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HYDROSTATIC APPROXIMATION OF THE 2D PRIMITIVE EQUATIONS IN A THIN STRIP

NACER AARACH AND VAN-SANG NGO

ABSTRACT. We prove the global wellposedness of the 2D non-rotating primitive equations with no-slip boundary conditions in a thin strip of width ε for small data which are analytic in the tangential direction. We also prove that the hydrostatic limit (when $\varepsilon \rightarrow 0$) is a couple of a Prandtl-like system for the velocity with a transport-diffusion equation for the temperature.

1. INTRODUCTION

In this paper, we study the two-dimensional Navier-Stokes system coupled with an evolution equation of the temperature in the thin-striped domain and provided with Dirichlet boundary conditions. Let $\mathcal{S}^\varepsilon = \{(x, y) \in \mathbb{R}^2 : 0 < y < \varepsilon\}$ where ε is the width of the strip. Then, our system writes

$$(1.1) \quad \begin{cases} \partial_t U^\varepsilon + U^\varepsilon \cdot \nabla U^\varepsilon - \varepsilon^2 \Delta U^\varepsilon + \nabla P^\varepsilon = \begin{pmatrix} 0 \\ \frac{T^\varepsilon}{Fr} \end{pmatrix} & \text{in } \mathcal{S}^\varepsilon \times]0, \infty[\\ \partial_t T^\varepsilon + U^\varepsilon \cdot \nabla T^\varepsilon - \Delta_\varepsilon T^\varepsilon = 0 & \text{in } \mathcal{S}^\varepsilon \times]0, \infty[\\ \operatorname{div} U^\varepsilon = 0 & \text{in } \mathcal{S}^\varepsilon \times]0, \infty[, \end{cases}$$

where $U^\varepsilon(t, x, y) = (U_1^\varepsilon(t, x, y), U_2^\varepsilon(t, x, y))$ denotes the velocity of the fluid and $P^\varepsilon(t, x, y)$ the scalar pressure function which guarantees the divergence-free property of the velocity field U^ε ; $T^\varepsilon(t, x, y)$ is the temperature of the system, and Fr is the Froude number measuring the importance of stratification, which is supposed to be εF where $F = 1$, as in the formulation introduced by Majda (see [15]). The system (1.1) is complemented by the no-slip boundary condition

$$U_{|y=0}^\varepsilon = U_{|y=\varepsilon}^\varepsilon = 0 \quad \text{and} \quad T_{|y=0}^\varepsilon = T_{|y=\varepsilon}^\varepsilon = 0.$$

Here, in the equation of the velocity, the Laplacian is $\Delta = \partial_x^2 + \partial_y^2$ and in the equation of the temperature, the anisotropic Laplacian $\Delta_\varepsilon = \partial_x^2 + \varepsilon^2 \partial_y^2$ reflects the difference between the horizontal and the vertical scales.

1.1. Physical motivations. For a geophysical fluid in a large volume scale (compared to the earth scale, for example, an ocean or the atmosphere), two main phenomena are important: the earth rotation and the density vertical stratification. The earth rotation induces two additional accelerations in the fluid equations: the centrifugal force which is included in the gravity gradient term and the Coriolis force which is characterized by the so-called Rossby number. The stratification forces the fluid masses to have a vertical distribution: heavier layers lay under lighter ones. Internal movements in the fluid tend to disturb this structure and the gravity basically tries to restore it constantly. The estimate of the importance of this rigidity on the movement leads to the comparison between the typical time scale of the system with the Brunt-Väisälä frequency and the definition of the Froude number Fr . For more details and physical considerations, we refer to [9], [12], [27], and [4] for example. In this paper, we will neglect the effect of the rotation and only focus on the effect of the vertical

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stratification as in the system (1.1). The combined effect of the rotation and the stratification in the full primitive equations will be studied in a forthcoming paper.

In order to describe hydrodynamical flows on the earth, in geophysics, it is usually assumed that vertical motion is much smaller than horizontal motion and that the fluid layer depth is small compared to the radius of the sphere, thus they are good approximation of global atmospheric flow and oceanic flow. The thin-stripped domain in the system (1.1) is considered to take into account this anisotropy between horizontal and vertical directions. Under this assumption, it is believed that the dynamics of fluids in large scale tends towards a geostrophic balance (see [19], [21] or [28]). In a formal way, as in [26], taking into account this anisotropy, we also consider the initial data in the following form,

$$U_{|t=0}^\varepsilon = U_0^\varepsilon = \left(u_0 \left(x, \frac{y}{\varepsilon} \right), \varepsilon v_0 \left(x, \frac{y}{\varepsilon} \right) \right) \quad \text{in } \mathcal{S}^\varepsilon$$

and

$$T_{|t=0}^\varepsilon = T_0^\varepsilon \left(x, \frac{y}{\varepsilon} \right).$$

In our paper, we look for solutions in the form

$$(1.2) \quad \begin{cases} U^\varepsilon(t, x, y) = \left(u^\varepsilon \left(t, x, \frac{y}{\varepsilon} \right), \varepsilon v^\varepsilon \left(t, x, \frac{y}{\varepsilon} \right) \right) \\ T^\varepsilon(t, x, y) = T^\varepsilon \left(t, x, \frac{y}{\varepsilon} \right) \\ P^\varepsilon(t, x, y) = p^\varepsilon \left(t, x, \frac{y}{\varepsilon} \right). \end{cases}$$

Performing the scaling change $\bar{y} = \frac{y}{\varepsilon}$ and let $\mathcal{S} = \{(x, \bar{y}) \in \mathbb{R}^2 : 0 < \bar{y} < 1\}$, we can rewrite the system (1.1) as follows

$$(1.3) \quad \begin{cases} \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_{\bar{y}} u^\varepsilon - \varepsilon^2 \partial_x^2 u^\varepsilon - \partial_{\bar{y}}^2 u^\varepsilon + \partial_x p^\varepsilon = 0 & \text{in } \mathcal{S} \times]0, \infty[\\ \varepsilon^2 (\partial_t v^\varepsilon + u^\varepsilon \partial_x v^\varepsilon + v^\varepsilon \partial_{\bar{y}} v^\varepsilon - \varepsilon^2 \partial_x^2 v^\varepsilon - \partial_{\bar{y}}^2 v^\varepsilon) + \partial_{\bar{y}} p^\varepsilon = T^\varepsilon & \text{in } \mathcal{S} \times]0, \infty[\\ \partial_t T^\varepsilon + u^\varepsilon \partial_x T^\varepsilon + v^\varepsilon \partial_{\bar{y}} T^\varepsilon - \Delta T^\varepsilon = 0 & \text{in } \mathcal{S} \times]0, \infty[\\ \partial_x u^\varepsilon + \partial_{\bar{y}} v^\varepsilon = 0 & \text{in } \mathcal{S} \times]0, \infty[\\ (u^\varepsilon, v^\varepsilon, T^\varepsilon)|_{t=0} = (u_0, v_0, T_0) & \text{in } \mathcal{S} \\ (u^\varepsilon, v^\varepsilon, T^\varepsilon)|_{\bar{y}=0} = (u^\varepsilon, v^\varepsilon, T^\varepsilon)|_{\bar{y}=1} = 0. \end{cases}$$

Formally taking $\varepsilon \rightarrow 0$ in the system (1.3), and writing y instead of \bar{y} when there is no confusion, we obtain the following hydrostatic primitive equations, which are the couple of a Prandtl-like system with a transport-diffusion equation of the temperature

$$(1.4) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = 0 & \text{in } \mathcal{S} \times]0, \infty[\\ \partial_y p = T & \text{in } \mathcal{S} \times]0, \infty[\\ \partial_t T + u \partial_x T + v \partial_y T - \Delta T = 0 & \text{in } \mathcal{S} \times]0, \infty[\\ \partial_x u + \partial_y v = 0 & \text{in } \mathcal{S} \times]0, \infty[\\ u|_{t=0} = u_0 & \text{in } \mathcal{S} \\ T|_{t=0} = T_0 & \text{in } \mathcal{S}, \end{cases}$$

where the velocity $U = (u, v)$ and the temperature T satisfy the Dirichlet no-slip boundary condition

$$(1.5) \quad (u, v, T)|_{y=0} = (u, v, T)|_{y=1} = 0.$$

We remark that in the system (1.4), we have to deal with the same difficulty as for Prandtl equations due to its degenerate form and the nonlinear term $v \partial_y u$ which will lead to the loss of one derivative in the tangential direction in the process of energy estimates. For a more complete

survey on this very challenging problem and we suggest the reader to the works [1, 13, 14, 16, 23] and references therein. To overcome this difficulty, one has to impose a monotonic hypothesis on the normal derivative of the velocity or an analytic regularity on the velocity. After the pioneer works of Oleinik [24] using the Crocco transformation under the monotonic hypothesis, Sammartino and Caffisch [29] solved the problem for analytic solutions on a half space and later, the analyticity in normal variable y was removed by Lombardo, Cannone and Sammartino in [22]. The main argument used in [29, 22] is to apply the abstract Cauchy-Kowalewskaya (CK) theorem. We also mention a well-posedness result of Prandtl system for a class of data with Gevrey regularity [17]. Lately, for a class of convex data, Gérard-Varet, Masmoudi and Vicol [18] proved the well-posedness of the Prandtl system in the Gevrey class. We also want to remark that unlike the case of Prandtl equations, in the system (1.4), the pressure term is not defined by the outer flows using Bernoulli's law but by temperature via the relation $\partial_y p = T$. One of the novelties of the paper is to find a way to treat the pressure term using the temperature equation.

We also want to recall some results concerning the system (1.3). This system was studied in the 90's by Lions-Temam-Wang [30, 31, 32], where the authors considered full viscosity and diffusivity, and establish the global existence of weak solutions. Concerning the strong solutions for the 2D case, the locale existence result was established by Guillén-González, Masmoudi and Rodriguez-Bellido [20], while the global existence for 2D case was achieved by Bresch, Kazhikhov and Lemoine in [5] and by Temam and Ziane in [33]. In our paper we also want to establish the global well posedness of the system (1.3) in 2D case but in a thin strip.

1.2. Functional framework. In order to introduce our results, we will briefly recall some elements of the Littlewood-Paley theory and introduce the function spaces and techniques using throughout our paper. Let ψ be an even smooth function in $C_0^\infty(\mathbb{R})$ such that the support is contained in the ball $B_{\mathbb{R}}(0, \frac{4}{3})$ and ψ is equal to 1 on a neighborhood of the ball $B_{\mathbb{R}}(0, \frac{3}{4})$. Let $\varphi(z) = \psi(\frac{z}{2}) - \psi(z)$. Thus, the support of φ is contained in the ring $\{z \in \mathbb{R} : \frac{3}{4} \leq |z| \leq \frac{8}{3}\}$, and φ is identically equal to 1 on the ring $\{z \in \mathbb{R} : \frac{4}{3} \leq |z| \leq \frac{3}{2}\}$. The functions ψ and φ enjoy the very important properties

$$(1.6) \quad \forall z \in \mathbb{R}, \quad \psi(z) + \sum_{j \in \mathbb{N}} \varphi(2^{-j}z) = 1,$$

and

$$\forall j, j' \in \mathbb{N}, |j - j'| \geq 2, \quad \text{supp } \varphi(2^{-j}\cdot) \cap \text{supp } \varphi(2^{-j'}\cdot) = \emptyset.$$

Let \mathcal{F}_h and \mathcal{F}_h^{-1} be the Fourier transform and the inverse Fourier transform respectively in the horizontal direction. We will also use the notation $\widehat{u} = \mathcal{F}_h u$. We introduce the following definitions of the homogeneous dyadic cut-off operators.

Definition 1.1. *For all tempered distribution u in the horizontal direction (of x variable) and for all $q \in \mathbb{Z}$, we set*

$$\begin{aligned} \Delta_q^h u(x, y) &= \mathcal{F}_h^{-1} (\varphi(2^{-q}|\xi|)\widehat{u}(\xi, y)), \\ S_q^h u(x, y) &= \mathcal{F}_h^{-1} (\psi(2^{-q}|\xi|)\widehat{u}(\xi, y)). \end{aligned}$$

We refer to [2] and [3] for a more detailed construction of the dyadic decomposition. This definition, combined with the equality (1.6), implies that all tempered distributions can be decomposed with respect to the horizontal frequencies as

$$u = \sum_{q \in \mathbb{Z}} \Delta_q^v u.$$

The following Bernstein lemma gives important properties of a distribution u when its Fourier transform is well localized. We refer the reader to [7] for the proof of this lemma.

Lemma 1.2. *Let $k \in \mathbb{N}$, $d \in \mathbb{N}^*$ and $r_1, r_2 \in \mathbb{R}$ satisfy $0 < r_1 < r_2$. There exists a constant $C > 0$ such that, for any $a, b \in \mathbb{R}$, $1 \leq a \leq b \leq +\infty$, for any $\lambda > 0$ and for any $u \in L^a(\mathbb{R}^d)$, we have*

$$\text{supp } (\widehat{u}) \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leq r_1 \lambda\} \implies \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^k \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a},$$

and

$$\text{supp } (\widehat{u}) \subset \{\xi \in \mathbb{R}^d \mid r_1 \lambda \leq |\xi| \leq r_2 \lambda\} \implies C^{-k} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^k \lambda^k \|u\|_{L^a}.$$

We now introduce the function spaces used throughout the paper. As in [26], we define the Besov-type spaces \mathcal{B}^s , $s \in \mathbb{R}$ as follows.

Definition 1.3. *Let $s \in \mathbb{R}$ and $\mathcal{S} = \mathbb{R} \times]0, 1[$. For any $u \in S'_h(\mathcal{S})$, i.e., u belongs to $S'(\mathcal{S})$ and $\lim_{q \rightarrow -\infty} \|S_q^h u\|_{L^\infty} = 0$, we set*

$$\|u\|_{\mathcal{B}^s} \stackrel{\text{def}}{=} \sum_{q \in \mathbb{Z}} 2^{qs} \|\Delta_q^h u\|_{L^2}.$$

(i) For $s \leq \frac{1}{2}$, we define

$$\mathcal{B}^s(\mathcal{S}) \stackrel{\text{def}}{=} \{u \in S'_h(\mathcal{S}) : \|u\|_{\mathcal{B}^s} < +\infty\}.$$

(ii) For $s \in]k - \frac{1}{2}, k + \frac{1}{2}[$, with $k \in \mathbb{N}^*$, we define $\mathcal{B}^s(\mathcal{S})$ as the subset of distributions u in $S'_h(\mathcal{S})$ such that $\partial_x^k u \in \mathcal{B}^{s-k}(\mathcal{S})$.

For a better use of the smoothing effect given by the diffusion terms, we will work in the following Chemin-Lerner type spaces and also the time-weighted Chemin-Lerner type spaces.

Definition 1.4. *Let $p \in [1, +\infty]$ and $T \in]0, +\infty]$. Then, the space $\tilde{L}_T^p(\mathcal{B}^s(\mathcal{S}))$ is the closure of $C([0, T]; S(\mathcal{S}))$ under the norm*

$$\|u\|_{\tilde{L}_T^p(\mathcal{B}^s(\mathcal{S}))} \stackrel{\text{def}}{=} \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T \|\Delta_q^h u(t)\|_{L^2}^p dt \right)^{\frac{1}{p}},$$

with the usual change if $p = +\infty$.

Definition 1.5. *Let $p \in [1, +\infty]$ and let $f \in L_{loc}^1(\mathbb{R}_+)$ be a nonnegative function. Then, the space $\tilde{L}_{t,f(t)}^p(\mathcal{B}^s(\mathcal{S}))$ is the closure of $C([0, T]; S(\mathcal{S}))$ under the norm*

$$\|u\|_{\tilde{L}_{t,f(t)}^p(\mathcal{B}^s(\mathcal{S}))} \stackrel{\text{def}}{=} \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^t f(t') \|\Delta_q^h u(t')\|_{L^2}^p dt' \right)^{\frac{1}{p}}.$$

1.3. Main results. Our main difficulty relies in finding a way to estimate the nonlinear terms, which allows to exploit the smoothing effect given by the above function spaces. Using the method introduced by Chemin in [8] (see also [10], [25] or [26]), for any $f \in L^2(\mathcal{S})$, we define the following auxiliary function, which allows to control the analyticity of f in the horizontal variable x ,

$$(1.7) \quad f_\phi(t, x, y) = e^{\phi(t, D_x)} f(t, x, y) \stackrel{\text{def}}{=} \mathcal{F}_h^{-1}(e^{\phi(t, \xi)} \widehat{f}(t, \xi, y)) \quad \text{with} \quad \phi(t, \xi) = (a - \lambda\theta(t))|\xi|,$$

where the quantity $\theta(t)$, which describes the evolution of the analytic band of f , satisfies

$$(1.8) \quad \forall t > 0, \dot{\theta}(t) \geq 0 \quad \text{and} \quad \theta(0) = 0.$$

The main idea of this technique consists in the fact that if we differentiate, with respect to the time variable, a function of the type $e^{\phi(t, D_x)} f(t, x, y)$, we obtain an additional “good term” which plays the smoothing role. More precisely, we have

$$\frac{d}{dt} (e^{\phi(t, D_x)} f(t, x, y)) = -\dot{\theta}(t) |D_x| e^{\phi(t, D_x)} f(t, x, y) + e^{\phi(t, D_x)} \partial_t f(t, x, y),$$

where $-\dot{\theta}(t) |D_x| e^{\phi(t, D_x)} f(t, x, y)$ gives a smoothing effect if $\dot{\theta}(t) \geq 0$. This smoothing effect allows to obtain our global existence and stability results in the analytic framework. Remark that the existence in the Prandtl case, we only have the local existence and the convergence is still an open question! Besides, Prandtl system is known to be very unstable.

Our main results are the following theorems.

Theorem 1.6 (Global wellposedness of the hydrostatic limit system). *Let $a > 0$. There exists a constant $c_0 > 0$ sufficiently small, independent of ε and there holds the compatibility condition $\int_0^1 u_0 dy = 0$, such that, for any data (u_0, v_0, T_0) satisfying*

$$\|e^{a|D_x|} u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|} T_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_0 a,$$

the system (1.4) has a unique global solution (u, v, T) satisfying

$$(1.9) \quad \|e^{\mathcal{R}t} (u_\phi, T_\phi)\|_{\tilde{L}^\infty(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})} + \frac{1}{2} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})} \leq 2C \|e^{a|D_x|} (u_0, T_0)\|_{\mathcal{B}^{\frac{1}{2}}},$$

where u_ϕ is determined by (1.7). Furthermore, if $e^{a|D_x|} u_0 \in \mathcal{B}^{\frac{5}{2}}$, $e^{a|D_x|} T_0 \in \mathcal{B}^{\frac{3}{2}}$, $e^{a|D_x|} \partial_y u_0 \in \mathcal{B}^{\frac{3}{2}}$ and

$$(1.10) \quad \|e^{a|D_x|} u_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{c_1 a}{1 + \|e^{a|D_x|} u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|} T_0\|_{\mathcal{B}^{\frac{3}{2}}}}$$

for some c_1 sufficiently small, then there exists a positive constant C so that for $\lambda = C^2(1 + \|e^{a|D_x|} u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|} T_0\|_{\mathcal{B}^{\frac{3}{2}}})$, and $1 \leq s \leq \frac{5}{2}$, one has

$$(1.11) \quad \|e^{\mathcal{R}t} (\partial_t u)_\phi\|_{\tilde{L}^2_t(\mathcal{B}^{\frac{3}{2}})} + \frac{1}{2} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}^\infty_t(\mathcal{B}^{\frac{3}{2}})} \lesssim C \left(\|e^{a|D_x|} \partial_y u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|} \partial_y u_0\|_{\mathcal{B}^{\frac{5}{2}}} + \|e^{a|D_x|} \partial_y T_0\|_{\mathcal{B}^{\frac{3}{2}}} \right).$$

Theorem 1.7 (Global wellposedness of the primitive system). *Let $a > 0$. There exists a constant $c_2 > 0$ sufficiently small, independent of ε , such that, for any data (u_0, v_0, T_0) satisfying*

$$\|e^{a|D_x|} (u_0, \varepsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|} T_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_2 a,$$

then the system (1.4) has a unique global solution (u, v) satisfying

$$\begin{aligned} & \|e^{\mathcal{R}t} (u_\Theta, \varepsilon v_\Theta, T_\Theta)\|_{\tilde{L}^\infty(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t} \partial_y (u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})} + \varepsilon \|e^{\mathcal{R}t} \partial_x (u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})} \\ & + \|e^{\mathcal{R}t} \nabla T_\Theta\|_{\tilde{L}^2_t(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})} \leq C \|e^{a|D_x|} (u_0, \varepsilon v_0, T_0)\|_{\mathcal{B}^{\frac{1}{2}}}, \end{aligned}$$

where $(u_\Theta^\varepsilon, v_\Theta^\varepsilon)$ is determined by (3.31) and the constant \mathcal{R} is determined by Poincaré inequality on the strip \mathcal{S} .

Theorem 1.8 (Convergence to the hydrostatic limit system). *Let $a > 0$ and $(u_\Theta^\varepsilon, v_\Theta^\varepsilon)$ satisfy the initial condition of the theorem (1.7). Let u_0 satisfy*

$$e^{a|D_x|} u_0 \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{5}{2}}, \quad e^{a|D_x|} \partial_y u_0 \in \mathcal{B}^{\frac{3}{2}}, \quad e^{a|D_x|} T_0 \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{3}{2}},$$

and there holds (1.10) for some c_1 sufficiently small and the compatibility condition $\int_0^1 u_0 dy = 0$. Then we have

$$(1.12) \quad \begin{aligned} & \| (w_\varphi^1, \varepsilon w_\varphi^2) \|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \| \partial_y (w_\varphi^1, \varepsilon w_\varphi^2) \|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \varepsilon \| (w_\varphi^1, \varepsilon w_\varphi^2) \|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \\ & \leq C \left(\| e^{a|D_x|} (u_0^\varepsilon - u_0, \varepsilon(v_0^\varepsilon - v_0)) \|_{\mathcal{B}^{\frac{1}{2}}} + C \| e^{a|D_x|} (T_0^\varepsilon - T_0) \|_{\mathcal{B}^{\frac{1}{2}}} + M\varepsilon \right). \end{aligned}$$

Where $w^1 = u^\varepsilon - u$, $w^2 = v^\varepsilon - v$, $\theta = T^\varepsilon - T$ and v_0 is determined from u_0 via the free divergence and the boundary condition of the initial data with respect y , and $(w_\varphi^1, \varepsilon w_\varphi^2)$ is given by (4.51) and $M \geq 1$ is a constant independent to ε .

The proofs of our main theorems rely on the following lemmas which will be proved in the appendix.

Lemma 1.9. *Let $s \in]0, 1]$, $T > 0$ and ϕ be defined as in (1.7), with $\dot{\theta}(t) = \|\partial_y u_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$. There exist $C \geq 1$ such that, for any $t > 0$, $\phi(t, \xi) > 0$ and for any $w \in \tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})$, we have*

$$(1.13) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x w)_\phi, e^{\mathcal{R}t'} \Delta_q^h w_\phi \right\rangle_{L^2} \right| dt' \leq C \| e^{\mathcal{R}t} w_\phi \|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

Lemma 1.10. *For any $s \in]0, 1]$ and $t \leq T^*$, there exist $C \geq 1$ such that,*

$$(1.14) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle_{L^2} \right| dt' \leq C \| e^{\mathcal{R}t} u_\phi \|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

and

$$(1.15) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y T)_\phi, e^{\mathcal{R}t'} \Delta_q^h T_\phi \right\rangle_{L^2} \right| dt' \leq C \| u_\phi \|_{\mathcal{B}^{\frac{1}{2}}} \| e^{\mathcal{R}t} \nabla T_\phi \|_{\tilde{L}_t^2(\mathcal{B}^s)}.$$

1.4. Organisation of the paper. Our paper will be divided into several sections as follows. In Section 2, we prove the global wellposedness of the system (1.4) for small data in analytic framework. Section 3 is devoted to the study of the system (1.3) and the proof of Theorem 1.7. In Section 4, we prove the convergence of the system (1.3) towards the system (1.4) when $\varepsilon \rightarrow 0$. Finally, in the appendix, we give the proofs of Lemmas 1.9 and 1.10.

2. GLOBAL WELLPOSEDNESS OF THE HYDROSTATIC LIMIT SYSTEM

The goal of this section is to prove Theorem 1.6. We remark that the construction of a local smooth solution of the system (1.4) follows a standard parabolic regularization method, similar to the case of Prandtl system, which consists of adding an addition horizontal smoothing term of the type $\delta \partial_x^2$ and then taking $\delta \rightarrow 0$. The difficulty here consists in the presence of the unknown pressure term $\partial_x p$ in the first equation of (1.4). However, as in [6], we can reformulate the problem by writing v and $\partial_x p$ as functions of u and T . First, we remark that the Dirichlet boundary condition $(u, v)|_{t=0} = (u, v)|_{t=1} = 0$ and the incompressibility condition $\operatorname{div} u = \partial_x u + \partial_y v = 0$ imply

$$(2.16) \quad v(t, x, y) = \int_0^y \partial_y v(t, x, s) ds = - \int_0^y \partial_x u(t, x, s) ds.$$

We want now to find the equation for the pressure. Due to the Dirichlet boundary condition $(u, v, T)|_{y=0} = 0$, we deduce from the incompressibility condition $\partial_x u + \partial_y v = 0$ that

$$(2.17) \quad \partial_x \int_0^1 u(t, x, y) dy = - \int_0^1 \partial_y v(t, x, y) dy = v(t, x, 1) - v(t, x, 0) = 0.$$

Integrating the equation $\partial_y p = T$ with respect y in $[0, y]$, we obtain

$$(2.18) \quad p(t, x, y) = p(t, x, 0) + \int_0^y T(t, x, s) ds.$$

Next, differentiating (2.18) with respect to x and using the first equation of the system (1.4), we get

$$\begin{aligned} \partial_x p(t, x, 0) &= - \int_0^y \partial_x T(t, x, y') dy' + \partial_x p(t, x, y) \\ &= - \int_0^y \partial_x T(t, x, y') dy' - (\partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u)(t, x, y) \end{aligned}$$

Integrating the above equation with respect to $y \in [0, 1]$ and performing integration by part lead to

$$\partial_x p(t, x, 0) = - \int_0^1 \int_0^y \partial_x T(t, x, y') dy' dy + \partial_y u(t, x, 1) - \partial_y u(t, x, 0) - \dot{c}(t) - \partial_x \int_0^1 u^2(t, x, y) dy$$

with $c(t) = \int_0^t u(t, x, y) dy$, then we replace we get

$$\begin{aligned} \partial_x p(t, x, y) &= \int_0^y \partial_x T(t, x, s) - \int_0^1 \int_0^y \partial_x T(t, x, y') dy' dy \\ &\quad + \partial_y u(t, x, 1) - \partial_y u(t, x, 0) - \dot{c}(t) - \partial_x \int_0^1 u^2(t, x, y) dy. \end{aligned}$$

Let (u_ϕ, v_ϕ, T_ϕ) be defined as in (1.7) and (1.8). Direct calculations from (1.4) show that (u_ϕ, v_ϕ, T_ϕ) satisfy the system

$$(2.19) \quad \begin{cases} \partial_t u_\phi + \lambda \dot{\theta}(t) |D_x| u_\phi + (u \partial_x u)_\phi + (v \partial_y u)_\phi - \partial_y^2 u_\phi + \partial_x p_\phi = 0 & \text{in } \mathcal{S} \times]0, \infty[, \\ \partial_y p_\phi = T_\phi \\ \partial_t T_\phi + \lambda \dot{\theta}(t) |D_x| T_\phi + (u \partial_x T)_\phi + (v \partial_y T)_\phi - \Delta T_\phi = 0 \\ \partial_x u_\phi + \partial_y v_\phi = 0, \\ u_\phi|_{t=0} = u_0, \\ T_\phi|_{t=0} = T_0, \end{cases}$$

where $|D_x|$ denotes the Fourier multiplier of symbol $|\xi|$. In what follows, we recall that we use “ C ” to denote a generic positive constant which can change from line to line.

Applying the dyadic operator Δ_q^h to the system (2.19), then taking the $L^2(\mathcal{S})$ scalar product of the first and the third equations of the obtained system with $\Delta_q^h u_\phi$ and $\Delta_q^h T_\phi$ respectively, we get

$$(2.20) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q^h u_\phi(t)\|_{L^2}^2 + \lambda \dot{\theta}(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi \right\|_{L^2}^2 + \|\Delta_q^h \partial_y u_\phi(t)\|_{L^2}^2 \\ = - \langle \Delta_q^h (u \partial_x u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} - \langle \Delta_q^h (v \partial_y u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h u_\phi \rangle_{L^2}, \end{aligned}$$

and

$$(2.21) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q^h T_\phi(t)\|_{L^2}^2 + \lambda \dot{\theta}(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h T_\phi \right\|_{L^2}^2 + \|\Delta_q^h \partial_y T_\phi(t)\|_{L^2}^2 + \|\Delta_q^h \partial_x T_\phi(t)\|_{L^2}^2 \\ = - \langle \Delta_q^h (u \partial_x T)_\phi, \Delta_q^h T_\phi \rangle_{L^2} - \langle \Delta_q^h (v \partial_y T)_\phi, \Delta_q^h T_\phi \rangle_{L^2}. \end{aligned}$$

Multiplying (2.20) and (2.21) with $e^{2\mathcal{R}t}$ and then integrating with respect to the time variable, we have

$$(2.22) \quad \begin{aligned} & \|e^{\mathcal{R}t} \Delta_q^h u_\phi(t)\|_{L_t^\infty L^2}^2 + \lambda \int_0^t \dot{\theta}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi \right\|_{L^2}^2 dt' + \|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\phi(t)\|_{L_t^2 L^2}^2 \\ & = \|\Delta_q^h u_\phi(0)\|_{L^2}^2 + D_1 + D_2 + D_3, \end{aligned}$$

and

$$(2.23) \quad \begin{aligned} & \|e^{\mathcal{R}t} \Delta_q^h T_\phi(t)\|_{L_t^\infty L^2}^2 + \lambda \int_0^t \dot{\theta}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h T_\phi \right\|_{L^2}^2 dt' + \|e^{\mathcal{R}t} \Delta_q^h \nabla T_\phi(t)\|_{L_t^2 L^2}^2 \\ & = \|\Delta_q^h T_\phi(0)\|_{L^2}^2 + D_4 + D_5. \end{aligned}$$

Next, Lemmas 1.9 and 1.10 yield

$$\begin{aligned} |D_1| & = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \\ |D_2| & = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \\ |D_4| & = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x T)_\phi, e^{\mathcal{R}t'} \Delta_q^h T_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} T_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \end{aligned}$$

and

$$|D_5| = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y T)_\phi, e^{\mathcal{R}t'} \Delta_q^h T_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2.$$

Now, for the pressure term, using the Dirichlet boundary condition $(u, v, T)|_{y=0} = 0$, and the incompressibility condition $\partial_x u + \partial_y v = 0$ and the relation $\partial_y p = T$, we can perform integrations by parts, use Poincaré's inequality and get

$$\begin{aligned} |\langle \Delta_q^h \partial_x p_\phi, \Delta_q^h u_\phi \rangle| & = |\langle \Delta_q^h p_\phi, \Delta_q^h \partial_x u_\phi \rangle| = |\langle \Delta_q^h p_\phi, \Delta_q^h \partial_y v_\phi \rangle| = |\langle \Delta_q^h \partial_y p_\phi, \Delta_q^h v_\phi \rangle| \\ & = |\langle \Delta_q^h T_\phi, \Delta_q^h v_\phi \rangle| = \left| \left\langle \Delta_q^h T_\phi, \Delta_q^h \int_0^y \partial_x u_\phi dy \right\rangle \right| = \left| \left\langle \Delta_q^h \partial_x T_\phi, \Delta_q^h \int_0^y u_\phi dy \right\rangle \right| \\ & \leq \|\Delta_q^h \partial_x T_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} \leq C \|\Delta_q^h \partial_x T_\phi\|_{L^2}^2 + \frac{1}{2} \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2. \end{aligned}$$

Thus,

$$|D_3| = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h \partial_x p_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\phi(t)\|_{L_t^2 L^2}^2.$$

Multiplying (2.22) and (2.23) by 2^{2qs} and summing with respect to $q \in \mathbb{Z}$, we obtain

$$(2.24) \quad \begin{aligned} & \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}^2 + \lambda \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \\ & \leq \|u_\phi(0)\|_{\mathcal{B}^s}^2 + C \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + C \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \frac{1}{2} \|e^{\mathcal{R}t} \partial_y u_\phi(t)\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2, \end{aligned}$$

and

$$(2.25) \quad \begin{aligned} & \|e^{\mathcal{R}t} T_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}^2 + \lambda \|e^{\mathcal{R}t} T_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|e^{\mathcal{R}t} \nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \\ & \leq \|T_\phi(0)\|_{\mathcal{B}^s}^2 + C \|e^{\mathcal{R}t} T_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + C \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2. \end{aligned}$$

From (2.25), we remark that

$$\|e^{\mathcal{R}t}\partial_x T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \leq \|T_\phi(0)\|_{\mathcal{B}^s}^2 + C\|e^{\mathcal{R}t}T_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + C\|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}\|e^{\mathcal{R}t}\nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2.$$

Thus, choosing

$$(2.26) \quad C \geq \max\left\{4, \frac{1}{2\mathcal{R}}\right\},$$

and taking the sum of (2.24) and (2.25), we have

$$(2.27) \quad \begin{aligned} & \|e^{\mathcal{R}t}(u_\phi, T_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}^2 + \lambda\|e^{\mathcal{R}t}(u_\phi, T_\phi)\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \frac{1}{2}\|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \|e^{\mathcal{R}t}\nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \\ & \leq 2C\|e^{a|D_x|}(u_0, T_0)\|_{\mathcal{B}^s}^2 + 2C^2\|e^{\mathcal{R}t}(u_\phi, T_\phi)\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + 2C^2\|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}\|e^{\mathcal{R}t}\nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2. \end{aligned}$$

We set

$$(2.28) \quad T^* \stackrel{\text{def}}{=} \sup\left\{t > 0 : \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{1}{2C^2} \text{ and } \theta(t) \leq \frac{a}{\lambda}\right\}.$$

We choose initial data such that

$$C\left(\|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}T_0\|_{\mathcal{B}^{\frac{1}{2}}}\right) < \min\left\{\frac{1}{2C^2}, \frac{a}{2\lambda}\right\},$$

then, combining with the fact that $\theta(0) = 0$, we deduce that $T^* > 0$. We choose now $\lambda = 2C^2$. For any $0 < t < T^*$, we deduce from (2.27) that

$$(2.29) \quad \|e^{\mathcal{R}t}(u_\phi, T_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}^2 + \frac{1}{2}\|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \leq 2C\|e^{a|D_x|}(u_0, T_0)\|_{\mathcal{B}^s}^2.$$

We then deduce from (2.29), using (2.26), that, for any $0 < t < T^*$,

$$\|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \leq \|e^{\mathcal{R}t}(u_\phi, T_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \leq C\|e^{a|D_x|}(u_0, T_0)\|_{\mathcal{B}^s} \leq C\left(\|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}T_0\|_{\mathcal{B}^{\frac{1}{2}}}\right) < \frac{1}{2C^2}.$$

Now, we recall that we already defined $\theta(t) = \|\partial_y u_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$ with $\theta(0) = 0$. Then, for any $0 < t < T^*$, Inequality (2.29) yields

$$\begin{aligned} \theta(t) &= \int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \leq \int_0^t e^{-\mathcal{R}t'} \|e^{\mathcal{R}t'} \partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \\ &\leq \left(\int_0^t e^{-2\mathcal{R}t'} dt'\right)^{\frac{1}{2}} \left(\int_0^t \|e^{\mathcal{R}t'} \partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 dt'\right)^{\frac{1}{2}} \\ &\leq C\left(\|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}T_0\|_{\mathcal{B}^{\frac{1}{2}}}\right) < \frac{a}{2\lambda}. \end{aligned}$$

A continuity argument implies that $T^* = +\infty$ and we have (2.29) is valid for any $t \in \mathbb{R}_+$.

In order to end the proof of Theorem 1.6, we only need to prove Inequality (1.11). For that, we apply Δ_q^h to (2.19) and take the L^2 inner product of resulting equation with $\Delta_q^h(\partial_t u)_\phi$. That yields

$$\begin{aligned} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 &= \langle \Delta_q^h \partial_y^2 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} - \langle \Delta_q^h(u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \\ &\quad - \langle \Delta_q^h(v \partial_y u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2}. \end{aligned}$$

The fact that $(\partial_t u)_\phi = \partial_t u_\phi + \lambda \dot{\theta}(t)|D_x|u_\phi$ implies

$$\langle \Delta_q^h \partial_y^2 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} = -\left(\frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2 + \lambda \dot{\theta}(t) 2^q \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2\right),$$

from which, we deduce that

$$\|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2 \leq I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \left| \langle \Delta_q^h(u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \\ I_2 &= \left| \langle \Delta_q^h(v \partial_y u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \\ I_3 &= \left| \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right|. \end{aligned}$$

Since $\partial_x u + \partial_y v = 0$, using (2.16) and integrations by parts, we find

$$I_3 = \left| \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \leq \|\Delta_q^h \partial_x T_\phi\|_{L^2} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2} \leq \frac{1}{2} \|\Delta_q^h \partial_x T_\phi\|_{L^2}^2 + \frac{1}{2} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2.$$

For I_1, I_2 we have

$$\begin{aligned} I_1 &= \left| \langle \Delta_q^h(u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \leq \frac{1}{2} \|\Delta_q^h(u \partial_x u)_\phi\|_{L^2}^2 + \frac{1}{10} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 \\ I_2 &= \left| \langle \Delta_q^h(v \partial_y u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \leq \frac{1}{2} \|\Delta_q^h(v \partial_y u)_\phi\|_{L^2}^2 + \frac{1}{10} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2. \end{aligned}$$

Then, we deduce that

$$\|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2 \leq C \left(\|\Delta_q^h(u \partial_x u)_\phi\|_{L^2}^2 + \|\Delta_q^h(v \partial_y u)_\phi\|_{L^2}^2 + \|\Delta_q^h \partial_x T_\phi\|_{L^2}^2 \right).$$

Multiplying the result by $e^{2\mathcal{R}t}$ and integrating over $[0, t]$, we get

$$\begin{aligned} &\|e^{\mathcal{R}t} \Delta_q^h(\partial_t u)_\phi\|_{L_t^2(L^2)}^2 + \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\phi\|_{L_t^\infty(L^2)}^2 \\ &\leq C \left(\|\Delta_q^h \partial_y e^{a|D_x|} u_0\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h(u \partial_x u)_\phi\|_{L_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h(v \partial_y u)_\phi\|_{L_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_x T_\phi\|_{L_t^2(L^2)}^2 \right). \end{aligned}$$

Multiplying the above inequality by 2^{3q} , then taking the square root of the resulting estimate, and finally summing up the obtained equations with respect to $q \in \mathbb{Z}$, we obtain

$$(2.30) \quad \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \frac{1}{2} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \leq C \left(\|e^{a|D_x|} \partial_y u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{\mathcal{R}t}(u \partial_x u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(v \partial_y u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right).$$

Next, it follows from the law of product in anisotropic Besov spaces and Poincaré inequality that

$$\begin{aligned} \|e^{\mathcal{R}t}(u \partial_x u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} &\leq C \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})}; \\ \|e^{\mathcal{R}t}(v \partial_y u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} &\leq C \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} + \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{5}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}. \end{aligned}$$

Inserting the above estimates into (2.30) and then using the smallness condition $\|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{1}{2C^2}$, we finally obtain

$$\|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \frac{1}{2} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \leq C \left(\|e^{a|D_x|} \partial_y u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|} \partial_y u_0\|_{\mathcal{B}^{\frac{5}{2}}} + \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right).$$

Theorem 1.6 is then proved. \square

3. GLOBAL WELL-POSEDNESS OF THE 2D NON-ROTATING PRIMITIVE EQUATIONS IN A THIN STRIP

The goal of this section is to prove Theorem 1.7 and to establish the global well-posedness of the system (1.3) with small analytic data. As in Section 2, for any locally bounded function Θ on $\mathbb{R}_+ \times \mathbb{R}$ and any $f \in L^2(\mathcal{S})$, we define the analyticity in the horizontal variable x by means of the following auxiliary function

$$(3.31) \quad f_\Theta^\varepsilon(t, x, y) = \mathcal{F}_{\xi \rightarrow x}^{-1}(\varepsilon^{\Theta(t, \xi)} \widehat{f}^\varepsilon(t, \xi, y)).$$

The width of the analyticity band Θ is defined by

$$\Theta(t, \xi) = (a - \lambda\tau(t))|\xi|,$$

where $\lambda > 0$ with be precised later and $\tau(t)$ will be chosen in such a way that $\Theta(t, \xi) > 0$, for any $(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}$ and $\dot{\tau}(t) = \tau'(t) = -\lambda\dot{\Theta}(t) \geq 0$. In our paper, we will choose

$$(3.32) \quad \dot{\tau}(t) = \|\partial_y u_\Theta^\varepsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \varepsilon \|\partial_y v_\Theta^\varepsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}} \quad \text{with} \quad \tau(0) = 0.$$

In what follows, for the sake of the simplicity, we will neglect the script ε and write $(u_\Theta, v_\Theta, T_\Theta)$ instead of $(u_\Theta^\varepsilon, v_\Theta^\varepsilon, T_\Theta^\varepsilon)$. Direct calculations from (1.3) and (3.31) show that $(u_\Theta, v_\Theta, T_\Theta)$ satisfies the system:

$$(3.33) \quad \begin{cases} \partial_t u_\Theta + \lambda\dot{\tau}(t)|D_x|u_\Theta + (u\partial_x u)_\Theta + (v\partial_y u)_\Theta - \varepsilon^2 \partial_x^2 u_\Theta - \partial_y^2 u_\Theta + \partial_x p_\Theta = 0, \\ \varepsilon^2 (\partial_t v_\Theta + (u\partial_x v)_\Theta + (v\partial_y v)_\Theta - \varepsilon^2 \partial_x^2 v_\Theta - \partial_y^2 v_\Theta) + \partial_y p_\Theta = T_\Theta, \\ \partial_t T_\Theta + (u\partial_x T)_\Theta + (v\partial_y T)_\Theta - \Delta T_\Theta = 0, \\ \partial_x u_\Theta + \partial_y v_\Theta = 0, \\ (u_\Theta, v_\Theta, T_\Theta)|_{y=0} = (u_\Theta, v_\Theta, T_\Theta)|_{y=1} = 0, \\ (u_\Theta, v_\Theta, T_\Theta)|_{t=0} = (u_0, v_0, T_0). \end{cases}$$

We remark that the pressure term is not really an unknown and can be determined as functions of (u_Θ, T_Θ) as we did for the hydrostatic limit system (see also [6] for more details). In what follows, we recall that we use ‘‘C’’ to denote a generic positive constant which can change from line to line. Before we give the proof of Theorem 1.7, we will introduce the following lemma, which allows to control the term $v\partial_y v$.

Lemma 3.1. *For any $s \in]0, 1]$ and $t \leq T^*$, and Θ be defined as in (3.31), with*

$$\dot{\tau}(t) = \|\partial_y u_\Theta^\varepsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \varepsilon \|\partial_y v_\Theta^\varepsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}}.$$

Then, there exists $C \geq 1$ such that, for any $t > 0$, $\Theta(t, \xi) > 0$ and for any $u \in \tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})$, we have

$$\varepsilon^2 \sum_{q \in \mathbb{Z}} 2^{2qs} \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (v\partial_y v)_\Theta, e^{\mathcal{R}t'} \Delta_q^h v_\Theta \right\rangle_{L^2} \right| dt' \leq C \|e^{\mathcal{R}t} (u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

We will postpone the proof of this lemma to the end of this section. Applying the dyadic operator in the horizontal variable Δ_q^h to the system (3.33), then taking the $L^2(\mathcal{S})$ scalar product of the first three equations of the obtained system with $\Delta_q^h u_\phi$, $\Delta_q^h v_\phi$ and $\Delta_q^h T_\phi$ respectively, we get

$$(3.34) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_q^h (u_\Theta, \varepsilon v_\Theta)(t)\|_{L^2}^2 + \lambda\dot{\tau}(t) (\|D_x| \Delta_q^h (u_\Theta, \varepsilon v_\Theta)\|_{L^2}^2 \\ & + \|\partial_y \Delta_q^h (u_\Theta, \varepsilon v_\Theta)\|_{L^2}^2 + \varepsilon^2 \|\partial_x \Delta_q^h (u_\Theta, \varepsilon v_\Theta)\|_{L^2}^2 \\ & = - \langle \Delta_q^h (u\partial_x u)_\Theta, \Delta_q^h u_\Theta \rangle_{L^2} - \langle \Delta_q^h (v\partial_y u)_\Theta, \Delta_q^h u_\Theta \rangle_{L^2} - \langle \nabla \Delta_q^h p_\Theta, \Delta_q^h (u_\Theta, v_\Theta) \rangle_{L^2} \\ & - \varepsilon^2 \langle \Delta_q^h (u\partial_x v)_\Theta, \Delta_q^h v_\Theta \rangle_{L^2} - \varepsilon^2 \langle \Delta_q^h (v\partial_y v)_\Theta, \Delta_q^h v_\Theta \rangle_{L^2} + \varepsilon^2 \langle \Delta_q^h T_\Theta, \Delta_q^h v_\Theta \rangle_{L^2}, \end{aligned}$$

and

$$(3.35) \quad \frac{1}{2} \frac{d}{dt} \|\Delta_q^h T_\Theta(t)\|_{L^2}^2 + \lambda \dot{\tau}(t) (\|D_x |\Delta_q^h T_\Theta, \Delta_q^h T_\Theta\rangle_{L^2} + \|\nabla \Delta_q^h T_\Theta\|_{L^2}^2) \\ = - \langle \Delta_q^h (u \partial_x T)_\Theta, \Delta_q^h T_\Theta \rangle_{L^2} - \langle \Delta_q^h (v \partial_y T)_\Theta, \Delta_q^h T_\Theta \rangle_{L^2}.$$

Multiplying (3.34) and (3.35) by $e^{2\mathcal{R}t}$ and integrating the obtained equations with respect to the time variable lead to

$$(3.36) \quad \|e^{\mathcal{R}t} \Delta_q^h (u_\Theta, \varepsilon v_\Theta)(t)\|_{L_t^\infty(L^2)}^2 + \lambda \int_0^t \dot{\tau}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h (u_\Theta, \varepsilon v_\Theta)(t') \right\|_{L^2}^2 dt' \\ + \|e^{\mathcal{R}t'} \partial_y \Delta_q^h (u_\Theta, \varepsilon v_\Theta)\|_{L_t^2(L^2)}^2 + \varepsilon^2 \|e^{\mathcal{R}t'} \partial_x \Delta_q^h (u_\Theta, \varepsilon v_\Theta)\|_{L_t^2(L^2)}^2 \\ = \|\Delta_q^h (u_\Theta, \varepsilon v_\Theta)(0)\|_{L^2}^2 + F_1 + F_2 + F_3 + F_4,$$

and

$$(3.37) \quad \|e^{\mathcal{R}t} \Delta_q^h T_\Theta(t)\|_{L_t^\infty L^2}^2 + \lambda \int_0^t \dot{\tau}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h T_\Theta \right\|_{L^2}^2 dt' + \|e^{\mathcal{R}t} \Delta_q^h \nabla T_\Theta(t)\|_{L_t^2 L^2}^2 \\ = \|\Delta_q^h T_\Theta(0)\|_{L^2}^2 + F_5 + F_6.$$

Next, Lemmas 1.9, 1.10 and 3.1 yield

$$|F_1| = \left| \int_0^t \langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x u)_\Theta, e^{\mathcal{R}t'} \Delta_q^h u_\Theta \rangle_{L^2} + \langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y u)_\Theta, e^{\mathcal{R}t'} \Delta_q^h u_\Theta \rangle_{L^2} dt' \right| \\ \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

$$|F_2| = \varepsilon^2 \left| \int_0^t \langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x v)_\Theta, e^{\mathcal{R}t'} \Delta_q^h v_\Theta \rangle_{L^2} + \langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y v)_\Theta, e^{\mathcal{R}t'} \Delta_q^h v_\Theta \rangle_{L^2} dt' \right| \\ \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} (u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2$$

and

$$|F_5| = \left| \int_0^t \langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x T)_\Theta, e^{\mathcal{R}t'} \Delta_q^h T_\Theta \rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} T_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \\ |F_6| = \left| \int_0^t \langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y T)_\Theta, e^{\mathcal{R}t'} \Delta_q^h T_\Theta \rangle dt' \right| \leq C d_q^2 2^{-2qs} \|u_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \nabla T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2.$$

The term F_3 can be calculated using the divergence-free property $\partial_x u_\Theta + \partial_y v_\Theta = 0$, and integrating by part

$$|F_3| = \left| \int_0^t \langle e^{\mathcal{R}t'} \nabla \Delta_q^h p_\Theta, e^{\mathcal{R}t'} \Delta_q^h (u_\Theta, v_\Theta) \rangle_{L^2} dt' \right| = 0.$$

In order to estimate the last term F_4 , we first use the boundary condition $(u_\Theta, v_\Theta)|_{y \in \{0,1\}} = 0$, and the fact that

$$v_\Theta(t, x, y) = - \int_0^y \partial_x u_\Theta(t, x, s) ds,$$

we get

$$\varepsilon^2 \left| \langle e^{\mathcal{R}t} \Delta_q^h T_\Theta, e^{\mathcal{R}t} \Delta_q^h v_\Theta \rangle_{L^2} \right| = \varepsilon^2 \left| \left\langle e^{\mathcal{R}t} \Delta_q^h T_\Theta, e^{\mathcal{R}t} \Delta_q^h \int_0^y -\partial_x u_\Theta ds \right\rangle_{L^2} \right| \\ \leq \varepsilon^2 \|\Delta_q^h e^{\mathcal{R}t} T_\Theta\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t} \partial_x u_\Theta\|_{L^2} \\ \leq C \|\Delta_q^h e^{\mathcal{R}t} T_\Theta\|_{L^2}^2 + \varepsilon^2 \|\Delta_q^h e^{\mathcal{R}t} \partial_x u_\Theta\|_{L^2}^2.$$

Then,

$$(3.38) \quad |F_4| = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h T_\Theta, e^{\mathcal{R}t'} \Delta_q^h v_\Theta \right\rangle_{L^2} dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} \Delta_q^h T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \varepsilon^2 \|e^{\mathcal{R}t} \Delta_q^h \partial_x u_\Theta\|_{\tilde{L}_t^2(L^2)}^2.$$

Multiplying now (3.36) and (3.37) by 2^{2qs} and summing with respect $q \in \mathbb{Z}$, we obtain

$$(3.39) \quad \|e^{\mathcal{R}t}(u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}^2 + \lambda \|e^{\mathcal{R}t}(u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|e^{\mathcal{R}t} \partial_y(u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \\ + \varepsilon^2 \|e^{\mathcal{R}t} \partial_x(u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \leq \|(u_\Theta, \varepsilon v_\Theta)(0)\|_{\mathcal{B}^s}^2 + C \|e^{\mathcal{R}t}(u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \\ + C \|e^{\mathcal{R}t} T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \frac{\varepsilon^2}{10} \|e^{\mathcal{R}t} \partial_x u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2.$$

and

$$(3.40) \quad \|e^{\mathcal{R}t} T_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}^2 + \lambda \|e^{\mathcal{R}t} T_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|e^{\mathcal{R}t} \nabla T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \\ \leq \|T_\Theta(0)\|_{\mathcal{B}^s}^2 + C \|e^{\mathcal{R}t} T_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + C \|u_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \nabla T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2,$$

which also implies that

$$\|e^{\mathcal{R}t} T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \leq \|T_\Theta(0)\|_{\mathcal{B}^s}^2 + C \|e^{\mathcal{R}t} T_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + C \|u_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \nabla T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2.$$

In what follows, we choose

$$(3.41) \quad C \geq \max \left\{ 4, \frac{1}{2\mathcal{R}} \right\}.$$

Taking the sum of (3.39) and (3.40), we have

$$\|e^{\mathcal{R}t}(u_\Theta, \varepsilon v_\Theta, T_\Theta)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}^2 + \lambda \|e^{\mathcal{R}t}(u_\Theta, \varepsilon v_\Theta, T_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|e^{\mathcal{R}t} \nabla T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \\ + \|e^{\mathcal{R}t} \partial_y(u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \varepsilon^2 \|e^{\mathcal{R}t} \partial_x(u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \leq 2C \|(u_\Theta, \varepsilon v_\Theta)(0)\|_{\mathcal{B}^s}^2 + 2C \|T_\Theta(0)\|_{\mathcal{B}^s}^2 \\ + 2C^2 \|e^{\mathcal{R}t}(u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + 2C^2 \|e^{\mathcal{R}t} T_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + 2C^2 \|u_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \nabla T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2,$$

from which we deduce

$$(3.42) \quad \|e^{\mathcal{R}t}(u_\Theta, \varepsilon v_\Theta, T_\Theta)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}^2 + \lambda \|e^{\mathcal{R}t}(u_\Theta, \varepsilon v_\Theta, T_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|e^{\mathcal{R}t} \nabla T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \\ + \|e^{\mathcal{R}t} \partial_y(u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \varepsilon^2 \|e^{\mathcal{R}t} \partial_x(u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \leq 2C \|e^{a|D_x|}(u_0, \varepsilon v_0, T_0)\|_{\mathcal{B}^s}^2 \\ + 2C^2 \|e^{\mathcal{R}t}(u_\Theta, \varepsilon v_\Theta, T_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + 2C^2 \|u_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \nabla T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2.$$

We set

$$(3.43) \quad T^* \stackrel{def}{=} \sup \left\{ t > 0 : \|u_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{1}{2C^2} \text{ and } \tau(t) \leq \frac{a}{\lambda} \right\},$$

and we choose initial data such that

$$C \left(\|e^{a|D_x|}(u_0, \varepsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|} T_0\|_{\mathcal{B}^{\frac{1}{2}}} \right) < \min \left\{ \frac{1}{2C^2}, \frac{a}{2\lambda} \right\}.$$

The fact that $\tau(0) = 0$ implies already that $T^* > 0$. If $\lambda = 2C^2$, for any $0 < t < T^*$, we have

$$(3.44) \quad \|e^{\mathcal{R}t}(u_\Theta, \varepsilon v_\Theta, T_\Theta)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}^2 + \|e^{\mathcal{R}t}\nabla T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \|e^{\mathcal{R}t}\partial_y(u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \\ + \varepsilon^2 \|e^{\mathcal{R}t}\partial_x(u_\Theta, \varepsilon v_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \leq 2C \|e^{a|D_x|}(u_0, T_0)\|_{\mathcal{B}^s}^2.$$

From (3.44) and (3.41), we get that, for any $0 < t < T^*$,

$$\|u_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \leq \|e^{\mathcal{R}t}(u_\Theta, \varepsilon v_\Theta, T_\Theta)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \leq C (\|e^{a|D_x|}(u_0, \varepsilon v_0, T_0)\|_{\mathcal{B}^s}) \\ \leq C \left(\|e^{a|D_x|}(u_0, \varepsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}T_0\|_{\mathcal{B}^{\frac{1}{2}}} \right) < \frac{1}{2C^2}.$$

Now, we recall that we already defined $\dot{\tau}(t) = \|\partial_y u_\Theta^\varepsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \varepsilon \|\partial_y v_\Theta^\varepsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}}$ with $\tau(0) = 0$. Then, for any $0 < t < T^*$, Inequality (3.44) yields

$$\tau(t) = \int_0^t \|\partial_y u_\Theta^\varepsilon(t')\|_{\mathcal{B}^{\frac{1}{2}}} + \varepsilon \|\partial_y v_\Theta^\varepsilon(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \\ \leq \int_0^t e^{-\mathcal{R}t'} \left(\|e^{\mathcal{R}t'} \partial_y u_\Theta^\varepsilon(t')\|_{\mathcal{B}^{\frac{1}{2}}} + \varepsilon \|e^{\mathcal{R}t'} \partial_y v_\Theta^\varepsilon(t')\|_{\mathcal{B}^{\frac{1}{2}}} \right) dt' \\ \leq \left(\int_0^t e^{-2\mathcal{R}t'} dt' \right)^{\frac{1}{2}} \left(\int_0^t (\|e^{\mathcal{R}t'} \partial_y u_\Theta^\varepsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \varepsilon \|e^{\mathcal{R}t'} \partial_y v_\Theta^\varepsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}})^2 dt' \right)^{\frac{1}{2}} \\ \leq C \|e^{\mathcal{R}t}(\varepsilon \partial_y v_\Theta^\varepsilon, \partial_y u_\Theta^\varepsilon)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\ \leq C \left(\|e^{a|D_x|}(u_0, \varepsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + C \|e^{a|D_x|}T_0\|_{\mathcal{B}^{\frac{1}{2}}} \right) < \frac{a}{2\lambda}.$$

A continuity argument implies that $T^* = +\infty$ and we have (3.44) is valid for any $t \in \mathbb{R}_+$.

Proof of Lemme 3.1. Using Bony's homogeneous decomposition into paraproducts and remainders as in Definition 1.1, we can write

$$v\partial_y v = T_v^h \partial_y v + T_{\partial_y v}^h v + R^h(v, \partial_y v),$$

where

$$T_a b = \sum_{q \in \mathbb{Z}} S_{q-1}^h a \Delta_q^h b \quad \text{and} \quad R^h(a, b) = \sum_{|q'-q| \leq 1} \Delta_q^h a \Delta_{q'}^h b.$$

So, we have

$$\varepsilon^2 \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (v\partial_y v)_\Theta, e^{\mathcal{R}t'} \Delta_q^h v_\Theta \right\rangle_{L^2} \right| dt' \leq I_{1,q} + I_{2,q} + I_{3,q},$$

where

$$I_{1,q} = \varepsilon^2 \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_v^h \partial_x v)_\Theta, e^{\mathcal{R}t'} \Delta_q^h v_\Theta \right\rangle_{L^2} \right| dt' \\ I_{2,q} = \varepsilon^2 \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y v}^h v)_\Theta, e^{\mathcal{R}t'} \Delta_q^h v_\Theta \right\rangle_{L^2} \right| dt' \\ I_{3,q} = \varepsilon^2 \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(v, \partial_x v))_\Theta, e^{\mathcal{R}t'} \Delta_q^h v_\Theta \right\rangle_{L^2} \right| dt'.$$

Using the support properties given in [[3], Proposition 2.10], the definition of $T_v^h \partial_y v$ and the fact that $\partial_y v = -\partial_x u$, we infer

$$\begin{aligned}
I_{1,q} &= \varepsilon^2 \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_v^h \partial_x v)_\Theta, e^{\mathcal{R}t'} \Delta_q^h v_\Theta \right\rangle_{L^2} \right| dt' \\
&\lesssim \varepsilon^2 \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h v_\Theta(t)\|_{L^\infty} \|\Delta_{q'}^h \partial_y v_\Theta(t)\|_{L^2} \|\Delta_q^h v_\Theta(t)\|_{L^2} \\
&\lesssim \varepsilon^2 \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h v_\Theta(t)\|_{L^\infty} \|\Delta_{q'}^h \partial_x u_\Theta(t)\|_{L^2} \|\Delta_q^h v_\Theta(t)\|_{L^2} \\
&\lesssim \varepsilon^2 \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} 2^{-\frac{q'}{2}} \|S_{q'-1}^h v_\Theta(t)\|_{L^\infty} 2^{\frac{q'}{2}} \|\Delta_{q'}^h \partial_x u_\Theta(t)\|_{L^2} \|\Delta_q^h v_\Theta(t)\|_{L^2} \\
&\lesssim \varepsilon^2 \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} 2^{-\frac{q'}{2}} \|S_{q'-1}^h v_\Theta(t)\|_{L^\infty} \|\partial_x u_\Theta(t)\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h v_\Theta(t)\|_{L^2}.
\end{aligned}$$

Estimate (2.16) implies

$$\begin{aligned}
\|\Delta_q^h v_\Theta(t)\|_{L^\infty} &\leq \int_0^1 \|\Delta_q^h \partial_x u_\Theta(t, \cdot, y')\|_{L_h^\infty} dy' \\
&\lesssim 2^{\frac{q}{2}} \times \int_0^1 \|\Delta_q^h \partial_x u_\Theta(t, \cdot, y')\|_{L_h^2} dy' \\
&\lesssim 2^{\frac{q}{2}} \times 2^q \|\Delta_q^h u_\Theta(t)\|_{L^2} \\
&\lesssim 2^{\frac{3q}{2}} \|\Delta_q^h u_\Theta(t)\|_{L^2}.
\end{aligned}$$

For $s \leq 1$, we have

$$\begin{aligned}
I_{1,q} &\lesssim \varepsilon \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} 2^{-\frac{q'}{2}} \|S_{q'-1}^h v_\Theta(t)\|_{L^\infty} \varepsilon \|\partial_x u_\Theta(t)\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h v_\Theta(t)\|_{L^2} \\
&\lesssim \varepsilon \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} 2^{q'} \|\Delta_{q'}^h u_\Theta(t)\|_{L^2} \varepsilon \|\partial_x u_\Theta(t)\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h v_\Theta(t)\|_{L^2} \\
&\lesssim \varepsilon \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^t \varepsilon \|\partial_x u_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_{q'}^h u_\Theta(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \varepsilon \|\partial_x u_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_q^h v_\Theta(t)\|_{L^2}^2 dt' \right)^{\frac{1}{2}}
\end{aligned}$$

The definition of $\dot{\tau}(t)$ and Definition 1.5 allow to write

$$\left(\int_0^t \dot{\tau}(t') e^{2\mathcal{R}t'} \|\Delta_{q'}^h u_\Theta(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q(s+\frac{1}{2})} d_{q'}(u_\Theta) \|e^{\mathcal{R}t} u_\Theta\|_{\tilde{L}_{t,\dot{\tau}}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Then,

$$I_{1,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} u_\Theta\|_{\tilde{L}_{t,\dot{\tau}}^2(\mathcal{B}^{s+\frac{1}{2}})} \varepsilon \|e^{\mathcal{R}t} v_\Theta\|_{\tilde{L}_{t,\dot{\tau}}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(v_\Theta) \left(\sum_{|q-q'| \leq 4} d_{q'}(u_\Theta) 2^{(q-q')(s-\frac{1}{2})} \right)$$

form a summable sequence, which implies

$$(3.45) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} I_{1,q} \lesssim \|e^{\mathcal{R}t} u_{\Theta}\|_{\tilde{L}_{t,\dot{\tau}}^2(\mathcal{B}^{s+\frac{1}{2}})} \varepsilon \|e^{\mathcal{R}t} v_{\Theta}\|_{\tilde{L}_{t,\dot{\tau}}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

The second term $I_{2,q}$ can be controlled in a similar way. Using the support properties given in [[3], Proposition 2.10], the definition of $T_v^h \partial_y v$ and the fact that $\partial_y v = -\partial_x u$, we have

$$\begin{aligned} I_{2,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y v}^h v)_{\Theta}, e^{\mathcal{R}t'} \Delta_q^h v_{\Theta} \right\rangle_{L^2} \right| dt' \\ &\leq \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_y v_{\Theta}(t')\|_{L^\infty} \|\Delta_{q'}^h v_{\Theta}(t')\|_{L^2} \|\Delta_q^h v_{\Theta}(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_x u_{\Theta}(t')\|_{L^\infty} \|\Delta_{q'}^h v_{\Theta}(t')\|_{L^2} \|\Delta_q^h v_{\Theta}(t')\|_{L^2} dt', \end{aligned}$$

By Poincaré's inequality, on the interval $\{0 < y < 1\}$, we have the inclusion $\dot{H}_y^1 \hookrightarrow L_y^\infty$ and,

$$(3.46) \quad \|\Delta_q^h \partial_x u_{\Theta}(t')\|_{L^\infty} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h \partial_x u_{\Theta}(t')\|_{L_h^2(L_y^\infty)} \lesssim 2^{\frac{3q}{2}} \|\Delta_q^h \partial_y u_{\Theta}(t')\|_{L^2} \lesssim d_q(u_{\Theta}) 2^q \|\partial_y u_{\Theta}(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

where $\{d_q(u_{\Theta})\}$ is a square-summable sequence with $\sum d_q(u_{\Theta})^2 = 1$. As a consequence,

$$\|S_{q'-1}^h u_{\Theta}(t')\|_{L^\infty} \lesssim 2^q \|\partial_y u_{\Theta}(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

which implies

$$\begin{aligned} I_{2,q} &\lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} 2^{q'} \|\partial_y u_{\Theta}(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h v_{\Theta}(t')\|_{L^2} \|\Delta_q^h v_{\Theta}(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^t \|\partial_y u_{\Theta}(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_{q'}^h v_{\Theta}(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_{\Theta}(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_q^h v_{\Theta}(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

Using the definition of $\dot{\tau}(t)$ and Definition 1.5, we write

$$\left(\int_0^t \dot{\tau}(t') e^{2\mathcal{R}t'} \|\Delta_{q'}^h v_{\Theta}(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q(s+\frac{1}{2})} d_{q'}(v_{\Theta}) \|e^{\mathcal{R}t} v_{\Theta}\|_{\tilde{L}_{t,\dot{\tau}}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Then,

$$I_{2,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} v_{\Theta}\|_{\tilde{L}_{t,\dot{\tau}}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

where

$$d_q^2 = d_q(v_{\Theta}) \left(\sum_{|q'-q| \leq 4} d_{q'}(v_{\Theta}) 2^{(q-q')(s-\frac{1}{2})} \right)$$

form a summable sequence, which implies

$$(3.47) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} I_{2,q} \lesssim \|e^{\mathcal{R}t} v_{\Theta}\|_{\tilde{L}_{t,\dot{\tau}}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

To end this proof, it remains to estimate $I_{3,q}$. Using the support properties given in [[3], Proposition 2.10], the definition of $R^h(v, \partial_y v)$, the divergence-free property and Bernstein lemma 1.2, we can write

$$\begin{aligned} I_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(v, \partial_y v))_\Theta, e^{\mathcal{R}t'} \Delta_q^h v_\Theta \right\rangle_{L^2} \right| dt' \\ &\leq \int_0^t 2^{\frac{q}{2}} \sum_{q' \geq q-3} e^{2\mathcal{R}t'} \|\Delta_{q'}^h v_\Theta(t')\|_{L^2} \|\tilde{\Delta}_{q'}^h \partial_y v_\Theta(t')\|_{L_h^2(L_v^\infty)} \|\Delta_q^h v_\Theta(t')\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} \|\Delta_{q'}^h v_\Theta(t')\|_{L^2} \|\tilde{\Delta}_{q'}^h \partial_x u_\Theta(t')\|_{L_h^2(L_v^\infty)} \|\Delta_q^h v_\Theta(t')\|_{L^2} dt'. \end{aligned}$$

The fact that

$$\|\tilde{\Delta}_{q'}^h \partial_x u_\Theta(t')\|_{L_h^2(L_v^\infty)} \leq 2^{q'} \|\tilde{\Delta}_{q'}^h \partial_y u_\Theta(t')\|_{L^2},$$

implies,

$$\begin{aligned} I_3 &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} \|\Delta_{q'}^h v_\Theta(t)\|_{L^2} 2^{q'} \|\tilde{\Delta}_{q'}^h \partial_y u_\Theta(t)\|_{L^2} \|\Delta_q^h v_\Theta(t)\|_{L^2} \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} \|\Delta_{q'}^h v_\Theta(t)\|_{L^2} 2^{\frac{q'}{2}} \|\partial_y u_\Theta(t)\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h v_\Theta(t)\|_{L^2} \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_{q'}^h v_\Theta(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_q^h v_\Theta(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \end{aligned}$$

Next, the definition of $\dot{\tau}(t)$ and Definition 1.5 yield

$$\left(\int_0^t \dot{\tau}(t') e^{2\mathcal{R}t'} \|\Delta_{q'}^h v_\Theta(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q(s+\frac{1}{2})} d_{q'}(v_\Theta) \|e^{\mathcal{R}t} v_\Theta\|_{\tilde{L}_{\dot{\tau}, \tau}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Thus,

$$I_{3,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} v_\Theta\|_{\tilde{L}_{\dot{\tau}, \tau}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

where

$$d_q^2 = d_q(v_\Theta) \left(\sum_{|q-q'| \leq 4} d_{q'}(v_\Theta) 2^{(q-q')(s-\frac{1}{2})} \right)$$

form a summable sequence, which implies

$$(3.48) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} I_{3,q} \lesssim \|e^{\mathcal{R}t} v_\Theta\|_{\tilde{L}_{\dot{\tau}, \tau}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

Lemma 3.1 is then proved by summing Estimates (3.45), (3.47) and (3.48). \square

4. THE CONVERGENCE TO THE HYDRO-STATIC NAVIER-STOKES COUPLED WITH TEMPERATURE

The goal of this section is to prove Theorem 1.8, which justifies the convergence of the scaled anisotropic Navier-Stokes coupled with temperature towards the hydro-static Navier-Stokes coupled with temperature in a 2D striped domain. To this end, we introduce

$$w_1^\varepsilon = u^\varepsilon - u \quad \text{and} \quad w_2^\varepsilon = v^\varepsilon - v,$$

$$q^\varepsilon = p^\varepsilon - p \quad \text{and} \quad \theta^\varepsilon = T^\varepsilon - T.$$

Then, systems (1.3) and (1.4) imply that $(w_\varepsilon^1, w_\varepsilon^2, q_\varepsilon, \theta^\varepsilon)$ verifies

$$(4.49) \quad \begin{cases} \partial_t w_1^\varepsilon - \varepsilon^2 \partial_x^2 w_1^\varepsilon - \partial_y^2 w_1^\varepsilon + \partial_x q^\varepsilon = R^{1,\varepsilon} & \text{in } \mathcal{S} \times]0, \infty[, \\ \varepsilon^2 (\partial_t w_2^\varepsilon - \varepsilon^2 \partial_x^2 w_2^\varepsilon - \partial_y^2 w_2^\varepsilon) + \partial_y q^\varepsilon = \theta^\varepsilon + R^{2,\varepsilon}, \\ \partial_t \theta^\varepsilon - \partial_x^2 \theta^\varepsilon - \partial_y^2 \theta^\varepsilon = R^{3,\varepsilon} \\ \partial_x w_1^\varepsilon + \partial_y w_2^\varepsilon = 0, \\ (w_1^\varepsilon, w_2^\varepsilon, \theta^\varepsilon)|_{t=0} = (u_0^\varepsilon - u_0, v_0^\varepsilon - v_0, T_0^\varepsilon - T_0), \\ (w_1^\varepsilon, w_2^\varepsilon, \theta^\varepsilon)|_{y=0} = (w_1^\varepsilon, w_2^\varepsilon, \theta^\varepsilon)|_{y=1} = 0, \end{cases}$$

where v_0 is a function of u_0 , using (2.16) and the remaining terms $R^{i,\varepsilon}$, with $i = 1, 2, 3$, are given by

$$(4.50) \quad \begin{cases} R^{1,\varepsilon} = \varepsilon^2 \partial_x^2 u - (u^\varepsilon \partial_x u^\varepsilon - v^\varepsilon \partial_y u^\varepsilon) - (u \partial_x u - v \partial_y u), \\ R^{2,\varepsilon} = -\varepsilon^2 (\partial_t v - \varepsilon^2 \partial_x^2 v - \partial_y^2 v + u^\varepsilon \partial_x v^\varepsilon + v^\varepsilon \partial_y v^\varepsilon), \\ R^{3,\varepsilon} = -(u^\varepsilon \partial_x T^\varepsilon + v^\varepsilon \partial_y T^\varepsilon) + (u \partial_x T + v \partial_y T). \end{cases}$$

Now for suitable function f , we define

$$(4.51) \quad f_\varphi(t, x, y) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(e^{\varphi(t, \xi)} \widehat{f}(t, \xi, y) \right) \quad \text{where} \quad \varphi(t, \xi) = (a - \mu \eta(t)) |\xi|,$$

where $\mu \geq \lambda$ will be determined later, and $\eta(t)$ is given by

$$\eta(t) = \int_0^t \left(\|(\partial_y u_\Theta^\varepsilon, \varepsilon \partial_x u_\Theta^\varepsilon)(t')\|_{\mathcal{B}^{\frac{1}{2}}} + \|\partial_y u_\Phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \right) dt'.$$

We can observe that, if we take c_0 and c_1 small enough in Theorems 1.6 and 1.7 then $\varphi(t) \geq 0$ and

$$0 \leq \varphi(t, \xi) \leq \min(\Phi(t, \xi), \Theta(t, \xi)).$$

In what follows, for simplicity, we drop the script ε and we will write $(w_\varphi^1, w_\varphi^2, q_\varphi, \theta_\varphi, R_\varphi^i)$ instead of $(w_\varphi^{\varepsilon,1}, w_\varphi^{\varepsilon,2}, q_\varphi^\varepsilon, \theta_\varphi^\varepsilon, R_\varphi^{i,\varepsilon})$. Direct calculations show that $(w_\varphi^1, w_\varphi^2, q_\varphi, \theta_\varphi)$ satisfies

$$(4.52) \quad \begin{cases} \partial_t w_\varphi^1 + \mu |D_x| \dot{\eta}(t) w_\varphi^1 - \varepsilon^2 \partial_x^2 w_\varphi^1 - \partial_y^2 w_\varphi^1 + \partial_x q_\varphi = R_\varphi^1 & \text{in } \mathcal{S} \times]0, \infty[, \\ \varepsilon^2 (\partial_t w_\varphi^2 + \mu |D_x| \dot{\eta}(t) w_\varphi^2 - \varepsilon^2 \partial_x^2 w_\varphi^2 - \partial_y^2 w_\varphi^2) + \partial_y q_\varphi = \theta_\varphi + R_\varphi^2, \\ \partial_t \theta_\varphi + \mu |D_x| \dot{\eta}(t) \theta_\varphi - \partial_x^2 \theta_\varphi - \partial_y^2 \theta_\varphi = R_\varphi^3 \\ \partial_x w_\varphi^1 + \partial_y w_\varphi^2 = 0, \\ (w_\varphi^1, w_\varphi^2, \theta_\varphi)|_{t=0} = (u_0^\varepsilon - u_0, v_0^\varepsilon - v_0, T_0^\varepsilon - T_0), \\ (w_\varphi^1, w_\varphi^2, \theta_\varphi)|_{y=0} = (w_\varphi^1, w_\varphi^2, \theta_\varphi)|_{y=1} = 0. \end{cases}$$

As in the previous sections, we will use “ C ” to denote a generic positive constant which can change from line to line.

Applying the dyadic operator in the horizontal variable Δ_q^h to the system (4.52), then taking the $L^2(\mathcal{S})$ scalar product of the first, second and the third equations of the obtained system with $\Delta_q^h w_\varphi^1$, $\Delta_q^h w_\varphi^2$ and $\Delta_q^h \theta_\varphi$ respectively, we obtain

$$(4.53) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2)(t)\|_{L^2}^2 + \mu \dot{\eta}(t) \langle |D_x| \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2), \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2) \rangle_{L^2} \\ & + \|\partial_y \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2)\|_{L^2}^2 + \varepsilon^2 \|\partial_x \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2)\|_{L^2}^2 = \langle \Delta_q^h R_\varphi^1, \Delta_q^h w_\varphi^1 \rangle_{L^2} \\ & + \langle \Delta_q^h R_\varphi^2, \Delta_q^h w_\varphi^2 \rangle_{L^2} - \langle \Delta_q^h \nabla q_\varphi, \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2) \rangle_{L^2} + \langle \Delta_q^h \theta_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \end{aligned}$$

and

$$(4.54) \quad \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \theta_\varphi(t)\|_{L^2}^2 + \mu \dot{\eta}(t) \langle |D_x| \Delta_q^h \theta_\varphi, \Delta_q^h \theta_\varphi \rangle_{L^2} + \|\nabla \Delta_q^h \theta_\varphi\|_{L^2}^2 = \langle \Delta_q^h R_\varphi^3, \Delta_q^h \theta_\varphi \rangle_{L^2}.$$

Integrating (4.53) and (4.54) with respect to the time variable, we have

$$(4.55) \quad \begin{aligned} & \|\Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2)(t)\|_{L_t^\infty(L^2)}^2 + \mu \int_0^t \dot{\eta}(t') \langle |D_x| \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2), \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2) \rangle_{L^2} dt' \\ & \quad + \|\partial_y \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2)\|_{L_t^2(L^2)}^2 + \varepsilon^2 \|e^{\mathcal{R}t'} \partial_x \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2)\|_{L_t^2(L^2)}^2 \\ & \leq \|\Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2)(0)\|_{L^2}^2 + G_1^q + G_2^q + G_3^q + G_4^q, \end{aligned}$$

and

$$(4.56) \quad \|\Delta_q^h \theta_\varphi(t)\|_{L^2}^2 + \lambda \int_0^t \dot{\eta}(t') \langle |D_x| \Delta_q^h \theta_\Theta, \Delta_q^h \theta_\varphi \rangle_{L^2} dt' + \|\nabla \Delta_q^h \theta_\varphi\|_{L^2}^2 \leq \|\Delta_q^h \theta_\varphi(0)\|_{L^2}^2 + G_5^q,$$

where the terms G_i^q , $i = 1, \dots, 5$, will be precised and controlled in what follows.

We will start with G_3^q for that, using the fact that $\partial_x w_\varphi^1 + \partial_y w_\varphi^2 = 0$ and integrating by parts, we have

$$G_3^q = \int_0^t \left| \langle \nabla \Delta_q^h q_\varphi, \Delta_q^h(w_\varphi^1, w_\varphi^2) \rangle_{L^2} \right| dt' = \int_0^t \left| \langle \Delta_q^h q_\varphi, \operatorname{div}(\Delta_q^h(w_\varphi^1, w_\varphi^2)) \rangle_{L^2} \right| dt' = 0.$$

Boundary condition $(u_\Theta, v_\Theta)|_{y=0} = (u_\Theta, v_\Theta)|_{y=1} = 0$, incompressibility property $\partial_x w_\varphi^1 + \partial_y w_\varphi^2 = 0$ and Poincaré inequality with respect to $y \in]0, 1[$ yield

$$\begin{aligned} G_4^q &= \int_0^t \left| \langle \Delta_q^h \theta_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' = \int_0^t \left| \left\langle \Delta_q^h \theta_\varphi, -\Delta_q^h \int_0^y \partial_x w_\varphi^1(x, s) ds \right\rangle_{L^2} \right| dt' \\ &= \int_0^t \left| \left\langle \Delta_q^h \partial_x \theta_\varphi, \Delta_q^h \int_0^y w_\varphi^1(x, s) ds \right\rangle_{L^2} \right| dt' \leq \int_0^t \|\Delta_q^h \partial_x \theta_\varphi\|_{L^2} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \\ &\leq C \|\Delta_q^h \partial_x \theta_\varphi\|_{L_t^2(L^2)}^2 + \frac{\varepsilon}{100} \|\Delta_q^h w_\varphi^1\|_{L_t^2(L^2)}^2 \end{aligned}$$

Using the two above estimates, we will rewrite (4.55) as follows

$$(4.57) \quad \begin{aligned} & \|\Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2)(t)\|_{L_t^\infty(L^2)}^2 + \mu \int_0^t \dot{\eta}(t') \langle |D_x| \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2), \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2) \rangle_{L^2} dt' \\ & \quad + \|\partial_y \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2)\|_{L_t^2(L^2)}^2 + \varepsilon^2 \|e^{\mathcal{R}t'} \partial_x \Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2)\|_{L_t^2(L^2)}^2 \\ & = \|\Delta_q^h(w_\varphi^1, \varepsilon w_\varphi^2)(0)\|_{L^2}^2 + G_1^q + G_2^q + C \|\Delta_q^h \partial_x \theta_\varphi\|_{L^2}^2, \end{aligned}$$

where

$$\begin{aligned} G_1^q &= \int_0^t \left| \langle \Delta_q^h R_\varphi^1, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\ G_2^q &= \int_0^t \left| \langle \Delta_q^h R_\varphi^2, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt', \end{aligned}$$

with R_φ^i being defined in (4.50).

The remaining of this section consists in controlling the two terms G_1^q et G_2^q . We will start with G_1^q by observing that

$$\begin{aligned} R_\varphi^1 &= (\varepsilon^2 \partial_x^2 u - (u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon) - (u \partial_x u - v \partial_y u))_\varphi \\ &= (\varepsilon^2 \partial_x^2 u - (u^\varepsilon \partial_x w^1 + w^1 \partial_x u) - (v^\varepsilon \partial_y w^1 + w^2 \partial_y u))_\varphi, \end{aligned}$$

so,

$$\begin{aligned}
G_1^q &= \int_0^t \left| \langle \Delta_q^h R_\varphi^1, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\
&= \int_0^t \left| \langle \Delta_q^h (\varepsilon^2 \partial_x^2 u - (u^\varepsilon \partial_x w^1 + w^1 \partial_x u) - (v^\varepsilon \partial_y w^1 + w^2 \partial_y u))_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\
&\leq I_1^q + I_2^q + I_3^q,
\end{aligned}$$

where

$$\begin{aligned}
I_1^q &= \int_0^t \left| \langle \Delta_q^h (\varepsilon^2 \partial_x^2 u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\
I_2^q &= \int_0^t \left| \langle \Delta_q^h (u^\varepsilon \partial_x w^1 + w^1 \partial_x u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\
I_3^q &= \int_0^t \left| \langle v^\varepsilon \partial_y w^1 + w^2 \partial_y u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt'.
\end{aligned}$$

Using integration by parts and Poicaré inequality, we have

$$\begin{aligned}
I_1^q &= \varepsilon^2 \int_0^t \left| \langle \Delta_q^h (\partial_x^2 u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' = \varepsilon^2 \int_0^t \left| \langle \Delta_q^h (\partial_x u)_\varphi, \Delta_q^h \partial_x w_\varphi^1 \rangle_{L^2} \right| dt' \\
&\leq C d_q^2 2^{-q} \varepsilon \|\partial_x u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|\varepsilon \partial_x w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \leq C d_q^2 2^{-q} \varepsilon \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \|\varepsilon w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})},
\end{aligned}$$

Summing with respect to $q \in \mathbb{Z}$, we obtain

$$(4.58) \quad \sum_{q \in \mathbb{Z}} 2^q I_1^q = \varepsilon^2 \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h (\partial_x^2 u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \lesssim C \varepsilon \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \|\varepsilon w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})}.$$

For I_2^q , we write

$$I_2^q = \int_0^t \left| \langle \Delta_q^h (u^\varepsilon \partial_x w^1 + w^1 \partial_x u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \leq I_{21}^q + I_{22}^q,$$

where

$$\begin{aligned}
I_{21}^q &= \int_0^t \left| \langle \Delta_q^h (u^\varepsilon \partial_x w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\
I_{22}^q &= \int_0^t \left| \langle \Delta_q^h (w^1 \partial_x u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt'.
\end{aligned}$$

Lemma (1.9) implies

$$I_{21}^q = \int_0^t \left| \langle \Delta_q^h (u^\varepsilon \partial_x w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \leq C d_q^2 2^{-q} \|w_\varphi^1\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)},$$

and then,

$$(4.59) \quad \sum_{q \in \mathbb{Z}} 2^q I_{21}^q = \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h (u^\varepsilon \partial_x w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \leq C d_q^2 2^{-q} \|w_\varphi^1\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)},$$

For I_{22}^q , using Bony's decomposition for the horizontal variable, we write

$$w^1 \partial_x u = T_{\partial_x u}^h w^1 + T_{w^1}^h \partial_x u + R^h(w^1, \partial_x u),$$

and then, we have the following bound

$$I_{22}^q = \int_0^t \left| \langle \Delta_q^h (T_{\partial_x u}^h w^1 + T_{w^1}^h \partial_x u + R^h(w^1, \partial_x u))_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \leq I_{22,1}^q + I_{22,2}^q + I_{22,3}^q$$

with

$$\begin{aligned} I_{22,1}^q &= \int_0^t \left| \langle \Delta_q^h (T_{\partial_x u}^h w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\ I_{22,2}^q &= \int_0^t \left| \langle \Delta_q^h (T_{w^1}^h \partial_x u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\ I_{22,3}^q &= \int_0^t \left| \langle \Delta_q^h (R^h(w^1, \partial_x u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt'. \end{aligned}$$

Using the support properties given in [[3], Proposition 2.10] and the definition of $T_{\partial_x}^h w^1$, we have

$$\begin{aligned} I_{22,1}^q &\leq \sum_{|q-q'|\leq 4} \int_0^t \|S_{q'-1}^h w_\varphi^1\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h \partial_x u_\varphi\|_{L_h^2(L_v^\infty)} \|\Delta_q^h w_\varphi^1\|_{L^2} \\ &\leq \sum_{|q-q'|\leq 4} \int_0^t 2^{\frac{q'}{2}} \|S_{q'-1}^h w_\varphi^1\|_{L^2} \|\Delta_{q'}^h \partial_x u_\varphi\|_{L_h^2(L_v^\infty)} \|\Delta_q^h w_\varphi^1\|_{L^2}. \end{aligned}$$

Since

$$\|\Delta_{q'}^h \partial_x u_\varphi\|_{L_h^2(L_v^\infty)} \lesssim d_{q'}(u_\varphi) \|u_\varphi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}},$$

then,

$$\begin{aligned} I_{22,1}^q &= \int_0^t \left| \langle \Delta_q^h (T_{\partial_x u}^h w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle \right| \lesssim \sum_{|q-q'|\leq 4} \int_0^t 2^{\frac{q'}{2}} \|S_{q'-1}^h w_\varphi^1\|_{L^2} d_{q'}(u_\varphi) \|u_\varphi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\varphi^1\|_{L^2} \\ &\lesssim \sum_{|q-q'|\leq 4} \int_0^t d_{q'}(u_\varphi) 2^{\frac{q'}{2}} 2^{-\frac{q'}{2}} \|\partial_y w_\varphi^1\|_{\mathcal{B}^{\frac{1}{2}}} \|u_\varphi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} 2^{-q} d_q(w_\varphi^1) \|w_\varphi^1\|_{\mathcal{B}^1} \\ &\lesssim d_q^2 2^{-q} \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^1\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^1)} \end{aligned}$$

where $\{d_q^2\}$ is a summable sequence, which implies

$$(4.60) \quad \sum_{q \in \mathbb{Z}} 2^q I_{22,1}^q = \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h (T_{\partial_x u}^h w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle \right| \lesssim \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^1\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^1)}.$$

Now, we recall that

$$\|\Delta_q^h \partial_x u_\varphi\|_{L^\infty} \leq \sum_{l \leq q-2} 2^{\frac{3l}{2}} \|\Delta_l^h u_\varphi\|_{L^2}^{\frac{1}{2}} \|\Delta_l^h \partial_y u_\varphi\|_{L^2}^{\frac{1}{2}} \lesssim 2^q \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}},$$

so we can deduce

$$\begin{aligned} I_{22,2}^q &= \int_0^t \left| \langle \Delta_q^h (T_{w^1}^h \partial_x u)_\varphi, \Delta_q^h w_\varphi^1 \rangle \right| \leq \sum_{|q-q'|\leq 4} \int_0^t \|S_{q'-1}^h \partial_x u_\varphi\|_{L^\infty} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} \|\Delta_q^h w_\varphi^1\|_{L^2} \\ &\lesssim \sum_{|q-q'|\leq 4} \int_0^t 2^{q'} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} \|\Delta_q^h w_\varphi^1\|_{L^2} \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \end{aligned}$$

Using the definition of $\tilde{\eta}(t)$ and Definition 1.5 we have

$$\left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q} d_q(w_\varphi^1) \|w_\varphi^1\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^1)}.$$

Then,

$$I_{22,2}^q \lesssim 2^{-q} d_q^2 \|w_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}^2,$$

where

$$d_q^2 = d_q(w_\varphi^1) \left(\sum_{|q-q'|\leq 4} d_{q'}(w_\varphi^1) \right)$$

form a summable sequence, which implies

$$(4.61) \quad \sum_{q \in \mathbb{Z}} 2^q I_{22,2}^q \lesssim \|w_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}^2.$$

In a similar way, we have

$$\begin{aligned} I_{22,3}^q &= \int_0^t |\langle \Delta_q^h (R^h(w^1, \partial_x u))_\varphi, \Delta_q^h w_\varphi^1 \rangle| dt' \lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t \|\Delta_{q'}^h w_\varphi^1\|_{L^2} \|\tilde{\Delta}_{q'}^h \partial_x u_\varphi\|_{L_h^2(L_v^\infty)} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t 2^{\frac{q'}{2}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} dt' \right) \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \right) \\ &\lesssim d_q^2 2^{-q} \|w_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}^2, \end{aligned}$$

form a summable sequence, which implies

$$(4.62) \quad \sum_{q \in \mathbb{Z}} 2^q I_{22,3}^q \lesssim \|w_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}^2.$$

Summing the estimates (4.60), (4.61) and (4.62) we obtain

$$(4.63) \quad \sum_{q \in \mathbb{Z}} 2^q I_{22}^q \lesssim \|w_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \left(\|w_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} + \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \right).$$

For the term I_3^q , we write

$$(4.64) \quad I_3^q = \int_0^t \left| \langle (v^\varepsilon \partial_y w^1 + w^2 \partial_y u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \leq I_{31}^q + I_{32}^q,$$

where

$$\begin{aligned} I_{31}^q &= \int_0^t \left| \langle \Delta_q^h (v^\varepsilon \partial_y w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\ I_{32}^q &= \int_0^t \left| \langle \Delta_q^h (w^2 \partial_y u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt'. \end{aligned}$$

Since

$$v^\varepsilon \partial_y w^1 = (w^2 + v) \partial_y w^1 = w^2 \partial_y w^1 + v \partial_y w^1,$$

we get

$$I_{31}^q \leq I_{31,1}^q + I_{31,2}^q,$$

with

$$\begin{aligned} I_{31,1}^q &= \int_0^t \left| \langle \Delta_q^h(w^2 \partial_y w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\ I_{31,2}^q &= \int_0^t \left| \langle \Delta_q^h(v \partial_y w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt'. \end{aligned}$$

Lemma (1.10) implies

$$(4.65) \quad \sum_{q \in \mathbb{Z}} 2^q I_{31,1}^q = \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h(w^2 \partial_y w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \lesssim \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

For the term $I_{31,2}^q$, we apply Bony's decomposition with respect to the horizontal variable

$$v \partial_y w^1 = T_v^h \partial_y w^1 + T_{\partial_y w^1}^h v + R^h(v, \partial_y w^1).$$

Using (2.16), we have

$$\begin{aligned} \|S_{q'-1}^h v_\varphi\|_{L^\infty} &= \|S_{q'-1}^h \int_0^y \partial_x u_\varphi(t, x, s) ds\|_{L^\infty} \lesssim \sum_{l \leq q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h u_\varphi\|_{L^2}^{\frac{1}{2}} \|\Delta_l^h \partial_y u_\varphi\|_{L^2}^{\frac{1}{2}} \\ &\lesssim 2^{\frac{q'}{2}} \|u_\varphi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}}, \end{aligned}$$

from which, we infer

$$\begin{aligned} \int_0^t \left| \langle \Delta_q^h(T_v^h \partial_y w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h v_\varphi\|_{L^\infty} \|\Delta_{q'}^h \partial_y w_\varphi^1\|_{L^2} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t 2^{\frac{q'}{2}} \|u_\varphi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_y w_\varphi^1\|_{L^2} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} dt' \\ &\leq d_q^2 2^{-q} \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}, \end{aligned}$$

where $\{d_q^2\}$ forms a summable sequence. Thus,

$$(4.66) \quad \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h(T_v^h \partial_y w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| \lesssim \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

In the same way, we have

$$\|\Delta_{q'}^h v_\varphi(t, x, y)\|_{L_h^2(L_v^\infty)} \lesssim \|u_\varphi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}},$$

from which, we infer

$$\begin{aligned} \int_0^t \left| \langle \Delta_q^h(T_{\partial_y w^1}^h v)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h \partial_y w_\varphi^1\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h v_\varphi\|_{L_h^2(L_v^\infty)} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|u_\varphi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|S_{q'-1}^h \partial_y w_\varphi^1\|_{L_h^\infty(L_v^2)} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} dt' \\ &\leq d_q^2 2^{-q} \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}, \end{aligned}$$

where $\{d_q^2\}$ forms a summable sequence. Multiply the above estimates by 2^q , and summing over $q \in \mathbb{Z}$, we obtain

$$(4.67) \quad \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h(T_{\partial_y w^1}^h v)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \lesssim \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Finally, we have

$$\begin{aligned}
\int_0^t \left| \left\langle \Delta_q^h (R^h(v, \partial_y w^1))_\varphi, \Delta_q^h w_\varphi^1 \right\rangle_{L^2} \right| dt' &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t \|\Delta_{q'}^h v_\varphi\|_{L_h^2(L_v^\infty)} \|\tilde{\Delta}_{q'}^h \partial_y w_\varphi^1\|_{L^2} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t \|u_\varphi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\tilde{\Delta}_{q'}^h \partial_y w_\varphi^1\|_{L^2} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} dt' \\
&\leq d_q^2 2^{-q} \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)},
\end{aligned}$$

which implies

$$(4.68) \quad \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \left\langle \Delta_q^h (R^h(v, \partial_y w^1))_\varphi, \Delta_q^h w_\varphi^1 \right\rangle_{L^2} \right| dt' \lesssim \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Taking the sum of (4.66), (4.67) and (4.68), we arrive to

$$(4.69) \quad \sum_{q \in \mathbb{Z}} 2^q I_{31,2}^q \lesssim \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

We now estimate the term I_{32}^q in (4.64). Bony's decomposition for the horizontal variable implies

$$I_{32}^q = \int_0^t \left| \left\langle \Delta_q^h (T_{w^2}^h \partial_y u + T_{\partial_y u}^h w^2 + R^h(w^2, \partial_y u))_\varphi, \Delta_q^h w_\varphi^1 \right\rangle_{L^2} \right| dt' \leq I_{32,1}^q + I_{32,2}^q + I_{32,3}^q,$$

where

$$\begin{aligned}
I_{32,1}^q &= \int_0^t \left| \left\langle \Delta_q^h (T_{w^2}^h \partial_y u)_\varphi, \Delta_q^h w_\varphi^1 \right\rangle_{L^2} \right| dt' \\
I_{32,2}^q &= \int_0^t \left| \left\langle \Delta_q^h (T_{\partial_y u}^h w^2)_\varphi, \Delta_q^h w_\varphi^1 \right\rangle_{L^2} \right| dt' \\
I_{32,3}^q &= \int_0^t \left| \left\langle \Delta_q^h (R^h(w^2, \partial_y u))_\varphi, \Delta_q^h w_\varphi^1 \right\rangle_{L^2} \right| dt'.
\end{aligned}$$

We first observe that

$$\begin{aligned}
I_{32,1}^q &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h w_\varphi^2\|_{L^\infty} \|\Delta_{q'}^h \partial_y u_\varphi\|_{L^2} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} dt' \\
&\lesssim \sum_{|q'-q| \leq 4} \int_0^t 2^{\frac{-q'}{2}} \|S_{q'-1}^h w_\varphi^2\|_{L^\infty} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} dt'.
\end{aligned}$$

Due to the fact that $w^2(t, x, y) = -\int_0^y \partial_x w^1(t, x, s) ds$, we deduce

$$\begin{aligned}
\|S_{q'-1}^h w_\varphi^2\|_{L^\infty} &\lesssim \int_0^y \|S_{q'-1}^h \partial_x w_\varphi^1(t, x, s)\|_{L_h^\infty} ds \\
&\lesssim 2^{\frac{3q'}{2}} \|S_{q'-1}^h w_\varphi^1\|_{L^2},
\end{aligned}$$

and then from which, we have

$$\begin{aligned}
I_{32,1}^q &\lesssim \sum_{|q'-q|\leq 4} \int_0^t 2^{-\frac{q'}{2}} \|S_{q'-1}^h w_\varphi^2\|_{L^\infty} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\varphi^1\|_{L^2} \\
&\lesssim \sum_{|q'-q|\leq 4} \int_0^t 2^{-\frac{q'}{2}} 2^{\frac{3q'}{2}} \|S_{q'-1}^h w_\varphi^1\|_{L^2} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \\
&\lesssim d_q^2 2^{-q} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|S_{q'-1}^h w_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}},
\end{aligned}$$

Taking into account the definition of $\dot{\eta}(t)$ and Definition 1.5 we obtain

$$\left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q} d_q(w_\varphi^1) \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Then,

$$I_{32,1}^q \lesssim 2^{-q} d_q^2 \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}^2,$$

where

$$d_q^2 = d_q(w_\varphi^1) \left(\sum_{|q-q'|\leq 4} d_{q'}(w_\varphi^1) \right)$$

form a summable sequence, which implies

$$(4.70) \quad \sum_{q \in \mathbb{Z}} 2^q I_{32,1}^q \lesssim \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}^2.$$

Now, for $I_{32,2}^q$, we have

$$\begin{aligned}
I_{32,2}^q &\lesssim \sum_{|q'-q|\leq 4} \int_0^t \|S_{q'-1}^h \partial_y u_\varphi\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h w_\varphi^1\|_{L_h^2(L_v^\infty)} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \\
&\lesssim \sum_{|q'-q|\leq 4} \int_0^t 2^{\frac{q'}{2}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} 2^{\frac{q'}{2}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \\
&\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim d_q^2 2^{-q} \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}^2,
\end{aligned}$$

which implies

$$(4.71) \quad \sum_{q \in \mathbb{Z}} 2^q I_{32,2}^q \lesssim \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}^2.$$

Estimates for the last term $I_{32,3}^q$ are also similar. We have

$$\begin{aligned}
I_{32,3}^q &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t \|\Delta_{q'}^h w_\varphi^2\|_{L_h^2(L_v^\infty)} \|\tilde{\Delta}_{q'}^h \partial_y u_\varphi\|_{L^2} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t 2^{\frac{q'}{2}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim d_q^2 2^{-q} \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)},
\end{aligned}$$

which implies

$$(4.72) \quad \sum_{q \in \mathbb{Z}} 2^q I_{32,3}^q \lesssim \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}^2.$$

Summing Estimate (4.70), (4.71) and (4.72) we achieve

$$(4.73) \quad \sum_{q \in \mathbb{Z}} 2^q I_{32}^q \lesssim \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}^2.$$

Summing Estimate (4.58), (4.59), (4.63), (4.65), (4.69) and (4.73), we obtain the following estimate for the term G_1^q in (4.57)

$$\begin{aligned}
(4.74) \quad \sum_{q \in \mathbb{Z}} 2^q G_1^q &= \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h R_\varphi^1, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \lesssim \varepsilon \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \|\varepsilon w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \\
&\quad + \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)} + \|w_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}^2.
\end{aligned}$$

We will now study the second term G_2^q in (4.57). Using the definition of R_φ^2 , we write G_2^q as follows

$$G_2^q \leq J_1^q + J_2^q + J_3^q + J_4^q + J_5^q,$$

where divergence-free property and and Poincaré inequality already imply that

$$\begin{aligned}
\sum_{q \in \mathbb{Z}} 2^q J_1^q &= \varepsilon^2 \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h (\partial_t v)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \lesssim \varepsilon^2 \|(\partial_t u)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \|\partial_y w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}, \\
\sum_{q \in \mathbb{Z}} 2^q J_2^q &= \varepsilon^2 \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h (\partial_y^2 v)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \lesssim \varepsilon^2 \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \|\partial_y w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}, \\
\sum_{q \in \mathbb{Z}} 2^q J_3^q &= \varepsilon^4 \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h (\partial_x^2 v)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \lesssim \varepsilon^4 \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} \|w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})}.
\end{aligned}$$

In order to give an estimate of G_2^q , we will only need to control

$$J_4^q = \varepsilon^2 \int_0^t \left| \langle \Delta_q^h (u^\varepsilon \partial_x v^\varepsilon)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt'$$

and

$$J_5^q = \varepsilon^2 \int_0^t \left| \langle \Delta_q^h (v^\varepsilon \partial_y v^\varepsilon)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt'.$$

We write

$$J_4^q \leq \varepsilon^2 (J_{41}^q + J_{42}^q),$$

where

$$J_{41}^q = \int_0^t \left| \langle \Delta_q^h (u^\varepsilon \partial_x w^2)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt'$$

$$J_{42}^q = \int_0^t \left| \langle \Delta_q^h (u^\varepsilon \partial_x v)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt'.$$

It follows from Lemma 1.9 that

$$\sum_{q \in \mathbb{Z}} 2^q J_{41}^q = \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h (u^\varepsilon \partial_x w^2)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \lesssim \|w_\varphi^2\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}^2.$$

For the second term, Bony's decomposition for the horizontal variable gives

$$J_{42}^q = \int_0^t \left| \langle \Delta_q^h (u^\varepsilon \partial_x v)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \leq J_{421}^q + J_{422}^q + J_{423}^q,$$

with

$$J_{421}^q = \int_0^t \left| \langle \Delta_q^h (T_{u^\varepsilon}^h \partial_x v)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt'$$

$$J_{422}^q = \int_0^t \left| \langle \Delta_q^h (T_{\partial_x v}^h u^\varepsilon)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt'$$

$$J_{423}^q = \int_0^t \left| \langle \Delta_q^h (R^h(u^\varepsilon, \partial_x v))_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt'.$$

Using the estimate

$$\|S_{q'-1}^h u_\varphi^\varepsilon\|_{L^\infty} \lesssim \|u_\varphi^\varepsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_y u_\varphi^\varepsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}},$$

and the relation (2.16), we have

$$J_{421}^q = \int_0^t \left| \langle \Delta_q^h (T_{u^\varepsilon}^h \partial_x v)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h u_\varphi^\varepsilon\|_{L^\infty} \|\Delta_{q'}^h \partial_x v_\varphi\|_{L^2} \|\Delta_q^h w_\varphi^2\|_{L^2} dt'$$

$$\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|u_\varphi^\varepsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_y u_\varphi^\varepsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} 2^{2q'} \|\Delta_{q'}^h u_\varphi\|_{L^2} \|\Delta_q^h w_\varphi^2\|_{L^2} dt'$$

$$\lesssim d_q^2 2^{-q} \|u_\varphi^\varepsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|w_\varphi^2\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Multiply the above inequality by 2^q and summing over $q \in \mathbb{Z}$, we obtain

$$(4.75) \quad \sum_{q \in \mathbb{Z}} 2^q J_{421}^q \lesssim \|u_\varphi^\varepsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|w_\varphi^2\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

In a similar way, the fact that

$$\|S_{q'-1}^h \partial_x v_\varphi\|_{L^\infty} \lesssim \int_0^y \|S_{q'-1}^h \partial_x (\partial_x u_\varphi(t, x, s))\|_{L^\infty} ds \lesssim 2^{\frac{q'}{2}} \|\partial_y u_\varphi\|_{\mathcal{B}^2},$$

leads to

$$J_{422}^q = \int_0^t \left| \langle \Delta_q^h (T_{\partial_x v}^h u^\varepsilon)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-q} \|u_\varphi^\varepsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|w_\varphi^2\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

So multiply by 2^q and summing over $q \in \mathbb{Z}$ imply

$$(4.76) \quad \sum_{q \in \mathbb{Z}} 2^q J_{422}^q \lesssim \|u_\varphi^\varepsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|w_\varphi^2\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

For the last term J_{423}^q , we have

$$J_{423}^q = \int_0^t \left| \langle \Delta_q^h (R^h(\partial_x v, u^\varepsilon))_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| \lesssim d^2 2^{-q} \|u_\varphi^\varepsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|w_\varphi^2\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Multiplying the result by 2^q , and summing over $q \in \mathbb{Z}$, we get

$$(4.77) \quad \sum_{q \in \mathbb{Z}} 2^q J_{423}^q \lesssim \|u_\varphi^\varepsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|w_\varphi^2\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Summing Inequalities (4.75), (4.76) and (4.77) finally yields

$$\sum_{q \in \mathbb{Z}} 2^q J_{42}^q \lesssim \|u_\varphi^\varepsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|w_\varphi^2\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Now for J_5^q , we use the following identity

$$v^\varepsilon \partial_y v^\varepsilon = v \partial_y w^2 + w^2 \partial_y w^2 + v \partial_y v + w^2 \partial_y v.$$

Lemma 1.10 yields

$$\varepsilon^2 \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h (w^2 \partial_y w^2)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \lesssim \|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}^2.$$

From (4.69), we have

$$\sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h (v \partial_y w^2)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \lesssim \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^2\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

As for (4.63), we obtain

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h (w^2 \partial_x u)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \\ & \lesssim \|w_\varphi^2\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)} \left(\|w_\varphi^2\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)} + \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \right). \end{aligned}$$

Then, we deduce from the proof of (4.69) that

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h (v \partial_y v)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| \lesssim \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y v_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^2\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)} \\ & \lesssim d^2 2^{-q} \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \|w_\varphi^2\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

As a result, it come out

$$(4.78) \quad \sum_{q \in \mathbb{Z}} 2^q J_5^q \lesssim \|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}^2 + \varepsilon^2 \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \left(\|\partial_y w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right) \|w_\varphi^2\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Summing all the above result finally gives

$$\begin{aligned}
(4.79) \quad \sum_{q \in \mathbb{Z}} 2^q G_2^q &= \sum_{q \in \mathbb{Z}} 2^q \int_0^t \left| \langle \Delta_q^h R_\varphi^2, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \\
&\lesssim \|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} + \varepsilon^2 \|w_\varphi^2\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} (\|w_\varphi^2\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \\
&\quad + \varepsilon^2 \|(\partial_y w_\varphi^2, \varepsilon \partial_x w_\varphi^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \left(\|(\partial_t u)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} \right) \\
&\quad + \|u_\varphi^\varepsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^2)} + \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} (\|\partial_y w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})}).
\end{aligned}$$

Proof of Theorem 1.8. First, we remark that, in this paragraph, we will not drop the index ε anymore. Thanks to the results obtained in Section 3 and 4, we have

$$(4.80) \quad \|u_\Theta^\varepsilon\|_{\tilde{L}^\infty(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})} + \|u_\phi\|_{\tilde{L}^\infty(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{5}{2}})} + \|\partial_y u_\phi\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{5}{2}})} + \|(\partial_t u)_\phi\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{\frac{3}{2}})} \leq M,$$

where u_Θ^ε and u_ϕ are respectively determined by (3.33) and (2.19) and $M \geq 1$ is a constant independent to ε . Then, Estimates (4.74), (4.79) and (4.80) imply

$$\begin{aligned}
(4.81) \quad \sum_{q \in \mathbb{Z}} 2^q \int_0^t (I^q + J^q)(t') dt' &\lesssim (M\varepsilon \|(\varepsilon \partial_x(w_\varphi^1, \varepsilon w_\varphi^2), \varepsilon \partial_y w_\varphi^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\
&\quad + M^{\frac{1}{2}} \|\partial_y(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \\
&\quad + M^{\frac{3}{2}} \varepsilon \|\varepsilon w_\varphi^2\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} + \|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}).
\end{aligned}$$

Multiplying the above inequality by 2^q , and summing the obtained inequalities with respect to $q \in \mathbb{Z}$, we come to

$$\begin{aligned}
(4.82) \quad &\|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \mu^{\frac{1}{2}} \|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} + \|\partial_y(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \varepsilon \|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \\
&\leq C \|e^{a|D_x|}(u_0^\varepsilon - u_0, \varepsilon(v_0^\varepsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} + C \|\partial_x \theta_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + C(\sqrt{M}\varepsilon \|(\varepsilon \partial_x(w_\varphi^1, \varepsilon w_\varphi^2), \varepsilon \partial_y w_\varphi^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \\
&\quad + M^{\frac{1}{4}} \|\partial_y(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}^{\frac{1}{2}} + M^{\frac{3}{4}} \varepsilon \|\varepsilon w_\varphi^2\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}^{\frac{1}{2}} + \|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}).
\end{aligned}$$

Young's inequality leads to

$$\begin{aligned}
(4.83) \quad &\|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \mu^{\frac{1}{2}} \|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} + \|\partial_y(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \varepsilon \|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \\
&\leq C \left(\|e^{a|D_x|}(u_0^\varepsilon - u_0, \varepsilon(v_0^\varepsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(T_0^\varepsilon - T_0)\|_{\mathcal{B}^{\frac{1}{2}}} + M(\varepsilon + \|(w_\varphi^1, \varepsilon w_\varphi^2)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}) \right).
\end{aligned}$$

Then by taking $\mu = CM$, we can complete the proof Theorem 1.8. \square

APPENDIX A. PROOF OF ESTIMATES FOR BI-LINEAR TERMS

A.1. Proof of Lemma 1.9. As in [2], using Bony's homogeneous decomposition into paraproducts and remainders as in Definition 1.1, we can write

$$u \partial_x w = T_u^h \partial_x w + T_{\partial_x w}^h u + R^h(u, \partial_x w),$$

where

$$T_a b = \sum_{q \in \mathbb{Z}} S_{q-1}^h a \Delta_q^h b \quad \text{and} \quad R^h(a, b) = \sum_{|q'-q| \leq 1} \Delta_q^h a \Delta_q^h b.$$

We have the following bound

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x w)_\phi, e^{\mathcal{R}t'} \Delta_q^h w_\phi \right\rangle_{L^2} \right| dt' \leq A_{1,q} + A_{2,q} + A_{3,q},$$

where

$$\begin{aligned} A_{1,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_u^h \partial_x w)_\phi, e^{\mathcal{R}t'} \Delta_q^h w_\phi \right\rangle_{L^2} \right| dt' \\ A_{2,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_x w}^h u)_\phi, e^{\mathcal{R}t'} \Delta_q^h w_\phi \right\rangle_{L^2} \right| dt' \\ A_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(u, \partial_x w))_\phi, e^{\mathcal{R}t'} \Delta_q^h w_\phi \right\rangle_{L^2} \right| dt'. \end{aligned}$$

Using the support properties given in [[3], Proposition 2.10] and the definition of $T_u^h \partial_x w$, we infer

$$A_{1,q} \leq \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h u_\phi(t')\|_{L^\infty} \|\Delta_{q'}^h \partial_x w_\phi(t')\|_{L^2} \|\Delta_q^h w_\phi(t')\|_{L^2} dt'.$$

By Poincaré's inequality, on the interval $\{0 < y < 1\}$, we have the inclusion $\dot{H}_y^1 \hookrightarrow L_y^\infty$ and,

$$(A.84) \quad \|\Delta_q^h u_\phi(t')\|_{L^\infty} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h u_\phi(t')\|_{L_h^2(L_v^\infty)} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h \partial_y u_\phi(t')\|_{L^2} \lesssim d_q(u_\phi) \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

where $\{d_q(u_\phi)\}$ is a square-summable sequence with $\sum d_q(u_\phi)^2 = 1$. Then,

$$\|S_{q'-1}^h u_\phi(t')\|_{L^\infty} \lesssim \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

which, combining with Hölder inequality, implies that

$$\begin{aligned} A_{1,q} &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{\mathcal{R}t'} \|\Delta_{q'}^h w_\phi(t')\|_{L^2} e^{\mathcal{R}t'} \|\Delta_q^h w_\phi(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_{q'}^h w_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_q^h w_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

We already define $\dot{\theta}(t) = \|\partial_y u_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$ and using Definition 1.5 we have

$$\left(\int_0^t \dot{\theta}(t') e^{2\mathcal{R}t'} \|\Delta_{q'}^h w_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q(s+\frac{1}{2})} d_q(w_\phi) \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\dot{\theta}}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Then,

$$A_{1,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\dot{\theta}}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

where

$$d_q^2 = d_q(w_\phi) \left(\sum_{|q-q'|\leq 4} d_{q'}(w_\phi) 2^{(q-q')(s-\frac{1}{2})} \right)$$

form a summable sequence, which implies

$$(A.85) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} A_{1,q} \lesssim \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\dot{\theta}}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

Using the support properties given in [[3], Proposition 2.10] and the definition of $T_u^h \partial_x w$, we can estimate $A_{2,q}$ in a similar way as we did for $A_{1,q}$. As in (A.84), we can write

$$\|\Delta_{q'}^h u_\phi\|_{L_h^2(L_v^\infty)} \lesssim \|\Delta_{q'}^h \partial_y u_\phi\|_{L^2} \lesssim 2^{-\frac{q'}{2}} d_{q'}(u_\phi) \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}.$$

Then,

$$\begin{aligned} I_q &= \left| \langle \Delta_q^h (T_{\partial_x w}^h u)_\phi, \Delta_q^h w_\phi \rangle_{L^2} \right| \leq \sum_{|q-q'| \leq 4} \|S_{q'-1}^h \partial_x w_\phi\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h u_\phi\|_{L_h^2(L_v^\infty)} \|\Delta_q^h w_\phi\|_{L^2} \\ &\leq \sum_{|q-q'| \leq 4} 2^{-\frac{q'}{2}} d_{q'}(u_\phi) \|S_{q'-1}^h \partial_x w_\phi\|_{L_h^\infty(L_v^2)} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\phi\|_{L^2}. \end{aligned}$$

Since $0 < s \leq 1$, we have

$$\begin{aligned} e^{2\mathcal{R}t} I_q &\leq \sum_{|q-q'| \leq 4} e^{2\mathcal{R}t} 2^{-\frac{q'}{2}} d_{q'}(u_\phi) \|S_{q'-1}^h \partial_x w_\phi\|_{L_h^\infty(L_v^2)} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\phi\|_{L^2} \\ &\leq \sum_{|q-q'| \leq 4} e^{2\mathcal{R}t} 2^{-\frac{q'}{2}} d_{q'}(u_\phi) \sum_{l \lesssim q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h w_\phi\|_{L^2} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\phi\|_{L^2} \\ &\leq \sum_{|q-q'| \leq 4} 2^{-\frac{q'}{2}} d_{q'}(u_\phi) \sum_{l \lesssim q'-2} 2^{l(1-s)} d_l(w_\phi) \|e^{\mathcal{R}t} w_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t} w_\phi\|_{L^2} \\ &\leq \sum_{|q-q'| \leq 4} 2^{-\frac{q'}{2}} d_{q'}(u_\phi) 2^{q'(1-s)} \|e^{\mathcal{R}t} w_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t} w_\phi\|_{L^2} \\ &\leq \sum_{|q-q'| \leq 4} 2^{-\frac{q'}{2}} d_{q'}(u_\phi) 2^{q'(1-s)} 2^{q(s+\frac{1}{2})} 2^{-q(s+\frac{1}{2})} \|e^{\mathcal{R}t} w_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t} w_\phi\|_{L^2} \\ &\leq d_q^2 2^{-2qs} \|e^{\mathcal{R}t} w_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}}^2 \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}. \end{aligned}$$

where

$$d_q^2 = d_q(w_\phi) \left(\sum_{|q-q'| \leq 4} d_{q'} 2^{(q-q')(s-\frac{1}{2})} \right)$$

is a summable sequence of positive constants. Summing with respect to $q \in \mathbb{Z}$, integrating over $[0, t]$ and using Fubini's theorem, we get

$$(A.86) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} A_{2,q} = \int_0^t \left(\sum_{q \in \mathbb{Z}} 2^{2qs} e^{2\mathcal{R}t'} I_q \right) dt' \lesssim \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

where we recall that $\dot{\theta}(t) = \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}$.

To end this proof, it remains to estimate $A_{3,q}$. Using the support properties given in [[3], Proposition 2.10], the definition of $R^h(u, \partial_x w)$ and Bernstein lemma 1.2, we can write

$$\begin{aligned} J_q &= \left| \langle \Delta_q^h (R^h(u, \partial_x w))_\phi, \Delta_q^h w_\phi \rangle_{L^2} \right| \leq 2^{\frac{q}{2}} \sum_{q' \geq k-3} \|\Delta_{q'}^h u_\phi\|_{L_h^2(L_v^\infty)} \|\Delta_{q'}^h \partial_x w_\phi\|_{L^2} \|\Delta_q^h w_\phi\|_{L^2} \\ &\leq 2^{\frac{q}{2}} \sum_{q' \geq k-3} 2^{q'(1-\frac{1}{2})} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h w_\phi\|_{L^2} \|\Delta_q^h w_\phi\|_{L^2} \\ &\leq 2^{\frac{q}{2}} \sum_{q' \geq k-3} 2^{\frac{q'}{2}} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h w_\phi\|_{L^2} \|\Delta_q^h w_\phi\|_{L^2}. \end{aligned}$$

Since $0 < s \leq 1$, we have

$$\begin{aligned}
e^{2\mathcal{R}t} J_q &\leq 2^{\frac{q}{2}} \sum_{q' \geq k-3} 2^{\frac{q'}{2}} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h e^{\mathcal{R}t} w_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t} w_\phi\|_{L^2} \\
&\leq 2^{\frac{q}{2}} \sum_{q' \geq k-3} 2^{\frac{q'}{2}} d_{q'}(w_\phi) 2^{-q'(s+\frac{1}{2})} \|e^{\mathcal{R}t} w_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} d_q(w_\phi) 2^{-q(s+\frac{1}{2})} \|e^{\mathcal{R}t} w_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}} \\
&\leq d_q(w_\phi) 2^{-2qs} \|e^{\mathcal{R}t} w_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}}^2 \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \left(\sum_{q' \geq k-3} d_{q'}(w_\phi) 2^{(q-q')s} \right) \\
&\leq d_q^2 2^{-2qs} \|e^{\mathcal{R}t} w_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}}^2 \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}},
\end{aligned}$$

where

$$d_q^2 = d_q(w_\phi) \left(\sum_{q' \geq k-3} d_{q'}(w_\phi) 2^{(q-q')s} \right)$$

is a summable sequence of positive constants. Summing with respect to $q \in \mathbb{Z}$, integrating over $[0, t]$ and using Fubini's theorem, we finally obtain

$$(A.87) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} A_{3,q} = \int_0^t \left(\sum_{q \in \mathbb{Z}} 2^{2qs} e^{2\mathcal{R}t'} J_q \right) dt' \lesssim \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

Lemma 1.9 is then proved by summing Estimates (A.85), (A.86) and (A.87). \square

A.2. Proof of Lemma 1.10. At first, we will prove Estimate (1.14) of Lemma 1.10. Bony's decomposition for the horizontal variable implies

$$(A.88) \quad \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle_{L^2} \right| dt' \leq B_{1,q} + B_{2,q} + B_{3,q},$$

with

$$\begin{aligned}
B_{1,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_v^h \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle_{L^2} \right| dt' \\
B_{2,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y u}^h v)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle_{L^2} \right| dt' \\
B_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(v, \partial_y u))_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle_{L^2} \right| dt'.
\end{aligned}$$

As for the term $A_{1,q}$ in the proof of Lemma 1.9, we have the following estimate

$$\begin{aligned}
K_{1,q} &= \left| \left\langle e^{\mathcal{R}t} \Delta_q^h (T_v^h \partial_y u)_\phi, e^{\mathcal{R}t} \Delta_q^h u_\phi \right\rangle_{L^2} \right| \\
&\lesssim \sum_{|q'-q| \leq 4} e^{\mathcal{R}t} \|S_{q'-1}^h v_\phi\|_{L^\infty} \|\Delta_{q'}^h \partial_y u_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t} u_\phi\|_{L^2} \\
&\lesssim \sum_{|q'-q| \leq 4} d_{q'}(u_\phi) 2^{-\frac{q'}{2}} e^{\mathcal{R}t} \|S_{q'-1}^h v_\phi\|_{L^\infty} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t} u_\phi\|_{L^2}.
\end{aligned}$$

Identity (2.16) and Bernstein lemma imply

$$(A.89) \quad \|\Delta_q^h v_\phi(t)\|_{L^\infty} \leq \int_0^1 \|\Delta_q^h \partial_x u_\phi(t, \cdot, y')\|_{L_h^\infty} dy' \lesssim 2^{\frac{3q}{2}} \int_0^1 \|\Delta_q^h u_\phi(t, \cdot, y')\|_{L_h^2} dy' \lesssim 2^{\frac{3q}{2}} \|\Delta_q^h u_\phi(t)\|_{L^2},$$

from which and the fact that $s \leq 1$, we infer

$$\begin{aligned}
K_{1,q} &\lesssim \sum_{|q'-q|\leq 4} d_{q'}(u_\phi) 2^{-\frac{q'}{2}} \sum_{l\leq q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h e^{\mathcal{R}t} u_\phi(t)\|_{L^2} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t} u_\phi\|_{L^2} \\
&\lesssim \sum_{|q'-q|\leq 4} d_{q'}(u_\phi) 2^{-\frac{q'}{2}} \sum_{l\leq q'-2} d_l 2^{\frac{3l}{2}} 2^{-l(s+\frac{1}{2})} \|e^{\mathcal{R}t} u_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t} u_\phi\|_{L^2} \\
&\lesssim \sum_{|q'-q|\leq 4} d_{q'}(u_\phi) 2^{-\frac{q'}{2}} 2^{q'(1-s)} \|e^{\mathcal{R}t} u_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t} u_\phi\|_{L^2} \\
&\lesssim d_q(u_\phi) \sum_{|q'-q|\leq 4} d_{q'}(u_\phi) 2^{-\frac{q'}{2}} 2^{q'(1-s)} 2^{-q(s+\frac{1}{2})} \|e^{\mathcal{R}t} u_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} u_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}} \\
&\lesssim d_q^2 2^{-2qs} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} u_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}}^2,
\end{aligned}$$

where

$$d_q^2 = d_q(u_\phi) \left(\sum_{|q'-q|\leq 4} d_{q'}(u_\phi) 2^{(q-q')(s-\frac{1}{2})} \right)$$

is a summable sequence of positive constants. Then, summing with respect to $q \in \mathbb{Z}$, integrating over $[0, t]$ and using Fubini's theorem lead to

$$(A.90) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} B_{1,q} = \int_0^t \left(\sum_{q \in \mathbb{Z}} 2^{2qs} K_{1,q} \right) dt' \lesssim \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

For the second term on the right-hand side of (A.88), we first have

$$K_{2,q} = \left| \left\langle e^{\mathcal{R}t} \Delta_q^h (T_{\partial_y u}^h v)_\phi, e^{\mathcal{R}t} \Delta_q^h u_\phi \right\rangle_{L^2} \right| \lesssim \sum_{|q'-q|\leq 4} e^{2\mathcal{R}t} \|S_{q'-1}^h \partial_y u_\phi\|_{L_h^\infty(L_h^2)} \|\Delta_{q'}^h v_\phi\|_{L_h^2(L_h^\infty)} \|\Delta_q^h u_\phi\|_{L^2}$$

Using (A.89) we can write

$$\begin{aligned}
K_{2,q} &\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t} u_\phi\|_{L^2} \\
&\lesssim d_q(u_\phi) \sum_{|q'-q|\leq 4} d_{q'}(u_\phi) 2^{q'} 2^{-q(s+\frac{1}{2})} 2^{-q'(s+\frac{1}{2})} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} u_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}}^2 \\
&\lesssim d_q^2 2^{-2qs} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|u_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}}^2,
\end{aligned}$$

where

$$d_q^2 = d_q(u_\phi) \left(\sum_{|q'-q|\leq 4} d_{q'}(u_\phi) 2^{(q-q')(s-\frac{1}{2})} \right)$$

is a summable sequence of positive constants. Summing with respect to $q \in \mathbb{Z}$, integrating over $[0, t]$ and using Fubini's theorem, we will get

$$(A.91) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} B_{2,q} = \int_0^t \left(\sum_{q \in \mathbb{Z}} 2^{2qs} K_{2,q} \right) dt' \lesssim \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

Now, for the third term on the right-hand side of (A.88), we have

$$K_{3,q} = \left| \left\langle e^{\mathcal{R}t} \Delta_q^h (R^h(v, \partial_y u))_\phi, e^{\mathcal{R}t} \Delta_q^h u_\phi \right\rangle_{L^2} \right| \lesssim 2^{\frac{q}{2}} e^{2\mathcal{R}t} \sum_{q' \geq q-3} \|\Delta_{q'}^h v_\phi\|_{L_h^\infty(L_h^2)} \|\Delta_{q'}^h \partial_y u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2}$$

Similar calculations as in (A.89) imply

$$\|\Delta_q^h v_\phi(t)\|_{L^\infty(L_h^2)} \leq \int_0^1 \|\Delta_q^h \partial_x u_\phi(t, \cdot, y')\|_{L_h^2} dy' \lesssim 2^{\frac{q}{2}} \int_0^1 \|\Delta_q^h u_\phi(t, \cdot, y')\|_{L_h^2} dy' \lesssim 2^{\frac{q}{2}} \|\Delta_q^h u_\phi(t)\|_{L^2},$$

which yields next

$$\begin{aligned} K_{3,q} &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \|\Delta_{q'}^h e^{\mathcal{R}t} u_\phi\|_{L^2} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t} u_\phi\|_{L^2} \\ &\lesssim d_q^2 2^{-2qs} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|u_\phi\|_{\mathcal{B}^{s+\frac{1}{2}}}^2, \end{aligned}$$

where

$$d_q^2 = d_q(u_\phi) \left(\sum_{q' \geq q-3} d_{q'}(u_\phi) 2^{(q-q')s} \right)$$

is a summable sequence of positive constants. It remains to take the sum with respect to $q \in \mathbb{Z}$, integrate it over $[0, t]$ and use Fubini's theorem to get

$$(A.92) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} B_{3,q} = \int_0^t \left(\sum_{q \in \mathbb{Z}} 2^{2qs} K_{3,q} \right) dt' \lesssim \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

The proof of Estimate (1.14) is completed.

We will now prove Estimate (1.15). Using Bony's decomposition for the horizontal variable, we have

$$(A.93) \quad \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y T)_\phi, e^{\mathcal{R}t'} \Delta_q^h T_\phi \right\rangle_{L^2} \right| dt' \leq C_{1,q} + C_{2,q} + C_{3,q},$$

where

$$\begin{aligned} C_{1,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_v^h \partial_y T)_\phi, e^{\mathcal{R}t'} \Delta_q^h T_\phi \right\rangle_{L^2} \right| dt' \\ C_{2,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y T}^h v)_\phi, e^{\mathcal{R}t'} \Delta_q^h T_\phi \right\rangle_{L^2} \right| dt' \\ C_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(v, \partial_y T))_\phi, e^{\mathcal{R}t'} \Delta_q^h T_\phi \right\rangle_{L^2} \right| dt'. \end{aligned}$$

From (A.89), we have

$$(A.94) \quad \|\Delta_q^h v_\phi\|_{L^\infty} \lesssim 2^{\frac{3q}{2}} \|\Delta_q^h u_\phi\|_{L^2} \lesssim d_q(u_\phi) 2^q \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}},$$

where $d_q(u_\phi)$ is a square-summable sequence of positive constants. Then, we deduce that

$$\|S_{q'-1}^h v_\phi\|_{L^\infty} \lesssim \sum_{l \leq q'-2} \|\Delta_l^h v_\phi\|_{L^\infty} \leq 2^{q'} \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}},$$

and we get

$$\begin{aligned} \left| \left\langle e^{\mathcal{R}t} \Delta_q^h (T_v^h \partial_y T)_\phi, e^{\mathcal{R}t} \Delta_q^h T_\phi \right\rangle_{L^2} \right| &\lesssim \sum_{|q'-q| \leq 4} e^{2\mathcal{R}t} \|S_{q'-1}^h v_\phi\|_{L^\infty} \|\Delta_{q'}^h \partial_y T_\phi\|_{L^2} \|\Delta_q^h T_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q| \leq 4} e^{2\mathcal{R}t} 2^{q'-q} \|u_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_y T_\phi\|_{L^2} \|\Delta_q^h \partial_x T_\phi\|_{L^2} \\ &\lesssim d_q^2 2^{-2qs} \|u_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \nabla T_\phi\|_{\mathcal{B}^s}^2, \end{aligned}$$

where

$$d_q^2 = d_q(T_\phi) \left(\sum_{|q'-q|\leq 4} d_{q'}(T_\phi) 2^{(q-q')(s-1)} \right),$$

is a summable sequence of positive constants. Taking the sum with respect to $q \in \mathbb{Z}$, integrating it over $[0, t]$ and using Fubini's theorem, we arrive to

$$(A.95) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} C_{1,q} \lesssim \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}.$$

The second term on the right-hand side of (A.88) can be controlled in a similar way as we did for $B_{1,q}$. We have

$$\begin{aligned} \left| \left\langle e^{\mathcal{R}t} \Delta_q^h (T_{\partial_y T}^h)_\phi, e^{\mathcal{R}t} \Delta_q^h T_\phi \right\rangle_{L^2} \right| &\lesssim \sum_{|q'-q|\leq 4} e^{2\mathcal{R}t} \|\Delta_{q'}^h v_\phi\|_{L_h^2(L_h^\infty)} \|S_{q'-1}^h \partial_y T_\phi\|_{L_h^\infty(L_h^2)} \|\Delta_q^h T_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{\frac{q'}{2}} \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} 2^{\frac{q'}{2}} \|\Delta_{q'}^h e^{\mathcal{R}t} \partial_y T_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t} T_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q|\leq 4} 2^{q'-q} \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h e^{\mathcal{R}t} \partial_y T_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t} \partial_x T_\phi\|_{L^2} \\ &\lesssim d_q^2 2^{-2qs} \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \partial_y T_\phi\|_{\mathcal{B}^s} \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\mathcal{B}^s} \end{aligned}$$

where

$$d_q^2 = d_q(T_\phi) 2^{-2qs} \left(\sum_{|q'-q|\leq 4} d_{q'}(T_\phi) 2^{(q-q')(s-1)} \right)$$

is a summable sequence of positive constants. Taking the sum with respect to $q \in \mathbb{Z}$, integrating it over $[0, t]$ and using Fubini's theorem, we obtain

$$(A.96) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} C_{2,q} \lesssim \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}.$$

For the last term on the right-hand side of (A.88), we first write

$$\begin{aligned} \Delta_q^h R^h(v, \partial_y T) &= \Delta_q^h \left(\sum_{q' \geq q} \Delta_{q'}^h v \Delta_{q'}^h \partial_y T \right) \\ &= \Delta_q^h \partial_y R^h(v, T) - \Delta_q^h R(\partial_y v, T) = \Delta_q^h \partial_y R^h(v, T) + \Delta_q^h R(\partial_x u, T). \end{aligned}$$

Thus,

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(v, \partial_y T))_\phi, e^{\mathcal{R}t'} \Delta_q^h T_\phi \right\rangle_{L^2} \right| dt' \leq E_{1,q} + E_{2,q},$$

where

$$\begin{aligned} E_{1,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h \partial_y (R^h(v, T))_\phi, e^{\mathcal{R}t'} \Delta_q^h T_\phi \right\rangle_{L^2} \right| dt' \\ E_{2,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(\partial_x u, T))_\phi, e^{\mathcal{R}t'} \Delta_q^h T_\phi \right\rangle_{L^2} \right| dt'. \end{aligned}$$

Since $s > 0$, using (A.94) and Bernstein lemma (1.2), we have

$$\begin{aligned}
\left| \langle e^{\mathcal{R}t} \Delta_q^h \partial_y (R^h(v, T))_\phi, e^{\mathcal{R}t} \Delta_q^h T_\phi \rangle_{L^2} \right| &= \left| \langle e^{\mathcal{R}t} \Delta_q^h (R^h(v, T))_\phi, e^{\mathcal{R}t} \Delta_q^h \partial_y T_\phi \rangle_{L^2} \right| \\
&\lesssim \sum_{q' \geq q-3} e^{2\mathcal{R}t} \|\Delta_{q'}^h v_\phi\|_{L_h^2(L_v^\infty)} \|\Delta_{q'}^h T_\phi\|_{L_h^\infty(L_v^2)} \|\Delta_q^h \partial_y T_\phi\|_{L^2} \\
&\lesssim \sum_{q' \geq q-3} 2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2} 2^{\frac{q'}{2}} 2^{-q'} \|\Delta_{q'}^h e^{\mathcal{R}t} \partial_x T_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t} \partial_y T_\phi\|_{L^2} \\
&\lesssim \sum_{q' \geq q-3} \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} 2^{-q's} d_{q'}(T_\phi) \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\mathcal{B}^s} 2^{-qs} d_q(T_\phi) \|e^{\mathcal{R}t} \partial_y T_\phi\|_{\mathcal{B}^s} \\
&\lesssim d_q^2 2^{-2qs} \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\mathcal{B}^s} \|e^{\mathcal{R}t} \partial_y T_\phi\|_{\mathcal{B}^s}
\end{aligned}$$

where

$$d_q^2 = d_q(T_\phi) \left(\sum_{q' \geq q-3} d_{q'}(T_\phi) 2^{(q-q')s} \right)$$

is a summable sequence of positive constants. We deduce that

$$(A.97) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} E_{1,q} \lesssim \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}.$$

For $E_{2,q}$, using Poincaré inequality with respect to $y \in [0, 1]$, we write

$$\begin{aligned}
\left| \langle e^{\mathcal{R}t} \Delta_q^h (R^h(\partial_x u, T))_\phi, e^{\mathcal{R}t} \Delta_q^h T_\phi \rangle_{L^2} \right| &\lesssim \sum_{q' \geq q-3} e^{2\mathcal{R}t} \|\Delta_{q'}^h \partial_x u_\phi\|_{L_h^2(L_v^\infty)} \|\Delta_{q'}^h T_\phi\|_{L_h^\infty(L_v^2)} \|\Delta_q^h T_\phi\|_{L^2} \\
&\lesssim \sum_{q' \geq q-3} 2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2} 2^{\frac{q'}{2}} \|\Delta_{q'}^h e^{\mathcal{R}t} T_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t} T_\phi\|_{L^2} \\
&\lesssim \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \|\Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_{q'}^h e^{\mathcal{R}t} \partial_x T_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t} \partial_y T_\phi\|_{L^2} \\
&\lesssim d_q^2 2^{-2qs} \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\mathcal{B}^s} \|e^{\mathcal{R}t} \partial_y T_\phi\|_{\mathcal{B}^s},
\end{aligned}$$

where

$$d_q^2 = d_q(T_\phi) \left(\sum_{q' \geq q-3} d_{q'}(T_\phi) 2^{(q-q')s} \right)$$

is a summable sequence of positive constants. Thus,

$$(A.98) \quad \sum_{q \in \mathbb{Z}} 2^{2qs} E_{2,q} \lesssim \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t} \nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}.$$

The proof of Lemma 1.10 is then completed by summing Estimates (A.95), (A.96), (A.97) and (A.98). \square

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