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► To cite this version:

Gerrit Coddens. The exact theory of the Stern-Gerlach experiment and why it does not imply that a fermion can only have its spin up or down. 2020. hal-02882969v3

HAL Id: hal-02882969

<https://hal.science/hal-02882969v3>

Preprint submitted on 15 Jul 2020 (v3), last revised 29 Aug 2020 (v5)

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The exact theory of the Stern-Gerlach experiment and why it does not imply that a fermion can only have its spin up or down

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15th July 2020

Abstract. The Stern-Gerlach [1] experiment is notoriously counter-intuitive. The official explanation has it that the spin remains always aligned with the magnetic field such that the directions of space would be quantized: A fermion can only have its spin up or down. But that theory is based on several blatant mathematical errors in the way it (mis)treats spinors and group theory. We present here a mathematically rigorous theory for a fermion in a magnetic field, which is all but beyond human intuition. It is based on an understanding of spinors in $SU(2)$ which as explained in [2] is only Euclidean geometry. Contrary to what Pauli [3] has been reading into the Stern-Gerlach experiment, the directions of space are not quantized. The new and corrected paradigm, which solves all the conceptual problems, is that the fermions precess around the magnetic-field lines just like Einstein had conjectured. But surprisingly this leads to only two energy states, which should be qualified as precession-up and precession-down rather than spin-up and spin down as has been claimed. Indeed, despite the presence of the many different possible angles θ between the spin axis \mathbf{s} and the magnetic field \mathbf{B} , the fermions can only have two possible energies $m_0 c^2 \pm \mu B$. The values $\pm \mu B$ are at variance with the continuum of values $-\boldsymbol{\mu} \cdot \mathbf{B}$ Einstein had anticipated. What is wrong in what Einstein had expected is that the energy term $V = -\boldsymbol{\mu} \cdot \mathbf{B}$ is a macroscopic quantity. It is a statistical average over a large ensemble of fermions distributed over the two microscopic energy states $\pm \mu B$, and as such not valid for individual fermions. The two fermion states $\pm \mu B$ are not potential-energy states, but they are stable, just like a precessing spinning top without friction in a gravitational field is stable. We also spell out the mathematically rigorous meaning of the up and down spinors. They represent left-handed and right-handed reference frames, such that now everything is intuitively clear and understandable in simple geometrical terms.

PACS. 02.20.-a, 03.65.Ta, 03.65.Ca Group theory, Quantum Mechanics

1 Confusion reigns

1.1 Preamble

In this section¹ we want to point out the total lack of intuition and the total lack of theory which prevail in the traditional presentation of the Stern-Gerlach experiment. We have shown in many examples on many occasions in the past [4] that the “geometry” that corresponds to the algebra of quantum mechanics, is the group theory of the rotation and Lorentz groups. We have in this respect often used the analogy of the correspondence between algebra and geometry in algebraic geometry to explain that the calculus of quantum mechanics, its algebra, is exact but that we do not know what its correct intuitive interpretation, its “geometry” should be. In this respect Villani uses the qualifiers “analytic” for what we call algebraic and “synthetic” for what we call “geometric” [5]. Perhaps this terminology is more accurate than ours. In this paper we will prove once more that the meaning of quantum mechanics is provided by group theory. The group theory permits even to spot and correct flaws in the traditional theory.

When you know the algebraic part of the spinor formalism and you know that the corresponding synthetic part must be the group theory of the rotation and Lorentz groups, then you might expect that explaining the Stern-Gerlach experiment synthetically should not be too difficult. But lo and behold, this is certainly not the case. One reason for this is that the textbook algebra is very egregiously wrong.

¹ In a Stern-Gerlach experiment [1] neutral particles with spin 1/2 are used, e.g. Ag atoms. In our description we will all the time focus our attention on electrons, even if a Stern-Gerlach experiment on electrons might be extremely difficult to perform. The real problem we want to discuss is the case of an electron with spin 1/2 in a magnetic field, for which we have learned that the electron spin can be only up or down, and never tilted as we will assume in the attempt to describe precession, reported in Section 1.3.

1.2 Total absence of theory

We have pointed out many times before, especially in [2, 4, 6], that the terms $\mathbf{B} \cdot \boldsymbol{\sigma}$ or $\mathbf{B} \cdot \boldsymbol{\gamma}$ that occur in the equations are not the scalar product of the magnetic field with the spin vector \mathbf{s} . As a matter of fact $\mathbf{B} \cdot \boldsymbol{\sigma}$ or $\mathbf{B} \cdot \boldsymbol{\gamma}$ just express the magnetic field in the formalism of the Clifford algebra, because $\sigma_x, \sigma_y, \sigma_z$ or $\gamma_x, \gamma_y, \gamma_z$ just represent the basis vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ (see [2]). There is absolutely no elbow room for eluding this death sentence. It is just an undeniable mathematical fact. Furthermore, the spin vector \mathbf{s} is not represented by $\boldsymbol{\sigma}$ or $\boldsymbol{\gamma}$ but by $\mathbf{s} \cdot \boldsymbol{\sigma}$ or $\mathbf{s} \cdot \boldsymbol{\gamma}$, which often remains hidden inside the notation for the spinor ψ . When $\mathbf{s} \cdot \boldsymbol{\sigma}$ or $\mathbf{s} \cdot \boldsymbol{\gamma}$ do not explicitly occur in the equations, there cannot be any form of algebraic chemistry between $\mathbf{s} \cdot \boldsymbol{\sigma}$ and $\mathbf{B} \cdot \boldsymbol{\sigma}$ in those equations. Similar remarks apply for $\mathbf{s} \cdot \boldsymbol{\gamma}$ and $\mathbf{B} \cdot \boldsymbol{\gamma}$ in the Dirac formalism, but from now on we will only formulate things in the SU(2) formalism.

The textbook theory exploits the mathematical errors mentioned to claim that the “spin vector” $\boldsymbol{\sigma}$, after multiplication by $\frac{\hbar q}{2m_0}$ defines the magnetic dipole $\boldsymbol{\mu} = \frac{\hbar q}{2m_0} \boldsymbol{\sigma}$. This slight of hand replaces the axial vector $\frac{\hbar q}{2m_0} \mathbf{B} \cdot \boldsymbol{\sigma}$ by a scalar $\mathbf{B} \cdot \boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is now considered to be a magnetic dipole², and $V = -\mathbf{B} \cdot \boldsymbol{\mu}$ becomes a “potential energy”. This balderdash of messing around with mathematical symbols is not a theory! However, the expression for this “potential energy” corresponds conveniently to our classical intuition, such that it can be accepted in blissful ignorance. It is then still difficult to understand within this picture why the spin should select two orientations in order to align with \mathbf{B} , rather than just one, *viz.* the one that would minimize its energy within the picture of a potential. Can the spin then also maximize its potential energy?

Despite its appeal, the ansatz $V = -\boldsymbol{\mu} \cdot \mathbf{B}$ is also problematic. We are talking here about the hypothetical potential energy of a charged spinning object in a field \mathbf{B} , but this field \mathbf{B} is not a force like the gravitational force $m\mathbf{g}$ exerted on a spinning top. Any analogy with the potential energy of a spinning top in a gravitational field is potentially misleading and conceptually wanting, as a magnetic field just cannot do any work on a charge. It can exert a force $\mathbf{F} = q(\mathbf{v} \wedge \mathbf{B})$, but this force is always perpendicular to the displacement $d\mathbf{r} = \mathbf{v}dt$ and therefore the work $-\mathbf{F} \cdot d\mathbf{r} = 0$.

1.3 Total absence of intuition

For a top which is precessing in a gravitational field, the energy of the top remains constant (if we assume that the dissipation of energy due to the friction is negligible, which of course can become wrong in the long run). But if you describe a precessing top within the spinor formalism of quantum mechanics, then the formalism says the energy is not constant and oscillates between two extreme values (see e.g. [4], p.307; [7]). We are referring here of course to the description of an electron in a magnetic field. That the energy could oscillate is really incomprehensible. We could imagine that the electron loses energy by e.g. radiation, but not how it could regain the energy lost, and what is more, exactly by the same amount.³

If we dare to be heretic and assume that there is something wrong with that calculation, and that the energy is constant like for a spinning top anyway, we may get a constant-energy term that has not the correct value, because it will contain an extra factor $\cos\theta$, where θ is the tilt of the spin axis with respect to the magnetic field, at least if you follow the common-sense arguments you have been taught. None of these speculations leads to a calculation that agrees with the startling experimental result, which seems to indicate that the spin of a fermion can only point up or down.

The cheap way out of these puzzling contradictions, is the textbook dogma that space would be quantized, and that this would be a quantum mystery. Whereas I fully agree that I do not understand the first world of it, such that calling this a mystery could be appropriate, I nevertheless think that this is logically and mathematically completely ramshackle. First of all we should refuse dogmatic mysteries. But there is something far worse at work than just a weird paradox. In fact, there is a fierce contradiction hidden within that statement.⁴ The contradiction at stake here is that the formalism is completely based on SU(2), wherein the allowed axes of rotation explore all directions of \mathbb{R}^3 while it derives from this that the directions would be quantized in the sense that quantum mechanics would only allow for two directions, spin-up and spin-down! This brutal contradiction between the starting point of the reasoning and its final result shows that something must be badly wrong.

Gaslighting destroys the ability to think and in the end its effect is that nobody knows the truth anymore. This is what has happened here. We have been forced with so many wrong images, wrong maths and apparent internal contradictions that we do not know on which leg to stand anymore. We do no longer know how to winnow the chaff from the wheat. We even do not have the correct algebra.

The wrong images create even more puzzles in the light of the way we could derive the Dirac equation from the assumption that the electron spins in [4]. In developing the Dirac equation by expressing the rotational motion of a spinning

² Multiplying an axial vector $\mathbf{B} \cdot \boldsymbol{\sigma}$ by the constant $-\frac{\hbar q}{2m_0}$ can only yield another axial vector, such that identifying the result $-\frac{\hbar q}{2m_0} \mathbf{B} \cdot \boldsymbol{\sigma}$ with a scalar $-\mathbf{B} \cdot \boldsymbol{\mu}$ is a glaring error.

³ The derivation of the Dirac equation in [4] relies on the assumption that the spin axis remains fixed ($\frac{d\mathbf{s}}{dt} = 0$). Therefore the motion of a precessing top cannot be studied with the traditional Dirac equation. One should first derive a generalized equation following the same methods as used for the Dirac equation in [4] and illustrated in Subsection 3.2.

⁴ What the current theory does is forcefully claiming in an authoritarian way that this contradiction would be correct. To sweeten the pill it is admitted that this is indeed weird but that quantum mechanics is beyond human understanding. In reality, there is no theory at all, and all this is just gaslighting.

electron with the aid of spinors, at a certain stage we must put $m_0 c^2 = \hbar \omega_0 / 2$ in order to obtain the Dirac equation. Here the electron spins with angular frequency ω_0 around the spin axis \mathbf{s} , and m_0 is its rest mas. This is analogous to the equations for the angular momentum $L = I\omega$ and energy $E = I\omega^2 / 2$ of a top spinning at a frequency ω , which lead to $E = L\omega / 2$. Here I is the moment of inertia. This means that the complete rest energy of the electron is rotational energy. Consider now the statement that in a magnetic field the spin axis aligns with the magnetic field, because the spin can only be up or down. We could e.g. imagine that the spin axis \mathbf{s} is pointing in a given direction and that we turn on the magnetic field in a completely different direction.⁵ There must then exist a really fast mechanism for the spin to align. This is puzzling, because the magnetic energy $\frac{\hbar q B}{2m_0}$ is much smaller than the energy $m_0 c^2$. It is as though you would be able to align the rotation axis of a very heavy fast-spinning top along a completely different direction with a very small torque. Intuitively, this looks like utter nonsense. It does not comply with our daily-life experience and the conservation of angular momentum. What you expect on applying some external torque is precession, not alignment. If you tried to touch a fast-spinning heavy object you would find out that you cannot bring about such an alignment. With my apologies for the irony, you could rather become more or less aligned yourself. How is this then possible? Moreover, we do not understand how this alignment process is supposed to work. Is there some radiation emitted, and if so should this have been observed?

2 Tabula rasa approach based on spinors

In view of all this confusion, we must rebuild a theory from scratch. It will be based on a good understanding of spinors. This should not dishearten the reader. Just remember, dear reader: SU(2) and spinors are only about rotations, i.e. Euclidean geometry. How could this possibly be difficult? We have shown that it is indeed not difficult in our account of spinors in [2]. Despite the fact that the author understands spinors quite well, the wrong pictorial arguments that are living on in the intuitive folk lore about the spin in a magnetic field amount to a formidable conceptual obstacle. They are a smoke screen of totally misleading hints that kept me in the dark for a very long time and rendered it extremely difficult to find the correct solution. I am confident that I am not the only one who has been running in circles for years in trying to make sense of this spin-up and spin-down narrative.⁶

We must thus warn the reader that he is in for a rough ride whereby a lot of what he has become used to take for granted will be ripped apart. Such a statement may cause irritation, but I think that if you pick up the basics about spinors from [2] and then read the present paper, you will feel rewarded for your efforts. We start from something we derived in [4] (see e.g. [4], p.142), *viz.* that you can write a spinning motion in SU(2) in terms of two components. We will interpret this for the moment as a simultaneous description of left-handed and right-handed frames (see [2], p.33; we have worked this out in the Appendix). E.g. if some spinning motion were to be described by $e^{-i\omega_0 \tau}$ in the right-handed frame, then it would be described by $e^{+i\omega_0 \tau}$ in the left-handed frame. It is just a matter of algebraic frequencies. The decomposition of a rotation into two components is as follows:

$$\mathbf{R}(\tau) = \frac{1}{2} \left[[\mathbb{1} + \mathbf{s} \cdot \boldsymbol{\sigma}] e^{-i\omega_0 \tau / 2} + [\mathbb{1} - \mathbf{s} \cdot \boldsymbol{\sigma}] e^{+i\omega_0 \tau / 2} \right]. \quad (1)$$

Eq. 1 is a direct consequence of the Rodrigues formula for a rotation over an angle φ around the axis \mathbf{s} :

$$\mathbf{R}(\mathbf{s}, \varphi) = \cos(\varphi/2) \mathbb{1} - i \sin(\varphi/2) [\mathbf{s} \cdot \boldsymbol{\sigma}], \quad (2)$$

after putting $\varphi = \omega_0 \tau$. Note that this is just Euclidean geometry. And here is then my question: What if these two components could correspond to a mixture of two beams? In fact, using Ehrenfest's interpretation of superposition states (see [2], p.10, complemented by [8], p.2, for a group-theoretical justification), Eq. 1 means that a spinning frame can be left-handed or right-handed with equal probability.⁷ Let us now write down Eq. 1 for a rotation with an axis \mathbf{s} that is different from the z -axis:

⁵ Note that this implies also the temporary presence of an electric field.

⁶ As we will see it is focusing the attention on the supposed aligning of the spin axis that sends us irrevocably down the rabbit hole. It is the unshakable belief that the experiment unmistakably tells us that the spin must be aligned which keeps us in the total impossibility of breaking away from the conceptual death trap of space quantization. The fact that this problem has remained unsolved for almost a century illustrates how difficult it was.

⁷ Due to the negative frequencies, it is customary to interpret this rather by saying that an electron is a superposition of a particle and an anti-particle, but the possibility of negative frequencies is a trivial feature in SU(2) whose axioms do not accommodate for the existence of anti-particles. What you do not put in cannot come out by magic. Also the gauge symmetry from which the idea of anti-particles is derived absolutely does not play any rôle in the derivation of the Dirac equation. Hence, once again, what does not go in cannot come out. One can introduce *a posteriori* anti-particles into the theory by adding the gauge symmetry, but in order to avoid ambiguity between the two types of negative frequencies one should then use the matrix γ_5 to define charge coordinates. Hence the hype of the antiparticle interpretation is just flawed within the present context.

$$\mathbf{R}(\tau) = \begin{bmatrix} \cos^2(\theta/2) & e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \sin^2(\theta/2) \end{bmatrix} e^{-i\omega_0 \tau/2} + \begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ -e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \cos^2(\theta/2) \end{bmatrix} e^{i\omega_0 \tau/2}. \quad (3)$$

Here (θ, ϕ) are the spherical coordinates of the spin axis \mathbf{s} . Note that we are using ϕ and φ as two different symbols in this document. Let us now inspect the two components. The $e^{-i\omega_0 \tau/2}$ component is:

$$\begin{bmatrix} \cos^2(\theta/2) & e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \sin^2(\theta/2) \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{+i\phi/2} \end{bmatrix} \otimes \begin{bmatrix} \cos(\theta/2) e^{i\phi/2} & \sin(\theta/2) e^{-i\phi/2} \end{bmatrix}. \quad (4)$$

We recover here the result $\mathbb{1} + \mathbf{s} \cdot \boldsymbol{\sigma} = 2\psi_1 \otimes \psi_1^\dagger$ from [4] (See Eqs. 3.28, 5.25). The $e^{+i\omega_0 \tau/2}$ component is:

$$\begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ -e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \cos^2(\theta/2) \end{bmatrix} = \begin{bmatrix} \sin(\theta/2) e^{-i\phi/2} \\ -\cos(\theta/2) e^{+i\phi/2} \end{bmatrix} \otimes \begin{bmatrix} \sin(\theta/2) e^{i\phi/2} & -\cos(\theta/2) e^{-i\phi/2} \end{bmatrix}. \quad (5)$$

This corresponds to $\mathbb{1} - \mathbf{s} \cdot \boldsymbol{\sigma} = 2\psi_2 \otimes \psi_2^\dagger$. Note that ψ_1 and ψ_2 are orthogonal. Now the idea is that a magnetic field would make the spin vector precess. We have explained this in earlier work (see [6]). For different radii of the circular motion within a magnetic field the cyclotron frequency remains the same in the non-relativistic limit. Every local co-traveling frame will spin at the same frequency, just like your horse on a merry-go-round does not only move along a circle but also spins around its own axis with respect to the frame of the observers on the ground.⁸ If you shrink the circular orbit in the magnetic field to a point the spinning motion with the cyclotron frequency around the axis will remain. Therefore a pointlike charged particle at rest in a magnetic field will be spinning even if it were initially spinless. But if it initially already spins and its spin axis is tilted, then this axis will be precessing, which corresponds to the intuitive narrative based on the analogy with a spinning top. The final test of this merry-go-round scenario will be whether it reproduces the experimental results. We have no *a priori* knowledge that would help us in deciding if this is correct or otherwise (except perhaps our remark in Footnote 8). We encounter this merry-go-round scenario also in Purcell's explanation of the Thomas precession [9]. We obtain then the motion:

$$\begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{i\Omega\tau/2} \end{bmatrix} \mathbf{R}(\tau), \quad (6)$$

where $\Omega = \frac{qB}{m_0}$ is the cyclotron frequency. Let us write the effect of this precession on both components of $\mathbf{R}(\tau)$:

$$\begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{+i\Omega\tau/2} \end{bmatrix} \begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ -e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \cos^2(\theta/2) \end{bmatrix} e^{+i\omega_0 \tau/2} = \begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ 0 & 0 \end{bmatrix} e^{i(\omega_0 - \Omega)\tau/2} + \begin{bmatrix} 0 & 0 \\ -e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \cos^2(\theta/2) \end{bmatrix} e^{i(\omega_0 + \Omega)\tau/2}. \quad (7)$$

The matrices are here again tensor products. But they are now of a novel type $\psi \otimes \chi^\dagger$, which no longer provides a familiar link with some rotation axis as in the equation $\mathbb{1} + \mathbf{s} \cdot \boldsymbol{\sigma} = 2\psi_1 \otimes \psi_1^\dagger$. This is quite normal because a precession has no fixed rotation axis. We could write actually the first term in the right-hand side of this equation as:

$$\begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ 0 & 0 \end{bmatrix} e^{i(\omega_0 - \Omega)\tau/2} = e^{-i\Omega\tau/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \end{bmatrix} e^{i\omega_0 \tau/2}, \quad (8)$$

which interestingly contains traces of the history of what we have done. Actually we can write:

$$\begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{i\Omega\tau/2} \end{bmatrix} = e^{-i\Omega\tau/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} + e^{+i\Omega\tau/2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix}. \quad (9)$$

Therefore the term that goes with $e^{i(\omega_0 - \Omega)\tau/2}$ is:

⁸ Note that this is different from what happens with a gyroscope in a space ship. There the gyroscope is not subject to gravitational forces because it is in free fall, such that it cannot precess. It is much harder to erase the electromagnetic field by an acceleration. One can erase the magnetic field by a rotating frame but then an electric field enters the scene. Note that in the rotating frame, the local co-moving frames are spinning like the horse in the merry-go-round. A magnetic field *is* a rotating frame. A good argument for the merry-go-round scenario is that the Lorentz transformation between the instantaneous boosts in two points along the orbit of a uniform circular motion contains also a rotation.

$$e^{-i\Omega\tau} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \otimes \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \sin(\theta/2)e^{-i\phi/2} \\ -\cos(\theta/2)e^{+i\phi/2} \end{bmatrix}}_{\sin(\theta/2)e^{-i\phi/2}} \otimes \begin{bmatrix} \sin(\theta/2)e^{+i\phi/2} & -\cos(\theta/2)e^{-i\phi/2} \end{bmatrix}. \quad (10)$$

We can multiply the underbraced matrices in the middle, which can be shown to be a correct procedure. We obtain then the scalar $\sin(\theta/2)e^{-i\phi/2}$. And this way we obtain again the same result as in Eq. 8. We are multiplying all the time with matrices that can be written as tensor products because they have determinant zero. This is a specificity of $SU(2)$ as explained in [2]. The result of multiplying two such matrices will still have a zero determinant, such that it can be written as a tensor product, but it will no longer have the structure $\psi \otimes \psi^\dagger$. We should not worry that the expressions could be meaningless, because this is just Euclidean geometry. They are definitely not obvious to interpret, but they are exact. This calculus is very handsome because it reduces the calculations to a minimum. The other component yields:

$$\begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{+i\Omega\tau/2} \end{bmatrix} \begin{bmatrix} \cos^2(\theta/2) & e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ e^{+i\phi} \sin(\theta/2) \cos(\theta/2) & \sin^2(\theta/2) \end{bmatrix} e^{-i\omega_0\tau/2} = \\ \begin{bmatrix} \cos^2(\theta/2) & e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ 0 & 0 \end{bmatrix} e^{-i(\omega_0+\Omega)\tau/2} + \begin{bmatrix} 0 & 0 \\ e^{+i\phi} \sin(\theta/2) \cos(\theta/2) & \sin^2(\theta/2) \end{bmatrix} e^{-i(\omega_0-\Omega)\tau/2}. \quad (11)$$

We can now rearrange the terms according to their energies:

$$\begin{bmatrix} \cos^2(\theta/2) & e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ 0 & 0 \end{bmatrix} e^{-i(\omega_0+\Omega)\tau/2} + \begin{bmatrix} 0 & 0 \\ -e^{+i\phi} \sin(\theta/2) \cos(\theta/2) & \cos^2(\theta/2) \end{bmatrix} e^{+i(\omega_0+\Omega)\tau/2}, \quad (12)$$

where we can factorize out the probability amplitude $\cos(\theta/2)$, and:

$$\begin{bmatrix} 0 & 0 \\ e^{+i\phi} \sin(\theta/2) \cos(\theta/2) & \sin^2(\theta/2) \end{bmatrix} e^{-i(\omega_0-\Omega)\tau/2} + \begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ 0 & 0 \end{bmatrix} e^{+i(\omega_0-\Omega)\tau/2}, \quad (13)$$

where we can factorize out the probability amplitude $\sin(\theta/2)$. We see thus that there are two possible energies for the electron within the magnetic field. All the fuss of interpreting the formalism with the energy operator $-\frac{\hbar}{i} \frac{\partial}{\partial \tau}$ was thus self-defeating bogus.⁹ This is because we apply it to a mixed state with four different frequencies in all. This is the reason why we found a non-constant oscillating energy with the traditional operator. Now we have found an analysis that yields the correct energies.¹⁰ It also explains the whole Stern-Gerlach experiment, provided we can still explain how these two energies lead to different trajectories (see below). Let us note that we have presented the effect of the magnetic field on the charge by Eq. 9. This is not something we find in textbooks, but we have explained this in [6] in terms of vorticity. The algebra does not contain a current loop. It just contains a rotating point charge.¹¹ The whole puzzle why the magnetic moment would have to align with the field has now disappeared. We find the right energy without having to invoke alignments of axes with the magnetic field. Such alignments are just no longer part of the story. Furthermore, there is simply no longer a well-defined single fixed axis as transpires from the weird terms $\psi \otimes \chi^\dagger$ in the formalism. Eq. 12 describes a motion with energy $\hbar(\omega_0 + \Omega)/2 = m_0 c^2 + \frac{\hbar q B}{2m_0}$ and which occurs with probability $\cos^2(\theta/2)$, while Eq. 13 describes a motion with energy $\hbar(\omega_0 - \Omega)/2 = m_0 c^2 - \frac{\hbar q B}{2m_0}$ and which occurs with probability $\sin^2(\theta/2)$, in agreement with the experimental results. These are both complex motions that we cannot describe in simple terms like a rotation around some axis. We can safely assume that these two components just describe precession. The Stern-Gerlach filter separates these two energies into two different beams. It is one of those two rearranged combinations that in general will be fed into a next Stern-Gerlach apparatus. Note that the average energy is $\hbar(\omega_0 + \Omega \cos \theta)/2$, such that $V = -\boldsymbol{\mu} \cdot \mathbf{B}$ is a macroscopic energy term, but not a potential energy.

Most textbooks calculate the force exerted on the fermion starting from an equation for a potential energy $V = -\boldsymbol{\mu} \cdot \mathbf{B}$ and then using $\mathbf{F} = -\nabla V$. But the physical existence of such a potential energy is doubtful, because a magnetic field cannot do any work. The equation $V = -\boldsymbol{\mu} \cdot \mathbf{B}$ suggests that all directions of space are allowed which is actually what, according

⁹ This operator has been derived by educated guesses from the de Broglie *ansatz*. It is obviously not universal and cannot be generalized to more complicated situations with non-scalar wave functions. We will elaborate this in Subsection 3.1. See also Footnote 3 and the discussion of Eq. 26.

¹⁰ In the matrix for a spinning motion with frequency Ω around the z -axis in Eq. 9, the two frequencies $-\Omega$ and $+\Omega$ occur neatly in two different columns. When we work with spinors, i.e. columns of the rotation matrices, we avoid this way getting confronted with multiple frequencies such that we can use $-\frac{\hbar}{i} \frac{\partial}{\partial \tau}$ to obtain a meaningful result. But with four frequencies we can no longer avoid that there will be more than one frequency projected out of a spinor by $-\frac{\hbar}{i} \frac{\partial}{\partial \tau}$. We end up with more than one frequency within a spinor, which conjures up the image of a varying energy. But it is the use of the operator $-\frac{\hbar}{i} \frac{\partial}{\partial \tau}$ which is then no longer correct, as will be discussed in Subsection 3.2.

¹¹ Let us for a while make the error of thinking that Eq. 9 represents the magnetic field. We see then that the spinning electron is expressed completely in the same way as the magnetic field. From this we could then conclude that the electron spin represents a magnetic field. See also our remark in Footnote 8.

to the traditional theory, the experiment proves to be conceptually wrong. It is therefore better to base the analysis on the expression for the energy $E = \hbar\omega/2$ and then to use $\mathbf{F} = -\nabla E$. This will lead then to the same result as in the textbook analysis of the trajectories. Note that the traditional theory for the trajectories is classical because the aim is to show that our classical notions are wrong. To fully validate the theory one should also calculate the trajectories quantum mechanically. In the new theory, classical geometry still prevails and the mysterious quantum effects disappear. Let us finally note that it is not angular momentum which is quantized in physics but the representations of the three-dimensional rotation group $\text{SO}(3)$. This is analogous to what transpires from Gell-Mann's theory of $\text{SU}(3)$.

3 More traditional formulation in terms of a differential equation

In this Section, we will reformulate everything again in the more familiar differential formalism of standard textbook quantum mechanics. In Subsection 3.1 we will justify in more detail our criticism formulated in Footnote 9 of the way the energy operator has been used on mixed states. In Subsection 3.2 we will elaborate the remark given in Footnote 3.

3.1 The apparent failure of the energy operator

We must learn the lessons from the errors that have been made in the traditional approach to the Stern-Gerlach experiment. In the case of precession we have:

$$\mathbf{R}(\tau)\mathbf{R}(0) = \begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{+i\Omega\tau/2} \end{bmatrix} \mathbf{R}_{\omega_0}(\tau)\mathbf{R}(0) = \mathbf{R}_{\Omega}(\tau)\mathbf{R}_{\omega_0}(\tau)\mathbf{R}(0) = \underbrace{\mathbf{R}_{\Omega}(\tau)\mathbf{R}_{\omega_0}(\tau)\mathbf{R}_{\Omega}^{-1}(\tau)}_{\text{part 1}} \underbrace{\mathbf{R}_{\Omega}(\tau)\mathbf{R}(0)}_{\text{part 2}}. \quad (14)$$

where $\mathbf{R}_{\omega_0}(\tau)$ is the matrix in Eq. 3, which is Eq. 2 where we have put $\varphi = \omega_0\tau$. Note that the unit matrix is invariant under the similarity transformation in part 1. Therefore, part 1 becomes:

$$\mathbf{R}_{\Omega}(\tau) \{ \cos(\omega_0\tau/2) \mathbb{1} - i \sin(\omega_0\tau/2) [\mathbf{s} \cdot \boldsymbol{\sigma}] \} [\mathbf{R}_{\Omega}(\tau)]^{-1}, \quad (15)$$

which is a rotation over the angle $\omega_0\tau$ around the rotated axis $\mathbf{R}_{\Omega}(\tau)[\mathbf{s} \cdot \boldsymbol{\sigma}][\mathbf{R}_{\Omega}(\tau)]^{-1}$. Part 1 in Eq. 14 is therefore the rotation with angular frequency ω_0 around the spin axis which has been rotated by the precession. Part 2 is this rotation of the spin axis by the precession. In [4], Eq. 9.34 we have found that the time derivative of Eq. 14 projects out $(\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau))/2$.¹²

Because we have learned that $-\frac{\hbar}{i}\frac{\partial}{\partial\tau}$ is the energy operator, we might be inclined to think that $\hbar(\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau))/2$ is the energy of the system. But this would imply that the energy is time-dependent, which is incomprehensible as we pointed out in Subsection 1.3. Furthermore, the time derivative leads to $(\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau))/2$ which is a vector, while the energy is not a vector but a scalar. This shows clearly that there is something wrong. We encountered already this problem in [4] in differentiating the Rodrigues equation Eq. 2 (after putting $\varphi = \omega_0\tau$): $\frac{d}{d\tau}\mathbf{R} = -i(\omega_0/2)[\mathbf{s} \cdot \boldsymbol{\sigma}]\mathbf{R}$, or $\frac{d}{d\tau}\chi = -i(\omega_0/2)[\mathbf{s} \cdot \boldsymbol{\sigma}]\chi$, for the spinor χ which is the first column of \mathbf{R} . This problem was solved by replacing χ by the superposition state $\psi = (\mathbb{1} + [\mathbf{s} \cdot \boldsymbol{\sigma}])\chi$, which permits to recover $\frac{d}{d\tau}\psi = -i(\omega_0/2)\psi$. We have to introduce such a superposition also in the derivation of the Dirac equation. In the present situation, there is no such simple solution due to the apparent time dependence of the time derivative.

Making the calculation of the temporal derivative seems therefore to be madness because it moves us away from our goal of calculating the energies, by sidetracking us onto a trail of vector calculations (see Footnote 6). The scalar we want is certainly not the norm of the vector sum. Indeed, when we have a system with energy $\hbar\omega_0/2$ and we add an energy $\hbar\Omega/2$ then we can only obtain a total energy $\hbar(\omega_0 + \Omega)/2$. The vector model makes you think erroneously it would be different, and that you have to calculate the norm of the vector $\hbar(\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau))/2$, which, by the way, is constant. We will now show how we can avoid making these strategic errors and save the energy operator.

¹² Imagine a spinning motion represented by the constant axial vector $\boldsymbol{\omega}_1$. Consider now a second spinning motion represented by the constant axial vector $\boldsymbol{\omega}_2$. Textbooks explain you then that the combined motion is given by the axial vector $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. And as both $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are constants, what else could we think than that $\boldsymbol{\omega}$ is also a constant. Well, this is wrong, which is the reason why we have noted $\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau)$ in the main text. In textbooks one proves $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ by considering infinitesimal displacements at a “point” $\mathbf{R}(\tau)$ of the rotation group. The “point” $\mathbf{R}(\tau)$ in question is of course a rotation. These displacements in an infinitesimal neighbourhood \mathcal{O} of $\mathbf{R}(\tau)$ add up as vectors $\boldsymbol{\omega}_j = \omega_j \mathbf{e}_j$, because \mathbf{e}_j behave as infinitesimal generators of the Lie algebra. They are small vectors in the tangent plane to the group manifold. Therefore they are even commuting in \mathcal{O} . Of course for larger displacements beyond \mathcal{O} this is no longer true because the rotation group is a curved manifold. By these methods one proves $\boldsymbol{\omega}(\tau) = \boldsymbol{\omega}_1(\tau) + \boldsymbol{\omega}_2(\tau)$, the validity of which is restricted to the infinitesimal neighbourhood \mathcal{O} of $\mathbf{R}(\tau)$, a snapshot of the motion at the instant τ . But of course, because for $\tau' > \tau$, $\mathbf{R}(\tau')$ will move out of \mathcal{O} , the identity should not be written by omitting the time dependence on τ , which is what many textbooks do. They write: $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ (see e.g. [10]). This kind of presentation may fool you by making you think that you can make the algebra just once at time τ and that the result $\boldsymbol{\omega}$ will remain constant with time. This is wrong, as we signal by the correct notation $\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau)$, which shows that the result is not time-independent as very clearly illustrated by a precessing top: $\boldsymbol{\Omega}$ remains fixed, and $\boldsymbol{\omega}_0(\tau)$ precesses. The vector notation may also make you think that the operations are commuting, while with the correct notations $\boldsymbol{\Omega}(\tau) + \boldsymbol{\omega}_0 \neq \boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau)$.

3.2 Not spin-up and spin-down but precession-up and precession-down

As announced, we will follow up in this Subsection on the remark made in Footnote 3. The motion described by Eq. 12 can be condensed into the form:

$$\mathbf{P}(\tau) = \begin{bmatrix} \cos(\theta/2) e^{-i(\omega_0 + \Omega)\tau/2} & e^{-i\phi} \sin(\theta/2) e^{-i(\omega_0 + \Omega)\tau/2} \\ -e^{i\phi} \sin(\theta/2) e^{+i(\omega_0 + \Omega)\tau/2} & \cos(\theta/2) e^{+i(\omega_0 + \Omega)\tau/2} \end{bmatrix}, \quad \text{with } \det(\mathbf{P}(\tau)) = 1. \quad (16)$$

Of course, $(\forall \tau \in \mathbb{R}) (\mathbf{P}(\tau) \in \text{SU}(2))$, such that we could use again a spinor formalism to calculate with these motions. Derivation with respect to τ yields:

$$\frac{d}{d\tau} \mathbf{P}(\tau) = -i((\omega_0 + \Omega)/2) \begin{bmatrix} \cos(\theta/2) e^{-i(\omega_0 + \Omega)\tau/2} & e^{-i\phi} \sin(\theta/2) e^{-i(\omega_0 + \Omega)\tau/2} \\ +e^{i\phi} \sin(\theta/2) e^{+i(\omega_0 + \Omega)\tau/2} & -\cos(\theta/2) e^{+i(\omega_0 + \Omega)\tau/2} \end{bmatrix}. \quad (17)$$

The inverse matrix of $\mathbf{P}(\tau)$ is:

$$[\mathbf{P}(\tau)]^{-1} = \begin{bmatrix} \cos(\theta/2) e^{+i(\omega_0 + \Omega)\tau/2} & -e^{-i\phi} \sin(\theta/2) e^{-i(\omega_0 + \Omega)\tau/2} \\ e^{i\phi} \sin(\theta/2) e^{+i(\omega_0 + \Omega)\tau/2} & \cos(\theta/2) e^{-i(\omega_0 + \Omega)\tau/2} \end{bmatrix}. \quad (18)$$

Hence $[\frac{d}{d\tau} \mathbf{P}(\tau)] [\mathbf{P}(\tau)]^{-1}$ is given by $-i((\omega_0 + \Omega)/2) \mathbf{V}(\tau)$ where $\mathbf{V}(\tau)$ is given by :

$$\begin{bmatrix} \cos(\theta/2) e^{-i(\omega_0 + \Omega)\tau/2} & e^{-i\phi} \sin(\theta/2) e^{-i(\omega_0 + \Omega)\tau/2} \\ +e^{i\phi} \sin(\theta/2) e^{+i(\omega_0 + \Omega)\tau/2} & -\cos(\theta/2) e^{+i(\omega_0 + \Omega)\tau/2} \end{bmatrix} \begin{bmatrix} \cos(\theta/2) e^{+i(\omega_0 + \Omega)\tau/2} & -e^{-i\phi} \sin(\theta/2) e^{-i(\omega_0 + \Omega)\tau/2} \\ e^{i\phi} \sin(\theta/2) e^{+i(\omega_0 + \Omega)\tau/2} & \cos(\theta/2) e^{-i(\omega_0 + \Omega)\tau/2} \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = [\mathbf{e}_z \cdot \boldsymbol{\sigma}]. \quad (19)$$

We have thus:

$$\frac{d}{d\tau} \mathbf{P}(\tau) = [\frac{d}{d\tau} \mathbf{P}(\tau)] [\mathbf{P}(\tau)]^{-1} [\mathbf{P}(\tau)] = -i[(\omega_0 + \Omega)/2] \mathbf{V}(\tau) \mathbf{P}(\tau) = -i[(\omega_0 + \Omega)/2] [\mathbf{e}_z \cdot \boldsymbol{\sigma}] \mathbf{P}(\tau). \quad (20)$$

First of all this shows that if we introduce again a mixed state (ψ instead of χ), then this mixed state will have indeed again a fixed energy $\hbar(\omega_0 + \Omega)/2$: i.e. $\frac{d}{d\tau} \psi = -i((\omega_0 + \Omega)/2) \psi$. *This saves the energy operator!* The result is rather amazing, because we have obtained in Eq. 20 the same type of differential equation as $\frac{d}{d\tau} \mathbf{R}(\tau) = -i(\omega_0/2) [\mathbf{e}_z \cdot \boldsymbol{\sigma}] \mathbf{R}(\tau)$ for the Rodrigues formula expressing a simple spinning motion around the z -axis, although the form of $\mathbf{P}(\tau)$ is different from the form of $\mathbf{R}(\tau)$ because it is not a diagonal matrix, whereas the matrix $\mathbf{R}(\tau)$ that describes a spinning motion around the z -axis is diagonal. With hindsight we can see that we could have anticipated all this. The equations $\frac{d}{d\tau} \chi = -i(\omega/2) [\mathbf{e}_z \cdot \boldsymbol{\sigma}] \chi$ or $\frac{d}{d\tau} \mathbf{R} = -i(\omega/2) [\mathbf{e}_z \cdot \boldsymbol{\sigma}] \mathbf{R}$ describe any type of object that rotates with an angular frequency ω around the z -axis. In the usual approach, the object is a spinless electron that we rotate with a frequency $\omega = \omega_0$ around the z -axis to give the electron its spin. In the new situation the object is an electron which is already spinning with a frequency ω_0 around an axis \mathbf{s} , and we rotate this object bodily with a frequency $\omega = \Omega$ around the z -axis, to describe the precession of the spinning electron within a magnetic field. That the new object is different from the initial one can be seen from the expression of the intervening matrix which is different from the diagonal form we had before. This result shows that whatever the level of complication in some hierarchy of precessions, we will always be able to treat a fixed-energy component this way.

The development for the equation of motion in Eq.13 is analogous. It can be condensed in the form:

$$\mathbf{M}(\tau) = \begin{bmatrix} \sin(\theta/2) e^{+i(\omega_0 - \Omega)\tau/2} & -e^{-i\phi} \cos(\theta/2) e^{+i(\omega_0 - \Omega)\tau/2} \\ e^{i\phi} \cos(\theta/2) e^{-i(\omega_0 - \Omega)\tau/2} & \sin(\theta/2) e^{-i(\omega_0 - \Omega)\tau/2} \end{bmatrix}, \quad \det(\mathbf{M}(\tau)) = 1. \quad (21)$$

$$\frac{d}{d\tau} \mathbf{M}(\tau) = -i((\omega_0 - \Omega)/2) \begin{bmatrix} -\sin(\theta/2) e^{+i(\omega_0 - \Omega)\tau/2} & e^{-i\phi} \cos(\theta/2) e^{+i(\omega_0 - \Omega)\tau/2} \\ e^{i\phi} \cos(\theta/2) e^{-i(\omega_0 - \Omega)\tau/2} & \sin(\theta/2) e^{-i(\omega_0 - \Omega)\tau/2} \end{bmatrix}. \quad (22)$$

The inverse matrix of $\mathbf{M}(\tau)$ is:

$$\mathbf{M}^{-1}(\tau) = \begin{bmatrix} \sin(\theta/2) e^{-i(\omega_0 - \Omega)\tau/2} & e^{-i\phi} \cos(\theta/2) e^{+i(\omega_0 - \Omega)\tau/2} \\ -e^{i\phi} \cos(\theta/2) e^{-i(\omega_0 - \Omega)\tau/2} & \sin(\theta/2) e^{+i(\omega_0 - \Omega)\tau/2} \end{bmatrix}. \quad (23)$$

We can again construct a matrix $\mathbf{W}(\tau) = [\frac{d}{d\tau} \mathbf{M}(\tau)] [\mathbf{M}^{-1}(\tau)]$, which is now given by:

$$\begin{bmatrix} -\sin(\theta/2) e^{+i(\omega_0 - \Omega)\tau/2} & e^{-i\phi} \cos(\theta/2) e^{+i(\omega_0 - \Omega)\tau/2} \\ e^{i\phi} \cos(\theta/2) e^{-i(\omega_0 - \Omega)\tau/2} & \sin(\theta/2) e^{-i(\omega_0 - \Omega)\tau/2} \end{bmatrix} \begin{bmatrix} \sin(\theta/2) e^{-i(\omega_0 - \Omega)\tau/2} & e^{-i\phi} \cos(\theta/2) e^{+i(\omega_0 - \Omega)\tau/2} \\ -e^{i\phi} \cos(\theta/2) e^{-i(\omega_0 - \Omega)\tau/2} & \sin(\theta/2) e^{+i(\omega_0 - \Omega)\tau/2} \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix}. \quad (24)$$

We have thus:

$$\frac{d}{d\tau} \mathbf{M}(\tau) = -i [(\omega_0 - \Omega)/2] [-\mathbf{e}_z \cdot \boldsymbol{\sigma}] \mathbf{M}(\tau). \quad (25)$$

This is now like the equation for a spinning motion around the negative z -axis.

The situation in the Eqs. 20 and 25 corresponds thus actually exactly to a physical picture of up and down states, but these states are different from what we have been told. It is no longer the same type of object, viz. the spin, that has its rotation axis aligned up or down. In the old context we started from a spinless electron and made it spin around an axis, in the new context we start from an already spinning electron whose axis is not aligned and we make the whole thing bodily spin around a precession axis. It is this precession axis which can be up or down, not the spin axis. *We should therefore have qualified the states as precession-up and precession-down rather than as spin-up and spin-down.* Pauli just introduced pragmatically the experimental result of the Stern-Gerlach experiment into the theory under the form of an *ad hoc* postulate, without any true justification. He replaced explaining by describing. All questions and demands for further explanations were given a first-class funeral by claiming that the experiment had proved that the theory was correct and that the directions in space were quantized. In view of the underlying heuristics claiming that the experiment had *proved* the theory amounts really to arm-twisting. All this was so highly counter-intuitive that it could only provoke intense bewilderment, as described in Section 1. After almost a century, we have now the theoretical justification for Pauli's *ad hoc* procedure, and we can appreciate that the directions in space are absolutely not quantized as he had pulled out of a hat.

All the difficulties we encountered before with the term $(\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau))/2$ (see Subsections 1.3 and 3.1) are due to the fact that we were describing a mixed state of two angular frequencies $\omega_1 = \omega_0 - \Omega$ and $\omega_2 = \omega_0 + \Omega$. If we note these two energy states as ψ_1 and ψ_2 , we have $\frac{d\psi_j}{d\tau} = -i \frac{\omega_j}{2} [\mathbf{e}_z \cdot \boldsymbol{\sigma}] \psi_j$, $\forall j \in \{1, 2\}$, but we cannot obtain a constant energy $\hbar\omega/2$ this way:

$$\neg(\exists \omega \in \mathbb{R})(\omega_1 \psi_1 + \omega_2 \psi_2 = \omega(\psi_1 + \psi_2)), \quad (26)$$

because there do not exist constants A, B, C , of any type such that $Ae^{i\omega_1\tau} + Be^{i\omega_2\tau} \equiv C(e^{i\omega_1\tau} + e^{i\omega_2\tau})$. Therefore using the energy operator $-\frac{\hbar}{i} \frac{\partial}{\partial \tau}$ on mixed states containing different energies projects out a meaningless “fluctuating energy”, as we pointed out in Subsection 1.3. But who will not have concluded in great dismay on the basis of the wrong use of the energy operator $-\frac{\hbar}{i} \frac{\partial}{\partial \tau}$ on such a mixed state that quantum mechanics is beyond human understanding? What this shows is that working on guessed equations under the motto “shut up and calculate!” cannot be the ultimate goal of true science. Figuring out subsequently what the successful algebra means is absolutely necessary to free the theory of conceptual deadlocks.

3.3 Quantum mechanics allowing for precession

We cannot apply Lorentz transformations to Eqs. 20 and 25 to describe orbital motion, because the magnetic field will not be Lorentz transformed while the spinor will. We must thus separate out the magnetic contribution, to prepare the derivation of a Dirac-like equation to treat precession. We can write:

$$\begin{aligned} \mathbf{P}(\tau) &= \begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{i\Omega\tau/2} \end{bmatrix} \begin{bmatrix} \cos(\theta/2)e^{-i\omega_0\tau/2} & e^{-i\phi} \sin(\theta/2)e^{-i\omega_0\tau/2} \\ -e^{i\phi} \sin(\theta/2)e^{+i\omega_0\tau/2} & \cos(\theta/2)e^{+i\omega_0\tau/2} \end{bmatrix}. \quad (27) \\ \frac{d}{d\tau} \mathbf{P}(\tau) &= -i(\Omega/2) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{i\Omega\tau/2} \end{bmatrix} \begin{bmatrix} \cos(\theta/2)e^{-i\omega_0\tau/2} & e^{-i\phi} \sin(\theta/2)e^{-i\omega_0\tau/2} \\ -e^{i\phi} \sin(\theta/2)e^{+i\omega_0\tau/2} & \cos(\theta/2)e^{+i\omega_0\tau/2} \end{bmatrix} + \\ &\quad \underbrace{\begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{i\Omega\tau/2} \end{bmatrix}}_{\text{term 1}} \underbrace{(-i\omega_0/2) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}}_{\text{term 2}} \begin{bmatrix} \cos(\theta/2)e^{-i\omega_0\tau/2} & e^{-i\phi} \sin(\theta/2)e^{-i\omega_0\tau/2} \\ -e^{i\phi} \sin(\theta/2)e^{+i\omega_0\tau/2} & \cos(\theta/2)e^{+i\omega_0\tau/2} \end{bmatrix}. \quad (28) \end{aligned}$$

The terms 1 and 2 commute, such that we obtain the same result as beforehand. In absence of a magnetic field \mathbf{B} we can thus write:

$$\lim_{\Omega \rightarrow +0} \frac{d\mathbf{P}}{d\tau} = \frac{d}{d\tau} \mathcal{P}(\tau) = -i(\omega_0/2) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \cos(\theta/2)e^{-i\omega_0\tau/2} & e^{-i\phi} \sin(\theta/2)e^{-i\omega_0\tau/2} \\ -e^{i\phi} \sin(\theta/2)e^{+i\omega_0\tau/2} & \cos(\theta/2)e^{+i\omega_0\tau/2} \end{bmatrix}}_{\mathcal{P}(\tau) \text{ (definition)}}. \quad (29)$$

We can imagine e.g. that there is a magnetic field along \mathbf{e}_z that is negligibly small, such that $\Omega \rightarrow +0$, to justify this writing. The matrix $\mathcal{P}(\tau)$ and the corresponding pure and mixed spinors χ and ψ are then just a special expression for the spinning

motion of the electron around the axis \mathbf{s} in a form that prepares for calculations of precession around \mathbf{e}_z . Isolating the part depending on Ω prepares a differential equation with a substitution introducing \mathbf{B} , that will this time not be the minimal substitution. When we add a stronger magnetic field along \mathbf{e}_z , we have then the substitution;

$$\frac{d}{d\tau} \rightarrow \frac{d}{d\tau} + i(\Omega/2)[\mathbf{e}_z \cdot \boldsymbol{\sigma}]. \quad (30)$$

In the $SL(2, \mathbb{C})$ representation this becomes:

$$\left[-\frac{\hbar}{i} \frac{\partial}{\partial \tau} \mathbb{1} - \frac{\hbar q}{2m_0 c} \mathbf{B} \cdot \boldsymbol{\sigma} \right] \chi = m_0 c \chi. \quad (31)$$

for a description of the state with energy $m_0 c^2 + \frac{\hbar q B}{2m_0}$ when the electron is at rest in the observer's frame. We can then lift this to the Dirac representation and try to use covariance to generalize this equation. We must note however that in the traditional Dirac theory this covariance is not correctly formulated, because it only addresses the boost part of the Lorentz transformation, while a general Lorentz transformation consists of a boost and a rotation. It is a major flaw to think that we could address accelerated motion by just considering instantaneous boosts, because it neglects the rotations. These rotations change the clock rates by precession and correspond to transverse accelerations.

Appendix. The missing link: the geometrical meaning of the spin-up and spin-down states

Let us note the up and down spinors as:

$$\chi_{\uparrow} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \chi_{\downarrow} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (32)$$

The tensor products $\chi_{\uparrow} \otimes \chi_{\uparrow}^{\dagger}$, $\chi_{\downarrow} \otimes \chi_{\uparrow}^{\dagger}$, $\chi_{\uparrow} \otimes \chi_{\downarrow}^{\dagger}$, and $\chi_{\downarrow} \otimes \chi_{\downarrow}^{\dagger}$, are a basis for the vector space $M(2, \mathbb{C})$ of complex 2×2 matrices. These four basis vectors are matrices which have each only one non-zero entry which is equal to 1. The manifold $F(G, G)$ of group automorphisms of the rotation group G , which is isomorphic to the rotation group G , is embedded in $M(2, \mathbb{C})$. The vector space of linear mappings $M(2, \mathbb{C})$ contains thus the representation of the group G by automorphisms. And when we extend G to larger groups, such as to contain also reflections and reversals, these will be also be embedded in $M(2, \mathbb{C})$. But $M(2, \mathbb{C})$ can also account for more complicated motions such as precession, nutation, etc... The four basis vectors of $M(2, \mathbb{C})$, $\chi_{\alpha} \otimes \chi_{\beta}^{\dagger}$, with $(\alpha, \beta) \in \{\uparrow, \downarrow\}^2$, transform β into α . The matrices $\frac{1}{2}(\mathbb{1} \pm \mathbf{e}_z \cdot \boldsymbol{\sigma})$ are then the basis vectors $\chi_{\uparrow} \otimes \chi_{\uparrow}^{\dagger}$ and $\chi_{\downarrow} \otimes \chi_{\downarrow}^{\dagger}$. In fact, $\frac{1}{2}(\mathbb{1} + \mathbf{e}_z \cdot \boldsymbol{\sigma}) = \chi_{\uparrow} \otimes \chi_{\uparrow}^{\dagger}$ transforms then χ_{\uparrow} into χ_{\uparrow} , while $\frac{1}{2}(\mathbb{1} - \mathbf{e}_z \cdot \boldsymbol{\sigma}) = \chi_{\downarrow} \otimes \chi_{\downarrow}^{\dagger}$ transforms then χ_{\downarrow} into χ_{\downarrow} . The spinning motion:

$$\begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{i\Omega\tau/2} \end{bmatrix}, \quad (33)$$

has two components in this basis because it transforms simultaneously χ_{\uparrow} into $e^{-i\Omega\tau/2} \chi_{\uparrow}$ and χ_{\downarrow} into $e^{i\Omega\tau/2} \chi_{\downarrow}$. The reflection operator $[\mathbf{e}_x \cdot \boldsymbol{\sigma}]$ transforms simultaneously χ_{\uparrow} into χ_{\downarrow} and χ_{\downarrow} into χ_{\uparrow} . We can identify χ_{\uparrow} as the representation of the identity element in a right-handed frame, in conformity with the way we have defined a spinor as the first column of the $SU(2)$ matrix. But χ_{\downarrow} can not be identified with the first column of a simple rotation matrix. In fact, the general expression for a rotation by an angle $\varphi = \omega_0 \tau$ around the axis \mathbf{s} with spherical coordinates (θ, ϕ) is according to the Rodrigues formula:

$$\begin{bmatrix} \cos(\omega_0 \tau/2) - i \cos \theta \sin(\omega_0 \tau/2) & -i \sin(\theta) e^{-i\phi} \sin(\omega_0 \tau/2) \\ -i \sin(\theta) e^{i\phi} \sin(\omega_0 \tau/2) & \cos(\omega_0 \tau/2) + i \cos \theta \sin(\omega_0 \tau/2) \end{bmatrix}. \quad (34)$$

Therefore obtaining χ_{\downarrow} as the first column of this rotation matrix would require $\cos(\omega_0 \tau/2) = 0$, implying $\sin(\omega_0 \tau/2) = 1$. This would then further require $\cos \theta = 0$, such that $\sin \theta = 1$. All these conditions are necessary just to make sure that the first entry of the spinor is zero. Because $\omega_0 \tau/2$ must have the fixed value $\pi/2$ we cannot have dynamical spinning motion associated with χ_{\downarrow} . Let us now check what follows from the condition that the second entry of the spinor must be 1. We must then have $-i e^{i\phi} = 1$, such that $\phi = \pi/2$. We have thus $(\theta, \phi) = (\pi/2, \pi/2)$ and $\varphi/2 = \omega_0 \tau/2 = \pi/2$. This spinor cannot be used to describe spinning motion. An out-of-the-box solution would be $\phi = \omega \tau + \pi/2$. We would obtain then the “spinor”:

$$\begin{bmatrix} 0 \\ e^{i\omega\tau} \end{bmatrix}. \quad (35)$$

This pseudo-solution would then represent the rotation of a non-spinning electron whose rotation axis would be in the Oxy plane and precessing around the z -axis with an angular frequency ω . The net result would be like an electron spinning around the z -axis. But this is somewhat of a cheat because it transgresses the domain of the original definitions, and we can represent such a motion already by means of χ_{\uparrow} . Hence the two sets of “spinors” generated by the rotation group by

operating on $\chi_{\downarrow} e^{-i\omega_0 \tau/2}$ and $\chi_{\uparrow} e^{i\omega_0 \tau/2}$ are physically disjoint. The quantity $\chi_{\downarrow} e^{i\omega_0 \tau/2}$ is not a spinor that corresponds to a spinning motion.

We can therefore adopt without ambiguity the convention to rather use $\chi_{\downarrow} e^{i\omega_0 \tau/2}$ in order to describe reversals, which are rotations of left-handed frames. Note that in a left-handed frame $\mathbf{a} \wedge \mathbf{b}$ is now defined according to the left-hand rule, such that $\omega| - \omega$ if we stick to the right-hand rule. The convention implies that χ_{\downarrow} represents a reference element with a left-handed frame, obtained from χ_{\uparrow} by the reflection operator $[\mathbf{e}_x \cdot \boldsymbol{\sigma}]$. We can in our convention consider the reference element χ_{\downarrow} as the identity element for the left-handed frames. In fact its triad of basis vectors consists of $(-\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$, which is a left-handed triad of basis vectors, when $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is a right-handed triad. The reflection operator σ_x transforms then a right-handed frame or triad into a left-handed frame or triad and vice versa. We can thus identify χ_{\uparrow} with a right-handed representation of the identity element and χ_{\downarrow} with a left-handed representation of the identity element.

Let us now call \mathbf{Q} the rotation around the axis parallel to $\mathbf{e}_z \wedge \mathbf{s}$ that rotates \mathbf{e}_z to \mathbf{s} . Under this rotation vectors are transformed “quadratically” according to: $[\mathbf{s} \cdot \boldsymbol{\sigma}] = \mathbf{Q}[\mathbf{e}_z \cdot \boldsymbol{\sigma}]\mathbf{Q}^\dagger$. This transforms $\frac{1}{2}(\mathbb{1} - \mathbf{e}_z \cdot \boldsymbol{\sigma})$ into $\frac{1}{2}(\mathbb{1} - \mathbf{s} \cdot \boldsymbol{\sigma})$ and $\frac{1}{2}(\mathbb{1} + \mathbf{e}_z \cdot \boldsymbol{\sigma})$ into $\frac{1}{2}(\mathbb{1} + \mathbf{s} \cdot \boldsymbol{\sigma})$. The operators $\frac{1}{2}(\mathbb{1} \pm \mathbf{s} \cdot \boldsymbol{\sigma})$ play thus locally the same rôle for rotations around \mathbf{s} as $\frac{1}{2}(\mathbb{1} \pm \mathbf{e}_z \cdot \boldsymbol{\sigma})$ for rotations around \mathbf{e}_z . They represent thus really left- and right-handed frames in the spinning motion around \mathbf{s} as claimed in the main text.¹³ We have always worked with the idea that a spinor is the first column of a SU(2) matrix. It is a right-handed formulation and shows how the spin-up spinor has been rotated. The second column of a SU(2) matrix corresponds to the left-handed formulation and shows how the spin-down spinor has been rotated. This action on the spinors of the two types of handedness can be clearly seen in the example of the rotation matrix in Eq. 33.

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¹³ Note that the basis vectors $\chi_\alpha \otimes \chi_\beta^\dagger$, with $(\alpha, \beta) \in \{\uparrow, \downarrow\}^2$ of $M(2, \mathbb{C})$ acquire a second meaning within the multivector formalism of the Clifford algebra discussed in [2]. Here two of the basis vectors are isotropic vectors, which can be considered as representing oriented planes and defining complete triads. These interpretations are of no use here because our spinors χ_{\uparrow} and χ_{\downarrow} must represent states of spinning motion, such that we need the interpretation of the vector space $M(2, \mathbb{C})$ in terms of rotations, reflections and reversals rather than multivectors.