Lur'e Postnikov Lyapunov functional technique to global Mittag-Leffler stability of fractional-order neural networks with piecewise constant argument

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Abstract

In this paper, the global Mittag-Leffler stability issue of fractional-order neural networks (FNNs) with piecewise constant argument is investigated. Firstly, a new inequality with respect to the fractional derivative of integer-order variable upper limit integral is proposed, which not only is favorable to the construction of Lyapunov function but also enriches the fractional order calculus theory. Secondly, based on topological degree theory, the existence and uniqueness of equilibrium point is certified. In addition, under some suitable assumptions, by applying Picard successive approximation technique, the proof of the existence and uniqueness of solution with the initial value is given. Moreover, by introducing integral term into Lyapunov functional, Lur'e Postnikov type Lyapunov functional is constructed, and a sufficient condition is addressed in terms of linear matrix inequalities (LMIs) to guarantee that the considered FNNs are Mittag-Leffler stable. Finally, an example is given to demonstrate the validity of the obtained results.

Keywords: Fractional neural network; Mittag-Leffler stability; Piecewise constant argument; Lur'e Postnikov type Lyapunov function; LMIs;

1. Introduction

Since Mcculloch and Pitts firstly put forward the mathematical model of neural networks on the basis of the characteristic analysis of the neurons [1], neural networks have become the hottest issues and have been applied widely in many fields, such as the artificial signal and image processing, switching in electronic circuits, automatic control and so on. It deserves to point that, in 1982 [2], Hopfield proposed a famous Hopfield neural networks model, and analyzed the stability of networks by utilizing energy function approach firstly, which set off a wave of worldwide researches on neural networks, such as the research on dynamic characteristics of neural networks [3].

Compared with integer-order calculus, fractional-order calculus provides a more accurate instrument for the description of memory and hereditary properties of various processes. Nowadays, it has found that fractional calculus can be applied in viscoelastic system, electric polarization, electromagnetic waves, colored noise, heat conduction, robotics, biological system, finance and so on [4-11], which can not be done by integer order calculus.

It is well known that neural networks can be utilized to deal with many computational problems, which requires the networks to accomplish some dynamical behaviors such as the stability. Only when neural networks is stable, can it operates normally. So, the studying stability of neural networks is essential. Recently, some excellent results in the stability analysis of FNNs have been derived. In [12], Wu et al discussed Quasi-uniform stability for FNNs with mixed time-varying delays. In [13-15], the researchers discussed different FNNs and established some sufficient conditions for global Mittag-Leffler stability and synchronization. In [16], Zhang et al addressed the stability analysis for fractional-order Hopfield neural networks with discontinuous activation functions. In [17-18], Rakkiyappan et al discussed the stability and asymptotical periodicity of a fractional-order complex-valued neural network with constant delays and time varying delays.

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The dynamic study of differential equations with piecewise constant argument was initiated by Cooke and Wiener in [22]. Subsequently, an increasing number of scholars are inspired to investigate differential equations with piecewise constant argument, owing to the fact that it represents a hybrid of continuous and discrete dynamical systems, and combines with the properties of both the differential and difference equations. Based on the above reason, differential equations with piecewise constant argument can play a crucial role in handling many real world problems in [23-26], such as oscillatory motion, mechanical, biological system and so on. Generally, the differential equation with piecewise constant argument is investigated by reducing the equation to an equivalent discrete equation [22,27]. However, this method has heavy limitations to non-integer piecewise constant argument. Recently, in [28,29], Akhmet reduced the differential equation with piecewise constant argument to an equivalent integral equation, which greatly promotes the application of this equation. It is worthy to mention that the arguments in the differential equation with piecewise constant argument are not only delayed but advanced. In [30, 31], Akhmet discussed the global stability of recurrent neural networks and cellular neural networks with piecewise constant argument, respectively. In [32], Yu et al studied the stability of neural networks with periodic coefficients and piecewise constant argument. In [33], the robust stability of interval fuzzy Cohen-Grossberg neural networks with piecewise constant argument of generalized type was investigated. The synchronization issue of complex dynamical networks with piecewise constant argument of generalized type was considered in [34]. Very recently, in [35], Wu et al studied the Mittag-Leffler stability of FNNs in the presence of generalized piecewise constant arguments, the sufficient condition is given to guarantee that the considered FNNs is Mittag-Leffler stable.

It should be pointed out that, in the aforementioned literature with respect to the stability of FNNs, the Lyapunov function for solving the stability issue is the absolute value function $V(t) = \sum_{i=0}^{n} |x_i|$, see [13-19], or the positive definite quadratic function $V(t) = x^T Px$, see [20-21, 35]. It is easy to see, the information from neuron activation functions of the network is not applied in the Lyapunov function construction. Hence, the obtained stability results in the above papers have a certain degree of conservatism. However, the introduction of Lur'e Postnikov type Lyapunov functional could overcome this shortcoming.

Motivated by the above discussions, in this paper, our aim is to investigate the gtlobal Mittag-Leffler stability of FNNs with piecewise constant argument of generalized type by applying Lur'e Postnikov type Lyapunov functional approach . The crucial novelties of our contribution lie in following aspects: 1. A new inequality with respect to the fractional derivative of integer-order variable upper limit integral is developed; 2. The existence and uniqueness of equilibrium point and solution for FNNs with piecewise constant argument are certified, respectively; 3. The integral term $\int_0^{z_i(t)} f_i(s)ds$ is introduced into Lur'e Postnikov Lyapunov functional, and the Mittag-Leffler stability condition is presented in terms of LMIs.

The rest of this paper is organized as follows. In Section 2, some definitions and lemmas are given, and model description is introduced. In Section 3, the existence and uniqueness of equilibrium point and solution with initial value are proved, respectively. Some sufficient criteria for the global Mittag-Leffler stability are derived. In Section 4, a numerical example is offered to verify the validity of the obtained results. The conclusion is drawn in Section 5.

Notation: R denotes the set of real numbers, N and R^+ represent the sets of natural and nonnegative real numbers, respectively. R^n denotes the n-dimensional Euclidean space, $R^{m\times n}$ denotes the set of all $m \times n$ real matrices. Given column vectors $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ and $y = (y_1, y_2, \cdots, y_n)^T \in R^n$, $x^T y = \sum_{i=1}^{n} x_i y_i$ denotes the scalar product of x and y, where the superscript T represents the transpose operator. $\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$ denotes the vector norm of x. For real symmetric P, $P > 0$ ($P < 0$) means that the matrix P is a real symmetric positive (negative) definite matrix, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ stand for the maximal and minimal eigenvalues of P, respectively. We take two real-valued sequence θ_i, ξ_i, $i \in N$ such that $0 < \theta_i < \theta_{i+1}$, $\theta_i < \xi_i < \theta_{i+1}$ for all $i \in N, \theta_i \to +\infty$.

2. Preliminaries and model description

2.1. Caputo fractional-order derivative

In this subsection, we recall some basic definitions and lemmas with respect to fractional calculus.

Definition 2.1. *([36]) The fractional integral of order α for an integrable function $f(t)$: $[0, +\infty) \to R$ is defined as*

$$ {}_0^R I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, $$

where $0 < \alpha < 1$, and $\Gamma(\cdot)$ is the Gamma function which is defined by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ ($Re(z) > 0$), where $Re(z)$ is the real part of z.

Definition 2.2. *([36]) Caputo's fractional derivative of order α for a function $f \in C^n([0,+\infty], R)$ is defined by*

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, $$

where $t \geqslant 0$ and n is a positive integer such that $n-1 < \alpha < n \in N$. Particularly, when $0 < \alpha < 1$,

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau. $$

Caputo's fractional derivative ${}_0^C D_t^\alpha$ has the following properties:

Lemma 2.1. *([37]).*

(i) ${}_0^C D_t^\alpha \, {}_0^R I_t^\beta x(t) = {}_0^C D_t^{\alpha-\beta} x(t)$, where $\alpha \geq \beta \geq 0$. Especially, when $\alpha = \beta$, ${}_0^C D_t^\alpha \, {}_0^R I_t^\beta x(t) = x(t)$.

(ii) Let $n = [\alpha] + 1$ for $\alpha \notin N$ or $n = \alpha$ for $\alpha \in N$. If $x(t) \in C^n[0,+\infty)$, then

$$ {}_0^R I_t^\alpha \, {}_0^C D_t^\alpha x(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k, \quad t > 0. $$

In particular, if $0 < \alpha \leq 1$ and $x(t) \in C^1[0,+\infty]$, then ${}_t^R I_0^\alpha \, {}_t^C D_0^\alpha x(t) = x(t) - x(0)$.

(iii) Let c be any constant, then ${}_0^C D_t^\alpha c = 0$.

(iv) ${}_0^C D_t^\alpha (v_1 f(t) + v_2 g(t)) = v_1 \, {}_0^C D_t^\alpha f(t) + v_2 \, {}_0^C D_t^\alpha g(t)$, where v_1 and v_2 are any constants.

Similar to the exponential function frequently used in the solutions of integer-order systems, a function frequently used in the solution of fractional-order systems is the Mittag-Leffler function.

Definition 2.3. *([37]). (Mittag-Leffler function) A two-parameter Mittag-Leffler function is defined as*

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, $$

where $\alpha > 0$, $\beta > 0$ and $z \in C$. For $\beta = 1$, its one-parameter form is shown as

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} = E_{\alpha,1}(z), $$

especially $E_{1,1}(z) = e^z$, where $\alpha = 1, \beta = 1$.

Lemma 2.2. *([13]) If $V(t)$ is a continuous function on $[0,+\infty)$ and satisfies*

$$ {}_0^C D_t^\alpha V(t) \leq \theta V(t), $$

where $0 < \alpha < 1$ and θ is a constant, then

$$ V(t) \leq V(0) E_\alpha(\theta t^\alpha), \quad t \geq 0. $$

Lemma 2.3. *Given any scalar $\epsilon > 0$, x, $y \in R^n$ and matrix A, then*

$$ x^T A y \leq \frac{1}{2\epsilon} x^T A A^T x + \frac{\epsilon}{2} y^T y. $$

2.2. Model description

In this subsection, we give the network model description and propose some rational assumptions which are needed in the development of this paper.

Consider the following FNNs described by differential equations with piecewise constant argument of generalized type:

$$ {}_0^C D_t^\alpha x(t) = -Ax(t) + Bf(x(t)) + Cg(x(\gamma(t))) + I, \quad (1) $$

or equivalently

$$ \begin{aligned} {}_0^C D_t^\alpha x_i(t) = &- a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(\gamma(t))) \\ &+ I_i, \quad i = 1, 2, \cdots, n, \end{aligned} \quad (2) $$

where $0 < \alpha < 1$. If $t \in [\theta_k, \theta_{k+1})$, $k \in N$, $t \in R^+$, then $\gamma(t) = \zeta_k$. n corresponds to the number of units in a neural network. $x(t) = (x_1(t), \cdots, x_n(t))^T \in R^n$, $x_i(t)$ stands for the state of the *ith* unit at time t. $A = \mathrm{diag}(a_1, a_2, \cdots, a_n)$ represents self connection weight, where $a_i > 0$. $B = (b_{ij})_{n\times n}$ represents interconnection weight matrix, where b_{ij} denotes the strength of the *jth* unit on the *ith* unit at time t. $C = (c_{ij})_{n\times n}$ also represents interconnection weight matrix, where c_{ij} infers the strength of the *jth* unit on the *ith* unit at time $\gamma(t)$. f, $g : R^n \to R^n$ are diagonal mapping, $g(x(\gamma(t))) = (g_1(x_1(\gamma(t))), g_2(x_2(\gamma(t))), \cdots, g_n(x_n(\gamma(t))))^T$, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t)))^T$, where f_j and g_j denote the measures of activation to its incoming potential of the units j at time $\gamma(t)$ and t, respectively. $I = (I_1, I_2, \cdots, I_n)^T$ is the external input vector.

The network system (1) is of mixed type system, that is, the argument $\gamma(t)$ can change its deviation character during the motion. Fix $k \in N$, and consider system (1) on the interval $[\theta_k, \theta_{k+1})$, the identification function $\gamma(t)$ is equal to ζ_k. If t satisfies $\theta_k \leq t < \zeta_k$, then $\gamma(t) > t$, and system (1) is an equation with advanced argument. Similarly, if $\zeta_k \leq t < \theta_{k+1}$, then $\gamma(t) < t$, and system (1) is an equation with delayed argument. Consequently, system (1) changes the type of deviation of the argument during the process.

The following assumptions will be needed throughout the paper:

(H1) the activation functions f_j, g_j satisfy $f_j(0) = 0$, $g_j(0) = 0$ for each $j = 1, 2, \cdots n$.

(H2) there exists constants $L_i^1, L_i^2, L_i^3, L_i^4 > 0$ such that

$$L_i^1 < \frac{f_i(u) - f_i(v)}{u - v} < L_i^2, L_i^3 < \frac{g_i(u) - g_i(v)}{u - v} < L_i^4$$

for all $u, v \in R^n, i = 1, 2, \cdots, n$. $L^2 = \text{diag}(L_1^2, L_2^2, \cdots, L_n^2)$, $L^4 = \text{diag}(L_1^4, L_2^4, \cdots, L_n^4)$.

(H3) there exists a positive number θ such that $\theta_{i+1} - \theta_i \leq \theta, i \in N$.

(H4) $\frac{\theta^\alpha(m_1+m_2)}{\Gamma(1+\alpha)}\left(\frac{1}{E_\alpha(m_1\theta^\alpha)} - \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)}\right)^{-1} < 1$.

(H5) $\frac{m_2\theta^\alpha}{\Gamma(1+\alpha)} + \frac{m_1\theta^\alpha}{\Gamma(1+\alpha)}\left(1 + \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)}\right)E_\alpha(m_1\theta^\alpha) < 1$, where

$$m_1 = \max_{1\leq i\leq n}\left(a_i + L_i^2\sum_{j=1}^{n}|b_{ji}|\right), m_2 = \max_{1\leq i\leq n}\left(L_i^4\sum_{j=1}^{n}|c_{ji}|\right).$$

Definition 2.4. *The constant vector $x^* \in R^n$ is a equilibrium point of the neural network system* (1)*, if x^* satisfies the equation*

$$-Ax^* + Bf(x^*) + Cg(x^*) + I = 0.$$

Definition 2.5. *([37] Mittag-Leffler stabillity). An equilibrium point x^* of the neural network system* (1) *is said to be Mittag-Leffler stable, if there exist constants $M > 0$, $\lambda > 0$ and $b > 0$, such that, the solution $x(t)$ with initial value $x(0) = x^0$ of system* (1) *satisfies*

$$\| x(t) - x^* \| \leq \{M \| x^0 - x^* \| E_\alpha(-\lambda t^\alpha)\}^b, \ t \geq 0.$$

In this paper, the deviating argument $\gamma(t)$ is supposed to be discontinuous, thus the right-hand side of system (1) has discontinuities at moments θ_i, $i \in N$. As a result, we define a solution $x(t)$ of system (1) as a continuous function on R^+, and the derivative $x'(t)$ exists at each point $t \in R^+$, with the possible exception of the points θ_i, $i \in N$, where a one-sided derivative exists. And system (1) is satisfied by $x(t)$ on each interval (θ_i, θ_{i+1}) as well.

2.3. Topological degree properties and Generalized Gronwall-Bellman inequality

Let Ω be a nonempty, bounded and open subset of R^n. The closure of Ω is denoted by $\overline{\Omega}$, and the boundary of Ω is denoted by $\partial\Omega$. Lemma 2.4 and Lemma 2.5 demonstrate two properties of the topological degree.

Lemma 2.4. *([38])Let $H(\lambda, x) : [0, 1] \times \overline{\Omega} \to R^n$ be a continuous homotopy mapping. If $H(\lambda, x) = y$ has no solution in $\partial\Omega$ for $\lambda \in [0, 1]$ and $y \in R^n \setminus H(\lambda, \partial\Omega)$, then the topological degree $deg(H(\lambda, x), \Omega, y)$ of $H(\lambda, x)$ is a constant which is independent of λ. In this case, $deg(H(0, x), \Omega, y) = deg(H(1, x), \Omega, y)$.*

Lemma 2.5. *([38])Let $H(x) : \overline{\Omega} \to R^n$ be a continuous mapping. If $deg(H(x), \Omega, y) \neq 0$, then there exists at least one solution of $H(x) = y$ in Ω.*

Lemma 2.6. *([20])Suppose $x(t) \in R^n$ is continuous and differentiable function, $P \in R^{n\times n}$ is a positive definite matrix. Then, for $\alpha \in (0, 1)$, the following inequality holds*

$$\frac{1}{2}\,{}_0^C D_t^\alpha x^T(t)Px(t) \leq x^T(t)P\,{}_0^C D_t^\alpha x(t).$$

Lemma 2.7. *([39]Generalized Gronwall-Bellman inequality). Suppose that $x(t)$, $a(t)$ are non-negative, non-decreasing and local integrable on $0 \leq t < T$, and $g(t)$ is a non-negative, non-decreasing continuous function defined on $0 \leq t < T$, $g(t) \leq M$ and M is a constant, $\alpha > 0$ with*

$$x(t) \leq a(t) + g(t)\int_0^t (t - s)^{\alpha-1}x(s)ds$$

on this interval. Then

$$x(t) \leq a(t)E_\alpha(g(t)\Gamma(\alpha)t^\alpha),$$

where E_α is the Mittag-Leffler function.

2.4. Fractional derivative of integer-order variable upper limit integral

In this subsection, we propose a new lemma about fractional derivative of integer-order variable upper limit integral, which will be used in the proof of the main result.

Lemma 2.8. *Suppose that $f, g : [0, +\infty) \to R$ are continuous function, and g is satisfied with $\tau_1 < g(x) < \tau_2$, where τ_1, τ_2 are positive constants. Then*

$$Sign(\int_0^t f(x)dx) = Sign(\int_0^t g(x)f(x)ds), \ \ t > 0,$$

where $Sign(\cdot)$ is the sign function.

Proof. According to $0 < \tau_1 < g(x) < \tau_2$, it follows that the value of $g(x)f(x)$ is between $\tau_1 f(x)$ and $\tau_2 f(x)$. Take t as a fixed point. It is easily obtain that $\int_0^t g(x)f(x)dx$ is between $\int_0^t \tau_1 f(x)dx$ and $\int_0^t \tau_2 f(x)dx$. Due to τ_1, $\tau_2 > 0$, we can derive that

$$Sign(\int_0^t f(x)dx) = Sign(\int_0^t g(x)f(x)dx).$$

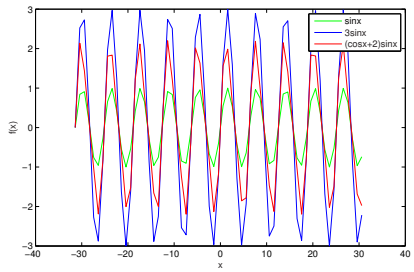


Figure 1: the graph of $sinx$, $3sinx$ and $(cosx+2)sinx$

Remark 2.1. *Let* $f(x) = \sin x$, $g(s) = \cos x + 2$,$x \in [0, +\infty)$. *It is easy to check that* $1 < g(x) < 3$. *Hence,* $g(x)$ *satisfies the assumption of Lemma 2.8. In the following, we apply Matlab tool to simulate and verify the validity of Lemma 2.8.*

In Fig.1, we depict the graphs of $\sin x$, $3\sin x$ *and* $(\cos x+2)\sin x$. *It is easy to see that the trace of* $(\cos x+2)\sin x$ *is between* $\sin x$ *and* $3\sin x$.

In Fig.2, we depict the graphs of $\int_0^t \sin x dx$ *and* $\int_0^t (\cos x+2)\sin x dx$. *Obviously, the sign of* $\int_0^t \sin x dx$ *is the same as the sign of* $\int_0^t (\cos x + 2)\sin x dx$.

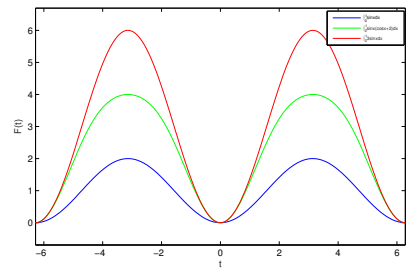


Figure 2: the graph of $\int_0^t \sin x dx$, $\int_0^t 3\sin x dx$ and $\int_0^t (\cos x+2)\sin x dx$

Lemma 2.9. *Suppose that* $z(t)$ *and* $f(t) \in R$ *are continuous functions, and* $f(t)$ *satisfies*

$$\tau_1 < \frac{f(z(t)) - f(z(s))}{z(t) - z(s)} < \tau_2,$$

where τ_1, τ_2 *are positive constants. Then*

$${}_0^C D_t^\alpha \left[\int_0^{z(s)} f(t)dt\right] \le f(z(t)) {}_0^C D_t^\alpha z(t), \quad \forall \alpha \in (0,1). \quad (3)$$

Proof. The inequality (3) is equivalent to

$$f(z(t)) {}_0^C D_t^\alpha z(t) - {}_0^C D_t^\alpha \left[\int_0^{z(s)} f(t)dt\right] \ge 0, \quad \forall \alpha \in (0,1). \quad (4)$$

According to Definition 2.2, we have

$${}_0^C D_t^\alpha z(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z'(s)}{(t-s)^\alpha} ds. \quad (5)$$

$$\begin{aligned} {}_0^C D_t^\alpha \left[\int_0^{z(s)} f(t)dt\right] &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(\int_0^{z(s)} f(t)dt)'}{(t-s)^\alpha} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(z(s))z'(s)}{(t-s)^\alpha} ds. \end{aligned} \quad (6)$$

Substituting (5) and (6) into (4), we obtain that

$$\frac{f(z(t))}{\Gamma(1-\alpha)} \int_0^t \frac{z'(s)}{(t-s)^\alpha} ds - \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(z(s))z'(s)}{(t-s)^\alpha} ds \ge 0. \quad (7)$$

That is,

$$-\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(z(t)) - f(z(s))}{(t-s)^\alpha} (-z'(s)) ds \ge 0. \quad (8)$$

(8) can be changed as

$$\begin{aligned} &\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(z(t)) - f(z(s))}{(t-s)^\alpha (z(t) - z(s))} (z(t) - z(s))(-z'(s)) ds \\ &\le 0. \end{aligned} \quad (9)$$

Next, we prove that the inequality

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(z(t) - z(s))(-z'(s))}{(t-s)^\alpha} ds \le 0 \quad (10)$$

is true. For convenient, by taking variable transformation $y(s) = z(t) - z(s)$, we discuss

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y(s)y'(s)}{(t-s)^\alpha} ds \le 0. \quad (11)$$

Applying integration by parts to (11) yields

$$\begin{aligned} &\frac{y^2(s)}{2\Gamma(1-\alpha)(t-s)^\alpha}|_{s=t} - \frac{y^2(s)}{2\Gamma(1-\alpha)(t-s)^\alpha}|_{s=0} \\ &- \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^t \frac{y^2(s)}{(t-s)^{\alpha+1}} ds \le 0, \end{aligned} \quad (12)$$

which is equivalent to

$$\begin{aligned} &- \frac{y^2(s)}{2\Gamma(1-\alpha)(t-s)^{\alpha}}|_{s=t} + \frac{y^2(0)}{2\Gamma(1-\alpha)t^{\alpha}} \\ &+ \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^t \frac{y^2(s)}{(t-s)^{\alpha+1}}ds \geq 0. \end{aligned} \tag{13}$$

The first term of (13) is singular when $s = t$, we consider the corresponding limitation

$$\begin{aligned} \lim_{s\to t}\frac{y(s)y(s)}{2\Gamma(1-\alpha)(t-s)^{\alpha}} &= \lim_{s\to t}\frac{(z(t)-z(s))(z(t)-z(s))}{2\Gamma(1-\alpha)(t-s)^{\alpha}} \\ &= \lim_{s\to t}\frac{z^2(t)-2z(t)z(s)+z^2(s)}{2\Gamma(1-\alpha)(t-s)^{\alpha}}. \end{aligned} \tag{14}$$

By utilizing *L'hôpital* law to (14), it follows that

$$\begin{aligned} &\lim_{s\to t}\frac{y(s)y(s)}{2\Gamma(1-\alpha)(t-s)^{\alpha}} \\ =&\lim_{s\to t}\frac{2z(t)z'(s)-2z(s)z'(s)}{2\Gamma(1-\alpha)\alpha(t-s)^{\alpha-1}} \\ =&\lim_{s\to t}\frac{[2z(t)z'(s)-2z(s)z'(s)](t-s)^{1-\alpha}}{2\Gamma(1-\alpha)\alpha} = 0. \end{aligned} \tag{15}$$

Thus, (13) is reduced to

$$\frac{y^2(0)}{2\Gamma(1-\alpha)t^{\alpha}} + \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^t \frac{y^2(s)}{(t-s)^{\alpha+1}}ds \geq 0. \tag{16}$$

Obviously, (16) is true. This proves the validity of (10).

Noting $0 < \tau_1 < \frac{f(z(t))-f(z(s))}{z(t)-z(s)} < \tau_2$, and applying Lemma 2.8, we conclude that (9) is true, which implies

$${}_0^C D_t^{\alpha}\left[\int_0^{z(s)} f(t)dt\right] \leq f(z(t)){}_0^C D_t^{\alpha} z(t), \quad \forall \alpha \in (0,1).$$

The proof is completed.

3. Main results

3.1. Existence and uniqueness of solution

In this subsection, we give the proof for the existence and uniqueness of equilibrium point and solution of the network system (1) on $t \geq 0$.

Theorem 3.1. *Suppose that (H1) and (H2) hold, if there exist diagonal matrix $P > 0$ and scalar $\varepsilon > 0$ such that*

$$\begin{aligned} \Delta =& PB + (PB)^T + \frac{1}{\varepsilon}(PC)(PC)^T \\ &+ \varepsilon((L^1)^{-1}L^4)^T(L^1)^{-1}L^4 < 0, \end{aligned}$$

then the neural network system (1) has a unique equilibrium point.

Proof. (Existence) Set

$$U(x) = Ax - Bf(x) - Cg(x) - I. \tag{17}$$

Obviously, $x^* \in R^n$ is an equilibrium point of system (1), if and only if, $U(x^*) = 0$. Construct a mapping $H(\lambda, x) : R^n \to R^n$ as

$$H(\lambda, x) = Ax - Bf(x) - Cg(x) - I, \quad \lambda \in [0,1].$$

It is easy to check that $H(\lambda, x)$ is a continuous homotopic mapping.

Next, we will construct a convex region Ω, such that $H(\lambda, x) \neq 0, x \in \partial\Omega, \lambda \in [0,1]$.

By the assumption (H1), it follows that $x_i f_i(x_i) = x_i(f_i(x_i) - f_i(0)) > 0$. Combining Lemma 2.3 with assumption (H2) yields

$$\begin{aligned} &f^T(x)PH(\lambda, x) \\ =&f^T(x)P\left(Ax - \lambda(Bf(x) + Cg(x)) - \lambda I\right) \\ \geq& f^T(x)(PA)x - \lambda f^T(x)(PI) - \lambda\Big[f^T(x)(PB)f(x) + \\ &\frac{1}{2\varepsilon}f^T(x)(PC)(PC)^T f(x) + \frac{\varepsilon}{2}g^T(x)g(x)\Big] \\ \geq& f^T(x)P(Ax - \lambda I) - \frac{\lambda}{2}f^T(x)\Big[(PB) + (PB)^T \\ &\frac{1}{\varepsilon}(PC)(PC)^T + \varepsilon((L^1)^{-1}L^4)^T(L^1)^{-1}L^4\Big]f(x) \\ \geq& f^T(x)P(Ax - \lambda I) \\ \geq& \sum_{i=1}^n [|f_i(x_i)|p_i a_i|x_i| - |f_i(x_i)|p_i|I_i|] \\ \geq& \sum_{i=1}^n |f_i(x_i)|p_i a_i[|x_i| - |\frac{I_i}{a_i}|]. \end{aligned} \tag{18}$$

Under the assumption (H_2), we have

$$\lim_{|x_i|\to+\infty} f_i(x_i)p_i a_i x_i \geq \lim_{|x_i|\to+\infty} L_i^1 p_i a_i x_i^2 = +\infty.$$

This implies that, for each $i \in \{1, 2, \cdots, n\}$,

$$\lim_{|x_i|\to+\infty} |\ f_i(x_i)\ |\ p_i a_i(|\ x_i\ | - |\ \frac{I_i}{a_i}\ |) = +\infty. \tag{19}$$

Let

$$M_i = \sum_{j=1, j\neq i}^n \sup\{|\ f_j(x_j)\ |\ p_j a_j(|\ x_j\ | + |\ \frac{I_j}{a_j}\ |) :|\ x_j\ | \leq |\ \frac{I_j}{a_j}\ |\}.$$

By (19), there exists scalar m_i, $m_i > |\frac{I_j}{a_j}|$, such that for any $|\ x_i\ | > m_i$, we have

$$|\ f_i(x_i)\ |\ p_i a_i(|\ x_i\ | - |\ \frac{I_i}{a_i}\ |) - M_i > 0. \tag{20}$$

Set $\Omega = \{x \in R^n : |x_i| < m_i + 1, i = 1, 2, \cdots, n\}$, Ω is an open convex bounded set independent of parameter λ. If $x \in \partial\Omega$, there must exist a $i_0 \in \{1, 2, \cdots, n\}$ such that $x_{i_0} = m_{i_0} + 1$. For any $\lambda \in [0, 1]$, we define following two index sets:

$$\Theta_+ = \{i \in \{1, 2, \cdots, n\} : |x_i| \geq |\frac{I_i}{a_i}|\},$$

$$\Theta_- = \{i \in \{1, 2, \cdots, n\} : |x_i| < |\frac{I_i}{a_i}|\} = \{1, 2, \cdots, n\} \setminus \Theta_+.$$

By the expression of M_i and (20), for any $x \in \partial\Omega$ and $\lambda \in [0, 1]$, it follows that

$$\begin{aligned}
& f^T(x)PH(\lambda, x) \\
\geq & \sum_{i=1}^{n} \mid f_i(x_i) \mid p_i a_i(\mid x_i \mid - \mid \frac{I_i}{a_i} \mid) \\
= & \sum_{i \in \Theta_+ \setminus \{i_0\}} \mid f_i(x_i) \mid p_i a_i(\mid x_i \mid - \mid \frac{I_i}{a_i} \mid) \\
& + \mid f_{i_0}(x_{i_0}) \mid p_{i_0} a_{i_0}(\mid x_{i_0} \mid - \mid \frac{I_{i_0}}{a_{i_0}} \mid) \\
& + \sum_{i \in \Theta_-} \mid f_i(x_i) \mid p_i a_i(\mid x_i \mid - \mid \frac{I_i}{a_i} \mid) \\
> & 0 + \mid f_{i_0}(x_{i_0}) \mid p_{i_0} a_{i_0}(\mid x_{i_0} \mid - \mid \frac{I_{i_0}}{a_{i_0}} \mid) - M_{i_0} \\
> & 0.
\end{aligned}$$

This implies that $H(\lambda, x) \neq 0$ for any $x \in \partial\Omega$, $\lambda \in [0, 1]$.

By Lemma 2.4, we obtain that $\deg(H(1, x), \Omega, 0) = \deg(H(0, x), \Omega, 0)$, i.e., $\deg(U(x), \Omega, 0) = \deg(Ax, \Omega, 0) = \mathrm{Sgn}|A| \neq 0$, where $|A|$ is the determinant of A. By Lemma 2.5, $U(x) = 0$ has at least one solution in Ω. This implies that the network system (1) has at least one equilibrium point in Ω.

(Uniqueness) Suppose that x_1^* and x_2^* are two different equilibrium points of the system (1). Then

$$A(x_1^* - x_2^*) = B[f(x_1^*) - f(x_2^*)] + C[g(x_1^*) - g(x_2^*)]$$

Multiplying by $2[f(x_1^*) - f(x_2^*)]^T P$, we obtain that

$$\begin{aligned}
& 0 < 2[f(x_1^*) - f(x_2^*)]^T PA(x_1^* - x_2^*) \\
= & 2[f(x_1^*) - f(x_2^*)]^T P(B[f(x_1^*) - f(x_2^*)] + C[g(x_1^*) - g(x_2^*)]) \\
\leq & [f(x_1^*) - f(x_2^*)]^T \{PB + (PB)^T + \frac{1}{\varepsilon}(PC)(PC)^T \\
& + \varepsilon((L^1)^{-1}L^4)^T(L^1)^{-1}L^4\}[f(x_1^*) - f(x_2^*)] \\
\leq & [f(x_1^*) - f(x_2^*)]^T \Delta[f(x_1^*) - f(x_2^*)] \\
< & 0.
\end{aligned}$$

This is a contradiction. The above discussion shows that system (1) has a unique equilibrium point. This completes the proof.

Theorem 3.2. *Suppose that assumptions (H1)-(H4) are established. Then, for any $(t_0, x^0) \in R^+ \times R^n$, there exists a unique solution $x(t) = x(t, t_0, x^0) = (x_1(t), \cdots, x_n(t))^T$ on $[t_0, +\infty)$ for system (1), such that $x(t_0) = x^0$.*

Proof. (Existence) Fix $k \in N$. In generally, we assume that $\theta_k \leq \zeta_k < t_0 \leq \theta_{k+1}$. First of all, for every $(t_0, x^0) \in R^+ \times R^n$, we shall prove that there exists a solution $x(t) = x(t, t_0, x^0) = (x_1(t), \cdots, x_n(t))$ of (1) under the initial value $x(t_0) = x^0 = (x_1^0, \cdots, x_n^0)^T$.

By Lemma 2.1, the fractional order differential equation (2) is equivalent to the corresponding fractional-order integral equation

$$\begin{aligned}
x_i(t) = & x_i^0 + {}_{t_0}^{R} I_t^{\alpha} \Big[- a_i x_i(s) + \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) \\
& + \sum_{j=1}^{n} c_{ij} g_j(x_j(\zeta_k)) + I_i \Big].
\end{aligned}$$

Define a norm $\|x(t)\|_0 = \max_{[\zeta_k, t_0]} \|x(t)\|$ and construct the sequence of iterations $x_i^m(t)$, $x_i^0(t) \equiv x_i^0$, $i = 1, 2, \cdots, n$, $m \geq 0$ as

$$\begin{aligned}
x_i^{m+1}(t) = & x_i^0 + {}_{t_0}^{R} I_t^{\alpha} \Big[- a_i x_i^m(s) + \sum_{j=1}^{n} b_{ij} f_j(x_j^m(s)) \\
& + \sum_{j=1}^{n} c_{ij} g_j(x_j^m(\zeta_k)) + I_i \Big].
\end{aligned}$$

It follows from (H2) that

$$\begin{aligned}
& \|x^{m+1}(t) - x^m(t)\| \\
\leq & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} \Big\{ \int_{t_0}^{t} (t-s)^{\alpha-1} \Big[\Big(a_i + L_i^2 \sum_{j=1}^{n} |b_{ji}| \Big) |x_i^m(s) - x_i^{m-1}(s)| \\
& + \Big(L_i^4 \sum_{j=1}^{n} |c_{ji}| \Big) |x_i^m(\zeta_k) - x_i^{m-1}(\zeta_k)| \Big] ds \Big\}.
\end{aligned}$$

By the assumption (H3) and (H4), we obtain that

$$\begin{aligned}
& \|x^{m+1}(t) - x^m(t)\|_0 \\
\leq & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} \Big\{ \int_{t_0}^{t} (t-s)^{\alpha-1} \Big[m_1 |x_i^m(s) - x_i^{m-1}(s)| \\
& + m_2 |x_i^m(\zeta_k) - x_i^{m-1}(\zeta_k)| \Big] ds \Big\} \\
\leq & \frac{1}{\Gamma(\alpha)} \Big\{ \int_{t_0}^{t} (t-s)^{\alpha-1} \Big[m_1 \|x^m(s) - x^{m-1}(s)\| \\
& + m_2 \|x^m(\zeta_k) - x^{m-1}(\zeta_k)\| \Big] ds \Big\} \\
\leq & \frac{\theta^{\alpha}}{\Gamma(1+\alpha)} (m_1 + m_2) \|x^m(t) - x^{m-1}(t)\|_0.
\end{aligned}$$

It is easy to get that

$$\begin{aligned}\|x^{m+1}(t)-x^m(t)\|_0 \le& \left(\frac{(m_1+m_2)\theta^\alpha}{\Gamma(1+\alpha)}\right)^m \|x^1(t)-x^0(t)\|\\ \le& \left(\frac{(m_1+m_2)\theta^\alpha}{\Gamma(1+\alpha)}\right)^m \tau,\end{aligned}$$

where

$$\tau = \frac{\theta^\alpha}{\Gamma(1+\alpha)}\left[(m_1+m_2)\|x^0\| + \sum_{i=1}^n I_i\right].$$

Thus, we conclude that there exists a solution $x(t) = x(t,t_0,x(t_0))$ for system (2) on $[\zeta_k, t_0]$. By the assumption (H2), it follows that $x(t)$ can be continued to θ_{k+1}. The above operation can be done once again on interval $(\theta_{k+1}, \theta_{k+2}]$. By the mathematical induction, we conclude that there exists a solution $x(t,t_0,x^0)$ with $x(t_0) = x^0$ on $[t_0, +\infty)$ for system (1).

(Uniqueness) First, let us discuss the $\theta_k \le t_0 \le \theta_{k+1}$. Define by $x^1(t) = x(t,t_0,x^1)$, $x^2(t) = x(t,t_0,x^2)$ the solution of system (1).

By reduction to absurdity, for every $t \in [\theta_k, \theta_{k+1}]$, we suppose that $x^1 = (x_1^1, \cdots, x_n^1)$, $x^2 = (x_1^2, \cdots, x_n^2)$, $x^1 \ne x^2$, which means that $x^1(t) \ne x^2(t)$. Then

$$\begin{aligned}&\|x^1(t)-x^2(t)\|\\ \le&\|x^1-x^2\| + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^n\left\{\int_{t_0}^t (t-s)^{\alpha-1}\right.\\ &\left[\left(a_i + L_i^2\sum_{j=1}^n |b_{ji}|\right)|x_i^1(s)-x_i^2(s)|\right.\\ &\left.\left.+\left(L_i^4\sum_{j=1}^n |c_{ji}|\right)|x_i^1(\zeta_k)-x_i^2(\zeta_k)|\right]ds\right\}\\ \le&(\|x^1-x^2\| + \frac{\theta^\alpha}{\Gamma(1+\alpha)}m_2\|x^1(\zeta_k)-x^2(\zeta_k)\|)\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}m_1\|x^1(s)-x^2(s)\|ds.\end{aligned}$$

By Lemma 2.7, it follows that

$$\begin{aligned}&\|x^1(t)-x^2(t)\|\\ \le&\left(\|x^1-x^2\| + \frac{\theta^\alpha}{\Gamma(1+\alpha)}m_2\|x^1(\zeta_k)-x^2(\zeta_k)\|\right)E_\alpha(m_1\theta^\alpha).\end{aligned}$$

In particular,

$$\begin{aligned}&\|x^1(\zeta_k)-x^2(\zeta_k)\|\\ \le&\left(\|x^1-x^2\| + \frac{\theta^\alpha}{\Gamma(1+\alpha)}m_2\|x^1(\zeta_k)-x^2(\zeta_k)\|\right)E_\alpha(m_1\theta^\alpha).\end{aligned}$$

Therefore,

$$\|x^1(t)-x^2(t)\| \le \left(\frac{1}{E_\alpha(m_1\theta^\alpha)} - \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)}\right)^{-1}\|x^1-x^2\|. \tag{21}$$

On the other hand, assume opposite that there exists $t \in [\theta_k, \theta_{k+1}]$ such that $x^1(t) = x^2(t)$. Then, we have

$$\begin{aligned}&\|x^1-x^2\|\\ =&\frac{1}{\Gamma(\alpha)}\sum_{i=1}^n\left\{\int_{t_0}^t (t-s)^{\alpha-1}\right.\Big[-a_i(x_i^2(s)-x_i^1(s))\\ &+\sum_{j=1}^n b_{ij}[f_j(x_j^2(s)-f_j(x_j^1(s))\\ &\left.+\sum_{j=1}^n c_{ij}[g_j(x_j^2(\zeta_k)-g_j(x_j^1(\zeta_k))\Big]ds\right\}\\ =&\frac{1}{\Gamma(\alpha)}\sum_{i=1}^n\left\{\int_{t_0}^t (t-s)^{\alpha-1}\left(a_i + L_i^2\sum_{j=1}^n|b_{ji}|\right)\right.\\ &\left.\times|x_i^1(s)-x_i^2(s)| + \left(L_i^4\sum_{j=1}^n|c_{ji}|\right)|x_i^1(\zeta_k)-x_i^2(\zeta_k)|ds\right\}\\ \le&\frac{\theta^\alpha}{\Gamma(1+\alpha)}m_2\|x^1(\zeta_k)-x^2(\zeta_k)\|\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}m_1\|x^1(s)-x^2(s)\|ds.\end{aligned} \tag{22}$$

By substituting (21) into (22) and applying assumption (H4), we obtain that

$$\begin{aligned}\|x^1-x^2\| \le& \frac{\theta^\alpha(m_1+m_2)}{\Gamma(1+\alpha)}\left(\frac{1}{E_\alpha(m_1\theta^\alpha)} - \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)}\right)^{-1}\|x^1-x^2\|\\ <& \|x^1-x^2\|.\end{aligned} \tag{23}$$

This is a contradiction. This proves that the uniqueness of solution for $t \in [\theta_k, \theta_{k+1}]$. The extension of the unique solution on R^+ is obvious. The proof is completed.

By substituting $x(t) = y(t) + x^*$, system (2) can be simplified as

$$\begin{aligned}{}_0^C D_t^\alpha y_i(t) =& -a_i y_i(t) + \sum_{j=1}^n b_{ij}\varphi_j(y_j(t))\\ &+\sum_{j=1}^n c_{ij}\psi_j(y_j(\gamma(t))),\quad i=1,\cdots,n.\end{aligned} \tag{24}$$

where $\varphi_j(y_j(t)) = f_j(y_j(t)+x_j^*) - f_j(x_j^*)$, $\psi_j(y_j(\gamma(t))) = g_j(y_j(\gamma(t)+x_j^*)) - g_j(x_j^*)$ with $\varphi_j(0) = 0$, $\psi_j(0) = 0$. It is

easy to see that the equilibrium point of system (24) is the original point. The stability of system (24) is equal to the stability of equilibrium x^* of system (2). Therefore, in the following, we only discuss the stability of the zero solution of system (24).

First of all, we address the following lemma, which is one of the important auxiliary results in this paper.

Lemma 3.1. *Suppose that assumptions* $(H1)-(H5)$ *are satisfied. Let* $y(t) = (y_1(t), \cdots, y_n(t))^T$ *be a solution of system (24). Then, the following inequality*

$$\|y(\gamma(t))\| \le \eta \|y(t)\| \tag{25}$$

holds for $t \in R^+$, *where* $\eta = \left(1 - \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)} - \frac{m_1\theta^\alpha}{\Gamma(1+\alpha)}\left(1 + \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)}\right)E_\alpha(m_1\theta^\alpha)\right)^{-1}$.

Proof. Fix $k \in N$. Then, for $t \in [\theta_k, \theta_{k+1})$, we have

$$\begin{aligned} y_i(t) =& y_i(\zeta_k) + {}^R_{\zeta_k}I^\alpha_t\Bigg[- a_i y_i(s) + \sum_{j=1}^n b_{ij}\varphi_j(y_j(s)) \\ &+ \sum_{j=1}^n c_{ij}\psi_j(y_j(\zeta_k))\Bigg], \end{aligned}$$

where $\gamma(t) = \zeta_k$, if $t \in [\theta_k, \theta_{k+1})$, $t \in R^+$. Taking absolute value of both sides of above expression for each $i = 1, 2, \cdots, n$, we obtain that

$$\begin{aligned} \|y(t)\| \le& \|y(\zeta_k)\| + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^n\Bigg\{\int_{\zeta_k}^t (t-s)^{\alpha-1}\Bigg[a_i|y_i(s)| \\ &+ \sum_{j=1}^n |b_{ij}\|\varphi_j(y_j(s))| + \sum_{j=1}^n |c_{ij}\|\psi_j(y_j(\zeta_k))|\Bigg]ds\Bigg\} \\ \le& \|y(\zeta_k)\| + \frac{1}{\Gamma(\alpha)}\Bigg\{\int_{\zeta_k}^t (t-s)^{\alpha-1}\Bigg[\sum_{i=1}^n\Bigg(a_i + L_i^2\sum_{j=1}^n |b_{ji}|\Bigg) \\ &|y_i(s)| + \sum_{i=1}^n\Bigg(L_i^4\sum_{j=1}^n |c_{ji}|\Bigg)|y_i(\zeta_k)|\Bigg]ds\Bigg\} \\ \le& \|y(\zeta_k)\| + \frac{1}{\Gamma(\alpha)}\Bigg\{\int_{\zeta_k}^t (t-s)^{\alpha-1}\Bigg[m_1\|y(s)\| \\ &+ m_2\|y(\zeta_k)\|\Bigg]ds\Bigg\} \\ =& \|y(\zeta_k)\| + \frac{m_2\|y(\zeta_k)\|}{\Gamma(\alpha)} \times \frac{(t-\zeta_k)^\alpha}{\alpha} \\ &+ \frac{m_1}{\Gamma(\alpha)}\int_{\zeta_k}^t (t-s)^{\alpha-1}\|y(s)\|ds \\ =& \left(1 + \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)}\right)\|y(\zeta_k)\| \\ &+ \frac{m_1}{\Gamma(\alpha)}\int_{\zeta_k}^t (t-s)^{\alpha-1}\|y(s)\|ds. \end{aligned}$$

By applying Lemma 2.7, it yields

$$\|y(t)\| \le \left(1 + \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)}\right)\|y(\zeta_k)\|E_\alpha(m_1 t^\alpha). \tag{26}$$

Furthermore, for $t \in [\theta_k, \theta_{k+1})$, we have

$$\begin{aligned} \|y(\zeta_k)\| \le& \|y(t)\| + \frac{m_2\|y(\zeta_k)\|}{\Gamma(\alpha)} \times \frac{(t-\zeta_k)^\alpha}{\alpha} \\ &+ \frac{m_1}{\Gamma(\alpha)}\int_{\zeta_k}^t (t-s)^{\alpha-1}\|y(s)\|ds \\ \le& \|y(t)\| + \frac{m_2\theta^\alpha\|y(\zeta_k)\|}{\Gamma(1+\alpha)} \\ &+ \frac{m_1}{\Gamma(\alpha)}\int_{\zeta_k}^t (t-s)^{\alpha-1}\|y(s)\|ds. \end{aligned} \tag{27}$$

Combining (26) with (27), we get

$$\begin{aligned} \|y(\zeta_k)\| \le& \|y(t)\| + \frac{m_2\theta^\alpha\|y(\zeta_k)\|}{\Gamma(1+\alpha)} + \frac{m_1}{\Gamma(\alpha)} \\ &\times \int_{\zeta_k}^t (t-s)^{\alpha-1}\left(1 + \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)}\right)\|y(\zeta_k)\|E_\alpha(m_1 t^\alpha)ds \\ \le& \|y(t)\| + \frac{m_2\theta^\alpha\|y(\zeta_k)\|}{\Gamma(1+\alpha)} + \frac{m_1}{\Gamma(\alpha)} \times \frac{\theta^\alpha}{\alpha} \\ &\times \left(1 + \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)}\right)\|y(\zeta_k)\|E_\alpha(m_1 t^\alpha) \\ \le& \|y(t)\| + \left[\frac{m_2\theta^\alpha}{\Gamma(1+\alpha)} + \frac{m_1\theta^\alpha}{\Gamma(1+\alpha)}\right. \\ &\left.\times \left(1 + \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)}\right)E_\alpha(m_1 t^\alpha)\right]\|y(\zeta_k)\|. \end{aligned}$$

Therefore, by assumption (H5), it follows that

$$\|y(\zeta_k)\| \le \eta\|y(t)\|, \quad t \in [\theta_k, \theta_{k+1}),$$

where $\eta = \left(1 - \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)} - \frac{m_1\theta^\alpha}{\Gamma(1+\alpha)}\left(1 + \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)}\right)E_\alpha(m_1\theta^\alpha)\right)^{-1}$. This implies that (25) holds for $t \in R^+$. The proof is completed.

3.2. Stability analysis

In this subsection, we establish some criteria for global Mittag-Leffler stability of the network system (1).

Theorem 3.3. *Supposed that assumptions (H1)-(H5) hold. If there exist a scalar* $\varepsilon > 0$ *and matrix* $P = diag(p_1, p_2, \cdots, p_n) > 0$, *such that*

$$\begin{aligned} \widetilde{\Delta} =& PB + (PB)^T + \frac{1}{\varepsilon}(PC)(PC)^T \\ &+ \varepsilon\eta^2((L^1)^{-1}L^4)^T(L^1)^{-1}L^4 < 0, \end{aligned}$$

then the network system (1) is globally Mittag-Leffler stable.

Proof. Consider Lur'e Postnikov type Lyapunov functional

$$V(t) = \frac{1}{2}y^T(t)Py(t) + \beta\sum_{i=1}^{n} p_i \int_{y_i(\gamma(t))}^{y_i(t)} \varphi_i(s)ds.$$

Taking Caputo's derivative of order α of $V(t)$ at t along the trajectories of system (24), and applying Lemma 2.6 and 2.9, we have

$$\begin{aligned}
{}_0^C D_t^\alpha V(t) \le & y^T(t)P\,{}_0^C D_t^\alpha y(t) + \beta\sum_{i=1}^{n} p_i\varphi_i(y_i(t)){}_0^C D_t^\alpha y_i(t)\\
\le & y^T(t)P\,{}_0^C D_t^\alpha y(t) + \beta\sum_{i=1}^{n} p_i\varphi_i(y_i(t))\times\Big[-a_i y_i(t)\\
& + \sum_{j=1}^{n} b_{ij}\varphi_j(y_j(t)) + \sum_{j=1}^{n} c_{ij}\psi_j(y_j(t))\Big]\\
= & y^T(t)P\,{}_0^C D_t^\alpha y(t) + \beta\Big[\varphi^T(y(t))(-PA)y(t))\\
& + \varphi^T(y(t))(PB)\varphi(y(t)) + \varphi^T(y(t))(PC)\psi(y(\gamma(t)))\Big]\\
= & -\frac{1}{2}y^T(t)\left(PA + (PA)^T\right)y(t) + y^T(t)(PB)\varphi(y(t)\\
& + y^T(t)(PC)\psi(y(\gamma(t))) + \beta\Big[\varphi^T(y(t))(-PA)y(t))\\
& + \frac{1}{2}\varphi^T(y(t))\left(PB + (PB)^T\right)\varphi(y(t))\\
& + \varphi^T(y(t))(PC)\psi(y(\gamma(t)))\Big].
\end{aligned} \tag{28}$$

Applying Lemma 2.3 yields

$$\begin{aligned}
y^T(t)PB\varphi(y(t)) \le & \frac{1}{2\varepsilon}y^T(t)(PB)(PB)^T y(t)\\
& + \frac{\varepsilon}{2}\varphi^T(y(t))\varphi(y(t)).
\end{aligned} \tag{29}$$

$$\begin{aligned}
y^T(t)(PC)\psi(y(\gamma(t))) \le & \frac{1}{2\varepsilon}y^T(t)(PC)(PC)^T y(t)\\
& + \frac{\varepsilon}{2}\psi^T(y(\gamma(t)))\psi(y(\gamma(t))).
\end{aligned} \tag{30}$$

$$\begin{aligned}
\varphi^T(y(t))(PC)\psi(y(\gamma(t))) \le & \frac{1}{2\varepsilon}\varphi^T(y(t))(PC)(PC)^T\varphi(y(t))\\
& + \frac{\varepsilon}{2}\psi^T(y(\gamma(t)))\psi(y(\gamma(t))).
\end{aligned} \tag{31}$$

According to assumption (H2), it follows that

$$\begin{aligned}
& \varphi^T(y(t))\varphi(y(t))\\
= & [f(y(t)+x^*) - f(x^*)]^T[f(y(t)+x^*) - f(x^*)]\\
\le & y^T(t)(L^2)^T L^2 y(t).
\end{aligned} \tag{32}$$

$$\psi^T(y(\gamma(t)))\psi(y(\gamma(t))) \le y^T(\gamma(t))(L^4)^T L^4 y(\gamma(t)). \tag{33}$$

By Lemma 3.1, we have

$$y^T(\gamma(t))y(\gamma(t)) \le \eta^2 y^T(t)y(t). \tag{34}$$

Substituting (29)-(34) into (28), we have

$$\begin{aligned}
& {}_0^C D_t^\alpha V(t)\\
\le & -\frac{1}{2}y^T(t)(PA)y(t) + \frac{1}{2}y^T(t)\Big(\frac{1}{\varepsilon}(PB)(PB)^T\\
& + \varepsilon(L^2)^T L^2 + \frac{1}{\varepsilon}(PC)(PC)^T + \varepsilon\eta^2(L^4)^T L^4\Big)y(t)\\
& + \beta\varphi^T(y(t))(-PA)y(t)) + \frac{1}{2}\beta y^T(t)(L^2)^T\Big(PB + (PB)^T\\
& + \frac{1}{\varepsilon}(PC)(PC)^T + \varepsilon\eta^2((L^1)^{-1}L^4)^T(L^1)^{-1}L^4\Big)L^2 y(t).
\end{aligned}$$

Let $\Sigma = \frac{1}{\varepsilon}(PB)(PB)^T + \varepsilon(L^2)^T L^2 + \frac{1}{\varepsilon}(PC)(PC)^T + \varepsilon\eta^2(L^4)^T L^4$. Obviously, $\Sigma > 0$.
Take $\beta > \frac{\lambda_{\max}(\Sigma)}{\lambda_{\min}(-(L^2)^T\widetilde{\Delta}L^2)} > 0$. Then

$$\begin{aligned}
& {}_0^C D_t^\alpha V(t)\\
\le & -\frac{1}{2}y^T(t)(PA)y(t) + \beta\varphi^T(y(t))(-PA)y(t))\\
& + \frac{1}{2}y^T(t)\Sigma y(t) + \frac{1}{2}\beta y^T(t)((L^2)^T\widetilde{\Delta}L^2)y(t)\\
\le & -\frac{1}{2}y^T(t)(PA)y(t) + \beta\varphi^T(y(t))(-PA)y(t))\\
\le & -\lambda_{\min}(A)\Big(\frac{1}{2}y^T(t)Py(t) + \beta\sum_{i=1}^{n} p_i\int_{y_i(\gamma(t))}^{y_i(t)}\varphi_i(s)ds\Big).
\end{aligned}$$

This implies that

$${}_0^C D_t^\alpha V(t) \le -\lambda V(t), \tag{35}$$

where $\lambda = \lambda_{\min}(A)$. By Lemma 2.2, from (35), we get

$$V(t) \le V(0)E_\alpha(-\lambda t^\alpha), \quad t > 0.$$

Noting that

$$\begin{aligned}
V(t) = & \frac{1}{2}y^T(t)Py(t) + \beta\sum_{i=1}^{n} p_i\int_{y_i(\gamma(t))}^{y_i(t)}\varphi_i(s)ds\\
\ge & \frac{1}{2}y^T(t)Py(t)\\
\ge & \frac{1}{2}\lambda_{\min}(P)\|y(t)\|^2,
\end{aligned}$$

we have

$$\|y(t)\|^2 \leq \frac{2}{\lambda_{\min}(P)} V(0) E_\alpha(-\lambda t^\alpha), \quad t > 0.$$

Hence, we obtain that

$$\begin{aligned}
&\|x(t) - x^*\|^2 \\
\leq &\frac{2}{\lambda_{\min}(P)} \Bigg(\frac{\lambda_{\max}(P)}{2} \|x(0) - x^*\|^2 \\
&+ \beta \sum_{i=1}^{n} p_i \int_0^{x(0)-x^*} \varphi_i(s) ds \Bigg) E_\alpha(-\lambda t^\alpha) \\
\leq &\frac{2}{\lambda_{\min}(P)} \Bigg(\frac{\lambda_{\max}(P)}{2} \|x(0) - x^*\|^2 + \\
&\beta \lambda_{\max}(P) \varphi^T(x(0) - x^*)(x(0) - x^*) \Bigg) E_\alpha(-\lambda t^\alpha) \\
\leq &\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \Big(1 + 2\beta L^4 \Big) \|x(0) - x^*\|^2 E_\alpha(-\lambda t^\alpha),
\end{aligned}$$

i.e.,

$$\|x(t) - x^*\| \leq \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \|x(0) - x^*\| (E_\alpha(-\lambda t^\alpha))^{\frac{1}{2}},$$

This shows that the network system (1) is global Mittag-Leffler stable. This completes the proof.

Remark 3.1. *It is easy to verify that $\eta > 1$ in Theorem 3.3. Hence, from $\widetilde{\Delta} < 0$, we can conclude that $\Delta < 0$ holds. This shows that, under the assumption of Theorem 3.3, there exist a unique equilibrium point and a unique solution with the initial value for the network system (1).*

Remark 3.2. *In[13-19], Lyapunov function is constructed by using the absolute value function $V(t) = \sum_{i=1}^{n} | x_i(t) |$. In [21-25, 35], Lyapunov function is considered by applying a positive definite quadratic function $V(t) = x^T(t)Px(t)$. However, in the proof of Theorem 3.3., Lur'e Postnikov type Lyapunov functional, which is composed of a positive definite quadratic function $\frac{1}{2}y^T(t)Py(t)$ and the integral term $\beta \sum_{i=1}^{n} \int_0^{z_i(t)} f_i(s)ds$, is utilized. It is easy to see that if $\beta = 0$, Lur'e Postnikov type Lyapunov functional is changed into the common positive definite quadratic function. Moreover, in Lur'e Postnikov type Lyapunov functional, neuron activation function $f(x)$ is used, which makes the obtained results possess less conservatism.*

In addition, compared with algebraic criteria in [13-19], the stability condition established in terms of LMIs in Theorem 3.3 has a great application value in practical engineering.

4. An illustrative example

In this section, an example is provided to illustrate the validity with respect to the results in this paper.

Example 1. *In the network system (1) with the piecewise constant argument $\gamma(t)$, we take the system parameters $\alpha = 0.98$,*
$A = \begin{pmatrix} 2 & 0 \\ 0 & 1.5 \end{pmatrix}$. $B = \begin{pmatrix} 0.02 & -0.03 \\ 0.01 & -1 \end{pmatrix}$. $C = \begin{pmatrix} 0.08 & 1 \\ 0.01 & 1 \end{pmatrix}$. $I = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $\gamma(t) = \zeta_k$, $\zeta_k = \frac{2k+1}{18}$, $k \in N$. $\theta_k = \frac{k}{9}$.

The active functions $f_i(x_i(t)) = \tanh(x_i(t)) + 0.5x_i(t)$, $i = 1, 2$. $g_1(x_1(t)) = \tanh(\frac{x_1(t)}{7}) + 0.5x_1(t)$, $g_2(x_2(t)) = \tanh(\frac{x_2(t)}{6}) + 0.5x_2(t)$.

It is easy to verify that $L_1^2 = L_2^2 = 3/2$, $L_1^4 = 9/14$, $L_2^4 = 2/3$, $m_1 = 2.0450$, $m_2 = 1.3333$, $\theta = 1/9$, $E_\alpha(m_1\theta^\alpha) = 1.2712$, $\eta = 2.0321$, and
$\frac{\theta^\alpha(m_1+m_2)}{\Gamma(1+\alpha)} \left(\frac{1}{E_\alpha(m_1\theta^\alpha)} - \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)} \right)^{-1} = 0.6272 < 1$,
$\frac{m_2\theta^\alpha}{\Gamma(1+\alpha)} + \frac{m_1\theta^\alpha}{\Gamma(1+\alpha)} \left(1 + \frac{m_2\theta^\alpha}{\Gamma(1+\alpha)} \right) E_\alpha(m_1\theta^\alpha) = 0.5709 < 1$.
Obviously, the assumption (H1)-(H5) hold.

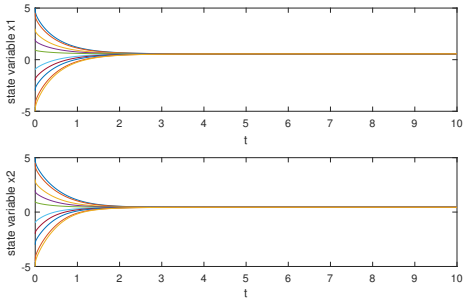


Figure 3: the state trajectory with different initial values of the network system (1) in Example 1

Set $\varepsilon = 1$. In the following, we check the valid of Theorem 3.3. To do so, by using LMI toolbox for solving $\widetilde{\Delta} < 0$ in Theorem 3.3, we get that

$$P = \begin{pmatrix} 0.0041 & 0 \\ 0 & 0.5196 \end{pmatrix}.$$

According to Theorem 3.3, we conclude that the unique equilibrium $x = (0.5385, 0.4424)$ of system (1) is Mittag-leffler stable, which is verified in Fig.3.*

Remark 4.1. *It is worthwhile pointing out that, due to the discontinuity of argument $\gamma(t)$, the solution of the network system (1) has the non-smoothness at the switching points θ_k, $k \in N$, however, this is not seen in Fig.3. The reason is that we choose Lipschitz constants with respect to the activation functions and a sufficiently*

small θ to fulfill the assumptions of Theorem 3.3. So, the sufficient small θ "hides" the turning point.

By taking appropriate parameters of the network system (1), it enables that non-smoothness can be seen to implement. Consider the network system

$$
\begin{aligned}
{}_0^C D_t^{\alpha} x(t) = &- \begin{pmatrix} 20 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\
&+ \begin{pmatrix} 2 & 1 \\ 8 & 0.2 \end{pmatrix} \begin{pmatrix} tanh(x_1(t)) + 0.1x_1(t) \\ tanh(x_2(t)) + 0.1x_2(t) \end{pmatrix} \\
&+ \begin{pmatrix} 1 & 20 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} tanh(x_1(t)) + 0.1x_1(t) \\ tanh(\frac{x_2(t)}{2}) + 0.1x_2(t) \end{pmatrix}
\end{aligned} \tag{36}
$$

where $\theta_k = \frac{k}{2}$, $\zeta_k = \frac{2k+1}{4}$, $k \in N$. Obviously, in system (36), Lipschitz coefficient with respect to the activation function is bigger than the aforementioned in Example 1, and $\theta = \frac{1}{2} > \frac{1}{9}$. The state of system (36) is depicted in Fig.4. From Fig.4, we can see clearly that the non-smoothness of the solution has emerged at the switching point $\theta_k, k \in N$.

In addition, from Fig.4, we can see that, the solution of system (36) converges to the unique equilibrium $x^* = (0.7598, 0.9944)^T$. However, it is easy to verify that the assumptions of Theorem 3.3 are not satisfied. This indicates that the assumption of Theorem 3.3 is only sufficient. That is to say, there exist some parameters, which do not satisfy the condition of Theorem 3.3, such that the network system (1) can also achieve the global Mittag-Leffler stability.

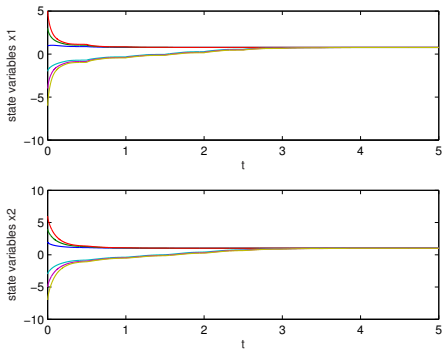


Figure 4: the state trajectory of the neural network system (36) with $\theta = 1/2$

5. Conclusion

In this paper, the global Mittag-Leffler stability has be investigated for FNNs with piecewise constant argument. An inequality with respect to the fractional derivative of integer-order variable upper limit integral has been developed. The existence and uniqueness of equilibrium point and solution of FNNs with piecewise constant argument have been proved, respectively. Lur'e Postnikov type Lyapunov functional has be applied to address the Mittag-Leffler stability condition in terms of LMIs. Numerical example has shown the validity of the obtained results via Matlab toolbox.

It would be interesting to extend the results proposed in this paper to the synchronization of FNNs with piecewise constant argument. This issue will be the topic of our future research.

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