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## Hopf-like Bifurcation and Mixed Mode Oscillation in a Fractional-Order FitzHugh-Nagumo Model

Mohammed Salah Abdelouahab <sup>a)</sup> and René Lozi <sup>b)</sup>

### Abstract

In this work we investigate the emergence of mixed-mode oscillations and canard explosion, in a planar fractional order FitzHugh-Nagumo model (FFHN). An algorithm, called Global-Local Canard Explosion Search Algorithm (GLCESA) is developed and used to investigate the existence of canard oscillations in the neighborhoods of Hopf-like bifurcation points. The appearance of various patterns of solutions is revealed, with an increasing number of small-amplitude oscillations when two key parameters of the FFHN model are varied. The numbers of such oscillations versus the two parameters, respectively, are perfectly fitted using exponential functions. Finally, it is conjectured that chaos could occur in a 2-dimensional fractional-order autonomous dynamical system, with a fractional order close to one. After all, the article demonstrates that the FFHN Model is a very simple 2-dimensional model with an incredible ability to present the complex dynamics of neurons.

**Keywords:** FitzHugh-Nagumo model, canard explosion, fractional-order system, mixed-mode oscillation, chaos.

PACS: 2010 Mathematics Subject Classification: 34D15, 34D20, 34C23.

*a)* Laboratory of Mathematics and Their Interactions, Mila University Center, Mila, Algeria, [medsalah3@yahoo.fr](mailto:medsalah3@yahoo.fr)  
*b)* Laboratoire de Mathématiques J.A. Dieudonné, Université Côte d'Azur, Nice, France, [René.lozi@univ-cotedazur.fr](mailto:René.lozi@univ-cotedazur.fr)

### 1. Introduction

The 4-dimensional Hodgkin-Huxley (HH) model of electric circuits, which reproduces fairly the action potential of many types of neurons [8], have several simplifications as a 2-dimensional systems, one this important simplification is the FitzHugh-Nagumo (FHN) system [5]. But due to the Poincaré-Bendixon theorem, a 2-dimensional dynamical systems highlighted by the FHN model is unable to reproduce some complex dynamics of the corresponding 4-dimensional system, such as chaos, hyperchaos and mixed-mode oscillations (MMO), which are very common in electroencephalography (EEG) data. Nevertheless, it is possible to find a resolution to this concerned issue between too simple and too complex systems, using fractional derivatives. The idea of fractional calculus goes back to the early development of the regular calculus [10], as a generalization of integration and differentiation to non-integer orders. There are several definitions of fractional derivatives [13, 4]. A common one is the Riemann–Liouville definition of fractional derivatives [13], given by

$${}^R D_t^\alpha x(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} x(\tau) d\tau = \frac{d^m}{dt^m} ({}_a J_t^{m-\alpha} x(t)), \quad t > a, \quad m-1 \leq \alpha < m$$

Where  $\Gamma$  is the gamma function and  ${}_a J_t^\beta$  is the Riemann-Liouville integral operator defined

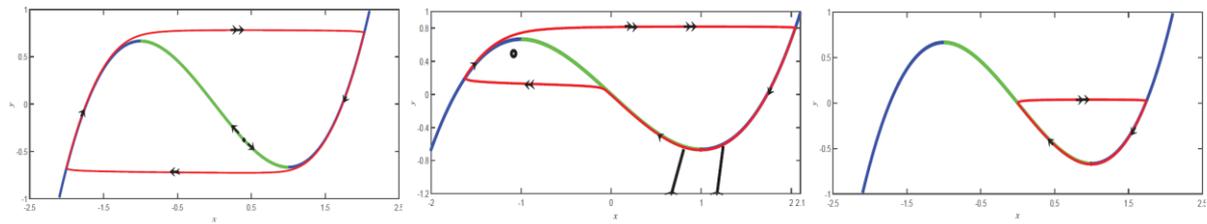
$$\text{as } {}_a J_t^\beta x(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-\tau)^{\beta-1} x(\tau) d\tau.$$

Another definition is the Caputo definition of fractional derivatives [4], given by

$${}_a^c D_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau = j^{n-\alpha} \left( \frac{d^n}{dt^n} x(t) \right), \quad t > a$$

where  $n = \lceil \alpha \rceil$  is the value of  $\alpha$  rounded up to the nearest integer.

In fact, the fractional-order FitzHugh-Nagumo (FFHN) model is introduced in [11]. This article further investigates the MMO and the complex canard explosion in the FFHN model. Specifically, the appearance of patterns, from the solution of the FFHN model with a fractional-order close to one, is studied as one system parameter is varied, where the number of small-amplitude oscillations increases. Such a phenomenon is impossible to appear from a system with the order of derivative being equal to one.



**Figure 1.** Canard explosion of the van der Pol oscillator [17] for  $\varepsilon = 0.01$  happens in an exponentially small parameter interval near  $a = a^* \approx 0.998740451245$ , where the transition from relaxation oscillations for  $a \leq 0.998740451244$  (a), to small amplitude limit cycles for  $a \geq 0.998740451246$  (c), and it happens via canard cycles (b).

## 2. Canard solutions and mixed-mode oscillations

Canard cycles were first discovered and investigated in 1981 by a team of French mathematicians in their pioneering work [3], who coined the French name of *canard* for such unexpected complex dynamical behavior. The conventional canard phenomena highlight the very fast transition (called canard explosion) with respect to a varying parameter, from a large-amplitude limit cycle (relaxation) Fig. 1-a to a small-amplitude one Fig. 1-c, in a slow-fast ODE, which is also referred to as singularly perturbed systems.

**Definition 1** (Canard trajectory) [15].

*A trajectory of a singularly perturbed system, first moving along the attractive branches  $S^a$  and then continuing for a while along the repulsive branch  $S^r$ , is called a canard (or duck) trajectory.*

The canard phenomenon is important for better understanding and analyzing the slow-fast dynamics. For example, the coupling of local passage near a folded singularity, around which

canard solutions emerge, with the global return mechanism via relaxation spikes that resets the local dynamics, can explain complex oscillatory patterns called mixed mode oscillations (MMO). The MMO consists of  $L$  large-amplitude (relaxation) oscillations followed by  $s$  small amplitude (sub-threshold) oscillations, simply denoted by  $L^s$  [14].

### 3. Hopf-Like Bifurcation in the Fractional-Order FitzHugh-Nagumo Model

Using the fractional-order constitutive equations of capacitor and inductor [19, 18] the system introduced in [5], can be transformed to its fractional version as

$$\begin{cases} D^{\alpha_1} x = x - \frac{1}{3} x^3 - y + I, \\ D^{\alpha_2} y = \varepsilon(x + a - by), \end{cases} \quad (1)$$

with parameters  $I$ ,  $a$ ,  $b$  and  $\varepsilon$ , where  $\alpha_1$  and  $\alpha_2$  are constants related to the loss of the capacitor and the proximity effect of the inductor, respectively. Only the case of  $\alpha_1 = \alpha_2 = \alpha \in (0, 2)$  is analyzed in this paper.

For all  $a, b, I \in \mathbb{R}$  satisfying  $-4\left(1 - \frac{1}{b}\right)^3 + 9\left(I - \frac{a}{b}\right)^3 > 0$ , the system (1) has a unique equilibrium point  $E = (x_e(b), y_e(b))$ .

The Jacobian matrix of system (1) at the equilibrium point  $E$  is

$$J_E = \begin{pmatrix} 1 - x_e(b) & -1 \\ \varepsilon & -b\varepsilon \end{pmatrix}$$

For  $(x_e^2(b) - b\varepsilon - 1)^2 < 4\varepsilon$ , the Jacobian matrix  $J_E$  has a pair of complex conjugate

eigenvalues:  $\lambda_{\pm} = \frac{-(x_e^2(b) + b\varepsilon - 1) \pm i\sqrt{-(x_e^2(b) - b\varepsilon - 1)^2 + 4\varepsilon}}{2}$ .

According stability criterion of nonlinear fractional systems [1], the fixed point  $E$  of (1) is

locally asymptotically stable if  $\left| \arctan\left(\frac{\sqrt{-(x_e^2(b) - b\varepsilon - 1)^2 + 4\varepsilon}}{x_e^2(b) + b\varepsilon - 1}\right) \right| > \alpha \frac{\pi}{2}$ .

Since exact periodic solutions are not expected in fractional-order autonomous systems [16], it is natural to introduce a new notion of the Hopf bifurcation that will be meaningful for this kind of systems. The idea is to define Hopf-Like bifurcation (HLB) in fractional-order systems as a local bifurcation, where a fixed point of the underlying dynamical system changes its stability property as a pair of complex conjugate eigenvalues  $\lambda_{\pm}$  of the Jacobian matrix at

the fixed point cross the boundary of an angular sector  $|\arg(\lambda_{\pm})| = \alpha \frac{\pi}{2}$  of the complex plane, giving rise to a small amplitude  $S$ -asymptotically  $T$ -periodic solution [7]. In fact, some criteria

of HLB in fractional-order systems were already introduced in [2], although it was not called ‘‘Hopf-like’’ therein. To analyze HLB in system (1) at its unique fixed point  $E = (x_e(b), y_e(b))$  with respect to the parameter  $b$  and the parameter  $\alpha$ , respectively, a function  $M(b, \alpha)$  is defined as follows:

$$M(b, \alpha) = \alpha \frac{\pi}{2} - \left| \arctan \left( \frac{\sqrt{-(x_e^2(b) - b\varepsilon - 1)^2 + 4\varepsilon}}{x_e^2(b) + b\varepsilon - 1} \right) \right|.$$

Then, some conditions on the parameters to generate HLB are derived.

**Proposition 1** (HLB with respect to the parameter  $b$ ) *Let the fractional order  $\alpha$  be fixed and  $b^*$  be the value of solution  $b$  to  $M(b, \alpha) = 0$ . If  $(x_e^2(b) - b\varepsilon - 1)^2 < 4\varepsilon$  and*

$$\left( 2x_e(b) \frac{dx_e(b)}{db} (b^2\varepsilon - b(x_e^2(b) - 1) - 2) + (x_e^2(b) - 1)^2 - b\varepsilon(x_e^2(b) - 1) - 2\varepsilon \right) \Big|_{b=b^*} \neq 0 \text{ then system (1)}$$

*undergoes an HLB at the unique equilibrium point  $E$ , when  $b = b^*$ .*

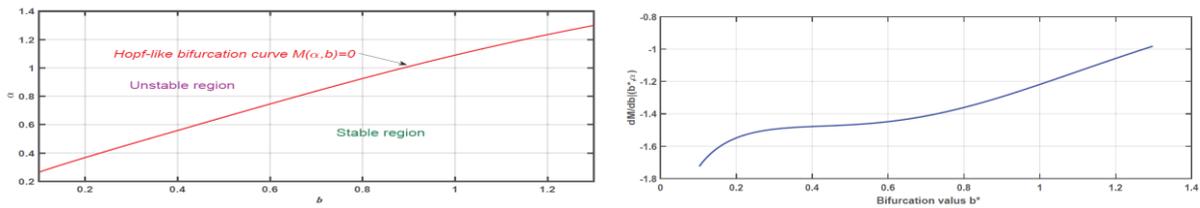
#### 4. Complex canard explosion and mixed-mode oscillations versus parameter $b$

The parameter values are chosen as  $a = 0.75$ ,  $I = 0.41$ ,  $\varepsilon = 0.05$  and the two other parameters  $0 < b \leq \bar{b}$  and  $\alpha \in (0, 2)$  are considered as control parameters, where  $\bar{b} \approx 1.4371898$ . Figure 2-(a) shows the critical curve  $\gamma$  of the following equation:

$$M(b, \alpha) = \alpha \frac{\pi}{2} - \min_i |\arg(\lambda_i(b))| = 0, \text{ which separates stable and unstable regions in the}$$

$(b, \alpha)$  parameter space. Figure 2-(b) shows the curve of the derivative function  $\frac{\partial M(b, \alpha)}{\partial b} \Big|_{b=b^*}$

versus the bifurcation value  $b^*$ , which is strictly negative.



**Figure 2.** (a) HLB curve in the  $(b, \alpha)$  parameter space. (b) Curve of the derivative  $\frac{\partial M(b, \alpha)}{\partial b} \Big|_{b=b^*}$  versus the bifurcation value  $b^*$

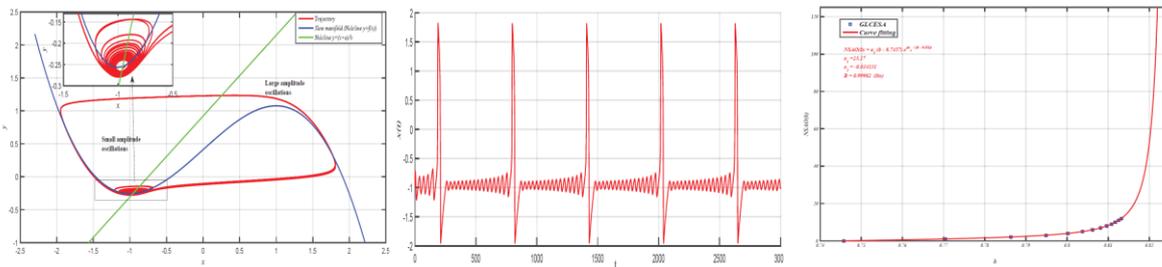
All conditions for HLB [2] are satisfied at each point in the curve  $\gamma$ , implying that when parameters move from stable to unstable regions in the  $(b, \alpha)$  parameter space, the fixed point

$E$  loses its stability near the critical curve  $\mathcal{V}$ . This gives rise to small-amplitude oscillatory behavior and allows the possibility of developing fractional-order canard solutions. To investigate the canard phenomenon in the FFHN model, the theory of singularly perturbed system has been applied.

In order to numerically illustrate the complex canard explosion versus the parameter  $b$ , fix  $\alpha = 0.95$  and consider  $b$  as the bifurcation parameter in interval  $(0, 1.2)$ . Then, HLB occurs at  $b^* \approx 0.83$ , while oscillations can be observed for  $b < 0.83$ . Figure 3 displays the phase portrait and the time evolution of system (1) at  $b = 0.815$ . From this figure, a new phenomenon can be observed that cannot be observed from the integer-order setting. There is an alternation between oscillations of distinct large- and small-amplitude “mixed-mode oscillations”. This phenomenon cannot occur in smooth 2-dimensional autonomous integer-order systems, thanks to the semi-group property of the flow  $\phi$ . This property does not allow any trajectory to cross itself without giving periodic orbits (due to the Cauchy-Lipschitz Theorem). But, this property is not verified in the fractional-order flow because of the memory dependency.

For  $b = 0.7$ , one can only observe large-amplitude oscillations. For  $b = 0.83$ , one can only observe small amplitude oscillations (with amplitude close to zero). When the parameter  $b$  is varied from  $b = 0.7$  to  $b = 0.83$ , the number of small-amplitude oscillations,  $NSAO(b)$ , which occurs between every two successive large-amplitude oscillations, changes from  $NSAO(0.7) = 0$  to  $NSAO(0.83) = +\infty$ .

To localize infinitesimal subintervals for which  $NSAO(b)$  increases by 1, where canard cycle can be developed, a ‘Global Local Canard Explosion Search Algorithm’ (GLCESA) is developed and applied, with two search steps.



**Figure 3.** Mixed-mode oscillations observed in the 2-dimensional fractional-order system (1), with  $b = 0.815$ : (a) Phase portrait. (b) Time evolution of  $x$ . (c) Curve fitting points  $(NSAO(\bar{b}_i), \bar{b}_i)$ , generated using the proposed GLCESA. The fitted function is  $NSAO(b) = 23.27(b - 0.7)e^{(-0.034131/(\alpha - 0.83))}$ .

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