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STABILIZATION OF NON-SMOOTH TRANSMISSION PROBLEM INVOLVING BRESSE SYSTEMS

STÉPHANE GERBI, CHIRAZ KASSEM, AND ALI WEHBE

ABSTRACT. We consider an elastic/viscoelastic transmission problem for the Bresse system with fully Dirichlet or Dirichlet-Neumann-Neumann boundary conditions. The physical model consists of three wave equations coupled in certain pattern. The system is damped directly or indirectly by global or local Kelvin-Voigt damping. Actually, the number of the dampings, their nature of distribution (locally or globally) and the smoothness of the damping coefficient at the interface play a crucial role in the type of the stabilization of the corresponding semigroup. Indeed, using frequency domain approach combined with multiplier techniques and the construction of a new multiplier function, we establish different types of energy decay rate (see the table of stability results below). Our results generalize and improve many earlier ones in the literature (see [6]) and in particular some studies done on the Timoshenko system with Kelvin-Voigt damping (see for instance [12], [37] and [39]).

1. INTRODUCTION

1.1. The Bresse system with Kelvin-Voigt damping. Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. There are several mathematical models representing physical damping. The most often encountered type of damping in vibration studies are linear viscous damping and Kelvin-Voigt damping which are special cases of proportional damping. Viscous damping usually models external friction forces such as air resistance acting on the vibrating structures and is thus called “external damping”, while Kelvin-Voigt damping originates from the internal friction of the material of the vibrating structures and thus called “internal damping”. The stabilization evolution systems (Wave equation, coupled wave equations, Timoshenko system ...) with viscoelastic Kelvin-Voigt type damping has attracted the attention of many authors. In particular, it was proved that the stabilization of wave equation with local Kelvin-Voigt damping is greatly influenced by the smoothness of the damping coefficient and the region where the damping is localized (near or faraway from the boundary) even in the one-dimensional case, see [22, 24]. This surprising result initiated the study of an elastic system with local Kelvin-Voigt damping (see Subsection 1.4 for more details). There are a few number of publications concerning the stabilization of Bresse or Timoshenko systems with viscoelastic Kelvin-Voigt damping (see Subsection 1.2 below).

In this paper, we study the stability of Bresse system with local or global Kelvin-Voigt damping and smooth or non-smooth coefficient at interface. We establish different types of energy decay rate which generalize and improve many earlier ones in the literature. The Bresse system is usually considered in studying elastic structures of the arcs type (see [21]). It can be expressed by the equations of motion:

$$\begin{aligned}\rho_1 \varphi_{tt} &= Q_x + lN \\ \rho_2 \psi_{tt} &= M_x - Q \\ \rho_1 w_{tt} &= N_x - lQ\end{aligned}$$

where

$$\begin{aligned}N &= k_3(w_x - l\varphi) + F_3, & Q &= k_1(\varphi_x + \psi + lw) + F_1, & M &= k_2\psi_x + F_2 \\ F_1 &= D_1(\varphi_{xt} + \psi_t + lw_t), & F_2 &= D_2\psi_{xt}, & F_3 &= D_3(w_{xt} - l\varphi_t)\end{aligned}$$

and where F'_1 , F'_2 and F'_3 are the Kelvin-Voigt dampings. When $F_1 = F_2 = F_3 = 0$, N , Q and M denote the axial force, the shear force and the bending moment. The functions φ , ψ , and w model the vertical, shear angle, and longitudinal displacements of the filament. Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k_1 = k'GA$, $k_3 = EA$, $k_2 = EI$, $l = R^{-1}$ where ρ is the density of the material, E is the modulus of elasticity, G is the shear modulus, k' is the shear factor, A is the cross-sectional area, I is the second moment of area of the cross-section, and R is the radius of curvature. The damping coefficients D_1 , D_2 and D_3 are bounded positive functions over $(0, L)$.

Key words and phrases. Bresse system, Kelvin-Voigt damping, polynomial stability, non uniform stability, frequency domain approach.

So we will consider the system of partial differential equations given on $(0, L) \times (0, +\infty)$ by the following form:

$$(1.1) \quad \begin{cases} \rho_1 \varphi_{tt} - [k_1 (\varphi_x + \psi + lw) + D_1 (\varphi_{xt} + \psi_t + lw_t)]_x - lk_3 (w_x - l\varphi) - lD_3 (w_{xt} - l\varphi_t) = 0, \\ \rho_2 \psi_{tt} - [k_2 \psi_x + D_2 \psi_{xt}]_x + k_1 (\varphi_x + \psi + lw) + D_1 (\varphi_{xt} + \psi_t + lw_t) = 0, \\ \rho_1 w_{tt} - [k_3 (w_x - l\varphi) + D_3 (w_{xt} - l\varphi_t)]_x + lk_1 (\varphi_x + \psi + lw) + lD_1 (\varphi_{xt} + \psi_t + lw_t) = 0, \end{cases}$$

with fully Dirichlet boundary conditions:

$$(1.2) \quad \varphi(0, \cdot) = \varphi(L, \cdot) = \psi(0, \cdot) = \psi(L, \cdot) = w(0, \cdot) = w(L, \cdot) = 0 \quad \text{in } \mathbb{R}_+,$$

or with Dirichlet-Neumann-Neumann boundary conditions:

$$(1.3) \quad \varphi(0, \cdot) = \varphi(L, \cdot) = \psi_x(0, \cdot) = \psi_x(L, \cdot) = w_x(0, \cdot) = w_x(L, \cdot) = 0 \quad \text{in } \mathbb{R}_+,$$

in addition to the following initial conditions:

$$(1.4) \quad \begin{aligned} \varphi(\cdot, 0) &= \varphi_0(\cdot), \quad \psi(\cdot, 0) = \psi_0(\cdot), \quad w(\cdot, 0) = w_0(\cdot), \\ \varphi_t(\cdot, 0) &= \varphi_1(\cdot), \quad \psi_t(\cdot, 0) = \psi_1(\cdot), \quad w_t(\cdot, 0) = w_1(\cdot), \quad \text{in } (0, L). \end{aligned}$$

We note that when $R \rightarrow \infty$, then $l \rightarrow 0$ and the Bresse model reduces, by neglecting w , to the well-known Timoshenko beam equations:

$$(1.5) \quad \begin{cases} \rho_1 \varphi_{tt} - [k_1 (\varphi_x + \psi) + D_1 (\varphi_{xt} + \psi_t)]_x = 0, \\ \rho_2 \psi_{tt} - [k_2 \psi_x + D_2 \psi_{xt}]_x + k_1 (\varphi_x + \psi) + D_1 (\varphi_{xt} + \psi_t) = 0 \end{cases}$$

with different types of boundary conditions and with initial data.

1.2. Motivation, aims and main results. The stability of elastic Bresse system with different types of damping (frictional, thermoelastic, Cattaneo, ...) has been intensively studied (see Subsection 1.3), but there are a few number of papers concerning the stability of Bresse or Timoshenko systems with local or global viscoelastic Kelvin-Voigt damping. In fact, in [6], El Arwadi and Youssef studied the theoretical and numerical stability on a Bresse system with Kelvin-Voigt damping under fully Dirichlet boundary conditions. Using multiplier techniques, they established an exponential energy decay rate provided that the system is subject to three global Kelvin-Voigt damping. Later, a numerical scheme based on the finite element method was introduced to approximate the solution. Zhao et al. in [39], considered a Timoshenko system with Dirichlet-Neumann boundary conditions. They obtained the exponential stability under certain hypotheses of the smoothness and structural condition of the coefficients of the system, and obtain the strong asymptotic stability under weaker hypotheses of the coefficients. Tian and Zhang in [37] considered a Timoshenko system under fully Dirichlet boundary conditions and with two locally or globally Kelvin-Voigt dampings. First, in the case when the two Kelvin-Voigt dampings are globally distributed, they showed that the corresponding semigroup is analytic. On the contrary, they proved that the energy of the system decays exponentially or polynomially and the decay rate depends on properties of material coefficient function. In [12], Ghader and Wehbe generalized the results of [39] and [37]. Indeed, they considered the Timoshenko system with only one locally or globally distributed Kelvin-Voigt damping and subject to fully Dirichlet or to Dirichlet-Neumann boundary conditions. They established a polynomial energy decay rate of type t^{-1} for smooth initial data. Moreover, they proved that the obtained energy decay rate is in some sense optimal. In [31], Maryati et al. considered the transmission problem of a Timoshenko beam composed by N components, each of them being either purely elastic, or a Kelvin-Voigt viscoelastic material, or an elastic material inserted with a frictional damping mechanism. They proved that the energy decay rate depends on the position of each component. In particular, they proved that the model is exponentially stable if and only if all the elastic components are connected with one component with frictional damping. Otherwise, only a polynomial energy decay rate is established. So, the stability of the Bresse system with local viscoelastic Kelvin-Voigt damping still an open problem.

The purpose of this paper is to study the Bresse system in the presence of local or global viscoelastic Kelvin-Voigt damping with smooth or non-smooth coefficient at interface and under fully Dirichlet or Dirichlet-Neumann-Neumann boundary conditions. The system is given by (1.1)-(1.2) or (1.1)-(1.3) with initial data (1.4). First, we prove that the corresponding semigroup is analytic provided that our system is subject to three global viscoelastic Kelvin-Voigt dampings *i.e.* there exists $d_0 > 0$ such that $D_1, D_2, D_3 \geq d_0$ over $(0, L)$ (see Theorem 4.2). This result generalize that of [6] where only an exponential energy decay is obtained. On the contrary, when the viscoelastic dampings are locally distributed, we prove that the number of the dampings and the

smoothness of the damping coefficients D_1, D_2, D_3 at the interface play a crucial role in the type of the stabilization of the corresponding semigroup. Indeed, if $D_1, D_2, D_3 \in W^{1,\infty}(0, L)$, using the frequency domain approach combined with multiplier techniques, we prove that the Bresse system (1.1)-(1.2) (or (1.1)-(1.3)) is exponentially stable (see Theorem 5.1). Otherwise, if $D_1, D_2, D_3 \in L^\infty(0, L)$, using frequency domain approach combined with multiplier techniques and the construction of new multiplier functions, we establish a polynomial stability of type $\frac{1}{t}$ (see Theorem 6.1). Moreover, in the absence of at least one damping, we prove the lack of uniform stability for the system (1.1)-(1.3) even with smoothness of damping coefficients. Finally, in the presence of only one local damping D_2 acting on the shear angle displacement ($D_1 = D_3 = 0$), we establish a polynomial energy decay estimate of type $\frac{1}{\sqrt{t}}$ (see Theorem 7.1). In these cases, we conjecture the optimality of the obtained decay rate. For clarity, let

$$\emptyset \neq \omega = (\alpha, \beta) \subset (0, L).$$

The following table summarizes the main results of this article:

Regularity of D_1	Regularity of D_2	Regularity of D_3	Localization	Energy decay rate
$L^\infty(0, L)$	$L^\infty(0, L)$	$L^\infty(0, L)$	$D_i \geq d_0 > 0$ in $(0, L)$ $i = 1, 2, 3$	Analytic stability
$W^{1,\infty}(0, L)$	$W^{1,\infty}(0, L)$	$W^{1,\infty}(0, L)$	$D_i \geq d_0 > 0$ in ω $i = 1, 2, 3$	Exponential stability
$L^\infty(0, L)$	$L^\infty(0, L)$	$L^\infty(0, L)$	$\bigcap_{i=1}^3 \text{supp} D_i = \bar{\omega}$	Polynomial of type $\frac{1}{t}$
0	$L^\infty(0, L)$	0	$D_2 \geq d_0 > 0$ in ω	Polynomial of type $\frac{1}{\sqrt{t}}$

1.3. Literature concerning the Bresse system. In [29], Liu and Rao considered the Bresse system with two thermal dissipation laws. The authors proved an exponential decay rate when the wave speed of the vertical displacement coincides with the wave speed of longitudinal displacement or of the shear angle displacement. Otherwise, they showed polynomial decays depending on the boundary conditions. These results are improved by Fatori and Rivera in [10] where they considered the case of one thermal dissipation law globally distributed on the displacement equation. Wehbe and Najdi in [32] extended and improved the results of [10], when the thermal dissipation is locally distributed. Wehbe and Youssef in [38] considered an elastic Bresse system subject to two locally internal dissipation laws. They proved that the system is exponentially stable if and only if the waves propagate at the same speed. Otherwise, a polynomial decay holds. Alabau et al in [1] considered the same system with one globally distributed dissipation law. The authors proved the existence of polynomial decays with rates that depend on some particular relation between the coefficients. In [13], Guesmia et al. considered Bresse system with infinite memories acting in the three equations of the system. They established asymptotic stability results under some conditions on the relaxation functions regardless the speeds of propagation. These results are improved by Abdallah et al. in [7] where they considered the Bresse system with infinite memory type control and/or with heat conduction given by Cattaneo's law acting in the shear angle displacement. The authors established an exponential energy decay rate when the waves propagate at same speed. Otherwise, they showed polynomial decays. In [5], Benaissa and Kasmi, considered the Bresse system with three control boundary conditions of fractional derivative type. They established a polynomial decay estimate.

1.4. Literature concerning the stability of some systems with viscoelastic damping. The stabilization of evolution systems with viscoelastic Kelvin-Voigt damping retains the attention of many authors. Huang in [19] considered a wave equation with globally distributed Kelvin-Voigt damping, i.e. the damping coefficient is strictly positive on the entire spatial domain. He proved that the corresponding semigroup is not only exponentially stable, but also is analytic. In [23], K. Liu and Z. Liu considered a wave equation with localized Kelvin-Voigt damping in the 1-dimensional case. The dissipation is distributed on any subinterval of the region

occupied by the beam and the damping coefficient is the characteristic function of the subinterval. They proved that the semigroup associated with the equation for the longitudinal motion of the beam is not exponentially stable. This result is due to the discontinuity of the viscoelastic materials. Later, in the 1-dimensional case, it was found that the smoothness of the damping coefficient at the interface is an essential factor for the stability and the regularity of the solutions (see in [22, 24, 25, 26, 27, 40]). For a system of coupled equations, in addition to references cited in Subsection 1.2, we recall the following results. In [14], Hassine considered the longitudinal and transversal vibrations of the transmission Euler-Bernoulli beam with Kelvin-Voigt damping distributed locally on any subinterval of the region occupied by the beam and only in one side of the transmission point. He proved that the semigroup associated with the equation for the transversal motion of the beam is exponentially stable, although the semigroup associated with the equation for the longitudinal motion of the beam is polynomially stable. Hassine in [15] discussed the asymptotic behavior of the transmission Euler-Bernoulli plate and wave equation with a localized Kelvin-Voigt damping. He proved that sufficiently smooth solutions decay logarithmically at infinity even when the feedback affects a small open subset of the interior. Also, in [16], Hassine considered a beam and a wave equations coupled by an elastic beam through transmission condition. The damping which is locally distributed acts only at one equation. Firstly he considered the case when the dissipation acts through the beam equation, he showed a precise polynomial energy decay rate. Secondly, in the case when the damping acts through the wave equation and he provided a precise polynomial energy decay rate. In both cases, he proved the lack of exponential stability. In [17], Hassine studied the asymptotic behavior of the energy decay of a transmission plate equation with locally distributed Kelvin-Voigt damping. More precisely, he proved that the energy decay at least logarithmically over the time. Recently, Ammari et al in [2] considered the wave equation with Kelvin-Voigt damping in a bounded domain. In their work, they proposed to deal with the geometrical condition by considering a singular Kelvin-Voigt damping which is localized far away from the boundary. In this particular case, they showed that the energy of the wave equation decreases logarithmically to zero as time goes to infinity.

1.5. Organization of the paper. This paper is organized as follows: In Section 2, we prove the well-posedness of system (1.1) with either the boundary conditions (1.2) or (1.3). Next, in Section 3, we prove the strong stability of the system in the lack of the compactness of the resolvent of the generator. In Section 4, we prove the analytic stability when the three Kelvin-Voigt dampings are globally distributed. Later, Sections 5 and 6 are devoted to analyze the stability of the system provided the existence of three local dampings by distinguishing two cases: in the first one, in section 5, when the coefficient functions D_1 , D_2 , and D_3 are smooth, we prove the exponential stability of the system. In the second one, in section 6, when the coefficient functions D_1 , D_2 , and D_3 are non smooth, we prove the polynomial stability of type $\frac{1}{t}$. Last but not least, in section 7, we prove the polynomial energy decay rate of type $\frac{1}{\sqrt{t}}$ for the system in the case of only one local non-smooth damping D_2 acting on the shear angle displacement.

Finally, in Section 8, under boundary conditions (1.3), we prove the lack of uniform (exponential) stability of the system in the absence of at least one damping.

2. WELL-POSEDNESS OF THE PROBLEM

In this part, using a semigroup approach, we establish well-posedness result for the systems (1.1)-(1.2) and (1.1)-(1.3). Let (φ, ψ, w) be a regular solution of system (1.1)-(1.2), its associated energy is given by:

$$(2.1) \quad E(t) = \frac{1}{2} \left\{ \int_0^L \left(\rho_1 |\varphi_t|^2 + \rho_2 |\psi_t|^2 + \rho_3 |w_t|^2 + k_1 |\varphi_x + \psi + lw|^2 \right) dx + \int_0^L \left(k_2 |\psi_x|^2 + k_3 |w_x - l\varphi|^2 \right) dx \right\},$$

and it is dissipated according to the following law:

$$(2.2) \quad E'(t) = - \int_0^L \left(D_1 |\varphi_{xt} + \psi_t + lw_t|^2 + D_2 |\psi_{xt}|^2 + D_3 |w_{xt} - l\varphi_t|^2 \right) dx \leq 0.$$

Now, we define the following energy spaces:

$$\mathcal{H}_1 = (H_0^1(0, L) \times L^2(0, L))^3 \quad \text{and} \quad \mathcal{H}_2 = H_0^1(0, L) \times L^2(0, L) \times (H_*^1(0, L) \times L_*^2(0, L))^2,$$

where

$$L_*^2(0, L) = \{f \in L^2(0, L) : \int_0^L f(x)dx = 0\} \text{ and } H_*^1(0, L) = \{f \in H^1(0, L) : \int_0^L f(x)dx = 0\}.$$

Both spaces \mathcal{H}_1 and \mathcal{H}_2 are equipped with the inner product which induces the energy norm:

$$(2.3) \quad \begin{aligned} \|U\|_{\mathcal{H}_j}^2 &= \|(v^1, v^2, v^3, v^4, v^5, v^6)\|_{\mathcal{H}_j}^2 \\ &= \rho_1 \|v^2\|^2 + \rho_2 \|v^4\|^2 + \rho_1 \|v^6\|^2 + k_1 \|v_x^1 + v^3 + lv^5\|^2 \\ &\quad + k_2 \|v_x^3\|^2 + k_3 \|v_x^5 - lv^1\|^2, \quad j = 1, 2 \end{aligned}$$

here and after $\|\cdot\|$ denotes the norm of $L^2(0, L)$.

Remark 2.1. *In the case of boundary condition (1.2), it is easy to see that expression (2.3) defines a norm on the energy space \mathcal{H}_1 . But in the case of boundary condition (1.3) the expression (2.3) define a norm on the energy space \mathcal{H}_2 if $L \neq \frac{n\pi}{1}$ for all positive integer n . Then, here and after, we assume that there exists no $n \in \mathbb{N}$ such that $L = \frac{n\pi}{1}$ when $j = 2$.*

Next, we define the linear operator \mathcal{A}_j in \mathcal{H}_j by:

$$\begin{aligned} D(\mathcal{A}_1) &= \left\{ U \in \mathcal{H}_1 \mid v^2, v^4, v^6 \in H_0^1(0, L), [k_1(v_x^1 + v^3 + lv^5) + D_1(v_x^2 + v^4 + lv^6)]_x \in L^2(0, L), \right. \\ &\quad \left. [k_2v_x^3 + D_2v_x^4]_x \in L^2(0, L), [k_3(v_x^5 - lv^1) + D_3(v_x^6 - lv^2)]_x \in L^2(0, L) \right\}, \\ D(\mathcal{A}_2) &= \left\{ U \in \mathcal{H}_2 \mid v^2 \in H_0^1(0, L), v^4, v^6 \in H_*^1(0, L), v_x^3|_{0,L} = v_x^5|_{0,L} = 0, \right. \\ &\quad \left. [k_1(v_x^1 + v^3 + lv^5) + D_1(v_x^2 + v^4 + lv^6)]_x \in L^2(0, L), \right. \\ &\quad \left. [k_2v_x^3 + D_2v_x^4]_x \in L_*^2(0, L), [k_3(v_x^5 - lv^1) + D_3(v_x^6 - lv^2)]_x \in L_*^2(0, L) \right\} \end{aligned}$$

and

$$(2.4) \quad \mathcal{A}_j \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \\ v^5 \\ v^6 \end{pmatrix} = \begin{pmatrix} v^2 \\ \rho_1^{-1} ([k_1(v_x^1 + v^3 + lv^5) + D_1(v_x^2 + v^4 + lv^6)]_x + lk_3(v_x^5 - lv^1) + lD_3(v_x^6 - lv^2)) \\ v^4 \\ \rho_2^{-1} ((k_2v_x^3 + D_2v_x^4)_x - k_1(v_x^1 + v^3 + lv^5) - D_1(v_x^2 + v^4 + lv^6)) \\ v^6 \\ \rho_1^{-1} ([k_3(v_x^5 - lv^1) + D_3(v_x^6 - lv^2)]_x - lk_1(v_x^1 + v^3 + lv^5) - lD_1(v_x^2 + v^4 + lv^6)) \end{pmatrix}$$

for all $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A}_j)$. So, if $U = (\varphi, \varphi_t, \psi, \psi_t, w, w_t)^\top$ is the state of (1.1)-(1.2) or (1.1)-(1.3), then the Bresse beam system is transformed into a first order evolution equation on the Hilbert space \mathcal{H}_j :

$$(2.5) \quad \begin{cases} U_t = \mathcal{A}_j U, & j = 1, 2 \\ U(0) = U_0(x), \end{cases}$$

where

$$U_0(x) = (\varphi_0(x), \varphi_1(x), \psi_0(x), \psi_1(x), w_0(x), w_1(x))^\top.$$

Remark 2.2. *It is easy to see that there exists a positive constant c_0 such that for any $(\varphi, \psi, w) \in (H_0^1(0, L))^3$ for $j = 1$ and for any $(\varphi, \psi, w) \in H_0^1(0, L) \times (H_*^1(0, L))^2$ for $j = 2$,*

$$(2.6) \quad k_1 \|\varphi_x + \psi + lw\|^2 + k_2 \|\psi_x\|^2 + k_3 \|w_x - l\varphi\|^2 \leq c_0 (\|\varphi_x\|^2 + \|\psi_x\|^2 + \|w_x\|^2).$$

On the other hand, we can show by a contradiction argument the existence of a positive constant c_1 such that, for any $(\varphi, \psi, w) \in (H_0^1(0, L))^3$ for $j = 1$ and for any $(\varphi, \psi, w) \in H_0^1(0, L) \times (H_^1(0, L))^2$ for $j = 2$,*

$$(2.7) \quad c_1 (\|\varphi_x\|^2 + \|\psi_x\|^2 + \|w_x\|^2) \leq k_1 \|\varphi_x + \psi + lw\|^2 + k_2 \|\psi_x\|^2 + k_3 \|w_x - l\varphi\|^2.$$

Therefore the norm on the energy space \mathcal{H}_j given in (2.3) is equivalent to the usual norm on \mathcal{H}_j .

Proposition 2.3. *Assume that coefficients functions D_1 , D_2 and D_3 are non negative. Then, the operator \mathcal{A}_j is m -dissipative in the energy space \mathcal{H}_j , for $j = 1, 2$.*

Proof. Let $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A}_j)$. By a straightforward calculation, we have:

$$(2.8) \quad \operatorname{Re}(\mathcal{A}_j U, U)_{\mathcal{H}_j} = - \int_0^L \left(D_1 |v_x^2 + v^4 + lv^6|^2 + D_2 |v_x^4|^2 + D_3 |v_x^6 - lv^2|^2 \right) dx.$$

As $D_1 \geq 0$, $D_2 \geq 0$ and $D_3 \geq 0$, we get: that \mathcal{A}_j is dissipative.

Now, we will check the maximality of \mathcal{A}_j . For this purpose, let $F = (f^1, f^2, f^3, f^4, f^5, f^6)^\top \in \mathcal{H}_j$, we have to prove the existence of $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A}_j)$ unique solution of the equation $-\mathcal{A}_j U = F$. Equivalently, we have the following system:

$$(2.9) \quad -v^2 = f^1,$$

$$(2.10) \quad -[k_1(v_x^1 + v^3 + lv^5) + D_1(v_x^2 + v^4 + lv^6)]_x - lk_3(v_x^5 - lv^1) - lD_3(v_x^6 - lv^2) = \rho_1 f^2,$$

$$(2.11) \quad -v^4 = f^3,$$

$$(2.12) \quad -(k_2 v_x^3 + D_2 v_x^4)_x + k_1(v_x^1 + v^3 + lv^5) + D_1(v_x^2 + v^4 + lv^6) = \rho_2 f^4,$$

$$(2.13) \quad -v^6 = f^5,$$

$$(2.14) \quad -[k_3(v_x^5 - lv^1) + D_3(v_x^6 - lv^2)]_x + lk_1(v_x^1 + v^3 + lv^5) + lD_1(v_x^2 + v^4 + lv^6) = \rho_1 f^6.$$

Let $(\varphi^1, \varphi^3, \varphi^5) \in (H_0^1(0, L))^3$ for $j = 1$ and $(\varphi^1, \varphi^3, \varphi^5) \in (H_0^1(0, L) \times (H_*^1(0, L))^2)$ for $j = 2$ be a test function. Multiplying (2.10), (2.12) and (2.14) by φ^1 , φ^3 and φ^5 respectively, (2.9)-(2.14) can be written after integrating by parts in the following form:

$$(2.15) \quad \begin{cases} k_1(v_x^1 + v^3 + lv^5) \varphi_x^1 - lk_3(v_x^5 - lv^1) \varphi^1 = h^1, \\ k_2 v_x^3 \varphi_x^3 + k_1(v_x^1 + v^3 + lv^5) \varphi^3 = h^3, \\ k_3(v_x^5 - lv^1) \varphi_x^5 + lk_1(v_x^1 + v^3 + lv^5) \varphi^5 = h^5, \end{cases}$$

where

$$\begin{aligned} h^1 &= \rho_1 f^2 \varphi^1 + D_1(f_x^1 + f^3 + lf^5) \varphi_x^1 - lD_3(f_x^5 - lf^1) \varphi^1, \\ h^3 &= \rho_2 f^4 \varphi^3 + D_2 f_x^3 \varphi_x^3 + D_1(f_x^1 + f^3 + lf^5) \varphi^3, \text{ and} \\ h^5 &= \rho_1 f^6 \varphi^5 + D_3(f^5 - lf^1) \varphi_x^5 + lD_1(f_x^1 + f^3 + lf^5) \varphi^5. \end{aligned}$$

Using Lax-Milgram Theorem (see [33]), we deduce that (2.15) admits a unique solution in $(H_0^1(0, L))^3$ for $j = 1$ and in $(H_0^1(0, L) \times (H_*^1(0, L))^2)$ for $j = 2$. Thus, $-\mathcal{A}_j U = F$ admits an unique solution $U \in D(\mathcal{A}_j)$ and consequently $0 \in \rho(\mathcal{A}_j)$. Then, \mathcal{A}_j is closed and consequently $\rho(\mathcal{A}_j)$ is open set of \mathbb{C} (see Theorem 6.7 in [20]). Hence, we easily get $R(\lambda I - \mathcal{A}_j) = \mathcal{H}_j$ for sufficiently small $\lambda > 0$. This, together with the dissipativeness of \mathcal{A}_j , imply that $D(\mathcal{A}_j)$ is dense in \mathcal{H}_j and that \mathcal{A}_j is m -dissipative in \mathcal{H}_j (see Theorems 4.5, 4.6 in [33]). Thus, the proof is complete. \square

Thanks to Lumer-Phillips Theorem (see [30, 33]), we deduce that \mathcal{A}_j generates a C_0 -semigroup of contraction $e^{t\mathcal{A}_j}$ in \mathcal{H}_j and therefore problem (2.5) is well-posed. Then, we have the following result.

Theorem 2.4. *For any $U_0 \in \mathcal{H}_j$, problem (2.5) admits a unique weak solution*

$$U \in C(\mathbb{R}_+; \mathcal{H}_j).$$

Moreover, if $U_0 \in D(\mathcal{A}_j)$, then

$$U \in C(\mathbb{R}_+; D(\mathcal{A}_j)) \cap C^1(\mathbb{R}_+; \mathcal{H}_j).$$

3. STRONG STABILITY OF THE SYSTEM

In this part, we use a general criteria of Arendt-Batty in [3] to show the strong stability of the C_0 -semigroup $e^{t\mathcal{A}_j}$ associated to the Bresse system (1.1) in the absence of the compactness of the resolvent of \mathcal{A}_j . Before, we state our main result, we need the following stability condition:

$$(SSC) \quad \text{There exist } i \in \{1, 2, 3\}, d_0 > 0 \text{ and } \alpha < \beta \in [0, L] \text{ such that } D_i \geq d_0 > 0 \text{ on } (\alpha, \beta).$$

Theorem 3.1. *Assume that condition (SSC) holds. Then the C_0 -semigroup $e^{t\mathcal{A}_j}$ is strongly stable in \mathcal{H}_j , $j = 1, 2$, i.e., for all $U_0 \in \mathcal{H}_j$, the solution of (2.5) satisfies*

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_j} U_0\|_{\mathcal{H}_j} = 0.$$

For the proof of Theorem 3.1, we need the following two lemmas.

Lemma 3.2. *Under the same condition of Theorem 3.1, we have*

$$(3.1) \quad \ker (i\lambda - \mathcal{A}_j) = \{0\}, \quad j = 1, 2, \quad \text{for all } \lambda \in \mathbb{R}.$$

Proof. We will prove Lemma 3.2 in the case $D_1 = D_3 = 0$ on $(0, L)$ and $D_2 \geq d_0 > 0$ on $(\alpha, \beta) \subset (0, L)$. The other cases are similar to prove.

First, from Proposition 2.3, we claim that $0 \in \rho(\mathcal{A}_j)$. We still have to show the result for $\lambda \in \mathbb{R}^*$. Suppose that there exist a real number $\lambda \neq 0$ and $0 \neq U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A}_j)$ such that:

$$(3.2) \quad \mathcal{A}_j U = i\lambda U.$$

Our goal is to find a contradiction by proving that $U = 0$. Taking the real part of the inner product in \mathcal{H}_j of $\mathcal{A}_j U$ and U , we get:

$$(3.3) \quad \operatorname{Re}(\mathcal{A}_j U, U)_{\mathcal{H}_j} = - \int_0^L D_2 |v_x^4|^2 dx = 0.$$

Since by assumption $D_2 \geq d_0 > 0$ on (α, β) , it follows from equality (3.3) that:

$$(3.4) \quad D_2 v_x^4 = 0 \quad \text{in } (0, L) \quad \text{and} \quad v_x^4 = 0 \quad \text{in } (\alpha, \beta).$$

Detailing (3.2) we get:

$$(3.5) \quad v^2 = i\lambda v^1,$$

$$(3.6) \quad k_1 (v_x^1 + v^3 + lv^5)_x + lk_3 (v_x^5 - lv^1) = i\rho_1 \lambda v^2,$$

$$(3.7) \quad v^4 = i\lambda v^3,$$

$$(3.8) \quad (k_2 v_x^3 + D_2 v_x^4)_x - k_1 (v_x^1 + v^3 + lv^5) = i\rho_2 \lambda v^4,$$

$$(3.9) \quad v^6 = i\lambda v^5,$$

$$(3.10) \quad k_3 (v_x^5 - lv^1)_x - lk_1 (v_x^1 + v^3 + lv^5) = i\rho_1 \lambda v^6.$$

Next, inserting (3.4) in (3.7) and using the fact that $\lambda \neq 0$, we get:

$$(3.11) \quad v_x^3 = 0 \quad \text{in } (\alpha, \beta).$$

Moreover, substituting equations (3.5), (3.7) and (3.9) into equations (3.6), (3.8) and (3.10), we get:

$$(3.12) \quad \begin{cases} \rho_1 \lambda^2 v^1 + k_1 (v_x^1 + v^3 + lv^5)_x + lk_3 (v_x^5 - lv^1) = 0, \\ \rho_2 \lambda^2 v^3 + (k_2 v_x^3 + iD_2 \lambda v_x^3)_x - k_1 (v_x^1 + v^3 + lv^5) = 0, \\ \rho_1 \lambda^2 v^5 + k_3 (v_x^5 - lv^1)_x - lk_1 (v_x^1 + v^3 + lv^5) = 0. \end{cases}$$

Now, we introduce the functions \widehat{v}^i , for $i = 1, \dots, 6$ by $\widehat{v}^i = v_x^i$. It is easy to see that $\widehat{v}^i \in H^1(0, L)$. It follows from equations (3.4) and (3.11) that:

$$(3.13) \quad \widehat{v}^3 = \widehat{v}^4 = 0 \quad \text{in } (\alpha, \beta)$$

and consequently system (3.12) will be, after differentiating it with respect to x , given by:

$$(3.14) \quad \rho_1 \lambda^2 \widehat{v}^1 + k_1 (\widehat{v}_x^1 + l\widehat{v}^5)_x + lk_3 (\widehat{v}_x^5 - l\widehat{v}^1) = 0 \quad \text{in } (\alpha, \beta),$$

$$(3.15) \quad \widehat{v}_x^1 + l\widehat{v}^5 = 0 \quad \text{in } (\alpha, \beta),$$

$$(3.16) \quad \rho_1 \lambda^2 \widehat{v}^5 + k_3 (\widehat{v}_x^5 - l\widehat{v}^1)_x - lk_1 (\widehat{v}_x^1 + l\widehat{v}^5) = 0 \quad \text{in } (\alpha, \beta).$$

Furthermore, substituting equation (3.15) into (3.14) and (3.16), we get:

$$(3.17) \quad \rho_1 \lambda^2 \widehat{v}^1 + lk_3 (\widehat{v}_x^5 - l\widehat{v}^1) = 0 \quad \text{in } (\alpha, \beta),$$

$$(3.18) \quad \widehat{v}_x^1 + l\widehat{v}^5 = 0 \quad \text{in } (\alpha, \beta),$$

$$(3.19) \quad \rho_1 \lambda^2 \widehat{v}^5 + k_3 (\widehat{v}_x^5 - l\widehat{v}^1)_x = 0 \quad \text{in } (\alpha, \beta).$$

Differentiating equation (3.17) with respect to x , a straightforward computation with equation (3.19) yields:

$$\rho_1 \lambda^2 (\widehat{v}_x^1 - l\widehat{v}^5) = 0 \quad \text{in } (\alpha, \beta).$$

Equivalently

$$(3.20) \quad \widehat{v}_x^1 - l\widehat{v}^5 = 0 \quad \text{in } (\alpha, \beta).$$

Hence, from equations (3.18) and (3.20), we get:

$$(3.21) \quad \widehat{v}^5 = 0 \quad \text{and} \quad \widehat{v}_x^1 = 0 \quad \text{in } (\alpha, \beta).$$

Plugging $\widehat{v}^5 = 0$ in (3.17), we get:

$$(3.22) \quad (\rho_1 \lambda^2 - l^2 k_3) \widehat{v}^1 = 0.$$

In order to finish our proof, we have to distinguish two cases:

Case 1: $\lambda \neq 1 \sqrt{\frac{k_3}{\rho_1}}$.

Using equation (3.22), we deduce that:

$$\widehat{v}^1 = 0 \quad \text{in } (\alpha, \beta).$$

Setting $V = (\widehat{v}^1, \widehat{v}_x^1, \widehat{v}^3, \widehat{v}_x^3, \widehat{v}^5, \widehat{v}_x^5)^\top$. By continuity of \widehat{v}^i on $(0, L)$, we deduce that $V(\alpha) = 0$. Then system (3.12) could be given as:

$$(3.23) \quad \begin{cases} V_x &= BV, & \text{in } (0, \alpha) \\ V(\alpha) &= 0, \end{cases}$$

where

$$(3.24) \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-\lambda^2 \rho_1 + l^2 k_3}{k_1} & 0 & 0 & -1 & \frac{-l(k_1 + k_3)}{k_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{k_1}{k_2 + i\lambda D_2} & \frac{k_1 - \lambda^2 \rho_2}{k_2 + i\lambda D_2} & 0 & \frac{l k_1}{k_2 + i\lambda D_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{l(k_3 + k_1)}{k_3} & \frac{l k_1}{k_3} & 0 & \frac{l^2 k_1 - \lambda^2 \rho_1}{k_3} & 0 \end{pmatrix}.$$

Using ordinary differential equation theory, we deduce that system (3.23) has the unique trivial solution $V = 0$ in $(0, \alpha)$. The same argument as above leads us to prove that $V = 0$ on (β, L) . Consequently, we obtain $\widehat{v}^1 = \widehat{v}^3 = \widehat{v}^5 = 0$ on $(0, L)$. It follows that $\widehat{v}^2 = \widehat{v}^4 = \widehat{v}^6 = 0$ on $(0, L)$, thus $\widehat{U} = 0$. This gives that $U = C$, where C is a constant. Finally, from the boundary condition (1.2) or (1.3), we deduce that $U = 0$.

Case 2: $\lambda = 1 \sqrt{\frac{k_3}{\rho_1}}$.

The fact that $\widehat{v}_x^1 = 0$ on (α, β) , we get: $\widehat{v}^1 = c$ on (α, β) , where c is a constant. By continuity of \widehat{v}^1 on $(0, L)$, we deduce that $\widehat{v}^1(\alpha) = c$. We know also that $\widehat{v}^3 = \widehat{v}^5 = 0$ on (α, β) from (3.13) and (3.21). Hence, setting $V(\alpha) = (c, 0, 0, 0, 0, 0)^\top = V_0$, we can rewrite system (3.12) on $(0, \alpha)$ under the form:

$$\begin{cases} V_x &= \widehat{B}V, \\ V(\alpha) &= V_0, \end{cases}$$

where

$$\widehat{B} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \frac{-l(k_1 + k_3)}{k_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{k_1}{k_2 + i l \sqrt{\frac{k_3}{\rho_1}} D_2} & \frac{k_1 - \lambda^2 \rho_2}{k_2 + i l \sqrt{\frac{k_3}{\rho_1}} \lambda D_2} & 0 & \frac{l k_1}{k_2 + i l \sqrt{\frac{k_3}{\rho_1}} D_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{l(k_3 + k_1)}{k_3} & \frac{l k_1}{k_3} & 0 & \frac{l^2(k_1 - k_3)}{k_3} & 0 \end{pmatrix}.$$

Introducing $\tilde{V} = (\tilde{v}_x^1, \tilde{v}^3, \tilde{v}_x^3, \tilde{v}^5, \tilde{v}_x^5)^\top$ and

$$\tilde{B} = \begin{pmatrix} 0 & 0 & -1 & \frac{-l(k_1 + k_3)}{k_1} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{k_1}{k_2 + il\sqrt{\frac{k_3}{\rho_1}}D_2} & \frac{k_1 - \lambda^2\rho_2}{k_2 + il\sqrt{\frac{k_3}{\rho_1}}\lambda D_2} & 0 & \frac{lk_1}{k_2 + il\sqrt{\frac{k_3}{\rho_1}}D_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{l(k_3 + k_1)}{k_3} & \frac{lk_1}{k_3} & 0 & \frac{l^2(k_1 - k_3)}{k_3} & 0 \end{pmatrix}.$$

Then system (3.12) could be given as:

$$(3.25) \quad \begin{cases} \tilde{V}_x &= \tilde{B}\tilde{V}, \text{ in } (0, \alpha), \\ \tilde{V}(\alpha) &= 0. \end{cases}$$

Using ordinary differential equation theory, we deduce that system (3.25) has the unique trivial solution $\tilde{V} = 0$ in $(0, \alpha)$. This implies that on $(0, \alpha)$, we have $\tilde{v}^3 = \tilde{v}^5 = 0$. Consequently, $v^3 = c_3$ and $v^5 = c_5$ where c_3 and c_5 are constants. But using the fact that $v^3(0) = v^5(0) = 0$, we deduce that $v^3 = v^5 = 0$ on $(0, \alpha)$.

Substituting v^3 and v^5 by their values in the second equation of system (3.12), we get: that $v_x^1 = 0$. This yields $v^1 = c_1$, where c_1 is a constant. But as $v^1(0) = 0$, we get: $v^1 = 0$ on $(0, \alpha)$. Thus $U = 0$ on $(0, \alpha)$. The same argument as above leads us to prove that $U = 0$ on (β, L) and therefore $U = 0$ on $(0, L)$. Thus the proof is complete. \square

Lemma 3.3. *Under the same condition of Theorem 3.1, $(i\lambda I - \mathcal{A}_j), j = 1, 2$ is surjective for all $\lambda \in \mathbb{R}$.*

Proof. We will prove Lemma 3.3 in the case $D_1 = D_3 = 0$ on $(0, L)$ and $D_2 \geq d_0 > 0$ on $(\alpha, \beta) \subset (0, L)$ and the other cases are similar to prove.

Since $0 \in \rho(\mathcal{A}_j)$, we still need to show the result for $\lambda \in \mathbb{R}^*$. For any

$$F = (f^1, f^2, f^3, f^4, f^5, f^6)^\top \in \mathcal{H}_j, \lambda \in \mathbb{R}^*,$$

we prove the existence of

$$U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A}_j)$$

solution of the following equation:

$$(3.26) \quad (i\lambda I - \mathcal{A}_j)U = F.$$

Equivalently, we have the following system:

$$(3.27) \quad i\lambda v^1 - v^2 = f^1,$$

$$(3.28) \quad \rho_1 i\lambda v^2 - k_1 (v_x^1 + v^3 + lv^5)_x - lk_3 (v_x^5 - lv^1) = \rho_1 f^2,$$

$$(3.29) \quad i\lambda v^3 - v^4 = f^3,$$

$$(3.30) \quad \rho_2 i\lambda v^4 - (k_2 v_x^3 + D_2 v_x^4)_x + k_1 (v_x^1 + v^3 + lv^5) = \rho_2 f^4,$$

$$(3.31) \quad i\lambda v^5 - v^6 = f^5,$$

$$(3.32) \quad \rho_1 i\lambda v^6 - k_3 (v_x^5 - lv^1)_x + lk_1 (v_x^1 + v^3 + lv^5) = \rho_1 f^6.$$

From (3.27), (3.29) and (3.31), we have:

$$(3.33) \quad v^2 = i\lambda v^1 - f^1, \quad v^3 = i\lambda v^3 - f^3, \quad v^6 = i\lambda v^5 - f^5.$$

Inserting (3.33) in (3.28), (3.30) and (3.32), we get:

$$(3.34) \quad \begin{cases} -\lambda^2 v^1 - k_1 \rho_1^{-1} (v_x^1 + v^3 + lv^5)_x - lk_3 \rho_1^{-1} (v_x^5 - lv^1) = h^1, \\ -\lambda^2 v^3 - \rho_2^{-1} (k_2 + i\lambda D_2) v_{xx}^3 + k_1 \rho_2^{-1} (v_x^1 + v^3 + lv^5) = h^3, \\ -\lambda^2 v^5 - k_3 \rho_1^{-1} (v_x^5 - lv^1)_x + lk_1 \rho_1^{-1} (v_x^1 + v^3 + lv^5) = h^5, \end{cases}$$

where

$$h^1 = f^2 + i\lambda f^1, \quad h^3 = f^4 + i\lambda f^3 - \rho_2^{-1} D_2 f_{xx}^3, \quad h^5 = f^6 + i\lambda f^5.$$

For all $v = (v^1, v^3, v^5)^\top \in (H_0^1(0, L))^3$ for $j = 1$ and $v = (v^1, v^3, v^5)^\top \in H_0^1(0, L) \times H_*^1(0, L)^2$ for $j = 2$, we define the linear operator \mathcal{L} by:

$$\mathcal{L}v = \begin{pmatrix} -k_1\rho_1^{-1}(v_x^1 + v^3 + lv^5)_x - lk_3\rho_1^{-1}(v_x^5 - lv^1) \\ -\rho_2^{-1}(k_2 + i\lambda D_2)v_{xx}^3 + k_1\rho_2^{-1}(v_x^1 + v^3 + lv^5) \\ -k_3\rho_1^{-1}(v_x^5 - lv^1)_x + lk_1\rho_1^{-1}(v_x^1 + v^3 + lv^5) \end{pmatrix}.$$

For clarity, we consider the case $j = 1$. The proof in the case $j = 2$ is very similar. Using Lax-Milgram theorem, it is easy to show that \mathcal{L} is an isomorphism from $(H_0^1(0, L))^3$ onto $(H^{-1}(0, L))^3$. Let $v = (v^1, v^3, v^5)^\top$ and $h = (-h^1, -h^3, -h^5)^\top$, then we transform system (3.34) into the following form:

$$(3.35) \quad (\lambda^2 \mathcal{I} - \mathcal{L})v = h.$$

Since the operator \mathcal{L} is an isomorphism from $(H_0^1(0, L))^3$ onto $(H^{-1}(0, L))^3$ and \mathcal{I} is a compact operator from $(H_0^1(0, L))^3$ onto $(H^{-1}(0, L))^3$, then using Fredholm's Alternative theorem, problem (3.35) admits a unique solution in $(H_0^1(0, L))^3$ if and only if $\lambda^2 \mathcal{I} - \mathcal{L}$ is injective. For that purpose, let: $\tilde{v} = (\tilde{v}^1, \tilde{v}^3, \tilde{v}^5)^\top$ in $\ker(\lambda^2 \mathcal{I} - \mathcal{L})$. Then, if we set $\tilde{v}^2 = i\lambda \tilde{v}^1$, $\tilde{v}^4 = i\lambda \tilde{v}^3$ and $\tilde{v}^6 = i\lambda \tilde{v}^5$, we deduce that $\tilde{V} = (\tilde{v}^1, \tilde{v}^2, \tilde{v}^3, \tilde{v}^5, \tilde{v}^6)$ belongs to $D(\mathcal{A}_1)$ and it is solution of:

$$(i\lambda \mathcal{I} - \mathcal{A}_1)\tilde{V} = 0.$$

Using Lemma 3.2, we deduce that $\tilde{v}^1 = \tilde{v}^3 = \tilde{v}^5 = 0$. This implies that equation (3.35) admits a unique solution in $v = (v^1, v^3, v^5) \in (H_0^1(0, L))^3$ and

$$\begin{aligned} -k_1\rho_1^{-1}(v_x^1 + v^3 + lv^5)_x - lk_3\rho_1^{-1}(v_x^5 - lv^1) &\in L^2(0, L), \\ -\rho_2^{-1}(k_2 v_x^3 + i\lambda D_2 v_x^3 - D_2 f_x^3)_x + k_1\rho_2^{-1}(v_x^1 + v^3 + lv^5) &\in L^2(0, L), \\ -k_3\rho_1^{-1}(v_x^5 - lv^1)_x + lk_1\rho_1^{-1}(v_x^1 + v^3 + lv^5) &\in L^2(0, L). \end{aligned}$$

By setting $v^2 = i\lambda v^1 - f^1$, $v^3 = i\lambda v^3 - f^3$ and $v^6 = i\lambda v^5 - f^5$, we deduce that $V = (v^1, v^2, v^3, v^4, v^5, v^6)$ belongs to $D(\mathcal{A}_1)$ and it is the unique solution of equation (3.26) and the proof is thus complete. \square

Proof of Theorem 3.1. Following a general criteria of Arendt-Batty in [3], the C_0 -semigroup $e^{t\mathcal{A}_j}$ of contractions is strongly stable if \mathcal{A}_j has no pure imaginary eigenvalues and $\sigma(\mathcal{A}_j) \cap i\mathbb{R}$ is countable. By Lemma 3.2, the operator \mathcal{A}_j has no pure imaginary eigenvalues and by Lemma 3.3, $R(i\lambda - \mathcal{A}_j) = \mathcal{H}_j$ for all $\lambda \in \mathbb{R}$. Therefore the closed graph theorem of Banach implies that $\sigma(\mathcal{A}_j) \cap i\mathbb{R} = \emptyset$. Thus, the proof is complete.

4. ANALYTIC STABILITY IN THE CASE OF THREE GLOBAL DAMPINGS

In this part, we prove the analytic stability of the Bresse systems (1.1)-(1.2) and (1.1)-(1.3) provided that there exists a positive constant d_0 such that:

$$(4.1) \quad D_1, D_2, D_3 \geq d_0 > 0 \quad \text{for every } x \in (0, L).$$

Before we state our main result, we recall the following results (see [18], [34] for part i), [4] for ii) and [33] for iii)).

Theorem 4.1. *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ be an unbounded operator generating a C_0 -semigroup of contractions $e^{t\mathcal{A}}$ on \mathcal{H} . Assume that $i\lambda \in \rho(\mathcal{A})$, for all $\lambda \in \mathbb{R}$. Then, the C_0 -semigroup $e^{t\mathcal{A}}$ is:*

i) *Exponentially stable if and only if*

$$\lim_{|\lambda| \rightarrow +\infty} \left\{ \sup_{\lambda \in \mathbb{R}} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \right\} < +\infty.$$

ii) *Polynomially stable of order $\frac{1}{l}$ ($l > 0$) if and only if*

$$\lim_{|\lambda| \rightarrow +\infty} \left\{ \sup_{\lambda \in \mathbb{R}} |\lambda|^{-l} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \right\} < +\infty.$$

iii) *Analytically stable if and only if*

$$\lim_{|\lambda| \rightarrow +\infty} \left\{ \sup_{\lambda \in \mathbb{R}} |\lambda| \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \right\} < +\infty.$$

Now, we are in a position to establish the main result of this part by the following stability estimate.

Theorem 4.2. *Assume that condition (4.1) holds. Then, the C_0 -semigroup $e^{t\mathcal{A}_j}$, for $j = 1, 2$, is analytically stable.*

To prove Theorem 4.2, we have to check if the following conditions:

$$(A1) \quad i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$$

and

$$(A2) \quad \lim_{|\lambda| \rightarrow +\infty} \left\{ \sup_{\lambda \in \mathbb{R}} |\lambda| \| (i\lambda I - \mathcal{A})^{-1} \|_{\mathcal{L}(\mathcal{H})} \right\} = O(1),$$

hold.

Condition (A1) is already proved in Lemma 3.2 and Lemma 3.3.

To prove condition (A2), we use a contradiction argument. For this aim, suppose that there exist a sequence of real numbers $(\lambda_n)_n$, with $|\lambda_n| \rightarrow +\infty$ and a sequence of vectors

$$(4.2) \quad U_n = (v_n^1, v_n^2, v_n^3, v_n^4, v_n^5, v_n^6)^\top \in D(\mathcal{A}_j) \quad \text{with} \quad \|U_n\|_{\mathcal{H}_j} = 1$$

such that:

$$(4.3) \quad \lambda_n^{-1} (i\lambda_n U_n - \mathcal{A}_j U_n) = (f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6)^\top \rightarrow 0 \quad \text{in} \quad \mathcal{H}_j, \quad j = 1, 2.$$

We will check the condition (A2) by finding a contradiction with (4.2)-(4.3) such as $\|U_n\|_{\mathcal{H}_j} = o(1)$.

Equation (4.3) is detailed as:

$$(4.4) \quad iv_n^1 - \lambda_n^{-1} v_n^2 = f_n^1,$$

$$(4.5) \quad i\rho_1 v_n^2 - \lambda_n^{-1} [k_1 ((v_n^1)_x + v_n^3 + lv_n^5) + D_1 ((v_n^2)_x + v_n^4 + lv_n^6)]_x - \lambda_n^{-1} k_3 [(v_n^5)_x - lv_n^1] - l\lambda_n^{-1} D_3 [(v_n^6)_x - lv_n^2] = \rho_1 f_n^2,$$

$$(4.6) \quad iv_n^3 - \lambda_n^{-1} v_n^4 = f_n^3,$$

$$(4.7) \quad i\rho_2 v_n^4 - \lambda_n^{-1} [k_2 (v_n^3)_x + D_2 (v_n^4)_x]_x + \lambda_n^{-1} k_1 [(v_n^1)_x + v_n^3 + lv_n^5] + \lambda_n^{-1} D_1 [(v_n^2)_x + v_n^4 + lv_n^6] = \rho_2 f_n^4,$$

$$(4.8) \quad iv_n^5 - \lambda_n^{-1} v_n^6 = f_n^5,$$

$$(4.9) \quad i\rho_1 v_n^6 - \lambda_n^{-1} [k_3 ((v_n^5)_x - lv_n^1) + D_3 ((v_n^6)_x - lv_n^2)]_x + \lambda_n^{-1} k_1 [(v_n^1)_x + v_n^3 + lv_n^5] + \lambda_n^{-1} l D_1 [(v_n^2)_x + v_n^4 + lv_n^6] = \rho_1 f_n^6.$$

For clarity, we divide the proof into several lemmas. From now on, for simplicity, we drop the index n .

First, remark that using (4.2) and (4.4) (respectively (4.6) and (4.8)), we deduce that:

$$(4.10) \quad \|v^1\| = o(1), \quad \|v^3\| = o(1), \quad \|v^5\| = o(1).$$

Lemma 4.3. *Under all the above assumptions, we have:*

$$(4.11) \quad \lambda^{-1/2} \|v_x^2\| = o(1), \quad \lambda^{-1/2} \|v_x^4\| = o(1) \quad \text{and} \quad \lambda^{-1/2} \|v_x^6\| = o(1).$$

Proof. Taking the inner product of (4.3) with U in \mathcal{H}_j , then using the fact that U is uniformly bounded in \mathcal{H}_j , we get:

$$\lambda^{-1} \int_0^L \left(D_1 |v_x^2|^2 + D_2 |v_x^4|^2 + D_3 |v_x^6|^2 \right) dx = \lambda^{-1} \operatorname{Re} (\mathcal{A}_j U, U)_{\mathcal{H}_j} = -\lambda^{-1} \operatorname{Re} (i\lambda U - \mathcal{A}_j U, U)_{\mathcal{H}_j} = o(1).$$

Thanks to condition (4.1), we obtain the desired asymptotic equation (4.11). Thus, the proof is complete. \square

Lemma 4.4. *Under all the above assumptions, we have:*

$$(4.12) \quad \|v_x^1\| = o(1), \quad \|v_x^3\| = o(1) \quad \text{and} \quad \|v_x^5\| = o(1).$$

Proof. Differentiating (4.4), (4.6) and (4.8) with respect to the variable x , we get:

$$(4.13) \quad v_x^1 = \frac{\lambda^{-1/2} v_x^2}{i\lambda^{1/2}} - if_x^1, \quad v_x^3 = \frac{\lambda^{-1/2} v_x^4}{i\lambda^{1/2}} - if_x^3 \quad \text{and} \quad v_x^5 = \frac{\lambda^{-1/2} v_x^6}{i\lambda^{1/2}} - if_x^5.$$

Using the asymptotic estimate (4.11) and the fact that f^1 , f^3 and f^5 converge to zero in $H_0^1(0, L)$ (or in $H_*^1(0, L)$), we obtain the desired estimate (4.12). \square

Lemma 4.5. *Under all the above assumptions, we have:*

$$(4.14) \quad \|v^2\| = o(1), \quad \|v^4\| = o(1) \quad \text{and} \quad \|v^6\| = o(1).$$

Proof. i) Multiplying (4.5) by $-i\overline{v^2}$ in $L^2(0, L)$ and after integrating over x , we get:

$$\begin{aligned} \rho_1 \int_0^L |v^2| dx - i\lambda^{-1}k_1 \int_0^L (v_x^1 + v^3 + lv^5) \overline{v_x^2} dx - i\lambda^{-1} \int_0^L D_1 (v_x^2 + v^4 + lv^6) \overline{v_x^2} dx \\ + i\lambda^{-1}k_3 \int_0^L (v_x^5 - lv^1) \overline{v^2} dx + i\lambda^{-1} \int_0^L D_3 (v_x^6 - lv^2) \overline{v^2} dx = -i \int_0^L \rho_1 f^2 \overline{v^2} dx. \end{aligned}$$

Using (4.10), the first asymptotic estimate of (4.11), (4.12), the fact that v^2 , v^4 and v^6 are uniformly bounded in $L^2(0, L)$ and f^2 converges to zero in $L^2(0, L)$ in the above equation, we obtain that $\|v^2\| = o(1)$.

ii) Similarly, multiplying (4.7) by $-i\overline{v^4}$ in $L^2(0, L)$, we get:

$$\begin{aligned} \rho_2 \int_0^L |v^4| dx - i\lambda^{-1} \int_0^L (k_2 v_x^3 + D_2 v_x^4) \overline{v_x^4} dx - i\lambda^{-1}k_1 \int_0^L (v_x^1 + v^3 + lv^5) \overline{v^4} dx \\ - i\lambda^{-1} \int_0^L D_1 (v_x^2 + v^4 + lv^6) \overline{v^4} dx = -i\rho_2 \int_0^L f^4 \overline{v^4} dx. \end{aligned}$$

Using (4.10), (4.11), (4.12), the fact that v^4 and v^6 are uniformly bounded in $L^2(0, L)$ and f^4 converges to zero in $L^2(0, L)$ in the preceding equation, we deduce that $\|v^4\| = o(1)$.

iii) Finally, multiplying (4.9) by $-i\overline{v^6}$ in $L^2(0, L)$, we get:

$$\begin{aligned} \rho_1 \int_0^L |v^6| dx - i\lambda^{-1}k_3 \int_0^L (v_x^5 - lv^1) \overline{v_x^6} dx - i\lambda^{-1} \int_0^L D_3 (v_x^6 - lv^2) \overline{v_x^6} dx \\ - i\lambda^{-1}k_1 \int_0^L (v_x^1 + v^3 + lv^5) \overline{v^6} dx - i\lambda^{-1} \int_0^L D_1 (v_x^2 + v^4 + lv^6) \overline{v^6} dx = -i\rho_1 \int_0^L f^6 \overline{v^6} dx. \end{aligned}$$

Using (4.10), (4.11), (4.12), the fact that v^2 , v^4 and v^6 are uniformly bounded in $L^2(0, L)$ and f^6 converges to zero in $L^2(0, L)$ in the previous equation, we deduce that $\|v^6\| = o(1)$. Thus the proof is complete. \square

Proof of Theorem 4.2. Using Lemma 4.4 and Lemma 4.5, we obtain $\|U\|_{\mathcal{H}_j} = o(1)$ which contradicts (4.2). Therefore, (A2) holds and consequently we deduce the analytic stability of the system (1.1) in the case of three global dampings.

Remark 4.6. *It is clear that the analytic stability implies the exponential one (see Theorem 4.1). So, Theorem 4.2 generalizes the results of [6] using simpler technique.*

5. EXPONENTIAL STABILITY: THE CASE OF THREE LOCAL DAMPINGS WITH SMOOTH COEFFICIENTS AT THE INTERFACE

The smoothness of the coefficient at the interface plays a crucial role in the stabilization of wave equation. In this section, we consider the Bresse systems (1.1)-(1.2) and (1.1)-(1.3) subject to three local viscoelastic Kelvin-Voigt dampings with smooth coefficients at the interface. We establish uniform (exponential) stability of the C_0 -semigroup $e^{t\mathcal{A}_j}$, $j = 1, 2$. For this purpose, let $\emptyset \neq \omega = (\alpha, \beta) \subset (0, L)$ be the biggest nonempty open subset of $(0, L)$ satisfying:

$$(5.1) \quad \exists d_0 > 0 \text{ such that } D_i \geq d_0, \text{ for almost every } x \in \omega, i = 1, 2, 3.$$

Our main result in this part is the following stability estimate.

Theorem 5.1. *Assume that condition (5.1) holds. Assume also that $D_1, D_2, D_3 \in W^{1,\infty}(0, L)$. Then, the C_0 -semigroup $e^{t\mathcal{A}_j}$ is exponentially stable in \mathcal{H}_j , $j = 1, 2$, i.e., for all $U_0 \in \mathcal{H}_j$, there exist constants $M \geq 1$ and $\delta > 0$ independent of U_0 such that:*

$$\|e^{t\mathcal{A}_j} U_0\|_{\mathcal{H}_j} \leq M e^{-\delta t} \|U_0\|_{\mathcal{H}_j}, \quad t \geq 0, j = 1, 2.$$

According to [18] and [34], we have to check if the following conditions:

$$(H1) \quad i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$$

and

$$(H2) \quad \lim_{|\lambda| \rightarrow +\infty} \left\{ \sup_{\lambda \in \mathbb{R}} \left\| (i\lambda I - \mathcal{A}_j)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_j)} \right\} = O(1)$$

hold. Condition $i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$ is already proved in Lemma 3.2 and Lemma 3.3.

We will establish (H2) by contradiction. Suppose that there exist a sequence of real numbers $(\lambda_n)_n$, with $|\lambda_n| \rightarrow +\infty$ and a sequence of vectors

$$(5.2) \quad U_n = (v_n^1, v_n^2, v_n^3, v_n^4, v_n^5, v_n^6)^\top \in D(\mathcal{A}_j) \quad \text{with} \quad \|U_n\|_{\mathcal{H}_j} = 1$$

such that:

$$(5.3) \quad i\lambda_n U_n - \mathcal{A}_j U_n = (f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6)^\top \rightarrow 0 \quad \text{in} \quad \mathcal{H}_j, \quad j = 1, 2.$$

We will check the condition (H2) by finding a contradiction with (5.2)-(5.3) such as $\|U_n\|_{\mathcal{H}_j} = o(1)$.

Equation (5.2) is detailed as:

$$(5.4) \quad i\lambda_n v_n^1 - v_n^2 = f_n^1,$$

$$(5.5) \quad i\rho_1 \lambda_n v_n^2 - k_1 \left[((v_n^1)_x + v_n^3 + lv_n^5) + \frac{D_1}{k_1} ((v_n^2)_x + v_n^4 + lv_n^6) \right]_x - 1k_3 [(v_n^5)_x - lv_n^1] - 1D_3 [(v_n^6)_x - lv_n^2] = \rho_1 f_n^2,$$

$$(5.6) \quad i\lambda_n v_n^3 - v_n^4 = f_n^3,$$

$$(5.7) \quad i\rho_2 \lambda_n v_n^4 - k_2 \left[(v_n^3)_x + \frac{D_2}{k_2} (v_n^4)_x \right]_x + k_1 [(v_n^1)_x + v_n^3 + lv_n^5] + D_1 [(v_n^2)_x + v_n^4 + lv_n^6] = \rho_2 f_n^4,$$

$$(5.8) \quad i\lambda_n v_n^5 - v_n^6 = f_n^5,$$

$$(5.9) \quad i\rho_1 \lambda_n v_n^6 - k_3 \left[((v_n^5)_x - lv_n^1) + \frac{D_3}{k_3} ((v_n^6)_x - lv_n^2) \right]_x + 1k_1 [(v_n^1)_x + v_n^3 + lv_n^5] + 1D_1 [(v_n^2)_x + v_n^4 + lv_n^6] = \rho_1 f_n^6.$$

For clarity, we divide the proof into several lemmas. From now on, for simplicity, we drop the index n .

Lemma 5.2. *Under all the above assumptions, we have*

$$(5.10) \quad \|D_1^{1/2} (v_x^2 + v^4 + lv^6)\| = o(1), \quad \|D_2^{1/2} v_x^4\| = o(1), \quad \|D_3^{1/2} (v_x^6 - lv^2)\| = o(1)$$

and

$$(5.11) \quad \|(v_x^2 + v^4 + lv^6)\| = o(1), \quad \|v_x^4\| = o(1), \quad \|(v_x^6 - lv^2)\| = o(1) \quad \text{in} \quad (\alpha, \beta).$$

Proof. Taking the inner product of (5.3) with U in \mathcal{H}_j , we get:

$$(5.12) \quad \operatorname{Re}(i\lambda U - \mathcal{A}_j U, U)_{\mathcal{H}_j} = -\operatorname{Re}(\mathcal{A}_j U, U)_{\mathcal{H}_j} = \int_0^L (D_1 |v_x^2 + v^4 + lv^6|^2 + D_2 |v_x^4|^2 + D_3 |v_x^6 - lv^2|^2) dx = o(1).$$

Thanks to (5.1), we obtain the desired asymptotic equation (5.10) and (5.11). Thus the proof is complete. \square

Remark 5.3. *These estimates are crucial for the rest of the proof and they will be used to prove each point of the global proof divided in several lemmas.*

Lemma 5.4. *Under all the above assumptions, we have:*

$$(5.13) \quad \|v_x^1 + v^3 + lv^5\| = \frac{o(1)}{\lambda}, \quad \|v_x^3\| = \frac{o(1)}{\lambda}, \quad \|v_x^5 - lv^1\| = \frac{o(1)}{\lambda} \quad \text{in} \quad (\alpha, \beta).$$

Proof. First, using equations (5.4), (5.6) and (5.8), we obtain:

$$(5.14) \quad \lambda (v_x^1 + v^3 + lv^5) = -i(v_x^2 + f_x^1 + v^4 + f^3 + lv^6 + lf^5).$$

Consequently,

$$(5.15) \quad \int_\alpha^\beta \lambda^2 |v_x^1 + v^3 + lv^5|^2 dx \leq 2 \int_\alpha^\beta |v_x^2 + v^4 + lv^6|^2 dx + 2 \int_\alpha^\beta |f_x^1 + f^3 + lf^5|^2 dx.$$

Using the first estimate of (5.11) and the fact that f^1, f^3, f^5 converge to zero in $H_0^1(0, L)$ (or in $H_\star^1(0, L)$) in equation (5.15), we deduce:

$$(5.16) \quad \int_\alpha^\beta \lambda^2 |v_x^1 + v^3 + lv^5|^2 dx = o(1).$$

In a similar way, one can prove:

$$(5.17) \quad \int_\alpha^\beta \lambda^2 |v_x^3|^2 dx = o(1) \quad \text{and} \quad \int_\alpha^\beta \lambda^2 |v_x^5 - lv^1|^2 dx = o(1).$$

The proof is thus complete. \square

Here and after ϵ designates a fixed positive real number such that $0 < \alpha + \epsilon < \beta - \epsilon < L$. Then, we define the cut-off function $\eta \in C_c^\infty(\mathbb{R})$ by:

$$\eta = 1 \text{ on } [\alpha + \epsilon, \beta - \epsilon], \quad 0 \leq \eta \leq 1, \quad \eta = 0 \text{ on } (0, L) \setminus (\alpha, \beta).$$

Lemma 5.5. *Under all the above assumptions, we have:*

$$(5.18) \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^1|^2 dx = o(1), \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^3|^2 dx = o(1), \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^5|^2 dx = o(1).$$

Proof. First, multiplying equation (5.4) by $i\lambda\eta\bar{v}^1$ in $L^2(0, L)$ and integrating by parts, we get:

$$(5.19) \quad - \int_0^L \eta |\lambda v^1|^2 dx - i \int_0^L \lambda \eta v^2 \bar{v}^1 dx = i \int_0^L \lambda f^1 \eta \bar{v}^1 dx.$$

As λv^1 is uniformly bounded in $L^2(0, L)$ and f^1 converges to zero in $H_0^1(0, L)$ (or in $H_x^1(0, L)$), we get: that the term on the right hand side of (5.19) converges to zero and consequently:

$$(5.20) \quad - \int_0^L \eta |\lambda v^1|^2 dx - i \int_0^L \lambda \eta v^2 \bar{v}^1 dx = o(1).$$

Moreover, multiplying (5.5) by $\rho_1^{-1}\eta\bar{v}^1$ in $L^2(0, L)$ and then integrating by parts, we obtain:

$$(5.21) \quad i \int_0^L \lambda \eta v^2 \bar{v}^1 dx + k_1 \rho_1^{-1} \int_0^L \left((v_x^1 + v^3 + lv^5) + \frac{D_1}{k_1} (v_x^2 + v^4 + lv^6) \right) (\eta \bar{v}^1)_x dx \\ - lk_3 \rho_1^{-1} \int_0^L (v_x^5 - lv^1) \eta \bar{v}^1 dx - l \rho_1^{-1} \int_0^L D_3 (v_x^6 - lv^2) \eta \bar{v}^1 dx = \int_0^L f^2 \eta \bar{v}^1 dx.$$

Using (5.10), (5.13), the fact that f^2 converges to zero in $L^2(0, L)$ and λv^1 is uniformly bounded in $L^2(0, L)$ in (5.21), we get:

$$(5.22) \quad i \int_0^L \lambda \eta v^2 \bar{v}^1 dx = o(1).$$

Finally, using (5.22) in (5.20), we get:

$$\int_0^L \eta |\lambda v^1|^2 dx = o(1), \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^1|^2 dx = o(1).$$

In a same way, we show:

$$\int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^3|^2 dx = o(1), \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^5|^2 dx = o(1).$$

□

Lemma 5.6. *Under all the above assumptions, we have:*

$$(5.23) \quad \|\sqrt{D_2} \lambda v^4\| = O(1).$$

Proof. First, multiplying (5.7) by $iD_2\lambda\bar{v}^4$ and integrating by parts, we get:

$$(5.24) \quad \rho_2 \int_0^L D_2 |\lambda v^4|^2 dx = \text{Re} \left\{ k_2 \int_0^L i \lambda v_x^3 (D_2 \bar{v}^4)_x dx + \int_0^L D_2 v_x^4 i \lambda (D_2 \bar{v}^4)_x dx \right. \\ \left. + \int_0^L i k_1 (v_x^1 + v^3 + lv^5) D_2 \lambda \bar{v}^4 dx + \int_0^L i D_1 (v_x^2 + v^4 + lv^6) D_2 \lambda \bar{v}^4 dx \right. \\ \left. - \int_0^L i \rho_2 f^4 D_2 \lambda \bar{v}^4 dx \right\}.$$

(i) **Estimation of the second term of (5.24).** Using (5.6), we have:

$$(5.25) \quad k_2 \int_0^L (i \lambda v^3)_x (D_2 \bar{v}^4)_x dx = k_2 \int_0^L (v_x^4 + f_x^3) (D_2' \bar{v}^4 + D_2 v_x^4) dx.$$

Using the fact that v^4 is uniformly bounded in $L^2(0, L)$, f^3 converges to zero in $H_0^1(0, L)$ (or in $H_x^1(0, L)$) and $\|D_2 v_x^4\| = o(1)$ due to (5.10) in the above equation, we deduce that:

$$(5.26) \quad k_2 \int_0^L (i \lambda v^3)_x (D_2 \bar{v}^4)_x dx = o(1).$$

(ii) **Estimation of the third term of (5.24).** We have:

$$\begin{aligned}
\operatorname{Re}\left\{\int_0^L D_2 v_x^4 i\lambda \left(D_2 \overline{v^4}\right)_x dx\right\} &= \operatorname{Re}\left\{\int_0^L D_2 v_x^4 i\lambda \left(D_2' \overline{v^4} + D_2 \overline{v_x^4}\right) dx\right\} \\
(5.27) \qquad \qquad \qquad &= \operatorname{Re}\left\{\int_0^L i\lambda D_2 v_x^4 \overline{v^4} D_2' dx\right\} + \operatorname{Re}\left\{\int_0^L |D_2|^2 i\lambda |v_x^4|^2 dx\right\} \\
&= \operatorname{Re}\left\{\int_0^L i\lambda D_2 v_x^4 \overline{v^4} D_2' dx\right\}.
\end{aligned}$$

Let ϵ_1 be a positive constant. Using Young's inequality in the above equation and then using the second estimate of (5.10), we get:

$$\begin{aligned}
\operatorname{Re}\left\{\int_0^L D_2 v_x^4 i\lambda \left(D_2 \overline{v^4}\right)_x dx\right\} &\leq \epsilon_1 \int_0^L D_2 |\lambda v^4|^2 dx + \frac{1}{\epsilon_1} \int_0^L D_2 |v_x^4|^2 |D_2'|^2 dx \\
(5.28) \qquad \qquad \qquad &\leq \epsilon_1 \int_0^L D_2 |\lambda v^4|^2 dx + o(1).
\end{aligned}$$

(iii) **Estimation of the fourth term of (5.24).** Let $\epsilon_2 > 0$. Using Young's inequality and the fact that $v_x^1 + v^3 + lv^5$ is uniformly bounded in $L^2(0, L)$ due to (5.2) in the fourth term of (5.24), we obtain:

$$\begin{aligned}
\operatorname{Re}\left\{\int_0^L ik_1 (v_x^1 + v^3 + lv^5) D_2 \lambda \overline{v^4} dx\right\} &\leq \epsilon_2 \int_0^L D_2 |\lambda v^4|^2 dx + \frac{1}{\epsilon_2} \int_0^L k_1^2 D_2 |v_x^1 + v^3 + lv^5|^2 dx \\
(5.29) \qquad \qquad \qquad &\leq \epsilon_2 \int_0^L D_2 |\lambda v^4|^2 dx + O(1).
\end{aligned}$$

(iv) **Estimation of the fifth term of (5.24).** Let $\epsilon_3 > 0$. Using Young's inequality and the first estimate of (5.10) in the fifth term of (5.24), we obtain:

$$\begin{aligned}
\operatorname{Re}\left\{\int_0^L iD_1 (v_x^2 + v^4 + lv^6) D_2 \lambda \overline{v^4} dx\right\} &\leq \epsilon_3 \int_0^L D_2 |\lambda v^4|^2 dx + \frac{1}{\epsilon_3} \int_0^L |D_1|^2 |v_x^2 + v^4 + lv^6|^2 D_2 dx \\
(5.30) \qquad \qquad \qquad &\leq \epsilon_3 \int_0^L D_2 |\lambda v^4|^2 dx + o(1).
\end{aligned}$$

(V) **Estimation of the last term of (5.24).** Let $\epsilon_4 > 0$. Using Young's inequality and the fact that f^4 converges to zero in $L^2(0, L)$, we get:

$$\begin{aligned}
\operatorname{Re}\left\{\int_0^L i\rho_2 f^4 D_2 \lambda \overline{v^4} dx\right\} &\leq \epsilon_4 \int_0^L D_2 |\lambda v^4|^2 dx + \frac{1}{\epsilon_4} \int_0^L D_2 \rho_2^2 |f^4|^2 dx \\
(5.31) \qquad \qquad \qquad &\leq \epsilon_4 \int_0^L D_2 |\lambda v^4|^2 dx + o(1).
\end{aligned}$$

Main estimate. Finally, inserting (5.26), (5.28), (5.29), (5.30) and (5.31) into (5.24), we get:

$$(5.32) \qquad \qquad \qquad (\rho_2 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) \int_0^L D_2 |\lambda v^4|^2 dx \leq O(1).$$

Taking $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \frac{\rho_2}{8}$ in the above equation, we get: the desired estimate (5.23). The proof is thus complete. \square

Lemma 5.7. *Under all the above assumptions, we have:*

$$(5.33) \qquad \qquad \qquad \|\sqrt{D_1} \lambda v^2\| = O(1).$$

Proof. Multiplying (5.5) by $iD_1 \lambda \overline{v^2}$ and integrating by parts, we get:

$$\begin{aligned}
\rho_1 \int_0^L D_1 |\lambda v^2|^2 dx &= \operatorname{Re}\left\{k_1 \int_0^L ((i\lambda v^1)_x + i\lambda v^3 + li\lambda v^5)(D_1 \overline{v^2})_x dx + \int_0^L D_1 (i\lambda v^2)_x (D_1 \overline{v^2})_x dx\right. \\
(5.34) \qquad \qquad \qquad &+ \int_0^L D_1 (v^4 + lv^6) i\lambda (D_1 \overline{v^2})_x dx - \int_0^L ik_3 (v_x^5 - lv^1) D_1 \lambda \overline{v^2} dx \\
&\left. - \int_0^L iD_3 (v_x^6 - lv^2) D_1 \lambda \overline{v^2} - \int_0^L \rho_1 f^2 i\lambda D_1 \overline{v^2} dx\right\}.
\end{aligned}$$

(i) **Estimation of the second term of (5.34).** Using equations (5.4), (5.6) and (5.8), we get:

$$\operatorname{Re}\left\{k_1 \int_0^L ((i\lambda v^1)_x + i\lambda v^3 + li\lambda v^5)(D_1 \overline{v^2})_x dx\right\} = \operatorname{Re}\left\{k_1 \int_0^L (v_x^2 + v^4 + lv^6 + f_x^1 + f^3 + lf^5)(D_1' \overline{v^2} + D_1 \overline{v_x^2}) dx\right\}.$$

Consequently, using the fact that v^2 is uniformly bounded in $L^2(0, L)$, f^1 , f^3 , f^5 converge to zero in $H_0^1(0, L)$ (or in and $H_*^1(0, L)$) and equation (5.10), we obtain:

$$(5.35) \quad \operatorname{Re}\left\{k_1 \int_0^L ((i\lambda v^1)_x + i\lambda v^3 + li\lambda v^5)(D_1 \overline{v^2})_x dx\right\} = o(1).$$

(ii) **Estimation of the third term of (5.34).** We have:

$$(5.36) \quad \begin{aligned} \operatorname{Re}\left\{\int_0^L D_1 v_x^2 i\lambda (D_1 \overline{v^2})_x dx\right\} &= \operatorname{Re}\left\{\int_0^L D_1 v_x^2 i\lambda (D_1' \overline{v^2} + D_1 \overline{v_x^2}) dx\right\} \\ &= \operatorname{Re}\left\{\int_0^L D_1 v_x^2 i\lambda \overline{v^2} D_1' dx\right\} + \operatorname{Re}\left\{\int_0^L i\lambda |D_1|^2 |v_x^2|^2 dx\right\} \\ &= \operatorname{Re}\left\{\int_0^L D_1 v_x^2 i\lambda \overline{v^2} D_1' dx\right\}. \end{aligned}$$

Let $\epsilon_1 > 0$. Using Young's inequality in the above equation and then using the first estimate of (5.10), we get:

$$(5.37) \quad \begin{aligned} \operatorname{Re}\left\{\int_0^L D_1 v_x^2 i\lambda (D_1 \overline{v^2})_x dx\right\} &\leq \epsilon_1 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_1} \int_0^L D_1 |v_x^2|^2 |D_1'|^2 dx \\ &\leq \epsilon_1 \int_0^L D_1 |\lambda v^2|^2 dx + o(1). \end{aligned}$$

(iii) **Estimation of the fourth term of (5.34).** We have:

$$(5.38) \quad \begin{aligned} \operatorname{Re}\left\{\int_0^L D_1 (v^4 + lv^6) i\lambda (D_1 \overline{v^2})_x dx\right\} &= \operatorname{Re}\left\{\int_0^L D_1 (v^4 + lv^6) i\lambda (D_1' \overline{v^2} + D_1 \overline{v_x^2}) dx\right\} \\ &= \operatorname{Re}\left\{\int_0^L D_1 (v^4 + lv^6) D_1' i\lambda \overline{v^2} dx\right\} + \operatorname{Re}\left\{\int_0^L |D_1|^2 (v^4 + lv^6) i\lambda \overline{v_x^2} dx\right\}. \end{aligned}$$

Now, we need to estimate each term of (5.38) as follows:

1) Let $\epsilon_2 > 0$ and using Young's inequality and the fact that v^4 and v^6 are uniformly bounded in $L^2(0, L)$, we get:

$$(5.39) \quad \begin{aligned} \operatorname{Re}\left\{\int_0^L D_1 (v^4 + lv^6) D_1' i\lambda \overline{v^2} dx\right\} &\leq \epsilon_2 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_2} \int_0^L D_1 |D_1'|^2 |v^4 + lv^6|^2 dx \\ &\leq \epsilon_2 \int_0^L D_1 |\lambda v^2|^2 dx + O(1). \end{aligned}$$

2) We have:

$$(5.40) \quad \operatorname{Re}\left\{\int_0^L |D_1|^2 (v^4 + lv^6) i\lambda \overline{v_x^2} dx\right\} = \operatorname{Re}\left\{\int_0^L D_1 i\lambda v^4 D_1 \overline{v_x^2} dx\right\} + \operatorname{Re}\left\{\int_0^L |D_1|^2 i\lambda lv^6 \overline{v_x^2} dx\right\}.$$

Hence, from condition (5.1), using Lemma 5.6 and estimate (5.10), we get:

$$(5.41) \quad \operatorname{Re}\left\{\int_0^L D_1 i\lambda v^4 D_1 \overline{v_x^2} dx\right\} = o(1).$$

On the other hand, after integrating by parts, then using Young's inequality, the fact that v^6 is uniformly bounded in $L^2(0, L)$ and from condition (5.1), the last estimate of (5.10), we get for $\epsilon_3 > 0$, $\epsilon_4 > 0$:

$$(5.42) \quad \begin{aligned} \operatorname{Re}\left\{\int_0^L |D_1|^2 i\lambda lv^6 \overline{v_x^2} dx\right\} &= -\operatorname{Re}\left\{\int_0^L 2D_1 D_1' lv^6 i\lambda \overline{v^2} dx\right\} - \operatorname{Re}\left\{\int_0^L D_1 lv_x^6 D_1 i\lambda \overline{v^2} dx\right\} \\ &\leq \epsilon_3 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_3} \int_0^L 4D_1 |D_1'|^2 |v^6|^2 dx + \epsilon_4 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_4} \int_0^L |D_1|^3 |v_x^6|^2 dx \\ &\leq (\epsilon_3 + \epsilon_4) \int_0^L D_1 |\lambda v^2|^2 dx + O(1). \end{aligned}$$

Finally, inserting (5.39), (5.41), (5.42) into (5.38), we obtain:

$$(5.43) \quad \operatorname{Re} \left\{ \int_0^L D_1(v^4 + lv^6) i \lambda (D_1 \overline{v^2})_x dx \right\} \leq (\epsilon_2 + \epsilon_3 + \epsilon_4) \int_0^L D_1 |\lambda v^2|^2 dx + O(1).$$

(iv) **Estimation of the fifth term of (5.34).** For $\epsilon_5 > 0$, by using Young's inequality and the fact that $v_x^5 - lv^1$ is uniformly bounded in $L^2(0, L)$, we get:

$$(5.44) \quad \begin{aligned} \operatorname{Re} \left\{ \int_0^L i l k_3 (v_x^5 - lv^1) D_1 \lambda \overline{v^2} dx \right\} &\leq \epsilon_5 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_5} \int_0^L l^2 k_3^2 D_1 |v_x^5 - lv^1|^2 dx \\ &\leq \epsilon_5 \int_0^L D_1 |\lambda v^2|^2 dx + O(1). \end{aligned}$$

(v) **Estimation of the sixth term of (5.34).** For $\epsilon_6 > 0$, by using Young's inequality and the third estimate of (5.10), we obtain:

$$(5.45) \quad \begin{aligned} \operatorname{Re} \left\{ \int_0^L i l D_3 (v_x^6 - lv^2) D_1 \lambda \overline{v^2} dx \right\} &\leq \epsilon_6 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_6} \int_0^L l^2 D_1 |D_3|^2 |v_x^6 - lv^2|^2 dx \\ &\leq \epsilon_6 \int_0^L D_1 |\lambda v^2|^2 dx + o(1). \end{aligned}$$

(vi) **Estimation of the last term of (5.34).** For $\epsilon_7 > 0$, by using Young's inequality and the fact that f^2 converges to zero in $L^2(0, L)$, we get:

$$(5.46) \quad \begin{aligned} \operatorname{Re} \left\{ \int_0^L \rho_1 f^2 i \lambda D_1 \overline{v^2} dx \right\} &\leq \epsilon_7 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_7} \int_0^L D_1 \rho_1^2 |f^2|^2 dx \\ &\leq \epsilon_7 \int_0^L D_1 |\lambda v^2|^2 dx + o(1). \end{aligned}$$

Main estimate. Inserting (5.35), (5.37), (5.43), (5.44), (5.45) and (5.46) into (5.34), we obtain:

$$(5.47) \quad (\rho_1 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7) \int_0^L D_1 |\lambda v^6|^2 dx \leq O(1).$$

Taking $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \frac{\rho_1}{14}$ in the above equation, we get: the desired estimate (5.33). The proof is thus complete. \square

Lemma 5.8. *Under all the above assumptions, we have:*

$$(5.48) \quad \|\sqrt{D_3} \lambda v^6\| = O(1).$$

Proof. First, multiplying (5.9) by $i D_3 \lambda \overline{v^6}$ and integrating by parts, we get:

$$(5.49) \quad \begin{aligned} \rho_1 \int_0^L D_3 |\lambda v^6|^2 dx &= \operatorname{Re} \left\{ k_3 \int_0^L ((i \lambda v^5)_x - li \lambda v^1) (D_3 \overline{v^6})_x dx + \int_0^L D_3 v_x^6 i \lambda (D_3 \overline{v^6})_x dx \right. \\ &\quad - \int_0^L lv^2 D_3 i \lambda (D_3 \overline{v^6})_x dx + \int_0^L i l k_1 (v_x^1 + v^3 + lv^5) D_3 \lambda \overline{v^6} dx \\ &\quad \left. + \int_0^L i l D_1 (v_x^2 + v^4 + lv^6) D_3 \lambda \overline{v^6} dx - \int_0^L i \rho_1 f^6 D_3 \lambda \overline{v^6} dx \right\}. \end{aligned}$$

(i) **Estimation of the second term of (5.49).** Using (5.4) and (5.8), we get:

$$(5.50) \quad \operatorname{Re} \left\{ k_3 \int_0^L ((i \lambda v^5)_x - li \lambda v^1) (D_3 \overline{v^6})_x dx \right\} = \operatorname{Re} \left\{ k_3 \int_0^L (v_x^6 + f_x^5 - lv^2 - lf^1) (D_3 \overline{v^6} + D_3 \overline{v_x^6}) dx \right\},$$

consequently, by using the fact that v^6 is uniformly bounded in $L^2(0, L)$, f^1, f^5 converge to zero in $H_0^1(0, L)$ (or in $H_*^1(0, L)$) and the third estimate of (5.10), we get:

$$(5.51) \quad \operatorname{Re} \left\{ k_3 \int_0^L ((i \lambda v^5)_x - li \lambda v^1) (D_3 \overline{v^6})_x dx \right\} = o(1).$$

(ii) **Estimation of the third term of (5.49).** We have:

$$\begin{aligned}
(5.52) \quad \operatorname{Re} \left\{ \int_0^L D_3 v_x^6 i \lambda (D_3 \overline{v^6})_x dx \right\} &= \operatorname{Re} \left\{ \int_0^L D_3 v_x^6 i \lambda (D_3' \overline{v^6} + D_3 \overline{v_x^6}) dx \right\} \\
&= \operatorname{Re} \left\{ \int_0^L D_3 v_x^6 i \lambda \overline{v^6} D_3' dx \right\} + \operatorname{Re} \left\{ \int_0^L i \lambda |D_3|^2 |v_x^6|^2 dx \right\} \\
&= \operatorname{Re} \left\{ \int_0^L D_3 v_x^6 i \lambda \overline{v^6} D_3' dx \right\}.
\end{aligned}$$

Let $\epsilon_1 > 0$. Using Young's inequality in the above equation and then using the third estimate of (5.10), we get:

$$\begin{aligned}
(5.53) \quad \operatorname{Re} \left\{ \int_0^L D_3 v_x^6 i \lambda (D_3 \overline{v^6})_x dx \right\} &\leq \epsilon_1 \int_0^L D_3 |\lambda \overline{v^6}|^2 dx + \frac{1}{\epsilon_1} \int_0^L D_3 |v_x^6|^2 |D_3'|^2 dx \\
&\leq \epsilon_1 \int_0^L D_3 |\lambda \overline{v^6}|^2 dx + o(1).
\end{aligned}$$

(iii) **Estimation of the fourth term of (5.49).** We have:

$$(5.54) \quad \operatorname{Re} \left\{ \int_0^L l v^2 D_3 i \lambda (D_3 \overline{v^6})_x dx \right\} = \operatorname{Re} \left\{ \int_0^L l i \lambda v^2 D_3 (D_3' \overline{v^6} + D_3 \overline{v_x^6}) dx \right\},$$

from condition (5.1), using Lemma 5.7, the fact that v^6 is uniformly bounded in $L^2(0, L)$ and the third estimate of (5.10), we deduce that:

$$(5.55) \quad \operatorname{Re} \left\{ \int_0^L l v^2 D_3 i \lambda (D_3 \overline{v^6})_x dx \right\} = O(1).$$

(iv) **Estimation of the fifth term of (5.49).** Let $\epsilon_2 > 0$. Using Young's inequality and then the fact that $v_x^1 + v_3 + l v^5$ is uniformly bounded in $L^2(0, L)$, we get:

$$\begin{aligned}
(5.56) \quad \operatorname{Re} \left\{ \int_0^L i l k_1 (v_x^1 + v^3 + l v^5) D_3 \lambda \overline{v^6} dx \right\} &\leq \epsilon_2 \int_0^L D_3 |\lambda v^6|^2 dx + \frac{1}{\epsilon_2} \int_0^L l^2 k_1^2 D_3 |v_x^1 + v^3 + l v^5|^2 dx \\
&\leq \epsilon_2 \int_0^L D_3 |\lambda v^6|^2 dx + O(1).
\end{aligned}$$

(v) **Estimation of the sixth term of (5.49).** Let $\epsilon_3 > 0$. Using Young's inequality and then the first estimate of (5.10), we obtain:

$$\begin{aligned}
(5.57) \quad \operatorname{Re} \left\{ \int_0^L i l D_1 (v_x^2 + v^4 + l v^6) D_3 \lambda \overline{v^6} dx \right\} &\leq \epsilon_3 \int_0^L D_3 |\lambda v^6|^2 dx + \frac{1}{\epsilon_3} \int_0^L l^2 D_3 |D_1|^2 |v_x^2 + v^4 + l v^6|^2 dx \\
&\leq \epsilon_3 \int_0^L D_3 |\lambda v^6|^2 dx + o(1).
\end{aligned}$$

(vi) **Estimation of the last term of (5.49).** Let $\epsilon_4 > 0$. Using Young's inequality and then the fact that f^6 converges to zero in $L^2(0, L)$, we obtain:

$$\begin{aligned}
(5.58) \quad \operatorname{Re} \left\{ \int_0^L i \rho_1 f^6 D_3 \lambda \overline{v^6} dx \right\} &\leq \epsilon_4 \int_0^L D_3 |\lambda v^6|^2 dx + \frac{1}{\epsilon_4} \int_0^L D_3 \rho_2^2 |f^6|^2 dx \\
&\leq \epsilon_4 \int_0^L D_3 |\lambda v^6|^2 dx + o(1).
\end{aligned}$$

Main estimate. Inserting (5.51), (5.53), (5.55), (5.56), (5.57) and (5.58) into (5.49), we get:

$$(5.59) \quad (\rho_1 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) \int_0^L D_3 |\lambda v^6|^2 dx \leq O(1).$$

Taking $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \frac{\rho_1}{8}$ in the above equation, we get: the desired estimate (5.48). The proof is thus complete. \square

Now, let $h \in H_0^1(0, L)$.

Lemma 5.9. *Under all the above assumptions, we have:*

$$(5.60) \quad \begin{aligned} & \frac{1}{2}\rho_1 \int_0^L h'|v^2|^2 dx + \frac{k_1}{2} \int_0^L h'|v_x^1 + \frac{D_1}{k_1}(v_x^2 + v^4 + lv^6)|^2 dx \\ & + \operatorname{Re} \left\{ -k_1 \int_0^L h v_x^3 \overline{v_x^1} dx - lk_1 \int_0^L v_x^5 h \overline{v_x^1} dx - lk_3 \int_0^L v_x^5 h \overline{v_x^1} dx \right\} = o(1). \end{aligned}$$

Proof. First, let $M = v_x^1 + \frac{D_1}{k_1}(v_x^2 + v^4 + lv^6)$. Multiplying (5.5) by $h\overline{M}$ then integrating by parts and using (5.10), the fact that v_x^1 is uniformly bounded in $L^2(0, L)$ and f^2 converges to zero in $L^2(0, L)$, we get:

$$(5.61) \quad \begin{aligned} & \operatorname{Re} \left\{ \int_0^L i\lambda\rho_1 v^2 h \overline{M} dx - k_1 \int_0^L v_x^3 h \overline{M} dx - lk_1 \int_0^L v_x^5 h \overline{M} dx \right. \\ & \left. + \frac{k_1}{2} \int_0^L h'|M|^2 dx - lk_3 \int_0^L (v_x^5 - lv^1) h \overline{M} dx \right\} = o(1). \end{aligned}$$

(i) **Estimation of the first term of (5.61).** First, we have

$$(5.62) \quad \operatorname{Re} \left\{ \int_0^L i\lambda\rho_1 v^2 h \overline{M} dx \right\} = -\operatorname{Re} \left\{ \int_0^L \rho_1 v^2 h (\overline{i\lambda v^1})_x dx \right\} + \operatorname{Re} \left\{ \int_0^L i\lambda v^2 \rho_1 h \frac{D_1}{k_1} \overline{(v_x^2 + v^4 + lv^6)} dx \right\}.$$

Next, we need to estimate each term of (5.62). For this, by using equation (5.4) we get:

$$(5.63) \quad \begin{aligned} -\operatorname{Re} \left\{ \int_0^L \rho_1 v^2 h (\overline{i\lambda v^1})_x dx \right\} &= -\operatorname{Re} \left\{ \int_0^L \rho_1 v^2 h (\overline{v_x^2} + \overline{f_x^1}) dx \right\} \\ &= \frac{1}{2}\rho_1 \int_0^L h'|v^2|^2 dx - \operatorname{Re} \left\{ \int_0^L \rho_1 v^2 h \overline{f_x^1} dx \right\}. \end{aligned}$$

In addition, using the fact that v^2 is uniformly bounded in $L^2(0, L)$ and f^1 converges to zero in $H_0^1(0, L)$, we obtain:

$$(5.64) \quad \operatorname{Re} \left\{ \int_0^L \rho_1 v^2 h \overline{f_x^1} dx \right\} = o(1).$$

Inserting (5.64) into (5.63), we get:

$$(5.65) \quad -\operatorname{Re} \left\{ \int_0^L \rho_1 v^2 h (\overline{i\lambda v^1})_x dx \right\} = \frac{1}{2}\rho_1 \int_0^L h'|v^2|^2 dx + o(1).$$

Moreover, using Lemma 5.7 and the first estimate of (5.10), we deduce that:

$$(5.66) \quad \operatorname{Re} \left\{ \int_0^L i\lambda v^2 \rho_1 h \frac{D_1}{k_1} \overline{(v_x^2 + v^4 + lv^6)} dx \right\} = o(1).$$

Finally, inserting (5.65) and (5.66) into (5.62), we get:

$$(5.67) \quad \operatorname{Re} \left\{ \int_0^L i\lambda\rho_1 v^2 h \overline{M} dx \right\} = \frac{1}{2}\rho_1 \int_0^L h'|v^2|^2 dx + o(1).$$

(ii) **Estimation of the second and third terms of (5.61).** By using the fact that v_x^3 is uniformly bounded in $L^2(0, L)$ and the first estimate of (5.10), we get:

$$(5.68) \quad \begin{aligned} \operatorname{Re} \left\{ k_1 \int_0^L v_x^3 h \overline{M} dx \right\} &= \operatorname{Re} \left\{ k_1 \int_0^L v_x^3 h \overline{v_x^1} dx + \int_0^L v_x^3 D_1 \overline{(v_x^2 + v^4 + lv^6)} dx \right\} \\ &= \operatorname{Re} \left\{ k_1 \int_0^L v_x^3 h \overline{v_x^1} dx \right\} + o(1). \end{aligned}$$

Also, by using the fact that v_x^5 is uniformly bounded in $L^2(0, L)$ and the first estimate of (5.10), we get:

$$(5.69) \quad \begin{aligned} \operatorname{Re} \left\{ lk_1 \int_0^L v_x^5 h \overline{M} dx \right\} &= \operatorname{Re} \left\{ l \int_0^L v_x^5 h \overline{v_x^1} dx + lk_1 \int_0^L v_x^5 D_1 \overline{(v_x^2 + v^4 + lv^6)} dx \right\} \\ &= \operatorname{Re} \left\{ lk_1 \int_0^L v_x^5 h \overline{v_x^1} dx \right\} + o(1). \end{aligned}$$

(iii) **Estimation of the fifth term of (5.61).** Using the fact that v_x^1 and v_x^5 are uniformly bounded in $L^2(0, L)$, $v^1 = O(\frac{1}{\lambda})$ and the first estimate of (5.10), we get:

$$(5.70) \quad \begin{aligned} \operatorname{Re} \left\{ -lk_3 \int_0^L (v_x^5 - lv^1) h \overline{M} dx \right\} &= \operatorname{Re} \left\{ -lk_3 \int_0^L (v_x^5 - lv^1) h \left(\overline{v_x^1} + \frac{D_1}{k_1} \overline{(v_x^2 + v^4 + lv^6)} \right) \right\} \\ &= \operatorname{Re} \left\{ -lk_3 \int_0^L v_x^5 h \overline{v_x^1} \right\} + o(1). \end{aligned}$$

(iv) **Main estimate.** Inserting (5.67), (5.68), (5.69) and (5.70) into (5.61), we obtain the desired estimate (5.60). Thus the proof is complete. \square

Lemma 5.10. *Under all the above assumptions, we have:*

$$(5.71) \quad \frac{1}{2} \rho_2 \int_0^L h' |v^4|^2 dx + \frac{k_2}{2} \int_0^L h' |v_x^3 + \frac{D_2}{k_2} v_x^4|^2 dx + \operatorname{Re} \left\{ k_1 \int_0^L v_x^1 h \overline{v_x^3} dx \right\} = o(1).$$

Proof. Let $N = v_x^3 + \frac{D_2}{k_2} v_x^4$. Multiplying (5.7) by $h \overline{N}$ then integrating by parts and using (5.10), the fact that v_x^3 is uniformly bounded in $L^2(0, L)$ and f^4 converges to zero in $L^2(0, L)$, we get:

$$(5.72) \quad \operatorname{Re} \left\{ \int_0^L i \lambda \rho_2 v^4 h \overline{N} dx + \frac{k_2}{2} \int_0^L h' |N|^2 dx + k_1 \int_0^L (v_x^1 + v^3 - lv^5) h \overline{N} dx \right\} = o(1).$$

(i) **Estimation of the first term of (5.72).** First, we have:

$$(5.73) \quad \operatorname{Re} \left\{ \int_0^L i \lambda \rho_2 v^4 h \overline{N} dx \right\} = -\operatorname{Re} \left\{ \int_0^L \rho_2 v^4 h (\overline{i \lambda v^3})_x dx \right\} + \operatorname{Re} \left\{ \int_0^L i \lambda v^4 \rho_2 h \frac{D_2}{k_2} \overline{v_x^4} dx \right\}.$$

Next, we need to estimate each term of (5.73). For this, by using equation (5.6) and then integrating by parts, we get:

$$(5.74) \quad \begin{aligned} -\operatorname{Re} \left\{ \int_0^L \rho_1 v^4 h (\overline{i \lambda v^3})_x dx \right\} &= -\operatorname{Re} \left\{ \int_0^L \rho_1 v^4 h (\overline{v_x^4} + \overline{f_x^3}) dx \right\} \\ &= \frac{1}{2} \rho_2 \int_0^L h' |v^4|^2 dx - \operatorname{Re} \left\{ \int_0^L \rho_2 v^4 h \overline{f_x^3} dx \right\}. \end{aligned}$$

In addition, using the fact that v^4 is uniformly bounded in $L^2(0, L)$ and f^3 converges to zero in $H_0^1(0, L)$ (or in $H_*^1(0, L)$), we obtain:

$$(5.75) \quad \operatorname{Re} \left\{ \int_0^L \rho_2 v^4 h \overline{f_x^3} dx \right\} = o(1).$$

Inserting (5.75) into (5.74), we get:

$$(5.76) \quad -\operatorname{Re} \left\{ \int_0^L \rho_2 v^4 h (\overline{i \lambda v^3})_x dx \right\} = \frac{1}{2} \rho_2 \int_0^L h' |v^4|^2 dx + o(1).$$

Moreover, using Lemma 5.6 and the second estimate of (5.10), we deduce that:

$$(5.77) \quad \operatorname{Re} \left\{ \int_0^L i \lambda v^4 \rho_2 h \frac{D_2}{k_2} \overline{v_x^4} dx \right\} = o(1).$$

Finally, inserting (5.76) and (5.77) into (5.73), we get:

$$(5.78) \quad \operatorname{Re} \left\{ \int_0^L i \lambda \rho_2 v^4 h \overline{N} dx \right\} = \frac{1}{2} \rho_2 \int_0^L h' |v^4|^2 dx + o(1).$$

(ii) **Estimation of the third term of (5.72).** Using the fact that v_x^1 , v_x^3 are uniformly bounded in $L^2(0, L)$, $v^3 = O(\frac{1}{\lambda})$, $v^5 = O(\frac{1}{\lambda})$ and the second estimate of (5.10), we get:

$$(5.79) \quad \begin{aligned} \operatorname{Re} \left\{ k_1 \int_0^L (v_x^1 + v^3 - lv^5) h \overline{N} dx \right\} &= \operatorname{Re} \left\{ k_1 \int_0^L (v_x^1 + v^3 - lv^5) h \left(\overline{v_x^1} + \frac{D_1}{k_1} \overline{v_x^2} \right) dx \right\} \\ &= \operatorname{Re} \left\{ k_1 \int_0^L v_x^1 h \overline{v_x^2} dx \right\} + o(1). \end{aligned}$$

(iii) **Main estimate.** Inserting (5.78) and (5.79) into (5.72) we get: the desired estimate (5.71). Thus the proof is complete. \square

Lemma 5.11. *Under all the above assumptions, we have:*

$$(5.80) \quad \begin{aligned} & \frac{1}{2}\rho_1 \int_0^L h'|v^6|^2 dx + \frac{k_3}{2} \int_0^L h'|v_x^5 + \frac{D_3}{k_3}(v_x^6 - lv^2)|^2 dx \\ & + \operatorname{Re} \left\{ lk_1 \int_0^L v_x^1 h \overline{v_x^5} dx + lk_3 \int_0^L v_x^1 h \overline{v_x^5} dx \right\} = o(1). \end{aligned}$$

Proof. Let $T = v_x^5 + \frac{D_3}{k_3}(v_x^6 - lv^2)$. Multiplying (5.9) by $h\overline{T}$ then integrating by parts and using (5.10), the fact that v_x^5 is uniformly bounded in $L^2(0, L)$ and f^6 converges to zero in $L^2(0, L)$, we get:

$$(5.81) \quad \begin{aligned} & \operatorname{Re} \left\{ \int_0^L i\lambda\rho_1 v^6 h \overline{T} dx + \frac{k_3}{2} \int_0^L h'|T|^2 dx + lk_3 \int_0^L v_x^1 h \overline{T} dx \right. \\ & \left. + lk_1 \int_0^L (v_x^1 + v^3 + lv^5) h \overline{T} dx \right\} = o(1). \end{aligned}$$

(i) **Estimation of the first term of (5.81).** First, we have:

$$(5.82) \quad \operatorname{Re} \left\{ \int_0^L i\lambda\rho_1 v^6 h \overline{T} dx \right\} = -\operatorname{Re} \left\{ \int_0^L \rho_1 v^6 h (i\lambda v^5)_x dx \right\} + \operatorname{Re} \left\{ \int_0^L i\lambda v^6 \rho_1 h \frac{D_3}{k_3} \overline{(v_x^6 - lv^2)} dx \right\}.$$

Next, we need to estimate each term of (5.82). For this, by using equation (5.8) we get:

$$(5.83) \quad \begin{aligned} -\operatorname{Re} \left\{ \int_0^L \rho_1 v^6 h (i\lambda v^5)_x dx \right\} &= -\operatorname{Re} \left\{ \int_0^L \rho_1 v^6 h (\overline{v_x^6} + \overline{f_x^5}) dx \right\} \\ &= \frac{1}{2}\rho_1 \int_0^L h'|v^6|^2 dx - \operatorname{Re} \left\{ \int_0^L \rho_1 v^6 h \overline{f_x^5} dx \right\}. \end{aligned}$$

Moreover, using the fact that v^6 is uniformly bounded in $L^2(0, L)$ and f^5 converges to zero in $H_0^1(0, L)$ (or in $H_\star^1(0, L)$), we obtain:

$$(5.84) \quad \operatorname{Re} \left\{ \int_0^L \rho_1 v^6 h \overline{f_x^5} dx \right\} = o(1).$$

Inserting (5.84) into (5.83), we get:

$$(5.85) \quad -\operatorname{Re} \left\{ \int_0^L \rho_1 v^6 h (i\lambda v^5)_x dx \right\} = \frac{1}{2}\rho_1 \int_0^L h'|v^6|^2 dx + o(1).$$

Furthermore, using Lemma 5.8 and the third estimate of (5.10), we deduce that:

$$(5.86) \quad \operatorname{Re} \left\{ \int_0^L i\lambda v^6 \rho_1 h \frac{D_3}{k_3} \overline{(v_x^6 - lv^2)} dx \right\} = o(1).$$

Finally, inserting (5.85) and (5.86) into (5.82), we get:

$$(5.87) \quad \operatorname{Re} \left\{ \int_0^L i\lambda\rho_1 v^6 h \overline{T} dx \right\} = \frac{1}{2}\rho_1 \int_0^L h'|v^6|^2 dx + o(1).$$

(ii) **Estimation of the third and fourth terms of (5.81).** By using the fact that v_x^1 is uniformly bounded in $L^2(0, L)$ and the third estimate of (5.10), we get:

$$(5.88) \quad \begin{aligned} \operatorname{Re} \left\{ lk_3 \int_0^L v_x^1 h \overline{T} dx \right\} &= \operatorname{Re} \left\{ lk_3 \int_0^L v_x^1 h \overline{v_x^5} dx + l \int_0^L v_x^1 h D_3 \overline{(v_x^6 - lv^2)} dx \right\} \\ &= \operatorname{Re} \left\{ lk_3 \int_0^L v_x^1 h \overline{v_x^5} dx \right\} + o(1). \end{aligned}$$

Using the fact that v_x^1, v_x^5 are uniformly bounded in $L^2(0, L)$, $v^3 = O(\frac{1}{\lambda}), v^5 = O(\frac{1}{\lambda})$ and the third estimate of (5.10), we get:

$$(5.89) \quad \begin{aligned} \operatorname{Re} \left\{ lk_1 \int_0^L (v_x^1 + v^3 + lv^5) h \overline{T} dx \right\} &= \operatorname{Re} \left\{ lk_1 \int_0^L (v_x^1 + v^3 + lv^5) h \left(\overline{v_x^5} + \frac{D_3}{k_3} \overline{(v_x^6 - lv^2)} \right) dx \right\} \\ &= \operatorname{Re} \left\{ lk_1 \int_0^L v_x^1 h \overline{v_x^5} dx \right\} + o(1). \end{aligned}$$

(iii) **Main estimate.** Inserting (5.87), (5.88) and (5.89) into (5.81) we get: the desired estimate (5.80). Thus the proof is complete. \square

Lemma 5.12. *Under all the above assumptions, we have:*

$$(5.90) \quad \begin{aligned} & \frac{1}{2}\rho_1 \left(\int_0^{\alpha+\epsilon} |v^2|^2 dx + \int_{\beta-\epsilon}^L |v^2|^2 dx \right) + \frac{k_1}{2} \left(\int_0^{\alpha+\epsilon} |v_x^1|^2 dx + \int_{\beta-\epsilon}^L |v_x^1|^2 dx \right) \\ & \frac{1}{2}\rho_2 \left(\int_0^{\alpha+\epsilon} |v^4|^2 dx + \int_{\beta-\epsilon}^L |v^4|^2 dx \right) + \frac{k_2}{2} \left(\int_0^{\alpha+\epsilon} |v_x^3|^2 dx + \int_{\beta-\epsilon}^L |v_x^3|^2 dx \right) \\ & \frac{1}{2}\rho_1 \left(\int_0^{\alpha+\epsilon} |v^6|^2 dx + \int_{\beta-\epsilon}^L |v^6|^2 dx \right) + \frac{k_3}{2} \left(\int_0^{\alpha+\epsilon} |v_x^5|^2 dx + \int_{\beta-\epsilon}^L |v_x^5|^2 dx \right) = o(1). \end{aligned}$$

Proof. First, combining Lemma 5.9, Lemma 5.10 and Lemma 5.11, we get:

$$(5.91) \quad \begin{aligned} & \frac{1}{2}\rho_1 \int_0^L h' |v^2|^2 dx + \frac{k_1}{2} \int_0^L h' |v_x^1| + \frac{D_1}{k_1} (v_x^2 + v^4 + lv^6) |^2 dx \\ & \frac{1}{2}\rho_2 \int_0^L h' |v^4|^2 dx + \frac{k_2}{2} \int_0^L h' |v_x^3| + \frac{D_2}{k_2} v_x^4 |^2 dx \\ & \frac{1}{2}\rho_1 \int_0^L h' |v^6|^2 dx + \frac{k_3}{2} \int_0^L h' |v_x^5| + \frac{D_3}{k_3} (v_x^6 - lv^2) |^2 dx = o(1). \end{aligned}$$

Next, we define the cut-off function $\tilde{\eta} \in C_0^\infty(0, L)$ by

$$\tilde{\eta} = 1 \text{ on } (0, \alpha + \epsilon), \quad 0 \leq \tilde{\eta} \leq 1, \quad \tilde{\eta} = 0 \text{ on } (\alpha + 2\epsilon, L)$$

and the cut-off function $\hat{\eta} \in C_0^\infty(0, L)$ by

$$\hat{\eta} = 0 \text{ on } (0, \beta - 2\epsilon), \quad 0 \leq \hat{\eta} \leq 1, \quad \hat{\eta} = 1 \text{ on } (\beta - \epsilon, L).$$

Taking $h = x\tilde{\eta} + (x-L)\hat{\eta}$ in (5.91). Then using estimates (5.10), (5.13), and (5.18), we get: the desired estimate (5.90). Thus the proof is complete. \square

Proof of Theorem 5.1 By using (5.13), (5.18) and (5.90), we get: $\|U\|_{\mathcal{H}_j} = o(1)$ on $(0, L)$ which is a contradiction with (5.2). Therefore (H2) holds and so, by [18] and [34], we deduce the exponential stability of the system (1.1) in the case of three local smooth dampings. The proof is thus complete. \square

6. POLYNOMIAL STABILITY: THE CASE OF THREE LOCAL DAMPINGS WITH NON SMOOTH COEFFICIENTS AT THE INTERFACE

It was proved that, see [22, 24], the stabilization of wave equation with local Kelvin-Voigt damping is greatly influenced by the smoothness of the damping coefficient and the region where the damping is localized (near or faraway from the boundary) even in the one-dimensional case. So, in this section, we consider the Bresse systems (1.1)-(1.2) and (1.1)-(1.3) subject to three local viscoelastic Kelvin-Voigt dampings with non smooth coefficients at the interface. Using frequency domain approach combined with multiplier techniques and the construction of a new multiplier function, we establish the polynomial stability of the C_0 -semigroup $e^{t\mathcal{A}_j}$, $j = 1, 2$. For this purpose, let $\emptyset \neq (\alpha_i, \beta_i) \subset (0, L)$, $i = 1, 2, 3$, be an arbitrary nonempty open subsets of $(0, L)$. We consider the following stability condition:

$$(6.1) \quad \exists d_0^i > 0 \text{ such that } D_i \geq d_0^i > 0 \text{ in } (\alpha_i, \beta_i), \quad i = 1, 2, 3, \text{ and } \bigcap_{i=1}^3 (\alpha_i, \beta_i) = (\alpha, \beta) \neq \emptyset.$$

Our main result in this section can be given by the following theorem:

Theorem 6.1. *Assume that condition (6.1) holds. Then, there exists a positive constant $c > 0$ such that for all $U_0 \in D(\mathcal{A}_j)$, $j = 1, 2$, the energy of the system satisfies the following decay rate:*

$$(6.2) \quad E(t) \leq \frac{c}{t} \|U_0\|_{D(\mathcal{A}_j)}^2.$$

Referring to [4], (6.2) is verified if the following conditions

$$(H1) \quad i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$$

and

$$(H3) \quad \lim_{\lambda \rightarrow +\infty} \sup_{\lambda \in \mathbb{R}} \left\{ \frac{1}{\lambda^2} \left\| (i\lambda Id - \mathcal{A}_j)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_j)} \right\} = O(1)$$

hold.

Condition $i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$ is already proved in Lemma 3.2 and Lemma 3.3.

We will establish (H3) by contradiction. Suppose that there exist a sequence of real numbers $(\lambda_n)_n$, with $|\lambda_n| \rightarrow +\infty$ and a sequence of vectors

$$(6.3) \quad U_n = (v_n^1, v_n^2, v_n^3, v_n^4, v_n^5, v_n^6)^\top \in D(\mathcal{A}_j) \quad \text{with} \quad \|U_n\|_{\mathcal{H}_j} = 1$$

such that

$$(6.4) \quad \lambda_n^2 (i\lambda_n U_n - \mathcal{A}_j U_n) = (f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6)^\top \rightarrow 0 \quad \text{in} \quad \mathcal{H}_j, \quad j = 1, 2.$$

We will check the condition (H3) by finding a contradiction with (6.3)-(6.4) such as $\|U_n\|_{\mathcal{H}_j} = o(1)$.

Equation (6.4) is detailed as:

$$(6.5) \quad i\lambda_n v_n^1 - v_n^2 = \frac{f_n^1}{\lambda_n^2},$$

$$(6.6) \quad i\rho_1 \lambda_n v_n^2 - [k_1 ((v_n^1)_x + v_n^3 + lv_n^5) + D_1 ((v_n^2)_x + v_n^4 + lv_n^6)]_x - 1k_3 [(v_n^5)_x - lv_n^1] - 1D_3 [(v_n^6)_x - lv_n^2] = \rho_1 \frac{f_n^2}{\lambda_n^2},$$

$$(6.7) \quad i\lambda_n v_n^3 - v_n^4 = \frac{f_n^3}{\lambda_n^2},$$

$$(6.8) \quad i\rho_2 \lambda_n v_n^4 - [k_2 (v_n^3)_x + D_2 (v_n^4)_x]_x + k_1 [(v_n^1)_x + v_n^3 + lv_n^5] + D_1 [(v_n^2)_x + v_n^4 + lv_n^6] = \rho_2 \frac{f_n^4}{\lambda_n^2},$$

$$(6.9) \quad i\lambda_n v_n^5 - v_n^6 = \frac{f_n^5}{\lambda_n^2},$$

$$(6.10) \quad i\rho_1 \lambda_n v_n^6 - [k_3 ((v_n^5)_x - lv_n^1) + D_3 ((v_n^6)_x - lv_n^2)]_x + 1k_1 [(v_n^1)_x + v_n^3 + lv_n^5] + 1D_1 [(v_n^2)_x + v_n^4 + lv_n^6] = \rho_1 \frac{f_n^6}{\lambda_n^2}.$$

From (6.3), (6.5), (6.7) and (6.9), we deduce that:

$$(6.11) \quad \|v_n^1\| = O\left(\frac{1}{\lambda_n}\right), \quad \|v_n^3\| = O\left(\frac{1}{\lambda_n}\right), \quad \|v_n^5\| = O\left(\frac{1}{\lambda_n}\right).$$

For clarity, we divide the proof into several lemmas. From now on, for simplicity, we drop the index n .

Lemma 6.2. *Under all the above assumptions, we have:*

$$(6.12) \quad \|D_1^{1/2}(v_x^2 + v^4 + lv^6)\| = \frac{o(1)}{\lambda}, \quad \|D_2^{1/2}v_x^4\| = \frac{o(1)}{\lambda}, \quad \|D_3^{1/2}(v_x^6 - lv^2)\| = \frac{o(1)}{\lambda}$$

and

$$(6.13) \quad \|v_x^2 + v^4 + lv^6\| = \frac{o(1)}{\lambda}, \quad \|v_x^4\| = \frac{o(1)}{\lambda}, \quad \|v_x^6 - lv^2\| = \frac{o(1)}{\lambda} \quad \text{in} \quad (\alpha, \beta).$$

Proof. Taking the inner product of (6.4) with U in \mathcal{H}_j , we get:

$$(6.14) \quad \begin{aligned} \operatorname{Re}(i\lambda^3 \|U\|^2 - \lambda^2 (\mathcal{A}_j U, U))_{\mathcal{H}_j} &= -\lambda^2 \operatorname{Re}(\mathcal{A}_j U, U)_{\mathcal{H}_j} \\ &= \lambda^2 \int_0^L (D_1 |v_x^2 + v^4 + lv^6|^2 + D_2 |v_x^4|^2 + D_3 |v_x^6 - lv^2|^2) dx = o(1). \end{aligned}$$

Thanks to (6.1), we obtain the desired asymptotic equation (6.12) and (6.13). Thus the proof is complete. \square

Remark 6.3. *Again, these estimates are crucial for the rest of the proof and they will be used to prove each point of the global proof divided in several lemmas.*

Lemma 6.4. *Under all the above assumptions, we have:*

$$(6.15) \quad \|v_x^1 + v^3 + lv^5\| = \frac{o(1)}{\lambda^2}, \quad \|v_x^3\| = \frac{o(1)}{\lambda^2}, \quad \|v_x^5 - lv^1\| = \frac{o(1)}{\lambda^2} \quad \text{in} \quad (\alpha, \beta).$$

Proof. First, using equations (6.5), (6.7) and (6.9), we obtain:

$$(6.16) \quad \lambda (v_x^1 + v^3 + lv^5) = -i(v_x^2 + \frac{f_x^1}{\lambda^2} + v^4 + \frac{f^3}{\lambda^2} + lv^6 + 1\frac{f^5}{\lambda^2}).$$

Consequently,

$$(6.17) \quad \int_{\alpha}^{\beta} \lambda^2 |v_x^1 + v^3 + lv^5|^2 dx \leq 2 \int_{\alpha}^{\beta} |v_x^2 + v^4 + lv^6|^2 dx + 2 \int_{\alpha}^{\beta} \frac{|f_x^1 + f^3 + lf^5|^2}{\lambda^4} dx.$$

Using the first estimate of (6.13) and the fact that f^1, f^3, f^5 converge to zero in $H_0^1(0, L)$ (or in $H_{\star}^1(0, L)$) in (6.17), we deduce:

$$(6.18) \quad \int_{\alpha}^{\beta} \lambda^2 |v_x^1 + v^3 + lv^5|^2 dx = \frac{o(1)}{\lambda^2}.$$

In a similar way, one can prove:

$$(6.19) \quad \int_{\alpha}^{\beta} \lambda^2 |v_x^3|^2 dx = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_{\alpha}^{\beta} \lambda^2 |v_x^5 - lv^1|^2 dx = \frac{o(1)}{\lambda^2}.$$

The proof is thus complete. \square

Lemma 6.5. *Under all the above assumptions, we have:*

$$(6.20) \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^1|^2 dx = \frac{o(1)}{\lambda}, \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^3|^2 dx = \frac{o(1)}{\lambda}, \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^5|^2 dx = \frac{o(1)}{\lambda}.$$

Proof. First, multiplying equation (6.5) by $i\lambda\eta\bar{v}^1$ in $L^2(0, L)$ and integrating by parts, we get:

$$(6.21) \quad - \int_0^L \eta |\lambda v^1|^2 dx - i \int_0^L \lambda \eta v^2 \bar{v}^1 dx = i \int_0^L \frac{f^1}{\lambda^2} \eta \lambda v^1 dx.$$

As λv^1 is uniformly bounded in $L^2(0, L)$ and f^1 converges to zero in $H_0^1(0, L)$, we get: that the term on the right hand side of (6.21) converges to zero and consequently

$$(6.22) \quad - \int_0^L \eta |\lambda v^1|^2 dx - i \int_0^L \lambda \eta v^2 \bar{v}^1 dx = \frac{o(1)}{\lambda^2}.$$

Moreover, multiplying (6.6) by $\rho_1^{-1}\eta\bar{v}^1$ in $L^2(0, L)$, then integrating by parts we obtain:

$$(6.23) \quad \begin{aligned} & i \int_0^L \lambda \eta v^2 \bar{v}^1 dx + \rho_1^{-1} \int_0^L (k_1 (v_x^1 + v^3 + lv^5) + D_1 (v_x^2 + v^4 + lv^6)) (\eta \bar{v}^1)_x dx \\ & - lk_3 \rho_1^{-1} \int_0^L (v_x^5 - lv^1) \eta \bar{v}^1 dx - l\rho_1^{-1} \int_0^L D_3 (v_x^6 - lv^2) \eta \bar{v}^1 dx = \int_0^L \frac{f^2}{\lambda^2} \eta \bar{v}^1 dx. \end{aligned}$$

Using (6.12), (6.15), the fact that f^2 converges to zero in $L^2(0, L)$ and $\lambda v^1, v_x^1$ are uniformly bounded in $L^2(0, L)$ in (6.23), we get:

$$(6.24) \quad i \int_0^L \lambda \eta v^2 \bar{v}^1 dx = \frac{o(1)}{\lambda}.$$

Finally, using (6.24) in (6.21), we get:

$$\int_0^L \eta |\lambda v^1|^2 dx = \frac{o(1)}{\lambda}, \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^1|^2 dx = \frac{o(1)}{\lambda}.$$

In a same way, we show:

$$\int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^3|^2 dx = \frac{o(1)}{\lambda}, \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^5|^2 dx = \frac{o(1)}{\lambda}.$$

The proof is thus complete. \square

Now, we introduce new multiplier functions. For this purpose, let $\emptyset \neq \omega_{\epsilon} = (\alpha + \epsilon, \beta - \epsilon)$.

Lemma 6.6. *The solution (u, y, z) of the following system*

$$(6.25) \quad \begin{cases} \rho_1 \lambda^2 u + k_1 (u_x + y + lz)_x + lk_3 (z_x - lu) - i\lambda \mathbb{1}_{\omega_{\epsilon}} u & = v^1, \\ \rho_2 \lambda^2 y + k_2 y_{xx} - k_1 (u_x + y + lz) - i\lambda \mathbb{1}_{\omega_{\epsilon}} y & = v^3, \\ \rho_1 \lambda^2 z + k_3 (z_x - lu)_x - lk_1 (u_x + y + lz) - i\lambda \mathbb{1}_{\omega_{\epsilon}} z & = v^5 \end{cases}$$

with fully Dirichlet boundary conditions:

$$(6.26) \quad u(0) = u(L) = y(0) = y(L) = z(0) = z(L) = 0$$

or with Dirichlet-Neumann-Neumann boundary conditions:

$$(6.27) \quad u(0) = u(L) = y_x(0) = y_x(L) = z_x(0) = z_x(L) = 0$$

verifies the following inequality:

$$(6.28) \quad \int_0^L \left(\rho_1 |\lambda u|^2 + \rho_2 |\lambda y|^2 + \rho_1 |\lambda z|^2 + k_2 |y_x|^2 + k_1 |u_x + y + lz|^2 + k_3 |z_x - lu|^2 \right) dx \leq C \int_0^L (|v^1|^2 + |v^3|^2 + |v^5|^2) dx,$$

where C is a constant independent of n .

Proof. We consider the following Bresse system subject to three local viscous dampings:

$$(6.29) \quad \begin{cases} \rho_1 u_{tt} - k_1 (u_x + y + lz)_x - lk_3 (z_x - lu) + \mathbb{1}_{\omega_\epsilon} u_t & = 0, \\ \rho_2 y_{tt} - k_2 y_{xx} + k_1 (u_x + y + lz) + \mathbb{1}_{\omega_\epsilon} y_t & = 0, \\ \rho_1 z_{tt} - k_3 (z_x - lu)_x + lk_1 (u_x + y + lz) + \mathbb{1}_{\omega_\epsilon} z_t & = 0 \end{cases}$$

with fully Dirichlet or Dirichlet-Neumann-Neumann boundary conditions. Systems (6.29)-(6.26) and (6.29)-(6.27) are well posed in the space $H_1 = (H_0^1(0, L) \times L^2(0, L))^3$ and in the space $H_2 = (H_0^1(0, L) \times L^2(0, L)) \times (H_*^1(0, L) \times L_*^2(0, L))^2$ respectively. In addition, both are exponentially stable (see [38]). Therefore, following Huang [18] and Pruss [34], we deduce that the resolvent of the associated operator:

$$\mathcal{A}_{aux_j} : D(\mathcal{A}_{aux_j}) \subset H_j \rightarrow H_j$$

defined by

$$D(\mathcal{A}_{aux_1}) = (H_0^1(\Omega) \cap H^2(\Omega))^3 \times (H_0^1(\Omega))^3,$$

$$D(\mathcal{A}_{aux_2}) = \{U \in H_2 : u \in H_0^1 \cap H^2, y, z \in H_*^1 \cap H^2, \tilde{u}, y_x, z_x \in H_0^1, \tilde{y}, \tilde{z} \in H_*^1\}$$

and

$$\mathcal{A}_{aux_j} \begin{pmatrix} u \\ \tilde{u} \\ y \\ \tilde{y} \\ z \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \rho_1^{-1} [k_1 (u_x + y + lz)_x + lk_3 (z_x - lu) - \mathbb{1}_{\omega_\epsilon} \tilde{u}] \\ \tilde{y} \\ \rho_2^{-1} [k_2 y_{xx} - k_1 (u_x + y + lz) - \mathbb{1}_{\omega_\epsilon} \tilde{y}] \\ \tilde{z} \\ \rho_1^{-1} [k_3 (z_x - lu)_x - lk_1 (u_x + z + lz) - \mathbb{1}_{\omega_\epsilon} \tilde{z}] \end{pmatrix}$$

is uniformly bounded on the imaginary axis. So, by setting $\tilde{u} = i\lambda u$, $\tilde{y} = i\lambda y$ and $\tilde{z} = i\lambda z$, we deduce that:

$$\begin{pmatrix} u \\ \tilde{u} \\ y \\ \tilde{y} \\ z \\ \tilde{z} \end{pmatrix} = (i\lambda - \mathcal{A}_{aux_j})^{-1} \begin{pmatrix} 0 \\ \frac{-1}{\rho_1} v^1 \\ 0 \\ \frac{-1}{\rho_2} v^3 \\ 0 \\ \frac{-1}{\rho_1} v^1 \end{pmatrix}.$$

This yields:

$$(6.30) \quad \begin{aligned} \|(u, \tilde{u}, y, \tilde{y}, z, \tilde{z})\|_{H_j}^2 &\leq \|(i\lambda - \mathcal{A}_{aux_j})^{-1}\|_{\mathcal{L}(H_j)} \|(0, \frac{-1}{\rho_1} v^1, 0, \frac{-1}{\rho_2} v^3, 0, \frac{-1}{\rho_1} v^1)\|_{H_j} \\ &\leq C \int_0^L (|v^1|^2 + |v^3|^2 + |v^5|^2) dx, \end{aligned}$$

where C is a constant independent of n . Consequently, (6.28) holds. The proof is thus complete. \square

Lemma 6.7. *Under all the above assumptions, we have:*

$$(6.31) \quad \int_0^L |\lambda v^1|^2 dx = o(1), \quad \int_0^L |\lambda v^3|^2 dx = o(1), \quad \int_0^L |\lambda v^5|^2 dx = o(1).$$

Proof. For clarity of the proof, we divide the proof into several steps.

Step 1. First, multiplying (6.5) by $i\rho_1\lambda\bar{u}$, where u is a solution of system (6.25), we get:

$$(6.32) \quad - \int_0^L \rho_1 \lambda^2 \bar{u} v^1 dx - i \int_0^L \rho_1 \lambda \bar{u} v^2 dx = \rho_1 \int_0^L \frac{if^1}{\lambda} \bar{u} dx.$$

Moreover, multiplying (6.6) by \bar{u} and integrating by parts, we obtain:

$$(6.33) \quad \begin{aligned} & i \int_0^L \rho_1 \lambda \bar{u} v^2 dx - \int_0^L k_1 \bar{u}_{xx} v^1 dx - \int_0^L k_3 (-l\bar{u}) v^1 dx + \int_0^L k_1 \bar{u}_x v^3 dx + \int_0^L k_1 \bar{u}_x v^5 dx \\ & + \int_0^L k_3 \bar{u}_x v^5 dx + \int_0^L D_1 (v_x^2 + v^4 + lv^6) \bar{u}_x dx - \int_0^L l D_3 (v_x^6 - lv^2) \bar{u} dx = \rho_1 \int_0^L \frac{f^2}{\lambda^2} \bar{u} dx. \end{aligned}$$

Now, combining (6.32) and (6.33), we get:

$$(6.34) \quad \begin{aligned} & \int_0^L [\rho_1 \lambda^2 \bar{u} + k_1 \bar{u}_{xx} + k_3 (-l\bar{u})] v^1 dx - \int_0^L k_1 \bar{u}_x v^3 dx - \int_0^L k_1 \bar{u}_x v^5 dx - \int_0^L k_3 \bar{u}_x v^5 dx \\ & - \int_0^L D_1 (v_x^2 + v^4 + lv^6) \bar{u}_x dx + \int_0^L l D_3 (v_x^6 - lv^2) \bar{u} dx = -\rho_1 \int_0^L \left(\frac{if^1}{\lambda} + \frac{f^2}{\lambda^2} \right) \bar{u} dx. \end{aligned}$$

Step 2. Similarly to Step 1, multiplying (6.7) by $i\rho_2\lambda\bar{y}$ and (6.8) by \bar{y} , where y is a solution of system (6.25), we get:

$$(6.35) \quad \begin{aligned} & \int_0^L [\rho_2 \lambda^2 \bar{y} + k_2 \bar{y}_{xx} - k_1 \bar{y}] v^3 dx + \int_0^L k_1 \bar{y}_x v^1 dx - \int_0^L k_1 \bar{y} v^5 dx \\ & - \int_0^L D_2 v_x^4 \bar{y}_x dx - \int_0^L D_1 (v_x^2 + v^4 + lv^6) \bar{y} dx = -\rho_2 \int_0^L \left(\frac{if^3}{\lambda} + \frac{f^4}{\lambda^2} \right) \bar{y} dx. \end{aligned}$$

Step 3. As in Step 1 and Step 2, by multiplying (6.9) by $i\rho_1\lambda\bar{z}$ and (6.10) by \bar{z} , where z is a solution of system (6.25), we get:

$$(6.36) \quad \begin{aligned} & \int_0^L [\rho_1 \lambda^2 \bar{z} + k_3 \bar{z}_{xx} - k_1 (l\bar{z})] v^5 dx + \int_0^L k_3 \bar{z}_x v^1 dx + \int_0^L k_1 \bar{z}_x v^1 dx \\ & - \int_0^L k_1 \bar{z} v^3 dx - \int_0^L D_3 (v_x^6 - lv^2) \bar{z}_x dx - l \int_0^L D_1 (v_x^2 + v^4 + lv^6) \bar{z} dx = -\rho_1 \int_0^L \left(\frac{if^5}{\lambda} + \frac{f^6}{\lambda^2} \right) \bar{z} dx. \end{aligned}$$

Step 4. First, combining (6.34), (6.35) and (6.36), we obtain:

$$(6.37) \quad \begin{aligned} & \int_0^L [\rho_1 \lambda^2 \bar{u} + k_1 (\bar{u}_x + \bar{y} + l\bar{z})_x + k_3 (\bar{z}_x - l\bar{u})] v^1 dx + \int_0^L [\rho_2 \lambda^2 \bar{y} + k_2 \bar{y}_{xx} - k_1 (\bar{u}_x + \bar{y} + l\bar{z})] v^3 dx \\ & + \int_0^L [\rho_1 \lambda^2 \bar{z} + k_3 (\bar{z}_x - l\bar{u})_x - k_1 (\bar{u}_x + \bar{y} + l\bar{z})] v^5 dx - \int_0^L D_1 (v_x^2 + v^4 + lv^6) \bar{u}_x dx \\ & + \int_0^L l D_3 (v_x^6 - lv^2) \bar{u} dx - \int_0^L D_2 v_x^4 \bar{y}_x dx - \int_0^L D_1 (v_x^2 + v^4 + lv^6) \bar{y} dx - \int_0^L D_3 (v_x^6 - lv^2) \bar{z}_x dx \\ & - l \int_0^L D_1 (v_x^2 + v^4 + lv^6) \bar{z} dx = -\rho_1 \int_0^L \left(\frac{if^1}{\lambda} + \frac{f^2}{\lambda^2} \right) \bar{u} dx - \rho_2 \int_0^L \left(\frac{if^3}{\lambda} + \frac{f^4}{\lambda^2} \right) \bar{y} dx \\ & - \rho_1 \int_0^L \left(\frac{if^5}{\lambda} + \frac{f^6}{\lambda^2} \right) \bar{z} dx. \end{aligned}$$

Combining equation (6.25) and (6.37), multiplying by λ^2 , we get:

$$(6.38) \quad \begin{aligned} & \int_0^L |\lambda v^1|^2 dx + \int_0^L |\lambda v^3|^2 dx + \int_0^L |\lambda v^5|^2 dx = i \int_{\alpha+\epsilon}^{\beta-\epsilon} (\lambda^2 \bar{u} \lambda v^1 dx + \lambda^2 \bar{y} \lambda v^3 + \lambda^2 \bar{z} \lambda v^5) dx \\ & + \int_0^L \lambda D_1 (v_x^2 + v^4 + lv^6) \lambda \bar{u}_x dx - \int_0^L l D_3 (v_x^6 - lv^2) \lambda^2 \bar{u} dx + \int_0^L \lambda D_2 v_x^4 \lambda \bar{y}_x dx \\ & + \int_0^L D_1 (v_x^2 + v^4 + lv^6) \lambda^2 \bar{y} dx + \int_0^L \lambda D_3 (v_x^6 - lv^2) \lambda \bar{z}_x dx + l \int_0^L D_1 (v_x^2 + v^4 + lv^6) \lambda^2 \bar{z} dx \\ & - \rho_1 \int_0^L (if^1 \lambda + f^2) \bar{u} dx - \rho_2 \int_0^L (if^3 \lambda + f^4) \bar{y} dx - \rho_1 \int_0^L (if^5 \lambda + f^6) \bar{z} dx. \end{aligned}$$

Using estimates (6.20) and the fact that $\lambda^2 u$, $\lambda^2 y$ and $\lambda^2 z$ are uniformly bounded in $L^2(0, L)$ due to (6.28), we get:

$$(6.39) \quad i \int_{\alpha+\epsilon}^{\beta-\epsilon} (\lambda^2 \bar{u} \lambda v^1 dx + \lambda^2 \bar{y} \lambda v^3 + \lambda^2 \bar{z} \lambda v^5) dx = \frac{o(1)}{\lambda^{1/2}}.$$

In addition, using (6.12) and the fact that λu_x , λy_x and λz_x are uniformly bounded in $L^2(0, L)$ due to (6.28), we get:

$$(6.40) \quad \int_0^L \lambda D_1 (v_x^2 + v^4 + lv^6) \lambda \bar{u}_x dx + \int_0^L \lambda D_2 v_x^4 \lambda \bar{y}_x dx + \int_0^L \lambda D_3 (v_x^6 - lv^2) \lambda \bar{z}_x dx = o(1).$$

Also, by using (6.12) and the fact that $\lambda^2 u$, $\lambda^2 y$ and $\lambda^2 z$ are uniformly bounded in $L^2(0, L)$ due to (6.28), we obtain:

$$(6.41) \quad \int_0^L 1 D_3 (v_x^6 - lv^2) \lambda^2 \bar{u} dx + \int_0^L D_1 (v_x^2 + v^4 + lv^6) \lambda^2 \bar{y} dx + 1 \int_0^L D_1 (v_x^2 + v^4 + lv^6) \lambda^2 \bar{z} dx = \frac{o(1)}{\lambda}.$$

Moreover, we have:

$$(6.42) \quad -\rho_1 \int_0^L (i f^1 \lambda + f^2) \bar{u} dx - \rho_2 \int_0^L (i f^3 \lambda + f^4) \bar{y} dx - \rho_1 \int_0^L (i f^5 \lambda + f^6) \bar{z} dx = o(1),$$

since f^1, f^3, f^5 converges to zero in $H_0^1(0, L)$ (or in $H_*^1(0, L)$), f^2, f^4, f^6 converges to zero in $L^2(0, L)$, and $\lambda^2 u, \lambda^2 y, \lambda^2 z$ are uniformly bounded in $L^2(0, L)$.

Finally, inserting (6.39) - (6.42) into (6.38), we get: the desired estimate (6.31). Thus the proof is complete. \square

Lemma 6.8. *Under all the above assumptions, we have:*

$$(6.43) \quad \int_0^L |v_x^1|^2 dx = o(1), \quad \int_0^L |v_x^3|^2 dx = o(1), \quad \int_0^L |v_x^5|^2 dx = o(1).$$

Proof. First, multiplying (6.6) by \bar{v}^1 and then integrating by parts, we get:

$$(6.44) \quad i \int_0^L \rho_1 \lambda v^2 \bar{v}^1 dx + k_1 \int_0^L |v_x^1|^2 dx + k_1 \int_0^L (v^3 + lv^5) \bar{v}_x^1 dx + \int_0^L D_1 (v_x^2 + v^4 + lv^6) \bar{v}_x^1 dx \\ - 1 k_3 \int_0^L (v_x^5 - lv^1) \bar{v}^1 dx - 1 \int_0^L D_3 (v_x^6 - lv^2) \bar{v}^1 dx = \rho_1 \int_0^L \frac{f^2}{\lambda^2} \bar{v}^1 dx.$$

Then, using (6.11), (6.12) and the fact that $v_x^1, (v_x^5 - lv^1)$ are uniformly bounded in $L^2(0, L)$ due to (6.3), we obtain:

$$(6.45) \quad k_1 \int_0^L (v^3 + lv^5) \bar{v}_x^1 dx + \int_0^L D_1 (v_x^2 + v^4 + lv^6) \bar{v}_x^1 dx \\ - 1 k_3 \int_0^L (v_x^5 - lv^1) \bar{v}^1 dx - 1 \int_0^L D_3 (v_x^6 - lv^2) \bar{v}^1 dx = o(1).$$

As f^2 converges to zero in $L^2(0, L)$ and λv^1 is uniformly bounded in $L^2(0, L)$, we have:

$$(6.46) \quad \rho_1 \int_0^L \frac{f^2}{\lambda^2} \bar{v}^1 dx = o(1).$$

Next, inserting (6.45) and (6.46) into (6.44), we get:

$$(6.47) \quad i \int_0^L \rho_1 \lambda v^2 \bar{v}^1 dx + k_1 \int_0^L |v_x^1|^2 dx = o(1).$$

Using Lemma 6.7 and the fact that v^2 is uniformly bounded in $L^2(0, L)$ due to (6.47), we deduce=

$$\int_0^L |v_x^1|^2 dx = o(1).$$

Similarly, one can prove that:

$$\int_0^L |v_x^3|^2 dx = o(1), \quad \int_0^L |v_x^5|^2 dx = o(1).$$

Thus, the proof is complete. \square

Proof of Theorem 6.1 Using Lemma 6.7 and Lemma 6.8, we get: that $\|U\|_{\mathcal{H}_j} = o(1)$. Therefore, we get a contradiction with (6.3) and consequently (H3) holds. Thus the proof is complete. \square

7. POLYNOMIAL STABILITY: THE CASE OF ONLY ONE LOCAL VISCOELASTIC DAMPING WITH NON SMOOTH COEFFICIENT AT THE INTERFACE

In control theory, it is important to reduce the number of control such as damping terms. So, this section is devoted to show the polynomial stability of systems (1.1)-(1.2) and (1.1)-(1.3) subject to only one viscoelastic Kelvin-Voigt damping with non smooth coefficient at the interface. For this purpose, we consider the following condition:

$$(7.1) \quad D_1 = D_3 = 0 \text{ in } (0, L) \quad \text{and} \quad \exists d_0 > 0 \text{ such that } D_2 \geq d_0 > 0 \text{ in } \emptyset \neq (\alpha, \beta) \subset (0, L).$$

The main result of this section is given by the following theorem:

Theorem 7.1. *Assume that condition (7.1) is satisfied. Then, there exists a positive constant $c > 0$ such that for all $U_0 \in D(\mathcal{A}_j)$, $j = 1, 2$, the energy of system (1.1) satisfies the following decay rate:*

$$(7.2) \quad E(t) \leq \frac{c}{\sqrt{t}} \|U_0\|_{D(\mathcal{A}_j)}^2.$$

Referring to [4], (7.2) is verified if the following conditions

$$(H1) \quad i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$$

and

$$(H4) \quad \lim_{|\lambda| \rightarrow +\infty} \sup_{\lambda \in \mathbb{R}} \left\{ \frac{1}{\lambda^4} \left\| (i\lambda I - \mathcal{A}_j)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_j)} \right\} = O(1)$$

hold.

Condition $i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$ is already proved in Lemma 3.2 and Lemma 3.3.

We will establish (H4) by contradiction. Suppose that there exist a sequence of real numbers $(\lambda_n)_n$, with $|\lambda_n| \rightarrow +\infty$ and a sequence of vectors

$$(7.3) \quad U_n = (v_n^1, v_n^2, v_n^3, v_n^4, v_n^5, v_n^6)^\top \in D(\mathcal{A}_j) \quad \text{with} \quad \|U_n\|_{\mathcal{H}_j} = 1$$

such that

$$(7.4) \quad \lambda_n^4 (i\lambda_n U_n - \mathcal{A}_j U_n) = (f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6)^\top \rightarrow 0 \text{ in } \mathcal{H}_j, \quad j = 1, 2.$$

We will check the condition (H4) by finding a contradiction with (7.3)-(7.4) such as $\|U_n\|_{\mathcal{H}_j} = o(1)$.

Equation (7.4) is detailed as:

$$(7.5) \quad i\lambda_n v_n^1 - v_n^2 = \frac{f_n^1}{\lambda_n^4},$$

$$(7.6) \quad i\rho_1 \lambda_n v_n^2 - k_1 [(v_n^1)_x + v_n^3 + lv_n^5]_x - lk_3 [(v_n^5)_x - lv_n^1] = \rho_1 \frac{f_n^2}{\lambda_n^4},$$

$$(7.7) \quad i\lambda_n v_n^3 - v_n^4 = \frac{f_n^3}{\lambda_n^4},$$

$$(7.8) \quad i\rho_2 \lambda_n v_n^4 - [k_2 (v_n^3)_x + D_2 (v_n^4)_x]_x + k_1 [(v_n^1)_x + v_n^3 + lv_n^5] = \rho_2 \frac{f_n^4}{\lambda_n^4},$$

$$(7.9) \quad i\lambda_n v_n^5 - v_n^6 = \frac{f_n^5}{\lambda_n^4},$$

$$(7.10) \quad i\rho_1 \lambda_n v_n^6 - [k_3 ((v_n^5)_x - lv_n^1)]_x + lk_1 [(v_n^1)_x + v_n^3 + lv_n^5] = \rho_1 \frac{f_n^6}{\lambda_n^4}.$$

From (7.5), (7.7), (7.9) and (7.3), we deduce that:

$$(7.11) \quad \|v_n^1\| = O\left(\frac{1}{\lambda_n}\right), \quad \|v_n^3\| = O\left(\frac{1}{\lambda_n}\right), \quad \|v_n^5\| = O\left(\frac{1}{\lambda_n}\right).$$

For clarity, we divide the proof into several lemmas. From now on, for simplicity, we drop the index n .

Lemma 7.2. *Under all the above assumptions, we have:*

$$(7.12) \quad \|D_2^{1/2} v_x^4\| = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \|v_x^4\| = \frac{o(1)}{\lambda^2} \quad \text{in } (\alpha, \beta).$$

Proof. Taking the inner product of (7.4) with U in \mathcal{H}_j , we get:

$$(7.13) \quad \operatorname{Re} \left(i\lambda^5 \|U\|^2 - \lambda^4 (\mathcal{A}_j U, U) \right)_{\mathcal{H}_j} = -\lambda^4 \operatorname{Re} (\mathcal{A}_j U, U)_{\mathcal{H}_j} = \lambda^4 \int_0^L D_2 |v_x^4|^2 dx = o(1).$$

Thanks to (7.1), we obtain the desired asymptotic equation (7.12). Thus the proof is complete. \square

Remark 7.3. *Again, these estimates are crucial for the rest of the proof and they will be used to prove each point of the global proof divided in several lemmas.*

Lemma 7.4. *Under all the above assumptions, we have:*

$$(7.14) \quad \|v_x^3\| = \frac{o(1)}{\lambda^3} \quad \text{in } (\alpha, \beta).$$

Proof. Differentiating equation (7.7), we get:

$$i\lambda v_x^3 = v_x^4 + \frac{f_x^3}{\lambda^4},$$

and consequently

$$\int_{\alpha}^{\beta} |\lambda v_x^3|^2 dx \leq 2 \int_{\alpha}^{\beta} |v_x^4|^2 dx + 2 \int_{\alpha}^{\beta} \frac{|f_x^3|^2}{\lambda^8} dx.$$

Using (7.12) and the fact that f^3 converges to zero in $H_0^1(0, L)$ (or in $H_*^1(0, L)$) in the above equation, we get: the desired estimate (7.14). Thus the proof is complete. \square

Let ϵ be a positive constant such that $0 < \alpha + \epsilon < \beta - \epsilon$. We define the cut-off function η by

$$\eta(x) = 1 \text{ in } (\alpha + \epsilon, \beta - \epsilon), \quad 0 \leq \eta(x) \leq 1, \quad \eta(x) = 0 \text{ in } (0, 1) \setminus (\alpha, \beta).$$

Lemma 7.5. *Under all the above assumptions, we have:*

$$(7.15) \quad \|\sqrt{\eta} \lambda v^4\| = O(1).$$

Proof. First, multiplying (7.8) by $i\rho_2^{-1} \eta \lambda \overline{v^4}$ and after integrating by parts, we get:

$$(7.16) \quad \int_0^L \eta |\lambda v^4|^2 dx = \operatorname{Re} \left\{ \rho_2^{-1} \int_0^L k_2 i \lambda v_x^3 (\eta \overline{v_x^4} + \eta' \overline{v^4}) dx + \rho_2^{-1} \int_0^L D_2 v_x^4 i \lambda (\eta \overline{v_x^4} + \eta' \overline{v^4}) dx \right. \\ \left. + \rho_2^{-1} \int_0^L k_1 (v_x^1 + v^3 + lv^5) \eta i \lambda \overline{v^4} dx - i \int_0^L \frac{f^4}{\lambda^3} \eta \overline{v^4} dx \right\}.$$

Now, using (7.12), (7.14) and the fact that v^4 is uniformly bounded in $L^2(0, L)$, we get:

$$(7.17) \quad \operatorname{Re} \left\{ \rho_2^{-1} \int_0^L k_2 i \lambda v_x^3 (\eta \overline{v_x^4} + \eta' \overline{v^4}) dx \right\} = \frac{o(1)}{\lambda^2}.$$

Moreover, using (7.12) and the fact that v^4 is uniformly bounded in $L^2(0, L)$, we obtain:

$$(7.18) \quad \operatorname{Re} \left\{ \rho_2^{-1} \int_0^L D_2 v_x^4 i \lambda (\eta \overline{v_x^4} + \eta' \overline{v^4}) dx \right\} = \frac{o(1)}{\lambda}.$$

Then, using Young's inequality and the fact that $v_x^1 + v^3 + lv^5$ is uniformly bounded in $L^2(0, L)$, we get for $\epsilon_1 > 0$

$$(7.19) \quad \operatorname{Re} \left\{ \rho_2^{-1} \int_0^L k_1 (v_x^1 + v^3 + lv^5) \eta i \lambda \overline{v^4} dx \right\} \leq \frac{1}{\epsilon_1} \int_0^L k_1^2 \eta |v_x^1 + v^3 + lv^5|^2 dx + \epsilon_1 \int_0^L \eta |\lambda v^4|^2 dx \\ \leq \epsilon_1 \int_0^L \eta |\lambda v^4|^2 dx + O(1).$$

Also, using the fact that f^4 converges to zero in $L^2(0, L)$ and v^4 is uniformly bounded in $L^2(0, L)$, we get:

$$(7.20) \quad \operatorname{Re} \left\{ \int_0^L \frac{f^4}{\lambda^3} \eta \overline{v^4} dx \right\} = \frac{o(1)}{\lambda^3}.$$

Finally, inserting (7.17), (7.18), (7.19) and (7.20) into (7.16), we get:

$$(1 - \epsilon_1) \int_0^L \eta |\lambda v^4|^2 dx \leq O(1).$$

Consequently for $\epsilon_1 = \frac{1}{2}$, we get the desired estimate (7.15). Thus the proof is complete. \square

Lemma 7.6. *Under all the above assumptions, we have:*

$$(7.21) \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^3|^2 dx = o(1).$$

Proof. First, multiplying (7.7) by $i\eta\lambda\overline{v^3}$, we get:

$$(7.22) \quad - \int_0^L \eta |\lambda v^3|^2 dx - i \int_0^L \eta \lambda v^4 \overline{v^3} dx = \int_0^L i\eta \frac{f^3}{\lambda^3} \overline{v^3} dx.$$

Multiplying (7.8) by $\rho_2^{-1}\eta\overline{v^3}$, we obtain after integrating by parts:

$$(7.23) \quad \begin{aligned} & i \int_0^L \eta \lambda v^4 \overline{v^3} dx + \rho_2^{-1} \int_0^L (k_2 v_x^3 + D_2 v_x^4) (\eta' \overline{v^3} + \eta \overline{v_x^3}) dx \\ & + \rho_2^{-1} \int_0^L k_1 (v_x^1 + v^3 + lv^5) \eta \overline{v^3} dx = \int_0^L \frac{f^4}{\lambda^4} \eta \overline{v^3} dx. \end{aligned}$$

Now, combining (7.22) and (7.23), we get:

$$(7.24) \quad \begin{aligned} \int_0^L \eta |\lambda v^3|^2 dx &= \rho_2^{-1} \int_0^L (k_2 v_x^3 + D_2 v_x^4) (\eta' \overline{v^3} + \eta \overline{v_x^3}) dx + \rho_2^{-1} \int_0^L k_1 (v_x^1 + v^3 + lv^5) \eta \overline{v^3} dx \\ &- \int_0^L i\eta \frac{f^3}{\lambda^3} \overline{v^3} dx - \int_0^L \frac{f^4}{\lambda^4} \eta \overline{v^3} dx. \end{aligned}$$

Then, using (7.12) and (7.14), $\|v^3\| = O(\frac{1}{\lambda})$, the fact that $(v_x^1 + v^3 + lv^5)$ is uniformly bounded in $L^2(0, L)$ and the fact that f^3, f^4 converge to zero in $H_0^1(0, L)$ (or in $H_*^1(0, L)$), $L^2(0, L)$ respectively, we deduce that:

$$(7.25) \quad \begin{aligned} & \rho_2^{-1} \int_0^L (k_2 v_x^3 + D_2 v_x^4) (\eta' \overline{v^3} + \eta \overline{v_x^3}) dx + \rho_2^{-1} \int_0^L k_1 (v_x^1 + v^3 + lv^5) \eta \overline{v^3} dx \\ & - \int_0^L i\eta \frac{f^3}{\lambda^3} \overline{v^3} dx - \int_0^L \frac{f^4}{\lambda^4} \eta \overline{v^3} dx = o(1). \end{aligned}$$

Finally, inserting (7.25) into (7.24) and using the definition of η , we deduce:

$$\int_0^L \eta |\lambda v^3|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^3|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 7.7. *Under all the above assumptions, we have:*

$$(7.26) \quad \int_0^L \eta |v_x^1|^2 dx = \frac{o(1)}{\lambda^2}, \quad \int_0^L \eta |\lambda v^1|^2 dx = \frac{o(1)}{\lambda^2}$$

and

$$(7.27) \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |v_x^1|^2 dx = \frac{o(1)}{\lambda^2}, \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^1|^2 dx = \frac{o(1)}{\lambda^2}.$$

Proof. For the clarity of the proof, we divide the proof into several steps.

Step 1. Our first aim here is to prove:

$$(7.28) \quad \int_0^L \eta |v_x^1|^2 dx = o(1).$$

For this sake, multiplying (7.8) by $\eta\overline{v_x^1}$ and integrating by parts, we get:

$$(7.29) \quad \begin{aligned} & -i \int_0^L \lambda \rho_2 v^4 \eta' \overline{v_x^1} dx - i \int_0^L \lambda \rho_2 v_x^4 \eta \overline{v_x^1} dx + \int_0^L (k_2 v_x^3 + D_2 v_x^4) (\eta \overline{v_x^1}) dx + \int_0^L (k_2 v_x^3 + D_2 v_x^4) (\eta' \overline{v_x^1}) dx \\ & + \int_0^L \eta k_1 |v_x^1|^2 dx + \int_0^L \eta k_1 v^3 \overline{v_x^1} dx + \int_0^L k_1 \eta v^5 \overline{v_x^1} dx = \int_0^L \rho_2 \frac{f^4}{\lambda^4} \eta \overline{v_x^1} dx. \end{aligned}$$

Now, we need to estimate each term of (7.29):

• Using (7.11) and (7.15), we get:

$$(7.30) \quad -i \int_0^L \lambda \rho_2 v^4 \eta' \overline{v_x^1} dx = o(1).$$

- Using (7.12) and the fact that λv^1 is uniformly bounded in $L^2(0, L)$, we obtain:

$$(7.31) \quad -i \int_0^L \lambda \rho_2 v_x^4 \eta \overline{v^1} dx = \frac{o(1)}{\lambda^2}.$$

- From (7.6), we remark that $\frac{1}{\lambda} v_{xx}^1$ is uniformly bounded in $L^2(0, L)$. This fact combined with (7.12) and (7.14) yields:

$$(7.32) \quad \int_0^L (k_2 \lambda v_x^3 + D_2 \lambda v_x^4) (\eta \frac{\overline{v_{xx}^1}}{\lambda}) dx = \frac{o(1)}{\lambda}.$$

- Using (7.12), (7.14) and the fact that v_x^1 is uniformly bounded in $L^2(0, L)$, we get:

$$(7.33) \quad \int_0^L (k_2 v_x^3 + D_2 v_x^4) (\eta' \overline{v_x^1}) dx = \frac{o(1)}{\lambda^2}.$$

- Using (7.11) and the fact that v_x^1 is uniformly bounded in $L^2(0, L)$, we obtain:

$$(7.34) \quad \int_0^L \eta k_1 v^3 \overline{v_x^1} dx + \int_0^L \eta k_1 \eta v^5 \overline{v_x^1} dx = o(1).$$

- Using the fact that f^4 converges to zero in $L^2(0, L)$ and v_x^1 is uniformly bounded in $L^2(0, L)$, we get:

$$(7.35) \quad \int_0^L \rho_2 \frac{f^4}{\lambda^4} \eta \overline{v_x^1} dx = \frac{o(1)}{\lambda^4}.$$

Finally, inserting equations (7.30)-(7.35) into (7.29) we get: the desired estimate (7.28).

Next, our second aim is to prove:

$$(7.36) \quad \int_0^L \eta |\lambda v^1|^2 dx = o(1).$$

Multiplying (7.5) by $i \eta \lambda \overline{v^1}$, we get:

$$(7.37) \quad - \int_0^L \eta |\lambda v^1|^2 dx - i \int_0^L \eta \lambda \overline{v^1} v^2 dx = i \int_0^L \frac{f^1}{\lambda^3} \eta \overline{v^1} dx.$$

Then, multiplying (7.6) by $\rho_1^{-1} \eta \overline{v^1}$ and integrating by parts, we get:

$$(7.38) \quad \begin{aligned} & i \int_0^L \eta \lambda \overline{v^1} v^2 dx + \rho_1^{-1} \int_0^L k_1 (v_x^1 + v^3 + l v^5) (\eta' \overline{v^1} + \eta \overline{v_x^1}) dx \\ & - \rho_1^{-1} \int_0^L \eta k_3 (v_x^5 - l v^1) \eta \overline{v^1} = \int_0^L \frac{f^2}{\lambda^4} \eta \overline{v^1} dx. \end{aligned}$$

Combining (7.37) and (7.38), we get:

$$(7.39) \quad \begin{aligned} \int_0^L \eta |\lambda v^1|^2 dx &= \rho_1^{-1} \int_0^L k_1 (v_x^1 + v^3 + l v^5) (\eta' \overline{v^1} + \eta \overline{v_x^1}) dx - \rho_1^{-1} \int_0^L \eta k_3 (v_x^5 - l v^1) \eta \overline{v^1} \\ & - \int_0^L \left(\frac{f^2}{\lambda^4} + i \frac{f^1}{\lambda^3} \right) \eta \overline{v^1} dx. \end{aligned}$$

Finally, using (7.11), (7.28), the fact that $(v_x^1 + v^3 + l v^5)$, $(v_x^5 - l v^1)$ are uniformly bounded in $L^2(0, L)$ and f^1 , f^2 converge respectively to zero in $H_0^1(0, L)$, $L^2(0, L)$ in the right hand side of the above equation, we deduce that:

$$\int_0^L \eta |\lambda v^1|^2 dx = o(1).$$

Step 2. Our aim here is to prove:

$$(7.40) \quad \int_0^L \eta |v_x^1|^2 dx = \frac{o(1)}{\lambda}$$

and

$$(7.41) \quad \int_0^L \eta |\lambda v^1|^2 dx = \frac{o(1)}{\lambda}.$$

To prove (7.40), multiplying (7.29) by λ , we get:

$$(7.42) \quad \begin{aligned} & -i \int_0^L \lambda \rho_2 v^4 \eta' \lambda \overline{v^1} dx - i \int_0^L \lambda \rho_2 v_x^4 \eta \lambda \overline{v^1} dx + \int_0^L (k_2 \lambda v_x^3 + D_2 \lambda v_x^4) (\eta \overline{v_{xx}^1}) dx \\ & + \int_0^L (k_2 \lambda v_x^3 + D_2 \lambda v_x^4) (\eta' \overline{v_x^1}) dx + \int_0^L \eta k_1 \lambda |v_x^1|^2 dx + \int_0^L \eta k_1 \lambda v^3 \overline{v_x^1} dx \\ & + \int_0^L k_1 \eta \lambda v^5 \overline{v_x^1} dx = \int_0^L \rho_2 \frac{f^4}{\lambda^3} \eta \overline{v_x^1} dx. \end{aligned}$$

Now, we need to estimate each term of (7.42) as follows:

- Using (7.15) and (7.36), we get:

$$(7.43) \quad -i \int_0^L \lambda \rho_2 v^4 \eta' \lambda \overline{v^1} dx = o(1).$$

- Using (7.12) and the fact that λv^1 is uniformly bounded in $L^2(0, L)$, we obtain:

$$(7.44) \quad -i \int_0^L \lambda \rho_2 v_x^4 \eta \lambda \overline{v^1} dx = \frac{o(1)}{\lambda}.$$

- Using the fact $\frac{1}{\lambda} v_{xx}^1$ is uniformly bounded in $L^2(0, L)$ due to (7.6) combined with (7.12) and (7.14) yields:

$$(7.45) \quad \int_0^L (k_2 \lambda^2 v_x^3 + D_2 \lambda^2 v_x^4) (\eta \frac{\overline{v_{xx}^1}}{\lambda}) dx = o(1).$$

- Using (7.12), (7.14) and the fact that v_x^1 is uniformly bounded in $L^2(0, L)$, we get:

$$(7.46) \quad \int_0^L (k_2 \lambda v_x^3 + D_2 \lambda v_x^4) (\eta' \overline{v_x^1}) dx = \frac{o(1)}{\lambda}.$$

- Using (7.28) and the fact that λv^3 and λv^5 are uniformly bounded in $L^2(0, L)$, we obtain:

$$(7.47) \quad \int_0^L \eta k_1 \lambda v^3 \overline{v_x^1} dx + \int_0^L k_1 \eta \lambda v^5 \overline{v_x^1} dx = o(1).$$

- Using the fact that f^4 converges to zero in $L^2(0, L)$ and v_x^1 is uniformly bounded in $L^2(0, L)$, we get:

$$(7.48) \quad \int_0^L \rho_1 \frac{f^4}{\lambda^3} \eta \overline{v_x^1} dx = \frac{o(1)}{\lambda^3}.$$

Finally, inserting equations (7.43)-(7.48) into (7.42), we deduce that:

$$\int_0^L \eta |v_x^1|^2 dx = \frac{o(1)}{\lambda}.$$

On the other hand, our target now is to prove (7.41). For this, multiplying (7.39) by λ , we get:

$$(7.49) \quad \begin{aligned} \int_0^L \eta \lambda |v^1|^2 dx &= \rho_1^{-1} \int_0^L k_1 (v_x^1 + v^3 + l v^5) (\eta' \lambda \overline{v^1} + \eta \lambda \overline{v_x^1}) dx - \rho_1^{-1} \int_0^L k_3 (v_x^5 - l v^1) \eta \lambda \overline{v^1} \\ &\quad - \int_0^L \left(\frac{f^2}{\lambda^3} + i \frac{f^1}{\lambda^2} \right) \eta \overline{v^1} dx. \end{aligned}$$

Using (7.11), (7.36), (7.40), the fact that $(v_x^1 + v^3 + l v^5)$, $(v_x^5 - l v^1)$ are uniformly bounded in $L^2(0, L)$ and the fact that f^1, f^2 converge to zero respectively in $H_0^1(0, L)$, $L^2(0, L)$ in the right hand side of the above equation, we deduce that:

$$\int_0^L \eta \lambda |v^1|^2 dx = o(1).$$

Step 3. Our target is to prove:

$$(7.50) \quad \left\| \eta \frac{v_{xx}^1}{\sqrt{\lambda}} \right\| = O(1).$$

So, multiplying (7.5) by $\eta \sqrt{\lambda}$, we get:

$$(7.51) \quad \eta \sqrt{\lambda} v^2 = i \eta \sqrt{\lambda} \lambda v^1 - \eta \sqrt{\lambda} \frac{f^1}{\lambda^4}.$$

Then, integrating (7.51) over $(0, L)$, we get:

$$(7.52) \quad \int_0^L \eta^2 \lambda |v^2|^2 dx \leq 2 \int_0^L \eta^2 \lambda |\lambda v^1|^2 dx + 2 \int_0^L \eta^2 \frac{|f^1|^2}{\lambda^7} dx.$$

Using (7.41) and the fact that f^1 converges to zero in $H_0^1(0, L)$ in the previous equation, we deduce:

$$(7.53) \quad \|\eta \sqrt{\lambda} v^2\| = o(1).$$

Next, multiplying (7.6) by $\frac{\eta}{\sqrt{\lambda}}$, we get:

$$(7.54) \quad k_1 \eta \frac{v_{xx}^1}{\sqrt{\lambda}} = i \rho_1 \eta \sqrt{\lambda} v^2 - k_1 \frac{\eta}{\sqrt{\lambda}} (v_x^3 + l v_x^5) - l k_3 \frac{\eta}{\sqrt{\lambda}} (v_x^5 - l v^1) - \rho_1 \eta \frac{f^2}{\lambda^4 \sqrt{\lambda}}.$$

Finally, using (7.53), the fact that $v_x^3, v_x^5, (v_x^5 - l v^1)$ are uniformly bounded in $L^2(0, L)$ and f^2 converges to zero in $L^2(0, L)$ in the previous equation, we get: the desired estimate (7.50).

Step 4. Our aim is to prove:

$$(7.55) \quad \int_0^L \eta |v_x^1|^2 dx = \frac{o(1)}{\lambda^{1+1/2}}$$

and

$$(7.56) \quad \int_0^L \eta |\lambda v^1|^2 dx = \frac{o(1)}{\lambda^{1+1/2}}.$$

To prove (7.55), multiplying (7.29) by $\lambda^{1+1/2}$, we get:

$$(7.57) \quad \begin{aligned} & -i \int_0^L \lambda \rho_2 v^4 \eta' \lambda^{1+1/2} \overline{v^1} dx - i \int_0^L \lambda^2 \rho_2 v_x^4 \eta \lambda^{1/2} \overline{v^1} dx + \int_0^L (k_2 \lambda^{1+1/2} v_x^3 + D_2 \lambda^{1+1/2} v_x^4) (\eta \overline{v_{xx}^1}) dx \\ & + \int_0^L (k_2 \lambda v_x^3 + D_2 \lambda v_x^4) (\eta' \lambda^{1/2} \overline{v_x^1}) dx + \int_0^L \eta k_1 \lambda^{1+1/2} |v_x^1|^2 dx + \int_0^L \eta k_1 \lambda v^3 \lambda^{1/2} \overline{v_x^1} dx \\ & + \int_0^L l k_1 \eta \lambda v^5 \lambda^{1/2} \overline{v_x^1} dx = \int_0^L \rho_2 \frac{f^4}{\lambda^3} \eta \lambda^{1/2} \overline{v_x^1} dx. \end{aligned}$$

Next, we need to estimate each term of (7.57) as follows:

- Using (7.15) and (7.41), we get:

$$(7.58) \quad -i \int_0^L \lambda \rho_2 v^4 \eta' \lambda^{1+1/2} \overline{v^1} dx = o(1).$$

- Using (7.12) and λv^1 is uniformly bounded in $L^2(0, L)$, we obtain:

$$(7.59) \quad -i \int_0^L \lambda^2 \rho_2 v_x^4 \eta \lambda^{1/2} \overline{v^1} dx = \frac{o(1)}{\lambda^{1/2}}.$$

- Using (7.12), (7.14) and (7.50), we obtain:

$$(7.60) \quad \int_0^L (k_2 \lambda^{1+1/2} v_x^3 + D_2 \lambda^{1+1/2} v_x^4) (\eta \overline{v_{xx}^1}) dx = \int_0^L (k_2 \lambda^2 v_x^3 + D_2 \lambda^2 v_x^4) \left(\eta \frac{\overline{v_{xx}^1}}{\sqrt{\lambda}} \right) = o(1).$$

- Using (7.12), (7.14) and (7.40), we get:

$$(7.61) \quad \int_0^L (k_2 \lambda v_x^3 + D_2 \lambda v_x^4) (\eta' \lambda^{1/2} \overline{v_x^1}) dx = \frac{o(1)}{\lambda}.$$

- Using (7.40) and the fact that λv^3 and λv^5 are uniformly bounded in $L^2(0, L)$, we obtain:

$$(7.62) \quad \int_0^L \eta k_1 \lambda v^3 \lambda^{1/2} \overline{v_x^1} dx + \int_0^L l k_1 \eta \lambda v^5 \lambda^{1/2} \overline{v_x^1} dx = o(1).$$

- Using (7.40) and the fact that f^4 converges to zero in $H_0^1(0, L)$, we get:

$$(7.63) \quad \int_0^L \rho_1 \frac{f^4}{\lambda^3} \eta \lambda^{1/2} \overline{v_x^1} dx = \frac{o(1)}{\lambda^3}.$$

Finally, inserting equations (7.58)-(7.63) into (7.57), we deduce that:

$$\int_0^L \eta |v_x^1|^2 dx = \frac{o(1)}{\lambda^{1+1/2}}.$$

On the other side, our aim now is to prove (7.56). For this sake, multiplying (7.39) by $\lambda^{1+1/2}$, we get:

$$(7.64) \quad \int_0^L \eta \lambda^{1+1/2} |\lambda v^1|^2 dx = \rho_1^{-1} \int_0^L k_1 (v_x^1 + v^3 + lv^5) (\eta' \lambda^{1+1/2} \overline{v^1} + \eta \lambda^{1+1/2} \overline{v_x^1}) dx \\ - \rho_1^{-1} \int_0^L lk_3 (v_x^5 - lv^1) \eta \lambda^{1+1/2} \overline{v^1} - \int_0^L \left(\frac{f^2}{\lambda^{5/2}} + i \frac{f^1}{\lambda^{3/2}} \right) \eta \overline{v^1} dx.$$

Using (7.11), (7.41), (7.55), the fact that $(v_x^1 + v^3 + lv^5)$, $(v_x^5 - lv^1)$ are uniformly bounded in $L^2(0, L)$ and the fact that f^1, f^2 converge to zero respectively in $H_0^1(0, L)$, $L^2(0, L)$ in the right hand side of the above equation, we deduce that:

$$\int_0^L \eta \lambda^{1+1/2} |\lambda v^1|^2 dx = o(1).$$

Step 5. Using (7.28), (7.40), (7.55) and the definition of η , we deduce:

$$\int_0^L \eta |v_x^1|^2 dx = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |v_x^1|^2 dx = \frac{o(1)}{\lambda^2}.$$

Using (7.36), (7.41), (7.56) and the definition of η , we deduce:

$$\int_0^L \eta |\lambda v^1|^2 dx = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^1|^2 dx = \frac{o(1)}{\lambda^2}.$$

The proof is thus complete. \square

Lemma 7.8. *Under all the above assumptions, we have:*

$$(7.65) \quad \int_0^L \eta |v_x^5|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |v_x^5|^2 dx = o(1)$$

Proof. First, substituting (7.5) into (7.6), we get:

$$(7.66) \quad -\lambda^2 \rho_1 v^1 - k_1 (v_x^1 + v^3 + lv^5)_x - lk_3 (v_x^5 - lv^1) = \rho_1 \left(\frac{f^2}{\lambda^4} + i \frac{f^1}{\lambda^3} \right).$$

Multiplying (7.66) by $\overline{\eta v_x^5}$ and integrating over $(0, L)$, we get:

$$(7.67) \quad (lk_1 + lk_3) \int_0^L \eta |v_x^5|^2 dx = -\rho_1 \int_0^L \eta \lambda^2 v^1 \overline{v_x^5} dx + k_1 \int_0^L v_x^1 \eta' \overline{v_x^5} dx + k_1 \int_0^L \eta \lambda v_x^1 \frac{\overline{v_{xx}^5}}{\lambda} dx \\ - k_1 \int_0^L \eta v_x^3 \overline{v_x^5} dx + l^2 k_3 \int_0^L \eta v^1 \overline{v_x^5} dx + \frac{o(1)}{\lambda^3}.$$

Finally, using (7.14), (7.26), the fact that v_x^5 is uniformly bounded in $L^2(0, L)$ and $\frac{1}{\lambda} v_{xx}^5$ is uniformly bounded in $L^2(0, L)$ due to (7.10) in the right hand side of the previous equation, we get: the desired estimate (7.65). The proof is thus complete. \square

Lemma 7.9. *Under all the above assumptions, we have:*

$$(7.68) \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^5|^2 dx = o(1).$$

Proof. Multiplying (7.9) by $i\eta \lambda \overline{v^5}$, we get:

$$(7.69) \quad - \int_0^L \eta |\lambda v^5|^2 dx - i \int_0^L \eta \lambda \overline{v^5} v^6 dx = i \int_0^L \frac{f^5}{\lambda^3} \eta \overline{v^5} dx.$$

Then, multiplying (7.10) by $\eta \rho_1^{-1} \overline{v^5}$ and integrating by parts, we get:

$$(7.70) \quad i \int_0^L \eta \lambda \overline{v^5} v^6 dx + \rho_1^{-1} \int_0^L k_3 (v_x^5 - lv^1) (\eta' \overline{v^5} + \eta \overline{v_x^5}) dx \\ + \rho_1^{-1} \int_0^L lk_1 (v_x^1 + v^3 + lv^5) \eta \overline{v^5} dx = \int_0^L \frac{f^6}{\lambda^4} \eta \overline{v^5} dx.$$

Combining (7.69) and (7.70), we get:

$$(7.71) \quad \int_0^L \eta |\lambda v^5|^2 dx = \rho_1^{-1} \int_0^L k_3 (v_x^5 - lv^1) (\eta' \overline{v^5} + \eta \overline{v_x^5}) dx + \rho_1^{-1} \int_0^L lk_1 (v_x^1 + v^3 + lv^5) \eta \overline{v^1} \\ - \int_0^L \left(\frac{f^6}{\lambda^4} + i \frac{f^5}{\lambda^3} \right) \eta \overline{v^5} dx.$$

Using (7.11), (7.65), the fact that $(v_x^1 + v^3 + lv^5)$, $(v_x^5 - lv^1)$ are uniformly bounded in $L^2(0, L)$, f^5 , f^6 converge to zero respectively in $H_0^1(0, L)$ (or in $H_*^1(0, L)$), $L^2(0, L)$ in the right hand side of the above equation, we deduce:

$$\int_0^L \eta |\lambda v^5|^2 dx = o(1).$$

Finally, using the definition of η , we get: the desired estimate (7.68). The proof is thus complete. \square

Remark 7.10. *It is easy to see that the results of Lemmas 5.9, 5.10, 5.11 still hold here, and consequently one may get the estimate (5.90) of Lemma 5.12.* \square

Proof of Theorem 7.1 As we mention in Remark 7.10, the estimate (5.90) is also true here. It follows from estimates (7.14), (7.21), (7.26), (7.65), (7.68) and (5.90) that $\|U_n\|_{\mathcal{H}_j} = o(1)$ which is a contradiction with (7.3). Consequently, condition (H4) holds and the energy of smooth solutions of system (1.1) decays polynomially as t goes to infinity. \square

8. LACK OF EXPONENTIAL STABILITY

It was proved that the Bresse system subject to one or two viscous dampings is exponentially stable if and only if the wave propagate at the same speed (see [38] and [7]). In the case of viscoelastic damping, the situation is more delicate. In this section, we prove that the Bresse system (1.1)-(1.3) subject to two global viscoelastic dampings is not exponentially stable even if the waves propagate at same speed. So, we assume that:

$$(8.1) \quad D_1 = 0 \quad \text{and} \quad D_2 = D_3 = 1 \quad \text{in} \quad (0, L).$$

Theorem 8.1. *Under hypothesis (8.1), the Bresse system (1.1)-(1.3), is not exponentially stable in the energy space \mathcal{H}_2 .*

Proof. For the proof of Theorem 8.1, it suffices to show that there exists

- a sequence $(\lambda_n) \subset \mathbb{R}$ with $\lim_{n \rightarrow +\infty} |\lambda_n| = +\infty$, and
- a sequence $(V_n) \subset D(\mathcal{A}_2)$,

such that $(i\lambda_n I - \mathcal{A}_2)V_n$ is bounded in \mathcal{H}_2 and $\lim_{n \rightarrow +\infty} \|V_n\| = +\infty$. For the sake of clarity, we skip the index n .

Let $F = (0, 0, 0, f_4, 0, 0) \in \mathcal{H}_2$ with

$$f_4(x) = \cos\left(\frac{n\pi x}{L}\right), \quad \lambda = \frac{n\pi\sqrt{\rho_2 k_2}}{L\rho_2}, \quad n \in \mathbb{N}.$$

We solve the following equations:

$$(8.2) \quad i\lambda v^1 - v^2 = 0,$$

$$(8.3) \quad i\lambda\rho_1 v^2 - k_1 (v_{xx}^1 + v_x^3 + lv_x^5) - lk_3 (v_x^5 - lv^1) - l(v_x^6 - lv^2) = 0,$$

$$(8.4) \quad i\lambda v^3 - v^4 = 0,$$

$$(8.5) \quad i\lambda\rho_2 v^4 - k_2 v_{xx}^3 + k_1 (v_x^1 + v^3 + lv^5) = \rho_2 f_4,$$

$$(8.6) \quad i\lambda v^5 - v^6 = 0,$$

$$(8.7) \quad i\lambda\rho_1 v^6 - k_3 (v_{xx}^5 - lv_x^1) + lv_x^2 + lk_1 (v_x^1 + v^3 + lv^5) = 0.$$

Eliminating v^2 , v^4 and v^6 in (8.3), (8.5) and (8.7) by (8.2), (8.4) and (8.6), we get:

$$(8.8) \quad \lambda^2 \rho_1 v^1 + k_1 (v_{xx}^1 + v_x^3 + lv_x^5) + l(k_3 + i\lambda) (v_x^5 - lv^1) = 0,$$

$$(8.9) \quad \lambda^2 \rho_2 v^3 + k_2 v_{xx}^3 - k_1 (v_x^1 + v^3 + lv^5) = -\rho_2 f^4,$$

$$(8.10) \quad \lambda^2 \rho_1 v^5 + k_3 (v_{xx}^5 - lv_x^1) - i\lambda lv_x^1 - lk_1 (v_x^1 + v^3 + lv^5) = 0.$$

This can be solved by the ansatz:

$$(8.11) \quad v^1 = A \sin\left(\frac{n\pi x}{L}\right), \quad v^3 = B \cos\left(\frac{n\pi x}{L}\right), \quad v^5 = C \cos\left(\frac{n\pi x}{L}\right)$$

where A , B and C depend on λ are constants to be determined. Notice that $k_2 \left(\frac{n\pi}{L}\right)^2 - \rho_2 \lambda^2 = 0$, and inserting (8.11) in (8.8)-(8.10) we obtain that:

$$(8.12) \quad \left(\left(\frac{n\pi}{L}\right)^2 k_1 - \lambda^2 \rho_1 + (k_3 + i\lambda) l^2 \right) A + k_1 \left(\frac{n\pi}{L}\right) B + (k_1 + k_3 + i\lambda) l \left(\frac{n\pi}{L}\right) C = 0,$$

$$(8.13) \quad k_1 \left(\frac{n\pi}{L}\right) A + k_1 B + l k_1 C = \rho_2,$$

$$(8.14) \quad (k_1 + k_3 + i\lambda) l \left(\frac{n\pi}{L}\right) A + l k_1 B + \left[k_3 \left(\frac{n\pi}{L}\right)^2 - \lambda^2 \rho_1 + l^2 k_1 \right] C = 0.$$

Equivalently,

$$(8.15) \quad \begin{pmatrix} \left(\frac{n\pi}{L}\right)^2 k_1 - \lambda^2 \rho_1 + (k_3 + i\lambda) l^2 & k_1 \left(\frac{n\pi}{L}\right) & (k_1 + k_3 + i\lambda) l \left(\frac{n\pi}{L}\right) \\ k_1 \left(\frac{n\pi}{L}\right) & k_1 & l k_1 \\ (k_1 + k_3 + i\lambda) l \left(\frac{n\pi}{L}\right) & l k_1 & k_3 \left(\frac{n\pi}{L}\right)^2 - \lambda^2 \rho_1 + l^2 k_1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ \rho_2 \\ 0 \end{pmatrix}.$$

This implies that:

$$(8.16) \quad A = \frac{(k_2 \rho_1 - \rho_2 k_3) \rho_2^2 L}{\pi (k_2 \rho_1^2 - k_3 \rho_1 \rho_2 + \rho_2 l^2) k_2 n} + O(n^{-2}),$$

$$(8.17) \quad B = \frac{\rho_2 (k_1 k_3 \rho_2^2 + ((-k_1 - k_3) \rho_1 + l^2) k_2 \rho_2 + k_2^2 \rho_1^2)}{k_1 ((-k_3 \rho_1 + l^2) \rho_2 + k_2 \rho_1^2) k_2} + O(n^{-1}),$$

$$(8.18) \quad C = \frac{i l \rho_2^2 L \sqrt{\rho_2 k_2}}{\pi ((-k_3 \rho_1 + l^2) \rho_2 + k_2 \rho_1^2) k_2 n} + O(n^{-2}).$$

Now, let $V_n = (v^1, i\lambda v^1, v^3, i\lambda v^3, v^5, i\lambda v^5)$, where v^1, v^3 and v^5 are given by (8.11) and (8.16)-(8.18). It is easy to check that

$$\|V_n\|_{\mathcal{H}_2} \geq \sqrt{\rho_2} \|\lambda v^3\| \sim |B\lambda| \sim |n| \rightarrow +\infty \text{ as } n \rightarrow +\infty.$$

On the other hand, using (8.2)-(8.7), we deduce that

$$\|(i\lambda I - \mathcal{A}_2)V_n\|_{\mathcal{H}_2}^2 = \|(0, 0, 0, \rho^2 f^4 - i\lambda D_2 v_{xx}^3, 0, i\lambda D_3 v_{xx}^5)\|_{\mathcal{H}_2}^2 \leq c.$$

Consequently, $\|(i\lambda I - \mathcal{A}_2)V_n\|_{\mathcal{H}_2}^2$ is bounded as n tends to $+\infty$. Thus the proof is complete. \square

Remark 8.2. *By a similar way, we can prove that the Bresse system (1.1)-(1.3) subject to only one viscoelastic damping is also not exponentially stable even if the waves propagate at same speed.* \square

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