



HAL
open science

On p -Laplacian reaction-diffusion problems with dynamical boundary conditions in perforated media

María Anguiano

► **To cite this version:**

María Anguiano. On p -Laplacian reaction-diffusion problems with dynamical boundary conditions in perforated media. 2020. hal-02881568v1

HAL Id: hal-02881568

<https://hal.science/hal-02881568v1>

Preprint submitted on 25 Jun 2020 (v1), last revised 13 Aug 2023 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On p -Laplacian reaction-diffusion problems with dynamical boundary conditions in perforated media

María ANGUIANO¹

Abstract

This paper deals with the homogenization of the p -Laplacian reaction-diffusion problems in a domain containing periodically distributed holes of size ε , with a dynamical boundary condition of *pure-reactive* type. We generalize our previous results (see [2]) established in the case where the diffusion is modeled by the Laplacian operator, i.e., with $p = 2$. We prove the convergence of the homogenization process to a nonlinear p -Laplacian reaction-diffusion equation defined on a unified domain without holes with zero Dirichlet boundary condition and with extra terms coming from the influence of the nonlinear dynamical boundary conditions.

AMS classification numbers: 35B27, 35K57

Keywords: Homogenization, perforated media, reaction-diffusion systems, dynamical boundary conditions, p -Laplacian

1 Introduction and setting of the problem

Homogenization problems in perforated media for the p -Laplacian operator have been considered in the literature over the last decades. The homogenization of the equation

$$-\operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) = f \tag{1}$$

in a periodically perforated domain is considered by Labani and Picard in [19] with Dirichlet boundary conditions. Such a problem is a generalization of the linear problem for Laplace's equation, which corresponds to $p = 2$, and was studied by Cioranescu and Murat in [8]. Donato and Moscardiello in [13] study the homogenization of a class of nonlinear elliptic Neumann problems in perforated domains of \mathbb{R}^N . As a consequence of [13], we are able in particular to describe the homogenization of (1) with Neumann boundary conditions. This result is a generalization of earlier related works, for instance, Cioranescu and Donato [6] and Cioranescu and Saint Jean Paulin [9]. In [23] Shaposhnikova and Podol'skii study the homogenization of (1) in an ε -periodically perforated domain with a nonlinear boundary condition. In [11] Díaz *et al.* consider (1) with a nonlinear perturbed Robin-type boundary condition in an ε periodically perforated domain where the size of the particles is smaller than the period ε , and the asymptotic behavior of the solution is studied as $\varepsilon \rightarrow 0$. The closest articles to this one in the literature are [16, 17, 18], where Gómez *et al.* consider the case $2 < p \leq N$, and [12], where Díaz *et al.* study the case $p > N$. As far as we know, in the previous literature there is no study for the homogenization of p -Laplacian parabolic models associated with nonlinear dynamical boundary conditions in a periodically perforated domain, as we consider in this article.

In the context of reaction-diffusion equations, dynamical boundary conditions have been rigorously derived in Gal and Shomberg [14] based on first and second thermodynamical principles and their physical interpretation was also given in Goldstein [15]. In a recent article (see [2]) we addressed the problem of the homogenization of the reaction-diffusion equations with a dynamical boundary condition of *pure-reactive* type in a domain perforated with holes. More recently, in [3] we generalize this previous study to the case of a dynamical boundary condition of *reactive-diffusive* type, i.e., we add to the dynamical boundary condition a Laplace-Beltrami correction term.

¹Departamento de Análisis Matemático. Facultad de Matemáticas. Universidad de Sevilla, 41012 Sevilla (Spain) anguiano@us.es

In the present article, we consider p -Laplacian parabolic problems with nonlinear dynamical boundary conditions of *pure-reactive* type occurring as generalizations of semilinear reaction-diffusion equations with dynamical boundary conditions. Let us introduce the model we will be involved with in this article.

The geometrical setting. Let Ω be a bounded connected open set in \mathbb{R}^N ($N \geq 2$), with smooth enough boundary $\partial\Omega$. Let us introduce a set of periodically distributed holes. As a result, we obtain an open set Ω_ε , where ε represents a small parameter related to the characteristic size of the holes.

Let $Y = [0, 1]^N$ be the representative cell in \mathbb{R}^N and F an open subset of Y with smooth enough boundary ∂F , such that $\bar{F} \subset Y$. We denote $Y^* = Y \setminus \bar{F}$. For $k \in \mathbb{Z}^N$ and $\varepsilon \in (0, 1]$, each cell $Y_{k,\varepsilon} = \varepsilon k + \varepsilon Y$ is similar to the unit cell Y rescaled to size ε and $F_{k,\varepsilon} = \varepsilon k + \varepsilon F$ is similar to F rescaled to size ε . We denote $Y_{k,\varepsilon}^* = Y_{k,\varepsilon} \setminus \bar{F}_{k,\varepsilon}$. We denote by F_ε the set of all the holes contained in Ω , i.e. $F_\varepsilon = \cup_{k \in K} \{F_{k,\varepsilon} : \bar{F}_{k,\varepsilon} \subset \Omega\}$, where $K := \{k \in \mathbb{Z}^N : Y_{k,\varepsilon} \cap \Omega \neq \emptyset\}$.

Let $\Omega_\varepsilon = \Omega \setminus \bar{F}_\varepsilon$. By this construction, Ω_ε is a periodically perforated domain with holes of the same size as the period. Remark that the holes do not intersect the boundary $\partial\Omega$. Let $\partial F_\varepsilon = \cup_{k \in K} \{\partial F_{k,\varepsilon} : \bar{F}_{k,\varepsilon} \subset \Omega\}$. So

$$\partial\Omega_\varepsilon = \partial\Omega \cup \partial F_\varepsilon.$$

Position of the problem. The prototype of the p -Laplacian parabolic initial-boundary value problems that we consider in this article is

$$\left\{ \begin{array}{ll} \frac{\partial u_\varepsilon}{\partial t} - \Delta_p u_\varepsilon + \kappa |u_\varepsilon|^{p-2} u_\varepsilon = -f(u_\varepsilon) & \text{in } \Omega_\varepsilon \times (0, T), \\ \partial_{\nu_p} u_\varepsilon + \varepsilon \frac{\partial u_\varepsilon}{\partial t} = -\varepsilon g(u_\varepsilon) & \text{on } \partial F_\varepsilon \times (0, T), \\ u_\varepsilon = 0, & \text{on } \partial\Omega \times (0, T), \\ u_\varepsilon(x, 0) = u_\varepsilon^0(x), & \text{for } x \in \Omega_\varepsilon, \\ u_\varepsilon(x, 0) = \psi_\varepsilon^0(x), & \text{for } x \in \partial F_\varepsilon, \end{array} \right. \quad (2)$$

where $u_\varepsilon = u_\varepsilon(x, t)$, $x \in \Omega_\varepsilon$, $t \in (0, T)$, with $T > 0$, and $\kappa > 0$ is a given constant. Here the diffusion is modeled by the p -Laplacian operator $\Delta_p u_\varepsilon := \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon)$ with $p \in [2, N]$. Notice that $p = 2$ corresponds to the linear diffusion operator. The ‘‘normal derivative’’ must be understood as $\partial_{\nu_p} u_\varepsilon = |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nu$, where ν denotes the outward normal to ∂F_ε . The boundary equation (2)₂ is multiplied by ε to compensate the growth of the surface by shrinking ε , where the value of u_ε is assumed to be the trace of the function u_ε defined for $x \in \Omega_\varepsilon$. We also assume that the functions f and $g \in C(\mathbb{R})$ are given, and satisfy that there exist exponents q_1 and q_2 , such that

$$p \leq q_1 < +\infty, \text{ if } p = N \quad \text{and} \quad 2 \leq q_1 \leq \frac{Np}{N-p}, \text{ if } p \in [2, N), \quad (3)$$

$$2 \leq q_2 < +\infty, \text{ if } p = N \quad \text{and} \quad 2 \leq q_2 \leq \frac{(N-1)p}{N-p}, \text{ if } p \in [2, N), \quad (4)$$

and constants $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta > 0$, and $l > 0$, such that

$$\alpha_1 |s|^{q_1} - \beta \leq f(s)s \leq \alpha_2 |s|^{q_1} + \beta, \quad \text{for all } s \in \mathbb{R}, \quad (5)$$

$$\alpha_1 |s|^{q_2} - \beta \leq g(s)s \leq \alpha_2 |s|^{q_2} + \beta, \quad \text{for all } s \in \mathbb{R}, \quad (6)$$

$$(f(s_1) - f(s_2))(s_1 - s_2) \geq -l(s_1 - s_2)^2, \quad \text{for all } s_1, s_2 \in \mathbb{R}, \quad (7)$$

and

$$(g(s_1) - g(s_2))(s_1 - s_2) \geq -l(s_1 - s_2)^2, \quad \text{for all } s_1, s_2 \in \mathbb{R}. \quad (8)$$

Finally, we also assume that

$$u_\varepsilon^0 \in L^2(\Omega), \quad \psi_\varepsilon^0 \in L^2(\partial F_\varepsilon), \quad (9)$$

are given, and we suppose that

$$|u_\varepsilon^0|_{\Omega_\varepsilon}^2 + \varepsilon |\psi_\varepsilon^0|_{\partial F_\varepsilon}^2 \leq C, \quad (10)$$

where C is a positive constant, and we denote by $|\cdot|_{\Omega_\varepsilon}$ and $|\cdot|_{\partial F_\varepsilon}$ the norm in $L^2(\Omega_\varepsilon)$ and $L^2(\partial F_\varepsilon)$, respectively.

In [2], we consider the homogenization of the problem (2) with $p = 2$ and we obtain rigorously a nonlinear parabolic problem with zero Dirichlet boundary condition and with extra-terms coming from the influence of the dynamical boundary conditions as the homogenized model. We would like to highlight that analyzing a p -Laplacian problem involves additional (nontrivial) difficulties compared to the study for the Laplacian. Due to the presence of p -Laplacian operator in the domain, the variational formulation of the p -Laplacian reaction-diffusion equation is different than in [2]. We have to work in the space

$$V_p := \{(v, \gamma_0(v)) : v \in W^{1,p}(\Omega_\varepsilon), \gamma_0(v) = 0 \text{ on } \partial\Omega\}, \quad (11)$$

where γ_0 denotes the trace operator $v \in W^{1,p}(\Omega_\varepsilon) \mapsto v|_{\partial\Omega_\varepsilon} \in W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)$. New $W^{1,p}$ -estimates are needed to deal with problem (2). To prove these estimates rigorously, we use the Galerkin approximations where we have to introduce a special basis consisting of functions in the space

$$\mathcal{V}_s := \{(v, \gamma_0(v)) : v \in W^{s,2}(\Omega_\varepsilon), \gamma_0(v) = 0 \text{ on } \partial\Omega\}, \quad s \geq \frac{N(p-2)}{2p} + 1,$$

in the sense of Lions [20, p. 161]. Therefore, thanks to the assumption made on s , we have $\mathcal{V}_s \subset V_p$. We use the so-called energy method introduced by Tartar [24] and considered by many authors (see, for instance, Cioranescu and Donato [6]). Finally, in order to identify the limit equation, it is necessary to use monotonicity arguments. We prove the following theorem:

Theorem 1.1 (Main Theorem). *Under the assumptions (3), (5)–(8) and (10), assume that $g \in C^1(\mathbb{R})$, the exponent q_2 satisfies that*

$$p \leq q_2 < +\infty \text{ if } p = N \quad \text{and} \quad 2 \leq q_2 \leq \frac{(N-1)p}{N-p} \text{ if } p \in [2, N), \quad (12)$$

and $(u_\varepsilon^0, \psi_\varepsilon^0) \in V_p$. Let $(u_\varepsilon, \psi_\varepsilon)$ be the unique solution of the problem (2), where $\psi_\varepsilon(t) = \gamma_0(u_\varepsilon(t))$ a.e. $t \in (0, T]$. Then, as $\varepsilon \rightarrow 0$, we have

$$\tilde{u}_\varepsilon(t) \rightarrow u(t) \quad \text{strongly in } L^p(\Omega), \quad \forall t \in [0, T],$$

where $\tilde{\cdot}$ denotes the $W^{1,p}$ -extension to $\Omega \times (0, T)$ and u is the unique solution of the following problem

$$\begin{cases} \left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \frac{\partial u}{\partial t} - \operatorname{div} b(\nabla u) + \frac{|Y^*|}{|Y|} (\kappa |u|^{p-2} u + f(u)) + \frac{|\partial F|}{|Y|} g(u) = 0, & \text{in } \Omega \times (0, T), \\ u(x, 0) = u_0(x), & \text{for } x \in \Omega, \\ u = 0, & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (13)$$

For any $\zeta \in \mathbb{R}^N$, if $w(y)$ is the solution of the problem

$$\begin{cases} \int_{Y^*} |\nabla_y w(y)|^{p-2} \nabla_y w(y) \cdot \nabla_y \varphi(y) dy = 0 & \forall \varphi \in \mathbb{H}_{\text{per}}(Y^*), \\ w \in \zeta \cdot y + \mathbb{H}_{\text{per}}(Y^*), \end{cases} \quad (14)$$

then b is defined by

$$b(\zeta) = \frac{1}{|Y|} \int_{Y^*} |\nabla v|^{p-2} \nabla v dy, \quad (15)$$

where $\mathbb{H}_{\text{per}}(Y^*)$ is the space of functions from $W^{1,p}(Y^*)$ which has the same trace on the opposite faces of Y .

Remark 1.2. Note that in the case $p = 2$ (i.e., if the diffusion is modeled by the Laplacian operator), the homogenized equation (13) is exactly the equation obtained in [2].

Remark 1.3. In this setting, a typical nonlinearity in the applications is a odd degree polynomial, $\sum_{j=0}^{2k+1} c_j s^j$, where $c_{2k+1} > 0$.

The article is organized as follows. In Section 2, we introduce suitable functions spaces for our considerations. We recall here some well-know facts on the Sobolev spaces $W^{s,p}$. To prove the main result, in Section 3 we prove the existence and uniqueness of solution of (2), *a priori* estimates are established in Section 4 and some compactness results are proved in Section 5. Finally, the proof of Theorem 1.1 is established in Section 6.

2 Functional setting

Notation: We denote by $(\cdot, \cdot)_{\Omega_\varepsilon}$ (respectively, $(\cdot, \cdot)_{\partial F_\varepsilon}$) the inner product in $L^2(\Omega_\varepsilon)$ (respectively, in $L^2(\partial F_\varepsilon)$), and by $|\cdot|_{\Omega_\varepsilon}$ (respectively, $|\cdot|_{\partial F_\varepsilon}$) the associated norm. We also denote by $(\cdot, \cdot)_{\Omega_\varepsilon}$ the inner product in $(L^2(\Omega_\varepsilon))^N$.

If $r \neq 2$, we will also denote by $(\cdot, \cdot)_{\Omega_\varepsilon}$ (respectively, $(\cdot, \cdot)_{\partial F_\varepsilon}$) the duality product between $L^{r'}(\Omega_\varepsilon)$ and $L^r(\Omega_\varepsilon)$ (respectively, the duality product between $L^{r'}(\partial F_\varepsilon)$ and $L^r(\partial F_\varepsilon)$). We will denote by $|\cdot|_{r, \Omega_\varepsilon}$ (respectively $|\cdot|_{r, \partial F_\varepsilon}$) the norm in $L^r(\Omega_\varepsilon)$ (respectively in $L^r(\partial F_\varepsilon)$).

We denote by $(\cdot, \cdot)_\Omega$ the inner product in $L^2(\Omega)$, and by $|\cdot|_\Omega$ the associated norm. If $r \neq 2$, we will also denote by $(\cdot, \cdot)_\Omega$ the duality product between $L^{r'}(\Omega)$ and $L^r(\Omega)$. We will denote by $|\cdot|_{r, \Omega}$ the norm in $L^r(\Omega)$.

The Sobolev spaces $W^{s,p}(\Omega_\varepsilon)$:

We recall here, for the reader's convenience, some well-know facts on the Sobolev spaces $W^{s,p}$. We refer to Adams and Fournier [1], Brezis [4, Chapter 9] and Nečas [21, Chapter 2] for more details and proofs.

For any positive integer s and $p \geq 1$, we define the Sobolev space $W^{s,p}(\Omega_\varepsilon)$ to be the completion of $C^s(\overline{\Omega_\varepsilon})$, with respect to the norm

$$\|v\|_{s,p,\Omega_\varepsilon} = \left(\sum_{0 \leq |\alpha| \leq s} |D^\alpha v|_{p,\Omega_\varepsilon}^p \right)^{1/p}.$$

Observe that $W^{s,p}(\Omega_\varepsilon)$ is a Banach space.

By the Sobolev Imbedding Theorem (see [1, Chapter 4, p. 99]), we have the imbedding

$$W^{s,p}(\Omega_\varepsilon) \subset W^{1,r}(\Omega_\varepsilon), \quad (16)$$

where $s \geq 1$ and $\frac{1}{p} - \frac{s-1}{N} \leq \frac{1}{r} \leq \frac{1}{p}$.

In particular, for $p \geq 1$, we define the Sobolev space $W^{1,p}(\Omega_\varepsilon)$ to be the completion of $C^1(\overline{\Omega_\varepsilon})$, with respect to the norm

$$\|v\|_{p,\Omega_\varepsilon} := \left(|v|_{p,\Omega_\varepsilon}^p + |\nabla v|_{p,\Omega_\varepsilon}^p \right)^{1/p}.$$

We set $H^1(\Omega_\varepsilon) = W^{1,2}(\Omega_\varepsilon)$.

By $\|\cdot\|_{p,\Omega_\varepsilon,T}$ we denote the norm in $L^p(0, T; W^{1,p}(\Omega_\varepsilon))$. By $|\cdot|_{r,\Omega_\varepsilon,T}$ (respectively, $|\cdot|_{r,\partial F_\varepsilon,T}$), we denote the norm in $L^r(0, T; L^r(\Omega_\varepsilon))$ (respectively, $L^r(0, T; L^r(\partial F_\varepsilon))$).

By $\|\cdot\|_{p,\Omega}$ we denote the norm in $W^{1,p}(\Omega)$, by $\|\cdot\|_{p,\Omega,T}$ we denote the norm in $L^p(0, T; W^{1,p}(\Omega))$ and we denote by $|\cdot|_{r,\Omega,T}$ the norm in $L^r(0, T; L^r(\Omega))$.

We define

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p = N. \end{cases}$$

We have the continuous embedding

$$W^{1,p}(\Omega_\varepsilon) \subset \begin{cases} L^r(\Omega_\varepsilon) & \text{if } p < N \text{ and } r = p^*, \\ L^r(\Omega_\varepsilon) & \text{if } p = N \text{ and } p \leq r < p^*. \end{cases} \quad (17)$$

By Rellich-Kondrachov Theorem (see [4, Chapter 9, Theorem 9.16]), we have the compact embedding

$$W^{1,p}(\Omega_\varepsilon) \subset \begin{cases} L^r(\Omega_\varepsilon) & \text{if } p < N \text{ and } 1 \leq r < p^*, \\ L^r(\Omega_\varepsilon) & \text{if } p = N \text{ and } p \leq r < p^*. \end{cases} \quad (18)$$

In particular, we have the compact embedding

$$W^{1,p}(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon), \quad \forall 2 \leq p \leq N. \quad (19)$$

One can define a family of spaces intermediate between L^p and $W^{1,p}$. More precisely for $p \geq 1$ we define the fractional order Sobolev space

$$W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) := \left\{ v \in L^p(\partial\Omega_\varepsilon); \frac{|v(x) - v(y)|}{|x - y|^{1-\frac{1}{p} + \frac{N}{p}}} \in L^p(\partial\Omega_\varepsilon \times \partial\Omega_\varepsilon) \right\},$$

equipped with the natural norm. We set $H^{1/2}(\partial\Omega_\varepsilon) = W^{\frac{1}{2},2}(\partial\Omega_\varepsilon)$. These spaces play an important role in the theory of traces.

We denote by γ_0 the trace operator $v \mapsto v|_{\partial\Omega_\varepsilon}$. The trace operator belongs to $\mathcal{L}(W^{1,p}(\Omega_\varepsilon), W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon))$, and we will use $\|\gamma_0\|$ to denote the norm of γ_0 in this space.

We will use $\|\cdot\|_{p,\partial\Omega_\varepsilon}$ to denote the norm in $W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)$, which is given by

$$\|\phi\|_{p,\partial\Omega_\varepsilon} = \inf\{\|v\|_{p,\Omega_\varepsilon} : \gamma_0(v) = \phi\}.$$

We define

$$p_b^* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p = N. \end{cases}$$

We have the continuous embedding

$$W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) \subset \begin{cases} L^r(\partial\Omega_\varepsilon) & \text{if } p < N \quad \text{and} \quad r = p_b^*, \\ L^r(\partial\Omega_\varepsilon) & \text{if } p = N \quad \text{and} \quad 1 \leq r < p_b^*. \end{cases} \quad (20)$$

By [21, Chaper 2, Theorem 6.2], we have the compact embedding

$$W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) := \gamma_0(W^{1,p}(\Omega_\varepsilon)) \subset L^r(\partial\Omega_\varepsilon) \quad \text{if } 1 \leq r < p_b^*. \quad (21)$$

In particular, we have the compact embedding

$$W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) \subset L^2(\partial\Omega_\varepsilon) \quad \forall 2 \leq p \leq N. \quad (22)$$

Some notations: Finally, for any positive integer s and $p \geq 1$, we denote by $W_{\partial\Omega}^{s,p}(\Omega_\varepsilon)$ and $W_{\partial\Omega}^{s,p}(\partial\Omega_\varepsilon)$ the standard Sobolev spaces which are closed subspaces of $W^{s,p}(\Omega_\varepsilon)$ and $W^{s,p}(\partial\Omega_\varepsilon)$, respectively, and the subscript $\partial\Omega$ means that, respectively, traces or functions in $\partial\Omega_\varepsilon$, vanish on this part of the boundary of Ω_ε , i.e.

$$W_{\partial\Omega}^{s,p}(\Omega_\varepsilon) = \{v \in W^{s,p}(\Omega_\varepsilon) : \gamma_0(v) = 0 \text{ on } \partial\Omega\},$$

and

$$W_{\partial\Omega}^{s,p}(\partial\Omega_\varepsilon) = \{v \in W^{s,p}(\partial\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\}.$$

Let us notice that, in fact, we can consider an element of $W^{1-\frac{1}{p},p}(\partial F_\varepsilon)$ as an element of $W_{\partial\Omega}^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)$.

Analogously, for $r \geq 2$, we denote

$$L_{\partial\Omega}^r(\partial\Omega_\varepsilon) := \{v \in L^r(\partial\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\}.$$

Let us notice that, in fact, we can consider the given ψ_ε^0 as an element of $L_{\partial\Omega}^2(\partial\Omega_\varepsilon)$.

Let us consider the space

$$H_r := L^r(\Omega_\varepsilon) \times L_{\partial\Omega}^r(\partial\Omega_\varepsilon), \quad \forall r \geq 2,$$

with the natural inner product $((v, \phi), (w, \varphi))_{H_r} = (v, w)_{\Omega_\varepsilon} + \varepsilon(\phi, \varphi)_{\partial F_\varepsilon}$, which in particular induces the norm $|(\cdot, \cdot)|_{H_r}$ given by

$$|(v, \phi)|_{H_r}^r = |v|_{r,\Omega_\varepsilon}^r + \varepsilon|\phi|_{r,\partial F_\varepsilon}^r, \quad (v, \phi) \in H_r.$$

For the sake of clarity, we shall omit to write explicitly the index r if $r = 2$, so we denote by H the Hilbert space

$$H := L^2(\Omega_\varepsilon) \times L_{\partial\Omega}^2(\partial\Omega_\varepsilon).$$

For each $p > 1$, we consider the space

$$V_p := \left\{ (v, \gamma_0(v)) : v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon) \right\}.$$

We note that V_p is a closed vector subspace of $W_{\partial\Omega}^{1,p}(\Omega_\varepsilon) \times W_{\partial\Omega}^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)$, and we endow it with the norm $\|(\cdot, \cdot)\|_{V_p}$ given by

$$\|(v, \gamma_0(v))\|_{V_p}^p = \|v\|_{p, \Omega_\varepsilon}^p + \|\gamma_0(v)\|_{p, \partial F_\varepsilon}^p, \quad (v, \gamma_0(v)) \in V_p.$$

Moreover, V_p is a reflexive and separable space.

Let p' , q'_1 and q'_2 be the conjugate exponents of p , q_1 and q_2 , respectively. Taking into account the continuous embeddings (17) and (20), and the assumptions (3)-(4), we have the following useful continuous inclusions

$$V_p \subset W^{1,p}(\Omega_\varepsilon) \subset L^{q_1}(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon), \quad V_p \subset W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) \subset L^{q_2}(\partial\Omega_\varepsilon) \subset L^2(\partial\Omega_\varepsilon), \quad (23)$$

and

$$L^2(\Omega_\varepsilon) \subset L^{q'_1}(\Omega_\varepsilon) \subset (W^{1,p}(\Omega_\varepsilon))' \subset V'_p, \quad (24)$$

where $(W^{1,p}(\Omega_\varepsilon))'$ and V'_p denote the dual of $W^{1,p}(\Omega_\varepsilon)$ and V_p , respectively. Note that $(W^{1,p}(\Omega_\varepsilon))'$ is a subspace of $W^{-1,p'}(\Omega_\varepsilon)$, where $W^{-1,p'}(\Omega_\varepsilon)$ denotes the dual of the Sobolev space $W_0^{1,p}(\Omega_\varepsilon) := \overline{\mathcal{D}(\Omega_\varepsilon)}^{W^{1,p}(\Omega_\varepsilon)}$.

Taking into account the compact embeddings (19) and (22), we have the compact embedding

$$V_p \subset H \quad \forall 2 \leq p \leq N. \quad (25)$$

Finally, for $s \geq 1$, we consider the space

$$\mathcal{V}_s := \left\{ (v, \gamma_0(v)) : v \in W_{\partial\Omega}^{s,2}(\Omega_\varepsilon) \right\}.$$

We note that \mathcal{V}_s is a closed vector subspace of $W_{\partial\Omega}^{s,2}(\Omega_\varepsilon) \times W_{\partial\Omega}^{s-\frac{1}{2},2}(\partial\Omega_\varepsilon)$.

Observe that by (16), we have that $W^{s,2}(\Omega_\varepsilon) \subset W^{1,2}(\Omega_\varepsilon)$ and by Rellich's Theorem (see [21, Chapter 1, Theorem 1.4]), we obtain the compact embedding

$$W^{1,2}(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon),$$

and by [21, Chapter 1, Exercise 1.2], we have the compact embedding

$$W^{s-\frac{1}{2},2}(\partial\Omega_\varepsilon) := \gamma_0(W^{s,2}(\Omega_\varepsilon)) \subset L^2(\partial\Omega_\varepsilon).$$

Thus, we can deduce the compact embedding

$$\mathcal{V}_s \subset H. \quad (26)$$

3 Existence and uniqueness of solution

Along this paper, we shall denote by C different constants which are independent of ε . We state in this section a result on the existence and uniqueness of solution of problem (2). It is easy to see from (5) and (6) that there exists a constant $C > 0$ such that

$$|f(s)| \leq C \left(1 + |s|^{q_1-1}\right), \quad |g(s)| \leq C \left(1 + |s|^{q_2-1}\right), \quad \text{for all } s \in \mathbb{R}. \quad (27)$$

Definition 3.1. A weak solution of (2) is a pair of functions $(u_\varepsilon, \psi_\varepsilon)$, satisfying

$$u_\varepsilon \in C([0, T]; L^2(\Omega_\varepsilon)), \quad \psi_\varepsilon \in C([0, T]; L^2_{\partial\Omega}(\partial\Omega_\varepsilon)), \quad \text{for all } T > 0, \quad (28)$$

$$u_\varepsilon \in L^p(0, T; W^{1,p}(\Omega_\varepsilon)), \quad \text{for all } T > 0, \quad (29)$$

$$\psi_\varepsilon \in L^p(0, T; W_{\partial\Omega}^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)), \quad \text{for all } T > 0, \quad (30)$$

$$\gamma_0(u_\varepsilon(t)) = \psi_\varepsilon(t), \quad \text{a.e. } t \in (0, T], \quad (31)$$

$$\left\{ \begin{array}{l} \frac{d}{dt}(u_\varepsilon(t), v)_{\Omega_\varepsilon} + \varepsilon \frac{d}{dt}(\psi_\varepsilon(t), \gamma_0(v))_{\partial F_\varepsilon} + (|\nabla u_\varepsilon(t)|^{p-2} \nabla u_\varepsilon(t), \nabla v)_{\Omega_\varepsilon} + \kappa(|u_\varepsilon(t)|^{p-2} u_\varepsilon(t), v)_{\Omega_\varepsilon} \\ + (f(u_\varepsilon(t)), v)_{\Omega_\varepsilon} + \varepsilon (g(\psi_\varepsilon(t)), \gamma_0(v))_{\partial F_\varepsilon} = 0 \\ \text{in } \mathcal{D}'(0, T), \text{ for all } v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon), \end{array} \right. \quad (32)$$

$$u_\varepsilon(0) = u_\varepsilon^0, \quad \text{and} \quad \psi_\varepsilon(0) = \psi_\varepsilon^0. \quad (33)$$

We have the following result.

Theorem 3.2. Under the assumptions (3)–(9), there exists a unique solution $(u_\varepsilon, \psi_\varepsilon)$ of the problem (2). Moreover, this solution satisfies the energy equality

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (|(u_\varepsilon(t), \psi_\varepsilon(t))|_H^2) + |\nabla u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + \kappa |u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p \\ & + (f(u_\varepsilon(t)), u_\varepsilon(t))_{\Omega_\varepsilon} + \varepsilon (g(\psi_\varepsilon(t)), \psi_\varepsilon(t))_{\partial F_\varepsilon} = 0, \quad \text{a.e. } t \in (0, T). \end{aligned} \quad (34)$$

Proof. The proof of this result is standard. For the sake of completeness, we give a sketch of a proof.

First, we prove that V_p is densely embedded in H . In fact, if we consider $(w, \phi) \in H$ such that

$$(v, w)_{\Omega_\varepsilon} + \varepsilon (\gamma_0(v), \phi)_{\partial F_\varepsilon} = 0, \quad \text{for all } v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon),$$

in particular, we have

$$(v, w)_{\Omega_\varepsilon} = 0, \quad \text{for all } v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon),$$

and therefore $w = 0$. Consequently,

$$(\gamma_0(v), \phi)_{\partial F_\varepsilon} = 0, \quad \text{for all } v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon),$$

and then, as $W_{\partial\Omega}^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) = \gamma_0(W_{\partial\Omega}^{1,p}(\Omega_\varepsilon))$ is dense in $L^2_{\partial\Omega}(\partial\Omega_\varepsilon)$, we have that $\phi = 0$.

Now, on the space V_p we define the nonlinear monotone operator $A_p : V_p \rightarrow V'_p$, given by

$$\langle A_p((v, \gamma_0(v))), (w, \gamma_0(w)) \rangle = (|\nabla v|^{p-2} \nabla v, \nabla w)_{\Omega_\varepsilon} + \kappa(|v|^{p-2} v, w)_{\Omega_\varepsilon}, \quad (35)$$

for all $v, w \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon)$.

We observe that A_p is coercive. In fact, we have

$$\begin{aligned} \langle A_p((v, \gamma_0(v))), (v, \gamma_0(v)) \rangle & \geq \min\{1, \kappa\} \|v\|_{p, \Omega_\varepsilon}^p \\ & = \frac{1}{1 + \|\gamma_0\|^p} \min\{1, \kappa\} \|v\|_{p, \Omega_\varepsilon}^p + \frac{\|\gamma_0\|^p}{1 + \|\gamma_0\|^p} \min\{1, \kappa\} \|v\|_{p, \Omega_\varepsilon}^p \\ & \geq \frac{1}{1 + \|\gamma_0\|^p} \min\{1, \kappa\} \|(v, \gamma_0(v))\|_{V_p}^p, \end{aligned} \quad (36)$$

for all $v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon)$.

Let us denote

$$\begin{aligned} V_1 &= V_p, & V_2 &= L^{q_1}(\Omega_\varepsilon) \times L^2_{\partial\Omega}(\partial\Omega_\varepsilon), & V_3 &= L^2(\Omega_\varepsilon) \times L^{q_2}_{\partial\Omega}(\partial\Omega_\varepsilon), \\ A_1 &= A_p, & A_2(v, \phi) &= (f(v), 0), & A_3(v, \phi) &= (0, \varepsilon g(\phi)). \end{aligned}$$

From (27) one deduces that $A_i : V_i \rightarrow V'_i$ for $i = 2, 3$.

With this notation, and denoting $V = \cap_{i=1}^3 V_i$, $p_1 = p$, $p_2 = q_1$, $p_3 = q_2$, $\vec{u}_\varepsilon = (u_\varepsilon, \psi_\varepsilon)$, one has that (28)–(33) is equivalent to

$$\vec{u}_\varepsilon \in C([0, T]; H), \quad \vec{u}_\varepsilon \in \bigcap_{i=1}^3 L^{p_i}(0, T; V_i), \quad \text{for all } T > 0, \quad (37)$$

$$(\vec{u}_\varepsilon)'(t) + \sum_{i=1}^3 A_i(\vec{u}_\varepsilon(t)) = 0 \quad \text{in } \mathcal{D}'(0, T; V'), \quad (38)$$

$$\vec{u}_\varepsilon(0) = (u_\varepsilon^0, \psi_\varepsilon^0). \quad (39)$$

Taking into account (23)–(24) and applying a slight modification of [20, Ch.2, Th.1.4], it is not difficult to see that problem (37)–(39) has a unique solution. Moreover, \vec{u}_ε satisfies the energy equality

$$\frac{1}{2} \frac{d}{dt} |\vec{u}_\varepsilon(t)|_H^2 + \sum_{i=1}^3 \langle A_i(\vec{u}_\varepsilon(t)), \vec{u}_\varepsilon(t) \rangle_i = 0 \quad \text{a.e. } t \in (0, T),$$

where $\langle \cdot, \cdot \rangle_i$ denotes the duality product between V'_i and V_i . This last equality turns out to be just (34). \square

4 A priori estimates

In this section we obtain some energy estimates for the solution of (2). By (34) and taking into account (5)–(6), we have

$$\begin{aligned} \frac{d}{dt} (|(u_\varepsilon(t), \psi_\varepsilon(t))|_H^2) + 2|\nabla u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + 2\kappa|u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + 2\alpha_1|u_\varepsilon(t)|_{q_1, \Omega_\varepsilon}^{q_1} \\ + 2\alpha_1\varepsilon|\psi_\varepsilon(t)|_{q_2, \partial F_\varepsilon}^{q_2} \leq 2\beta(|\Omega_\varepsilon| + \varepsilon|\partial F_\varepsilon|), \end{aligned} \quad (40)$$

where $|\Omega_\varepsilon|$ and $|\partial F_\varepsilon|$ denotes the measure of Ω_ε and ∂F_ε , respectively.

Observe that the number of holes is given by

$$N(\varepsilon) = \frac{|\Omega|}{(2\varepsilon)^N} (1 + o(1)),$$

then using the change of variable

$$y = \frac{x}{\varepsilon}, \quad d\sigma(y) = \varepsilon^{-(N-1)} d\sigma(x),$$

we can deduce

$$|\partial F_\varepsilon| = N(\varepsilon)|\partial F_{k, \varepsilon}| = N(\varepsilon)\varepsilon^{N-1}|\partial F| \leq \frac{C}{\varepsilon}.$$

And since $|\Omega_\varepsilon| \leq |\Omega|$, we have that

$$|\Omega_\varepsilon| + \varepsilon|\partial F_\varepsilon| \leq C. \quad (41)$$

Let us denote

$$\mathcal{F}(s) := \int_0^s f(r) dr \quad \text{and} \quad \mathcal{G}(s) := \int_0^s g(r) dr.$$

Then, there exist positive constants $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, and $\tilde{\beta}$ such that

$$\tilde{\alpha}_1|s|^{q_1} - \tilde{\beta} \leq \mathcal{F}(s) \leq \tilde{\alpha}_2|s|^{q_1} + \tilde{\beta} \quad \forall s \in \mathbb{R}, \quad (42)$$

and

$$\tilde{\alpha}_1|s|^{q_2} - \tilde{\beta} \leq \mathcal{G}(s) \leq \tilde{\alpha}_2|s|^{q_2} + \tilde{\beta} \quad \forall s \in \mathbb{R}. \quad (43)$$

Lemma 4.1. *Assume the assumptions (3)–(8) and (10). Then, for any initial condition $(u_\varepsilon^0, \psi_\varepsilon^0) \in V_p$, there exists a constant C independent of ε , such that the solution $(u_\varepsilon, \psi_\varepsilon)$ of the problem (2) satisfies*

$$\|u_\varepsilon\|_{p, \Omega_\varepsilon, T} \leq C, \quad \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{p, \Omega_\varepsilon} \leq C. \quad (44)$$

Proof. Taking into account (41) in (40), we obtain

$$\frac{d}{dt} (|(u_\varepsilon(t), \psi_\varepsilon(t))|_H^2) + 2|\nabla u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + 2\kappa|u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + 2\alpha_1|u_\varepsilon(t)|_{q_1, \Omega_\varepsilon}^{q_1} + 2\alpha_1\varepsilon|\psi_\varepsilon(t)|_{q_2, \partial F_\varepsilon}^{q_2} \leq C. \quad (45)$$

From (35)–(36), in particular, we can deduce

$$\frac{d}{dt} (|(u_\varepsilon(t), \psi_\varepsilon(t))|_H^2) + \frac{2 \min\{1, \kappa\}}{1 + \|\gamma_0\|^p} \|(u_\varepsilon(t), \gamma_0(u_\varepsilon(t)))\|_{V_p}^p \leq C. \quad (46)$$

Integrating between 0 and t and taking into account (10), in particular, we have the first estimate in (44).

Now, if we want to take the inner product in the problem (2) with u'_ε , we need that $u'_\varepsilon \in L^p(0, T; W_{\partial\Omega}^{1,p}(\Omega_\varepsilon)) \cap L^{q_1}(0, T; L^{q_1}(\Omega_\varepsilon))$ with $\gamma_0(u'_\varepsilon) \in L^{q_2}(0, T; L^{q_2}(\partial\Omega_\varepsilon))$. However, we do not have it for our weak solution. Therefore, we use the Galerkin method in order to prove, rigorously, new *a priori* estimates for u_ε .

At first, we introduce a special basis consisting of functions $(w_j, \gamma_0(w_j)) \in \mathcal{V}_s$ with $s \geq \frac{N(p-2)}{2p} + 1$ in the sense of [20, Chapter 2, Remark 1.6, p. 161]. Therefore, thanks to the assumption made on s , taking into account (16), we have $\mathcal{V}_s \subset V_p$. The scalar product in H generates on $\mathcal{V}_s \subset H$ the bilinear functional $((v, \gamma_0(v)), (w, \gamma_0(w)))_H$ which can be represented in the form $((v, \gamma_0(v)), (w, \gamma_0(w)))_H = \langle L((v, \gamma_0(v))), (w, \gamma_0(w)) \rangle_{\mathcal{V}_s}$, where L is a self-adjoint operator. The compact embedding (26) implies the compactness of the operator L . Hence, L has a complete system of eigenvectors $\{(w_j, \gamma_0(w_j)) : j \geq 1\}$. These vectors are orthonormal in H and orthogonal in \mathcal{V}_s . Observe that $\text{span}\{(w_j, \gamma_0(w_j)) : j \geq 1\}$ is dense in V_p .

Taking into account the above facts, we denote by

$$(u_{\varepsilon, m}(t), \gamma_0(u_{\varepsilon, m}(t))) = (u_{\varepsilon, m}(t; 0, u_\varepsilon^0, \psi_\varepsilon^0), \gamma_0(u_{\varepsilon, m}(t; 0, u_\varepsilon^0, \psi_\varepsilon^0)))$$

the Galerkin approximation of the solution $(u_\varepsilon(t; 0, u_\varepsilon^0, \psi_\varepsilon^0), \gamma_0(u_\varepsilon(t; 0, u_\varepsilon^0, \psi_\varepsilon^0)))$ to (2) for each integer $m \geq 1$, which is given by

$$(u_{\varepsilon, m}(t), \gamma_0(u_{\varepsilon, m}(t))) = \sum_{j=1}^m \delta_{\varepsilon m j}(t) (w_j, \gamma_0(w_j)), \quad (47)$$

and is the solution of

$$\begin{aligned} & \frac{d}{dt} ((u_{\varepsilon, m}(t), \gamma_0(u_{\varepsilon, m}(t))), (w_j, \gamma_0(w_j)))_H + \langle A_p((u_{\varepsilon, m}(t), \gamma_0(u_{\varepsilon, m}(t))), (w_j, \gamma_0(w_j))) \\ & + (f(u_{\varepsilon, m}(t)), w_j)_{\Omega_\varepsilon} + \varepsilon(g(\gamma_0(u_{\varepsilon, m}(t))), \gamma_0(w_j))_{\partial F_\varepsilon} = 0, \quad j = 1, \dots, m, \end{aligned} \quad (48)$$

with initial data

$$(u_{\varepsilon, m}(0), \gamma_0(u_{\varepsilon, m}(0))) = (u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0)), \quad (49)$$

and

$$\delta_{\varepsilon m j}(t) = (u_{\varepsilon, m}(t), w_j)_{\Omega_\varepsilon} + (\gamma_0(u_{\varepsilon, m}(t)), \gamma_0(w_j))_{\partial F_\varepsilon},$$

and $(u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0)) \in \text{span}\{(w_j, \gamma_0(w_j)) : j = 1, \dots, m\}$ converge (when $m \rightarrow \infty$) to $(u_\varepsilon^0, \psi_\varepsilon^0)$ in a suitable sense which will be specified below.

Let $(u_\varepsilon^0, \psi_\varepsilon^0) \in V_p$. For all $m \geq 1$, since $\text{span}\{(w_j, \gamma_0(w_j)) : j \geq 1\}$ is densely embedded in V_p , there exists $(u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0)) \in \text{span}\{(w_j, \gamma_0(w_j)) : 1 \leq j \leq m\}$, such that the sequence $\{(u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0))\}$ converges to $(u_\varepsilon^0, \psi_\varepsilon^0)$ in V_p . Then, in particular we know that there exists a constant C such that

$$\|(u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0))\|_{V_p} \leq C. \quad (50)$$

For each integer $m \geq 1$, we consider the sequence $\{(u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))\}$ defined by (47)-(49) with these initial data.

Multiplying by the derivative $\delta'_{\varepsilon m j}$ in (48), and summing from $j = 1$ to m , we obtain

$$\begin{aligned} & |(u'_{\varepsilon,m}(t), \gamma_0(u'_{\varepsilon,m}(t)))|_H^2 + \frac{1}{p} \frac{d}{dt} (\langle A_p((u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))) \rangle, (u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))) \rangle) \\ & + (f(u_{\varepsilon,m}(t)), u'_{\varepsilon,m}(t))_{\Omega_\varepsilon} + \varepsilon (g(\gamma_0(u_{\varepsilon,m}(t))), \gamma_0(u'_{\varepsilon,m}(t)))_{\partial F_\varepsilon} = 0. \end{aligned} \quad (51)$$

We observe that

$$(f(u_{\varepsilon,m}(t)), u'_{\varepsilon,m}(t))_{\Omega_\varepsilon} = \frac{d}{dt} \int_{\Omega_\varepsilon} \mathcal{F}(u_{\varepsilon,m}(t)) dx,$$

and

$$(g(\gamma_0(u_{\varepsilon,m}(t))), \gamma_0(u'_{\varepsilon,m}(t)))_{\partial F_\varepsilon} = \frac{d}{dt} \int_{\partial F_\varepsilon} \mathcal{G}(\gamma_0(u_{\varepsilon,m}(t))) d\sigma(x).$$

Then, integrating (51) between 0 and t , and taking into account (36) and (41)-(43), we obtain

$$\begin{aligned} & \int_0^t |(u'_{\varepsilon,m}(s), \gamma_0(u'_{\varepsilon,m}(s)))|_H^2 ds + \frac{\min\{1, \kappa\}}{1 + \|\gamma_0\|^p} \frac{1}{p} \|(u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))\|_{V_p}^p \\ & + \tilde{\alpha}_1 \left(|u_{\varepsilon,m}(t)|_{q_1, \Omega_\varepsilon}^{q_1} + \varepsilon |\gamma_0(u_{\varepsilon,m}(t))|_{q_2, \partial F_\varepsilon}^{q_2} \right) \leq \frac{\max\{1, \kappa\}}{p} \|(u_{\varepsilon,m}^0, \gamma_0(u_{\varepsilon,m}^0))\|_{V_p}^p \\ & + \tilde{\alpha}_2 \left(|u_{\varepsilon,m}^0|_{q_1, \Omega_\varepsilon}^{q_1} + \varepsilon |\gamma_0(u_{\varepsilon,m}^0)|_{q_2, \partial F_\varepsilon}^{q_2} \right) + 2\tilde{\beta}C, \end{aligned}$$

for all $t \in (0, T)$. In order to estimate the right hand side of the last inequality we use (23) together with (50) and $\varepsilon \ll 1$. In particular, we can deduce

$$\int_0^t |(u'_{\varepsilon,m}(s), \gamma_0(u'_{\varepsilon,m}(s)))|_H^2 ds + \frac{\min\{1, \kappa\}}{1 + \|\gamma_0\|^p} \frac{1}{p} \|(u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))\|_{V_p}^p \leq C, \quad (52)$$

for all $t \in (0, T)$. We have proved that the sequence $\{(u_{\varepsilon,m}, \gamma_0(u_{\varepsilon,m}))\}$ is bounded in $C([0, T]; V_p)$, and $\{(u'_{\varepsilon,m}, \gamma_0(u'_{\varepsilon,m}))\}$ is bounded in $L^2(0, T; H)$, for all $T > 0$.

If we work with the truncated Galerkin equations (47)-(49) instead of the full PDE, we note that the calculations of the proof of (46) can be following identically to show that $\{(u_{\varepsilon,m}, \gamma_0(u_{\varepsilon,m}))\}$ is bounded in $L^p(0, T; V_p)$, for all $T > 0$.

Moreover, taking into account the uniqueness of solution to (2), using Aubin-Lions compactness lemma (e.g., cf. Lions [20]) and the fact that A_p is a monotone operator, it is not difficult to conclude that the sequence $\{(u_{\varepsilon,m}, \gamma_0(u_{\varepsilon,m}))\}$ converges weakly in $L^p(0, T; V_p)$ to the solution $(u_\varepsilon, \gamma_0(u_\varepsilon))$ to (2). Since the inclusion $V_p \subset H$ is compact (see (25)) and $(u_\varepsilon, \gamma_0(u_\varepsilon)) \in C([0, T]; H)$, it follows using [22, Lemma 11.2] that

$$\sup_{t \in [0, T]} \|(u_\varepsilon(t), \gamma_0(u_\varepsilon(t)))\|_{V_p} \leq C,$$

and, in particular, the second estimate in (44) is proved. □

The extension of u_ε to the whole $\Omega \times (0, T)$: since the solution u_ε of the problem (2) is defined only in $\Omega_\varepsilon \times (0, T)$, we need to extend it to the whole $\Omega \times (0, T)$ to be able to state the convergence result. In order to do that, we shall use the following well-known extension result given by Donato and Moscardello [13, Lemma 2.4]: Let $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ be a $W^{1,p}$ -extension of u_ε , that satisfies the following condition

$$|\nabla \tilde{u}_\varepsilon|_{p, \Omega} \leq C |\nabla u_\varepsilon|_{p, \Omega_\varepsilon}. \quad (53)$$

Using (53) and taking into account Lemma 4.1, we obtain some *a priori* estimates for the extension of u_ε to the whole $\Omega \times (0, T)$.

Corollary 4.2. *Assume the assumptions in Lemma 4.1. Then, there exists a $W^{1,p}$ -extension \tilde{u}_ε of the solution u_ε of the problem (2) into $\Omega \times (0, T)$, such that*

$$\|\tilde{u}_\varepsilon\|_{p,\Omega,T} \leq C, \quad \sup_{t \in [0,T]} \|\tilde{u}_\varepsilon(t)\|_{p,\Omega} \leq C, \quad (54)$$

where the constant C does not depend on ε .

Proof. Using Poincaré's inequality, (53) and the first estimate in (44), we obtain

$$\|\tilde{u}_\varepsilon\|_{p,\Omega,T}^p = |\tilde{u}_\varepsilon|_{p,\Omega,T}^p + |\nabla \tilde{u}_\varepsilon|_{p,\Omega,T}^p \leq C |\nabla \tilde{u}_\varepsilon|_{p,\Omega,T}^p \leq C |\nabla u_\varepsilon|_{p,\Omega_\varepsilon,T}^p \leq C,$$

and the first estimate in (54) is proved. Similarly, using Poincaré's inequality, (53) and the second estimate in (44), we have the second estimate in (54). \square

5 A compactness result

In this section, we obtain some compactness results about the behavior of the sequence \tilde{u}_ε satisfying the *a priori* estimates given in Corollary 4.2.

By $\chi_{\Omega_\varepsilon}$ we denote the characteristic function of the domain Ω_ε . Due to the periodicity of the domain Ω_ε , from [7, Theorem 2.6] one has, for $\varepsilon \rightarrow 0$, that

$$\chi_{\Omega_\varepsilon} \xrightarrow{*} \frac{|Y^*|}{|Y|} \quad \text{weakly-star in } L^\infty(\Omega), \quad (55)$$

where the limit is the proportion of the material in the cell Y .

Let $\xi_\varepsilon := |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon$ in $\Omega_\varepsilon \times (0, T)$ and let us denote by $\tilde{\xi}_\varepsilon$ its extension with zero to the whole of $\Omega \times (0, T)$, i.e.

$$\tilde{\xi}_\varepsilon = \begin{cases} \xi_\varepsilon & \text{in } \Omega_\varepsilon \times (0, T), \\ 0 & \text{in } (\Omega \setminus \overline{\Omega_\varepsilon}) \times (0, T). \end{cases} \quad (56)$$

Proposition 5.1. *Under the assumptions in Lemma 4.1, there exists a function $u \in L^p(0, T; W_0^{1,p}(\Omega))$ (u will be the unique solution of the limit system (13)) and a function $\xi \in L^{p'}(0, T; L^{p'}(\Omega))$ such that, at least after extraction of a subsequence, we have the following convergences for all $T > 0$,*

$$\tilde{u}_\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly in } W_0^{1,p}(\Omega), \quad \forall t \in [0, T], \quad (57)$$

$$\tilde{u}_\varepsilon(t) \rightarrow u(t) \quad \text{strongly in } L^p(\Omega), \quad \forall t \in [0, T], \quad (58)$$

$$|\tilde{u}_\varepsilon(t)|^{p-2} \tilde{u}_\varepsilon(t) \rightarrow |u(t)|^{p-2} u(t) \quad \text{strongly in } L^{p'}(\Omega), \quad \forall t \in [0, T], \quad (59)$$

$$f(\tilde{u}_\varepsilon(t)) \rightarrow f(u(t)) \quad \text{strongly in } L^{q_1}(\Omega), \quad \forall t \in [0, T], \quad (60)$$

$$\tilde{\xi}_\varepsilon \rightharpoonup \xi \quad \text{weakly in } L^{p'}(0, T; L^{p'}(\Omega)), \quad (61)$$

where $\tilde{\xi}_\varepsilon$ is given by (56).

Let q_2 be the exponent satisfying (12). Let $\bar{q} > 1$ given by

$$\bar{q} \in (1, p) \quad \text{if } p = N, \quad \bar{q} = \frac{Np}{(N-p)(q_2-2) + N} \quad \text{if } p < N.$$

Assume that $g \in C^1(\mathbb{R})$. Then, we have the following convergences for all $T > 0$,

$$g(\tilde{u}_\varepsilon(t)) \rightarrow g(u(t)) \quad \text{strongly in } L^{\bar{q}}(\Omega), \quad \forall t \in [0, T], \quad (62)$$

$$g(\tilde{u}_\varepsilon(t)) \rightharpoonup g(u(t)) \quad \text{weakly in } W_0^{1,\bar{q}}(\Omega), \quad \forall t \in [0, T]. \quad (63)$$

Proof. By the first estimate in (54), we observe that the sequence $\{\tilde{u}_\varepsilon\}$ is bounded in the space $L^p(0, T; W_0^{1,p}(\Omega))$, for all $T > 0$. Let us fix $T > 0$. Then, there exists a subsequence $\{\tilde{u}_{\varepsilon'}\} \subset \{\tilde{u}_\varepsilon\}$ and function $u \in L^p(0, T; W_0^{1,p}(\Omega))$ such that

$$\tilde{u}_{\varepsilon'} \rightharpoonup u \quad \text{weakly in } L^p(0, T; W_0^{1,p}(\Omega)). \quad (64)$$

By the second estimate in (54), for each $t \in [0, T]$, we have that $\{\tilde{u}_\varepsilon(t)\}$ is bounded in $W_0^{1,p}(\Omega)$, and since we have (64), we can deduce (57). The conclusions of the Rellich-Kondrachov Theorem are also valid substituting $W^{1,p}(\Omega_\varepsilon)$ for $W_0^{1,p}(\Omega)$, so using (18) and (57), we obtain

$$\tilde{u}_{\varepsilon'}(t) \rightarrow u(t) \quad \text{strongly in } L^r(\Omega), \quad \forall t \in [0, T], \quad (65)$$

where $1 \leq r < p^*$ if $p < N$ and $p \leq r < p^*$ if $p = N$. Observe that, in particular, we have (58).

On the other hand, observe that

$$||v|^{p-2}v| \leq C(1 + |v|^{p-1}),$$

then, applying [10, Theorem 2.4] for $G(x, v) = |v|^{p-2}v$, $t = p'$ and $r = p$, we have that the map $v \in L^p(\Omega) \mapsto |v|^{p-2}v \in L^{p'}(\Omega)$ is continuous in the strong topologies. Then, taking into account (65) with $r = p$, we get (59).

Thanks to (27), applying [10, Theorem 2.4] for $G(x, v) = f(v)$, $t = q_1'$ and $r = q_1$, we have that the map $v \in L^{q_1}(\Omega) \mapsto f(v) \in L^{q_1'}(\Omega)$ is continuous in the strong topologies. Then, taking into account (65) with $r = q_1$, we get (60).

From the first estimate in (44) and (56), we have $|\tilde{\xi}_\varepsilon|_{p', \Omega, T} \leq C$, and hence, up a sequence, there exists $\xi \in L^{p'}(0, T, L^{p'}(\Omega))$ such that we obtain (61).

By the arbitrariness of $T > 0$, all the convergences are satisfied, as we wanted to prove.

Now, we analyze the convergences for the nonlinear term g . We separate the cases $p < N$ and $p = N$.

Case 1: $p < N$. Since

$$\frac{p^*}{\bar{q}} = \frac{(N-p)(q_2-2) + N}{N-p} = q_2 - 1 + \frac{p}{N-p} > q_2 - 1,$$

there exists $r \in [2, p^*)$ such that $\frac{r}{\bar{q}} \geq q_2 - 1$ and

$$|g(s)| \leq C \left(1 + |s|^{q_2-1}\right) \leq C \left(1 + |s|^{\frac{r}{\bar{q}}}\right).$$

Then, applying [10, Theorem 2.4] for $G(x, v) = g(v)$, $t = \bar{q}$ and $r \in [2, p^*)$ such that $\frac{r}{\bar{q}} \geq q_2 - 1$, we have that the map $v \in L^r(\Omega) \mapsto g(v) \in L^{\bar{q}}(\Omega)$ is continuous in the strong topologies. Then, taking into account (65), we get (62).

Finally, we prove (63). Observe that as we are assuming that $g \in C^1(\mathbb{R})$, it is easy to see from (6) that there exists a constant $C > 0$ such that

$$|g'(s)| \leq C \left(1 + |s|^{q_2-2}\right).$$

Then, we get

$$\begin{aligned} \int_\Omega \left| \frac{\partial g}{\partial x_i}(\tilde{u}_\varepsilon(t)) \right|^{\bar{q}} dx &\leq C \int_\Omega \left(1 + |\tilde{u}_\varepsilon(t)|^{(q_2-2)\bar{q}}\right) \left| \frac{\partial \tilde{u}_\varepsilon(t)}{\partial x_i} \right|^{\bar{q}} dx \\ &\leq C \left(1 + \left(\int_\Omega |\tilde{u}_\varepsilon(t)|^{(q_2-2)\bar{q}\gamma} dx\right)^{1/\gamma}\right) \left(\int_\Omega |\nabla \tilde{u}_\varepsilon(t)|^{\bar{q}\eta} dx\right)^{1/\eta}, \end{aligned} \quad (66)$$

where we took γ and η such that $\bar{q}\eta = p$, $1/\gamma + 1/\eta = 1$ and $(q_2-2)\bar{q}\gamma = p^*$. Note that from here we get

$$\bar{q} = \frac{Np}{(N-p)(q_2-2) + N}.$$

Observe that $\bar{q} > 1$. Indeed,

$$q_2 \leq \frac{(N-1)p}{N-p} = \frac{N(p-1)}{N-p} + 1 < \frac{N(p-1)}{N-p} + 2 \Rightarrow (N-p)(q_2-2) + N < Np \Rightarrow \frac{Np}{(N-p)(q_2-2) + N} > 1.$$

Then, we have

$$\int_{\Omega} \left| \frac{\partial g}{\partial x_i}(\tilde{u}_\varepsilon(t)) \right|^{\bar{q}} dx \leq C \left(1 + |\tilde{u}_\varepsilon|_{p^*, \Omega}^{p^*/\gamma} \right) |\nabla \tilde{u}_\varepsilon|_{p, \Omega}^{p/\eta}.$$

The conclusions of (17) are also valid substituting $W^{1,p}(\Omega_\varepsilon)$ for $W_0^{1,p}(\Omega)$, so using (17) and the second estimate in (54), we get

$$|\nabla g(\tilde{u}_\varepsilon(t))|_{\bar{q}, \Omega} \leq C. \quad (67)$$

Then, from (62) and (67), we can deduce (63).

Case 2: $p = N$. We consider $s \in [\frac{p(q_2-1)}{p-1}, +\infty)$ and

$$\bar{q} = \frac{sp}{p(q_2-2) + s}. \quad (68)$$

Taking into account (12), we can deduce

$$\frac{s}{\bar{q}} = \frac{p(q_2-2) + s}{p} = q_2 - 1 + \frac{s-p}{p} \geq q_2 - 1,$$

and, we have

$$|g(s)| \leq C \left(1 + |s|^{q_2-1} \right) \leq C \left(1 + |s|^{\frac{s}{\bar{q}}} \right).$$

Then, applying [10, Theorem 2.4] for $G(x, v) = g(v)$, $t = \bar{q}$ and $r = s$, we have that the map $v \in L^s(\Omega) \mapsto g(v) \in L^{\bar{q}}(\Omega)$ is continuous in the strong topologies. Thanks to (12), we can consider (65) with $r = s$, and we get (62).

Finally, we prove (63). In (66) we took γ and η such that $\bar{q}\eta = p$, $1/\gamma + 1/\eta = 1$ and $(q_2-2)\bar{q}\gamma = s$. Note that from here we get \bar{q} given by (68).

Observe that $\bar{q} \in (1, p)$. Indeed, taking into account that $\frac{1}{\bar{q}} = \frac{q_2-2}{s} + \frac{1}{p}$, we can deduce

$$\frac{p(q_2-1)}{p-1} \leq s < +\infty \Rightarrow 0 < \frac{1}{s} \leq \frac{p-1}{p(q_2-1)} \Rightarrow \frac{1}{p} < \frac{1}{\bar{q}} \leq \frac{(q_2-2)(p-1)}{p(q_2-1)} + \frac{1}{p} \Rightarrow \frac{p(q_2-1)}{(q_2-2)(p-1) + (q_2-1)} \leq \bar{q} < p,$$

and using that $\frac{p(q_2-1)}{(q_2-2)(p-1) + (q_2-1)} > 1$, we have that $\bar{q} \in (1, p)$.

Then, we have

$$\int_{\Omega} \left| \frac{\partial g}{\partial x_i}(\tilde{u}_\varepsilon(t)) \right|^{\bar{q}} dx \leq C \left(1 + |\tilde{u}_\varepsilon|_{s, \Omega}^{s/\gamma} \right) |\nabla \tilde{u}_\varepsilon|_{p, \Omega}^{p/\eta}.$$

The conclusions of (17) are also valid substituting $W^{1,p}(\Omega_\varepsilon)$ for $W_0^{1,p}(\Omega)$, so using (17) and the second estimate in (54), we get

$$|\nabla g(\tilde{u}_\varepsilon(t))|_{\bar{q}, \Omega} \leq C. \quad (69)$$

Then, from (62) and (69), we can deduce (63).

□

6 Homogenized model: proof of the main Theorem

In this section, we identify the homogenized model.

We multiply system (2) by a test function $v \in \mathcal{D}(\Omega)$, and integrating by parts, we have

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(t) v dx \right) + \varepsilon \frac{d}{dt} \left(\int_{\partial F_{\varepsilon}} \gamma_0(u_{\varepsilon}(t)) v d\sigma(x) \right) + \int_{\Omega} \tilde{\xi}_{\varepsilon} \cdot \nabla v dx \\ & + \kappa \int_{\Omega} \chi_{\Omega_{\varepsilon}} |\tilde{u}_{\varepsilon}(t)|^{p-2} \tilde{u}_{\varepsilon}(t) v dx + \int_{\Omega} \chi_{\Omega_{\varepsilon}} f(\tilde{u}_{\varepsilon}(t)) v dx + \varepsilon \int_{\partial F_{\varepsilon}} g(\gamma_0(u_{\varepsilon}(t))) v d\sigma(x) = 0, \end{aligned}$$

in $\mathcal{D}'(0, T)$.

We consider $\varphi \in C_c^1([0, T])$ such that $\varphi(T) = 0$ and $\varphi(0) \neq 0$. Multiplying by φ and integrating between 0 and T , we have

$$\begin{aligned} & -\varphi(0) \left(\int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(0) v dx \right) - \int_0^T \frac{d}{dt} \varphi(t) \left(\int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(t) v dx \right) dt \\ & - \varepsilon \varphi(0) \left(\int_{\partial F_{\varepsilon}} \gamma_0(u_{\varepsilon}(0)) v d\sigma(x) \right) - \varepsilon \int_0^T \frac{d}{dt} \varphi(t) \left(\int_{\partial F_{\varepsilon}} \gamma_0(u_{\varepsilon}(t)) v d\sigma(x) \right) dt \\ & + \int_0^T \varphi(t) \int_{\Omega} \tilde{\xi}_{\varepsilon} \cdot \nabla v dx dt + \kappa \int_0^T \varphi(t) \int_{\Omega} \chi_{\Omega_{\varepsilon}} |\tilde{u}_{\varepsilon}(t)|^{p-2} \tilde{u}_{\varepsilon}(t) v dx dt \\ & + \int_0^T \varphi(t) \int_{\Omega} \chi_{\Omega_{\varepsilon}} f(\tilde{u}_{\varepsilon}(t)) v dx dt + \varepsilon \int_0^T \varphi(t) \int_{\partial F_{\varepsilon}} g(\gamma_0(u_{\varepsilon}(t))) v d\sigma(x) dt = 0. \end{aligned} \tag{70}$$

For the sake of clarity, we split the proof in five parts. Firstly, we pass to the limit, as $\varepsilon \rightarrow 0$, in (70) in order to get the limit equation satisfied by u . In the first step, we pass to the limit in the integrals on Ω , using Proposition 5.1 and (55), in the second step we pass to the limit in the integrals on the boundary of the holes, where we use a convergence result based on a technique introduced by Vanninathan [25], and in the third step we deduce the limit equation satisfied by u . In the fourth step we identify ξ making use of the solutions of the problems (14), and finally we prove that u is uniquely determined.

Step 1. Passing to the limit, as $\varepsilon \rightarrow 0$, in the integrals on Ω :

From (58)-(60) and (55), we have respectively, for $\varepsilon \rightarrow 0$,

$$\begin{aligned} & \int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(t) v dx \rightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} u(t) v dx, \quad \forall v \in \mathcal{D}(\Omega), \\ & \int_{\Omega} \chi_{\Omega_{\varepsilon}} |\tilde{u}_{\varepsilon}(t)|^{p-2} \tilde{u}_{\varepsilon}(t) v dx \rightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} |u(t)|^{p-2} u(t) v dx, \quad \forall v \in \mathcal{D}(\Omega), \end{aligned}$$

and

$$\int_{\Omega} \chi_{\Omega_{\varepsilon}} f(\tilde{u}_{\varepsilon}(t)) v dx \rightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} f(u(t)) v dx, \quad \forall v \in \mathcal{D}(\Omega),$$

which integrating in time and using Lebesgue's Dominated Convergence Theorem, gives

$$\begin{aligned} & \int_0^T \frac{d}{dt} \varphi(t) \left(\int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(t) v dx \right) dt \rightarrow \frac{|Y^*|}{|Y|} \int_0^T \frac{d}{dt} \varphi(t) \left(\int_{\Omega} u(t) v dx \right) dt, \\ & \kappa \int_0^T \varphi(t) \int_{\Omega} \chi_{\Omega_{\varepsilon}} |\tilde{u}_{\varepsilon}(t)|^{p-2} \tilde{u}_{\varepsilon}(t) v dx dt \rightarrow \kappa \frac{|Y^*|}{|Y|} \int_0^T \varphi(t) \int_{\Omega} |u(t)|^{p-2} u(t) v dx dt, \end{aligned}$$

and

$$\int_0^T \varphi(t) \int_{\Omega} \chi_{\Omega_\varepsilon} f(\tilde{u}_\varepsilon(t)) v dx dt \rightarrow \frac{|Y^*|}{|Y|} \int_0^T \varphi(t) \int_{\Omega} f(u(t)) v dx dt.$$

By (58) and (55), we have

$$\varphi(0) \left(\int_{\Omega} \chi_{\Omega_\varepsilon} \tilde{u}_\varepsilon(0) v dx \right) \rightarrow \varphi(0) \frac{|Y^*|}{|Y|} \int_{\Omega} u(0) v dx, \quad \forall v \in \mathcal{D}(\Omega).$$

On the other hand, using (61), we obtain, for $\varepsilon \rightarrow 0$

$$\int_0^T \varphi(t) \int_{\Omega} \tilde{\xi}_\varepsilon \cdot \nabla v dx dt \rightarrow \int_0^T \varphi(t) \int_{\Omega} \xi \cdot \nabla v dx dt.$$

Step 2. Passing to the limit, as $\varepsilon \rightarrow 0$, in the surface integrals on the boundary of the holes:

We make use of the technique introduced by Vanninathan [25] for the Steklov problem which transforms surface integrals into volume integrals. This technique was already used as a main tool to homogenize the non homogeneous Neumann problem for the elliptic case by Cioranescu and Donato [6].

By [6, Definition 3.2], let us introduce, for any $h \in L^{s'}(\partial F)$, $1 \leq s' \leq \infty$, the linear form μ_h^ε on $W_0^{1,s}(\Omega)$ defined by

$$\langle \mu_h^\varepsilon, \varphi \rangle = \varepsilon \int_{\partial F_\varepsilon} h \left(\frac{x}{\varepsilon} \right) \varphi(x) d\sigma(x), \quad \forall \varphi \in W_0^{1,s}(\Omega),$$

with $1/s + 1/s' = 1$. It is proved in [6, Lemma 3.3] that

$$\mu_h^\varepsilon \rightarrow \mu_h \quad \text{strongly in } (W_0^{1,s}(\Omega))', \quad (71)$$

where $\langle \mu_h, \varphi \rangle = \mu_h \int_{\Omega} \varphi(x) dx$, with

$$\mu_h = \frac{1}{|Y|} \int_{\partial F} h(y) d\sigma(y).$$

In the particular case in which $h \in L^\infty(\partial F)$ or even when h is constant, we have

$$\mu_h^\varepsilon \rightarrow \mu_h \quad \text{strongly in } W^{-1,\infty}(\Omega).$$

We denote by μ_1^ε the above introduced measure in the particular case in which $h = 1$. Notice that in this case μ_h becomes $\mu_1 = |\partial F|/|Y|$.

Observe that from (57) and (71) with $s = p$, we can deduce, for $\varepsilon \rightarrow 0$,

$$\varepsilon \int_{\partial F_\varepsilon} \gamma_0(u_\varepsilon(t)) v d\sigma(x) = \langle \mu_1^\varepsilon, \tilde{u}_{\varepsilon|\Omega_\varepsilon}(t)v \rangle \rightarrow \mu_1 \int_{\Omega} u(t) v dx = \frac{|\partial F|}{|Y|} \int_{\Omega} u(t) v dx,$$

for all $v \in \mathcal{D}(\Omega)$, which integrating in time and using Lebesgue's Dominated Convergence Theorem, gives

$$\varepsilon \int_0^T \frac{d}{dt} \varphi(t) \left(\int_{\partial F_\varepsilon} \gamma_0(u_\varepsilon(t)) v d\sigma(x) \right) dt \rightarrow \frac{|\partial F|}{|Y|} \int_0^T \frac{d}{dt} \varphi(t) \left(\int_{\Omega} u(t) v dx \right) dt.$$

Moreover, from (57) and (71) with $s = p$, we can deduce, for $\varepsilon \rightarrow 0$,

$$\varepsilon \int_{\partial F_\varepsilon} \gamma_0(u_\varepsilon(0)) v d\sigma(x) = \langle \mu_1^\varepsilon, \tilde{u}_{\varepsilon|\Omega_\varepsilon}(0)v \rangle \rightarrow \mu_1 \int_{\Omega} u(0) v dx = \frac{|\partial F|}{|Y|} \int_{\Omega} u(0) v dx,$$

for all $v \in \mathcal{D}(\Omega)$.

On the other hand, from (63) and (71) with $s = \bar{q}$, we conclude

$$\varepsilon \int_{\partial F_\varepsilon} g(\gamma_0(u_\varepsilon(t))) v d\sigma(x) = \langle \mu_1^\varepsilon, g(\tilde{u}_\varepsilon(t)) v \rangle \rightarrow \frac{|\partial F|}{|Y|} \int_\Omega g(u(t)) v dx,$$

for all $v \in \mathcal{D}(\Omega)$, which integrating in time and using Lebesgue's Dominated Convergence Theorem, gives

$$\varepsilon \int_0^T \varphi(t) \int_{\partial F_\varepsilon} g(\gamma_0(u_\varepsilon(t))) v d\sigma(x) dt \rightarrow \frac{|\partial F|}{|Y|} \int_0^T \varphi(t) \int_\Omega g(u(t)) v dx dt.$$

Step 3. Passing to the limit, as $\varepsilon \rightarrow 0$, in (70):

All the terms in (70) pass to the limit, as $\varepsilon \rightarrow 0$, and therefore taking into account the previous steps, we get

$$\begin{aligned} -\varphi(0) \left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \left(\int_\Omega u(0) v dx \right) - \left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \int_0^T \frac{d}{dt} \varphi(t) \left(\int_\Omega u(t) v dx \right) dt \\ + \int_0^T \varphi(t) \int_\Omega \xi \cdot \nabla v dx dt + \kappa \frac{|Y^*|}{|Y|} \int_0^T \varphi(t) \int_\Omega |u(t)|^{p-2} u(t) v dx dt \\ + \frac{|Y^*|}{|Y|} \int_0^T \varphi(t) \int_\Omega f(u(t)) v dx dt + \frac{|\partial F|}{|Y|} \int_0^T \varphi(t) \int_\Omega g(u(t)) v dx dt = 0. \end{aligned}$$

Hence, ξ verifies

$$\left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \frac{\partial u}{\partial t} - \operatorname{div} \xi + \frac{|Y^*|}{|Y|} (\kappa |u|^{p-2} u + f(u)) + \frac{|\partial F|}{|Y|} g(u) = 0, \quad \text{in } \Omega \times (0, T). \quad (72)$$

Step 4. It remains now to identify ξ . For the sake of completeness, we give a sketch of a proof, following, for instance, the proof of [13, Theorem 3.1].

Let us consider $\forall \nu \in \mathbb{N}$ and for any $i = 1, \dots, N$, a partition of \mathbb{R}^N by intervals $\{Q_{i\nu}\}_i$ of side $2^{-\nu}$. Since the holes F_ε do not intersect $\partial\Omega$, we can assume that $\forall \varepsilon > 0$, F_ε does not intersect $\partial Q_{i\nu}$. Let us denote by $x_{i\nu}$ and $\chi_{i\nu}$ respectively the center and the characteristic function of $Q_{i\nu}$. Then, we define $I_\nu = \{i : Q_{i\nu} \subset \Omega\}$, $\Omega_\nu = \cup_{i \in I_\nu} Q_{i\nu}$. For any i , let us denote

$$\langle \nabla u \rangle_{i\nu} := \frac{1}{|Q_{i\nu}|} \int_{Q_{i\nu}} \nabla u(x) dx.$$

By the continuity of b (see [13, Lemma 2.11]), we have, if $\nu \rightarrow +\infty$, that

$$\sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle \nabla u \rangle_{i\nu}) \rightarrow b(\nabla u(x)) \quad \text{a.e. in } \Omega, \quad (73)$$

where b is given by (15).

Moreover, from [13, Lemma 2.10], we have for any measurable set $E \subset \Omega$ that

$$\int_E \left| \sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle \nabla u \rangle_{i\nu}) \right|^{p'} dx \leq C \int_E \left(1 + \left| \sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle \nabla u \rangle_{i\nu}) \right|^p \right) dx.$$

So, from the equi-absolute continuity of the integral on the left-hand side and from (73), we deduce that

$$\sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle \nabla u \rangle_{i\nu}) \rightarrow b(\nabla u(x)) \quad \text{in } L^{p'}(\Omega), \quad \text{as } \nu \rightarrow +\infty. \quad (74)$$

Let $w_{i\nu} \in \langle \nabla u \rangle_{i\nu} \cdot y + \mathbb{H}_{\text{per}}(Y^*)$ the solution of the problem (14) corresponding to $\langle \nabla u \rangle_{i\nu}$. Then

$$v_{i\nu} = (w_{i\nu} - \langle \nabla u \rangle_{i\nu} \cdot y) \in \mathbb{H}_{\text{per}}(Y^*),$$

and so, by [13, Lemma 2.3], we can consider $\tilde{v}_{i\nu} \in \mathbb{H}_{\text{per}}(Y)$, where $\tilde{\cdot}$ denotes the $W^{1,p}$ -extension to Y . If we extend $\tilde{v}_{i\nu}$, by periodicity, the resulting function (still denoted by $\tilde{v}_{i\nu}$) is in $W^{1,p}(\Omega)$.

Let us define

$$\Psi_{i\nu}^\varepsilon(x) = \varepsilon \tilde{v}_{i\nu} \left(\frac{x}{\varepsilon} \right) - \langle \nabla u \rangle_{i\nu} \cdot x, \quad (75)$$

and for $\zeta \in \mathbb{R}^N$

$$\tilde{a}_p(\zeta) = \begin{cases} |\zeta|^{p-2} \zeta & \text{for } y \in Y^*, \\ 0 & \text{for } y \in \bar{F}. \end{cases}$$

Arguing as in the proof of [13, Lemma 2.13], we obtain

$$\Psi_{i\nu}^\varepsilon \rightharpoonup \langle \nabla u \rangle_{i\nu} \cdot x \quad \text{weakly in } W^{1,p}(\Omega), \quad (76)$$

$$\tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \rightharpoonup b(\langle \nabla u \rangle_{i\nu}) \quad \text{weakly in } L^{p'}(\Omega), \quad (77)$$

and

$$\text{div } \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) = 0. \quad (78)$$

By Rellich-Kondrachov Theorem and (76), we can deduce

$$\Psi_{i\nu}^\varepsilon \rightarrow \langle \nabla u \rangle_{i\nu} \cdot x \quad \text{strongly in } L^p(\Omega). \quad (79)$$

Using the periodicity of $\tilde{v}_{i\nu}$ and [13, Lemma 2.3], we have

$$\begin{aligned} \sum_{i \in I_\nu} \int_{Q_{i\nu}} |\nabla \Psi_{i\nu}^\varepsilon|^p dx &\leq \sum_{i \in I_\nu} 2^{-\nu N} \varepsilon^N (1/\varepsilon + 2^\nu)^N \int_Y |\nabla \Psi_{i\nu}^\varepsilon(y)|^p dy \\ &\leq C \sum_{i \in I_\nu} 2^{-\nu N} (1 + \varepsilon^N 2^{\nu N}) \left(\int_{Y^*} |\nabla w_{i\nu}|^p dy + \langle \nabla u \rangle_{i\nu} \right)^p. \end{aligned}$$

Then, due to the structure properties of b (see [13, Lemma 2.10]), writing the last term as an integral over Ω_ν , we have

$$\sum_{i \in I_\nu} \int_{Q_{i\nu}} |\nabla \Psi_{i\nu}^\varepsilon|^p dx \leq C (1 + \varepsilon^N 2^{\nu N}) \int_\Omega (1 + |\nabla u|)^p dx. \quad (80)$$

Let now $\eta \in C_0^1(Q_{i\nu})$, $0 \leq \eta \leq 1$, and extend it by periodicity to the whole \mathbb{R}^N . We consider $\varphi \in C_c^1([0, T])$ such that $\varphi(T) = 0$ and $\varphi(0) \neq 0$. If $\psi \in C^0(\bar{\Omega})$, set $M_\psi = \sup_\Omega |\psi|$ and $M_\varphi = \sup_{[0, T]} |\varphi|$, we have

$$\begin{aligned} &\left| \int_0^T \varphi(t) \int_\Omega \tilde{\xi}_\varepsilon \psi \eta dx dt - \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \psi \eta dx dt \right| \quad (81) \\ &\leq CM_\psi M_\varphi |\Omega \setminus \Omega_\nu|^{1/p} + \left| \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} (|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - |\nabla \Psi_{i\nu}^\varepsilon|^{p-2} \nabla \Psi_{i\nu}^\varepsilon) \psi \eta dx dt \right| \\ &\leq CM_\psi M_\varphi |\Omega \setminus \Omega_\nu|^{1/p} + \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} M_\psi \eta \left((|\nabla u_\varepsilon| + |\nabla \Psi_{i\nu}^\varepsilon|)^{p-2} \cdot |\nabla u_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon| \right) dx dt \\ &\leq CM_\psi M_\varphi |\Omega \setminus \Omega_\nu|^{1/p} + CM_\psi^{p/(p-1)} \delta^{p/(p-1)} (1 + \varepsilon^N 2^{\nu N}) + \delta^{-p} \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |\nabla u_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon|^p dx dt, \end{aligned}$$

where the last inequality is obtained by applying Young inequality with $\delta > 0$ and the first estimate in (44), the first estimate in (54) and (80).

On the other hand, from monotonicity, we can deduce

$$\begin{aligned}
 \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |\nabla u_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon|^p dx dt &\leq \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \left(\tilde{\xi}_\varepsilon - \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \right) \cdot (\nabla \tilde{u}_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon) dx dt \\
 &= \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \left(\tilde{\xi}_\varepsilon - \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \right) \cdot \nabla \rho_{i\nu}^\varepsilon dx dt \\
 &+ \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \left(\tilde{\xi}_\varepsilon - \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \right) \cdot (\nabla u - \langle \nabla u \rangle_{i\nu}) dx dt,
 \end{aligned} \tag{82}$$

where $\rho_{i\nu}^\varepsilon := \tilde{u}_\varepsilon - \Psi_{i\nu}^\varepsilon - u + \langle \nabla u \rangle_{i\nu} \cdot x$. Taking into account (58) and (79), we have

$$\rho_{i\nu}^\varepsilon \rightarrow 0 \quad \text{strongly in } L^p(\Omega). \tag{83}$$

Now, we multiply system (2) by the test function $\eta \rho_{i\nu}^\varepsilon$ and integrating by parts, we get

$$\begin{aligned}
 \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \tilde{\xi}_\varepsilon \cdot \nabla \rho_{i\nu}^\varepsilon dx dt &= - \sum_{i \in I_\nu} \int_{Q_{i\nu}} \varphi(0) \chi_{\Omega_\varepsilon} \tilde{u}_\varepsilon(0) \eta \rho_{i\nu}^\varepsilon dx \\
 - \int_0^T \frac{d}{dt} \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \chi_{\Omega_\varepsilon} \tilde{u}_\varepsilon(t) \eta \rho_{i\nu}^\varepsilon dx dt &- \varphi(0) \langle \mu_1^\varepsilon, \tilde{u}_{\varepsilon|\Omega_\varepsilon}(0) \eta \rho_{i\nu}^\varepsilon \rangle - \int_0^T \frac{d}{dt} \varphi(t) \langle \mu_1^\varepsilon, \tilde{u}_{\varepsilon|\Omega_\varepsilon}(t) \eta \rho_{i\nu}^\varepsilon \rangle dt \\
 - \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{\xi}_\varepsilon \cdot \nabla \eta \rho_{i\nu}^\varepsilon dx dt &- \kappa \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \chi_{\Omega_\varepsilon} |\tilde{u}_\varepsilon(t)|^{p-2} \tilde{u}_\varepsilon(t) \eta \rho_{i\nu}^\varepsilon dx dt \\
 - \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \chi_{\Omega_\varepsilon} f(\tilde{u}_\varepsilon(t)) \eta \rho_{i\nu}^\varepsilon dx dt &- \int_0^T \varphi(t) \langle \mu_1^\varepsilon, g(\tilde{u}_\varepsilon(t)) \rho_{i\nu}^\varepsilon \rangle dt.
 \end{aligned}$$

We pass to the limit, as $\varepsilon \rightarrow 0$. Taking into account (83), we reason as in the first two steps and we can deduce

$$\int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \tilde{\xi}_\varepsilon \cdot \nabla \rho_{i\nu}^\varepsilon dx dt \rightarrow 0. \tag{84}$$

On the other hand, integrating by parts, taking into account (77)-(78) and (83), as $\varepsilon \rightarrow 0$, by Lebesgue's Dominated Convergence Theorem, we obtain

$$\int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \cdot \nabla \rho_{i\nu}^\varepsilon \eta dx dt = \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \cdot \nabla \eta \rho_{i\nu}^\varepsilon dx dt \rightarrow 0. \tag{85}$$

Then, passing to the limit, as $\varepsilon \rightarrow 0$, in (82), by (61), (77), (84)-(85), we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |\nabla u_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon|^p dx dt \leq \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta (\xi - b(\langle \nabla u \rangle_{i\nu})) (\nabla u - \langle \nabla u \rangle_{i\nu}) dx dt. \tag{86}$$

If we first pass to the limit in (81), as $\varepsilon \rightarrow 0$, then $\eta \rightarrow 1$, $\nu \rightarrow +\infty$, by (61), (74), (77) and (86) we get

$$\left| \int_0^T \varphi(t) \int_\Omega \xi \psi dx dt - \int_0^T \varphi(t) \int_\Omega b(\nabla u) \psi dx dt \right| \leq CM_\psi^{p/(p-1)} \delta^{p/(p-1)}.$$

So, letting $\delta \rightarrow 0$, from the arbitrariness of φ and ψ , we deduce

$$\xi = b(\nabla u) \quad \text{a.e. in } \Omega \times (0, T). \quad (87)$$

Step 5. Finally, thanks to (72) and (87), we observe that u satisfies the first equation in (13). A weak solution of (13) is any function u , satisfying

$$u \in C([0, T]; L^2(\Omega)), \quad \text{for all } T > 0,$$

$$u \in L^p(0, T; W_0^{1,p}(\Omega)), \quad \text{for all } T > 0,$$

$$\left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \frac{d}{dt} (u(t), v) + (b(\nabla u(t)), \nabla v) + \frac{|Y^*|}{|Y|} \kappa(|u(t)|^{p-2}u(t) + f(u(t)), v) + \frac{|\partial F|}{|Y|} (g(u(t)), v) = 0, \quad \text{in } \mathcal{D}'(0, T),$$

for all $v \in W_0^{1,p}(\Omega)$, and

$$u(0) = u_0.$$

Due to the structure properties of b (see [13, Lemmas 2.10-2.11-2.13]), applying a slight modification of [20, Chapter 2, Theorem 1.4], we obtain that the problem (13) has a unique solution, and therefore Theorem 1.1 is proved.

References

- [1] R. Adams and J. Fournier, Sobolev spaces, Vol. 140, 2nd Edition, Academic Press, Elsevier, 2003.
- [2] M. Anguiano, Existence, uniqueness and homogenization of nonlinear parabolic problems with dynamical boundary conditions in perforated media, *Mediterr. J. Math.* (2020) 17:18.
- [3] M. Anguiano, Homogenization of parabolic problems with dynamical boundary conditions of reactive-diffusive type in perforated media, *Z. Angew. Math. Mech.* 2020; e202000088. <https://doi.org/10.1002/zamm.202000088>
- [4] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Springer-Verlag New York, 2011.
- [5] D. Cioranescu, A. Dalmlamian and G. Griso, The periodic Unfolding Method in Homogenization, *SIAM Journal on Mathematical Analysis*, 40, No. 4 (2008) 1585-1620.
- [6] D. Cioranescu and P. Donato, Homogénéisation du problème de Neumann non homogène dans des ouverts perforés, *Asymptotic Analysis*, 1, (1988) 115-138.
- [7] D. Cioranescu and P. Donato, An Introduction to Homogenization, Oxford Lectures Series in Mathematics and its Applications, 17, New York, 1999.
- [8] D. Cioranescu and F. Murat, Un terme étrange venu d'ailleurs, II, *Nonlinear partial differential equations and their applications*, Collège de France Seminar, vol. III (Paris 1980/1981), pp. 154-178, Res. Notes in Math., 70, Pitman, Boston, MA, 1982.
- [9] D. Cioranescu and J. Saint Jean Paulin, Homogenization in open sets with holes, *J. Math. Anal. Appl.*, 71, (1979) 590-607.
- [10] C. Conca, J.I. Díaz, A. Liñán and C. Timofte, Homogenization in chemical reactive flows, *Electronic Journal of Differential Equations*. 40 (2004), 1-22.
- [11] J.I. Díaz, D. Gómez-Castro, A.V. Podol'skii and T.A. Shaposhnikova, On the asymptotic limit of the effectiveness of reaction-diffusion equations in periodically structured media, *J. Math. Anal. Appl.* 455 (2017) 1597-1613.

- [12] J.I. Díaz, D. Gómez-Castro, A.V. Podol'skii and T.A. Shaposhnikova, Non existence of critical scales in the homogenization of the problem with p -Laplace diffusion and nonlinear reaction in the boundary of periodically distributed particles in n -dimensional domains when $p > n$, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM* (2018) 112, 331-340.
- [13] P. Donato and G. Moscarillo, On the homogenization of some nonlinear problems in perforated domains, *Rendiconti del Seminario Matematico della Università di Padova*, 84 (1990) 91-108.
- [14] C.G. Gal and J. Shomberg, Coleman-Gurtin type equations with dynamic boundary conditions, *Phys. D.* 292, (2015) 29-45.
- [15] G.R. Goldstein, Derivation and physical interpretation of general boundary conditions, *Adv. Differential Equations*, 11, No. 4 (2006) 457-480.
- [16] D. Gómez, M. Lobo, E. Pérez, A.V. Podol'skii and T.A. Shaposhnikova, Homogenization for the p -Laplace operator and nonlinear Robin boundary conditions in perforated media along $(n-1)$ -dimensional manifolds, *Doklady Mathematics* 89(1) (2014) 11-15.
- [17] D. Gómez, M. Lobo, E. Pérez, A.V. Podol'skii and T.A. Shaposhnikova, Unilateral problems for the p -Laplace operator in perforated media involving large parameters, *ESAIM Control Opti. Calc. Var.* 24(3) (2018) 921-964.
- [18] D. Gómez, M.E. Pérez, A.V. Podol'skii and T.A. Shaposhnikova, Homogenization for the p -Laplace operator in perforated media with nonlinear restrictions on the boundary of the perforations: a critical case, *Doklady Mathematics*, 92(1) (2015) 433-438.
- [19] N. Labani and C. Picard, Homogenization of a nonlinear Dirichlet problem in a periodically perforated domain, in *Recent advances in nonlinear elliptic and parabolic problems*, Pitman Res. Notes Math. Ser. 208, Longman Sci. Tech. (1989).
- [20] J.L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non linéaires*, Dunod, 1969.
- [21] J. Nečas, *Direct methods in the theory of elliptic equations*, Springer-Verlag Berlin Heidelberg, 2012.
- [22] J.C. Robinson, *Infinite-dimensional dynamical systems*. Cambridge University Press, 2001.
- [23] T.A. Shaposhnikova and A.V. Podol'skii, Homogenization limit for the boundary value problem with the p -Laplace operator and a nonlinear third boundary condition on the boundary of the holes in a perforated domain, *Funct. Differ. Equ.* 19 (2012) 351-370.
- [24] L. Tartar, *Problèmes d'homogénéisation dans les équations aux dérivées partielles*, Cours Peccot Collège de France, 1977.
- [25] M. Vanninathan, Homogenization of eigenvalues problems in perforated domains, *Proc. Indian Acad. of Sciences*, 90, (1981) 239-271.