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# A KERNEL-BASED CONSENSUAL REGRESSION AGGREGATION METHOD

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## Abstract

In this article, we introduce a kernel-based consensual aggregation method for regression problems. We aim to flexibly combine individual regression estimators  $r_1, r_2, \dots, r_M$  using a weighted average where the weights are defined based on some kernel function. It may be seen as a kernel smoother method implemented on the predictions given by all the individual estimators instead of the original input. This kernel-based configuration asymptotically inherits the consistency property of the basic consistent estimators. Moreover, numerical experiments carried out on several simulated and real datasets confirm that the proposed method mostly outperforms or at least biases towards the best candidate estimator of the combination.

*Keywords:* Consensual aggregation, kernel, regression.

*2010 Mathematics Subject Classification:* 62G08, 62J99, 62P30

## 1 Introduction

Aggregation methods, given the high diversity of available estimation strategies, are now of great interest in constructing predictive models. To this goal, several aggregation methods consisting of building linear or convex combination of a bunch of initial estimators have been introduced, for instance in [Catoni \(2004\)](#), [Juditsky and Nemirovski \(2000\)](#), [Nemirovski \(2000\)](#), [Yang \(2000, 2001\)](#); [Yang et al. \(2004\)](#), [Györfi et al. \(2002\)](#), [Wegkamp \(2003\)](#), [Audibert \(2004\)](#), [Bunea et al. \(2006, 2007a,b\)](#), and [Dalalyan and Tsybakov \(2008\)](#). Another approach of model selection, which aims at selecting the best estimator among the candidate estimators, has also been proposed (see, for instance, [Massart \(2007\)](#)).

Apart from the usual linear combination and model selection methods, a different technique has been introduced in classification problems by [Mojirsheibani \(1999\)](#). In his paper, the combination is the *majority vote* among all the points for which their predicted classes given by all the basic classifiers *coincide* with the predicted classes of the query point. Roughly speaking, instead of predicting a new point based on the structure of the original input, we look at the topology defined by the predictions of the candidate estimators. Each estimator was constructed differently so may be able to capture different features of the input data and useful in defining “closeness”. Consequently, two points having similar predictions seem reasonably having similar actual response values. For instance in classification, two points having the same predicted classes should actually belong to the same class.

Later, [Mojirsheibani \(2000\)](#) and [Mojirsheibani and Kong \(2016\)](#) introduced an exponential and general kernel-based versions of the primal idea to improve the smoothness in selecting and weighting individual points in the combination. In this context, the kernel function transforms the number of *disagreements* between the predicted classes of a training point  $x_i$  and the query point  $x$  into a contributed weight given to the corresponding point in the vote. Analogously, [Biau et al. \(2016\)](#) configured the original idea of [Mojirsheibani \(1999\)](#) as regression framework where a training point  $x_i$  is “close” to the query point  $x$  if each of their predictions given by all the basic regression estimators is close. Each of the close neighbors of  $x$  will be given a uniformly 0-1 weight. It was shown theoretically in these former papers that the combinations inherit the consistency property of consistent basic estimators.

Up to now, the kernel-based version of consensual regression aggregation method has not been considered. To fill the gap, we present in this paper such a framework where the weight given to an individual point of the combination is defined associated to some *regular* kernel. Similarly, this kernel function converts the level of “closeness” between the predictions of a training point  $x_i$  and the query point  $x$  into a weight given to the corresponding training point in the combination. More precisely, a point that is farther away from  $x$  may be given a relatively small kernel-based weight rather than being completely ignored as in the classical case. We provide in this study the theoretical and numerical evidence of the consistency property and outperformance of the proposed method.

This paper is organized as follows. Section 2 introduces some notation, the definition of the proposed method and presents the theoretical results,

namely consistency and the convergence rate of variance term for a subclass of kernel functions. Section 3 illustrates the performances of the proposed method through several numerical examples of simulated and real datasets. Section ?? collects all the proofs of the theoretical results given in Section 2.

## 2 The kernel-based combining regression

### 2.1 Notation

We consider a training sample  $\mathcal{D}_n = \{(X_i, Y_i)_{i=1}^n\}$  where  $(X_i, Y_i), i = 1, 2, \dots, n$ , are assumed to be independent and identically distributed with the same realization as  $(X, Y)$ . We assume that  $(X, Y)$  is an  $\mathbb{R}^d \times \mathbb{R}$ -valued random variable with a suitable integrability which will be specified later.

We randomly split the training data  $\mathcal{D}_n$  into two parts of size  $\ell$  and  $k$  such that  $\ell + k = n$ , which are denoted by  $\mathcal{D}_\ell = \{(X_i^{(\ell)}, Y_i^{(\ell)})_{i=1}^\ell\}$  and  $\mathcal{D}_k = \{(X_i^{(k)}, Y_i^{(k)})_{i=1}^k\}$  respectively (a common choice is  $k = \ell = n/2$ ). We construct the  $M$  basic regression estimators or machines  $r_{k,1}, r_{k,2}, \dots, r_{k,M}$  using only the data points in  $\mathcal{D}_k$ . These basic machines can be any regression estimators such as linear regression,  $k$ NN, kernel smoother, SVR, Lasso, Ridge, neural networks, naive Bayes, or random forests... They could be parametric, nonparametric or semi-parametric with their possible tuning parameters. For the combination, we only need the predictions of the remaining part  $\mathcal{D}_\ell$  given by all the basic machines.

In the sequel, for any  $x \in \mathbb{R}^d$ , the following notations are used:

- $\mathbf{r}_k(x) = (r_{k,1}(x), r_{k,2}(x), \dots, r_{k,M}(x))$ : the vector of predictions of  $x$ .
- $\|x\| = \|x\|_2 = \sqrt{\sum_{i=1}^d x_i^2}$ : Euclidean norm on  $\mathbb{R}^d$ .
- $\|x\|_1 = \sum_{i=1}^d |x_i|$ :  $\ell^1$  norm on  $\mathbb{R}^d$ .

The consensual aggregation is the weighted average defined by,

$$g_n(\mathbf{r}_k(x)) = \sum_{i=1}^{\ell} W_{n,i}(x) Y_i^{(\ell)}. \quad (1)$$

Recall that given all the basic machines  $r_{k,1}, r_{k,2}, \dots, r_{k,M}$ , the aggregation method proposed by [Biau et al. \(2016\)](#) corresponds to the following naive weight:

$$W_{n,i}(x) = \frac{\prod_{m=1}^M \mathbb{1}_{\{|r_{k,m}(X_i) - r_{k,m}(x)| < \varepsilon\}}}{\sum_{j=1}^{\ell} \prod_{m=1}^M \mathbb{1}_{\{|r_{k,m}(X_j) - r_{k,m}(x)| < \varepsilon\}}}, i = 1, 2, \dots, \ell. \quad (2)$$

Moreover, the condition of “closeness” for all predictions, can be relaxed to “some” predictions, which corresponds to the following weight:

$$W_{n,i}(x) = \frac{\mathbb{1}_{\{\sum_{m=1}^M \mathbb{1}_{\{|r_{k,m}(X_i) - r_{k,m}(x)| < \varepsilon\}} \geq \alpha M\}}}{\sum_{j=1}^{\ell} \mathbb{1}_{\{\sum_{m=1}^M \mathbb{1}_{\{|r_{k,m}(X_j) - r_{k,m}(x)| < \varepsilon\}} \geq \alpha M\}}}, i = 1, 2, \dots, \ell \quad (3)$$

where  $\alpha \in \{1/M, 2/M, \dots, 1\}$  is the proportion of consensual predictions required and  $\varepsilon > 0$  is the smoothing parameter to be determined.

In the present paper,  $K : \mathbb{R}^d \rightarrow \mathbb{R}_+$  denotes a regular kernel which is a decreasing function satisfying:

$$\exists b, \kappa_0, \rho > 0 \text{ such that } \begin{cases} \forall x \in \mathbb{R}^d : b \mathbb{1}_{B_d(0, \rho)}(x) \leq K(x) \leq 1 \\ \int_{\mathbb{R}^d} \sup_{u \in B_d(x, \rho)} K(u) dx = \kappa_0 < +\infty \end{cases} \quad (4)$$

where  $B_d(x, r) = \{z \in \mathbb{R}^d : \|x - z\|_2 < r\}$  denotes the open ball of center  $x \in \mathbb{R}^d$  and radius  $r > 0$  of  $\mathbb{R}^d$ . The proposed method evaluated at a point  $x \in \mathbb{R}^d$  corresponds to the following weight:

$$W_{n,i}(x) = \frac{K_h(\mathbf{r}_k(X_i^{(\ell)}) - \mathbf{r}_k(x))}{\sum_{j=1}^{\ell} K_h(\mathbf{r}_k(X_j^{(\ell)}) - \mathbf{r}_k(x))}, i = 1, 2, \dots, \ell \quad (5)$$

where  $K_h(z) = K(z/h)$  for some smoothing parameter  $h > 0$  with the convention of  $0/0 = 0$ . Note that the combination is based only on  $\mathcal{D}_\ell$  but the whole construction of this method definitely depends on the whole training data  $\mathcal{D}_n$  as the basic machines are all constructed using  $\mathcal{D}_k$ .

## 2.2 Theoretical performance

The performance of the combining estimation  $g_n$  is measured using the quadratic risk defined by,

$$\mathbb{E} \left[ |g_n(\mathbf{r}_k(X)) - g^*(X)|^2 \right]$$

where the expectation is taken with respect to both  $X$  and the training sample  $\mathcal{D}_n$ .

The first result of nonasymptotic-type inequality, controlling the performance of the proposed method is given in the following proposition.

**Proposition 1** Let  $\mathbf{r}_k = (r_{k,1}, r_{k,2}, \dots, r_{k,M})$  be the collection of all basic estimators and  $g_n(\mathbf{r}_k(x))$  be the combined estimator computed at point  $x \in \mathbb{R}^d$ . Then, for all distributions of  $(X, Y)$  with  $\mathbb{E}[|Y|^2] < +\infty$ ,

$$\begin{aligned} \mathbb{E}\left[|g_n(\mathbf{r}_k(X)) - g^*(X)|^2\right] &\leq \inf_{f \in \mathcal{G}} \mathbb{E}\left[|f(\mathbf{r}_k(X)) - g^*(X)|^2\right] \\ &\quad + \mathbb{E}\left[|g_n(\mathbf{r}_k(X)) - g^*(\mathbf{r}_k(X))|^2\right]. \end{aligned}$$

In particular,

$$\begin{aligned} \mathbb{E}\left[|g_n(\mathbf{r}_k(X)) - g^*(X)|^2\right] &\leq \min_{1 \leq m \leq M} \mathbb{E}\left[|r_{k,m}(X) - g^*(X)|^2\right] \\ &\quad + \mathbb{E}\left[|g_n(\mathbf{r}_k(X)) - g^*(\mathbf{r}_k(X))|^2\right]. \end{aligned}$$

Proposition 1 ensures the performance of  $g_n$  with respect to the performance of best basic regression estimator and the variation of the combination itself for whatever the distribution of  $(X, Y)$  such that  $Y$  is square-integrable. The two terms of the last bound can be seen as a bias-variance decomposition where the first term  $\min_{1 \leq m \leq M} \mathbb{E}[|r_{k,m}(X) - g^*(X)|^2]$  can be regarded as the bias and  $\mathbb{E}[|g_n(\mathbf{r}_k(X)) - g^*(\mathbf{r}_k(X))|^2]$  can be seen as the variance-type term, which can be asymptotically neglected, as shown in the following proposition.

**Proposition 2** Assume that  $r_{k,m}$  is bounded for all  $m = 1, 2, \dots, M$ . Let  $h \rightarrow 0$  and  $\ell \rightarrow +\infty$  such that  $h^M \ell \rightarrow +\infty$ . Then

$$\mathbb{E}\left[|g_n(\mathbf{r}_k(X)) - g^*(\mathbf{r}_k(X))|^2\right] \rightarrow 0 \text{ as } \ell \rightarrow +\infty$$

for all distribution of  $(X, Y)$  with  $\mathbb{E}[|Y|^2] < +\infty$ . Thus,

$$\limsup_{\ell \rightarrow +\infty} \mathbb{E}\left[|g_n(\mathbf{r}_k(X)) - g^*(X)|^2\right] \leq \inf_{f \in \mathcal{G}} \mathbb{E}\left[|f(\mathbf{r}_k(X)) - g^*(X)|^2\right].$$

And in particular,

$$\limsup_{\ell \rightarrow +\infty} \mathbb{E}\left[|g_n(\mathbf{r}_k(X)) - g^*(X)|^2\right] \leq \min_{1 \leq m \leq M} \mathbb{E}\left[|r_{k,m}(X) - g^*(X)|^2\right].$$

This result is remarkable as it guarantees the asymptotic outperformance of the proposed method with respect to the best basic estimator in term of

quadratic risk. Moreover, the result is nearly universal in a sense that it is true for all distributions of  $(X, Y)$  such that  $\mathbb{E}[Y^2] < +\infty$ .

The result of Proposition 2 holds for all regular kernels. However, the price to pay for the universality is the lack of convergence rate of the variance term. Under a weak smoothness assumption of the optimal estimator  $g^*$  and by considering a smaller class of kernel functions, the convergence rate of the method can be derived in the following theorem which is the main result of the paper.

**Theorem 1** *Assume that the response variable  $Y$  and all the basic machines  $r_{k,m}, m = 1, 2, \dots, M$ , are bounded by some constant  $R$ . Suppose that there exists a constant  $L \geq 0$  such that, for every  $k \geq 1$ ,*

$$|g^*(\mathbf{r}_k(x)) - g^*(\mathbf{r}_k(y))| \leq L \|\mathbf{r}_k(x) - \mathbf{r}_k(y)\|, \forall x, y \in \mathbb{R}^d.$$

*We assume moreover that the kernel  $K$  is compactly supported i.e.,*

$$\exists \rho_1 > 0 : K(z) \leq \mathbb{1}_{\{\|z\| \leq \rho_1\}}, \forall z \in \mathbb{R}^M.$$

*Then, with the choice of  $h \propto \ell^{-\frac{1}{M+2}}$ , one has*

$$\mathbb{E}[|g_n(\mathbf{r}_k(X)) - g^*(X)|^2] \leq \min_{1 \leq m \leq M} \mathbb{E}[|r_{k,m}(X) - g^*(X)|^2] + C\ell^{-\frac{2}{M+2}} \quad (6)$$

*for some positive constant  $C = C(b, L, R, \rho_1)$  independent of  $\ell$ .*

Moreover,  $g_n$  inherits the consistent property of consistent estimators if there exists one in the combination. This means that if there is a consistent estimator  $r_{k,m_0}$  among  $\{r_{k,m}\}_{m=1}^M$  i.e.,

$$\mathbb{E}[|r_{k,m_0}(X) - g^*(X)|^2] \rightarrow 0 \text{ as } k \rightarrow +\infty$$

for all distribution of  $(X, Y)$  in some class  $\mathcal{M}$ . Then, under the assumption of Theorem 1, one has

$$\lim_{k, \ell \rightarrow +\infty} \mathbb{E}[|g_n(\mathbf{r}_k(X)) - g^*(X)|^2] = 0.$$

Theorem 1 presents the control of the performance of our method by the minimal risk among all the basic estimators, and the convergence rate of order  $\ell^{-\frac{2}{M+2}}$  of the variance term, which is the price to pay for combing  $M$  individual estimators. Note that this convergence rate of the second term coincides with the one in [Biau et al. \(2016\)](#).

### 3 Numerical examples

This section is devoted to numerical experiments to illustrate the performance of our proposed method. It is shown in [Biau et al. \(2016\)](#) that the classical method mostly outperforms the basic machines of the combination. In this experiments, we compare the performances of the proposed methods with the classical one (naive kernel) and all the basic machines. We consider several options of kernel functions. Most kernels are compactly supported on  $[-1, 1]$ , taking nonzero values only on  $[-1, 1]$ , except for the case of compactly supported Gaussian and 4-exponential kernels which are supported on  $[-\rho_1, \rho_1]$  for some  $\rho_1 > 0$ . To take a broader view, we also present the results of non compactly supported case such as classical Gaussian and Cauchy kernels. All kernels considered in this paper are listed in Table 1 and the shapes of some kernels in univariate case are displayed in Figure 1.

Kernel	Formula
Naive <sup>1</sup>	$K(x) = \prod_{i=1}^d \mathbb{1}_{\{ x_i  \leq 1\}}$
Epanechnikov	$K(x) = (1 - \ x\ ^2) \mathbb{1}_{\{\ x\  \leq 1\}}$
Bi-weight	$K(x) = (1 - \ x\ ^2)^2 \mathbb{1}_{\{\ x\  \leq 1\}}$
Tri-weight	$K(x) = (1 - \ x\ ^2)^3 \mathbb{1}_{\{\ x\  \leq 1\}}$
Triangular	$K(x) = (1 - \ x\ _1) \mathbb{1}_{\{\ x\ _1 \leq 1\}}$
Compact 4-Exponential	$K(x) = \exp\{-\ x\ ^4/(2\sigma^2)\} \mathbb{1}_{\{\ x\  \leq \rho_1\}}, \sigma, \rho_1 > 0$
Compact-support Gaussian	$K(x) = \exp\{-\ x\ ^2/(2\sigma^2)\} \mathbb{1}_{\{\ x\  \leq \rho_1\}}, \sigma, \rho_1 > 0$
Gaussian	$K(x) = \exp\{-\ x\ ^2/(2\sigma^2)\}, \sigma > 0$
Cauchy	$K(x) = 1/(1 + \ x\ ^2)$

Table 1: Kernel functions used.

#### 3.1 Simulated datasets

In this subsection, we study the performances of our proposed method on the same set of simulated datasets of size  $n$  given in [Biau et al. \(2016\)](#). The input data is either uniformly distributed over  $(-1, 1)^d$  (*uncorrelated case*) or distributed from a Gaussian distribution  $\mathcal{N}(0, \Sigma)$  where the covariance matrix  $\Sigma$  is defined by  $\Sigma_{ij} = 2^{-|i-j|}$  for  $1 \leq i, j \leq d$  (*correlated case*). We consider the following models,

<sup>1</sup>The naive kernel corresponds to the classical method by [Biau et al. \(2016\)](#).



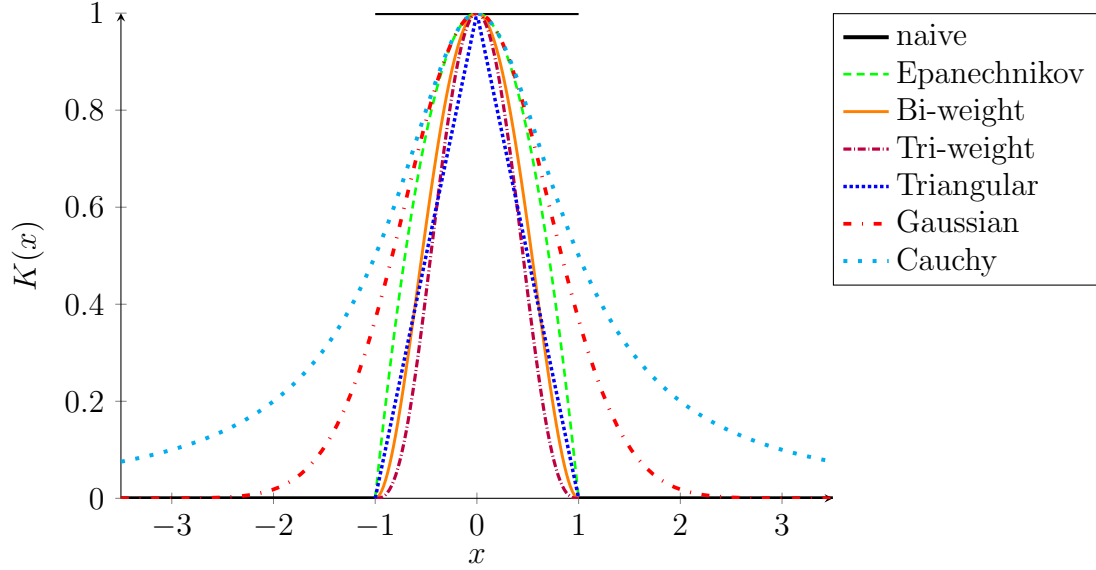


Figure 1: The shapes of some kernels.

**Model 1 :**  $n = 800, d = 50, Y = X_1^2 + \exp(-X_2^2)$ .

**Model 2 :**  $n = 600, d = 100, Y = X_1X_2 + X_3^2 - X_4X_7 + X_8X_{10} - X_6^2 + \mathcal{N}(0, 0.5)$ .

**Model 3 :**  $n = 600, d = 100, Y = -\sin(2X_1) + X_2^2 + X_3 - \exp(-X_4) + \mathcal{N}(0, 0.5)$ .

**Model 4 :**  $n = 600, d = 100, Y = X_1 + (2X_2 - 1)^2 + \sin(2\pi X_3)/(2 - \sin(2\pi X_3)) + \sin(2\pi X_4) + 2 \cos(2\pi X_4) + 3 \sin^2(2\pi X_4) + 4 \cos^2(2\pi X_4) + \mathcal{N}(0, 0.5)$ .

**Model 5 :**  $n = 700, d = 20, Y = \mathbb{1}_{\{X_1 > 0\}} + X_2^3 + \mathbb{1}_{\{X_4 + X_6 - X_8 - X_9 > 1 + X_{14}\}} + \exp(-X_2^2) + \mathcal{N}(0, 0.05)$ .

**Model 6 :**  $n = 500, d = 30, Y = \sum_{k=1}^{10} \mathbb{1}_{\{X_k < 0\}} - \mathbb{1}_{\{\mathcal{N}(0,1) > 1.25\}}$ .

**Model 7 :**  $n = 600, d = 300, Y = X_1^2 + X_2^2 X_3 \exp(-|X_4|) + X_6 - X_8 + \mathcal{N}(0, 0.5)$ .

**Model 8 :**  $n = 600, d = 50, Y = \mathbb{1}_{\{X_1 + X_4^3 + X_9 + \sin(X_{12}X_{18}) + \mathcal{N}(0,0.01) > 0.38\}}$ .

Moreover, it is interesting to consider high-dimensional datasets as many real problems involve these kind of datasets such as image and signal processing. Therefore, we also consider the following two high-dimensional models:

**Model 9** :  $n = 500, d = 1000, Y = X_1 + 3X_3^2 - 2 \exp(-X_5) + X_6$ .

**Model 10** :  $n = 500, d = 1500, Y = \exp(-X_1) + \exp(X_1) + \sum_{j=2}^d X_j^{j/100}$ .

For each model, the proposed method is implemented over 100 replications. We randomly split 80% of each simulated dataset into two equal parts,  $\mathcal{D}_\ell$  and  $\mathcal{D}_k$  where  $\ell = k = 0.8 \times n/2$ , and the remaining 20% will be treated as validation data. We measure the performance of an estimation method  $f$  using the *root mean square error* (RMSE) evaluated on the 20%-testing data defined by,

$$\text{RMSE}(f) = \sqrt{\frac{1}{n_{\text{test}}} \sum_{i=1}^{n_{\text{test}}} (y_i - f(x_i))^2} \quad (7)$$

Table 2, 3 and 4 below contain the average RMSEs and the corresponding standard errors (into brackets) over 100 runs of *uncorrelated* and *correlated* cases respectively. In each table, the first block contains five columns corresponding to the following five basic machines  $\mathbf{r}_k = (r_{k,m})_{m=1}^5$ :

- **Rid**: Ridge regression (R package `glmnet`, see [Friedman et al. \(2010\)](#)).
- **Las**: Lasso regression (R package `glmnet`).
- **k-NN**:  $k$ -nearest neighbors regression (R package `FNN`, see [Li \(2019\)](#)).
- **P-tr**: Pruned tree regression (R package `tree`, see [Ripley \(2019\)](#)).
- **RF**: Random Forest regression (R package `randomForest`, see [Liaw and Wiener \(2002\)](#)).

For ridge and lasso regression, we consider 100 values of penalty parameter  $\lambda$  on a uniform grid  $\{10^{-2}, \dots, 10^2\}$ . We choose  $k = 10$  for  $k$ -NN, and the last two methods are implemented using default parameters. The out-performance of each model in this block is highlighted in **boldface**. The second block contains the last nine columns corresponding to the kernel

functions used in the combining method where **Naiv**<sup>2</sup>, **Epan**, **Bi-wgt**, **Tri-wgt**, **Triang**, **Exp4**, **C-Gaus**, **Gaus** and **Cauchy** respectively stand for the Naive, Epanechnikov, Bi-weight, Tri-weight, Triangular, Compact 4-Exponential, Compact-support Gaussian, Gaussian and Cauchy kernels as listed in Table 1. In this block, the outperformance is highlighted in **bold-faced blue**. For each kernel, constructing the associated combining regression method is equivalent to searching for a smoothing parameter  $h$  minimizing cross-validation risk of the training predictions given by all the five basic machines. We consider 500 values of  $h$  in a uniform grid  $\{10^{-300}, \dots, h_{\max}\}$  where  $h_{\max} = 10$  for all models except for model 10 of the correlated case where  $h_{\max} = 50$ . For the compactly supported Gaussian and 4-Exponential, we set  $\rho_1 = 5$  and  $\sigma = 1$  therefore the support is  $[-5, 5]$ .

In each table, we are interested in comparing the smallest average RMSE of the first block (**boldface**) to the first column (naive) and the smallest one of the second block (**blue**). We observe that the combining estimation methods mostly outperform the best basic machine (random forest). In the uncorrelated case, the classical Gaussian and triangular kernel seem to be the outstanding ones. However in the correlated case, the proposed method with classical Gaussian kernel is the best method comparing to others. In addition, as we set its support to be around 5 times of its standard deviation which is significantly wide, we observe that the compactly supported Gaussian kernel does the job almost as good as the classical one in both cases with slightly larger standard errors. However, Cauchy kernel outperforms others in Model 10 of both cases (Table 2 and Table 3) while Gaussian kernel is the worst one in Table 4.

## 3.2 Real public datasets

In this part, we consider three public datasets which are available and easily accessible on the internet. The first dataset (**Abalone**, available at [Dua and Graff \(2017a\)](#)) contains 4177 rows and 9 columns of measurements of abalones observed in Tasmania, Australia. We are interested in predicting the age of each abalone through the number of the ring using its gender, size, weight, etc. The second dataset (**House**, available at [Kaggle \(2016\)](#)) comprises house sale prices for King County including Seattle. It contains homes sold between May 2014 and May 2015. The dataset consists of 21613

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<sup>2</sup>We use the relaxed version of [Biau et al. \(2016\)](#) with the weight given in equation (3).

Table 2: Average RMSEs of the uncorrelated case.

Model	Rid	Las	kNN	P-tr	RF	Naiv	Epan	Bi-wgt	Tri-wgt	Triang	Exp4	C-Gaus	Gaus	Cauchy
1	0.358	0.358	0.361	0.157	<b>0.142</b>	0.122	0.124	0.120	0.116	0.121	0.119	0.111	<b>0.096</b>	0.120
	(0.018)	(0.017)	(0.018)	(0.013)	(0.010)	(0.024)	(0.029)	(0.029)	(0.028)	(0.028)	(0.028)	(0.023)	(0.008)	(0.010)
2	0.876	0.876	0.894	0.833	<b>0.814</b>	0.839	0.836	0.836	0.836	0.836	0.837	0.835	0.835	<b>0.828</b>
	(0.056)	(0.056)	(0.054)	(0.058)	(0.057)	(0.059)	(0.060)	(0.061)	(0.060)	(0.060)	(0.061)	(0.060)	(0.060)	(0.058)
3	0.699	<b>0.635</b>	1.164	0.790	0.720	0.665	0.648	0.644	0.644	<b>0.640</b>	0.650	0.643	0.643	0.702
	(0.049)	(0.043)	(0.072)	(0.056)	(0.054)	(0.066)	(0.050)	(0.047)	(0.047)	(0.047)	(0.047)	(0.048)	(0.046)	(0.056)
4	2.279	2.107	2.957	1.662	<b>1.671</b>	1.611	1.581	1.574	1.567	1.566	1.590	1.566	<b>1.557</b>	1.661
	(0.130)	(0.116)	(0.181)	(0.130)	(0.115)	(0.176)	(0.151)	(0.144)	(0.140)	(0.144)	(0.146)	(0.141)	(0.134)	(0.130)
5	0.691	0.689	0.783	0.686	<b>0.625</b>	0.639	0.631	0.630	0.629	<b>0.616</b>	0.632	0.628	0.628	0.652
	(0.043)	(0.042)	(0.052)	(0.048)	(0.042)	(0.043)	(0.043)	(0.043)	(0.044)	(0.043)	(0.044)	(0.044)	(0.043)	(0.044)
6	0.879	<b>0.871</b>	1.270	1.413	1.037	0.972	0.896	0.887	0.873	<b>0.855</b>	0.910	0.869	0.860	0.977
	(0.063)	(0.061)	(0.082)	(0.101)	(0.077)	(0.171)	(0.124)	(0.117)	(0.093)	(0.099)	(0.116)	(0.091)	(0.063)	(0.071)
7	0.773	<b>0.606</b>	0.985	0.706	0.675	<b>0.695</b>	0.700	0.704	0.705	0.697	0.704	0.703	0.705	0.706
	(0.050)	(0.040)	(0.065)	(0.050)	(0.043)	(0.051)	(0.052)	(0.053)	(0.051)	(0.053)	(0.052)	(0.052)	(0.052)	(0.045)
8	0.342	<b>0.329</b>	0.424	0.365	0.331	0.315	0.314	0.314	0.314	<b>0.313</b>	0.315	0.314	<b>0.313</b>	0.316
	(0.017)	(0.016)	(0.017)	(0.038)	(0.019)	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)	(0.030)
9	1.614	<b>0.983</b>	1.783	1.088	1.009	1.193	1.289	1.277	1.266	1.232	1.287	1.141	<b>1.114</b>	1.158
	(0.111)	(0.062)	(0.120)	(0.093)	(0.077)	(0.256)	(0.190)	(0.197)	(0.192)	(0.214)	(0.182)	(0.156)	(0.146)	(0.101)
10	8.845	8.859	11.292	9.651	<b>8.831</b>	10.010	12.192	12.033	12.051	12.313	11.908	12.041	9.861	<b>9.255</b>
	(0.622)	(0.617)	(0.874)	(0.680)	(0.635)	(4.058)	(7.261)	(7.089)	(7.084)	(7.897)	(6.709)	(7.084)	(0.779)	(0.678)

Table 3: Average RMSEs of the correlated case.

Model	Rid	Las	kNN	P-tr	RF	Naiv	Epan	Bi-wgt	Tri-wgt	Triang	Exp4	C-Gaus	Gaus	Cauchy
1	1.392	1.394	1.354	<b>0.515</b>	0.593	0.536	0.531	0.515	0.509	0.511	0.529	0.507	<b>0.456</b>	0.612
	(0.214)	(0.213)	(0.204)	(0.125)	(0.185)	(0.193)	(0.193)	(0.191)	(0.192)	(0.192)	(0.196)	(0.194)	(0.144)	(0.186)
2	2.846	2.850	2.779	2.299	<b>2.015</b>	2.138	2.108	2.101	2.098	2.096	2.106	<b>2.093</b>	2.096	2.192
	(0.272)	(0.273)	(0.255)	(0.246)	(0.208)	(0.210)	(0.213)	(0.210)	(0.208)	(0.211)	(0.209)	(0.206)	(0.200)	(0.227)
3	2.205	2.072	2.726	1.703	<b>1.541</b>	1.653	1.548	1.525	1.511	1.514	1.548	1.515	<b>1.480</b>	1.939
	(0.459)	(0.453)	(0.515)	(0.374)	(0.493)	(0.383)	(0.371)	(0.373)	(0.369)	(0.366)	(0.369)	(0.376)	(0.367)	(0.510)
4	6.374	6.010	6.650	<b>2.765</b>	3.241	3.003	3.124	3.048	3.019	3.188	3.127	2.970	<b>2.588</b>	3.813
	(0.892)	(0.871)	(0.926)	(0.465)	(0.715)	(0.900)	(1.038)	(1.000)	(0.986)	(1.106)	(0.972)	(0.995)	(0.483)	(0.790)
5	2.443	2.422	3.024	<b>1.555</b>	1.761	1.717	1.510	1.460	1.460	1.443	1.545	1.441	<b>1.264</b>	2.272
	(0.720)	(0.722)	(0.846)	(0.647)	(0.787)	(0.917)	(0.970)	(0.970)	(0.984)	(1.011)	(0.927)	(0.918)	(0.560)	(0.931)
6	1.064	1.054	1.358	1.471	<b>0.980</b>	1.024	0.948	<b>0.898</b>	0.918	0.941	0.927	1.036	0.972	0.995
	(0.070)	(0.070)	(0.080)	(0.115)	(0.060)	(0.151)	(0.124)	(0.063)	(0.108)	(0.127)	(0.099)	(0.069)	(0.134)	(0.126)
7	1.996	1.751	2.217	1.758	<b>1.513</b>	1.552	1.504	1.499	1.490	1.477	1.509	1.489	<b>1.462</b>	1.702
	(0.247)	(0.251)	(0.252)	(0.258)	(0.243)	(0.285)	(0.294)	(0.294)	(0.292)	(0.266)	(0.290)	(0.292)	(0.254)	(0.265)
8	0.348	0.335	0.423	0.364	<b>0.324</b>	0.311	0.310	0.310	0.310	0.307	0.312	0.310	0.310	<b>0.306</b>
	(0.015)	(0.013)	(0.018)	(0.038)	(0.017)	(0.037)	(0.035)	(0.035)	(0.035)	(0.035)	(0.035)	(0.035)	(0.034)	(0.026)
9	5.697	4.734	6.104	3.316	<b>3.185</b>	3.590	3.660	3.564	3.550	3.661	3.600	3.463	<b>3.053</b>	4.349
	(0.941)	(0.826)	(0.946)	(0.666)	(0.828)	(0.788)	(1.113)	(1.134)	(1.131)	(1.150)	(1.081)	(0.991)	(0.783)	(0.937)

Table 4: Average RMSEs of model 10 of the correlated case (1unit = 10<sup>8</sup>).

Model	Rid	Las	kNN	P-tr	RF	Naiv	Epan	Bi-wgt	Tri-wgt	Triang	Exp4	C-Gaus	Gaus	Cauchy
10	3.272	3.270	<b>1.269</b>	1.854	1.287	1.285	1.334	1.338	1.339	1.251	1.495	1.567	2.038	<b>1.221</b>
	(5.044)	(5.044)	(1.031)	(1.568)	(1.005)	(0.986)	(0.937)	(0.937)	(0.936)	(0.961)	(0.931)	(0.932)	(1.429)	(0.965)

rows of houses and 21 columns of characteristics of each house including ID, Year of sale, Size, Location, etc. In this case, we want to predict the price of each house using all of its quantitative characteristics.

Notice that Model 6 and 8 of the simulation part are about predicting labels of response variable. Analogously, the last dataset (**Wine**, see [Dua and Graff \(2017b\)](#); [Cortez et al. \(2009\)](#)), which was also considered in [Biau et al. \(2016\)](#), containing 1599 rows of different types of wines and 12 columns corresponding to different substances of red wines including the amount of different types of acids, sugar, chlorides, PH, etc. The variable of interest is *quality* which scales from 3 to 8 where 8 represents the best quality. We aim at predicting the quality of each wine, which is treated as a continuous variable, using all other criteria.

With the same set of parameters as in the previous subsection, the five basic machines are Ridge, LASSO, 10-NN, Pruned tree and Random Forest regression. We combine the five basic machines using the proposed method with all the introduced kernel functions. The results obtained from 50 independent runs, evaluated on 20%-testing data of the three public datasets, are provided in Table 5 and 6 below. Random forest is the outstanding method among all the basic machines, and Gaussian kernels (classical and compactly supported) are the best ones among all combining estimation methods in this case. We observe that the performances of the proposed method with classical Gaussian kernel are quite close to the ones of random forest on **House** and **Wine** datasets. Moreover, the compactly supported and classical Gaussian kernels beat random forest on **Abalone** dataset.

### 3.3 Real private dataset: Air compressor

The result presented in this subsection comes from a private dataset provided by [Cadet et al. \(2005\)](#). This dataset contains six columns corresponding to the six variables including *Air temperature*, *Input Pressure*, *Output Pressure*, *Flow*, *Water Temperature* and *Power Consumption* along with 2026 rows of hourly observations of these measurements of an air compressor machine. The goal is to predict Power Consumption of this machine using the remaining ones as explanatory variables. We run our method over 100 random partitions of 80%-training sample. The average RMSEs and the corresponding standard errors evaluated on the remaining 20%-testing data are presented in Table 7. With the same set of parameters as in the previous part, the performances of the proposed method are very satisfactory. The classical

Table 5: Average RMSEs of House dataset (1 unit =  $10^4$ ).

Data	Rid	Las	kNN	P-tr	RF	Naiv	Epan	Bi-wgt	Tri-wgt	Triang	Exp4	C-Gaus	Gaus	Cauchy
House	24.087 (1.366)	24.315 (1.370)	24.317 (1.886)	25.902 (1.326)	20.245 (1.056)	23.143 (2.234)	23.733 (1.849)	23.257 (1.971)	22.948 (1.976)	22.774 (2.068)	22.908 (1.925)	22.225 (1.975)	21.326 (1.318)	27.573 (2.461)

Table 6: Average RMSEs of Wine and Abalone datasets.

Data	Rid	Las	kNN	P-tr	RF	Naiv	Epan	Bi-wgt	Tri-wgt	Triang	Exp4	C-Gaus	Gaus	Cauchy
Wine	0.660 (0.025)	0.660 (0.025)	0.749 (0.034)	0.684 (0.026)	0.578 (0.027)	0.646 (0.056)	0.638 (0.052)	0.634 (0.051)	0.633 (0.052)	0.631 (0.041)	0.630 (0.044)	0.623 (0.038)	0.614 (0.026)	0.632 (0.028)
Abalone	2.246 (0.092)	2.246 (0.092)	2.177 (0.067)	2.444 (0.084)	2.163 (0.068)	2.211 (0.087)	2.178 (0.091)	2.174 (0.070)	2.169 (0.087)	2.171 (0.088)	2.174 (0.087)	2.151 (0.087)	2.153 (0.070)	2.246 (0.084)

Table 7: Average RMSEs of Air Compressor machine.

Data	Rid	Las	kNN	P-tr	RF	Naiv	Epan	Bi-wgt	Tri-wgt	Triang	Exp4	C-Gaus	Gaus	Cauchy
Air	178.885 (4.298)	178.885 (4.298)	303.399 (10.868)	411.002 (22.514)	211.525 (10.688)	171.739 (9.955)	159.192 (9.272)	154.308 (7.559)	151.657 (6.674)	154.020 (6.627)	157.000 (9.219)	150.918 (6.421)	147.298 (5.765)	193.919 (10.368)

Gaussian kernel is the outstanding method, followed by the compactly supported one and triangular kernel. The proposed method outperforms the best basic machine (Lasso) for almost all kernels except for Cauchy case.

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