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► **To cite this version:**

María Anguiano. On p -Laplacian reaction-diffusion problems with dynamical boundary conditions in perforated media. 2020. hal-02881568v1

HAL Id: hal-02881568

<https://hal.science/hal-02881568v1>

Preprint submitted on 25 Jun 2020 (v1), last revised 13 Aug 2023 (v3)

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On p -Laplacian reaction-diffusion problems with dynamical boundary conditions in perforated media

María ANGUIANO¹

Abstract

This paper deals with the homogenization of the p -Laplacian reaction-diffusion problems in a domain containing periodically distributed holes of size ε , with a dynamical boundary condition of *pure-reactive* type. We generalize our previous results (see [2]) established in the case where the diffusion is modeled by the Laplacian operator, i.e., with $p = 2$. We prove the convergence of the homogenization process to a nonlinear p -Laplacian reaction-diffusion equation defined on a unified domain without holes with zero Dirichlet boundary condition and with extra terms coming from the influence of the nonlinear dynamical boundary conditions.

AMS classification numbers: 35B27, 35K57

Keywords: Homogenization, perforated media, reaction-diffusion systems, dynamical boundary conditions, p -Laplacian

1 Introduction and setting of the problem

Homogenization problems in perforated media for the p -Laplacian operator have been considered in the literature over the last decades. The homogenization of the equation

$$-\operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) = f \tag{1}$$

in a periodically perforated domain is considered by Labani and Picard in [19] with Dirichlet boundary conditions. Such a problem is a generalization of the linear problem for Laplace's equation, which corresponds to $p = 2$, and was studied by Cioranescu and Murat in [8]. Donato and Moscarriello in [13] study the homogenization of a class of nonlinear elliptic Neumann problems in perforated domains of \mathbb{R}^N . As a consequence of [13], we are able in particular to describe the homogenization of (1) with Neumann boundary conditions. This result is a generalization of earlier related works, for instance, Cioranescu and Donato [6] and Cioranescu and Saint Jean Paulin [9]. In [23] Shaposhnikova and Podol'skii study the homogenization of (1) in an ε -periodically perforated domain with a nonlinear boundary condition. In [11] Díaz *et al.* consider (1) with a nonlinear perturbed Robin-type boundary condition in an ε periodically perforated domain where the size of the particles is smaller than the period ε , and the asymptotic behavior of the solution is studied as $\varepsilon \rightarrow 0$. The closest articles to this one in the literature are [16, 17, 18], where Gómez *et al.* consider the case $2 < p \leq N$, and [12], where Díaz *et al.* study the case $p > N$. As far as we know, in the previous literature there is no study for the homogenization of p -Laplacian parabolic models associated with nonlinear dynamical boundary conditions in a periodically perforated domain, as we consider in this article.

In the context of reaction-diffusion equations, dynamical boundary conditions have been rigorously derived in Gal and Shomberg [14] based on first and second thermodynamical principles and their physical interpretation was also given in Goldstein [15]. In a recent article (see [2]) we addressed the problem of the homogenization of the reaction-diffusion equations with a dynamical boundary condition of *pure-reactive* type in a domain perforated with holes. More recently, in [3] we generalize this previous study to the case of a dynamical boundary condition of *reactive-diffusive* type, i.e., we add to the dynamical boundary condition a Laplace-Beltrami correction term.

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In the present article, we consider p -Laplacian parabolic problems with nonlinear dynamical boundary conditions of *pure-reactive* type occurring as generalizations of semilinear reaction-diffusion equations with dynamical boundary conditions. Let us introduce the model we will be involved with in this article.

The geometrical setting. Let Ω be a bounded connected open set in \mathbb{R}^N ($N \geq 2$), with smooth enough boundary $\partial\Omega$. Let us introduce a set of periodically distributed holes. As a result, we obtain an open set Ω_ε , where ε represents a small parameter related to the characteristic size of the holes.

Let $Y = [0, 1]^N$ be the representative cell in \mathbb{R}^N and F an open subset of Y with smooth enough boundary ∂F , such that $\bar{F} \subset Y$. We denote $Y^* = Y \setminus \bar{F}$. For $k \in \mathbb{Z}^N$ and $\varepsilon \in (0, 1]$, each cell $Y_{k,\varepsilon} = \varepsilon k + \varepsilon Y$ is similar to the unit cell Y rescaled to size ε and $F_{k,\varepsilon} = \varepsilon k + \varepsilon F$ is similar to F rescaled to size ε . We denote $Y_{k,\varepsilon}^* = Y_{k,\varepsilon} \setminus \bar{F}_{k,\varepsilon}$. We denote by F_ε the set of all the holes contained in Ω , i.e. $F_\varepsilon = \cup_{k \in K} \{F_{k,\varepsilon} : \bar{F}_{k,\varepsilon} \subset \Omega\}$, where $K := \{k \in \mathbb{Z}^N : Y_{k,\varepsilon} \cap \Omega \neq \emptyset\}$.

Let $\Omega_\varepsilon = \Omega \setminus \bar{F}_\varepsilon$. By this construction, Ω_ε is a periodically perforated domain with holes of the same size as the period. Remark that the holes do not intersect the boundary $\partial\Omega$. Let $\partial F_\varepsilon = \cup_{k \in K} \{\partial F_{k,\varepsilon} : \bar{F}_{k,\varepsilon} \subset \Omega\}$. So

$$\partial\Omega_\varepsilon = \partial\Omega \cup \partial F_\varepsilon.$$

Position of the problem. The prototype of the p -Laplacian parabolic initial-boundary value problems that we consider in this article is

$$\left\{ \begin{array}{ll} \frac{\partial u_\varepsilon}{\partial t} - \Delta_p u_\varepsilon + \kappa |u_\varepsilon|^{p-2} u_\varepsilon = -f(u_\varepsilon) & \text{in } \Omega_\varepsilon \times (0, T), \\ \partial_{\nu_p} u_\varepsilon + \varepsilon \frac{\partial u_\varepsilon}{\partial t} = -\varepsilon g(u_\varepsilon) & \text{on } \partial F_\varepsilon \times (0, T), \\ u_\varepsilon = 0, & \text{on } \partial\Omega \times (0, T), \\ u_\varepsilon(x, 0) = u_\varepsilon^0(x), & \text{for } x \in \Omega_\varepsilon, \\ u_\varepsilon(x, 0) = \psi_\varepsilon^0(x), & \text{for } x \in \partial F_\varepsilon, \end{array} \right. \quad (2)$$

where $u_\varepsilon = u_\varepsilon(x, t)$, $x \in \Omega_\varepsilon$, $t \in (0, T)$, with $T > 0$, and $\kappa > 0$ is a given constant. Here the diffusion is modeled by the p -Laplacian operator $\Delta_p u_\varepsilon := \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon)$ with $p \in [2, N]$. Notice that $p = 2$ corresponds to the linear diffusion operator. The ‘‘normal derivative’’ must be understood as $\partial_{\nu_p} u_\varepsilon = |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nu$, where ν denotes the outward normal to ∂F_ε . The boundary equation (2)₂ is multiplied by ε to compensate the growth of the surface by shrinking ε , where the value of u_ε is assumed to be the trace of the function u_ε defined for $x \in \Omega_\varepsilon$. We also assume that the functions f and $g \in C(\mathbb{R})$ are given, and satisfy that there exist exponents q_1 and q_2 , such that

$$p \leq q_1 < +\infty, \text{ if } p = N \quad \text{and} \quad 2 \leq q_1 \leq \frac{Np}{N-p}, \text{ if } p \in [2, N), \quad (3)$$

$$2 \leq q_2 < +\infty, \text{ if } p = N \quad \text{and} \quad 2 \leq q_2 \leq \frac{(N-1)p}{N-p}, \text{ if } p \in [2, N), \quad (4)$$

and constants $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta > 0$, and $l > 0$, such that

$$\alpha_1 |s|^{q_1} - \beta \leq f(s)s \leq \alpha_2 |s|^{q_1} + \beta, \quad \text{for all } s \in \mathbb{R}, \quad (5)$$

$$\alpha_1 |s|^{q_2} - \beta \leq g(s)s \leq \alpha_2 |s|^{q_2} + \beta, \quad \text{for all } s \in \mathbb{R}, \quad (6)$$

$$(f(s_1) - f(s_2))(s_1 - s_2) \geq -l(s_1 - s_2)^2, \quad \text{for all } s_1, s_2 \in \mathbb{R}, \quad (7)$$

and

$$(g(s_1) - g(s_2))(s_1 - s_2) \geq -l(s_1 - s_2)^2, \quad \text{for all } s_1, s_2 \in \mathbb{R}. \quad (8)$$

Finally, we also assume that

$$u_\varepsilon^0 \in L^2(\Omega), \quad \psi_\varepsilon^0 \in L^2(\partial F_\varepsilon), \quad (9)$$

are given, and we suppose that

$$|u_\varepsilon^0|_{\Omega_\varepsilon}^2 + \varepsilon |\psi_\varepsilon^0|_{\partial F_\varepsilon}^2 \leq C, \quad (10)$$

where C is a positive constant, and we denote by $|\cdot|_{\Omega_\varepsilon}$ and $|\cdot|_{\partial F_\varepsilon}$ the norm in $L^2(\Omega_\varepsilon)$ and $L^2(\partial F_\varepsilon)$, respectively.

In [2], we consider the homogenization of the problem (2) with $p = 2$ and we obtain rigorously a nonlinear parabolic problem with zero Dirichlet boundary condition and with extra-terms coming from the influence of the dynamical boundary conditions as the homogenized model. We would like to highlight that analyzing a p -Laplacian problem involves additional (nontrivial) difficulties compared to the study for the Laplacian. Due to the presence of p -Laplacian operator in the domain, the variational formulation of the p -Laplacian reaction-diffusion equation is different than in [2]. We have to work in the space

$$V_p := \{(v, \gamma_0(v)) : v \in W^{1,p}(\Omega_\varepsilon), \gamma_0(v) = 0 \text{ on } \partial\Omega\}, \quad (11)$$

where γ_0 denotes the trace operator $v \in W^{1,p}(\Omega_\varepsilon) \mapsto v|_{\partial\Omega_\varepsilon} \in W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)$. New $W^{1,p}$ -estimates are needed to deal with problem (2). To prove these estimates rigorously, we use the Galerkin approximations where we have to introduce a special basis consisting of functions in the space

$$\mathcal{V}_s := \{(v, \gamma_0(v)) : v \in W^{s,2}(\Omega_\varepsilon), \gamma_0(v) = 0 \text{ on } \partial\Omega\}, \quad s \geq \frac{N(p-2)}{2p} + 1,$$

in the sense of Lions [20, p. 161]. Therefore, thanks to the assumption made on s , we have $\mathcal{V}_s \subset V_p$. We use the so-called energy method introduced by Tartar [24] and considered by many authors (see, for instance, Cioranescu and Donato [6]). Finally, in order to identify the limit equation, it is necessary to use monotonicity arguments. We prove the following theorem:

Theorem 1.1 (Main Theorem). *Under the assumptions (3), (5)–(8) and (10), assume that $g \in C^1(\mathbb{R})$, the exponent q_2 satisfies that*

$$p \leq q_2 < +\infty \text{ if } p = N \quad \text{and} \quad 2 \leq q_2 \leq \frac{(N-1)p}{N-p} \text{ if } p \in [2, N), \quad (12)$$

and $(u_\varepsilon^0, \psi_\varepsilon^0) \in V_p$. Let $(u_\varepsilon, \psi_\varepsilon)$ be the unique solution of the problem (2), where $\psi_\varepsilon(t) = \gamma_0(u_\varepsilon(t))$ a.e. $t \in (0, T]$. Then, as $\varepsilon \rightarrow 0$, we have

$$\tilde{u}_\varepsilon(t) \rightarrow u(t) \quad \text{strongly in } L^p(\Omega), \quad \forall t \in [0, T],$$

where $\tilde{\cdot}$ denotes the $W^{1,p}$ -extension to $\Omega \times (0, T)$ and u is the unique solution of the following problem

$$\begin{cases} \left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \frac{\partial u}{\partial t} - \operatorname{div} b(\nabla u) + \frac{|Y^*|}{|Y|} (\kappa |u|^{p-2} u + f(u)) + \frac{|\partial F|}{|Y|} g(u) = 0, & \text{in } \Omega \times (0, T), \\ u(x, 0) = u_0(x), & \text{for } x \in \Omega, \\ u = 0, & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (13)$$

For any $\zeta \in \mathbb{R}^N$, if $w(y)$ is the solution of the problem

$$\begin{cases} \int_{Y^*} |\nabla_y w(y)|^{p-2} \nabla_y w(y) \cdot \nabla_y \varphi(y) dy = 0 & \forall \varphi \in \mathbb{H}_{\text{per}}(Y^*), \\ w \in \zeta \cdot y + \mathbb{H}_{\text{per}}(Y^*), \end{cases} \quad (14)$$

then b is defined by

$$b(\zeta) = \frac{1}{|Y|} \int_{Y^*} |\nabla v|^{p-2} \nabla v dy, \quad (15)$$

where $\mathbb{H}_{\text{per}}(Y^*)$ is the space of functions from $W^{1,p}(Y^*)$ which has the same trace on the opposite faces of Y .

Remark 1.2. Note that in the case $p = 2$ (i.e., if the diffusion is modeled by the Laplacian operator), the homogenized equation (13) is exactly the equation obtained in [2].

Remark 1.3. In this setting, a typical nonlinearity in the applications is a odd degree polynomial, $\sum_{j=0}^{2k+1} c_j s^j$, where $c_{2k+1} > 0$.

The article is organized as follows. In Section 2, we introduce suitable functions spaces for our considerations. We recall here some well-know facts on the Sobolev spaces $W^{s,p}$. To prove the main result, in Section 3 we prove the existence and uniqueness of solution of (2), *a priori* estimates are established in Section 4 and some compactness results are proved in Section 5. Finally, the proof of Theorem 1.1 is established in Section 6.

2 Functional setting

Notation: We denote by $(\cdot, \cdot)_{\Omega_\varepsilon}$ (respectively, $(\cdot, \cdot)_{\partial F_\varepsilon}$) the inner product in $L^2(\Omega_\varepsilon)$ (respectively, in $L^2(\partial F_\varepsilon)$), and by $|\cdot|_{\Omega_\varepsilon}$ (respectively, $|\cdot|_{\partial F_\varepsilon}$) the associated norm. We also denote by $(\cdot, \cdot)_{\Omega_\varepsilon}$ the inner product in $(L^2(\Omega_\varepsilon))^N$.

If $r \neq 2$, we will also denote by $(\cdot, \cdot)_{\Omega_\varepsilon}$ (respectively, $(\cdot, \cdot)_{\partial F_\varepsilon}$) the duality product between $L^{r'}(\Omega_\varepsilon)$ and $L^r(\Omega_\varepsilon)$ (respectively, the duality product between $L^{r'}(\partial F_\varepsilon)$ and $L^r(\partial F_\varepsilon)$). We will denote by $|\cdot|_{r, \Omega_\varepsilon}$ (respectively $|\cdot|_{r, \partial F_\varepsilon}$) the norm in $L^r(\Omega_\varepsilon)$ (respectively in $L^r(\partial F_\varepsilon)$).

We denote by $(\cdot, \cdot)_\Omega$ the inner product in $L^2(\Omega)$, and by $|\cdot|_\Omega$ the associated norm. If $r \neq 2$, we will also denote by $(\cdot, \cdot)_\Omega$ the duality product between $L^{r'}(\Omega)$ and $L^r(\Omega)$. We will denote by $|\cdot|_{r, \Omega}$ the norm in $L^r(\Omega)$.

The Sobolev spaces $W^{s,p}(\Omega_\varepsilon)$:

We recall here, for the reader's convenience, some well-know facts on the Sobolev spaces $W^{s,p}$. We refer to Adams and Fournier [1], Brezis [4, Chapter 9] and Nečas [21, Chapter 2] for more details and proofs.

For any positive integer s and $p \geq 1$, we define the Sobolev space $W^{s,p}(\Omega_\varepsilon)$ to be the completion of $C^s(\overline{\Omega_\varepsilon})$, with respect to the norm

$$\|v\|_{s,p,\Omega_\varepsilon} = \left(\sum_{0 \leq |\alpha| \leq s} |D^\alpha v|_{p,\Omega_\varepsilon}^p \right)^{1/p}.$$

Observe that $W^{s,p}(\Omega_\varepsilon)$ is a Banach space.

By the Sobolev Imbedding Theorem (see [1, Chapter 4, p. 99]), we have the imbedding

$$W^{s,p}(\Omega_\varepsilon) \subset W^{1,r}(\Omega_\varepsilon), \quad (16)$$

where $s \geq 1$ and $\frac{1}{p} - \frac{s-1}{N} \leq \frac{1}{r} \leq \frac{1}{p}$.

In particular, for $p \geq 1$, we define the Sobolev space $W^{1,p}(\Omega_\varepsilon)$ to be the completion of $C^1(\overline{\Omega_\varepsilon})$, with respect to the norm

$$\|v\|_{p,\Omega_\varepsilon} := \left(|v|_{p,\Omega_\varepsilon}^p + |\nabla v|_{p,\Omega_\varepsilon}^p \right)^{1/p}.$$

We set $H^1(\Omega_\varepsilon) = W^{1,2}(\Omega_\varepsilon)$.

By $\|\cdot\|_{p,\Omega_\varepsilon,T}$ we denote the norm in $L^p(0, T; W^{1,p}(\Omega_\varepsilon))$. By $|\cdot|_{r,\Omega_\varepsilon,T}$ (respectively, $|\cdot|_{r,\partial F_\varepsilon,T}$), we denote the norm in $L^r(0, T; L^r(\Omega_\varepsilon))$ (respectively, $L^r(0, T; L^r(\partial F_\varepsilon))$).

By $\|\cdot\|_{p,\Omega}$ we denote the norm in $W^{1,p}(\Omega)$, by $\|\cdot\|_{p,\Omega,T}$ we denote the norm in $L^p(0, T; W^{1,p}(\Omega))$ and we denote by $|\cdot|_{r,\Omega,T}$ the norm in $L^r(0, T; L^r(\Omega))$.

We define

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p = N. \end{cases}$$

We have the continuous embedding

$$W^{1,p}(\Omega_\varepsilon) \subset \begin{cases} L^r(\Omega_\varepsilon) & \text{if } p < N \text{ and } r = p^*, \\ L^r(\Omega_\varepsilon) & \text{if } p = N \text{ and } p \leq r < p^*. \end{cases} \quad (17)$$

By Rellich-Kondrachov Theorem (see [4, Chapter 9, Theorem 9.16]), we have the compact embedding

$$W^{1,p}(\Omega_\varepsilon) \subset \begin{cases} L^r(\Omega_\varepsilon) & \text{if } p < N \text{ and } 1 \leq r < p^*, \\ L^r(\Omega_\varepsilon) & \text{if } p = N \text{ and } p \leq r < p^*. \end{cases} \quad (18)$$

In particular, we have the compact embedding

$$W^{1,p}(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon), \quad \forall 2 \leq p \leq N. \quad (19)$$

One can define a family of spaces intermediate between L^p and $W^{1,p}$. More precisely for $p \geq 1$ we define the fractional order Sobolev space

$$W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) := \left\{ v \in L^p(\partial\Omega_\varepsilon); \frac{|v(x) - v(y)|}{|x - y|^{1-\frac{1}{p} + \frac{N}{p}}} \in L^p(\partial\Omega_\varepsilon \times \partial\Omega_\varepsilon) \right\},$$

equipped with the natural norm. We set $H^{1/2}(\partial\Omega_\varepsilon) = W^{\frac{1}{2},2}(\partial\Omega_\varepsilon)$. These spaces play an important role in the theory of traces.

We denote by γ_0 the trace operator $v \mapsto v|_{\partial\Omega_\varepsilon}$. The trace operator belongs to $\mathcal{L}(W^{1,p}(\Omega_\varepsilon), W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon))$, and we will use $\|\gamma_0\|$ to denote the norm of γ_0 in this space.

We will use $\|\cdot\|_{p,\partial\Omega_\varepsilon}$ to denote the norm in $W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)$, which is given by

$$\|\phi\|_{p,\partial\Omega_\varepsilon} = \inf\{\|v\|_{p,\Omega_\varepsilon} : \gamma_0(v) = \phi\}.$$

We define

$$p_b^* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p = N. \end{cases}$$

We have the continuous embedding

$$W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) \subset \begin{cases} L^r(\partial\Omega_\varepsilon) & \text{if } p < N \quad \text{and} \quad r = p_b^*, \\ L^r(\partial\Omega_\varepsilon) & \text{if } p = N \quad \text{and} \quad 1 \leq r < p_b^*. \end{cases} \quad (20)$$

By [21, Chaper 2, Theorem 6.2], we have the compact embedding

$$W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) := \gamma_0(W^{1,p}(\Omega_\varepsilon)) \subset L^r(\partial\Omega_\varepsilon) \quad \text{if } 1 \leq r < p_b^*. \quad (21)$$

In particular, we have the compact embedding

$$W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) \subset L^2(\partial\Omega_\varepsilon) \quad \forall 2 \leq p \leq N. \quad (22)$$

Some notations: Finally, for any positive integer s and $p \geq 1$, we denote by $W_{\partial\Omega}^{s,p}(\Omega_\varepsilon)$ and $W_{\partial\Omega}^{s,p}(\partial\Omega_\varepsilon)$ the standard Sobolev spaces which are closed subspaces of $W^{s,p}(\Omega_\varepsilon)$ and $W^{s,p}(\partial\Omega_\varepsilon)$, respectively, and the subscript $\partial\Omega$ means that, respectively, traces or functions in $\partial\Omega_\varepsilon$, vanish on this part of the boundary of Ω_ε , i.e.

$$W_{\partial\Omega}^{s,p}(\Omega_\varepsilon) = \{v \in W^{s,p}(\Omega_\varepsilon) : \gamma_0(v) = 0 \text{ on } \partial\Omega\},$$

and

$$W_{\partial\Omega}^{s,p}(\partial\Omega_\varepsilon) = \{v \in W^{s,p}(\partial\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\}.$$

Let us notice that, in fact, we can consider an element of $W^{1-\frac{1}{p},p}(\partial F_\varepsilon)$ as an element of $W_{\partial\Omega}^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)$.

Analogously, for $r \geq 2$, we denote

$$L_{\partial\Omega}^r(\partial\Omega_\varepsilon) := \{v \in L^r(\partial\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\}.$$

Let us notice that, in fact, we can consider the given ψ_ε^0 as an element of $L_{\partial\Omega}^2(\partial\Omega_\varepsilon)$.

Let us consider the space

$$H_r := L^r(\Omega_\varepsilon) \times L_{\partial\Omega}^r(\partial\Omega_\varepsilon), \quad \forall r \geq 2,$$

with the natural inner product $((v, \phi), (w, \varphi))_{H_r} = (v, w)_{\Omega_\varepsilon} + \varepsilon(\phi, \varphi)_{\partial F_\varepsilon}$, which in particular induces the norm $|(\cdot, \cdot)|_{H_r}$ given by

$$|(v, \phi)|_{H_r}^r = |v|_{r,\Omega_\varepsilon}^r + \varepsilon|\phi|_{r,\partial F_\varepsilon}^r, \quad (v, \phi) \in H_r.$$

For the sake of clarity, we shall omit to write explicitly the index r if $r = 2$, so we denote by H the Hilbert space

$$H := L^2(\Omega_\varepsilon) \times L_{\partial\Omega}^2(\partial\Omega_\varepsilon).$$

For each $p > 1$, we consider the space

$$V_p := \left\{ (v, \gamma_0(v)) : v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon) \right\}.$$

We note that V_p is a closed vector subspace of $W_{\partial\Omega}^{1,p}(\Omega_\varepsilon) \times W_{\partial\Omega}^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)$, and we endow it with the norm $\|(\cdot, \cdot)\|_{V_p}$ given by

$$\|(v, \gamma_0(v))\|_{V_p}^p = \|v\|_{p,\Omega_\varepsilon}^p + \|\gamma_0(v)\|_{p,\partial F_\varepsilon}^p, \quad (v, \gamma_0(v)) \in V_p.$$

Moreover, V_p is a reflexive and separable space.

Let p' , q'_1 and q'_2 be the conjugate exponents of p , q_1 and q_2 , respectively. Taking into account the continuous embeddings (17) and (20), and the assumptions (3)-(4), we have the following useful continuous inclusions

$$V_p \subset W^{1,p}(\Omega_\varepsilon) \subset L^{q_1}(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon), \quad V_p \subset W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) \subset L^{q_2}(\partial\Omega_\varepsilon) \subset L^2(\partial\Omega_\varepsilon), \quad (23)$$

and

$$L^2(\Omega_\varepsilon) \subset L^{q'_1}(\Omega_\varepsilon) \subset (W^{1,p}(\Omega_\varepsilon))' \subset V'_p, \quad (24)$$

where $(W^{1,p}(\Omega_\varepsilon))'$ and V'_p denote the dual of $W^{1,p}(\Omega_\varepsilon)$ and V_p , respectively. Note that $(W^{1,p}(\Omega_\varepsilon))'$ is a subspace of $W^{-1,p'}(\Omega_\varepsilon)$, where $W^{-1,p'}(\Omega_\varepsilon)$ denotes the dual of the Sobolev space $W_0^{1,p}(\Omega_\varepsilon) := \overline{\mathcal{D}(\Omega_\varepsilon)}^{W^{1,p}(\Omega_\varepsilon)}$.

Taking into account the compact embeddings (19) and (22), we have the compact embedding

$$V_p \subset H \quad \forall 2 \leq p \leq N. \quad (25)$$

Finally, for $s \geq 1$, we consider the space

$$\mathcal{V}_s := \left\{ (v, \gamma_0(v)) : v \in W_{\partial\Omega}^{s,2}(\Omega_\varepsilon) \right\}.$$

We note that \mathcal{V}_s is a closed vector subspace of $W_{\partial\Omega}^{s,2}(\Omega_\varepsilon) \times W_{\partial\Omega}^{s-\frac{1}{2},2}(\partial\Omega_\varepsilon)$.

Observe that by (16), we have that $W^{s,2}(\Omega_\varepsilon) \subset W^{1,2}(\Omega_\varepsilon)$ and by Rellich's Theorem (see [21, Chapter 1, Theorem 1.4]), we obtain the compact embedding

$$W^{1,2}(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon),$$

and by [21, Chapter 1, Exercise 1.2], we have the compact embedding

$$W^{s-\frac{1}{2},2}(\partial\Omega_\varepsilon) := \gamma_0(W^{s,2}(\Omega_\varepsilon)) \subset L^2(\partial\Omega_\varepsilon).$$

Thus, we can deduce the compact embedding

$$\mathcal{V}_s \subset H. \quad (26)$$

3 Existence and uniqueness of solution

Along this paper, we shall denote by C different constants which are independent of ε . We state in this section a result on the existence and uniqueness of solution of problem (2). It is easy to see from (5) and (6) that there exists a constant $C > 0$ such that

$$|f(s)| \leq C \left(1 + |s|^{q_1-1}\right), \quad |g(s)| \leq C \left(1 + |s|^{q_2-1}\right), \quad \text{for all } s \in \mathbb{R}. \quad (27)$$

Definition 3.1. A weak solution of (2) is a pair of functions $(u_\varepsilon, \psi_\varepsilon)$, satisfying

$$u_\varepsilon \in C([0, T]; L^2(\Omega_\varepsilon)), \quad \psi_\varepsilon \in C([0, T]; L^2_{\partial\Omega}(\partial\Omega_\varepsilon)), \quad \text{for all } T > 0, \quad (28)$$

$$u_\varepsilon \in L^p(0, T; W^{1,p}(\Omega_\varepsilon)), \quad \text{for all } T > 0, \quad (29)$$

$$\psi_\varepsilon \in L^p(0, T; W_{\partial\Omega}^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)), \quad \text{for all } T > 0, \quad (30)$$

$$\gamma_0(u_\varepsilon(t)) = \psi_\varepsilon(t), \quad \text{a.e. } t \in (0, T], \quad (31)$$

$$\left\{ \begin{array}{l} \frac{d}{dt}(u_\varepsilon(t), v)_{\Omega_\varepsilon} + \varepsilon \frac{d}{dt}(\psi_\varepsilon(t), \gamma_0(v))_{\partial F_\varepsilon} + (|\nabla u_\varepsilon(t)|^{p-2} \nabla u_\varepsilon(t), \nabla v)_{\Omega_\varepsilon} + \kappa(|u_\varepsilon(t)|^{p-2} u_\varepsilon(t), v)_{\Omega_\varepsilon} \\ + (f(u_\varepsilon(t)), v)_{\Omega_\varepsilon} + \varepsilon (g(\psi_\varepsilon(t)), \gamma_0(v))_{\partial F_\varepsilon} = 0 \\ \text{in } \mathcal{D}'(0, T), \text{ for all } v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon), \end{array} \right. \quad (32)$$

$$u_\varepsilon(0) = u_\varepsilon^0, \quad \text{and} \quad \psi_\varepsilon(0) = \psi_\varepsilon^0. \quad (33)$$

We have the following result.

Theorem 3.2. Under the assumptions (3)–(9), there exists a unique solution $(u_\varepsilon, \psi_\varepsilon)$ of the problem (2). Moreover, this solution satisfies the energy equality

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (|(u_\varepsilon(t), \psi_\varepsilon(t))|_H^2) + |\nabla u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + \kappa |u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p \\ & + (f(u_\varepsilon(t)), u_\varepsilon(t))_{\Omega_\varepsilon} + \varepsilon (g(\psi_\varepsilon(t)), \psi_\varepsilon(t))_{\partial F_\varepsilon} = 0, \quad \text{a.e. } t \in (0, T). \end{aligned} \quad (34)$$

Proof. The proof of this result is standard. For the sake of completeness, we give a sketch of a proof.

First, we prove that V_p is densely embedded in H . In fact, if we consider $(w, \phi) \in H$ such that

$$(v, w)_{\Omega_\varepsilon} + \varepsilon (\gamma_0(v), \phi)_{\partial F_\varepsilon} = 0, \quad \text{for all } v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon),$$

in particular, we have

$$(v, w)_{\Omega_\varepsilon} = 0, \quad \text{for all } v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon),$$

and therefore $w = 0$. Consequently,

$$(\gamma_0(v), \phi)_{\partial F_\varepsilon} = 0, \quad \text{for all } v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon),$$

and then, as $W_{\partial\Omega}^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) = \gamma_0(W_{\partial\Omega}^{1,p}(\Omega_\varepsilon))$ is dense in $L^2_{\partial\Omega}(\partial\Omega_\varepsilon)$, we have that $\phi = 0$.

Now, on the space V_p we define the nonlinear monotone operator $A_p : V_p \rightarrow V'_p$, given by

$$\langle A_p((v, \gamma_0(v))), (w, \gamma_0(w)) \rangle = (|\nabla v|^{p-2} \nabla v, \nabla w)_{\Omega_\varepsilon} + \kappa(|v|^{p-2} v, w)_{\Omega_\varepsilon}, \quad (35)$$

for all $v, w \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon)$.

We observe that A_p is coercive. In fact, we have

$$\begin{aligned} \langle A_p((v, \gamma_0(v))), (v, \gamma_0(v)) \rangle & \geq \min\{1, \kappa\} \|v\|_{p, \Omega_\varepsilon}^p \\ & = \frac{1}{1 + \|\gamma_0\|^p} \min\{1, \kappa\} \|v\|_{p, \Omega_\varepsilon}^p + \frac{\|\gamma_0\|^p}{1 + \|\gamma_0\|^p} \min\{1, \kappa\} \|v\|_{p, \Omega_\varepsilon}^p \\ & \geq \frac{1}{1 + \|\gamma_0\|^p} \min\{1, \kappa\} \|(v, \gamma_0(v))\|_{V_p}^p, \end{aligned} \quad (36)$$

for all $v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon)$.

Let us denote

$$\begin{aligned} V_1 &= V_p, & V_2 &= L^{q_1}(\Omega_\varepsilon) \times L^2_{\partial\Omega}(\partial\Omega_\varepsilon), & V_3 &= L^2(\Omega_\varepsilon) \times L^{q_2}_{\partial\Omega}(\partial\Omega_\varepsilon), \\ A_1 &= A_p, & A_2(v, \phi) &= (f(v), 0), & A_3(v, \phi) &= (0, \varepsilon g(\phi)). \end{aligned}$$

From (27) one deduces that $A_i : V_i \rightarrow V'_i$ for $i = 2, 3$.

With this notation, and denoting $V = \bigcap_{i=1}^3 V_i$, $p_1 = p$, $p_2 = q_1$, $p_3 = q_2$, $\vec{u}_\varepsilon = (u_\varepsilon, \psi_\varepsilon)$, one has that (28)–(33) is equivalent to

$$\vec{u}_\varepsilon \in C([0, T]; H), \quad \vec{u}_\varepsilon \in \bigcap_{i=1}^3 L^{p_i}(0, T; V_i), \quad \text{for all } T > 0, \quad (37)$$

$$(\vec{u}_\varepsilon)'(t) + \sum_{i=1}^3 A_i(\vec{u}_\varepsilon(t)) = 0 \quad \text{in } \mathcal{D}'(0, T; V'), \quad (38)$$

$$\vec{u}_\varepsilon(0) = (u_\varepsilon^0, \psi_\varepsilon^0). \quad (39)$$

Taking into account (23)–(24) and applying a slight modification of [20, Ch.2, Th.1.4], it is not difficult to see that problem (37)–(39) has a unique solution. Moreover, \vec{u}_ε satisfies the energy equality

$$\frac{1}{2} \frac{d}{dt} |\vec{u}_\varepsilon(t)|_H^2 + \sum_{i=1}^3 \langle A_i(\vec{u}_\varepsilon(t)), \vec{u}_\varepsilon(t) \rangle_i = 0 \quad \text{a.e. } t \in (0, T),$$

where $\langle \cdot, \cdot \rangle_i$ denotes the duality product between V'_i and V_i . This last equality turns out to be just (34). \square

4 A priori estimates

In this section we obtain some energy estimates for the solution of (2). By (34) and taking into account (5)–(6), we have

$$\begin{aligned} \frac{d}{dt} (|(u_\varepsilon(t), \psi_\varepsilon(t))|_H^2) + 2|\nabla u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + 2\kappa|u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + 2\alpha_1|u_\varepsilon(t)|_{q_1, \Omega_\varepsilon}^{q_1} \\ + 2\alpha_1\varepsilon|\psi_\varepsilon(t)|_{q_2, \partial F_\varepsilon}^{q_2} \leq 2\beta(|\Omega_\varepsilon| + \varepsilon|\partial F_\varepsilon|), \end{aligned} \quad (40)$$

where $|\Omega_\varepsilon|$ and $|\partial F_\varepsilon|$ denotes the measure of Ω_ε and ∂F_ε , respectively.

Observe that the number of holes is given by

$$N(\varepsilon) = \frac{|\Omega|}{(2\varepsilon)^N} (1 + o(1)),$$

then using the change of variable

$$y = \frac{x}{\varepsilon}, \quad d\sigma(y) = \varepsilon^{-(N-1)} d\sigma(x),$$

we can deduce

$$|\partial F_\varepsilon| = N(\varepsilon)|\partial F_{k, \varepsilon}| = N(\varepsilon)\varepsilon^{N-1}|\partial F| \leq \frac{C}{\varepsilon}.$$

And since $|\Omega_\varepsilon| \leq |\Omega|$, we have that

$$|\Omega_\varepsilon| + \varepsilon|\partial F_\varepsilon| \leq C. \quad (41)$$

Let us denote

$$\mathcal{F}(s) := \int_0^s f(r) dr \quad \text{and} \quad \mathcal{G}(s) := \int_0^s g(r) dr.$$

Then, there exist positive constants $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, and $\tilde{\beta}$ such that

$$\tilde{\alpha}_1|s|^{q_1} - \tilde{\beta} \leq \mathcal{F}(s) \leq \tilde{\alpha}_2|s|^{q_1} + \tilde{\beta} \quad \forall s \in \mathbb{R}, \quad (42)$$

and

$$\tilde{\alpha}_1|s|^{q_2} - \tilde{\beta} \leq \mathcal{G}(s) \leq \tilde{\alpha}_2|s|^{q_2} + \tilde{\beta} \quad \forall s \in \mathbb{R}. \quad (43)$$

Lemma 4.1. *Assume the assumptions (3)–(8) and (10). Then, for any initial condition $(u_\varepsilon^0, \psi_\varepsilon^0) \in V_p$, there exists a constant C independent of ε , such that the solution $(u_\varepsilon, \psi_\varepsilon)$ of the problem (2) satisfies*

$$\|u_\varepsilon\|_{p, \Omega_\varepsilon, T} \leq C, \quad \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{p, \Omega_\varepsilon} \leq C. \quad (44)$$

Proof. Taking into account (41) in (40), we obtain

$$\frac{d}{dt} (|(u_\varepsilon(t), \psi_\varepsilon(t))|_H^2) + 2|\nabla u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + 2\kappa|u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + 2\alpha_1|u_\varepsilon(t)|_{q_1, \Omega_\varepsilon}^{q_1} + 2\alpha_1\varepsilon|\psi_\varepsilon(t)|_{q_2, \partial F_\varepsilon}^{q_2} \leq C. \quad (45)$$

From (35)–(36), in particular, we can deduce

$$\frac{d}{dt} (|(u_\varepsilon(t), \psi_\varepsilon(t))|_H^2) + \frac{2 \min\{1, \kappa\}}{1 + \|\gamma_0\|^p} \|(u_\varepsilon(t), \gamma_0(u_\varepsilon(t)))\|_{V_p}^p \leq C. \quad (46)$$

Integrating between 0 and t and taking into account (10), in particular, we have the first estimate in (44).

Now, if we want to take the inner product in the problem (2) with u'_ε , we need that $u'_\varepsilon \in L^p(0, T; W_{\partial\Omega}^{1,p}(\Omega_\varepsilon)) \cap L^{q_1}(0, T; L^{q_1}(\Omega_\varepsilon))$ with $\gamma_0(u'_\varepsilon) \in L^{q_2}(0, T; L^{q_2}(\partial\Omega_\varepsilon))$. However, we do not have it for our weak solution. Therefore, we use the Galerkin method in order to prove, rigorously, new *a priori* estimates for u_ε .

At first, we introduce a special basis consisting of functions $(w_j, \gamma_0(w_j)) \in \mathcal{V}_s$ with $s \geq \frac{N(p-2)}{2p} + 1$ in the sense of [20, Chapter 2, Remark 1.6, p. 161]. Therefore, thanks to the assumption made on s , taking into account (16), we have $\mathcal{V}_s \subset V_p$. The scalar product in H generates on $\mathcal{V}_s \subset H$ the bilinear functional $((v, \gamma_0(v)), (w, \gamma_0(w)))_H$ which can be represented in the form $((v, \gamma_0(v)), (w, \gamma_0(w)))_H = \langle L((v, \gamma_0(v))), (w, \gamma_0(w)) \rangle_{\mathcal{V}_s}$, where L is a self-adjoint operator. The compact embedding (26) implies the compactness of the operator L . Hence, L has a complete system of eigenvectors $\{(w_j, \gamma_0(w_j)) : j \geq 1\}$. These vectors are orthonormal in H and orthogonal in \mathcal{V}_s . Observe that $\text{span}\{(w_j, \gamma_0(w_j)) : j \geq 1\}$ is dense in V_p .

Taking into account the above facts, we denote by

$$(u_{\varepsilon, m}(t), \gamma_0(u_{\varepsilon, m}(t))) = (u_{\varepsilon, m}(t; 0, u_\varepsilon^0, \psi_\varepsilon^0), \gamma_0(u_{\varepsilon, m}(t; 0, u_\varepsilon^0, \psi_\varepsilon^0)))$$

the Galerkin approximation of the solution $(u_\varepsilon(t; 0, u_\varepsilon^0, \psi_\varepsilon^0), \gamma_0(u_\varepsilon(t; 0, u_\varepsilon^0, \psi_\varepsilon^0)))$ to (2) for each integer $m \geq 1$, which is given by

$$(u_{\varepsilon, m}(t), \gamma_0(u_{\varepsilon, m}(t))) = \sum_{j=1}^m \delta_{\varepsilon m j}(t)(w_j, \gamma_0(w_j)), \quad (47)$$

and is the solution of

$$\begin{aligned} & \frac{d}{dt} ((u_{\varepsilon, m}(t), \gamma_0(u_{\varepsilon, m}(t))), (w_j, \gamma_0(w_j)))_H + \langle A_p((u_{\varepsilon, m}(t), \gamma_0(u_{\varepsilon, m}(t))), (w_j, \gamma_0(w_j))) \\ & + (f(u_{\varepsilon, m}(t)), w_j)_{\Omega_\varepsilon} + \varepsilon(g(\gamma_0(u_{\varepsilon, m}(t))), \gamma_0(w_j))_{\partial F_\varepsilon} = 0, \quad j = 1, \dots, m, \end{aligned} \quad (48)$$

with initial data

$$(u_{\varepsilon, m}(0), \gamma_0(u_{\varepsilon, m}(0))) = (u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0)), \quad (49)$$

and

$$\delta_{\varepsilon m j}(t) = (u_{\varepsilon, m}(t), w_j)_{\Omega_\varepsilon} + (\gamma_0(u_{\varepsilon, m}(t)), \gamma_0(w_j))_{\partial F_\varepsilon},$$

and $(u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0)) \in \text{span}\{(w_j, \gamma_0(w_j)) : j = 1, \dots, m\}$ converge (when $m \rightarrow \infty$) to $(u_\varepsilon^0, \psi_\varepsilon^0)$ in a suitable sense which will be specified below.

Let $(u_\varepsilon^0, \psi_\varepsilon^0) \in V_p$. For all $m \geq 1$, since $\text{span}\{(w_j, \gamma_0(w_j)) : j \geq 1\}$ is densely embedded in V_p , there exists $(u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0)) \in \text{span}\{(w_j, \gamma_0(w_j)) : 1 \leq j \leq m\}$, such that the sequence $\{(u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0))\}$ converges to $(u_\varepsilon^0, \psi_\varepsilon^0)$ in V_p . Then, in particular we know that there exists a constant C such that

$$\|(u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0))\|_{V_p} \leq C. \quad (50)$$

For each integer $m \geq 1$, we consider the sequence $\{(u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))\}$ defined by (47)-(49) with these initial data.

Multiplying by the derivative $\delta'_{\varepsilon m j}$ in (48), and summing from $j = 1$ to m , we obtain

$$\begin{aligned} & |(u'_{\varepsilon,m}(t), \gamma_0(u'_{\varepsilon,m}(t)))|_H^2 + \frac{1}{p} \frac{d}{dt} (\langle A_p((u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))) \rangle, (u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))) \rangle) \\ & + (f(u_{\varepsilon,m}(t)), u'_{\varepsilon,m}(t))_{\Omega_\varepsilon} + \varepsilon (g(\gamma_0(u_{\varepsilon,m}(t))), \gamma_0(u'_{\varepsilon,m}(t)))_{\partial F_\varepsilon} = 0. \end{aligned} \quad (51)$$

We observe that

$$(f(u_{\varepsilon,m}(t)), u'_{\varepsilon,m}(t))_{\Omega_\varepsilon} = \frac{d}{dt} \int_{\Omega_\varepsilon} \mathcal{F}(u_{\varepsilon,m}(t)) dx,$$

and

$$(g(\gamma_0(u_{\varepsilon,m}(t))), \gamma_0(u'_{\varepsilon,m}(t)))_{\partial F_\varepsilon} = \frac{d}{dt} \int_{\partial F_\varepsilon} \mathcal{G}(\gamma_0(u_{\varepsilon,m}(t))) d\sigma(x).$$

Then, integrating (51) between 0 and t , and taking into account (36) and (41)-(43), we obtain

$$\begin{aligned} & \int_0^t |(u'_{\varepsilon,m}(s), \gamma_0(u'_{\varepsilon,m}(s)))|_H^2 ds + \frac{\min\{1, \kappa\}}{1 + \|\gamma_0\|^p} \frac{1}{p} \|(u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))\|_{V_p}^p \\ & + \tilde{\alpha}_1 \left(|u_{\varepsilon,m}(t)|_{q_1, \Omega_\varepsilon}^{q_1} + \varepsilon |\gamma_0(u_{\varepsilon,m}(t))|_{q_2, \partial F_\varepsilon}^{q_2} \right) \leq \frac{\max\{1, \kappa\}}{p} \|(u_{\varepsilon,m}^0, \gamma_0(u_{\varepsilon,m}^0))\|_{V_p}^p \\ & + \tilde{\alpha}_2 \left(|u_{\varepsilon,m}^0|_{q_1, \Omega_\varepsilon}^{q_1} + \varepsilon |\gamma_0(u_{\varepsilon,m}^0)|_{q_2, \partial F_\varepsilon}^{q_2} \right) + 2\tilde{\beta}C, \end{aligned}$$

for all $t \in (0, T)$. In order to estimate the right hand side of the last inequality we use (23) together with (50) and $\varepsilon \ll 1$. In particular, we can deduce

$$\int_0^t |(u'_{\varepsilon,m}(s), \gamma_0(u'_{\varepsilon,m}(s)))|_H^2 ds + \frac{\min\{1, \kappa\}}{1 + \|\gamma_0\|^p} \frac{1}{p} \|(u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))\|_{V_p}^p \leq C, \quad (52)$$

for all $t \in (0, T)$. We have proved that the sequence $\{(u_{\varepsilon,m}, \gamma_0(u_{\varepsilon,m}))\}$ is bounded in $C([0, T]; V_p)$, and $\{(u'_{\varepsilon,m}, \gamma_0(u'_{\varepsilon,m}))\}$ is bounded in $L^2(0, T; H)$, for all $T > 0$.

If we work with the truncated Galerkin equations (47)-(49) instead of the full PDE, we note that the calculations of the proof of (46) can be following identically to show that $\{(u_{\varepsilon,m}, \gamma_0(u_{\varepsilon,m}))\}$ is bounded in $L^p(0, T; V_p)$, for all $T > 0$.

Moreover, taking into account the uniqueness of solution to (2), using Aubin-Lions compactness lemma (e.g., cf. Lions [20]) and the fact that A_p is a monotone operator, it is not difficult to conclude that the sequence $\{(u_{\varepsilon,m}, \gamma_0(u_{\varepsilon,m}))\}$ converges weakly in $L^p(0, T; V_p)$ to the solution $(u_\varepsilon, \gamma_0(u_\varepsilon))$ to (2). Since the inclusion $V_p \subset H$ is compact (see (25)) and $(u_\varepsilon, \gamma_0(u_\varepsilon)) \in C([0, T]; H)$, it follows using [22, Lemma 11.2] that

$$\sup_{t \in [0, T]} \|(u_\varepsilon(t), \gamma_0(u_\varepsilon(t)))\|_{V_p} \leq C,$$

and, in particular, the second estimate in (44) is proved. \square

The extension of u_ε to the whole $\Omega \times (0, T)$: since the solution u_ε of the problem (2) is defined only in $\Omega_\varepsilon \times (0, T)$, we need to extend it to the whole $\Omega \times (0, T)$ to be able to state the convergence result. In order to do that, we shall use the following well-known extension result given by Donato and Moscardello [13, Lemma 2.4]: Let $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ be a $W^{1,p}$ -extension of u_ε , that satisfies the following condition

$$|\nabla \tilde{u}_\varepsilon|_{p, \Omega} \leq C |\nabla u_\varepsilon|_{p, \Omega_\varepsilon}. \quad (53)$$

Using (53) and taking into account Lemma 4.1, we obtain some *a priori* estimates for the extension of u_ε to the whole $\Omega \times (0, T)$.

Corollary 4.2. *Assume the assumptions in Lemma 4.1. Then, there exists a $W^{1,p}$ -extension \tilde{u}_ε of the solution u_ε of the problem (2) into $\Omega \times (0, T)$, such that*

$$\|\tilde{u}_\varepsilon\|_{p,\Omega,T} \leq C, \quad \sup_{t \in [0,T]} \|\tilde{u}_\varepsilon(t)\|_{p,\Omega} \leq C, \quad (54)$$

where the constant C does not depend on ε .

Proof. Using Poincaré's inequality, (53) and the first estimate in (44), we obtain

$$\|\tilde{u}_\varepsilon\|_{p,\Omega,T}^p = |\tilde{u}_\varepsilon|_{p,\Omega,T}^p + |\nabla \tilde{u}_\varepsilon|_{p,\Omega,T}^p \leq C |\nabla \tilde{u}_\varepsilon|_{p,\Omega,T}^p \leq C |\nabla u_\varepsilon|_{p,\Omega_\varepsilon,T}^p \leq C,$$

and the first estimate in (54) is proved. Similarly, using Poincaré's inequality, (53) and the second estimate in (44), we have the second estimate in (54). \square

5 A compactness result

In this section, we obtain some compactness results about the behavior of the sequence \tilde{u}_ε satisfying the *a priori* estimates given in Corollary 4.2.

By $\chi_{\Omega_\varepsilon}$ we denote the characteristic function of the domain Ω_ε . Due to the periodicity of the domain Ω_ε , from [7, Theorem 2.6] one has, for $\varepsilon \rightarrow 0$, that

$$\chi_{\Omega_\varepsilon} \xrightarrow{*} \frac{|Y^*|}{|Y|} \quad \text{weakly-star in } L^\infty(\Omega), \quad (55)$$

where the limit is the proportion of the material in the cell Y .

Let $\xi_\varepsilon := |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon$ in $\Omega_\varepsilon \times (0, T)$ and let us denote by $\tilde{\xi}_\varepsilon$ its extension with zero to the whole of $\Omega \times (0, T)$, i.e.

$$\tilde{\xi}_\varepsilon = \begin{cases} \xi_\varepsilon & \text{in } \Omega_\varepsilon \times (0, T), \\ 0 & \text{in } (\Omega \setminus \overline{\Omega_\varepsilon}) \times (0, T). \end{cases} \quad (56)$$

Proposition 5.1. *Under the assumptions in Lemma 4.1, there exists a function $u \in L^p(0, T; W_0^{1,p}(\Omega))$ (u will be the unique solution of the limit system (13)) and a function $\xi \in L^{p'}(0, T; L^{p'}(\Omega))$ such that, at least after extraction of a subsequence, we have the following convergences for all $T > 0$,*

$$\tilde{u}_\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly in } W_0^{1,p}(\Omega), \quad \forall t \in [0, T], \quad (57)$$

$$\tilde{u}_\varepsilon(t) \rightarrow u(t) \quad \text{strongly in } L^p(\Omega), \quad \forall t \in [0, T], \quad (58)$$

$$|\tilde{u}_\varepsilon(t)|^{p-2} \tilde{u}_\varepsilon(t) \rightarrow |u(t)|^{p-2} u(t) \quad \text{strongly in } L^{p'}(\Omega), \quad \forall t \in [0, T], \quad (59)$$

$$f(\tilde{u}_\varepsilon(t)) \rightarrow f(u(t)) \quad \text{strongly in } L^{q_1}(\Omega), \quad \forall t \in [0, T], \quad (60)$$

$$\tilde{\xi}_\varepsilon \rightharpoonup \xi \quad \text{weakly in } L^{p'}(0, T; L^{p'}(\Omega)), \quad (61)$$

where $\tilde{\xi}_\varepsilon$ is given by (56).

Let q_2 be the exponent satisfying (12). Let $\bar{q} > 1$ given by

$$\bar{q} \in (1, p) \quad \text{if } p = N, \quad \bar{q} = \frac{Np}{(N-p)(q_2-2) + N} \quad \text{if } p < N.$$

Assume that $g \in C^1(\mathbb{R})$. Then, we have the following convergences for all $T > 0$,

$$g(\tilde{u}_\varepsilon(t)) \rightarrow g(u(t)) \quad \text{strongly in } L^{\bar{q}}(\Omega), \quad \forall t \in [0, T], \quad (62)$$

$$g(\tilde{u}_\varepsilon(t)) \rightharpoonup g(u(t)) \quad \text{weakly in } W_0^{1,\bar{q}}(\Omega), \quad \forall t \in [0, T]. \quad (63)$$

Proof. By the first estimate in (54), we observe that the sequence $\{\tilde{u}_\varepsilon\}$ is bounded in the space $L^p(0, T; W_0^{1,p}(\Omega))$, for all $T > 0$. Let us fix $T > 0$. Then, there exists a subsequence $\{\tilde{u}_{\varepsilon'}\} \subset \{\tilde{u}_\varepsilon\}$ and function $u \in L^p(0, T; W_0^{1,p}(\Omega))$ such that

$$\tilde{u}_{\varepsilon'} \rightharpoonup u \quad \text{weakly in } L^p(0, T; W_0^{1,p}(\Omega)). \quad (64)$$

By the second estimate in (54), for each $t \in [0, T]$, we have that $\{\tilde{u}_\varepsilon(t)\}$ is bounded in $W_0^{1,p}(\Omega)$, and since we have (64), we can deduce (57). The conclusions of the Rellich-Kondrachov Theorem are also valid substituting $W^{1,p}(\Omega_\varepsilon)$ for $W_0^{1,p}(\Omega)$, so using (18) and (57), we obtain

$$\tilde{u}_{\varepsilon'}(t) \rightarrow u(t) \quad \text{strongly in } L^r(\Omega), \quad \forall t \in [0, T], \quad (65)$$

where $1 \leq r < p^*$ if $p < N$ and $p \leq r < p^*$ if $p = N$. Observe that, in particular, we have (58).

On the other hand, observe that

$$||v|^{p-2}v| \leq C(1 + |v|^{p-1}),$$

then, applying [10, Theorem 2.4] for $G(x, v) = |v|^{p-2}v$, $t = p'$ and $r = p$, we have that the map $v \in L^p(\Omega) \mapsto |v|^{p-2}v \in L^{p'}(\Omega)$ is continuous in the strong topologies. Then, taking into account (65) with $r = p$, we get (59).

Thanks to (27), applying [10, Theorem 2.4] for $G(x, v) = f(v)$, $t = q_1'$ and $r = q_1$, we have that the map $v \in L^{q_1}(\Omega) \mapsto f(v) \in L^{q_1'}(\Omega)$ is continuous in the strong topologies. Then, taking into account (65) with $r = q_1$, we get (60).

From the first estimate in (44) and (56), we have $|\tilde{\xi}_\varepsilon|_{p', \Omega, T} \leq C$, and hence, up a sequence, there exists $\xi \in L^{p'}(0, T, L^{p'}(\Omega))$ such that we obtain (61).

By the arbitrariness of $T > 0$, all the convergences are satisfied, as we wanted to prove.

Now, we analyze the convergences for the nonlinear term g . We separate the cases $p < N$ and $p = N$.

Case 1: $p < N$. Since

$$\frac{p^*}{\bar{q}} = \frac{(N-p)(q_2-2) + N}{N-p} = q_2 - 1 + \frac{p}{N-p} > q_2 - 1,$$

there exists $r \in [2, p^*)$ such that $\frac{r}{\bar{q}} \geq q_2 - 1$ and

$$|g(s)| \leq C \left(1 + |s|^{q_2-1}\right) \leq C \left(1 + |s|^{\frac{r}{\bar{q}}}\right).$$

Then, applying [10, Theorem 2.4] for $G(x, v) = g(v)$, $t = \bar{q}$ and $r \in [2, p^*)$ such that $\frac{r}{\bar{q}} \geq q_2 - 1$, we have that the map $v \in L^r(\Omega) \mapsto g(v) \in L^{\bar{q}}(\Omega)$ is continuous in the strong topologies. Then, taking into account (65), we get (62).

Finally, we prove (63). Observe that as we are assuming that $g \in C^1(\mathbb{R})$, it is easy to see from (6) that there exists a constant $C > 0$ such that

$$|g'(s)| \leq C \left(1 + |s|^{q_2-2}\right).$$

Then, we get

$$\begin{aligned} \int_\Omega \left| \frac{\partial g}{\partial x_i}(\tilde{u}_\varepsilon(t)) \right|^{\bar{q}} dx &\leq C \int_\Omega \left(1 + |\tilde{u}_\varepsilon(t)|^{(q_2-2)\bar{q}}\right) \left| \frac{\partial \tilde{u}_\varepsilon(t)}{\partial x_i} \right|^{\bar{q}} dx \\ &\leq C \left(1 + \left(\int_\Omega |\tilde{u}_\varepsilon(t)|^{(q_2-2)\bar{q}\gamma} dx\right)^{1/\gamma}\right) \left(\int_\Omega |\nabla \tilde{u}_\varepsilon(t)|^{\bar{q}\eta} dx\right)^{1/\eta}, \end{aligned} \quad (66)$$

where we took γ and η such that $\bar{q}\eta = p$, $1/\gamma + 1/\eta = 1$ and $(q_2-2)\bar{q}\gamma = p^*$. Note that from here we get

$$\bar{q} = \frac{Np}{(N-p)(q_2-2) + N}.$$

Observe that $\bar{q} > 1$. Indeed,

$$q_2 \leq \frac{(N-1)p}{N-p} = \frac{N(p-1)}{N-p} + 1 < \frac{N(p-1)}{N-p} + 2 \Rightarrow (N-p)(q_2-2) + N < Np \Rightarrow \frac{Np}{(N-p)(q_2-2) + N} > 1.$$

Then, we have

$$\int_{\Omega} \left| \frac{\partial g}{\partial x_i}(\tilde{u}_\varepsilon(t)) \right|^{\bar{q}} dx \leq C \left(1 + |\tilde{u}_\varepsilon|_{p^*, \Omega}^{p^*/\gamma} \right) |\nabla \tilde{u}_\varepsilon|_{p, \Omega}^{p/\eta}.$$

The conclusions of (17) are also valid substituting $W^{1,p}(\Omega_\varepsilon)$ for $W_0^{1,p}(\Omega)$, so using (17) and the second estimate in (54), we get

$$|\nabla g(\tilde{u}_\varepsilon(t))|_{\bar{q}, \Omega} \leq C. \quad (67)$$

Then, from (62) and (67), we can deduce (63).

Case 2: $p = N$. We consider $s \in [\frac{p(q_2-1)}{p-1}, +\infty)$ and

$$\bar{q} = \frac{sp}{p(q_2-2) + s}. \quad (68)$$

Taking into account (12), we can deduce

$$\frac{s}{\bar{q}} = \frac{p(q_2-2) + s}{p} = q_2 - 1 + \frac{s-p}{p} \geq q_2 - 1,$$

and, we have

$$|g(s)| \leq C \left(1 + |s|^{q_2-1} \right) \leq C \left(1 + |s|^{\frac{s}{\bar{q}}} \right).$$

Then, applying [10, Theorem 2.4] for $G(x, v) = g(v)$, $t = \bar{q}$ and $r = s$, we have that the map $v \in L^s(\Omega) \mapsto g(v) \in L^{\bar{q}}(\Omega)$ is continuous in the strong topologies. Thanks to (12), we can consider (65) with $r = s$, and we get (62).

Finally, we prove (63). In (66) we took γ and η such that $\bar{q}\eta = p$, $1/\gamma + 1/\eta = 1$ and $(q_2-2)\bar{q}\gamma = s$. Note that from here we get \bar{q} given by (68).

Observe that $\bar{q} \in (1, p)$. Indeed, taking into account that $\frac{1}{\bar{q}} = \frac{q_2-2}{s} + \frac{1}{p}$, we can deduce

$$\frac{p(q_2-1)}{p-1} \leq s < +\infty \Rightarrow 0 < \frac{1}{s} \leq \frac{p-1}{p(q_2-1)} \Rightarrow \frac{1}{p} < \frac{1}{\bar{q}} \leq \frac{(q_2-2)(p-1)}{p(q_2-1)} + \frac{1}{p} \Rightarrow \frac{p(q_2-1)}{(q_2-2)(p-1) + (q_2-1)} \leq \bar{q} < p,$$

and using that $\frac{p(q_2-1)}{(q_2-2)(p-1) + (q_2-1)} > 1$, we have that $\bar{q} \in (1, p)$.

Then, we have

$$\int_{\Omega} \left| \frac{\partial g}{\partial x_i}(\tilde{u}_\varepsilon(t)) \right|^{\bar{q}} dx \leq C \left(1 + |\tilde{u}_\varepsilon|_{s, \Omega}^{s/\gamma} \right) |\nabla \tilde{u}_\varepsilon|_{p, \Omega}^{p/\eta}.$$

The conclusions of (17) are also valid substituting $W^{1,p}(\Omega_\varepsilon)$ for $W_0^{1,p}(\Omega)$, so using (17) and the second estimate in (54), we get

$$|\nabla g(\tilde{u}_\varepsilon(t))|_{\bar{q}, \Omega} \leq C. \quad (69)$$

Then, from (62) and (69), we can deduce (63).

□

6 Homogenized model: proof of the main Theorem

In this section, we identify the homogenized model.

We multiply system (2) by a test function $v \in \mathcal{D}(\Omega)$, and integrating by parts, we have

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(t) v dx \right) + \varepsilon \frac{d}{dt} \left(\int_{\partial F_{\varepsilon}} \gamma_0(u_{\varepsilon}(t)) v d\sigma(x) \right) + \int_{\Omega} \tilde{\xi}_{\varepsilon} \cdot \nabla v dx \\ & + \kappa \int_{\Omega} \chi_{\Omega_{\varepsilon}} |\tilde{u}_{\varepsilon}(t)|^{p-2} \tilde{u}_{\varepsilon}(t) v dx + \int_{\Omega} \chi_{\Omega_{\varepsilon}} f(\tilde{u}_{\varepsilon}(t)) v dx + \varepsilon \int_{\partial F_{\varepsilon}} g(\gamma_0(u_{\varepsilon}(t))) v d\sigma(x) = 0, \end{aligned}$$

in $\mathcal{D}'(0, T)$.

We consider $\varphi \in C_c^1([0, T])$ such that $\varphi(T) = 0$ and $\varphi(0) \neq 0$. Multiplying by φ and integrating between 0 and T , we have

$$\begin{aligned} & -\varphi(0) \left(\int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(0) v dx \right) - \int_0^T \frac{d}{dt} \varphi(t) \left(\int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(t) v dx \right) dt \\ & - \varepsilon \varphi(0) \left(\int_{\partial F_{\varepsilon}} \gamma_0(u_{\varepsilon}(0)) v d\sigma(x) \right) - \varepsilon \int_0^T \frac{d}{dt} \varphi(t) \left(\int_{\partial F_{\varepsilon}} \gamma_0(u_{\varepsilon}(t)) v d\sigma(x) \right) dt \\ & + \int_0^T \varphi(t) \int_{\Omega} \tilde{\xi}_{\varepsilon} \cdot \nabla v dx dt + \kappa \int_0^T \varphi(t) \int_{\Omega} \chi_{\Omega_{\varepsilon}} |\tilde{u}_{\varepsilon}(t)|^{p-2} \tilde{u}_{\varepsilon}(t) v dx dt \\ & + \int_0^T \varphi(t) \int_{\Omega} \chi_{\Omega_{\varepsilon}} f(\tilde{u}_{\varepsilon}(t)) v dx dt + \varepsilon \int_0^T \varphi(t) \int_{\partial F_{\varepsilon}} g(\gamma_0(u_{\varepsilon}(t))) v d\sigma(x) dt = 0. \end{aligned} \tag{70}$$

For the sake of clarity, we split the proof in five parts. Firstly, we pass to the limit, as $\varepsilon \rightarrow 0$, in (70) in order to get the limit equation satisfied by u . In the first step, we pass to the limit in the integrals on Ω , using Proposition 5.1 and (55), in the second step we pass to the limit in the integrals on the boundary of the holes, where we use a convergence result based on a technique introduced by Vanninathan [25], and in the third step we deduce the limit equation satisfied by u . In the fourth step we identify ξ making use of the solutions of the problems (14), and finally we prove that u is uniquely determined.

Step 1. Passing to the limit, as $\varepsilon \rightarrow 0$, in the integrals on Ω :

From (58)-(60) and (55), we have respectively, for $\varepsilon \rightarrow 0$,

$$\begin{aligned} & \int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(t) v dx \rightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} u(t) v dx, \quad \forall v \in \mathcal{D}(\Omega), \\ & \int_{\Omega} \chi_{\Omega_{\varepsilon}} |\tilde{u}_{\varepsilon}(t)|^{p-2} \tilde{u}_{\varepsilon}(t) v dx \rightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} |u(t)|^{p-2} u(t) v dx, \quad \forall v \in \mathcal{D}(\Omega), \end{aligned}$$

and

$$\int_{\Omega} \chi_{\Omega_{\varepsilon}} f(\tilde{u}_{\varepsilon}(t)) v dx \rightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} f(u(t)) v dx, \quad \forall v \in \mathcal{D}(\Omega),$$

which integrating in time and using Lebesgue's Dominated Convergence Theorem, gives

$$\begin{aligned} & \int_0^T \frac{d}{dt} \varphi(t) \left(\int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(t) v dx \right) dt \rightarrow \frac{|Y^*|}{|Y|} \int_0^T \frac{d}{dt} \varphi(t) \left(\int_{\Omega} u(t) v dx \right) dt, \\ & \kappa \int_0^T \varphi(t) \int_{\Omega} \chi_{\Omega_{\varepsilon}} |\tilde{u}_{\varepsilon}(t)|^{p-2} \tilde{u}_{\varepsilon}(t) v dx dt \rightarrow \kappa \frac{|Y^*|}{|Y|} \int_0^T \varphi(t) \int_{\Omega} |u(t)|^{p-2} u(t) v dx dt, \end{aligned}$$

and

$$\int_0^T \varphi(t) \int_{\Omega} \chi_{\Omega_\varepsilon} f(\tilde{u}_\varepsilon(t)) v dx dt \rightarrow \frac{|Y^*|}{|Y|} \int_0^T \varphi(t) \int_{\Omega} f(u(t)) v dx dt.$$

By (58) and (55), we have

$$\varphi(0) \left(\int_{\Omega} \chi_{\Omega_\varepsilon} \tilde{u}_\varepsilon(0) v dx \right) \rightarrow \varphi(0) \frac{|Y^*|}{|Y|} \int_{\Omega} u(0) v dx, \quad \forall v \in \mathcal{D}(\Omega).$$

On the other hand, using (61), we obtain, for $\varepsilon \rightarrow 0$

$$\int_0^T \varphi(t) \int_{\Omega} \tilde{\xi}_\varepsilon \cdot \nabla v dx dt \rightarrow \int_0^T \varphi(t) \int_{\Omega} \xi \cdot \nabla v dx dt.$$

Step 2. Passing to the limit, as $\varepsilon \rightarrow 0$, in the surface integrals on the boundary of the holes:

We make use of the technique introduced by Vanninathan [25] for the Steklov problem which transforms surface integrals into volume integrals. This technique was already used as a main tool to homogenize the non homogeneous Neumann problem for the elliptic case by Cioranescu and Donato [6].

By [6, Definition 3.2], let us introduce, for any $h \in L^{s'}(\partial F)$, $1 \leq s' \leq \infty$, the linear form μ_h^ε on $W_0^{1,s}(\Omega)$ defined by

$$\langle \mu_h^\varepsilon, \varphi \rangle = \varepsilon \int_{\partial F_\varepsilon} h \left(\frac{x}{\varepsilon} \right) \varphi(x) d\sigma(x), \quad \forall \varphi \in W_0^{1,s}(\Omega),$$

with $1/s + 1/s' = 1$. It is proved in [6, Lemma 3.3] that

$$\mu_h^\varepsilon \rightarrow \mu_h \quad \text{strongly in } (W_0^{1,s}(\Omega))', \quad (71)$$

where $\langle \mu_h, \varphi \rangle = \mu_h \int_{\Omega} \varphi(x) dx$, with

$$\mu_h = \frac{1}{|Y|} \int_{\partial F} h(y) d\sigma(y).$$

In the particular case in which $h \in L^\infty(\partial F)$ or even when h is constant, we have

$$\mu_h^\varepsilon \rightarrow \mu_h \quad \text{strongly in } W^{-1,\infty}(\Omega).$$

We denote by μ_1^ε the above introduced measure in the particular case in which $h = 1$. Notice that in this case μ_h becomes $\mu_1 = |\partial F|/|Y|$.

Observe that from (57) and (71) with $s = p$, we can deduce, for $\varepsilon \rightarrow 0$,

$$\varepsilon \int_{\partial F_\varepsilon} \gamma_0(u_\varepsilon(t)) v d\sigma(x) = \langle \mu_1^\varepsilon, \tilde{u}_{\varepsilon|\Omega_\varepsilon}(t)v \rangle \rightarrow \mu_1 \int_{\Omega} u(t) v dx = \frac{|\partial F|}{|Y|} \int_{\Omega} u(t) v dx,$$

for all $v \in \mathcal{D}(\Omega)$, which integrating in time and using Lebesgue's Dominated Convergence Theorem, gives

$$\varepsilon \int_0^T \frac{d}{dt} \varphi(t) \left(\int_{\partial F_\varepsilon} \gamma_0(u_\varepsilon(t)) v d\sigma(x) \right) dt \rightarrow \frac{|\partial F|}{|Y|} \int_0^T \frac{d}{dt} \varphi(t) \left(\int_{\Omega} u(t) v dx \right) dt.$$

Moreover, from (57) and (71) with $s = p$, we can deduce, for $\varepsilon \rightarrow 0$,

$$\varepsilon \int_{\partial F_\varepsilon} \gamma_0(u_\varepsilon(0)) v d\sigma(x) = \langle \mu_1^\varepsilon, \tilde{u}_{\varepsilon|\Omega_\varepsilon}(0)v \rangle \rightarrow \mu_1 \int_{\Omega} u(0) v dx = \frac{|\partial F|}{|Y|} \int_{\Omega} u(0) v dx,$$

for all $v \in \mathcal{D}(\Omega)$.

On the other hand, from (63) and (71) with $s = \bar{q}$, we conclude

$$\varepsilon \int_{\partial F_\varepsilon} g(\gamma_0(u_\varepsilon(t))) v d\sigma(x) = \langle \mu_1^\varepsilon, g(\tilde{u}_\varepsilon(t)) v \rangle \rightarrow \frac{|\partial F|}{|Y|} \int_\Omega g(u(t)) v dx,$$

for all $v \in \mathcal{D}(\Omega)$, which integrating in time and using Lebesgue's Dominated Convergence Theorem, gives

$$\varepsilon \int_0^T \varphi(t) \int_{\partial F_\varepsilon} g(\gamma_0(u_\varepsilon(t))) v d\sigma(x) dt \rightarrow \frac{|\partial F|}{|Y|} \int_0^T \varphi(t) \int_\Omega g(u(t)) v dx dt.$$

Step 3. Passing to the limit, as $\varepsilon \rightarrow 0$, in (70):

All the terms in (70) pass to the limit, as $\varepsilon \rightarrow 0$, and therefore taking into account the previous steps, we get

$$\begin{aligned} -\varphi(0) \left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \left(\int_\Omega u(0) v dx \right) - \left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \int_0^T \frac{d}{dt} \varphi(t) \left(\int_\Omega u(t) v dx \right) dt \\ + \int_0^T \varphi(t) \int_\Omega \xi \cdot \nabla v dx dt + \kappa \frac{|Y^*|}{|Y|} \int_0^T \varphi(t) \int_\Omega |u(t)|^{p-2} u(t) v dx dt \\ + \frac{|Y^*|}{|Y|} \int_0^T \varphi(t) \int_\Omega f(u(t)) v dx dt + \frac{|\partial F|}{|Y|} \int_0^T \varphi(t) \int_\Omega g(u(t)) v dx dt = 0. \end{aligned}$$

Hence, ξ verifies

$$\left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \frac{\partial u}{\partial t} - \operatorname{div} \xi + \frac{|Y^*|}{|Y|} (\kappa |u|^{p-2} u + f(u)) + \frac{|\partial F|}{|Y|} g(u) = 0, \quad \text{in } \Omega \times (0, T). \quad (72)$$

Step 4. It remains now to identify ξ . For the sake of completeness, we give a sketch of a proof, following, for instance, the proof of [13, Theorem 3.1].

Let us consider $\forall \nu \in \mathbb{N}$ and for any $i = 1, \dots, N$, a partition of \mathbb{R}^N by intervals $\{Q_{i\nu}\}_i$ of side $2^{-\nu}$. Since the holes F_ε do not intersect $\partial\Omega$, we can assume that $\forall \varepsilon > 0$, F_ε does not intersect $\partial Q_{i\nu}$. Let us denote by $x_{i\nu}$ and $\chi_{i\nu}$ respectively the center and the characteristic function of $Q_{i\nu}$. Then, we define $I_\nu = \{i : Q_{i\nu} \subset \Omega\}$, $\Omega_\nu = \cup_{i \in I_\nu} Q_{i\nu}$. For any i , let us denote

$$\langle \nabla u \rangle_{i\nu} := \frac{1}{|Q_{i\nu}|} \int_{Q_{i\nu}} \nabla u(x) dx.$$

By the continuity of b (see [13, Lemma 2.11]), we have, if $\nu \rightarrow +\infty$, that

$$\sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle \nabla u \rangle_{i\nu}) \rightarrow b(\nabla u(x)) \quad \text{a.e. in } \Omega, \quad (73)$$

where b is given by (15).

Moreover, from [13, Lemma 2.10], we have for any measurable set $E \subset \Omega$ that

$$\int_E \left| \sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle \nabla u \rangle_{i\nu}) \right|^{p'} dx \leq C \int_E \left(1 + \left| \sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle \nabla u \rangle_{i\nu}) \right|^p \right) dx.$$

So, from the equi-absolute continuity of the integral on the left-hand side and from (73), we deduce that

$$\sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle \nabla u \rangle_{i\nu}) \rightarrow b(\nabla u(x)) \quad \text{in } L^{p'}(\Omega), \quad \text{as } \nu \rightarrow +\infty. \quad (74)$$

Let $w_{i\nu} \in \langle \nabla u \rangle_{i\nu} \cdot y + \mathbb{H}_{\text{per}}(Y^*)$ the solution of the problem (14) corresponding to $\langle \nabla u \rangle_{i\nu}$. Then

$$v_{i\nu} = (w_{i\nu} - \langle \nabla u \rangle_{i\nu} \cdot y) \in \mathbb{H}_{\text{per}}(Y^*),$$

and so, by [13, Lemma 2.3], we can consider $\tilde{v}_{i\nu} \in \mathbb{H}_{\text{per}}(Y)$, where $\tilde{\cdot}$ denotes the $W^{1,p}$ -extension to Y . If we extend $\tilde{v}_{i\nu}$, by periodicity, the resulting function (still denoted by $\tilde{v}_{i\nu}$) is in $W^{1,p}(\Omega)$.

Let us define

$$\Psi_{i\nu}^\varepsilon(x) = \varepsilon \tilde{v}_{i\nu} \left(\frac{x}{\varepsilon} \right) - \langle \nabla u \rangle_{i\nu} \cdot x, \quad (75)$$

and for $\zeta \in \mathbb{R}^N$

$$\tilde{a}_p(\zeta) = \begin{cases} |\zeta|^{p-2} \zeta & \text{for } y \in Y^*, \\ 0 & \text{for } y \in \bar{F}. \end{cases}$$

Arguing as in the proof of [13, Lemma 2.13], we obtain

$$\Psi_{i\nu}^\varepsilon \rightharpoonup \langle \nabla u \rangle_{i\nu} \cdot x \quad \text{weakly in } W^{1,p}(\Omega), \quad (76)$$

$$\tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \rightharpoonup b(\langle \nabla u \rangle_{i\nu}) \quad \text{weakly in } L^{p'}(\Omega), \quad (77)$$

and

$$\text{div } \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) = 0. \quad (78)$$

By Rellich-Kondrachov Theorem and (76), we can deduce

$$\Psi_{i\nu}^\varepsilon \rightarrow \langle \nabla u \rangle_{i\nu} \cdot x \quad \text{strongly in } L^p(\Omega). \quad (79)$$

Using the periodicity of $\tilde{v}_{i\nu}$ and [13, Lemma 2.3], we have

$$\begin{aligned} \sum_{i \in I_\nu} \int_{Q_{i\nu}} |\nabla \Psi_{i\nu}^\varepsilon|^p dx &\leq \sum_{i \in I_\nu} 2^{-\nu N} \varepsilon^N (1/\varepsilon + 2^\nu)^N \int_Y |\nabla \Psi_{i\nu}^\varepsilon(y)|^p dy \\ &\leq C \sum_{i \in I_\nu} 2^{-\nu N} (1 + \varepsilon^N 2^{\nu N}) \left(\int_{Y^*} |\nabla w_{i\nu}|^p dy + \langle \nabla u \rangle_{i\nu} \right)^p. \end{aligned}$$

Then, due to the structure properties of b (see [13, Lemma 2.10]), writing the last term as an integral over Ω_ν , we have

$$\sum_{i \in I_\nu} \int_{Q_{i\nu}} |\nabla \Psi_{i\nu}^\varepsilon|^p dx \leq C (1 + \varepsilon^N 2^{\nu N}) \int_\Omega (1 + |\nabla u|)^p dx. \quad (80)$$

Let now $\eta \in C_0^1(Q_{i\nu})$, $0 \leq \eta \leq 1$, and extend it by periodicity to the whole \mathbb{R}^N . We consider $\varphi \in C_c^1([0, T])$ such that $\varphi(T) = 0$ and $\varphi(0) \neq 0$. If $\psi \in C^0(\bar{\Omega})$, set $M_\psi = \sup_\Omega |\psi|$ and $M_\varphi = \sup_{[0, T]} |\varphi|$, we have

$$\begin{aligned} &\left| \int_0^T \varphi(t) \int_\Omega \tilde{\xi}_\varepsilon \psi \eta dx dt - \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \psi \eta dx dt \right| \quad (81) \\ &\leq CM_\psi M_\varphi |\Omega \setminus \Omega_\nu|^{1/p} + \left| \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} (|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - |\nabla \Psi_{i\nu}^\varepsilon|^{p-2} \nabla \Psi_{i\nu}^\varepsilon) \psi \eta dx dt \right| \\ &\leq CM_\psi M_\varphi |\Omega \setminus \Omega_\nu|^{1/p} + \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} M_\psi \eta \left((|\nabla u_\varepsilon| + |\nabla \Psi_{i\nu}^\varepsilon|)^{p-2} \cdot |\nabla u_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon| \right) dx dt \\ &\leq CM_\psi M_\varphi |\Omega \setminus \Omega_\nu|^{1/p} + CM_\psi^{p/(p-1)} \delta^{p/(p-1)} (1 + \varepsilon^N 2^{\nu N}) + \delta^{-p} \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |\nabla u_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon|^p dx dt, \end{aligned}$$

where the last inequality is obtained by applying Young inequality with $\delta > 0$ and the first estimate in (44), the first estimate in (54) and (80).

On the other hand, from monotonicity, we can deduce

$$\begin{aligned}
 \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |\nabla u_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon|^p dx dt &\leq \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \left(\tilde{\xi}_\varepsilon - \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \right) \cdot (\nabla \tilde{u}_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon) dx dt \\
 &= \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \left(\tilde{\xi}_\varepsilon - \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \right) \cdot \nabla \rho_{i\nu}^\varepsilon dx dt \\
 &+ \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \left(\tilde{\xi}_\varepsilon - \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \right) \cdot (\nabla u - \langle \nabla u \rangle_{i\nu}) dx dt,
 \end{aligned} \tag{82}$$

where $\rho_{i\nu}^\varepsilon := \tilde{u}_\varepsilon - \Psi_{i\nu}^\varepsilon - u + \langle \nabla u \rangle_{i\nu} \cdot x$. Taking into account (58) and (79), we have

$$\rho_{i\nu}^\varepsilon \rightarrow 0 \quad \text{strongly in } L^p(\Omega). \tag{83}$$

Now, we multiply system (2) by the test function $\eta \rho_{i\nu}^\varepsilon$ and integrating by parts, we get

$$\begin{aligned}
 \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \tilde{\xi}_\varepsilon \cdot \nabla \rho_{i\nu}^\varepsilon dx dt &= - \sum_{i \in I_\nu} \int_{Q_{i\nu}} \varphi(0) \chi_{\Omega_\varepsilon} \tilde{u}_\varepsilon(0) \eta \rho_{i\nu}^\varepsilon dx \\
 - \int_0^T \frac{d}{dt} \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \chi_{\Omega_\varepsilon} \tilde{u}_\varepsilon(t) \eta \rho_{i\nu}^\varepsilon dx dt &- \varphi(0) \langle \mu_1^\varepsilon, \tilde{u}_{\varepsilon|\Omega_\varepsilon}(0) \eta \rho_{i\nu}^\varepsilon \rangle - \int_0^T \frac{d}{dt} \varphi(t) \langle \mu_1^\varepsilon, \tilde{u}_{\varepsilon|\Omega_\varepsilon}(t) \eta \rho_{i\nu}^\varepsilon \rangle dt \\
 - \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{\xi}_\varepsilon \cdot \nabla \eta \rho_{i\nu}^\varepsilon dx dt &- \kappa \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \chi_{\Omega_\varepsilon} |\tilde{u}_\varepsilon(t)|^{p-2} \tilde{u}_\varepsilon(t) \eta \rho_{i\nu}^\varepsilon dx dt \\
 - \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \chi_{\Omega_\varepsilon} f(\tilde{u}_\varepsilon(t)) \eta \rho_{i\nu}^\varepsilon dx dt &- \int_0^T \varphi(t) \langle \mu_1^\varepsilon, g(\tilde{u}_\varepsilon(t)) \rho_{i\nu}^\varepsilon \rangle dt.
 \end{aligned}$$

We pass to the limit, as $\varepsilon \rightarrow 0$. Taking into account (83), we reason as in the first two steps and we can deduce

$$\int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \tilde{\xi}_\varepsilon \cdot \nabla \rho_{i\nu}^\varepsilon dx dt \rightarrow 0. \tag{84}$$

On the other hand, integrating by parts, taking into account (77)-(78) and (83), as $\varepsilon \rightarrow 0$, by Lebesgue's Dominated Convergence Theorem, we obtain

$$\int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \cdot \nabla \rho_{i\nu}^\varepsilon \eta dx dt = \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \cdot \nabla \eta \rho_{i\nu}^\varepsilon dx dt \rightarrow 0. \tag{85}$$

Then, passing to the limit, as $\varepsilon \rightarrow 0$, in (82), by (61), (77), (84)-(85), we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |\nabla u_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon|^p dx dt \leq \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta (\xi - b(\langle \nabla u \rangle_{i\nu})) (\nabla u - \langle \nabla u \rangle_{i\nu}) dx dt. \tag{86}$$

If we first pass to the limit in (81), as $\varepsilon \rightarrow 0$, then $\eta \rightarrow 1$, $\nu \rightarrow +\infty$, by (61), (74), (77) and (86) we get

$$\left| \int_0^T \varphi(t) \int_\Omega \xi \psi dx dt - \int_0^T \varphi(t) \int_\Omega b(\nabla u) \psi dx dt \right| \leq CM_\psi^{p/(p-1)} \delta^{p/(p-1)}.$$

So, letting $\delta \rightarrow 0$, from the arbitrariness of φ and ψ , we deduce

$$\xi = b(\nabla u) \quad \text{a.e. in } \Omega \times (0, T). \quad (87)$$

Step 5. Finally, thanks to (72) and (87), we observe that u satisfies the first equation in (13). A weak solution of (13) is any function u , satisfying

$$u \in C([0, T]; L^2(\Omega)), \quad \text{for all } T > 0,$$

$$u \in L^p(0, T; W_0^{1,p}(\Omega)), \quad \text{for all } T > 0,$$

$$\left(\frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \frac{d}{dt} (u(t), v) + (b(\nabla u(t)), \nabla v) + \frac{|Y^*|}{|Y|} \kappa(|u(t)|^{p-2}u(t) + f(u(t)), v) + \frac{|\partial F|}{|Y|} (g(u(t)), v) = 0, \quad \text{in } \mathcal{D}'(0, T),$$

for all $v \in W_0^{1,p}(\Omega)$, and

$$u(0) = u_0.$$

Due to the structure properties of b (see [13, Lemmas 2.10-2.11-2.13]), applying a slight modification of [20, Chapter 2, Theorem 1.4], we obtain that the problem (13) has a unique solution, and therefore Theorem 1.1 is proved.

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