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The exact theory of the Stern-Gerlach experiment and why it does not imply that a fermion can only have its spin up or down

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Abstract. The Stern-Gerlach experiment is notoriously counter-intuitive. The official theory is that the spin of a fermion remains always aligned with the magnetic field. Its directions are thus quantized: It can only be spin up or down. But that theory is based on mathematical errors in the way it (mis)treats spinors and group theory. We present here a mathematically rigorous theory for a fermion in a magnetic field, which is no longer counter-intuitive. It is based on an understanding of spinors in $SU(2)$ which is only Euclidean geometry. Contrary to what Pauli has been reading into the Stern-Gerlach experiment, the spin directions are not quantized. The new corrected paradigm, which solves all conceptual problems, is that the fermions precess around the magnetic-field just like Einstein and Ehrenfest had conjectured. Surprisingly this leads to only two energy states, which should be qualified as precession-up and precession-down rather than spin-up and spin down. Indeed, despite the presence of the many different possible angles θ between the spin axis \mathbf{s} and the magnetic field \mathbf{B} , the fermions can only have two possible energies $m_0 c^2 \pm \mu B$. The values $\pm \mu B$ do thus not correspond to the continuum of values $-\boldsymbol{\mu} \cdot \mathbf{B}$ Einstein and Ehrenfest had conjectured. The energy term $V = -\boldsymbol{\mu} \cdot \mathbf{B}$ is a macroscopic quantity. It is a statistical average over a large ensemble of fermions distributed over the two microscopic energy states $\pm \mu B$, and as such not valid for individual fermions. The two fermion states $\pm \mu B$ are not potential-energy states. We also explain the mathematically rigorous meaning of the up and down spinors. They represent left-handed and right-handed reference frames, such that now everything is intuitively clear and understandable in simple geometrical terms. The paradigm shift does not affect the Pauli principle.

PACS. 02.20.-a, 03.65.Ta, 03.65.Ca Group theory, Quantum Mechanics

1 Confusion reigns

1.1 Preamble

In this section¹ we want to point out the total lack of intuition and the total lack of theory which prevail in the traditional presentation of the Stern-Gerlach experiment. This experiment is a choice example of what happens all the time in quantum mechanics. The theory agrees perfectly with the experimental data but we cannot possibly make sense of what that theory means. After reading the present paper you will be able to perfectly make sense of the Stern-Gerlach experiment.

I use often the analogy of the correspondence between algebra and geometry in algebraic geometry to explain that the calculus of quantum mechanics, its algebra, is exact but that we do not know what its correct intuitive interpretation, its “geometry” should be. In this respect Villani uses the qualifiers “analytic” for what we call algebraic and “synthetic” for what we call “geometric” [2]. Perhaps this terminology is more accurate than ours. The purpose of making this difference between algebra and “geometry” is to make very clear right from the start that in general I am not questioning the algebra because it is correct. All I want to do is to find an intelligible “geometry”.

In this respect my previous work has shown time and again on the basis of many examples that the synthetic counterpart of the algebra of quantum mechanics, is the geometry behind the group theory of the rotation and Lorentz groups.² In the

¹ In a Stern-Gerlach experiment [1] neutral spin-1/2 particles are used, e.g. Ag atoms. In our description we will all the time focus our attention on electrons, even if a Stern-Gerlach experiment on electrons might be extremely difficult to perform. The real problem we want to discuss is the case of an electron with spin 1/2 in a magnetic field (the anomalous Zeeman effect), for which we have been taught that the electron spin can be only up or down, and never tilted as we will assume in the attempt to describe precession, reported in Section 1.3.

² In other words, if you master this group theory, you can derive many results of quantum mechanics by just classical reasoning. The quantum mysteries disappear and the theory becomes intuitive and intelligible. Some salient examples of our results are the derivation of

present paper we will prove this once more. The group theory permits even to spot and correct flaws in the traditional theory. To develop the argument I will have to rely from time to time on some earlier results, as it is just impossible to render the text completely self-contained. Due to the interconnections trying to solve one issue introduces a next one and this way one would end up writing a whole book. In moving along I will therefore drop a plethora of footnotes and remarks in order to address (incompletely) many questions that may naturally arise.

Traditional quantum mechanics has been discovered with rather stunning serendipity. Dirac just guessed his equation and many other rules were introduced *ad hoc*. Despite the shifting grounds of these shaky foundations, quantum mechanics has proved extremely successful. However, I must insist on warning the over-sceptical reader that he cannot attack my work by using the traditional textbook wisdom as the ultimate touchstone for the truth, e.g. when my work flies in the face of accepted notions or if it draws him out of his comfort zone. This is because in my approach based on group theory the results are mathematically derived and proved. A viewpoint developed from guesses cannot seriously pretend to prevail with authority over an approach based on mathematical derivations and proofs.

When you know the algebraic part of the spinor formalism and you know that the corresponding synthetic part must be the group theory of the rotation and Lorentz groups, then you might expect that explaining the Stern-Gerlach experiment synthetically should not be too difficult. But lo and behold, this is here certainly not the case. One reason for this is that the textbook algebra is very egregiously wrong.³

1.2 Total absence of theory

We have indeed pointed out many times before, especially in [3,6,7], that the shorthand notations $\mathbf{B}\cdot\boldsymbol{\sigma}$ or $\mathbf{B}\cdot\boldsymbol{\gamma}$ that occur in the equations are not the scalar product of the magnetic field \mathbf{B} with some “vector” $\boldsymbol{\sigma}$, where $\frac{\hbar}{2}\boldsymbol{\sigma}$ would be the “spin vector”. As a matter of fact $\mathbf{B}\cdot\boldsymbol{\sigma}$ or $\mathbf{B}\cdot\boldsymbol{\gamma}$ just express the magnetic field $\mathbf{B} = B_x\mathbf{e}_x + B_y\mathbf{e}_y + B_z\mathbf{e}_z$ in the formalism of the Clifford algebra, because $\sigma_x, \sigma_y, \sigma_z$ or $\gamma_x, \gamma_y, \gamma_z$ just represent the basis vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ (see [6]), such that $\boldsymbol{\sigma}$ is a trivector, corresponding to the triad of canonical basis vectors. There is absolutely no elbow room for eluding this undeniable mathematical fact. Furthermore, the (non-relativistic) unit vector \mathbf{s} which is parallel to the spin is not represented by $\boldsymbol{\sigma}$ or $\boldsymbol{\gamma}$ but by $\mathbf{s}\cdot\boldsymbol{\sigma}$ or $\mathbf{s}\cdot\boldsymbol{\gamma}$, which often remains hidden inside the notation for the spinor ψ . When $\mathbf{s}\cdot\boldsymbol{\sigma}$ or $\mathbf{s}\cdot\boldsymbol{\gamma}$ do not explicitly occur in the equations, there cannot be any form of algebraic chemistry between $\mathbf{s}\cdot\boldsymbol{\sigma}$ and $\mathbf{B}\cdot\boldsymbol{\sigma}$ in those equations. Similar remarks apply for $\mathbf{s}\cdot\boldsymbol{\gamma}$ and $\mathbf{B}\cdot\boldsymbol{\gamma}$ in the Dirac formalism, but from now on we will only formulate things in the SU(2) formalism.

The textbook theory exploits the mathematical errors mentioned to claim that the “spin vector” $\frac{\hbar}{2}\boldsymbol{\sigma}$, after multiplication by $\frac{q}{m_0}$ defines the “magnetic dipole” $\boldsymbol{\mu} = \frac{\hbar q}{2m_0}\boldsymbol{\sigma}$. This slight of hand transforms the axial vector $\frac{\hbar q}{2m_0}\mathbf{B}\cdot\boldsymbol{\sigma}$ by magic into a scalar $\mathbf{B}\cdot\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is now considered to be a magnetic dipole,⁴ and $V = -\mathbf{B}\cdot\boldsymbol{\mu}$ becomes a “potential energy”. I am sorry for the tone, but this balderdash of messing around with mathematical symbols is not a theory! However, the expression for this “potential energy” corresponds conveniently to our classical intuition, such that it can be accepted in blissful ignorance. It remains then still difficult to understand within this picture why the spin should select two orientations in order to align with

the Dirac equation from scratch in [3], the solution of the particle-wave duality, the solution of the paradox of Schrödinger’s cat, and an explanation for the double-slit experiment in [4], but there are many more. We have not addressed tunneling because the work of Hansen and Ravndal [5] already explains it perfectly.

³ The fact that I insist on pointing out errors in this article may upset some readers. But my work is an alternative approach to quantum mechanics, which makes a lot of things that looked mysterious intelligible. It is absolutely crucial to delineate what is wrong and what is right in this approach when it contradicts accepted notions. Else this will lead to confusion. It would also become very easy to nip the novel approach in the bud in the following way. Imagine a deeply entrenched belief A that at a certain stage interferes with my development, and could easily be seized upon to object that my approach “must be wrong because it is at variance with A , while everybody knows that A is true”. There is absolutely no other solution for avoiding such an instant death than rebutting the objection proactively by showing why it is A which is wrong instead. Unfortunately this will happen a few of times in going along. People like Elsevier editor Matteo Paris take advantage of this to turn the tables by claiming that there is too much negativity in my style of presentation, that it will hurt people’s feelings and that I therefore should rewrite the paper in a more lenient style. This is a very shallow and unfair appraisal of the real situation. In fact, yielding to the pressure of the allegations by complying with the injunctions would immediately clear the way for inflicting the death sentence evoked above. Furthermore, it must be very clear that *you cannot have your cake and eat it too*. Solving paradoxes requires pinpointing and neutralizing subliminal logical errors with surgical precision and quantum mechanics is fraught with paradoxes. Nobody is served with keeping the truth about the errors under wraps by censorship. If people want to understand quantum mechanics they will have to accept that it takes correcting mistakes, which can be the very reason why after almost a century we still do not succeed in wrapping our head around it. I have to break away from all this and I do not think that I should be criticized for cleaning up while I am only trying to bring enlightenment. For many people the frustration of being force-fed during their studies with the contradictions in the Copenhagen interpretation has been just a living hell. It made some of the brighter among them just quit! What about their hurt feelings?

⁴ Multiplying an axial vector $\mathbf{B}\cdot\boldsymbol{\sigma}$ by the constant $-\frac{\hbar q}{2m_0}$ can only yield another axial vector, such that identifying the result $-\frac{\hbar q}{2m_0}\mathbf{B}\cdot\boldsymbol{\sigma}$ with a scalar $-\mathbf{B}\cdot\boldsymbol{\mu}$ is wrong. It is absolutely essential that the reader gets the point that he cannot override or talk a way out of this mathematical verdict by belittling it as inconsequential, even if this comes as a slap in the face and even if this error has never been corrected. The trivector $\boldsymbol{\mu} = \frac{\hbar q}{2m_0}\boldsymbol{\sigma}$ even does not contain the spin vector $\frac{\hbar}{2}\mathbf{s}$. That $\mathbf{B}\cdot\boldsymbol{\sigma}$ represents a vector can be checked in [6], p.12 and [8], p.43.

\mathbf{B} , rather than just one, *viz.* the one that would minimize its energy within the picture of a potential. Can the spin then also maximize its potential energy?

Despite its appeal, the ansatz $V = -\boldsymbol{\mu} \cdot \mathbf{B}$ is also problematic. There is no dipole in its mathematics. We are talking here about the hypothetical potential energy of a charged spinning point particle in a field \mathbf{B} , but this field \mathbf{B} is not a force like the gravitational force $m\mathbf{g}$ exerted on a spinning top. Any analogy with the potential energy of a spinning top in a gravitational field is potentially misleading and conceptually wanting, as a magnetic field just cannot do any work on a charge. It can exert a force $\mathbf{F} = q(\mathbf{v} \wedge \mathbf{B})$, but this force is always perpendicular to the displacement $d\mathbf{r} = \mathbf{v}dt$ and therefore the work $-\mathbf{F} \cdot d\mathbf{r} = 0$.

1.3 Total absence of intuition

For a top which is precessing in a gravitational field, the energy of the top remains constant.⁵ But if you describe a precessing top within the spinor formalism of quantum mechanics, then the formalism says that the energy is not constant and oscillates between two extreme values (see e.g. [3], p.307; [9]). We are referring here of course to the description of an electron in a magnetic field. That the energy could oscillate is really incomprehensible. We could imagine that the electron loses energy by e.g. radiation, but not how it could regain the energy lost, and what is more, exactly by the same amount.⁶

If we dare to be heretic and assume that there is something wrong with that calculation, and that the energy is constant like for a spinning top anyway, we may get a constant-energy term that has not the correct value, because it will contain an extra factor $\cos\theta$, where θ is the tilt of the spin axis with respect to the magnetic field, at least if you follow the common-sense arguments you have been taught. None of these speculations leads to a calculation that agrees with the startling experimental result, which seems to indicate that the spin of a fermion can only point up or down.

The traditional way out of these puzzling contradictions is the textbook dogma that space would be quantized, and that this would be a quantum mystery. Whereas I fully agree that I do not understand the first word of it, such that calling this a mystery could be appropriate, I nevertheless think that this is logically and mathematically completely ramshackle. First of all we should refuse dogmatic mysteries. But there is something far worse at work than just a weird paradox. In fact, there is a fierce contradiction hidden within that statement. The contradiction at stake here is that the formalism is completely based on the use of $SU(2)$, wherein the allowed axes of rotation explore all directions of \mathbb{R}^3 while it claims that the directions would be quantized in the sense that quantum mechanics would only allow for two directions, spin-up and spin-down! Such a claim is not compatible with the geometry of $SU(2)$.

The wrong images create even more puzzles in the light of the way we could derive the Dirac equation from the assumption that the electron spins in [3]. In developing the Dirac equation by expressing the rotational motion of a spinning electron with the aid of spinors, at a certain stage we must put $m_0c^2 = \hbar\omega_0/2$ in order to obtain the Dirac equation. Here the electron spins with angular frequency ω_0 around the spin axis \mathbf{s} , and m_0 is its rest mass. This means that the complete rest energy of the electron is rotational energy. Consider now the statement that in a magnetic field the spin axis aligns with the magnetic field, because the spin can only be up or down. We could e.g. imagine that the spin axis \mathbf{s} is pointing in a given direction and that we turn on the magnetic field in a completely different direction.⁷ There must then exist a really fast mechanism for the spin to align. This is puzzling, because the magnetic energy $\frac{\hbar qB}{2m_0}$ is dwarfed by the energy m_0c^2 . How could this small magnetic energy possibly succeed in imposing alignment? It does not comply with our daily-life experience and the conservation of angular momentum. If you tried to touch a fast-spinning heavy object (e.g. a fast spinning skyscraper) you would find out that you cannot bring about such an alignment. With my apologies for the irony, you could rather become more or less aligned yourself. How is this then possible? Moreover, we do not understand how this alignment process is supposed to work. Is there some radiation emitted, and if so should this have been observed? Einstein and Ehrenfest have even calculated that the realignment would take hundreds of years [10].⁸

A final stark example illustrating the ambient confusion and ambivalence is the following. In the Dirac theory, the electron spin is always taken as perpendicular to the plane of motion. When there is a magnetic field, then it is always chosen to be parallel to the z -axis such that the spin can only have the values up or down along the z -axis, in conformity with the theoretical interpretation of the Stern-Gerlach experiment. But in the explanation of the neutron spin echo technique in solid-state physics [11], you will be told that the natural state of affairs is that the neutron spin is always initially aligned with its direction of motion. In order to make this spin perpendicular to its direction of motion, one has to apply a magnetic field at an angle of 45 degrees with respect to this direction of motion. After a Larmor precession over an angle of 180 degrees around this applied field the neutron spin will then have become perpendicular to its direction of motion. The rest of the explanation of the method is also entirely based on further Larmor precession of the neutron spin around a guide field. But as pointed out by Einstein and Ehrenfest, this precession scenario is in contradiction with the results of the Stern-Gerlach experiment. And

⁵ If we assume that the dissipation of energy due to the friction is negligible, which of course can become wrong in the long run.

⁶ The derivation of the Dirac equation in [3] relies on the assumption that the spin axis remains fixed: $\frac{d\mathbf{s}}{d\tau} = 0$. (We use throughout the notation τ for the proper time). Therefore the motion of a precessing top cannot be studied with the traditional Dirac equation. One should first derive a generalized equation following the same methods as used for the Dirac equation in [3] and illustrated in Subsection 3.2.

⁷ Note that this implies also the temporary presence of an electric field.

⁸ Einstein and Ehrenfest had anticipated that the spin would precess around the magnetic field (Larmor precession).

at the very end of the spin echo protocol the polarization of the spin is measured, and then it is assumed again that the spin can be only up or down. To make the puzzle complete, neutron spin echo has been tried and proved. It works! This reveals how the literature is rife with mutually contradictory scenarios about the way spin behaves. These contradictions are silently hiding under a rug. Sometimes one assumes that the spin just aligns and one invokes then often the paradigm of a torque exerted on a current loop to explain how this can happen, sometimes one assumes that the spin must precess and then one often wonders about the mechanism that might eventually align it. Enjoy the paradox: the two mental representations are mutually exclusive. They cannot possibly be both right at the same time. How can we possibly sort this out?

2 Tabula rasa approach based on spinors

In view of all this confusion, we must rebuild a theory from scratch and try to solve the paradox. It will be mathematically rigorous and based on a good understanding of spinors. This should not dishearten the reader. Just remember, dear reader: SU(2) and spinors are only about rotations, i.e. Euclidean geometry. How could this possibly be difficult? We have shown that it is indeed not difficult in our account of spinors in [6]. Despite the fact that the author understands spinors quite well, the many contradicting images that are living on in the intuitive folk lore about the spin in a magnetic field amount to a formidable conceptual obstacle. They are a smoke screen that kept me in the dark for a very long time and rendered it extremely difficult to find the correct solution. I am confident that I am not the only one who has been running in circles for years in trying to make sense of this spin-up and spin-down narrative.⁹

We must thus warn the reader that he is in for a rough ride whereby a lot of what he has become used to take for granted will be ripped apart. Such a statement may cause irritation (see Footnote 3), but I think that if you pick up the basics about spinors from [6] and then read the present paper, you will feel rewarded for your efforts. We start from something we derived in [3] (see e.g. [3], p.142), *viz.* that you can write a spinning motion in SU(2) in terms of two components. We can understand this as a simultaneous description of left-handed and right-handed frames (see [6], p.33; we have worked this out completely in the Appendix). E.g. if some spinning motion were to be described by $e^{-i\omega_0\tau}$ in the right-handed frame, then it would be described by $e^{+i\omega_0\tau}$ in the left-handed frame. It is just a matter of algebraic frequencies. The decomposition of a rotation into two components is as follows:

$$\mathbf{R}(\tau) = \frac{1}{2} \left[[\mathbb{1} + \mathbf{s} \cdot \boldsymbol{\sigma}] e^{-i\omega_0\tau/2} + [\mathbb{1} - \mathbf{s} \cdot \boldsymbol{\sigma}] e^{+i\omega_0\tau/2} \right]. \quad (1)$$

Eq. 1 is a direct consequence of the well-known Rodrigues formula for a rotation over an angle φ around the axis \mathbf{s} :

$$\mathbf{R}(\mathbf{s}, \varphi) = \cos(\varphi/2) \mathbb{1} - \iota \sin(\varphi/2) [\mathbf{s} \cdot \boldsymbol{\sigma}], \quad (2)$$

after putting $\varphi = \omega_0\tau$. Note that this is just Euclidean geometry. And here is then my question: What if these two components could correspond to a mixture of two beams? In fact, using Ehrenfest's interpretation of superposition states (see [6], p.10, complemented by [12], p.2, for a group-theoretical justification), the presence of the two frequencies in Eq. 1 means that we are describing left-handed and right-handed frames simultaneously.¹⁰ In fact, in the Appendix we will show that the SU(2) formalism describes left-handed and right-handed frames simultaneously. Let us now write down Eq. 1 for a rotation with an axis \mathbf{s} that is different from the z -axis:

$$\mathbf{R}(\tau) = \begin{bmatrix} \cos^2(\theta/2) & e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \sin^2(\theta/2) \end{bmatrix} e^{-i\omega_0\tau/2} + \begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ -e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \cos^2(\theta/2) \end{bmatrix} e^{i\omega_0\tau/2}. \quad (3)$$

Here (θ, ϕ) are the spherical coordinates of the spin axis \mathbf{s} . Note that we are using ϕ and φ as two different symbols in this document. Let us now inspect the two components. The $e^{-i\omega_0\tau/2}$ component is:

⁹ As we will see it is focusing the attention on the supposed aligning of the spin axis that sends us irrevocably down the rabbit hole. It is the unshakable belief that the experiment unmistakably tells us that the spin must be aligned which keeps us in the total impossibility of breaking away from the conceptual death trap of space quantization. The fact that this problem has remained unsolved for almost a century illustrates how difficult it was.

¹⁰ Due to the negative frequencies, it is customary to interpret this rather by saying that an electron is a superposition of a particle and an anti-particle, but the possibility of negative frequencies is a trivial feature in SU(2) whose axioms do not accommodate for the existence of anti-particles. What you do not put into a formalism cannot come out by magic. The signs of the frequencies just represent clockwise and counterclockwise motion, or frequencies expressed in left-handed and right-handed frames. Both algebraic frequencies ω correspond to a positive energy $E = |\hbar\omega/2|$. Also the gauge symmetry from which the idea of anti-particles is derived absolutely does not play any rôle in the derivation of the Dirac equation in [3]. Hence, once again, what does not go in cannot come out. One can introduce *a posteriori* anti-particles into the theory by adding the gauge symmetry, but negative frequencies can then mean two different things. In order to avoid ambiguity between the two types of negative frequencies one should then use the matrix γ_5 to define charge coordinates. The antiparticle interpretation will not be relied upon within the present context.

$$\begin{bmatrix} \cos^2(\theta/2) & e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \sin^2(\theta/2) \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{+i\phi/2} \end{bmatrix} \otimes \begin{bmatrix} \cos(\theta/2) e^{i\phi/2} & \sin(\theta/2) e^{-i\phi/2} \end{bmatrix}. \quad (4)$$

We recover here the result $\mathbb{1} + \mathbf{s} \cdot \boldsymbol{\sigma} = 2\psi_1 \otimes \psi_1^\dagger$ from [3] (See Eqs. 3.28, 5.25). The $e^{+i\omega_0\tau/2}$ component is:

$$\begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ -e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \cos^2(\theta/2) \end{bmatrix} = \begin{bmatrix} \sin(\theta/2) e^{-i\phi/2} \\ -\cos(\theta/2) e^{+i\phi/2} \end{bmatrix} \otimes \begin{bmatrix} \sin(\theta/2) e^{i\phi/2} & -\cos(\theta/2) e^{-i\phi/2} \end{bmatrix}. \quad (5)$$

This corresponds to $\mathbb{1} - \mathbf{s} \cdot \boldsymbol{\sigma} = 2\psi_2 \otimes \psi_2^\dagger$. Note that ψ_1 and ψ_2 are orthogonal.

Now the idea is that a magnetic field would make the spin vector precess, based on the following heuristics.¹¹ For different radii of the circular motion within a magnetic field the cyclotron frequency remains the same in the non-relativistic limit. Every local co-traveling frame will spin at the same frequency, just like your horse on a merry-go-round does not only move along a circle but also spins around its own axis with respect to the frame of the observers on the ground.¹² If you shrink the circular orbit in the magnetic field to a point the spinning motion with the cyclotron frequency around the axis will remain. Therefore a pointlike charged particle at rest in a magnetic field will be spinning even if it were initially spinless.¹³ But if it initially already spins and its spin axis is tilted, then this axis will be precessing, which corresponds to the intuitive narrative based on the analogy with a spinning top. These are mere heuristics. The final test of this merry-go-round scenario will be whether it reproduces the experimental results. We have no *a priori* knowledge that would help us in deciding if this is correct or otherwise (except perhaps our remark in Footnote 12). We encounter this merry-go-round scenario also in Purcell's explanation of the Thomas precession [13]. For a magnetic field \mathbf{B} aligned with the z -axis, we obtain then the motion:

$$\begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{i\Omega\tau/2} \end{bmatrix} \mathbf{R}(\tau), \quad (6)$$

where $\Omega = \frac{qB}{m_0}$ is the cyclotron frequency. Let us write the effect of this precession on both components of $\mathbf{R}(\tau)$:

$$\begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{+i\Omega\tau/2} \end{bmatrix} \begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ -e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \cos^2(\theta/2) \end{bmatrix} e^{+i\omega_0\tau/2} = \begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ 0 & 0 \end{bmatrix} e^{i(\omega_0-\Omega)\tau/2} + \begin{bmatrix} 0 & 0 \\ -e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \cos^2(\theta/2) \end{bmatrix} e^{i(\omega_0+\Omega)\tau/2}. \quad (7)$$

The matrices are here again tensor products. But they are now of a novel type $\psi \otimes \chi^\dagger$, which no longer provides a familiar link with some rotation axis as in the equation $\mathbb{1} + \mathbf{s} \cdot \boldsymbol{\sigma} = 2\psi_1 \otimes \psi_1^\dagger$. This is quite normal because a precession has no fixed rotation axis. We could write actually the first term in the right-hand side of this equation as:

$$\begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ 0 & 0 \end{bmatrix} e^{i(\omega_0-\Omega)\tau/2} = e^{-i\Omega\tau/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \end{bmatrix} e^{i\omega_0\tau/2}, \quad (8)$$

which interestingly contains traces of the history of what we have done. Actually we can write:

$$\begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{i\Omega\tau/2} \end{bmatrix} = e^{-i\Omega\tau/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} + e^{+i\Omega\tau/2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix}. \quad (9)$$

Therefore the term that goes with $e^{i(\omega_0-\Omega)\tau/2}$ is:

¹¹ We have actually explained these heuristics already in earlier work (see [7]).

¹² Note that this is different from what happens with a gyroscope in a space ship. There the gyroscope is not subject to gravitational forces because it is in free fall, such that it cannot precess. It is much harder to erase the electromagnetic field by an acceleration. One can erase the magnetic field by a rotating frame but then an electric field enters the scene. Note that in the rotating frame, the local co-moving frames are spinning like the horse in the merry-go-round. A magnetic field in a rotating frame. A good argument for the merry-go-round scenario is that the Lorentz transformation between the instantaneous boosts in two points along the orbit of a uniform circular motion contains also a rotation.

¹³ This gives some feeling for the anomalous Zeeman effect, but of course this is all purely classical. The heuristics of shrinking the orbit are thwarted by the fact that the orbits are quantized (but in the calculus of variations one considers virtual orbits). Nevertheless, the image of a magnetic field introducing a rotation is correct.

$$e^{-i\Omega\tau} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \sin(\theta/2)e^{-i\phi/2} \\ -\cos(\theta/2)e^{+i\phi/2} \end{bmatrix}}_{\sin(\theta/2)e^{-i\phi/2}} \otimes \begin{bmatrix} \sin(\theta/2)e^{i\phi/2} & -\cos(\theta/2)e^{-i\phi/2} \end{bmatrix}. \quad (10)$$

We can multiply the underbraced matrices in the middle, which can be shown to be a correct procedure. We obtain then the scalar $\sin(\theta/2)e^{-i\phi/2}$. And this way we obtain again the same result as in Eq. 8. We are working all the time with matrices that can be written as tensor products because they have determinant zero. That a matrix with zero determinant can be written as a tensor product is a specificity of 2×2 matrices. The result of multiplying such a matrix with determinant zero with another matrix will lead to a new matrix that still has determinant zero, such that it can be written again as a tensor product, but it will no longer have the structure $\psi \otimes \psi^\dagger$. We should not worry that the expressions could be meaningless, because this is just Euclidean geometry. They are definitely not obvious to interpret, but they are exact. This calculus is very handsome because it reduces the calculations to a minimum. The other component yields:

$$\begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{+i\Omega\tau/2} \end{bmatrix} \begin{bmatrix} \cos^2(\theta/2) & e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \sin^2(\theta/2) \end{bmatrix} e^{-i\omega_0\tau/2} = \\ \begin{bmatrix} \cos^2(\theta/2) & e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ 0 & 0 \end{bmatrix} e^{-i(\omega_0+\Omega)\tau/2} + \begin{bmatrix} 0 & 0 \\ e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \sin^2(\theta/2) \end{bmatrix} e^{-i(\omega_0-\Omega)\tau/2}. \quad (11)$$

We can now rearrange the terms according to their energies:

$$\begin{bmatrix} \cos^2(\theta/2) & e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ 0 & 0 \end{bmatrix} e^{-i(\omega_0+\Omega)\tau/2} + \begin{bmatrix} 0 & 0 \\ -e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \cos^2(\theta/2) \end{bmatrix} e^{+i(\omega_0+\Omega)\tau/2}, \quad (12)$$

where we can factorize out the probability amplitude $\cos(\theta/2)$, and:

$$\begin{bmatrix} 0 & 0 \\ e^{i\phi} \sin(\theta/2) \cos(\theta/2) & \sin^2(\theta/2) \end{bmatrix} e^{-i(\omega_0-\Omega)\tau/2} + \begin{bmatrix} \sin^2(\theta/2) & -e^{-i\phi} \sin(\theta/2) \cos(\theta/2) \\ 0 & 0 \end{bmatrix} e^{+i(\omega_0-\Omega)\tau/2}, \quad (13)$$

where we can factorize out the probability amplitude $\sin(\theta/2)$. We see thus that there are two possible energies for the electron within the magnetic field. All the fuss of interpreting the formalism with the energy operator $-\frac{\hbar}{i} \frac{\partial}{\partial \tau}$ was thus self-defeating bogus.¹⁴ This is because we apply it to a mixed state with four different frequencies in all. This is the reason why we found a non-constant oscillating energy with the traditional operator (see Section 3 for further details). Now we have found an analysis that yields the correct energies.¹⁵ It also explains the whole Stern-Gerlach experiment, provided we can still explain how these two energies lead to different trajectories (see below). Let us note that we have presented the effect of the magnetic field on the charge by Eq. 6. This is not something we find in textbooks, but as mentioned in Footnote 11 we have explained this in [7] in terms of vorticity. The algebra does not contain a current loop or a magnetic dipole.¹⁶ It just contains a rotating point charge.¹⁷ The whole puzzle why the magnetic moment would have to align with the field has now disappeared. We find the right energy without having to invoke alignments of axes with the magnetic field. Such alignments are just no longer part of the story. Furthermore, there is simply no longer a well-defined single fixed axis as transpires from the weird terms $\psi \otimes \chi^\dagger$ in the formalism. Eq. 12 describes a motion with energy $\hbar(\omega_0 + \Omega)/2 = m_0 c^2 + \frac{\hbar q B}{2m_0}$ and which occurs with probability $\cos^2(\theta/2)$, while Eq. 13 describes a motion with energy $\hbar(\omega_0 - \Omega)/2 = m_0 c^2 - \frac{\hbar q B}{2m_0}$ and which occurs with probability $\sin^2(\theta/2)$, in agreement with the experimental results. These are both complex motions that we cannot describe in simple terms like a rotation around some axis. We can safely assume that these two components just describe precession (see Section 3). The Stern-Gerlach filter separates these two energies into two different beams. It is one of those two rearranged combinations

¹⁴ This operator has been derived by educated guesses from the de Broglie *ansatz*. It is obviously not universal and can *a priori* not be generalized to more complicated situations with non-scalar wave functions. We will elaborate this in Subsection 3.1. See also Footnote 6 and the discussion of Eq. 27.

¹⁵ In the matrix for a spinning motion with frequency Ω around the z -axis in Eq. 6, the two frequencies $-\Omega$ and $+\Omega$ occur neatly in two different columns. When we work with spinors, i.e. columns of the rotation matrices, we avoid this way getting confronted with multiple frequencies such that we can use $-\frac{\hbar}{i} \frac{\partial}{\partial \tau}$ to obtain a meaningful result. But with four frequencies we can no longer avoid that there will be more than one frequency projected out of a spinor by $-\frac{\hbar}{i} \frac{\partial}{\partial \tau}$. We end up with more than one frequency within a spinor, which conjures up the image of a varying energy. But it is the use of the operator $-\frac{\hbar}{i} \frac{\partial}{\partial \tau}$ which is then no longer correct, as will be discussed in Subsection 3.2.

¹⁶ The symmetry of the anomalous Zeeman effect is completely different from that of the orbital Zeeman effect such that it is wrong to interpret them the same way.

¹⁷ Let us for a while make the error of thinking that Eq. 6 represents the magnetic field. We see then that the spinning electron is expressed completely in the same way as the magnetic field. From this we could then conclude that the electron spin represents a magnetic field. See also our remark in Footnote 12. That the magnetism produced by the spin does not need to be of the dipole type is shown by the exchange mechanism proposed by Heisenberg and Majorana, which is based on the Coulomb interaction and the exclusion principle.

that in general will be fed into a next Stern-Gerlach apparatus. Note that the average energy is $\hbar(\omega_0 + \Omega \cos\theta)/2$, such that $V = -\boldsymbol{\mu} \cdot \mathbf{B}$ is a macroscopic energy term, which is not applicable to individual fermions. It is not a potential energy.

Most textbooks calculate the force exerted on the fermion starting from an equation for a “potential energy” $V = -\boldsymbol{\mu} \cdot \mathbf{B}$ and then using $\mathbf{F} = -\nabla V$. But the physical existence of such a potential energy is doubtful, because a magnetic field cannot do any work. The equation $V = -\boldsymbol{\mu} \cdot \mathbf{B}$ suggests that all directions of space are allowed which is actually what, according to the traditional theory, the experiment proves to be conceptually wrong. It is therefore better to base the analysis on the expression for the energy $E = \hbar\omega/2$ and then to use $\mathbf{F} = -\nabla E$. This will lead then to the same result as in the textbook analysis of the trajectories. Note that the traditional theory for the trajectories is classical because the aim is to show that our classical notions are wrong. To fully validate the theory one should also calculate the trajectories quantum mechanically. In the new theory, classical geometry still prevails and the mysterious quantum effects disappear. Our explanation is entirely classical.

3 More traditional formulation in terms of a differential equation

In this Section, we will reformulate everything again in the more familiar differential calculus of standard textbook quantum mechanics. In Subsection 3.1 we will justify in more detail our criticism formulated in Footnote 14 of the way the energy operator has been used on mixed states. In Subsection 3.2 we will elaborate the remark given in Footnote 6.

3.1 The apparent failure of the energy operator

We must learn the lessons from the errors that have been made in the traditional approach to the Stern-Gerlach experiment. In the case of precession we have:

$$\mathbf{R}(\tau)\mathbf{R}(0) = \left[\begin{array}{c} e^{-i\Omega\tau/2} \\ e^{+i\Omega\tau/2} \end{array} \right] \mathbf{R}_{\omega_0}(\tau)\mathbf{R}(0) = \mathbf{R}_{\Omega}(\tau)\mathbf{R}_{\omega_0}(\tau)\mathbf{R}(0) = \underbrace{\mathbf{R}_{\Omega}(\tau)\mathbf{R}_{\omega_0}(\tau)\mathbf{R}_{\Omega}^{-1}(\tau)}_{\text{part 1}} \underbrace{\mathbf{R}_{\Omega}(\tau)\mathbf{R}(0)}_{\text{part 2}}. \quad (14)$$

where $\mathbf{R}_{\omega_0}(\tau)$ is the matrix in Eq. 3, which is Eq. 2 where we have put $\varphi = \omega_0\tau$. Note that the unit matrix is invariant under the similarity transformation in part 1. Therefore, part 1 becomes:

$$\mathbf{R}_{\Omega}(\tau) \{ \cos(\omega_0\tau/2)\mathbb{1} - i \sin(\omega_0\tau/2) [\mathbf{s} \cdot \boldsymbol{\sigma}] \} [\mathbf{R}_{\Omega}(\tau)]^{-1}, \quad (15)$$

which is a rotation over the angle $\omega_0\tau$ around the rotated axis $\mathbf{R}_{\Omega}(\tau)[\mathbf{s} \cdot \boldsymbol{\sigma}][\mathbf{R}_{\Omega}(\tau)]^{-1}$. Part 1 in Eq. 14 is therefore the rotation with angular frequency ω_0 around the spin axis which has been rotated by the precession. Part 2 is this rotation of the spin axis by the precession. In [3], Eq. 9.34 we have found that the time derivative of Eq. 14 projects out $(\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau))/2$.¹⁸

Because we have learned that $-\frac{\hbar}{i} \frac{\partial}{\partial \tau}$ is the energy operator, we might be inclined to think that $\hbar(\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau))/2$ is the energy of the system. But this would imply that the energy is time-dependent, which is incomprehensible as we pointed out in Subsection 1.3. Furthermore, the time derivative leads to $(\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau))/2$ which is a vector, while the energy is not a vector but a scalar. This shows clearly that there is something wrong.¹⁹

¹⁸ Imagine a spinning motion represented by the constant axial vector $\boldsymbol{\omega}_1$. Consider now a second spinning motion represented by the constant axial vector $\boldsymbol{\omega}_2$. Textbooks explain you then that the combined motion is given by the axial vector $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. And as both $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are constants, what else could we think than that $\boldsymbol{\omega}$ is also a constant. Well, this is wrong, which is the reason why we have noted $\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau)$ in the main text. In textbooks one proves $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ by considering infinitesimal displacements at a “point” $\mathbf{R}(\tau)$ of the rotation group. The “point” $\mathbf{R}(\tau)$ in question is of course a rotation. These displacements in an infinitesimal neighbourhood \mathcal{O} of $\mathbf{R}(\tau)$ add up as vectors $\boldsymbol{\omega}_j = \omega_j \mathbf{e}_j$, because \mathbf{e}_j behave as infinitesimal generators of the Lie algebra. They are small vectors in the tangent plane to the group manifold. Therefore they are even commuting in \mathcal{O} . Of course for larger displacements beyond \mathcal{O} this is no longer true because the rotation group is a curved manifold. By these methods one proves $\boldsymbol{\omega}(\tau) = \boldsymbol{\omega}_1(\tau) + \boldsymbol{\omega}_2(\tau)$, the validity of which is restricted to the infinitesimal neighbourhood \mathcal{O} of $\mathbf{R}(\tau)$, a snapshot of the motion at the instant τ . But of course, because for $\tau' > \tau$, $\mathbf{R}(\tau')$ will move out of \mathcal{O} , the identity should not be written by omitting the time dependence on τ , which is what many textbooks do. They write: $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ (see e.g. [14]). This kind of presentation may fool you by making you think that you can make the algebra just once at time τ and that the result $\boldsymbol{\omega}$ will remain constant with time. This is wrong, as we signal by the correct notation $\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau)$, which shows that the result is not time-independent as very clearly illustrated by a precessing top: $\boldsymbol{\Omega}$ remains fixed, and $\boldsymbol{\omega}_0(\tau)$ precesses. The vector notation may also make you think that the operations are commuting, while with the correct notations $\boldsymbol{\Omega}(\tau) + \boldsymbol{\omega}_0 \neq \boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau)$.

¹⁹ We encountered already this problem in [3] in differentiating the Rodrigues equation Eq. 2 (after putting $\varphi = \omega_0\tau$), where we obtained: $\frac{d}{d\tau} \mathbf{R} = -i(\omega_0/2) [\mathbf{s} \cdot \boldsymbol{\sigma}] \mathbf{R}$, or $\frac{d}{d\tau} \chi = -i(\omega_0/2) [\mathbf{s} \cdot \boldsymbol{\sigma}] \chi$, for the spinor χ which is the first column of \mathbf{R} . The presence of $[\mathbf{s} \cdot \boldsymbol{\sigma}]$ precludes deriving the equation $\frac{d}{d\tau} \chi = -i(\omega_0/2) \chi$ that would lead to the Dirac equation. It should be noted that $[\mathbf{s} \cdot \boldsymbol{\sigma}] \chi$ is a reversal, while χ is a rotation, such that they can never be proportional. This problem is solved by replacing χ by the mixed state $\psi = (\mathbb{1} + [\mathbf{s} \cdot \boldsymbol{\sigma}]) \chi$, which permits to recover $\frac{d}{d\tau} \psi = -i(\omega_0/2) \psi$. This transforms quantum mechanics automatically into a statistical theory. We have to introduce such a superposition also in the derivation of the Dirac equation, which is why one qualifies the Dirac spinors as bispinors. In the present situation, there is

Making the calculation of the temporal derivative seems therefore to be madness because it moves us away from our goal of calculating the energies, by sidetracking us onto a trail of vector calculations (see Footnote 9). The scalar we want is certainly not the norm of the vector sum. Indeed, when we have a system with energy $\hbar\omega_0/2$ and we add an energy $\hbar\Omega/2$ then we can only obtain a total energy $\hbar(\omega_0 + \Omega)/2$.²⁰ The vector model makes you think erroneously it would be different, and that you have to calculate the norm of the vector $\hbar(\boldsymbol{\Omega} + \boldsymbol{\omega}_0(\tau))/2$, which, by the way, is constant. We will now show how we can avoid making these strategic errors and save the energy operator.

3.2 Not spin-up and spin-down but precession-up and precession-down

As announced, we will follow up in this Subsection on the remark made in Footnote 6. The motion described by Eq. 12 can be condensed into the form:

$$\mathbf{P}(\tau) = \begin{bmatrix} \cos(\theta/2)e^{-i(\omega_0+\Omega)\tau/2} & e^{-i\phi}\sin(\theta/2)e^{-i(\omega_0+\Omega)\tau/2} \\ -e^{i\phi}\sin(\theta/2)e^{+i(\omega_0+\Omega)\tau/2} & \cos(\theta/2)e^{+i(\omega_0+\Omega)\tau/2} \end{bmatrix}, \quad \text{with } \det(\mathbf{P}(\tau)) = 1. \quad (16)$$

Of course, $(\forall \tau \in \mathbb{R})(\mathbf{P}(\tau) \in \text{SU}(2))$, such that we could use again a spinor formalism to calculate with these motions. Derivation with respect to τ yields:

$$\frac{d}{d\tau}\mathbf{P}(\tau) = -i((\omega_0 + \Omega)/2) \begin{bmatrix} \cos(\theta/2)e^{-i(\omega_0+\Omega)\tau/2} & e^{-i\phi}\sin(\theta/2)e^{-i(\omega_0+\Omega)\tau/2} \\ +e^{i\phi}\sin(\theta/2)e^{+i(\omega_0+\Omega)\tau/2} & -\cos(\theta/2)e^{+i(\omega_0+\Omega)\tau/2} \end{bmatrix}. \quad (17)$$

The inverse matrix of $\mathbf{P}(\tau)$ is:

$$[\mathbf{P}(\tau)]^{-1} = \begin{bmatrix} \cos(\theta/2)e^{+i(\omega_0+\Omega)\tau/2} & -e^{-i\phi}\sin(\theta/2)e^{-i(\omega_0+\Omega)\tau/2} \\ e^{i\phi}\sin(\theta/2)e^{+i(\omega_0+\Omega)\tau/2} & \cos(\theta/2)e^{-i(\omega_0+\Omega)\tau/2} \end{bmatrix}. \quad (18)$$

Hence $[\frac{d}{d\tau}\mathbf{P}(\tau)][\mathbf{P}(\tau)]^{-1}$ is given by $-i((\omega_0 + \Omega)/2)\mathbf{V}(\tau)$ where $\mathbf{V}(\tau)$ is given by :

$$\begin{bmatrix} \cos(\theta/2)e^{-i(\omega_0+\Omega)\tau/2} & e^{-i\phi}\sin(\theta/2)e^{-i(\omega_0+\Omega)\tau/2} \\ +e^{i\phi}\sin(\theta/2)e^{+i(\omega_0+\Omega)\tau/2} & -\cos(\theta/2)e^{+i(\omega_0+\Omega)\tau/2} \end{bmatrix} \begin{bmatrix} \cos(\theta/2)e^{+i(\omega_0+\Omega)\tau/2} & -e^{-i\phi}\sin(\theta/2)e^{-i(\omega_0+\Omega)\tau/2} \\ e^{i\phi}\sin(\theta/2)e^{+i(\omega_0+\Omega)\tau/2} & \cos(\theta/2)e^{-i(\omega_0+\Omega)\tau/2} \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = [\mathbf{e}_z \cdot \boldsymbol{\sigma}]. \quad (19)$$

We have thus:

$$\frac{d}{d\tau}\mathbf{P}(\tau) = [\frac{d}{d\tau}\mathbf{P}(\tau)][\mathbf{P}(\tau)]^{-1}[\mathbf{P}(\tau)] = -i[(\omega_0 + \Omega)/2]\mathbf{V}(\tau)\mathbf{P}(\tau) = -i[(\omega_0 + \Omega)/2][\mathbf{e}_z \cdot \boldsymbol{\sigma}]\mathbf{P}(\tau). \quad (20)$$

First of all this shows that if we introduce again a mixed state (ψ instead of χ , see Footnote 19), then this mixed state will have indeed again a fixed energy $\hbar(\omega_0 + \Omega)/2$: i.e. $\frac{d}{d\tau}\psi = -i((\omega_0 + \Omega)/2)\psi$. *We recover thus the traditional use of the energy operator!* The result is rather amazing, because we have obtained in Eq. 20 the same type of differential equation as $\frac{d}{d\tau}\mathbf{R}(\tau) = -i(\omega_0/2)[\mathbf{e}_z \cdot \boldsymbol{\sigma}]\mathbf{R}(\tau)$ for the Rodrigues formula expressing a simple spinning motion around the z -axis, although the form of $\mathbf{P}(\tau)$ is different from the form of $\mathbf{R}(\tau)$ because it is not a diagonal matrix, whereas the matrix $\mathbf{R}(\tau)$ that describes a spinning motion around the z -axis is diagonal. With hindsight we can see that we could have anticipated all this. The equations $\frac{d}{d\tau}\chi = -i(\omega/2)[\mathbf{e}_z \cdot \boldsymbol{\sigma}]\chi$ or $\frac{d}{d\tau}\mathbf{R} = -i(\omega/2)[\mathbf{e}_z \cdot \boldsymbol{\sigma}]\mathbf{R}$ describe any type of object that rotates with an angular frequency ω around the z -axis. In the usual approach, the object is a spinless electron that we rotate with a frequency $\omega = \omega_0$ around the z -axis to give the electron its spin. In the new situation the object is an electron which is already spinning with a frequency ω_0 around an axis \mathbf{s} , and we rotate this object bodily with a frequency $\omega = \Omega$ around the z -axis, to describe the precession of the spinning electron within a magnetic field. That the new object is different from the initial one can be seen from the expression of the intervening matrix which is different from the diagonal form we had before. This result shows that whatever the level of complication in some hierarchy of precessions, we will always be able to treat a fixed-energy component this way. We could have reached these conclusions also by observing that:

no such simple solution due to the apparent time dependence of the time derivative. Note that when one writes the Dirac equation this superposition has already been introduced. In the present approach this is not the case because we have not written the dynamics under the form of a differential equation and it is thus normal that we are confronted with this problem. One could propose to use $-\frac{\hbar}{i}[\mathbf{s} \cdot \boldsymbol{\sigma}]\frac{\partial}{\partial t}$ as the energy operator, but this is not practical because it requires knowing the value of $[\mathbf{s} \cdot \boldsymbol{\sigma}]$.

²⁰ In the spinning top, we must consider three contributions to the energy: the potential energy, and the kinetic energies of the spinning and the precessing motions. For the electron in a magnetic field, there are only two kinetic-energy terms.

$$\mathbf{P}(\tau) = \begin{bmatrix} e^{-i(\omega_0+\Omega)\tau/2} & \\ & e^{+i(\omega_0+\Omega)\tau/2} \end{bmatrix} \begin{bmatrix} \cos(\theta/2) & e^{-i\phi} \sin(\theta/2) \\ -e^{i\phi} \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}. \quad (21)$$

A surprising fact is that the whole energy is attributed to a rotation around the precession axis. But this illustrates what we noted, *viz.* that the energy is not a vector. We have an object that bodily rotates around the precession axis and its energy is $\hbar(\omega_0 + \Omega)\tau/2$. The development for the equation of motion in Eq.13 is analogous. It can be condensed in the form:

$$\mathbf{M}(\tau) = \begin{bmatrix} \sin(\theta/2) e^{+i(\omega_0-\Omega)\tau/2} & -e^{-i\phi} \cos(\theta/2) e^{+i(\omega_0-\Omega)\tau/2} \\ e^{i\phi} \cos(\theta/2) e^{-i(\omega_0-\Omega)\tau/2} & \sin(\theta/2) e^{-i(\omega_0-\Omega)\tau/2} \end{bmatrix}, \quad \det(\mathbf{M}(\tau)) = 1. \quad (22)$$

$$\frac{d}{d\tau} \mathbf{M}(\tau) = -i(\omega_0 - \Omega)/2 \begin{bmatrix} -\sin(\theta/2) e^{+i(\omega_0-\Omega)\tau/2} & e^{-i\phi} \cos(\theta/2) e^{+i(\omega_0-\Omega)\tau/2} \\ e^{i\phi} \cos(\theta/2) e^{-i(\omega_0-\Omega)\tau/2} & \sin(\theta/2) e^{-i(\omega_0-\Omega)\tau/2} \end{bmatrix}. \quad (23)$$

The inverse matrix of $\mathbf{M}(\tau)$ is:

$$\mathbf{M}^{-1}(\tau) = \begin{bmatrix} \sin(\theta/2) e^{-i(\omega_0-\Omega)\tau/2} & e^{-i\phi} \cos(\theta/2) e^{+i(\omega_0-\Omega)\tau/2} \\ -e^{i\phi} \cos(\theta/2) e^{-i(\omega_0-\Omega)\tau/2} & \sin(\theta/2) e^{+i(\omega_0-\Omega)\tau/2} \end{bmatrix}. \quad (24)$$

We can again construct a matrix $\mathbf{W}(\tau) = [\frac{d}{d\tau} \mathbf{M}(\tau)] [\mathbf{M}^{-1}(\tau)]$, which is now given by:

$$\begin{bmatrix} -\sin(\theta/2) e^{+i(\omega_0-\Omega)\tau/2} & e^{-i\phi} \cos(\theta/2) e^{+i(\omega_0-\Omega)\tau/2} \\ e^{i\phi} \cos(\theta/2) e^{-i(\omega_0-\Omega)\tau/2} & \sin(\theta/2) e^{-i(\omega_0-\Omega)\tau/2} \end{bmatrix} \begin{bmatrix} \sin(\theta/2) e^{-i(\omega_0-\Omega)\tau/2} & e^{-i\phi} \cos(\theta/2) e^{+i(\omega_0-\Omega)\tau/2} \\ -e^{i\phi} \cos(\theta/2) e^{-i(\omega_0-\Omega)\tau/2} & \sin(\theta/2) e^{+i(\omega_0-\Omega)\tau/2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix}. \quad (25)$$

We have thus:

$$\frac{d}{d\tau} \mathbf{M}(\tau) = -i [(\omega_0 - \Omega)/2] [-\mathbf{e}_z \cdot \boldsymbol{\sigma}] \mathbf{M}(\tau). \quad (26)$$

This is now like the equation for a spinning motion around the negative z -axis.

The situation in the Eqs. 20 and 26 corresponds thus actually exactly to a physical picture of up and down states, but these states are different from what we have been told. It is no longer the same type of object, *viz.* the spin, that has its rotation axis aligned up or down. In the old context we started from a spinless electron and made it spin around an axis, in the new context we start from an already spinning electron whose axis is not aligned and we make the whole thing bodily spin around a precession axis. It is this precession axis which can now be up or down, not the spin axis. *We should therefore have qualified the states as precession-up and precession-down rather than as spin-up and spin-down.* Pauli [15] just introduced pragmatically the experimental result of the Stern-Gerlach experiment into the theory under the form of an *ad hoc* postulate, without any true justification. He replaced explaining by describing. All questions and demands for further explanations are given here a first-class funeral by claiming that the experiment proves that the theory is correct and that the directions in space are quantized. In view of the underlying heuristics claiming that the experiment *confirms* the theory amounts actually to cyclic reasoning. All this was so highly counter-intuitive that it could only provoke intense bewilderment, as described in Section 1. After almost a century, we have now the theoretical justification for Pauli's *ad hoc* procedure, and we can appreciate that the directions in space are absolutely not "quantized".

3.3 Conclusion about the energy operator

All the difficulties we encountered with the calculation of the energy (see Subsections 1.3, 3.1) are due to the fact that we were describing a mixed state of two angular frequencies $\omega_1 = \omega_0 - \Omega$ and $\omega_2 = \omega_0 + \Omega$, whose absolute values are different. If we note these two energy states as ψ_1 and ψ_2 , we have $\frac{d\psi_j}{d\tau} = -i \frac{\omega_j}{2} [\mathbf{e}_z \cdot \boldsymbol{\sigma}] \psi_j, \forall j \in \{1,2\}$, but we cannot obtain a constant energy $\hbar\omega/2$ this way:

$$\neg(\exists \omega \in \mathbb{R})(\omega_1 \psi_1 + \omega_2 \psi_2 = \omega(\psi_1 + \psi_2)), \quad (27)$$

because there do not exist constants A, B, C , of any type such that $Ae^{i\omega_1\tau} + Be^{i\omega_2\tau} \equiv C(e^{i\omega_1\tau} + e^{i\omega_2\tau})$. Therefore using the energy operator $-\frac{\hbar}{i} \frac{\partial}{\partial \tau}$ on mixed states containing different energies projects out a meaningless "fluctuating energy", as we pointed out in Subsection 1.3.

3.4 Quantum mechanics allowing for precession

We cannot apply Lorentz transformations to Eqs. 20 and 26 to describe orbital motion, because the magnetic field will not be Lorentz transformed while the spinor will. We must thus separate out the magnetic contribution, to prepare the derivation of a Dirac-like equation to treat precession. We can write:

$$\mathbf{P}(\tau) = \begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{i\Omega\tau/2} \end{bmatrix} \begin{bmatrix} \cos(\theta/2)e^{-i\omega_0\tau/2} & e^{-i\phi}\sin(\theta/2)e^{-i\omega_0\tau/2} \\ -e^{i\phi}\sin(\theta/2)e^{+i\omega_0\tau/2} & \cos(\theta/2)e^{+i\omega_0\tau/2} \end{bmatrix}. \quad (28)$$

$$\begin{aligned} \frac{d}{d\tau}\mathbf{P}(\tau) = -i(\Omega/2) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{i\Omega\tau/2} \end{bmatrix} \begin{bmatrix} \cos(\theta/2)e^{-i\omega_0\tau/2} & e^{-i\phi}\sin(\theta/2)e^{-i\omega_0\tau/2} \\ -e^{i\phi}\sin(\theta/2)e^{+i\omega_0\tau/2} & \cos(\theta/2)e^{+i\omega_0\tau/2} \end{bmatrix} + \\ \underbrace{\begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{i\Omega\tau/2} \end{bmatrix}}_{\text{term 1}} \underbrace{\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}}_{\text{term 2}} \begin{bmatrix} \cos(\theta/2)e^{-i\omega_0\tau/2} & e^{-i\phi}\sin(\theta/2)e^{-i\omega_0\tau/2} \\ -e^{i\phi}\sin(\theta/2)e^{+i\omega_0\tau/2} & \cos(\theta/2)e^{+i\omega_0\tau/2} \end{bmatrix}. \end{aligned} \quad (29)$$

The terms 1 and 2 commute, such that we obtain the same result as beforehand. In absence of a magnetic field \mathbf{B} we can thus write:

$$\lim_{\Omega \rightarrow +0} \frac{d\mathbf{P}}{d\tau} = \frac{d}{d\tau}\mathcal{P}(\tau) = -i(\omega_0/2) \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \cos(\theta/2)e^{-i\omega_0\tau/2} & e^{-i\phi}\sin(\theta/2)e^{-i\omega_0\tau/2} \\ -e^{i\phi}\sin(\theta/2)e^{+i\omega_0\tau/2} & \cos(\theta/2)e^{+i\omega_0\tau/2} \end{bmatrix}}_{\mathcal{P}(\tau) \text{ (definition)}}. \quad (30)$$

We can imagine e.g. that there is a magnetic field along \mathbf{e}_z that is negligibly small, such that $\Omega \rightarrow +0$, to justify this writing. The matrix $\mathcal{P}(\tau)$ and the corresponding pure and mixed spinors χ and ψ are then just a special expression for the spinning motion of the electron around the axis \mathbf{s} in a form that prepares for calculations of precession around \mathbf{e}_z . Isolating the part depending on Ω prepares a differential equation with a substitution introducing \mathbf{B} , that will this time not be the minimal substitution. When we add a stonger magnetic field along \mathbf{e}_z , we have then the substitution;

$$\frac{d}{d\tau} \rightarrow \frac{d}{d\tau} + i(\Omega/2)[\mathbf{e}_z \cdot \boldsymbol{\sigma}]. \quad (31)$$

In the $SL(2, \mathbb{C})$ representation this becomes:

$$\left[-\frac{\hbar}{i} \frac{\partial}{\partial c\tau} \mathbb{1} - \frac{\hbar q}{2m_0 c} \mathbf{B} \cdot \boldsymbol{\sigma} \right] \chi = m_0 c \chi. \quad (32)$$

for a description of the state with energy $m_0 c^2 + \frac{\hbar q B}{2m_0}$ when the electron is at rest in the observer's frame. We can then lift this to the Dirac representation and try to use covariance to generalize this equation. We must note however that in the traditional Dirac theory this covariance is not correctly formulated, because it only addresses the boost part of the Lorentz transformation, while a general Lorentz transformation consists of a boost and a rotation. It is a major flaw to think that we could address accelerated motion by just considering instantaneous boosts, because it neglects the rotations. These rotations change the clock rates by precession and correspond to transverse accelerations.

4 The Pauli exclusion principle remains valid

Feynman [16] has given an intuitive explanation for the Pauli principle. However, he did not write down his idea under algebraic form, such that a detailed proof is lacking.²¹ Due to its historical context, one may suspect that the Pauli principle relies on the assumption that the spins can only be up and down, i.e. on parallelism. Now that we have discovered that the energy states must rather be characterized in terms of precession-up and precession-down one may formulate some concerns if the Pauli principle remains valid. As the spins are no longer parallel we might just have destroyed the Pauli principle. Certainly, there are still only two possible states for the energy, but there are now many more possible states of

²¹ Intuitively, when you exchange two electrons, each of them makes a turn over an angle of π . You may think that this will multiply their spinors by i and therefore the tensor product of the two spinors by -1 . But the moves involved in the exchange are, at least in appearance, taking place in space rather than inside the electron. They are of the position type such that they and the angle ζ which characterizes them (see below) should in principle not intervene in the argument, because the position coordinates do not belong to the set of parameters that define a spin state. The real exchange is thus not the swap of the positions but that of the spin states. However, these moves are accompanied by the rotation of the co-moving Fresnel frame, which is also characterized by ζ . And this rotational motion is of the spin type. In our development, the phase ζ which intervenes is obtained by Lorentz transformation of the spin variable $\omega_0\tau$, and therefore really of the spin type (see below).

motion. The motion is no longer characterized by Ω but by (Ω, θ) . In fact the spins no longer need to be parallel in order to resort to the same energy state. Could the change of paradigm cause the meltdown of the Pauli principle?

We will show that the Pauli principle is not under fire, but let us first try to write Feynman's argument algebraically (in the non-relativistic limit), rendering our proof open to a detailed scrutiny of the effects of the change. Let us take for the spin-up and spin-down functions, the wave functions for non-relativistic electrons moving on a circle:

$$\psi_{\uparrow} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i[(\omega_0 t - k\ell)/2]} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i[(\omega_0 t - \zeta)/2]}, \quad \psi_{\downarrow} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{+i[(\omega_0 t - k\ell)/2]} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{+i[(\omega_0 t - \zeta)/2]}. \quad (33)$$

The expressions in the exponentials come from integrating $\int \omega_0 dt - \mathbf{k} \cdot d\mathbf{r} = \int \omega_0 dt - kd\ell$ along the circle, The expression $\omega_0 dt - \mathbf{k} \cdot d\mathbf{r}$ is the Lorentz invariant $\omega dt - \mathbf{k} \cdot d\mathbf{r} = \omega_0 d\tau$ whereby we have dropped the factor $\gamma \approx 1$ in $\omega = \gamma\omega_0$ in the non-relativistic limit. Here ℓ is the curvilinear distance travelled along the circle, and $k = 1/r$.²² The tangent vector \mathbf{k} permits to follow the Thomas precession of the Fresnel basis on the merry-go-round which embodies the true rigid-body rotation of the whole two-electron configuration. We also note $\varphi = \omega_0 t$ for the spin angle, in contrast with ϕ which is the precession angle (and which does not intervene here). In fact, the electron does not have to move. When we freeze the time, we can still move around the circle geometrically. We have introduced the angle ζ in order to specify the position of the electron on the circle. We have $\ell = \zeta r$, such that $k\ell = \zeta$. The angle ζ is related to k and we can understand the value of ζ also as the rotation angle of the co-moving Fresnel basis. Consider now two spin-up electrons at diametrically opposed positions on a circle of radius r . We can consider then two spin-up electrons positioned in $\mathbf{r}_1 = \mathbf{r}$, $\mathbf{r}_2 = -\mathbf{r}$, $\zeta_1 = 0$, $\zeta_2 = \pi$. The phase difference $\zeta_2 = \zeta_1 + \pi$ just translates the different position on the circle.

We have then:

$$\psi_1 \otimes \psi_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i[(\omega_0 t - \zeta_1)/2]} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i[(\omega_0 t - \zeta_2)/2]} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i[\omega_0 t/2]} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i[(\omega_0 t - \pi)/2]}. \quad (34)$$

The expressions are pure spin functions. We consider Eq. 34 as the canonical situation. We will treat other situations later on. An exchange of the two electrons can be obtained by a rotation over an angle of π around the centre of the circle. Under such a rotation \mathbf{R} over π we obtain, $\mathbf{r}_1 \rightarrow \mathbf{r}_2$, $\mathbf{r}_2 \rightarrow \mathbf{r}_1$, $\zeta_j \rightarrow \zeta_j + \pi$.

$$\mathbf{R}(\psi_1 \otimes \psi_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i[(\omega_0 t - \pi)/2]} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i[(\omega_0 t - 2\pi)/2]} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i[(\omega_0 t - \pi)/2]} \otimes (-1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i[\omega_0 t/2]}. \quad (35)$$

Hence the rotation induces the substitutions $\psi_1 \rightarrow \psi_2$, $\psi_2 \rightarrow -\psi_1$, and $\psi_1 \otimes \psi_2 \rightarrow -\psi_2 \otimes \psi_1$. We can see that the cause for the minus sign is the fact that position angles ζ occur under the form $\zeta/2$ in the spinor calculus. After this rotation \mathbf{R} , the physical situation is indistinguishable from the situation before, because \mathbf{R} transforms the electron 1 into the electron 2 and vice versa. This means that the wave function must be invariant under the rotation. This implies that $\psi_1 \otimes \psi_2$ can not be the wave function Ψ . In fact, $\mathbf{R}(\Psi)$ would lead to $\mathbf{R}(\Psi) = \mathbf{R}(\psi_1 \otimes \psi_2) = \psi_2 \otimes \psi_1$, where we express the exchange $\psi_1 \leftrightarrow \psi_2$. But we have also calculated in Eq. 35 that $\mathbf{R}(\Psi) = -\psi_2 \otimes \psi_1$. This leads to $\mathbf{R}(\Psi) = -\mathbf{R}(\Psi)$, such that $\mathbf{R}(\Psi) = 0$ and $\Psi = 0$. Similarly, if we take:

$$\Psi = \underbrace{\psi_1 \otimes \psi_2}_{p_1} + \underbrace{\psi_2 \otimes \psi_1}_{p_2}. \quad (36)$$

then we obtain also $\mathbf{R}(\Psi) = -\Psi$ because \mathbf{R} transforms p_1 into $-p_2$ and p_2 into $-p_1$, while we have also $\mathbf{R}(\Psi) = \Psi$ because \mathbf{R} is an exchange. It follows then again that $\Psi = 0$. But if we rather take:

$$\Psi = \underbrace{\psi_1 \otimes \psi_2}_{p_1} - \underbrace{\psi_2 \otimes \psi_1}_{p_2}. \quad (37)$$

we obtain $\mathbf{R}(\Psi) = \Psi$ because now \mathbf{R} transforms p_1 into p_2 and p_2 into p_1 . This is now consistent with the fact that \mathbf{R} is an exchange. Hence Ψ in Eq. 37 is a wave function that takes into account the exchange correctly. The wave function has to be antisymmetric. The configuration of two electrons with parallel spins in the same place, can be obtained by considering the special case $\mathbf{r} = 0$. When the spins are parallel, we have then $\psi_1 = -\psi_2$ and $\Psi = 0$. We are thus obliged to take the spins antiparallel if we want to succeed to have them in the same place. This is the Pauli exclusion principle for spin-up and spin-down states.

Let us now investigate what this becomes with the new paradigm of precession-up and precession-down states. We can consider this as the non-canonical counterpart of the canonical state described above. We start from $\exists(\mathbf{R}_1, \mathbf{R}_2)$:

$$\chi_1 = \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} e^{-i[(\omega_0 t - \zeta_1)/2]} = \mathbf{R}_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i[(\omega_0 t - \zeta_1)/2]}, \quad \chi_2 = \begin{bmatrix} \eta_0 \\ \eta_1 \end{bmatrix} e^{-i[(\omega_0 t - \zeta_2)/2]} = \mathbf{R}_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i[(\omega_0 t - \zeta_2)/2]}. \quad (38)$$

²² By noting $w = c^2/v$ for the superluminal phase velocity w and putting $w = \omega_0 r$ we obtain $\omega_0 v d\ell/c^2 = \omega_0 d\ell/w = d\ell/r$, which must be $k d\ell$. Therefore $k = 1/r$.

We are thus considering the rotations \mathbf{R}_1 and \mathbf{R}_2 that relate the wave functions $\chi_j = \mathbf{R}_j(\psi_j)$ to the wave functions ψ_j of the canonical configuration. We have again $\zeta_2 = \zeta_1 + \pi$, where we can take $\zeta_1 = 0$. Then:

$$\psi_1 = \mathbf{R}_1^{-1}\chi_1, \quad \psi_2 = \mathbf{R}_2^{-1}\chi_2. \quad (39)$$

The two exponentials still exhibit a phase difference π leading to a factor -1 such that:

$$\psi_1 \rightarrow \psi_2, \quad \psi_2 \rightarrow -\psi_1, \quad \text{and therefore: } \mathbf{R}_1^{-1}\chi_1 \rightarrow \mathbf{R}_2^{-1}\chi_2, \quad \mathbf{R}_2^{-1}\chi_2 \rightarrow -\mathbf{R}_1^{-1}\chi_1, \quad (40)$$

or:

$$\chi_1 \rightarrow \mathbf{R}_1\mathbf{R}_2^{-1}\chi_2, \quad \mathbf{R}_1\mathbf{R}_2^{-1}\chi_2 \rightarrow -\chi_1. \quad (41)$$

In other words: $\exists \mathbf{R} = \mathbf{R}_2\mathbf{R}_1^{-1} \parallel \chi_2 = \mathbf{R}\chi_1$ & $\chi_1 = -\mathbf{R}^{-1}\chi_2$. Combining these two identities leads to $\chi_1 = -\chi_1$. Therefore the wave function must still be antisymmetrical. Hence Pauli's principle remains even valid when the two spins are not parallel. Furthermore, we prove in the Appendix that $\neg \exists \mathbf{R} \parallel \psi_1 = \mathbf{R}\psi_1$. This impossibility to connect an up state to a down state by rotation implies that a configuration of two electrons with antiparallel precessions is an allowed state.

5 Epilogue

We have thus explained the Stern-Gerlach experiment based on pure Euclidean geometry. But the algebra in one-to-one correspondence with the geometry is of an unusual type we are not familiar with. It is an algebra of group elements rather than of vectors. It is nevertheless perfectly understandable. Quantum mechanics is written in that type of algebra, not the vector algebra of classical theories. When we inspect quantum mechanics through the correct lens of the group theory the mysteries disappear and everything becomes intelligible. We have proved it once more. The mysteries were only due to the way we tried to interpret the group elements in the new algebra as vectors in the traditional algebra. There is no incompatibility between relativity and quantum mechanics. Of course it requires effort to get acquainted with the algebra of the group representation theories. Students are often given the mischievous advice to "*Shut up and calculate!*". I would rather go for a whimsical: "*Get a heart attack! It is all written in Gruppenpest!*".

Appendix. The missing link: the geometrical meaning of the spin-up and spin-down states

Let us note the up and down spinors as:

$$\chi_\uparrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \chi_\downarrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (42)$$

The tensor products $\chi_\uparrow \otimes \chi_\uparrow^\dagger$, $\chi_\downarrow \otimes \chi_\downarrow^\dagger$, $\chi_\uparrow \otimes \chi_\downarrow^\dagger$, and $\chi_\downarrow \otimes \chi_\uparrow^\dagger$ are a basis for the vector space $M(2, \mathbb{C})$ of complex 2×2 matrices. These four basis vectors are matrices which have each only one non-zero entry which is equal to 1. The manifold $F(G, G)$ of group automorphisms of the rotation group G , which is isomorphic to the rotation group G , is embedded in $M(2, \mathbb{C})$. The vector space of linear mappings $M(2, \mathbb{C})$ contains thus the representation of the group G by automorphisms. And when we extend G to larger groups, such as to contain also reflections and reversals, these will be also embedded in $M(2, \mathbb{C})$. But $M(2, \mathbb{C})$ can also account for more complicated motions such as precession, nutation, etc... The four basis vectors of $M(2, \mathbb{C})$, $\chi_\alpha \otimes \chi_\beta^\dagger$, with $(\alpha, \beta) \in \{\uparrow, \downarrow\}^2$, transform β into α . The matrices $\frac{1}{2}(\mathbb{1} \pm \mathbf{e}_z \cdot \boldsymbol{\sigma})$ are then the basis vectors $\chi_\uparrow \otimes \chi_\uparrow^\dagger$ and $\chi_\downarrow \otimes \chi_\downarrow^\dagger$. In fact, $\frac{1}{2}(\mathbb{1} + \mathbf{e}_z \cdot \boldsymbol{\sigma}) = \chi_\uparrow \otimes \chi_\uparrow^\dagger$ transforms then χ_\uparrow into χ_\uparrow , while $\frac{1}{2}(\mathbb{1} - \mathbf{e}_z \cdot \boldsymbol{\sigma}) = \chi_\downarrow \otimes \chi_\downarrow^\dagger$ transforms then χ_\downarrow into χ_\downarrow . The spinning motion:

$$\begin{bmatrix} e^{-i\Omega\tau/2} & \\ & e^{i\Omega\tau/2} \end{bmatrix}, \quad (43)$$

has two components in this basis because it transforms simultaneously χ_\uparrow into $e^{-i\Omega\tau/2}\chi_\uparrow$ and χ_\downarrow into $e^{+i\Omega\tau/2}\chi_\downarrow$. The reflection operator $[\mathbf{e}_x \cdot \boldsymbol{\sigma}]$ transforms simultaneously χ_\uparrow into χ_\downarrow and χ_\downarrow into χ_\uparrow . We can identify χ_\uparrow as the representation of the identity element in a right-handed frame, in conformity with the way we have defined a spinor as the first column of the $SU(2)$ matrix. But χ_\downarrow can not be identified with the first column of a simple rotation matrix. In fact, the general expression for a rotation by an angle $\varphi = \omega_0\tau$ around the axis \mathbf{s} with spherical coordinates (θ, ϕ) is according to the Rodrigues formula:

$$\begin{bmatrix} \cos(\omega_0\tau/2) - \imath \cos\theta \sin(\omega_0\tau/2) & -\imath \sin(\theta) e^{-i\phi} \sin(\omega_0\tau/2) \\ -\imath \sin(\theta) e^{i\phi} \sin(\omega_0\tau/2) & \cos(\omega_0\tau/2) + \imath \cos\theta \sin(\omega_0\tau/2) \end{bmatrix}. \quad (44)$$

Therefore obtaining χ_\downarrow as the first column of this rotation matrix would require $\cos(\omega_0\tau/2) = 0$, implying $\sin(\omega_0\tau/2) = 1$. This would then further require $\cos\theta = 0$, such that $\sin\theta = 1$. All these conditions are necessary just to make sure that the

first entry of the spinor is zero. Because $\omega_0\tau/2$ must have the fixed value $\pi/2$ we cannot have dynamical spinning motion associated with χ_{\downarrow} . Let us now check what follows from the condition that the second entry of the spinor must be 1. We must then have $-ie^{i\phi} = 1$, such that $\phi = \pi/2$. We have thus $(\theta, \phi) = (\pi/2, \pi/2)$ and $\varphi/2 = \omega_0\tau/2 = \pi/2$. This spinor cannot be used to describe spinning motion. An out-of-the-box solution would be $\phi = \omega\tau + \pi/2$. We would obtain then the “spinor”:

$$\begin{bmatrix} 0 \\ e^{i\omega\tau} \end{bmatrix}. \quad (45)$$

This pseudo-solution would then represent the rotation of a non-spinning electron whose rotation axis would be in the Oxy plane and precessing around the z -axis with an angular frequency ω . The net result would be like an electron spinning around the z -axis. But this is somewhat of a cheat because it transgresses the domain of the original definitions, and we can represent such a motion already by means of χ_{\uparrow} . Hence the two sets of “spinors” generated by the rotation group by operating on $\chi_{\uparrow}e^{-i\omega_0\tau/2}$ and $\chi_{\downarrow}e^{i\omega_0\tau/2}$ are physically disjoint. The quantity $\chi_{\downarrow}e^{i\omega_0\tau/2}$ is not a spinor that corresponds to a spinning motion.

We can therefore adopt without ambiguity the convention to rather use $\chi_{\downarrow}e^{i\omega_0\tau/2}$ in order to describe reversals, which are rotations of left-handed frames. Note that in a left-handed frame $\mathbf{a} \wedge \mathbf{b}$ is now defined according to the left-hand rule, such that $\omega|-\omega$ if we stick to the right-hand rule. The convention implies that χ_{\downarrow} represents a reference element with a left-handed frame, obtained from χ_{\uparrow} by the reflection operator $[\mathbf{e}_x \cdot \boldsymbol{\sigma}]$. We can in our convention consider the reference element χ_{\downarrow} as the identity element for the left-handed frames. In fact its triad of basis vectors consists of $(-\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$, which is a left-handed triad of basis vectors, when $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is a right-handed triad. The reflection operator σ_x transforms then a right-handed frame or triad into a left-handed frame or triad and vice versa. We can thus identify χ_{\downarrow} with a right-handed representation of the identity element and χ_{\uparrow} with a left-handed representation of the identity element.

Let us now call \mathbf{Q} the rotation around the axis parallel to $\mathbf{e}_z \wedge \mathbf{s}$ that rotates \mathbf{e}_z to \mathbf{s} (This is actually the second matrix in Eq. 21). Under this rotation vectors are transformed “quadratically” according to: $[\mathbf{s} \cdot \boldsymbol{\sigma}] = \mathbf{Q}[\mathbf{e}_z \cdot \boldsymbol{\sigma}]\mathbf{Q}^{\dagger}$. This transforms $\frac{1}{2}(\mathbb{1} - \mathbf{e}_z \cdot \boldsymbol{\sigma})$ into $\frac{1}{2}(\mathbb{1} - \mathbf{s} \cdot \boldsymbol{\sigma})$ and $\frac{1}{2}(\mathbb{1} + \mathbf{e}_z \cdot \boldsymbol{\sigma})$ into $\frac{1}{2}(\mathbb{1} + \mathbf{s} \cdot \boldsymbol{\sigma})$. The operators $\frac{1}{2}(\mathbb{1} \pm \mathbf{s} \cdot \boldsymbol{\sigma})$ play thus locally the same rôle for rotations around \mathbf{s} as $\frac{1}{2}(\mathbb{1} \pm \mathbf{e}_z \cdot \boldsymbol{\sigma})$ for rotations around \mathbf{e}_z . They represent thus really left- and right-handed frames in the spinning motion around \mathbf{s} as claimed in the main text.²³ We have always worked with the idea that a spinor is the first column of a SU(2) matrix. It is a right-handed formulation and shows how the spin-up spinor has been rotated. The second column of a SU(2) matrix corresponds to the left-handed formulation and shows how the spin-down spinor has been rotated. This action on the spinors of the two types of handedness can be clearly seen in the example of the rotation matrix in Eq. 43.

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²³ Note that the basis vectors $\chi_{\alpha} \otimes \chi_{\beta}^{\dagger}$, with $(\alpha, \beta) \in \{\uparrow, \downarrow\}^2$ of $M(2, \mathbb{C})$ acquire a second meaning within the multivector formalism of the Clifford algebra discussed in [6]. Here two of the basis vectors are isotropic vectors, which can be considered as representing oriented planes and defining complete triads. These interpretations are of no use here because our spinors χ_{\uparrow} and χ_{\downarrow} must represent states of spinning motion, such that we need the interpretation of the vector space $M(2, \mathbb{C})$ in terms of rotations, reflections and reversals rather than multivectors.