



**HAL**  
open science

## Higher bifurcations for polynomial skew-products

Matthieu Astorg, Fabrizio Bianchi

► **To cite this version:**

Matthieu Astorg, Fabrizio Bianchi. Higher bifurcations for polynomial skew-products. 2020. hal-02882110v1

**HAL Id: hal-02882110**

**<https://hal.science/hal-02882110v1>**

Preprint submitted on 26 Jun 2020 (v1), last revised 20 Apr 2021 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# HIGHER BIFURCATIONS FOR POLYNOMIAL SKEW PRODUCTS

MATTHIEU ASTORG AND FABRIZIO BIANCHI

We continue our investigation of the parameter space of families of polynomial skew-products. We study the self-intersections of the bifurcation current, and in particular the bifurcation measure, through the simultaneous bifurcations of multiple critical points. Our main result is the equality of the supports of the bifurcation current and the bifurcation measure for families of polynomial skew-products over a fixed base. This is a striking difference with respect to the one-dimensional case.

Combined with results by Dujardin and Taflin, this also implies that the support of the bifurcation measure in these families has non-empty interior. It also provides a new proof of the existence of holomorphic families of arbitrarily large dimension whose bifurcation locus has non empty interior. Finally, it shows that the Hausdorff dimension of the support of the bifurcation measure is maximal at any point of its support.

Our proof is based on an analytical criterion for the non-vanishing of the bifurcation currents and on a geometric method to create multiple bifurcations at a common parameter. The latter is a variant of the inclination lemma, applied to the postcritical set at a Misiurewicz parameter.

## 1. INTRODUCTION

Polynomial skew products are regular polynomial endomorphisms of  $\mathbb{C}^2$  of the form  $f(z, w) = (p(z), q(z, w))$ , for  $p$  and  $q$  polynomials of a given degree  $d \geq 2$ . *Regular* here means that the coefficient of  $w^d$  in  $q$  is non zero, which is equivalent to the extendibility of these maps as holomorphic self maps of  $\mathbb{P}^2$ . Despite their specific forms, these maps already provided examples of new phenomena with respect to the established theory of one-variable polynomials or rational maps, see for instance [ABD<sup>+</sup>16, Duj16, Taf17]. We started in [AB18] a detailed study of the parameter space of such maps.

We will denote in what follows by  $\mathbf{Sk}(p, d)$  the family of all polynomial skew products of a given degree  $d$  over a fixed base polynomial  $p$  up to affine conjugacy, and denote by  $D_d$  its dimension. Following [BBD18] it is possible to divide the parameter space of the family  $\mathbf{Sk}(p, d)$  (identified with  $\mathbb{C}^{D_d}$ ) into two dynamically defined subsets: the *stability locus* and the *bifurcation locus*. The bifurcation locus coincides with the support of  $dd^c L_v$ , where  $L_v(f)$  denotes the vertical Lyapunov function of  $f$ , see [Jon99, AB18]. We gave in [AB18] a description of the bifurcation locus and current in terms of natural bifurcation loci and currents associated to the vertical fibres, and a classification of unbounded hyperbolic components in the quadratic case.

For families of rational maps, the study of the powers of the bifurcation current (which are meaningful because of the continuity of its potential) was started in [BB07], see also [DF08, Duj11]. A geometric interpretation of the support of these currents is the following: the support of  $T_{\text{bif}}^k$  is the locus where  $k$  critical points bifurcate *independently*. They also detect the distribution of certain dynamically defined parameters, for instance the loci of rational maps with  $k$  periodic cycles with prescribed multipliers. This gives rise to a natural stratification of the bifurcation locus as

$$\text{Supp } T_{\text{bif}} \supseteq \text{Supp } T_{\text{bif}}^2 f \supseteq \cdots \supseteq \text{Supp } T_{\text{bif}}^{k_{\text{max}}}$$

where  $k_{max}$  is the dimension of the parameter space. The inclusions above are not equalities in general, and are for instance strict when considering the family of all polynomial or rational maps of a given degree (where  $k_{max}$  is equal to  $d - 1$  and  $2d - 2$ , respectively). It is worth pointing out that this stratification has often been compared with an analogous stratification for the Julia set of endomorphisms of  $\mathbb{P}^k$  (given by the support of the self-intersections of the Green current, see for instance [DS10]). We refer to [Duj11] for a more detailed exposition.

The goal of this paper is to show that the situation in families of higher dimensional dynamical systems is completely different. Namely, we establish the following result. A *moderately repelling cycle* for a polynomial  $p$  of degree  $d \geq 2$  is a repelling cycle of period  $n_0$  and multiplier strictly smaller than  $d^{n_0}$ , see Definition 2.8.

**Theorem 1.1.** *Let  $\mathbf{Sk}(p, d)$  denote the family of polynomial skew products of degree  $d \geq 2$  up to affine conjugacy, let  $D_d$  be its dimension and assume that  $p$  has a moderately repelling cycle. Then the associated bifurcation current  $T_{\text{bif}}$  satisfies*

$$\text{Supp } T_{\text{bif}} \equiv \text{Supp } T_{\text{bif}}^2 \equiv \dots \equiv \text{Supp } T_{\text{bif}}^{D_d}.$$

The assumption that  $p$  has a moderately repelling cycle is relatively mild: we prove that it is satisfied whenever the Julia set of  $p$  is not totally disconnected and  $p$  is neither a power map nor a Chebyshev polynomial, see Proposition 2.10.

The proof of Theorem 1.1 essentially consists of two ingredients, respectively of analytical and geometrical flavours.

The first is an analytical sufficient condition for a parameter to be in the support of  $T_{\text{bif}}^k$ . This is inspired by analogous results by Buff-Epstein [BE09] and Gauthier [Gau12] in the context of rational maps, and is based on the notion of *large scale condition* introduced in [AGMV19]. It is a way to give a quantified meaning to the *simultaneous independent bifurcation of multiple critical points*, and to exploit this condition to prove the non-vanishing of  $T_{\text{bif}}^k$ . This part does not require essentially new arguments.

The second ingredient is a procedure to build these multiple independent bifurcations at a common parameter starting from a simple one. The idea is to start with a parameter with a *Misiurewicz* bifurcation, i.e., a non-persistent collision between a critical orbit and a repelling point, and to construct a new parameter nearby where two (and actually, any arbitrary large number of) Misiurewicz bifurcations occur. This geometrical construction is our main technical result, and the main contribution of this paper.

Our main theorem and the method developed for its proof have a number of consequences and corollaries. First of all, results analogous to Theorem 1.1 (in fact, stronger) hold for many algebraic hypersurfaces of the family  $\mathbf{Sk}(p, d)$ . More precisely, as a by-product of the proof of Theorem 1.1, we obtain the following.

**Corollary 1.2.** *Assume that  $p$  has a moderately repelling cycle. Near any bifurcating parameter in  $\mathbf{Sk}(p, d)$ , there exist algebraic subfamilies  $M^k$  of  $\mathbf{Sk}(p, d)$  of any dimension  $k < D_d$  such that the support of the bifurcation measure of  $M^k$  has non-empty interior in  $M^k$ .*

These families are given by the maps satisfying a given critical relation. Notice that  $d$  (and thus  $D_d$ ) can be taken arbitrarily large. This result is for instance an improvement of the main result in [BT17], where 1-parameter families with the same property are constructed.

More strikingly, in [Duj17, Taf17], Dujardin and Taffin construct open sets in the bifurcation locus in the family  $\mathcal{H}_d(\mathbb{P}^k)$  of all endomorphisms of  $\mathbb{P}^k$  of a given degree  $d \geq 2$  (see also [Bie19] for further examples). Their strategy also works when considering the subfamily of polynomial skew products (and actually these open sets are built close to this family). Combining Theorem 1.1 with their result we thus get the following consequence.

**Corollary 1.3.** *Assume that  $p$  has a moderately repelling cycle. The support of the bifurcation measure in  $\mathbf{Sk}(p, d)$  has non empty interior.*

Notice that it is not known whether the bifurcation locus is the closure of its interior (see the last paragraph in [Duj17]). Hence, a priori, the open sets as above could exist only in some regions of the parameter space. The last consequence of our main Theorem 1.1 is a uniform and optimal bound for the Hausdorff dimension of the support of the bifurcation measure, which is a generalization to this setting of the main result in [Gau12].

**Corollary 1.4.** *Assume that  $p$  has a moderately repelling cycle. The Hausdorff dimension of the support of the bifurcation measure in  $\mathbf{Sk}(p, d)$  is maximal at all points of its support.*

Notice that, in the family of all endomorphisms of a given degree, such a uniform estimate is not known even for the bifurcation locus, see [BB18] for some local estimates.

The paper is organized as follows. In Section 2 we set the notations and prove some preliminary general results that will be needed in the sequel. In Section 3 we give the analytical criterion for the non-vanishing of  $T_{\text{bif}}^k$ , and in Section 4 we develop our main construction to build multiple bifurcations starting from a simple one. As a by-product of our procedure, we deduce Corollary 1.2. The proof of Theorem 1.1 is then concluded in Section 5.

**Acknowledgements.** This project has received funding from the French government through the programs I-SITE ULNE / ANR-16-IDEX-0004 ULNE, LabEx CEMPI / ANR-11-LABX-0007-01, and ANR JCJC Fatou ANR-17-CE40-0002-01, from the CNRS through the program PEPS JCJC 2019, and from the Louis D. Foundation through the project "Jeunes Géomètres".

## 2. NOTATIONS AND PRELIMINARY RESULTS

**2.1. Notations.** We collect here the main notations that we will use through all the paper. We refer to [AB18] and [Jon99] for more details.

Given a polynomial skew product of degree  $d \geq 2$  of the form  $f(z, w) = (p(z), q(z, w)) =: (p(z), q_z(w))$ , we will write the  $n$ -th iterate as

$$f^n(z, w) = (p^n(z), q_{p^{n-1}(z)} \circ \cdots \circ q_z(w)) =: (p^n(z), Q_z^n(w)).$$

In particular, if  $z_0$  is a  $n_0$ -periodic point for  $p$ , the map  $Q_{z_0}^{n_0}$  is the return map to the vertical fibre  $\{z_0\} \times \mathbb{C}$  and is a polynomial of degree  $d^{n_0}$ .

Let us now denote by  $(f_\lambda)_{\lambda \in M}$  a holomorphic family of polynomial skew products of a given degree  $d \geq 2$ , that is a holomorphic map  $F: M \times \mathbb{C}^2 \rightarrow \mathbb{C}^2$  such that  $f_\lambda := F(\lambda, \cdot)$  is a polynomial skew product of degree  $d$  for all  $\lambda \in M$ . We will denote by  $\mathbf{Sk}(d)$  the family of all polynomial skew products of degree  $d$ , and by  $\mathbf{Sk}(p, d)$  the subfamily of

those with the given polynomial  $p$  as first component, both up to affine conjugacy. An explicit description of these families in the case  $d = 2$  is given in [AB18, Lemma 2.9], the general case can be treated similarly.

We are interested in *bifurcations* within families of polynomial skew products. Following [BBD18], the *bifurcation locus*  $\text{Bif}$  is defined as the support of the  $(1, 1)$ -positive closed current  $T_{\text{bif}} := dd_{\lambda}^c L(\lambda)$  on  $M$ , where  $L(\lambda)$  is the Lyapunov function associated to  $f_{\lambda}$  with respect to its measure of maximal entropy. In the case of polynomial skew products, the function  $L$  has a quite explicit description. Indeed, by [Jon99] we have  $L(\lambda) = L_p(\lambda) + L_v(\lambda)$ , where

$$(1) \quad L_p(\lambda) = \log d + \sum_{z \in C_{p_{\lambda}}} G_{p_{\lambda}}(z) \quad \text{and} \quad L_v(\lambda) = \log d + \int \left( \sum_{w: q'_{\lambda, z}(w)=0} G_{\lambda}(z, w) \right) \mu_{p_{\lambda}}(z).$$

Here  $\mu_{p_{\lambda}}, G_{p_{\lambda}}, C_{p_{\lambda}}$  are the measure of maximal entropy, the Green function and the critical set (whose points are counted with multiplicity) of  $f_{\lambda}$  and  $p_{\lambda}$  respectively, and  $G_{\lambda}(z, w)$  is the *non-autonomous Green function* for the family  $\{Q_z^n\}_{n \in \mathbb{N}}$ . The current  $T_p := dd_{\lambda}^c L_p(\lambda)$  is positive and closed. We proved in [AB18, Proposition 3.1] that  $T_v := dd_{\lambda}^c L_v = T_{\text{bif}} - T_p$  is also positive and closed. This allowed us to define the *vertical bifurcation* in any family of polynomial skew products. This was generalized in [DT18] to cover families of endomorphisms of  $\mathbb{P}^k(\mathbb{C})$  preserving a fibration. Of course, when  $p$  is constant we have  $T_{\text{bif}} = T_v$ .

**2.2. The unicritical subfamily**  $U_d \subset \mathbf{Sk}(p, d)$ . We consider here the *unicritical subfamily*  $U_d \subset \mathbf{Sk}(p, d)$  given by

$$(2) \quad U_d := \{f(z, w) = (p(z), w^d + a(z))\}, \quad a(z) \in \mathbb{C}_d[z] \sim \mathbb{C}^{d+1}.$$

Thus,  $U_d$  has dimension  $d + 1$ . We parametrize it with  $\lambda := (a_0, \dots, a_d)$ , where the  $a_i$  are the coefficients of  $a(z)$ . We can compactify this parameter space to  $\mathbb{P}^{d+1}$  and we denote by  $\mathbb{P}_{\infty}^d$  the hyperplane at infinity. Given  $z_0 \in J_p$ , we set

$$(3) \quad \mathcal{B}_{z_0} := \{ \lambda \in U_d : \{f_{\lambda}^n(z_0, 0)\} \text{ is bounded} \}.$$

Notice that  $(z_0, 0)$  is the only critical point for  $f_{\lambda}$  in the fibre  $\{z = z_0\}$  (this justifies the name chosen for this family, coherently with the name of the one dimensional family  $f_{\lambda}(z) = z^d + \lambda$ ). The following results are proved in [AB18] for the case  $d = 2$ . The proofs in the general case are similar, we include a sketch for Theorem 2.1 for the reader's convenience.

**Theorem 2.1.** *The accumulation on  $\mathbb{P}_{\infty}^d$  of  $\mathcal{B}_{z_0}$  is precisely equal to  $E_{z_0} := \{[\lambda] : a(z_0) = 0\}$ .*

*Proof.* Denote  $A(\lambda) := \sup_{z \in J_p} |a(z)|$ . We prove that, for  $\lambda$  sufficiently large and any  $z_0 \in J_p$ ,

$$(4) \quad G_{f_{\lambda}}(z_0, 0) = 0 \Rightarrow |a(z_0)| \leq 2A(\lambda)^{1/d}$$

This implies that the accumulation on  $\mathbb{P}_{\infty}^d$  of  $\mathcal{B}_{z_0}$  is included in  $E_{z_0}$ . Indeed, take a sequence  $\lambda_{(n)} \rightarrow \infty$  with  $[\lambda_{(n)}] \rightarrow [\lambda]$  and associated polynomials  $a_{(n)}$ . If  $[\lambda] \notin E_{z_0}$ , both  $|a_{(n)}(z_0)|$  and  $A(\lambda_{(n)})$  grow linearly with  $|\lambda_{(n)}|$ . This proves the inclusion.

In order to prove (4), set  $(z_n, w_n) := f_\lambda^n(z_0, 0) = (p^n(z_0), Q_{z_0}^n(0))$  and  $\rho_n := a(z_n)$ . Notice that  $w_{n+1} = w_n^d + \rho_n$ . Since  $G_{f_\lambda}(z_0, 0) = 0$  we have  $|w_n| \leq 2A(\lambda)^{1/d}$  for all  $n$ . We deduce (4) taking  $n = 0$ .

For the opposite inclusion, we can restrict ourselves to generic 2-dimensional subfamilies  $L$  of  $U_d$ , with the property that the intersection of their closure in  $\mathbb{P}^{d+1}$  with the hyperplane  $E_{z_0}$  is one point. It is then enough to prove that the restriction of  $\mathcal{B}_{z_0}$  to  $L$  is not compact. This can be seen explicitly when  $z_0$  is periodic, and follows from the density of the periodic points and the continuity of the Green function for all other  $z_0 \in J_p$ .  $\square$

The following is a consequence of Theorem 2.1, see [AB18, Corollary 4.3] for a proof.

**Corollary 2.2.** *Let  $z_0, \dots, z_d \in J_p$  be  $d + 1$  distinct points. Then  $\mathcal{B}_{z_0} \cap \dots \mathcal{B}_{z_d}$  is compact.*

**2.3. Families defined by Misiurewicz relations.** By [BBD18, Bia19] the bifurcation locus of a family  $(f_\lambda)_{\lambda \in M}$  coincides with the closure of the set of *Misiurewicz parameters*, i.e., parameters for which we have a non-persistent intersection between some component of the post critical set and the motion of some repelling point. More precisely, in our setting take  $\lambda_0 \in \mathbf{Sk}(p, d)$  and let  $M$  be an algebraic subfamily of  $\mathbf{Sk}(p, d)$  such that  $\lambda_0 \in M$ . A *Misiurewicz relation* for  $f_{\lambda_0}$  is an equation of the form  $f_{\lambda_0}^{n_0}(z_0, c_0) = (z_1, w_1)$  where  $(z_1, w_1)$  is a repelling periodic point of period  $m$  for  $f_{\lambda_0}$ , and  $q'_{z_0, \lambda_0}(c_0) = 0$ .

Assume that  $c_0$  is a simple root of  $q'_{z_0, \lambda_0}$  (this assumption could be removed, but we keep it here for the sake of simplicity). Then there is a unique holomorphic map  $\lambda \mapsto c(\lambda)$  defined on a neighbourhood of  $\lambda_0$  in  $\mathbf{Sk}(p, d)$  such that  $c(\lambda_0) = c_0$ . Similarly, it is possible to locally follow holomorphically the repelling point  $(z_1, w_1)$  as  $\lambda \mapsto (z_1, w_1(\lambda))$ .

The Misiurewicz relation  $f_{\lambda_0}^{n_0}(z_0, c_0) = (z_1, w_1)$  is said to be *locally persistent in  $M$*  if  $f_\lambda^{n_0}(z_0, c(\lambda)) = (z_1, w_1(\lambda))$  for all  $\lambda$  in a neighbourhood of  $\lambda_0$  in  $M$ . If this is not the case, the equation  $f_\lambda^{n_0}(z_0, c(\lambda)) = (z_1, w_1(\lambda))$  defines a germ of analytic hypersurface in  $M$  at  $\lambda_0$ , which is open inside the algebraic hypersurface of  $M$  given by  $\{\lambda \in M : \text{Res}_w(q'_{z_0, \lambda}, Q_{z_0, \lambda}^{n_0+m} - Q_{z_0, \lambda}^m) = 0\}$ . Here,  $\text{Res}_w(P, Q)$  denotes the resultant of two polynomials  $P, Q \in A[w]$ , where  $A := \mathbb{C}[\lambda]$ ; it is therefore an element of  $A$ . Notice that this algebraic hypersurface consists of all  $\lambda \in M$  such that some critical point in the fibre at  $z_0$  lands after  $n_0$  iterations on some periodic point of period dividing  $m$ . We also say in this case that  $\lambda_0$  is a *Misiurewicz parameter in  $M$* .

If the Misiurewicz relation is non-persistent in  $M$ , we denote by  $M_{(z_0, c), (z_1, w_1), n_0}$  (or by  $M_{(z_0, c_0), (z_1, w_1(\lambda_0)), n_0}$  if we wish to emphasize the starting parameter  $\lambda_0$ ) this irreducible component and we call it the locus where the relation is locally preserved. We may avoid mentioning the periodic point if this does not create confusion.

The following lemma, although quite elementary, provides a property of the families  $M_{(z, c), n}$  that will be of crucial use in this paper.

**Lemma 2.3.** *Let  $\lambda_0 \in \mathbf{Sk}(p, d)$  be a Misiurewicz parameter:  $(z_1, w_1(\lambda_0)) = f_{\lambda_0}^{n_0}(z_0, c_0)$ , where  $q'_{z_0, \lambda_0}(c_0) = 0$  and  $(z_1, w_1(\lambda_0))$  is a repelling point of period  $m$  for  $f_{\lambda_0}$ . Assume that  $z_0 \notin \{p^k(z_1) : 0 \leq k \leq m-1\}$ . Then the vertical multiplier of the cycle of  $(z_1, w_1(\lambda))$  is non-constant on the algebraic hypersurface  $M := M_{(z_0, c_0), n}$ .*

*Proof.* Let  $U_d$  be the *unicritical subfamily* of  $\mathbf{Sk}(p, d)$  as defined in (2) and denote by  $L$  the algebraic hypersurface in  $U_d$  given by  $L := M \cap U_d$ . It is enough to prove the statement in restriction to  $L$ . We compactify  $U_d$  to  $\mathbb{P}^{d+1} = \mathbb{C}^d \cup \mathbb{P}_\infty^d$  as in Section 2.2 and  $L$  to  $\hat{L}$ , respectively. Since  $L \subset \mathcal{B}_{z_0}$  (where  $\mathcal{B}_{z_0}$  is defined as in (3)), we have  $\hat{L} \cap \mathbb{P}_\infty^d \subseteq \{[\lambda] : a(z_0) = 0\}$ . Since  $\hat{L} \cap \mathbb{P}_\infty^d$  is an algebraic hypersurface of  $\mathbb{P}_\infty^d$ , the inclusion is actually an equality.

Pick any  $[\hat{\lambda}] \in \hat{L} \cap \mathbb{P}_\infty^d$  such that  $z_0$  is the only (simple) root in  $J_p$  of the associated polynomial  $a(z)$ , and consider a sequence of points  $\lambda_j \in L$ , for  $j \in \mathbb{N}$ , such that  $\|\lambda_j\| \rightarrow \infty$  and  $[\lambda_j] \rightarrow [\hat{\lambda}]$  as  $j \rightarrow \infty$ . In order to prove the statement, it is enough to prove that the vertical eigenvalue of  $(z_1, w_1(\lambda_j))$  diverges as  $j \rightarrow \infty$ .

Fix  $\alpha, \beta$  such that  $0 < \alpha < \frac{1}{d} < \beta < 1$ . First note that for  $j$  large enough,  $R_j := \|\lambda_j\|^\beta$  is an escape radius for  $f_{\lambda_j}$  over  $J_p$ , meaning that  $K_z \subset \mathbb{D}(0, R_j)$  for all  $z \in J_p$ . Indeed, if  $|w| \geq R_j$  then  $|q_z(w)| \asymp |w|^d \geq R_j^d$ , hence the orbit of  $(z, w)$  (for  $z \in J_p$ ) is unbounded. Similarly, set  $r_j := \|\lambda_j\|^\alpha$ ; we claim that, for any  $z \in J_p$  distinct from  $z_0$ ,  $K_z \cap \mathbb{D}(0, r_j) = \emptyset$  for  $j$  large enough. Indeed, we have  $|a_j(z)| \asymp \|\lambda_j\|$  and so  $|q_{z, \lambda_j}(w)| \asymp \|\lambda_j\| > R_j$  for any  $w \in \mathbb{D}(0, r_j)$ . The claim follows from the previous statement for  $R_j$ .

In particular, since we assumed  $z_0 \neq z_1$ , we have that  $w_{k,j} := Q_{z_1, \lambda_j}^k(w_1(\lambda_j))$  satisfies  $|q'_{p^k(z_1)}(w_{k,j})| \gtrsim r_j^{(d-1)}$ . This means that the vertical eigenvalue for  $f_{\lambda_j}$  is larger than  $r_j^{m(d-1)}$ , so in particular cannot be constant in  $\lambda$ . The assertion follows.  $\square$

By another intersection argument with the well-understood unicritical family  $U_d$  as in (2) we also obtain the following Lemma.

**Lemma 2.4.** *For every  $(n, m) \in \mathbb{N} \times \mathbb{N}^*$  the set*

$$\Lambda_{n,m} := \{ \lambda \in \mathbf{Sk}(p, d) : f_\lambda \text{ has a critical component of preperiod } n \text{ and period } m \}$$

*is contained in an algebraic set of codimension at least 2 of  $\mathbf{Sk}(p, d)$ .*

*Proof.* For any given  $(n, m)$ , the set  $\Lambda_{n,m}$  is algebraic. Let  $\tilde{\Lambda}$  be a component. In order to prove the statement, it is enough to show that the intersection  $\tilde{\Lambda} \cap U_d$  is bounded (recall that  $\dim U_d = d + 1$ ). This is a consequence of Corollary 2.2.  $\square$

Note that the argument above actually proves that  $\Lambda_{n,m}$  has codimension at least  $d + 1$  in  $\mathbf{Sk}(p, d)$ , and in fact this is likely still not sharp. However, for our purposes it will be enough that the codimension is at least 2.

**Lemma 2.5.** *Let  $\lambda_0 \in \mathbf{Sk}(p, d)$  and assume that  $f_{\lambda_0}$  has a Misiurewicz relation  $f_{\lambda_0}^n(z_0, c_0) = (z_1, w_1)$ . Then the set of  $\lambda \in M_{(z_0, c_0), n}$  for which  $q_{z_0, \lambda}$  has only simple critical points is Zariski open in  $M_{(z_0, c_0), n}$ .*

*Proof.* Again, it is enough to check the property on the intersection between  $M_{(z_0, c_0), n}$  and any algebraic subfamily of  $\mathbf{Sk}(p, d)$ . This time we can take the subfamily given by the product maps, of the form  $\{(z, w) \mapsto (p(z), q(w))\}$  for  $q: \mathbb{C} \rightarrow \mathbb{C}$  of degree  $d$ . The assertion then follows from the corresponding property for the family  $P_d$  of one-dimensional polynomials of degree  $d$ .  $\square$

**2.4. Higher bifurcations currents and loci.** Higher bifurcation currents for families of polynomials (or rational maps) in one variable were introduced in [BB07] with the aim of understanding the loci where simultaneous and independent bifurcations happen, from an analytical point of view. Since the Lyapunov function is continuous with respect to the parameters [DS10], it is indeed meaningful to consider the self-intersections  $T_{\text{bif}}^k$  of the bifurcation current, for every  $k$  up to the dimension of the parameter space. The measure obtained by taking the maximal power is usually referred to as the *bifurcation measure*. We denote by  $\text{Bif}_k$  the support of  $T_{\text{bif}}^k$ .

While in dimension one it is quite natural to associate a geometric meaning to  $\text{Bif}_k$  (as, for instance, the points where  $k$  independent Misiurewicz relations happens, in a quite precise sense, see, e.g., [Duj11]), in higher dimensions the critical set is of positive dimension and thus this interpretation is far less clear.

The following result gives a first step in the interpretation of the higher bifurcations as average of non-autonomous counterparts of the classical one-dimensional objects, valid in any family of polynomial skew products over a fixed base  $p$ . An interpretation of the non-autonomous factors will be the object of the next Section. The case of general polynomial skew products is completely analogous, and the following should be read as a decomposition for the vertical bifurcation  $T_v^k = (dd^c L_v)^k$ , see Section 2.1. Given  $\underline{z} := (z_1, \dots, z_k) \in J_p^k$ , we denote by  $T_{\underline{z}}$  the current  $T_{\underline{z}} = T_{\text{bif}_{z_1}} \wedge \dots \wedge T_{\text{bif}_{z_k}}$ .

**Proposition 2.6.** *Let  $(f_\lambda)_{\lambda \in M}$  be a family of polynomial skew products over a fixed base  $p$ . Then*

$$T_{\text{bif}}^k = \int_{J_p^k} T_z \mu^{\otimes k} \quad \text{and} \quad \text{Bif}_k = \text{Supp}(T_{\text{bif}}^k) = \overline{\cup_{\underline{z}} \text{Supp } T_{\underline{z}}}.$$

*Proof.* The case  $k = 1$  (which is a consequence of the explicit formula for  $L_v$  in (1)) is proved in [AB18, Theorem 3.3]. The first formula in the statement is a consequence of the continuity of the potentials of the bifurcation currents  $T_{\text{bif},z}$ . The continuity of the potentials (in both  $z$  and the parameter) also implies that the currents  $T_{\underline{z}}$  are continuous in  $\underline{z} \in J_p^k$ . We can thus apply the general Lemma 2.7 below to the family of currents  $R_a = T_{\underline{z}}$  and  $a = \underline{z} \in J_p^k = A$ . This concludes the proof.  $\square$

**Lemma 2.7.** *Let  $A$  be a compact metric space,  $\nu$  a positive measure on  $A$  and  $R_a$  a family of positive closed currents on  $\mathbb{C}^N$  depending continuously on  $a \in A$ . Set  $R := \int_A R_a \nu(a)$ . Then*

- (1) *the support of  $R_a$  depends lower semicontinuously from  $a$  (in the Hausdorff topology);*
- (2) *the support of  $R$  is included in  $\overline{\cup_a \text{Supp } R_a}$ ;*
- (3) *for every  $a \in \text{Supp } \nu$ , we have  $\text{Supp } R_a \subseteq \text{Supp } R$ .*

Recall that the current  $R = \int_A R_a \nu(a)$  is defined by the identity  $\langle R, \varphi \rangle = \int_A \langle R_a, \varphi \rangle \nu(a)$ , for  $\varphi$  test form of the right degree.

*Proof.* The first property is classical and the second is a direct consequence. Let us prove the last item. Fix  $a \in A$  and take  $p \in \text{Supp } R_a$ . There exists an (arbitrarily small) ball  $B$  centred at  $p$  such that the mass of  $R_a$  on  $B$  is larger than some  $\eta > 0$ . By the continuity of  $R_a$ , the mass of  $R_{a'}$  on  $B$  is larger than  $\eta/2$  for every  $a'$  sufficiently close to  $a$ . In particular, this is true for all  $a'$  in a ball  $B'$  centred at  $a$ . Since  $a \in \text{Supp } \nu$ , we



have  $\nu(B') > \eta'$  for some positive  $\eta'$ . Thus,  $R$  has mass  $> \eta\eta'/2$  on  $B$ , which in turn gives  $p \in \text{Supp } R$ .  $\square$

**2.5. Moderate and vertical-like repelling points.** Our method of proof of Theorem 1.1 requires that the base polynomial  $p$  admits a repelling cycle satisfying the following technical assumption.

**Definition 2.8.** *Let  $p$  be a polynomial of degree  $d$  and let  $z_0$  be a  $n_0$ -periodic point for  $p$ . We say that  $z_0$  is moderately repelling if  $|(p^{n_0})'(z_0)| < d^{n_0}$ .*

Although the two next results are not necessary for the proof of our main theorem, they illustrate the existence of many polynomials satisfying this condition.

**Lemma 2.9.** *The set of degree  $d$  polynomials  $p \in P_d$  with a moderately repelling periodic point is an open set whose closure contains  $\text{Bif}(P_d)$ .*

*Proof.* The openness is clear. For the second part of the assertion, note that arbitrarily close to any  $p_0 \in \text{Bif}(P_d)$  one can find some polynomial with a neutral cycle. A suitable perturbation makes that cycle become moderately repelling.  $\square$

More interestingly, we also have the following result.

**Proposition 2.10.** *Let  $p$  be a polynomial with Julia set not totally disconnected, which is neither conjugated to  $z \mapsto z^d$  nor to a Chebyshev polynomial. Then  $p$  has a moderately repelling cycle.*

*Proof.* By a result of Przytycki and Zdunik [PZ20] (see also [Prz85, Zdu90] for previous results in the connected case), since  $p$  is neither conjugated to  $z \mapsto z^d$  nor to a Chebyshev polynomial, there exists a compact hyperbolic invariant set  $K \subset J(p)$ , with  $\delta := \dim_H K > 1$ . By the general theory of the thermodynamical formalism, there exists a unique ergodic invariant probability measure  $\nu$  supported on  $K$  that is absolutely continuous with respect to the  $\delta$ -dimensional Hausdorff measure (see for instance [PU10, Prz18]). By Manning's formula,  $\chi_\nu = \frac{h_\nu}{\delta}$ , where  $\chi_\nu$  is the Lyapunov exponent of  $\nu$ , and  $h_\nu$  is its metric entropy. Since  $\delta > 1$  and  $h_\nu < \log d$ , we deduce that  $\chi_\nu < \log d$ .

We will now approximate  $\nu$  by a sequence of discrete invariant measures supported by periodic cycles, in the following way. Fix  $\varepsilon > 0$  and let  $\delta_1 > 0$  be small enough so that for all  $x, y \in J_p$ ,  $|x - y| \leq \delta_1$  implies  $|\log |f'(x)| - \log |f'(y)|| \leq \varepsilon$  (the existence of such  $\delta_1$  follows from the uniform continuity of  $\log |f'|$  on  $K$ , since  $p$  has no critical point on  $K$ ). Chose  $z \in K$  generic for  $\nu$ , in the sense that its forward orbit equidistributes on  $K$  according to  $\nu$ . In particular, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ :

$$(5) \quad \left| \left\langle \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(z)}, \log |f'| \right\rangle - \langle \nu, \log |f'| \rangle \right| \leq \varepsilon,$$

where  $\delta_y$  is the Dirac mass at  $y \in \mathbb{C}$ . Since  $p : K \rightarrow K$  is hyperbolic, as a consequence of the Shadowing Lemma (see, e.g., [Jon97], Theorem 2.4) there exists  $\delta_2 > 0$  such that any  $\delta_2$ -pseudo-orbit can be  $\delta_1$ -shadowed by an actual orbit. Since  $z$  is recurrent, we may chose  $n \geq N$  such that  $|f^{n-1}(z) - z| \leq \delta_2$ ; then the Shadowing Lemma gives the existence of a cycle  $y_0, \dots, y_{n-1}$  such that for all  $0 \leq j \leq n-1$ ,  $|f^j(z) - y_j| \leq \delta_1$ . With our choice of  $\delta_1$  and (5), this means that

$$|\langle \nu_n, \log |f'| \rangle - \chi_\nu| \leq 2\varepsilon,$$

where  $\nu_n := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{y_k}$ . In particular, if  $\varepsilon$  is small enough, we have  $\langle \nu_n, \log |f'| \rangle < \log d$ , which exactly means that the cycle  $y_0, \dots, y_{n-1}$  is moderately repelling.  $\square$

**Definition 2.11.** Let  $f(z) = (p(z), q(z, w))$  be a polynomial skew product of degree  $\geq 2$  and let  $(z_1, w_1)$  be a  $n_1$ -periodic point for  $f$ . Let  $A := (p^{n_1})'(z_1)$  and  $B := (Q_{z_1}^{n_1})'(w_1)$  be the two eigenvalues of  $Df_{(z_1, w_1)}^{n_1}$ . We say that  $(z_1, w_1)$  is vertical-like if  $|B| > |A|$ .

**Lemma 2.12.** Let  $f(z, w) = (p(z), q(z, w))$  be a polynomial skew product. Let  $z_1$  be moderately repelling  $n_1$ -periodic point for  $p$ . Then  $(z_1, w_1)$  is a vertical-like repelling periodic point for  $f$  for asymptotically all  $w_1 \in \mathbb{C}$  that are periodic for  $Q_{z_1}^{n_1}$ .

The above means that if  $P_n := \{w_1 \in \mathbb{C} : Q_{z_1}^{nn_1}(w) = w\}$  and  $V_n := \{w_1 \in P_n : (z_1, w_1) \text{ is vertical-like}\}$ , then  $\lim_{n \rightarrow \infty} \frac{\text{card } V_n}{\text{card } P_n} = 1$ .

*Proof.* Since by assumption  $z_1$  is moderately repelling for  $p$ , there exists  $\varepsilon > 0$  such that  $|(p^{n_1})'(z_1)| < (d - \varepsilon)^{n_1}$ . The return map  $Q := Q_{z_1}^{n_1}$  associated to the vertical fibre of  $z_1$  is a polynomial of degree  $d^{n_1}$ , thus its Lyapunov exponent is larger than or equal to  $\log(d^{n_1})$ . By [BDM08, Lemma 4.5] for any  $\eta > 0$ , asymptotically all of the  $d^{nn_1}$  repelling periodic points of period dividing  $n$  for  $Q$  have multiplier larger than  $(d^{n_1} - \eta)^n$ . Choosing  $\eta > 0$  small enough that  $d^{n_1} - \eta > (d - \varepsilon)^{n_1}$ , this proves that for asymptotically all periodic point  $w_1$  for  $Q$ , the point  $(z_1, w_1)$  is vertical-like, as desired.  $\square$

### 3. HIGHER BIFURCATIONS: AN ANALYTIC CRITERION

In this section we establish the following technical result, which gives an analytic sufficient condition for a point to lie in the support of the higher bifurcation currents. Recall that, given a simple critical point  $c(\lambda)$  for  $q_{z_0, \lambda}$  and a repelling point  $r(\lambda)$  for  $f_\lambda$ , we denote by  $M_{(z_0, c), r, n_0}$  the analytic subset of  $M$  given by the equation  $f_\lambda^{n_0}(c(\lambda)) = r(\lambda)$ .

**Proposition 3.1.** Let  $(f_\lambda)_{\lambda \in M}$  be a holomorphic family of polynomial skew products over a given base  $p$ . Let  $\lambda_0 \in M$  and  $z_1, \dots, z_k \in J_p$  satisfy the following properties:

- (1) there exist simple critical points  $c_i$  for  $q_{z_i, \lambda_0}$  such that  $r_i := f_{\lambda_0}^{n_i}(z_i, c_i)$  is a repelling periodic point for  $f_{\lambda_0}$ ;
- (2)  $\text{codim} \cap_{i=1}^k M_{(z_i, c_i), r_i, n_i} = k$ .

Then  $\lambda_0 \in \text{Bif}_k(M)$ .

In the case of families of rational maps, this result is due to Buff-Epstein [BE09] when the hypersurfaces defined by the critical relations are transversal, and to Gauthier [Gau12] when the intersections are just proper, as is the case above. A more general condition (called the *generalized large scale condition*) was introduced in [AGMV19] as a sufficient condition for a point to lie in the support of  $T_{\text{bif}}^k$  (for a family of rational maps). We give an adapted version of this notion in our non-autonomous setting, and deduce that a parameter  $\lambda_0$  as in the statement satisfies such condition. This will prove Proposition 3.1.

In the following we assume that  $z_1, \dots, z_k \in J_p$  and that  $c_j(\lambda)$  are holomorphic maps such that  $c_j(\lambda)$  is a critical point for  $q_{z_j, \lambda}$  for all  $\lambda \in M$ . We denote by  $\underline{c} : M \rightarrow \mathbb{C}^k$  the map  $\underline{c}(\lambda) = (c_1(\lambda), \dots, c_k(\lambda))$ . For a  $k$ -uple  $\underline{n} := (n_1, \dots, n_k)$ , we define

$$\xi_{n_j}^j(\lambda) := Q_{z_j}^{n_j}(c_j(\lambda)) \quad \text{and} \quad \Xi_{\underline{n}}^c(\lambda) := (\xi_{n_1}^1(\lambda), \dots, \xi_{n_k}^k(\lambda))$$

Notice that  $\Xi_{\underline{n}}^c : M \rightarrow \mathbb{C}^k$ . Denote by  $V_{\underline{n}}$  the graph of  $\Xi_{\underline{n}}^c$  in  $M \times \mathbb{C}^k$ .

**Definition 3.2** (Fibred large scale condition). *We say that  $\lambda_0 \in M$  satisfies the fibred large scale condition for the critical points  $(z_1, c_1), \dots, (z_k, c_k)$  if there exist  $z'_1, \dots, z'_k \in J_p$ , disks  $D_1, \dots, D_k \subset \mathbb{C}$  with  $D_i \cap J_{z'_i} \neq \emptyset$ , a sequence  $\underline{n}_l = (n_{l,1}, \dots, n_{l,k})$  of  $k$ -uples with  $n_{l,i} \rightarrow \infty$  and a nested sequence of open subsets  $\Omega_l$  such that*

- $\cap_l \overline{\Omega}_l = \{\lambda_0\}$ , and
- $\Xi_{\underline{n}_l}^c : \Omega_l \rightarrow D_1 \times \dots \times D_k$  is a proper surjective map.

**Proposition 3.3.** *Let  $\lambda_0$  satisfy the fibred large scale condition for some  $(z_1, c_1), \dots, (z_k, c_k)$ , with  $q'_{z_j}(c_j) = 0$  for every  $j$  and such that the  $z_j$  are preperiodic for  $p$ . Then  $\lambda_0 \in \text{Supp } T_{\text{bif}_{z_1}} \wedge \dots \wedge T_{\text{bif}_{z_k}}$ .*

*Proof.* The proof is a direct consequence of the analogous result [AGMV19, Theorem 3.2]. Notice that the first step in the proof of that result actually consists in building a new product dynamical space. In our case, we just directly consider the return maps on the periodic fibres, near the repelling periodic points. The fact that the critical points may lie in some pre-periodic fibre can be addressed by considering the orbit of their first image on a periodic fibre.  $\square$

We can now prove Proposition 3.1

*Proof of Proposition 3.1.* By Proposition 2.6 it is enough to prove that  $\lambda_0 \in \text{Supp } T_{\text{bif}_{z_1}} \wedge T_{\text{bif}_{z_k}}$ . It is thus enough to prove that any  $\lambda_0$  as in the statement satisfies the fibred large scale condition above.

Fix  $\eta > 0$  and an open neighbourhood  $\Omega_0$  of  $\lambda_0$  such that the following holds:

- (1) for all  $r_i$  as in the statement,  $r_i(\lambda) \in \mathbb{D}(r_i, \eta/2)$  for all  $\lambda \in \Omega_0$ ;
- (2) for every  $i$  and every  $\lambda \in \Omega_0$ , the map  $f_\lambda$  is uniformly expanding on  $D_i := \mathbb{D}(r_i, \eta)$ .

To verify the fibred large scale condition it is enough that every ball centred at  $\lambda_0$  has open image for the map

$$\lambda \mapsto (f_\lambda^{n_1}(z_1, c_1(\lambda)) - r_1(\lambda), \dots, f_\lambda^{n_k}(z_k, c_k(\lambda)) - r_k(\lambda)).$$

But this is implied by the second assumption in the statement. Indeed, if this were not the case, the image would be contained in an analytic set of dimension  $\leq k - 1$ , and the preimage of any point would be (empty or of) codimension  $\leq k - 1$ . Since the origin is in the image, and is equal to the image of  $\cap_{i=1}^k M_{(z_i, c_i), r_i, n_i}$ , this would contradict the second assumption. The proof is complete.  $\square$

**Remark 3.4.** *As is the case in [AGMV19], it is enough to make a weaker assumption in Proposition 3.1, namely that the critical orbits fall in the motion of some hyperbolic set. The proof is slightly more involved in that situation (as is the case in [AGMV19]). We prefer to state only the simple criterion based on repelling periodic orbits since this simpler version will be enough to deduce our main result.*

#### 4. CREATING MULTIPLE BIFURCATIONS: A GEOMETRIC METHOD

In this section we develop our method to construct multiple bifurcations (in the form of Misiurewicz parameters) starting from a simple one. In the next section we will ensure the applicability of this method. First, let us introduce the following definition.

**Definition 4.1.** Let  $(f_\lambda)_{\lambda \in M}$  be a holomorphic family of polynomial skew-products. We say that  $M$  is a good Misiurewicz family, or that  $M$  has a persistently good Misiurewicz relation  $f_\lambda^{m_0}(z, c(\lambda)) = (z_1, w_1(\lambda))$  if the Misiurewicz relation  $f_\lambda^{m_0}(z, c(\lambda)) = (z_1, w_1(\lambda))$  is persistent in  $M$ , and if moreover

- (G1) the vertical eigenvalue  $B(\lambda) := (Q_{z,\lambda}^m)'(w_1(\lambda))$  is non-constant on  $M$ ;
- (G2)  $(z_1, w_1(\lambda))$  is vertical-like for all  $\lambda \in M$ ;
- (G3) for all  $\lambda$  in a residual subset of  $M$ , at least one component of  $f_\lambda^n(\text{Crit}(f_\lambda))$  passing through  $(z, w_\lambda)$  is not preperiodic;
- (G4)  $c(\lambda)$  is a simple root of  $q'_{z,\lambda}$  for a generic  $\lambda \in M$ .

A parameter  $\lambda_0 \in M$  satisfying all the conditions above will be called a good Misiurewicz parameter.

The next Proposition is the key technical result of our argument:

**Proposition 4.2.** Let  $(f_\lambda)_{\lambda \in M}$  be a holomorphic family of polynomial skew products with a persistently good Misiurewicz relation  $(z_1, w_1(\lambda)) := f_\lambda^N(z_0, c_0(\lambda))$ . There exists a dense subset  $S \subset M$  such that for all  $\lambda_\infty \in S$ , for any non-exceptional graph  $\lambda \mapsto (z, w(\lambda))$  defined in some neighbourhood of  $\lambda_\infty$ , there exists a sequence  $\lambda_n \rightarrow \lambda_\infty$  such that  $(z, w(\lambda_n))$  is non-persistently in the forward orbit of a critical point  $(z_n, c_n(\lambda_n))$ , with  $(z_n, c_n(\lambda_n)) \rightarrow (z_0, c_0(\lambda_\infty))$ .

We call *exceptional graph* a holomorphic map  $\lambda \mapsto (z, w(\lambda))$  such that  $(z, w(\lambda))$  belongs to the exceptional set of  $f_\lambda$  for some  $\lambda$ . The following is a simple consequence of Proposition 4.2. Together with Proposition 5.1, it will give Corollary 1.2.

**Corollary 4.3.** If  $(f_\lambda)_{\lambda \in M}$  has a persistently good Misiurewicz relation, then  $\text{Bif}(M) = M$ .

*Proof.* Let  $S$  be given by Proposition 4.2, take  $\lambda_\infty \in S$  and let  $\Omega_\infty$  be any neighbourhood of  $\lambda_\infty$ . All possible motions on  $\Omega_\infty$  of repelling points for  $f_{\lambda_\infty}$  are of the form  $(z, w(\lambda))$ . Since the exceptional set has zero mass for the equilibrium measure, there is at least one among these graphs (and actually, many of them) which is not persistently contained in the exceptional set. Let  $\gamma(\lambda)$  be one such graph. If  $\gamma$  is exceptional, since the exceptional set is contained in the postcritical set, we get a Misiurewicz parameter at a parameter  $\lambda \in \Omega_\infty$  such that  $\gamma(\lambda)$  is an exceptional point for  $f_\lambda$ . If  $\gamma$  is not exceptional, we apply Proposition 4.2 and still get a Misiurewicz parameters  $\lambda \in \Omega_\infty$ . The assertion follows since Misiurewicz parameters belong to the bifurcation locus, see [BBD18, Bia19].  $\square$

The remaining part of this section is devoted to proving Proposition 4.2. We state and prove the following lemma before defining the subset  $S$ .

**Lemma 4.4.** Let  $\alpha(z) = \alpha_p z^p + O(z^{p+1})$  be a holomorphic germ of  $(\mathbb{C}, 0)$ , with  $\alpha_p \neq 0$  and  $p > 0$ . Let  $|A|, |B| > 1$  be such that  $\log A, \log B$  and  $2i\pi$  are rationally independent, and fix  $z_0 \in \mathbb{C}^*$ . Then  $\{\alpha(z_0 A^{-n}) B^m : m, n \in \mathbb{N}\}$  is dense in  $\mathbb{C}$ .

*Proof.* We first prove that  $X := \{A^{-n} B^m : m, n \in \mathbb{N}\}$  is dense in  $\mathbb{C}$ . This corresponds to the case  $\alpha(z) = z$  and  $z_0 = 1$ .

Working in logarithmic coordinates, we define  $Y := \exp^{-1}(\{A^{-n} B^m : m \in \mathbb{N}, n \in \mathbb{Z}\})$ . Notice that  $Y$  is invariant under the translations by integer multiples of the vectors  $2i\pi$  and  $\log A$ , and positive integer multiples of  $\log B$  (we can choose any branch of  $\log$  here).

Therefore, the set  $Y$  is dense in  $\mathbb{C}$  if and only if orbits under translations by positive integer multiples of the vector  $\log B$  are dense in the complex torus  $\mathbb{C}/\langle 2i\pi, \log A \rangle$ . Since we assumed that  $\log A, \log B$  and  $2i\pi$  are rationally independent, this follows from Kronecker's Theorem. This implies that  $\hat{X} := \{A^{-n}B^m : m \in \mathbb{N}, n \in \mathbb{Z}\}$  is dense in  $\mathbb{C}$ .

Take now any  $w \in \mathbb{C}$ . By the above, there exist  $n_k \in \mathbb{Z}$  and  $m_k \in \mathbb{N}$  such that  $\lim_{k \rightarrow \infty} A^{-n_k}B^{m_k} = w$ . We may assume without loss of generality that  $\lim m_k = +\infty$ . Since  $|B| > 1$ , this implies that  $\lim n_k = +\infty$ , and in particular that  $n_k \in \mathbb{N}$  for  $k$  large enough. So  $X$  is indeed dense in  $\mathbb{C}$ . Note that since  $p \log A, \log B$  and  $2i\pi$  are also rationally independent, we may replace  $\log A$  by  $p \log A$  in the argument above.

Let us now treat the general case, with  $\alpha(z) = \alpha_p z^p + O(z^{p+1})$  and  $z_0 \in \mathbb{C}^*$  as in the statement. Take  $w \in \mathbb{C}$ . By the density  $X$  proved above, there exist sequences  $n_k$  and  $m_k$  of integers, with  $\lim_k n_k = \lim_k m_k = +\infty$ , such that  $\lim_k A^{-pn_k}B^{m_k} = \frac{w}{\alpha_p z_0^p}$ . This implies that  $\alpha_p \cdot (z_0 A^{-n_k})^p B^{m_k} = w + o_{k \rightarrow +\infty}(1)$ , and thus that

$$\left( \alpha(z_0 A^{-n_k}) + O(z_0^{p+1} A^{-(p+1)n_k}) \right) B^{m_k} = w + o_{k \rightarrow +\infty}(1).$$

Since  $A^{-pn_k}B^{m_k} = O(1)$  by construction and  $|A| > 1$  by assumption, we have

$$\alpha(z_0 A^{-n_k}) B^{m_k} + O(A^{-n_k}) = w + o_{k \rightarrow +\infty}(1).$$

We deduce that  $\lim_k \alpha(z_0 A^{-n_k}) B^{m_k} = w$ . The lemma is proved.  $\square$

**Definition 4.5.** *Let  $M$  be a good Misiurewicz family, and let  $f_\lambda^{n_0}(z, c(\lambda)) = (z_1, w_1(\lambda))$  be a persistent Misiurewicz relation satisfying the requirements of Definition 4.1.*

*We define the set  $S \subset M$  to be the set of  $\lambda_\infty \in M$  for which each of the following properties holds:*

- (S1)  $\log A, \log B(\lambda_\infty)$  and  $2i\pi$  are rationally independent over  $\mathbb{Q}$ ;
- (S2)  $d_\lambda B(\lambda_\infty) \neq 0$ ;
- (S3)  $\log B(\lambda_\infty) \notin \mathbb{R} \log A$ ;
- (S4)  $f_{\lambda_\infty}$  has no preperiodic critical component;
- (S5)  $c(\lambda_\infty)$  is a simple root of  $q'_{z, \lambda_\infty}$ .

Note that  $S$  is indeed dense (in fact, residual) in  $M$ . Indeed, since by the assumption (G1) in Definition 4.1 the map  $\lambda \mapsto B(\lambda)$  is non-constant on  $M$ , conditions (S1), (S2) and (S3) are satisfied in a residual subset of  $M$ . The same also holds for condition (S4) by the assumption (G3). Condition (S5) is also a Zariski dense condition once (G4) holds, as observed after Definition 4.1.

From now on, we fix an arbitrary  $\lambda_\infty \in S$ , and we choose a one-dimensional disk in local coordinates in  $M$  in which  $\lambda_\infty = 0$  (hence  $B'(0) \neq 0$ ). The proof of Proposition 4.2 will mostly use local arguments in phase space. Therefore, we will work in local linearizing coordinates near  $(z_1, w_1)$ ; in particular, in the rest of this section we will take  $(z_1, w_1) = (0, 0)$ , and we will treat it as a fixed point (which we can do up to passing to an iterate).

By item (S3) of the definition of  $S$ , there are no resonances between eigenvalues of this fixed point  $(0, 0)$  for  $\lambda$  close to  $\lambda_\infty = 0$ . We may therefore assume that the fixed point  $(0, 0)$  is linearizable for  $f_\lambda$ ; moreover the linearizing map can be chosen to depend holomorphically on the parameter. More precisely, we can fix a neighbourhood  $U$  of  $(0, 0)$  such that such linearising coordinates are defined for  $(z, w) \in U$  for all  $f_\lambda$  with  $|\lambda|$  small enough. So  $f_\lambda$  acts in those coordinates as the linear map  $(z, w) \mapsto (Az, B(\lambda)w)$ .

Since the graph  $\gamma(\lambda) = (z, w(\lambda))$  is not exceptional, by the equidistribution of preimages we can also assume that  $\gamma(\lambda) \in U$  for all  $\lambda$  with  $|\lambda|$  small enough.

Let  $L_0$  be a non-periodic postcritical component of  $f_0$  passing through  $(0, 0)$  (as in the assumption **(G3)** in Definition 4.1). It follows from the Implicit Function Theorem and **(G3)** that  $L_0$  is locally a smooth graph over  $z$ , and that there exists a local holomorphic function  $\beta$  such that for all  $(\lambda, z)$  small enough, the graph  $w = \beta(\lambda, z)$  is a local parametrization of a non-periodic component  $L_\lambda$  of  $f_\lambda^k(\text{Crit}(f_\lambda))$ . The assumption that  $L_0$  is not invariant implies that the holomorphic map  $z \mapsto \beta(z, \lambda)$  is non-constantly equal to 0.

**Lemma 4.6.** *Let  $K \subset \mathbb{C}^*$  be compact and such that  $\{0\} \times K \subset U$ , let  $z \in J_p$  and  $(n_k), (m_k)$  be sequences such that  $\beta(zA^{-n_k}, 0)B(0)^{m_k} \in K$ . Define  $\varphi_k(\lambda) := \beta(zA^{-n_k}, \lambda)B(\lambda)^{m_k}$ . Then  $\lim_k |\varphi_k'(0)| = \infty$ .*

*Proof.* We have

$$\varphi_k'(0) = \frac{\partial \beta}{\partial \lambda}(A^{-n_k}z, 0)B(0)^{m_k} + \beta(A^{-n_k}z, 0)m_k B_1 B(0)^{m_k-1}$$

where  $B_1 := \frac{d}{d\lambda}|_{\lambda=0} B(\lambda) \neq 0$  by assumption. Set  $\alpha(z) := \beta(z, 0)$ . Writing  $\alpha(z)$  as  $\alpha(z) = \alpha_p z^p + O(z^{p+1})$ , we see that the assumption  $\alpha(zA^{-n_k})B(0)^{m_k} \in K$  implies that

$$(6) \quad A^{n_k} \asymp B(0)^{m_k/p}.$$

Moreover, since  $\beta(A^{-n_k}z, 0)B(0)^{m_k} \in K$ , and  $B_1 \neq 0$ , we have

$$(7) \quad \beta(A^{-n_k}z, 0)m_k B_1 B(0)^{m_k-1} \asymp m_k.$$

Finally, since the Misiurewicz relation at  $(0, 0)$  is preserved for all  $\lambda \in M$ , we have  $\beta(0, \lambda) = 0$  for all  $\lambda$ , and thus  $\frac{\partial \beta}{\partial \lambda}(0, 0) = 0$ . Hence, we can write  $\frac{\partial \beta}{\partial \lambda}(z, 0) = c_\ell z^\ell + O(z^{\ell+1})$  with  $c_\ell \neq 0$ , for some  $\ell \geq 1$ . This implies that

$$\frac{\partial \beta}{\partial \lambda}(A^{-n_k}z, 0)B(0)^{m_k} \asymp A^{-\ell n_k} B(0)^{m_k}.$$

We now have two possibilities:

*Case 1:*  $\ell \geq p$ . In this case,  $\frac{\partial \beta}{\partial \lambda}(A^{-n_k}z, 0)B^{m_k} = O(1)$  (in fact, it tends to zero if  $\ell > p$ ). Together with (7), this gives  $\varphi_k'(0) \asymp m_k$ .

*Case 2:*  $\ell < p$ . In this case, by (6) we have  $\frac{\partial \beta}{\partial \lambda}(A^{-n_k}z, 0)B^{m_k} \asymp B^{m_k(1-\ell/p)}$ . By (7) this gives  $\varphi_k'(0) \asymp B^{m_k(1-\ell/p)}$ , too.

In either case, we have  $\lim_k \varphi_k'(0) = +\infty$ , and the lemma is proved.  $\square$

*Proof of Proposition 4.2.* We work in the setting described before Lemma 4.6. Let  $x \in J_0$  be small enough so that  $(0, x) \in U$ . By Lemma 4.4 applied to  $\alpha := \beta(\cdot, 0)$ , there exist sequences  $n_k, m_k \rightarrow +\infty$  such that  $\beta(A^{-n_k}z, 0)B(0)^{m_k} \rightarrow x$ . Let  $z_k := zA^{m_k-n_k}$ ; since  $\lim_k n_k - m_k = +\infty$ , we have that  $\lim_k z_k = 0$ . Observe that

$$(8) \quad f_\lambda^{m_k}(A^{-n_k}z, \beta(A^{-n_k}z, \lambda)) = (z_k, \varphi_k(\lambda)),$$

i.e.,  $(z_k, \varphi_k(\lambda))$  is in the postcritical set of  $f_\lambda$ .

**Step 1.** For  $k$  large enough, there exist two distinct points  $y_{k,i}$ ,  $i = 1, 2$ , such that  $(z_k, y_{k,i}) \in U$  and  $f_0^{n_k - m_k}(z_k, y_{k,i}) = (z, w)$ .

(Note that the orbits  $\{f_0^\ell(z_k, y_{k,i}) : 1 < \ell < n_k - m_k\}$  will not a priori remain in  $U$ .) Consider  $s_0$  such that  $q_{zA^{-s_0}}^{-s_0}(\{w(0)\})$  contains at least 3 points  $y'_i$ . By the lower semicontinuity of  $z \mapsto J_z$  we have  $\mathbb{D}(x, \delta) \cap J_{z_k} \neq \emptyset$  for  $k$  large enough, so the family  $\{Q_{z_k A^{s_0}}^{n_k - s_0} : \mathbb{D}(x, \delta) \rightarrow \mathbb{C}\}$  is not normal. Indeed, for all  $k$  large enough we can find  $w_k \in \mathbb{D}(x, \delta)$  such that  $G_{z_k}(w_k) \geq c > 0$ , for some constant  $c$  independent from  $k$ . By Montel's theorem, this sequence then has to take at least two of the values  $y'_i$ .

**Step 2.** We can assume that, for any  $\varepsilon > 0$  and  $k \in \mathbb{N}$  large enough, there exist holomorphic maps  $\lambda \mapsto y_{k,i}(\lambda)$  defined on  $\mathbb{D}(0, \varepsilon)$  such that  $f_\lambda^{n_k - m_k}(z_k, y_{k,i}(\lambda)) = (z, w(\lambda))$ .

Indeed, if there are  $\lambda \in \mathbb{D}(0, \varepsilon)$  and  $k \in \mathbb{N}^*$  such that  $\{f_\lambda^\ell(z_k, y_{k,i}(\lambda)) : 0 \leq \ell \leq n_k - m_k\}$  meets (non persistently) the critical set of  $f_\lambda$ , then  $(z, w(\lambda))$  is a non-persistent intersection with the postcritical set of  $f_\lambda$  and Proposition 4.2 is proved. Otherwise, Step 2 follows from the Implicit Function Theorem and analytic continuation.

**Step 3.** There exists  $\lambda \in \mathbb{D}(0, \varepsilon)$ ,  $i \in \{1, 2\}$  and  $k \in \mathbb{N}^*$  such that  $\varphi_k(\lambda) = y_{k,i}(\lambda)$  and the intersection is not persistent in  $\lambda$ .

This follows from Montel Theorem applied to  $\{\varphi_k : \mathbb{D}(0, \varepsilon) \rightarrow \mathbb{C}\}$ : indeed, by Lemma 4.6, this sequence is not normal, so it cannot avoid both of the two moving points  $\lambda \mapsto y_{k,i}(\lambda)$ . For the same reason, the intersection cannot be persistent, since this would also contradict the non-normality.

In conclusion, by the steps above and (8) there is a parameter  $\lambda \in \mathbb{D}(0, \varepsilon)$  such that

$$f_\lambda^{m_k}(zA^{-n_k}, \beta(zA^{-n_k}, \lambda)) = (z_k, y_{k,i}(\lambda)),$$

so by definition of  $y_{k,i}(\lambda)$ , we have

$$f_\lambda^{n_k}(zA^{-n_k}, \beta(zA^{-n_k}, \lambda)) = (z, w(\lambda)).$$

By the definition of  $\beta$ , the point  $(zA^{-n_k}, \beta(zA^{-n_k}, \lambda))$  belongs to the postcritical set of  $f_\lambda$ . By construction the intersection above is not persistent. Moreover, we have  $zA^{-n_k} \rightarrow 0$  and  $\beta(zA^{-n_k}, \lambda) \rightarrow \beta(0, \lambda) = \alpha(\lambda)$  as  $k \rightarrow \infty$ . The last assertion in the statement follows by considering a suitable preimage. The proof is complete.  $\square$

## 5. PROOF OF THE MAIN RESULTS

**5.1. Finding a first good Misiurewicz parameter.** In this subsection, we prove that there exists a good Misiurewicz parameter arbitrarily close to any  $\lambda_0 \in \text{Bif}(\mathbf{Sk}(p, d))$  (under the assumption that  $p$  admits a moderately repelling periodic point). This provides the first step in the proof of Theorem 1.1, that will be concluded in the next subsection. Combined with Corollary 4.3, this fact already gives Corollary 1.2.

**Proposition 5.1.** *Let  $\lambda_0 \in \text{Bif}(\mathbf{Sk}(p, d))$  for some polynomial  $p$  admitting a moderately repelling point  $z_1$ . For all  $\varepsilon > 0$ , there exists  $\lambda_1 \in B(\lambda_0, \varepsilon)$  with a Misiurewicz relation  $f_{\lambda_1}^{n_0}(z, c) = (z_1, w_1)$  such that  $M_{(z,c),n_0}$  is a good Misiurewicz family.*

We emphasize here that we only prove the existence of a good Misiurewicz family in  $\mathbf{Sk}(p, d)$ ; we do not claim that the arguments used here work in arbitrary subfamilies of  $\mathbf{Sk}(p, d)$ . In order to prove Proposition 5.1, we need to find  $\lambda_1$  satisfying all the requirements **(G1-4)** of Definition 4.1. We will need the following elementary lemma.

**Lemma 5.2.** *Let  $(f_\lambda)_{\lambda \in M}$  be a holomorphic family of skew-products, and take  $\lambda_0 \in \text{Bif}(M)$ . Let  $(z_1, w_1)$  be any repelling periodic point of  $f_{\lambda_0}$ , which we locally follow as  $(z_1, w_1(\lambda))$ . Then, arbitrarily close to  $\lambda_0$ , there exists  $\lambda_1 \in M$  such that  $f_{\lambda_1}$  has a Misiurewicz relation of the form  $f_{\lambda_1}^n(z, c) = (z_1, w_1(\lambda_1))$  with  $z \neq z_1$ .*

*Proof.* By [AB18, Proposition 3.5], we may find  $\lambda'_0$  arbitrarily close to  $\lambda_0$  for which a critical point of the form  $(y, c)$  is active, where  $y$  is in the strict backward orbit of  $z_1$  by  $p$ . By Montel Theorem, we can further slightly perturb  $\lambda'_0$  to a  $\lambda_1$  with the property that some iterate of  $(y, c)$  by  $f_{\lambda_1}$  coincides with  $(z, w_1(\lambda_1))$ . This completes the proof.  $\square$

*Proof of Proposition 5.1.* We will replace  $\lambda_0$  by a finite sequence of arbitrarily small perturbations, making sure that at every step we stay inside  $\text{Bif}(\mathbf{Sk}(p, d))$ .

By assumption,  $p$  has a moderately repelling periodic point  $z_1$ . Lemma 2.12 gives a vertical-like repelling point (that we can follow holomorphically in a neighbourhood of  $\lambda_0$ ) and Lemma 5.2 a Misiurewicz relation satisfying item **(G2)**.

Consider now the algebraic hypersurface  $M$  of  $\mathbf{Sk}(p, d)$  where the Misiurewicz relation constructed above is preserved. By Lemma 2.3, the map  $\lambda \mapsto B(\lambda)$  is non-constant on  $M$ . This takes care of item **(G1)**. Moreover, by Lemma 2.5, we can perturb  $\lambda_0$  inside  $M$  to ensure that **(G4)** holds, too.

Finally, Lemma 2.4 implies that the set of  $\lambda \in M$  for which  $f_\lambda$  has a periodic critical component is meager in the sense of Baire. This ensures that **(G3)** holds, and completes the proof.  $\square$

**5.2. Conclusion.** We can now conclude the proof of Theorem 1.1. We will need two further lemmas. The first is an improvement of Lemma 2.12.

**Lemma 5.3.** *Let  $(f_\lambda)_{\lambda \in M}$  be an algebraic family of polynomial skew products over a fixed base  $p$  with a moderately repelling periodic point  $z_1$ , and let  $\lambda_0 \in M$ . Then there exists  $w_1 \in \mathbb{C}$  such that  $(z_1, w_1)$  is a vertical-like repelling periodic point for  $f_{\lambda_0}$ . Moreover, if  $\dim M \geq 2$  and  $\mathcal{F}$  is a given algebraic holomorphic codimension 1 foliation of  $M$ , we may choose  $w_1$  such that  $\mathcal{F}$  and the foliation given by the level sets of  $(Q_{z_1, \lambda}^n)'(w_1(\lambda))$  do not have common leaves near  $\lambda_1$ .*

*Proof.* Lemma 2.12 gives the first part of the statement. More precisely, for asymptotically all periodic point  $w_1$  for  $Q := Q_{z_1, \lambda_0}^m$ , the point  $(z_1, w_1)$  is vertical-like. We need to prove the second assertion.

First, assume for the sake of contradiction that each of the vertical-like repelling periodic points constructed above have vertical multipliers that are constant on the leaf  $L$  of  $\mathcal{F}$  passing through  $\lambda_0$ . Then, by [BDM08], the Lyapunov exponent of the first return maps  $Q_{z_1, \lambda}^n$  is constant for  $\lambda \in L$ . In particular, the family  $(Q_{z_1, \lambda}^n)_{\lambda \in L}$  is a non-isotrivial stable algebraic family of polynomials. This is impossible by [McM87].

Therefore, we may choose  $w_1$  so that  $(z_1, w_1)$  is vertical-like and its vertical multiplier is non-constant on  $L$ . In other words, the leaves passing through  $\lambda_0$  of  $\mathcal{F}$  and of the foliation induced by this vertical multiplier are different; so in a small enough



neighbourhood of  $\lambda_0$ , these two foliations cannot have common leaves. The proof is complete.  $\square$

**Lemma 5.4.** *Let  $(f_\lambda)_{\lambda \in M}$  be a holomorphic family of polynomial skew-product over a fixed base  $p$  and assume that there exists  $\lambda_0 \in M$  and  $c \in \mathbb{C}$  such that  $c$  is an escaping simple critical point for  $q_{\lambda_0, z_0}$  for some  $n_0$ -periodic repelling point  $z_0$  for  $p$ . Then there exist an open neighbourhood  $\Omega$  of  $\lambda_0$ , a positive  $\varepsilon$ , and a holomorphic map  $c : \Omega \times \mathbb{D}(z_0, \varepsilon) \rightarrow \mathbb{C}$  such that*

- (1)  $c(\lambda_0, z_0) = c$ ;
- (2) for all  $\lambda \in \Omega$  and  $z \in \mathbb{D}(z_0, \varepsilon)$ ,  $q'_{\lambda, z}(c(\lambda, z)) = 0$ ;
- (3) for all  $\lambda \in \Omega$ , the critical component of  $f_\lambda$  containing all  $c(\lambda, z)$  ( $z \in \mathbb{D}(z_0, \varepsilon)$ ) is not preperiodic.

*Proof.* The existence of  $c$  satisfying the first two properties follows from the Implicit Function Theorem. Let us prove the last item. Denote by  $\tilde{C}_\lambda$  the critical component for  $f_\lambda$  containing all  $c(\lambda, z)$ . By the continuity of the Green function  $G$ , we may assume that for all  $\lambda \in \Omega$ ,  $c(\lambda, z_0)$  is also escaping. Assume for the purpose of a contradiction that there is  $\lambda \in \Omega$  such that  $f_\lambda^{n+k}(\tilde{C}_\lambda) = f_\lambda^k(\tilde{C}_\lambda)$  for some  $(n, k) \in \mathbb{N} \times \mathbb{N}^*$ . Then the projection  $\pi_z : \bigcup_{j \geq 0} f_\lambda^j(\tilde{C}_\lambda) \rightarrow \mathbb{C}$  on the first coordinate has finite degree. In particular, the sequence  $(f_\lambda^{n_0 j} \varphi(\lambda, z_0))_{j \geq 0}$  is finite. This contradicts the fact that  $c(\lambda, z_0)$  is escaping.  $\square$

*Proof of Theorem 1.1.* Fix  $\lambda_0 \in \text{Bif}(\mathbf{Sk}(p, d))$  and  $\varepsilon > 0$ . Set  $M^0 := \mathbf{Sk}(p, d)$ . We will prove by induction on  $1 \leq k \leq \dim V$  that there exists a family  $M^k$  which is  $\varepsilon$ -close to  $\lambda_0$  and satisfies the following properties:

- (I1)  $M^k = \bigcap_{1 \leq j \leq k} M_{(y_j, c_j), n_j}$  is the intersection of  $k$  distinct Misiurewicz loci
- (I2)  $M^k$  has codimension  $k$  in  $\mathbf{Sk}(p, d)$
- (I3) if  $k < \dim \mathbf{Sk}(p, d)$ , among the  $k$  persistent Misiurewicz relations defining  $M^k$ , at least one is good in the sense of Definition 4.1.

The case  $k = 1$  is exactly the content of Proposition 5.1.

Before going into the argument of the induction, we start by choosing  $\lambda_1 \in M^1$  ( $\varepsilon$ -close to  $\lambda_0$ ) such that for all  $\lambda$  in a neighbourhood of  $\lambda_1$  in  $M^1$ , all maps  $f_\lambda$  have at least one non-periodic post-critical component passing through the repelling cycle. To find such  $\lambda_1$ , we choose a point  $(z, w)$ , where  $z$  is a periodic repelling point of  $p$  and such that  $G_{z, \lambda_0}(w) > 0$ . Applying Proposition 4.2 to the constant graph  $\lambda \mapsto (z, w)$ , we find  $\lambda_1 \in B(\lambda_0, \varepsilon) \cap M^1$  such that  $(z, w)$  is still escaping for  $f_{\lambda_1}$ , and is also a postcritical point. Then Lemma 5.4 gives the desired result.

Now assume that  $M^k$  is constructed, and let  $\lambda_k \in M^k$  be  $k\varepsilon$  close to  $\lambda_0$ . Let  $f_\lambda^n(z, c(\lambda)) = (z_k, w_k(\lambda))$  be the persistently good Misiurewicz relation on  $M^k$ . By Lemma 5.3, there exists a vertical-like repelling periodic point  $(z_{k+1}, w_{k+1})$  for  $f_{\lambda_k}$ , which we may locally follow as  $(z_{k+1}, w_{k+1}(\lambda))$ . Moreover, we choose it so that the foliations given by the level sets of the vertical multipliers of  $(z_k, w_k(\lambda))$  and  $(z_{k+1}, w_{k+1}(\lambda))$  do not share leaves near  $\lambda_k$ . By Proposition 4.2, there exists  $\lambda_{k+1} \in B(\lambda_k, \varepsilon)$  such that  $f_{\lambda_{k+1}}$  has a Misiurewicz relation of the form  $f_{\lambda_{k+1}}^{n_{k+1}}(y_{k+1}, c_{k+1}) = (z_{k+1}, w_{k+1}(\lambda_{k+1}))$  that is not persistent on  $M^k$ . We let  $M^{k+1} := M^k \cap M_{(y_{k+1}, c_{k+1}), n_{k+1}}$ . Moreover, we may assume that  $z_{k+1}$  is close enough to  $z_k$  (and hence  $z_0$ ) that  $q_{z_{k+1}, \lambda_{k+1}}$  still has only

simple critical points. By construction  $\text{codim } M^{k+1} = k + 1$ . Thus the requirements **(I1)** and **(I2)** are satisfied. We need to check **(I3)**.

If  $k = \dim \mathbf{Sk}(p, d) - 1$  the condition is empty. Otherwise, it only remains to prove that  $M^{k+1}$  has at least one locally good persistent Misiurewicz relation at  $\lambda_{k+1}$ . Items **(G2)**, **(G3)** and **(G4)** of Definition 4.1 hold on  $M^{k+1}$  for both  $(z_k, w_k(\lambda))$  and  $(z_{k+1}, w_{k+1}(\lambda))$ , provided  $\varepsilon$  is small enough (this is clear for **(G2)** and **(G4)**, and it was proved above for item **(G3)**). The difficulty here is that it may be the case that the vertical eigenvalue of either  $(z_k, w_k(\lambda))$  or  $(z_{k+1}, w_{k+1}(\lambda))$  is constant on  $M^{k+1}$ . But we will prove that our choice of  $(z_{k+1}, w_{k+1})$  implies that it cannot be the case for both. This will complete the induction step.

If we denote by  $L_k$  and  $L_{k+1}$  the level sets passing through  $\lambda_{k+1}$  of the vertical multipliers of  $(z_k, w_k(\lambda))$  and  $(z_{k+1}, w_{k+1}(\lambda))$  respectively, then we know that  $L_k$  and  $L_{k+1}$  intersect properly at  $\lambda_{k+1}$ . Therefore,  $M^k$  cannot be contained in  $L_k \cap L_{k+1}$ , since  $\text{codim}(L_k \cap L_{k+1} \cap M^k) = k + 2$  and  $\text{codim}(M^{k+1}) = k + 1$ . In other words, at least one of the two repelling cycles has non-constant vertical eigenvalue on  $M^{k+1}$ . This ensures that **(G1)** in Definition 4.1 holds. The induction is complete.

Finally, once  $M^{D_d = \dim \mathbf{Sk}(p, d)}$  is constructed, we can apply Proposition 3.1. The proof is complete.  $\square$

*Proof of Corollary 1.4.* By Theorem 1.1 it is enough to check that the same property is true for the bifurcation locus. By [AB18], the bifurcation loci associated to periodic fibres are dense in the bifurcation locus. By [McM00] all these loci have full Hausdorff dimensions. The assertion follows.  $\square$

## REFERENCES

- [AB18] Matthieu Astorg and Fabrizio Bianchi. Hyperbolicity and bifurcations in holomorphic families of polynomial skew products. *arXiv preprint arXiv:1801.01460*, 2018.
- [ABD<sup>+</sup>16] Matthieu Astorg, Xavier Buff, Romain Dujardin, Han Peters, and Jasmin Raissy. A two-dimensional polynomial mapping with a wandering Fatou component. *Ann. of Math.*, 184:263–313, 2016.
- [AGMV19] Matthieu Astorg, Thomas Gauthier, Nicolae Mihalache, and Gabriel Vigny. Collet, Eckmann and the bifurcation measure. *Inventiones mathematicae*, 217(3):749–797, 2019.
- [BB07] Giovanni Bassanelli and François Berteloot. Bifurcation currents in holomorphic dynamics on. *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, 2007(608):201–235, 2007.
- [BB18] François Berteloot and Fabrizio Bianchi. Perturbations d'exemples de Lattès et dimension de Hausdorff du lieu de bifurcation. *Journal de Mathématiques Pures et Appliquées*, 116:161–173, 2018.
- [BBD18] François Berteloot, Fabrizio Bianchi, and Christophe Dupont. Dynamical stability and Lyapunov exponents for holomorphic endomorphisms of  $\mathbb{P}^k$ . *Annales scientifiques de l'ENS*, 51(1):215–262, 2018.
- [BDM08] François Berteloot, Christophe Dupont, and Laura Molino. Normalization of bundle holomorphic contractions and applications to dynamics (normalisation de contractions holomorphes fibrées et applications en dynamique). *Annales de l'institut Fourier*, 58(6):2137–2168, 2008.
- [BE09] Xavier Buff and Adam L. Epstein. Bifurcation measure and postcritically finite rational maps. *Complex dynamics: families and friends*, pages 491–512, 2009.
- [Bia19] Fabrizio Bianchi. Misiurewicz parameters and dynamical stability of polynomial-like maps of large topological degree. *Mathematische Annalen*, 373(3-4):901–928, 2019.
- [Bie19] Sébastien Biebler. Lattès maps and the interior of the bifurcation locus. *Journal of Modern Dynamics*, 15:95–130, 2019.

- [BT17] Fabrizio Bianchi and Johan Taflin. Bifurcations in the elementary Desboves family. *Proceedings of the AMS*, 145(10):4337–4343, 2017.
- [DF08] Romain Dujardin and Charles Favre. Distribution of rational maps with a preperiodic critical point. *American Journal of Mathematics*, 130(4):979–1032, 2008.
- [DS10] Tien-Cuong Dinh and Nessim Sibony. Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings. In *Holomorphic dynamical systems*, pages 165–294. Springer, 2010.
- [DT18] Christophe Dupont and Johan Taflin. Dynamics of fibered endomorphisms of  $\mathbb{P}^k$ . *to appear on Ann. Scuola Normale Sup. Pisa, arXiv preprint arXiv:1811.06909*, 2018.
- [Duj11] Romain Dujardin. Bifurcation currents and equidistribution on parameter space. *Frontiers in Complex Dynamics*, 2011.
- [Duj16] Romain Dujardin. A non-laminar dynamical green current. *Mathematische Annalen*, 365(1-2):77–91, 2016.
- [Duj17] Romain Dujardin. Non density of stability for holomorphic mappings on  $\mathbb{P}^k$ . *Journal de l'École Polytechnique*, 4:813–843, 2017.
- [Gau12] Thomas Gauthier. Strong bifurcation loci of full Hausdorff dimension. *Ann. Sci. Éc. Norm. Sup.(4)*, 45(6):947–984, 2012.
- [Jon97] Mattias Jonsson. Hyperbolic dynamics of endomorphisms. *preprint*, 1997.
- [Jon99] Mattias Jonsson. Dynamics of polynomial skew products on  $\mathbb{C}^2$ . *Mathematische Annalen*, 314(3):403–447, 1999.
- [McM87] Curtis McMullen. Families of rational maps and iterative root-finding algorithms. *Annals of mathematics*, 125(3):467–493, 1987.
- [McM00] Curtis McMullen. The Mandelbrot set is universal. In Tan Lei, editor, *The Mandelbrot set, themes and variations*, pages 1–18. Cambridge University Press, 2000.
- [Prz85] Feliks Przytycki. Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map. *Inventiones mathematicae*, 80(1):161–179, 1985.
- [Prz18] Feliks Przytycki. Thermodynamic formalism methods in one-dimensional real and complex dynamics. In *Proceedings of International Congress of Mathematicians*, pages 2081–2106. World Scientific, 2018.
- [PU10] Feliks Przytycki and Mariusz Urbański. *Conformal fractals: ergodic theory methods*, volume 371. Cambridge University Press, 2010.
- [PZ20] Feliks Przytycki and Anna Zdunik. On hausdorff dimension of polynomial not totally disconnected julia sets. *arXiv preprint arXiv:2003.12612*, 2020.
- [Taf17] Johan Taflin. Blenders near polynomial product maps of  $\mathbb{C}^2$ . *to appear on the Journal of the European Mathematical Society, arXiv preprint arXiv:1702.02115*, 2017.
- [Zdu90] Anna Zdunik. Parabolic orbifolds and the dimension of the maximal measure for rational maps. *Inventiones mathematicae*, 99(1):627–649, 1990.

UNIVERSITÉ D'ORLÉANS, INSTITUT DENIS POISSON, UMR CNRS 7013, 45067 ORLÉANS CEDEX 2, FRANCE

*E-mail address:* `matthieu.astorg@univ-orleans.fr`

CNRS, UNIV. LILLE, UMR 8524 - LABORATOIRE PAUL PAINLEVÉ, F-59000 LILLE, FRANCE

*E-mail address:* `fabrizio.bianchi@univ-lille.fr`