



**HAL**  
open science

# Numerical algorithm to solve generalized fractional pantograph equations with variable coefficients based on shifted Chebyshev polynomials

Li-Ping Wang, Yi-Ming Chen, Da-Yan Liu, Driss Boutat

## ► To cite this version:

Li-Ping Wang, Yi-Ming Chen, Da-Yan Liu, Driss Boutat. Numerical algorithm to solve generalized fractional pantograph equations with variable coefficients based on shifted Chebyshev polynomials. *International Journal of Computer Mathematics*, 2019, 96 (12), pp.2487-2510. 10.1080/00207160.2019.1573992 . hal-02884254

**HAL Id: hal-02884254**

**<https://hal.science/hal-02884254v1>**

Submitted on 4 May 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Numerical solutions of generalized fractional pantograph equations with variable coefficients using shifted Chebyshev polynomials

Li-Ping Wang<sup>a</sup>, Yi-Ming Chen<sup>a,b,c</sup>, Da-Yan Liu<sup>d,\*</sup>, Driss Boutat<sup>d</sup>

<sup>a</sup>College of Sciences, Yanshan University, Qinhuangdao, Hebei, P.R.China, 066004.

<sup>b</sup>LE STUDIUM RESEARCH PROFESSOR, Loire Valley Institute for Advanced Studies, Orléans, France

<sup>c</sup>PRISME (INSA-Institut National des sciences appliquées) -88, Boulevard Lahitolle, 18000 Bourges, France

<sup>d</sup>INSA Centre Val de Loire, Université d'Orléans, PRISME EA 4229, Bourges Cedex 18022, France

---

## Abstract

In this paper, an efficient numerical technique based on the shifted Chebyshev polynomials (SCPs) is established to obtain numerical solutions of generalized fractional pantograph equations with variable coefficients. These polynomials are orthogonal and have compact support on  $[0, L]$ . We use these polynomials to approximate the unknown function. Using the properties of the SCPs, we derive the generalized pantograph operational matrix of SCPs and the one of fractional-order differentiation. Then the original problems can be transformed to a system of algebraic equations based on these matrices. By solving these algebraic equations, we can obtain numerical solutions. In addition, we investigate the error analysis and introduce the process of error correction for improving the precision of numerical solutions. Lastly, by giving some examples and comparing with other existing methods, the validity and efficiency of our method is demonstrated.

*Keywords:* Shifted Chebyshev polynomials, Operational matrix, Generalized fractional pantograph equations, Numerical solution, Error analysis

---

\*Corresponding author

Email address: [dayan.liu@insa-cvl.fr](mailto:dayan.liu@insa-cvl.fr) (Da-Yan Liu)

URL: [Wangliping2500@163.com](mailto:Wangliping2500@163.com) (Li-Ping Wang), [chenym@ysu.edu.cn](mailto:chenym@ysu.edu.cn) (Yi-Ming Chen), [driss.boutat@insa-cvl.fr](mailto:driss.boutat@insa-cvl.fr) (Driss Boutat)

---

## 1. Introduction

Fractional calculus is a branch of calculus theory, which makes calculus theory more perfect. In recent decades, fractional calculus have been widely used in various areas, such as viscoelasticity [1, 2], economics [3], control theory [4, 5], and fractals dynamics [6]. One of the interesting research topics is the design of fractional differentiators [7] to compute fractional differentials of unknown signals in a noisy environment. With the application of fractional differential equations in more and more scientific fields, the study of numerical calculations of the differential equations of fractional order is particularly important. At present, the majority of scholars have studied different kind of vigorous numerical methods to obtained an approximate solution of fractional differential equations. These methods include Chebyshev collocation method [8], Laplace transform method [9], differential transform method [10, 11], Adomian decomposition method [12], Legendre operational matrix [13], and CAS wavelet method [14], etc.

Delay differential equations have many applications in different fields, such as biological, industrial, electronic, chemical and transportation systems [15, 16, 17]. To obtain the numerical solutions of delay differential equations, Many researchers have studied different kind of vigorous techniques [18]. The functional differential equations with proportional delay are generally called to pantograph equations or generalized pantograph equations. As the one of the most important types of delay differential equations, the pantograph equation or the generalized pantograph equation can explaining various physical phenomena. And they are used in many fields. In recent years, there have been many numerical methods for solving pantograph differential equations or generalized pantograph equations of integer order, such as Chebyshev polynomials [19], Bernoulli polynomials [20], variational iteration method [21], etc. Further, [22] introduces the stability properties of many numerical techniques for nonlinear generalized pantograph equations.

30 The fractional delay differential equation is a generalization of the delay differential equation to arbitrary non-integer order. Fractional delay differential equations are adapted to many fields, such as hydraulic networks, automatic control, long transmission lines, economy and biology [23]. The numerical calculation of fractional delay differential equations has also attracted the attention  
35 of many scholars. Because the fractional delay differential equations cannot be analytically solved, different numerical methods [24, 25, 26] have been devoted to obtain the approximate solutions. Sherif et al. [27] considered the Spline functions to solve fractional delay differential equation. Authors of [28] investigated modified Laguerre wavelets method. Modified Chebyshev wavelet methods and  
40 a operational matrix based on Bernoulli wavelets are utilized in [29, 30]. However, there are few scholars that pay attention to study the numerical methods of fractional pantograph delay differential equations. From these works we can mention, Y Yang and Y Huang [32] have studied the existence of solutions of nonlinear fractional pantograph equations with the order of the derivative is in  
45  $[0, 1]$ ; Using spectral-collocation methods, Yang and Huang [31] obtained the approximate solution for fractional pantograph delay-integro-differential equations; the approximate solution of fractional pantograph differential equations can be obtained by using the explicit formula of the generalized fractional-order Bernoulli wavelet in [33].

50 The polynomial approximation theory is an important branch of the function approximation theory. As the name suggests, polynomials are used to approximate a function whose analytical form is more complex or whose analytical form is unknown. In general, the polynomial has many advantages, such as its structure is clear, its calculation is simple and it is relatively easy  
55 to integral and derivative. For some complex problems, applying polynomials to approximate function , and then studying the laws of actual problems, the problems can be simplified. At present, polynomial approximation theory has been widely used in different fields, such as numerical approximation theory, engineering calculation, and practical life. In this paper, based on the  
60 properties of the shifted Chebyshev polynomials, we derive SCPs generalized

pantograph operational matrix. And with the aid of the operational matrix of fractional differentiation of SCPs, generalized pantograph operational matrix of fractional-order differentiation is obtained. We combine polynomial approximation theory and operational matrix to solve the following generalized fractional  
65 pantograph equation with variable coefficients

$${}^c D^\beta u(t) = b(t) u(t) + \sum_{j=0}^J \sum_{n=0}^{r-1} v_{j,n}(t) {}^c D^{\alpha_n} u(q_{j,n}t - r_{j,n}) + g(t), \quad 0 \leq t \leq L, \quad (1)$$

subject to the initial conditions

$$u^{(n)}(0) = d_n, \quad n = 0, 1, \dots, r-1, \quad (2)$$

where  $d_n$ ,  $q_{j,n}$  and  $r_{j,n}$  are real or complex coefficients,  $r-1 < \alpha \leq r$ ,  $0 < \alpha_0 < \alpha_1 < \dots < \alpha_{r-1} < \beta$ , while  $b(t)$ ,  $v_{j,n}(t)$  and  $g(t)$  are continuous functions in the interval  $[0, L]$ ,  ${}^c D^\beta$  and  ${}^c D^{\alpha_n}$  denote fractional derivatives in the Caputo's  
70 sense.

The rest of the paper is organized as follows: Section 2 introduces some mathematical preliminaries of fractional calculus. In Section 3, we review the basic definitions of shifted Chebyshev polynomials and discuss the polynomial approximation theory. In Section 4, we derive the SCPs generalized pantograph  
75 operational matrix and the one of fractional-order differentiation. In Section 5, we apply the proposed method to solve the generalized fractional pantograph equations. The error correction and error analysis are given in Section 6. In Section 7, the proposed approach is tested through several numerical examples. Finally, a conclusion is given in Section 8.

## 80 2. Basic definitions of fractional calculus

In this section, we review the necessary definitions and preliminaries of fractional calculus theory that will be used in this article.

**Definition 1.** *The Riemann-Liouville fractional integral operator of  $f(t)$  is defined as*

$${}^{RL}I^\beta f(t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t (t-T)^{\beta-1} f(T) dT, & \beta > 0, t > 0, \\ f(t), & \beta = 0. \end{cases} \quad (3)$$

The Riemann-Liouville fractional differential operator of order  $\beta$  is derived by the definition of the Riemann-Liouville fractional integral operator

$${}^{RL}D^\beta f(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t \frac{f(T)}{(t-T)^{\beta-n+1}} dT, & \beta > 0, n-1 \leq \beta < n, \\ \frac{d^n f(t)}{dt^n}, & \beta = n, t > 0. \end{cases} \quad (4)$$

**Definition 2.** The fractional differential operator of order  $\beta$  in the Caputo sense is defined as

$${}^cD^\beta f(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^{(n)}(T)}{(t-T)^{\beta-n+1}} dT, & n-1 \leq \beta < n, \\ \frac{d^n f(t)}{dt^n}, & \beta = n, t > 0. \end{cases} \quad (5)$$

For the Caputo differential operator, we have

$${}^cD_t^\beta t^m = \begin{cases} 0, & \text{for } m \in \mathbb{N}_0 \text{ and } m < [\beta], \\ \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} t^{m-\beta}, & \text{for } m \in \mathbb{N}_0 \text{ and } m \geq [\beta] \text{ or } m \notin \mathbb{N}_0 \text{ and } m > \lfloor \beta \rfloor. \end{cases} \quad (6)$$

where  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ .

### 95 3. Shifted Chebyshev polynomials

#### 3.1. Properties of the shifted Chebyshev polynomials

The well-known Chebyshev polynomials are defined on the interval  $[-1, 1]$ , and are derived by orthogonalizing the sequence  $\{1, t, \dots, t^n, \dots\}$ .

The specific form of the Chebyshev polynomials can be obtained by the following recurrence relation

$$T_{i+1}(z) = 2(2z-1)T_i(z) - T_{i-1}(z), \quad i = 1, 2, \dots$$

where  $T_0(z) = 1$  and  $T_1(z) = z$ . To use the Chebyshev polynomials on the extended interval  $[0, L]$ , it is necessary to shift the defining domain  $[-1, 1]$ . The shifted Chebyshev polynomials on  $[0, L]$  can easily be derived by introducing the change of variables  $z = \frac{2t}{L} - 1$ . We can denote the shifted Chebyshev polynomials  $T_i\left(\frac{2t}{L} - 1\right)$  by  $H_{L,i}(t)$ , then  $H_{L,i}(t)$  can be determined with the aid of the following recurrence formula

$$H_{L,i+1}(t) = 2\left(\frac{2t}{L} - 1\right)H_{L,i}(t) - H_{L,i-1}(t), \quad i = 1, 2, \dots$$

where  $H_{L,0}(t) = 1$  and  $H_{L,1}(t) = \frac{2t}{L} - 1$ . The analytical form of the shifted Chebyshev polynomials  $H_{L,i}(t)$  of degree  $i$  is given by

$$H_{L,i}(t) = T_i\left(\frac{2t}{L} - 1\right) = i \sum_{k=0}^i (-1)^{i-k} \frac{(i+k-1)!}{(2k)!(i-k)!} \frac{(2)^{2k}}{L^k} (t)^k, \quad i = 1, 2, \dots \quad (7)$$

where  $H_{L,i}(0) = (-1)^i$  and  $H_{L,i}(L) = 1$ .

The shifted Chebyshev polynomials satisfy the following orthogonality relation

$$\int_0^L H_{L,j}(t) H_{L,k}(t) \omega_L(t) dt = h_k, \quad (8)$$

where the weight function  $\omega_L(t) = \frac{1}{\sqrt{Lt-t^2}}$  and  $h_k = \begin{cases} \frac{b_k}{2}\pi, & k = j, \\ 0, & k \neq j, \end{cases} b_0 = 2, b_k = 1, k \geq 1$ .

### 3.2. Function approximation

A function  $u(t) \in L^2([0, L])$  can be expanded in terms of the shifted Chebyshev polynomials as follows

$$u(t) = \sum_{i=0}^{\infty} c_i H_{L,i}(t), \quad (9)$$

where the coefficients  $c_i$  are obtained by

$$c_i = \frac{1}{h_i} \int_0^L u(t) H_{L,i}(t) \omega_L(t) dt, \quad i = 0, 1, 2, \dots$$

If we consider truncated series in Eq. (9), we can get

$$u(t) \approx \sum_{i=0}^m c_i H_{L,i}(t) = C^T \Phi_m(t), \quad (10)$$

where

$$C = [c_0, c_1, \dots, c_m]^T, \Phi_m(t) = [H_{L,0}(t), H_{L,1}(t), \dots, H_{L,m}(t)]^T. \quad (11)$$

#### 120 4. SCPs operational matrix for solving the pantograph equations

In this part, we derive the necessary SCPs generalized pantograph operational matrix and the fractional generalized pantograph operational matrix .

From Eq. (7),  $\Phi_m(t)$  can be denoted by the product of two matrices

$$\Phi_m(t) = AZ_m(t), \quad (12)$$

where

$$Z_m(t) = [1, t, \dots, t^m]^T.$$

125 The matrix  $A$  is SCPs coefficient matrix and we assume each item of the matrix  $A$  can be write as follows

$$A = \begin{bmatrix} P_{0,0} & 0 & \dots & 0 \\ P_{1,0} & P_{1,1} & \dots & 0 \\ \dots & \dots & \ddots & \vdots \\ P_{m,0} & P_{m,1} & \dots & P_{m,m} \end{bmatrix}, \quad (13)$$

where

$$\begin{cases} P_{0,0} = 1, \\ P_{i,j} = 2 \left( \frac{2}{L} P_{i-1,j-1} - P_{i-1,j} \right) - P_{i-2,j}, \\ P_{i,j} = 0, \quad \text{for } i < j \text{ or } i < 0 \text{ or } j < 0. \end{cases}$$

According to [34], inverse matrix of the coefficient matrix  $A$  can be expressed as follows

$$A^{-1} = \begin{bmatrix} P_{0,0}^{-1} & 0 & 0 & \dots & 0 & 0 \\ P_{0,0}^{-1} a_{1,2} & P_{1,1}^{-1} & 0 & \dots & 0 & 0 \\ P_{0,0}^{-1} a_{1,3} & P_{1,1}^{-1} a_{2,3} & P_{2,2}^{-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{0,0}^{-1} a_{1,m} & P_{1,1}^{-1} a_{2,m} & P_{2,2}^{-1} a_{3,m} & \dots & P_{m-1,m-1}^{-1} & 0 \\ P_{0,0}^{-1} a_{1,m+1} & P_{1,1}^{-1} a_{2,m+1} & P_{2,2}^{-1} a_{3,m+1} & \dots & P_{m-1,m-1}^{-1} a_{m,m+1} & P_{m,m}^{-1} \end{bmatrix}, \quad (14)$$



130 where

$$\begin{cases} a_{i,i+1} = -P_{i,i}^{-1}P_{i,i-1}, & i = 1, 2, \dots, m, \\ a_{i,j} = -P_{j-1,j-1}^{-1} \left( P_{j-1,i-1} + \sum_{i < k < j} a_{i,k}P_{j-1,k-1} \right), & i = 1, 2, \dots, m-1, j = 3, 4, \dots, m+1. \end{cases}$$

#### 4.1. SCPs generalized pantograph operational matrix

The shifted Chebyshev vector with delay parameter  $r$  ( $0 < r < 1$ ) and pantograph coefficient  $q$  ( $0 < q < 1$ ) is given as follows

$$\Phi_m(qt-r) = [H_{L,0}(qt-r), H_{L,1}(qt-r), \dots, H_{L,i}(qt-r), \dots, H_{L,m}(qt-r)]^T. \quad (15)$$

135 **Theorem 1.** Let  $\Phi_m(qt)$  be the special case of  $r = 0$  in Eq. (15) and suppose  $0 < q < 1$ , then

$$\Phi_m(qt) = F\Phi_m(t), \quad (16)$$

where the matrix  $F$  is called the pantograph operational matrix of SCPs, and it is defined as follows

$$F = \begin{bmatrix} f_{0,0} & 0 & \cdots & 0 & 0 \\ f_{1,0} & f_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{m-1,0} & f_{m-1,1} & \cdots & f_{m-1,m-1} & 0 \\ f_{m,0} & f_{m,1} & \cdots & f_{m,m-1} & f_{m,m} \end{bmatrix}$$

where

$$f_{i,j} = \begin{cases} P_{j,j}^{-1} \left( P_{i,j}q^j + \sum_{l=j+1}^i P_{i,l}q^l a_{j+1,l+1} \right), & i \neq j \\ q^i, & i = j \end{cases}$$

$$i = 0, 1, \dots, m.$$

140

**Proof.** The  $H_{L,i}(qt)$ ,  $i = 0, 1, \dots, m$  must be expanded in terms of  $(H_{L,j}(t))_{j=0,1,\dots,i}$ .

Let

$$H_{L,i}(qt) = \sum_{j=0}^i f_{i,j}H_{L,j}(t).$$

And then we can get

$$\Phi_m(qt) = \begin{bmatrix} f_{0,0} & 0 & \cdots & 0 & 0 \\ f_{1,0} & f_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{m-1,0} & f_{m-1,1} & \cdots & f_{m-1,m-1} & 0 \\ f_{m,0} & f_{m,1} & \cdots & f_{m,m-1} & f_{m,m} \end{bmatrix} \Phi_m(t) = F\Phi_m(t), \quad (17)$$

where

$$F = \begin{bmatrix} f_{0,0} & 0 & \cdots & 0 & 0 \\ f_{1,0} & f_{1,1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{m-1,0} & f_{m-1,1} & \cdots & f_{m-1,m-1} & 0 \\ f_{m,0} & f_{m,1} & \cdots & f_{m,m-1} & f_{m,m} \end{bmatrix}.$$

145 According to Eq. (12), we get

$$\begin{aligned} \Phi_m(qt) &= AZ_m(qt) = A \begin{bmatrix} 1 \\ qt \\ q^2t^2 \\ \vdots \\ q^mt^m \end{bmatrix} \\ &= A \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q & 0 & \cdots & 0 \\ 0 & 0 & q^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q^m \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix} \\ &= A \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q & 0 & \cdots & 0 \\ 0 & 0 & q^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q^m \end{bmatrix} A^{-1}\Phi_m(t). \end{aligned}$$

By substituting the formula of matrix  $A^{-1}$  into the above equation, we obtain

$$\Phi_m(qt) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ P_{0,0}^{-1}(P_{1,0} + P_{1,1}qa_{1,2}) & q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_{0,0}^{-1}\left(P_{m,0} + \sum_{i=1}^m P_{m,i}q^i a_{1,i+1}\right) & P_{1,1}^{-1}\left(P_{m,1}q + \sum_{i=2}^m P_{m,i}q^i a_{2,i+1}\right) & \cdots & q^m \end{bmatrix} \Phi_m(t),$$

Combining the above equation with Eq. (17), we can get the recurrence formula of  $f_{i,j}$

$$f_{i,j} = \begin{cases} P_{j,j}^{-1} \left( P_{i,j} q^j + \sum_{l=j+1}^i P_{i,l} q^l a_{j+1,l+1} \right), & i \neq j \\ q^i, & i = j \end{cases}$$

Theorem. 1 is proved . ■

150 **Theorem 2.** Consider  $\Phi_m(qt - r)$  is the shifted Chebyshev vector defined in Eq. (15) and suppose  $0 < q < 1$ ,  $0 < r < 1$ , then

$$\Phi_m(qt - r) = W \Phi_m(t), \quad (18)$$

where the matrix  $W$  is called the generalized pantograph operational matrix of SCPs, and it is defined as follows

$$W = [w_0, w_1, \dots, w_i, \dots, w_m]^T, \quad i = 0, 1, \dots, m,$$

where

$$w_i = \sum_{j=0}^i P_{i,j} N_j,$$

155

$$N_j = \sum_{k=0}^j \binom{j}{k} (-r)^{j-k} [\alpha_0, \alpha_1, \dots, \alpha_n, \dots, \alpha_k, 0, \dots, 0],$$

where

$$\alpha_n = \sum_{l=n}^k P_{l,l}^{-1} a_{l+1,k+1} f_{l,n},$$

where

$$f_{l,n} = \begin{cases} P_{n,n}^{-1} \left( P_{l,n} q^n + \sum_{s=n+1}^l P_{l,s} q^s a_{n+1,s+1} \right), & l \neq n, \\ q^l, & l = n. \end{cases}$$

**Proof.** By Eq. (12), we have

$$\Phi_m (qt - r) = AZ_m (qt - r) = A \begin{bmatrix} 1 \\ qt - r \\ (qt - r)^2 \\ \vdots \\ (qt - r)^i \\ \vdots \\ (qt - r)^m \end{bmatrix}. \quad (19)$$

We expand the formula  $(qt - r)^i$  in Eq. (19)

$$(qt - r)^i = \sum_{k=0}^i \binom{i}{k} (-r)^{i-k} (qt)^k. \quad (20)$$

160 And by Eq. (12), we get

$$Z_m (qt) = \begin{bmatrix} 1 \\ qt \\ (qt)^2 \\ \vdots \\ (qt)^i \\ \vdots \\ (qt)^m \end{bmatrix} = A^{-1} \Phi_m (qt),$$

therefore

$$(qt)^i = A_{[i+1]}^{-1} \Phi_m (qt), \quad (21)$$

where  $A_{[i+1]}^{-1}$  is the  $(i + 1)^{th}$  row of  $A^{-1}$ ,  $i = 0, 1, \dots, m$ .

By substituting Eqs. (16) and (21) into Eq. (20), we obtain

$$(qt - r)^i = \sum_{k=0}^i \binom{i}{k} (-r)^{i-k} A_{[k+1]}^{-1} F \Phi_m (t).$$

Using the formula of  $A^{-1}$  and Eq. (16),  $A_{[k+1]}^{-1} F$  can be written as follows

$$A_{[k+1]}^{-1} F = [\alpha_0, \alpha_1, \dots, \alpha_n, \dots, \alpha_k, 0, \dots, 0],$$

165 where

$$\alpha_n = \sum_{l=n}^k P_{l,l}^{-1} a_{l+1,k+1} f_{l,n}.$$

In conclude

$$\Phi_m(qt - r) = A \begin{bmatrix} N_0 \\ N_1 \\ \vdots \\ N_i \\ \vdots \\ N_m \end{bmatrix} \Phi_m(t) = W \Phi_m(t),$$

where

$$N_i = \sum_{k=0}^i \binom{i}{k} (-r)^{i-k} [\alpha_0, \alpha_1, \dots, \alpha_n, \dots, \alpha_k, 0, \dots, 0]. \quad (22)$$

$$W = A \begin{bmatrix} N_0 \\ N_1 \\ \vdots \\ N_i \\ \vdots \\ N_m \end{bmatrix} = \begin{bmatrix} P_{0,0}N_0 \\ P_{1,0}N_0 + P_{1,1}N_1 \\ \vdots \\ P_{i,0}N_0 + P_{i,1}N_1 + \dots + P_{i,i}N_i \\ \vdots \\ P_{m,0}N_0 + P_{m,1}N_1 + \dots + P_{m,m}N_m \end{bmatrix}.$$

Let  $W = [w_0, w_1, \dots, w_i, \dots, w_m]^T$ , hence the precise expression of  $w_i$  can be  
170 concluded as follows

$$w_i = \sum_{j=0}^i P_{i,j} N_j. \quad (23)$$

By substituting Eq. (22) into Eq. (23), Theorem. 2 is proved. ■

#### 4.2. SCPs operational matrix of derivative

In order to build the SCPs operational matrix of derivative, the differentiation of vector  $\Phi_m(t)$  can be expressed by

$$\Phi_m^{(1)}(t) = P^{(1)} \Phi_m(t), \quad (24)$$

175 where  $P^{(1)}$  is called the  $(m+1) \times (m+1)$  SCPs operational matrix of derivative.

According to Eq. (12), we can get

$$\Phi_m^{(1)}(t) = A \begin{bmatrix} 0 \\ 1 \\ \vdots \\ mt^{m-1} \end{bmatrix} = AV_{(m+1) \times m} Z_m^*(t), \quad (25)$$

where

$$V_{(m+1) \times m} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m \end{bmatrix},$$

$$Z_m^*(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{m-2} \\ t^{m-1} \end{bmatrix}.$$

We now expand vector  $Z_m^*(t)$  in terms of  $\Phi_m(t)$ . From Eq. (12), we have

$$Z_m^*(t) = B^* \Phi_m(t), \quad (26)$$

180 where

$$B^* = \begin{bmatrix} A_{[1]}^{-1} \\ A_{[2]}^{-1} \\ \vdots \\ A_{[m]}^{-1} \end{bmatrix},$$

$A_{[k]}^{-1}$  is the  $k^{th}$  row of  $A^{-1}$ ,  $k = 1, 2, \dots, m$ .

Then Eq. (25) can be rewritten as

$$\Phi_m^{(1)}(t) = AV_{(m+1) \times m} B^* \Phi_m(t). \quad (27)$$

Therefore we have the operational matrix of derivative as

$$P^{(1)} = AV_{(m+1) \times m} B^*.$$

Further, we can get

$$\Phi_m^{(n)}(t) = \left(P^{(1)}\right)^n \Phi_m(t), \quad n = 1, 2, \dots \quad (28)$$

185 When  $n = 1$ , from Eq. (24), we get

$$\Phi_m^{(1)}(t) = P^{(1)} \Phi_m(t).$$

Suppose Eq. (28) is correct, when  $n = s$ . Then we obtain

$$\Phi_m^{(s)}(t) = \left(P^{(1)}\right)^s \Phi_m(t).$$

Thus, when  $n = s + 1$ , we have

$$\begin{aligned} \Phi_m^{(s+1)}(t) &= \frac{\partial^s}{\partial t^s} \left( \frac{\partial \Phi_m(t)}{\partial t} \right) = P^{(1)} \frac{\partial^s}{\partial t^s} \Phi_m(t) \\ &= P^{(1)} \left(P^{(1)}\right)^s \Phi_m(t) = \left(P^{(1)}\right)^{s+1} \Phi_m(t) \end{aligned}$$

For any integer  $s$ , the Eq. (28) holds.

Therefore, Eq. (28) can be proved.

190 *4.3. SCPs generalized pantograph operational matrix of fractional-order differentiation*

In order to build the operational matrix of fractional-order differentiation of SCPs. Let

$${}^c D^\beta \Phi_m(t) = P^\beta(t) \Phi_m(t), \quad \beta > 0, \quad (29)$$

where  $\Phi_m(t)$  is the shifted Chebyshev vector defined in Eq. (10) and the matrix

195  $P^\beta$  is called the SCPs operational matrix of fractional derivatives.

**Theorem 3.** *Suppose  $P^\beta$  is the SCPs operational matrix of Caputo fractional-*

order differentiation of order  $\beta > 0$ , then the elements of  $P^\beta$  are given as follows

$$P^\beta(t) = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ S_\beta([\beta], 0) & \cdots & S_\beta([\beta], [\beta]) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ S_\beta(i, 0) & \cdots & S_\beta(i, [\beta]) & \cdots & S_\beta(i, i) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ S_\beta(m, 0) & \cdots & S_\beta(m, [\beta]) & \cdots & S_\beta(m, i) & \cdots & S_\beta(m, m) \end{bmatrix},$$

where

$$S_\beta(i, j) = \sum_{k=\lceil\beta\rceil}^i t^{-\beta} P_{j,j}^{-1} a_{j+1,k+1} P_{i,k} \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}, \quad i = \lceil\beta\rceil, \lceil\beta\rceil + 1, \dots, m.$$

200 **Proof.** From Eq. (12), we get

$${}^c D^\beta \Phi_m(t) = A {}^c D^\beta Z_m(t) = A {}^c D^\beta \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix}. \quad (30)$$

Using Eq. (6), we can derive  ${}^c D^\beta Z_m(t)$  in Eq. (30) as

$${}^c D^\beta Z_m(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{\Gamma([\beta]+1)}{\Gamma([\beta]+1-\beta)} t^{[\beta]-\beta} \\ \vdots \\ \frac{\Gamma(i+1)}{\Gamma(i+1-\beta)} t^{i-\beta} \\ \vdots \\ \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} t^{m-\beta} \end{bmatrix}, \quad i = \lceil\beta\rceil, \lceil\beta\rceil + 1, \dots, m. \quad (31)$$



Define the  $(m+1) \times (m+1)$  matrix  $V_{(m+1) \times (m+1)}^*(t)$  as

$$V_{(m+1) \times (m+1)}^*(t) = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\Gamma([\beta]+1)}{\Gamma([\beta]+1-\beta)} t^{-\beta} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{\Gamma(i+1)}{\Gamma(i+1-\beta)} t^{-\beta} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} t^{-\beta} \end{bmatrix}.$$

Eq. (31) may be restated as

$${}^c D^\beta Z_m(t) = V_{(m+1) \times (m+1)}^*(t) Z_m(t). \quad (32)$$

Using Eq. (12), Eq. (28) can be rewritten as

$${}^c D^\beta Z_m(t) = V_{(m+1) \times (m+1)}^*(t) A^{-1} \Phi_m(t).$$

205 Therefore, we have

$${}^c D^\beta \Phi_m(t) = AV_{(m+1) \times (m+1)}^*(t) A^{-1} \Phi_m(t) = P^\beta(t) \Phi_m(t) \quad (33)$$

Substituting the formulas of  $A$  and  $A^{-1}$  into Eq. (33), we get

$$P^\beta(t) = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ S_\beta([\beta], 0) & \cdots & S_\beta([\beta], [\beta]) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ S_\beta(i, 0) & \cdots & S_\beta(i, [\beta]) & \cdots & S_\beta(i, i) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ S_\beta(m, 0) & \cdots & S_\beta(m, [\beta]) & \cdots & S_\beta(m, i) & \cdots & S_\beta(m, m) \end{bmatrix},$$

where

$$S_\beta(i, j) = \sum_{k=[\beta]}^i t^{-\beta} P_{j,j}^{-1} a_{j+1, k+1} P_{i,k} \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}.$$

Theorem. 3 is proved. ■

**Theorem 4.** Let  $\Phi_m(qt-r)$  be the shifted Chebyshev vector defined in Eq. (15) and suppose  $0 < q < 1$ ,  $0 < r < 1$ , then

$${}^c D^\beta \Phi_m(qt-r) = K^\beta \Phi_m(t), \quad (34)$$

where the matrix  $K^\beta$  is called the generalized pantograph operational matrix of fractional-order differentiation, and the elements are given

$$K^\beta = \left[ 0, 0, \dots, 0, \tau_{\lceil \beta \rceil}, \dots, \tau_i, \dots, \tau_m \right]^\top, \quad (35)$$

where

$$\tau_i = \sum_{j=0}^i x_\beta(i, j) \omega_j, \quad i = \lceil \beta \rceil, \lceil \beta \rceil + 1, \dots, m,$$

where

$$x_\beta(i, j) = \sum_{k=\lceil \beta \rceil}^i (qt-r)^{-\beta} P_{j,j}^{-1} a_{j+1, k+1} P_{i,k} \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)},$$

The formula of  $\omega_j$  can be represented as Eq. (18).

**Proof.** By Eqs. (18) and (29), we have

$${}^c D^\beta \Phi_m(qt-r) = P^\beta (qt-r) \Phi_m(qt-r) = P^\beta (qt-r) W \Phi_m(t) = K^\beta \Phi_m(t), \quad (36)$$

where  $K^\beta = P^\beta (qt-r) W$ .

According to the formula of  $P^\beta(t)$ , we can derive the expression of  $P^\beta(qt-r)$

$$P^\beta(qt-r) = \begin{bmatrix} 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ x_\beta(\lceil \beta \rceil, 0) & \dots & x_\beta(\lceil \beta \rceil, \lceil \beta \rceil) & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_\beta(i, 0) & \dots & x_\beta(i, \lceil \beta \rceil) & \dots & x_\beta(i, i) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_\beta(m, 0) & \dots & x_\beta(m, \lceil \beta \rceil) & \dots & x_\beta(m, i) & \dots & x_\beta(m, m) \end{bmatrix}, \quad (37)$$

220 where

$$x_\beta(i, j) = \sum_{k=\lceil\beta\rceil}^i (qt-r)^{-\beta} P_{j,j}^{-1} a_{j+1,k+1} P_{i,k} \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}, \quad i = \lceil\beta\rceil, \lceil\beta\rceil+1, \dots, m. \quad (38)$$

Substituting the concrete formulas of  $W$  and  $P^\beta(qt-r)$  into Eq. (36), we get

$$K^\beta = P^\beta(qt-r)W = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \tau_{\lceil\beta\rceil} \\ \vdots \\ \tau_i \\ \vdots \\ \tau_m \end{bmatrix},$$

where

$$\tau_i = \sum_{j=0}^i x_\beta(i, j) \omega_j, \quad i = \lceil\beta\rceil, \lceil\beta\rceil+1, \dots, m.$$

Theorem. 4 is proved by substituting Eqs. (18) and (38) into the above equation. ■

#### 225 4.4. SCPs operational matrix of product

Let  $C$  be the vector with the parameters  $c_i$  given in Eq. (11). By multiplying  $C$  with the outer product of two shifted orthonormal Chebyshev polynomial vectors, we can get the row vector. And this row vector can be approximated based on the shifted Chebyshev polynomial vector.

230 Let

$$C^T \Phi_m(t) \Phi_m^T(t) \approx \Phi_m^T(t) \bar{C}. \quad (39)$$

$\bar{C}$  which satisfies in the above relation is called the operational matrix of product of two shifted Chebyshev polynomial vectors.

To derive the operational matrix of product, inserting Eq. (12) into Eq. (39), we can get

$$\begin{aligned}
C^T \Phi_m(t) \Phi_m^T(t) &= C^T \Phi_m(t) Z_m(t)^T A^T \\
&= [C^T \Phi_m(t), t(C^T \Phi_m(t)), t^2(C^T \Phi_m(t)), \dots, t^m(C^T \Phi_m(t))] A^T \\
&= \left[ \sum_{i=0}^m c_i H_{L,i}(t), \sum_{i=0}^m c_i t H_{L,i}(t), \sum_{i=0}^m c_i t^2 H_{L,i}(t), \dots, \sum_{i=0}^m c_i t^m H_{L,i}(t) \right] A^T.
\end{aligned} \tag{40}$$

235 For each of  $t^y H_{L,i}(t)$ ,  $y = 0, 1, \dots, m$ , it can be approximated by the shifted Chebyshev polynomials, then

$$t^y H_{L,i}(t) \approx \sum_{k=0}^m e_k^{y,i} H_{L,k}(t) = E_{y,i}^T \Phi_m(t),$$

where  $E_{y,i}^T = [e_0^{y,i}, e_1^{y,i}, \dots, e_m^{y,i}]$ ,  $y = 0, 1, \dots, m$ ,  $i = 0, 1, \dots, m$ ,  $e_k^{y,i} = \frac{1}{h_k} \int_0^L t^y H_{L,i}(t) H_{L,k}(t) \omega_L(t) dt$ .

Thus, we obtain

$$\begin{aligned}
\sum_{i=0}^m c_i t^y H_{L,i}(t) &\simeq \sum_{i=0}^m c_i \left( \sum_{k=0}^m e_k^{y,i} H_{L,k}(t) \right) \\
&= \sum_{k=0}^m H_{L,k}(t) \left( \sum_{i=0}^m c_i e_k^{y,i} \right) \\
&= \Phi_m^T(t) [E_{y,0}, E_{y,1}, \dots, E_{y,m}] C \\
&= \Phi_m^T(t) U_{y,c},
\end{aligned} \tag{41}$$

240 where  $U_{y,c} = [E_{y,0}, E_{y,1}, \dots, E_{y,m}] C$ .

By defining matrix  $U_c = [U_{0,c}, U_{1,c}, \dots, U_{m,c}]$ , and substituting Eq. (41) into Eq. (40), we can get

$$C^T \Phi_m(t) \Phi_m^T(t) = \Phi_m^T(t) U_c A^T, \tag{42}$$

therefore

$$\bar{C} \approx U_c A^T.$$

## 5. Numerical algorithms

245 For the generalized fractional pantograph equation Eq. (1) that satisfies the initial condition Eq. (2), we first approximate

$$u(t) \approx C^T \Phi_m(t), \quad (43)$$

$$b(t) \approx B^T \Phi_m(t), \quad (44)$$

$$g(t) \approx G^T \Phi_m(t), \quad (45)$$

$$v_{j,n}(t) \approx C_{vjn}^T \Phi_m(t), \quad (46)$$

250 where  $G^T = \{g_i\}_{i=0}^m$ ,  $C_{vjn}^T = \{c_{i,vjn}\}_{i=0}^m$ ,  $B^T = \{b_i\}_{i=0}^m$ .

Now, using Eqs. (29) and (43), we have

$${}^c D^\beta u(t) \approx {}^c D^\beta C^T \Phi_m(t) = C^T P^\beta(t) \Phi_m(t). \quad (47)$$

For solving  ${}^c D^{\alpha_n} u(q_{j,n}t - r_{j,n})$  in Eq. (1), by using Eqs. (34) and (43), we obtain

$${}^c D^{\alpha_n} u(q_{j,n}t - r_{j,n}) \approx C^T K_{j,n}^{\alpha_n} \Phi_m(t). \quad (48)$$

Moreover, by the product operational matrix of SCPs and Eq. (46), we have

$$\begin{aligned} v_{j,n}(t) {}^c D^{\alpha_n} u(q_{j,n}t - r_{j,n}) &\approx C_{vjn}^T \Phi_m(t) C^T K_{j,n}^{\alpha_n} \Phi_m(t) \\ &= C_{vjn}^T \Phi_m(t) \Phi_m^T(t) \left( C^T K_{j,n}^{\alpha_n} \right)^T \\ &\approx \Phi_m^T(t) \overline{C_{vjn}} \left( C^T K_{j,n}^{\alpha_n} \right)^T \\ &= C^T K_{j,n}^{\alpha_n} \left( \overline{C_{vjn}} \right)^T \Phi_m(t) \\ &= D^{(jn)T} \Phi_m(t), \end{aligned} \quad (49)$$

255 where  $\overline{C_{vjn}}$  is product operational matrix for the vector  $C_{vjn}$ ,  $D^{(jn)T} = C^T K_{j,n}^{\alpha_n} \left( \overline{C_{vjn}} \right)^T$ .

And we also have

$$\begin{aligned}
b(t) u(t) &\approx B^T \Phi_m(t) C^T \Phi_m(t) \\
&= B^T \Phi_m(t) \Phi_m^T(t) C \\
&\approx \Phi_m^T(t) \bar{B} C \\
&= C^T (\bar{B})^T \Phi_m(t) \\
&= R^T \Phi_m(t),
\end{aligned} \tag{50}$$

where  $\bar{B}$  is product operational matrix for the vector  $B$ ,  $R^T = C^T (\bar{B})^T$ .

Substituting Eqs. (45), (47), (49) and (50) into Eq. (1), we obtain

$$C^T P^\beta(t) \Phi_m(t) = R^T \Phi_m(t) + \sum_{j=0}^J \sum_{n=0}^{r-1} D^{(jn)^T} \Phi_m(t) + G^T \Phi_m(t). \tag{51}$$

For the initial conditions, we can write

$$f = C^T \left( P^{(1)} \right)^n \Phi_m(0). \tag{52}$$

260 We collocate this system at the following points

$$t_i = \frac{2i-1}{2(m+1)}, \quad i = 1, 2, \dots, m+1.$$

These equations can be transferred to algebraic equations. Combining Matlab soft-ware and least square method, the unknown vector  $C$  can be solved.

## 6. Error analysis and error correction

### 6.1. Error analysis

265 **Lemma 1.** We assume that  $u \in C^{m+1}[0, L]$  with  $m \in N^*$ , and  $\beta < m$  with  $\beta \in R_+ \setminus N$ . Let  $\xi = -r + qt$ , then we have

$${}^c D^\beta u(\xi) = \sum_{i=n}^m \frac{(\xi)^{i-\beta}}{\Gamma(i-\beta+1)} u^{(i)}(0) + \frac{1}{\Gamma(m-\beta+1)} \int_0^\xi (\xi-T)^{m-\beta} u^{(m+1)}(T) dT$$

where  $n-1 < \beta < n \leq m$  with  $n \in N^*$ .

**Proof.** Let  $\xi = -r + qt$ . Using Eq. (5), we obtain

$${}^c D^\beta u(\xi) = \frac{1}{\Gamma(n-\beta)} \int_0^\xi (\xi-T)^{n-1-\beta} u^{(n)}(T) dT. \tag{53}$$

By applying  $m - n + 1$  times integration by parts in Eq. (53), we have

$${}^c D^\beta u(\xi) = \sum_{i=n}^m \frac{(\xi)^{i-\beta}}{\Gamma(i-\beta+1)} u^{(i)}(0) + \frac{1}{\Gamma(m-\beta+1)} \int_0^\xi (\xi-T)^{m-\beta} u^{(m+1)}(T) dT.$$

270 Lemma. 1 is proved . ■

**Lemma 2.** Assume that  $u \in C^{m+1}[0, L]$  with  $m \in \mathbb{N}^*$ , and  $\beta < m$  with  $\beta \in \mathbb{R}_+ \setminus \mathbb{N}$ . Let  $Y = \text{span}\{H_{L,0}, H_{L,1}, \dots, H_{L,m}\}$ ,  $u_m = C^T \Phi_m$  is the approximate function to  $u$  from  $Y$ . Let  $\xi = -r + qt$ , then we have

$${}^c D^\beta u_m(\xi) = \int_0^L {}^c D^\beta Q(t, \xi) u(t) dt, \quad (54)$$

where

$$Q(t, \xi) = \sum_{i=0}^m \frac{1}{h_i} H_{L,i}(t) \omega_L(t) H_{L,i}(\xi).$$

275

**Proof.** Let  $\xi = -r + qt$ . From Eqs. (9) and (10), we get

$$\begin{aligned} u_m(\xi) &= \sum_{i=0}^m c_i H_{L,i}(\xi) \\ &= \sum_{i=0}^m \frac{1}{h_i} \int_0^L u(t) H_{L,i}(t) \omega_L(t) dt H_{L,i}(\xi) \\ &= \int_0^L \left( \sum_{i=0}^m \frac{1}{h_i} H_{L,i}(t) \omega_L(t) H_{L,i}(\xi) \right) u(t) dt \\ &= \int_0^L Q(t, \xi) u(t) dt, \end{aligned}$$

where

$$Q(t, \xi) = \sum_{i=0}^m \frac{1}{h_i} H_{L,i}(t) \omega_L(t) H_{L,i}(\xi).$$

And we can obtain

$${}^c D^\beta u_m(\xi) = \int_0^L {}^c D^\beta Q(t, \xi) u(t) dt.$$

Lemma. 2 is proved . ■

280 **Theorem 5.** Suppose that  $u \in C^{m+1}[0, L]$  with  $m \in \mathbb{N}^*$ , and  $\beta < m$  with  $\beta \in \mathbb{R}_+ \setminus \mathbb{N}$ . Let  $Y = \text{span}\{H_{L,0}, H_{L,1}, \dots, H_{L,m}\}$ ,  $u_m = C^T \Phi_m$  is the approximate

function to  $u$  from  $Y$ .  $u^{(i)}$  ( $i = 0, 1, 2, \dots, m + 1$ ) are continuous functions. Let  $\xi = -r + qt$ , the error of  ${}^c D^\beta u(\xi)$  and  ${}^c D^\beta u_m(\xi)$  is represented as follows

$$e(t) = \int_0^L {}^c D^\beta Q(t, \xi) I_1(t) dt + I_2(\xi), \quad (55)$$

where  $Q(t, \xi)$  is given by Lemma. 2, and

$$I_1(t) = t^{m+1} \int_0^1 \frac{(1-T)^m}{m!} u^{(m+1)}(tT) dT,$$

285

$$I_2(\xi) = \frac{\xi^{m-\beta+1}}{\Gamma(m-\beta+1)} \int_0^1 (1-T)^{m-\beta} u^{(m+1)}(\xi T) dT.$$

If  $M_m = \|u^{(m+1)}\|_\infty = \sup\{|u^{(m+1)}(t)|, t \in R\}$  exists, then  $e(t)$  can be bounded as follows

$$|e(t)| \leq M_{m+1} \left( \left| \int_0^L {}^c D^\beta Q(t, \xi) \frac{t^{m+1}}{(m+1)!} dt \right| + \frac{\xi^{m-\beta+1}}{\Gamma(m-\beta+2)} \right).$$

**Proof.**  $u(t)$  can be expanded into Taylor formula as

$$u(t) = \sum_{i=0}^m \frac{(t)^i}{i!} u^{(i)}(0) + \int_0^t \frac{(t-T)^m}{m!} u^{(m+1)}(T) dT. \quad (56)$$

290 Then, we define the following truncated Taylor series expansion

$$\widehat{u}_m(t) = \sum_{i=0}^m \frac{(t)^i}{i!} u^{(i)}(0). \quad (57)$$

Let  $\xi = -r + qt$ , from Eqs. (56) and (57), we get

$$u(\xi) = \sum_{i=0}^m \frac{(\xi)^i}{i!} u^{(i)}(0) + \int_0^\xi \frac{(\xi-T)^m}{m!} u^{(m+1)}(T) dT, \quad (58)$$

$$\widehat{u}_m(\xi) = \sum_{i=0}^m \frac{(\xi)^i}{i!} u^{(i)}(0). \quad (59)$$

Hence, the  $\beta^{th}$  order derivative of Eq. (59) can be calculated as follows

$${}^c D^\beta \widehat{u}_m(\xi) = \sum_{i=n}^m \frac{(\xi)^{i-\beta}}{\Gamma(i-\beta+1)} u^{(i)}(0). \quad (60)$$



Let us consider the following equality

$$e(\xi) = ({}^c D^\beta u_m(\xi) - {}^c D^\beta \widehat{u}_m(\xi)) + ({}^c D^\beta \widehat{u}_m(\xi) - {}^c D^\beta u(\xi)).$$

295

Similar to the proof process of Lemma. 2, we get

$$\begin{aligned} \widehat{u}_m(\xi) &= \sum_{i=0}^m g_i H_{L,i}(\xi) \\ &= \sum_{i=0}^m \frac{1}{h_i} \int_o^L \widehat{u}_m(t) H_{L,i}(t) \omega_L(t) dt H_{L,i}(\xi) \\ &= \int_o^L \left( \sum_{i=0}^m \frac{1}{h_i} H_{L,i}(t) \omega_L(t) H_{L,i}(\xi) \right) \widehat{u}_m(t) dt \\ &= \int_o^L Q(t, \xi) \widehat{u}_m(t) dt. \end{aligned}$$

The  $\beta^{th}$  order derivative of  $\widehat{u}_m(\xi)$  can be calculated as follows

$${}^c D^\beta \widehat{u}_m(\xi) = \int_o^L {}^c D^\beta Q(t, \xi) \widehat{u}_m(t) dt.$$

Therefore

$$\begin{aligned} {}^c D^\beta u_m(\xi) - {}^c D^\beta \widehat{u}_m(\xi) &= \int_o^L {}^c D^\beta Q(t, \xi) u(t) dt - \int_o^L {}^c D^\beta Q(t, \xi) \widehat{u}_m(t) dt \\ &= \int_o^L {}^c D^\beta Q(t, \xi) (u(t) - \widehat{u}_m(t)) dt. \end{aligned} \quad (61)$$

According to Eqs. (56) and (57), we obtain

$$I_1(t) = u(t) - \widehat{u}_m(t) = \int_0^t \frac{(t-T)^m}{m!} u^{(m+1)}(T) dT. \quad (62)$$

Applying the following change of variables  $T \rightarrow tT$  in Eq. (62)

$$\begin{aligned} I_1(t) &= \int_0^t \frac{(t-T)^m}{m!} u^{(m+1)}(T) dT \\ &= t^{m+1} \int_0^1 \frac{(1-T)^m}{m!} u^{(m+1)}(tT) dT. \end{aligned}$$

300

The  $\beta^{th}$  order derivative of Eq. (58) can be represented as follows

$${}^c D^\beta \widehat{u}_m(\xi) - {}^c D^\beta u(\xi) = I_2.$$

According to Eq. (60) and Lemma. 1, we get

$$I_2(\xi) = \frac{1}{\Gamma(m - \beta + 1)} \int_0^\xi (\xi - T)^{m-\beta} u^{(m+1)}(T) dT \quad (63)$$

Applying the following change of variables  $T \rightarrow \xi T$  in Eq. (63)

$$I_2(\xi) = \frac{\xi^{m-\beta+1}}{\Gamma(m - \beta + 1)} \int_0^1 (1 - T)^{m-\beta} u^{(m+1)}(\xi T) dT.$$

Using Eqs. (61), (62) and (63), we get

$$e(t) = \int_0^L {}^c D^\beta Q(t, \xi) I_1(t) dt + I_2(\xi).$$

Finally, this proof can be completed by taking the absolute value of  $e(t)$  and the following inequalities

$$|I_1(t)| \leq M_{m+1} \frac{t^{m+1}}{(m+1)!}$$

$$|I_2(t)| \leq M_{m+1} \frac{\xi^{m-\beta+1}}{\Gamma(m - \beta + 2)}.$$

Theorem. 5 is proved . ■

## 6.2. Error correction

For Eq. (1), we consider the following residual function

$$R_m(t) = L[u_m(t)] - g(t), \quad (64)$$

where

$$L[u_m(t)] = {}^c D^\beta u_m(t) - b(t) u_m(t) - \sum_{j=0}^J \sum_{n=0}^r v_{j,n}(t) {}^c D^{\alpha_n} u_m(q_{j,n}t - r_{j,n}).$$

Eq. (64) satisfies the following form

$$L[u_m(t)] = R_m(t) + g(t).$$

It needs to be pointed out,  $u_m(t)$  is the approximate solution for  $u(t)$ , and  $u(t)$  is the exact solution of Eq. (1).

315 Defining the error function, as follows

$$e_m(t) = u(t) - u_m(t),$$

then we can get the differential equation about the error function

$$\begin{aligned} L[e_m(t)] &= L[u(t)] - L[u_m(t)] \\ &= g(t) - R_m(t) - g(t) \\ &= -R_m(t). \end{aligned}$$

The formula of the error function is giving

$$\begin{aligned} L[e_m(t)] &= {}^c D^\beta e_m(t) - b(t) e_m(t) - \sum_{j=0}^J \sum_{n=0}^r v_{j,n}(t) {}^c D^{\alpha_n} e_m(q_{j,n}t - r_{j,n}) \\ &= -R_m(t). \end{aligned} \tag{65}$$

In order to construct the approximate  $e_m^*(x, t)$  to  $e_m(t)$ , only Eq. (65) needs to be recalculated in the same way as we did before for the solution of Eq. (1).

320 And we define the  $e_m^*(t)$  as the approximate error function.

According to the numerical solution  $u_m(t)$  of Eq. (1) and the numerical solution  $e_m^*(t)$  of Eq. (65), corrective solution  $u^*(t)$  is obtained

$$u^*(t) = u_m(t) + e_m^*(t). \tag{66}$$

From Eq.(66), we can get the corrective error function

$$e_r(t) = e_m(t) - e_m^*(t) = u(t) - u_m(t) - e_m^*(t).$$

## 7. Numerical experiments

325 In this section, to demonstrate the applicability and accuracy of our method, we shows some numerical examples in the form Eq. (1) with intial conditions. All the numerical computations have been done using Matlab.

**Example 1.** Consider the fractional pantograph differential equation with variable coefficients

$$D^\beta u(t) = \frac{1}{2}e^{\frac{t}{2}}u\left(\frac{t}{2}\right) + \frac{1}{2}u(t), \quad 0 < \beta \leq 1, \quad 0 \leq t \leq 1,$$

$$u(0) = 1.$$

330 In the case, when  $\beta = 1$ , the exact solution is  $u(t) = e^t$ .

Table 1 shows the comparison of the absolute errors of the proposed method for  $m = 9, 10, 11$  with that of the Taylor method [35] for  $N = 9$ . Also we do correction for the numerical solutions with  $m = 9$ , and obtain the absolute corrective errors for  $m = 9, me = 11$ . We see that the approximation solutions  
 335 obtained by the present method have good agreement with the exact solution, and the absolute errors of corrective solutions are smaller than the absolute errors of numerical solutions.

Table 1: Absolute errors at some points for Example 1.

t	Taylor method	Present method with			
		$m = 9$	$m = 9, me = 11$	$m = 10$	$m = 11$
0.2	$0.70 \times 10^{-14}$	$3.03 \times 10^{-11}$	$3.18 \times 10^{-14}$	$1.03 \times 10^{-12}$	$2.82 \times 10^{-14}$
0.4	$0.10 \times 10^{-10}$	$3.76 \times 10^{-11}$	$3.86 \times 10^{-14}$	$1.26 \times 10^{-12}$	$3.42 \times 10^{-14}$
0.6	$0.29 \times 10^{-9}$	$4.69 \times 10^{-11}$	$4.69 \times 10^{-14}$	$1.53 \times 10^{-12}$	$4.29 \times 10^{-14}$
0.8	$0.38 \times 10^{-8}$	$5.75 \times 10^{-11}$	$5.73 \times 10^{-14}$	$1.88 \times 10^{-12}$	$5.20 \times 10^{-14}$
1	$0.29 \times 10^{-7}$	$7.10 \times 10^{-11}$	$7.11 \times 10^{-14}$	$2.26 \times 10^{-12}$	$6.35 \times 10^{-14}$

**Example 2.** Consider the generalized fractional pantograph differential equation

$$D^\beta u(t) = -u(t) - u\left(\frac{t}{2} - 0.3\right) + g(t), \quad 0 < \beta \leq 3,$$

$$u(0) = 1, u^{(1)}(0) = -1, u^{(2)}(0) = 1.$$

340 In the case, when  $\beta = 3$ ,  $g(t) = e^{-\frac{1}{2}t+0.3}$ , the exact solution is  $u(t) = e^{-t}$ .

Table 2 shows the absolute errors between the exact solution and approximate solutions of our method for different values of  $m$ . From Table 2, we can

Table 2: Absolute errors for different values of  $m$  for Example 2.

$t$	$m = 6$	$m = 8$	$m = 10$	$m = 12$
0.2	$5.18 \times 10^{-7}$	$1.45 \times 10^{-9}$	$2.86 \times 10^{-12}$	$5.22 \times 10^{-15}$
0.4	$1.93 \times 10^{-6}$	$5.24 \times 10^{-9}$	$1.21 \times 10^{-11}$	$2.39 \times 10^{-14}$
0.6	$3.27 \times 10^{-6}$	$1.21 \times 10^{-8}$	$2.77 \times 10^{-11}$	$5.67 \times 10^{-14}$
0.8	$5.34 \times 10^{-6}$	$2.13 \times 10^{-8}$	$4.99 \times 10^{-11}$	$1.03 \times 10^{-13}$
1	$8.62 \times 10^{-6}$	$3.29 \times 10^{-8}$	$7.80 \times 10^{-11}$	$1.62 \times 10^{-13}$

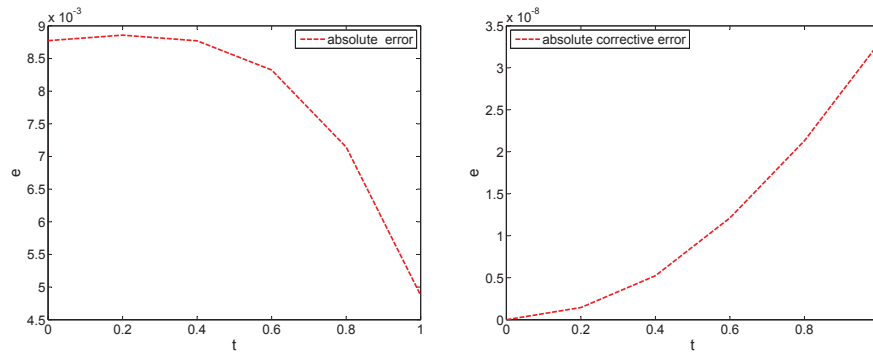


Figure 1: The absolute errors for  $m = 4$ , and the absolute corrective errors for  $m = 4$ ,  $me = 8$  for Example 2.

say that the numerical solutions come close to the exact solution with the increasing value of  $m$ . In Figure 1, the absolute errors of our method are given  
345 for  $m = 4$ . And the errors are not perfect enough. By doing correction for the numerical solutions for  $m = 4$ ,  $me = 8$ , the absolute corrective errors achieve about  $10^{-8}$ . Figure 2 displays the computational results for  $m = 7$  on  $[0, 2]$  when  $\beta$  takes different values. By comparing these computational results with exact solution, it is evident from Figure 2 that as  $\beta$  approaches 3, the numerical  
350 solutions converge to those of integer order differential equations.

**Example 3.** Consider the fractional multi-pantograph differential equation [33]

$$D^\beta u(t) = -\frac{5}{6}u(t) + 4u\left(\frac{1}{2}t\right) + 9u\left(\frac{1}{3}t\right) + t^2 - 1, \quad 0 < \beta \leq 1,$$

$$u(0) = 1.$$

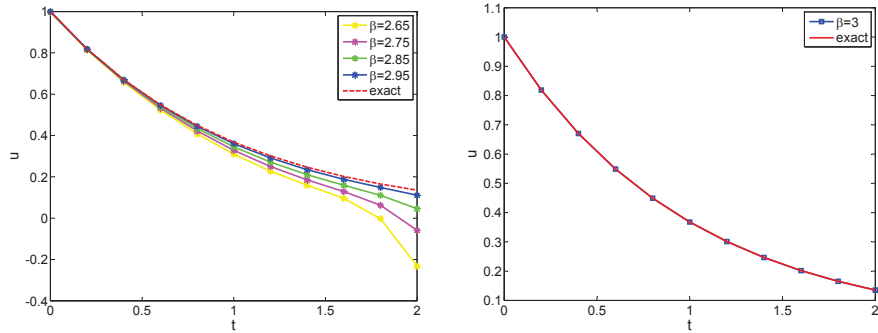


Figure 2: The comparison of  $u(t)$  for  $m = 7$ , with  $\beta = 2.65, 2.75, 2.85, 2.95, 1$ , and the exact solution for Example 2.

In the case, when  $\beta = 1$ , the exact solution is  $u(t) = 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3$ .

Table 3: The absolute errors for various intervals for Example 3.

t	[0, 5]		[0, 10]	
	GFBWFs method	Present method	GFBWFs method	Present method
0	$1.60 \times 10^{-10}$	$5.34 \times 10^{-12}$	$1.31 \times 10^{-9}$	$7.28 \times 10^{-12}$
1	$5.49 \times 10^{-9}$	$4.31 \times 10^{-10}$	$5.31 \times 10^{-8}$	$4.13 \times 10^{-10}$
2	$2.33 \times 10^{-8}$	$1.93 \times 10^{-9}$	$2.26 \times 10^{-7}$	$1.48 \times 10^{-9}$
3	$3.31 \times 10^{-6}$	$5.36 \times 10^{-9}$	$5.85 \times 10^{-7}$	$3.05 \times 10^{-9}$
4	$1.20 \times 10^{-5}$	$1.16 \times 10^{-8}$	$1.19 \times 10^{-6}$	$4.98 \times 10^{-9}$
5	$5.18 \times 10^{-7}$	$2.14 \times 10^{-8}$	$2.13 \times 10^{-6}$	$7.12 \times 10^{-9}$

In Table 3, we compare the absolute errors of our method for  $m = 3$  with those of the GFBWFs method of [33] for  $k = 2, M = 4$  on various intervals. Figure 3 gives the numerical results for different choices of  $\beta$  with  $m = 3$  on the interval  $[0, 10]$ . The Figure 4 shows the absolute errors between the exact solution and approximate solutions for  $m = 3, \beta = 1$  on  $[0, 10]$ , and the absolute corrective errors between the exact solution and corrective solutions for  $m = 3, me = 3$ . These results explained that as  $\beta$  approaches 1, the numerical solutions converge to the exact solution. And it is evident from Figure 4 that the corrective solutions obtained by doing correction have better convergence

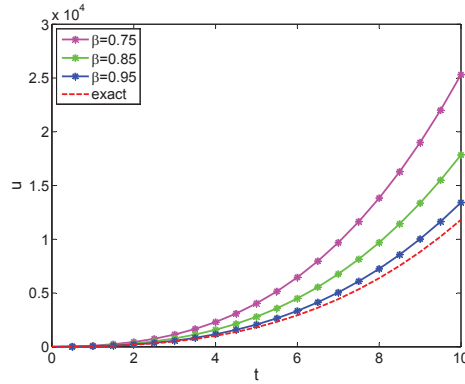


Figure 3: The comparison of  $u(t)$  for  $m = 3$ , with  $\beta = 0.75, 0.85, 0.95$ , and the exact solution for Example 3.

to exact solution than numerical solutions.

**Example 4.** Consider the fractional neutral pantograph differential equation

$$D^\beta u(t) = -u(t) + 0.1u(qt) + 0.5D^\beta u(qt) + (0.32t - 0.5)e^{-0.8t} + e^{-t}, 0 < \beta \leq 1, \\ u(0) = 0.$$

In the case, when  $\beta = 1, q = 0.8$ , the exact solution is  $u(t) = te^{-t}$ .

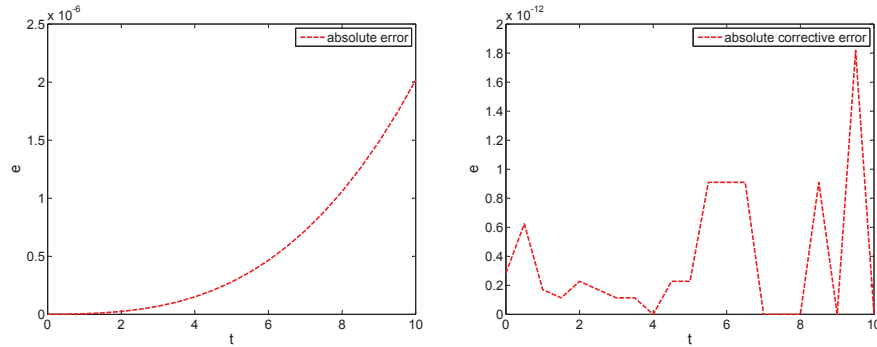


Figure 4: The absolute errors for  $m = 3$ , and the absolute corrective errors for  $m = 3, me = 3$  for Example 3.

365 In Table 4, the comparison, the absolute errors of the proposed method for  $m = 6, 8, 10$  with those of the one-Leg  $\theta$  [36] with  $\theta = 0.8, h = 0.01$ , the

370 variational iteration method (V-I method) [37] for  $\widehat{m} = 6$ , and the GFBWFs method (G method) [33] for  $k = 2, M = 6$  on the interval  $[0, 1]$ , is given. In Figure 5, we do correction for the numerical solutions with  $m = 6$ , and obtain the absolute corrective errors for  $m = 6, me = 10$ . Also, the Figure 6 displays the numerical results for  $m = 10, \beta = 1$  on  $[0, 2]$ , when  $q$  takes different values. And by comparing these results and exact solution, we can see that, as  $q$  approaches 0.8, the numerical solutions converge to the exact solution.

Table 4: The comparison of the absolute errors with other methods for Example 4.

t	$\theta$ method	V-I method	G method	Present method with		
				$m = 6$	$m = 8$	$m = 10$
0.1	$4.65 \times 10^{-3}$	$1.30 \times 10^{-3}$	$1.98 \times 10^{-8}$	$2.13 \times 10^{-6}$	$5.08 \times 10^{-9}$	$7.33 \times 10^{-12}$
0.3	$2.57 \times 10^{-2}$	$2.63 \times 10^{-3}$	$7.78 \times 10^{-9}$	$8.65 \times 10^{-7}$	$2.70 \times 10^{-9}$	$4.31 \times 10^{-12}$
0.5	$4.43 \times 10^{-2}$	$2.83 \times 10^{-3}$	$6.34 \times 10^{-5}$	$7.01 \times 10^{-7}$	$1.68 \times 10^{-9}$	$2.82 \times 10^{-12}$
0.7	$5.37 \times 10^{-2}$	$2.39 \times 10^{-3}$	$4.36 \times 10^{-5}$	$3.14 \times 10^{-7}$	$1.10 \times 10^{-9}$	$1.79 \times 10^{-12}$
0.9	$5.35 \times 10^{-2}$	$1.64 \times 10^{-3}$	$2.80 \times 10^{-5}$	$2.94 \times 10^{-7}$	$7.32 \times 10^{-10}$	$1.09 \times 10^{-12}$

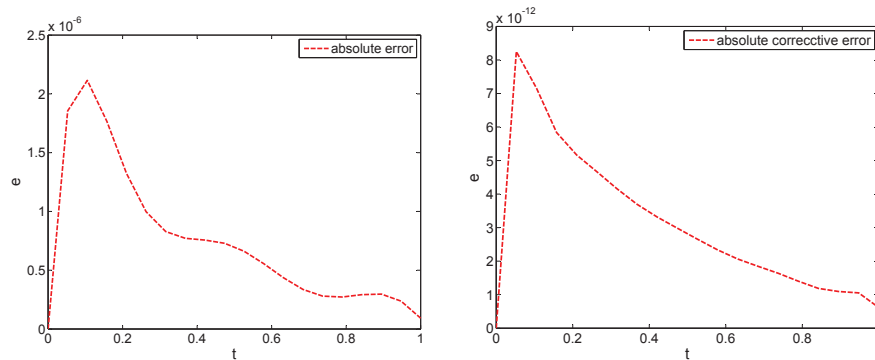


Figure 5: The absolute errors for  $m = 6$ , and the absolute corrective errors for  $m = 6, me = 10$  for Example 4.



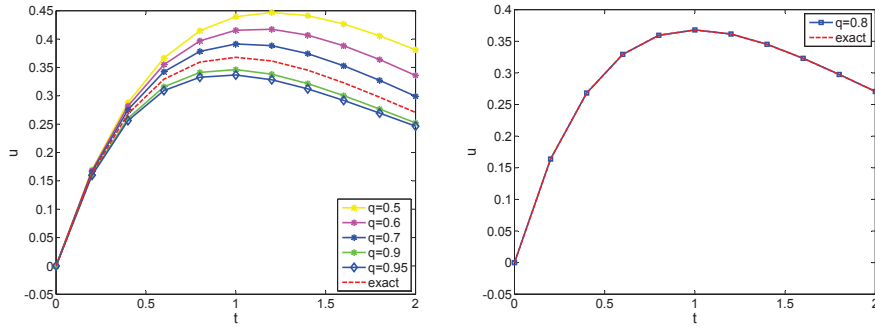


Figure 6: The comparison of  $u(t)$  for  $m = 10$ ,  $\beta = 1$  with  $q = 0.5, 0.6, 0.7, 0.8, 0.9, 0.95$ , and the exact solution for Example 4.

## 8. Conclusion

375 In this article, applying the properties of the shifted Chebyshev polynomials, we have derived the generalized pantograph operational matrix. Also, according to the SCPs fractional differential operational matrix, the generalized pantograph operational matrix of fractional-order differentiation is introduced. These matrices combined with collocation method are used to simplify and effectively calculate the numerical solutions of the generalized fractional pantograph delay equations. By constructing the generalized fractional pantograph delay equations of error function, we obtain the approximate error function to correct numerical solutions. Numerical examples show our method is effective. From examples, it is seen that with the increasing value of  $m$ , the absolute error is smaller and the convergence effect between the numerical solutions and the exact solution is better. The corrective solutions have better convergence to exact solutions than the numerical solutions. In addition, we find that the present method is an excellent mathematical method, when the function defined on the interval  $[0, L]$  and various order  $\beta > 0$ .

## 390 Acknowledgements

This work is supported by the Natural Science Foundation of Hebei Province (A2017203100) in China and the LE STUDIUM RESEARCH PROFESSOR-

SHIP award of Centre-Val de Loire region in France.

## References

- 395 [1] F. Mainardi, Fractional calculus and waves in linear viscoelasticity, Imperial College Press, London. 2010.
- [2] R.L. Bagley, P.J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, *J. Rheology*. 27 (3) (1983) 201-210.
- [3] R.T. Baillie, Long memory processes and fractional integration in econometrics, *J. Econometrics*. 73 (1996) 5-59.
- 400 [4] D.Y. Liu, G. Zheng, D. Boutat, H.R. Liu, Non-asymptotic fractional order differentiator for a class of fractional order linear systems, *Automatica*. 78 (2017) 61-71.
- [5] X. Wei, D.Y. Liu, D. Boutat, Non-asymptotic state estimation for a class of fractional order linear systems, *IEEE Trans. Autom. Control*. 62 (3) (2017) 1150-1164.
- 405 [6] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos Solitons Fract*. 7 (9) (1996) 1461-1477.
- [7] D.Y. Liu, O. GIBARU, W. Perruquetti, T.M. Laleg-Kirati, Fractional order differentiation by integration and error analysis in noisy environment, *IEEE Trans. Autom. Control*. 60 (11) (2015) 2945-2960.
- 410 [8] M.M. Khader, N.H. Sweilam, A.M.S. Mahdy, Numerical study for the fractional differential equations generated by optimization problem using Chebyshev collocation method and FDM, *Int. J. Pure Appl. Math*. 7 (2013) 2011-2018.
- 415 [9] S. Kaze, Exact solution of some linear fractional differential equations by Laplace transform, *Int. J. Nonlinear Sci*. 16 (2013) 3-11.

- [10] V.S. Ertrk, S. Momani, Solving systems of fractional differential equations using differential transform method, *J. Comput. Appl. Math.* 215 (1) (2007) 142-151.
- 420
- [11] X.J. Yang, J.A.T. Machado, H.M. Srivastava, A new numerical technique for solving the local fractional diffusion equation: two-dimensional extended differential transform approach, *Appl. Math. Comput.* 274 (2016) 143-151.
- [12] S. Momani, K. Al-Khaled, Numerical solutions for systems of fractional differential equations by the decomposition method, *Appl. Math. Comput.* 162 (3) (2005) 1351-1365.
- 425
- [13] A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractional-order differential equations, *Comput. Math. Appl.* 59 (3) (2010) 1326-1336.
- [14] H. Saeedi, M. Mohseni Moghadam, Numerical solution of nonlinear Volterra integro-differential equations of arbitrary order by CAS wavelets, *Nonlinear Sci. Numer. Simul.* 16 (3) (2011) 1216-1226.
- 430
- [15] W.G. Ajello, H.I. Freedman, J. Wu, A model of stage structured population growth with density depended time delay, *SIAM J. Appl. Math.* 52 (1992) 855-869.
- 435
- [16] M.D. Buhmann, A. Iserles, Stability of the discretized pantograph differential equation, *Math. Comp.* 60 (1993) 575-589.
- [17] E.H. Doha, A.H. Bhrawy, S.S. Ezz-Eldien, A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order, *Comput. Math. Appl.* 62 (2011) 2364-2373.
- 440
- [18] M.A. Ramadan, A.E.A. El-Sherbeiny, M.N. Sherif, Numerical solution of system of first-order delay differential equations using polynomial spline functions, *Int. J. Comput. Math.* 83 (12) (2006) 925-937.

- 445 [19] S. Sedaghat, Y. Ordokhani, M. Dehghan, Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials, *Commun. Nonlinear Sci. Numer. Simul.* 17 (12) (2012) 4815-4830.
- [20] E. Tohidi, A.H. Bhrawy, K.A. Erfani, A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation, *Appl. Math. Model.* 37 (6) (2013) 4283-4294.
- 450 [21] Z.H. Yu, Variational iteration method for solving the multi-pantograph delay equation, *Phys. Lett. A.* 372 (43) (2008) 6475-6479.
- [22] C.M. Huang, S. Vandewalle, Discretized stability and error growth of the nonautonomous pantograph equation, *SIAM J. Numer. Anal.* 42 (5) (2005) 2020-2042.
- 455 [23] R.I. Magin, Fractional calculus models of complex dynamics in biological tissues, *Comput. Math. Appl.* 59 (2010) 1586-1593.
- [24] B.P. Moghaddam, Z.S. Mostaghim, A numerical method based on finite difference for solving fractional delay differential equations, *J. Taibah Univ. Sci.* 7 (3) (2013) 120-127.
- 460 [25] M.L. Morgado, N.J. Ford, P.M. Lima, Analysis and numerical methods for fractional differential equations with delay, *J. Comput. Appl. Math.* 252 (2013) 159-168.
- [26] U. Saeed, M.U. Rehman, Hermite wavelet method for fractional delay differential equations, *J. Diff. Equa.* 2014 (1) (2014) 1-8.
- 465 [27] M.N. Sherif, I. Abouelfarag, T.S. Amer, Numerical solution of fractional delay differential equations using Spline functions, *Int. J. Pure Appl. Math.* 90 (1) (2014) 73-83.
- [28] M.A. Iqbal, U. Saeed, S.T. Mohyud-Din, Modified Laguerre Wavelets method for delay differential equations of fractional-order, *Egypt. J. Basic Appl. Sci.* 2 (1) (2015) 50-54.
- 470

- [29] U. Saeed, M.U. Rehman, M.A. Iqbal, Modified Chebyshev wavelet methods for fractional delay-type equations, *Appl. Math. Comput.* 264 (2015) 431-442.
- [30] P. Rahimkhani, Y. Ordokhani, E. Babolian, A new operational matrix based on Bernoulli wavelets for solving fractional delay differential equations, *Numer. Algor.* 74 (1) (2016) 223-245.
- [31] K. Balachandran, S. Kiruthika. Existence of solutions of nonlinear fractional pantograph equations, *Acta Math. Sci.* 33 (2013) 712-720.
- [32] Y. Yang, Y.Q. Huang, Spectral-collocation methods for fractional pantograph delay-integrodifferential equations, *Adv. Math. Phys.* 2013 (2013) 1-14.
- [33] P. Rahimkhani, Y. Ordokhani, E. Babolian, Numerical solution of fractional pantograph differential equations by using generalized fractional-order Bernoulli wavelet, *J. Comput. Appl. Math.* 309 (2016) 493-510.
- [34] Z.J. Hei, A method of solving inverse of a matrix, *Journal of Harbin Institute of Technology.* 36 (10) (2004) 1351-1353.
- [35] M. Sezer, S. Yalcinbas, M.Culsu, A Taylor polynomial approach for solving generalized pantograph equations with nonhomogeneous term, *Int. J. Comput. Math.* 85 (7) (2008) 1055-1063.
- [36] W.S. Wang, S.F. Li, On the one-leg  $\theta$ -methods for solving nonlinear neutral functional differential equations, *Appl. Math. Comput.* 193 (1) (2007) 285-301.
- [37] X. Chen, L. Wang, The variational iteration method for solving a neutral functional-differential equation with proportional delays, *Comput. Math. Appl.* 59 (8) (2010) 2696-2702.