

Kuhn’s Equivalence Theorem for Games in Intrinsic Form

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Abstract

We state and prove Kuhn’s equivalence theorem for a new representation of games, the intrinsic form. First, we introduce games in intrinsic form where information is represented by σ -fields over a product set. For this purpose, we adapt to games the intrinsic representation that Witsenhausen introduced in control theory. Those intrinsic games do not require an explicit description of the play temporality, as opposed to extensive form games on trees. Second, we prove, for this new and more general representation of games, that behavioral and mixed strategies are equivalent under perfect recall (Kuhn’s theorem). As the intrinsic form replaces the tree structure with a product structure, the handling of information is easier. This makes the intrinsic form a new valuable tool for the analysis of games with information.

Keywords. Games with information, Kuhn’s equivalence theorem, Witsenhausen intrinsic model.

1 Introduction

From the origin, games in extensive form have been formulated on a tree. In his seminal paper *Extensive Games and the Problem of Information* [7], Kuhn claimed that “The use of a geometrical model (...) clarifies the delicate problem of information”. This tells us that the proper handling of information was a strong motivation for Kuhn’s extensive games. On the game tree, moves are those vertices that possess alternatives, then moves are partitioned into players moves, themselves partitioned into information sets (with the constraint that no two moves in an information set can be on the same play). Kuhn mentions agents, one agent per information set, to “personalize the interpretation” but the notion is not central (to the point that his definition of perfect recall “obviates the use of agents”).

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By contrast, in the so-called Witsenhausen’s intrinsic model [9, 10], agents play a central role. Each agent is equipped with a decision set and a σ -field, and the same for Nature. Then, Witsenhausen introduces the product set and the product σ -field. This product set hosts the agents’ information subfields. The Witsenhausen’s intrinsic model was elaborated in the control theory setting, in order to handle how information is distributed among agents and how it impacts their strategies. Although not explicitly designed for games, Witsenhausen’s intrinsic model had, from the start, the potential to be adapted to games. Indeed, in [9] Witsenhausen places his own model in the context of game theory by referring to von Neuman and Morgenstern [8], Kuhn [7] and Aumann [2].

In this paper, we introduce a new representation of games that we call *games in intrinsic form*. Game representations play a key role in their analysis (see the illuminating introduction of the book [1]), and we claim that games in intrinsic form display appealing features. In the philosophy of the tree-based extensive form (Kuhn’s view), the temporal ordering is hardcoded in the tree structure: one goes from the root to the leaves, making decisions at the moves, contingent on information, chance and strategies. For Kuhn, the time arrow (tree) comes first; information comes second (partition of the move vertices). By contrast, for Witsenhausen, information comes first; the time arrow comes (possibly) second, under a proper causality assumption contingent to the information structure.

Not having a hardcoded temporal ordering makes mathematical representations less constrained, hence more general. Moreover, Witsenhausen’s framework makes representations more intrinsic. As an illustration, let us consider a game where two players play once but at the same time. To formulate it on a tree requires to arbitrarily decide which of the two plays first. This is not the case for games in intrinsic form, where each player/agent is equipped with an information subfield and strategies that are measurable with respect to the latter; writing the system of two equations that express decisions as the output of strategies leads to a unique outcome, without having to solve one equation first, and the other second.

The tree representation of games has its pros and cons. On the one hand, trees are perfect to follow step by step how a game is played as any strategy profile induces a unique play: one goes from the root to the leaves, passing from one node to the next by an edge that depends on the strategy profile. On the other hand, in games with information, information sets are represented as “union” of tree nodes that must satisfy restrictive axioms, and such unions do not comply in a natural way with the tree structure, which can render the game analysis delicate [1, 4, 5]. By contrast, the notion of Witsenhausen’s intrinsic games (W-games) does not require an explicit description of the play temporality, and the intrinsic form replaces the tree structure with a product structure, more amenable to mathematical analysis. If the introduction of the model may seem involved, we argue that the resulting structure is a powerful mathematical tool, because there are many situations in which it is easier to reason and discuss with mathematical formulas than with trees.

We illustrate our claim with a proof of the celebrated Kuhn’s equivalence theorem for games in intrinsic form. Indeed, as a first step in a broader research program, we show that equivalence between mixed and behavioral strategies holds under perfect recall for W-games. More precisely, our proof relies on an equivalence between behavioral, mixed and a

new notion of product-mixed strategies. These latter form a subclass of mixed strategies. In the spirit of [2], in a product-mixed strategy, each agent (corresponding to time index in [2]) generates strategies from a random device that is independent of all the other agents. We prove that, under perfect recall for W-games, any mixed strategy of a player is not only equivalent to a behavioral strategy, but also to a product-mixed strategy where all the agents under control of the player randomly select their pure strategy independently of the other agents.

The paper is organized as follows. In Sect. 2, we present the finite version of Witsenhausen’s intrinsic model. Then, in Sect. 3, we propose a formal definition of games in intrinsic form (W-games), and then discuss three notions of “randomization” of pure strategies — mixed, product-mixed and behavioral. Finally, we derive an equivalent of Kuhn’s equivalence theorem for games in intrinsic form in Sect. 4. In Appendix A, we present background material on fields, atoms and partitions, as these notions lay at the core of Witsenhausen’s intrinsic model in the finite case. In all the paper, we adopt the convention that a player is female (hence using “she” and “her”), whereas an agent is male (“he”, “his”).

2 Witsenhausen’s intrinsic model (the finite case)

In this paper, we tackle the issue of information in the context of finite games. For this purpose, we will present the so-called intrinsic model of Witsenhausen [10, 6] but with finite sets rather than with infinite ones as in the original exposition. We refer the reader to Appendix A for background material on fields, atoms and partitions.

In §2.1, we present the finite version of Witsenhausen’s intrinsic model, where we highlight the role of the configuration field that contains the information subfields of all agents. In §2.2, we illustrate, on a few examples, the ease with which one can model information in strategic contexts, using subfields of the configuration field. Finally, we present in §2.3 the notions of solvability and causality.

2.1 Finite Witsenhausen’s intrinsic model (W-model)

We present the finite version of Witsenhausen’s intrinsic model, introduced some five decades ago in the control community [9, 10].

Definition 1. (adapted from [9, 10])

A finite W-model is a collection $(\mathbb{A}, (\Omega, \mathcal{F}), (\mathbb{U}_a, \mathcal{U}_a)_{a \in \mathbb{A}}, (\mathcal{J}_a)_{a \in \mathbb{A}})$, where

- \mathbb{A} is a finite set, whose elements are called agents;
- Ω is a finite set which represents all uncertainties; any $\omega \in \Omega$ is called a state of Nature; \mathcal{F} is the complete field over Ω ;
- for any $a \in \mathbb{A}$, \mathbb{U}_a is a finite set, the set of decisions for agent a ; \mathcal{U}_a is the complete field over \mathbb{U}_a ;

- for any $a \in \mathbb{A}$, \mathcal{J}_a is a subfield of the following product field

$$\mathcal{J}_a \subset \mathcal{F} \otimes \bigotimes_{b \in \mathbb{A}} \mathcal{U}_b, \quad \forall a \in \mathbb{A} \quad (1)$$

and is called the information field of the agent a .

The *configuration space* is the product space (called *hybrid space* by Witsenhausen, hence the \mathbb{H} notation)

$$\mathbb{H} = \Omega \times \prod_{a \in \mathbb{A}} \mathcal{U}_a. \quad (2a)$$

As all fields \mathcal{F} and $(\mathcal{U}_a)_{a \in \mathbb{A}}$ are complete, the product *configuration field*

$$\mathcal{H} = \mathcal{F} \otimes \bigotimes_{a \in \mathbb{A}} \mathcal{U}_a \quad (2b)$$

is also the complete field of \mathbb{H} . A *configuration* $h \in \mathbb{H}$ is denoted by

$$h = (\omega, (u_a)_{a \in \mathbb{A}}) \iff h_\emptyset = \omega \text{ and } h_a = u_a, \quad \forall a \in \mathbb{A}. \quad (2c)$$

In lieu of the information field \mathcal{J}_a in (1), it will be convenient to consider the equivalence relation \sim_a , on the configuration space \mathbb{H} , defined in such a way that the equivalence classes $[\cdot]_a \subset \mathbb{H}$ coincide with the atoms of \mathcal{J}_a , that is, with the elements of the partition $\langle \mathcal{J}_a \rangle$ in (33):

$$(\forall h', h'' \in \mathbb{H}) \quad h' \sim_a h'' \iff h'' \in [h']_a \iff \exists G \in \langle \mathcal{J}_a \rangle, \{h', h''\} \subset G. \quad (3)$$

Thus defined, the subset $[h]_a \subset \mathbb{H}$ is the unique atom G in $\langle \mathcal{J}_a \rangle \subset \mathcal{J}_a$ that contains the configuration h .

We will need the following equivalent characterization of measurable mappings, which is a slight reformulation of [6, Proposition 3.35].

Proposition 2. (adapted from [6, Proposition 3.35]) *Let $\rho : (\mathbb{H}, \mathcal{H}) \rightarrow (\mathbb{D}, \mathcal{D})$ be a mapping, where \mathbb{D} is a set and \mathcal{D} is a σ -field over \mathbb{D} . We suppose that the σ -field \mathcal{D} contains all the singletons. Then, for any agent $a \in \mathbb{A}$, the following statements are equivalent:*

$$\rho^{-1}(\mathcal{D}) \subset \mathcal{J}_a, \quad (4a)$$

$$(\forall h', h'' \in \mathbb{H}) \quad h' \sim_a h'' \implies \rho(h') = \rho(h''), \quad (4b)$$

$$\begin{aligned} & \text{the set-valued mapping } \hat{\rho} : \langle \mathcal{J}_a \rangle \rightrightarrows \mathbb{D}, \text{ defined by} \\ & \hat{\rho}(G_a) = \{\rho(h) \mid h \in G_a\}, \quad \forall G_a \in \langle \mathcal{J}_a \rangle, \text{ is a mapping.} \end{aligned} \quad (4c)$$

In any of these equivalent cases, we say that the mapping ρ is \mathcal{J}_a -measurable, and, for all $G_a \in \langle \mathcal{J}_a \rangle$, we denote by $\rho(G_a)$ the unique element of \mathbb{D} in $\hat{\rho}(G_a)$, that is,

$$(\forall r \in \rho(\mathbb{H}), \forall G_a \in \langle \mathcal{J}_a \rangle) \quad \rho(G_a) = r \iff \hat{\rho}(G_a) = \{r\}. \quad (5)$$

Then, using the extended notation above (3), we have the property

$$\rho \text{ is } \mathcal{J}_a\text{-measurable} \implies \rho([h]_a) = \rho(h), \quad \forall h \in \mathbb{H}. \quad (6)$$

Now that we have explicated measurable mappings with respect to agents information subfields, we introduce the notion of pure W-strategy.

Definition 3. ([9, 10]) A pure W-strategy of agent $a \in \mathbb{A}$ is a mapping

$$\lambda_a : (\mathbb{H}, \mathcal{H}) \rightarrow (\mathbb{U}_a, \mathcal{U}_a) \quad (7a)$$

from configurations to decisions, which is measurable with respect to the information field \mathcal{J}_a of agent a , that is,

$$\lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{J}_a . \quad (7b)$$

We denote by Λ_a the set of all pure W-strategies of agent $a \in \mathbb{A}$. A pure W-strategies profile λ is a family

$$\lambda = (\lambda_a)_{a \in \mathbb{A}} \in \prod_{a \in \mathbb{A}} \Lambda_a \quad (8a)$$

of pure W-strategies, one per agent $a \in \mathbb{A}$. The set of pure W-strategies profiles is

$$\Lambda = \prod_{a \in \mathbb{A}} \Lambda_a . \quad (8b)$$

Condition (7b) expresses the property that any (pure) W-strategy of agent a may only depend upon the information \mathcal{J}_a available to the agent.

In what follows, we will need some notations. For any nonempty subset $\mathbb{B} \subset \mathbb{A}$ of agents, we define

$$\mathcal{U}_{\mathbb{B}} = \bigotimes_{b \in \mathbb{B}} \mathcal{U}_b \otimes \bigotimes_{a \notin \mathbb{B}} \{\emptyset, \mathbb{U}_a\} \subset \bigotimes_{a \in \mathbb{A}} \mathcal{U}_a , \quad (9a)$$

$$\mathcal{H}_{\mathbb{B}} = \mathcal{F} \otimes \mathcal{U}_{\mathbb{B}} = \mathcal{F} \otimes \bigotimes_{b \in \mathbb{B}} \mathcal{U}_b \otimes \bigotimes_{a \notin \mathbb{B}} \{\emptyset, \mathbb{U}_a\} \subset \mathcal{H} , \quad (9b)$$

$$h_{\mathbb{B}} = (h_b)_{b \in \mathbb{B}} \in \prod_{b \in \mathbb{B}} \mathbb{U}_b , \quad \forall h \in \mathbb{H} , \quad (9c)$$

$$\lambda_{\mathbb{B}} = (\lambda_b)_{b \in \mathbb{B}} \in \prod_{b \in \mathbb{B}} \Lambda_b , \quad \forall \lambda \in \Lambda . \quad (9d)$$

2.2 Examples

We illustrate, on a few examples, the ease with which one can model information in strategic contexts, using subfields of the configuration field. Even if we have presented the finite version of Witsenhausen's intrinsic model in §2.1, we take the opportunity here to show its potential to describe infinite decision and Nature sets.

Sequential decisions. Suppose an individual has to take decisions (say, an element of \mathbb{R}^n) at every discrete time step in the set¹ $\llbracket 1, T-1 \rrbracket$, where $T \geq 1$ is an integer. The situation will be modeled with (possibly) Nature set and field (Ω, \mathcal{F}) , and with T agents in $\mathbb{A} = \llbracket 0, T-1 \rrbracket$, and their corresponding sets, $\mathbb{U}_t = \mathbb{R}^n$, and fields, $\mathcal{U}_t = \mathcal{B}_{\mathbb{R}^n}$ (the Borel σ -field of \mathbb{R}^n), for $t \in \mathbb{A}$. Then, one builds up the product set $\mathbb{H} = \Omega \times \prod_{t=0}^{T-1} \mathbb{U}_t$ and the product field $\mathcal{H} = \mathcal{F} \otimes \bigotimes_{t=0}^{T-1} \mathcal{U}_t$. Every agent $t \in \llbracket 0, T-1 \rrbracket$ is equipped with an information field $\mathcal{J}_t \subset \mathcal{H}$. Then, we show how we can express four information patterns: sequentiality, memory of past information, memory of past actions, perfect recall. The inclusions $\mathcal{J}_t \subset \mathcal{H}_{\{0, \dots, t-1\}} = \mathcal{F} \otimes \bigotimes_{s=0}^{t-1} \mathcal{U}_s \otimes \bigotimes_{s=t}^{T-1} \{\emptyset, \mathbb{U}_s\}$, for $t \in \llbracket 0, T-1 \rrbracket$, express that every agent can remember no more than his past actions (sequentiality); memory of past information is represented by the inclusions $\mathcal{J}_{t-1} \subset \mathcal{J}_t$, for $t \in \llbracket 1, T-1 \rrbracket$; memory of past actions is represented by the inclusions $\{\emptyset, \Omega\} \otimes \bigotimes_{s=0}^{t-1} \mathcal{U}_s \otimes \bigotimes_{s=t}^{T-1} \{\emptyset, \mathbb{U}_s\} = \{\emptyset, \Omega\} \otimes \mathcal{U}_{\{0, \dots, t-1\}} \subset \mathcal{J}_t$, for $t \in \llbracket 1, T-1 \rrbracket$; perfect recall is represented by the inclusions $\mathcal{J}_{t-1} \vee (\{\emptyset, \Omega\} \otimes \mathcal{U}_{\{0, \dots, t-1\}}) \subset \mathcal{J}_t$, for $t \in \llbracket 1, T-1 \rrbracket$.

To represent N players — each p of whom makes a sequence of decisions, one for each period $t \in \llbracket 0, T_p-1 \rrbracket$ — we use $\prod_{p=1}^N T_p$ agents, labelled by $(p, t) \in \bigcup_{p'=1}^N (\{p'\} \times \llbracket 0, T_{p'}-1 \rrbracket)$. With obvious notations, the inclusions $\mathcal{J}_{(p, t-1)} \subset \mathcal{J}_{(p, t)}$ express memory of one's own past information, whereas the inclusions $\bigvee_{p'=1}^N (\{\emptyset, \Omega\} \otimes \bigotimes_{s=0}^{t-1} \mathcal{U}_s \otimes \bigotimes_{s=t}^{T_{p'}-1} \{\emptyset, \mathbb{U}_s\}) \subset \mathcal{J}_{(p, t)}$, express memory of all players past actions.

Principal-Agent models. A branch of Economics studies so-called *Principal-Agent* models with two decision makers (agents) — the Principal **Pr** (*leader*) who makes decisions $u_{\text{Pr}} \in \mathbb{U}_{\text{Pr}}$, where the set \mathbb{U}_{Pr} is equipped with a σ -field \mathcal{U}_{Pr} , and the Agent **Ag** (*follower*) who makes decisions $u_{\text{Ag}} \in \mathbb{U}_{\text{Ag}}$, where the set \mathbb{U}_{Ag} is equipped with a σ -field \mathcal{U}_{Ag} — and with Nature, corresponding to *private information (or type)* of the Agent **Ag**, taking values in a set Ω , equipped with a σ -field \mathcal{F} .

Hidden type (leading to adverse selection or to signaling) is represented by any information structure with the property that, on the one hand,

$$\mathcal{J}_{\text{Pr}} \subset \underbrace{\{\emptyset, \Omega\}}_{\text{Ag type not observed}} \otimes \underbrace{\{\emptyset, \mathbb{U}_{\text{Pr}}\}}_{\text{Ag's action possibly observed}} \otimes \underbrace{\mathcal{U}_{\text{Ag}}}_{\text{Ag's action possibly observed}}, \quad (10)$$

that is, the Principal **Pr** does not know the Agent **Ag** type, but can possibly observe the Agent **Ag** action, and, on the other hand, that

$$\underbrace{\mathcal{F}}_{\text{known inner type}} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\} \subset \mathcal{J}_{\text{Ag}}, \quad (11)$$

that is, the Agent **Ag** knows the state of nature (his type).

Hidden action (leading to moral hazard) is represented by any information structure with the property that, on the one hand,

$$\mathcal{J}_{\text{Pr}} \subset \underbrace{\mathcal{F}}_{\text{possibly knows Ag type}} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\} \otimes \underbrace{\{\emptyset, \mathbb{U}_{\text{Ag}}\}}_{\text{cannot observe Ag's action}}, \quad (12)$$

¹For any integers $a \leq b$, $\llbracket a, b \rrbracket$ denotes the subset $\{a, a+1, \dots, b-1, b\}$.

that is, the Principal **Pr** does not know the Agent **Ag** action, but can possibly observe the Agent **Ag** type and, on the other hand, that the inclusion (11) holds true, that is, the agent **Ag** knows the state of nature (his type).

Stackelberg leadership model. In Stackelberg games, the leader **Pr** makes a decision $u_{\text{Pr}} \in \mathbb{U}_{\text{Pr}}$ — based at most upon the partial observation of the state $\omega \in \Omega$ of Nature — and the the follower **Ag** makes a decision $u_{\text{Ag}} \in \mathbb{U}_{\text{Ag}}$ — based at most upon the partial observation of the state of Nature $\omega \in \Omega$, and upon the leader decision $u_{\text{Pr}} \in \mathbb{U}_{\text{Pr}}$. This kind of information structure is expressed with the following inclusions of fields:

$$\mathcal{J}_{\text{Pr}} \subset \mathcal{F} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\} \text{ and } \mathcal{J}_{\text{Ag}} \subset \mathcal{F} \otimes \mathbb{U}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\}. \quad (13)$$

Even if the players are called leader and follower, there is no explicit time arrow in (13). It is the information structure that reveals the time arrow. Indeed, if we label the leader **Pr** as $t = 0$ (first player) and the follower **Ag** as $t = 1$ (second player), the inclusions (13) become the inclusions $\mathcal{J}_0 \subset \mathcal{F} \otimes \{\emptyset, \mathbb{U}_0\} \otimes \{\emptyset, \mathbb{U}_1\}$, and $\mathcal{J}_1 \subset \mathcal{F} \otimes \mathbb{U}_0 \otimes \{\emptyset, \mathbb{U}_1\}$: the sequence $\mathcal{J}_0, \mathcal{J}_1$ of information fields is “adapted” to the filtration $\mathcal{F} \otimes \{\emptyset, \mathbb{U}_0\} \otimes \{\emptyset, \mathbb{U}_1\} \subset \mathcal{F} \otimes \mathbb{U}_0 \otimes \{\emptyset, \mathbb{U}_1\}$. But if we label the leader **Pr** as $t = 1$ and the follower **Ag** as $t = 0$, the new sequence of information fields would not be “adapted” to the new filtration. It is the information structure that prevents the follower to play first, but that makes possible the leader to play first and the follower to play second.

2.3 Solvability and causality

In the Kuhn formulation, Witsenhausen says that “For any combination of policies one can find the corresponding outcome by following the tree along selected branches, and this is an explicit procedure” [9]. In the Witsenhausen formulation, there is no such explicit procedure as, for any combination of policies, there may be none, one or many solutions to the closed-loop equations; these equations express the decision of one agent as the output of his strategy, supplied with Nature outcome and with all agents decisions. This is why Witsenhausen needs a property of solvability, whereas Kuhn does not need it as it is hardcoded in the tree structure. Then, Witsenhausen defines the notion of causality (which parallels that of tree) and proves in [9] that solvability holds true under causality. Yet, in [9, Theorem 2], Witsenhausen exhibits an example of noncausal W -model that is solvable.

2.3.1 Solvability

With any given pure W -strategies profile $\lambda = (\lambda_a)_{a \in \mathbb{A}} \in \prod_{a \in \mathbb{A}} \Lambda_a$ we associate the set-valued mapping

$$\begin{aligned} \mathcal{M}_\lambda : \Omega &\rightrightarrows \prod_{b \in \mathbb{A}} \mathbb{U}_b \\ \omega &\mapsto \left\{ (u_b)_{b \in \mathbb{A}} \in \prod_{b \in \mathbb{A}} \mathbb{U}_b \mid u_a = \lambda_a(\omega, (u_b)_{b \in \mathbb{A}}), \forall a \in \mathbb{A} \right\}. \end{aligned} \quad (14)$$

With this definition, we slightly reformulate below how Witsenhausen introduced the property of solvability.

Definition 4. ([9, 10]) *The solvability property holds true for the W-model of Definition 1 when, for any pure W-strategies profile $\lambda = (\lambda_a)_{a \in \mathbb{A}} \in \prod_{a \in \mathbb{A}} \Lambda_a$, the set-valued mapping \mathcal{M}_λ in (14) is a mapping whose domain is Ω , that is, the cardinal of $\mathcal{M}_\lambda(\omega)$ is equal to one, for any state of nature $\omega \in \Omega$.*

Thus, under the solvability property, for any state of nature $\omega \in \Omega$, there exists one, and only one, decision profile $(u_b)_{b \in \mathbb{A}} \in \prod_{b \in \mathbb{A}} \mathbb{U}_b$ which is a solution of the closed-loop equations

$$u_a = \lambda_a(\omega, (u_b)_{b \in \mathbb{A}}), \quad \forall a \in \mathbb{A}. \quad (15a)$$

In this case, we define the solution map

$$M_\lambda : \Omega \rightarrow \prod_{b \in \mathbb{A}} \mathbb{U}_b \quad (15b)$$

as the unique element contained in the image set $\mathcal{M}_\lambda(\omega)$ that is, for all $(u_b)_{b \in \mathbb{A}} \in \prod_{b \in \mathbb{A}} \mathbb{U}_b$, $M_\lambda(\omega) = (u_b)_{b \in \mathbb{A}} \iff \mathcal{M}_\lambda(\omega) = \{(u_b)_{b \in \mathbb{A}}\}$.

2.3.2 Configuration-orderings

In his articles [9, 10], Witsenhausen introduces a notion of causality that relies on suitable configuration-orderings. Here, we introduce our own notations, because they make possible a compact formulation of the causality property and, later, of perfect recall.

For any finite set \mathbb{D} , let $|\mathbb{D}|$ denote the cardinal of \mathbb{D} . Thus, $|\mathbb{A}|$ denotes the cardinal of the set \mathbb{A} , that is, $|\mathbb{A}|$ is the number of agents. For $k \in \llbracket 1, |\mathbb{A}| \rrbracket$, let Σ^k denote the set of k -orderings, that is, injective mappings from $\llbracket 1, k \rrbracket$ to \mathbb{A} :

$$\Sigma^k = \{ \kappa : \llbracket 1, k \rrbracket \rightarrow \mathbb{A}; \kappa \text{ is an injection} \}. \quad (16a)$$

The set $\Sigma^{|\mathbb{A}|}$ is the set of *total orderings* of agents in \mathbb{A} , that is, bijective mappings from $\llbracket 1, |\mathbb{A}| \rrbracket$ to \mathbb{A} (in contrast with *partial orderings* in Σ^k for $k < |\mathbb{A}|$). For any $k \in \llbracket 1, |\mathbb{A}| \rrbracket$, any ordering $\kappa \in \Sigma^k$, and any integer $\ell \leq k$, $\kappa|_{\{1, \dots, \ell\}}$ is the restriction of the ordering κ to the first ℓ integers. For any $k \in \llbracket 1, |\mathbb{A}| \rrbracket$, there is a natural mapping ψ_k

$$\psi_k : \Sigma^{|\mathbb{A}|} \rightarrow \Sigma^k, \quad \rho \mapsto \rho|_{\{1, \dots, k\}}, \quad (16b)$$

which is the restriction of any (total) ordering of \mathbb{A} to $\llbracket 1, k \rrbracket$. We define the *set of orderings* by

$$\Sigma = \bigcup_{k \in \llbracket 0, |\mathbb{A}| \rrbracket} \Sigma^k \quad \text{where } \Sigma^0 = \{\emptyset\}. \quad (16c)$$

For any $k \in \llbracket 1, |\mathbb{A}| \rrbracket$, and any k -ordering $\kappa \in \Sigma^k$, we define the *range* $\|\kappa\|$ of the ordering κ as the subset

$$\|\kappa\| = \{ \kappa(1), \dots, \kappa(k) \} \subset \mathbb{A}, \quad \forall \kappa \in \Sigma^k, \quad (16d)$$

the *cardinal* $|\kappa|$ of the ordering κ as the integer

$$|\kappa| = k \in \llbracket 1, |\mathbb{A}| \rrbracket, \quad \forall \kappa \in \Sigma^k, \quad (16e)$$

the *last element* κ^* of the ordering κ as the agent

$$\kappa^* = \kappa(k) \in \mathbb{A}, \quad \forall \kappa \in \Sigma^k, \quad (16f)$$

the *restriction* κ^- of the ordering κ to the first $k-1$ elements

$$\kappa^- = \kappa|_{\{1, \dots, k-1\}} \in \Sigma^{k-1}, \quad \forall \kappa \in \Sigma^k. \quad (16g)$$

With the notations introduced, any ordering $\kappa \in \Sigma \setminus \{\emptyset\}$ can be written as $\kappa = (\kappa^-, \kappa^*)$, with the convention that $\kappa = (\kappa^*)$ when $\kappa \in \Sigma^1$.

Definition 5. (*[9, 10]*) A configuration-ordering is a mapping $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ from configurations towards total orderings. With any configuration-ordering φ , and any ordering $\kappa \in \Sigma$, we associate the subset $\mathbb{H}_\kappa^\varphi \subset \mathbb{H}$ of configurations defined by

$$\mathbb{H}_\kappa^\varphi = \{h \in \mathbb{H}; \psi_{|\kappa|}(\varphi(h)) = \kappa\}, \quad \forall \kappa \in \Sigma. \quad (17)$$

By convention, we put $\mathbb{H}_\emptyset^\varphi = \mathbb{H}$.

Along each configuration $h \in \mathbb{H}$, the agents are ordered by $\varphi(h) \in \Sigma^{|\mathbb{A}|}$. The set $\mathbb{H}_\kappa^\varphi$ in (17) contains all the configurations for which the agent $\kappa(1)$ is acting first, the agent $\kappa(2)$ is acting second, \dots , till the last agent $\kappa^* = \kappa(|\kappa|)$ acting at stage $|\kappa|$.

2.3.3 Causality

In his article [9], Witsenhausen introduces a notion of causality and he proves that causal systems are solvable.

The following definition can be interpreted as follows. In a causal W-model, there exists a configuration-ordering with the following property: when an agent is called to play — as he is the last one in an ordering — what he knows cannot depend on decisions made by agents that are not his predecessors (in the range of the ordering under consideration).

Definition 6. (*[9, 10]*) A W-model (as in Definition 1) is causal if there exists (at least) one configuration-ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ with the property that

$$\mathbb{H}_\kappa^\varphi \cap H \in \mathcal{H}_{\|\kappa^-\|}, \quad \forall H \in \mathcal{J}_{\kappa^*}, \quad \forall \kappa \in \Sigma. \quad (18)$$

Otherwise said, once we know the first $|\kappa|$ agents, the information of the (last) agent κ^* depends at most on the decisions of the (previous) agents in the range $\|\kappa^-\|$. In (18), the subset $\mathbb{H}_\kappa^\varphi \subset \mathbb{H}$ of configurations has been defined in (17), the last agent κ^* in (16f), the partial ordering κ^- in (16g), the range $\|\kappa^-\|$ in (16d), and — using the definition (9b) of

the subfield $\mathcal{H}_{\mathbb{B}}$ of \mathcal{H} , with the subset $\mathbb{B} = \|\kappa^-\|$ of agents defined in (16g) and (16d) — the subfield $\mathcal{H}_{\|\kappa^-\|}$ of \mathcal{H} is

$$\mathcal{H}_{\|\kappa^-\|} = \mathcal{F} \otimes \bigotimes_{a \in \|\kappa^-\|} \mathcal{U}_a \otimes \bigotimes_{b \notin \|\kappa^-\|} \{\emptyset, \mathbb{U}_b\} \subset \mathcal{H}. \quad (19)$$

Witsenhausen’s intrinsic model deals with agents, information and strategies, but not with players and preferences. We now turn to extending the Witsenhausen’s intrinsic model to games.

3 Finite games in intrinsic form

We are now ready to embed Witsenhausen’s intrinsic model into game theory. In §3.1, we introduce a formal definition of a finite game in intrinsic form (W-game), and in §3.2 we introduce three notions of “randomization” of pure strategies — mixed, product-mixed and behavioral. In §3.3, we discuss relations between product-mixed and behavioral W-strategies.

In what follows, when \mathbb{D} is a finite set, we denote by $\Delta(\mathbb{D})$ the set of probability distributions over \mathbb{D} . When needed, the set $\Delta(\mathbb{D})$ can be equipped with the Borel topology and the Borel σ -field, as $\Delta(\mathbb{D})$ is homeomorphic to the simplex $\Sigma_{|\mathbb{D}|}$ of $\mathbb{R}^{|\mathbb{D}|}$, and is thus homeomorphic to a closed subset of a finite dimensional space.

3.1 Definition of a finite game in intrinsic form (W-game)

We introduce a formal definition of a finite game in intrinsic form (W-game).

Definition 7. A finite W-game $\left(((\mathbb{A}^p)_{p \in P}, (\Omega, \mathcal{F}), (\mathbb{U}_a, \mathcal{U}_a, \mathcal{J}_a)_{a \in \bigcup_{p \in P} \mathbb{A}^p}), (\preceq^p)_{p \in P} \right)$, or a finite game in intrinsic form, is a made of

- a family $(\mathbb{A}^p)_{p \in P}$, where the set P of players is finite, of two by two disjoint nonempty sets whose union $\mathbb{A} = \bigcup_{p \in P} \mathbb{A}^p$ is the set of agents; each subset \mathbb{A}^p is interpreted as the subset of executive agents of the player $p \in P$,
- a finite W-model $(\mathbb{A}, (\Omega, \mathcal{F}), (\mathbb{U}_a, \mathcal{U}_a, \mathcal{J}_a)_{a \in \mathbb{A}})$, as in Definition 1,
- for each player $p \in P$, a preference relation \preceq^p on the set of mappings $\Omega \rightarrow \Delta(\prod_{b \in \mathbb{A}} \mathbb{U}_b)$.

A finite W-game is said to be solvable (resp. causal) if the underlying W-model is solvable as in Definition 4 (resp. causal as in Definition 6).

We comment on the preference relations \preceq^p on the set of mappings $\Omega \rightarrow \Delta(\prod_{b \in \mathbb{A}} \mathbb{U}_b)$. Our definition covers (like in [3]) the most traditional preference relation \preceq^p , which is the numerical *expected utility* preference. In this latter, each player $p \in P$ is endowed, on the one hand, with a *criterion* (payoff), that is, a measurable function $j_p : (\mathbb{H}, \mathcal{H}) \rightarrow [-\infty, +\infty[$, and, on the other hand, with a *belief*, that is, a probability distribution $\nu^p : \mathcal{F} \rightarrow [0, 1]$ over

the states of Nature (Ω, \mathcal{F}) . Then, given $K_i : \Omega \rightarrow \Delta(\prod_{b \in \mathbb{A}} \mathbb{U}_b)$, $i = 1, 2$, one says that $K_1 \preceq^p K_2$ if

$$\begin{aligned} & \int_{\Omega} \nu^p(d\omega) \int_{\prod_{b \in \mathbb{A}} \mathbb{U}_b} j_p(\omega, (u_b)_{b \in \mathbb{A}}) K_1(\omega, d(u_b)_{b \in \mathbb{A}}) \\ & \leq \int_{\Omega} \nu^p(d\omega) \int_{\prod_{b \in \mathbb{A}} \mathbb{U}_b} j_p(\omega, (u_b)_{b \in \mathbb{A}}) K_2(\omega, d(u_b)_{b \in \mathbb{A}}) . \end{aligned}$$

Note also that the Definition 7 includes Bayesian games, by specifying a product structure for Ω — where some factors represent types of players, and one factor represents chance — and by considering additional probability distributions.

3.2 Mixed, product-mixed and behavioral strategies

We introduce three notions of “randomization” of pure strategies: mixed, product-mixed and behavioral.

The notion of mixed strategy comes from the study of games in normalized form, where each player has to select a pure strategy, the collection of which determines a unique outcome. If we allow the players to select their pure strategy at random, the lottery they use is called a mixed strategy. For an extensive game, a mixed strategy can be interpreted in the following sense. First, the player selects a pure strategy using the lottery. Second, the game is played. When the player is called by the umpire, she plays the action specified by the selected pure strategy for the current information set.

Observe that there is only one dice roll per player. This dice roll determines the reactions of the player for every situation of the game. It would be more natural to let the player roll a dice every time she has to play, leading to the notion of behavioral strategy.

A fundamental question in game theory is to identify settings in which those two views (mixed strategy and behavioral strategy) are equivalent. To formulate this question in the W-game framework, we will give formal definitions of these two notions of randomization. We will also add a third one, that we call product-mixed strategy, and which is in the spirit of Aumann [2], as each agent (corresponding to time index in [2]) “generates” strategies from a random device that is independent of all the other agents.

3.2.1 Mixed W-strategies

For any agent $a \in \mathbb{A}$, the set Λ_a of pure W-strategies for agent a (see Definition 3) is finite, hence the set $\Delta(\Lambda_a)$ of probability distributions over Λ_a is homeomorphic to $\Sigma_{|\Lambda_a|}$, the simplex of $\mathbb{R}^{|\Lambda_a|}$, and is thus homeomorphic to a closed subset of a finite dimensional space. So is the space $\Delta(\Lambda)$ of probability distributions over the set Λ of pure W-strategies profiles. We will also consider the sets

$$\Lambda^p = \prod_{a \in \mathbb{A}^p} \Lambda_a , \quad \forall p \in P \tag{20}$$

of pure W-strategies profiles, player by player, and the set $\Delta(\Lambda^p)$ of probability distributions over Λ^p .

Definition 8. We consider a finite W-game, as in Definition 7. A mixed W-strategy for player $p \in P$ is an element μ^p of $\Delta(\Lambda^p)$, the set of probability distributions over the set Λ^p in (20) of W-strategies of the executive agents in \mathbb{A}^p . The set of mixed W-strategies profiles is $\prod_{p \in P} \Delta(\Lambda^p)$.

A mixed W-strategies profile is denoted by

$$\mu = (\mu^p)_{p \in P} \in \prod_{p \in P} \Delta(\Lambda^p), \quad (21a)$$

and, when we focus on player p , we write

$$\mu = (\mu^p, \mu^{-p}) \in \Delta(\Lambda^p) \times \prod_{p' \neq p} \Delta(\Lambda^{p'}). \quad (21b)$$

Definition 9. We consider a solvable finite W-game (see Definition 7), and $\mu = (\mu^p)_{p \in P} \in \prod_{p \in P} \Delta(\Lambda^p)$ a mixed W-strategies profile as in (21a). For any $\omega \in \Omega$, we denote by

$$\mathbb{Q}_\mu^\omega = \mathbb{Q}_{(\mu^p)_{p \in P}}^\omega = \left(\bigotimes_{p \in P} \mu^p \right) \circ \left(M(\omega, \cdot) \right)^{-1} \in \Delta \left(\prod_{b \in \mathbb{A}} \mathbb{U}_b \right) \quad (22a)$$

the pushforward probability, on the space $(\prod_{b \in \mathbb{A}} \mathbb{U}_b, \bigotimes_{b \in \mathbb{A}} \mathcal{U}_b)$ of the product probability distribution $\bigotimes_{p \in P} \mu^p$ on $\prod_{p \in P} \Lambda^p = \Lambda$ by the mapping

$$M(\omega, \cdot) : \Lambda \rightarrow \prod_{b \in \mathbb{A}} \mathbb{U}_b, \quad \lambda \mapsto M_\lambda(\omega), \quad (22b)$$

where M_λ is the solution map (15b), which exists by the solvability assumption.

By (15a), which defines the solution map, and by definition of a pushforward probability, we have, for any configuration $(\omega, (u_b)_{b \in \mathbb{A}}) \in \mathbb{H}$,

$$\begin{aligned} \mathbb{Q}_\mu^\omega((u_b)_{b \in \mathbb{A}}) &= \left(\bigotimes_{p \in P} \mu^p \right) \left(M(\omega, \cdot)^{-1}((u_b)_{b \in \mathbb{A}}) \right) \\ &= \prod_{p \in P} \mu^p \left(\left\{ (\lambda_a)_{a \in \mathbb{A}^p} \in \Lambda^p \mid \lambda_a(\omega, (u_b)_{b \in \mathbb{A}}) = u_a, \forall a \in \mathbb{A}^p \right\} \right). \end{aligned}$$

3.2.2 Product-mixed W-strategies

In a mixed W-strategy, the executive agents of player $p \in P$ can be correlated because the probability μ^p in Definition 8 is a joint probability on the product space $\Lambda^p = \prod_{a \in \mathbb{A}^p} \Lambda_a$. We now introduce product-mixed W-strategies, where the executive agents of player $p \in P$ are independent in the sense that the probability μ^p is the product of individual probabilities, each of them on the individual space Λ_a of the strategies of one agent a .

Definition 10. We consider a finite W-game, as in Definition 7. A product-mixed W-strategy for player $p \in P$ is an element $\pi^p = (\pi_a^p)_{a \in \mathbb{A}^p}$ of $\prod_{a \in \mathbb{A}^p} \Delta(\Lambda_a)$.

The product-mixed W-strategy $(\pi_a^p)_{a \in \mathbb{A}^p}$ induces a product probability² $\bigotimes_{a \in \mathbb{A}^p} \pi_a^p$ on the

²By an abuse of notation, we will sometimes write $\pi^p = \bigotimes_{a \in \mathbb{A}^p} \pi_a^p$.

set Λ^p , which is a mixed W-strategy as in Definition 8.

3.2.3 Behavioral W-strategies

We formalize the intuition of behavioral strategies in W-games by the following definition of behavioral W-strategies.

Definition 11. *We consider a finite W-game, as in Definition 7. A behavioral W-strategy for player $p \in P$ is a family $\beta^p = (\beta_a^p)_{a \in \mathbb{A}^p}$, where*

$$\beta_a^p : \mathbb{H} \times \mathcal{U}_a \rightarrow [0, 1], \quad (h, U_a) \mapsto \beta_a^p(U_a | h) \quad (23)$$

is an \mathcal{J}_a -measurable stochastic kernel for each $a \in \mathbb{A}^p$, that is, if one of the two equivalent statements holds true:

1. *on the one hand, the function $h \in \mathbb{H} \mapsto \beta_a^p(\{u_a\} | h)$ is \mathcal{J}_a -measurable, for any $u_a \in \mathcal{U}_a$ and, on the other hand, each $\beta_a^p(\cdot | h)$ is a probability distribution on the finite set \mathcal{U}_a , for any $h \in \mathbb{H}$,*
2. *on the one hand, $h' \sim_a h'' \implies \beta_a^p(\{u_a\} | h') = \beta_a^p(\{u_a\} | h'')$, for any $u_a \in \mathcal{U}_a$ and, on the other hand, for any $h \in \mathbb{H}$, we have $\beta_a^p(\{u_a\} | h) \geq 0$, $\forall u_a \in \mathcal{U}_a$, and $\sum_{u_a \in \mathcal{U}_a} \beta_a^p(\{u_a\} | h) = 1$.*

The equivalences come from the fact that the sets \mathbb{H} and \mathcal{U}_a are finite and equipped with their respective complete fields, and by Proposition 2, and especially (4b).

3.3 Relations between product-mixed and behavioral W-strategies

Here, we show that product-mixed and behavioral W-strategies are “equivalent” in the sense that a product-mixed W-strategy naturally induces a behavioral W-strategy, and that a behavioral W-strategy can be “realized” as a product-mixed W-strategy (see Figure 1).

From product-mixed to behavioral W-strategies

We prove that a product-mixed W-strategy naturally induces a behavioral W-strategy.

Proposition 12. *We consider a finite W-game, as in Definition 7, and a player $p \in P$.*

For any product-mixed W-strategy $\pi^p = (\pi_a^p)_{a \in \mathbb{A}^p} \in \prod_{a \in \mathbb{A}^p} \Delta(\Lambda_a)$, as in Definition 10, we define, for any agent $a \in \mathbb{A}^p$,

$$\hat{\pi}_a^p(\{u_a\} | h) = \pi_a^p\left(\{\lambda_a \in \Lambda_a; \lambda_a(h) = u_a\}\right), \quad \forall u_a \in \mathcal{U}_a, \quad \forall h \in \mathbb{H}. \quad (24)$$

Then, $\hat{\pi}^p = (\hat{\pi}_a^p)_{a \in \mathbb{A}^p}$ is a behavioral W-strategy, as in Definition 11.

Proof. Let be given a product-mixed W-strategy $\pi^p = (\pi_a^p)_{a \in \mathbb{A}^p} \in \prod_{a \in \mathbb{A}^p} \Delta(\Lambda_a)$.

To prove that (24) defines a behavioral W-strategy, we have to show (see Item 1 in Definition 11), on the one hand, that the function $h \in \mathbb{H} \mapsto \hat{\pi}_a^p(\{u_a\} | h)$ is \mathcal{J}_a -measurable, for any $u_a \in \mathbb{U}_a$ and, on the other hand, that each $\beta_a^p(\cdot | h)$ is a probability on the finite set \mathbb{U}_a , for any $h \in \mathbb{H}$. For this purpose, we will use the more practical characterization of Item 2 in Definition 11.

Let us fix $u_a \in \mathbb{U}_a$. Let $h', h'' \in \mathbb{H}$ be such that $h' \sim_a h''$, where we recall that the classes of the equivalence relation \sim_a in (3) are exactly the atoms in $\langle \mathcal{J}_a \rangle$. By (6), we have that $\{\lambda_a \in \Lambda_a; \lambda_a(h') = u_a\} = \{\lambda_a \in \Lambda_a; \lambda_a(h'') = u_a\}$. Therefore, from the expression (24), we have obtained that $h' \sim_a h'' \implies \hat{\pi}_a^p(\{u_a\} | h') = \hat{\pi}_a^p(\{u_a\} | h'')$, hence that the function $h \in \mathbb{H} \mapsto \hat{\pi}_a^p(\{u_a\} | h)$ is \mathcal{J}_a -measurable, by Proposition 2, and especially (4b).

By the expression (24), we have that $\hat{\pi}_a^p(\{u_a\} | h) \geq 0$, $\forall u_a \in \mathbb{U}_a$ and, since π_a^p is a probability on $\Delta(\Lambda_a)$, that $\sum_{u_a \in \mathbb{U}_a} \hat{\pi}_a^p(\{u_a\} | h) = 1$.

This ends the proof. \square

From behavioral to product-mixed W-strategies

We prove that a behavioral W-strategy can be “realized” as a product-mixed W-strategy.

Proposition 13. *We consider a finite W-game, as in Definition 7, and a player $p \in P$.*

For any behavioral W-strategy $\beta^p = (\beta_a^p)_{a \in \mathbb{A}^p}$, as in Definition 11, there exists a product-mixed W-strategy $\check{\beta}^p = (\check{\beta}_a^p)_{a \in \mathbb{A}^p}$, as in Definition 10, with the property that, for any agent a in \mathbb{A}^p we have

$$\check{\beta}_a^p(\{\lambda_a \in \Lambda_a; \lambda_a(h) = u_a\}) = \beta_a^p(\{u_a\} | h), \quad \forall u_a \in \mathbb{U}_a, \quad \forall h \in \mathbb{H}. \quad (25)$$

Proof. We consider a fixed agent $a \in \mathbb{A}^p$.

On the one hand, by Proposition 2, we get that

$$\lambda_a \in \Lambda_a \iff \exists (u_a^{G_a})_{G_a \in \langle \mathcal{J}_a \rangle} \in \mathbb{U}_a^{\langle \mathcal{J}_a \rangle}, \quad \lambda_a(h) = u_a^{G_a}, \quad \forall G_a \in \langle \mathcal{J}_a \rangle, \quad \forall h \in G_a,$$

where $\mathbb{U}_a^{\langle \mathcal{J}_a \rangle}$ is the set of mappings from $\langle \mathcal{J}_a \rangle$ to \mathbb{U}_a . Therefore, the following mapping is a bijection:

$$\Psi : \Lambda_a \rightarrow \mathbb{U}_a^{\langle \mathcal{J}_a \rangle}, \quad \lambda_a \mapsto (\lambda_a(G_a))_{G_a \in \langle \mathcal{J}_a \rangle}. \quad (26a)$$

We denote the inverse bijection by

$$\Phi = \Psi^{-1} : \mathbb{U}_a^{\langle \mathcal{J}_a \rangle} \rightarrow \Lambda_a. \quad (26b)$$

On the other hand, by Item 1 in Definition 11, each $\beta_a^p(\cdot | h)$ is a probability on the finite set \mathbb{U}_a , for any $h \in \mathbb{H}$. As the mapping $h \in \mathbb{H} \mapsto \beta_a^p(\cdot | h)$ is \mathcal{J}_a -measurable, by Definition 11, the notation $\beta_a^p(\cdot | G_a)$ makes sense by (5).

We equip the finite set $\mathbb{U}_a^{\langle \mathcal{J}_a \rangle}$ with the product probability $\bigotimes_{G_a \in \langle \mathcal{J}_a \rangle} \beta_a^p(\cdot | G_a)$, and we define the pushforward probability

$$\check{\beta}_a^p = \left(\bigotimes_{G_a \in \langle \mathcal{J}_a \rangle} \beta_a^p(\cdot | G_a) \right) \circ \Phi^{-1}, \quad (26c)$$

on the finite set Λ_a . Then, we calculate, for any $h \in \mathbb{H}$,

$$\begin{aligned} & \check{\beta}_a^p \left(\{ \lambda_a \in \Lambda_a; \lambda_a(h) = u_a \} \right) \\ &= \left(\bigotimes_{G_a \in \langle \mathcal{J}_a \rangle} \beta_a^p(\cdot | G_a) \right) \left(\Phi^{-1} \left(\{ \lambda_a \in \Lambda_a; \lambda_a(h) = u_a \} \right) \right) && \text{(by definition (26c))} \\ &= \left(\bigotimes_{G_a \in \langle \mathcal{J}_a \rangle} \beta_a^p(\cdot | G_a) \right) \left(\Psi \left(\{ \lambda_a \in \Lambda_a; \lambda_a(h) = u_a \} \right) \right) && \text{(as } \Phi^{-1} = \Psi \text{ by (26b))} \\ &= \left(\bigotimes_{G_a \in \langle \mathcal{J}_a \rangle} \beta_a^p(\cdot | G_a) \right) \left(\{ u_a \} \times \mathbb{U}_a^{\langle \mathcal{J}_a \rangle \setminus [h]_a} \right) \end{aligned}$$

because, by definition of the mapping Ψ in (26a), any $\lambda_a \in \{ \lambda'_a \in \Lambda_a; \lambda'_a(h) = u_a \}$ takes the value u_a on the atom $[h]_a$ (by definition of the set Λ_a in Definition 3 and by (6)), and any possible value in \mathbb{U}_a for all the other atoms in $\langle \mathcal{J}_a \rangle \setminus [h]_a$

$$\begin{aligned} &= \beta_a^p(\{ u_a \} | [h]_a) \times \prod_{G_a \in \langle \mathcal{J}_a \rangle \setminus [h]_a} \beta_a^p(\mathbb{U}_a | G_a) \\ & \quad \text{(by definition of the product probability } \bigotimes_{G_a \in \langle \mathcal{J}_a \rangle} \beta_a^p(\cdot | G_a)) \\ &= \beta_a^p(\{ u_a \} | [h]_a) \times \prod_{G_a \in \langle \mathcal{J}_a \rangle \setminus [h]_a} 1 && \text{(as } \beta_a^p(\mathbb{U}_a | G_a) = 1 \text{ for all } G_a \in \langle \mathcal{J}_a \rangle) \\ &= \beta_a^p(\{ u_a \} | [h]_a) \\ &= \beta_a^p(\{ u_a \} | h) \end{aligned}$$

as the function $h \in \mathbb{H} \mapsto \beta_a^p(\{ u_a \} | h)$ is \mathcal{J}_a -measurable, by Item 1 in Definition 11, and using (6).

This ends the proof. \square

4 Kuhn's equivalence theorem

Now, we are equipped to give, for games in intrinsic form, a statement and a proof of the celebrated Kuhn's equivalence theorem: when a player enjoys perfect recall, for any mixed W-strategy, there is an equivalent behavioral strategy.

In this Section, we consider a causal finite W-game (see Definition 7), that is, the underlying W-model (as in Definition 1) is causal (see Definition 6), with suitable configuration-ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$. In §4.1, we introduce a formal definition of perfect recall in a causal

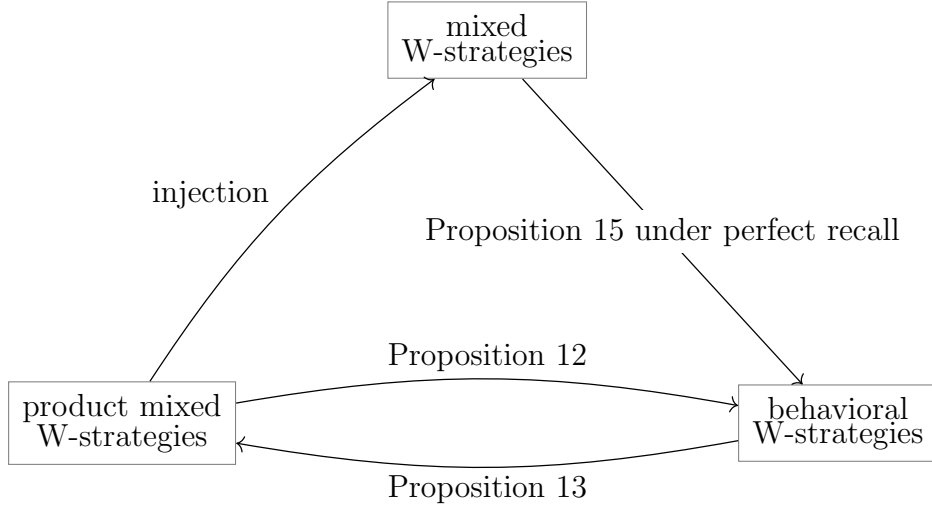


Figure 1: Three Propositions that relate three notions of randomization of strategies

finite game in intrinsic form. In §4.2 (Proposition 15), we show that any mixed W-strategy induces a behavioral W-strategy under perfect recall (see Figure 1). Finally, in §4.3 (Theorem 16), we give a statement and a proof of Kuhn’s equivalence theorem for games in intrinsic form.

4.1 Definition of perfect recall for causal W-games

For any agent $a \in \mathbb{A}$, we define the *choice field* $\mathcal{C}_a \subset \mathcal{H}$ by

$$\mathcal{C}_a = \mathcal{U}_a \bigvee \mathcal{J}_a, \quad \forall a \in \mathbb{A}. \quad (27)$$

Thus defined, the choice field of an agent contains both what the agent did and what he knew when making the decision.

The following definition of perfect recall is new.

Definition 14. *We consider a causal finite W-game for which the underlying W-model is causal with the configuration-ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$.*

We say that a player $p \in P$ enjoys perfect recall if, for any ordering $\kappa \in \Sigma$ such that $\kappa^ \in \mathbb{A}^p$ (that is, the last agent is an executive of the player), we have*

$$\mathbb{H}_\kappa^\varphi \cap H \in \mathcal{J}_{\kappa^*}, \quad \forall H \in \mathcal{C}_{\|\kappa^-\| \cap \mathbb{A}^p}, \quad (28a)$$

where the subset $\mathbb{H}_\kappa^\varphi \subset \mathbb{H}$ of configurations has been defined in (17), the last agent κ^ in (16f), the partial ordering κ^- in (16g), the range $\|\kappa^-\|$ in (16d), and where*

$$\mathcal{C}_{\|\kappa^-\| \cap \mathbb{A}^p} = \bigvee_{\substack{a \in \|\kappa^-\| \\ a \in \mathbb{A}^p}} \mathcal{C}_a, \quad (28b)$$

with the choice subfield $\mathcal{C}_a \subset \mathcal{H}$ given by (27).

We interpret the above definition as follows. A player enjoys perfect recall when any of her executive agents — when called to play as the last one in an ordering — knows at least what did and knew those of the executive agents that are both his predecessors (in the range of the ordering under consideration) and that are executive agents of the player.

4.2 Mixed W-strategy induces behavioral W-strategy under perfect recall

As a preparatory result for the proof of Kuhn's equivalence theorem, we show that any mixed W-strategy induces a behavioral W-strategy under perfect recall.

Proposition 15. *We consider a causal finite W-game (see Definition 7), for which the underlying W-model (see Definition 1) is causal (see Definition 6) with the configuration-ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$.*

We consider a player $p \in P$, equipped with a mixed W-strategy $\mu^p \in \Delta(\Lambda^p)$ and supposed to enjoy perfect recall (see Definition 14), and an agent $a \in \mathbb{A}^p$.

Then, for each agent $a \in \mathbb{A}^p$, the following formula³

$$\tilde{\mu}_a^p(\{u_a\} | h) = \frac{\mu^p\{\lambda \in \Lambda^p; \exists \kappa \in \Sigma, h \in \mathbb{H}_{\kappa}^{\varphi}, \kappa^* = a, \lambda_a(h) = u_a, \lambda_{\|\kappa^-\| \cap \mathbb{A}^p}(h) = h_{\|\kappa^-\| \cap \mathbb{A}^p}\}}{\mu^p\{\lambda \in \Lambda^p; \exists \kappa \in \Sigma, h \in \mathbb{H}_{\kappa}^{\varphi}, \kappa^* = a, \lambda_{\|\kappa^-\| \cap \mathbb{A}^p}(h) = h_{\|\kappa^-\| \cap \mathbb{A}^p}\}} \quad (29)$$

defines an \mathcal{J}_a -measurable stochastic kernel $\tilde{\mu}_a^p$. As a consequence, the family $\tilde{\mu}^p = (\tilde{\mu}_a^p)_{a \in \mathbb{A}^p}$, is a behavioral W-strategy, as in Definition 11.

Proof. We consider a player $p \in P$, equipped with a mixed W-strategy $\mu^p \in \Delta(\Lambda^p)$ and supposed to enjoy perfect recall as in Definition 14, and an agent $a \in \mathbb{A}^p$.

By (29), it is easy to see that, for any $h \in \mathbb{H}$, we have $\tilde{\mu}_a^p(\{u_a\} | h) \geq 0$, $\forall u_a \in \mathbb{U}_a$, and $\sum_{u_a \in \mathbb{U}_a} \tilde{\mu}_a^p(\{u_a\} | h) = 1$. Therefore, by Item 1 in Definition 11, there remains to prove that the function $h \in \mathbb{H} \mapsto \tilde{\mu}_a^p(\{u_a\} | h)$ is \mathcal{J}_a -measurable. The proof is in several steps.

- As the agent $a \in \mathbb{A}$ is fixed, it is easily seen that the family $(\mathbb{H}_{\kappa}^{\varphi})_{\kappa \in \Sigma, \kappa^* = a}$, where the subset $\mathbb{H}_{\kappa}^{\varphi} \subset \mathbb{H}$ of configurations has been defined in (17), is made of (possibly empty) disjoint sets whose union is \mathbb{H} . Indeed, on the one hand, for any $h \in \mathbb{H}$, we have that $h \in \mathbb{H}_{\kappa_0}^{\varphi}$ where $\kappa_0 = \psi_k(\rho)$, with $\rho = \varphi(h)$ and k the unique integer such that $\rho(k) = a$. On the other hand, if we had $\mathbb{H}_{\kappa_1}^{\varphi} \cap \mathbb{H}_{\kappa_2}^{\varphi} \neq \emptyset$, with $\kappa_1^* = \kappa_2^* = a$, then $h \in \mathbb{H}_{\kappa_1}^{\varphi} \cap \mathbb{H}_{\kappa_2}^{\varphi}$ would be such that $\kappa_1 = \kappa_2 = \psi_k(\rho)$, with $\rho = \varphi(h)$ and k the unique integer such that $\rho(k) = a$. Thus, $\kappa_1 \neq \kappa_2 \implies \mathbb{H}_{\kappa_1}^{\varphi} \cap \mathbb{H}_{\kappa_2}^{\varphi} = \emptyset$.

³With the convention that $\tilde{\mu}_a^p(\{u_a\} | h) = 0$ if the denominator is zero (in which case the numerator, which is smaller, is also zero), and using the notations $\lambda = (\lambda_a)_{a \in \mathbb{A}^p} \in \Lambda^p$, (9c) for $h_{\mathbb{B}}$ and (9d) for $\lambda_{\mathbb{B}}$, with $\mathbb{B} = \|\kappa^-\| \cap \mathbb{A}^p$, where the last agent κ^* has been defined in (16f), the partial ordering κ^- in (16g) and the range $\|\kappa^-\|$ in (16d).

- As the family $(\mathbb{H}_\kappa^\varphi)_{\kappa \in \Sigma, \kappa^* = a}$ is made of disjoint sets whose union is \mathbb{H} , we rewrite (29) as

$$\begin{aligned}
& \tilde{\mu}_a^p(\{u_a\} | h) \\
&= \frac{\sum_{\substack{\kappa \in \Sigma \\ \kappa^* = a}} \mu^p \{ \lambda \in \Lambda^p ; h \in \mathbb{H}_\kappa^\varphi, \lambda_a(h) = u_a, \lambda_{\|\kappa^-\| \cap \mathbb{A}^p}(h) = h_{\|\kappa^-\| \cap \mathbb{A}^p} \}}{\sum_{\kappa \in \Sigma, \kappa^* = a} \mu^p \{ \lambda \in \Lambda^p ; h \in \mathbb{H}_\kappa^\varphi, \lambda_{\|\kappa^-\| \cap \mathbb{A}^p}(h) = h_{\|\kappa^-\| \cap \mathbb{A}^p} \}} \\
&= \frac{\sum_{\kappa \in \Sigma, \kappa^* = a} \mu^p \{ \Phi_\kappa(h, u_a) \}}{\sum_{\kappa \in \Sigma, \kappa^* = a} \sum_{u_a \in \mathbb{U}_a} \mu^p \{ \Phi_\kappa(h, u_a) \}},
\end{aligned}$$

where, for any $h \in \mathbb{H}$, $u_a \in \mathbb{U}_a$ and $\kappa \in \Sigma$ such that $\kappa^* = a$, we have defined the following subset of strategies

$$\begin{aligned}
\Phi_\kappa(h, u_a) &= \\
& \{ \lambda \in \Lambda^p ; h \in \mathbb{H}_\kappa^\varphi, \lambda_a(h) = u_a, \lambda_{\|\kappa^-\| \cap \mathbb{A}^p}(h) = h_{\|\kappa^-\| \cap \mathbb{A}^p} \} \subset \Lambda^p.
\end{aligned} \tag{30a}$$

We will prove, in three steps, that $\Phi_\kappa(h, u_a)$ in (30a) takes the same (set) value for any $h \in G_a$, where G_a is an atom of \mathcal{J}_a .

- Let $G_a \subset \mathbb{H}$ be an atom of \mathcal{J}_a . We prove that there exists a unique $\kappa_0 \in \Sigma$ such that $\kappa_0^* = a$ and $G_a \subset \mathbb{H}_{\kappa_0}^\varphi$, that is, we prove that

$$G_a \in \langle \mathcal{J}_a \rangle \implies \exists! \kappa_0 \in \Sigma, \kappa_0^* = a, G_a \subset \mathbb{H}_{\kappa_0}^\varphi. \tag{30b}$$

First, we show that, for any $\kappa \in \Sigma$ such that $\kappa^* = a$, we have either $G_a \subset \mathbb{H}_\kappa^\varphi$ or $G_a \cap \mathbb{H}_\kappa^\varphi = \emptyset$. Indeed, by (28a) with $H = \mathbb{H} \in \mathcal{C}_{\|\kappa^-\| \cap \mathbb{A}^p}$, we obtain that $\mathbb{H}_\kappa^\varphi \in \mathcal{J}_a$. Therefore, either $G_a \cap \mathbb{H}_\kappa^\varphi = \emptyset$ or $G_a \subset \mathbb{H}_\kappa^\varphi$ by (34) since G_a is an atom of \mathcal{J}_a . Second, we have seen that the family $(\mathbb{H}_\kappa^\varphi)_{\kappa \in \Sigma, \kappa^* = a}$ is made of disjoint sets whose union is \mathbb{H} .

By combining both results, we conclude that there exists a unique $\kappa_0 \in \Sigma$ such that $\kappa_0^* = a$ and $G_a \subset \mathbb{H}_{\kappa_0}^\varphi$.

- As a consequence of (30b), we have that $\Phi_\kappa(h, u_a) = \emptyset$, for any $h \in G_a$, and for any $\kappa \in \Sigma$ such that $\kappa \neq \kappa_0$ and $\kappa^* = a$. There only remains to prove that $\Phi_{\kappa_0}(h, u_a)$ in (30a) takes the same (set) value for any $h \in G_a$. For this purpose, we consider $h', h'' \in \mathbb{H}$ which belong to the atom $G_a \in \langle \mathcal{J}_a \rangle$, that is, $\{h', h''\} \subset G_a$, and we establish two preliminary results.

First, we prove that $h'_{\|\kappa_0^-\| \cap \mathbb{A}^p} = h''_{\|\kappa_0^-\| \cap \mathbb{A}^p}$. For this purpose, we define the subset $H' = \{h \in \mathbb{H} ; h_{\|\kappa_0^-\| \cap \mathbb{A}^p} = h'_{\|\kappa_0^-\| \cap \mathbb{A}^p}\} \subset \mathbb{H}$ and we show in two steps that $h'' \in H'$, hence that $h'_{\|\kappa_0^-\| \cap \mathbb{A}^p} = h''_{\|\kappa_0^-\| \cap \mathbb{A}^p}$, that is, we show that

$$\{h', h''\} \subset G_a \implies h'_{\|\kappa_0^-\| \cap \mathbb{A}^p} = h''_{\|\kappa_0^-\| \cap \mathbb{A}^p}. \tag{30c}$$

- We show that $\mathbb{H}_{\kappa_0}^\varphi \cap H' \in \mathcal{J}_a$. By definition of the field $\mathcal{U}_{\|\kappa_0^-\| \cap \mathbb{A}^p}$ in (9a) with $\mathbb{B} = \|\kappa_0^-\| \cap \mathbb{A}^p$, and because each field \mathcal{U}_b , for $b \in \|\kappa_0^-\| \cap \mathbb{A}^p$, is complete, hence has the singletons for atoms, we have that $H' \in \mathcal{U}_{\|\kappa_0^-\| \cap \mathbb{A}^p}$. As $\mathcal{U}_{\|\kappa_0^-\| \cap \mathbb{A}^p} \subset \mathcal{C}_{\|\kappa_0^-\| \cap \mathbb{A}^p}$ by (28b), we use the perfect recall assumption (28a) with $H = H'$, and obtain that $\mathbb{H}_{\kappa_0}^\varphi \cap H' \in \mathcal{J}_{\kappa_0^*} = \mathcal{J}_a$ since $\kappa_0^* = a$.

- As $\mathbb{H}_{\kappa_0}^\varphi \cap H' \in \mathcal{J}_a$ and G_a is an atom of \mathcal{J}_a , we have either $G_a \cap \mathbb{H}_{\kappa_0}^\varphi \cap H' = \emptyset$ or $G_a \subset \mathbb{H}_{\kappa_0}^\varphi \cap H'$ by (34). Now, as $h' \in H'$ and $h' \in G_a \subset \mathbb{H}_{\kappa_0}^\varphi$, we get that $G_a \cap \mathbb{H}_{\kappa_0}^\varphi \cap H' \neq \emptyset$, and therefore $G_a \subset \mathbb{H}_{\kappa_0}^\varphi \cap H'$. Since $h'' \in G_a$, we deduce that $h'' \in H'$, hence that $h'_{\|\kappa_0^-\| \cap \mathbb{A}^p} = h''_{\|\kappa_0^-\| \cap \mathbb{A}^p}$.

Second, we prove that, for any $b \in \|\kappa_0^-\| \cap \mathbb{A}^p$ and any atom G_b of \mathcal{J}_b , we have either $\{h', h''\} \cap G_b = \emptyset$ or $\{h', h''\} \subset G_b$, that is, we prove that

$$\begin{aligned} \left(b \in \|\kappa_0^-\| \cap \mathbb{A}^p \text{ and } G_b \in \langle \mathcal{J}_b \rangle \right) &\implies \\ \left(\{h', h''\} \cap G_b = \emptyset \text{ or } \{h', h''\} \subset G_b \right). & \end{aligned} \quad (30d)$$

For this purpose, we consider $b \in \|\kappa_0^-\| \cap \mathbb{A}^p$ and $G_b \in \langle \mathcal{J}_b \rangle$.

As $\mathcal{J}_b \subset \mathcal{C}_{\|\kappa_0^-\| \cap \mathbb{A}^p}$ by (28b), we use the perfect recall assumption (28a) with $H = G_b$, and obtain that $\mathbb{H}_{\kappa_0}^\varphi \cap G_b \in \mathcal{J}_{\kappa_0^*} = \mathcal{J}_a$ since $\kappa_0^* = a$ by (30b). As a consequence, as $\mathbb{H}_{\kappa_0}^\varphi \cap G_b \in \mathcal{J}_a$ and G_a is an atom of \mathcal{J}_a , we have either $G_a \cap \mathbb{H}_{\kappa_0}^\varphi \cap G_b = \emptyset$ or $G_a \subset \mathbb{H}_{\kappa_0}^\varphi \cap G_b$ by (34). Since $\{h', h''\} \subset G_a$ by assumption, where $G_a \subset \mathbb{H}_{\kappa_0}^\varphi$ by (30b), we conclude that either $\{h', h''\} \subset G_b$ or $\{h', h''\} \cap G_b = \emptyset$.

- We consider an atom $G_a \in \langle \mathcal{J}_a \rangle$, and we finally prove that $\Phi_{\kappa_0}(h, u_a)$ in (30a) takes the same (set) value for any $h \in G_a$. For this purpose, we consider h' and h'' which belong to the atom G_a of \mathcal{J}_a , that is, $\{h', h''\} \subset G_a$, and we show that $\lambda \in \Phi_{\kappa_0}(h', u_a) \implies \lambda \in \Phi_{\kappa_0}(h'', u_a)$.

Thus, we take $\lambda \in \Phi_{\kappa_0}(h', u_a)$ in (30a) — that is, $\lambda \in \Lambda^p$ is such that $h' \in \mathbb{H}_{\kappa_0}^\varphi$, $\lambda_a(h') = u_a$ and $\lambda_{\|\kappa_0^-\| \cap \mathbb{A}^p}(h') = h'_{\|\kappa_0^-\| \cap \mathbb{A}^p}$ — and we are going to prove in three steps that $h'' \in \mathbb{H}_{\kappa_0}^\varphi$, $\lambda_a(h'') = u_a$ and $\lambda_{\|\kappa_0^-\| \cap \mathbb{A}^p}(h'') = h''_{\|\kappa_0^-\| \cap \mathbb{A}^p}$.

- First, we have that $h'' \in \mathbb{H}_{\kappa_0}^\varphi$ since $\{h', h''\} \subset G_a$ by assumption, and $G_a \subset \mathbb{H}_{\kappa_0}^\varphi$ by (30b).
- Second, since $\{h', h''\} \subset G_a$ by assumption and since the strategy λ_a is \mathcal{J}_a -measurable, we have that $\lambda_a(h') = \lambda_a(h'')$ by (4b), hence $\lambda_a(h'') = u_a$ since $\lambda_a(h') = u_a$ by assumption.
- Third, we have shown that, for any $b \in \|\kappa_0^-\| \cap \mathbb{A}^p$ and any atom G_b of \mathcal{J}_b , we have either $\{h', h''\} \subset G_b$ or $\{h', h''\} \cap G_b = \emptyset$ by (30d). As a consequence, the pair $\{h', h''\}$ is included in one of the atoms that make the partition $\langle \mathcal{J}_b \rangle$. Therefore, we obtain that $\lambda_b(h') = \lambda_b(h'')$ by (4b), hence $\lambda_{\|\kappa_0^-\| \cap \mathbb{A}^p}(h') = \lambda_{\|\kappa_0^-\| \cap \mathbb{A}^p}(h'')$, using the notation (9d) for $\lambda_{\mathbb{B}}$, with $\mathbb{B} = \|\kappa_0^-\| \cap \mathbb{A}^p$. As we had obtained in (30c) that $h'_{\|\kappa_0^-\| \cap \mathbb{A}^p} = h''_{\|\kappa_0^-\| \cap \mathbb{A}^p}$, we conclude that $\lambda_{\|\kappa_0^-\| \cap \mathbb{A}^p}(h'') = \lambda_{\|\kappa_0^-\| \cap \mathbb{A}^p}(h') = h'_{\|\kappa_0^-\| \cap \mathbb{A}^p} = h''_{\|\kappa_0^-\| \cap \mathbb{A}^p}$.

As a consequence, we have just proved that $\lambda \in \Phi_{\kappa_0}(h'', u_a)$, hence that $\Phi_{\kappa_0}(h', u_a) = \Phi_{\kappa_0}(h'', u_a)$ whenever $\{h', h''\} \subset G_a$.

- Let G_a be an atom of \mathcal{J}_a . Finally, since $\Phi_{\kappa}(h, u_a)$ in (30a) takes the same (set) value for any $h \in G_a$, the expression (29) takes the same value for any $h \in G_a$, and thus the function $h \in \mathbb{H} \mapsto \tilde{\mu}_a^p(\{u_a\} | h)$ is \mathcal{J}_a -measurable.

This ends the proof. □

4.3 Kuhn's equivalence theorem for causal finite games in intrinsic form

Finally, we give a statement and a proof of Kuhn's equivalence theorem for games in intrinsic form.

Theorem 16. *We consider a causal finite W-game (see Definition 7), and a player $p \in P$ supposed to enjoy perfect recall (see Definition 14).*

Then, for any mixed W-strategy $\mu^p \in \Delta(\Lambda^p)$, there exists a product-mixed W-strategy $\pi^p = (\pi_a^p)_{a \in \mathbb{A}^p} \in \prod_{a \in \mathbb{A}^p} \Delta(\Lambda_a)$, as in Definition 10, such that⁴

$$\mathbb{Q}_{(\mu^p, \mu^{-p})}^\omega = \mathbb{Q}_{(\pi^p, \mu^{-p})}^\omega, \quad \forall \mu^{-p} \in \prod_{p' \neq p} \Delta(\Lambda^{p'}), \quad \forall \omega \in \Omega, \quad (31)$$

where the probability distribution $\mathbb{Q}_\mu^\omega \in \Delta(\prod_{b \in \mathbb{A}} \mathbb{U}_b)$ has been defined in (22).

Proof. The proof is in three steps.

- First, as all the assumptions of Proposition 15 are satisfied, there exists a behavioral W-strategy $\tilde{\mu}^p = (\tilde{\mu}_a^p)_{a \in \mathbb{A}^p}$, as in Definition 11, which satisfies (29). By Proposition 13, we define the product-mixed W-strategy $\pi^p = \tilde{\mu}^p$, that is, with the property (25) that, for any agent $a \in \mathbb{A}^p$,

$$\pi_a^p \{ \lambda_a \in \Lambda_a; \lambda_a(h) = u_a \} = \tilde{\mu}_a^p(\{u_a\} | h), \quad \forall u_a \in \mathbb{U}_a, \quad \forall h \in \mathbb{H}. \quad (32a)$$

- Second, we prove that⁵

$$\begin{aligned} & \mu^p \{ (\lambda_a)_{a \in \mathbb{A}^p} \in \Lambda^p; \lambda_a(h) = h_a, \quad \forall a \in \mathbb{A}^p \} \\ & = \pi^p \{ (\lambda_a)_{a \in \mathbb{A}^p} \in \Lambda^p; \lambda_a(h) = h_a, \quad \forall a \in \mathbb{A}^p \}, \quad \forall h \in \mathbb{H}. \end{aligned} \quad (32b)$$

In what follows, we consider a configuration $h \in \mathbb{H}$ and the total ordering $\rho = \varphi(h) \in \Sigma^{|\mathbb{A}|}$. We label the set \mathbb{A}^p of agents of the player p by the stage at which each of them plays as follows:

$$\mathbb{A}^p = \{ \rho(j_1), \dots, \rho(j_N) \} \text{ with } j_1 < \dots < j_N. \quad (32c)$$

⁴See Footnote 2 for the abuse of notation $\pi^p = \otimes_{a \in \mathbb{A}^p} \pi_a^p$.

⁵See Footnote 2 for the abuse of notation $\pi^p = \otimes_{a \in \mathbb{A}^p} \pi_a^p$.

With this, we have⁶

$$\begin{aligned}
& \mu^p \{ (\lambda_a)_{a \in \mathbb{A}^p} \in \Lambda^p ; \lambda_a(h) = h_a , \forall a \in \mathbb{A}^p \} \\
&= \mu^p \{ (\lambda_{\rho(j_k)})_{k \in \llbracket N \rrbracket} \in \prod_{k=1}^N \Lambda_{\rho(j_k)} ; \lambda_{\rho(j_k)}(h) = h_{\rho(j_k)} , \forall k \in \llbracket N \rrbracket \} \\
& \hspace{15em} \text{(as } \mathbb{A}^p = \{ \rho(j_1), \dots, \rho(j_N) \} \text{ by (32c))} \\
&= \prod_{n=1}^N \frac{\mu^p \{ (\lambda_{\rho(j_k)})_{k \in \llbracket n \rrbracket} \in \prod_{k=1}^n \Lambda_{\rho(j_k)} ; \lambda_{\rho(j_k)}(h) = h_{\rho(j_k)} , \forall k \in \llbracket n \rrbracket \}}{\mu^p \{ (\lambda_{\rho(j_k)})_{k \in \llbracket n-1 \rrbracket} \in \prod_{k=1}^{n-1} \Lambda_{\rho(j_k)} ; \lambda_{\rho(j_k)}(h) = h_{\rho(j_k)} , \forall k \in \llbracket n-1 \rrbracket \}}
\end{aligned}$$

where, if the smaller term (the one to be found two equality lines above) is zero, every fraction is supposed to take the value zero, and, if the smaller term is positive, so are all the terms and no denominator is zero

$$= \prod_{n=1}^N \tilde{\mu}_{\rho(j_n)}^p(\{h_{\rho(j_n)}\} | h)$$

by (29), because $\|\psi_{j_n}(\rho)\| \cap \mathbb{A}^p = \{\rho(j_n)\} \cup (\|\psi_{j_{n-1}}(\rho)\| \cap \mathbb{A}^p)$ by definition (16b) of the restriction mapping ψ , and by definition of the sequence $j_1 < \dots < j_N$ in (32c), which is such that $\mathbb{A}^p = \{\rho(j_1), \dots, \rho(j_N)\}$

$$\begin{aligned}
&= \prod_{n=1}^N \pi_{\rho(j_n)}^p \{ \lambda_{\rho(j_n)} \in \Lambda_{\rho(j_n)} ; \lambda_{\rho(j_n)}(h) = h_{\rho(j_n)} \} \hspace{5em} \text{(by (32a))} \\
&= \left(\bigotimes_{n=1}^N \pi_{\rho(j_n)}^p \right) \{ (\lambda_{\rho(j_k)})_{k \in \llbracket N \rrbracket} \in \prod_{k=1}^N \Lambda_{\rho(j_k)} ; \lambda_{\rho(j_k)}(h) = h_{\rho(j_k)} , \forall k \in \llbracket N \rrbracket \} \\
& \hspace{15em} \text{(by definition of the product probability)} \\
&= \left(\bigotimes_{n=1}^N \pi_{\rho(j_n)}^p \right) \{ (\lambda_a)_{a \in \mathbb{A}^p} \in \Lambda^p ; \lambda_a(h) = h_a \} \quad \text{(as } \mathbb{A}^p = \{ \rho(j_1), \dots, \rho(j_N) \} \text{ in (32c))}
\end{aligned}$$

Thus, we have proved (32b).

- Third, for any configuration $h = (\omega, (u_b)_{b \in \mathbb{A}}) \in \mathbb{H}$, we have

⁶Using the notation $\llbracket n \rrbracket = \{1, \dots, n\}$ to shorten some expressions.

$$\begin{aligned}
\mathbb{Q}_\mu^\omega((u_b)_{b \in \mathbb{A}}) &= \left(\bigotimes_{p \in P} \mu^p \right) \left(M(\omega, \cdot)^{-1}((u_b)_{b \in \mathbb{A}}) \right) && \text{(by definition (22a) of } \mathbb{Q}_\mu^\omega) \\
&= \left(\bigotimes_{p \in P} \mu^p \right) \{ \lambda \in \Lambda ; M(\omega, \lambda) = (u_b)_{b \in \mathbb{A}} \} \\
&&& \text{(by definition of a pushforward probability)} \\
&= \left(\bigotimes_{p \in P} \mu^p \right) \{ \lambda \in \Lambda ; \lambda_a(\omega, (u_b)_{b \in \mathbb{A}}) = u_a , \forall a \in \mathbb{A} \} \\
&&& \text{(by (22b) and (15a) which define the mapping } M(\omega, \cdot)) \\
&= \prod_{p \in P} \mu^p \{ (\lambda_a)_{a \in \mathbb{A}^p} \in \Lambda^p ; \lambda_a(\omega, (u_b)_{b \in \mathbb{A}}) = u_a , \forall a \in \mathbb{A}^p \}
\end{aligned}$$

by definition of the product probability $\bigotimes_{p \in P} \mu^p$ on the product space $\prod_{p \in P} \Lambda^p$

$$= \prod_{p \in P} \mu^p \{ (\lambda_a)_{a \in \mathbb{A}^p} \in \Lambda^p ; \lambda_a(h) = h_a , \forall a \in \mathbb{A}^p \}$$

(because $h = (\omega, (u_b)_{b \in \mathbb{A}})$ and where we have used notation (2c))

$$\begin{aligned}
&= \prod_{p' \neq p} \mu^{p'} \{ (\lambda_a)_{a \in \mathbb{A}^{p'}} \in \Lambda^{p'} ; \lambda_a(h) = h_a , \forall a \in \mathbb{A}^{p'} \} \\
&\quad \times \mu^p \{ (\lambda_a)_{a \in \mathbb{A}^p} \in \Lambda^p ; \lambda_a(h) = h_a , \forall a \in \mathbb{A}^p \} \\
&\quad \text{(where we have singled out the player } p) \\
&= \prod_{p' \neq p} \mu^{p'} \{ (\lambda_a)_{a \in \mathbb{A}^{p'}} \in \Lambda^{p'} ; \lambda_a(h) = h_a , \forall a \in \mathbb{A}^{p'} \} \\
&\quad \times \pi^p \{ (\lambda_a)_{a \in \mathbb{A}^p} \in \Lambda^p ; \lambda_a(h) = h_a , \forall a \in \mathbb{A}^p \} && \text{(by (32b))} \\
&= \mathbb{Q}_{(\pi^p, \mu^{-p})}^\omega((u_b)_{b \in \mathbb{A}}) ,
\end{aligned}$$

by reverting to the top equality with μ^p replaced by π^p .

This ends the proof. \square

5 Discussion

Most games in extensive form are formulated on a tree. However, whereas trees are perfect to follow step by step how a game is played, they can be delicate to manipulate when information sets are added and must satisfy restrictive axioms to comply with the tree structure [1, 4, 5]. In this paper, we have introduced the notion of games in intrinsic form, where the tree structure is replaced with a product structure, more amenable to mathematical analysis. For this, we have adapted Witsenhausen's intrinsic model — a model with Nature, agents

and their decision sets, and where information is represented by σ -fields — to games. In contrast to games in extensive form formulated on a tree, Witsenhausen’s intrinsic games (W-games) do not require an explicit description of the play temporality. Not having a hardcoded temporal ordering makes mathematical representations more intrinsic.

As part of a larger research program, we have focused here on Kuhn’s equivalence theorem. For this purpose, we have defined the property of perfect recall for a player of a causal W-game (that is, without referring to a tree structure), and we have introduced three different definitions of “randomized” strategies in the W-games setting — mixed, product-mixed and behavioral. Then, we have shown that, under perfect recall for a player, any of her possible mixed strategies can be replaced by a behavioral strategy, which is the statement of Kuhn’s equivalence theorem. Moreover, we have shown that any of her possible mixed strategies can also be replaced by a product-mixed strategy, that is, a mixed strategy under which her executive agents are probabilistically independent.

We add to the existing literature on extensive games representation by proposing a representation that is more general than the tree-based ones as, for instance, it allows to describe noncausal situations. Indeed, Witsenhausen showed that there are noncausal W-models that yet are solvable.

Furthermore, our paper illustrates that the intrinsic form is well equipped to handle proofs with mathematical formulas, without resorting to tree-based arguments that can be cumbersome when handling information. We hence believe that the intrinsic form constitutes a new valuable tool for the analysis of games with information.

The current work is the first output of a larger research program that addresses games in intrinsic form. We are currently working on the embedding of tree-based games in extensive form into W-games (by a mapping that associates each information set with an agent), and on the restricted class of W-games that can be embedded in tree-based games. Further research includes extensions to measurable decision sets, and to infinite number of agents or players. We will also investigate what can be said about subgame perfect equilibria and backward induction, as well as Bayesian games.

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A Background on fields, atoms and partitions

In this paper, we present the intrinsic model of Witsenhausen [10, 6] but with finite sets rather than with infinite ones as in the original exposition. Witsenhausen used the language of σ -fields⁷ to handle the concept of information in control theory. Because sets are finite, we consider a restricted subclass of σ -fields. In what follows, \mathbb{D} is a finite set.

⁷Recall that a σ -field over the set \mathbb{D} is a subset $\mathcal{D} \subset 2^{\mathbb{D}}$, containing \mathbb{D} , and which is stable under complementation and countable union.

An *algebra*, or a *field*, over the finite set \mathbb{D} is a subset $\mathcal{D} \subset 2^{\mathbb{D}}$, containing \mathbb{D} , and which is stable under complementation and union. The trivial field over the finite set \mathbb{D} is the field $\{\emptyset, \mathbb{D}\}$. The *complete field* over the finite set \mathbb{D} is the field $2^{\mathbb{D}}$.

An *atom* of the field \mathcal{D} (over the finite set \mathbb{D}) is a minimal element for the inclusion \subset , that is, an atom is a nonempty subset $D \in \mathcal{D}$ such that $K \in \mathcal{D}$ and $K \subset D$ imply that $K = \emptyset$ or $K = D$. We denote by $\langle \mathcal{D} \rangle$ the set of atoms of the field \mathcal{D} :

$$\langle \mathcal{D} \rangle = \{D \in \mathcal{D} \setminus \{\emptyset\}; (K \in \mathcal{D} \text{ and } K \subset D) \Rightarrow (K = \emptyset \text{ or } K = D)\}. \quad (33)$$

For instance, a complete field has the singletons for atoms. It can be shown that the atoms of \mathcal{D} form a partition of \mathbb{D} , that is, they consist of mutually disjoint nonempty subsets whose union is \mathbb{D} [6, Proposition 3.18]. As a consequence, any element of the field \mathcal{D} is necessarily written as the union of atoms that it contains, and we have the useful property

$$(K \in \mathcal{D} \text{ and } D \in \langle \mathcal{D} \rangle) \Rightarrow (K \cap D = \emptyset \text{ or } K \subset D). \quad (34)$$

Consider two fields \mathcal{D} and \mathcal{D}' over the finite set \mathbb{D} . We say that the field \mathcal{D} is *finer* than the field \mathcal{D}' if $\mathcal{D} \supset \mathcal{D}'$ (notice the reverse inclusion). We also say that \mathcal{D}' is a *subfield* of \mathcal{D} . As an illustration, the complete field is finer than any field or, equivalently, any field is a subfield of the complete field.

The *least upper bound* of two fields \mathcal{D} and \mathcal{D}' , denoted by $\mathcal{D} \vee \mathcal{D}'$, is the smallest field that contains \mathcal{D} and \mathcal{D}' . The atoms of $\mathcal{D} \vee \mathcal{D}'$ are all the nonempty intersections between an atom of \mathcal{D} and an atom of \mathcal{D}' . The least upper bound of two fields is finer than any of the two.

Consider a field \mathcal{D} over the finite set \mathbb{D} , and a field \mathcal{D}' over the finite set \mathbb{D}' . The *product field* $\mathcal{D} \otimes \mathcal{D}'$ is the smallest field, over the finite product set $\mathbb{D} \times \mathbb{D}'$, that contains all the rectangles, that is, that contains all the products of an element of \mathcal{D} with an element of \mathcal{D}' .

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