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**The differential calculus, mathematicians and economists in the nineteenth century
K. Marx and H. Laurent, readers of J. L. Boucharlat.**

Alain Alcouffe

University of Toulouse

Résumé

Parmi les sources des Manuscrits mathématiques de Marx figure notamment le livre sur le calcul différentiel de Jean-Louis Boucharlat, qui connut de multiples éditions jusqu'en 1926. Ce livre fera aussi l'objet de remarques de la part de Herman Laurent, un actuaire et relation de Léon Walras. Le traitement L'article montre que la méthode génétique (histoire des concepts) et dialectique (Marx s'est intéressé aux transformations que provoquait le développement même des différentes approches du calcul différentiel) mérite de retenir l'attention et éclairer l'histoire du calcul différentiel jusqu'à ces développements du 20^e siècle.

Mots-clés: Marx, Boucharlat, Laurent, calcul différentiel, méthodologie.

Abstract

Among the references used by Marx when he wrote his Mathematical Manuscripts, the book of Jean-Louis Boucharlat on Calculus is remarkable by its multiple editions till 1926. It was also criticized by Herman Laurent, an actuary and one of the first disciple of Léon Walras. The paper shows that the genetic method (history of the concepts) and dialectic (Marx scrutinized the transformations of the various approaches used in the calculus) deserves attention and highlights the history of calculus till the XXth century.

Keywords : Marx, Boucharlat, Laurent, calculus, methodology.

JEL : B1, B5

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The differential calculus, mathematicians and economists in the nineteenth century

K. Marx and H. Laurent, readers of J. L. Boucharlat.

In 1881, two old men in London, German emigrants, without any particular mathematical training, were exchanging reflections on the differential calculus. That these men were Marx and Engels, and that these reflections were the result of an enormous amount of reading and elaboration work on Marx over a period of nearly thirty years, would not be enough to make these exchanges escape from the caricature immortalized by Flaubert in *Bouvard et Pécuchet*¹(and published in 1881, after Flaubert's death) if these "mathematical manuscripts" were nothing more than fatuous². These manuscripts contain about a thousand pages of mathematics written in instalments from 1846 or at least 1852 to his death in 1883. In this research, some have wanted to see preparatory work on economics, but it seems that Marx relentlessly took a real interest in mathematics itself, probably as an extension of his interest in Hegel's texts on differential calculus. Indeed, the MMMs, apart from the interest they may have for the marxologist or philosopher of sciences, make interesting contributions, both direct and indirect, to the history of mathematics in the nineteenth century. Their interest is, in fact, twofold: on the one hand, Marx's reading of the history of differential calculus and, on the other hand, they are an incentive to read again his sources, which we can compare with other contemporary authors and place in the history of mathematics. From this point of view, one of Marx's sources, Jean-Louis Boucharlat's *Elements of Differential Calculus and Integral Calculus*, occupies a prominent place. It is, first of all, one of the most frequently used references by Marx, in particular, in his most elaborate text (the memoir on the differential calculus written for Engels, which they may have intended to publish³). There is no indication as to the reasons for Marx's interest in this particular book, a rather late choice apparently, since Marx had previously used a book by another French mathematician, L. B. Francoeur⁴, but it must be admitted that he had made a choice ratified by many people concerned by this field of mathematics: here is a book first published in 1813 which was translated into many languages (including a translation

¹ Cf. G. Flaubert, *Bouvard and Pecuchet: a Tragi-Comic Novel of Bourgeois Life*. The Floating Press. 2016, <http://public.eblib.com/choice/publicfullrecord.aspx?p=4585572>. The book details the adventures of two Parisian copy-clerks, that decide to move to the countryside. Their search for intellectual stimulation leads them, over the course of years, to flounder through almost every branch of knowledge Flaubert uses their ridiculous quest to expose the hidden weaknesses of the sciences.

² The (most) complete edition of Marx's Mathematical Manuscripts was published in 1968, in a bilingual German-Russian presentation (Nauka, Moscou). It strives to present Marx's work in a systematic way, for the most part, readings and book reviews. The most elaborate passages were published in West Germany in 1974, (facsimile of the 1968 edition). Italian and English translations were published in 1975 and 1983 before the French translation of 1985.

³ Engels wrote to Marx on August 18, 1881 "You need have no fear that in all this you will have been anticipated by any mathematician.". Engels stressed also Marx's original discoveries in mathematics in his Speech at the grave of Marx, March 17, 1883.

⁴ L. B. Francoeur, *Cours complet de mathématiques pures*, Paris, Bachelier, (3rd edition, 1828).

into Hindustani), published in many editions without major revision, and still be reprinted more than a century after its first publication (until 1926 at least!). We propose to examine the presentation of Boucharlat's differential calculus concepts, taking as a guide, on the one hand, Marx's criticism and, on the other hand, the 1886 version of Boucharlat's book which includes a revision of the treatment of differentials by Hermann Laurent.

1°) General presentation of J. L. Boucharlat's *Elémens de Calcul différentiel et de Calcul Intégral*.

Jean Louis Boucharlat was born in Lyon in 1775 to a wealthy family who gave him a solid education⁵. At a very young age, he developed a strong interest in both poetry and mathematics. Worried during the Terror, he took refuge in Saint-Etienne but then, equipped with letters of recommendation for Monge and Lagrange, he arrived in Paris, took the Polytechnique exams and was admitted. After graduating, he obtained a chair in mathematics in Lyon. He published mathematical works including a *Theory of Curves and Surfaces of the Second Order* "with the help of Lagrange and dedicated to him" (1810). In the meantime he had been appointed assistant repétitor⁶ at the Ecole Polytechnique, from which he resigned, unhappy to have been ousted when the chair he taught was assigned to another.

Thanks to Lagrange's support, he finally obtained a chair in mathematics at the Prytané de la Flèche, whose mathematical course (artillery division) had been elevated to the level of that of Polytechnique. It was precisely the notes prepared for this course that were to constitute the *Elémens de Calcul Différentiel*, the first edition of which appeared in 1813. Under the Restoration, he resigned any function in the teaching of mathematics to devote himself to literature. He taught the literature course at the Athenaeum as an extension of La Harpe⁷. He died in Paris in 1848.

In his biography of 1846, *Theory of Curves* is presented as an introduction to the *Elémens*, while the *Elémens de Mécanique* published in 1815 would be a sequel. The revisions Boucharlat carried out in subsequent editions show that Boucharlat's interest in mathematics was not linked to his employment as a professor of mathematics but to a genuine intellectual inclination.

⁵ We follow the fairly consistent biographies of E. Garay de Monglavé (1846), the *Grande Encyclopédie* (1893) and the *Dictionnaire de Biographie Française* (1954).

⁶ A repétitor was an adjunct lecturer, usually the first step in an academic career at Polytechnique.

⁷ The Athénée of Paris was a 'learned society' founded in 1781 under the name of Lycée. It became the Republican Athénée, then the Royal Athénée. La Harpe, (1739 – 1803), a French playwright, writer and literary critic, taught there a *Cours de Littérature* which has been considered as « the emergence of the discourse of the history of ideas ». Cf. Goulemot Jean-Marie. Le Cours de littérature de La Harpe, ou l'émergence du discours de l'histoire des idées. In: *Littérature*, n°24, 1976. pp. 51-62

Hermann Laurent, born in 1841, was the son of the chemist Auguste Laurent⁸. He graduated from the Ecole Polytechnique in 1862. In 1865 he was awarded a doctorate in mathematical sciences with two theses entitled *De la continuité dans les séries* and *Sur les lignes isothermes* and was appointed, the following year, as a repetitor for analysis at the Ecole Polytechnique. After resuming his service during the War of 1870,⁹ he left the army again in 1872, but he had to continue his life as a repetitor at the Ecole Polytechnique. From 1883 onwards, he performed the tasks of admission examiner and from 1889 onwards he was also professor of mathematics at the Institut Agronomique. He died in 1908. Hermann Laurent had an intellectual production of the first rank: he wrote many articles in the *Journal de Liouville*, the *Proceedings of the Academy of Sciences*, etc.. He also published several mathematical works among which we will examine in particular a *Traité d'Analyse* in 7 volumes (1885-91) which allows us to better situate the revisions he introduced in the 9th edition of Boucharlat. He was equally interested in statistics and financial mathematics and hence in political economy. Thus he was in contact with Léon Walras and published in 1902 a *Petit traité d'économie politique mathématique*, in accordance with the precepts of the Ecole de Lausanne.¹⁰

Although he was not credited with exceptional discoveries, he was nevertheless a very representative author of mathematics in the last quarter of the 19th century in France. This last point is not restrictive: H. Laurent in his *Traité d'Analyse* quotes Weierstrass and "verbatim translates" his demonstration that function, $f(x) = \cos \pi x + b \cos a \pi x + \dots + b^n \cos a^n \pi x$ although continuous has no derivative. H. Laurent adds a note to his translation of Weierstrass stating that "these lines were written several years ago, at a time when many geometers refused to accept the previous conclusions which seem generally accepted today » (P.414). C. Méray, a French mathematician who shared with G. Cantor, Weierstrass and Dedekind the invention of the modern definition of numbers, was to congratulate H. Laurent on the earliness of his "rallying" to the new conceptions.¹¹

II) The History of Differentials:

⁸ Biographical information on H. Laurent is taken from the Grande Encyclopédie (H. Laurent was responsible for mathematical articles) and from the obituary published in 1908 in *L'Enseignement Mathématique*.

⁹ Polytechnique was created during the Revolution as a training school for the army. French graduates are automatically officers and serve for a period at least equal to the length of their studies in the army.

¹⁰ Laurent's book was published in 1902, Paris, C. Schmid. The relations between Laurent and Walras were essentially epistolary; see *Correspondence of Léon Walras and related Papers*, edited and annotated by W. Jaffé, N. H. 1965.

¹¹ Cf. C. Méray, *Leçons nouvelles sur l'analyse infinitésimale et ses applications géométriques*, Paris, Gauthier-Villars, 1894. On the history of numbers at the end of the 19th century, see N. Bourbaki, *Eléments d'histoire des mathématiques*, Paris, Hermann, 2nd edition 1969. On the relationship between this history and the differential calculus, cf. C. E. Edwards, *The Historical Development of the Calculus*, Springer, New York, Berlin, 1979.

Before examining the concepts of differential calculus in Boucharlat, it is interesting to carefully read his short preface in which he presents a periodization of the history of differential calculus, a periodization on which Marx had to rely. He distinguishes three periods. First of all, he locates the origin in Newton and Leibniz, inventors of an "analysis far superior to Descartes' geometry. But there were still remaining some "mysteries". Newton, the first one, penetrated this mystery by considering the differential calculus as the method of the first and final ratios of the quantities or otherwise as the method of the limits of their relationship. The method of limits was perfected by D'Alembert so that it was "freed from all consideration of motion, ideas foreign to the differential calculus. Finally, Lagrange's method in turn complements that of limits by attaching the differential coefficients to pure Algebra". In this sketch of the history of the differential calculus, the dominant impression is that of a very modern guideline, the passage from one stage to the next being linked to an "instillation of rigour", to quote Kline¹², but the end of the preface blurs this image, in fact, for Boucharlat, "these three methods form, so to speak, only one". This point puzzled Marx who rejected Boucharlat's eclecticism, seeking in the perspective of a Lakatosian internal history the reasons for the succession of the three methods.

It is not possible to compare the history of the differential calculus of Boucharlat-Marx with that of Laurent, because on the one hand, the latter purely and simply eliminated the preface of Boucharlat from his edition of the *Elements* (the 9°). On the other hand, Laurent has expressed his views on the origins of the differential calculus in passing comments in his correspondence with Walras and in articles in *L'enseignement mathématique*. It can be inferred from this that he shared the idea of a growing instillation of rigour: "When Leibniz created this admirable instrument called the differential calculus, he was unfairly criticized, and the proof is that differential notation is now universally adopted, but the criticisms that were directed at him had their raison d'être."¹³

But before establishing a periodization of the history of the differential calculus and the articulation of the different conceptions, it is necessary to examine one by one the concepts of the differential calculus to see the movement that can connect the different approaches. For this reason, it seemed preferable to give a direct reading of Boucharlat's presentation of differential calculus before

¹² Mr. Kline entitled his chapter on the XVIII° "Instillation of rigour", *Mathematical Thought from Ancient to Modern Times*, New York, Oxford University Press, 1972.

¹³ H. Laurent, "Considerations sur l'enseignement des mathématiques dans les classes de spéciales en France" in *L'enseignement mathématiques*, 1899; (Laurent opposes "the notation of Lagrange's accents, born out of a false idea, a barbaric notation rejected by all civilized peoples, rejected by Lagrange himself, who never used it in his memoirs, and the notation of Leibniz' differentials, universally used, as the simplest, most expressive and most convenient." p.41). and in the correspondence Walras-Laurent (in Jaffé, letter 1452), Laurent writes to Walras: "My opinion is that in your theories, there is still a lot of vagueness, which may be said without offending you because one cannot demand more from a science such as yours, more than one has demanded from the founders of differential calculus. You have had the merit of posing problems and providing undoubtedly imperfect solutions from the point of view of rigour, as Leibnitz did". The quotation from the text is taken from the *Petit traité d'économie*, 1902, p. 3.

confronting it with the readings of Marx and Laurent, on the one hand, and the evolution of differential calculus, on the other hand.

III) The concepts of function and limit:

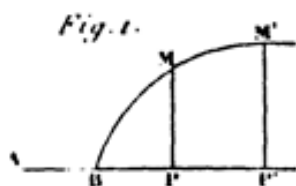
(Boucharlat, 1826, pp.1, 2, 3).

DIFFERENTIAL CALCULUS.

On the differentiation of algebraic quantities

1. ONE variable is said to be a function of another variable, when the first is equal to a certain analytical expression composed of the second: for example, y is a function of x in the following equations :

$$y = \sqrt{a^2 - x^2}, \quad y = a^3 + 3bx^2, \quad y = \frac{x^2}{a}, \quad y = b + cx^3$$



2. Let us consider a function when in its state of increase, by reason of the increase of the variable which it contains ; and since every function of a variable x may be represented by the ordinate of a curve BMM' , fig.1, let $AP=a$ and $PM=y$ be the coordinates of a point M in that curve, and suppose that the abscissa AP receives an increment $PP=h$, then the ordinate PM will become $P'M'=y'$. In order, therefore, to obtain the value of this new ordinate, we see that we must change x into $x+h$ in the equation of the curve, and the value which the equation shall determine for y will be that of y' .

For example, if we had the equation $y = mx^2$, we should obtain y' by changing x into $x+h$, and y into y' , and we should have $y' = m(x+h)^2$.

Or, by developing $y' = mx^2 + 2mxh + mh^2$

3. Let us take also the equation $y = x^3$ (1)

And suppose that when x becomes $x+h$, y becomes y' , we shall have then

$$y' = (x+h)^3;$$

Or by expanding, : $y' = x^3 + 3x^2h + 3xh^2 + h^3$;

If from the equation we subtract (1) there will remain

$$y' - y = 3x^2h + 3xh^2 + h^3$$

And by dividing by h ,

$$\frac{y' - y}{h} = 3x^2h + 3xh^2 + h^3 \quad (2)$$

Let us see now what is to be learnt from this result : $y' - y$, being the difference between the new value of y and its primitive one, represents the increment of the function y in consequence of the increment h given to

w ; and the increment of w , on the other hand, being h , it follows that the expression $\frac{y' - y}{h}$ is the ratio of

the increment of the function y to that of the variable x . By attending to the second side of equation (2), we see that this ratio is diminished the more h is diminished, and that when h become 0 this ratio is reduced to

$3x^2$. This term $3x^2$ is therefore the limit of the ratio $\frac{y' - y}{h}$, being the term to which it tends as we diminish h .

Boucharlat's formulations are very characteristic of early 19th-century mathematics. Thus, the definition of the function remains very similar to that given by Bernoulli in 1718: "Here, the function of a variable quantity is called a quantity composed in any way whatsoever of this variable quantity and constants", taken up by L. Euler in 1730, who was to develop it in 1748 into "an analytical

expression made up of the variable and constants"¹⁴ In particular, it does not distinguish between the function (relationship between two variables, one of which depends on the other) and the dependent variable. In $y = f(x)$ the "function" for Boucharlat is "y" and not "f()".

Marx was interested in the evolution of the concept of function and the 1968 edition of the MM included the translation/transposition into German of the history of the concept he had found in Lacroix' Treatise (1810), as well as an analysis of the concept of function used by Lagrange. Lacroix did not mention any particular mathematician in his historical sketch, but presented the successive "developments" of the concept from the "ancient analysts (who) generally included under the term function of a quantity all the powers of that quantity" (Lacroix, 1810, p. 1) to the algebraic and transcendental functions. Lacroix was particularly interested in the development of a function in series which he contrasted with the value of a function, a distinction which testifies to the dissociation between "y" and "f()"¹⁵ Marx tried to theorize this distinction by opposing 'function of x' and 'function in x': the former designating y's dependence on x, the latter a pure expression of x. He pointed out precisely the additional generalization in this area carried out by Lagrange and the subsequent confusion of meanings of the term. He notes that:

For Lagrange, on the other hand, the expression $f(x)$, as soon as it is found in the left side of the algebraic expression in x, refers only to the general expression, hence indeterminate as opposed to the particular expression; and $f(x+h)$ refers to the meaning of the general undeveloped expression as opposed to its developed expression of development in series, such as for example in algebra, $((x + a)^m$ is the general, undeveloped expression, whereas on the right side, we have the development in series, $x^m + etc.$

This is quite satisfactory and appropriate in certain well-defined cases, but the distinction between the function of x and the function in x cannot yet be emptied of its content because it alone implies that the function of x can have a concrete existence differentiated from the function in x, such as that of the ordinate if x is the abscissa [...]. (Marx, 1968, pp. 502-4)

But if the proposed distinction was intended to clarify the "confusions" between the multiple meanings of the word function, it does not seem to have found any application in the differential calculus where Marx uses an elliptical formula "function x".

For Laurent, it will be noted first of all that he repeats without modification in the 9th edition of Boucharlat the formulation given in earlier editions. On the other hand, in his *Traité*, he felt it

¹⁴ J. Bernoulli, *Opera Omnia*, t. II, Lausanne and Geneva, 1742, p. 241, cited in A. P. Juschkewitsch, p. 241. "Euler und Lagrange über die Grundlagen der Analysis", in *Sammelband zu Ehren des 250. Geburtstages Leonhard Eulers*, Akademie-Verlag, Berlin, 1959.

¹⁵ S.F. Lacroix, *Traité du calcul différentiel et du calcul intégral*, 2^{de} édition, Paris, Courcier, 1810.

necessary to start with some discussion of the definition of the term function. After having pointed out that the term has seen its meanings changed, he gives "the more precise definition » which it seems to him should be adopted and which he will adopt in what follows: « Two quantities are a function of each other when one remains constant, the other also remains constant. » (Laurent, 1881, p. 1). This definition seems to be directly inspired by that of Cauchy, whose disciple Laurent always proclaimed himself to be; indeed, Cauchy presents the functions in the following way:

When variable quantities are so closely linked that, given the value of one variable, the values of all the others can be inferred, one usually conceives of the various quantities expressed by means of one of them, which then takes the name of an independent variable; and the other quantities, expressed by means of the independent variable, are what are called functions of that variable. (Cauchy, 1829, p. 5)

Laurent himself points out some of the disadvantages of his own definition, which erases the distinction between dependent and independent variables and further increases confusion. It is this situation that Frege referred to in 1891 in a paper entitled "Function and Concept" when he stated that the usual definition of function was not satisfactory because "no distinction is made between form and content, sign and thing signified [Bezeichnetes]; a mistake, admittedly, that is very often met with in mathematical works, even those of celebrated authors"¹⁶.

This imprecision is reflected in the definition of the limit we find in paragraph 3 of the Boucharlat text. But here the delimitation of concepts has immediate consequences on the implementation of the differential calculus. It was Newton who had introduced the idea of limit using what Marx called a "metaphysical and non-mathematical hypothesis". Newton defended himself against those who objected to the "the first and last sums and ratios of nascent and evanescent quantities" by specifying that it was necessary to understand "by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives; that is, that velocity with which the body arrives at its last place, and with which the motion ceases."¹⁷ The whole mystery seems to reside in this moment when the movement ceases. More precisely, if we give, as Boucharlat does, to an indeterminate variable x an increase h and we make $x+h$ tend to x , does "h becomes null" mean anything other than "the 'limit' is reached for $h=0$ "? We know that Lagrange's project was precisely to establish the differential calculus without evanescent quantities, limits and the infinitely small i. e. by resolutely placing itself

¹⁶ G. Frege, "Funktion und Begriff", reprinted in *Funktion, Begriff, Bedeutung*, Vandenhoeck & Ruprecht, Göttingen, 1969. (Peter; Black, Max, eds. (1952). Translations from the philosophical writings of Gottlob Frege (1st ed.). Oxford: Basil Blackwell)

¹⁷ CAJORI, F., & NEWTON, I. (1934). *Sir Isaac Newton's mathematical principles of natural philosophy and his system of the world*. Los Angeles, Calif, University of California Press, pp.38-9.

at point $h=0$. On this point, it seems that Boucharlat's synthesis was purely verbal: he uses the term "limit" but in reality it is placed in the perspective of Lagrange and "h becomes null" simply means: $h=0$.

This "ambiguity" of Boucharlat's notion of limits was the starting point of Marxian criticism. Marx notes, indeed, in the case of x^3 that:

If one argues that that the more the value of h decreases on the right-hand side, so much the more does the value of the terms $3xh + h^2$ decrease, so that the value as well of the entire right-hand side $3x^2 + 3xh + h^2$ more and more closely approaches the value $3x^2$, we then must add, however : 'yet without being able to coincide with it'. [...] In this sense $3x^2$ becomes the limit value of the series $3x^2 + 3xh + h^2$ [...]

However as soon as we set $h=0$, the terms on the right-hand side vanish, making $3x^2$ the limit of its value; now $3x^2$ is the first derivative of x^3 and so $= f'(x)$. As $f'(x)$ it indicates that an $f''(x)$ is also derivable from it [...] and thus that the increment $f'(x)$ or $3x^2$ is not =the sum of the increments which can be developed from $f(x) = x^3$ [...] In this sense, however, the developed series of increments becomes, as soon as I break it off, the *limit value* of the development [...] This has absolutely nothing in common with the limit value in the first sense (Marx, 1983, pp.124-5)

It is this ambiguity in the post-Lagrangian use of the concept of limit that will lead Marx to reject it as misleading and constantly misleading. More deeply, we can see here a consequence of Marx's concern to deal with infinity because, on the one hand, he insists on the possibility that h decreases without limit, and, on the other hand, Marx wants to reason by posing $h=0$ that is, to master the actual infinity. In this sense, one can assume that he would not have been satisfied with Cauchy's reasoning, even if he had known Cauchy, which does not seem to have been the case. He would undoubtedly have appreciated the way in which Cauchy clarified the concept of limits, but he avoided dealing with infinite quantities in the following way.

When the values successively assigned to the same variable approach indefinitely a fixed value so as to end up differing as little as one wishes, the latter is called the limit of all the others¹⁸.

At this point in examining the concepts of the differential calculus, it may be useful to verify that it is not just a matter of words, but that the implementation of different concepts leads to significant differences in demonstrations, some of which may be wrong. To this end, we choose Boucharlat's study of $\frac{\sin h}{h}$, a study that attracted Marx's attention.

¹⁸ Augustin Cauchy, *Leçons sur le calcul différentiel*, Paris, Debure, 1829, pp. 1,2.

39. The arc is greater than the sine, and less than the tangent.

To prove this, let AB, fig. 2, be an arc, which has BE for its sine, and DA for its tangent, and take the arc AB' equal to the arc AB. Then the chord BB' being a straight line, BB' is less than the arc AB'; and therefore the straight line BE, which is the half of the chord BB', is less than the arc BA, the half of arc BAB'; whence it follows that the sine is less than the arc.

To prove that the tangent is greater than the arc, we have

Area of triangle DD'C > area of sector BAB'C;

or, putting for these areas their geometrical values,

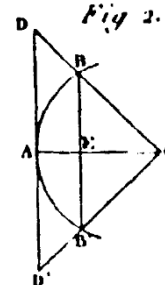
$$DD' \times \frac{1}{2} AC > BAB' \times \frac{1}{2} AC;$$

suppressing the common factor $\frac{1}{2} AC$, there remains

$$DD' > BAB',$$

and taking the halves, we have

$$DA > \text{arc } BA.$$



40. It follows from this, that the limit of the ratio of the sine to the arc is unity; for since, when the arc h , represented by AB, becomes nothing, the sine coincides with the tangent; much more does the sine coincide with the arc, which lies between the tangent and sine; and, consequently, we have, in the case of the limit, $\frac{\sin h}{\text{arc } h}$ or rather, $\frac{\sin h}{h} = 1$

Boucharlat, Blakelock, 1828, pp. 24-5,

The ambiguity that surrounds the notion of limit allows us to juggle away¹⁹ the passage from h tends to 0 to $h=0$. Conversely, we can find a confirmation of the role played by ambiguity in the definition of the limit by considering the precautions Cauchy uses to proceed with the same demonstration:

The limits, which the 2 variable expressions $\frac{\sin \alpha}{\alpha}$, $(1 + \alpha)^{1/\alpha}$ are indefinitely approaching, while

α converges towards zero, present themselves in indeterminate forms, $\frac{0}{0}$, $1^{\pm\infty}$ and yet both

limits have fixed values that can be calculated as follows.

¹⁹ Jacques Derrida has devoted a chapter of his 1993 book *Specters of Marx the state of the debt, the work of mourning and the New international* to Marx' use of the French words « escamotage » or « escamoter ». Marx used several times these words that can be found in the German Ideology, written in 1841-1846. See especially the chapter Sankt Max). Derrida's chapter is curiously entitled "The apparition of the unapparent". It includes a note by the translator stipulating that The English translation of The German Ideology retains "conjuring trick" for Marx and Engels' term Eskamotage. We will do so as well for obvious reasons, although the French word "escamotage" could also be translated by "dodge," "evasion," "filching," "pinching," and so forth" (p.). To be more specific, we can consider the definition to be found in the Dictionnaire de Trévoux. The original word seems to be "escamote" which is the name of a little cork ball which is taken subtly between fingers. Hence "escamoter is the verb used by players of cups and ball in taking the cork ball (escamote) subtly between fingers to realize some tricks. It means also: to make something disappear by a legerdemain unnoticed. Trévoux adds that "escamoter" means also, to steal subtly and deftly. Marx uses repeatedly these words but nowhere more often than his mathematical manuscripts to explain how mathematicians obtain the derivative eliminating the infinitesimals. The English translation of the MMM uses various expressions to translate "escamoter" (to juggle away, conjuring tricks, etc.).

Obviously, for very-small numerical values of α : $\frac{\sin \alpha}{\sin \alpha} > \frac{\sin \alpha}{\alpha} > \frac{\sin \alpha}{\tan \alpha}$

Therefore, the ratio $\frac{\sin \alpha}{\alpha}$ always lies between the two quantities $\frac{\sin \alpha}{\sin \alpha} = 1$ and $\frac{\sin \alpha}{\tan \alpha} = \cos \alpha$, the first of which serves as a limit to the second, will itself have the unit as limit ". (Cauchy, pp. 1-2)

It is striking to note that the precautions deployed here by Cauchy seem to have totally escaped H. Laurent, who adopts unchanged the first three paragraphs of Boucharlat, as well as the paragraph devoted to the limit of $\frac{\sin h}{h}$

IV) Derivatives and differentials:

It is in his § 4 that Boucharlat introduces the concepts of derivative and differential. This paragraph is reproduced below in its entirety in the 1826 edition (right-hand column) and in the revised edition by H. Laurent (right-hand column).

4. Since, on the hypothesis of $h = 0$, the increment of y becomes also 0, $\frac{y' - y}{h}$ is reduced to $\frac{0}{0}$ and consequently the equation (2) becomes $\frac{0}{0} = 3x^2$

This equation involves in it nothing absurd, for from Algebra we know that $\frac{0}{0}$ may represent every sort of quantity; besides which it will be easily seen, that since by dividing the two terms of a fraction by the same number the fraction is not altered in value, it follows that the smallness of the terms of a fraction does not at all affect its value, and that, consequently, it may remain the same when its terms are diminished to the last degree, that is to say, when they become each of them 0.

The fraction $\frac{0}{0}$ which appears in the equation (3), is a symbol which has expressed the ratio of the increment of the function to that of the variable: since this symbol retains no trace of that variable, we will represent it by $\frac{dy}{dx}$

∴ and then $\frac{dy}{dx}$ will remind us that the function was y and the variable x ; but dy and dx will be no less evanescent quantities, and we shall have

$$\frac{dy}{dx} = 3x^2 \dots\dots\dots (4).$$

$\frac{dy}{dx}$ or rather its value $3x^2$ is the differential coefficient of the function y .

We may observe that $\frac{dy}{dx}$ being the symbol which represents the limit $3x^2$, [as is shown by equation (4)], dx ought properly to be always placed under dy . In order, however, to facilitate operations in algebra, we may for a time clear equation (4) of its denominator, and we shall have $dy = 3x^2 dx$. The expression $dy = 3x^2 dx$ is what we call the differential of the function y .

J. L. Boucharlat, § 4, p. 4

4. It is often necessary in Analysis to find the limit of the ratio of the increase that a function takes to the corresponding increase h of its variable: we give it the name of derivative or differential coefficient.

Thus, the derivative of a function is the limit of the ratio of the increase that the function takes to the corresponding increase of its variable, when the latter tends towards 0.

We represent the derivative of a function y of x by the notation $\frac{dy}{dx}$, an expressive notation which reminds us that y is the function, that x is the variable, and that we have taken a ratio of increases relative to each of these quantities.

Note that $\frac{dy}{dx}$ being the sign representing the limit $3x^2$, dx ought properly to be always placed under dy . However, in order to facilitate operations in algebra, we may for a time disregard and clear equation (4) of the denominator dx and we shall have $dy = 3x^2 dx$. This expression $dy = 3x^2 dx$ or dy is what is called the differential of the function y ; dx is then arbitrary and dy is nothing else than the product of dx by the derivative of the function. The differential of a function can therefore be defined as follows:

The differential of a function is the product of the derivative of the function by an arbitrary quantity called dx .

It goes without saying that if we take $dx = 1$, dy is reduced to the very derivative of y

Boucharlat-Laurent, § 4.

Here we can note a revision by Laurent which rules out what was more obscure in Boucharlat's synthesis between the Lagrange method and that of limits: it is clear here that, for Boucharlat, h effectively becomes null when it decreases. On the contrary, Laurent, without explicitly introducing limits in the sense of Cauchy, gives a formulation that is not incompatible with them.

This passage from Boucharlat seems to have drawn Marx's attention to the nature of the differential, and he cites it repeatedly, Indeed, he was particularly shocked by the way in which the separation of numerator and denominator is presented in the ratio $\frac{dy}{dx}$. He writes: " Thus, in order

to facilitate algebraic operations" one introduces a false formula which one baptises the "differential ", (Marx, 1983:25)

It is this inadequacy that will lead Marx to specify his own conception of the differential calculus. To do this, he will carefully compare the calculation of the derivative of a function of a single independent variable in two cases:

- a) The simple case : $y = f(x)$
- b) The product of two functions : $y = u \cdot z$ with $u = f(x)$ and $z = \phi(x)$

Calculating the derivative of $f(x)$, we get : $\frac{dy}{dx} = f'(x)$

Now in this expression, $\frac{dy}{dx}$ is only a symbol, a sign that represents $\frac{0}{0}$ and Marx uses images: "ceremonial uniform", "Sunday dress". At the same time, he notes the absence in the right-hand side of the equation of symbols proper to the differential calculus indicating the operations to be carried out.

On the contrary, in the second case, Marx presents the differentiation in the following way:

$$y' = u' \cdot z' \cdot z$$

$$y' - y = u' \cdot z' - u \cdot z = z' (u' - u) + u (z' - z)$$

$$\frac{y' - y}{x' - x} \text{ or } \frac{\Delta y}{\Delta x} = z' \frac{u' - u}{x' - x} + u \frac{z' - z}{x' - x} = z' \frac{\Delta u}{\Delta x} + u \frac{\Delta z}{\Delta x}$$

Then, putting on the right-hand side $x' = x$, Marx draws from it: $\frac{dy}{dx} = z \frac{du}{dx} + u \frac{dz}{dx}$ or by eliminating

the denominator: dy or $d(uz) = zdu + u dz$

In analysing the legitimacy of these operations, Marx insists on the difference in nature between $\frac{dy}{dx}$

which is the symbolic expression of $f'(x)$ in both cases and the $\frac{du}{dx}$ and $\frac{dz}{dx}$. These thus appear as

"symbolic differential coefficients without real differential coefficients, i. e. without corresponding equivalent 'derivatives'. The symbolic coefficient thus becomes an independent starting point, the real equivalent of which is to be found first. [...] Whereas, initially, the differential coefficient is the expression of the derivative already carried out, it now plays the role of symbols of differentiation operations to be carried out. » For Marx, this "reversal of the method" marks the birth of the

differential calculus "which then operates independently on its own ground. For its starting points $\frac{du}{dx}$, $\frac{dz}{dx}$ belong only to it and are mathematical quantities characteristics of it. (Marx, 1983, pp. 20-21)

Marx then examines the separation between dx and dy and the definition of the differential, always distinguishing the same two kinds of function of x . He states, first of all, that in the first case, separation is legitimate when the substitution of dx for $(x' - x)$ causes the latter to lose "the form that manifests it as a vanished difference of the independent variable and likewise for dy [...]. Regardless of where dx is now located, such a change of location leaves the relationship between dy and dy unchanged; $dy=f'(x)dx$ appears to us as another form of $\frac{dy}{dx} = f'(x)$ and is always likely to be written in this form ».

In the second case, he goes on to say, "In the second case, we saw that the differential coefficients were transformed into" independent starting points "or" symbols of operations "and likewise," symbolic equations are transformed into equations of symbolic operations. Thus, we have a double basis for treating the differential $dy = f'(x) dx$ as a symbolic equation of operation. [...] If the differential operation on $f(x)$ designated by $df(x)$ is carried out, the result will be $dy = f'(x) dx$ and this gives finally $\frac{dy}{dx} = f'(x)$ ». And Marx concludes that "it is only when the differential functions as a starting point of the calculation that the reversal of the algebraic differentiation method is completed and that the differential calculation itself would then appear a unique, specific method of calculating with variable quantities". (Marx, 1983, p. 26).

This 1881 text follows various drafts and reflects the conception of the differential calculus he had arrived at. It was the conception of the differential that attracted the attention of the commentators, but it is interesting to note that for Marx, the differential calculus is really founded from the moment the differential is conceived as a symbol of operation. Also before emphasizing the similarity between this conception and that of other mathematicians, it should be noted that this is precisely the approach adopted by Cauchy in his 1829 book. Cauchy first defines and studies the limits in "Preliminaries", then he begins his "1st Lesson" on "the object of the differential calculus" with a provisional definition of the differential:

x, y, z, \dots being variables « determined by one or more equations, we call the differential of x, y, z, \dots and we indicate it by means of the characteristic letter d in the notations: dx, dy, dz quantities whose ratios are equivalent to the final ratios of the infinitely small increases in these variables. The purpose of the differential calculus is to determine the ratios of the differentials dx, dy, dz, \dots when we know the relationships that exist between the variables, x, y, z, \dots ; [...]

« To fix ideas », Cauchy then studies the case: $y = f(x)$.

If the function $f(x)$ remains continuous between two given limits of the variable x and if a value between the two limits of the variable x is assigned to this variable, an infinitely small increase, attributed to the variable, will produce an infinitely small increase in the function itself. Therefore, if then $x = i$, the two terms of the ratio of the differences

$$\frac{dy}{dx} = \frac{f(x+i) - f(x)}{i}$$

will be infinitely small quantities. But while these two terms will indefinitely and simultaneously approach the zero limit, the ratio itself will be able to converge towards another limit, either positive or negative, which will be the ratio of the the infinitely small differences Δy and Δx . This limit, or final value, when it exists, has a specific value for each particular value of x ; but it varies with x . [...] The form of the new function that will serve as a limit to the relationship will depend on the form of the proposed function $y = f(x)$. To indicate this dependency, the new function is given the name of the derived function, [...]

Having said that, the differentials dx , dy of the independent variable x and the function $y = f(x)$ will be the quantities so chosen that their ratio $\frac{dy}{dx}$ coincides with the last reason of the infinitely

small quantities Δy , Δx i. e. with the limit $y' = f'(x)$ of the ratio. $\frac{\Delta y}{\Delta x}$ These differentials will

therefore be linked by the equation $\frac{dy}{dx} = y'$ or $dy = y'dx$ which can also be presented in one of the

following forms : $\frac{df(x)}{dx} = f'(x)$ $df(x) = f'(x) dx$ (Cauchy, 1829, pp. 17-8)

Cauchy's approach is a good illustration of the reversal in the differential calculus that Marx had anticipated. The question then arises as to whether Marx was aware of Cauchy's works or works of the same type. There is no evidence that he had read Cauchy, yet one might think that Marx was at least aware of his work. In fact, on a list of books that Marx proposed to consult, one can find the *Leçons de calcul différentiel et de calcul integral, written according to the methods and published or unpublished works of M. A. -L Cauchy* by Abbé Moigno (Paris, Bachelier, 1840).. But Abbé Moigno's fidelity to Cauchy is marred by notable elisions: first of all, we do not find in Moigno the care taken by Cauchy to define in his "Préliminaires" either infinitesimals or the limit of a variable. Moigno defines the former in a vague way, and for the limit he merely indicates that one "usually calls the limit of a function, the value towards which it converges when the variable on which it depends itself converges towards a given value". (Moigno, 1840, p. 3). Moigno then explicitly

departs from Cauchy on the question of the definition of the differential. He writes in his "Introduction":

In recent years, Mr. Cauchy believed that he had to give a direct, immediate definition of the differential, independent of the consideration of derived functions. Similarly to the approach of Maclaurin and D' Alembert, he calls differentials quantities whose ratios are equivalent to the final ratios of the increases that variables can take simultaneously. Starting from this definition one can calculate the differentials without going through the derivatives [...] but I did not want to take it as a starting point [...] I would have feared, by using it too soon to throw some obscurities on the principles of the Differential Calculus that it was so important to clarify". (Moigno, pp. XX- XXI)

So when Marx writes:"I remark further that from the early part of the eighteenth century, right down to the present day, apart from a few rare exceptions, the general task of the differential calculus has usually been formulated as follows : to find the real equivalent of the symbolic differential coefficient". (Marx, 1983, pp. 22) Perhaps he counts Cauchy in the exceptions as a result of his reading of Moigno.

V) Various concepts of differentials :

In 1928, the Russian mathematician V. Glivenko reported in 1928 the resemblance between the concept of the differential in Marx and in Hadamard. Fréchet studied different conceptions of the differential article celebrating Hadamard's achievements ²⁰. It can be used to highlight the inaccuracies or even errors into which H. Laurent has fallen, due to the lack of a correct definition. Fréchet writes:

Assuming that the definition of the differential of a function of a variable is well known, M Hadamard proposed to define the differential of a function of several variables, for example, $f(x, y)$, as the classic expression

$$(1) \quad dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

but for him, this expression is only a symbol of operations.

Fréchet continues: "what does the equality (1) mean?. That if x, y , and hence $z = f(x, y)$ are expressed as a function of an auxiliary variable u , we have whatever these expressions are

$$(2) \quad \frac{\partial z}{\partial u} = \frac{\delta z}{\delta x} \frac{dx}{du} + \frac{\delta z}{\delta y} \frac{dy}{du}$$

²⁰ M. Fréchet, "Sur la notion de différentielle", *Journal de Mathématiques Pures et Appliquées*, July-September 1937.

This is the only meaning of equality (1). Since equality (2) takes place regardless of the variable by which the other two variables are expressed, the reference to "u" is deleted. (M. Fréchet, 1937, pp.233-4)

Fréchet notes that « Hadamard's definition emphasizes the operative properties of the differential. Other definitions rather highlight its property of being a kind of principal component of the increase of the function ». (Fréchet, 1937, p. 235) We can justly find an example of such conceptions in Laurent's *Treatise of Analysis*. H. Laurent does not clearly distinguish the two concepts of the differential. In fact, in his *Treatise*, after a chapter 3 devoted to derivatives, he deals in turn in chapters 5 and 6 with the differentials of functions of one variable and those of the functions of several variables. In Chapter 5, after studying the "various orders of infinitesimals", he introduces the differentials in the following way: "It is advantageous to substitute $f'(x) \Delta x$ to $\Delta f(x)$ in such a way that one has expressed the need to give a name to this quantity $f'(x) \Delta x$; it is called the differential of $f(x)$ ". He then takes the case where $f(x) = x$, $\Delta x = dx$ and writes: $dy = f'(x)dx$; (H. Laurent, 1881, pp. 122-3). On the other hand, in Chapter 6, he seems to agree with an operative definition in paragraph I "on the calculation of symbolic expressions": "We have already seen that an operation symbol can be represented by a letter. Thus the operation which aims to take the derivative of φ and multiply it by h is represented by $d\varphi$. (ibidem, p. 133). But finally, the idea that differentiation is the main part of the infinitely small Δf seems to be much more characteristic than Laurent's conception, as can be seen precisely in the presentation of the derivative of a compound function. It assumes, for this reason, that "in function $f(x, y, z)$, the variables x, y, z cease to be independent, and that y and z are functions of x ; according to the theorem of **the composite functions**, the derivative of f relative to x , which we will call the total derivative relative to x , will be given by the formula [...] $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial z} \frac{dz}{dx}$ and he concludes a detailed discussion of the different differential notations with the remark: "It is almost useless to add that this is an infinitely small increase and that the terms of the second order are neglected" (ibidem, p. 141)

One can see in the use of this definition of the differential of a function of several variables the errors to which it can lead. H. Laurent believes that it is indeed possible to establish the "fundamental theorem":

The increase that a function of several variables undergoes when one gives increments of the same order to its variables is, except for the infinitesimals of higher order, equal to its differential, and consequently to a sufficient degree of accuracy, the increase of a function can be replaced by its differential without changing the result. (H. Laurent, p. 148).

However, following the work of various mathematicians (Otto Stolz and W. H. Young) of the late 19th and early 20th centuries, as M. Fréchet writes²¹

it is easy to see that even for very simple $f(x, y)$ functions, the ratio $\frac{\Delta f}{df}$, i.e. $\frac{f(x + \Delta x, y + \Delta y) - f(x, y)}{f'_x \Delta x + f'_y \Delta y}$ does not necessarily tend towards unity in any way when Δx and Δy tend towards zero. (M. Fréchet, 1937, p. 235)

Engels used ironic words about mathematicians in general after becoming familiar with the critical reading of the history of differential calculus that Marx presented to him²². They were probably not justified for all mathematicians. Similarly an overall assessment of Marx's mathematical work would require a careful analysis of his history of Taylor's and Maclaurin's theorems and their role in the differential calculus. But it seems legitimate, as of now, to bring Marx's concerns closer to the modern presentation of the differential. Let us recall that:

a) Taylor's formula approximates the difference $f(x+h) - f(x)$ by $f'(x)h$. We call the differential of $f(x)$ in x , the application noted $df(x)$:

$$\begin{aligned} R & \text{-----} R \\ h & \text{-----} df(x)(h) = f'(x)h \\ df(x) & \text{ is linear.} \end{aligned}$$

b) the differential of $f(x)$ is then defined as the application noted df of R in the set $L(R)$ of the linear applications of R in R .

$$\begin{aligned} df : R & \text{-----} \rightarrow L(R) \\ x & \text{-----} \rightarrow df(x) \end{aligned}$$

However, what explains the difficulties encountered by mathematicians of the 19th century (and after) and the confusion in this area is that:

$$L(R) = R.$$

²¹ Mr Fréchet quotes Otto Stolz and W. H. Young. He's probably has in mind O. Stolz: "B. Bolzano's Bedeutung in der Geschichte der Infinitesimalrechnung", *Mathematischen Annalen*, XVIII, 1881, 255-279. Bourbaki's *Elements* mentions various works on the integral calculus by one W. H. Young published in 1911.

²² « The thing is as clear as daylight, so that we cannot wonder enough at how stubbornly mathematicians insist on mystifying it » (Letter from Engels to Marx, August 18th, 1881).

Marx saw these difficulties and his concern for dialectics led him to carefully distinguish the process (the operation of differentiation) from the result (the derivative), but, not having at his disposal the modern tools of set theory, he was led to propose different formulations and distinctions (e. g. the symbols of operation). It appears, however, that Marx's interest in the transformations brought about by the development of different approaches to the differential calculus suggests that the genetic and dialectic method (the history of concepts) deserves attention

Alain Alcouffe

Autumn 1986.

University of Social Sciences of Toulouse.

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