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# On $p$ -Laplacian reaction-diffusion problems with dynamical boundary conditions in perforated media

María ANGUIANO<sup>1</sup>

## Abstract

This paper deals with the homogenization of the  $p$ -Laplacian reaction-diffusion problems in a domain containing periodically distributed holes of size  $\varepsilon$ , with a dynamical boundary condition of *pure-reactive* type. We generalize our previous results (see [2]) established in the case where the diffusion is modeled by the Laplacian operator, i.e., with  $p = 2$ . We prove the convergence of the homogenization process to a nonlinear  $p$ -Laplacian reaction-diffusion equation defined on a unified domain without holes with zero Dirichlet boundary condition and with extra terms coming from the influence of the nonlinear dynamical boundary conditions.

**AMS classification numbers:** 35B27, 35K57

**Keywords:** Homogenization, perforated media, reaction-diffusion systems, dynamical boundary conditions,  $p$ -Laplacian

## 1 Introduction and setting of the problem

Homogenization problems in perforated media for the  $p$ -Laplacian operator have been considered in the literature over the last decades. The homogenization of the equation

$$-\operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) = f \tag{1}$$

in a periodically perforated domain is considered by Labani and Picard in [19] with Dirichlet boundary conditions. Such a problem is a generalization of the linear problem for Laplace's equation, which corresponds to  $p = 2$ , and was studied by Cioranescu and Murat in [8]. Donato and Moscarello in [13] study the homogenization of a class of nonlinear elliptic Neumann problems in perforated domains of  $\mathbb{R}^N$ . As a consequence of [13], we are able in particular to describe the homogenization of (1) with Neumann boundary conditions. This result is a generalization of earlier related works, for instance, Cioranescu and Donato [6] and Cioranescu and Saint Jean Paulin [9]. In [23] Shaposhnikova and Podol'skii study the homogenization of (1) in an  $\varepsilon$ -periodically perforated domain with a nonlinear boundary condition. In [11] Díaz *et al.* consider (1) with a nonlinear perturbed Robin-type boundary condition in an  $\varepsilon$  periodically perforated domain where the size of the particles is smaller than the period  $\varepsilon$ , and the asymptotic behavior of the solution is studied as  $\varepsilon \rightarrow 0$ . The closest articles to this one in the literature are [16, 17, 18], where Gómez *et al.* consider the case  $2 < p \leq N$ , and [12], where Díaz *et al.* study the case  $p > N$ . As far as we know, in the previous literature there is no study for the homogenization of  $p$ -Laplacian parabolic models associated with nonlinear dynamical boundary conditions in a periodically perforated domain, as we consider in this article.

In the context of reaction-diffusion equations, dynamical boundary conditions have been rigorously derived in Gal and Shomberg [14] based on first and second thermodynamical principles and their physical interpretation was also given in Goldstein [15]. In a recent article (see [2]) we addressed the problem of the homogenization of the reaction-diffusion equations with a dynamical boundary condition of *pure-reactive* type in a domain perforated with holes. More recently, in [3] we generalize this previous study to the case of a dynamical boundary condition of *reactive-diffusive* type, i.e., we add to the dynamical boundary condition a Laplace-Beltrami correction term.

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In the present article, we consider  $p$ -Laplacian parabolic problems with nonlinear dynamical boundary conditions of *pure-reactive* type occurring as generalizations of semilinear reaction-diffusion equations with dynamical boundary conditions. Let us introduce the model we will be involved with in this article.

**The geometrical setting.** Let  $\Omega$  be a bounded connected open set in  $\mathbb{R}^N$  ( $N \geq 2$ ), with smooth enough boundary  $\partial\Omega$ . Let us introduce a set of periodically distributed holes. As a result, we obtain an open set  $\Omega_\varepsilon$ , where  $\varepsilon$  represents a small parameter related to the characteristic size of the holes.

Let  $Y = [0, 1]^N$  be the representative cell in  $\mathbb{R}^N$  and  $F$  an open subset of  $Y$  with smooth enough boundary  $\partial F$ , such that  $\bar{F} \subset Y$ . We denote  $Y^* = Y \setminus \bar{F}$ . For  $k \in \mathbb{Z}^N$  and  $\varepsilon \in (0, 1]$ , each cell  $Y_{k,\varepsilon} = \varepsilon k + \varepsilon Y$  is similar to the unit cell  $Y$  rescaled to size  $\varepsilon$  and  $F_{k,\varepsilon} = \varepsilon k + \varepsilon F$  is similar to  $F$  rescaled to size  $\varepsilon$ . We denote  $Y_{k,\varepsilon}^* = Y_{k,\varepsilon} \setminus \bar{F}_{k,\varepsilon}$ . We denote by  $F_\varepsilon$  the set of all the holes contained in  $\Omega$ , i.e.  $F_\varepsilon = \cup_{k \in K} \{F_{k,\varepsilon} : \bar{F}_{k,\varepsilon} \subset \Omega\}$ , where  $K := \{k \in \mathbb{Z}^N : Y_{k,\varepsilon} \cap \Omega \neq \emptyset\}$ .

Let  $\Omega_\varepsilon = \Omega \setminus \bar{F}_\varepsilon$ . By this construction,  $\Omega_\varepsilon$  is a periodically perforated domain with holes of the same size as the period. Remark that the holes do not intersect the boundary  $\partial\Omega$ . Let  $\partial F_\varepsilon = \cup_{k \in K} \{\partial F_{k,\varepsilon} : \bar{F}_{k,\varepsilon} \subset \Omega\}$ . So

$$\partial\Omega_\varepsilon = \partial\Omega \cup \partial F_\varepsilon.$$

**Position of the problem.** The prototype of the  $p$ -Laplacian parabolic initial-boundary value problems that we consider in this article is

$$\left\{ \begin{array}{ll} \frac{\partial u_\varepsilon}{\partial t} - \Delta_p u_\varepsilon + \kappa |u_\varepsilon|^{p-2} u_\varepsilon = -f(u_\varepsilon) & \text{in } \Omega_\varepsilon \times (0, T), \\ \partial_{\nu_p} u_\varepsilon + \varepsilon \frac{\partial u_\varepsilon}{\partial t} = -\varepsilon g(u_\varepsilon) & \text{on } \partial F_\varepsilon \times (0, T), \\ u_\varepsilon = 0, & \text{on } \partial\Omega \times (0, T), \\ u_\varepsilon(x, 0) = u_\varepsilon^0(x), & \text{for } x \in \Omega_\varepsilon, \\ u_\varepsilon(x, 0) = \psi_\varepsilon^0(x), & \text{for } x \in \partial F_\varepsilon, \end{array} \right. \quad (2)$$

where  $u_\varepsilon = u_\varepsilon(x, t)$ ,  $x \in \Omega_\varepsilon$ ,  $t \in (0, T)$ , with  $T > 0$ , and  $\kappa > 0$  is a given constant. Here the diffusion is modeled by the  $p$ -Laplacian operator  $\Delta_p u_\varepsilon := \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon)$  with  $p \in [2, N]$ . Notice that  $p = 2$  corresponds to the linear diffusion operator. The ‘‘normal derivative’’ must be understood as  $\partial_{\nu_p} u_\varepsilon = |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nu$ , where  $\nu$  denotes the outward normal to  $\partial F_\varepsilon$ . The boundary equation (2)<sub>2</sub> is multiplied by  $\varepsilon$  to compensate the growth of the surface by shrinking  $\varepsilon$ , where the value of  $u_\varepsilon$  is assumed to be the trace of the function  $u_\varepsilon$  defined for  $x \in \Omega_\varepsilon$ . We also assume that the functions  $f$  and  $g \in C(\mathbb{R})$  are given, and satisfy that there exist exponents  $q_1$  and  $q_2$ , such that

$$p \leq q_1 < +\infty, \text{ if } p = N \quad \text{and} \quad 2 \leq q_1 \leq \frac{Np}{N-p}, \text{ if } p \in [2, N), \quad (3)$$

$$2 \leq q_2 < +\infty, \text{ if } p = N \quad \text{and} \quad 2 \leq q_2 \leq \frac{(N-1)p}{N-p}, \text{ if } p \in [2, N), \quad (4)$$

and constants  $\alpha_1 > 0$ ,  $\alpha_2 > 0$ ,  $\beta > 0$ , and  $l > 0$ , such that

$$\alpha_1 |s|^{q_1} - \beta \leq f(s)s \leq \alpha_2 |s|^{q_1} + \beta, \quad \text{for all } s \in \mathbb{R}, \quad (5)$$

$$\alpha_1 |s|^{q_2} - \beta \leq g(s)s \leq \alpha_2 |s|^{q_2} + \beta, \quad \text{for all } s \in \mathbb{R}, \quad (6)$$

$$(f(s_1) - f(s_2))(s_1 - s_2) \geq -l(s_1 - s_2)^2, \quad \text{for all } s_1, s_2 \in \mathbb{R}, \quad (7)$$

and

$$(g(s_1) - g(s_2))(s_1 - s_2) \geq -l(s_1 - s_2)^2, \quad \text{for all } s_1, s_2 \in \mathbb{R}. \quad (8)$$

Finally, we also assume that

$$u_\varepsilon^0 \in L^2(\Omega), \quad \psi_\varepsilon^0 \in L^2(\partial F_\varepsilon), \quad (9)$$

are given, and we suppose that

$$|u_\varepsilon^0|_{\Omega_\varepsilon}^2 + \varepsilon |\psi_\varepsilon^0|_{\partial F_\varepsilon}^2 \leq C, \quad (10)$$

where  $C$  is a positive constant, and we denote by  $|\cdot|_{\Omega_\varepsilon}$  and  $|\cdot|_{\partial F_\varepsilon}$  the norm in  $L^2(\Omega_\varepsilon)$  and  $L^2(\partial F_\varepsilon)$ , respectively.

In [2], we consider the homogenization of the problem (2) with  $p = 2$  and we obtain rigorously a nonlinear parabolic problem with zero Dirichlet boundary condition and with extra-terms coming from the influence of the dynamical boundary conditions as the homogenized model. We would like to highlight that analyzing a  $p$ -Laplacian problem involves additional (nontrivial) difficulties compared to the study for the Laplacian. Due to the presence of  $p$ -Laplacian operator in the domain, the variational formulation of the  $p$ -Laplacian reaction-diffusion equation is different than in [2]. We have to work in the space

$$V_p := \{(v, \gamma_0(v)) : v \in W^{1,p}(\Omega_\varepsilon), \gamma_0(v) = 0 \text{ on } \partial\Omega\}, \quad (11)$$

where  $\gamma_0$  denotes the trace operator  $v \in W^{1,p}(\Omega_\varepsilon) \mapsto v|_{\partial\Omega_\varepsilon} \in W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)$ . New  $W^{1,p}$ -estimates are needed to deal with problem (2). To prove these estimates rigorously, we use the Galerkin approximations where we have to introduce a special basis consisting of functions in the space

$$\mathcal{V}_s := \{(v, \gamma_0(v)) : v \in W^{s,2}(\Omega_\varepsilon), \gamma_0(v) = 0 \text{ on } \partial\Omega\}, \quad s \geq \frac{N(p-2)}{2p} + 1,$$

in the sense of Lions [20, p. 161]. Therefore, thanks to the assumption made on  $s$ , we have  $\mathcal{V}_s \subset V_p$ . We use the so-called energy method introduced by Tartar [24] and considered by many authors (see, for instance, Cioranescu and Donato [6]). Finally, in order to identify the limit equation, it is necessary to use monotonicity arguments. We prove the following theorem:

**Theorem 1.1** (Main Theorem). *Under the assumptions (3), (5)–(8) and (10), assume that  $g \in C^1(\mathbb{R})$ , the exponent  $q_2$  satisfies that*

$$p \leq q_2 < +\infty \text{ if } p = N \quad \text{and} \quad 2 \leq q_2 \leq \frac{(N-1)p}{N-p} \text{ if } p \in [2, N), \quad (12)$$

and  $(u_\varepsilon^0, \psi_\varepsilon^0) \in V_p$ . Let  $(u_\varepsilon, \psi_\varepsilon)$  be the unique solution of the problem (2), where  $\psi_\varepsilon(t) = \gamma_0(u_\varepsilon(t))$  a.e.  $t \in (0, T]$ . Then, as  $\varepsilon \rightarrow 0$ , we have

$$\tilde{u}_\varepsilon(t) \rightarrow u(t) \quad \text{strongly in } L^p(\Omega), \quad \forall t \in [0, T],$$

where  $\tilde{\cdot}$  denotes the  $W^{1,p}$ -extension to  $\Omega \times (0, T)$  and  $u$  is the unique solution of the following problem

$$\begin{cases} \left( \frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \frac{\partial u}{\partial t} - \operatorname{div} b(\nabla u) + \frac{|Y^*|}{|Y|} (\kappa |u|^{p-2} u + f(u)) + \frac{|\partial F|}{|Y|} g(u) = 0, & \text{in } \Omega \times (0, T), \\ u(x, 0) = u_0(x), & \text{for } x \in \Omega, \\ u = 0, & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (13)$$

For any  $\zeta \in \mathbb{R}^N$ , if  $w(y)$  is the solution of the problem

$$\begin{cases} \int_{Y^*} |\nabla_y w(y)|^{p-2} \nabla_y w(y) \cdot \nabla_y \varphi(y) dy = 0 & \forall \varphi \in \mathbb{H}_{\text{per}}(Y^*), \\ w \in \zeta \cdot y + \mathbb{H}_{\text{per}}(Y^*), \end{cases} \quad (14)$$

then  $b$  is defined by

$$b(\zeta) = \frac{1}{|Y|} \int_{Y^*} |\nabla v|^{p-2} \nabla v dy, \quad (15)$$

where  $\mathbb{H}_{\text{per}}(Y^*)$  is the space of functions from  $W^{1,p}(Y^*)$  which has the same trace on the opposite faces of  $Y$ .

**Remark 1.2.** Note that in the case  $p = 2$  (i.e., if the diffusion is modeled by the Laplacian operator), the homogenized equation (13) is exactly the equation obtained in [2].

**Remark 1.3.** In this setting, a typical nonlinearity in the applications is a odd degree polynomial,  $\sum_{j=0}^{2k+1} c_j s^j$ , where  $c_{2k+1} > 0$ .

The article is organized as follows. In Section 2, we introduce suitable functions spaces for our considerations. We recall here some well-know facts on the Sobolev spaces  $W^{s,p}$ . To prove the main result, in Section 3 we prove the existence and uniqueness of solution of (2), *a priori* estimates are established in Section 4 and some compactness results are proved in Section 5. Finally, the proof of Theorem 1.1 is established in Section 6.

## 2 Functional setting

**Notation:** We denote by  $(\cdot, \cdot)_{\Omega_\varepsilon}$  (respectively,  $(\cdot, \cdot)_{\partial F_\varepsilon}$ ) the inner product in  $L^2(\Omega_\varepsilon)$  (respectively, in  $L^2(\partial F_\varepsilon)$ ), and by  $|\cdot|_{\Omega_\varepsilon}$  (respectively,  $|\cdot|_{\partial F_\varepsilon}$ ) the associated norm. We also denote by  $(\cdot, \cdot)_{\Omega_\varepsilon}$  the inner product in  $(L^2(\Omega_\varepsilon))^N$ .

If  $r \neq 2$ , we will also denote by  $(\cdot, \cdot)_{\Omega_\varepsilon}$  (respectively,  $(\cdot, \cdot)_{\partial F_\varepsilon}$ ) the duality product between  $L^{r'}(\Omega_\varepsilon)$  and  $L^r(\Omega_\varepsilon)$  (respectively, the duality product between  $L^{r'}(\partial F_\varepsilon)$  and  $L^r(\partial F_\varepsilon)$ ). We will denote by  $|\cdot|_{r, \Omega_\varepsilon}$  (respectively  $|\cdot|_{r, \partial F_\varepsilon}$ ) the norm in  $L^r(\Omega_\varepsilon)$  (respectively in  $L^r(\partial F_\varepsilon)$ ).

We denote by  $(\cdot, \cdot)_\Omega$  the inner product in  $L^2(\Omega)$ , and by  $|\cdot|_\Omega$  the associated norm. If  $r \neq 2$ , we will also denote by  $(\cdot, \cdot)_\Omega$  the duality product between  $L^{r'}(\Omega)$  and  $L^r(\Omega)$ . We will denote by  $|\cdot|_{r, \Omega}$  the norm in  $L^r(\Omega)$ .

**The Sobolev spaces  $W^{s,p}(\Omega_\varepsilon)$ :**

We recall here, for the reader's convenience, some well-know facts on the Sobolev spaces  $W^{s,p}$ . We refer to Adams and Fournier [1], Brezis [4, Chapter 9] and Nečas [21, Chapter 2] for more details and proofs.

For any positive integer  $s$  and  $p \geq 1$ , we define the Sobolev space  $W^{s,p}(\Omega_\varepsilon)$  to be the completion of  $C^s(\overline{\Omega_\varepsilon})$ , with respect to the norm

$$\|v\|_{s,p,\Omega_\varepsilon} = \left( \sum_{0 \leq |\alpha| \leq s} |D^\alpha v|_{p,\Omega_\varepsilon}^p \right)^{1/p}.$$

Observe that  $W^{s,p}(\Omega_\varepsilon)$  is a Banach space.

By the Sobolev Imbedding Theorem (see [1, Chapter 4, p. 99]), we have the imbedding

$$W^{s,p}(\Omega_\varepsilon) \subset W^{1,r}(\Omega_\varepsilon), \quad (16)$$

where  $s \geq 1$  and  $\frac{1}{p} - \frac{s-1}{N} \leq \frac{1}{r} \leq \frac{1}{p}$ .

In particular, for  $p \geq 1$ , we define the Sobolev space  $W^{1,p}(\Omega_\varepsilon)$  to be the completion of  $C^1(\overline{\Omega_\varepsilon})$ , with respect to the norm

$$\|v\|_{p,\Omega_\varepsilon} := \left( |v|_{p,\Omega_\varepsilon}^p + |\nabla v|_{p,\Omega_\varepsilon}^p \right)^{1/p}.$$

We set  $H^1(\Omega_\varepsilon) = W^{1,2}(\Omega_\varepsilon)$ .

By  $\|\cdot\|_{p,\Omega_\varepsilon,T}$  we denote the norm in  $L^p(0, T; W^{1,p}(\Omega_\varepsilon))$ . By  $|\cdot|_{r,\Omega_\varepsilon,T}$  (respectively,  $|\cdot|_{r,\partial F_\varepsilon,T}$ ), we denote the norm in  $L^r(0, T; L^r(\Omega_\varepsilon))$  (respectively,  $L^r(0, T; L^r(\partial F_\varepsilon))$ ).

By  $\|\cdot\|_{p,\Omega}$  we denote the norm in  $W^{1,p}(\Omega)$ , by  $\|\cdot\|_{p,\Omega,T}$  we denote the norm in  $L^p(0, T; W^{1,p}(\Omega))$  and we denote by  $|\cdot|_{r,\Omega,T}$  the norm in  $L^r(0, T; L^r(\Omega))$ .

We define

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p = N. \end{cases}$$

We have the continuous embedding

$$W^{1,p}(\Omega_\varepsilon) \subset \begin{cases} L^r(\Omega_\varepsilon) & \text{if } p < N \text{ and } r = p^*, \\ L^r(\Omega_\varepsilon) & \text{if } p = N \text{ and } p \leq r < p^*. \end{cases} \quad (17)$$

By Rellich-Kondrachov Theorem (see [4, Chapter 9, Theorem 9.16]), we have the compact embedding

$$W^{1,p}(\Omega_\varepsilon) \subset \begin{cases} L^r(\Omega_\varepsilon) & \text{if } p < N \text{ and } 1 \leq r < p^*, \\ L^r(\Omega_\varepsilon) & \text{if } p = N \text{ and } p \leq r < p^*. \end{cases} \quad (18)$$

In particular, we have the compact embedding

$$W^{1,p}(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon), \quad \forall 2 \leq p \leq N. \quad (19)$$

One can define a family of spaces intermediate between  $L^p$  and  $W^{1,p}$ . More precisely for  $p \geq 1$  we define the fractional order Sobolev space

$$W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) := \left\{ v \in L^p(\partial\Omega_\varepsilon); \frac{|v(x) - v(y)|}{|x - y|^{1-\frac{1}{p} + \frac{N}{p}}} \in L^p(\partial\Omega_\varepsilon \times \partial\Omega_\varepsilon) \right\},$$

equipped with the natural norm. We set  $H^{1/2}(\partial\Omega_\varepsilon) = W^{\frac{1}{2},2}(\partial\Omega_\varepsilon)$ . These spaces play an important role in the theory of traces.

We denote by  $\gamma_0$  the trace operator  $v \mapsto v|_{\partial\Omega_\varepsilon}$ . The trace operator belongs to  $\mathcal{L}(W^{1,p}(\Omega_\varepsilon), W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon))$ , and we will use  $\|\gamma_0\|$  to denote the norm of  $\gamma_0$  in this space.

We will use  $\|\cdot\|_{p,\partial\Omega_\varepsilon}$  to denote the norm in  $W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)$ , which is given by

$$\|\phi\|_{p,\partial\Omega_\varepsilon} = \inf\{\|v\|_{p,\Omega_\varepsilon} : \gamma_0(v) = \phi\}.$$

We define

$$p_b^* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p = N. \end{cases}$$

We have the continuous embedding

$$W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) \subset \begin{cases} L^r(\partial\Omega_\varepsilon) & \text{if } p < N \quad \text{and} \quad r = p_b^*, \\ L^r(\partial\Omega_\varepsilon) & \text{if } p = N \quad \text{and} \quad 1 \leq r < p_b^*. \end{cases} \quad (20)$$

By [21, Chaper 2, Theorem 6.2], we have the compact embedding

$$W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) := \gamma_0(W^{1,p}(\Omega_\varepsilon)) \subset L^r(\partial\Omega_\varepsilon) \quad \text{if } 1 \leq r < p_b^*. \quad (21)$$

In particular, we have the compact embedding

$$W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) \subset L^2(\partial\Omega_\varepsilon) \quad \forall 2 \leq p \leq N. \quad (22)$$

**Some notations:** Finally, for any positive integer  $s$  and  $p \geq 1$ , we denote by  $W_{\partial\Omega}^{s,p}(\Omega_\varepsilon)$  and  $W_{\partial\Omega}^{s,p}(\partial\Omega_\varepsilon)$  the standard Sobolev spaces which are closed subspaces of  $W^{s,p}(\Omega_\varepsilon)$  and  $W^{s,p}(\partial\Omega_\varepsilon)$ , respectively, and the subscript  $\partial\Omega$  means that, respectively, traces or functions in  $\partial\Omega_\varepsilon$ , vanish on this part of the boundary of  $\Omega_\varepsilon$ , i.e.

$$W_{\partial\Omega}^{s,p}(\Omega_\varepsilon) = \{v \in W^{s,p}(\Omega_\varepsilon) : \gamma_0(v) = 0 \text{ on } \partial\Omega\},$$

and

$$W_{\partial\Omega}^{s,p}(\partial\Omega_\varepsilon) = \{v \in W^{s,p}(\partial\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\}.$$

Let us notice that, in fact, we can consider an element of  $W^{1-\frac{1}{p},p}(\partial F_\varepsilon)$  as an element of  $W_{\partial\Omega}^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)$ .

Analogously, for  $r \geq 2$ , we denote

$$L_{\partial\Omega}^r(\partial\Omega_\varepsilon) := \{v \in L^r(\partial\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\}.$$

Let us notice that, in fact, we can consider the given  $\psi_\varepsilon^0$  as an element of  $L_{\partial\Omega}^2(\partial\Omega_\varepsilon)$ .

Let us consider the space

$$H_r := L^r(\Omega_\varepsilon) \times L_{\partial\Omega}^r(\partial\Omega_\varepsilon), \quad \forall r \geq 2,$$

with the natural inner product  $((v, \phi), (w, \varphi))_{H_r} = (v, w)_{\Omega_\varepsilon} + \varepsilon(\phi, \varphi)_{\partial F_\varepsilon}$ , which in particular induces the norm  $|(\cdot, \cdot)|_{H_r}$  given by

$$|(v, \phi)|_{H_r}^r = |v|_{r,\Omega_\varepsilon}^r + \varepsilon|\phi|_{r,\partial F_\varepsilon}^r, \quad (v, \phi) \in H_r.$$

For the sake of clarity, we shall omit to write explicitly the index  $r$  if  $r = 2$ , so we denote by  $H$  the Hilbert space

$$H := L^2(\Omega_\varepsilon) \times L_{\partial\Omega}^2(\partial\Omega_\varepsilon).$$

For each  $p > 1$ , we consider the space

$$V_p := \left\{ (v, \gamma_0(v)) : v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon) \right\}.$$

We note that  $V_p$  is a closed vector subspace of  $W_{\partial\Omega}^{1,p}(\Omega_\varepsilon) \times W_{\partial\Omega}^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)$ , and we endow it with the norm  $\|(\cdot, \cdot)\|_{V_p}$  given by

$$\|(v, \gamma_0(v))\|_{V_p}^p = \|v\|_{p, \Omega_\varepsilon}^p + \|\gamma_0(v)\|_{p, \partial F_\varepsilon}^p, \quad (v, \gamma_0(v)) \in V_p.$$

Moreover,  $V_p$  is a reflexive and separable space.

Let  $p'$ ,  $q'_1$  and  $q'_2$  be the conjugate exponents of  $p$ ,  $q_1$  and  $q_2$ , respectively. Taking into account the continuous embeddings (17) and (20), and the assumptions (3)-(4), we have the following useful continuous inclusions

$$V_p \subset W^{1,p}(\Omega_\varepsilon) \subset L^{q_1}(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon), \quad V_p \subset W^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) \subset L^{q_2}(\partial\Omega_\varepsilon) \subset L^2(\partial\Omega_\varepsilon), \quad (23)$$

and

$$L^2(\Omega_\varepsilon) \subset L^{q'_1}(\Omega_\varepsilon) \subset (W^{1,p}(\Omega_\varepsilon))' \subset V'_p, \quad (24)$$

where  $(W^{1,p}(\Omega_\varepsilon))'$  and  $V'_p$  denote the dual of  $W^{1,p}(\Omega_\varepsilon)$  and  $V_p$ , respectively. Note that  $(W^{1,p}(\Omega_\varepsilon))'$  is a subspace of  $W^{-1,p'}(\Omega_\varepsilon)$ , where  $W^{-1,p'}(\Omega_\varepsilon)$  denotes the dual of the Sobolev space  $W_0^{1,p}(\Omega_\varepsilon) := \overline{\mathcal{D}(\Omega_\varepsilon)}^{W^{1,p}(\Omega_\varepsilon)}$ .

Taking into account the compact embeddings (19) and (22), we have the compact embedding

$$V_p \subset H \quad \forall 2 \leq p \leq N. \quad (25)$$

Finally, for  $s \geq 1$ , we consider the space

$$\mathcal{V}_s := \left\{ (v, \gamma_0(v)) : v \in W_{\partial\Omega}^{s,2}(\Omega_\varepsilon) \right\}.$$

We note that  $\mathcal{V}_s$  is a closed vector subspace of  $W_{\partial\Omega}^{s,2}(\Omega_\varepsilon) \times W_{\partial\Omega}^{s-\frac{1}{2},2}(\partial\Omega_\varepsilon)$ .

Observe that by (16), we have that  $W^{s,2}(\Omega_\varepsilon) \subset W^{1,2}(\Omega_\varepsilon)$  and by Rellich's Theorem (see [21, Chapter 1, Theorem 1.4]), we obtain the compact embedding

$$W^{1,2}(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon),$$

and by [21, Chapter 1, Exercise 1.2], we have the compact embedding

$$W^{s-\frac{1}{2},2}(\partial\Omega_\varepsilon) := \gamma_0(W^{s,2}(\Omega_\varepsilon)) \subset L^2(\partial\Omega_\varepsilon).$$

Thus, we can deduce the compact embedding

$$\mathcal{V}_s \subset H. \quad (26)$$

### 3 Existence and uniqueness of solution

Along this paper, we shall denote by  $C$  different constants which are independent of  $\varepsilon$ . We state in this section a result on the existence and uniqueness of solution of problem (2). It is easy to see from (5) and (6) that there exists a constant  $C > 0$  such that

$$|f(s)| \leq C \left(1 + |s|^{q_1-1}\right), \quad |g(s)| \leq C \left(1 + |s|^{q_2-1}\right), \quad \text{for all } s \in \mathbb{R}. \quad (27)$$

**Definition 3.1.** A weak solution of (2) is a pair of functions  $(u_\varepsilon, \psi_\varepsilon)$ , satisfying

$$u_\varepsilon \in C([0, T]; L^2(\Omega_\varepsilon)), \quad \psi_\varepsilon \in C([0, T]; L^2_{\partial\Omega}(\partial\Omega_\varepsilon)), \quad \text{for all } T > 0, \quad (28)$$

$$u_\varepsilon \in L^p(0, T; W^{1,p}(\Omega_\varepsilon)), \quad \text{for all } T > 0, \quad (29)$$

$$\psi_\varepsilon \in L^p(0, T; W_{\partial\Omega}^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon)), \quad \text{for all } T > 0, \quad (30)$$

$$\gamma_0(u_\varepsilon(t)) = \psi_\varepsilon(t), \quad \text{a.e. } t \in (0, T], \quad (31)$$

$$\begin{cases} \frac{d}{dt}(u_\varepsilon(t), v)_{\Omega_\varepsilon} + \varepsilon \frac{d}{dt}(\psi_\varepsilon(t), \gamma_0(v))_{\partial F_\varepsilon} + (|\nabla u_\varepsilon(t)|^{p-2} \nabla u_\varepsilon(t), \nabla v)_{\Omega_\varepsilon} + \kappa(|u_\varepsilon(t)|^{p-2} u_\varepsilon(t), v)_{\Omega_\varepsilon} \\ + (f(u_\varepsilon(t)), v)_{\Omega_\varepsilon} + \varepsilon (g(\psi_\varepsilon(t)), \gamma_0(v))_{\partial F_\varepsilon} = 0 \\ \text{in } \mathcal{D}'(0, T), \text{ for all } v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon), \end{cases} \quad (32)$$

$$u_\varepsilon(0) = u_\varepsilon^0, \quad \text{and} \quad \psi_\varepsilon(0) = \psi_\varepsilon^0. \quad (33)$$

We have the following result.

**Theorem 3.2.** Under the assumptions (3)–(9), there exists a unique solution  $(u_\varepsilon, \psi_\varepsilon)$  of the problem (2). Moreover, this solution satisfies the energy equality

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (|(u_\varepsilon(t), \psi_\varepsilon(t))|_H^2 + |\nabla u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + \kappa |u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p \\ & + (f(u_\varepsilon(t)), u_\varepsilon(t))_{\Omega_\varepsilon} + \varepsilon (g(\psi_\varepsilon(t)), \psi_\varepsilon(t))_{\partial F_\varepsilon}) = 0, \quad \text{a.e. } t \in (0, T). \end{aligned} \quad (34)$$

*Proof.* The proof of this result is standard. For the sake of completeness, we give a sketch of a proof.

First, we prove that  $V_p$  is densely embedded in  $H$ . In fact, if we consider  $(w, \phi) \in H$  such that

$$(v, w)_{\Omega_\varepsilon} + \varepsilon (\gamma_0(v), \phi)_{\partial F_\varepsilon} = 0, \quad \text{for all } v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon),$$

in particular, we have

$$(v, w)_{\Omega_\varepsilon} = 0, \quad \text{for all } v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon),$$

and therefore  $w = 0$ . Consequently,

$$(\gamma_0(v), \phi)_{\partial F_\varepsilon} = 0, \quad \text{for all } v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon),$$

and then, as  $W_{\partial\Omega}^{1-\frac{1}{p},p}(\partial\Omega_\varepsilon) = \gamma_0(W_{\partial\Omega}^{1,p}(\Omega_\varepsilon))$  is dense in  $L^2_{\partial\Omega}(\partial\Omega_\varepsilon)$ , we have that  $\phi = 0$ .

Now, on the space  $V_p$  we define the nonlinear monotone operator  $A_p : V_p \rightarrow V'_p$ , given by

$$\langle A_p((v, \gamma_0(v))), (w, \gamma_0(w)) \rangle = (|\nabla v|^{p-2} \nabla v, \nabla w)_{\Omega_\varepsilon} + \kappa(|v|^{p-2} v, w)_{\Omega_\varepsilon}, \quad (35)$$

for all  $v, w \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon)$ .

We observe that  $A_p$  is coercive. In fact, we have

$$\begin{aligned} \langle A_p((v, \gamma_0(v))), (v, \gamma_0(v)) \rangle & \geq \min\{1, \kappa\} \|v\|_{p, \Omega_\varepsilon}^p \\ & = \frac{1}{1 + \|\gamma_0\|^p} \min\{1, \kappa\} \|v\|_{p, \Omega_\varepsilon}^p + \frac{\|\gamma_0\|^p}{1 + \|\gamma_0\|^p} \min\{1, \kappa\} \|v\|_{p, \Omega_\varepsilon}^p \\ & \geq \frac{1}{1 + \|\gamma_0\|^p} \min\{1, \kappa\} \|(v, \gamma_0(v))\|_{V_p}^p, \end{aligned} \quad (36)$$

for all  $v \in W_{\partial\Omega}^{1,p}(\Omega_\varepsilon)$ .



Let us denote

$$\begin{aligned} V_1 &= V_p, & V_2 &= L^{q_1}(\Omega_\varepsilon) \times L^2_{\partial\Omega}(\partial\Omega_\varepsilon), & V_3 &= L^2(\Omega_\varepsilon) \times L^{q_2}_{\partial\Omega}(\partial\Omega_\varepsilon), \\ A_1 &= A_p, & A_2(v, \phi) &= (f(v), 0), & A_3(v, \phi) &= (0, \varepsilon g(\phi)). \end{aligned}$$

From (27) one deduces that  $A_i : V_i \rightarrow V'_i$  for  $i = 2, 3$ .

With this notation, and denoting  $V = \bigcap_{i=1}^3 V_i$ ,  $p_1 = p$ ,  $p_2 = q_1$ ,  $p_3 = q_2$ ,  $\vec{u}_\varepsilon = (u_\varepsilon, \psi_\varepsilon)$ , one has that (28)–(33) is equivalent to

$$\vec{u}_\varepsilon \in C([0, T]; H), \quad \vec{u}_\varepsilon \in \bigcap_{i=1}^3 L^{p_i}(0, T; V_i), \quad \text{for all } T > 0, \quad (37)$$

$$(\vec{u}_\varepsilon)'(t) + \sum_{i=1}^3 A_i(\vec{u}_\varepsilon(t)) = 0 \quad \text{in } \mathcal{D}'(0, T; V'), \quad (38)$$

$$\vec{u}_\varepsilon(0) = (u_\varepsilon^0, \psi_\varepsilon^0). \quad (39)$$

Taking into account (23)–(24) and applying a slight modification of [20, Ch.2, Th.1.4], it is not difficult to see that problem (37)–(39) has a unique solution. Moreover,  $\vec{u}_\varepsilon$  satisfies the energy equality

$$\frac{1}{2} \frac{d}{dt} |\vec{u}_\varepsilon(t)|_H^2 + \sum_{i=1}^3 \langle A_i(\vec{u}_\varepsilon(t)), \vec{u}_\varepsilon(t) \rangle_i = 0 \quad \text{a.e. } t \in (0, T),$$

where  $\langle \cdot, \cdot \rangle_i$  denotes the duality product between  $V'_i$  and  $V_i$ . This last equality turns out to be just (34).  $\square$

## 4 A priori estimates

In this section we obtain some energy estimates for the solution of (2). By (34) and taking into account (5)–(6), we have

$$\begin{aligned} \frac{d}{dt} (|(u_\varepsilon(t), \psi_\varepsilon(t))|_H^2) + 2|\nabla u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + 2\kappa|u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + 2\alpha_1|u_\varepsilon(t)|_{q_1, \Omega_\varepsilon}^{q_1} \\ + 2\alpha_1\varepsilon|\psi_\varepsilon(t)|_{q_2, \partial F_\varepsilon}^{q_2} \leq 2\beta(|\Omega_\varepsilon| + \varepsilon|\partial F_\varepsilon|), \end{aligned} \quad (40)$$

where  $|\Omega_\varepsilon|$  and  $|\partial F_\varepsilon|$  denotes the measure of  $\Omega_\varepsilon$  and  $\partial F_\varepsilon$ , respectively.

Observe that the number of holes is given by

$$N(\varepsilon) = \frac{|\Omega|}{(2\varepsilon)^N} (1 + o(1)),$$

then using the change of variable

$$y = \frac{x}{\varepsilon}, \quad d\sigma(y) = \varepsilon^{-(N-1)} d\sigma(x),$$

we can deduce

$$|\partial F_\varepsilon| = N(\varepsilon)|\partial F_{k, \varepsilon}| = N(\varepsilon)\varepsilon^{N-1}|\partial F| \leq \frac{C}{\varepsilon}.$$

And since  $|\Omega_\varepsilon| \leq |\Omega|$ , we have that

$$|\Omega_\varepsilon| + \varepsilon|\partial F_\varepsilon| \leq C. \quad (41)$$

Let us denote

$$\mathcal{F}(s) := \int_0^s f(r) dr \quad \text{and} \quad \mathcal{G}(s) := \int_0^s g(r) dr.$$

Then, there exist positive constants  $\tilde{\alpha}_1$ ,  $\tilde{\alpha}_2$ , and  $\tilde{\beta}$  such that

$$\tilde{\alpha}_1|s|^{q_1} - \tilde{\beta} \leq \mathcal{F}(s) \leq \tilde{\alpha}_2|s|^{q_1} + \tilde{\beta} \quad \forall s \in \mathbb{R}, \quad (42)$$

and

$$\tilde{\alpha}_1|s|^{q_2} - \tilde{\beta} \leq \mathcal{G}(s) \leq \tilde{\alpha}_2|s|^{q_2} + \tilde{\beta} \quad \forall s \in \mathbb{R}. \quad (43)$$

**Lemma 4.1.** *Assume the assumptions (3)–(8) and (10). Then, for any initial condition  $(u_\varepsilon^0, \psi_\varepsilon^0) \in V_p$ , there exists a constant  $C$  independent of  $\varepsilon$ , such that the solution  $(u_\varepsilon, \psi_\varepsilon)$  of the problem (2) satisfies*

$$\|u_\varepsilon\|_{p, \Omega_\varepsilon, T} \leq C, \quad \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{p, \Omega_\varepsilon} \leq C. \quad (44)$$

*Proof.* Taking into account (41) in (40), we obtain

$$\frac{d}{dt} (|(u_\varepsilon(t), \psi_\varepsilon(t))|_H^2) + 2|\nabla u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + 2\kappa|u_\varepsilon(t)|_{p, \Omega_\varepsilon}^p + 2\alpha_1|u_\varepsilon(t)|_{q_1, \Omega_\varepsilon}^{q_1} + 2\alpha_1\varepsilon|\psi_\varepsilon(t)|_{q_2, \partial F_\varepsilon}^{q_2} \leq C. \quad (45)$$

From (35)–(36), in particular, we can deduce

$$\frac{d}{dt} (|(u_\varepsilon(t), \psi_\varepsilon(t))|_H^2) + \frac{2 \min\{1, \kappa\}}{1 + \|\gamma_0\|^p} \|(u_\varepsilon(t), \gamma_0(u_\varepsilon(t)))\|_{V_p}^p \leq C. \quad (46)$$

Integrating between 0 and  $t$  and taking into account (10), in particular, we have the first estimate in (44).

Now, if we want to take the inner product in the problem (2) with  $u'_\varepsilon$ , we need that  $u'_\varepsilon \in L^p(0, T; W_{\partial\Omega}^{1,p}(\Omega_\varepsilon)) \cap L^{q_1}(0, T; L^{q_1}(\Omega_\varepsilon))$  with  $\gamma_0(u'_\varepsilon) \in L^{q_2}(0, T; L_{\partial\Omega}^{q_2}(\partial\Omega_\varepsilon))$ . However, we do not have it for our weak solution. Therefore, we use the Galerkin method in order to prove, rigorously, new *a priori* estimates for  $u_\varepsilon$ .

At first, we introduce a special basis consisting of functions  $(w_j, \gamma_0(w_j)) \in \mathcal{V}_s$  with  $s \geq \frac{N(p-2)}{2p} + 1$  in the sense of [20, Chapter 2, Remark 1.6, p. 161]. Therefore, thanks to the assumption made on  $s$ , taking into account (16), we have  $\mathcal{V}_s \subset V_p$ . The scalar product in  $H$  generates on  $\mathcal{V}_s \subset H$  the bilinear functional  $((v, \gamma_0(v)), (w, \gamma_0(w)))_H$  which can be represented in the form  $((v, \gamma_0(v)), (w, \gamma_0(w)))_H = \langle L((v, \gamma_0(v))), (w, \gamma_0(w)) \rangle_{\mathcal{V}_s}$ , where  $L$  is a self-adjoint operator. The compact embedding (26) implies the compactness of the operator  $L$ . Hence,  $L$  has a complete system of eigenvectors  $\{(w_j, \gamma_0(w_j)) : j \geq 1\}$ . These vectors are orthonormal in  $H$  and orthogonal in  $\mathcal{V}_s$ . Observe that  $\text{span}\{(w_j, \gamma_0(w_j)) : j \geq 1\}$  is dense in  $V_p$ .

Taking into account the above facts, we denote by

$$(u_{\varepsilon, m}(t), \gamma_0(u_{\varepsilon, m}(t))) = (u_{\varepsilon, m}(t; 0, u_\varepsilon^0, \psi_\varepsilon^0), \gamma_0(u_{\varepsilon, m}(t; 0, u_\varepsilon^0, \psi_\varepsilon^0)))$$

the Galerkin approximation of the solution  $(u_\varepsilon(t; 0, u_\varepsilon^0, \psi_\varepsilon^0), \gamma_0(u_\varepsilon(t; 0, u_\varepsilon^0, \psi_\varepsilon^0)))$  to (2) for each integer  $m \geq 1$ , which is given by

$$(u_{\varepsilon, m}(t), \gamma_0(u_{\varepsilon, m}(t))) = \sum_{j=1}^m \delta_{\varepsilon m j}(t)(w_j, \gamma_0(w_j)), \quad (47)$$

and is the solution of

$$\begin{aligned} & \frac{d}{dt} ((u_{\varepsilon, m}(t), \gamma_0(u_{\varepsilon, m}(t))), (w_j, \gamma_0(w_j)))_H + \langle A_p((u_{\varepsilon, m}(t), \gamma_0(u_{\varepsilon, m}(t))), (w_j, \gamma_0(w_j))) \\ & + (f(u_{\varepsilon, m}(t)), w_j)_{\Omega_\varepsilon} + \varepsilon(g(\gamma_0(u_{\varepsilon, m}(t))), \gamma_0(w_j))_{\partial F_\varepsilon} = 0, \quad j = 1, \dots, m, \end{aligned} \quad (48)$$

with initial data

$$(u_{\varepsilon, m}(0), \gamma_0(u_{\varepsilon, m}(0))) = (u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0)), \quad (49)$$

and

$$\delta_{\varepsilon m j}(t) = (u_{\varepsilon, m}(t), w_j)_{\Omega_\varepsilon} + (\gamma_0(u_{\varepsilon, m}(t)), \gamma_0(w_j))_{\partial F_\varepsilon},$$

and  $(u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0)) \in \text{span}\{(w_j, \gamma_0(w_j)) : j = 1, \dots, m\}$  converge (when  $m \rightarrow \infty$ ) to  $(u_\varepsilon^0, \psi_\varepsilon^0)$  in a suitable sense which will be specified below.

Let  $(u_\varepsilon^0, \psi_\varepsilon^0) \in V_p$ . For all  $m \geq 1$ , since  $\text{span}\{(w_j, \gamma_0(w_j)) : j \geq 1\}$  is densely embedded in  $V_p$ , there exists  $(u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0)) \in \text{span}\{(w_j, \gamma_0(w_j)) : 1 \leq j \leq m\}$ , such that the sequence  $\{(u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0))\}$  converges to  $(u_\varepsilon^0, \psi_\varepsilon^0)$  in  $V_p$ . Then, in particular we know that there exists a constant  $C$  such that

$$\|(u_{\varepsilon, m}^0, \gamma_0(u_{\varepsilon, m}^0))\|_{V_p} \leq C. \quad (50)$$

For each integer  $m \geq 1$ , we consider the sequence  $\{(u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))\}$  defined by (47)-(49) with these initial data.

Multiplying by the derivative  $\delta'_{\varepsilon m j}$  in (48), and summing from  $j = 1$  to  $m$ , we obtain

$$\begin{aligned} & |(u'_{\varepsilon,m}(t), \gamma_0(u'_{\varepsilon,m}(t)))|_H^2 + \frac{1}{p} \frac{d}{dt} (\langle A_p((u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))) \rangle, (u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))) \rangle) \\ & + (f(u_{\varepsilon,m}(t)), u'_{\varepsilon,m}(t))_{\Omega_\varepsilon} + \varepsilon (g(\gamma_0(u_{\varepsilon,m}(t))), \gamma_0(u'_{\varepsilon,m}(t)))_{\partial F_\varepsilon} = 0. \end{aligned} \quad (51)$$

We observe that

$$(f(u_{\varepsilon,m}(t)), u'_{\varepsilon,m}(t))_{\Omega_\varepsilon} = \frac{d}{dt} \int_{\Omega_\varepsilon} \mathcal{F}(u_{\varepsilon,m}(t)) dx,$$

and

$$(g(\gamma_0(u_{\varepsilon,m}(t))), \gamma_0(u'_{\varepsilon,m}(t)))_{\partial F_\varepsilon} = \frac{d}{dt} \int_{\partial F_\varepsilon} \mathcal{G}(\gamma_0(u_{\varepsilon,m}(t))) d\sigma(x).$$

Then, integrating (51) between 0 and  $t$ , and taking into account (36) and (41)-(43), we obtain

$$\begin{aligned} & \int_0^t |(u'_{\varepsilon,m}(s), \gamma_0(u'_{\varepsilon,m}(s)))|_H^2 ds + \frac{\min\{1, \kappa\}}{1 + \|\gamma_0\|^p} \frac{1}{p} \|(u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))\|_{V_p}^p \\ & + \tilde{\alpha}_1 \left( |u_{\varepsilon,m}(t)|_{q_1, \Omega_\varepsilon}^{q_1} + \varepsilon |\gamma_0(u_{\varepsilon,m}(t))|_{q_2, \partial F_\varepsilon}^{q_2} \right) \leq \frac{\max\{1, \kappa\}}{p} \|(u_{\varepsilon,m}^0, \gamma_0(u_{\varepsilon,m}^0))\|_{V_p}^p \\ & + \tilde{\alpha}_2 \left( |u_{\varepsilon,m}^0|_{q_1, \Omega_\varepsilon}^{q_1} + \varepsilon |\gamma_0(u_{\varepsilon,m}^0)|_{q_2, \partial F_\varepsilon}^{q_2} \right) + 2\tilde{\beta}C, \end{aligned}$$

for all  $t \in (0, T)$ . In order to estimate the right hand side of the last inequality we use (23) together with (50) and  $\varepsilon \ll 1$ . In particular, we can deduce

$$\int_0^t |(u'_{\varepsilon,m}(s), \gamma_0(u'_{\varepsilon,m}(s)))|_H^2 ds + \frac{\min\{1, \kappa\}}{1 + \|\gamma_0\|^p} \frac{1}{p} \|(u_{\varepsilon,m}(t), \gamma_0(u_{\varepsilon,m}(t)))\|_{V_p}^p \leq C, \quad (52)$$

for all  $t \in (0, T)$ . We have proved that the sequence  $\{(u_{\varepsilon,m}, \gamma_0(u_{\varepsilon,m}))\}$  is bounded in  $C([0, T]; V_p)$ , and  $\{(u'_{\varepsilon,m}, \gamma_0(u'_{\varepsilon,m}))\}$  is bounded in  $L^2(0, T; H)$ , for all  $T > 0$ .

If we work with the truncated Galerkin equations (47)-(49) instead of the full PDE, we note that the calculations of the proof of (46) can be following identically to show that  $\{(u_{\varepsilon,m}, \gamma_0(u_{\varepsilon,m}))\}$  is bounded in  $L^p(0, T; V_p)$ , for all  $T > 0$ .

Moreover, taking into account the uniqueness of solution to (2), using Aubin-Lions compactness lemma (e.g., cf. Lions [20]) and the fact that  $A_p$  is a monotone operator, it is not difficult to conclude that the sequence  $\{(u_{\varepsilon,m}, \gamma_0(u_{\varepsilon,m}))\}$  converges weakly in  $L^p(0, T; V_p)$  to the solution  $(u_\varepsilon, \gamma_0(u_\varepsilon))$  to (2). Since the inclusion  $V_p \subset H$  is compact (see (25)) and  $(u_\varepsilon, \gamma_0(u_\varepsilon)) \in C([0, T]; H)$ , it follows using [22, Lemma 11.2] that

$$\sup_{t \in [0, T]} \|(u_\varepsilon(t), \gamma_0(u_\varepsilon(t)))\|_{V_p} \leq C,$$

and, in particular, the second estimate in (44) is proved.  $\square$

**The extension of  $u_\varepsilon$  to the whole  $\Omega \times (0, T)$ :** since the solution  $u_\varepsilon$  of the problem (2) is defined only in  $\Omega_\varepsilon \times (0, T)$ , we need to extend it to the whole  $\Omega \times (0, T)$  to be able to state the convergence result. In order to do that, we shall use the following well-known extension result given by Donato and Moscarello [13, Lemma 2.4]: Let  $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$  be a  $W^{1,p}$ -extension of  $u_\varepsilon$ , that satisfies the following condition

$$|\nabla \tilde{u}_\varepsilon|_{p, \Omega} \leq C |\nabla u_\varepsilon|_{p, \Omega_\varepsilon}. \quad (53)$$

Using (53) and taking into account Lemma 4.1, we obtain some *a priori* estimates for the extension of  $u_\varepsilon$  to the whole  $\Omega \times (0, T)$ .

**Corollary 4.2.** *Assume the assumptions in Lemma 4.1. Then, there exists a  $W^{1,p}$ -extension  $\tilde{u}_\varepsilon$  of the solution  $u_\varepsilon$  of the problem (2) into  $\Omega \times (0, T)$ , such that*

$$\|\tilde{u}_\varepsilon\|_{p,\Omega,T} \leq C, \quad \sup_{t \in [0,T]} \|\tilde{u}_\varepsilon(t)\|_{p,\Omega} \leq C, \quad (54)$$

where the constant  $C$  does not depend on  $\varepsilon$ .

*Proof.* Using Poincaré's inequality, (53) and the first estimate in (44), we obtain

$$\|\tilde{u}_\varepsilon\|_{p,\Omega,T}^p = |\tilde{u}_\varepsilon|_{p,\Omega,T}^p + |\nabla \tilde{u}_\varepsilon|_{p,\Omega,T}^p \leq C |\nabla \tilde{u}_\varepsilon|_{p,\Omega,T}^p \leq C |\nabla u_\varepsilon|_{p,\Omega_\varepsilon,T}^p \leq C,$$

and the first estimate in (54) is proved. Similarly, using Poincaré's inequality, (53) and the second estimate in (44), we have the second estimate in (54).  $\square$

## 5 A compactness result

In this section, we obtain some compactness results about the behavior of the sequence  $\tilde{u}_\varepsilon$  satisfying the *a priori* estimates given in Corollary 4.2.

By  $\chi_{\Omega_\varepsilon}$  we denote the characteristic function of the domain  $\Omega_\varepsilon$ . Due to the periodicity of the domain  $\Omega_\varepsilon$ , from [7, Theorem 2.6] one has, for  $\varepsilon \rightarrow 0$ , that

$$\chi_{\Omega_\varepsilon} \xrightarrow{*} \frac{|Y^*|}{|Y|} \quad \text{weakly-star in } L^\infty(\Omega), \quad (55)$$

where the limit is the proportion of the material in the cell  $Y$ .

Let  $\xi_\varepsilon := |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon$  in  $\Omega_\varepsilon \times (0, T)$  and let us denote by  $\tilde{\xi}_\varepsilon$  its extension with zero to the whole of  $\Omega \times (0, T)$ , i.e.

$$\tilde{\xi}_\varepsilon = \begin{cases} \xi_\varepsilon & \text{in } \Omega_\varepsilon \times (0, T), \\ 0 & \text{in } (\Omega \setminus \overline{\Omega_\varepsilon}) \times (0, T). \end{cases} \quad (56)$$

**Proposition 5.1.** *Under the assumptions in Lemma 4.1, there exists a function  $u \in L^p(0, T; W_0^{1,p}(\Omega))$  ( $u$  will be the unique solution of the limit system (13)) and a function  $\xi \in L^{p'}(0, T; L^{p'}(\Omega))$  such that, at least after extraction of a subsequence, we have the following convergences for all  $T > 0$ ,*

$$\tilde{u}_\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly in } W_0^{1,p}(\Omega), \quad \forall t \in [0, T], \quad (57)$$

$$\tilde{u}_\varepsilon(t) \rightarrow u(t) \quad \text{strongly in } L^p(\Omega), \quad \forall t \in [0, T], \quad (58)$$

$$|\tilde{u}_\varepsilon(t)|^{p-2} \tilde{u}_\varepsilon(t) \rightarrow |u(t)|^{p-2} u(t) \quad \text{strongly in } L^{p'}(\Omega), \quad \forall t \in [0, T], \quad (59)$$

$$f(\tilde{u}_\varepsilon(t)) \rightarrow f(u(t)) \quad \text{strongly in } L^{q_1}(\Omega), \quad \forall t \in [0, T], \quad (60)$$

$$\tilde{\xi}_\varepsilon \rightharpoonup \xi \quad \text{weakly in } L^{p'}(0, T; L^{p'}(\Omega)), \quad (61)$$

where  $\tilde{\xi}_\varepsilon$  is given by (56).

Let  $q_2$  be the exponent satisfying (12). Let  $\bar{q} > 1$  given by

$$\bar{q} \in (1, p) \quad \text{if } p = N, \quad \bar{q} = \frac{Np}{(N-p)(q_2-2) + N} \quad \text{if } p < N.$$

Assume that  $g \in C^1(\mathbb{R})$ . Then, we have the following convergences for all  $T > 0$ ,

$$g(\tilde{u}_\varepsilon(t)) \rightarrow g(u(t)) \quad \text{strongly in } L^{\bar{q}}(\Omega), \quad \forall t \in [0, T], \quad (62)$$

$$g(\tilde{u}_\varepsilon(t)) \rightharpoonup g(u(t)) \quad \text{weakly in } W_0^{1,\bar{q}}(\Omega), \quad \forall t \in [0, T]. \quad (63)$$

*Proof.* By the first estimate in (54), we observe that the sequence  $\{\tilde{u}_\varepsilon\}$  is bounded in the space  $L^p(0, T; W_0^{1,p}(\Omega))$ , for all  $T > 0$ . Let us fix  $T > 0$ . Then, there exists a subsequence  $\{\tilde{u}_{\varepsilon'}\} \subset \{\tilde{u}_\varepsilon\}$  and function  $u \in L^p(0, T; W_0^{1,p}(\Omega))$  such that

$$\tilde{u}_{\varepsilon'} \rightharpoonup u \quad \text{weakly in } L^p(0, T; W_0^{1,p}(\Omega)). \quad (64)$$

By the second estimate in (54), for each  $t \in [0, T]$ , we have that  $\{\tilde{u}_\varepsilon(t)\}$  is bounded in  $W_0^{1,p}(\Omega)$ , and since we have (64), we can deduce (57). The conclusions of the Rellich-Kondrachov Theorem are also valid substituting  $W^{1,p}(\Omega_\varepsilon)$  for  $W_0^{1,p}(\Omega)$ , so using (18) and (57), we obtain

$$\tilde{u}_{\varepsilon'}(t) \rightarrow u(t) \quad \text{strongly in } L^r(\Omega), \quad \forall t \in [0, T], \quad (65)$$

where  $1 \leq r < p^*$  if  $p < N$  and  $p \leq r < p^*$  if  $p = N$ . Observe that, in particular, we have (58).

On the other hand, observe that

$$||v|^{p-2}v| \leq C(1 + |v|^{p-1}),$$

then, applying [10, Theorem 2.4] for  $G(x, v) = |v|^{p-2}v$ ,  $t = p'$  and  $r = p$ , we have that the map  $v \in L^p(\Omega) \mapsto |v|^{p-2}v \in L^{p'}(\Omega)$  is continuous in the strong topologies. Then, taking into account (65) with  $r = p$ , we get (59).

Thanks to (27), applying [10, Theorem 2.4] for  $G(x, v) = f(v)$ ,  $t = q_1'$  and  $r = q_1$ , we have that the map  $v \in L^{q_1}(\Omega) \mapsto f(v) \in L^{q_1'}(\Omega)$  is continuous in the strong topologies. Then, taking into account (65) with  $r = q_1$ , we get (60).

From the first estimate in (44) and (56), we have  $|\tilde{\xi}_\varepsilon|_{p', \Omega, T} \leq C$ , and hence, up a sequence, there exists  $\xi \in L^{p'}(0, T, L^{p'}(\Omega))$  such that we obtain (61).

By the arbitrariness of  $T > 0$ , all the convergences are satisfied, as we wanted to prove.

Now, we analyze the convergences for the nonlinear term  $g$ . We separate the cases  $p < N$  and  $p = N$ .

**Case 1:**  $p < N$ . Since

$$\frac{p^*}{\bar{q}} = \frac{(N-p)(q_2-2) + N}{N-p} = q_2 - 1 + \frac{p}{N-p} > q_2 - 1,$$

there exists  $r \in [2, p^*)$  such that  $\frac{r}{\bar{q}} \geq q_2 - 1$  and

$$|g(s)| \leq C \left(1 + |s|^{q_2-1}\right) \leq C \left(1 + |s|^{\frac{r}{\bar{q}}}\right).$$

Then, applying [10, Theorem 2.4] for  $G(x, v) = g(v)$ ,  $t = \bar{q}$  and  $r \in [2, p^*)$  such that  $\frac{r}{\bar{q}} \geq q_2 - 1$ , we have that the map  $v \in L^r(\Omega) \mapsto g(v) \in L^{\bar{q}}(\Omega)$  is continuous in the strong topologies. Then, taking into account (65), we get (62).

Finally, we prove (63). Observe that as we are assuming that  $g \in C^1(\mathbb{R})$ , it is easy to see from (6) that there exists a constant  $C > 0$  such that

$$|g'(s)| \leq C \left(1 + |s|^{q_2-2}\right).$$

Then, we get

$$\begin{aligned} \int_\Omega \left| \frac{\partial g}{\partial x_i}(\tilde{u}_\varepsilon(t)) \right|^{\bar{q}} dx &\leq C \int_\Omega \left(1 + |\tilde{u}_\varepsilon(t)|^{(q_2-2)\bar{q}}\right) \left| \frac{\partial \tilde{u}_\varepsilon(t)}{\partial x_i} \right|^{\bar{q}} dx \\ &\leq C \left(1 + \left(\int_\Omega |\tilde{u}_\varepsilon(t)|^{(q_2-2)\bar{q}\gamma} dx\right)^{1/\gamma}\right) \left(\int_\Omega |\nabla \tilde{u}_\varepsilon(t)|^{\bar{q}\eta} dx\right)^{1/\eta}, \end{aligned} \quad (66)$$

where we took  $\gamma$  and  $\eta$  such that  $\bar{q}\eta = p$ ,  $1/\gamma + 1/\eta = 1$  and  $(q_2-2)\bar{q}\gamma = p^*$ . Note that from here we get

$$\bar{q} = \frac{Np}{(N-p)(q_2-2) + N}.$$

Observe that  $\bar{q} > 1$ . Indeed,

$$q_2 \leq \frac{(N-1)p}{N-p} = \frac{N(p-1)}{N-p} + 1 < \frac{N(p-1)}{N-p} + 2 \Rightarrow (N-p)(q_2-2) + N < Np \Rightarrow \frac{Np}{(N-p)(q_2-2) + N} > 1.$$

Then, we have

$$\int_{\Omega} \left| \frac{\partial g}{\partial x_i}(\tilde{u}_\varepsilon(t)) \right|^{\bar{q}} dx \leq C \left( 1 + |\tilde{u}_\varepsilon|_{p^*, \Omega}^{p^*/\gamma} \right) |\nabla \tilde{u}_\varepsilon|_{p, \Omega}^{p/\eta}.$$

The conclusions of (17) are also valid substituting  $W^{1,p}(\Omega_\varepsilon)$  for  $W_0^{1,p}(\Omega)$ , so using (17) and the second estimate in (54), we get

$$|\nabla g(\tilde{u}_\varepsilon(t))|_{\bar{q}, \Omega} \leq C. \quad (67)$$

Then, from (62) and (67), we can deduce (63).

**Case 2:**  $p = N$ . We consider  $s \in [\frac{p(q_2-1)}{p-1}, +\infty)$  and

$$\bar{q} = \frac{sp}{p(q_2-2) + s}. \quad (68)$$

Taking into account (12), we can deduce

$$\frac{s}{\bar{q}} = \frac{p(q_2-2) + s}{p} = q_2 - 1 + \frac{s-p}{p} \geq q_2 - 1,$$

and, we have

$$|g(s)| \leq C \left( 1 + |s|^{q_2-1} \right) \leq C \left( 1 + |s|^{\frac{s}{\bar{q}}} \right).$$

Then, applying [10, Theorem 2.4] for  $G(x, v) = g(v)$ ,  $t = \bar{q}$  and  $r = s$ , we have that the map  $v \in L^s(\Omega) \mapsto g(v) \in L^{\bar{q}}(\Omega)$  is continuous in the strong topologies. Thanks to (12), we can consider (65) with  $r = s$ , and we get (62).

Finally, we prove (63). In (66) we took  $\gamma$  and  $\eta$  such that  $\bar{q}\eta = p$ ,  $1/\gamma + 1/\eta = 1$  and  $(q_2-2)\bar{q}\gamma = s$ . Note that from here we get  $\bar{q}$  given by (68).

Observe that  $\bar{q} \in (1, p)$ . Indeed, taking into account that  $\frac{1}{\bar{q}} = \frac{q_2-2}{s} + \frac{1}{p}$ , we can deduce

$$\frac{p(q_2-1)}{p-1} \leq s < +\infty \Rightarrow 0 < \frac{1}{s} \leq \frac{p-1}{p(q_2-1)} \Rightarrow \frac{1}{p} < \frac{1}{\bar{q}} \leq \frac{(q_2-2)(p-1)}{p(q_2-1)} + \frac{1}{p} \Rightarrow \frac{p(q_2-1)}{(q_2-2)(p-1) + (q_2-1)} \leq \bar{q} < p,$$

and using that  $\frac{p(q_2-1)}{(q_2-2)(p-1) + (q_2-1)} > 1$ , we have that  $\bar{q} \in (1, p)$ .

Then, we have

$$\int_{\Omega} \left| \frac{\partial g}{\partial x_i}(\tilde{u}_\varepsilon(t)) \right|^{\bar{q}} dx \leq C \left( 1 + |\tilde{u}_\varepsilon|_{s, \Omega}^{s/\gamma} \right) |\nabla \tilde{u}_\varepsilon|_{p, \Omega}^{p/\eta}.$$

The conclusions of (17) are also valid substituting  $W^{1,p}(\Omega_\varepsilon)$  for  $W_0^{1,p}(\Omega)$ , so using (17) and the second estimate in (54), we get

$$|\nabla g(\tilde{u}_\varepsilon(t))|_{\bar{q}, \Omega} \leq C. \quad (69)$$

Then, from (62) and (69), we can deduce (63).

□

## 6 Homogenized model: proof of the main Theorem

In this section, we identify the homogenized model.

We multiply system (2) by a test function  $v \in \mathcal{D}(\Omega)$ , and integrating by parts, we have

$$\begin{aligned} & \frac{d}{dt} \left( \int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(t) v dx \right) + \varepsilon \frac{d}{dt} \left( \int_{\partial F_{\varepsilon}} \gamma_0(u_{\varepsilon}(t)) v d\sigma(x) \right) + \int_{\Omega} \tilde{\xi}_{\varepsilon} \cdot \nabla v dx \\ & + \kappa \int_{\Omega} \chi_{\Omega_{\varepsilon}} |\tilde{u}_{\varepsilon}(t)|^{p-2} \tilde{u}_{\varepsilon}(t) v dx + \int_{\Omega} \chi_{\Omega_{\varepsilon}} f(\tilde{u}_{\varepsilon}(t)) v dx + \varepsilon \int_{\partial F_{\varepsilon}} g(\gamma_0(u_{\varepsilon}(t))) v d\sigma(x) = 0, \end{aligned}$$

in  $\mathcal{D}'(0, T)$ .

We consider  $\varphi \in C_c^1([0, T])$  such that  $\varphi(T) = 0$  and  $\varphi(0) \neq 0$ . Multiplying by  $\varphi$  and integrating between 0 and  $T$ , we have

$$\begin{aligned} & -\varphi(0) \left( \int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(0) v dx \right) - \int_0^T \frac{d}{dt} \varphi(t) \left( \int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(t) v dx \right) dt \\ & - \varepsilon \varphi(0) \left( \int_{\partial F_{\varepsilon}} \gamma_0(u_{\varepsilon}(0)) v d\sigma(x) \right) - \varepsilon \int_0^T \frac{d}{dt} \varphi(t) \left( \int_{\partial F_{\varepsilon}} \gamma_0(u_{\varepsilon}(t)) v d\sigma(x) \right) dt \\ & + \int_0^T \varphi(t) \int_{\Omega} \tilde{\xi}_{\varepsilon} \cdot \nabla v dx dt + \kappa \int_0^T \varphi(t) \int_{\Omega} \chi_{\Omega_{\varepsilon}} |\tilde{u}_{\varepsilon}(t)|^{p-2} \tilde{u}_{\varepsilon}(t) v dx dt \\ & + \int_0^T \varphi(t) \int_{\Omega} \chi_{\Omega_{\varepsilon}} f(\tilde{u}_{\varepsilon}(t)) v dx dt + \varepsilon \int_0^T \varphi(t) \int_{\partial F_{\varepsilon}} g(\gamma_0(u_{\varepsilon}(t))) v d\sigma(x) dt = 0. \end{aligned} \tag{70}$$

For the sake of clarity, we split the proof in five parts. Firstly, we pass to the limit, as  $\varepsilon \rightarrow 0$ , in (70) in order to get the limit equation satisfied by  $u$ . In the first step, we pass to the limit in the integrals on  $\Omega$ , using Proposition 5.1 and (55), in the second step we pass to the limit in the integrals on the boundary of the holes, where we use a convergence result based on a technique introduced by Vanninathan [25], and in the third step we deduce the limit equation satisfied by  $u$ . In the fourth step we identify  $\xi$  making use of the solutions of the problems (14), and finally we prove that  $u$  is uniquely determined.

**Step 1.** Passing to the limit, as  $\varepsilon \rightarrow 0$ , in the integrals on  $\Omega$ :

From (58)-(60) and (55), we have respectively, for  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned} & \int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(t) v dx \rightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} u(t) v dx, \quad \forall v \in \mathcal{D}(\Omega), \\ & \int_{\Omega} \chi_{\Omega_{\varepsilon}} |\tilde{u}_{\varepsilon}(t)|^{p-2} \tilde{u}_{\varepsilon}(t) v dx \rightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} |u(t)|^{p-2} u(t) v dx, \quad \forall v \in \mathcal{D}(\Omega), \end{aligned}$$

and

$$\int_{\Omega} \chi_{\Omega_{\varepsilon}} f(\tilde{u}_{\varepsilon}(t)) v dx \rightarrow \frac{|Y^*|}{|Y|} \int_{\Omega} f(u(t)) v dx, \quad \forall v \in \mathcal{D}(\Omega),$$

which integrating in time and using Lebesgue's Dominated Convergence Theorem, gives

$$\begin{aligned} & \int_0^T \frac{d}{dt} \varphi(t) \left( \int_{\Omega} \chi_{\Omega_{\varepsilon}} \tilde{u}_{\varepsilon}(t) v dx \right) dt \rightarrow \frac{|Y^*|}{|Y|} \int_0^T \frac{d}{dt} \varphi(t) \left( \int_{\Omega} u(t) v dx \right) dt, \\ & \kappa \int_0^T \varphi(t) \int_{\Omega} \chi_{\Omega_{\varepsilon}} |\tilde{u}_{\varepsilon}(t)|^{p-2} \tilde{u}_{\varepsilon}(t) v dx dt \rightarrow \kappa \frac{|Y^*|}{|Y|} \int_0^T \varphi(t) \int_{\Omega} |u(t)|^{p-2} u(t) v dx dt, \end{aligned}$$

and

$$\int_0^T \varphi(t) \int_{\Omega} \chi_{\Omega_\varepsilon} f(\tilde{u}_\varepsilon(t)) v dx dt \rightarrow \frac{|Y^*|}{|Y|} \int_0^T \varphi(t) \int_{\Omega} f(u(t)) v dx dt.$$

By (58) and (55), we have

$$\varphi(0) \left( \int_{\Omega} \chi_{\Omega_\varepsilon} \tilde{u}_\varepsilon(0) v dx \right) \rightarrow \varphi(0) \frac{|Y^*|}{|Y|} \int_{\Omega} u(0) v dx, \quad \forall v \in \mathcal{D}(\Omega).$$

On the other hand, using (61), we obtain, for  $\varepsilon \rightarrow 0$

$$\int_0^T \varphi(t) \int_{\Omega} \tilde{\xi}_\varepsilon \cdot \nabla v dx dt \rightarrow \int_0^T \varphi(t) \int_{\Omega} \xi \cdot \nabla v dx dt.$$

**Step 2.** Passing to the limit, as  $\varepsilon \rightarrow 0$ , in the surface integrals on the boundary of the holes:

We make use of the technique introduced by Vanninathan [25] for the Steklov problem which transforms surface integrals into volume integrals. This technique was already used as a main tool to homogenize the non homogeneous Neumann problem for the elliptic case by Cioranescu and Donato [6].

By [6, Definition 3.2], let us introduce, for any  $h \in L^{s'}(\partial F)$ ,  $1 \leq s' \leq \infty$ , the linear form  $\mu_h^\varepsilon$  on  $W_0^{1,s}(\Omega)$  defined by

$$\langle \mu_h^\varepsilon, \varphi \rangle = \varepsilon \int_{\partial F_\varepsilon} h \left( \frac{x}{\varepsilon} \right) \varphi(x) d\sigma(x), \quad \forall \varphi \in W_0^{1,s}(\Omega),$$

with  $1/s + 1/s' = 1$ . It is proved in [6, Lemma 3.3] that

$$\mu_h^\varepsilon \rightarrow \mu_h \quad \text{strongly in } (W_0^{1,s}(\Omega))', \quad (71)$$

where  $\langle \mu_h, \varphi \rangle = \mu_h \int_{\Omega} \varphi(x) dx$ , with

$$\mu_h = \frac{1}{|Y|} \int_{\partial F} h(y) d\sigma(y).$$

In the particular case in which  $h \in L^\infty(\partial F)$  or even when  $h$  is constant, we have

$$\mu_h^\varepsilon \rightarrow \mu_h \quad \text{strongly in } W^{-1,\infty}(\Omega).$$

We denote by  $\mu_1^\varepsilon$  the above introduced measure in the particular case in which  $h = 1$ . Notice that in this case  $\mu_h$  becomes  $\mu_1 = |\partial F|/|Y|$ .

Observe that from (57) and (71) with  $s = p$ , we can deduce, for  $\varepsilon \rightarrow 0$ ,

$$\varepsilon \int_{\partial F_\varepsilon} \gamma_0(u_\varepsilon(t)) v d\sigma(x) = \langle \mu_1^\varepsilon, \tilde{u}_{\varepsilon|\Omega_\varepsilon}(t)v \rangle \rightarrow \mu_1 \int_{\Omega} u(t) v dx = \frac{|\partial F|}{|Y|} \int_{\Omega} u(t) v dx,$$

for all  $v \in \mathcal{D}(\Omega)$ , which integrating in time and using Lebesgue's Dominated Convergence Theorem, gives

$$\varepsilon \int_0^T \frac{d}{dt} \varphi(t) \left( \int_{\partial F_\varepsilon} \gamma_0(u_\varepsilon(t)) v d\sigma(x) \right) dt \rightarrow \frac{|\partial F|}{|Y|} \int_0^T \frac{d}{dt} \varphi(t) \left( \int_{\Omega} u(t) v dx \right) dt.$$

Moreover, from (57) and (71) with  $s = p$ , we can deduce, for  $\varepsilon \rightarrow 0$ ,

$$\varepsilon \int_{\partial F_\varepsilon} \gamma_0(u_\varepsilon(0)) v d\sigma(x) = \langle \mu_1^\varepsilon, \tilde{u}_{\varepsilon|\Omega_\varepsilon}(0)v \rangle \rightarrow \mu_1 \int_{\Omega} u(0) v dx = \frac{|\partial F|}{|Y|} \int_{\Omega} u(0) v dx,$$

for all  $v \in \mathcal{D}(\Omega)$ .



On the other hand, from (63) and (71) with  $s = \bar{q}$ , we conclude

$$\varepsilon \int_{\partial F_\varepsilon} g(\gamma_0(u_\varepsilon(t))) v d\sigma(x) = \langle \mu_1^\varepsilon, g(\tilde{u}_\varepsilon(t)) v \rangle \rightarrow \frac{|\partial F|}{|Y|} \int_\Omega g(u(t)) v dx,$$

for all  $v \in \mathcal{D}(\Omega)$ , which integrating in time and using Lebesgue's Dominated Convergence Theorem, gives

$$\varepsilon \int_0^T \varphi(t) \int_{\partial F_\varepsilon} g(\gamma_0(u_\varepsilon(t))) v d\sigma(x) dt \rightarrow \frac{|\partial F|}{|Y|} \int_0^T \varphi(t) \int_\Omega g(u(t)) v dx dt.$$

**Step 3.** Passing to the limit, as  $\varepsilon \rightarrow 0$ , in (70):

All the terms in (70) pass to the limit, as  $\varepsilon \rightarrow 0$ , and therefore taking into account the previous steps, we get

$$\begin{aligned} -\varphi(0) \left( \frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \left( \int_\Omega u(0) v dx \right) - \left( \frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \int_0^T \frac{d}{dt} \varphi(t) \left( \int_\Omega u(t) v dx \right) dt \\ + \int_0^T \varphi(t) \int_\Omega \xi \cdot \nabla v dx dt + \kappa \frac{|Y^*|}{|Y|} \int_0^T \varphi(t) \int_\Omega |u(t)|^{p-2} u(t) v dx dt \\ + \frac{|Y^*|}{|Y|} \int_0^T \varphi(t) \int_\Omega f(u(t)) v dx dt + \frac{|\partial F|}{|Y|} \int_0^T \varphi(t) \int_\Omega g(u(t)) v dx dt = 0. \end{aligned}$$

Hence,  $\xi$  verifies

$$\left( \frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \frac{\partial u}{\partial t} - \operatorname{div} \xi + \frac{|Y^*|}{|Y|} (\kappa |u|^{p-2} u + f(u)) + \frac{|\partial F|}{|Y|} g(u) = 0, \quad \text{in } \Omega \times (0, T). \quad (72)$$

**Step 4.** It remains now to identify  $\xi$ . For the sake of completeness, we give a sketch of a proof, following, for instance, the proof of [13, Theorem 3.1].

Let us consider  $\forall \nu \in \mathbb{N}$  and for any  $i = 1, \dots, N$ , a partition of  $\mathbb{R}^N$  by intervals  $\{Q_{i\nu}\}_i$  of side  $2^{-\nu}$ . Since the holes  $F_\varepsilon$  do not intersect  $\partial\Omega$ , we can assume that  $\forall \varepsilon > 0$ ,  $F_\varepsilon$  does not intersect  $\partial Q_{i\nu}$ . Let us denote by  $x_{i\nu}$  and  $\chi_{i\nu}$  respectively the center and the characteristic function of  $Q_{i\nu}$ . Then, we define  $I_\nu = \{i : Q_{i\nu} \subset \Omega\}$ ,  $\Omega_\nu = \cup_{i \in I_\nu} Q_{i\nu}$ . For any  $i$ , let us denote

$$\langle \nabla u \rangle_{i\nu} := \frac{1}{|Q_{i\nu}|} \int_{Q_{i\nu}} \nabla u(x) dx.$$

By the continuity of  $b$  (see [13, Lemma 2.11]), we have, if  $\nu \rightarrow +\infty$ , that

$$\sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle \nabla u \rangle_{i\nu}) \rightarrow b(\nabla u(x)) \quad \text{a.e. in } \Omega, \quad (73)$$

where  $b$  is given by (15).

Moreover, from [13, Lemma 2.10], we have for any measurable set  $E \subset \Omega$  that

$$\int_E \left| \sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle \nabla u \rangle_{i\nu}) \right|^{p'} dx \leq C \int_E \left( 1 + \left| \sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle \nabla u \rangle_{i\nu}) \right|^p \right) dx.$$

So, from the equi-absolute continuity of the integral on the left-hand side and from (73), we deduce that

$$\sum_{i \in I_\nu} \chi_{i\nu}(x) b(\langle \nabla u \rangle_{i\nu}) \rightarrow b(\nabla u(x)) \quad \text{in } L^{p'}(\Omega), \quad \text{as } \nu \rightarrow +\infty. \quad (74)$$

Let  $w_{i\nu} \in \langle \nabla u \rangle_{i\nu} \cdot y + \mathbb{H}_{\text{per}}(Y^*)$  the solution of the problem (14) corresponding to  $\langle \nabla u \rangle_{i\nu}$ . Then

$$v_{i\nu} = (w_{i\nu} - \langle \nabla u \rangle_{i\nu} \cdot y) \in \mathbb{H}_{\text{per}}(Y^*),$$

and so, by [13, Lemma 2.3], we can consider  $\tilde{v}_{i\nu} \in \mathbb{H}_{\text{per}}(Y)$ , where  $\tilde{\cdot}$  denotes the  $W^{1,p}$ -extension to  $Y$ . If we extend  $\tilde{v}_{i\nu}$ , by periodicity, the resulting function (still denoted by  $\tilde{v}_{i\nu}$ ) is in  $W^{1,p}(\Omega)$ .

Let us define

$$\Psi_{i\nu}^\varepsilon(x) = \varepsilon \tilde{v}_{i\nu} \left( \frac{x}{\varepsilon} \right) - \langle \nabla u \rangle_{i\nu} \cdot x, \quad (75)$$

and for  $\zeta \in \mathbb{R}^N$

$$\tilde{a}_p(\zeta) = \begin{cases} |\zeta|^{p-2} \zeta & \text{for } y \in Y^*, \\ 0 & \text{for } y \in \bar{F}. \end{cases}$$

Arguing as in the proof of [13, Lemma 2.13], we obtain

$$\Psi_{i\nu}^\varepsilon \rightharpoonup \langle \nabla u \rangle_{i\nu} \cdot x \quad \text{weakly in } W^{1,p}(\Omega), \quad (76)$$

$$\tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \rightharpoonup b(\langle \nabla u \rangle_{i\nu}) \quad \text{weakly in } L^{p'}(\Omega), \quad (77)$$

and

$$\text{div } \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) = 0. \quad (78)$$

By Rellich-Kondrachov Theorem and (76), we can deduce

$$\Psi_{i\nu}^\varepsilon \rightarrow \langle \nabla u \rangle_{i\nu} \cdot x \quad \text{strongly in } L^p(\Omega). \quad (79)$$

Using the periodicity of  $\tilde{v}_{i\nu}$  and [13, Lemma 2.3], we have

$$\begin{aligned} \sum_{i \in I_\nu} \int_{Q_{i\nu}} |\nabla \Psi_{i\nu}^\varepsilon|^p dx &\leq \sum_{i \in I_\nu} 2^{-\nu N} \varepsilon^N (1/\varepsilon + 2^\nu)^N \int_Y |\nabla \Psi_{i\nu}^\varepsilon(y)|^p dy \\ &\leq C \sum_{i \in I_\nu} 2^{-\nu N} (1 + \varepsilon^N 2^{\nu N}) \left( \int_{Y^*} |\nabla w_{i\nu}|^p dy + \langle \nabla u \rangle_{i\nu} \right)^p. \end{aligned}$$

Then, due to the structure properties of  $b$  (see [13, Lemma 2.10]), writing the last term as an integral over  $\Omega_\nu$ , we have

$$\sum_{i \in I_\nu} \int_{Q_{i\nu}} |\nabla \Psi_{i\nu}^\varepsilon|^p dx \leq C (1 + \varepsilon^N 2^{\nu N}) \int_\Omega (1 + |\nabla u|)^p dx. \quad (80)$$

Let now  $\eta \in C_0^1(Q_{i\nu})$ ,  $0 \leq \eta \leq 1$ , and extend it by periodicity to the whole  $\mathbb{R}^N$ . We consider  $\varphi \in C_c^1([0, T])$  such that  $\varphi(T) = 0$  and  $\varphi(0) \neq 0$ . If  $\psi \in C^0(\bar{\Omega})$ , set  $M_\psi = \sup_\Omega |\psi|$  and  $M_\varphi = \sup_{[0, T]} |\varphi|$ , we have

$$\begin{aligned} &\left| \int_0^T \varphi(t) \int_\Omega \tilde{\xi}_\varepsilon \psi \eta dx dt - \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \psi \eta dx dt \right| \quad (81) \\ &\leq CM_\psi M_\varphi |\Omega \setminus \Omega_\nu|^{1/p} + \left| \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} (|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - |\nabla \Psi_{i\nu}^\varepsilon|^{p-2} \nabla \Psi_{i\nu}^\varepsilon) \psi \eta dx dt \right| \\ &\leq CM_\psi M_\varphi |\Omega \setminus \Omega_\nu|^{1/p} + \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} M_\psi \eta \left( (|\nabla u_\varepsilon| + |\nabla \Psi_{i\nu}^\varepsilon|)^{p-2} \cdot |\nabla u_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon| \right) dx dt \\ &\leq CM_\psi M_\varphi |\Omega \setminus \Omega_\nu|^{1/p} + CM_\psi^{p/(p-1)} \delta^{p/(p-1)} (1 + \varepsilon^N 2^{\nu N}) + \delta^{-p} \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |\nabla u_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon|^p dx dt, \end{aligned}$$

where the last inequality is obtained by applying Young inequality with  $\delta > 0$  and the first estimate in (44), the first estimate in (54) and (80).

On the other hand, from monotonicity, we can deduce

$$\begin{aligned}
 \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |\nabla u_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon|^p dx dt &\leq \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \left( \tilde{\xi}_\varepsilon - \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \right) \cdot (\nabla \tilde{u}_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon) dx dt \\
 &= \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \left( \tilde{\xi}_\varepsilon - \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \right) \cdot \nabla \rho_{i\nu}^\varepsilon dx dt \\
 &+ \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \left( \tilde{\xi}_\varepsilon - \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \right) \cdot (\nabla u - \langle \nabla u \rangle_{i\nu}) dx dt,
 \end{aligned} \tag{82}$$

where  $\rho_{i\nu}^\varepsilon := \tilde{u}_\varepsilon - \Psi_{i\nu}^\varepsilon - u + \langle \nabla u \rangle_{i\nu} \cdot x$ . Taking into account (58) and (79), we have

$$\rho_{i\nu}^\varepsilon \rightarrow 0 \quad \text{strongly in } L^p(\Omega). \tag{83}$$

Now, we multiply system (2) by the test function  $\eta \rho_{i\nu}^\varepsilon$  and integrating by parts, we get

$$\begin{aligned}
 \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \tilde{\xi}_\varepsilon \cdot \nabla \rho_{i\nu}^\varepsilon dx dt &= - \sum_{i \in I_\nu} \int_{Q_{i\nu}} \varphi(0) \chi_{\Omega_\varepsilon} \tilde{u}_\varepsilon(0) \eta \rho_{i\nu}^\varepsilon dx \\
 - \int_0^T \frac{d}{dt} \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \chi_{\Omega_\varepsilon} \tilde{u}_\varepsilon(t) \eta \rho_{i\nu}^\varepsilon dx dt &- \varphi(0) \langle \mu_1^\varepsilon, \tilde{u}_{\varepsilon|\Omega_\varepsilon}(0) \eta \rho_{i\nu}^\varepsilon \rangle - \int_0^T \frac{d}{dt} \varphi(t) \langle \mu_1^\varepsilon, \tilde{u}_{\varepsilon|\Omega_\varepsilon}(t) \eta \rho_{i\nu}^\varepsilon \rangle dt \\
 - \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{\xi}_\varepsilon \cdot \nabla \eta \rho_{i\nu}^\varepsilon dx dt &- \kappa \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \chi_{\Omega_\varepsilon} |\tilde{u}_\varepsilon(t)|^{p-2} \tilde{u}_\varepsilon(t) \eta \rho_{i\nu}^\varepsilon dx dt \\
 - \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \chi_{\Omega_\varepsilon} f(\tilde{u}_\varepsilon(t)) \eta \rho_{i\nu}^\varepsilon dx dt &- \int_0^T \varphi(t) \langle \mu_1^\varepsilon, g(\tilde{u}_\varepsilon(t)) \rho_{i\nu}^\varepsilon \rangle dt.
 \end{aligned}$$

We pass to the limit, as  $\varepsilon \rightarrow 0$ . Taking into account (83), we reason as in the first two steps and we can deduce

$$\int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta \tilde{\xi}_\varepsilon \cdot \nabla \rho_{i\nu}^\varepsilon dx dt \rightarrow 0. \tag{84}$$

On the other hand, integrating by parts, taking into account (77)-(78) and (83), as  $\varepsilon \rightarrow 0$ , by Lebesgue's Dominated Convergence Theorem, we obtain

$$\int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \cdot \nabla \rho_{i\nu}^\varepsilon \eta dx dt = \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \tilde{a}_p(\nabla \Psi_{i\nu}^\varepsilon) \cdot \nabla \eta \rho_{i\nu}^\varepsilon dx dt \rightarrow 0. \tag{85}$$

Then, passing to the limit, as  $\varepsilon \rightarrow 0$ , in (82), by (61), (77), (84)-(85), we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu} \cap \Omega_\varepsilon} \eta |\nabla u_\varepsilon - \nabla \Psi_{i\nu}^\varepsilon|^p dx dt \leq \int_0^T \varphi(t) \sum_{i \in I_\nu} \int_{Q_{i\nu}} \eta (\xi - b(\langle \nabla u \rangle_{i\nu})) (\nabla u - \langle \nabla u \rangle_{i\nu}) dx dt. \tag{86}$$

If we first pass to the limit in (81), as  $\varepsilon \rightarrow 0$ , then  $\eta \rightarrow 1$ ,  $\nu \rightarrow +\infty$ , by (61), (74), (77) and (86) we get

$$\left| \int_0^T \varphi(t) \int_\Omega \xi \psi dx dt - \int_0^T \varphi(t) \int_\Omega b(\nabla u) \psi dx dt \right| \leq CM_\psi^{p/(p-1)} \delta^{p/(p-1)}.$$

So, letting  $\delta \rightarrow 0$ , from the arbitrariness of  $\varphi$  and  $\psi$ , we deduce

$$\xi = b(\nabla u) \quad \text{a.e. in } \Omega \times (0, T). \quad (87)$$

**Step 5.** Finally, thanks to (72) and (87), we observe that  $u$  satisfies the first equation in (13). A weak solution of (13) is any function  $u$ , satisfying

$$u \in C([0, T]; L^2(\Omega)), \quad \text{for all } T > 0,$$

$$u \in L^p(0, T; W_0^{1,p}(\Omega)), \quad \text{for all } T > 0,$$

$$\left( \frac{|Y^*|}{|Y|} + \frac{|\partial F|}{|Y|} \right) \frac{d}{dt} (u(t), v) + (b(\nabla u(t)), \nabla v) + \frac{|Y^*|}{|Y|} \kappa(|u(t)|^{p-2}u(t) + f(u(t)), v) + \frac{|\partial F|}{|Y|} (g(u(t)), v) = 0, \quad \text{in } \mathcal{D}'(0, T),$$

for all  $v \in W_0^{1,p}(\Omega)$ , and

$$u(0) = u_0.$$

Due to the structure properties of  $b$  (see [13, Lemmas 2.10-2.11-2.13]), applying a slight modification of [20, Chapter 2, Theorem 1.4], we obtain that the problem (13) has a unique solution, and therefore Theorem 1.1 is proved.

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