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# A BRAID-LIKE PRESENTATION OF THE INTEGRAL STEINBERG GROUP OF TYPE $C_2$

CHRISTIAN KASSEL

ABSTRACT. We show that the Steinberg group  $\text{St}(C_2, \mathbb{Z})$  associated with the Lie type  $C_2$  and with integer coefficients can be realized as a quotient of the braid group  $B_6$  by one relation. As an application we give a new braid-like presentation of the symplectic modular group  $\text{Sp}_4(\mathbb{Z})$ .

*In memoriam amici*  
*Patrick Dehornoy*  
*(1952–2019)*

## 1. INTRODUCTION

Let  $B_6$  be the braid group on six strands. It has a standard presentation with five generators  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ , and the following ten relations ( $1 \leq i, j \leq 5$ ):

$$(1.1) \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| > 1$$

and

$$(1.2) \quad \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i - j| = 1.$$

The purpose of this note is to show that if we add the single relation

$$(1.3) \quad (\sigma_1 \sigma_2 \sigma_1)^2 (\sigma_1 \sigma_3^{-1} \sigma_5) (\sigma_1 \sigma_2 \sigma_1)^{-2} (\sigma_1 \sigma_3^{-1} \sigma_5) = 1,$$

to Relations (1.1) and (1.2), we obtain a presentation of the Steinberg group  $\text{St}(C_2, \mathbb{Z})$  associated with the Lie type  $C_2$  over the ring of integers (see Theorem 4.1). This group was defined for general commutative rings in [15].

If we further add the relation

$$(1.4) \quad (\sigma_1 \sigma_2 \sigma_1)^4 = 1,$$

we obtain a presentation of the symplectic modular group  $\text{Sp}_4(\mathbb{Z})$  (see Corollary 4.2). It is known that the Steinberg group  $\text{St}(C_2, \mathbb{Z})$  is a group extension of  $\text{Sp}_4(\mathbb{Z})$  with infinite cyclic kernel.

In order to prove these results we construct a surjective group homomorphism  $f : B_6 \rightarrow \text{St}(C_2, \mathbb{Z})$ , show that  $f$  vanishes on the normal subgroup  $N$  of  $B_6$  generated by the element represented by the braid word in (1.3), and construct a group homomorphism  $\varphi : \text{St}(C_2, \mathbb{Z}) \rightarrow B_6/N$  such that  $\varphi \circ f = \text{id}$ . The homomorphism  $f$  is a lifting of a map  $B_6 \rightarrow \text{Sp}_4(\mathbb{Z})$ , which is a special case of the homomorphism  $\bar{f} : B_{2g+2} \rightarrow \text{Sp}_{2g}(\mathbb{Z})$  constructed in [12, Sect. 4] for any  $g \geq 1$ .

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The note is organized as follows. In Section 2 we give a presentation of the Steinberg group  $\text{St}(C_2, \mathbb{Z})$  and we prove a number of relations between special elements of  $\text{St}(C_2, \mathbb{Z})$ . In Section 3 we construct the homomorphism  $f$  from the braid group to the Steinberg group. In Section 4 we state and prove our results. In Appendix A we give a presentation of the Steinberg group  $\text{St}(C_2, R)$  with coefficients in an arbitrary commutative ring  $R$ .

**Notation.** All the groups we consider are noted multiplicatively. We denote their identity elements by 1 and we use brackets for the commutators:

$$[x, y] = xyx^{-1}y^{-1}.$$

Recall that  $[y, x] = [x, y]^{-1}$ .

## 2. THE STEINBERG GROUP $\text{St}(C_2, \mathbb{Z})$

The positive roots of the root system  $C_2$  consist of four vectors  $\alpha$ ,  $\beta$ ,  $\alpha + \beta$  and  $2\alpha + \beta$  of the Euclidean plane  $\mathbb{R}^2$ , where the roots  $\alpha$  and  $\alpha + \beta$  are of length  $\sqrt{2}$ , the roots  $\beta$  and  $2\alpha + \beta$  are of length 2, and  $\alpha$  is orthogonal to  $\alpha + \beta$ . Together with  $-\alpha$ ,  $-\beta$ ,  $-(\alpha + \beta)$  and  $-(2\alpha + \beta)$ , they form the root system  $\Phi$  of type  $C_2$  (see e.g. [6, 7]).

**2.1. Defining relations for the Steinberg group.** We have the following presentation for the Steinberg group  $\text{St}(C_2, \mathbb{Z})$  of type  $C_2$  over the ring  $\mathbb{Z}$  of integers.

**Proposition 2.1.** *The Steinberg group  $\text{St}(C_2, \mathbb{Z})$  has a presentation with eight generators*

$$x_\alpha, x_\beta, x_{\alpha+\beta}, x_{2\alpha+\beta}, x_{-\alpha}, x_{-\beta}, x_{-(\alpha+\beta)}, x_{-(2\alpha+\beta)}$$

and the following 24 relations:

$$(2.1) \quad [x_\alpha, x_{2\alpha+\beta}] = [x_\beta, x_{\alpha+\beta}] = [x_\beta, x_{2\alpha+\beta}] = [x_{\alpha+\beta}, x_{2\alpha+\beta}] = 1,$$

$$(2.2) \quad [x_\alpha, x_{-\beta}] = [x_\beta, x_{-\alpha}] = [x_\beta, x_{-(2\alpha+\beta)}] = [x_{-\beta}, x_{2\alpha+\beta}] = 1,$$

$$(2.3) \quad [x_{-\alpha}, x_{-(2\alpha+\beta)}] = [x_{-\beta}, x_{-(\alpha+\beta)}] \\ = [x_{-\beta}, x_{-(2\alpha+\beta)}] = [x_{-(\alpha+\beta)}, x_{-(2\alpha+\beta)}] = 1,$$

$$(2.4) \quad [x_\alpha, x_\beta] = x_{\alpha+\beta} x_{2\alpha+\beta} = x_{2\alpha+\beta} x_{\alpha+\beta},$$

$$(2.5) \quad [x_\alpha, x_{\alpha+\beta}] = x_{2\alpha+\beta}^2,$$

$$(2.6) \quad [x_\alpha, x_{-(\alpha+\beta)}] = x_{-\beta}^{-2},$$

$$(2.7) \quad [x_\alpha, x_{-(2\alpha+\beta)}] = x_{-\beta} x_{-(\alpha+\beta)}^{-1} = x_{-(\alpha+\beta)}^{-1} x_{-\beta},$$

$$(2.8) \quad [x_\beta, x_{-(\alpha+\beta)}] = x_{-\alpha} x_{-(2\alpha+\beta)} = x_{-(2\alpha+\beta)} x_{-\alpha},$$

$$(2.9) \quad [x_{\alpha+\beta}, x_{-\alpha}] = x_\beta^{-2},$$

$$(2.10) \quad [x_{\alpha+\beta}, x_{-\beta}] = x_\alpha x_{2\alpha+\beta}^{-1} = x_{2\alpha+\beta}^{-1} x_\alpha,$$

$$(2.11) \quad [x_{\alpha+\beta}, x_{-(2\alpha+\beta)}] = x_{-\alpha} x_\beta^{-1} = x_\beta^{-1} x_{-\alpha},$$

$$(2.12) \quad [x_{2\alpha+\beta}, x_{-\alpha}] = x_{\alpha+\beta}^{-1} x_{\beta}^{-1} = x_{\beta}^{-1} x_{\alpha+\beta}^{-1},$$

$$(2.13) \quad [x_{2\alpha+\beta}, x_{-(\alpha+\beta)}] = x_{\alpha} x_{-\beta} = x_{-\beta} x_{\alpha},$$

$$(2.14) \quad [x_{-\alpha}, x_{-\beta}] = x_{-(\alpha+\beta)}^{-1} x_{-(2\alpha+\beta)} = x_{-(2\alpha+\beta)} x_{-(\alpha+\beta)}^{-1},$$

$$(2.15) \quad [x_{-\alpha}, x_{-(\alpha+\beta)}] = x_{-(2\alpha+\beta)}^{-2}.$$

Note that in view of Relations (2.10), (2.11) and (2.14) the generators  $x_{\alpha}$ ,  $x_{-\alpha}$  and  $x_{-(\alpha+\beta)}$  can be expressed in terms of the five remaining generators  $x_{\beta}$ ,  $x_{\alpha+\beta}$ ,  $x_{2\alpha+\beta}$ ,  $x_{-\beta}$ ,  $x_{-(2\alpha+\beta)}$ .

Proposition 2.1 will be proved in Section 2.3 below.

**2.2. The surjection  $\pi : \text{St}(C_2, \mathbb{Z}) \rightarrow \text{Sp}_4(\mathbb{Z})$ .** Recall that the symplectic modular group  $\text{Sp}_4(\mathbb{Z})$  is the group of automorphisms of the free abelian rank 4 group  $\mathbb{Z}^4$  preserving the standard alternating form, namely the group of  $4 \times 4$ -matrices  $M$  with integer entries such that

$$M^T \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

where  $M^T$  is the transpose of  $M$  and  $I_2$  is the identity  $2 \times 2$ -matrix.

It is known that the group  $\text{Sp}_4(\mathbb{Z})$  is generated by the following eight matrices (see [1] or [11]):

$$(2.16) \quad X_{\alpha} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad X_{-\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(2.17) \quad X_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{-\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$(2.18) \quad X_{\alpha+\beta} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{-(\alpha+\beta)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

$$(2.19) \quad X_{2\alpha+\beta} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{-(2\alpha+\beta)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Observe that each matrix  $X_{-\gamma}$  ( $\gamma \in \Phi$ ) is the transpose of  $X_{\gamma}$ .

It is easy to check that the matrices  $X_{\gamma}$  ( $\gamma \in \Phi$ ) satisfy the relations (2.1)–(2.15) defining the Steinberg group  $\text{St}(C_2, \mathbb{Z})$ . Therefore there exists a unique homomorphism

$$(2.20) \quad \pi : \text{St}(C_2, \mathbb{Z}) \rightarrow \text{Sp}_4(\mathbb{Z})$$

such that  $\pi(x_{\gamma}) = X_{\gamma}$  for all  $\gamma \in \Phi$ . Since the matrices  $X_{\gamma}$  ( $\gamma \in \Phi$ ) generate  $\text{Sp}_4(\mathbb{Z})$ , the homomorphism  $\pi$  is surjective.

**2.3. Proof of Proposition 2.1.** By [2, Sect. 3] the Steinberg group  $\text{St}(C_2, \mathbb{Z})$  has a presentation with the set of generators  $\{x_\gamma : \gamma \in \Phi\}$  subject to the following relations: if  $\gamma, \delta \in \Phi$  such that  $\gamma + \delta \neq 0$ , then

$$(2.21) \quad [x_\gamma, x_\delta] = \prod x_{i\gamma+j\delta}^{c_{i,j}^{\gamma,\delta}},$$

where  $i$  and  $j$  are positive integers such that  $i\gamma + j\delta$  belongs to  $\Phi$  and the exponents  $c_{i,j}^{\gamma,\delta}$  are integers depending only on the structure of  $\text{Sp}_4(\mathbb{Z})$ . In order to find the structure constants  $c_{i,j}^{\gamma,\delta}$  it is enough to apply the homomorphism  $\pi$  of (2.20) and to compute the integers  $c_{i,j}^{\gamma,\delta}$  appearing in the relations

$$[X_\gamma, X_\delta] = \prod X_{i\gamma+j\delta}^{c_{i,j}^{\gamma,\delta}},$$

where  $X_\gamma = \pi(x_\gamma)$  are the elements of  $\text{Sp}_4(\mathbb{Z})$  defined by (2.16)–(2.19).

In particular, when  $X_\gamma X_\delta = X_\delta X_\gamma$  ( $\gamma, \delta \in \Phi$ ), then  $x_\gamma x_\delta = x_\delta x_\gamma$ . Relations (2.1)–(2.3) follow from this remark. We next have the following equalities in  $\text{Sp}_4(\mathbb{Z})$ :

$$\begin{aligned} [X_\alpha, X_\beta] &= X_{\alpha+\beta} X_{2\alpha+\beta}, & [X_\alpha, X_{\alpha+\beta}] &= X_{2\alpha+\beta}^2, \\ [X_\alpha, X_{-(\alpha+\beta)}] &= X_{-\beta}^{-2}, & [X_\alpha, X_{-(2\alpha+\beta)}] &= X_{-\beta} X_{-(\alpha+\beta)}^{-1}, \\ [X_\beta, X_{-(\alpha+\beta)}] &= X_{-\alpha} X_{-(2\alpha+\beta)}, & [X_{\alpha+\beta}, X_{-\alpha}] &= X_\beta^{-2}, \\ [X_{\alpha+\beta}, X_{-\beta}] &= X_\alpha X_{2\alpha+\beta}^{-1}, & [X_{\alpha+\beta}, X_{-(2\alpha+\beta)}] &= X_{-\alpha} X_\beta^{-1}, \\ [X_{2\alpha+\beta}, X_{-\alpha}] &= X_{\alpha+\beta}^{-1} X_\beta^{-1}, & [X_{2\alpha+\beta}, X_{-(\alpha+\beta)}] &= X_\alpha X_{-\beta}, \\ [X_{-\alpha}, X_{-\beta}] &= X_{-(\alpha+\beta)}^{-1} X_{-(2\alpha+\beta)}, & [X_{-\alpha}, X_{-(\alpha+\beta)}] &= X_{-(2\alpha+\beta)}^{-2}. \end{aligned}$$

From these relations the remaining relations of Proposition 2.1 follow.

**2.4. The elements  $w_\gamma$ .** For any root  $\gamma \in \Phi$ , set

$$(2.22) \quad w_\gamma = x_\gamma x_{-\gamma}^{-1} x_\gamma.$$

In Steinberg's notation (see [16, 17]), we have  $x_\gamma = x_\gamma(1)$  and  $w_\gamma = w_\gamma(1)$ .

In the sequel we need to know that the kernel of  $\pi : \text{St}(C_2, \mathbb{Z}) \rightarrow \text{Sp}_4(\mathbb{Z})$  is generated by  $w_\gamma^4$  for any long root  $\gamma$ , for instance by  $w_\beta^4$  or by  $w_{2\alpha+\beta}^4$ ; this follows from [2, Kor. 3.2]. The fact that such a generator of the kernel is of infinite order was established in [14, Th. 6.3].

We now list a few properties of the elements  $w_\gamma$ .

**Lemma 2.2.** *We have  $w_\gamma = w_{-\gamma}^{-1}$  for all  $\gamma \in \Phi$ .*

*Proof.* By [16] (see also [14, Lemme 5.2 (g)]) we have

$$\begin{aligned} w_\gamma &= w_\gamma(1) = w_{-\gamma}(-1) \\ &= x_{-\gamma}(-1) x_\gamma(1) x_{-\gamma}(-1) = x_{-\gamma}^{-1} x_\gamma x_{-\gamma}^{-1} \\ &= (x_{-\gamma} x_\gamma^{-1} x_{-\gamma})^{-1} = w_{-\gamma}^{-1}, \end{aligned}$$

which was to be proved.  $\square$

**Lemma 2.3.** *We have*

$$\begin{aligned}
w_\beta x_\alpha w_\beta^{-1} &= x_{\alpha+\beta}^{-1}, & w_\beta x_{-\alpha} w_\beta^{-1} &= x_{-(\alpha+\beta)}^{-1}, \\
w_\beta x_{\alpha+\beta} w_\beta^{-1} &= x_\alpha, & w_\beta x_{-(\alpha+\beta)} w_\beta^{-1} &= x_{-\alpha}, \\
w_\beta x_{2\alpha+\beta} w_\beta^{-1} &= x_{2\alpha+\beta}, & w_\beta x_{-(2\alpha+\beta)} w_\beta^{-1} &= x_{-(2\alpha+\beta)}, \\
w_{2\alpha+\beta} x_\beta w_{2\alpha+\beta}^{-1} &= x_\beta, & w_{2\alpha+\beta} x_{-\beta} w_{2\alpha+\beta}^{-1} &= x_{-\beta}, \\
w_{2\alpha+\beta} x_\alpha w_{2\alpha+\beta}^{-1} &= x_{-(\alpha+\beta)}^{-1}, & w_{2\alpha+\beta} x_{-(\alpha+\beta)} w_{2\alpha+\beta}^{-1} &= x_\alpha, \\
w_{2\alpha+\beta} x_{\alpha+\beta} w_{2\alpha+\beta}^{-1} &= x_{-\alpha}, & w_{2\alpha+\beta} x_{-\alpha} w_{2\alpha+\beta}^{-1} &= x_{\alpha+\beta}^{-1}, \\
w_\alpha x_\beta w_\alpha^{-1} &= x_{2\alpha+\beta}, & w_\alpha x_{-\beta} w_\alpha^{-1} &= x_{-(2\alpha+\beta)}, \\
w_\alpha x_{\alpha+\beta} w_\alpha^{-1} &= x_{\alpha+\beta}^{-1}, & w_\alpha x_{-(\alpha+\beta)} w_\alpha^{-1} &= x_{-(\alpha+\beta)}^{-1}, \\
w_\alpha x_{2\alpha+\beta} w_\alpha^{-1} &= x_\beta, & w_\alpha x_{-(2\alpha+\beta)} w_\alpha^{-1} &= x_{-\beta}, \\
w_{\alpha+\beta} x_\alpha w_{\alpha+\beta}^{-1} &= x_\alpha^{-1}, & w_{\alpha+\beta} x_{-\alpha} w_{\alpha+\beta}^{-1} &= x_{-\alpha}^{-1}, \\
w_{\alpha+\beta} x_\beta w_{\alpha+\beta}^{-1} &= x_{-(2\alpha+\beta)}^{-1}, & w_{\alpha+\beta} x_{-(2\alpha+\beta)} w_{\alpha+\beta}^{-1} &= x_{\alpha+\beta}^{-1}, \\
w_{\alpha+\beta} x_{2\alpha+\beta} w_{\alpha+\beta}^{-1} &= x_{-\beta}^{-1}, & w_{\alpha+\beta} x_{-\beta} w_{\alpha+\beta}^{-1} &= x_{2\alpha+\beta}^{-1},
\end{aligned}$$

*Proof.* By Relation (R7) in [17, Chap. 3, p. 23], for any two roots  $\gamma$  and  $\delta$  such that  $\gamma + \delta \neq 0$  we have  $w_\gamma x_\delta w_\gamma^{-1} = x_{\delta'}^\varepsilon$ , where  $\delta'$  is the image of  $\delta$  under the reflection in the line orthogonal to  $\gamma$  and  $\varepsilon = \pm 1$ . To determine the root  $\delta'$  and the sign  $\varepsilon$  it is enough to compute the image  $\pi(w_\gamma x_\delta w_\gamma^{-1})$  in  $\mathrm{Sp}_4(\mathbb{Z})$ .  $\square$

### 3. FROM THE BRAID GROUP TO THE STEINBERG GROUP

We now construct a homomorphism from the braid group  $B_6$  to the Steinberg group  $\mathrm{St}(C_2, \mathbb{Z})$ .

**Proposition 3.1.** *There exists a unique homomorphism  $f : B_6 \rightarrow \mathrm{St}(C_2, \mathbb{Z})$  such that*

$$\begin{aligned}
f(\sigma_1) &= x_{2\alpha+\beta}, & f(\sigma_2) &= x_{-(2\alpha+\beta)}^{-1}, \\
f(\sigma_3) &= x_\beta x_{\alpha+\beta}^{-1} x_{2\alpha+\beta}, \\
f(\sigma_4) &= x_{-\beta}^{-1}, & f(\sigma_5) &= x_\beta.
\end{aligned}$$

*The homomorphism  $f$  is surjective.*

The homomorphism  $f$  lifts the homomorphism  $\bar{f} : B_6 \rightarrow \mathrm{Sp}_4(\mathbb{Z})$  constructed in [12, Sect. 4.2] in the sense that  $\bar{f} = \pi \circ f$ , where  $\pi$  is the natural surjection  $\mathrm{St}(C_2, \mathbb{Z}) \rightarrow \mathrm{Sp}_4(\mathbb{Z})$  defined in (2.20).

*Proof.* For the existence and the uniqueness of  $f$  it suffices to check that the five elements  $f(\sigma_i)$  ( $1 \leq i \leq 5$ ) of the Steinberg group  $\mathrm{St}(C_2, \mathbb{Z})$  satisfy the braid relations (1.1) and (1.2).

(i) Let us first check the commutation relations (1.1).

- *Commutation of  $f(\sigma_1)$  with  $f(\sigma_3)$ ,  $f(\sigma_4)$  and  $f(\sigma_5)$ .* This follows from the fact that  $x_{2\alpha+\beta}$  commutes with  $x_\beta$  and  $x_{\alpha+\beta}$  by (2.1), and with  $x_{-\beta}$  by (2.2).
- *Commutation of  $f(\sigma_2)$  with  $f(\sigma_4)$  and  $f(\sigma_5)$ .* It follows from (2.2) and (2.3).

• *Commutation of  $f(\sigma_3)$  with  $f(\sigma_5)$ .* It follows from (2.1).

(ii) The relation  $f(\sigma_1)f(\sigma_2)f(\sigma_1) = f(\sigma_2)f(\sigma_1)f(\sigma_2)$  reads as

$$x_{2\alpha+\beta} x_{-(2\alpha+\beta)}^{-1} x_{2\alpha+\beta} = x_{-(2\alpha+\beta)}^{-1} x_{2\alpha+\beta} x_{-(2\alpha+\beta)}^{-1},$$

which is equivalent to  $w_{2\alpha+\beta} = w_{-(2\alpha+\beta)}^{-1}$ . The latter holds by Lemma 2.2.

(iii) The relation  $f(\sigma_2)f(\sigma_3)f(\sigma_2) = f(\sigma_3)f(\sigma_2)f(\sigma_3)$  reads as

$$x_{-(2\alpha+\beta)}^{-1} x_\beta x_{\alpha+\beta}^{-1} x_{2\alpha+\beta} x_{-(2\alpha+\beta)}^{-1} = x_\beta x_{\alpha+\beta}^{-1} x_{2\alpha+\beta} x_{-(2\alpha+\beta)}^{-1} x_\beta x_{\alpha+\beta}^{-1} x_{2\alpha+\beta}.$$

Let LHS (resp. RHS) be the element of  $\text{St}(C_2, \mathbb{Z})$  represented by the left-hand (resp. right-hand) side of the previous equation. By (2.1), (2.2), (2.3), (2.11), (2.22) we obtain

$$\begin{aligned} \text{LHS} &= x_\beta \underbrace{x_{-(2\alpha+\beta)}^{-1} x_{\alpha+\beta}^{-1}}_{\text{}} x_{-(2\alpha+\beta)} w_{-(2\alpha+\beta)}^{-1} \\ &= x_\beta x_{\alpha+\beta}^{-1} \underbrace{x_{-(2\alpha+\beta)}^{-1} x_{-\alpha}^{-1} x_\beta x_{-(2\alpha+\beta)}}_{\text{}} w_{-(2\alpha+\beta)}^{-1} \\ &= \underbrace{x_\beta x_{\alpha+\beta}^{-1} x_{-\alpha}^{-1} x_\beta}_{\text{}} w_{-(2\alpha+\beta)}^{-1} \\ &= x_\beta^2 x_{\alpha+\beta}^{-1} x_{-\alpha}^{-1} w_{-(2\alpha+\beta)}^{-1}. \end{aligned}$$

(The above underbraces  $\underbrace{\quad}$  mark the places to which we apply the relations we refer to.)

Let us now deal with RHS. Since by (2.1) and (2.2)  $x_\beta$  commutes with  $x_{\alpha+\beta}$  and with  $x_{\pm(2\alpha+\beta)}$ , and  $x_{\alpha+\beta}$  commutes with  $x_{2\alpha+\beta}$ , we obtain

$$\text{RHS} = x_\beta^2 x_{\alpha+\beta}^{-1} w_{2\alpha+\beta} x_{\alpha+\beta}^{-1}.$$

Now by Lemma 2.3, we have  $w_{2\alpha+\beta} x_{\alpha+\beta}^{-1} = x_{-\alpha}^{-1} w_{2\alpha+\beta}$ . Therefore,

$$\text{RHS} = x_\beta^2 x_{\alpha+\beta}^{-1} x_{-\alpha}^{-1} w_{2\alpha+\beta}.$$

It follows that LHS = RHS is equivalent to  $w_{-(2\alpha+\beta)}^{-1} = w_{2\alpha+\beta}$ , which again holds by Lemma 2.2.

(iv) The relation  $f(\sigma_3)f(\sigma_4)f(\sigma_3) = f(\sigma_4)f(\sigma_3)f(\sigma_4)$  reads as

$$x_\beta x_{\alpha+\beta}^{-1} x_{2\alpha+\beta} x_{-\beta}^{-1} x_\beta x_{\alpha+\beta}^{-1} x_{2\alpha+\beta} = x_{-\beta}^{-1} x_\beta x_{\alpha+\beta}^{-1} x_{2\alpha+\beta} x_{-\beta}^{-1}.$$

Let LHS' (resp. RHS') be the element of  $\text{St}(C_2, \mathbb{Z})$  represented by the left-hand (resp. right-hand) side of the previous equation. Since by (2.1) and (2.2)  $x_{2\alpha+\beta}$  commutes with  $x_\alpha$ , with  $x_{\alpha+\beta}$  and with  $x_{\pm\beta}$ , hence with  $w_\beta$ , and  $x_\beta$  commutes with  $x_{\alpha+\beta}$ , we obtain

$$\text{LHS}' = x_{\alpha+\beta}^{-1} x_{2\alpha+\beta}^2 w_\beta x_{\alpha+\beta}^{-1} = x_{\alpha+\beta}^{-1} x_{2\alpha+\beta}^2 x_\alpha^{-1} w_\beta = x_{\alpha+\beta}^{-1} x_\alpha^{-1} x_{2\alpha+\beta}^2 w_\beta,$$

the second equality holding by Lemma 2.3. For RHS', by Lemma 2.2 we have

$$\text{RHS}' = w_\beta x_{-\beta} x_{\alpha+\beta}^{-1} x_{-\beta}^{-1} x_{2\alpha+\beta}.$$

Now by (2.10), we have  $x_{-\beta} x_{\alpha+\beta}^{-1} x_{-\beta}^{-1} = x_{\alpha+\beta}^{-1} x_\alpha x_{2\alpha+\beta}^{-1}$ . Therefore,

$$\text{RHS}' = w_\beta x_{\alpha+\beta}^{-1} x_\alpha.$$

It follows from Lemma 2.3 that  $w_\beta x_{\alpha+\beta}^{-1} = x_\alpha^{-1} w_\beta$  and  $w_\beta x_\alpha = x_{\alpha+\beta}^{-1} w_\beta$ . Moreover by (2.5) we have  $x_\alpha^{-1} x_{\alpha+\beta}^{-1} = x_{\alpha+\beta}^{-1} x_\alpha^{-1} x_{2\alpha+\beta}^2$ . Therefore,

$$\text{RHS}' = x_\alpha^{-1} w_\beta x_\alpha = x_\alpha^{-1} x_{\alpha+\beta}^{-1} w_\beta = x_{\alpha+\beta}^{-1} x_\alpha^{-1} x_{2\alpha+\beta}^2 w_\beta = \text{LHS}'.$$

(v) The relation  $f(\sigma_4)f(\sigma_5)f(\sigma_4) = f(\sigma_4)f(\sigma_5)f(\sigma_4)$  reads as

$$x_{-\beta}^{-1} x_\beta x_{-\beta}^{-1} = x_\beta x_{-\beta}^{-1} x_\beta,$$

which is equivalent to  $w_{-\beta}^{-1} = w_\beta$ . The last equality holds by Lemma 2.2.

As we noted after stating Proposition 2.1, the five elements  $x_\beta, x_{\alpha+\beta}, x_{2\alpha+\beta}, x_{-\beta}, x_{-(2\alpha+\beta)}$  generate  $\text{St}(C_2, \mathbb{Z})$ . They clearly are in the image of  $f$ , which implies the surjectivity of the latter.  $\square$

#### 4. RESULTS

Let  $f : B_6 \rightarrow \text{St}(C_2, \mathbb{Z})$  be the surjective homomorphism defined in Proposition 3.1. We now state our main result.

**Theorem 4.1.** *The kernel of the homomorphism  $f : B_6 \rightarrow \text{St}(C_2, \mathbb{Z})$  is the normal subgroup of  $B_6$  generated by*

$$(\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5).$$

This means that the Steinberg group  $\text{St}(C_2, \mathbb{Z})$  has a presentation with five generators  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ , and eleven relations consisting of the ten braid relations (1.1)–(1.2) and the additional relation

$$(4.1) \quad (\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2} = (\sigma_1\sigma_3^{-1}\sigma_5)^{-1}.$$

This relation is clearly equivalent to (1.3).

As mentioned at the beginning of Section 2.4, the kernel of the projection  $\pi : \text{St}(C_2, \mathbb{Z}) \rightarrow \text{Sp}_4(\mathbb{Z})$  is generated by  $w_{2\alpha+\beta}^4$ . Since

$$(4.2) \quad w_{2\alpha+\beta} = x_{2\alpha+\beta} x_{-(2\alpha+\beta)}^{-1} x_{2\alpha+\beta} = f(\sigma_1\sigma_2\sigma_1),$$

we deduce the following presentation<sup>1</sup> of  $\text{Sp}_4(\mathbb{Z})$ .

**Corollary 4.2.** *The symplectic modular group  $\text{Sp}_4(\mathbb{Z})$  has a presentation with five generators  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ , and twelve relations consisting of Relations (1.1)–(1.2) and the two relations*

$$(\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2} = (\sigma_1\sigma_3^{-1}\sigma_5)^{-1}$$

and

$$(\sigma_1\sigma_2\sigma_1)^4 = 1.$$

Let  $N$  be the normal subgroup of  $B_6$  generated by

$$\beta = (\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5).$$

Theorem 4.1 is a consequence of the following two propositions.

**Proposition 4.3.** *We have  $f(N) = 1$ .*

<sup>1</sup>Group presentations of  $\text{Sp}_4(\mathbb{Z})$  of a different kind have been given in [1, 3].

*Proof.* It suffices to check that  $f(\beta) = 1$ . By (4.2) we have  $f(\sigma_1\sigma_2\sigma_1) = w_{2\alpha+\beta}$ . We also have

$$f(\sigma_1\sigma_3^{-1}\sigma_5) = x_{2\alpha+\beta}(x_{2\alpha+\beta}^{-1}x_{\alpha+\beta}x_{\beta}^{-1})x_{\beta} = x_{\alpha+\beta}.$$

Therefore, using Lemma 2.3, we obtain

$$f(\beta) = w_{2\alpha+\beta}^2 x_{\alpha+\beta} w_{2\alpha+\beta}^{-2} x_{\alpha+\beta} = w_{2\alpha+\beta} x_{-\alpha} w_{2\alpha+\beta}^{-1} x_{\alpha+\beta} = x_{\alpha+\beta}^{-1} x_{\alpha+\beta} = 1. \quad \square$$

It follows from the previous proposition that  $f : B_6 \rightarrow \text{St}(C_2, \mathbb{Z})$  factors through a homomorphism  $B_6/N \rightarrow \text{St}(C_2, \mathbb{Z})$ , which we still denote by  $f$ .

**Proposition 4.4.** *There exists a homomorphism  $\varphi : \text{St}(C_2, \mathbb{Z}) \rightarrow B_6/N$  such that*

$$\begin{aligned} \varphi(x_{\alpha}) &= (\sigma_5\sigma_4)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_5\sigma_4)^{-1}, \\ \varphi(x_{\beta}) &= \sigma_5, \\ \varphi(x_{\alpha+\beta}) &= \sigma_1\sigma_3^{-1}\sigma_5, \\ \varphi(x_{2\alpha+\beta}) &= \sigma_1, \\ \varphi(x_{-\alpha}) &= (\sigma_1\sigma_2)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2)^{-1}, \\ \varphi(x_{-\beta}) &= \sigma_4^{-1}, \\ \varphi(x_{-(\alpha+\beta)}) &= (\sigma_1\sigma_2\sigma_5\sigma_4)(\sigma_1^{-1}\sigma_3\sigma_5^{-1})(\sigma_1\sigma_2\sigma_5\sigma_4)^{-1}, \\ \varphi(x_{-(2\alpha+\beta)}) &= \sigma_2^{-1} \end{aligned}$$

modulo  $N$ . Moreover,  $\varphi \circ f = \text{id}$ .

Before we prove Proposition 4.4, let us record the following four equalities in the braid group  $B_6$ . One may check them using one's favorite algorithm for solving the word problem in braid groups (see for instance the monographs [5, 8, 9, 10, 13]).

**Lemma 4.5.** *The following equalities hold in  $B_6$ :*

$$\begin{aligned} [\varphi(x_{\alpha}), \varphi(x_{\beta})]^{-1} \varphi(x_{\alpha+\beta}) \varphi(x_{2\alpha+\beta}) &= (\sigma_4\sigma_5\sigma_4)^2 (\sigma_1\sigma_3^{-1}\sigma_5) (\sigma_4\sigma_5\sigma_4)^{-2} (\sigma_1\sigma_3^{-1}\sigma_5), \\ [\varphi(x_{\alpha}), \varphi(x_{\alpha+\beta})]^{-1} \varphi(x_{2\alpha+\beta}^2) &= (\sigma_1\sigma_3^{-1}\sigma_5) (\sigma_4\sigma_5\sigma_4)^2 (\sigma_1\sigma_3^{-1}\sigma_5) (\sigma_4\sigma_5\sigma_4)^{-2}, \\ [\varphi(x_{\alpha}), \varphi(x_{-(\alpha+\beta)})] \varphi(x_{-\beta}^2) &= (\sigma_5\sigma_4\sigma_1\sigma_2) (\sigma_1\sigma_2\sigma_3) \gamma^{-1} (\sigma_1\sigma_2\sigma_3)^{-1} (\sigma_5\sigma_4\sigma_1\sigma_2)^{-1}, \end{aligned}$$

where  $\gamma = (\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^2$ , and

$$[\varphi(x_{\alpha+\beta}), \varphi(x_{-\alpha})] \varphi(x_{\beta}^2) = (\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}.$$

We make also the following observation. Let

$$\Delta = (\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)(\sigma_1\sigma_2\sigma_3\sigma_4)(\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2)\sigma_1$$

be the ‘half-twist’ in the braid group  $B_6$ . It has the following important property (see for instance [5] or [13, Sect. 1.3.3]):

$$(4.3) \quad \Delta\sigma_i\Delta^{-1} = \sigma_{6-i}, \quad (i = 1, \dots, 5)$$

which implies that its square  $\Delta^2$  is central (actually  $\Delta^2$  generates the center of the braid group). Note that the braids  $\sigma_1\sigma_5$ ,  $\sigma_2\sigma_4$  and  $\sigma_3$  are invariant under conjugation by  $\Delta$  so that

$$\Delta(\sigma_1\sigma_3^{-1}\sigma_5)\Delta^{-1} = \sigma_5\sigma_3^{-1}\sigma_1 = \sigma_1\sigma_3^{-1}\sigma_5,$$

whereas  $\sigma_1\sigma_2\sigma_1$  and  $\sigma_4\sigma_5\sigma_4$  are interchanged:

$$(4.4) \quad \Delta(\sigma_1\sigma_2\sigma_1)\Delta^{-1} = \sigma_5\sigma_4\sigma_5 = \sigma_4\sigma_5\sigma_4.$$

In view of (4.3) and of the definition of  $\varphi$  we have the following ‘‘symmetries’’:

$$(4.5) \quad \Delta \varphi(x_\beta) \Delta^{-1} = \varphi(x_{2\alpha+\beta}),$$

$$(4.6) \quad \Delta \varphi(x_{-\beta}) \Delta^{-1} = \varphi(x_{-(2\alpha+\beta)}),$$

$$(4.7) \quad \Delta \varphi(x_\alpha) \Delta^{-1} = \varphi(x_{-\alpha}),$$

$$(4.8) \quad \Delta \varphi(x_{\pm(\alpha+\beta)}) \Delta^{-1} = \varphi(x_{\pm(\alpha+\beta)}).$$

Geometrically speaking, conjugating the elements  $\varphi(x_\gamma)$  ( $\gamma \in \Phi$ ) by  $\Delta$  corresponds in the root system  $C_2$  to the reflection in the one-dimensional vector space spanned by  $\alpha + \beta$ .

**Remark 4.6.** It follows from [4, Th. 0.2] that the centralizer of  $\Delta$  is an Artin group of type  $C_3$ . Actually it is generated by the above-mentioned elements  $\beta_1 = \sigma_1\sigma_5$ ,  $\beta_2 = \sigma_2\sigma_4$  and  $\beta_3 = \sigma_3$ , which satisfy the relations

$$\beta_1\beta_2\beta_1 = \beta_2\beta_1\beta_2, \quad \beta_1\beta_3 = \beta_3\beta_1, \quad \beta_2\beta_3\beta_2\beta_3 = \beta_3\beta_2\beta_3\beta_2.$$

*Proof of Proposition 4.4.* To prove the existence of  $\varphi : \text{St}(C_2, \mathbb{Z}) \rightarrow B_6/N$  it suffices to check that the elements  $\varphi(x_\gamma)$  ( $\gamma \in \Phi$ ) satisfy the 24 relations (2.1)–(2.15) in the quotient group  $B_6/N$ .

(a) Relations (2.1): in view of the braid relations (1.1), we have

$$[\varphi(x_\alpha), \varphi(x_{2\alpha+\beta})] = [(\sigma_5\sigma_4)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_5\sigma_4)^{-1}, \sigma_1] = 1,$$

$$[\varphi(x_\beta), \varphi(x_{\alpha+\beta})] = [\sigma_5, \sigma_1\sigma_3^{-1}\sigma_5] = 1,$$

$$[\varphi(x_\beta), \varphi(x_{2\alpha+\beta})] = [\sigma_5, \sigma_1] = 1,$$

$$[\varphi(x_{\alpha+\beta}), \varphi(x_{2\alpha+\beta})] = [\sigma_1\sigma_3^{-1}\sigma_5, \sigma_1] = 1.$$

(b) Relations (2.2): again in view of the commutation relations (1.1), but also of the braid relations (1.2), we have

$$\begin{aligned} [\varphi(x_\alpha), \varphi(x_{-\beta})] &= [(\sigma_5\sigma_4)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_5\sigma_4)^{-1}, \sigma_4^{-1}] \\ &= \sigma_5\sigma_4(\sigma_1\sigma_3^{-1}\sigma_5) \underbrace{\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1}} \sigma_5\sigma_4(\sigma_1\sigma_3^{-1}\sigma_5)^{-1} \underbrace{\sigma_4^{-1}\sigma_5^{-1}\sigma_4} \\ &= \sigma_5\sigma_4(\sigma_1\sigma_3^{-1}\sigma_5) \underbrace{\sigma_5^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_5\sigma_4(\sigma_1\sigma_3^{-1}\sigma_5)^{-1}\sigma_5\sigma_4^{-1}\sigma_5^{-1}} \\ &= \sigma_5\sigma_4(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_3^{-1}\sigma_5)^{-1}\sigma_4^{-1}\sigma_5^{-1} = 1. \end{aligned}$$

(Here and below the underbraces indicate the places where we apply the braid relations and the trivial relations  $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$ .)

Similarly, by (1.1) we have

$$[\varphi(x_\beta), \varphi(x_{-\alpha})] = [\sigma_5, (\sigma_1\sigma_2)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2)^{-1}] = 1,$$

$$\begin{aligned} [\varphi(x_\beta), \varphi(x_{-(2\alpha+\beta)})] &= [\sigma_5, \sigma_2^{-1}] = 1, \\ [\varphi(x_{-\beta}), \varphi(x_{2\alpha+\beta})] &= [\sigma_4^{-1}, \sigma_1] = 1. \end{aligned}$$

Note that only  $[\varphi(x_\alpha), \varphi(x_{-\beta})] = 1$  requires one of the braid relations (1.2).

(c) Relations (2.3): for the first one, using (4.6) and (4.7), we have

$$\begin{aligned} [\varphi(x_{-\alpha}), \varphi(x_{-(2\alpha+\beta)})] &= [\Delta \varphi(x_\alpha) \Delta^{-1}, \Delta \varphi(x_{-\beta}) \Delta^{-1}] \\ &= \Delta [\varphi(x_\alpha), \varphi(x_{-\beta})] \Delta^{-1} = 1 \end{aligned}$$

since  $[\varphi(x_\alpha), \varphi(x_{-\beta})] = 1$ , as we have just proved in Item (b).

For the second one we use the equality  $\varphi(x_{-(\alpha+\beta)}) = (\sigma_1\sigma_2) \varphi(x_\alpha)^{-1} (\sigma_1\sigma_2)^{-1}$ , which we derive from the definition of  $\varphi$ . Then

$$\begin{aligned} [\varphi(x_{-\beta}), \varphi(x_{-(\alpha+\beta)})] &= [\varphi(x_{-\beta}), (\sigma_1\sigma_2) \varphi(x_\alpha)^{-1} (\sigma_1\sigma_2)^{-1}] \\ &= [\sigma_4^{-1}, (\sigma_1\sigma_2) \varphi(x_\alpha)^{-1} (\sigma_1\sigma_2)^{-1}] \\ &= [(\sigma_1\sigma_2)\sigma_4^{-1}(\sigma_1\sigma_2)^{-1}, (\sigma_1\sigma_2) \varphi(x_\alpha)^{-1} (\sigma_1\sigma_2)^{-1}] \\ &= (\sigma_1\sigma_2) [\varphi(x_{-\beta}), \varphi(x_\alpha)^{-1}] (\sigma_1\sigma_2)^{-1}. \end{aligned}$$

Since  $\varphi(x_{-\beta})$  commutes with  $\varphi(x_\alpha)$  by Item (b), it commutes with its inverse  $\varphi(x_\alpha)^{-1}$ . Hence,  $[\varphi(x_{-\beta}), \varphi(x_{-(\alpha+\beta)})] = (\sigma_1\sigma_2)(\sigma_1\sigma_2)^{-1} = 1$ .

For the third one, we have  $[\varphi(x_{-\beta}), \varphi(x_{-(2\alpha+\beta)})] = [\sigma_4^{-1}, \sigma_2^{-1}] = 1$  by (1.1).

For the fourth one, conjugating by  $\Delta$ , and using (4.6) and (4.8), we reduce the desired equality  $[\varphi(x_{-(\alpha+\beta)}), \varphi(x_{-(2\alpha+\beta)})] = 1$  to the previous equality  $[\varphi(x_{-\beta}), \varphi(x_{-(\alpha+\beta)})] = 1$ .

(d) Relation (2.4): using the first equality in Lemma 4.5 and Equation (4.4), we have

$$\begin{aligned} &[\varphi(x_\alpha), \varphi(x_\beta)]^{-1} \varphi(x_{\alpha+\beta}) \varphi(x_{2\alpha+\beta}) \\ &= (\sigma_4\sigma_5\sigma_4)^2 (\sigma_1\sigma_3^{-1}\sigma_5) (\sigma_4\sigma_5\sigma_4)^{-2} (\sigma_1\sigma_3^{-1}\sigma_5) \\ &= \Delta (\sigma_1\sigma_2\sigma_1)^2 (\sigma_1\sigma_3^{-1}\sigma_5) (\sigma_1\sigma_2\sigma_1)^{-2} (\sigma_1\sigma_3^{-1}\sigma_5) \Delta^{-1}. \end{aligned}$$

It follows that  $[\varphi(x_\alpha), \varphi(x_\beta)]^{-1} \varphi(x_{\alpha+\beta}) \varphi(x_{2\alpha+\beta})$  belongs to the normal subgroup  $N$ , hence is trivial in  $B_6/N$ .

(e) Relation (2.5): we use the second equality in Lemma 4.5 and argue as in Item (d). It follows that  $[\varphi(x_\alpha), \varphi(x_{\alpha+\beta})]^{-1} \varphi(x_{2\alpha+\beta}^2)$  is trivial in  $B_6/N$ .

(f) Relation (2.6): consider the third equality in Lemma 4.5; the braid  $\gamma$  is trivial in  $B_6/N$ ; hence, so is  $[\varphi(x_\alpha), \varphi(x_{-(\alpha+\beta)})] \varphi(x_{-\beta}^2)$ .

(g) Relation (2.7): we have

$$\begin{aligned} &[\varphi(x_\alpha), \varphi(x_{-(2\alpha+\beta)})]^{-1} \varphi(x_{-(\alpha+\beta)})^{-1} \varphi(x_{-\beta}) \\ &= \underbrace{\sigma_2^{-1} \sigma_5 \sigma_4 \sigma_1 \sigma_3^{-1} \sigma_5 \sigma_4^{-1} \sigma_5^{-1} \sigma_2 \sigma_5 \sigma_4 \sigma_1^{-1} \sigma_3 \sigma_5^{-1}}_{\times \underbrace{\sigma_4^{-1} \sigma_5^{-1} \sigma_1 \sigma_2 \sigma_5 \sigma_4 \sigma_1 \sigma_3^{-1} \sigma_5 \sigma_4^{-1} \sigma_5^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_4^{-1}}_{\sigma_5 \sigma_4 \sigma_5 \sigma_2^{-1} \sigma_1 \sigma_3^{-1} \sigma_2 \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_1 \sigma_2 \sigma_5 \sigma_3^{-1} \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_4^{-1} \sigma_5^{-1} \sigma_4^{-1}}} \\ &= \sigma_5 \sigma_4 \sigma_5 \sigma_2^{-1} \sigma_1 \sigma_3^{-1} \sigma_2 \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_1 \sigma_2 \sigma_5 \sigma_3^{-1} \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_4^{-1} \sigma_5^{-1} \sigma_4^{-1} \\ &= \sigma_5 \sigma_4 \sigma_5 \sigma_2^{-1} \sigma_1 \sigma_3^{-1} \sigma_2 \sigma_1^{-1} \sigma_3 \sigma_1 \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2 \sigma_5^{-1} \sigma_4^{-1} \sigma_5^{-1} \\ &= \sigma_5 \sigma_4 \sigma_5 \sigma_2^{-1} \sigma_1 \sigma_3^{-1} \sigma_2 \sigma_3 \sigma_3^{-1} \sigma_2^{-1} \sigma_3 \sigma_1^{-1} \sigma_2 \sigma_5^{-1} \sigma_4^{-1} \sigma_5^{-1} \\ &= (\sigma_5 \sigma_4 \sigma_5 \sigma_2^{-1} \sigma_1 \sigma_3^{-1} \sigma_2 \sigma_3) (\sigma_5 \sigma_4 \sigma_5 \sigma_2^{-1} \sigma_1 \sigma_3^{-1} \sigma_2 \sigma_3)^{-1} = 1. \end{aligned}$$

(h) Relation (2.8): we have

$$\begin{aligned}
& [\varphi(x_\beta), \varphi(x_{-(\alpha+\beta)})]^{-1} \varphi(x_{-\alpha}) \varphi(x_{-(2\alpha+\beta)}) \\
&= \sigma_1 \sigma_2 \sigma_5 \sigma_4 \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_4^{-1} \underbrace{\sigma_5^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_5}_{\sigma_1 \sigma_2 \sigma_5 \sigma_4 \sigma_1 \sigma_3^{-1} \sigma_5} \sigma_1 \sigma_2 \sigma_5 \sigma_4 \sigma_1 \sigma_3^{-1} \sigma_5 \\
&\quad \times \sigma_4^{-1} \sigma_5^{-1} \underbrace{\sigma_2^{-1} \sigma_1^{-1} \sigma_5^{-1} \sigma_1 \sigma_2}_{\sigma_1 \sigma_3^{-1} \sigma_5} \underbrace{\sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1}}_{\sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1}} \\
&= \sigma_1 \sigma_2 \sigma_5 \sigma_4 \sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_4^{-1} \underbrace{\sigma_2^{-1} \sigma_1^{-1} \sigma_1 \sigma_2}_{\sigma_5 \sigma_4 \sigma_1 \sigma_3^{-1} \sigma_5} \sigma_5 \sigma_4 \sigma_1 \sigma_3^{-1} \sigma_5 \\
&\quad \times \sigma_4^{-1} \sigma_5^{-1} \underbrace{\sigma_5^{-1} \sigma_1 \sigma_3^{-1} \sigma_5 \sigma_1^{-1}}_{\sigma_2^{-1} \sigma_1^{-1}} \sigma_2^{-1} \sigma_1^{-1} \\
&= \sigma_1 \sigma_2 \sigma_5 \sigma_4 \underbrace{\sigma_1^{-1} \sigma_3 \sigma_5^{-1} \sigma_4^{-1} \sigma_5 \sigma_4}_{\sigma_1 \sigma_3^{-1} \sigma_5 \sigma_4^{-1} \sigma_5^{-1} \sigma_3^{-1}} \sigma_1 \sigma_3^{-1} \sigma_5 \sigma_4^{-1} \underbrace{\sigma_5^{-1} \sigma_3^{-1}}_{\sigma_2^{-1} \sigma_1^{-1}} \sigma_2^{-1} \sigma_1^{-1} \\
&= \sigma_1 \sigma_2 \sigma_5 \sigma_4 \sigma_3 \underbrace{\sigma_5^{-1} \sigma_5}_{\sigma_4} \underbrace{\sigma_4^{-1} \sigma_3^{-1} \sigma_5}_{\sigma_4^{-1} \sigma_3^{-1} \sigma_5^{-1} \sigma_2^{-1} \sigma_1^{-1}} \sigma_4^{-1} \sigma_3^{-1} \sigma_5^{-1} \sigma_2^{-1} \sigma_1^{-1} \\
&= \sigma_1 \sigma_2 \sigma_5 \sigma_4 \sigma_3 \sigma_4 \underbrace{\sigma_3^{-1} \sigma_4^{-1} \sigma_3^{-1}}_{\sigma_5^{-1} \sigma_2^{-1} \sigma_1^{-1}} \sigma_5^{-1} \sigma_2^{-1} \sigma_1^{-1} \\
&= \sigma_1 \sigma_2 \sigma_5 \sigma_4 \sigma_3 \sigma_4 \sigma_4^{-1} \sigma_3^{-1} \sigma_4^{-1} \sigma_5^{-1} \sigma_2^{-1} \sigma_1^{-1} \\
&= (\sigma_1 \sigma_2 \sigma_5 \sigma_4 \sigma_3 \sigma_4) (\sigma_1 \sigma_2 \sigma_5 \sigma_4 \sigma_3 \sigma_4)^{-1} = 1.
\end{aligned}$$

(i) Relation (2.9): the fourth equality in Lemma 4.5 implies that the braid  $[\varphi(x_{\alpha+\beta}), \varphi(x_{-\alpha})] \varphi(x_\beta^2)$  belongs to  $N$ , hence is trivial in  $B_6/N$ .

(j) Relation (2.10): we have

$$\begin{aligned}
& [\varphi(x_{\alpha+\beta}), \varphi(x_{-\beta})]^{-1} \varphi(x_\alpha) \varphi(x_{2\alpha+\beta}^{-1}) \\
&= \sigma_4^{-1} \underbrace{\sigma_1 \sigma_3^{-1} \sigma_5 \sigma_4 \sigma_1^{-1}}_{\sigma_3 \sigma_5^{-1} \sigma_5 \sigma_4} \underbrace{\sigma_3 \sigma_5^{-1} \sigma_5 \sigma_4}_{\sigma_1 \sigma_3^{-1} \sigma_5 \sigma_4^{-1} \sigma_5^{-1} \sigma_1^{-1}} \sigma_1 \sigma_3^{-1} \sigma_5 \sigma_4^{-1} \sigma_5^{-1} \sigma_1^{-1} \\
&= \sigma_4^{-1} \sigma_3^{-1} \sigma_5^{-1} \underbrace{\sigma_4 \sigma_3 \sigma_4}_{\sigma_3^{-1} \sigma_4^{-1} \sigma_5^{-1}} \underbrace{\sigma_5 \sigma_4^{-1} \sigma_5^{-1}}_{\sigma_5^{-1} \sigma_4^{-1} \sigma_5^{-1}} \\
&= \sigma_4^{-1} \underbrace{\sigma_3^{-1} \sigma_5 \sigma_3}_{\sigma_4 \sigma_3 \sigma_3^{-1} \sigma_4^{-1}} \underbrace{\sigma_4 \sigma_3 \sigma_3^{-1} \sigma_4^{-1}}_{\sigma_5^{-1} \sigma_4^{-1}} \sigma_5^{-1} \sigma_4^{-1} \\
&= \sigma_4^{-1} \sigma_5 \sigma_5^{-1} \sigma_4 = 1.
\end{aligned}$$

(k) Relation (2.11): using the computation in Item (j) and conjugating by  $\Delta$ , we obtain

$$\begin{aligned}
& [\varphi(x_{\alpha+\beta}), \varphi(x_{-(2\alpha+\beta)})]^{-1} \varphi(x_{-\alpha}) \varphi(x_\beta^{-1}) \\
&= \Delta [\varphi(x_{\alpha+\beta}), \varphi(x_{-\beta})]^{-1} \varphi(x_\alpha) \varphi(x_{2\alpha+\beta}^{-1}) \Delta^{-1} = 1.
\end{aligned}$$

(l) Relation (2.12): by (4.5), (4.7) and (4.8) we have

$$\begin{aligned}
& [\varphi(x_{2\alpha+\beta}), \varphi(x_{-\alpha})] \varphi(x_{\alpha+\beta}) \varphi(x_\beta) \\
&= \Delta [\varphi(x_\beta), \varphi(x_\alpha)] \varphi(x_{\alpha+\beta}) \varphi(x_{2\alpha+\beta}) \Delta^{-1},
\end{aligned}$$

which in view of Item (d) above implies that  $[\varphi(x_{2\alpha+\beta}), \varphi(x_{-\alpha})] \varphi(x_{\alpha+\beta}) \varphi(x_\beta)$  is trivial in  $B_6/N$ .

(m) Relation (2.13) reduces to Relation (2.8) after conjugating by  $\Delta$ . In the same way Relations (2.14) and (2.15) reduce to Relations (2.7) and (2.6) respectively.

To complete the proof it remains to check that  $\varphi \circ f = \text{id}$ . This is a consequence of the equalities  $(\varphi \circ f)(\sigma_i) = \sigma_i$  ( $1 \leq i \leq 5$ ), which follow immediately from the definitions of  $f$  and  $\varphi$ .  $\square$

As can be seen in the previous proof, 18 out of the 24 images under  $\varphi$  of the relations defining  $\text{St}(C_2, \mathbb{Z})$  hold in the braid group  $B_6$  itself. Only six of them hold modulo the normal subgroup  $N$ .

**Remark 4.7.** In [12, Cor. 7] we gave a braid-like presentation of the symplectic modular group  $\text{Sp}_4(\mathbb{Z})$  with four additional relations rather than the two additional ones in Corollary 4.2 above. In [12] the first relation in Equation (27) is equivalent (modulo the braid relations) to the relation  $(\sigma_1\sigma_2\sigma_1)^4 = 1$  above. Relation (28) in *loc. cit.* is equivalent to the relation  $(\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2} = (\sigma_1\sigma_3^{-1}\sigma_5)^{-1}$ . Our corollary 4.2 shows that the relation  $\Delta^2 = 1$  of [12, Eq. (26)] is not needed. Actually, one can prove

$$f(\Delta^2) = w_\beta^{12} = w_{2\alpha+\beta}^{12} = f((\sigma_4\sigma_5\sigma_4)^{12}) = f((\sigma_1\sigma_2\sigma_1)^{12}).$$

#### APPENDIX A. THE STEINBERG GROUP OF TYPE $C_2$ OVER A COMMUTATIVE RING

For completeness we give a presentation of the Steinberg group  $\text{St}(C_2, R)$  over an arbitrary commutative ring  $R$ .

By [15, Sect. 3] (see also [16] or [17, Chap. 6]) the group  $\text{St}(C_2, R)$  has a presentation with the set of generators  $\{x_\gamma(u) : \gamma \in \Phi, u \in R\}$  subject to the following relations:

- $x_\gamma(u+v) = x_\gamma(u)x_\gamma(v)$  for all  $u, v \in R$ ;
- if  $\gamma, \delta \in \Phi$  such that  $\gamma + \delta \neq 0$ , then

$$[x_\gamma(u), x_\delta(v)] = \prod x_{i\gamma+j\delta}(c_{i,j}^{\gamma,\delta} u^i v^j),$$

where  $i$  and  $j$  are positive integers such that  $i\gamma + j\delta$  belongs to  $\Phi$ .

The coefficients  $c_{i,j}^{\gamma,\delta}$  are the integers which have already come up in Relations (2.21) in the proof of Proposition 2.1.

In this way we obtain the following 24 defining relations for  $\text{St}(C_2, R)$  (where  $u, v \in R$ ):

$$(A.1) \quad [x_\alpha(u), x_{2\alpha+\beta}(v)] = [x_\beta(u), x_{\alpha+\beta}(v)] = [x_\beta(u), x_{2\alpha+\beta}(v)] = [x_{\alpha+\beta}(u), x_{2\alpha+\beta}(v)] = 1,$$

$$(A.2) \quad [x_\alpha(u), x_{-\beta}(v)] = [x_\beta(u), x_{-\alpha}(v)] = [x_\beta(u), x_{-(2\alpha+\beta)}(v)] = [x_{-\beta}(u), x_{2\alpha+\beta}(v)] = 1,$$

$$(A.3) \quad [x_{-\alpha}(u), x_{-(2\alpha+\beta)}(v)] = [x_{-\beta}(u), x_{-(\alpha+\beta)}(v)] = [x_{-\beta}(u), x_{-(2\alpha+\beta)}(v)] = [x_{-(\alpha+\beta)}(u), x_{-(2\alpha+\beta)}(v)] = 1,$$

$$(A.4) \quad [x_\alpha(u), x_\beta(v)] = x_{\alpha+\beta}(uv) x_{2\alpha+\beta}(u^2v),$$

$$(A.5) \quad [x_\alpha(u), x_{\alpha+\beta}(v)] = x_{2\alpha+\beta}(2uv),$$

$$(A.6) \quad [x_\alpha(u), x_{-(\alpha+\beta)}(v)] = x_{-\beta}(-2uv),$$

$$(A.7) \quad [x_\alpha(u), x_{-(2\alpha+\beta)}(v)] = x_{-\beta}(u^2v) x_{-(\alpha+\beta)}(-uv),$$

$$(A.8) \quad [x_\beta(u), x_{-(\alpha+\beta)}(v)] = x_{-\alpha}(uv) x_{-(2\alpha+\beta)}(uv^2),$$

$$(A.9) \quad [x_{\alpha+\beta}(u), x_{-\alpha}(v)] = x_\beta(-2uv),$$

$$(A.10) \quad [x_{\alpha+\beta}(u), x_{-\beta}(v)] = x_\alpha(uv) x_{2\alpha+\beta}(-u^2v),$$

$$(A.11) \quad [x_{\alpha+\beta}(u), x_{-(2\alpha+\beta)}(v)] = x_{-\alpha}(uv) x_\beta(-u^2v),$$

$$(A.12) \quad [x_{2\alpha+\beta}(u), x_{-\alpha}(v)] = x_{\alpha+\beta}(-uv) x_\beta(-uv^2),$$

$$(A.13) \quad [x_{2\alpha+\beta}(u), x_{-(\alpha+\beta)}(v)] = x_\alpha(uv) x_{-\beta}(uv^2),$$

$$(A.14) \quad [x_{-\alpha}(u), x_{-\beta}(v)] = x_{-(\alpha+\beta)}(-uv) x_{-(2\alpha+\beta)}(u^2v),$$

$$(A.15) \quad [x_{-\alpha}(u), x_{-(\alpha+\beta)}(v)] = x_{-(2\alpha+\beta)}(-2uv).$$

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