $d_q^s \geq 2$, or $d_q^s = 1$ but (31) is not satisfied. In both cases, the following fact holds:

Fact 1. The measure $\mathscr{H}_q^s$ is in the weak-star closure of the set of measures $\mathscr{H}_{q'}^{s'}$ for which $d_{q'}^{s'} = 1$ and (31) is satisfied.

Let us denote by $\mathbb{T}^{\mathcal{J}} = (\mathbb{R}/2\pi\mathbb{Z})^{\mathcal{J}}$ the Riemannian torus of dimension $\#\mathcal{J}$ equipped with the flat metric. Due to the structure of $\Sigma_{\mathcal{J}}$, proving Fact 1 is equivalent to proving the following fact, called Fact 2 in the sequel: if γ is a geodesic of $\mathbb{T}^{\mathcal{J}}$ and $\mathscr{H}_\gamma$ is the Hausdorff measure on γ, then $\mathscr{H}_\gamma$ is in the weak-star closure of the set of measures $\mathscr{H}_{\gamma'}$ with γ' a periodic geodesic of $\mathbb{T}^{\mathcal{J}}$ of slope $(s_1, \dots, s_J)$ verifying (31) for some J-tuple $(n_1, \dots, n_J)$. Let us prove Fact 2.

In case $d_q^s \geq 2$, possibly restricting to the flat torus given by the closure of γ, we can assume that γ is a dense geodesic in $\mathbb{T}^{\mathcal{J}}$. To prove Fact 2 in this elementary case, we take a sequence of geodesics $(\gamma_n')_{n\in\mathbb{N}^*}$ contained in $\mathbb{T}^{\mathcal{J}}$, with rational slopes given by J-tuples $(s_1^n, \dots, s_J^n)$ of the form (31), and which become dense in $\mathbb{T}^{\mathcal{J}}$ as $n \to +\infty$.

For the case $d_q^s = 1$ where (31) is not satisfied, similarly, we take a sequence of geodesics with rational slopes which converges to γ. This proves Fact 2 and hence Fact 1 follows.

Since the set of QLs is closed, Fact 1 implies Lemma 7. □

Remark 11. *Note that, following the proofs of Lemma 6 and Lemma 7, any measure $\mathscr{H}_q^s \otimes \delta_p$ is a Quantum Limit associated to an orthonormal sequence of eigenfunctions $(\varphi_k)_{k\in\mathbb{N}^*}$ such that, for any $k \in \mathbb{N}^*$, φ_k belongs to some eigenspace $\mathcal{H}^{\mathcal{J}}_{(n_{j,k}),(\alpha_{j,k})}$. In particular, φ_k is an eigenfunction of Ω_j for any $j \in \mathcal{J}$.*

Note also that to guarantee this last property, it is not sufficient to invoke, at the end of the proof of Lemma 7, the closedness of the set of QLs: it is necessary to follow the proof of this fact, which consists in a simple extraction argument.

Lemma 8. *Let $s \in \mathbf{S}_{\mathcal{J}}$ and $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$ having no mass outside $S\Sigma_{\mathcal{J}}$ and being invariant under $\vec{\rho}_s^{\mathcal{J}}$. Then $\nu_s^{\mathcal{J}}$ is a Quantum Limit.*[9]

Proof. Let us consider the set $\mathscr{P}_s^{\mathcal{J}} \subset \mathscr{P}(S^*\mathbf{H}^m)$ of probability measures

$$\nu_s^{\mathcal{J}} = \sum_{(q_i,p_i)\in\mathscr{E}} \beta_i \mathscr{H}_{q_i}^s \otimes \delta_{p_i} \tag{35}$$

where i ranges over some finite set $\mathcal{F}$, $\mathscr{E}$ is a set of pairs $(q_i, p_i) \in S\Sigma$, and $\beta_i \in \mathbb{R}$.

We consider $\nu_s^{\mathcal{J}} \in \mathscr{P}_s^{\mathcal{J}}$ defined by (35). Note that if $i \neq i'$, either $\mathscr{H}_{q_i}^s \otimes \delta_{p_i} = \mathscr{H}_{q_i'}^s \otimes \delta_{p_i'}$, or the supports of $\mathscr{H}_{q_i}^s \otimes \delta_{p_i}$ and $\mathscr{H}_{q_i'}^s \otimes \delta_{p_i'}$ are disjoint. Therefore, possibly gathering terms in the above sum, we assume that the supports of $\mathscr{H}_{q_i}^s \otimes \delta_{p_i}$ and $\mathscr{H}_{q_i'}^s \otimes \delta_{p_i'}$ are disjoint as soon as $i \neq i'$.

For $i \in \mathcal{F}$, using Lemma 6 and Lemma 7, we consider a sequence of eigenfunctions $(\varphi_k^i)_{k\in\mathbb{N}^*}$ with eigenvalues $(\lambda_k^i)_{k\in\mathbb{N}^*}$ and whose unique QL is $\mathscr{H}_{q_i}^s \otimes \delta_{p_i}$. According to the

[8] See Remark 11 for the description of the associated orthonormal basis of eigenfunctions.

[9] See Remark 12 for the description of the associated orthonormal basis of eigenfunctions.

proof of these lemmas (see also Remark 11), we can also assume that $\varphi_k^i \in \mathcal{H}^{\mathcal{J}}_{(n_{j,k}),(\alpha^i_{j,k})}$ for some J-tuples such that

$$\lambda_k^i := \sum_{j\in\mathcal{J}} (2n_{j,k}+1)|\alpha^i_{j,k}|$$

does not depend on $i \in \mathcal{F}$. In other words,

- for any $1 \leq j \leq J$, φ_k^i is also an eigenvalue of Ω_j with eigenvalue $n_{j,k}$ which does not depend on $i \in \mathcal{F}$;
- for any $i, i' \in \mathcal{F}$, $\lambda_k^i = \lambda_k^{i'}$ and we denote this common value by λ_k. This means that for any $i \in \mathcal{F}$, φ_k^i belongs to the eigenspace of $-\Delta$ corrresponding to the eigenvalue λ_k.

Since $\mathscr{H}^s_{q_i} \otimes \delta_{p_i}$ and $\mathscr{H}^s_{q'_i} \otimes \delta_{p'_i}$ have disjoint supports, the joint microlocal defect measure of $(\varphi_k^i)_{k\in\mathbb{N}^*}$ and $(\varphi_k^{i'})_{k\in\mathbb{N}^*}$ vanishes for $i \neq i'$ by Lemma 3. It follows that

$$\varphi_k := \sum_{i\in\mathcal{F}} \beta_i \varphi_k^i$$

is an eigenfunction of $-\Delta$ with eigenvalue λ_k, and that in the limit $k \to +\infty$, it admits $\nu_s^{\mathcal{J}}$ as unique Quantum Limit.

Finally, we note that any $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$ having all its mass contained in $S\Sigma_{\mathcal{J}}$ and being invariant under $\vec{\rho}_s^{\mathcal{J}}$ is in the closure of $\mathscr{P}_s^{\mathcal{J}}$. Since the set of QLs is closed, Lemma 8 is proved. □

Remark 12. *The above proof shows that $\nu_\infty = \nu_s^{\mathcal{J}}$ is a QL for an orthonormal sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ such that φ_k belongs to*

$$\bigoplus_{(\alpha_j)\in(\mathbb{Z}^*)^{\mathcal{J}}} \mathcal{H}^{\mathcal{J}}_{(n_{j',k'}),(\alpha_j)}$$

for some J-tuple $(n_{j',k'}) \in \mathbb{N}^{\mathcal{J}}$ which depends only on $k \in \mathbb{N}^*$.

Lemma 9. *Let $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, and*

$$\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \nu_s^{\mathcal{J}} dQ^{\mathcal{J}}(s)$$

for some $Q^{\mathcal{J}} \in \mathscr{P}(\mathbf{S}_{\mathcal{J}})$ and $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$ having no mass outside $S\Sigma_{\mathcal{J}}$ and such that, for $Q^{\mathcal{J}}$-almost any $s \in \mathbf{S}_{\mathcal{J}}$, $\vec{\rho}_s^{\mathcal{J}}\nu_s^{\mathcal{J}} = 0$. Then $\nu^{\mathcal{J}}$ is a Quantum Limit.[10]

Proof. As in the previous proofs, we assume without loss of generality that $\mathcal{J} = \{1, \dots, J\}$ for some $1 \leq J \leq m$. Let $(s^\ell)_{\ell\in\mathcal{L}}$ be a finite family of distinct elements of $\mathbf{S}_{\mathcal{J}}$ indexed by $\mathcal{L}$, and let $\gamma_\ell \in \mathbb{R}$ for $\ell \in \mathcal{L}$. For any $\ell \in \mathcal{L}$, let also ν_{s^ℓ}, with mass only in $S\Sigma_{\mathcal{J}}$, be invariant under the flow of $\vec{\rho}_{s^\ell}^{\mathcal{J}}$. Let us prove that

$$\nu^{\mathcal{J}} = \sum_{\ell\in\mathcal{L}} \gamma^\ell \nu_{s^\ell} \tag{36}$$

is a Quantum Limit. This corresponds to the case where the measure $Q^{\mathcal{J}}$ on $\mathbf{S}_{\mathcal{J}}$ is given by

$$Q^{\mathcal{J}} = \sum_{\ell\in\mathcal{L}} \gamma^\ell \delta_{s^\ell}.$$

For any $\ell \in \mathcal{L}$, we take $(\varphi_k^\ell)_{k\in\mathbb{N}^*}$ to be a sequence of eigenfunctions of $-\Delta$ whose unique QL is ν_{s^ℓ}. As emphasized in the proof of Lemma 8, it is possible to assume that φ_k^ℓ is an eigenfunction of Ω_j for any $1 \leq j \leq J$, with eigenvalue $2n_{j,k}^\ell + 1$ such that

$$\frac{2n_{j,k}^\ell + 1}{\sum_{i=1}^J (2n_{i,k}^\ell) + 1} \underset{k\to+\infty}{\longrightarrow} s_j^\ell \tag{37}$$

[10] See Remark 13 for the description of the associated orthonormal basis of eigenfunctions.

where $s^\ell = (s_1^\ell, \dots, s_J^\ell)$.

Let us prove that the joint microlocal defect measure $\nu_{\ell,\ell'}$ of $(\varphi_k^\ell)_{k\in\mathbb{N}^*}$ and $(\varphi_k^{\ell'})_{k\in\mathbb{N}^*}$ vanishes for $\ell \neq \ell'$: we note that for $\mathrm{Op}(a)$ commuting with $\Omega_1, \dots, \Omega_m$, with $a \in \mathscr{S}^0(\mathbf{H}^m)$,

$$\begin{aligned}(2n_{j,k}^\ell + 1)(\mathrm{Op}(a)\varphi_k^\ell, \varphi_k^{\ell'}) &= (\mathrm{Op}(a)\Omega_j\varphi_k^\ell, \varphi_k^{\ell'}) \\ &= (\mathrm{Op}(a)\varphi_k^\ell, \Omega_j\varphi_k^{\ell'}) \\ &= (2n_{j,k}^{\ell'} + 1)(\mathrm{Op}(a)\varphi_k^\ell, \varphi_k^{\ell'})\end{aligned}$$

From (37) and the fact that $s^\ell \neq s^{\ell'}$, we deduce that, for any sufficiently large $k \in \mathbb{N}^*$, there exists $1 \leq j \leq J$ such that $n_{j,k}^\ell \neq n_{j,k}^{\ell'}$. Hence, the above computation shows that $(\mathrm{Op}(a)\varphi_k^\ell, \varphi_k^{\ell'}) = 0$ for sufficiently large $k \in \mathbb{N}^*$. Therefore,

$$\int_{S^*\mathbf{H}^m} a d\nu_{\ell,\ell'} = 0.$$

Since ν_{s^ℓ} and $\nu_{s^{\ell'}}$ give no mass to the complementary set of $S\Sigma_{\mathcal{J}}$ in $S^*\mathbf{H}^m$, we know that it is also the case for $\nu_{\ell,\ell'}$ by Lemma 3. Therefore, if $b \in \mathscr{S}^0(\mathbf{H}^m)$ is arbitrary, averaging $\mathrm{Op}(b)$ with respect to the operators $\Omega_1, \dots, \Omega_J$ as in Lemma 4, we obtain an operator $A \in \Psi^0(\mathbf{H}^m)$ such that $\sigma_P(A)$ coincides with b on $\Sigma_{\mathcal{J}}$, and A commutes with $\Omega_1, \dots, \Omega_J$. Therefore,

$$\int_{S^*\mathbf{H}^m} b d\nu_{\ell,\ell'} = \int_{S\Sigma_{\mathcal{J}}} b d\nu_{\ell,\ell'} = \int_{S\Sigma_{\mathcal{J}}} \sigma_P(A) d\nu_{\ell,\ell'} = 0,$$

and since this is true for any $b \in \mathscr{S}^0(\mathbf{H}^m)$, we conclude that $\nu_{\ell,\ell'} = 0$.

This implies that the sequence given by

$$\varphi_k^{\mathcal{J}} = \sum_{\ell\in\mathcal{L}} \gamma^\ell \varphi_k^\ell$$

admits $\nu^{\mathcal{J}}$ as unique QL, where $\nu^{\mathcal{J}}$ is defined by (36). Note that to ensure that $\varphi_k^{\mathcal{J}}$ is still an eigenfunction of $-\Delta$, it is necessary, as in the proof of Lemma 8, to adjust the sequences $(n_{j,k}^\ell)$ and $(\alpha_{j,k}^\ell)$ in order to guarantee that all φ_k^ℓ (for $\ell \in \mathcal{L}$) are eigenfunctions of $-\Delta$ with same eigenvalue.

We notice that the closure of the set of Radon measures on $S\Sigma_{\mathcal{J}}$ which may be written as a finite linear combination (36) is exactly the subset of $\mathscr{P}_{S\Sigma}$ for which $Q^{\mathcal{J}'} = 0$ for any $\mathcal{J}' \neq \mathcal{J}$. Using that the set of QLs is closed, Lemma 9 is proved. □

Remark 13. *The above proof shows that $\nu_\infty = \nu^{\mathcal{J}}$ is a QL for an orthonormal sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ such that φ_k belongs to*

$$\bigoplus_{(n_j)\in\mathbb{N}^{\mathcal{J}}} \bigoplus_{(\alpha_j)\in(\mathbb{Z}^*)^{\mathcal{J}}} \mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}.$$

Let us now finish the proof of Theorem 2. Let $\nu_\infty \in \mathscr{P}_{S\Sigma}$,

$$\nu_\infty = \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} \nu^{\mathcal{J}}.$$

Note that the measures $\nu^{\mathcal{J}}$ are non-negative, but are not necessarily probability measures.

Let $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ be a sequence of eigenfunctions of $-\Delta$ whose unique microlocal defect measure is $\nu^{\mathcal{J}}$. The proof of Lemma 9 guarantees that, for any $k \in \mathbb{N}^*$, one may choose all $\varphi_k^{\mathcal{J}}$, for $\mathcal{J}$ running over $\mathcal{P}\setminus\{\emptyset\}$, to have the same eigenvalue with respect to $-\Delta$. Therefore,

$$\varphi_k = \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} \varphi_k^{\mathcal{J}}$$

is also an eigenfunction of $-\Delta$. Moreover, for any distinct $\mathcal{J}, \mathcal{J}' \in \mathcal{P} \setminus \{\emptyset\}$, the joint microlocal defect measure of $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ and $(\varphi_k^{\mathcal{J}'})_{k\in\mathbb{N}^*}$ vanishes (see Lemma 3). Computing $(\mathrm{Op}(a)\varphi_k, \varphi_k)$ for any $a \in \mathscr{S}^0(\mathbf{H}^m)$ in the limit $k \to +\infty$, we obtain that the unique Quantum Limit of $(\varphi_k)_{k\in\mathbb{N}^*}$ is ν_∞. Note that, as already explained in Remarks 11, 12 and 13, the orthonormal sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ is fully explicit in our construction.

Finally, we note that the invariance properties of ν_∞ may be established separately on each $S\Sigma_{\mathcal{J}}$ since $([A, R_s]\varphi_k^{\mathcal{J}}, \varphi_k^{\mathcal{J}'}) \to 0$ as $k \to +\infty$ for $\mathcal{J} \neq \mathcal{J}'$ (the bracket $[A, R_s]$ is the natural operator to consider for establishing invariance properties, see Section 2.3). This concludes the proof of Theorem 2.

A Classical pseudodifferential calculus

We briefly gather some basic facts of pseudodifferential calculus used along this paper (see also [Hör85, Chapter XVIII]).

Following our notations of Section 1, we denote by M a smooth compact manifold of dimension n. We denote by $\mathscr{S}^k(M)$ the space of smooth homogeneous functions of order k on the cone $T^*M \setminus \{0\}$. They are the *classical symbols* of order k.

The algebra $\Psi(M)$ of classical pseudodifferential operators on M is graded according to the chain of inclusions $\Psi^{-\infty}(M) \subset \ldots \subset \Psi^k(M) \subset \Psi^{k+1}(M) \subset \ldots$ where $k \in \mathbb{Z} \cup \{-\infty\}$ is called the order.

To a pseudodifferential operator $A \in \Psi^m(M)$, we can associate its *principal symbol* $\sigma_P(A)$, and the map $\sigma_P : \Psi^k(M)/\Psi^{k-1}(M) \to \mathscr{S}^k(M)$ is bijective. A *quantization* is a continuous linear mapping

$$\mathrm{Op} : \mathscr{S}^0(M) \to \Psi^0(M)$$

with $\sigma_P(\mathrm{Op}(a)) = a$. An example is obtained using partitions of unity and the Weyl quantization which is given in local coordinates by

$$\mathrm{Op}^W(a)f(q) = (2\pi)^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\langle q-q',p\rangle} a\left(\frac{q+q'}{2}, p\right) f(q')dq'dp.$$

Although we omitted the upper W index in the paper, this is the quantization we used by default in this paper.

We have the following properties:

- If $A \in \Psi^k(M)$ and $B \in \Psi^\ell(M)$, then $AB \in \Psi^{k+\ell}(M)$ and $\sigma_P(AB) = \sigma_P(A)\sigma_P(B)$.
- If $A \in \Psi^k(M)$ and $B \in \Psi^\ell(M)$, then $[A, B] \in \Psi^{k+\ell-1}(M)$ and

$$\sigma_P([A, B]) = \frac{1}{i}\{\sigma_P(A), \sigma_P(B)\},$$

  where the Poisson bracket is taken with respect to the canonical symplectic structure of T^*M.

Let us prove Lemma 3 of Section 2.2.

Proof of Lemma 3. If $a \in \mathscr{S}^0(M)$ is such that $a \geq 0$ and a is supported in a set where $\mu_{11} = 0$, then, setting $a_\varepsilon = a + \varepsilon$ for any $\varepsilon > 0$, we get

$$(\mathrm{Op}(a_\varepsilon)u_k, v_k) = (\mathrm{Op}(a_\varepsilon^{1/2})u_k, \mathrm{Op}(a_\varepsilon^{1/2})v_k) + o(1) \leq \|\mathrm{Op}(a_\varepsilon^{1/2})u_k\|_{L^2}\|\mathrm{Op}(a_\varepsilon^{1/2})v_k\|_{L^2} + o(1)$$

where $a_\varepsilon^{1/2} \in \mathscr{S}^0(M)$. We know that

$$\|\mathrm{Op}(a_\varepsilon^{1/2})u_k\|_{L^2}^2 = (\mathrm{Op}(a_\varepsilon)u_k, u_k) + o(1) = (\mathrm{Op}(a)u_k, u_k) + \varepsilon\|u_k\|_2^2 + o(1) = \varepsilon\|u_k\|^2 + o(1)$$

and that $\|\mathrm{Op}(a_\varepsilon^{1/2})v_k\|_{L^2}^2 \leq (C + \varepsilon)\|v_k\|^2$ where C does not depend on ε. Therefore $(\mathrm{Op}(a_\varepsilon)u_k, v_k) \lesssim \varepsilon$. Hence $(\mathrm{Op}(a)u_k, v_k) \to 0$. The same result holds for $a \leq 0$ supported in a set where $\mu_{11} = 0$. Therefore, decomposing any symbol as $a = a^+ + a^- + r$, where $a^+, a^-, r \in \mathscr{S}^0(M)$, $a^+ \geq 0$, $a^- \leq 0$, and $|r| \leq \delta$ for some small $\delta > 0$, we get that μ_{12} is absolutely continuous with respect to μ_{11}. The rest of the lemma follows by symmetry. □

Lemma 10. *Let us assume that $\ell \in \mathbb{N}$ and $P \in \Psi^\ell(M)$ is elliptic in any cone contained in the complementary of a closed conic set $F \subset T^*M$. Assume that $(u_k)_{k\in\mathbb{N}^*}$ is a bounded sequence in $L^2(M)$ weakly converging to 0 and such that $Pu_k \to 0$ strongly in $L^2(M)$. Then any microlocal defect measure of $(u_k)_{k\in\mathbb{N}^*}$ is supported in F.*

Proof. Let μ be a microlocal defect measure of $(u_k)_{k\in\mathbb{N}^*}$, i.e.,

$$(\mathrm{Op}(a)u_{\sigma(k)}, u_{\sigma(k)}) \underset{k\to+\infty}{\longrightarrow} \int_{S^*M} a d\mu$$

for any $a \in \mathscr{S}^0(M)$, where σ is an extraction. Let $a \in \mathscr{S}^0(M)$ be supported outside F. Let $Q \in \Psi^{-\ell}(M)$ be such that $PQ - I \in \Psi^{-1}(M)$ on the support of a. Then $Q\mathrm{Op}(a)P \in \Psi^0(M)$ has principal symbol a, and therefore

$$(Q\mathrm{Op}(a)Pu_{\sigma(k)}, u_{\sigma(k)}) \underset{k\to+\infty}{\longrightarrow} \int_{S^*M} a d\mu.$$

Using that $Pu_{\sigma(k)} \to 0$, we get $(Q\mathrm{Op}(a)Pu_{\sigma(k)}, u_{\sigma(k)}) \to 0$ as $k \to +\infty$, and therefore $\int_{S^*M} a d\mu = 0$. Hence, μ is supported in F. □

B Another view on the measures $Q^{\mathcal{J}}$ and $\nu_s^{\mathcal{J}}$

We explain an alternative way to obtain the measure $Q^{\mathcal{J}}$ on $\mathbf{S}_{\mathcal{J}}$ and the family of measures $(\nu_s^{\mathcal{J}})_{s\in\mathbf{S}_{\mathcal{J}}}$ on $S\Sigma_{\mathcal{J}}$, based on pure functional analysis. This way of obtaining $Q^{\mathcal{J}}$ and $(\nu_s^{\mathcal{J}})_{s\in\mathbf{S}_{\mathcal{J}}}$ does not allow to prove easily that $\vec{\rho}_s^{\mathcal{J}}\nu_s^{\mathcal{J}} = 0$, thus we did not use it in the core of the proof of Theorem 1. However, it sheds a different light on Point (2) of Theorem 1, therefore we decided to include it here.

In this Section, as in Section 2.3, we work under Assumption 1. In particular, $\mathcal{J} = \{1, \dots, J\}$ is fixed.

For $f \in C^0(\mathbf{S}_{\mathcal{J}})$ (the set of continuous functions on $\mathbf{S}_{\mathcal{J}}$), we define Π_f the operator

$$\Pi_f = f\left(\frac{\Omega_1}{\Omega_1 + \ldots + \Omega_J}, \ldots, \frac{\Omega_J}{\Omega_1 + \ldots + \Omega_J}\right) \tag{38}$$

defined through functional calculus. For any $k \in \mathbb{N}^*$ and any $f \in C^0(\mathbf{S}_{\mathcal{J}})$, there holds $\|\Pi_f\varphi_k\|_{L^2} \le \|f\|_{L^\infty}\|\varphi_k\|_{L^2}$.

Let us denote by $\mathscr{M}_+(\mathbf{S}_{\mathcal{J}}, \mathscr{M}(S\Sigma_{\mathcal{J}}))$ the set of non-negative measures on $\mathbf{S}_{\mathcal{J}}$ which are valued in the set $\mathscr{M}(S\Sigma_{\mathcal{J}})$ of Radon measures on $S\Sigma_{\mathcal{J}}$. More precisely, $\mu \in \mathscr{M}_+(\mathbf{S}_{\mathcal{J}}, \mathscr{M}(S\Sigma_{\mathcal{J}}))$ is defined as a family of Radon measures μ^f on $S\Sigma_{\mathcal{J}}$ (for any $f \in C^0(\mathbf{S}_{\mathcal{J}})$), which we also denote by $\int_{\mathbf{S}_{\mathcal{J}}} f\, d\mu$, such that $f \mapsto \mu^f$ is linear and continuous, and μ^f is non-negative if f is non-negative. Here and in the sequel, $C^0(\mathbf{S}_{\mathcal{J}})$ is equipped with the topology of uniform convergence and $\mathscr{M}(S\Sigma_{\mathcal{J}})$ is equipped with the weak-star topology.

Lemma 11. *There exists $\mu \in \mathscr{M}_+(\mathbf{S}_{\mathcal{J}}, \mathscr{M}(S\Sigma_{\mathcal{J}}))$ and an extraction $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ such that, for any symbol $a \in \mathscr{S}^0(\mathbf{H}^m)$ and any $f \in C^0(\mathbf{S}_{\mathcal{J}})$,*

$$(Op(a)\Pi_f\varphi_{\sigma(k)}, \varphi_{\sigma(k)})_{L^2} \underset{k\to+\infty}{\longrightarrow} \int_{S\Sigma_{\mathcal{J}}} a\; d\left(\int_{\mathbf{S}_{\mathcal{J}}} f\; d\mu\right). \tag{39}$$

Proof. Let $f \in C^0(\mathbf{S}_{\mathcal{J}})$ be non-negative. For any symbol $a \in \mathscr{S}^0(\mathbf{H}^m)$, we denote by $\tilde{a} \in \mathscr{S}^0(\mathbf{H}^m)$ the symbol satisfying

$$\mathrm{Op}(\tilde{a}) = \int_{(\mathbb{R}/2\pi\mathbb{Z})^J} U(-t)\mathrm{Op}(a)U(t)dt$$

where $U(t) = U(t_1, \dots, t_J) = e^{i(t_1\Omega_1 + \ldots + t_J\Omega_J)}$ for $t = (t_1, \dots, t_J) \in (\mathbb{R}/2\pi\mathbb{Z})^J$. Then, $\mathrm{Op}(\tilde{a})$ commutes with Ω_J for any $1 \le j \le J$. Therefore, for any $k \in \mathbb{N}^*$, since all Ω_j are selfadjoint,

$$(\mathrm{Op}(\tilde{a})\Pi_f\varphi_k, \varphi_k)_{L^2} = (\mathrm{Op}(\tilde{a})\Pi_{\sqrt{f}}\varphi_k, \Pi_{\sqrt{f}}\varphi_k)_{L^2}.$$

By [Gér91b], we know that there exists a non-negative Radon measure $\mu^{\sqrt{f}}$ on $S^*\mathbf{H}^m$ and an extraction $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ such that

$$(\mathrm{Op}(\tilde{a})\Pi_{\sqrt{f}}\varphi_{\sigma(k)}, \Pi_{\sqrt{f}}\varphi_{\sigma(k)})_{L^2} \underset{k\to+\infty}{\longrightarrow} \int_{S^*\mathbf{H}^m} \tilde{a}\ d\mu^{\sqrt{f}}.$$

Note that $\mu^{\sqrt{f}}$ has no mass outside $S\Sigma_{\mathcal{J}}$. Moreover, $\sigma_P(\tilde{a})$ and $\sigma_P(a)$ coincide on $S\Sigma_{\mathcal{J}}$. Therefore, for any $a \in \mathscr{S}^0(\mathbf{H}^m)$,

$$(\mathrm{Op}(a)\Pi_{\sqrt{f}}\varphi_{\sigma(k)}, \Pi_{\sqrt{f}}\varphi_{\sigma(k)})_{L^2} \underset{k\to+\infty}{\longrightarrow} \int_{S\Sigma_{\mathcal{J}}} a\ d\mu^{\sqrt{f}}.$$

We can do the same argument for non-positive $f \in C^0(\mathbf{S}_{\mathcal{J}})$, and, finally writing the decomposition of an arbitrary $f \in C^0(\mathbf{S}_{\mathcal{J}})$ into its positive and its negative part, we get (39). Note that using a classical separability argument, linearity and diagonal extraction, the extraction σ may be chosen to be the same for any $f \in C^0(\mathbf{S}_{\mathcal{J}})$. □

Lemma 12. *For any $\mu \in \mathscr{M}_+(\mathbf{S}_{\mathcal{J}}, \mathscr{M}(S\Sigma_{\mathcal{J}}))$, there exist $Q^{\mathcal{J}} \in \mathscr{M}_+(\mathbf{S}_{\mathcal{J}})$, and, for $Q^{\mathcal{J}}$-almost every $s \in \mathbf{S}_{\mathcal{J}}$, $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$ having no mass outside $S\Sigma_{\mathcal{J}}$ such that the equality*

$$\int_{S\Sigma_{\mathcal{J}}} a\ d\left(\int_{\mathbf{S}_{\mathcal{J}}} f\ d\mu\right) = \int_{\mathbf{S}_{\mathcal{J}}} f(s)\left(\int_{S\Sigma_{\mathcal{J}}} a\ d\nu_s^{\mathcal{J}}\right) dQ^{\mathcal{J}}(s) \tag{40}$$

holds for any $f \in C^0(\mathbf{S}_{\mathcal{J}})$ and any $a \in \mathscr{S}^0(\mathbf{H}^m)$.

Proof. Since both parts of (40) depend only on the part of a which lies in $S\Sigma_{\mathcal{J}}$, we call "symbol on $S\Sigma_{\mathcal{J}}$" any restriction to $S\Sigma_{\mathcal{J}}$ of some $a \in \mathscr{S}^0(\mathbf{H}^m)$. It follows from the usual Riesz representation theorem that for any symbol a on $S\Sigma_{\mathcal{J}}$, the functional

$$C^0(\mathbf{S}_{\mathcal{J}}) \to \mathbb{R}, \quad f \mapsto \int_{S\Sigma_{\mathcal{J}}} a\ d\left(\int_{\mathbf{S}_{\mathcal{J}}} f d\mu\right),$$

which is linear and continuous, may be written as

$$\int_{S\Sigma_{\mathcal{J}}} a\ d\left(\int_{\mathbf{S}_{\mathcal{J}}} f d\mu\right) = \int_{\mathbf{S}_{\mathcal{J}}} f(s)\ dQ_a^{\mathcal{J}}(s) \tag{41}$$

for some Radon measure $Q_a^{\mathcal{J}}$ on $\mathbf{S}_{\mathcal{J}}$ (which is unique for each symbol a on $S\Sigma$). In particular, for $a \equiv 1$, this formula defines a measure $Q_1^{\mathcal{J}}$ on $\mathbf{S}_{\mathcal{J}}$ which we denote by $Q^{\mathcal{J}}$. Note also that if a is non-negative, then $Q_a^{\mathcal{J}}$ is non-negative.

For any symbol a on $S\Sigma_{\mathcal{J}}$, $Q_a^{\mathcal{J}}$ is absolutely continuous with respect to $Q^{\mathcal{J}}$. To prove it, assume that $Q^{\mathcal{J}}(E) = 0$ for some measurable set $E \subset \mathbf{S}_{\mathcal{J}}$. Then, for any $f \in C^0(\mathbf{S}_{\mathcal{J}})$ supported in E, $\int_{\mathbf{S}_{\mathcal{J}}} f(s)\, dQ^{\mathcal{J}}(s) = 0$, which in turn implies that $\int_{\mathbf{S}_{\mathcal{J}}} f d\mu = 0$ thanks to (41). Therefore, for any symbol a on $S\Sigma_{\mathcal{J}}$, the left-hand side of (41) vanishes, hence the right-hand side also vanishes. From this, it follows that $Q_a^{\mathcal{J}}(E) = 0$, and that $Q_a^{\mathcal{J}}$ is absolutely continuous with respect to $Q^{\mathcal{J}}$.

By the Radon-Nikodym theorem, for any symbol a on $S\Sigma_{\mathcal{J}}$, there exists a measurable function θ_a on $\mathbf{S}_{\mathcal{J}}$ such that

$$dQ_a^{\mathcal{J}}(s) = \theta_a(s) dQ^{\mathcal{J}}(s) \tag{42}$$

for $Q^{\mathcal{J}}$-almost every $s \in \mathbf{S}_{\mathcal{J}}$. Moreover, if a is non-negative, then $\theta_a(s)$ is non-negative.

We note that, for $Q^{\mathcal{J}}$-almost every $s \in \mathbf{S}_{\mathcal{J}}$, $a \mapsto \theta_a(s)$ is linear. Let us prove that it is also continuous for $Q^{\mathcal{J}}$-almost every $s \in \mathbf{S}_{\mathcal{J}}$. Let $(a_n)_{n\in\mathbb{N}^*}$ be a sequence of non-negative symbols on $S\Sigma_{\mathcal{J}}$ tending to 0. Taking $f \equiv 1$ and using (42) and (41), we have

$$\int_{\mathbf{S}_{\mathcal{J}}} \theta_{a_n}(s)\ dQ^{\mathcal{J}}(s) = \int_{S\Sigma_{\mathcal{J}}} a_n\ d\left(\int_{\mathbf{S}_{\mathcal{J}}} 1\ d\mu\right) \underset{n\to+\infty}{\longrightarrow} 0.$$

Since $Q^{\mathcal{J}}$ is a non-negative measure and $\theta_{a_n}(s) \geq 0$ for $Q^{\mathcal{J}}$-almost every s (because $Q^{\mathcal{J}}_{a_n}$ is non-negative), by Lebesgue's dominated convergence theorem, it implies that $\theta^{\mathcal{J}}_{a_n}(s) \to 0$ for $Q^{\mathcal{J}}$-almost every $s \in \mathbf{S}_{\mathcal{J}}$. Similarly, if $(a_n)_{n\in\mathbb{N}^*}$ is a sequence of non-positive symbols on $S\Sigma_{\mathcal{J}}$ tending to 0, then $\theta_{a_n}(s) \to 0$ for $Q^{\mathcal{J}}$-almost every $s \in \mathbf{S}_{\mathcal{J}}$. Altogether, this implies that $a \mapsto \theta_a(s)$ is continuous for $Q^{\mathcal{J}}$-almost every $s \in \mathbf{S}_{\mathcal{J}}$.

Using Riesz representation theorem, we get that for $Q^{\mathcal{J}}$-almost every $s \in \mathbf{S}_{\mathcal{J}}$ (by a classical separability argument, this "$Q^{\mathcal{J}}$-almost every" does not depend on a), there exists a non-negative Radon measure $\nu^{\mathcal{J}}_s$ on $S\Sigma_{\mathcal{J}}$ such that

$$\theta_a(s) = \int_{S\Sigma_{\mathcal{J}}} a \; d\nu^{\mathcal{J}}_s \tag{43}$$

for any symbol a on $S\Sigma_{\mathcal{J}}$. Combining (41), (42) and (43), we obtain (40). □

Finally, combining Lemma 11 and Lemma 12, we get that for any symbol $a \in \mathscr{S}^0(\mathbf{H}^m)$ and any $f \in C^0(\mathbf{S}_{\mathcal{J}})$,

$$\begin{aligned}(\mathrm{Op}(a)\Pi_f \varphi_{\sigma(k)}, \varphi_{\sigma(k)})_{L^2} &\underset{k\to+\infty}{\longrightarrow} \int_{S\Sigma_{\mathcal{J}}} a \; d\left(\int_{\mathbf{S}_{\mathcal{J}}} f \; d\mu\right) \\ &= \int_{\mathbf{S}_{\mathcal{J}}} f(s) \left(\int_{S\Sigma_{\mathcal{J}}} a \; d\nu^{\mathcal{J}}_s\right) dQ^{\mathcal{J}}(s).\end{aligned}$$

Taking $f \equiv 1$, since $\nu^{\mathcal{J}}$ is the unique microlocal defect measure of $(\varphi_k)_{k\in\mathbb{N}^*}$ (see Assumption 1), we get that

$$\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \nu^{\mathcal{J}}_s \; dQ^{\mathcal{J}}(s).$$

Remark 14. *Approaching the characteristic function $f(s) = \mathbf{1}_{s\in\mathbf{S}^N_\ell}$ for some fixed $N \in \mathbb{N}^*$ and $0 \leq \ell \leq 2^N - 1$ (see (24) for notations) by continuous functions, and considering (38) and (39), we see that the disintegration of ν provided by the above argument coincides with the disintegration done in Section 2.3, i.e., the measures $Q^{\mathcal{J}}$ and $\nu^{\mathcal{J}}_s$ are the same.*

C Quantum Limits for tensorial bases

This section consists in a short remark concerning Quantum Limits in a tensor form. It says that if an orthonormal Hilbert basis of eigenfunctions is in a tensor form, then all associated QLs can also be written as tensorial products.

Proposition 13. *Let $\mathcal{B} = \{\psi_\ell, \ell \in \mathbb{N}^*\}$ be an orthonormal Hilbert basis of $L^2(\mathbf{H})$ of eigenfunctions of $-\Delta_{\mathbf{H}}$, and let $\mathcal{B}^{\otimes m}$ be the orthonormal Hilbert basis of $L^2(\mathbf{H}^m)$ consisting of all tensorial products of m elements of $\mathcal{B}$. Then any Quantum Limit associated to $\mathcal{B}^{\otimes m}$ is a tensorial product of Quantum Limits of $(\psi_\ell)_{\ell\in\mathbb{N}^*}$.*

Proof. We denote by $(\varphi_k)_{k\in\mathbb{N}^*}$ a subsequence of $\mathcal{B}^{\otimes m}$ (with increasing eigenvalues) having a unique Quantum Limit ν. We write

$$\varphi_k = \psi^1_k \otimes \ldots \otimes \psi^m_k$$

with $\psi^j_k \in \mathcal{B}$ for any $1 \leq j \leq m$ and any $k \in \mathbb{N}^*$. Then, for any sequence $(\psi^j_k)_{k\in\mathbb{N}^*}$, we denote by ν^j one of its Quantum Limits. Note that the linear combinations of tensorial products (with m components in the tensorial product) of symbols in $\mathscr{S}^0(\mathbf{H})$ are dense in the set $\mathscr{S}^0(\mathbf{H}^m)$. Therefore, $\nu = \nu^1 \otimes \ldots \otimes \nu^m$, which concludes the proof. □

D Quantum Limits of flat contact manifolds

The study of Quantum Limits of higher dimensional contact manifolds is also an interesting problem. In this section, we prove that for "flat" contact manifolds, typically a quotient of the Heisenberg group H_N of dimension $2N+1$ by one of its discrete cocompact subgroups, the invariance properties of Quantum Limits are much simpler than those described in Theorem 1, even though "frequencies" show up: the part of the QL which lies in $S\Sigma$ is invariant under the lift of the Reeb flow, as in the 3D case.

For $N \geq 1$, we consider the group law on $\mathbb{R}^{2N+1}$ given by

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' - x \cdot y')$$

where $x, x', y, y' \in \mathbb{R}^N$ and $z, z' \in \mathbb{R}$. The Heisenberg group $\tilde{H}_N$ is the group $\tilde{H}_N = (\mathbb{R}^{2N+1}, \star)$. We consider the subgroup $\Gamma_N = (\sqrt{2\pi\mathbb{Z}})^{2N} \times 2\pi\mathbb{Z}$ of $\tilde{H}_N$, and the left quotient $H_N = \Gamma_N \backslash \tilde{H}_N$. We also define the $2N$ left invariant vector fields on H_N given by

$$X_j = \partial_{x_j}, \qquad Y_j = \partial_{y_j} - x_j \partial_z$$

for $1 \leq j \leq N$. We fix $\beta_1, \dots, \beta_N > 0$ satisfying $\prod_{j=1}^N \beta_j = 1$, we set $\beta = (\beta_1, \dots, \beta_N)$ and we consider the sub-Laplacian

$$\Delta_\beta = \sum_{j=1}^N \beta_j (X_j^2 + Y_j^2)$$

which is an operator acting on functions on H_N. The positive real numbers β_j are sometimes called frequencies, see [Agr96].

We set $\rho = p_z|_\Sigma$, which is the Hamiltonian lift of the Reeb vector field $Z = \partial_z$ to Σ (see [CdVHT18, Section 2.3] for properties of the Reeb vector field).

Proposition 14. *Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be an orthonormal sequence of $L^2(H_N)$ consisting of eigenfunctions of $-\Delta_\beta$. Then, any Quantum Limit ν_∞ associated to $(\varphi_k)_{k\in\mathbb{N}^*}$ and supported in $S\Sigma$ is invariant under $e^{t\vec{\rho}}$, the lift of the Reeb flow.*

Remark 15. *We do not expect such a result to be true when the frequencies β_j are not constant on the manifold.*

Proof of Proposition 14. Denoting by (q, p) the canonical coordinates in T^*H_N, i.e., $q = (x_1, \dots, x_N, y_1, \dots, y_N, z)$ and $p = (p_{x_1}, \dots, p_{x_N}, p_{y_1}, \dots, p_{y_N}, p_z)$, we know that

$$\Sigma = \{(q, p) \in T^*H_N,\ p_{x_j} = p_{y_j} - x_j p_{z_j} = 0\}$$

is isomorphic to $H_N \times \mathbb{R}$.

Up to extraction of a subsequence, we may assume that $(\varphi_k)_{k\in\mathbb{N}^*}$ has a unique QL ν_∞, which is supported in $S\Sigma$. We set $R = \sqrt{\partial_z^* \partial_z}$ and, on its eigenspaces corresponding to non-zero eigenvalues, we define $\Omega_j = -R^{-1}(X_j^2 + Y_j^2) = -(X_j^2 + Y_j^2)R^{-1}$ for $1 \leq j \leq N$. On these eigenspaces, the sub-Laplacian acts as

$$-\Delta_\beta = R\Omega = \Omega R \qquad \text{with} \quad \Omega = \sum_{j=1}^N \beta_j \Omega_j$$

and $[R, \Omega] = 0$.

Let V be a (small) conic microlocal neighborhood of Σ, and let us consider R, Ω as acting on functions microlocally supported in V (meaning that their wave-front set is contained in

V). If $B \in \Psi^0(H_N)$ is microlocally supported in V and commutes with Ω, then

$$\begin{aligned}([B,R]\varphi_k,\varphi_k) &= \frac{1}{\lambda_k}(BR\varphi_k,-\Delta_\beta\varphi_k) - \frac{1}{\lambda_k}(RB(-\Delta_\beta)\varphi_k,\varphi_k)\\ &= \frac{1}{\lambda_k}(BR\varphi_k,R\Omega\varphi_k) - \frac{1}{\lambda_k}(RBR\Omega\varphi_k,\varphi_k)\\ &= \frac{1}{\lambda_k}([\Omega,RBR]\varphi_k,\varphi_k)\\ &= 0.\end{aligned}$$

Let $U(t) = U(t_1,\dots,t_N) = e^{i(t_1\Omega_1+\dots+t_N\Omega_N)}$ for $t = (t_1,\dots,t_N) \in (\mathbb{R}/2\pi\mathbb{Z})^N$. For $A \in \Psi^0(H_N)$ microlocally supported in V, we consider

$$\tilde{A} = \int_{(\mathbb{R}/2\pi\mathbb{Z})^N} U(-t)AU(t)dt$$

As in the proof of Lemma 4, we know that $[\tilde{A},\Omega] = 0$ and that $\sigma_P(A)$ and $\sigma_P(\tilde{A})$ coincide on Σ. Therefore, using the previous computation with $B = \tilde{A}$, we obtain

$$\int_\Sigma \{\sigma_P(A),\rho\}_{\omega_{|\Sigma}} d\nu_\infty = \int_\Sigma \{\sigma_P(\tilde{A}),\rho\}_{\omega_{|\Sigma}} d\nu_\infty = \lim_{k\to+\infty}([\tilde{A},R]\varphi_k,\varphi_k) = 0.$$

Since it is true for any A microlocally supported in V, this implies that ν_∞ is invariant under the flow $e^{t\vec{\rho}}$. □

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