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HYDRODYNAMIC LIMIT FOR A BOUNDARY DRIVEN SUPER-DIFFUSIVE SYMMETRIC EXCLUSION

CÉDRIC BERNARDIN, PEDRO CARDOSO, PATRÍCIA GONÇALVES, STEFANO SCOTTA

ABSTRACT. We study the hydrodynamic limit for a model of symmetric exclusion processes with heavy-tailed long jumps and in contact with infinitely extended reservoirs. We show how the corresponding hydrodynamic equations are affected by the parameters defining the model. The hydrodynamic equations are characterized by a class of super-diffusive operators that are given by the regional fractional Laplacian with some additional reaction terms and various boundary conditions. This work solves some questions left open in [2] about the same model.

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1. INTRODUCTION

The understanding of the macroscopic behavior of a physical system, from the microscopic description of its molecules, is the main goal of statistical mechanics. Stochastic interacting particle systems (IPS), introduced in the mathematics community by [29], have been intensively used to obtain, rigorously, the macroscopic laws which rule the space-time evolution of the conserved quantities of a system. These macroscopic laws take the form of partial differential equations (PDEs) whose nature depends on some features of the underlying IPS. The PDEs associated with an IPS are derived thanks to a scaling parameter $N \rightarrow \infty$, which connects the macroscopic space, where the solutions of the PDE live, with the microscopic space, a discrete space where the particles of the IPS evolve. The exclusion process is the most studied IPS in the mathematical literature. Its dynamics consists in a large system of particles evolving on the lattice \mathbb{Z} like independent continuous-time random walks, but with the rule that any jump, which would result in the occupation of a site by more than one particle, is suppressed. This “exclusion” condition characterizes the interactions between particles and gives its name to this Markov process. The number of particles is preserved and the goal is then to understand how does the particles’ density evolves in space and time. The transition probability of the underlying random walks is denoted by $p : \mathbb{Z} \times \mathbb{Z} \rightarrow [0, 1]$.

When the transition probability p is translation invariant, i.e. $p(x, y) = p(y - x)$, with zero mean and finite variance, the density is governed by the heat equation with a constant diffusion coefficient equal to the variance of p . Different boundary conditions (Dirichlet, Robin, Neumann) can be imposed by putting the system in contact with some stochastic reservoirs. The nature of the boundary conditions depends on the reservoirs’ model and their intensity via some scaling parameter. If p has heavy tails, in particular, if the variance is infinite, the density evolves superdiffusively according to a fractional diffusion equation [19], which is a non-local PDE. Putting the system in contact with reservoirs gives then rise to a bunch of boundary conditions which are much more difficult to describe than in the diffusive case. This is due to the fact that a boundary condition has usually a local nature while the PDE, in this case, does not have.

The exclusion process we analyze in this work was first introduced in [2, 3, 5]. We recall here how it is defined. Let $N > 1$ and call “bulk” the lattice $\Lambda_N = \{1, \dots, N - 1\}$. On the right and on the left of the bulk we put stochastic reservoirs, which means that, according to some rules that we will explain in details later, the sets $\{x \leq 0 ; x \in \mathbb{Z}\}$ and $\{x \geq N ; x \in \mathbb{Z}\}$ interact with the particles in the bulk, creating or annihilating them. The particles inside the bulk move according to an exclusion process with long jumps: this means that particles can perform jumps of any length but they can not jump to a site where there is already a particle

(exclusion rule). This kind of dynamics is determined by a symmetric probability transition

$$p : z \in \mathbb{Z} \rightarrow p(z) = \frac{c_\gamma}{|z|^{\gamma+1}} \mathbb{1}_{\{z \neq 0\}}, \quad (1.1)$$

depending only on the length of the jump and on a parameter $\gamma > 0$. Hence the constant c_γ is a normalizing constant equal to

$$c_\gamma = \sum_{z \neq 0} \frac{1}{|z|^{\gamma+1}}. \quad (1.2)$$

Associated with this transition probability we denote its second moment by $\sigma^2 := \sum_{z \in \mathbb{Z}} z^2 p(z) \in [0, \infty]$ and we also introduce the quantity $m := \sum_{z \geq 0} z p(z) \in [0, \infty]$. The strength of the reservoirs is regulated by a factor κ/N^θ depending on two parameters $\kappa > 0$ and $\theta \in \mathbb{R}$. We will see how the macroscopic behavior of the system changes drastically according to the values of θ . We call the reservoirs “slow” (or “weak”) if $\theta \geq 0$ and “fast” (or “strong”) if $\theta < 0$. Moreover, we introduce two more parameters $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ regulating, respectively, the density of the left reservoir and of the right reservoir. The precise description of the evolution of the process is given in Section 2.1. Figure 1 gives an illustration of the dynamics just described.

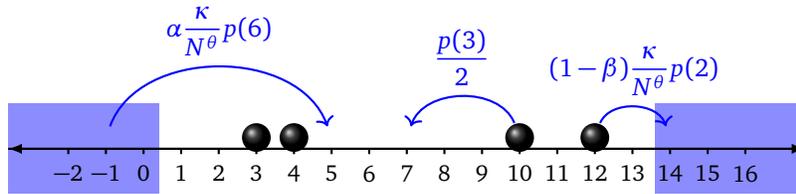


FIGURE 1. Example of the dynamics of the model with $N = 14$ and the configuration $\eta = (0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0)$; the sites in the region colored in blue act as reservoirs.

In [1] the exclusion process evolving on Λ_N , with nearest-neighbour jumps ($p(1) = p(-1) = 1/2$) and only one reservoir at each endpoint of the bulk is considered and depending on the value of θ , three different hydrodynamic equations are derived: the heat equation, either with Dirichlet ($0 \leq \theta < 1$), Robin ($\theta = 1$) or Neumann ($\theta > 1$) boundary conditions. The hydrodynamic behavior of the exclusion process with long jumps given by a symmetric p was studied in [3] when p has finite variance ($\gamma > 2$) for all the regimes of θ . There, it is shown that the hydrodynamic equations involve the standard Laplacian operator and in particular, they are very similar to the ones found in [1], where the reservoirs are present only at $\{0\}$ and at $\{N\}$. In both models, when the reservoirs are strong, (the rate of entering and exiting the system is high) the hydrodynamic equation is simply a reaction equation where the diffusive operator is not present since the dynamics caused by the reservoirs are influencing the system much more than the interactions between particles inside Λ_N .

In [2] the authors treat the case when p has infinite variance and m is finite ($1 < \gamma < 2$) and the reservoirs are quite strong, i.e. $\theta \leq 0$. As in the equilibrium setting of [19], in this regime, the standard Laplacian operator is replaced by the fractional Laplacian, but since

in [2] the process evolves in a macroscopic finite domain, the fractional operator on \mathbb{R} is replaced by the *regional fractional Laplacian*, which is defined in Subsection 2.2, with some specific boundary conditions. Also there, it is possible to see that when the reservoirs are very strong ($\theta < 0$) the fractional operator disappears and the PDE involved is a simple reaction equation. All these results are reviewed in the lecture notes [13]. Partial results in the case of *asymmetric* heavy tailed p with infinite mean and without boundary conditions have been obtained in [28] (see also [4]).

The aim of this work is to complete the analysis of this model for all the values of the parameters involved, namely, the regimes in which the variance is infinite and which were not treated in [2]. In [2] the authors proved that for $\gamma \in (1, 2)$ and $\theta < 0$, the hydrodynamic equation is a reaction equation, while for $\theta = 0$, the hydrodynamic equation is a “regional fractional reaction-diffusion equation” with Dirichlet boundary conditions. The other regimes ($\theta > 0$ and $1 < \gamma < 2$; and $0 < \gamma < 1$) were left open. Here we complete the scenario of the hydrodynamic limits for this process for all the values of θ and $0 < \gamma < 2$ except for the pathological cases $\gamma = 1, 2$. In particular, in [2], it was conjectured that the hydrodynamic equation, in a regime where $\theta > 0$ sufficiently small, is still a regional fractional diffusion equation with Dirichlet boundary conditions, by looking at the convergence as $\kappa \rightarrow 0$ of the solution of the hydrodynamic equation in the case $\theta = 0$. In this article, we prove this conjecture and identify precisely its domain of validity, which is $0 < \theta < \gamma - 1$. By the way, we propose some weak formulation of fractional diffusion equations with various boundary conditions which seem to be new in the PDE’s literature. We believe that this work could also be of interest to understand the non-equilibrium properties of systems of oscillators in contact with Langevin baths at their boundaries (see e.g. [6, 23]).

The hard part of the proof of the conjecture formulated in [2] is to derive the Dirichlet boundary conditions from the particle system, i.e. to show, for example, when $\gamma \in (1, 2)$ and $0 < \theta < \gamma - 1$, that the empirical density in a box of size εN around site 1 is close to α in the limit $N \rightarrow \infty$ and then $\varepsilon \rightarrow 0$. In [2] the authors manage to avoid a proof of this claim by imposing the boundary condition at the PDE level in some indirect way. This argument fails when $\theta > 0$. The derivation of the Dirichlet boundary condition we obtain in this paper, valid even for $\theta = 0$, is explained in Section 5.2 and differs drastically from the method used in [2]. It relies on the result that we present in Section 5.3 and on a renormalization procedure. In the first step of this renormalization procedure we show that we can replace the occupation number at site 1 by its average in a box of length ℓ_0 for a certain value of ℓ_0 , which is, unfortunately, not big enough to conclude the proof. Then, we use a multi-scale argument (reminiscent of [12, 14]) which has the following role: we increase the size of ℓ_0 by doubling it until we reach the needed size $\lfloor \varepsilon N \rfloor$ of the box. Finally, we show that the occupation number at site 1 can be replaced by α in a proper regime of θ . The error that we get by increasing the size of the box from ℓ_0 to $\lfloor \varepsilon N \rfloor$ is estimated in Section 5.3. In particular, it permits us to estimate the change of energy of the system when we change the position of particles. This result (Lemma 5.8) was difficult to prove since particles can perform long jumps and so there is a high number of different configurations contributing to the change of

energy. Hence, we had to carefully consider all these possibilities to prove the final estimate that we used to show that the PDE actually satisfies the Dirichlet boundary conditions.

Another difficulty we faced in the remaining regimes was to derive the convergence of the discrete version operator that appears from the random underlying dynamics to the regional fractional Laplacian. In [2], the authors prove a uniform convergence on smooth test functions with compact support. In this work, we show that this convergence holds in the L^1 sense for any smooth test function (even if not compactly supported) so that at the hydrodynamics level we may consider test functions that do not have compact support and therefore, see the boundary conditions. This result is proved in Subsection 5.1.

Our results are summarized in Figure 2 and the proper statement is given below in Theorem 2.18.

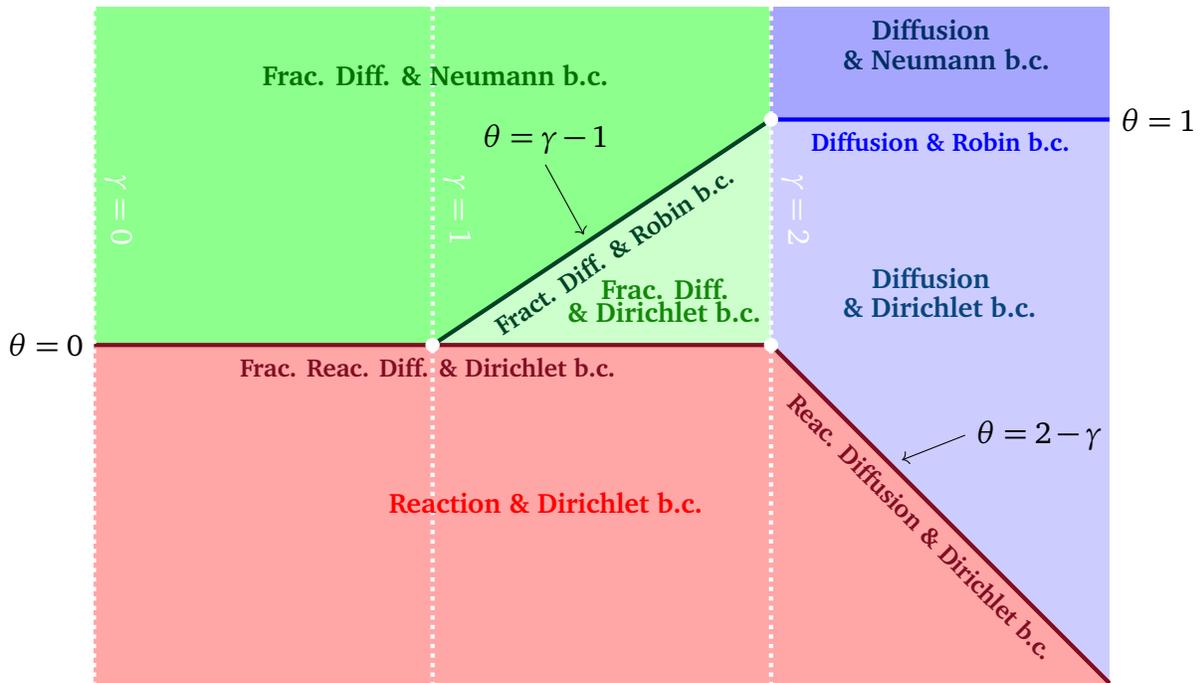


FIGURE 2. Hydrodynamic behavior depending on the values of θ (vertical axis) and γ (horizontal axis). The case $\gamma > 2$ is treated in [3]. The case $\gamma \in (1, 2)$ and $\theta \leq 0$ is treated in [2]. In this work we analyse all the other regimes, apart from the critical cases $\gamma = 1$ and $\gamma = 2$, that are left to a future work.

The paper is organized as follows: in Section 2, we define precisely the model and we present the hydrodynamic equations we obtained and the definition of weak solutions that we use. Moreover, in the same section, we state the main result of this paper. We also comment on the stationary profiles associated with the hydrodynamic equations. In order to derive these stationary profiles from the microscopic dynamics, we advise that it is possible

to follow the strategies presented in [30] and [5]. At the end of Section 2, we present also some heuristic results on possible variations of the model considered. Section 3 contains the proof of the convergence of the empirical measure to a weak solution of the PDEs introduced in Section 2. In Section 4 we prove that the weak solutions of the hydrodynamic equations we defined are unique, which completes the proof of the main result. Section 5 is devoted to the proof of some technical lemmas that we need during the proofs of these results.

2. STATEMENT OF RESULTS

2.1. The model. The Markov process we consider is analogous to the one introduced in [3] and [2], so we will use the same notation.

The dynamics of the process we are interested in is determined by the interactions between particles inside the set, defined for $N \geq 2$ by $\Lambda_N := \{1, \dots, N-1\}$ called “bulk”. The evolution of the configuration of particles is described by a function of time η_\cdot , taking values in the space $\Omega_N := \{0, 1\}^{\Lambda_N}$ of functions which associate to any point of Λ_N either the value 1 or 0. Thus, if we fix some horizon time $T > 0$, at time $t \in [0, T]$, the configuration η_t is fully described by the values of $\eta_t(x)$ for $x \in \Lambda_N$: if $\eta_t(x) = 1$ it means that there is a particle at site x at time t , otherwise if $\eta_t(x) = 0$ it means that the site x is empty at time t .

The dynamics of this process is explained in detail in [2, 3]. We recall it here briefly. It is easier to understand it separating bulk and reservoirs dynamics:

- **bulk dynamics:** each pair of sites $(x, y) \in \Lambda_N \times \Lambda_N$ carries a Poisson process of intensity one. When there is an occurrence in this process we exchange the value of $\eta(x)$ and $\eta(y)$ with probability $p(y-x)/2$;
- **reservoirs dynamics:** let us start describing the dynamics associated to the left reservoirs. Every pair of points $(x, y) \in \{x \in \mathbb{Z} : x \leq 0\} \times \Lambda_N$ carries a Poisson process of intensity one. When there is an occurrence in this process we exchange $\eta(y)$ with $1 - \eta(y)$ with rate $\alpha \frac{\kappa}{N^\theta} p(y-x)$ if $\eta(y) = 0$ and with rate $(1 - \alpha) \frac{\kappa}{N^\theta} p(x, y)$ if $\eta(y) = 1$. The right reservoirs act in an analogous way: every pair of points $(x, y) \in \{x \in \mathbb{Z} : x \geq N\} \times \Lambda_N$ carries a Poisson process of intensity one. When there is an occurrence in this process we exchange $\eta(y)$ with $1 - \eta(y)$ with rate $\beta \frac{\kappa}{N^\theta} p(y-x)$ if $\eta(y) = 0$ and with rate $(1 - \beta) \frac{\kappa}{N^\theta} p(y-x)$ if $\eta(y) = 1$.

The infinitesimal generator of the process considered, which depends on some parameter $\alpha, \beta \in (0, 1)$, $\kappa > 0$ and $\theta \in \mathbb{R}$, is defined as

$$L_N = L_N^0 + \kappa N^{-\theta} L_N^l + \kappa N^{-\theta} L_N^r, \quad (2.1)$$

and acts on functions $f : \Omega_N \rightarrow \mathbb{R}$ as

$$\begin{aligned}
 (L_N^0 f)(\eta) &= \frac{1}{2} \sum_{x,y \in \Lambda_N} p(x-y)[f(\sigma^{x,y} \eta) - f(\eta)], \\
 (L_N^l f)(\eta) &= \sum_{\substack{x \in \Lambda_N \\ y \leq 0}} p(x-y)c_x(\eta; \alpha)[f(\sigma^x \eta) - f(\eta)], \\
 (L_N^r f)(\eta) &= \sum_{\substack{x \in \Lambda_N \\ y \geq N}} p(x-y)c_x(\eta; \beta)[f(\sigma^x \eta) - f(\eta)]
 \end{aligned} \tag{2.2}$$

where

$$(\sigma^{x,y} \eta)(z) = \begin{cases} \eta(z), & \text{if } z \neq x, y, \\ \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y \end{cases}, \quad (\sigma^x \eta)(z) = \begin{cases} \eta(z), & \text{if } z \neq x, \\ 1 - \eta(x), & \text{if } z = x, \end{cases}$$

and for $\delta \in (0, 1)$ and $x \in \Lambda_N$, we define

$$c_x(\eta; \delta) := [\eta(x)(1 - \delta) + (1 - \eta(x))\delta]. \tag{2.3}$$

We will consider the Markov process in the time scale $t\Theta(N)$, where $\Theta(N)$ is defined in (2.26). In order to have a lighter notation we will denote $\eta_t^N := \eta_{t\Theta(N)}$, so that the process η_t^N has as infinitesimal generator $\Theta(N)L_N$.

2.2. Topological setting and fractional operators in bounded domains. In this subsection, we introduce the notation, the operators and the spaces of functions that we will use in the rest of this work.

First, let us fix some notation. For $h : [0, 1] \rightarrow [0, \infty)$ a Borel function we will consider frequently the Hilbert space $L^2([0, 1], h(u)du)$ that we denote simply by $L_h^2([0, 1])$. The inner product and the norm associated to this space will be denoted respectively by $\langle \cdot, \cdot \rangle_h$ and $\|\cdot\|_h$. When $h = 1$, we will work with the standard $L^2([0, 1], du)$ space, and in the norm and the inner product, we will omit the index $h = 1$. For any interval $I \subseteq \mathbb{R}$, we denote $C^k(I)$ (resp. $C_c^k(I)$) the space of continuous real-valued functions (resp. with compact support included in I) with the first k -th derivatives being continuous as well. Moreover, for $T > 0$, we will say that a function $H \in C^{m,n}([0, T] \times I)$ if $H(\cdot, x) \in C^m([0, T])$ and $H(t, \cdot) \in C^n(I)$ for any $x \in I$ and $t \in [0, T]$. Analogously $H \in C_c^{m,n}([0, T] \times I)$ if $H \in C^{m,n}([0, T] \times I)$ and $H(t, \cdot) \in C_c^n(I)$ for any $t \in [0, T]$. If one (or more) superscript is equal to ∞ it means that the function considered is smooth in the respective variable. We will use equivalently the notation $H_t(u)$ and $H(t, u)$, so the subscripts have not to be confused with partial derivatives. We will denote the derivatives of a function $H \in C^{m,n}([0, T] \times I)$, by $\partial_t H$ if the derivative is in the time variable and $\partial_u H$ if the derivative is in the space variable.

We will write hereinafter $f(u) \lesssim g(u)$ if there exists a constant C independent of u such that $f(u) \leq Cg(u)$ for every u ; moreover, we will write $f(u) = O(g(u))$ if the condition $|f(u)| \lesssim |g(u)|$ is satisfied.

We recall now the definition of the fractional operators and the spaces that we will use.

The fractional Laplacian $(-\Delta)^{\gamma/2}$ with exponent $\gamma/2$, for $\gamma \in (0, 2)$, is the operator acting on functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{|f(u)|}{(1+|u|)^{1+\gamma}} du < \infty \quad (2.4)$$

as

$$-(-\Delta)^{\gamma/2} f(u) = c_{\gamma} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \mathbb{1}_{|u-v| \geq \epsilon} \frac{f(v) - f(u)}{|u-v|^{1+\gamma}} dv, \quad (2.5)$$

for any $u \in \mathbb{R}$, if the previous limit exists and where c_{γ} is defined in (1.2). The interested reader may have a look at [26] for more details. In a finite domain we introduce the regional fractional Laplacian \mathbb{L} on the interval $I \subset \mathbb{R}$ which acts on functions $f : I \rightarrow \mathbb{R}$ such that

$$\int_I \frac{|f(u)|}{(1+|u|)^{1+\gamma}} du < \infty$$

as

$$(\mathbb{L}f)(u) = c_{\gamma} \lim_{\epsilon \rightarrow 0} \int_I \mathbb{1}_{|u-v| \geq \epsilon} \frac{f(v) - f(u)}{|u-v|^{1+\gamma}} dv, \quad (2.6)$$

for any $u \in I$ provided the limit exists. In this work we will always consider the case in which I is the interval $(0, 1)$ and in this particular case $\mathbb{L}f$ is well defined, for example, if $f \in C^2([0, 1])$.

On $(0, 1)$ we introduce the semi inner-product $\langle \cdot, \cdot \rangle_{\gamma/2}$, and the corresponding semi-norm $\|\cdot\|_{\gamma/2} = \langle \cdot, \cdot \rangle_{\gamma/2}^{1/2}$, defined by

$$\langle f, g \rangle_{\gamma/2} = \frac{c_{\gamma}}{2} \iint_{[0,1]^2} \frac{(f(u) - f(v))(g(u) - g(v))}{|u-v|^{1+\gamma}} dudv, \quad (2.7)$$

where $f, g : (0, 1) \rightarrow \mathbb{R}$ are functions such that $\|f\|_{\gamma/2} < \infty$ and $\|g\|_{\gamma/2} < \infty$.

We have the following integration by parts formula (Theorem 3.3 and 3.4) in [16]:

Proposition 2.1 ([16]). *If $f, g \in C^2([0, 1])$ then*

$$\langle f, -\mathbb{L}g \rangle = \langle -\mathbb{L}f, g \rangle = \langle f, g \rangle_{\gamma/2}.$$

If $f, g : [0, 1] \rightarrow \mathbb{R}$ are such that $\mathbb{L}f \in L^1([0, 1], du)$ and if g is such that

$$\iint_{[0,1]^2} \frac{|f(u) - f(v)| |g(u) - g(v)|}{|u-v|^{1+\gamma}} dudv < \infty \quad (2.8)$$

then

$$\langle -\mathbb{L}f, g \rangle = \langle f, g \rangle_{\gamma/2}.$$

As observed in [16], $\mathbb{L}f \in L^1([0, 1], du)$ as soon as $f \in C^1([0, 1])$ for $0 < \gamma < 1$ and $f \in C^2([0, 1])$ for $1 < \gamma < 2$.

In Corollary 7.6 of [16] the following generalization is also established for $\gamma \in (1, 2)$.

Proposition 2.2 ([16]). *Let $\gamma \in (1, 2)$ and $g \in C^2([0, 1])$. Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $u \rightarrow f(u)u^{1-\gamma}$ and $u \rightarrow f(u)(1-u)^{1-\gamma}$ are in $C^2([0, 1])$. Then,*

$$\langle -\mathbb{L}g, f \rangle = \chi_\gamma [g(1)D^\gamma f(1) - g(0)D^\gamma f(0)] + \langle -\mathbb{L}f, g \rangle \quad (2.9)$$

where χ_γ is a constant defined below equation (7.4) in [16] and

$$(D^\gamma f)(0) = \lim_{u \rightarrow 0^+} f'(u)u^{2-\gamma}, \quad (D^\gamma f)(1) = \lim_{u \rightarrow 1^-} f'(u)(1-u)^{2-\gamma}. \quad (2.10)$$

In order to have later a lighter notation it is useful to introduce a family of fractional operators $\{\mathbb{L}_\kappa\}_{\kappa \geq 0}$, defined in [2] as

$$\mathbb{L}_\kappa := \mathbb{L} - \kappa V_1, \quad \text{where} \quad V_1(u) := r^-(u) + r^+(u)$$

and, for any $u \in (0, 1)$,

$$r^-(u) := c_\gamma \gamma^{-1} u^{-\gamma}, \quad r^+(u) := c_\gamma \gamma^{-1} (1-u)^{-\gamma}. \quad (2.11)$$

We also define, for any $u \in (0, 1)$ and $\alpha, \beta \in (0, 1)$,

$$V_0(u) = \alpha r^-(u) + \beta r^+(u). \quad (2.12)$$

Remark 2.3. *Observe that the operators \mathbb{L}_κ defined above are symmetric in $L^2([0, 1], du)$. Moreover, for $\kappa = 1$, we recover the so-called restricted fractional Laplacian (see [11]), for any $f \in C_c^\infty((0, 1))$:*

$$\forall u \in (0, 1), \quad -(-\Delta)^{\gamma/2} f(u) = (\mathbb{L}f)(u) - V_1(u)f(u) := (\mathbb{L}_1 f)(u), \quad (2.13)$$

while for $\kappa = 0$ we get the regional fractional Laplacian \mathbb{L} defined in (2.6).

Remark 2.4. *It is well known that the fractional Laplacian $-(-\Delta)^{\gamma/2}$ is the infinitesimal generator of a γ -stable Lévy jump process on \mathbb{R} . In the finite domain $(0, 1)$, the regional fractional Laplacian \mathbb{L} is also associated to a Markov process which roughly corresponds to the Lévy jump process above reflected at the boundaries of $(0, 1)$. Because it is a jump process, making sense of the previous sentence is not trivial (see [16]). Following [16], the process with generator \mathbb{L} is called the “reflected symmetric γ -stable process”. Its behavior is strongly dependent on the exponent γ . For $\gamma \in (0, 1]$, it is essentially the same as the “symmetric γ -stable censored process” defined in [7] and which is obtained from the Lévy jump process on \mathbb{R} by suppressing its jumps from $(0, 1)$ to its complement. It is proved that for $\gamma \leq 1$, the censored (or equivalently the reflected) process will never touch the boundaries, so that in some sense the process does not see the boundaries. This is not the case if $\gamma > 1$.*

Definition 2.5. *Let us denote by $\mathcal{H}^{\gamma/2} := \mathcal{H}^{\gamma/2}([0, 1])$ the Sobolev space containing all the functions $g \in L^2([0, 1])$ such that $\|g\|_{\gamma/2} < \infty$. This is a Hilbert space endowed with the norm $\|\cdot\|_{\mathcal{H}^{\gamma/2}}$ defined by*

$$\|g\|_{\mathcal{H}^{\gamma/2}}^2 := \|g\|^2 + \|g\|_{\gamma/2}^2.$$

If $\gamma \in (1, 2)$ its elements coincide a.e. with continuous functions. The space $\mathcal{H}_0^{\gamma/2} := \mathcal{H}_0^{\gamma/2}([0, 1])$ is defined as the completion of $C_c^\infty((0, 1))$ for the norm just introduced. If $\gamma \in (1, 2)$, its elements coincide a.e. with continuous functions vanishing at 0 and 1 and as pointed out in [9], on

$\mathcal{H}_0^{\gamma/2}$, the norms $\|\cdot\|_{\mathcal{H}^{\gamma/2}}$ and $\|\cdot\|_{\gamma/2}$ are equivalent. If $\gamma \in (0, 1)$, $\mathcal{H}_0^{\gamma/2} = \mathcal{H}^{\gamma/2}$ (see Theorem 3, Section 2.4 of [27]).

Remark 2.6. The main difference between the case $\gamma \in (0, 1)$ and the case $\gamma \in (1, 2)$ is that the trace on $\{0, 1\}$ of a function $u \in \mathcal{H}^{\gamma/2}$ exists if $\gamma > 1$ while it does not exist (a priori) if $\gamma < 1$.

The weak solutions of the hydrodynamic equations that we will define live in the space $L^2(0, T; \mathcal{H}^{\gamma/2})$ which is the set of measurable functions $f : [0, T] \rightarrow \mathcal{H}^{\gamma/2}$ such that

$$\int_0^T \|f_t\|_{\mathcal{H}^{\gamma/2}}^2 dt < \infty.$$

The spaces $L^2(0, T; \mathcal{H}_0^{\gamma/2})$ and $L^2(0, T; L_h^2)$ are defined in an analogous way, using the spaces $L^2([0, 1], h(u)du)$ and $\mathcal{H}_0^{\gamma/2}$ instead of $\mathcal{H}^{\gamma/2}$.

The regional fractional Laplacian can be extended to the space $\mathcal{H}^{\gamma/2}$. Indeed for any $\rho \in \mathcal{H}^{\gamma/2}$ we can define the distribution $\mathbb{L}\rho$ on $(0, 1)$ by its action on functions $f \in C_c^\infty((0, 1))$:

$$\langle \mathbb{L}\rho, f \rangle = \langle \rho, \mathbb{L}f \rangle.$$

The proof that $\mathbb{L}\rho$ is indeed a well defined distribution can be found in [2].

2.3. Hydrodynamic equations. Now, that we have introduced all the notation and the spaces of functions that we will use, we can define the PDEs and respective notions of weak solutions, which are involved in the hydrodynamic limit of this model.

Definition 2.7. Let $\gamma \in (0, 2) \setminus \{1\}$ and $g : [0, 1] \rightarrow [0, 1]$ be a measurable function. We say that $\rho : [0, T] \times [0, 1] \rightarrow [0, 1]$ is a weak solution of the regional fractional diffusion equation with fractional Neumann boundary conditions and initial condition g :

$$\begin{cases} \partial_t \rho_t(u) = \mathbb{L}\rho_t(u), & (t, u) \in [0, T] \times (0, 1), \\ (D^\gamma \rho_t^{\hat{\kappa}})(0) = (D^\gamma \rho_t^{\hat{\kappa}})(1) = 0, \\ \rho_0(u) = g(u), & u \in (0, 1), \end{cases} \quad (2.14)$$

if:

i) $\rho \in L^2(0, T; \mathcal{H}^{\gamma/2})$.

ii) For all $t \in [0, T]$ and all functions $G \in C^{1,\infty}([0, T] \times (0, 1))$ we have that

$$F_{\text{Neu}}(t, \rho, G, g) := \langle \rho_t, G_t \rangle - \langle g, G_0 \rangle - \int_0^t \langle \rho_s, (\partial_s + \mathbb{L})G_s \rangle ds = 0. \quad (2.15)$$

Definition 2.8. Let $\gamma \in (0, 2) \setminus \{1\}$, $\hat{\kappa} > 0$ be some parameter and $g : [0, 1] \rightarrow [0, 1]$ be a measurable function. We say that $\rho^{\hat{\kappa}} : [0, T] \times [0, 1] \rightarrow [0, 1]$ is a weak solution of the regional fractional reaction-diffusion equation with non-homogeneous Dirichlet boundary conditions and initial condition g :

$$\begin{cases} \partial_t \rho_t^{\hat{\kappa}}(u) = \mathbb{L}_{\hat{\kappa}} \rho_t^{\hat{\kappa}}(u) + \hat{\kappa} V_0(u), & (t, u) \in [0, T] \times (0, 1), \\ \rho_t^{\hat{\kappa}}(0) = \alpha, \quad \rho_t^{\hat{\kappa}}(1) = \beta, & t \in (0, T], \\ \rho_0^{\hat{\kappa}}(u) = g(u), & u \in (0, 1), \end{cases} \quad (2.16)$$

if :

- i) $\rho^{\hat{\kappa}} \in L^2(0, T; \mathcal{H}^{\gamma/2})$.
- ii) $\int_0^T \int_0^1 \left\{ \frac{(\alpha - \rho_t^{\hat{\kappa}}(u))^2}{u^\gamma} + \frac{(\beta - \rho_t^{\hat{\kappa}}(u))^2}{(1-u)^\gamma} \right\} du dt < \infty$.
- iii) For all $t \in [0, T]$ and all functions $G \in C_c^{1,\infty}([0, T] \times (0, 1))$ we have that

$$F_{RD}(t, \rho^{\hat{\kappa}}, G, g) := \langle \rho_t^{\hat{\kappa}}, G_t \rangle - \langle g, G_0 \rangle - \int_0^t \langle \rho_s^{\hat{\kappa}}, (\partial_s + \mathbb{L}_{\hat{\kappa}}) G_s \rangle ds - \hat{\kappa} \int_0^t \langle G_s, V_0 \rangle ds = 0. \quad (2.17)$$

Remark 2.9. Observe that in Definition 2.8, the second item is automatically satisfied if $\gamma \in (0, 1)$ since $\rho^{\hat{\kappa}}$ is uniformly bounded by 1. If $\gamma \in (1, 2)$ the second item implies that for a.e. time t we have that $\rho_t^{\hat{\kappa}}(0) = \alpha$ and $\rho_t^{\hat{\kappa}}(1) = \beta$. We conjecture that this remains true for $\gamma \in (0, 1)$ even if the weak formulation (2.17) does not seem to imply trivially these boundary conditions.

Definition 2.10. Let $\gamma \in (1, 2)$. Let $g : [0, 1] \rightarrow [0, 1]$ be a measurable function. We say that $\rho : [0, T] \times [0, 1] \rightarrow [0, 1]$ is a weak solution of the regional fractional diffusion equation with non-homogeneous Dirichlet boundary conditions and initial condition g :

$$\begin{cases} \partial_t \rho_t(u) = \mathbb{L} \rho_t(u), & (t, u) \in [0, T] \times (0, 1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, & t \in (0, T], \\ \rho_0(u) = g(u), & u \in (0, 1), \end{cases} \quad (2.18)$$

if :

- i) $\rho \in L^2(0, T; \mathcal{H}^{\gamma/2})$.
- ii) For all $t \in [0, T]$ and all functions $G \in C_c^{1,\infty}([0, T] \times (0, 1))$ we have that

$$F_{Dir}(t, \rho, G, g) := \langle \rho_t, G_t \rangle - \langle g, G_0 \rangle - \int_0^t \langle \rho_s, (\partial_s + \mathbb{L}) G_s \rangle ds = 0. \quad (2.19)$$

iii) for t a.s. in $(0, T]$, $\rho_t(0) = \alpha$ and $\rho_t(1) = \beta$.

Definition 2.11. Let $\hat{\kappa} \geq 0$, $\gamma \in (1, 2)$ and let $g : [0, 1] \rightarrow [0, 1]$ be a measurable function. We say that $\rho^{\hat{\kappa}} : [0, T] \times [0, 1] \rightarrow [0, 1]$ is a weak solution of the regional fractional diffusion equation with fractional Robin boundary conditions and initial condition g :

$$\begin{cases} \partial_t \rho_t^{\hat{\kappa}}(u) = \mathbb{L} \rho_t^{\hat{\kappa}}(u), & (t, u) \in [0, T] \times (0, 1), \\ \chi_\gamma(D^\gamma \rho_t^{\hat{\kappa}})(0) = \hat{\kappa} m(\alpha - \rho_t^{\hat{\kappa}}(0)), \quad \chi_\gamma(D^\gamma \rho_t^{\hat{\kappa}})(1) = \hat{\kappa} m(\beta - \rho_t^{\hat{\kappa}}(1)), \\ \rho_0^{\hat{\kappa}}(u) = g(u), & u \in (0, 1), \end{cases} \quad (2.20)$$

if:

- i) $\rho^{\hat{\kappa}} \in L^2(0, T; \mathcal{H}^{\gamma/2})$.

ii) For all $t \in [0, T]$ and all functions $G \in C^{1,\infty}([0, T] \times (0, 1))$ we have that

$$\begin{aligned} F_{Rob}(t, \rho^{\hat{\kappa}}, G, g) &:= \langle \rho_t^{\hat{\kappa}}, G_t \rangle - \langle g, G_0 \rangle - \int_0^t \langle \rho_s^{\hat{\kappa}}, (\partial_s + \mathbb{L})G_s \rangle ds \\ &\quad - \hat{\kappa}m \int_0^t \{G_s(0)(\alpha - \rho_s^{\hat{\kappa}}(0)) + G_s(1)(\beta - \rho_s^{\hat{\kappa}}(1))\} ds = 0. \end{aligned} \quad (2.21)$$

Remark 2.12. Observe that in (2.21), the terms $\rho_s^{\hat{\kappa}}(0)$ and $\rho_s^{\hat{\kappa}}(1)$ are well defined for a.e. time s since $\rho^{\hat{\kappa}} \in L^2(0, T; \mathcal{H}^{\gamma/2})$ so that for a.e. time s $\rho_s^{\hat{\kappa}}$ has a continuous representative on $[0, 1]$.

Definition 2.13. Let $\gamma \in (0, 2) \setminus \{1\}$ and $\hat{\kappa} > 0$ be some parameter and let $g : [0, 1] \rightarrow [0, 1]$ be a measurable function. We say that $\rho^{\hat{\kappa}} : [0, T] \times [0, 1] \rightarrow [0, 1]$ is a weak solution of the reaction equation with non-homogeneous Dirichlet boundary condition and initial condition g :

$$\begin{cases} \partial_t \rho_t^{\hat{\kappa}}(u) = -\hat{\kappa} \rho_t^{\hat{\kappa}}(u) V_1(u) + \hat{\kappa} V_0(u), & (t, u) \in [0, T] \times (0, 1), \\ \rho_t^{\hat{\kappa}}(0) = \alpha, \quad \rho_t^{\hat{\kappa}}(1) = \beta, & t \in (0, T], \\ \rho_0^{\hat{\kappa}}(u) = g(u), & u \in (0, 1), \end{cases} \quad (2.22)$$

if:

i) $\int_0^T \int_0^1 \left\{ \frac{(\alpha - \rho_t^{\hat{\kappa}}(u))^2}{u^\gamma} + \frac{(\beta - \rho_t^{\hat{\kappa}}(u))^2}{(1-u)^\gamma} \right\} du dt < \infty.$

ii) For all $t \in [0, T]$ and all functions $G \in C_c^{1,\infty}([0, T] \times (0, 1))$ we have

$$\begin{aligned} F_{Reac}(t, \rho^{\hat{\kappa}}, G, g) &:= \langle \rho_t^{\hat{\kappa}}, G_t \rangle - \langle g, G_0 \rangle - \int_0^t \langle \rho_s^{\hat{\kappa}}, \partial_s G_s \rangle ds \\ &\quad + \hat{\kappa} \int_0^t \langle \rho_s^{\hat{\kappa}}, G_s \rangle_{V_1} ds - \hat{\kappa} \int_0^t \langle G_s, V_0 \rangle ds = 0. \end{aligned} \quad (2.23)$$

Remark 2.14. Observe that if $\gamma \in (0, 1)$ the first item is trivial since $\rho^{\hat{\kappa}}$ is uniformly bounded by 1. Moreover, as proved in [2], the equation (2.22) has an explicit solution in the sense of Definition 2.13 which is given by

$$\frac{V_0(u)}{V_1(u)} + \left[g(u) - \frac{V_0(u)}{V_1(u)} \right] e^{-t\hat{\kappa} \frac{V_0(u)}{V_1(u)}}, \quad (2.24)$$

for any $u \in [0, 1]$ and any $t \in [0, T]$. Therefore is trivial to show that $\rho_t^{\hat{\kappa}}(0) = \alpha$ and $\rho_t^{\hat{\kappa}}(1) = \beta$ for any $t \in (0, T]$.

Remark 2.15. The reader can easily check that thanks to Proposition 2.2 the notion of weak solution coincides formally ¹ with the corresponding strong PDE formulation for each of the definition given above.

Proposition 2.16. The weak solutions of equations (2.14), (2.16), (2.18), (2.20) and (2.22) given in the definitions above are unique.

Proof. It is detailed in Section 4. □

¹It is only formal since the regularity properties of the solution of the PDE are not known.

2.4. Hydrodynamic limit. Let \mathcal{M}^+ be the space of positive measures on $[0, 1]$ with total mass bounded by 1 equipped with the weak topology. For any configuration $\eta \in \Omega_N$, we define the empirical measure $\pi^N(\eta, du) \in \mathcal{M}^+$ by

$$\pi^N(\eta, du) = \frac{1}{N-1} \sum_{x \in \Lambda_N} \eta(x) \delta_{\frac{x}{N}}(du), \quad (2.25)$$

where δ_a is a Dirac mass on $a \in [0, 1]$ and we denote

$$\pi_t^N(du) := \pi^N(\eta_t^N, du).$$

Fix $T > 0$. We denote by \mathbb{P}_{μ_N} the probability measure in the Skorohod space $\mathcal{D}([0, T], \Omega_N)$ induced by the Markov process η^N with initial distribution μ_N and we denote by \mathbb{E}_{μ_N} the expectation with respect to \mathbb{P}_{μ_N} . Let $\{\mathbb{Q}^N\}_{N \geq 1}$ be the sequence of probability measures on $\mathcal{D}([0, T], \mathcal{M}^+)$ induced by the Markov process $\{\pi_t^N\}_{t \geq 0}$ and by \mathbb{P}_{μ_N} .

Definition 2.17. Let $\rho_0 : [0, 1] \rightarrow [0, 1]$ be a measurable function. We say that a sequence of probability measures $\{\mu_N\}_{N \geq 1}$ on Ω_N is associated with the profile $\rho_0(\cdot)$ if for any continuous function $G : [0, 1] \rightarrow \mathbb{R}$ and every $\delta > 0$

$$\lim_{N \rightarrow \infty} \mu_N \left(\eta \in \Omega_N : \left| \frac{1}{N} \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) \eta(x) - \int_0^1 G(q) \rho_0(q) dq \right| > \delta \right) = 0.$$

The next statement is the main theorem of this work. The result is summarized in Figure 2. The proof is given in details in the next sections.

Theorem 2.18 (Hydrodynamic limit).

Let $g : [0, 1] \rightarrow [0, 1]$ be a measurable function and let $\{\mu_N\}_{N \geq 1}$ be a sequence of probability measures in Ω_N associated to $g(\cdot)$. Then, for any $0 \leq t \leq T$,

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\mu_N} \left(\eta^N \in \mathcal{D}([0, T], \Omega_N) : \left| \frac{1}{N-1} \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) \eta_t^N(x) - \int_0^1 G(u) \rho_t^\kappa(u) du \right| > \delta \right) = 0,$$

where the time scale is given by

$$\Theta(N) = \begin{cases} N^{\gamma+\theta} & \theta < 0; \\ N^\gamma & \theta \geq 0; \end{cases} \quad (2.26)$$

and ρ_t^κ is the unique weak solution of:

- (2.14) if $\theta > 0$ and $\gamma \in (0, 1)$ or if $\theta > \gamma - 1$ and $\gamma \in (1, 2)$;
- (2.16) with $\hat{\kappa} = \kappa$, if $\theta = 0$ and $\gamma \in (0, 2) \setminus \{1\}$;
- (2.18) if $\theta \in (0, \gamma - 1)$ and $\gamma \in (1, 2)$;
- (2.20) with $\hat{\kappa} = \kappa$, if $\theta = \gamma - 1$ and $\gamma \in (1, 2)$;
- (2.22) with $\hat{\kappa} = \kappa$, if $\theta < 0$ and $\gamma \in (0, 2) \setminus \{1\}$.

To prove this theorem we will use the entropy method introduced in [17]. Therefore, we will prove that the sequence $\{\mathbb{Q}^N\}_{N \in \mathbb{N}}$ is tight and we will characterize uniquely the limiting point \mathbb{Q} by showing that it is a Dirac measure over the trajectory $\pi_t(du) = \rho(t, u)du$, where $\rho(t, u)$ is the unique weak solution of the corresponding hydrodynamic equation.

2.5. Few words about stationary solutions. In this subsection we discuss some facts about the study of stationary solutions $\lim_{t \rightarrow \infty} \rho_t^K$ of the hydrodynamic limits, presented in Theorem 2.18, and their possible derivations from the stationary state of the microscopic model (hydrostatic limit).

- If $\gamma \in (0, 1)$ and $\theta > 0$ or if $\gamma \in (1, 2)$ and $\theta > \gamma - 1$, we conjecture that the stationary solution depends on the initial condition g and it is given by the constant $\int_0^1 g(u) du$. On the other hand, in these regimes the hydrostatic profile is different and explicitly given by

$$\rho^\infty(u) = \int_0^1 \frac{V_0(w)}{V_1(w)} dw = \frac{\alpha + \beta}{2}$$

for any $u \in [0, 1]$. The hydrostatic limits can be derived by following the recent strategy developed in [30].

- If $\gamma \in (0, 2) \setminus \{1\}$ and $\theta < 0$ the stationary solution is given by

$$\rho^\infty(u) = \frac{V_0(u)}{V_1(u)}$$

for any $u \in [0, 1]$ and coincides with the hydrostatic profile which can probably be derived by following the strategy presented in [5] combined with the methods used in this paper.

- In the other regimes the stationary solution is expected to be unique, but not explicit, connecting α to β , apart from the case of Robin boundary conditions ($\gamma \in (1, 2)$, $\theta = \gamma - 1$). Here it should also coincide with the hydrostatic profile which can, likely, be derived by following the strategy presented in [5] and the results of the present paper.

The hydrostatic profiles obtained numerically are represented in Figure 3 below. We observe that apart from the flat case, they are nonlinear and not smooth at the boundaries.

2.6. Variations of the dynamics. In this section we present some results about three models with different dynamics but which can be studied with the techniques developed in this paper. We do not write the details of the proofs. We consider only the infinite variance case ($\gamma < 2$).

2.6.1. Super-diffusive model with reservoirs acting only on one site. This particular variant of our model has the same dynamics inside the bulk but the reservoirs are only located at the sites 0 and N and they act only on the sites 1 and $N - 1$, respectively. It means that the action of the reservoirs is exactly the same as the one presented in [1] while the bulk dynamics is the same as the one presented in Subsection 2.1. Therefore the generator associated to this

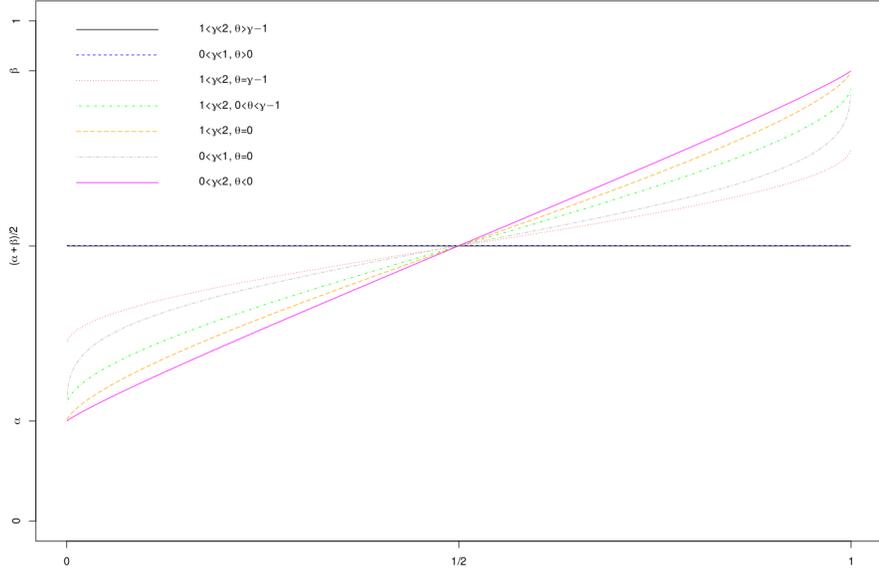


FIGURE 3. Here are represented some numerical simulations of the stationary solutions for the various PDEs, according to the value of γ and θ as in the legend.

model is defined by its action on functions $f : \Omega_N \rightarrow \mathbb{R}$ as:

$$\begin{aligned}
 L_N f(\eta) = & \frac{1}{2} \sum_{x,y \in \Lambda_N} p(x-y)[f(\sigma^{x,y} \eta) - f(\eta)] \\
 & + \frac{\kappa}{N^\theta} [\alpha(1 - \eta(1)) + (1 - \alpha)\eta(1)]f(\sigma^1 \eta) - f(\eta) \\
 & + \frac{\kappa}{N^\theta} [\beta(1 - \eta(N-1)) + (1 - \beta)\eta(N-1)]f(\sigma^{N-1} \eta) - f(\eta)].
 \end{aligned}
 \tag{2.27}$$

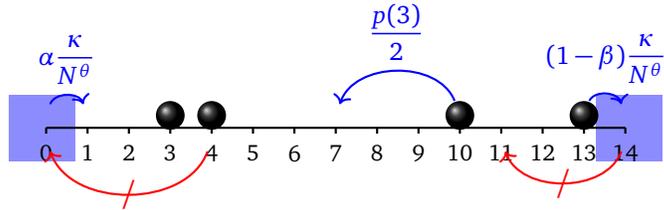


FIGURE 4. Example of the dynamics of the model presented in Section 2.6.1, with $N = 14$.

In this case we need to accelerate the system by a factor which does not depend on θ and which is constantly equal to $\Theta(N) = N^\gamma$. Then, following the same strategies used in this work, it is possible to prove that the hydrodynamic limits associated to this model are given by the following hydrodynamic equations (summarized in Figure 5):

- if $\theta < \gamma - 1$ we get the equation of Definition 2.7;
- if $\theta = \gamma - 1$ we get the equation of Definition 2.11 with $m = 1$ and $\hat{\kappa} = \kappa$;
- if $\theta > \gamma - 1$ we get the equation of Definition 2.7.

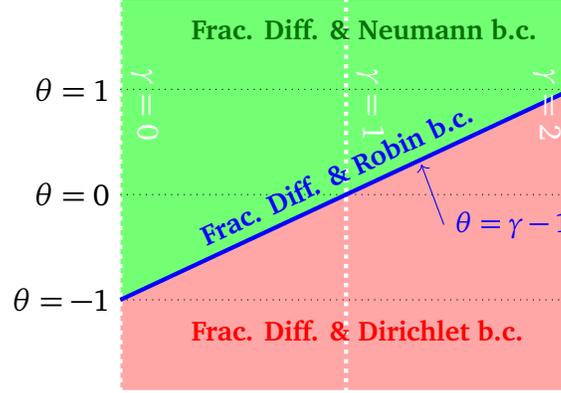


FIGURE 5. Summary of result on hydrodynamic limits for the model introduced in Section 2.6.1.

2.6.2. *Diffusive model with reservoirs acting on all the sites.* In this other variation of our model, the reservoirs have the same dynamics as the one described in Section 2.1, while in the bulk the particles move according to the nearest-neighbour dynamics defined in [1], that is in the bulk the transition probability $p(\cdot)$ has range 1. Therefore the generator associated to this model is given by its action on functions $f : \Omega_N \rightarrow \mathbb{R}$ as:

$$\begin{aligned}
 L_N f(\eta) &= \sum_{x=1}^{N-2} [f(\sigma^{x,x+1}\eta) - f(\eta)] \\
 &+ \frac{\kappa}{N^\theta} \sum_{x \in \Lambda_N} \sum_{y \leq 0} p(x-y) c_x(\eta; \alpha) [f(\sigma^x \eta) - f(\eta)] \\
 &+ \frac{\kappa}{N^\theta} \sum_{x \in \Lambda_N} \sum_{y \geq N} p(x-y) c_x(\eta; \beta) [f(\sigma^x \eta) - f(\eta)]
 \end{aligned} \tag{2.28}$$

where $c_x(\eta, \alpha)$ and $c_x(\eta, \beta)$ were defined in (2.3).

The hydrodynamic equations associated to this model (summarized in Figure 7) are given by:

- For $\theta > 2 - \gamma$, if we accelerate the system by a factor $\Theta(N) = N^2$, we get the heat equation with Neumann boundary conditions i.e.

$$\begin{cases} \partial_t \rho_t(u) = \Delta \rho_t(u) & u \times t \in (0, 1) \times [0, T]; \\ \partial_u \rho_t(0) = \partial_u \rho_t(1) = 0 & t \in [0, T]; \\ \rho_0(u) = g(u) & u \in (0, 1); \end{cases} \tag{2.29}$$

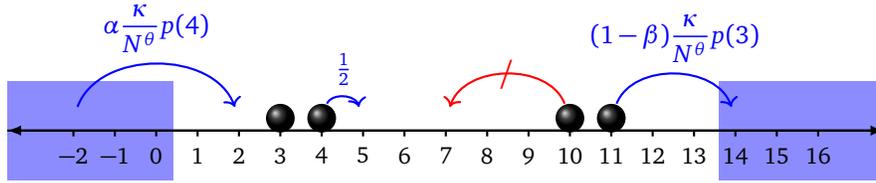


FIGURE 6. Example of the dynamics of the model presented in Section 2.6.2, with $N = 14$.

for some arbitrary initial condition $g : [0, 1] \rightarrow \mathbb{R}$ and with the notion of weak solution introduced in Definition 2.7 of [1];

- For $\theta = 2 - \gamma$, if we accelerate the system by a factor $\Theta(N) = N^2$, we get the reaction-diffusion equation with Dirichlet boundary conditions i.e.

$$\begin{cases} \partial_t \rho_t(u) = \Delta \rho_t(u) + V_0(u) \rho_t(u) & u \times t \in (0, 1) \times [0, T]; \\ \rho_t(0) = \alpha \text{ and } \rho_t(1) = \beta & t \in [0, T]; \\ \rho_0(u) = g(u) & u \in (0, 1); \end{cases} \quad (2.30)$$

for some arbitrary initial condition $g : [0, 1] \rightarrow \mathbb{R}$ and with the notion of weak solution introduced in Definition 2.2 of [3] but with $\sigma^2/2$ that now is equal to 1;

- For $\theta < 2 - \gamma$, if we accelerate the system by a factor $\Theta(N) = N^{\gamma + \theta}$ we get the reaction equation introduced in Definition 2.13 with $\hat{\kappa} = \kappa$.

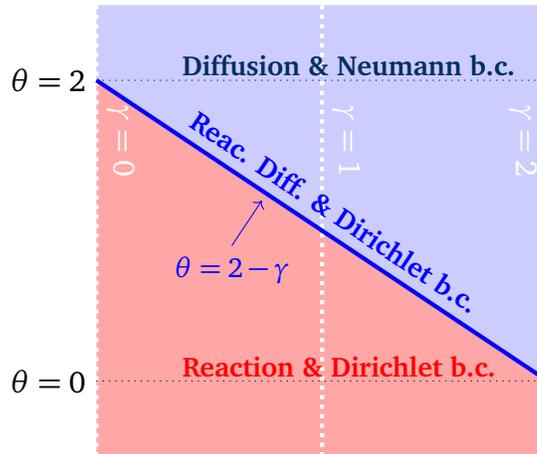


FIGURE 7. Summary of the results on hydrodynamic limit for the model introduced in Section 2.6.2.

2.6.3. *One site reservoirs' with long jumps.* Another interesting case which deserves a mention is the model with the same dynamics as the ones introduced in Subsection 2.1 but with reservoirs only at the sites 0 and N . This was studied in the lecture notes [13] and the

analysis is almost analogous to the one we do in this work and therefore we will not treat this case here.

3. PROOF OF THEOREM 2.18

3.1. Heuristics for the hydrodynamic equations. In this section we give the main ideas of the proof of the derivation of the weak solution of the hydrodynamic equation for each regime of θ and γ . Let us, by now, assume that the probability measure \mathbb{Q}^N on $\mathcal{D}([0, T], \mathcal{M}^+)$ defined above, converges to a measure \mathbb{Q} on the same space, concentrated on a trajectory of measures $\pi_t(dq)$, which is absolutely continuous with respect to the Lebesgue measure with a density $\rho(t, q)$. We will prove the aforementioned convergence, up to a subsequence, in the next subsections.

For simplicity we will forget the time dependence on the test functions, which would only make the notation heavier without bringing additional difficulties. Therefore, consider a test function $G : [0, 1] \rightarrow \mathbb{R}$ for which $\mathbb{L}G$ is well defined. From Dynkin's formula (Lemma A.5 of [21]) we have that

$$M_t^N(G) = \langle \pi_t^N, G \rangle - \langle \pi_0^N, G \rangle - \int_0^t \Theta(N) L_N \langle \pi_s^N, G \rangle ds, \quad (3.1)$$

is a martingale with respect to the natural filtration $\{\mathcal{F}_t^N\}_{t \geq 0}$, where $\mathcal{F}_t^N := \sigma(\{\eta_s^N\}_{s \leq t})$ for all $t \in [0, T]$. In the previous display we denoted by $\langle \pi_s^N, G \rangle$ the integration of the function G on $[0, 1]$ with respect to the empirical measure π_s^N , i.e.

$$\langle \pi_s^N, G \rangle = \frac{1}{N-1} \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) \eta_s^N(x).$$

Now, we compute the explicit value of the integrand function in (3.1), which is given by

$$\begin{aligned} \Theta(N) L_N \langle \pi_s^N, G \rangle &= \frac{\Theta(N)}{N-1} \sum_{x \in \Lambda_N} (\mathcal{L}_N G)\left(\frac{x}{N}\right) \eta_s^N(x) \\ &+ \frac{\kappa \Theta(N)}{(N-1)N^\theta} \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) \left(r_N^-\left(\frac{x}{N}\right) (\alpha - \eta_s^N(x)) + r_N^+\left(\frac{x}{N}\right) (\beta - \eta_s^N(x)) \right), \end{aligned} \quad (3.2)$$

where, $\mathcal{L}_N G$ is defined on test functions G by

$$(\mathcal{L}_N G)\left(\frac{x}{N}\right) = \sum_{y \in \Lambda_N} p(y-x) \left[G\left(\frac{y}{N}\right) - G\left(\frac{x}{N}\right) \right], \quad (3.3)$$

for all $x \in \Lambda_N$ with $(\mathcal{L}_N G)(0) = (\mathcal{L}_N G)(1) = 0$ and $r^\pm(x/N)$ are defined as

$$r_N^-\left(\frac{x}{N}\right) = \sum_{y \geq x} p(y) \quad \text{and} \quad r_N^+\left(\frac{x}{N}\right) = \sum_{y \leq N-x} p(y). \quad (3.4)$$

We have to study the various terms in (3.2) in the different regimes of $\gamma \in (0, 2) \setminus \{1\}$ and $\theta \in \mathbb{R}$. Recall that the cases $\theta \leq 0$ and $\gamma \in (1, 2)$ were already treated in [2] and, for that reason, they will not be considered in this work.

3.1.1. *Case $\theta < 0$ and $\gamma \in (0, 1)$.* In this regime the time scale is given by $\Theta(N) = N^{\gamma+\theta}$ and the test functions are in $C_c^\infty([0, 1])$. Let us first analyze the first term on the right-hand side of (3.2). Recalling (3.3) and performing a Taylor expansion of G around x/N , we can bound from above this term by a constant times

$$\frac{N^{\gamma+\theta}}{N^2} \sum_{x \in \Lambda_N} \sum_{\substack{y \in \Lambda_N \\ y \neq x}} \frac{1}{|y-x|^\gamma} \tag{3.5}$$

plus lower order terms with respect to N , since G' is uniformly bounded and $|\eta_s^N(x)| \leq 1$. Then, changing variables, that term can be rewritten as

$$2 \frac{N^{\gamma+\theta}}{N^2} \sum_{x=2}^{N-1} \sum_{z=1}^{x-1} \frac{1}{|z|^\gamma} \lesssim N^{\theta-1} \sum_{x=2}^{N-1} \int_{\frac{1}{N}}^{\frac{x-1}{N}} u^{-\gamma} du \lesssim N^{\theta-1} \sum_{x=2}^{N-1} \left(\frac{x-1}{N}\right)^{-\gamma+1} \lesssim N^\theta. \tag{3.6}$$

So, since $\theta < 0$, the term in the last display vanishes, as N goes to infinity.

Let us now analyze the second terms on the right-hand side of (3.2). We treat in details just the term involving r_N^- since the other one is completely analogous. That term can be rewritten as

$$\frac{\kappa}{N-1} \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) \left(N^\gamma r_N^-\left(\frac{x}{N}\right) - r^-\left(\frac{x}{N}\right)\right) (\alpha - \eta_s^N(x)) + \frac{\kappa}{N-1} \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) r^-\left(\frac{x}{N}\right) (\alpha - \eta_s^N(x)). \tag{3.7}$$

We will show now that the term on the left-hand side in the previous display vanishes as N goes to infinity. In order to do that, we use the estimate

$$|N^\gamma r_N^-\left(\frac{z}{N}\right) - r^-\left(\frac{z}{N}\right)| \leq c_\gamma N^{-1} \left(\frac{z}{N}\right)^{-\gamma-1}$$

when $z/N > a$ for some fixed $a \in (0, 1)$, proved in the Appendix of [5]. Using this estimate, the fact that $(\alpha - \eta_s^N(x))$ is uniformly bounded and G has compact support, we get that the term on the left-hand side of (3.7) is bounded from above by a constant times

$$\frac{\kappa}{N^2} \sum_{x \in \Lambda_N} \left(\frac{x}{N}\right)^{-\gamma-1} \lesssim \frac{1}{N} \int_{1/N}^1 u^{-\gamma-1} du \lesssim N^{\gamma-1}, \tag{3.8}$$

and since $\gamma < 1$, it vanishes as N goes to infinity. Finally, we prove that the second term of (3.7) converges to

$$\kappa \int_0^1 \alpha G(u) r^-(u) du - \kappa \int_0^1 G(u) r^-(u) \rho_s^\kappa(u) du, \tag{3.9}$$

provided that π_s^N converges to an absolutely continuous measure, with respect to the Lebesgue measure on $[0, 1]$, with density $\rho_s^\kappa(\cdot)$. In order to do that let us denote $H := G r^-$ which is a function in $C_c^\infty([0, 1])$. Then, the last term in (3.2) is equal to

$$\frac{\kappa}{N-1} \sum_{x \in \Lambda_N} H\left(\frac{x}{N}\right) (\alpha - \eta_s^N(x)), \tag{3.10}$$

plus terms vanishing as N goes to infinity. Which, under the hypothesis of the convergence of π^N that we just did, is exactly the Riemann series converging to (3.9).

Repeating the same proof for the term on the right-hand side of (3.2) involving r_N^+ , we get that the whole term converges to

$$\kappa \int_0^1 G(u)V_0(u)du - \kappa \int_0^1 G(u)V_1(u)\rho_s^k(u)du, \quad (3.11)$$

as N goes to infinity, which is the term involved in the definition of the weak solution for this regime of θ and γ .

3.1.2. *Case $\theta = 0$ and $\gamma \in (0, 1)$.* In this regime the time scale is $\Theta(N) = N^\gamma$ and the test functions are again in $C_c^\infty([0, 1])$. Then, the first term on the right-hand side of (3.2) can be written as

$$\frac{1}{N-1} \sum_{x \in \Lambda_N} (N^\gamma (\mathcal{L}_N G)(\frac{x}{N}) - \mathbb{L}G(\frac{x}{N})) \eta_s^N(x) + \frac{1}{N-1} \sum_{x \in \Lambda_N} \mathbb{L}G(\frac{x}{N}) \eta_s^N(x). \quad (3.12)$$

The first term on the left-hand side of the previous display vanishes as N goes to infinity, as a consequence of the fact that $|\eta_s^N(x)| \leq 1$ and Lemma 5.1. Now, using an argument similar to the one used in the proof of Lemma 5.1, it is possible to approximate the operator \mathbb{L} by an operator \mathbb{L}_ϵ in such a way that $\mathbb{L}_\epsilon G$ is a continuous function on $(0, 1)$ and $\lim_{\epsilon \rightarrow 0} \|\mathbb{L}_\epsilon G - \mathbb{L}G\|_{L^1(0,1)} = 0$.

Therefore we can rewrite the term on the right-hand side of (3.12) as

$$\frac{1}{N-1} \sum_{x \in \Lambda_N} [\mathbb{L}G(\frac{x}{N}) - \mathbb{L}_\epsilon G(\frac{x}{N})] \eta_s^N(x) + \frac{1}{N-1} \sum_{x \in \Lambda_N} \mathbb{L}_\epsilon G(\frac{x}{N}) \eta_s^N(x).$$

Then, reasoning as we did in the proof of Lemma 5.1, it is possible to show that the first term of the previous display vanishes as $\epsilon \rightarrow 0$ and the second one converges, as $N \rightarrow \infty$ and $\epsilon \rightarrow 0$, to

$$\int_0^1 \mathbb{L}G(u)\rho_s^k(u)du. \quad (3.13)$$

The second term on the right-hand side of (3.2) can be treated exactly as in the previous case, where we showed that it converges to (3.9) as N goes to infinity. Note that (3.13) and (3.9) are exactly the terms involved in the definition of weak solution in this regime of θ and γ .

3.1.3. *Case $\gamma \in (0, 1)$ and $\theta > 0$ or $\gamma \in (1, 2)$ and $\theta > \gamma - 1$.* In these regimes the time scale is again $\Theta(N) = N^\gamma$ and the test functions are in $C^\infty([0, 1])$. In both cases, the analysis of the first term in (3.2) is completely analogous to the one that we did in the previous subsection. Indeed, using Lemma 5.1 it is possible to show, with the assumption on the convergence of π^N , that this term converges to (3.13) with $G \in C^\infty([0, 1])$. This is the only term involved in the definition of weak solution in these regimes. Indeed, we show now that the second term on the right-hand side of (3.2) vanishes, as N goes to infinity. Since $G(\frac{x}{N})$ and $\alpha - \eta_s^N(x)$ are

bounded, we can rewrite the part involving r_N^- on the second term on the right-hand side of (3.2) as a constant times

$$N^{\gamma-\theta-1} \sum_{x \in \Lambda_N} r_N^-\left(\frac{x}{N}\right). \quad (3.14)$$

Now, observe that:

- in the case $\gamma \in (0, 1)$ the sum in (3.14) is diverging so we have to bound it in some way. Observing that $y^{-\gamma-1}$ is a decreasing function we can perform the following bound

$$\begin{aligned} \sum_{x \in \Lambda_N} r_N^-\left(\frac{x}{N}\right) &= c_\gamma \sum_{x \in \Lambda_N} \sum_{y \geq x} y^{-\gamma-1} \lesssim r_N^-\left(\frac{1}{N}\right) + \sum_{x=2}^{N-1} \sum_{y \geq x} y^{-\gamma-1} \\ &\leq \frac{1}{2} + \sum_{x=2}^{N-1} \int_{x-1}^{+\infty} y^{-\gamma-1} dy \leq C + \frac{1}{\gamma} \sum_{x=1}^{N-2} x^{-\gamma} \leq \frac{1}{2} + \frac{N^{-\gamma+1}}{\gamma}, \end{aligned} \quad (3.15)$$

since $r_N^-\left(\frac{1}{N}\right) = \frac{1}{2}$. So, we can bound (3.14) by

$$\frac{N^{\gamma-\theta-1}}{2} + \frac{N^{-\theta}}{\gamma} \lesssim N^{-\theta}$$

and it vanishes, as N goes to infinity;

- in the case $\gamma \in (1, 2)$ the sum in (3.14) is converging since, in this regime, $\sum_{x \in \Lambda_N} r_N^-\left(\frac{x}{N}\right)$ converges to m (see [2]). So the order of the whole term is $N^{\gamma-\theta-1}$, therefore it vanishes, as N goes to infinity, since $\theta > \gamma - 1$.

The term involving r_N^+ in (3.2) is treated in the same way. Notice that the integration by parts formula (Proposition 2.1) plus the boundary conditions of Definition 2.7 implies (2.15) (under the assumption of convergence of π^N).

3.1.4. Case $\gamma \in (1, 2)$ and $0 < \theta < \gamma - 1$. In this regime the time scale is $\Theta(N) = N^\gamma$ and the test functions are in $C_c^\infty([0, 1])$. The analysis of the first term at the right-hand side of (3.2) is analogous to the one of the previous case.

Let us then show how we treat the second term on the right-hand side of (3.2). We will use the fact that the test function G satisfies $G(x) = 0$ for $x \in \{0, 1\}$. Observe indeed that the term in (3.2) involving r_N^- can be rewritten as a constant times

$$N^{\gamma-\theta-1} \sum_{x \in \Lambda_N} x^{-\gamma} G\left(\frac{x}{N}\right),$$

since the order of $r_N^-(x/N)$ is $x^{-\gamma}$ when we are considering $x > aN$ for some fixed $a \in (0, 1)$ (see Lemma 3.3 of [5]) and this is true whenever the test function has compact support. Then, performing a Taylor expansion on G around 0 we can bound it from above by a constant times

$$N^{\gamma-\theta-2} \sum_{x \in \Lambda_N} x^{-\gamma+1},$$

plus lower order terms in N . Therefore, since the sum in the last display is of order $N^{-\gamma+2}$, the order of the whole term in the last display is $N^{-\theta}$, and it vanishes as N goes to infinity. The same analysis can be done for the term involving r_N^+ in (3.2).

The item (iii) of Definition 2.10 also holds, indeed the fact that $\rho_t(0) = \alpha$ and $\rho_t(1) = \beta$ follows by standard arguments thanks to Proposition 5.3 (see Appendix A.4 of [13] for details).

3.1.5. *Case $\gamma \in (1, 2)$ and $\theta = \gamma - 1$.* In this regime the time scale is $\Theta(N) = N^\gamma$ and the test functions are in $C^\infty([0, 1])$. The analysis of the first term on the right-hand side of (3.2) is analogous to the previous cases and is done using Lemma 5.1.

The analysis of the second term on the right-hand side of (3.2) is a bit different from the previous cases.

Consider first the part involving r_N^- . For any $\epsilon > 0$, we can rewrite it as

$$\frac{\kappa N^\gamma}{(N-1)N^{\gamma-1}}(\alpha - \overrightarrow{\eta}_s^{\epsilon N}(0)) \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right) + \frac{\kappa N^\gamma}{(N-1)N^{\gamma-1}} \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right) (\overrightarrow{\eta}_s^{\epsilon N}(0) - \eta_s^N(x)),$$

where $\overrightarrow{\eta}_s^{\epsilon N}(0)$ is defined in (5.28). On the other hand, the term on the right-hand side of the previous display vanishes, as N goes to infinity and then as ϵ goes to 0, thanks to Remark 5.10.

The term on the left-hand side of the last display, performing a Taylor expansion on G , can be written as

$$\frac{\kappa N^\gamma}{(N-1)N^{\gamma-1}} G(0)(\alpha - \overrightarrow{\eta}_s^{\epsilon N}(0)) \sum_{x \in \Lambda_N} r_N^-\left(\frac{x}{N}\right) + \frac{\kappa}{N-1} G'(0)(\alpha - \overrightarrow{\eta}_s^{\epsilon N}(0)) \sum_{x \in \Lambda_N} x r_N^-\left(\frac{x}{N}\right) \quad (3.16)$$

plus lower order terms in N . Observe that the term on the right-hand side of the previous display can be bounded by a constant times

$$\frac{1}{N-1} \sum_{x \in \Lambda_N} x^{-\gamma+1} = O(N^{-\gamma+1}),$$

which goes to 0 as N goes to infinity, since $\gamma > 1$. The remaining term in (3.16) converges in some sense to $G(0)(\alpha - \rho_s^\kappa(0))m$, as N goes to infinity, since $\lim_{N \rightarrow \infty} \sum_{x \in \Lambda_N} r_N^-\left(\frac{x}{N}\right) = m$ (see, for example, [2], for details about this computation) and since, in some sense, well explained in [13], $\overrightarrow{\eta}_s^{\epsilon N}(0) \rightarrow \rho_s^\kappa(0)$. Then, reasoning in a similar way as before, we can conclude that, for N going to infinity, the second term on the right-hand side of (3.2) converges to

$$mG(0)(\alpha - \rho_s^\kappa(0)) + mG(1)(\beta - \rho_s^\kappa(1)).$$

This is exactly the term involved in (2.21) and also that gives us, using the integration by parts formula (see Proposition 2.1), the boundary conditions in the Definition 2.11.

3.2. Tightness. In this subsection we prove that the sequence $\{\mathbb{Q}^N\}_{N \geq 1}$ is tight. We use Proposition 4.1.6 in [21], from where it is enough to show that, for all $\epsilon > 0$

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{\tau \in \mathcal{T}_T, \bar{\tau} \leq \delta} \mathbb{P}_{\mu_N} \left[\eta^N \in \mathcal{D}([0, T], \Omega_N) : \left| \langle \pi_{\tau+\bar{\tau}}^N, G \rangle - \langle \pi_{\tau}^N, G \rangle \right| > \epsilon \right] = 0, \quad (3.17)$$

for any function G belonging to $C^0([0, 1])$. Above \mathcal{T}_T is the set of stopping times bounded by T and we implicitly assume that all the stopping times are bounded by T , thus, $\tau + \bar{\tau}$ should be read as $(\tau + \bar{\tau}) \wedge T$. Indeed, we prove below that (3.17) is true for any function in $C_c^2([0, 1])$, and then, by using an L^1 approximation procedure (as it is done in [3]), we can extend the result to functions in $C^0([0, 1])$.

Proposition 3.1. *The sequence of measures $\{\mathbb{Q}^N\}_{N \geq 1}$ is tight with respect to the Skorohod topology of $\mathcal{D}([0, T], \mathcal{M}^+)$.*

Proof. The proof is almost analogous to the one done presented in [2] so we will omit most of the details. We are going to prove (3.17) for functions G in $C_c^2([0, 1])$. Recall now (3.1). Then, in order to prove (3.17), it is enough to show that

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{\tau \in \mathcal{T}_T, \bar{\tau} \leq \delta} \mathbb{E}_{\mu_N} \left[\left| \int_{\tau}^{\tau+\bar{\tau}} \Theta(N) L_N \langle \pi_s^N, G \rangle ds \right| \right] = 0 \quad (3.18)$$

and

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{\tau \in \mathcal{T}_T, \bar{\tau} \leq \delta} \mathbb{E}_{\mu_N} \left[(M_{\tau}^N(G) - M_{\tau+\bar{\tau}}^N(G))^2 \right] = 0. \quad (3.19)$$

The proof of (3.18) is completely analogous to the one in [2], therefore we omit it.

We have now to prove (3.19). By Dynkin's formula (see Appendix 1 in [21]) we know that

$$(M_t^N(G))^2 - \int_0^t \Theta(N) [L_N \langle \pi_s^N, G \rangle^2 - 2 \langle \pi_s^N, G \rangle L_N \langle \pi_s^N, G \rangle] ds,$$

is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Proceeding as in [2] it is possible to prove that the previous display is bounded from above by a constant times

$$\frac{\Theta(N)}{(N-1)^4} \sum_{x, y \in \Lambda_N} (x-y)^2 p(x-y) + \frac{\kappa \Theta(N)}{(N-1)^2 N^\theta} \sum_{x \in \Lambda_N} \left(G\left(\frac{x}{N}\right) \right)^2 \left(r_N^-\left(\frac{x}{N}\right) + r_N^+\left(\frac{x}{N}\right) \right). \quad (3.20)$$

Then, we can bound the term on the left-hand side of the previous display from above in the following way:

$$\frac{\Theta(N)}{(N-1)^4} \sum_{x, y \in \Lambda_N} (x-y)^2 p(x-y) \lesssim \frac{\Theta(N)}{(N-1)^4} \sum_{x, y \in \Lambda_N} (x-y)^{1-\gamma} \lesssim \frac{\Theta(N)}{N} N^{-\gamma} \leq N^{-1}.$$

The remaining term in (3.20) is of order less than an $\mathcal{O}(1)$, since performing a Taylor expansion on G around 0 we can rewrite it as a constant times

$$\frac{\kappa \Theta(N)}{(N-1)^2 N^{2+\theta}} (G'(0))^2 \sum_{x \in \Lambda_N} x^{-\gamma+2}$$

plus lower-order terms in N . Then, since the sum is diverging this term is at most of order $\Theta(N)N^{-\gamma-\theta-1}$, which, for any value of $\Theta(N)$ given by (2.26), vanishes as N goes to infinity.

Thus, since τ is a bounded stopping time we have that

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{\tau \in \mathcal{T}_T, \bar{\tau} \leq \delta} \mathbb{E}_{\mu^N} \left[(M_\tau^{N,G} - M_{\tau+\bar{\tau}}^{N,G})^2 \right] \\ &= \lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{\tau \in \mathcal{T}_T, \bar{\tau} \leq \delta} \mathbb{E}_{\mu^N} \left[\int_\tau^{\tau+\bar{\tau}} \Theta(N) [L_N \langle \pi_s^N, G \rangle^2 - 2 \langle \pi_s^N, G \rangle L_N \langle \pi_s^N, G \rangle] ds \right] \\ &= 0. \end{aligned}$$

Therefore, we have proved (3.17) for functions G in $C_c^2([0, 1])$ and, as we have recalled in the beginning of the proof, this is enough to conclude tightness. \square

3.3. Energy estimates. We prove in this subsection that any limit point \mathbb{Q} of the sequence $\{\mathbb{Q}^N\}_{N \geq 1}$ is concentrated on trajectories $\{\pi_t(u)du\}_{t \in [0, T]}$ with finite energy, i.e., π belongs to the space $L^2(0, T; \mathcal{H}^{\gamma/2})$. The latter is the content of Theorem 3.2 stated below. Fix a limit point \mathbb{Q} of the sequence $\{\mathbb{Q}^N\}_{N \geq 1}$ and assume, without loss of generality, that the whole sequence \mathbb{Q}^N converges to \mathbb{Q} , as N goes to infinity.

Theorem 3.2. *For $\theta \geq 0$ and $\gamma \in (0, 2) \setminus \{1\}$, the measure \mathbb{Q} is concentrated on trajectories of measures of the form $\{\pi_t(u)du\}_{t \in [0, T]}$, such that for any interval $I \subset [0, T]$ the density π satisfies \mathbb{Q} -a.s.*

$$\int_I \|\pi_t\|_{\gamma/2}^2 dt \lesssim |I|.$$

Before we prove Theorem 3.2, we establish some estimates on the Dirichlet form which are needed for the proof. Let $\rho : [0, 1] \rightarrow [0, 1]$ be a function such that $\alpha \leq \rho(u) \leq \beta$, for any $u \in [0, 1]$, and assume that $\rho(0) = \alpha$ and $\rho(1) = \beta$. Let $\nu_{\rho(\cdot)}^N$ be the inhomogeneous Bernoulli product measure on Ω_N with marginals given by

$$\nu_{\rho(\cdot)}^N \{\eta : \eta(x) = 1\} = \rho\left(\frac{x}{N}\right). \quad (3.21)$$

We denote by $H_N(\mu | \nu_{\rho(\cdot)}^N)$ the relative entropy of a probability measure μ on Ω_N with respect to the probability measure $\nu_{\rho(\cdot)}^N$. For exclusion processes on a finite domain and for any initial measure μ_N , it is easy to prove the existence of a constant C_0 , such that

$$H(\mu_N | \nu_{\rho(\cdot)}^N) \leq C_0 N, \quad (3.22)$$

see for example [3] for the proof. We remark here that the restriction $\alpha, \beta \notin \{0, 1\}$ comes from last estimate since the constant C_0 above is given by $C_0 = -\log(\alpha \wedge (1-\beta))$, when $\alpha < \beta$. On the other hand, for a probability measure μ on Ω_N and a density function $f : \Omega_N \rightarrow [0, \infty)$ with respect to μ we introduce

$$D_N^0(\sqrt{f}, \mu) := \frac{1}{2} \sum_{x, y \in \Lambda_N} p(y-x) I_{x,y}(\sqrt{f}, \mu), \quad (3.23)$$

$$D_N^l(\sqrt{f}, \mu) := \sum_{x \in \Lambda_N} \sum_{y \leq 0} p(y-x) I_x^\alpha(\sqrt{f}, \mu) = \sum_{x \in \Lambda_N} r_N^-(\frac{x}{N}) I_x^\alpha(\sqrt{f}, \mu), \quad (3.24)$$

$$D_N^r(\sqrt{f}, \mu) := \sum_{x \in \Lambda_N} \sum_{y \geq N} p(y-x) I_x^\beta(\sqrt{f}, \mu) = \sum_{x \in \Lambda_N} r_N^+(\frac{x}{N}) I_x^\beta(\sqrt{f}, \mu). \quad (3.25)$$

Above, we used the following notation

$$I_{x,y}(\sqrt{f}, \mu) := \int \left(\sqrt{f(\sigma^{x,y}\eta)} - \sqrt{f(\eta)} \right)^2 d\mu, \quad (3.26)$$

$$I_x^\delta(\sqrt{f}, \mu) := \int c_x(\eta; \delta) \left(\sqrt{f(\sigma^x\eta)} - \sqrt{f(\eta)} \right)^2 d\mu, \quad \delta \in \{\alpha, \beta\}.$$

Our goal is to express, for the measure $\nu_{\rho(\cdot)}^N$, a relation between the Dirichlet form defined by $\langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(\cdot)}^N}$ and the quantity

$$D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N) := (D_N^0 + \kappa N^{-\theta} D_N^l + \kappa N^{-\theta} D_N^r)(\sqrt{f}, \nu_{\rho(\cdot)}^N). \quad (3.27)$$

That relation is given by the following Lemma, already proved in Section 3.3.1 of [2].

Lemma 3.3. *There exists a constant $C > 0$ such that for any positive constant B and any density function f with respect to $\nu_{\rho(\cdot)}^N$, we have that*

$$\begin{aligned} \frac{\Theta(N)}{NB} \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(\cdot)}^N} &\leq -\frac{\Theta(N)}{4NB} D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N) + \frac{C\Theta(N)}{NB} \sum_{x,y \in \Lambda_N} p(y-x) \left(\rho(\frac{x}{N}) - \rho(\frac{y}{N}) \right)^2 \\ &\quad + \frac{C\kappa\Theta(N)}{N^{\theta+1}B} \sum_{x \in \Lambda_N} \left\{ \left(\rho(\frac{x}{N}) - \alpha \right)^2 r_N^-(\frac{x}{N}) + \left(\rho(\frac{x}{N}) - \beta \right)^2 r_N^+(\frac{x}{N}) \right\}. \end{aligned} \quad (3.28)$$

Remark 3.4. *Note that, as a consequence of the previous lemma, for a Lipschitz function $\rho(\cdot)$ such that $\alpha \leq \rho(u) \leq \beta$ and $\rho(0) = \alpha$ and $\rho(1) = \beta$, we have that*

$$\frac{\Theta(N)}{NB} \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(\cdot)}^N} \leq -\frac{\Theta(N)}{4NB} D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N) + \frac{C\Theta(N)(\kappa + N^\theta)}{BN^{\gamma+\theta}}. \quad (3.29)$$

Remark 3.5. *Recasting the bound found in Lemma 3.3 when $\rho(\cdot)$ is equal to a constant we get that: if $\gamma \in (1, 2)$, the bound is*

$$\frac{\Theta(N)}{NB} \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(\cdot)}^N} \leq -\frac{\Theta(N)}{4NB} D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N) + \frac{C\kappa\Theta(N)}{BN^{\theta+1}} \quad (3.30)$$

and for $\gamma \in (0, 1)$, the bound is

$$\frac{\Theta(N)}{NB} \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(\cdot)}^N} \leq -\frac{\Theta(N)}{4NB} D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N) + \frac{C\kappa\Theta(N)}{BN^{\gamma+\theta}}. \quad (3.31)$$

The difference between the two previous bounds comes from the behavior of the sum $\sum_{x \in \Lambda_N} r_N^\pm(\frac{x}{N})$ according to the value of γ .

Remark 3.6. Now we can establish the properties that we need to impose on the profile $\rho(\cdot)$ in order to have a good approximation of $\langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(\cdot)}^N}$ via $D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N)$. This is equivalent to find the condition on $\rho(\cdot)$ such that the last two terms on the right-hand side of (3.28) vanish when we send first N to infinity and then B to infinity. Thanks to the previous two remarks in order to have this property we need to take:

- $\rho(\cdot) = \rho \in (0, 1)$ constant if $\gamma \in (0, 1)$ and $\theta \geq 0$ or if $\gamma \in (1, 2)$ and $\theta \geq \gamma - 1$;
- $\rho(\cdot)$ Lipschitz such that $\rho(0) = \alpha$ and $\rho(1) = \beta$ in all the other regimes.

3.3.1. *Proof of Theorem 3.2.* The proof of this theorem is almost analogous to the proof of item i) of Theorem 3.2 in [2] (where the authors proved it for the regime $\gamma \in (1, 2)$ and $\theta \leq 0$). We write here the main steps. We want to show that $\pi_t \in L^2(0, T; \mathcal{H}^{\gamma/2})$ \mathbb{Q} -almost surely. In order to do that we will show the next bound

$$\mathbb{E}_{\mathbb{Q}} \left[\int_I \|\pi_t\|_{\gamma/2}^2 dt \right] \lesssim |I|, \quad (3.32)$$

which implies the result in the statement above.

Recall that the system is speeded up on the time scale $\Theta(N)$, defined by (2.26). Let $\rho(\cdot)$ be a macroscopic profile chosen according to Remark 3.6. Let $\epsilon > 0$ be a small real number. Let $F \in C_c^{0,\infty}(I \times [0, 1]^2)$, where I is a subinterval of $[0, T]$. By the entropy and Jensen's inequality and Feynman-Kac's formula (see Lemma A.7.2 in [21]), we have that

$$\begin{aligned} & \mathbb{E}_{\mu_N} \left[\int_I \Theta(N) N^{-1} \sum_{\substack{x, y \in \Lambda_N \\ |x-y| \geq \epsilon N}} F_t\left(\frac{x}{N}, \frac{y}{N}\right) p(y-x) (\eta_t^N(y) - \eta_t^N(x)) dt \right] \\ & \leq C_0 + \int_I \sup_f \left\{ \Theta(N) N^{-1} \sum_{\substack{x, y \in \Lambda_N \\ |x-y| \geq \epsilon N}} F_t\left(\frac{x}{N}, \frac{y}{N}\right) p(y-x) \int (\eta(y) - \eta(x)) f(\eta) d\nu_{\rho(\cdot)}^N \right. \\ & \quad \left. + \Theta(N) N^{-1} \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(\cdot)}^N} \right\} dt \end{aligned} \quad (3.33)$$

where the supremum is taken over all densities f on Ω_N with respect to $\nu_{\rho(\cdot)}^N$. Now, denoting the antisymmetric part of F_t by F_t^a with

$$F_t^a(u, v) = \frac{1}{2} [F_t(u, v) - F_t(v, u)]$$

for all $t \in I$ and $(u, v) \in [0, 1]^2$, we can rewrite the first term inside the supremum in (3.33) as

$$\begin{aligned}
 & \Theta(N)N^{-1} \sum_{\substack{x, y \in \Lambda_N \\ |x-y| \geq \epsilon N}} F_t^a\left(\frac{x}{N}, \frac{y}{N}\right) p(y-x) \int (\eta(y) - \eta(x)) f(\eta) d\nu_{\rho(\cdot)}^N \\
 &= \Theta(N)N^{-1} \sum_{\substack{x, y \in \Lambda_N \\ |x-y| \geq \epsilon N}} F_t^a\left(\frac{x}{N}, \frac{y}{N}\right) p(y-x) \int \eta(y) (f(\eta) - f(\sigma^{x,y}\eta)) d\nu_{\rho(\cdot)}^N \\
 &+ \Theta(N)N^{-1} \sum_{\substack{x, y \in \Lambda_N \\ |x-y| \geq \epsilon N}} F_t^a\left(\frac{x}{N}, \frac{y}{N}\right) p(y-x) \int \eta(x) f(\eta) (\theta^{x,y}(\eta) - 1) d\nu_{\rho(\cdot)}^N,
 \end{aligned} \tag{3.34}$$

where the equality follows from the change of variables $\eta \mapsto \sigma^{x,y}\eta$ and

$$\theta^{x,y}(\eta) = \frac{d\nu_{\rho(\cdot)}^N(\sigma^{x,y}\eta)}{d\nu_{\rho(\cdot)}^N(\eta)}.$$

Now, we have to split the proof in two cases: the case in which $\rho(\cdot) = \rho$ is constant (corresponding to the regimes $(\theta, \gamma) \in [0, \infty) \times (0, 1)$ and $(\theta, \gamma) \in [\gamma - 1, \infty) \times (1, 2)$) and the case in which $\rho(\cdot)$ is a Lipschitz profile (corresponding to the regime $(\theta, \gamma) \in (0, \gamma - 1) \times (1, 2)$).

In the case in which $\rho(\cdot) = \rho$ is constant we have that $\theta^{x,y}(\eta) - 1 = 0$. Therefore, by Young's inequality, the fact that f is a density and $|\eta(y)| \leq 1$, we have that, for any $A > 0$, (3.34) is bounded from above by a constant times

$$c_\gamma A \Theta(N) N^{-2+\gamma} \sum_{\substack{x, y \in \Lambda_N \\ |x-y| \geq \epsilon N}} \frac{\left(F_t^a\left(\frac{x}{N}, \frac{y}{N}\right)\right)^2}{\left|\frac{x}{N} - \frac{y}{N}\right|^{1+\gamma}} + \frac{2\Theta(N)N^{-1}}{A} D_N^0(\sqrt{f}, \nu_{\rho(\cdot)}^N).$$

Then, using (3.30) and (3.31), we can bound (3.33) in the following way:

$$\begin{aligned}
 & \mathbb{E}_{\mu_N} \left[\int_I \Theta(N)N^{-1} \sum_{\substack{x, y \in \Lambda_N \\ |x-y| \geq \epsilon N}} F_t^a\left(\frac{x}{N}, \frac{y}{N}\right) p(y-x) (\eta_t^N(y) - \eta_t^N(x)) dt \right] \\
 &= \mathbb{E}_{\mu_N} \left[\int_I -2c_\gamma \langle \pi_t^N, g_t^N \rangle dt \right] \lesssim \int_I N^{-2} \sum_{\substack{x, y \in \Lambda_N \\ |x-y| \geq \epsilon N}} \frac{c_\gamma \left(F_t^a\left(\frac{x}{N}, \frac{y}{N}\right)\right)^2}{\left|\frac{x}{N} - \frac{y}{N}\right|^{1+\gamma}} dt + |I|(\kappa + 1),
 \end{aligned} \tag{3.35}$$

where the function g^N is defined on $I \times [0, 1]$ by

$$g_t^N(u) = \frac{1}{N-1} \sum_{y \in \Lambda_N} \mathbf{1}_{\left|\frac{y}{N} - u\right| \geq \epsilon} \frac{F_t^a\left(u, \frac{y}{N}\right)}{\left|u - \frac{y}{N}\right|^{1+\gamma}}$$

and it is a discretization of the smooth function g defined on $(t, u) \in I \times [0, 1]$ by

$$g_t(u) = \int_0^1 \mathbf{1}_{\{|v-u| \geq \epsilon\}} \frac{F_t^a(u, v)}{|u-v|^{1+\gamma}} dv.$$

Now, we can proceed exactly in the same way as in the proof of Theorem 3.2 of [2] to conclude that

$$\mathbb{E}_{\mathbb{Q}} \left[\sup_F \left\{ \int_I \int \int_{Q_\epsilon} \frac{(\pi_t(v) - \pi_t(u)) F_t(u, v)}{|u-v|^{1+\gamma}} - C \frac{(F_t(u, v))^2}{|u-v|^{1+\gamma}} dudvdt \right\} \right] \lesssim |I|(\kappa + 1), \quad (3.36)$$

where $Q_\epsilon = \{(u, v) \in [0, 1]^2; |u-v| \geq \epsilon\}$. That bound, passing to the limit $\epsilon \rightarrow 0$, by the monotone convergence Theorem, implies the statement (3.32).

The case in which $\rho(\cdot) : [0, 1] \rightarrow [\alpha, \beta]$ is Lipschitz continuous and such that $\rho(0) = \alpha$ and $\rho(1) = \beta$ can be analyzed exactly in the same way as it was proved in Theorem 3.2 of [2] therefore we will omit this part of the proof.

3.4. Characterization of the limit points. In this subsection we characterize all limit points \mathbb{Q} of the sequence $\{\mathbb{Q}^N\}_{N \geq 1}$, which we know that exist from Proposition 3.1. Let us assume without loss of generality, that $\{\mathbb{Q}^N\}_{N \geq 1}$ converges to \mathbb{Q} , as N goes to infinity. Since there is at most one particle per site, it is easy to show that \mathbb{Q} is concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure, i.e. $\pi_t^k(du) = \rho_t^k(u)du$ (for details see [21]). In Proposition 3.7 below we prove, for each range of θ and γ , that \mathbb{Q} is concentrated on trajectories of measures whose density ρ_t^k satisfies a weak form of the corresponding hydrodynamic equation. Moreover, we have seen in Theorem 3.2 that \mathbb{Q} is concentrated on trajectories of measures whose density satisfies the energy estimate, i.e. $\rho^k \in L^2(0, T; \mathcal{H}^{\gamma/2})$ when $\theta \geq 0$ and $\gamma \in (0, 2) \setminus \{1\}$.

Proposition 3.7. *If \mathbb{Q} is a limit point of $\{\mathbb{Q}^N\}_{N \geq 1}$ then*

$$\mathbb{Q}(\pi. \in \mathcal{D}([0, T], \mathcal{M}^+) : F_{\theta, \gamma}(t, \rho^k, G, g)) = 0,$$

for each θ and each γ , where $F_{\theta, \gamma}$ is defined by

$$F_{\theta, \gamma}(t, \rho^k, G, g) := \begin{cases} F_{Reac}(t, \rho^k, G, g) & \text{if } \theta < 0, \gamma \in (0, 1); \\ F_{RD}(t, \rho^k, G, g) & \text{if } \theta = 0, \gamma \in (0, 1); \\ F_{Neu}(t, \rho^0, G, g) & \text{if } \theta > 0, \gamma \in (0, 1) \text{ or } \theta > \gamma - 1, \gamma \in (1, 2); \\ F_{Rob}(t, \rho, G, g) & \text{if } \theta = \gamma - 1, \gamma \in (1, 2); \\ F_{Dir}(t, \rho, G, g) & \text{if } \theta \in (0, \gamma - 1), \gamma \in (1, 2). \end{cases} \quad (3.37)$$

where $C_{\theta, \gamma} = C^{1,2}([0, T] \times [0, 1])$ if $\theta > 0$ and $\gamma \in (0, 1)$ or if $\theta \geq \gamma - 1$ and $\gamma \in (1, 2)$, while in all the other regimes it is $C_{\theta, \gamma} = C_c^{1,2}([0, T] \times [0, 1])$.

Proof. The proof is very similar to the one in [2] and for that reason some details are omitted. Note that in order to prove the proposition, it is enough to verify, for $\delta > 0$ and G in the

corresponding space of test functions, that

$$\mathbb{Q}\left(\pi. \in \mathcal{D}([0, T], \mathcal{M}^+) : \sup_{0 \leq t \leq T} |F_{\theta, \gamma}(t, \rho^\kappa, G, g)| > \delta\right) = 0, \quad (3.38)$$

for each θ and each γ in the respective domains.

We have that

$$\begin{aligned} F_{\theta, \gamma}(t, \rho^\kappa, G, g) &= \langle \rho_t^\kappa, G_t \rangle - \langle g, G_0 \rangle - \int_0^t \langle \rho_s^\kappa, (\partial_s + \mathbb{1}_{\{\theta \geq 0\}} \mathbb{L}) G_s \rangle ds \\ &\quad + \mathbb{1}_{\{\theta \leq 0\}} \mathbb{1}_{\{\gamma \in (0, 1)\}} \kappa \int_0^t \langle \rho_s^\kappa, G_s \rangle_{V_1} ds - \mathbb{1}_{\{\theta \leq 0\}} \mathbb{1}_{\{\gamma \in (0, 1)\}} \kappa \int_0^t \langle G_s, V_0 \rangle ds \\ &\quad - \mathbb{1}_{\{\theta = \gamma - 1\}} \mathbb{1}_{\{\gamma \in (1, 2)\}} \kappa \int_0^t m G_s(0) (\alpha - \rho_s^\kappa(0)) + m G_s(1) (\beta - \rho_s^\kappa(1)) ds = 0. \end{aligned} \quad (3.39)$$

From here on, in order to simplify notation, we will erase $\pi.$ from the sets that we have to look at. By (3.39) we can bound from above (3.38) by the sum of

$$\mathbb{Q}\left(\sup_{0 \leq t \leq T} |F_{\theta, \gamma}(t, \rho^\kappa, G, \rho_0)| > \frac{\delta}{2}\right) \quad (3.40)$$

and

$$\mathbb{Q}\left(|\langle \rho_0 - g, G_0 \rangle| > \frac{\delta}{2}\right). \quad (3.41)$$

We note that (3.41) is equal to zero since \mathbb{Q} is a limit point of $\{\mathbb{Q}^N\}_{N \geq 1}$ and \mathbb{Q}^N is induced by μ_N which is associated to $g(\cdot)$. Now we deal with (3.40).

Notice that by Proposition A.3 of [10], the set inside the probability in (3.40) is an open set in the Skorohod topology. Therefore, from Portmanteau's Theorem we bound (3.40) from above by

$$\liminf_{N \rightarrow \infty} \mathbb{Q}^N \left(\sup_{0 \leq t \leq T} |F_{\theta, \gamma}(t, \rho^\kappa, G, \rho_0)| > \frac{\delta}{2} \right).$$

Summing and subtracting $\int_0^t \Theta(N) L_N \langle \pi_s^N, G_s \rangle ds$ to the term inside the previous absolute value, recalling (3.1), (3.39) and the definition of \mathbb{Q}^N , we can bound the previous probability from above by the sum of

$$\mathbb{P}_{\mu_N} \left(\sup_{0 \leq t \leq T} |M_t^N(G)| > \frac{\delta}{4} \right)$$

and

$$\begin{aligned}
& \mathbb{P}_{\mu_N} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \Theta(N) L_N \langle \pi_s^N, G_s \rangle ds - \int_0^t \langle \rho_s^\kappa, \mathbb{1}_{\{\theta \geq 0\}} \mathbb{L} G_s \rangle ds \right. \right. \\
& \quad + \mathbb{1}_{\{\gamma \in (0,1)\}} \mathbb{1}_{\{\theta \leq 0\}} \kappa \int_0^t \langle \rho_s^\kappa, G_s \rangle_{V_1} ds - \mathbb{1}_{\{\gamma \in (0,1)\}} \mathbb{1}_{\{\theta \leq 0\}} \kappa \int_0^t \langle G_s, V_0 \rangle ds \\
& \quad \left. \left. - \mathbb{1}_{\{\theta = \gamma - 1\}} \mathbb{1}_{\{\gamma \in (1,2)\}} \kappa \int_0^t m G_s(0)(\alpha - \rho_s^\kappa(0)) + m G_s(1)(\beta - \rho_s^\kappa(1)) ds \right| > \frac{\delta}{4} \right). \tag{3.42}
\end{aligned}$$

By Doob's inequality we have that

$$\mathbb{P}_{\mu_N} \left(\sup_{0 \leq t \leq T} |M_t^N(G)| > \frac{\delta}{4} \right) \lesssim \frac{1}{\delta^2} \mathbb{E}_{\mu_N} \left[\int_0^T \Theta(N) [L_N \langle \pi_s^N, G \rangle^2 - 2 \langle \pi_s^N, G \rangle L_N \langle \pi_s^N, G \rangle] ds \right].$$

In the proof of Proposition 3.1 we have proved that the term inside the time integral in the previous expression vanishes, as N goes to infinity. It remains to prove that (3.42) vanishes as well when we send N to infinity. Recalling (3.2), we can bound (3.42) from above by the sum of the following terms

$$\mathbb{P}_{\mu_N} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \frac{\Theta(N)}{N-1} \sum_{x \in \Lambda_N} \mathcal{L}_N G_s \left(\frac{x}{N} \right) \eta_s^N(x) ds - \int_0^t \langle \rho_s^\kappa, \mathbb{1}_{\{\theta \geq 0\}} \mathbb{L} G_s \rangle ds \right| > \frac{\delta}{2^4} \right), \tag{3.43}$$

$$\begin{aligned}
& \mathbb{P}_{\mu_N} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \frac{\kappa \Theta(N)}{N^\theta (N-1)} \sum_{x \in \Lambda_N} (G_s r_N^-) \left(\frac{x}{N} \right) (\alpha - \eta_s^N(x)) \right. \right. \right. \\
& \quad - \mathbb{1}_{\{\gamma \in (0,1)\}} \mathbb{1}_{\{\theta \leq 0\}} \kappa \int_0^1 (G_s r^-)(u) (\alpha - \rho_s^\kappa(u)) du \\
& \quad \left. \left. - \mathbb{1}_{\{\theta = \gamma - 1\}} \mathbb{1}_{\{\gamma \in (1,2)\}} \kappa m G_s(0) (\alpha - \rho_s^\kappa(0)) \right\} ds \right| > \frac{\delta}{2^4} \right). \tag{3.44}
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{P}_{\mu_N} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \frac{\kappa \Theta(N)}{N^\theta (N-1)} \sum_{x \in \Lambda_N} (G_s r_N^+) \left(\frac{x}{N} \right) (\beta - \eta_s^N(x)) \right. \right. \right. \\
& \quad - \mathbb{1}_{\{\gamma \in (0,1)\}} \mathbb{1}_{\{\theta \leq 0\}} \kappa \int_0^1 (G_s r^+)(u) (\beta - \rho_s^\kappa(u)) du \\
& \quad \left. \left. - \mathbb{1}_{\{\theta = \gamma - 1\}} \mathbb{1}_{\{\gamma \in (1,2)\}} \kappa m G_s(1) (\beta - \rho_s^\kappa(1)) \right\} ds \right| > \frac{\delta}{2^4} \right). \tag{3.45}
\end{aligned}$$

For $\theta \geq 0$, since $\Theta(N) = N^\gamma$ and Lemma 5.1 holds for any G at least C^2 in the space variable, applying Markov's inequality we have that (3.43) goes to 0, as N goes to infinity. For $\theta < 0$ and $\gamma \in (0, 1)$ we also have that (3.43) vanishes as N goes to infinity thanks to Markov's

inequality and the same analysis that we did in Subsection 3.1.1. The boundary terms (3.44) and (3.45) can be treated in a similar way, therefore we explain in details why they vanish just referring to (3.44). In the case $\gamma \in (0, 1)$ and $\theta \leq 0$, it is enough to use Markov's inequality and what we did in Subsection 3.1.1 to show that (3.44) vanishes, as N goes to infinity. In the case $\gamma \in (1, 2)$ and $\theta = \gamma - 1$ we again use Markov's inequality and then we repeat the computations we did in Subsection 3.1.5 to see that also in this case (3.44) vanishes, as N goes to infinity. In all the other regimes that we are considering the only term we have to estimate using Markov's inequality is

$$\mathbb{E}_{\mu_N} \left[\left| \int_0^t \left\{ \frac{\kappa N^\gamma}{N^\theta(N-1)} \sum_{x \in \Lambda_N} (G_s r_N^-)(\frac{x}{N})(\beta - \eta_s^N(x)) ds \right\} \right| \right] \quad (3.46)$$

and it vanishes, as N goes to infinity, thanks to the analysis that we did in Section 3.1 relative to these regimes. This finishes the proof of Proposition 3.7. \square

3.5. Conclusion. Note that Proposition 3.1, Theorem 3.2 and Proposition 3.7 just proved, imply that the sequence $\{\mathbb{Q}^N\}_{N \geq 1}$ converges up to subsequences to a measure \mathbb{Q} concentrated on an absolutely continuous measure with respect to the Lebesgue measure of density ρ^κ , where ρ^κ is the weak solution of the hydrodynamic equation chosen accordingly to the values of the parameters of the model (as stated in Theorem 2.18). Moreover, thanks to Lemma 2.16 which grants the uniqueness of the weak solutions, we can conclude that this limiting measure is unique and this proves completely Theorem 2.18.

4. UNIQUENESS OF WEAK SOLUTIONS

In this section we prove uniqueness of the weak solutions we defined in Subsection 2.3. Since the PDEs are linear, the strategy is standard. We consider two weak solutions starting from the same initial condition, take their difference and write down the integral formulation for the difference of the two solutions. Then, chose a proper test function, plug it into the integral formulation and from there obtain that the difference is 0 in some topology, which implies the uniqueness result we need.

4.1. Uniqueness of weak solutions of (2.14). Fix $T > 0$ and $\gamma \in (0, 2) \setminus \{1\}$. Consider ρ^1 and ρ^2 two weak solutions of (2.14) starting from the same initial condition and denote by ρ their difference: $\rho = \rho^1 - \rho^2 \in L^2(0, T; \mathcal{H}^{\gamma/2})$. The set $C^{1,\infty}([0, T] \times (0, 1))$ is dense in $L^2(0, T; \mathcal{H}^{\gamma/2})$. Therefore we can consider a sequence $\{G_n\}_{n \in \mathbb{N}} \subset C^{1,\infty}([0, T] \times (0, 1))$ converging to ρ with respect to the norm of $L^2(0, T; \mathcal{H}^{\gamma/2})$. Now, we consider a special set of test functions, defined as $H_n(t, u) := \int_t^T G_n(s, u) ds$, for any $t \in [0, T]$ and any $u \in [0, 1]$. It is easy to check that H_n are suitable test functions. Now we state the Lemma 6.1 of [2] adapting its proof to our case.

Lemma 4.1. *Let $\{H_n\}_{n \in \mathbb{N}}$ defined as above. Then we have*

$$(1) \lim_{n \rightarrow \infty} \int_0^T \langle \rho_s, \partial_s H_n(s, \cdot) \rangle ds = - \int_0^T \|\rho_s\|^2 ds;$$

$$(2) \lim_{n \rightarrow \infty} \int_0^T \langle \rho_s, \mathbb{L} H_n(s, \cdot) \rangle ds = - \frac{1}{2} \left\| \int_0^T \rho_s ds \right\|_{\gamma/2}^2;$$

Proof. The proof of item (1) is completely analogous to the one which can be found in [2]. Let us now prove item (2). We want to apply the integration by parts formula stated in Proposition 2.1. Therefore let us check the hypothesis. First, we have that $\mathbb{L}H_n(s, \cdot) \in L^1([0, 1])$ because $H_n(s, \cdot)$ is at least $C^2([0, 1])$ (see the comment just after Proposition 2.1). Let us now check (2.8). Since $\rho(\cdot)$ is uniformly bounded by 1 and H_n uniformly Lipschitz, we have to show that

$$\int_0^1 \int_0^1 \frac{|H_n(s, v) - H_n(s, u)|}{|u - v|^{\gamma-1}} dudv \lesssim \int_0^1 \int_0^1 |u - v|^{2-\gamma} dudv < \infty \quad (4.1)$$

which trivially holds for $\gamma < 2$. Therefore we can apply Proposition 2.1 and get that

$$\int_0^T \langle \rho_s, \mathbb{L}H_n(s, \cdot) \rangle ds = \int_0^T \langle \rho_s, H_n(s, \cdot) \rangle_{\gamma/2} ds. \quad (4.2)$$

The rest of the proof of item (2) is then analogous to the one given in [2]. \square

Thanks to this Lemma, we just have to plug in (2.19) the test function H_n , and then pass to the limit when n goes to infinity, as done in [2], to get

$$\int_0^T \|\rho_s\|^2 ds + \frac{1}{2} \left\| \int_0^T \rho_s ds \right\|_{\gamma/2}^2 = 0.$$

This implies that for almost every time $s \in [0, T]$ the function $\rho_s = 0$ and this means that ρ^1 coincides with ρ^2 almost at every time. So, the weak solution of (2.14) is unique.

4.2. Uniqueness of weak solutions of (2.16). The uniqueness of the solution to (2.16) in the case $\gamma \in (1, 2)$ has been proved in [3]. Hence we assume now $\gamma \in (0, 1)$ and prove the uniqueness of the weak solution to (2.16). The proof relies on the following Hardy's inequality proved in [8]

$$\forall H \in C_c^\infty((0, 1)), \quad \|H\|_{V_1} \lesssim \|H\|_{\mathcal{H}^{\gamma/2}}. \quad (4.3)$$

Since $C_c^\infty((0, 1))$ is dense in $L_{V_1}^2([0, 1])$ and in $\mathcal{H}_0^{\gamma/2}$ and since for $\gamma < 1$, we have the equality $\mathcal{H}_0^{\gamma/2} = \mathcal{H}^{\gamma/2}$ we have in fact that (4.3) holds for any $H \in L_{V_1}^2([0, 1]) \cap \mathcal{H}^{\gamma/2}$. Once this observed, the proof follows the same arguments as in [3] for the case $\gamma \in (1, 2)$.

Remark 4.2. *Remark that the crucial difference between the case $\gamma \in (1, 2)$ and the case $\gamma \in (0, 1)$ is that in the case $\gamma \in (1, 2)$ the weak solutions are assumed to satisfy the Dirichlet boundary conditions so that the difference of two weak solutions belongs to $\mathcal{H}_0^{\gamma/2}$ and Hardy's inequality can be used for the difference. In the case $\gamma \in (0, 1)$, we do not know if weak solutions satisfy in the strong sense Dirichlet boundary conditions. Hence the difference of two weak solutions does not necessarily vanishes at the boundary (we do not even know if it has a continuous representative). Fortunately if $\gamma \in (0, 1)$, $\mathcal{H}_0^{\gamma/2} = \mathcal{H}^{\gamma/2}$, so that it is not a real problem.*

4.3. Uniqueness of weak solutions of (2.18). We have $\gamma \in (1, 2)$. Fix $T > 0$. Consider ρ^1 and ρ^2 two weak solutions of (2.18) starting from the same initial condition and denote by ρ their difference: $\rho = \rho^1 - \rho^2 \in L^2(0, T; \mathcal{H}^{\gamma/2})$. Since the continuous representatives of ρ^1 and ρ^2 satisfy Dirichlet boundary conditions, we have in fact that $\rho \in L^2(0, T; \mathcal{H}_0^{\gamma/2})$. The set $C_c^{1,\infty}([0, T] \times (0, 1))$ is dense in $L^2(0, T; \mathcal{H}_0^{\gamma/2})$. Therefore we can consider a sequence $\{G_n\}_{n \in \mathbb{N}} \subset C_c^{1,\infty}([0, T] \times (0, 1))$ converging to ρ with respect to the norm of $L^2(0, T; \mathcal{H}_0^{\gamma/2})$. Now, we consider a special set of test functions, defined as $H_n(t, u) := \int_t^T G_n(s, u) ds$, for any $t \in [0, T]$ and any $u \in [0, 1]$. It is easy to check that H_n are suitable test functions that can be used in (2.19). Lemma 4.1 holds and we get then

$$\int_0^T \|\rho_s\|^2 ds + \frac{1}{2} \left\| \int_0^T \rho_s ds \right\|_{\gamma/2}^2 = 0$$

which implies that $\rho = 0$ and shows uniqueness.

4.4. Uniqueness of weak solutions of (2.20). We have $\gamma \in (1, 2)$. Fix $T > 0$. Consider two weak solutions of (2.20) denoted by $\rho^{\hat{k},1}$ and $\rho^{\hat{k},2}$, starting from the same initial condition. Let the difference be $\rho^{\hat{k}} = \rho^{\hat{k},1} - \rho^{\hat{k},2}$. We have that $\rho^{\hat{k}} \in L^2(0, T; \mathcal{H}^{\gamma/2})$. Since $\gamma > 1$ we may consider a continuous representative (for a.e. time) of $\rho^{\hat{k}}$. The set $C^{1,\infty}([0, T] \times (0, 1))$ is dense in $L^2(0, T; \mathcal{H}^{\gamma/2})$. Therefore, we can consider a sequence $\{G_n^{\hat{k}}\}_{n \in \mathbb{N}} \subset C^{1,\infty}([0, T] \times (0, 1))$ converging to $\rho^{\hat{k}}$ with respect to the norm of $L^2(0, T; \mathcal{H}^{\gamma/2})$, and we use as test functions $H_n^{\hat{k}}(t, u) := \int_t^T G_n^{\hat{k}}(s, u) ds$, for any $t \in [0, T]$ and any $u \in [0, 1]$. Observe that Lemma 4.1 holds for these test functions. Plugging the test functions $H_n^{\hat{k}}$ in (2.21) and passing to the limit as n goes to infinity we get

$$\int_0^T \|\rho_s^{\hat{k}}\|^2 ds + \frac{1}{2} \left\| \int_0^T \rho_s^{\hat{k}} ds \right\|_{\gamma/2}^2 + \frac{\hat{k}m}{2} \left| \int_0^T \rho_s^{\hat{k}}(0) ds \right|^2 + \frac{\hat{k}m}{2} \left| \int_0^T \rho_s^{\hat{k}}(1) ds \right|^2 = 0, \quad (4.4)$$

if we can prove that

$$\lim_{n \rightarrow \infty} \int_0^T H_n^{\hat{k}}(s, 0) \rho_s^{\hat{k}}(0) ds = \frac{1}{2} \left| \int_0^T \rho_s^{\hat{k}}(0) ds \right|^2 \quad (4.5)$$

and similarly with the point 0 replaced by the point 1. This will imply the uniqueness of weak solutions for (2.20).

To prove (4.5) we observe, by using Fubini's Theorem, that

$$\int_0^T H_n^{\hat{k}}(s, 0) \rho_s^{\hat{k}}(0) ds = \iint_{0 \leq s \leq r \leq T} \hat{G}_n^{\hat{k}}(r, 0) \rho_s^{\hat{k}}(0) dr ds.$$

To conclude it is sufficient to prove that $G_n^{\hat{k}}(\cdot, 0)$ converges to $\rho^{\hat{k}}(0)$ in $L^2([0, T], dt)$. Let us denote $f_n = G_n^{\hat{k}} - \rho^{\hat{k}}$. We know that $\{f_n\}_{n \in \mathbb{N}}$ converges to 0 in $L^2([0, T]; \mathcal{H}^{\gamma/2})$, as n goes to

infinity. By Theorem 8.2 in [9] we have that for any $s \in [0, T]$

$$\sup_{u, v \in [0, 1]^2} |f_n(s, u) - f_n(s, v)| \lesssim |u - v|^{\frac{\gamma-1}{2}} \|f_n(s, \cdot)\|_{\mathcal{H}^{\gamma/2}} \lesssim \|f_n(s, \cdot)\|_{\mathcal{H}^{\gamma/2}}.$$

By the triangular inequality, averaging and Cauchy-Schwarz inequality, we get that for any $s \in [0, T]$ and any $0 < R < 1$

$$\begin{aligned} |f_n(s, 0)| &\lesssim \frac{1}{R} \int_0^R |f_n(s, v)| dv + \|f_n(s, \cdot)\|_{\mathcal{H}^{\gamma/2}} \\ &\lesssim \frac{1}{\sqrt{R}} \|f_n(s, \cdot)\| + \|f_n(s, \cdot)\|_{\mathcal{H}^{\gamma/2}} \lesssim \frac{1}{\sqrt{R}} \|f_n(s, \cdot)\|_{\mathcal{H}^{\gamma/2}}. \end{aligned}$$

It follows that

$$\int_0^T |f_n(s, 0)|^2 ds \lesssim \frac{1}{R} \int_0^T \|f_n(s, \cdot)\|_{\mathcal{H}^{\gamma/2}}^2 ds.$$

The right-hand side of the previous inequality goes to 0, as n goes to infinity, so that the same holds for the left-hand side. This completes the proof.

4.5. Uniqueness of weak solutions of (2.22). Here we assume $\gamma \in (0, 1)$ since the case $\gamma \in (1, 2)$ has been considered in [3]. The idea in this case is similar to the ones above. But, first we have to note that a weak solution $\rho^{\hat{\kappa}}$ belongs to $L^2(0, T; V_1)$, i.e for any $T > 0$ we have

$$\int_0^T \|\rho_s^{\hat{\kappa}}\|_{V_1}^2 < \infty. \quad (4.6)$$

This is trivial since $\gamma \in (0, 1)$ and $\rho^{\hat{\kappa}}$ is uniformly bounded by 1. Hence if we consider two different weak solutions $\rho^{\hat{\kappa}, 1}$ and $\rho^{\hat{\kappa}, 2}$, then their difference $\rho^{\hat{\kappa}} := \rho^{\hat{\kappa}, 1} - \rho^{\hat{\kappa}, 2}$ belongs to $L^2(0, T; V_1)$. This observation being done we can perform the same proof done in [2] for the uniqueness of (2.22) in the regime $\gamma \in (1, 2)$.

5. TECHNICAL LEMMAS

In this section we state and prove the main lemmas we used in this work.

5.1. Convergence of a discrete operator to the regional fractional Laplacian. The first lemma we prove in this section establishes the convergence in L^1 of the discrete operator $N^\gamma \mathcal{L}_N$ defined by (3.3) to the regional fractional laplacian \mathbb{L} when applied to smooth test functions. This result differs from the one proved in [2], where the authors showed that for a function $G \in C_c^\infty([0, 1])$, the limit

$$\lim_{N \rightarrow \infty} N^\gamma \mathcal{L}_N G(u) = \mathbb{L}G(u)$$

holds uniformly for any $u \in [a, 1 - a]$ for an arbitrary small $a > 0$. Hence, in the following lemma we consider a weaker form of convergence, but in this way we are able to prove the convergence result for a larger set of functions, namely for all functions in $C^\infty([0, 1])$.

Lemma 5.1. *For any $G \in C^\infty([0, 1])$ and $\gamma \in (0, 2)$, the following convergence holds*

$$\lim_{N \rightarrow \infty} N^{-1} \sum_{x \in \Lambda_N} |N^\gamma (\mathcal{L}_N G)(\frac{x}{N}) - (\mathbb{L}G)(\frac{x}{N})| = 0. \quad (5.1)$$

Proof. Fix a small $\epsilon > 0$, then the left-hand side of (5.1) is bounded from above by

$$\begin{aligned} & \limsup_{N \rightarrow \infty} N^{-1} \left\{ \sum_{1 \leq x \leq \epsilon N} |N^\gamma (\mathcal{L}_N G)(\frac{x}{N}) - (\mathbb{L}G)(\frac{x}{N})| + \sum_{\epsilon N < x \leq (N-1)/2} |N^\gamma (\mathcal{L}_N G)(\frac{x}{N}) - (\mathbb{L}G)(\frac{x}{N})| \right. \\ & \left. + \sum_{(N-1)/2 < x < N-1-\epsilon N} |N^\gamma (\mathcal{L}_N G)(\frac{x}{N}) - (\mathbb{L}G)(\frac{x}{N})| + \sum_{N-1-\epsilon N \leq x \leq N-1} |N^\gamma (\mathcal{L}_N G)(\frac{x}{N}) - (\mathbb{L}G)(\frac{x}{N})| \right\}. \end{aligned} \quad (5.2)$$

First, we focus on the sum over $1 \leq x \leq \epsilon N$. In this case, we can bound the term on the left-hand side of the previous display by using the triangular inequality and get that it is bounded from above by

$$\limsup_{N \rightarrow \infty} N^{-1} \sum_{1 \leq x \leq \epsilon N} |N^\gamma (\mathcal{L}_N G)(\frac{x}{N})| + \limsup_{N \rightarrow \infty} N^{-1} \sum_{1 \leq x \leq \epsilon N} |(\mathbb{L}G)(\frac{x}{N})|. \quad (5.3)$$

Now, notice that in this case ($1 \leq x \leq \epsilon N$) we can write

$$(\mathcal{L}_N G)(\frac{x}{N}) = \sum_{y=1}^{x-1} p(y) \theta_{\frac{x}{N}}(\frac{y}{N}) + \sum_{y=x}^{N-1-x} p(y) (G(\frac{x+y}{N}) - G(\frac{x}{N})), \quad (5.4)$$

where we introduced

$$\theta_{\frac{x}{N}}(z) := G(\frac{x}{N} + z) + G(\frac{x}{N} - z) - 2G(\frac{x}{N}). \quad (5.5)$$

Therefore, the term on the left-hand side of (5.3) is bounded from above by

$$\limsup_{N \rightarrow \infty} N^{-1} \sum_{1 \leq x \leq \epsilon N} \left| N^\gamma \sum_{y=1}^{x-1} p(y) \theta_{\frac{x}{N}}(\frac{y}{N}) \right| + \limsup_{N \rightarrow \infty} N^{-1} \sum_{1 \leq x \leq \epsilon N} \left| N^\gamma \sum_{y=x}^{N-1-x} p(y) (G(\frac{x+y}{N}) - G(\frac{x}{N})) \right|. \quad (5.6)$$

Performing a second order Taylor expansion in $\theta_{\frac{x}{N}}$, we can bound from above the term on the left-hand side of the previous display in the following way:

$$\begin{aligned} \limsup_{N \rightarrow \infty} N^{-1} \sum_{1 \leq x \leq \epsilon N} \left| N^\gamma \sum_{y=1}^{x-1} p(y) \theta_{\frac{x}{N}}(\frac{y}{N}) \right| & \lesssim \limsup_{N \rightarrow \infty} N^{-1} \sum_{1 \leq x \leq \epsilon N} N^{\gamma-2} \sum_{y=1}^{x-1} y^{1-\gamma} \\ & \lesssim \limsup_{N \rightarrow \infty} N^{\gamma-3} \sum_{1 \leq x \leq \epsilon N} x^{2-\gamma} \lesssim \epsilon^{3-\gamma} \end{aligned} \quad (5.7)$$

which goes to 0 as $\epsilon \rightarrow 0$. In order to prove that the first term of (5.3) goes to 0, we still have to treat the term on the right-hand side of (5.6). Performing a Taylor expansion on G ,

we get

$$\begin{aligned}
& \limsup_{N \rightarrow \infty} N^{-1} \sum_{1 \leq x \leq \epsilon N} \left| N^\gamma \sum_{y=x}^{N-1-x} p(y) (G(\frac{x+y}{N}) - G(\frac{x}{N})) \right| \\
& \lesssim \limsup_{N \rightarrow \infty} N^{-1} \sum_{1 \leq x \leq \epsilon N} N^{\gamma-1} \sum_{y=x}^{N-1-x} y^{-\gamma} \\
& \lesssim \limsup_{N \rightarrow \infty} N^{\gamma-2} \sum_{y=1}^{N-1} \sum_{\substack{1 \leq x \leq \epsilon N \\ x \leq \inf\{y, N-1-y\}}} y^{-\gamma} \\
& \lesssim \limsup_{N \rightarrow \infty} N^{\gamma-2} \left[\sum_{y \leq \epsilon N} y^{-\gamma+1} + \sum_{y > \epsilon N} \epsilon N y^{-\gamma} \right] \lesssim \epsilon^{-\gamma+2}
\end{aligned} \tag{5.8}$$

and so it goes to 0 for $\gamma < 2$ when we pass to the limit $\epsilon \rightarrow 0$.

Now, we have to estimate the term on the right-hand side of (5.3). In order to do that, we first get an upper bound for $|\mathbb{L}G(\frac{x}{N})|$ for $1 \leq x \leq \epsilon N$. By definition of the operator \mathbb{L} , we have that

$$\mathbb{L}G(\frac{x}{N}) := c_\gamma \lim_{\epsilon' \rightarrow 0} \int_0^1 \mathbb{1}_{|\frac{x}{N} - \nu| > \epsilon'} \frac{G(\nu) - G(\frac{x}{N})}{|\frac{x}{N} - \nu|^{\gamma+1}} d\nu. \tag{5.9}$$

We use a second order Taylor expansion to write

$$\int_0^1 \mathbb{1}_{|\frac{x}{N} - \nu| > \epsilon'} \frac{G(\nu) - G(\frac{x}{N})}{|\frac{x}{N} - \nu|^{\gamma+1}} d\nu = G'(\frac{x}{N}) \int_0^1 \mathbb{1}_{|\frac{x}{N} - \nu| > \epsilon'} \frac{\nu - \frac{x}{N}}{|\frac{x}{N} - \nu|^{\gamma+1}} d\nu + R_{N,x,\epsilon,\epsilon',G}, \tag{5.10}$$

where the remainder term can be bounded as follows

$$|R_{N,x,\epsilon,\epsilon',G}| \lesssim \int_0^1 \mathbb{1}_{|\frac{x}{N} - \nu| > \epsilon'} |\frac{x}{N} - \nu|^{1-\gamma} d\nu \lesssim 1.$$

Moreover, by changing variables and taking $\epsilon' \leq 1/N \leq \epsilon$, we can write the integral on the right-hand side of (5.10) as

$$\begin{aligned}
& \int_{-\frac{x}{N}}^{1-\frac{x}{N}} \mathbb{1}_{\{|z| > \epsilon'\}} z |z|^{-\gamma-1} dz = \int_{\epsilon'}^{1-\frac{x}{N}} z z^{-\gamma-1} dz + \int_{-\frac{x}{N}}^{-\epsilon'} z (-z)^{-\gamma-1} dz \\
& = \int_{\epsilon'}^{1-\frac{x}{N}} z^{-\gamma} dz - \int_{\frac{x}{N}}^{\epsilon'} z^{-\gamma} dz = \frac{(1-\frac{x}{N})^{-\gamma+1} - (\frac{x}{N})^{-\gamma+1}}{-\gamma+1}.
\end{aligned} \tag{5.11}$$

Therefore, since G' is uniformly bounded, if $\gamma \in (1, 2)$ we have the following bound:

$$\lim_{N \rightarrow \infty} N^{-1} \sum_{1 \leq x \leq \epsilon N} |(\mathbb{L}G)(\frac{x}{N})| \lesssim \lim_{N \rightarrow \infty} N^{-1} \sum_{1 \leq x \leq \epsilon N} \left[(\frac{x}{N})^{1-\gamma} + 1 \right] \leq \epsilon^{2-\gamma} \tag{5.12}$$

which goes to 0 if we send ϵ to 0. Otherwise, if $\gamma \in (0, 1)$ we have

$$\lim_{N \rightarrow \infty} N^{-1} \sum_{1 \leq x \leq \epsilon N} |(\mathbb{L}G)(\frac{x}{N})| \lesssim \lim_{N \rightarrow \infty} N^{-1} \sum_{1 \leq x \leq \epsilon N} 1 \lesssim \epsilon \tag{5.13}$$

which also goes to 0 as $\epsilon \rightarrow 0$. This concludes the proof of the fact that the term on the left-hand side of (5.2) vanishes.

Let us now focus on the second term of (5.2), namely the one involving the sites $\epsilon N < x < (N-1)/2$. We can bound this term by the limit for N going to infinity of

$$\begin{aligned} N^{-1} \sum_{\epsilon N < x < (N-1)/2} |N^\gamma (\mathcal{L}_N G)\left(\frac{x}{N}\right) - (\mathbb{L}G)\left(\frac{x}{N}\right)| &\leq N^{-1} \sum_{\epsilon N < x < (N-1)/2} |N^\gamma (\mathcal{L}_N G)\left(\frac{x}{N}\right) - (\mathbb{L}_\epsilon G)\left(\frac{x}{N}\right)| \\ &+ N^{-1} \sum_{\epsilon N < x < (N-1)/2} |(\mathbb{L}_\epsilon G)\left(\frac{x}{N}\right) - (\mathbb{L}G)\left(\frac{x}{N}\right)|, \end{aligned} \quad (5.14)$$

where we defined

$$\mathbb{L}_\epsilon G\left(\frac{x}{N}\right) := c_\gamma \int_0^1 \mathbb{1}_{|\frac{x}{N} - \nu| > \epsilon} \frac{G(\nu) - G\left(\frac{x}{N}\right)}{|\frac{x}{N} - \nu|^{\gamma+1}} d\nu. \quad (5.15)$$

By changing variables in $\mathbb{L}_\epsilon G\left(\frac{x}{N}\right)$ we can rewrite it as

$$\mathbb{L}_\epsilon G\left(\frac{x}{N}\right) = c_\gamma \int_\epsilon^{\frac{x}{N}} |z|^{-(1+\gamma)} \theta_{\frac{x}{N}}(z) dz + c_\gamma \int_{\frac{x}{N}-1}^{-\frac{x}{N}} \frac{G\left(\frac{x}{N} - z\right) - G\left(\frac{x}{N}\right)}{|z|^{\gamma+1}} dz, \quad (5.16)$$

since $\epsilon N < x < (N-1)/2$. Notice now that we can in the same way re-write

$$\mathbb{L}G\left(\frac{x}{N}\right) = \lim_{\epsilon' \rightarrow 0} c_\gamma \left\{ \int_{\epsilon'}^{\frac{x}{N}} |z|^{-(1+\gamma)} \theta_{\frac{x}{N}}(z) dz + \int_{\frac{x}{N}-1}^{-\frac{x}{N}} \frac{G\left(\frac{x}{N} - z\right) - G\left(\frac{x}{N}\right)}{|z|^{\gamma+1}} dz \right\}. \quad (5.17)$$

Thus the second term at the right-hand side of (5.14) can be bounded by

$$\begin{aligned} N^{-1} \sum_{\epsilon N < x < (N-1)/2} |(\mathbb{L}_\epsilon G)\left(\frac{x}{N}\right) - (\mathbb{L}G)\left(\frac{x}{N}\right)| &= N^{-1} \sum_{\epsilon N < x < (N-1)/2} |(\mathbb{L}_\epsilon G)\left(\frac{x}{N}\right) - \lim_{\substack{\epsilon' \rightarrow 0 \\ \epsilon' < \epsilon}} (\mathbb{L}_{\epsilon'} G)\left(\frac{x}{N}\right)| \\ &= \lim_{\substack{\epsilon' \rightarrow 0 \\ \epsilon' < \epsilon}} N^{-1} \sum_{\epsilon N < x < (N-1)/2} \left| \int_{\epsilon'}^{\epsilon} z^{-(1+\gamma)} \theta_{\frac{x}{N}}(z) dz \right| \\ &\lesssim \lim_{\substack{\epsilon' \rightarrow 0 \\ \epsilon' < \epsilon}} (\epsilon^{2-\gamma} - (\epsilon')^{2-\gamma}) = \epsilon^{2-\gamma}, \end{aligned} \quad (5.18)$$

where the last inequality is obtained by performing a Taylor expansion of the second order on G and using the fact that $\|G''\|_\infty$ is bounded. So, we have proved that this term goes to 0 when ϵ goes to 0 if $\gamma < 2$.

We estimate now the first term on the right-hand side of (5.14). Observe that, by changing variables, we can write in this case ($\epsilon N < x < (N-1)/2$)

$$(\mathcal{L}_N G)\left(\frac{x}{N}\right) = \sum_{\epsilon N \leq y \leq x-1} p(y) \theta_{\frac{x}{N}}\left(\frac{y}{N}\right) + \sum_{1 \leq y < \epsilon N} p(y) \theta_{\frac{x}{N}}\left(\frac{y}{N}\right) + \sum_{x \leq y \leq N-1-x} p(y) (G\left(\frac{x+y}{N}\right) - G\left(\frac{x}{N}\right)). \quad (5.19)$$

Then, recalling (5.16), we have to estimate the following quantity:

$$\begin{aligned}
& N^{-1} \sum_{\epsilon N < x < (N-1)/2} |N^\gamma (\mathcal{L}_N G)(\frac{x}{N}) - (\mathbb{L}_\epsilon G)(\frac{x}{N})| \\
& \lesssim N^{-1} \sum_{\epsilon N < x < (N-1)/2} \left\{ \left| N^\gamma \sum_{y=\epsilon N}^{x-1} p(y) \theta_{\frac{x}{N}}(\frac{y}{N}) - c_\gamma \int_\epsilon^{\frac{x}{N}} |z|^{-(1+\gamma)} \theta_{\frac{x}{N}}(z) dz \right| + \left| N^\gamma \sum_{y=1}^{\epsilon N-1} p(y) \theta_{\frac{x}{N}}(\frac{y}{N}) \right| \right. \\
& \left. + \left| N^\gamma \sum_{y=x}^{N-1-x} p(y) (G(\frac{x+y}{N}) - G(\frac{x}{N})) - c_\gamma \int_{\frac{x}{N}}^{1-\frac{x}{N}} |z|^{-(1+\gamma)} (G(\frac{x}{N} + z) - G(\frac{x}{N})) dz \right| \right\} \\
& = N^{-1} \sum_{\epsilon N < x < (N-1)/2} \{(I) + (II) + (III)\}.
\end{aligned} \tag{5.20}$$

We can treat the term (I) in the previous sum as

$$\begin{aligned}
(I) & = \left| \sum_{y=\epsilon N}^{x-1} c_\gamma \int_{\frac{y}{N}}^{\frac{y+1}{N}} \left[(\frac{y}{N})^{-(1+\gamma)} \theta_{\frac{x}{N}}(\frac{y}{N}) - z^{-(1+\gamma)} \theta_{\frac{x}{N}}(z) \right] dz \right| \\
& \lesssim \left| \sum_{y=\epsilon N}^{x-1} \int_{\frac{y}{N}}^{\frac{y+1}{N}} \left[(\frac{y}{N})^{-(1+\gamma)} - z^{-(1+\gamma)} \right] \theta_{\frac{x}{N}}(\frac{y}{N}) dz \right| + \left| \sum_{y=\epsilon N}^{x-1} \int_{\frac{y}{N}}^{\frac{y+1}{N}} z^{-(1+\gamma)} \left(\theta_{\frac{x}{N}}(\frac{y}{N}) - \theta_{\frac{x}{N}}(z) \right) dz \right| \\
& \lesssim \sum_{y=\epsilon N}^{x-1} \int_{\frac{y}{N}}^{\frac{y+1}{N}} \|\theta_{\frac{x}{N}}(\cdot)\|_\infty \left((\frac{y}{N})^{-(1+\gamma)} - z^{-(1+\gamma)} \right) dz + \sum_{y=\epsilon N}^{x-1} \int_{\frac{y}{N}}^{\frac{y+1}{N}} z^{-(1+\gamma)} \|\theta'_{\frac{x}{N}}(\cdot)\|_\infty |\frac{y}{N} - z| dz \\
& \lesssim \sum_{y=\epsilon N}^{x-1} \int_{\frac{y}{N}}^{\frac{y+1}{N}} \left((\frac{y}{N})^{-(1+\gamma)} - z^{-(1+\gamma)} \right) dz + N^{-1} \int_\epsilon^1 z^{-(1+\gamma)} dz.
\end{aligned} \tag{5.21}$$

Above we have used the fact that $\|\theta_{\frac{x}{N}}(\cdot)\|_\infty$ and $\|\theta'_{\frac{x}{N}}(\cdot)\|_\infty$ are uniformly bounded in x . Now, it is not difficult to check that

$$\int_{\frac{y}{N}}^{\frac{y+1}{N}} \left((\frac{y}{N})^{-(1+\gamma)} - z^{-(1+\gamma)} \right) dz \lesssim N^\gamma y^{-(2+\gamma)}. \tag{5.22}$$

Thanks to these computations we conclude that (recall that ϵ will go to 0 after N being sent to infinity)

$$\lim_{N \rightarrow \infty} N^{-1} \sum_{\epsilon N < x < (N-1)/2} (I) \lesssim \lim_{N \rightarrow \infty} N^{-1} \sum_{\epsilon N < x < (N-1)/2} (N^\gamma \sum_{y=\epsilon N}^{x-1} y^{-(2+\gamma)} + \epsilon^{-\gamma} N^{-1}) = 0. \tag{5.23}$$

Now we estimate the term involving (II):

$$\begin{aligned}
\lim_{N \rightarrow \infty} N^{-1} \sum_{\epsilon N < x < (N-1)/2} (II) &\leq \lim_{N \rightarrow \infty} N^{-1} \sum_{\epsilon N < x < (N-1)/2} \left| N^\gamma \sum_{y=1}^{\epsilon N} p(y) \theta_x \left(\frac{y}{N} \right) \right| \\
&\lesssim \lim_{N \rightarrow \infty} N^{-1} \sum_{\epsilon N < x < (N-1)/2} \left| N^{\gamma-2} \sum_{y=1}^{\epsilon N} y^{1-\gamma} \right| \\
&\lesssim \lim_{N \rightarrow \infty} N^{-1} \sum_{\epsilon N < x < (N-1)/2} \epsilon^{2-\gamma} \lesssim \epsilon^{2-\gamma},
\end{aligned} \tag{5.24}$$

which goes to 0, as ϵ goes to 0.

In order to conclude, we have to prove that the term involving (III) goes to 0. We have that

$$\begin{aligned}
(III) &= \left| \sum_{y=x}^{N-1-x} c_\gamma \int_{\frac{y}{N}}^{\frac{y+1}{N}} \left[\left(\frac{y}{N} \right)^{-(\gamma+1)} \left(G \left(\frac{x+y}{N} \right) - G \left(\frac{x}{N} \right) \right) - z^{-(\gamma+1)} \left(G \left(\frac{x}{N} + z \right) - G \left(\frac{x}{N} \right) \right) \right] dz \right| \\
&\lesssim \left| \sum_{y=x}^{N-1-x} \int_{\frac{y}{N}}^{\frac{y+1}{N}} \left[\left(\frac{y}{N} \right)^{-(\gamma+1)} - z^{-(\gamma+1)} \right] \left(G \left(\frac{x+y}{N} \right) - G \left(\frac{x}{N} \right) \right) dz \right| \\
&+ \left| \sum_{y=x}^{N-1-x} \int_{\frac{y}{N}}^{\frac{y+1}{N}} z^{-(\gamma+1)} \left[G \left(\frac{x+y}{N} \right) - G \left(\frac{x}{N} + z \right) \right] dz \right|
\end{aligned} \tag{5.25}$$

By the mean value theorem applied to the function $f : u \rightarrow u^{-(\gamma+1)}$ and to the function G we have

$$(III) \lesssim \frac{1}{N^2} \sum_{y=x}^{N-1-x} \left(\frac{y}{N} \right)^{-(\gamma+1)} \leq \frac{1}{N^2} \sum_{y=x}^{N-1-x} \int_{\frac{y-1}{N}}^{\frac{y}{N}} u^{-(\gamma+1)} du \lesssim \frac{1}{N^2} \left(\frac{x}{N} \right)^{-\gamma}. \tag{5.26}$$

It follows that

$$N^{-1} \sum_{\epsilon N < x < (N-1)/2} (III) \lesssim N^{\gamma-2} \sum_{\epsilon N < x < (N-1)/2} \frac{1}{x^\gamma}. \tag{5.27}$$

If $\gamma \in (1, 2)$ the series $\sum_{x \geq 1} x^{-\gamma}$ is converging and so the term to estimate is of order $N^{\gamma-2}$ (uniformly in ϵ) and goes to 0, as N goes to infinity. If $\gamma < 1$, for fixed $\epsilon > 0$ the sum is of order $N^{-\gamma+1}$ and so the global order is N^{-1} and then goes to 0, as N goes to infinity. This proves that the first two terms of (5.2) vanishes, as N goes to infinity and then ϵ goes to 0.

The analysis of the last two terms in (5.2) can be done in the same way of the first two, just performing the following change of variables $x \mapsto N - 1 - x$. \square

5.2. Fixing the value of the profile at the boundary. The aim of this subsection is to show that, in the case $0 \leq \theta < \gamma - 1$ and $\gamma \in (1, 2)$, the profile $\rho(\cdot)$ satisfies the Dirichlet boundary conditions stated in item iii) of Definition 2.10. This will be a consequence of Proposition 5.3 which is a consequence of a series of results that we state below. Some of these auxiliary lemmas will also be used in the case $\theta = \gamma - 1$. First, we define the following empirical averages:

$$\overrightarrow{\eta}_s^\ell(x) = \frac{1}{\ell} \sum_{y=x+1}^{x+\ell} \eta_s^N(y) \quad \text{and} \quad \overleftarrow{\eta}_s^\ell(x) = \frac{1}{\ell} \sum_{y=x-\ell}^{x-1} \eta_s^N(y), \quad (5.28)$$

where $x \in \{0, \dots, N\}$ and ℓ is a positive integer such that, in the sum on the left-hand side we have $x + \ell \leq N - 1$ and on the right-hand side we have $x - \ell \geq 1$.

Remark 5.2. We always assume that in the size of the boxes involved, the number of sites on which we compute the mean is an integer. When it is not we approximate with the closest greater integer, for example $\overrightarrow{\eta}^{\epsilon N}(0)$ shall be read as $\overrightarrow{\eta}^{\lceil \epsilon N \rceil}(0)$. We do not write it in each step in order to leave the computations more readable.

Proposition 5.3. In the regime $0 \leq \theta < \gamma - 1$ and $\gamma \in (1, 2)$, for any $T > 0$ and $\epsilon > 0$, the following convergences hold

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^T (\alpha - \overrightarrow{\eta}_s^{\epsilon N}(0)) ds \right| \right] = 0, \quad (5.29)$$

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^T (\beta - \overleftarrow{\eta}_s^{\epsilon N}(N)) ds \right| \right] = 0, \quad (5.30)$$

where $\overrightarrow{\eta}_s^{\epsilon N}(0)$ and $\overleftarrow{\eta}_s^{\epsilon N}(N)$ are defined in (5.28).

Proof. The proof of this lemma is a trivial consequence of Lemmas 5.4, 5.5 and 5.6 in which is proved (5.29). In Remark 5.7 we explain how to adapt this proof to show (5.30). \square

We present first how to pass from the occupation variable $\eta_s^N(1)$ to the mean of the values of the occupations variables in the first ϵN sites. It is done in two steps and it is the content of the next two lemmas. Finally, we show in Lemma 5.6 that $\eta_s^N(1)$, in some way, is a good approximation of α .

In the next lemma we show that it is possible to replace the occupation variable $\eta_s^N(1)$ by its empirical mean in a box of size $\ell_0 = \epsilon N^{\gamma-1}$ sites in the double ordered limit $N \rightarrow \infty$ and then $\epsilon \rightarrow 0$.

Lemma 5.4 (First approximation). In the regime ² $0 \leq \theta \leq \gamma - 1$ and $\gamma \in (1, 2)$, for any $T > 0$, $\epsilon > 0$ and $\ell_0 = \epsilon N^{\gamma-1}$, it holds

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^T (\eta_s^N(1) - \overrightarrow{\eta}_s^{\ell_0}(0)) ds \right| \right] = 0, \quad (5.31)$$

²We observe that this lemma is true for any value of $\theta \geq 0$ and $\gamma < 2$, but we restrict to these values of θ, γ since we only need the result in this regime.

where $\vec{\eta}_s^{\ell_0}(0)$ is defined in (5.28).

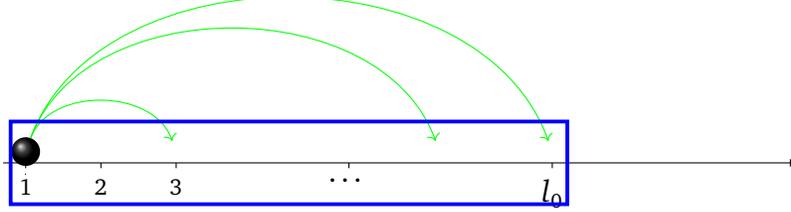


FIGURE 8. First step, we can replace $\eta_s(1)$ with its empirical mean in the box $\{1, \dots, \ell_0\}$.

Proof. Consider an arbitrary Lipschitz function $\rho : [0, 1] \rightarrow [\alpha, \beta]$ such that $\rho(0) = \alpha$, $\rho(1) = \beta$ and define the product measure $\nu_{\rho(\cdot)}^N$ on Ω_N by (3.21). By entropy and Jensen's inequalities we bound the expectation in the statement, for any $B > 0$, by

$$\frac{H(\mu_N | \nu_{\rho(\cdot)}^N)}{BN} + \frac{1}{BN} \log \mathbb{E}_{\nu_{\rho(\cdot)}^N} \left[e^{BN \left| \int_0^T (\eta_s^N(1) - \vec{\eta}_s^{\ell_0}(0)) ds \right|} \right]. \quad (5.32)$$

It is possible to remove the absolute value in the exponential above observing that $e^{|x|} \leq \max\{e^x, e^{-x}\}$. Now, using the entropy bound (3.22) and Feynman-Kac's formula, there exists a constant $C_0 > 0$ independent of N such that the last expression can be bounded by above by

$$\frac{C_0}{B} + \sup_f \left\{ \int (\eta(1) - \vec{\eta}^{\ell_0}(0)) f(\eta) d\nu_{\rho(\cdot)}^N + \frac{N^{\gamma-1}}{B} \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(\cdot)}^N} \right\} \quad (5.33)$$

where the supremum is carried on all the probability densities with respect to $\nu_{\rho(\cdot)}^N$. We use (3.29) which, in this case, becomes (for a constant $C > 0$ independent of N)

$$\frac{N^{\gamma-1}}{B} \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(\cdot)}^N} \leq -\frac{N^{\gamma-1}}{4B} D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N) + C \frac{\kappa N^{-\theta} + 1}{B}. \quad (5.34)$$

Observe that the integral inside the supremum in (5.33) can be rewritten, using the definition of $\vec{\eta}^{\ell_0}(0)$, as

$$\frac{1}{\ell_0} \int \sum_{x=1}^{\ell_0} (\eta(1) - \eta(x)) f(\eta) d\nu_{\rho(\cdot)}^N = \frac{1}{\ell_0} \int \sum_{x=1}^{\ell_0} \sum_{z=1}^{x-1} (\eta(z) - \eta(z+1)) f(\eta) d\nu_{\rho(\cdot)}^N \quad (5.35)$$

where the equality follows from a simple telescopic argument. Now, by writing (5.35) as twice its half and summing and subtracting appropriate terms, we can write it as the sum of the following two terms:

$$\frac{1}{2\ell_0} \sum_{x=1}^{\ell_0} \sum_{z=1}^{x-1} \int (\eta(z) - \eta(z+1)) (f(\eta) - f(\sigma^{z, z+1} \eta)) d\nu_{\rho(\cdot)}^N, \quad (5.36)$$

$$\frac{1}{2\ell_0} \sum_{x=1}^{\ell_0} \sum_{z=1}^{x-1} \int (\eta(z) - \eta(z+1))(f(\eta) + f(\sigma^{z,z+1}\eta)) d\nu_{\rho(\cdot)}^N. \quad (5.37)$$

Let us now focus on (5.36), which is equal to

$$\frac{1}{2\ell_0} \int \sum_{x=1}^{\ell_0} \sum_{z=1}^{x-1} (\eta(z+1) - \eta(z))(\sqrt{f}(\eta) - \sqrt{f}(\sigma^{z,z+1}\eta))(\sqrt{f}(\eta) + \sqrt{f}(\sigma^{z,z+1}\eta)) d\nu_{\rho(\cdot)}^N, \quad (5.38)$$

and from Young's inequality, it can be bounded from above, for any $A > 0$, by the sum of

$$\frac{1}{2A\ell_0} \int \sum_{x=1}^{\ell_0} \sum_{z=1}^{x-1} (\sqrt{f}(\eta) - \sqrt{f}(\sigma^{z,z+1}\eta))^2 d\nu_{\rho(\cdot)}^N \quad (5.39)$$

and

$$\frac{A}{2\ell_0} \int \sum_{x=1}^{\ell_0} \sum_{z=1}^{x-1} (\eta(z) - \eta(z+1))^2 (\sqrt{f}(\eta) + \sqrt{f}(\sigma^{z,z+1}\eta))^2 d\nu_{\rho(\cdot)}^N. \quad (5.40)$$

Observe now that if $D_{NN}(\sqrt{f}, \nu_{\rho(\cdot)}^N) := \sum_{z=1}^{N-2} \int (\sqrt{f}(\eta) - \sqrt{f}(\sigma^{z,z+1}\eta))^2 d\nu_{\rho(\cdot)}^N$ we have that $D_{NN}(\sqrt{f}, \nu_{\rho(\cdot)}^N) \lesssim D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N)$. This means that we can bound (5.39) from above by a constant times

$$\frac{1}{2A\ell_0} \sum_{x=1}^{\ell_0} D_N(\sqrt{f}, d\nu_{\rho(\cdot)}^N) \leq \frac{1}{2A} D_N(\sqrt{f}, d\nu_{\rho(\cdot)}^N).$$

Let us now treat (5.40). Using $(a+b)^2 \lesssim a^2 + b^2$, changing variables ($\eta \mapsto \sigma^{z,z+1}\eta$) and remembering that $\nu_{\rho(\cdot)}^N$ is a product measure with $\rho(\cdot)$ Lipschitz profile, we can bound from above (5.40) by a constant times

$$\frac{A}{2\ell_0} \int \sum_{x=1}^{\ell_0} \sum_{z=1}^{x-1} (\eta(z) - \eta(z+1))^2 f(\eta) d\nu_{\rho(\cdot)}^N.$$

Therefore, since f is a density and the difference $(\eta(z) - \eta(z+1))^2 \leq 1$, we can bound the previous display from above by a constant times

$$\frac{A}{2\ell_0} \int \sum_{x=1}^{\ell_0} \sum_{z=1}^{x-1} (\eta(z) - \eta(z+1))^2 f(\eta) d\nu_{\rho(\cdot)}^N \lesssim A\ell_0.$$

We have thus obtained that

$$(5.36) \lesssim \frac{1}{2A} D_N(\sqrt{f}, d\nu_{\rho(\cdot)}^N) + A\ell_0. \quad (5.41)$$

In order to treat (5.37), we introduce the configuration $\tilde{\eta}^z \in \Omega_N^{z,z+1} := \{0, 1\}^{\Lambda_N^{z,z+1}}$ where $\Lambda_N^{z,z+1} := \Lambda_N \setminus \{z, z+1\}$. Using the notation $f(\eta) = f(\tilde{\eta}, \eta(z), \eta(z+1))$ we can rewrite the sum of densities inside the integral in (5.37) as

$$\sum_{\tilde{\eta} \in \Omega_N^{z,z+1}} \{f(\tilde{\eta}, 0, 1) + f(\tilde{\eta}, 1, 0)\} \left[\rho\left(\frac{z}{N}\right)(1 - \rho\left(\frac{z+1}{N}\right)) - \rho\left(\frac{z+1}{N}\right)(1 - \rho\left(\frac{z}{N}\right)) \right] \tilde{\nu}_{\rho(\cdot)}^N(\tilde{\eta}), \quad (5.42)$$

where $\tilde{\nu}_{\rho(\cdot)}^N$ is the marginal density associated to $\nu_{\rho(\cdot)}^N$ restricted to the configurations $\tilde{\eta} \in \Omega_N^{z, z+1}$. Then, since $\rho(\cdot)$ was chosen to be Lipschitz continuous and f is a density, then (5.42) is of order $1/N$. Then we obtained that

$$(5.37) \lesssim \frac{1}{\ell_0} \sum_{x=1}^{\ell_0} \sum_{z=1}^{x-1} \frac{1}{N} \lesssim \frac{\ell_0}{N}. \tag{5.43}$$

Hence (5.34), (5.41) and (5.43) give that (5.33) is bounded from above by

$$C \left(\frac{1}{B} + \frac{1}{2A} D_N(\sqrt{f}, d \nu_{\rho(\cdot)}^N) + A \ell_0 + \frac{\ell_0}{N} \right) - \frac{N^{\gamma-1}}{4B} D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N) + C \frac{\kappa N^{-\theta} + 1}{B}.$$

where C is a constant depending on γ, α, β . By choosing $A = 2BN^{-\gamma+1}C^{-1}$, then, for the choice $\ell_0 = \epsilon N^{\gamma-1}$, the last display is less or equal than a constant times

$$\frac{1}{B} + B\epsilon + \epsilon N^{\gamma-2} + \frac{\kappa N^{-\theta} + 1}{B}.$$

Since $\gamma \in (1, 2)$ and $\theta \geq 0$, the previous display and thus (5.33) vanishes as $N \rightarrow \infty$, then $\epsilon \rightarrow 0$ and finally $B \rightarrow \infty$. This concludes the proof of the lemma. \square

Finally, the following lemma shows that we can approximate (in some sense) the empirical mean of the occupation variable with respect to the first ℓ_0 sites by the one with respect to the first ϵN sites. It is done using an iterative procedure, doubling the number of sites involved in the empirical mean at each step up to reach the final size ϵN .

Lemma 5.5 (Multi-scale argument). *Consider the case $0 \leq \theta \leq \gamma - 1$ and $\gamma \in (1, 2)$. Then, for any $T > 0$,*

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}_{\nu_N} \left[\left| \int_0^T (\vec{\eta}_s^{\ell_0}(0) - \vec{\eta}_s^{\epsilon N}(0)) ds \right| \right] = 0. \tag{5.44}$$

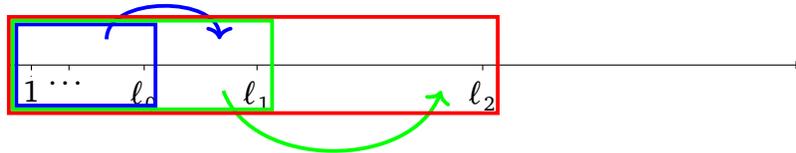


FIGURE 9. Multi-scale argument: we double the box $\{1, \dots, \ell_0\}$ a finite number of times M until reaching the final box of size $2^M \ell_0 = \ell_M = \epsilon N$. Here we illustrate the first two steps, until reaching $\ell_2 = 2^2 \ell_0$.

Proof. Consider a Bernoulli product measure $\nu_{\rho(\cdot)}^N$, as defined in (3.21), with a Lipschitz profile $\rho : [0, 1] \rightarrow (0, 1)$ such that $\rho(0) = \alpha$ and $\rho(1) = \beta$. Now, proceeding exactly as we did

in the beginning of the proof of Lemma 5.4, using entropy and Jensen's inequality and then Feynman-Kac's formula, we are reduced to prove that

$$\frac{C}{B} + \sup_f \left\{ \int (\vec{\eta}^{\ell_0}(0) - \vec{\eta}^{\epsilon N}(0)) f(\eta) d\nu_{\rho(\cdot)}^N + \frac{N^{\gamma-1}}{B} \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(\cdot)}^N} \right\}, \quad (5.45)$$

vanishes in the triple limit $N \rightarrow \infty$, then $\epsilon \rightarrow 0$ and finally $B \rightarrow \infty$. In the previous formula the supremum is carried over all the densities f with respect to $\nu_{\rho(\cdot)}^N$.

Let us call $\ell_i := 2^i \ell_0$, and choose $M := M_N$ such that $2^M \ell_0 = \epsilon N$. Note that this M is independent of ϵ and of order $\log N$ since $\ell_0 = \epsilon N^{\gamma-1}$. With this notation we can write

$$\begin{aligned} \vec{\eta}^{\ell_0}(0) - \vec{\eta}^{\epsilon N}(0) &= \sum_{i=1}^M [\vec{\eta}^{\ell_{i-1}}(0) - \vec{\eta}^{\ell_i}(0)] = \sum_{i=1}^M \left[\frac{1}{\ell_{i-1}} \sum_{y=1}^{\ell_{i-1}} \eta(y) - \frac{1}{\ell_i} \sum_{y=1}^{\ell_i} \eta(y) \right] \\ &= \sum_{i=1}^M \frac{1}{\ell_i} \sum_{y=1}^{\ell_{i-1}} [\eta(y) - \eta(y + \ell_{i-1})], \end{aligned} \quad (5.46)$$

where the last equality comes from the definition of the ℓ_i 's.

Thanks to the previous display, the first term in the supremum of (5.45) can be rewritten as the sum of

$$\frac{1}{2} \int \sum_{i=1}^M \frac{1}{\ell_i} \sum_{y=1}^{\ell_{i-1}} (\eta(y) - \eta(y + \ell_{i-1})) (f(\eta) - f(\sigma^{y, y+\ell_{i-1}} \eta)) d\nu_{\rho(\cdot)}^N \quad (5.47)$$

and

$$\frac{1}{2} \int \sum_{i=1}^M \frac{1}{\ell_i} \sum_{y=1}^{\ell_{i-1}} (\eta(y) - \eta(y + \ell_{i-1})) (f(\eta) + f(\sigma^{y, y+\ell_{i-1}} \eta)) d\nu_{\rho(\cdot)}^N. \quad (5.48)$$

Recall (3.26). Using Young's inequality as we did in Lemma 5.4, we bound from above (5.47) by the sum of

$$\int \frac{1}{4} \sum_{i=1}^M \frac{1}{A_i \ell_i} \sum_{y=1}^{\ell_{i-1}} (\sqrt{f}(\eta) - \sqrt{f}(\sigma^{y, y+\ell_{i-1}} \eta))^2 d\nu_{\rho(\cdot)}^N = \frac{1}{4} \sum_{i=1}^M \frac{1}{A_i \ell_i} \sum_{y=1}^{\ell_{i-1}} I_{y, y+\ell_{i-1}}(\sqrt{f}, \nu_{\rho(\cdot)}^N) \quad (5.49)$$

and

$$\int \frac{1}{4} \sum_{i=1}^M \frac{A_i}{\ell_i} \sum_{y=1}^{\ell_{i-1}} (\eta(y) - \eta(y + \ell_{i-1}))^2 (\sqrt{f}(\eta) + \sqrt{f}(\sigma^{y, y+\ell_{i-1}} \eta))^2 d\nu_{\rho(\cdot)}^N, \quad (5.50)$$

for some arbitrary positive A_i that we are going to properly choose. Choosing

$$A_i = A_0 \frac{B \ell_{i-1}^\gamma}{N^{\gamma-1} \ell_i}$$

with a suitable $A_0 > 0$ we can apply Lemma 5.8 to bound the right-hand side of (5.49) by ³

$$\frac{N^{\gamma-1}}{4B} D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N).$$

Let us estimate now (5.50). Since f is a density and the $\eta(y)$'s are uniformly bounded, recalling that $2^M \ell_0 = \epsilon N$, we can bound from above (5.50) by a constant times

$$\frac{B}{4N^{\gamma-1}} \sum_{i=1}^M \frac{\ell_{i-1}^\gamma}{\ell_i^2} \sum_{y=1}^{\ell_{i-1}} 1 = \frac{B}{4N^{\gamma-1}} \sum_{i=1}^M \frac{\ell_{i-1}^{\gamma+1}}{\ell_i^2} = \frac{B\ell_0^{\gamma-1}}{4N^{\gamma-1}} \frac{1}{2^{\gamma+1}} \sum_{i=1}^M 2^{i(\gamma-1)} \lesssim \frac{B(2^M \ell_0)^{\gamma-1}}{N^{\gamma-1}} \lesssim B\epsilon^{\gamma-1}.$$

Hence we obtained that there exists a constant $C > 0$ (independent of N, ϵ, B) such that

$$(5.47) \leq \frac{N^{\gamma-1}}{4B} D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N) + CB\epsilon^{\gamma-1}. \quad (5.51)$$

We still have to estimate (5.48). Reasoning as we did in (5.42), we can rewrite the sum of the densities inside the integral in (5.48) as

$$\sum_{\tilde{\eta} \in \Omega_N^{y, y+\ell_{i-1}}} \left\{ f(\tilde{\eta}, 0, 1) + f(\tilde{\eta}, 1, 0) \right\} \left[\rho\left(\frac{y}{N}\right)(1 - \rho\left(\frac{y+\ell_{i-1}}{N}\right)) - \rho\left(\frac{y+\ell_{i-1}}{N}\right)(1 - \rho\left(\frac{y}{N}\right)) \right] d\tilde{\nu}_{\rho(\cdot)}^N, \quad (5.52)$$

where now we denote $\tilde{\eta}$ as the configuration η restricted to the sites in $\Lambda_N \setminus \{y, y + \ell_{i-1}\}$. So, the configuration $(\tilde{\eta}, 0, 1)$ coincides with the configuration η where $\eta(y + \ell_{i-1}) = 1$ and $\eta(y) = 0$, and $\tilde{\nu}_{\rho(\cdot)}^N$ is the marginal density associated to $\nu_{\rho(\cdot)}^N$ restricted to the configurations $\tilde{\eta} \in \Omega_N^{y, y+\ell_{i-1}} := \{0, 1\}^{\Lambda_N \setminus \{y, y+\ell_{i-1}\}}$. Since $\rho(\cdot)$ was chosen to be Lipschitz continuous and f is a density, the previous display is of order ℓ_{i-1}/N . Therefore we get that

$$(5.48) \lesssim \sum_{i=1}^M \frac{1}{\ell_i} \sum_{y=1}^{\ell_{i-1}} \frac{\ell_{i-1}}{N} \lesssim \frac{1}{N} \sum_{i=1}^M \frac{\ell_i^2}{\ell_i} \lesssim \frac{1}{N} \sum_{i=1}^M 2^i \ell_0 \lesssim \frac{2^M \ell_0}{N} = \epsilon \quad (5.53)$$

since M is such that $2^M \ell_0 = \epsilon N$.

Hence, recalling (3.29), (5.51) and (5.53), we get that the term inside the supremum in (5.45) can be bounded from above by a constant times

$$B\epsilon^{\gamma-1} + \epsilon + \frac{N^{\gamma-1}(\kappa + N^\theta)}{BN^{\gamma+\theta}}.$$

It follows that

$$\limsup_{B \rightarrow \infty} \limsup_{\epsilon \rightarrow 0} \limsup_{N \rightarrow \infty} (5.45) = 0.$$

This concludes the proof. □

In the next lemma we prove that, in the regime $\theta \in (0, \gamma - 1)$ and $\gamma \in (1, 2)$, it is possible to replace, when integrated in time, the occupation variable $\eta_s^N(1)$ by α (and, analogously, $\eta_s^N(N - 1)$ by β), as N goes to infinity.

³We choose A_0 to be exactly the inverse of the constant which appear implicitly in Lemma 5.8.

Lemma 5.6. *For any $T > 0$, $0 \leq \theta < \gamma - 1$ and $\gamma \in (1, 2)$ it holds:*

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^T (\eta_s^N(1) - \alpha) ds \right| \right] = 0, \quad (5.54)$$

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^T (\eta_s^N(N-1) - \beta) ds \right| \right] = 0, \quad (5.55)$$

Proof. We present the proof for the first expectation, but the proof of the other one is exactly the same. The first step in this proof is analogous to the beginning of the proof of Lemma 5.4 therefore we omit some of the steps. We use Feynman-Kac's formula, entropy and Jensen's inequality to prove (5.54) it is sufficient to show that in the double limit ($N \rightarrow \infty$ and then $B \rightarrow \infty$) the term

$$\frac{C}{B} + \sup_f \left\{ \int (\eta(1) - \alpha) f(\eta) d\nu_{\rho(\cdot)}^N + \frac{N^{\gamma-1}}{B} \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\rho(\cdot)}^N} \right\} \quad (5.56)$$

vanishes. Here C is a constant independent from N and the supremum is carried over all the densities with respect to $\nu_{\rho(\cdot)}^N$ defined in (3.21) where $\rho(\cdot)$ is a Lipschitz continuous profile such that for all $u \in [0, 1]$, $\alpha \leq \rho(u) \leq \beta$ and $\rho(0) = \alpha$, $\rho(1) = \beta$. The first term inside the supremum above can be rewritten as

$$\frac{1}{2} \int (\eta(1) - \alpha) (f(\eta) - f(\sigma^1 \eta)) d\nu_{\rho(\cdot)}^N + \frac{1}{2} \int (\eta(1) - \alpha) (f(\eta) + f(\sigma^1 \eta)) d\nu_{\rho(\cdot)}^N. \quad (5.57)$$

Recall (3.26). From Young's inequality and $(a - b) = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$, we can bound the first term in the previous display from above by

$$\frac{A}{2} I_1^\alpha(\sqrt{f}, \nu_{\rho(\cdot)}^N) + \frac{1}{2A} \int (\eta(1) - \alpha)^2 (\sqrt{f}(\eta) + \sqrt{f}(\sigma^1 \eta))^2 d\nu_{\rho(\cdot)}^N$$

for any arbitrary positive constant A . Now observe first that by (3.24), $D_N^\ell(\sqrt{f}, \nu_{\rho(\cdot)}^N) \leq r_1^-(\frac{1}{N}) I_1^\alpha(\sqrt{f}, \nu_{\rho(\cdot)}^N)$ and secondly, that since f is a density and $\rho(\cdot)$ is Lipschitz the second term in the last display is bounded from above by a constant times $1/N + 1/A$ (see Lemma 5.5 of [3] for details). Hence we get that the first term in (5.57) is bounded from above by

$$\frac{A}{2r_1^-(\frac{1}{N})} D_N^\ell(\sqrt{f}, \nu_{\rho(\cdot)}^N) + C \left(\frac{1}{A} + \frac{1}{N} \right) \quad (5.58)$$

for some constant $C > 0$ independent of N, B, A . Therefore, if we chose

$$A = \frac{1}{2} \kappa B^{-1} N^{\gamma-\theta-1} r_1^-(1/N)$$

we can bound the previous display by

$$\frac{\kappa N^\gamma}{4BN^{\theta+1}} D_N^\ell(\sqrt{f}, \nu_{\rho(\cdot)}^N) + C \left[\frac{2BN^{\theta+1}}{\kappa N^\gamma r_1^-(1/N)} + \frac{1}{N} \right]. \quad (5.59)$$

Since $\kappa N^{-\theta} D_N^l(\sqrt{f}, \nu_{\rho(\cdot)}^N) \leq D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N)$, by using (3.29), we get that (5.56) is bounded from above by a constant times

$$\frac{1}{B} + \frac{2BN^{\theta+1}}{\kappa N^\gamma r_1^-(1/N)} + \frac{1}{N}.$$

Since $r_1^-(1/N)$ is bounded from below by a constant independent of N and $\theta < \gamma - 1$ the proof ends. \square

Remark 5.7. *The proof of (5.30) is completely analogous and it is a consequence of Lemma 5.6, Lemmas 5.4 and 5.5, when we replace α by β , $\eta_s^N(1)$ by $\eta_s^N(N-1)$ and $\vec{\eta}_s^\ell(0)$ by $\overleftarrow{\eta}_s^\ell(N)$. The details are left to the reader.*

5.3. Moving particle lemma. In this section we present a useful result to estimate the ‘cost’ of a change of position of particles in our model. In other words, how much the exchange of position of a particle from a site $y \in \Lambda_N$ to a site $y + \ell \in \Lambda_N$, for some $\ell > 0$, changes the ‘energy’ of the system in terms of the quantity (3.27).

Lemma 5.8. *Fix $\ell_0 \leq N - 1$ and define $\ell_i := 2^i \ell_0$ for any $i \in \{1, \dots, M\}$ where M is a positive integer such that $\ell_M < N - 1$. Let f be a density with respect to $\nu_{\rho(\cdot)}^N$ on Ω_N , where $\nu_{\rho(\cdot)}^N$ is defined as in (3.21) with $\rho : [0, 1] \rightarrow (0, 1)$ such that $\rho(0) = \alpha$ and $\rho(1) = \beta$. Then*

$$\sum_{i=1}^M \sum_{y=1}^{\ell_{i-1}} \frac{I_{y, y+\ell_{i-1}}(\sqrt{f}, \nu_{\rho(\cdot)}^N)}{\ell_{i-1}^\gamma} \lesssim D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N)$$

where $D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N)$ is defined in (3.27) and $I_{y, y+\ell_{i-1}}$ in (3.26).

Proof. We can assume without loss of generality that ℓ_0 is even (the argument is easy to extend to an odd ℓ_0) and then ℓ_{i-1} is an even number for any $i \in \{1, \dots, M\}$.

Fix $i \in \{1, \dots, M\}$. For every $y \in \{1, \dots, \ell_{i-1}\}$ consider the $\frac{\ell_{i-1}}{2}$ possibilities that a particle has to jump from y to $y + \ell_{i-1}$ in at most two steps. Hence for any $j \in \{1, \dots, \frac{\ell_{i-1}}{2}\}$, define

$$z_{0,j} := z_{0,j}^i(y) = y, \quad z_{1,j} := z_{1,j}^i(y) = y + \frac{\ell_{i-1}}{2} + j, \quad z_{2,j} := z_{2,j}^i(y) = y + \ell_{i-1}, \quad (5.60)$$

which correspond to one jump of length $\frac{\ell_{i-1}}{2} + j$ from $z_{0,j} = y$ to $z_{1,j}$ and one jump of length $\frac{\ell_{i-1}}{2} - j$ from $z_{1,j}$ to $z_{2,j} = y + \ell_{i-1}$. Observe that for $j = \frac{\ell_{i-1}}{2}$, we are moving from y to $y + \ell_{i-1}$ with only one jump and $z_{1,j} = z_{2,j}$.

Recall (3.26). Observe that

$$\sqrt{f(\sigma^{y, y+\ell_{i-1}} \eta)} - \sqrt{f(\eta)} = \sqrt{f(\sigma^{z_{0,j}, z_{2,j}} \eta)} - \sqrt{f(\eta)} \quad (5.61)$$

is not 0 if, and only if, $\eta(z_{0,j}) \neq \eta(z_{2,j})$. We want to rewrite (5.61) using the point $z_{1,j}$. To do that we consider separately the possible combinations of values of the $z_{i,j}$ with $i \in \{0, 1, 2\}$. First, assume that $\eta(z_{0,j}) = 1$ and $\eta(z_{2,j}) = 0$. In this case we have two possibilities:

- a. when $\eta(z_{1,j}) = 0$, we observe that the action of the operator $\sigma^{z_{0,j},z_{2,j}}$ on the configuration η is equivalent to the action in sequence (and in that order) of $\sigma^{z_{0,j},z_{1,j}}$ and then $\sigma^{z_{1,j},z_{2,j}}$. So, in this particular case we can write (5.61) as

$$\left[\sqrt{f(\sigma^{z_{1,j},z_{2,j}}(\sigma^{z_{0,j},z_{1,j}}\eta))} - \sqrt{f(\sigma^{z_{0,j},z_{1,j}}\eta)} \right] + \left[\sqrt{f(\sigma^{z_{0,j},z_{1,j}}\eta)} - \sqrt{f(\eta)} \right]. \quad (5.62)$$

- b. when $\eta(z_{1,j}) = 1$, the action of the operator $\sigma^{z_{0,j},z_{2,j}}$ on the configuration η is equivalent to the action in sequence (and in that order) of $\sigma^{z_{1,j},z_{2,j}}$ and then $\sigma^{z_{0,j},z_{1,j}}$. So, in this particular case we can write (5.61) as

$$\left[\sqrt{f(\sigma^{z_{0,j},z_{1,j}}(\sigma^{z_{1,j},z_{2,j}}\eta))} - \sqrt{f(\sigma^{z_{1,j},z_{2,j}}\eta)} \right] + \left[\sqrt{f(\sigma^{z_{1,j},z_{2,j}}\eta)} - \sqrt{f(\eta)} \right]. \quad (5.63)$$

Then, we have to consider the case in which $\eta(z_{1,j}) = 0$ and $\eta(z_{2,j}) = 1$. Then, reasoning similarly to what we did above, if $\eta(z_{1,j}) = 0$ we can rewrite (5.61) as (5.63), otherwise, if $\eta(z_{1,j}) = 1$ we can rewrite (5.61) as (5.62).

Let us now consider the configurations in which we can rewrite (5.61) as (5.62) and call the set of these configurations

$$\tilde{\Omega}_N^1 = \{\eta \in \Omega_N : \eta(z_{0,j}) = 1, \eta(z_{1,j}) = 0, \eta(z_{2,j}) = 0 \text{ or } \eta(z_{0,j}) = 0, \eta(z_{1,j}) = 1, \eta(z_{2,j}) = 1\}.$$

In a similar way let us define the following set of configurations

$$\tilde{\Omega}_N^2 = \{\eta \in \Omega_N : \eta(z_{0,j}) = 1, \eta(z_{1,j}) = 1, \eta(z_{2,j}) = 0 \text{ or } \eta(z_{0,j}) = 0, \eta(z_{1,j}) = 0, \eta(z_{2,j}) = 1\},$$

which are the ones corresponding to the case in which we can rewrite (5.61) as (5.63). Observe that $\tilde{\Omega}_N^1 \cap \tilde{\Omega}_N^2 = \emptyset$. Now, thanks to the reasoning that we did above and using the inequality $(a + b)^2 \leq 2(a^2 + b^2)$, we can write

$$\begin{aligned} I_{y,y+\ell_{i-1}}(\sqrt{f}, \nu_{\rho(\cdot)}^N) &= \int (\sqrt{f(\eta^{y,y+\ell_{i-1}})} - \sqrt{f(\eta)})^2 d\nu_{\rho(\cdot)}^N \\ &= \int_{\tilde{\Omega}_N^1} \left(\left[\sqrt{f(\sigma^{z_{1,j},z_{2,j}}(\sigma^{z_{0,j},z_{1,j}}\eta))} - \sqrt{f(\sigma^{z_{0,j},z_{1,j}}\eta)} \right] + \left[\sqrt{f(\sigma^{z_{0,j},z_{1,j}}\eta)} - \sqrt{f(\eta)} \right] \right)^2 d\nu_{\rho(\cdot)}^N \\ &\quad + \int_{\tilde{\Omega}_N^2} \left(\left[\sqrt{f(\sigma^{z_{0,j},z_{1,j}}(\sigma^{z_{1,j},z_{2,j}}\eta))} - \sqrt{f(\sigma^{z_{1,j},z_{2,j}}\eta)} \right] + \left[\sqrt{f(\sigma^{z_{1,j},z_{2,j}}\eta)} - \sqrt{f(\eta)} \right] \right)^2 d\nu_{\rho(\cdot)}^N \\ &\lesssim \int_{\tilde{\Omega}_N^1} \left(\sqrt{f(\sigma^{z_{1,j},z_{2,j}}(\sigma^{z_{0,j},z_{1,j}}\eta))} - \sqrt{f(\sigma^{z_{0,j},z_{1,j}}\eta)} \right)^2 d\nu_{\rho(\cdot)}^N \\ &\quad + \int_{\tilde{\Omega}_N^1} \left(\sqrt{f(\sigma^{z_{0,j},z_{1,j}}\eta)} - \sqrt{f(\eta)} \right)^2 d\nu_{\rho(\cdot)}^N \\ &\quad + \int_{\tilde{\Omega}_N^2} \left(\sqrt{f(\sigma^{z_{0,j},z_{1,j}}(\sigma^{z_{1,j},z_{2,j}}\eta))} - \sqrt{f(\sigma^{z_{1,j},z_{2,j}}\eta)} \right)^2 d\nu_{\rho(\cdot)}^N \\ &\quad + \int_{\tilde{\Omega}_N^2} \left(\sqrt{f(\sigma^{z_{1,j},z_{2,j}}\eta)} - \sqrt{f(\eta)} \right)^2 d\nu_{\rho(\cdot)}^N. \end{aligned} \quad (5.64)$$

Note that $\int_{\tilde{\Omega}_N^i} (\sqrt{f(\sigma^{x,y}\eta)} - \sqrt{f(\eta)})^2 d\nu_{\rho(\cdot)}^N \leq I_{x,y}(\sqrt{f}, \nu_{\rho(\cdot)}^N)$, for any $i = 1, 2$ and any $x, y \in \Omega_N$. Moreover, by the properties of $\nu_{\rho(\cdot)}^N$ (product measure associated to a Lipschitz profile which is never 0 nor 1), the Radon-Nikodym derivatives of the form $\frac{d\nu_{\rho(\cdot)}^N(\sigma^{x,y}\eta)}{d\nu_{\rho(\cdot)}^N(\eta)}$, for any $x, y \in \Lambda_N$, are bounded by a positive constant independent of x, y and N . Hence, last display can be bounded from above by a constant times

$$I_{z_{1,j}, z_{2,j}}(\sqrt{f}, \nu_{\rho(\cdot)}^N) + I_{z_{0,j}, z_{1,j}}(\sqrt{f}, \nu_{\rho(\cdot)}^N) + I_{z_{0,j}, z_{1,j}}(\sqrt{f}, \nu_{\rho(\cdot)}^N) + I_{z_{1,j}, z_{2,j}}(\sqrt{f}, \nu_{\rho(\cdot)}^N). \quad (5.65)$$

Therefore, we obtained that

$$I_{y, y+\ell_{i-1}}(\sqrt{f}, \nu_{\rho(\cdot)}^N) \lesssim \sum_{k=1}^2 I_{z_{k-1,j}, z_{k,j}}(\sqrt{f}, \nu_{\rho(\cdot)}^N). \quad (5.66)$$

Observe now that by construction we have $[p(z_{k,j} - z_{k-1,j})]^{-1} \lesssim \ell_{i-1}^{1+\gamma}$ (the longest possible jump is equal to ℓ_{i-1}). Hence, we have

$$I_{y, y+\ell_{i-1}}(\sqrt{f}, \nu_{\rho(\cdot)}^N) \lesssim \sum_{k=1}^2 I_{z_{k-1,j}, z_{k,j}}(\sqrt{f}, \nu_{\rho(\cdot)}^N) \lesssim \ell_{i-1}^{1+\gamma} \sum_{k=1}^2 p(z_{k,j} - z_{k-1,j}) I_{z_{k-1,j}, z_{k,j}}(\sqrt{f}, \nu_{\rho(\cdot)}^N). \quad (5.67)$$

This is true for any $j \in \{1, \dots, \frac{\ell_{i-1}}{2}\}$, so we can write

$$\ell_{i-1} I_{y, y+\ell_{i-1}}(\sqrt{f}, \nu_{\rho(\cdot)}^N) \lesssim \ell_{i-1}^{1+\gamma} \sum_{j=1}^{\ell_{i-1}/2} \sum_{k=1}^2 p(z_{k,j} - z_{k-1,j}) I_{z_{k-1,j}, z_{k,j}}(\sqrt{f}, \nu_{\rho(\cdot)}^N), \quad (5.68)$$

which implies that

$$\sum_{i=1}^M \sum_{y=1}^{\ell_{i-1}} \frac{I_{y, y+\ell_{i-1}}(\sqrt{f}, \nu_{\rho(\cdot)}^N)}{\ell_{i-1}^\gamma} \lesssim \sum_{i=1}^M \sum_{y=1}^{\ell_{i-1}} \sum_{j=1}^{\ell_{i-1}/2} \sum_{k=1}^2 p(z_{k,j} - z_{k-1,j}) I_{z_{k-1,j}, z_{k,j}}(\sqrt{f}, \nu_{\rho(\cdot)}^N). \quad (5.69)$$

Recall (5.60) and, in particular, that the $z_{k,j}$'s depend in fact on i and y . We claim that when i, y, j, k describe the sets involved in last sum, the couples $(z_{k-1,j}, z_{k,j}) := (z_{k-1,j}^i(y), z_{k,j}^i(y))$ are all different, i.e.

$$\Phi : (i, y, j, k) \rightarrow (z_{k-1,j}^i(y), z_{k,j}^i(y)) \in \Lambda_N \times \Lambda_N \quad \text{is injective.} \quad (5.70)$$

Therefore, recalling (3.27), we can bound from above the term on the right-hand side of (5.69) by

$$\sum_{\substack{v, w \in \Lambda_N \\ v \leq w}} p(w - v) I_{v,w}(\sqrt{f}, \nu_{\rho(\cdot)}^N) \lesssim D_N(\sqrt{f}, \nu_{\rho(\cdot)}^N). \quad (5.71)$$

Putting together (5.71) and (5.69) we get the statement.

We still have to prove (5.70) to conclude the proof. Let us assume that

$$\Phi(i, y, j, k) = \Phi(i', y', j', k')$$

and let us prove that $(i, y, j, k) = (i', y', j', k')$. We distinguish four cases according to the values of k and k' .

- $k = k' = 1$: then $z_{0,j}^i(y) = z_{0,j'}^{i'}(y')$ and $z_{1,j}^i(y) = z_{1,j'}^{i'}(y')$ imply that

$$y = y', \quad \frac{\ell_{i-1}}{2} + j = \frac{\ell_{i'-1}}{2} + j'.$$

Since $1 \leq j \leq \ell_{i-1}/2$ and $1 \leq j' \leq \ell_{i'-1}/2$ we have that

$$1 + \frac{\ell_{i-1}}{2} \leq \frac{\ell_{i-1}}{2} + j \leq \ell_{i-1} \quad \text{and} \quad 1 + \frac{\ell_{i'-1}}{2} \leq \frac{\ell_{i'-1}}{2} + j' \leq \ell_{i'-1}. \quad (5.72)$$

If $i \leq i' - 1$ then $\ell_{i-1} \leq \frac{\ell_{i'-1}}{2} < \frac{\ell_{i'-1}}{2} + 1$ and the equality $\frac{\ell_{i-1}}{2} + j = \frac{\ell_{i'-1}}{2} + j'$ is then in contradiction with (5.72). If $i' \leq i - 1$ then $\ell_{i'-1} \leq \frac{\ell_{i-1}}{2} < 1 + \frac{\ell_{i-1}}{2}$ the equality $\frac{\ell_{i-1}}{2} + j = \frac{\ell_{i'-1}}{2} + j'$ is then again in contradiction with (5.72). Hence $i = i'$ and consequently $j = j'$. We are done.

- $k = 1$ and $k' = 2$: then $z_{0,j}^i(y) = z_{1,j'}^{i'}(y')$ and $z_{1,j}^i(y) = z_{2,j'}^{i'}(y')$ imply that

$$y = y' + \frac{\ell_{i'-1}}{2} + j', \quad y + \frac{\ell_{i-1}}{2} + j = y' + \ell_{i'-1},$$

and hence by replacing y in the second equality by $y' + \frac{\ell_{i'-1}}{2} + j'$, we get

$$y = y' + \frac{\ell_{i'-1}}{2} + j', \quad \frac{\ell_{i-1}}{2} + j = \frac{\ell_{i'-1}}{2} - j'.$$

Since $1 \leq y \leq \ell_{i-1}$, $1 \leq y' \leq \ell_{i'-1}$, $1 \leq j \leq \ell_{i-1}/2$ and $1 \leq j' \leq \ell_{i'-1}/2$ we have that

$$\begin{aligned} 1 \leq y \leq \ell_{i-1} \quad \text{and} \quad 2 + \frac{\ell_{i'-1}}{2} \leq y \leq 2\ell_{i'-1}, \\ 1 + \frac{\ell_{i-1}}{2} \leq \frac{\ell_{i-1}}{2} + j \leq \ell_{i-1} \quad \text{and} \quad 0 \leq \frac{\ell_{i-1}}{2} + j \leq \frac{\ell_{i'-1}}{2} - 1. \end{aligned} \quad (5.73)$$

If $i \leq i' - 1$ then $\ell_{i-1} \leq \frac{\ell_{i'-1}}{2} < 2 + \frac{\ell_{i'-1}}{2}$ and there is a contradiction with the first line of (5.73). If $i' \leq i$ then $\ell_{i'-1} \leq \ell_{i-1}$, hence $\frac{\ell_{i-1}}{2} - 1 < 1 + \frac{\ell_{i-1}}{2}$, which is in contradiction with the second line of (5.73). Hence this case is not possible and we are done.

- $k = 2$ and $k' = 1$: by symmetry this case is equivalent to the previous one.
- $k' = 2$ and $k = 2$: then $z_{1,j}^i(y) = z_{1,j'}^{i'}(y')$ and $z_{2,j}^i(y) = z_{2,j'}^{i'}(y')$ imply that

$$\frac{\ell_{i-1}}{2} + j + y = \frac{\ell_{i'-1}}{2} + j' + y', \quad \ell_{i-1} + y = \ell_{i'-1} + y'.$$

The second equality and the fact that $1 \leq y \leq \ell_{i-1}$, resp. $1 \leq y' \leq \ell_{i'-1}$ implies that

$$1 + \ell_{i-1} \leq y + \ell_{i-1} \leq 2\ell_{i-1} \quad \text{and} \quad 1 + \ell_{i'-1} \leq y + \ell_{i-1} \leq 2\ell_{i'-1}. \quad (5.74)$$

If $i \leq i' - 1$ then $2\ell_{i-1} \leq \ell_{i'-1} < \ell_{i'-1} + 1$ which is in contradiction with (5.74). Similarly if $i' \leq i - 1$ then $2\ell_{i'-1} \leq \ell_{i-1} < \ell_{i-1} + 1$ is in contradiction with (5.74). Hence $i = i'$ and consequently we deduce that $y = y'$ and $j = j'$.

This concludes the proof of the lemma. □

5.4. Auxiliary result. The next proposition is used in order to define a notion of weak solution in the case $\theta = \gamma - 1$ and $\gamma \in (1, 2)$. The strategy of the proof is similar to the one used in the previous lemmas, so we will just sketch its proof.

Lemma 5.9. *Fix $\theta = \gamma - 1$, $\gamma \in (1, 2)$. For any $T > 0$, for any $G \in C^\infty([0, 1])$ it holds*

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^T \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right) (\eta_s^N(x) - \overrightarrow{\eta}_s^{\ell_0}(0)) ds \right| \right] = 0,$$

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^T \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) r_N^+\left(\frac{x}{N}\right) (\eta_s^N(x) - \overleftarrow{\eta}_s^{\ell_0}(N)) ds \right| \right] = 0,$$

where $\ell_0 = \epsilon N^{\gamma-1}$.

Proof. We prove only the part involving r_N^- , since the other one can be proved analogously. By the triangular inequality we can bound the expectation in the statement by the sum of

$$\mathbb{E}_{\mu_N} \left[\left| \int_0^T \sum_{\substack{x \in \Lambda_N \\ x \leq \ell_0}} G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right) (\eta_s^N(x) - \overrightarrow{\eta}_s^{\ell_0}(0)) ds \right| \right] \quad (5.75)$$

and

$$\mathbb{E}_{\mu_N} \left[\left| \int_0^T \sum_{\substack{x \in \Lambda_N \\ x > \ell_0}} G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right) (\eta_s^N(x) - \overrightarrow{\eta}_s^{\ell_0}(0)) ds \right| \right]. \quad (5.76)$$

The last display can be bounded from above by a constant times

$$\sum_{\substack{x \in \Lambda_N \\ x > \ell_0}} |G\left(\frac{x}{N}\right)| |r_N^-\left(\frac{x}{N}\right)| \leq \|G\|_\infty \sum_{\substack{x \in \Lambda_N \\ x > \ell_0}} |r_N^-\left(\frac{x}{N}\right)|.$$

Since $r_N^-\left(\frac{x}{N}\right) \lesssim x^{-\gamma}$ the right-hand side of the last display can be bounded from above, using Riemann integral approximation, by a constant times

$$(\ell_0)^{-\gamma+1} = \epsilon^{-\gamma+1} (N^{\gamma-1})^{-\gamma+1} = \epsilon^{-\gamma+1} N^{-(\gamma-1)^2} = 0,$$

and it vanishes in the limit $N \rightarrow \infty, \epsilon \rightarrow 0$. This means that in order to prove the statement of the Lemma we only need to show that the limit (5.75) is 0.

We proceed as we did in the proof of Lemma 5.4. Let $\rho \in (0, 1)$ be a constant. By Jensen's and entropy inequalities plus Feynman-Kac's formula, we are reduced to show that

$$\frac{C_0}{B} + \sup_f \left\{ \int \sum_{\substack{x \in \Lambda_N \\ x \leq \ell_0}} G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right) (\eta(x) - \overrightarrow{\eta}^{\ell_0}(0)) f(\eta) d\nu_\rho^N + \frac{N^{\gamma-1}}{B} \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_\rho^N} \right\} \quad (5.77)$$

vanishes in the limit $N \rightarrow \infty, \epsilon \rightarrow 0$ and then $B \rightarrow \infty$. As usual C_0 is a constant independent from N and the supremum is carried over all the densities with respect to the Bernoulli

product measure ν_ρ^N . From (3.30) and recalling that $\theta = \gamma - 1$, the term on the right-hand side of last display becomes bounded from above by

$$-\frac{N^\gamma}{4NB}D_N(\sqrt{f}, \nu_\rho^N) + \frac{C\kappa}{B} \quad (5.78)$$

where C is a constant independent of N, B (and ϵ). The first term inside the supremum of (5.77) can be written as

$$\frac{1}{\ell_0} \int \sum_{\substack{x \in \Lambda_N \\ x \leq \ell_0}} G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right) \left[\sum_{y=1}^x \sum_{z=y}^{x-1} (\eta(z+1) - \eta(z)) + \sum_{y=x}^{\ell_0} \sum_{z=x}^{y-1} (\eta(z) - \eta(z+1)) \right] f(\eta) d\nu_\rho^N.$$

Now, since the profile in the product measure ν_ρ^N is a constant we can rewrite the last term, by a change of variable as

$$\begin{aligned} \frac{1}{2\ell_0} \int \sum_{\substack{x \in \Lambda_N \\ x \leq \ell_0}} G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right) & \left[\sum_{y=1}^x \sum_{z=y}^{x-1} (\eta(z+1) - \eta(z))(f(\eta) - f(\sigma^{z,z+1}\eta)) \right. \\ & \left. + \sum_{y=x}^{\ell_0} \sum_{z=x}^{y-1} (\eta(z) - \eta(z+1))(f(\eta) - f(\sigma^{z,z+1}\eta)) \right] d\nu_\rho^N. \end{aligned} \quad (5.79)$$

Hence, using Young's inequality on both the terms in the last display with the same constant $A > 0$ we can bound from above the whole term by

$$\begin{aligned} \frac{1}{4\ell_0} \int \sum_{\substack{x \in \Lambda_N \\ x \leq \ell_0}} |G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right)| & \left[A \sum_{y=1}^x \sum_{z=y}^{x-1} (\eta(z+1) - \eta(z))^2 (\sqrt{f}(\eta) + \sqrt{f}(\sigma^{z,z+1}\eta))^2 \right. \\ & + A \sum_{y=x}^{\ell_0} \sum_{z=x}^{y-1} (\eta(z) - \eta(z+1))^2 (\sqrt{f}(\eta) + \sqrt{f}(\sigma^{z,z+1}\eta))^2 \\ & \left. + \frac{1}{A} \sum_{y=1}^{\ell_0} \sum_{z=1}^{\ell_0} (\sqrt{f}(\eta) - \sqrt{f}(\sigma^{z,z+1}\eta))^2 \right] d\nu_\rho^N. \end{aligned} \quad (5.80)$$

Since f is a density and $|\eta(z)| \leq 1$ for any z , we can bound from above the whole last term by (recall the definition (3.26))

$$\frac{1}{4\ell_0} \int \sum_{\substack{x \in \Lambda_N \\ x \leq \ell_0}} |G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right)| \left[A \sum_{y=1}^{\ell_0} \sum_{z=1}^{\ell_0} 2 + \frac{1}{A} \sum_{y=1}^{\ell_0} \sum_{z=1}^{\ell_0} I_{z,z+1}(\sqrt{f}, \nu_\rho^N) \right] \leq \frac{C'A\ell_0}{2} + \frac{1}{4A} D_N(\sqrt{f}, \nu_\rho^N), \quad (5.81)$$

where C' is a constant independent of N, ϵ, B (this because $\sum_{\substack{x \in \Lambda_N \\ x \leq \ell_0}} |G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right)| \lesssim 1$ and $\sum_{z=1}^{\ell_0} I_{z,z+1}(\sqrt{f}, \nu_\rho^N) \leq D_N(\sqrt{f}, \nu_\rho^N)$).

Then, taking $A = BN^{1-\gamma}$ the second term on the right-hand side of (5.81) cancels with the first of (5.78). Summarising we bounded (5.77) and hence the expectation in (5.75) by

$$\frac{C\kappa}{B} + C' \frac{BN^{1-\gamma}\ell_0}{2} \lesssim \frac{1}{B} + B\epsilon. \quad (5.82)$$

So, sending ϵ to 0 and then B to infinity, the proof is concluded. \square

Remark 5.10. Observe that if we apply Lemma 5.9 and then Lemma 5.5 it is easy to show that

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^T \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right) (\eta_s^N(x) - \vec{\eta}_s^{\epsilon N}(0)) ds \right| \right] = 0.$$

Indeed the expectation above can be bounded from above by

$$\begin{aligned} & \mathbb{E}_{\mu_N} \left[\left| \int_0^T \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right) (\eta_s^N(x) - \vec{\eta}_s^{\ell_0}(0)) ds \right| \right] \\ & + \mathbb{E}_{\mu_N} \left[\left| \int_0^T \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right) (\vec{\eta}_s^{\ell_0}(0) - \vec{\eta}_s^{\epsilon N}(0)) ds \right| \right]. \end{aligned} \quad (5.83)$$

The first expectation above vanishes, as N goes to infinity and ϵ goes to 0, thanks to Lemma 5.9. Now, call $C_N := \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) r_N^-\left(\frac{x}{N}\right)$, which is of order 1 since the sum is convergent. Then, the second expectation above vanishes, as N goes to infinity and ϵ goes to 0, thanks to Lemma 5.5. Moreover, not that by the symmetry property of this model, the results that we used works also when we study the right boundary (see Remark 5.7). So, it is possible to show in the same way as above that

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^T \sum_{x \in \Lambda_N} G\left(\frac{x}{N}\right) r_N^+\left(\frac{x}{N}\right) (\eta_s^N(x) - \overleftarrow{\eta}_s^{\epsilon N}(N)) ds \right| \right] = 0.$$

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Cédric Bernardin, UNIVERSITÉ CÔTE D'AZUR, CNRS, LJAD, PARC VALROSE, 06108 NICE CEDEX 02, FRANCE
& INTERDISCIPLINARY SCIENTIFIC CENTER PONCELET (CNRS IRL 2615), 119002 MOSCOW, RUSSIA
E-mail address: cbernard@unice.fr

Pedro Cardoso, CENTER FOR MATHEMATICAL ANALYSIS, GEOMETRY AND DYNAMICAL SYSTEMS
INSTITUTO SUPERIOR TÉCNICO, UNIVERSIDADE DE LISBOA
AV. ROVISCO PAIS, NO. 1, 1049-001 LISBOA, PORTUGAL
E-mail address: pedro.gondim@tecnico.ulisboa.pt

Patrícia Gonçalves, CENTER FOR MATHEMATICAL ANALYSIS, GEOMETRY AND DYNAMICAL SYSTEMS
INSTITUTO SUPERIOR TÉCNICO, UNIVERSIDADE DE LISBOA
AV. ROVISCO PAIS, NO. 1, 1049-001 LISBOA, PORTUGAL
E-mail address: pgoncalves@tecnico.ulisboa.pt

Stefano Scotta, CENTER FOR MATHEMATICAL ANALYSIS, GEOMETRY AND DYNAMICAL SYSTEMS
INSTITUTO SUPERIOR TÉCNICO, UNIVERSIDADE DE LISBOA
AV. ROVISCO PAIS, NO. 1, 1049-001 LISBOA, PORTUGAL
E-mail address: stefano.scotta@tecnico.ulisboa.pt