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A survey and new results

on Banach algebras of ultrametric continuous functions

by Monique Chicourrat, Bertin Diarra and Alain Escassut

Abstract Let \mathbb{K} be an ultrametric complete valued field and \mathbb{E} be an ultrametric space. We examine some Banach algebras S of bounded continuous functions from \mathbb{E} to \mathbb{K} with the use of ultrafilters, particularly the relation of stickness. We recall and deepen results obtained in a previous paper by N. Mainetti and the third author concerning the whole algebra \mathcal{A} of all bounded continuous functions from \mathbb{E} to \mathbb{K} . We show that every maximal ideal of finite codimension of \mathcal{A} is of codimension 1. Moreover, that property holds for every algebra S , provided \mathbb{K} is perfect. If S admits the uniform norm on \mathbb{E} as its spectral norm, then every maximal ideal is the kernel of only one multiplicative semi-norm, the Shilov boundary is equal to the whole multiplicative spectrum and the Banaschewski compactification of \mathbb{E} is homeomorphic to the multiplicative spectrum of S .

1. Introduction and basic results in topology

Let \mathbb{K} be an ultrametric complete valued field and \mathbb{E} be an ultrametric space. It is well known that the set of maximal ideals of a Banach \mathbb{K} -algebra is not sufficient to describe its spectral properties: we have to consider the set of continuous multiplicative semi-norms often called the multiplicative spectrum [2], [8], [11], [12], [14], [16]. In this paper, we generalize some of the results obtained in [15] to some Banach algebras of continuous bounded functions that we call *semi-admissible algebras* which concern maximal ideals, multiplicative spectrum, Shilov boundary and the Stone space of the Boolean ring of the clopen subsets of \mathbb{E} . The relation of *stickness* on ultrafilters, already considered in [14], here plays an important role.

Notations and definitions: Let \mathbb{K} be a complete valued field with respect to an ultrametric absolute value $|\cdot|$ and let \mathbb{E} denote a metric space whose distance δ is ultrametric. Given $a \in \mathbb{E}$ and $r > 0$, we denote by $d_{\mathbb{E}}(a, r^-)$ the open ball $\{x \in \mathbb{E} \mid \delta(a, x) < r\}$ and particularly in \mathbb{K} we denote by $d(a, r^-)$ the open disk $\{x \in \mathbb{K} \mid |x - a| < r\}$. In the same way, in \mathbb{E} , we denote by $d_{\mathbb{E}}(a, r)$ the closed ball $\{x \in \mathbb{E} \mid \delta(a, x) \leq r\}$ and we denote by $d(a, r)$ the closed disk $\{x \in \mathbb{K} \mid |x - a| \leq r\}$. Moreover, in \mathbb{K} , we denote by $C(a, r)$ the set $\{x \in \mathbb{K} \mid |x - a| = r\}$.

We denote by $|\cdot|_{\infty}$ the Archimedean absolute value of \mathbb{R} .

Given a subset H of \mathbb{E} , we denote by \overline{H} the closure of H in \mathbb{E} and the function u defined on \mathbb{E} by $u(x) = 1$ if $x \in H$ and $u(x) = 0$ otherwise is called the *characteristic function of H* .

We will denote by $\mathbb{B}(\mathbb{E})$ the Boolean ring of clopen subsets of \mathbb{E} with respect to the two classical laws Δ and \cap .

Let us recall this obvious lemma:

Lemma 1.1: *Let F be a subset of \mathbb{E} and let u be its characteristic function. Then u is continuous if and only if F is clopen.*

The following lemma is also clear since each ball of \mathbb{K} is clopen.

Lemma 1.2: *Let f be a continuous function from \mathbb{E} to \mathbb{K} and let $M > 0$. Given $M > 0$, the sets $E_1 = \{x \in \mathbb{E} \mid |f(x)| \geq M\}$ and $E_2 = \{x \in \mathbb{E} \mid |f(x)| \leq M\}$ are clopen.*

Corollary 1.2.a: *Let f be a continuous function from \mathbb{E} to \mathbb{K} , let $M > 0$ and let $h > 0$. Then $\{x \in \mathbb{E} \mid \left| |f(x)| - M \right|_{\infty} \leq h\}$ is clopen.*

Given a normed \mathbb{K} -algebra whose norm is $\| \cdot \|$, we denote by $\| \cdot \|_{sp}$ the spectral semi-norm that is associated and defined as $\|f\|_{sp} = \lim_{n \rightarrow +\infty} \left(\|f^n\| \right)^{\frac{1}{n}}$.

The Banach \mathbb{K} -algebra \mathcal{A} of all bounded continuous functions from \mathbb{E} to \mathbb{K} is provided with the norm $\| \cdot \|_0$ that we will call *the uniform norm* and which is defined as $\|f\|_0 = \sup\{|f(x)| \mid x \in \mathbb{E}\}$.

Recall the following:

Proposition 1.3: *Let T be a commutative unital Banach \mathbb{K} -algebra of bounded continuous functions defined from \mathbb{E} to \mathbb{K} . Then $\|f\|_0 \leq \|f\|_{sp} \leq \|f\| \quad \forall f \in T$. Moreover, given $f \in T$ satisfying $\|f\|_{sp} < 1$, then $\lim_{n \rightarrow +\infty} \|f^n\| = 0$.*

Proof: The norm $\| \cdot \|_0$ is power multiplicative and classically it is bounded by the norm $\| \cdot \|$ of T , it is then bounded by $\| \cdot \|_{sp}$. The last claim is immediate.

Definition. Let S be a \mathbb{K} -subalgebra of \mathcal{A} . We say that $(S, \| \cdot \|)$ is *semi-admissible* if it is a Banach algebra satisfying the following two properties:

- 1) For every $O \in \mathbb{B}(\mathbb{E})$, the characteristic function of O belongs to S .
- 2) For every $f \in S$ such that $\inf\{|f(x)| \mid x \in \mathbb{E}\} > 0$, f is invertible in S .

Moreover the semi-admissible algebra S will be called *admissible* if $\|f\|_0 = \|f\|_{sp} \quad \forall f \in S$.

Given a subset X of S , we call *spectral closure* of X denoted by \tilde{X} the closure of X with respect to the semi-norm $\| \cdot \|_{sp}$ and X will be said to be *spectrally closed* if $X = \tilde{X}$. Moreover, X will be said to be *uniformly closed* if it is closed with respect to the uniform norm and we call *uniform closure* of X the closure of X with respect to the semi-norm $\| \cdot \|_0$.

Throughout the paper the algebra S will be supposed to be semi-admissible.

Let $f \in \mathcal{A}$ be such that $\inf\{|f(x)| \mid x \in \mathbb{E}\} > 0$, it is clear that $\frac{1}{f}$ belongs to \mathcal{A} . On the other hand, \mathcal{A} is complete with respect to the uniform norm, then we have the following statement:

Proposition 1.4: *The Banach \mathbb{K} -algebra \mathcal{A} is admissible.*

The following theorem 1.6 shows another example of admissible algebra which is a bit less immediate. In Theorem 1.8, we can see that in various cases, this algebra is strictly included in \mathcal{A} .

Lemma 1.5. *Let $(O_i)_{i=1,\dots,n}$ be a finite cover of \mathbb{E} with clopen sets. Then there exists a finite cover $(B_j)_{j=1,\dots,p}$ of \mathbb{E} where the sets B_j are not empty, clopen, pairwise disjoint and such that every B_j is contained in some O_i .*

Proof. To the system $(O_i)_{i=1,\dots,n}$, associate the system $(O'_i)_{i=1,\dots,2n}$ where $O'_i = O_i$ if $1 \leq i \leq n$ and $O'_i = X \setminus O_{i-n}$ otherwise. For every $x \in \mathbb{E}$ define $I_x = \{i \in \{1, \dots, 2n\} : x \in O'_i\}$ and consider the following equivalence relation on \mathbb{E} : $x(R)y$ if and only if $I_x = I_y$. For any $x \in \mathbb{E}$ the equivalence class of x is equal to $\bigcap_{i \in I_x} O'_i$ and it is clopen since so are the O'_i . Then the equivalence classes $(B_j)_{j=1,\dots,p}$ satisfy the expected properties.

Theorem 1.6. *Let \mathcal{T} be the \mathbb{K} -subalgebra of \mathcal{A} generated by the characteristic functions of all clopen sets of \mathbb{E} and let \mathcal{G} be its closure in \mathcal{A} (for the uniform convergence $\| \cdot \|_0$ on \mathbb{E}). Then \mathcal{G} is admissible.*

Proof. One just has to prove Property 2) in the definition of a semi-admissible algebra.

First we check that if $g \in \mathcal{T}$ is such that $\inf\{|g(x)| : x \in \mathbb{E}\} = m > 0$ then $\frac{1}{g} \in \mathcal{T}$. Since $g \in \mathcal{T}$ there exists a finite cover $(O_i)_{i=1,\dots,n}$ of \mathbb{E} with clopen sets and scalars

$(\lambda_i)_{1 \leq i \leq n}$ in \mathbb{K} such that $g = \sum_{i=1}^n \lambda_i u_i$ where u_i is the characteristic function of the clopen

O_i . Using the preceding lemma we get a finite cover $(B_j)_{j=1,\dots,p}$ of \mathbb{E} where the sets B_j are not empty, clopen, pairwise disjoint and such that every B_j is contained in some

O_i . Then there exist scalars $(\beta_j)_{1 \leq j \leq p}$ in \mathbb{K} such that $g = \sum_{j=1}^p \beta_j e_j$ where e_j is the

characteristic function of the clopen B_j . For every j we get $|\beta_j| \geq m > 0$, then it is clear

that $\frac{1}{g} = \sum_{j=1}^p \frac{1}{\beta_j} e_j$ and $\frac{1}{g} \in \mathcal{T}$.

Now consider any $f \in \mathcal{G}$ such that $\inf\{|f(x)| : x \in \mathbb{E}\} = m > 0$. For every $\epsilon > 0$ such that $\epsilon < \frac{1}{m}$ we have $\epsilon m^2 < m$ and we can consider some $g \in \mathcal{T}$ such that $\|f - g\|_0 \leq \epsilon m^2$. Since $\epsilon m^2 < m$, we get $|f(x)| = |g(x)|$ for every $x \in \mathbb{E}$ and then $\inf\{|g(x)| : x \in \mathbb{E}\} = m$.

Next $\frac{1}{g} \in \mathcal{T}$ and we have for every $x \in \mathbb{E}$: $\left| \frac{1}{f(x)} - \frac{1}{g(x)} \right| = \frac{|f(x) - g(x)|}{|f(x)g(x)|} \leq \frac{\|f - g\|_0}{m^2} \leq \epsilon$.

This proves that $\frac{1}{f} \in \mathcal{G}$, which ends the proof.

The following proposition is Example 3E, chap.3 in [19].

Proposition 1.7: *The algebra \mathcal{G} is the \mathbb{K} -algebra of the continuous functions from \mathbb{E} to \mathbb{K} such that the closure of $f(\mathbb{E})$ in \mathbb{K} is compact. In particular when \mathbb{K} is locally compact or \mathbb{E} is compact then $\mathcal{G} = \mathcal{A}$.*

In order to prove Theorem 1.9, we must recall the following classical proposition:

Proposition 1.8 : *The field \mathbb{K} is locally compact if and only if its valuation group is discrete and its residue class field is finite. [1], (Proposition 2.3.3) .*

Theorem 1.9: *Suppose that \mathbb{E} contains a sequence $(a_n)_{n \in \mathbb{N}}$ such that $\inf_{n \neq m} (\delta(a_n, a_m)) > 0$ and that \mathbb{K} is not locally compact. Then \mathcal{G} is strictly included in \mathcal{A} .*

Proof: We put $s = \inf_{n \neq m} (\delta(a_n, a_m))$. Suppose first that the valuation group of \mathbb{K} is dense. We can consider a partition of \mathbb{E} by an infinite family of balls $d_{\mathbb{E}}(b_i, s^-)$.

Suppose first that the valuation group of \mathbb{K} is dense. Then we can define a bounded mapping ψ from \mathbb{E} into \mathbb{K} such that $\psi(x)$ is constant in each ball $d_{\mathbb{E}}(b_i, s^-)$, such that $|\psi(a_n) - \psi(a_m)| \geq 1$ and such that $|\psi(x)| \geq 1 \forall x \in \mathbb{E}$. Particularly, $|\psi(x)|$ takes infinitely many values. Suppose that \mathcal{T} is dense in \mathcal{A} . Following the same process as in the proof of Theorem 1.6 above, we can construct a function $g \in \mathcal{T}$ such that $|\psi(x)| = |g(x)| = \lambda_j \forall x \in \mathbb{E}$. But $|\psi(x)|$ then only takes finitely many values, a contradiction.

Similarly, suppose now that the residue class field of \mathbb{K} is infinite. Let us consider a sequence of distinct disks $(d(\mu_n, 1^-))_{n \in \mathbb{N}}$ in the unit circle and and put $B_n = d(\mu_n, 1^-)$. Now, consider a sequence of balls $d_{\mathbb{E}}(a_n, s^-)$ in \mathbb{E} and an element f of \mathcal{A} constant in $d_{\mathbb{E}}(a_n, s)$ and such that $f(a_n)$ belongs to B_n . Suppose that f is in the closure of \mathcal{T} . Then there exists $g \in \mathcal{T}$ such that $|f(x) - g(x)| < 1$ for any $x \in \mathbb{E}$. In particular, we get that $g(a_n) \in B_n$ for every n . Thus, g should take infinitely many values, a contradiction.

Definition: Recall that an element x of a normed \mathbb{K} -algebra A is called a *topological divisor of zero* if $\inf_{y \in A, \|y\| \geq 1} \|xy\| = 0$.

Theorem 1.10: *Suppose that \mathbb{E} has no isolated points. Let T be an admissible Banach \mathbb{K} -algebra complete with respect to the norm $\| \cdot \|_0$. An element of T is a topological divisor of zero if and only if it is not invertible.*

Proof: It is obvious that an invertible element of T is not a topological divisor of zero. Now, consider an element $f \in T$ that is not invertible. Then $\inf_{x \in \mathbb{E}} |f(x)| = 0$. Therefore, there exists a sequence of disks $(d_{\mathbb{E}}(a_n, r_n))_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} r_n = 0$, such that $|f(x)| \leq \frac{1}{n}, \forall x \in d_{\mathbb{E}}(a_n, r_n), \forall n \in \mathbb{N}^*$. For each $n \in \mathbb{N}$, let h_n be the characteristic function of $d_{\mathbb{E}}(a_n, r_n)$. Then h_n belongs to T and satisfies $\|h_n\|_0 = 1 \forall n \in \mathbb{N}^*$. On the other hand, we have $\|fh_n\|_0 \leq \frac{1}{n}$, hence $\lim_{n \rightarrow +\infty} fh_n = 0$.

More notations and definitions: Let \mathcal{F} be a filter on \mathbb{E} . Given a function f from \mathbb{E} to \mathbb{K} admitting a limit along \mathcal{F} , we will denote by $\lim_{\mathcal{F}} f(x)$ this limit.

Given a filter \mathcal{F} on \mathbb{E} , we will denote by $\mathcal{I}(\mathcal{F}, S)$ the ideal of the $f \in S$ such that $\lim_{\mathcal{F}} f(x) = 0$. Notice that the unity does not belong to $\mathcal{I}(\mathcal{F}, S)$, so $\mathcal{I}(\mathcal{F}, S) \neq S$.

Given $a \in \mathbb{E}$, we will denote by $\mathcal{I}(a, S)$ the ideal of the $f \in S$ such that $f(a) = 0$ and by $\mathcal{I}'(a, S)$ the ideal of the $f \in S$ such that there exists an open neighborhood L of a such that $f(x) = 0 \forall x \in L$.

We will denote by $Max(S)$ the set of maximal ideals of S and by $Max_{\mathbb{E}}(S)$ the set of maximal ideals of S of the form $\mathcal{I}(a, S)$, $a \in \mathbb{E}$.

Given a set F , we will denote by $U(F)$ the set of ultrafilters on F .

Two ultrafilters \mathcal{U}, \mathcal{V} on \mathbb{E} will be said to be *sticked* if for every closed subsets $H \in \mathcal{U}$, $G \in \mathcal{V}$, we have $H \cap G \neq \emptyset$.

We will denote by (\mathcal{R}) the relation defined on $U(\mathbb{E})$ as $\mathcal{U}(\mathcal{R})\mathcal{V}$ if \mathcal{U} and \mathcal{V} are stucked [14].

Remark 1: Relation (\mathcal{R}) is not the equality between ultrafilters, even when the ultrafilters are not convergent. In [17], Labib Haddad introduced the following equivalence relation (\mathcal{H}) on ultrafilters. Given two ultrafilters \mathcal{U}, \mathcal{V} we write $\mathcal{U}(\mathcal{H})\mathcal{V}$ if there exists an ultrafilter \mathcal{W} such that every closed set L lying in \mathcal{W} also lies in \mathcal{U} and similarly, every closed set L lying in \mathcal{W} also lies in \mathcal{V} . So, Relation (\mathcal{H}) is clearly thinner than Relation (\mathcal{R}) . However, it is shown that two ultrafilters \mathcal{U}, \mathcal{V} satisfying $\mathcal{U}(\mathcal{H})\mathcal{V}$ may be distinct without converging.

The following lemma is classical [7]:

Lemma 1.11: *Given $\mathcal{U} \in U(\mathbb{E})$ and a subset X of \mathbb{E} , then either $X \in \mathcal{U}$ or $(\mathbb{E} \setminus X) \in \mathcal{U}$.*

Theorem 1.12:

1) *if F and G are disjoint closed subsets of \mathbb{E} then there exists a clopen O such that $F \subset O$ and $G \subset (\mathbb{E} \setminus O)$.*

This is the case when $\delta(F, G) > 0$.

2) *If \mathcal{U} and \mathcal{V} are ultrafilters on \mathbb{E} then they are stucked if and only if they contain the same clopen sets.*

In particular if \mathcal{U}, \mathcal{V} are not stucked, there exist disjoint clopen subsets H and L of \mathbb{E} such that $H \in \mathcal{U}$, $H \notin \mathcal{V}$ and $L \in \mathcal{V}$, $L \notin \mathcal{U}$.

Proof:

1) For each $x \in F$ take $r_x > 0$ such that $d(x, r_x^-) \cap G = \emptyset$ and define the open set $O = \bigcup_{x \in F} d(x, r_x^-)$. We clearly have $F \subset O$ and $G \subset \mathbb{E} \setminus O$. Let us prove that O is closed.

Let $y \in \overline{O}$. For every $n \in \mathbb{N}^*$, there exists $x_n \in F$ such that $d(y, \frac{1}{n}^-) \cap d(x_n, r_{x_n}^-) \neq \emptyset$, then let $y_n \in d(y, \frac{1}{n}^-) \cap d(x_n, r_{x_n}^-)$.

First assume that $\inf\{r_{x_n} : n \in \mathbb{N}^*\} = m > 0$. Take $n \in \mathbb{N}^*$ such that $\frac{1}{n} < m$. Since the distance is ultrametric we then have: $d(y, \frac{1}{n}^-) = d(y_n, \frac{1}{n}^-) \subset d(y_n, r_{x_n}^-) = d(x_n, r_{x_n}^-)$. Finally $y \in O$.

Assume now that $\inf\{r_{x_n} : n \in \mathbb{N}^*\} = 0$. There exists a subsequence $(x_{n_k})_k$ such that $(r_{x_{n_k}})_k$ tends to 0. Then we immediately get that $(x_{n_k})_k$ tends to y since $(y_{n_k})_k$ tends to y . So $y \in \overline{F} = F$ and again $y \in O$.

2) If \mathcal{U} and \mathcal{V} are sticked then for every clopen $O \in \mathcal{U}$ we necessarily have $O \in \mathcal{V}$. Otherwise using the preceding lemma the clopen $\mathbb{E} \setminus O$ is in \mathcal{V} so \mathcal{U} and \mathcal{V} cannot be sticked. Conversely, if \mathcal{U} and \mathcal{V} contain the same clopen sets then using the preceding property 1), for every closed sets $F \in \mathcal{U}$ and $G \in \mathcal{V}$ we necessarily get $F \cap G \neq \emptyset$, otherwise taking a clopen O such in 1) we have $O \in \mathcal{U}$ and $O \notin \mathcal{V}$ since $\mathbb{E} \setminus O \in \mathcal{V}$.

In particular if \mathcal{U} and \mathcal{V} are not sticked then taking some clopen H in \mathcal{U} which is not in \mathcal{V} , we have $(\mathbb{E} \setminus H) \in \mathcal{V}$ and putting $L = \mathbb{E} \setminus H$, H and L are clopen sets satisfying the expected property.

Corollary 1.12.a: *Let \mathcal{U}, \mathcal{V} be two ultrafilters on E that are not sticked. There exist clopen subsets $H \in \mathcal{U}, L \in \mathcal{V}$ and $f \in S$ such that $f(x) = 1 \forall x \in H, f(x) = 0 \forall x \in L$.*

Lemma 1.13 is classical:

Lemma 1.13: *Let \mathcal{U} be an ultrafilter on \mathbb{E} . Let f be a bounded function from \mathbb{E} to \mathbb{K} . The function $|f|$ from \mathbb{E} to \mathbb{R}_+ defined as $|f|(x) = |f(x)|$ admits a limit along \mathcal{U} . Moreover, if \mathbb{K} is locally compact, then $f(x)$ admits a limit along \mathcal{U} .*

Recall that for any normed \mathbb{K} -algebra $(G, \|\cdot\|)$, the closure of an ideal of G is an ideal of G . Lemmas 1.14 and 1.15 are immediate:

Lemma 1.14: *The spectral closure of an ideal of S is an ideal of S .*

Lemma 1.15: *Let $X \subset S$ be spectrally closed. Then X is closed with respect to the norm of S . Let $Y \subset S$ be uniformly closed. Then it is spectrally closed.*

Now we can recall a classical result known in ultrametric analysis as in Archimedean analysis.

Proposition 1.16: *Every maximal ideal \mathcal{M} of S is spectrally closed.*

Proof: By Lemma 1.14 the spectral closure $\widetilde{\mathcal{M}}$ of \mathcal{M} is an ideal. If \mathcal{M} is not spectrally closed, then $\widetilde{\mathcal{M}} = S$, hence there exists $t \in S$ such that $1 - t \in \mathcal{M}$ and $\|t\|_{sp} < 1$.

Consequently, by Proposition 1.3 $\lim_{n \rightarrow +\infty} \|t^n\| = 0$, therefore the series $(\sum_{n=0}^{\infty} t^n)$ converges

and $(\sum_{n=0}^{\infty} t^n)(1 - t) = 1$ and hence the unity belongs to \mathcal{M} , a contradiction.

Proposition 1.17 now is easy:

Proposition 1.17: *Given an ultrafilter \mathcal{U} on \mathbb{E} , $\mathcal{I}(\mathcal{U}, S)$ is a prime ideal. Moreover, $\mathcal{I}(\mathcal{U}, S)$ is uniformly closed and hence is spectrally closed and closed for the topology of S .*

Proof: Since \mathcal{U} is an ultrafilter, it is obvious that $\mathcal{I}(\mathcal{U}, S)$ is prime. Indeed, given $f \in S$, by Lemma 1.13, $|f(x)|$ admits a limit along \mathcal{U} and hence, if $f, g \in S$ are such that

$\lim_{\mathcal{U}} f(x)g(x) = 0$, then either $\lim_{\mathcal{U}} f(x) = 0$ or $\lim_{\mathcal{U}} g(x) = 0$, hence either f or g belongs to $\mathcal{I}(\mathcal{U}, S)$.

Let us now check that $\mathcal{I}(\mathcal{U}, S)$ is uniformly closed. Indeed let g in the closure of $\mathcal{I}(\mathcal{U}, S)$ with respect to $\| \cdot \|_0$, let $b = \lim_{\mathcal{U}} |g(x)|$ and suppose $b > 0$. There exists $f \in \mathcal{I}(\mathcal{U}, S)$ such that $\|f - g\|_0 < b$ and then

$$b = \left| \lim_{\mathcal{U}} |f(x)| - \lim_{\mathcal{U}} |g(x)| \right|_{\infty} \leq \lim_{\mathcal{U}} |f(x) - g(x)| \leq \|f - g\|_0 < b,$$

a contradiction showing that $\mathcal{I}(\mathcal{U}, S)$ is uniformly closed. Therefore, it is spectrally closed and closed for the topology of S .

The following Theorem 1.18 is proved in [14].

Theorem 1.18: *Let \mathcal{U}, \mathcal{V} be two ultrafilters on \mathbb{E} . Then $\mathcal{I}(\mathcal{U}, S) = \mathcal{I}(\mathcal{V}, S)$ if and only if \mathcal{U} and \mathcal{V} are sticked.*

Proof: First, if \mathcal{U} and \mathcal{V} are not sticked, by Corollary 1.12.a we have $\mathcal{I}(\mathcal{U}, S) \neq \mathcal{I}(\mathcal{V}, S)$. Now, suppose that \mathcal{U}, \mathcal{V} are sticked. By Theorem 1.12, then they contain the same clopen sets. But for every $f \in S$ and $\epsilon > 0$ the set $L_{\epsilon} = \{x \in \mathbb{E} : |f(x)| \leq \epsilon\}$ is clopen and we have: $f \in \mathcal{I}(\mathcal{U}, S) \iff \forall \epsilon > 0, L_{\epsilon} \in \mathcal{U}$ and hence $L_{\epsilon} \in \mathcal{V}$. Consequently, $\forall \epsilon > 0, L_{\epsilon} \in \mathcal{V}$ and hence f belongs to $\mathcal{I}(\mathcal{V}, S)$. Thus $\mathcal{I}(\mathcal{U}, S) \subset \mathcal{I}(\mathcal{V}, S)$ and similarly, $\mathcal{I}(\mathcal{V}, S) \subset \mathcal{I}(\mathcal{U}, S)$, therefore $\mathcal{I}(\mathcal{V}, S) = \mathcal{I}(\mathcal{U}, S)$

Corollary 1.18.a: *Relation (\mathcal{R}) is an equivalence relation on $U(\mathbb{E})$.*

Theorem 1.19 looks like certain Bezout-Corona statements [18]:

Theorem 1.19: *Let $f_1, \dots, f_q \in S$ satisfy $\inf_{x \in \mathbb{E}} (\max_{1 \leq j \leq q} |f_j(x)|) > 0$. Then there exist*

$g_1, \dots, g_q \in S$ such that $\sum_{j=1}^q f_j(x)g_j(x) = 1 \ \forall x \in \mathbb{E}$.

Proof: Let $M = \inf_{x \in \mathbb{E}} (\max_{1 \leq j \leq q} |f_j(x)|)$. Let $E_j = \{x \in \mathbb{E} \mid |f_j(x)| \geq M\}$, $j = 1, \dots, q$ and let $F_j = \bigcup_{m=1}^j E_m$, $j = 1, \dots, q$. Let $g_1(x) = \frac{1}{f_1(x)} \forall x \in E_1$ and $g_1(x) = 0 \ \forall x \in \mathbb{E} \setminus E_1$. Since $|f_1(x)| \geq M \ \forall x \in E_1$, $|g_1(x)|$ is clearly bounded. By Lemma 1.2 each E_j is obviously clopen and so is each F_j . And since f_1 is continuous g_1 is continuous, hence belongs to S .

Suppose now we have constructed $g_1, \dots, g_k \in S$ satisfying $\sum_{j=1}^k f_j g_j(x) = 1 \ \forall x \in F_k$

and $\sum_{j=1}^k f_j g_j(x) = 0 \ \forall x \in \mathbb{E} \setminus F_k$. Let g_{k+1} be defined on \mathbb{E} by $g_{k+1}(x) = \frac{1}{f_{k+1}(x)} \ \forall x \in$

$F_{k+1} \setminus F_k$ and $g_{k+1}(x) = 0 \ \forall x \in \mathbb{E} \setminus (F_{k+1} \setminus F_k)$. Since F_k and F_{k+1} are clopens, so is $\mathbb{E} \setminus (F_{k+1} \setminus F_k)$ and consequently, g_{k+1} is continuous. Similarly as for g_1 , since

$|f_{k+1}(x)| \geq M \forall x \in E_{k+1}$, $|g_{k+1}(x)|$ is clearly bounded, hence g_{k+1} belongs to S . Now we can check that $\sum_{j=1}^{k+1} f_j g_j(x) = 1 \forall x \in F_{k+1}$ and $\sum_{j=1}^k f_j g_j(x) = 0 \forall x \in \mathbb{E} \setminus F_{k+1}$. So, by a finite induction, we get functions $g_1, \dots, g_q \in S$ such that $\sum_{j=1}^q f_j g_j(x) = 1 \forall x \in \mathbb{E}$, which ends the proof.

Notation: Let $f \in S$ and let $\epsilon > 0$. We set $D(f, \epsilon) = \{x \in \mathbb{E} \mid |f(x)| \leq \epsilon\}$.

Corollary 1.19.a: *Let I be an ideal of S different from S . The family of sets $\{D(f, \epsilon), f \in I, \epsilon > 0\}$ generates a filter $\mathcal{F}_{I,S}$ on \mathbb{E} such that $I \subset \mathcal{I}(\mathcal{F}_{I,S}, S)$.*

2. Maximal and prime ideals of S

Except Theorem 2.4 and its corollaries, most of the results of this paragraph were given in [14] for the algebra \mathcal{A} .

Theorem 2.1: *Let \mathcal{M} be a maximal ideal of S . There exists an ultrafilter \mathcal{U} on \mathbb{E} such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$. Moreover, \mathcal{M} is of codimension 1 if and only if every element of S converges along \mathcal{U} . In particular if \mathcal{U} is convergent, then \mathcal{M} is of codimension 1.*

Proof: Indeed, by Corollary 1.19.a, we can consider the filter $\mathcal{F}_{\mathcal{M},S}$ and we have $\mathcal{M} \subset \mathcal{I}(\mathcal{F}_{\mathcal{M},S}, S)$. Let \mathcal{U} be an ultrafilter thinner than $\mathcal{F}_{\mathcal{M},S}$. So, we have $\mathcal{M} \subset \mathcal{I}(\mathcal{F}_{\mathcal{M},S}, S) \subset \mathcal{I}(\mathcal{U}, S)$. But since \mathcal{M} is a maximal ideal, either $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$, or $\mathcal{I}(\mathcal{U}, S) = S$. But obviously, $\mathcal{I}(\mathcal{U}, S) \neq S$, hence $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$.

Now assume that \mathcal{M} is of codimension 1 and let χ be the \mathbb{K} -algebra homomorphism from S to \mathbb{K} admitting \mathcal{M} for kernel. Let $f \in S$ and let $b = \chi(f)$. Then $f - b$ belongs to the kernel of \mathcal{M} , hence $\lim_{\mathcal{U}} f(x) - b = 0$ that is $\lim_{\mathcal{U}} f(x) = b$ therefore every element of S converges along \mathcal{U} .

Conversely if every element of S admits a limit along \mathcal{U} then the mapping χ which associates to each $f \in S$ its limit along \mathcal{U} is a \mathbb{K} -algebra homomorphism from S to \mathbb{K} admitting \mathcal{M} for kernel, therefore \mathcal{M} is of codimension 1.

In particular if \mathcal{U} converges to a point a , then each f in S converges to $f(a)$ along \mathcal{U} .

By Lemma 1.13 and Theorem 2.1, the following corollary is immediate:

Corollary 2.1.a: *Let \mathbb{K} be a locally compact field. Then every maximal ideal of S is of codimension 1.*

Remark 3: If \mathbb{K} is locally compact, a maximal ideal of codimension 1 of S is not necessarily of the form $\mathcal{I}(\mathcal{U}, S)$ where \mathcal{U} is a Cauchy ultrafilter. Suppose that \mathbb{E} admits a sequence $(a_n)_{n \in \mathbb{N}}$ such that either it satisfies $|a_n - a_m| = r \forall n \neq m$, or the sequence $|a_{n+1} - a_n|$ is strictly increasing. Let \mathcal{U} be an ultrafilter thinner than the sequence $(a_n)_{n \in \mathbb{N}}$.

Consider now a function $f \in S$. Since \mathbb{K} is locally compact, $f(x)$ does converge along \mathcal{U} to a point $b \in \mathbb{K}$. In that way, we can define a homomorphism χ from S onto \mathbb{K} as $\chi(g) = \lim_{\mathcal{U}} g(x)$ and therefore \mathbb{K} is the quotient $\frac{S}{\text{Ker}(\chi)}$. So $\mathcal{I}(\mathcal{U}, S)$ is a maximal ideal of codimension 1.

Notation: Following notations of [14], we will denote by $Y_{(\mathcal{R})}(\mathbb{E})$ the set of equivalence classes on $U(\mathbb{E})$ with respect to Relation (\mathcal{R}) .

By Theorem 1.18, we can get Corollary 2.1.b:

Corollary 2.1.b: *Let \mathcal{M} be a maximal ideal of S . There exists a unique $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{E})$ such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$ for every $\mathcal{U} \in \mathcal{H}$.*

Conversely, Theorem 2.2 now characterizes all maximal ideals of S .

Theorem 2.2: *Let \mathcal{U} be an ultrafilter on \mathbb{E} . Then $\mathcal{I}(\mathcal{U}, S)$ is a maximal ideal of S .*

Proof: Let $I = \mathcal{I}(\mathcal{U}, S)$ and let \mathcal{M} be a maximal ideal of S containing I . Then by Theorem 2.1 there exists an ultrafilter \mathcal{V} such that $\mathcal{M} = \mathcal{I}(\mathcal{V}, S)$. Suppose now $\mathcal{I}(\mathcal{U}, S) \neq \mathcal{I}(\mathcal{V}, S)$. Then, \mathcal{U} and \mathcal{V} are not stucked. Consequently, by Theorem 1.12 there exists a clopen subset $F \in \mathcal{V}$ that does not belong to \mathcal{U} and hence its characteristic function $u \in S$ belongs to $\mathcal{I}(\mathcal{U}, S)$ but does not belong to $\mathcal{I}(\mathcal{V}, S)$. Thus, u belongs to I but does not belong to \mathcal{M} , a contradiction to the hypothesis.

By Corollary 2.1.b and Theorem 2.2, we can derive the following Corollary 2.2.a:

Corollary 2.2.a: *The mapping that associates to each maximal ideal \mathcal{M} of S the class with respect to (\mathcal{R}) of ultrafilters \mathcal{U} , such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$, is a bijection from $\text{Max}(S)$ onto $Y_{(\mathcal{R})}(\mathbb{E})$.*

Remark 4: Let \mathcal{F} be a Cauchy filter on \mathbb{E} admitting a limit $a \in \mathbb{E}$ and let $\mathcal{M} = \mathcal{I}(\mathcal{F}, S)$. Then every function $f \in S$ converges to a limit $\theta(f)$ along \mathcal{F} and \mathcal{M} is a maximal ideal of codimension 1. Indeed, let $f \in S$. Since f is continuous, then $f(x)$ converges to a point $\theta(f) = f(a)$ in \mathbb{K} . Consider now the mapping θ from S into \mathbb{K} : it is an algebra morphism whose kernel is \mathcal{M} and whose image is \mathbb{K} . Consequently, $\frac{S}{\mathcal{M}} = \mathbb{K}$, therefore \mathcal{M} is a maximal ideal of codimension 1.

Notation: For any subset F of \mathbb{E} , we denote by u_F its characteristic function. Let \mathcal{M} be a maximal ideal of S and let $\mathcal{U} \in U(\mathbb{E})$ be such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$. By Theorems 1.18 we can define the set $\mathcal{O}_{\mathcal{M}}$ of all clopen subsets of \mathbb{E} which belong to \mathcal{U} . We then denote by $\mathcal{C}_{\mathcal{M}}$ the set $\{u_{\mathbb{E} \setminus L} \mid L \in \mathcal{O}_{\mathcal{M}}\}$ and by $\mathcal{J}_{\mathcal{M}}$ the set of all functions $f \in S$ which are equal to 0 on some $L \in \mathcal{O}_{\mathcal{M}}$.

Given $a \in \mathbb{E}$, we will denote by $\mathcal{I}'(a, S)$ the ideal of the functions $f \in S$ equal to 0 on an open subset of \mathbb{E} containing a .

Theorem 2.3: *Let \mathcal{M} be a maximal ideal of S .*

- 1) \mathcal{J}_M is an ideal of S containing \mathcal{C}_M ,
- 2) \mathcal{J}_M is the ideal of S generated by \mathcal{C}_M and $\mathcal{J}_M = \{fu \mid f \in S, u \in \mathcal{C}_M\}$,
- 3) If \mathcal{P} is a prime ideal of S contained in \mathcal{M} , then $\mathcal{J}_M \subset \mathcal{P}$.
- 4) if $\mathcal{M} = \mathcal{I}(a, S)$, then $\mathcal{I}'(a, S) = \mathcal{J}_M$.

Proof: 1) Let us check that \mathcal{J}_M is an ideal of S . Let $f, g \in \mathcal{J}_M$. So, there exist $F, G \in \mathcal{O}_M$ such that $f(x) = 0 \forall x \in F$, $g(x) = 0 \forall x \in G$, hence $f(x) - g(x) = 0 \forall x \in F \cap G$. Since $F \cap G$ belongs to \mathcal{O}_M , $f - g$ lies in \mathcal{J}_M . And obviously, for every $h \in S$, we have $h(x)f(x) = 0 \forall x \in F$, hence fh lies in \mathcal{J}_M .

Next, \mathcal{J}_M contains \mathcal{C}_M because given $L \in \mathcal{O}_M$, the set $\mathbb{E} \setminus L$ is clopen, then $u_{\mathbb{E} \setminus L}$ belongs to S and is equal to 0 on L .

2) Notice that if $f \in S$ and $u \in \mathcal{C}_M$, then by 1) fu belongs to \mathcal{J}_M . Conversely, if $f \in \mathcal{J}_M$ and $L \in \mathcal{O}_M$ are such that $f(x)$ is equal to 0 on L , then $u_{\mathbb{E} \setminus L}$ belongs to \mathcal{C}_M and $f = fu_{\mathbb{E} \setminus L}$. This proves that $\mathcal{J}_M = \{fu \mid f \in S, u \in \mathcal{C}_M\}$ and that \mathcal{J}_M is the ideal generated by \mathcal{C}_M .

3) It is sufficient to prove that \mathcal{C}_M is included in \mathcal{P} . Indeed, let $\mathcal{U} \in U(\mathbb{E})$ be such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$ and let $L \in \mathcal{O}_M$. Then $L \in \mathcal{U}$ and $u_L \notin \mathcal{M}$. So, $u_L \notin \mathcal{P}$. But $u_L \cdot u_{\mathbb{E} \setminus L} = 0$. Thus $u_{\mathbb{E} \setminus L}$ belongs to \mathcal{P} since \mathcal{P} is prime.

4) Just notice that \mathcal{J}_M is the set of all functions in S which are equal to 0 on some clopen containing a and that each open neighborhood of a contains a disk $d_{\mathbb{E}}(a, r^-)$, which is clopen.

Corollary 2.3.a: *Let \mathcal{U} be an ultrafilter on \mathbb{E} and let \mathcal{P} be a prime ideal included in $\mathcal{I}(\mathcal{U}, S)$. Let $L \in \mathcal{U}$ be clopen and let $H = \mathbb{E} \setminus L$. Then the characteristic function u of H belongs to \mathcal{P} .*

Theorem 2.4: *Let \mathcal{M} be a maximal ideal of S . The uniform closure of \mathcal{J}_M is equal to \mathcal{M} .*

Proof: Let $f \in \mathcal{M} = \mathcal{I}(\mathcal{U}, S)$. Then for every $\epsilon > 0$ the set $L = D(f, \epsilon)$ belongs to \mathcal{U} and L is clopen. Therefore L belongs to \mathcal{O}_M and the characteristic function u of $\mathbb{E} \setminus L$ lies in \mathcal{C}_M , so that $fu \in \mathcal{J}_M$. But $f(x) - uf(x) = 0 \forall x \notin L$ and $|f(x) - uf(x)| = |f(x)| \leq \epsilon \forall x \in L$, so $\|f - uf\|_0 \leq \epsilon$. Hence \mathcal{M} is the uniform closure of \mathcal{J}_M since, by Proposition 1.16, \mathcal{M} is uniformly closed.

Corollary 2.4.a: *Let \mathcal{P} be a prime ideal contained in a maximal ideal \mathcal{M} . Then \mathcal{M} is the uniform closure of \mathcal{P} .*

Corollary 2.4.b: *The uniform closure of a prime ideal of S is a maximal ideal of S and a prime ideal of S is contained in a unique maximal ideal of S .*

Corollary 2.4.c: *A prime ideal of S is a maximal ideal if and only if it is uniformly closed.*

Using property 4 of theorem 2.3 we get Corollary 2.4.d:

Corollary 2.4.d: *The uniform closure of $\mathcal{I}'(a, S)$ is $\mathcal{I}(a, S)$.*

Corollary 2.4.e: *Let \mathcal{M} be a maximal ideal of S . If S is admissible then:*

- 1) \mathcal{M} is the spectral closure of $\mathcal{J}_{\mathcal{M}}$ and the spectral closure of any prime ideal contained in \mathcal{M} ;
- 2) a prime ideal is maximal if and only if it is spectrally closed.

3 Maximal ideals of finite codimension

The main results of this paragraph were already obtained in [5]. We recall them with all proofs in order to make easy the conclusions of this article.

Notation: Let \mathbb{L} be a finite algebraic extension of \mathbb{K} provided with the absolute value which extends that of \mathbb{K} and let $t = [\mathbb{L} : \mathbb{K}]$. Let \mathcal{A}^e be equal to the \mathbb{L} -algebra of the bounded continuous functions of \mathbb{E} into \mathbb{L} and $\widehat{\mathcal{A}} = \mathbb{L} \otimes_{\mathbb{K}} \mathcal{A}$. Since \mathbb{L} is of finite dimension over \mathbb{K} , one obtains an immediate identification of \mathcal{A}^e with $\widehat{\mathcal{A}}$

The following Theorem 3.2 holds on all complete valued fields and is proven in [6] (Lemma 7.2). First we must state Lemma 3.1.

Lemma 3.1: *Let \mathbb{L} be of the form $\mathbb{L} = \mathbb{K}[a]$. Let $f \in \mathcal{A}^e$. Then f is of the form $\sum_{j=0}^{t-1} a^j f_j$, $j = 0, \dots, t-1$, with $f_j \in \mathcal{A}$.*

Theorem 3.2: *Let $T = \mathcal{A}$. Suppose there exists a morphism of \mathbb{K} -algebra, χ from T onto \mathbb{L} . Then χ has continuation to a surjective morphism of \mathbb{L} -algebra $\widehat{\chi}$ from \widehat{T} to \mathbb{L} .*

Proof: Suppose first that \mathbb{L} is of the form $\mathbb{K}[a]$. Let $f, g \in \widehat{T}$. Then by Lemma 3.1, f is of the form $\sum_{j=0}^{t-1} a^j f_j$, $j = 0, \dots, t-1$ and g is of the form $\sum_{j=0}^{t-1} a^j g_j$, $j = 0, \dots, t-1$, where the f_j and the g_j are functions from \mathbb{E} to \mathbb{K} .

We can now define $\widehat{\chi}$ on \widehat{T} as $\widehat{\chi}(f) = \sum_{j=0}^{t-1} a^j \chi(f_j)$. Then obviously, $\widehat{\chi}$ is \mathbb{K} -linear. On the other hand, for each $q \in \mathbb{N}$, a^q is of the form $P_q(a)$ where $P_q \in \mathbb{K}[x]$, $\deg(P_q) \leq t-1$. Then $\widehat{\chi}(a^q) = \widehat{\chi}(P_q(a)) = P_q(\widehat{\chi}(a)) = P_q(a) = a^q$. Next,

$$\begin{aligned} \widehat{\chi}(fg) &= \widehat{\chi}\left(\left(\sum_{j=0}^{t-1} a^j f_j\right)\left(\sum_{j=0}^{t-1} a^j g_j\right)\right) = \widehat{\chi}\left(\sum_{\substack{0 \leq m \leq t-1 \\ 0 \leq n \leq t-1}} a^{m+n} f_m g_n\right) \\ &= \sum_{\substack{0 \leq m \leq t-1 \\ 0 \leq n \leq t-1}} a^{m+n} \chi(f_m) \chi(g_n) = \left(\sum_{j=0}^{t-1} a^j \chi(f_j)\right) \left(\sum_{j=0}^{t-1} a^j \chi(g_j)\right) = \chi(f) \chi(g). \end{aligned}$$

Thus, the extension of χ is proved whenever \mathbb{L} is of the form $\mathbb{K}[a]$. It is then immediate to check that $\widehat{\chi}$ is surjective: since \widehat{T} is a \mathbb{L} -algebra, it contains the field \mathbb{L} and every morphism $\widehat{\chi}$ from \widehat{T} obviously satisfies $\widehat{\chi}(c) = c \forall c \in \mathbb{L}$.

Consider now the general case. We can obviously write \mathbb{L} in the form $\mathbb{K}[b_1, \dots, b_q]$. Writing \mathbb{L}_j for the extension $\mathbb{K}[b_1, \dots, b_j]$ we have $\mathbb{L}_j = \mathbb{L}_{j-1}[b_j]$.

By induction on j , using the preceding just proved result, we get that for each $j = 1, \dots, q$, χ has continuation to a surjective morphism of \mathbb{L}_j -algebra, $\widehat{\chi}_j$, from $\mathbb{L}_j \otimes T$ onto \mathbb{L}_j . Taking $j = q$ ends the proof.

We are now able to prove that maximal ideals of finite codimension of S are of codimension 1 in two cases: when $S = \mathcal{A}$ and when the field \mathbb{K} is perfect.

Theorem 3.3: *Every maximal ideal \mathcal{M} of finite codimension of \mathcal{A} is of codimension 1.*

Proof: Let $S = \mathcal{A}$, let \mathbb{L} be the field $\frac{S}{\widehat{\mathcal{M}}}$ and let S' be the \mathbb{L} -algebra of bounded continuous functions from \mathbb{E} to \mathbb{L} . Then S' is semi-admissible. Now, let χ be the \mathbb{K} -algebra morphism from S onto \mathbb{L} whose kernel is \mathcal{M} . Let $g \in S$ and let $b = \chi(g) \in \mathbb{L}$. By Theorem 3.2 χ admits an extension to a morphism $\widehat{\chi}$ from S' to \mathbb{L} . Now, since S' is semi-admissible and since the kernel of $\widehat{\chi}$ is a maximal ideal $\widehat{\mathcal{M}}$ of S' , there exists an ultrafilter \mathcal{U} on \mathbb{E} such that $\widehat{\mathcal{M}} = \mathcal{I}(\mathcal{U}, S')$. Then we have $\widehat{\chi}(g-b) = 0$, hence $g-b$ belongs to $\widehat{\mathcal{M}}$, therefore $\lim_{\mathcal{U}} g(x) - b = 0$ i.e. $\lim_{\mathcal{U}} g(x) = b$. But since $g \in S$, $g(x)$ belongs to \mathbb{K} for all $x \in \mathbb{E}$. Therefore, since \mathbb{K} is complete, b belongs to \mathbb{K} . But by definition χ is a surjection from S onto \mathbb{L} , hence every value b of \mathbb{L} is the image of some $f \in S$ and hence it lies in \mathbb{K} , therefore $\mathbb{L} = \mathbb{K}$.

Theorem 3.3 can be generalized to all semi-admissible algebras provided \mathbb{K} is a perfect field [5].

Henceforth, for the rest of this Paragraph 3, the field \mathbb{K} is supposed to be perfect.

Proposition 3.4: *Let $\mathbb{L} = \mathbb{K}[a]$ be a finite extension of \mathbb{K} of degree t , provided with the unique absolute value extending that of \mathbb{K} and let a_2, \dots, a_t be the conjugates of a over \mathbb{K} , with $a_1 = a$. Let $\widehat{S} = \mathbb{L} \otimes_{\mathbb{K}} S$ and let $g = \sum_{j=0}^{t-1} a^j f_j$, $f_j \in S$ be such that $\inf_{\mathbb{E}} |g(x)| > 0$.*

For every $k = 1, \dots, t$, let $g_k = \sum_{j=0}^{t-1} a_k^j f_j$, $f_j \in S$. Then $\prod_{k=1}^t g_k$ belongs to S and $\prod_{k=2}^t g_k$ belongs to \widehat{S} .

Proof: Since $\mathbb{L} = \mathbb{K}[a_1]$ and since \mathbb{K} is perfect, the extension $N = \mathbb{K}[a_1, \dots, a_t]$ is of the form

$$N = \mathbb{K}[a_1][a_2, \dots, a_t] = \mathbb{L}[a_2, \dots, a_t]$$

and N is a normal extension of \mathbb{K} and of \mathbb{L} respectively.

Thus, assuming that a_1, \dots, a_s belong to \mathbb{L} , we have $\mathbb{L} = \mathbb{K}[a_1, \dots, a_s]$ and $N = \mathbb{K}[a_1, \dots, a_s, a_{s+1}, \dots, a_t] = \mathbb{K}[a_1, \dots, a_s][a_{s+1}, \dots, a_t] = \mathbb{L}[a_{s+1}, \dots, a_t]$.

Let G be the Galois group of N over \mathbb{K} and put $G' = \{\sigma \in G : \sigma(x) = x, \forall x \in \mathbb{L}\}$ where the extension N over \mathbb{L} is Galoisian, whose Galois group $G(N|\mathbb{L})$ is G' . The subfield \mathbb{L} of N then corresponds to the subgroup G' of G through the Galois correspondance.

Now, given $\sigma \in G$, set $\sigma(g) = \sum_{j=0}^{t-1} (\sigma(a))^j f_j$. Let $F = \prod_{k=1}^t g_k = \prod_{\sigma \in G} \sigma(g)$. Then F belongs to S if and only if for every $\tau \in G$, $\tau(F) = F$. Now, we have

$$\tau(F) = \prod_{\sigma \in G} \tau \circ \sigma(g) = \prod_{\zeta \in G} \zeta(g) = F,$$

therefore F belongs to S .

On the other hand the roots a_i , for $s+1 \leq i \leq t$, are conjugate over \mathbb{L} . Therefore if $s+1 \leq i \leq t$, there exists $\theta \in G'$ such that $a_i = \theta(a_{s+1})$. It follows that $g_i = \sum_{j=0}^{t-1} a_i^j f_j = \sum_{j=0}^{t-1} \theta(a_{s+1})^j f_j = \theta(\sum_{j=0}^{t-1} a_{s+1}^j f_j) = \theta(g_{s+1})$

Let $H = \prod_{i=s+1}^t g_i = \prod_{\theta \in G'} \theta(g_{s+1})$. Then H belongs to \widehat{S} if and only if $\tau(H) = H \forall \tau \in G'$. Now, we have $\tau(H) = \prod_{\theta \in G'} \tau \circ \theta(g_{s+1}) = \prod_{\zeta \in G'} \zeta(g_{s+1}) = H$, therefore H belongs to \widehat{S} . Consequently, since $\prod_{i=2}^s g_i$ belongs to \widehat{S} , one gets that $(\prod_{i=2}^s g_i) \cdot H = \prod_{i=2}^t g_i$ is an element of \widehat{S} .

We can now establish the following Proposition 3.5:

Proposition 3.5: *Let $\mathbb{L} = \mathbb{K}[a]$ be a finite extension of \mathbb{K} of degree t , provided with the unique absolute value extending that of \mathbb{K} and let a_2, \dots, a_t be the conjugates of a over \mathbb{K} , with $a_1 = a$. Let $\widehat{S} = \mathbb{L} \otimes_{\mathbb{K}} S$ and let $g \in \widehat{S}$ be such that $\inf_{\mathbb{E}} |g(x)| > 0$. Then g is invertible in \widehat{S} .*

Proof: Let $g = \sum_{j=0}^t a^j f_j$, $f_j \in S$ and for every $k = 1, \dots, t$, let $g_k = \sum_{j=0}^t a_k^j f_j$, $f_j \in S$. Then, by Proposition 3.4, $\prod_{k=1}^t g_k$ belongs to S and in the same way, $\prod_{k=2}^t g_k$ belongs to \widehat{S} .

Now, since $\inf_{\mathbb{E}} |g(x)|$ is a number $m > 0$, we have $|\prod_{k=1}^t g_k| \geq m^t$ because in \mathbb{L} , we have

$|g_k(x)| = |g_1(x)| \forall k = 1, 2, \dots, t, \forall x \in \mathbb{E}$. Consequently, $\prod_{k=1}^t g_k$ is invertible in S . Thus,

there exists $f \in S$ such that $\prod_{k=1}^t g_k \cdot f = 1$. But since, by Proposition 3.4, $\prod_{k=2}^t g_k$ belongs to

\widehat{S} , one sees that $\prod_{k=2}^t g_k \cdot f$ is an element of \widehat{S} . Hence $g = g_1$ is invertible in \widehat{S} with inverse

$$g^{-1} = \prod_{k=2}^t g_k \cdot f$$

Definition and notation: In the following Proposition 3.7 and in the theorems we will have to consider the *tensor product norm*. We remind here some general facts (for completeness one can see [19]). Let \mathbb{L} be a complete valued field extension of \mathbb{K} and A be a unital, ultrametric \mathbb{K} -Banach algebra. Given $z \in \mathbb{L} \otimes_{\mathbb{K}} A$, we put

$$\|z\|_{\otimes} = \inf \left\{ \max_{i \in I} |b_i| \cdot \|x_i\| \mid \sum_{i \in I} b_i \otimes_{\mathbb{K}} x_i = z, I \text{ finite} \right\}.$$

This norm $\|\cdot\|_{\otimes}$ will be called *the (projective) tensor product norm*. It is an ultrametric norm.

In any unital \mathbb{K} -algebra B , let 1_B be the unity of B . Then for $b \in \mathbb{L}$ and $x \in B$, one has $\|b \otimes x\|_{\otimes} = |b| \cdot \|x\|$. In particular for any $b \in \mathbb{L}$, (resp. $x \in B$), one has $\|b \otimes 1_B\|_{\otimes} = |b|$ (resp. $\|1_{\mathbb{K}} \otimes x\|_{\otimes} = \|x\|$.) Hence one has an isometric identification of \mathbb{L} (resp. B) with $\mathbb{L} \otimes_{\mathbb{K}} 1_B$ (resp. $1_{\mathbb{L}} \otimes_{\mathbb{K}} B$).

Furthermore, one verifies that with the tensor norm, the tensor product $\mathbb{L} \otimes_{\mathbb{K}} B$, of the two unital \mathbb{K} -algebras \mathbb{L} and B is a normed unital \mathbb{K} -algebra. It is also a unital \mathbb{L} -algebra (obtained by extension of scalars). The completion $\widehat{\mathbb{L} \otimes_{\mathbb{K}} B}$ of $\mathbb{L} \otimes_{\mathbb{K}} B$ with respect to the tensor product norm $\|\cdot\|_{\otimes}$ (called the topological tensor product) is a unital \mathbb{K} -Banach algebra as well as a \mathbb{L} -Banach algebra.

Now assume that \mathbb{L} is of finite dimension d over \mathbb{K} . Fix a \mathbb{K} -basis $(e_j)_{1 \leq j \leq d}$ of \mathbb{L} . It is readily seen that any $z \in \mathbb{L} \otimes_{\mathbb{K}} B$ can be written in the unique form $z = \sum_{j=1}^d e_j \otimes y_j$

$$\text{and } \|z\|_{\otimes} = \left\| \sum_{j=1}^d e_j \otimes y_j \right\|_{\otimes} \leq \max_{1 \leq j \leq d} |e_j| \cdot \|y_j\|.$$

On the other hand, given $b = \sum_{j=1}^d \beta_j e_j \in \mathbb{L}$, let us consider the norm $\|b\|_1 = \max_{1 \leq j \leq d} |\beta_j| \cdot |e_j|$. One has $|b| \leq \|b\|_1$ and since \mathbb{L} is finite dimensional, there exists $\alpha > 0$ such that $\alpha \max_{1 \leq j \leq d} |\beta_j| \cdot |e_j| \leq |b| \leq \max_{1 \leq j \leq d} |\beta_j| \cdot |e_j|$. Considering the dual basis $(e'_j)_{1 \leq j \leq d}$ of $(e_j)_{1 \leq j \leq d}$ and the continuous linear operators $e'_j \otimes id_B$ of $\mathbb{L} \otimes_{\mathbb{K}} B$ into $\mathbb{K} \otimes_{\mathbb{K}} B = B$, one proves that $\alpha \max_{1 \leq j \leq d} |e_j| \cdot \|y_j\| \leq \|z\|_{\otimes} \leq \max_{1 \leq j \leq d} |e_j| \cdot \|y_j\| = \|z\|_1$.

This means that the norms $\|\cdot\|_{\otimes}$ and $\|\cdot\|_1$ of $\mathbb{L} \otimes_{\mathbb{K}} B$ are equivalent. One immediately sees that $\mathbb{L} \otimes_{\mathbb{K}} B$ equipped with the norm $\|z\|_1 = \max_{1 \leq j \leq d} |e_j| \|y_j\|$ is isomorphic to the product \mathbb{K} -Banach space B^d and then it is complete. It follows that $(\mathbb{L} \otimes_{\mathbb{K}} B, \|\cdot\|_{\otimes})$ is complete and $\mathbb{L} \otimes_{\mathbb{K}} B = \widehat{\mathbb{L} \otimes_{\mathbb{K}} B}$.

One then has the following Theorem 3.6 contained in [19] (Chapter 4).

Theorem 3.6: *If \mathbb{L} is a finite extension of \mathbb{K} and B is a commutative unital \mathbb{K} -Banach algebra, then with the tensor product norm $\|\cdot\|_{\otimes}$, the tensor product $\mathbb{L} \otimes_{\mathbb{K}} B$ of the \mathbb{K} -algebras \mathbb{L} and B is a \mathbb{K} -Banach algebra as well as a Banach algebra over \mathbb{L} .*

Taking $B = S$, we can now conclude.

Proposition 3.7 : *Let $\mathbb{L} = \mathbb{K}[a]$ be a finite extension of \mathbb{K} of degree t , provided with the unique absolute value extending that of \mathbb{K} . Then the algebra $\widehat{S} = \mathbb{L} \otimes_{\mathbb{K}} S$ provided with the tensor product norm $\| \cdot \|_{\otimes}$, is complete.*

Moreover, \widehat{S} can be identified with the Banach \mathbb{L} -algebra of functions f from \mathbb{E} to \mathbb{L} of the form $f = \sum_{j=0}^{t-1} a^j f_j$ with $f_j \in S$ and \widehat{S} is a semi-admissible \mathbb{L} -algebra.

Proof: By construction, \widehat{S} is the set of functions $f = \sum_{j=0}^{t-1} a^j f_j$ with $f_j \in S$. Since each

f_j is continuous, so is f . By Theorem 3.6, \widehat{S} is a Banach \mathbb{L} -algebra. Next, given a clopen subset D of \mathbb{E} , the characteristic function u of D exists in S and hence it belongs to \widehat{S} . Finally, given an element $g \in \widehat{S}$ such that $\inf_{x \in \mathbb{E}} |g(x)| > 0$, by Proposition 3.5, g is invertible in \widehat{S} . Therefore, \widehat{S} is semi-admissible.

Theorem 3.8: *Let \mathbb{L} be a finite extension of \mathbb{K} of degree t , provided with the unique absolute value extending that of \mathbb{K} and let $\widehat{S} = \mathbb{L} \otimes_{\mathbb{K}} S$ be provided with the tensor product norm. Then \widehat{S} is a semi-admissible Banach \mathbb{L} -algebra.*

Proof: By definition, \mathbb{L} is of the form $\mathbb{K}[b_1, \dots, b_q]$ with $\mathbb{K}[b_1, \dots, b_j]$ strictly included in $\mathbb{K}[b_1, \dots, b_{j+1}]$, $j = 1, \dots, q-1$. Put $\mathbb{L}_j = \mathbb{K}[b_1, \dots, b_j]$, $j = 1, \dots, q$ and $\widehat{S}_j = \mathbb{L}_j \otimes_{\mathbb{K}} S$. Suppose we have proved that \widehat{S}_j is semi-admissible for some $j < q$. Next, since $\mathbb{L}_{j+1} = \mathbb{L}_j[b_{j+1}]$, by Proposition 3.7, \widehat{S}_{j+1} is semi-admissible. Therefore, by induction, $\widehat{S}_q = \widehat{S}$ is a semi-admissible Banach \mathbb{L} -algebra.

Theorem 3.9: *Let \mathcal{M} be a maximal ideal of finite codimension of S . Then \mathcal{M} is of codimension 1.*

Proof: Let \mathbb{L} be the field $\frac{S}{\mathcal{M}}$ and let $\widehat{S} = \mathbb{L} \otimes_{\mathbb{K}} S$ be provided with the tensor product norm. By Theorem 3.8, \widehat{S} is semi-admissible. Now, let χ be the morphism from S onto \mathbb{L} whose kernel is \mathcal{M} . Let $g \in S$ and let $b = \chi(g) \in \mathbb{L}$. By Theorem 3.5 χ admits an extension to a morphism $\widehat{\chi}$ from \widehat{S} to \mathbb{L} . But since \widehat{S} is semi-admissible and since the kernel of $\widehat{\chi}$ is a maximal ideal $\widehat{\mathcal{M}}$ of \widehat{S} , by Theorem 2.1 there exists an ultrafilter \mathcal{U} on \mathbb{E} such that $\widehat{\mathcal{M}} = \mathcal{I}(\mathcal{U}, \widehat{S})$. Take $g \in S$ and let $b = \chi(g)$. Then we have $\widehat{\chi}(g - b) = 0$, hence $g - b$ belongs to $\widehat{\mathcal{M}}$, therefore $\lim_{\mathcal{U}} g(x) - b = 0$ i.e. $\lim_{\mathcal{U}} g(x) = b$. But since $g \in S$, $g(x)$ belongs to \mathbb{K} for all $x \in \mathbb{E}$. Therefore, since \mathbb{K} is complete, b belongs to \mathbb{K} . But by definition χ is a surjection from S onto \mathbb{L} , hence every value b of \mathbb{L} actually lies in \mathbb{K} and hence $\mathbb{L} = \mathbb{K}$.

4. Multiplicative spectrum

The multiplicative spectrum of a Banach \mathbb{K} -algebra was first introduced by B. Guenebaude [16] and was at the basis of Berkovich's analytic space theory [2]. It was also much used in [12], [14], [15].

Notations and definitions: Let T be a normed \mathbb{K} -algebra. We denote by $Mult(T, \|\cdot\|)$ the set of continuous multiplicative algebra semi-norms of T provided with the topology of pointwise convergence [3], which means that a basic neighborhood of some $\psi \in Mult(T, \|\cdot\|)$ is a set of the form $W(\psi, f_1, \dots, f_q, \epsilon)$, with $f_j \in T$ and $\epsilon > 0$ and this is the set of $\phi \in Mult(T, \|\cdot\|)$ such that $|\psi(f_j) - \phi(f_j)|_\infty \leq \epsilon \forall j = 1, \dots, q$. The topological space $Mult(T, \|\cdot\|)$ is then compact (see [2], or Theorem 6.2 in [10]).

Given $\phi \in Mult(T, \|\cdot\|)$, we call *kernel of ϕ* the set of the $x \in T$ such that $\phi(x) = 0$ and we denote it by $Ker(\phi)$. It is a prime closed ideal of T with respect to the norm $\|\cdot\|$ [10].

We denote by $Mult_m(T, \|\cdot\|)$ the set of continuous multiplicative semi-norms of T whose kernel is a maximal ideal and by $Mult_1(T, \|\cdot\|)$ the set of continuous multiplicative semi-norms of T whose kernel is a maximal ideal of codimension 1. Particularly, considering the algebra S , we denote by $Mult_{\mathbb{E}}(S, \|\cdot\|)$ the set of continuous multiplicative semi-norms of S whose kernel is a maximal ideal of the form $\mathcal{I}(a, S)$, $a \in \mathbb{E}$.

We denote by $\Upsilon(T)$ the set of \mathbb{K} -algebra homomorphisms from S to \mathbb{K} .

Let us recall that in S , we have $\|\cdot\|_0 \leq \|\cdot\|_{sp}$ and that if S is admissible, then $\|\cdot\|_0 = \|\cdot\|_{sp}$. Theorem 4.1 is classical [10], [11]:

Theorem 4.1: *Let T be a unital commutative ultrametric Banach \mathbb{K} -algebra. For each $f \in T$, $\|f\|_{sp} = \sup\{\phi(f) \mid \phi \in Mult(T, \|\cdot\|)\}$. For every $\chi \in \Upsilon(T)$, we have $|\chi(f)| \leq \|f\|_{sp} \forall f \in T$.*

More notations: For any ultrafilter $\mathcal{U} \in U(\mathbb{E})$ and any $f \in S$, $|f(x)|$ has a limit along \mathcal{U} since f is bounded. Given $a \in \mathbb{E}$ we denote by φ_a the mapping from S to \mathbb{R} defined by $\varphi_a(f) = |f(a)|$ and for any ultrafilter $\mathcal{U} \in U(\mathbb{E})$, we denote by $\varphi_{\mathcal{U}}$ the mapping from S to \mathbb{R} defined by $\varphi_{\mathcal{U}}(f) = \lim_{\mathcal{U}} |f(x)|$ (see Lemma 1.13). These maps belong to $Mult(S, \|\cdot\|)$ since $\|\cdot\|_0 \leq \|\cdot\|_{sp} \leq \|\cdot\|$. Particularly, the elements of $Mult_{\mathbb{E}}(S, \|\cdot\|)$ are the φ_a , $a \in \mathbb{E}$.

Proposition 4.2: *Let $a \in \mathbb{E}$. Then $\mathcal{I}(a, S)$ is a maximal ideal of S of codimension 1 and φ_a belongs to $Mult_1(S, \|\cdot\|)$. Conversely, for every algebra homomorphism χ from S to \mathbb{K} , its kernel is a maximal ideal of the form $\mathcal{I}(a, S)$ with $a \in \mathbb{E}$ and χ is defined as $\chi(f) = f(a)$, while $\varphi_a(f) = |\chi(f)|$.*

Theorem 4.3: *Let \mathcal{U} be an ultrafilter on \mathbb{E} . Then $\varphi_{\mathcal{U}}$ belongs to the closure of $Mult_{\mathbb{E}}(S, \|\cdot\|)$, with respect to the topology of $Mult(S, \|\cdot\|)$.*

Proof: Let $\psi = \varphi_{\mathcal{U}}$, take $\epsilon > 0$ and let $f_1, \dots, f_q \in S$. There exists $L \in \mathcal{U}$ such that $|\psi(f_j) - |f_j(x)||_\infty \leq \epsilon \forall x \in L, \forall j = 1, \dots, q$. Therefore, taking $a \in L$, we have $|\varphi_a(f_j) - \psi(f_j)|_\infty \leq \epsilon \forall j = 1, \dots, q$ which shows that φ_a belongs to the neighborhood W of ψ of the form $\{\phi \mid |\psi(f_j) - \phi(f_j)|_\infty \leq \epsilon \forall j = 1, \dots, q\}$ and this proves the claim.

Remark 5: In the field \mathbb{K} , we call monotonous distances sequence a sequence $(a_n)_{n \in \mathbb{N}}$ of \mathbb{E} such that the sequence $\delta(a_n, a_{n+1})_{n \in \mathbb{N}}$ is strictly monotonous. We call constant distances sequence a sequence $(a_n)_{n \in \mathbb{N}}$ of \mathbb{E} such that $\delta(a_n, a_m)$ is constant when n, m

are big enough and $n \neq m$. According to Remark 3, if \mathbb{K} is locally compact and \mathbb{E} admits monotonous distances sequences or constant distances sequences, we can define $\varphi_{\mathcal{U}} \in \text{Mult}_1(S, \|\cdot\|)$ which does not belong to $\text{Mult}_{\mathbb{E}}(S, \|\cdot\|)$.

Theorem 4.4: *For each $\phi \in \text{Mult}(S, \|\cdot\|)$, $\text{Ker}(\phi)$ is a prime spectrally closed ideal.*

Proof: Let $\phi \in \text{Mult}(S, \|\cdot\|)$ and let f belong to the spectral closure of $\text{Ker}(\phi)$. There exists a sequence $(f_n)_{n \in \mathbb{N}}$ of $\text{Ker}(\phi)$ such that $\lim_{n \rightarrow \infty} \|f_n - f\|_{sp} = 0$. By Theorem 4.1, since $\phi(g) \leq \|g\|_{sp} \forall g \in S$, we have $\lim_{n \rightarrow \infty} \phi(f_n - f) = 0$. But $\phi(f_n) = 0 \forall n \in \mathbb{N}$, hence it follows that $\phi(f) = \lim_{n \rightarrow +\infty} \phi(f_n - f) = 0$. Therefore, f belongs to $\text{Ker}(\phi)$, which means that $\widetilde{\text{Ker}(\phi)} = \text{Ker}(\phi)$.

By Theorem 4.4 and Corollary 2.4.e, we have the following Corollary 4.4.a:

Corollary 4.4.a: *Suppose S is admissible. Then $\text{Mult}(S, \|\cdot\|) = \text{Mult}_m(S, \|\cdot\|)$.*

Theorem 4.5 is classical (Theorem 6.15 in [10]).

Theorem 4.5: *Let T be a commutative unital ultrametric Banach \mathbb{K} -algebra. For every maximal ideal \mathcal{M} of T , there exists $\phi \in \text{Mult}_m(S, \|\cdot\|)$ such that $\mathcal{M} = \text{Ker}(\phi)$.*

Recall that a unital commutative Banach \mathbb{K} -algebra is said to be *multibjective* if every maximal ideal is the kernel of only one continuous multiplicative semi-norm.

Remark 6: There exist some rare cases of ultrametric Banach algebras that are not multibjective [8], [9].

Theorem 4.6: *Suppose S is admissible. Then S is multibjective. Precisely if $\psi \in \text{Mult}(S, \|\cdot\|)$ and $\text{Ker}(\psi) = \mathcal{M}$ then $\psi = \varphi_{\mathcal{U}}$ for every ultrafilter \mathcal{U} such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$.*

Proof: Let $\psi \in \text{Mult}_m(S, \|\cdot\|)$, let $\mathcal{M} = \text{Ker}(\psi)$ and \mathcal{U} be an ultrafilter such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$.

Let $f \in S$. Notice that if $f \in \mathcal{M}$ then $\psi(f) = \varphi_{\mathcal{U}}(f) = 0$. Now we assume that $f \notin \mathcal{M}$. So $\psi(f)$ and $\varphi_{\mathcal{U}}(f)$ are both strictly positive. We prove that they are equal.

First let $\epsilon > 0$ and consider the set $L = \{x \in \mathbb{E} : |f(x)| \leq \varphi_{\mathcal{U}}(f) + \epsilon\}$. This set belongs to \mathcal{U} and by Lemma 1.2, it is clopen. Therefore its characteristic function u lies in S . We have $\varphi_{\mathcal{U}}(u) = 1$. Consequently, we can derive that $\psi(u) = 1$ because u is idempotent and does not belong to \mathcal{M} . Therefore $\psi(uf) = \psi(f)$ and $\varphi_{\mathcal{U}}(uf) = \varphi_{\mathcal{U}}(f)$. By Theorem 4.1, we have $\psi(f) = \psi(uf) \leq \|uf\|_{sp} = \|uf\|_0$ because S is admissible. But by definition of L we have: $\|uf\|_0 \leq \varphi_{\mathcal{U}}(f) + \epsilon$. Therefore, $\psi(f) \leq \varphi_{\mathcal{U}}(f) + \epsilon$. This holds for every $\epsilon > 0$. Consequently we may conclude that $\psi(f) \leq \varphi_{\mathcal{U}}(f)$ for every $f \in S$.

We prove now the inverse inequality. We have $\varphi_{\mathcal{U}}(f) > 0$, so consider the set $W = \{x \in \mathbb{E} : |f(x)| \geq \frac{\varphi_{\mathcal{U}}(f)}{2}\}$. This is a clopen set which belongs to \mathcal{U} . Let w be the characteristic function of W and put $g = wf + (1 - w)$. We have $\varphi_{\mathcal{U}}(w) = 1$ and $\varphi_{\mathcal{U}}(1 - w) = 0$ so

$w \notin \mathcal{M}$ and $1 - w \in \mathcal{M}$. Since $\mathcal{M} = \text{Ker}(\psi)$ we then have $\psi(1 - w) = 0$ and $\psi(w) = 1$ because w is idempotent. Finally $\psi(g) = \psi(f)$ and $\varphi_{\mathcal{U}}(g) = \varphi_{\mathcal{U}}(f)$.

On the other hand, we can check that $|g(x)| \geq \min(1, \frac{\varphi_{\mathcal{U}}(f)}{2})$ for all $x \in \mathbb{E}$, hence g is invertible in S . Putting $h = \frac{1}{g}$, using the first inequality yet proved, we have

$$\psi(f) = \psi(g) = \frac{1}{\psi(h)} \geq \frac{1}{\varphi_{\mathcal{U}}(h)} = \varphi_{\mathcal{U}}(g) = \varphi_{\mathcal{U}}(f).$$

That concludes the proof.

Remark 7: Thus, if S is admissible, $\text{Mult}(S, \| \cdot \|)$ can be identified to $\text{Mult}(S, \| \cdot \|_0)$

Corollary 4.6.a: Suppose S is admissible. For every $\phi \in \text{Mult}(S, \| \cdot \|)$ there exists a unique $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{E})$ such that $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in S, \forall \mathcal{U} \in \mathcal{H}$.

Moreover, the mapping Ψ that associates to each $\phi \in \text{Mult}(S, \| \cdot \|)$ the unique $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{E})$ such that $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in S, \forall \mathcal{U} \in \mathcal{H}$, is a bijection from $\text{Mult}(S, \| \cdot \|)$ onto $Y_{(\mathcal{R})}(\mathbb{E})$.

Assuming that S is admissible, since by Theorem 4.6 each element $\phi \in \text{Mult}(S, \| \cdot \|)$ is of the form $\varphi_{\mathcal{U}}$, Corollary 4.6.b is immediate from theorem 4.3:

Corollary 4.6.b: If S is admissible, then $\text{Mult}_{\mathbb{E}}(S, \| \cdot \|)$ is dense in $\text{Mult}(S, \| \cdot \|)$.

Theorem 4.7: The topological space \mathbb{E} , provided with its distance δ , is homeomorphic to $\text{Mult}_{\mathbb{E}}(S, \| \cdot \|)$ provided with the restricted topology from that of $\text{Mult}(S, \| \cdot \|)$.

Proof: For every $a \in \mathbb{E}$, put $\Lambda(a) = \varphi_a$, take $f_1, \dots, f_q \in S$ and $\epsilon > 0$. We set $W'(\varphi_a, f_1, \dots, f_q, \epsilon) = W(\varphi_a, f_1, \dots, f_q, \epsilon) \cap \text{Mult}_{\mathbb{E}}(S, \| \cdot \|)$. Considering the natural topology on \mathbb{E} , the filter of neighborhoods of a admits for basis the family of disks $d_{\mathbb{E}}(a, r^-)$, $r > 0$. We will show that it is induced through the mapping Λ by the filter admitting for basis the family of neighborhoods of φ_a in $\text{Mult}(S, \| \cdot \|)$. Indeed, take $r \in]0, 1[$ and let u be the characteristic function of $d_{\mathbb{E}}(a, r^-)$. Then $W'(\varphi_a, u, r)$ is the set of φ_b such that $|\varphi_a(u) - \varphi_b(u)|_{\infty} < \frac{1}{2}$, i.e. the $\Lambda(b)$ such that $b \in d_{\mathbb{E}}(a, r^-)$. Therefore the topology induced on \mathbb{E} by $\text{Mult}(S, \| \cdot \|)$ is thinner than its metric topology.

Conversely, take some neighborhood $W'(\varphi_a, f_1, \dots, f_q, \epsilon)$ of φ_a in $\text{Mult}_{\mathbb{E}}(S, \| \cdot \|)$. For each $j = 1, \dots, q$, the set of the $x \in \mathbb{E}$ such that $|\varphi_x(f_j) - \varphi_a(f_j)|_{\infty} \leq \epsilon$ is the set of the x such that $\left| |f_j(x)| - |f_j(a)| \right|_{\infty} \leq \epsilon$. But now, since each f_j is continuous, the set of the x such that $\left| |f_j(x)| - |f_j(a)| \right|_{\infty} \leq \epsilon \forall j = 1, \dots, q$ is a neighborhood of a in \mathbb{E} . Consequently, the metric topology of \mathbb{E} is thinner than the topology induced by $\text{Mult}_{\mathbb{E}}(S, \| \cdot \|)$ and that finishes proving that the two topological spaces are homeomorphic.

Corollary 4.7.a: $\text{Mult}(S, \| \cdot \|)$ is a compactification of the topological space \mathbb{E} .

Theorem 4.8: Let $\phi = \varphi_{\mathcal{U}} \in \text{Mult}_m(S, \|\cdot\|)$, with \mathcal{U} an ultrafilter on \mathbb{E} , let Γ be the field $\frac{S}{\text{Ker}(\phi)}$ and let θ be the canonical surjection from S onto Γ . Then, the mapping $\|\cdot\|'$ defined on Γ by $\|\theta(f)\|' = \phi(f) \forall f \in S$, is the quotient norm $\|\cdot\|'$ of $\|\cdot\|_0$ defined on Γ and is an absolute value on Γ . Moreover, if $\text{Ker}(\phi)$ is of codimension 1, then this absolute value is the one defined on \mathbb{K} and coincides with the quotient norm of the norm $\|\cdot\|$ of S .

Proof: Let $\mathcal{M} = \text{Ker}(\varphi_{\mathcal{U}})$. Let $t \in \Gamma$ and let $f \in S$ be such that $\theta(f) = t$. So, $\|t\|' \geq \lim_{\mathcal{U}} |f(s)|$. Conversely, take $\epsilon > 0$ and let $V = \{x \in \mathbb{E} : |f(x)| \leq \lim_{\mathcal{U}} |f(s)| + \epsilon\}$. By Lemma 1.2, the set V is clopen and belongs to \mathcal{U} . The characteristic function u of $\mathbb{E} \setminus V$ belongs to \mathcal{M} and so does uf . But by construction, $(f - uf)(x) = 0 \forall x \in \mathbb{E} \setminus V$ and $(f - uf)(x) = f(x) \forall x \in V$. Consequently, $\|f - uf\|_0 \leq \lim_{\mathcal{U}} |f(s)| + \epsilon$ and therefore $\|t\|' \leq \|f - uf\|_0 \leq \lim_{\mathcal{U}} |f(s)| + \epsilon$. This finishes proving the equality $\|\theta(f)\|' = \lim_{\mathcal{U}} |f(s)|$ and hence the mapping defined by $|\theta(f)| = \phi(f)$, $f \in S$ is the quotient norm $\|\cdot\|'$ of $\|\cdot\|_0$. Then it is multiplicative, hence it is an absolute value on Γ .

Now, suppose that \mathcal{M} is of codimension 1. Then Γ is isomorphic to \mathbb{K} and its absolute value $\|\cdot\|'$ is continuous with respect to the topology of \mathbb{K} , hence it is equal to the absolute value of \mathbb{K} . Finally consider the quotient norm $\|\cdot\|_q$ of the norm $\|\cdot\|$ of S : that quotient norm of course bounds the quotient norm $\|\cdot\|'$ which is the absolute value of \mathbb{K} . If $f \in S$ and $b = \theta(f)$, we have $f - b \in \mathcal{M}$ and $\|\theta(f)\|_q \leq \|b\| = |b| = |\theta(f)| = \|\theta(f)\|'$, which ends the proof.

Corollary 4.8.a: Suppose that S is admissible. Let $\phi \in \text{Mult}(S, \|\cdot\|)$, let Γ be the field $\frac{S}{\text{Ker}(\phi)}$ and let θ be the canonical surjection from S onto Γ . Then, the mapping defined on Γ by $|\theta(f)| = \phi(f), \forall f \in S$ is the quotient norm $\|\cdot\|'$ of $\|\cdot\|_0$ on Γ and is an absolute value on Γ . Moreover, if $\text{Ker}(\phi)$ is of codimension 1, then this absolute value is the one defined on \mathbb{K} and coincides with the quotient norm of the norm $\|\cdot\|$ of S .

Remark 9: It is not clear whether an algebra S admits a prime closed ideal \mathcal{P} (with respect to the norm $\|\cdot\|$) which is not a maximal ideal. If it admits such a prime closed ideal, then it is not the kernel of a continuous multiplicative semi-norm. In such a case, the quotient algebra by \mathcal{P} has no continuous absolute value extending that of \mathbb{K} , although it has no divisors of zero. Such a situation can happen in certain Banach algebras [4].

Definition and notation: Given a \mathbb{K} -normed algebra G , we call *Shilov boundary* of G a closed subset F of $\text{Mult}(G, \|\cdot\|)$ that is minimum with respect to inclusion, such that, for every $x \in G$, there exists $\phi \in F$ such that $\phi(x) = \|x\|_{sp}$ [12], [13].

Let us recall the following Theorem 4.9 given in [10] and [12]:

Theorem 4.9: Every normed \mathbb{K} -algebra admits a Shilov boundary.

Notation: Given a normed \mathbb{K} -algebra G , we denote by $\text{Shil}(G)$ the Shilov boundary of G .

Lemma 4.10: *Let us fix $a \in \mathbb{E}$. For every $r > 0$, let $Z(a, r)$ be the set of multiplicative semi-norms $\varphi_{\mathcal{U}}$, with $\mathcal{U} \in U(\mathbb{E})$, such that $d_{\mathbb{E}}(a, r)$ belongs to \mathcal{U} . The family $\{Z(a, r) \mid r \in]0, 1[\}$ forms a basis of the filter of neighborhoods of φ_a .*

Proof: Let $W(\varphi_a, f_1, \dots, f_q, \epsilon)$ be a neighborhood of φ_a in $Mult(S, \|\cdot\|)$. There exists $r > 0$ such that, whenever $\delta(a, x) \leq r$ we have $|f_j(x) - f_j(a)| \leq \epsilon \forall j = 1, \dots, q$ and therefore, clearly, $|\varphi_{\mathcal{U}}(f_j) - \varphi_a(f_j)|_{\infty} \leq \epsilon \forall j = 1, \dots, q$ for every \mathcal{U} containing $d_{\mathbb{E}}(a, r)$. Thus $Z(a, r)$ is included in $W(\varphi_a, f_1, \dots, f_q, \epsilon)$.

Conversely, consider a set $Z(a, r)$ with $r \in]0, 1[$, let u be the characteristic function of $d_{\mathbb{E}}(a, r)$ and consider $W(\varphi_a, u, r)$. Given $\psi = \varphi_{\mathcal{U}} \in W(\varphi_a, u, r)$, we have $|\psi(u) - \varphi_a(u)|_{\infty} \leq r$. But $|\psi(u) - \varphi_a(u)|_{\infty} = |\psi(u) - 1|_{\infty} = |\lim_{\mathcal{U}} |u(x)| - 1|_{\infty}$. If $d_{\mathbb{E}}(a, r)$ belongs to \mathcal{U} , then $\lim_{\mathcal{U}} |u(x)| = 1$ and therefore $|\lim_{\mathcal{U}} |u(x)| - 1|_{\infty} = 0$. But if $d_{\mathbb{E}}(a, r)$ does not belong to \mathcal{U} , then $\lim_{\mathcal{U}} |u(x)| = 0$ and therefore $|\lim_{\mathcal{U}} |u(x)| - 1|_{\infty} = 1$. Consequently, since $r < 1$, $W(\varphi_a, u, r)$ is included in $Z(a, r)$, which finishes proving that the family of $Z(a, r)$, $r \in]0, 1[$ is a basis of the filter of neighborhoods of φ_a .

Theorem 4.11: *Suppose S is admissible. The Shilov boundary of S is equal to $Mult(S, \|\cdot\|)$.*

Proof: We will show that for every $a \in \mathbb{E}$, φ_a belongs to $Shil(S)$. So, let us fix $a \in \mathbb{E}$ and suppose that φ_a does not belong to $Shil(S)$. Since $Shil(S)$ is a closed subset of $Mult(S, \|\cdot\|)$, there exists a neighborhood of φ_a that contains no element of $Shil(S)$. Therefore, by Lemma 4.10, there exists $s > 0$ such that $Z(a, s)$ contains no element of $Shil(S)$. Now, let $D = d_{\mathbb{E}}(a, s)$ and let u be the characteristic function of D . Since any $\phi \in Mult(S, \|\cdot\|)$ satisfies either $\phi(u) = 1$ or $\phi(u) = 0$, there exists $\theta \in Shil(S)$ such that $\theta(u) = \|\theta\|_{sp} = 1$. Then, θ is of the form $\varphi_{\mathcal{U}}$, with $\mathcal{U} \in U(\mathbb{E})$ and \mathcal{U} does not contain D . But since $u(x) = 0 \forall x \in \mathbb{E} \setminus D$, we have $\theta(u) = 0$, a contradiction. Consequently, for every $a \in \mathbb{E}$, φ_a belongs to $Shil(S)$ which is a closed subset of $Mult(S, \|\cdot\|)$ and since, by Corollary 4.6.b, $Mult_{\mathbb{E}}(S, \|\cdot\|)$ is dense in $Mult(S, \|\cdot\|)$, then $Shil(S)$ is equal to $Mult(S, \|\cdot\|)$.

5. The Stone space of $\mathbb{B}(\mathbb{E})$.

It was proved in [14] that for the algebra \mathcal{A} of continuous bounded functions from \mathbb{E} to \mathbb{K} , the Banaschewski compactification of \mathbb{E} is homeomorphic to $Mult(\mathcal{A}, \|\cdot\|_0)$. Here we get the same result for admissible algebras.

We denote by $\mathbb{B}(\mathbb{E})$ the Boolean ring of clopen subsets of \mathbb{E} provided with the laws Δ for the addition and \cap for the multiplication. As usually called the *Stone space of the Boolean ring* $\mathbb{B}(\mathbb{E})$ is the space $\Sigma(\mathbb{E})$ of non-zero ring homomorphisms from $\mathbb{B}(\mathbb{E})$ onto \mathbb{F}_2 , provided with the topology of pointwise convergence. This space is a compactification of \mathbb{E} and is called the *Banaschewski compactification* of \mathbb{E} (see for example [19] for further details).

For every $\mathcal{U} \in U(\mathbb{I})$, we denote by $\zeta_{\mathcal{U}}$ the ring homomorphism from $\mathbb{I}(\mathbb{I})$ onto \mathbb{F}_2 defined by $\zeta_{\mathcal{U}}(O) = 1$ for every $O \in \mathbb{I}(\mathbb{I})$ that belongs to \mathcal{U} and $\zeta_{\mathcal{U}}(O) = 0$ for every $O \in \mathbb{I}(\mathbb{I})$ that does not belong to \mathcal{U} .

Particularly, given $a \in \mathbb{I}$, we denote by ζ_a the ring homomorphism from $\mathbb{I}(\mathbb{I})$ onto \mathbb{F}_2 defined by $\zeta_a(O) = 1$ for every $O \in \mathbb{I}(\mathbb{I})$ that contains a and $\zeta_a(O) = 0$ for every $O \in \mathbb{I}(\mathbb{I})$ that does not contain a .

Throughout Paragraph 5, we suppose that S is an admissible algebra.

Remark 10: Let $\Sigma'(\mathbb{I})$ be the set of ζ_a , $a \in \mathbb{I}$. The mapping that associates $\zeta_a \in \mathbb{I}$ to $a \in \mathbb{I}$ defines a surjective mapping from \mathbb{I} onto $\Sigma'(\mathbb{I})$. That mapping is also injective because given $a, b \in \mathbb{I}$, there exists a clopen subset F such that $a \in F$ and $b \notin F$.

By Corollary 4.6.a, we have a bijection Ψ from $Mult(S, \|\cdot\|)$ onto $Y_{(\mathcal{R})}(\mathbb{I})$ associating to each $\phi \in Mult(S, \|\cdot\|)$ the unique $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{I})$ such that $\phi(f) = \lim_{\mathcal{U}} |f(x)|$, $\mathcal{U} \in \mathcal{H}$, $f \in S$, i.e. $\phi = \varphi_{\mathcal{U}}$ for every $\mathcal{U} \in \mathcal{H}$.

On the other hand, let us take some $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{I})$ and ultrafilters \mathcal{U}, \mathcal{V} in \mathcal{H} . Since \mathcal{U}, \mathcal{V} own the same clopen subsets of \mathbb{I} , we have $\zeta_{\mathcal{U}} = \zeta_{\mathcal{V}}$ and hence we can define a mapping Ξ from $Y_{\mathcal{R}}(\mathbb{I})$ into $\Sigma(\mathbb{I})$ which associates to each $\mathcal{H} \in Y_{\mathcal{R}}(\mathbb{I})$ the boolean homomorphism $\zeta_{\mathcal{U}}$ independant from $\mathcal{U} \in \mathcal{H}$

Lemma 5.1: Ξ is a bijection from $Y_{\mathcal{R}}(\mathbb{I})$ onto $\Sigma(\mathbb{I})$.

Proof: Indeed, let $\mathcal{H}, \mathcal{K} \in Y_{(\mathcal{R})}(\mathbb{I})$ and suppose that $\mathcal{H} \neq \mathcal{K}$. Take ultrafilters $\mathcal{U} \in \mathcal{H}$ and $\mathcal{V} \in \mathcal{K}$. They are not sticked, therefore by Theorem 1.12, there exists clopens $L \in \mathcal{H}$, $M \in \mathcal{K}$ such that $L \cap M = \emptyset$. Then, $\Xi(\mathcal{H}) \neq \Xi(\mathcal{K})$, which proves the injectivity.

Now, let us check that Ξ is surjective. Let $\theta \in \Sigma(\mathbb{I})$. Since θ is a ring homomorphism for the Boolean laws, the family of clopen sets X satisfying $\theta(X) = 1$ generates a filter \mathcal{F} . Let $\mathcal{U} \in U(\mathbb{I})$ be thinner than \mathcal{F} and let \mathcal{H} be the class of \mathcal{U} with respect to (\mathcal{R}) . We will check that $\theta = \Xi(\mathcal{H}) = \zeta_{\mathcal{U}}$. Let O be a clopen subset that belongs to \mathcal{U} . Then $\mathbb{I} \setminus O$ does not belong to \mathcal{U} and therefore it does not belong to \mathcal{F} , so $\theta(\mathbb{I} \setminus O) = 0$, consequently $\theta(O) = 1$. And now, let O be a clopen subset that does not belong to \mathcal{U} . Then O does not belong to \mathcal{F} , hence $\theta(O) = 0$, which ends the proof.

We put $\Phi = \Xi \circ \Psi$ and hence Φ is a bijection from $Mult(S, \|\cdot\|)$ onto $\Sigma(\mathbb{I})$. Notice that for every ultrafilter \mathcal{U} , $\Psi(\varphi_{\mathcal{U}})$ is the class \mathcal{H} of \mathcal{U} with respect to (\mathcal{R}) and $\Xi(\mathcal{H}) = \zeta_{\mathcal{U}}$ so $\Phi(\varphi_{\mathcal{U}}) = \zeta_{\mathcal{U}}$.

Theorem 5.2: Φ is a homeomorphism once $\Sigma(\mathbb{I})$ and $Mult(S, \|\cdot\|)$ are provided with topologies of pointwise convergence.

Proof: Recall that for any $\mathcal{U} \in U(\mathbb{I})$, a neighborhoods basis of $\varphi_{\mathcal{U}}$ in $Mult(S, \|\cdot\|)$ is given by the family of sets of the form $W(\varphi_{\mathcal{U}}, f_1, \dots, f_q, \epsilon)$ with $f_1, \dots, f_q \in S$, $\epsilon > 0$ and

$$W(\varphi_{\mathcal{U}}, f_1, \dots, f_q, \epsilon) = \{ \varphi_{\mathcal{V}} \mid \left| \lim_{\mathcal{U}} |f_j(x)| - \lim_{\mathcal{V}} |f_j(x)| \right|_{\infty} \leq \epsilon, j = 1, \dots, q \}.$$

On the other hand, for any $\mathcal{U} \in U(\mathbb{E})$, a neighborhood basis for $\zeta_{\mathcal{U}}$ in $\Sigma(\mathbb{E})$ is given by the family of sets $V(\zeta_{\mathcal{U}}, O_1, \dots, O_q)$ where O_1, \dots, O_q belong to $\mathbb{B}(E)$ and

$$V(\zeta_{\mathcal{U}}, O_1, \dots, O_q) = \{\zeta_{\mathcal{V}} \mid \zeta_{\mathcal{U}}(O_j) = \zeta_{\mathcal{V}}(O_j), j = 1, \dots, q\}.$$

Notice also that if F belongs to $\mathbb{B}(\mathbb{E})$ and if u is its characteristic function, then for any $\mathcal{U} \in U(\mathbb{E})$, we have $\zeta_{\mathcal{U}}(F) = 1$ if and only if $F \in \mathcal{U}$, i.e. if and only if $\lim_{\mathcal{U}} |u(x)| = 1$. Otherwise, both $\zeta_{\mathcal{U}}(F)$ and $\lim_{\mathcal{U}} |u(x)|$ are equal to 0. Therefore, the relation

$$\left| \lim_{\mathcal{U}} |u(x)| - \lim_{\mathcal{V}} |u(x)| \right|_{\infty} \leq \frac{1}{2}$$

holds if and only if $\zeta_{\mathcal{U}}(F) = \zeta_{\mathcal{V}}(F)$. Recall that for every $\mathcal{U} \in U(\mathbb{E})$ we have $\Phi(\varphi_{\mathcal{U}}) = \zeta_{\mathcal{U}}$.

We will show that Φ is continuous. Consider $O_1, \dots, O_q \in \mathbb{B}(\mathbb{E})$, $\mathcal{U} \in U(\mathbb{E})$ and the neighborhood $V(\zeta_{\mathcal{U}}, O_1, \dots, O_q)$ of $\zeta_{\mathcal{U}}$. From the preceding remark, $\zeta_{\mathcal{V}}$ belongs to $V(\zeta_{\mathcal{U}}, O_1, \dots, O_q)$ if and only if for every $j = 1, \dots, q$, $\zeta_{\mathcal{U}}(O_j) = \zeta_{\mathcal{V}}(O_j)$, i.e. if for every $j = 1, \dots, q$,

$$\left| \lim_{\mathcal{U}} |u_j(x)| - \lim_{\mathcal{V}} |u_j(x)| \right|_{\infty} \leq \frac{1}{2}$$

i.e. if $\varphi_{\mathcal{V}}$ belongs to $W(\varphi_{\mathcal{U}}, u_1, \dots, u_q, \frac{1}{2})$. Consequently, this proves that Φ is continuous.

We can now deduce that Φ is a homeomorphism because it is a continuous bijection between compact spaces (Corollaire 2 of Theoreme 2 in [3]).

Corollary 5.2.a: *The space $\Sigma(\mathbb{E})$ is a compactification of \mathbb{E} which is equivalent to the compactification $Mult(S, \|\cdot\|)$.*

Remark 11: For an admissible algebra S , the Banaschewski compactification $\Sigma(\mathbb{E})$ coincides with the Guennebaud-Berkovich multiplicative spectrum.

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