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1 **HIGH ORDER HOMOGENIZATION OF THE STOKES SYSTEM IN**
2 **A PERIODIC POROUS MEDIUM***

3 FLORIAN FEPPON[†]

4 **Abstract.** We derive high order homogenized models for the incompressible Stokes system in
5 a cubic domain filled with periodic obstacles. These models have the potential to unify the three
6 classical limit problems (namely the “unchanged” Stokes system, the Brinkman model, and the
7 Darcy’s law) corresponding to various asymptotic regimes of the ratio $\eta \equiv a_\varepsilon/\varepsilon$ between the radius
8 a_ε of the holes and the size ε of the periodic cell. What is more, a novel, rather surprising feature
9 of our higher order effective equations is the occurrence of odd order differential operators when the
10 obstacles are not symmetric. Our derivation relies on the method of two-scale power series expansions
11 and on the existence of a “criminal” ansatz, which allows to reconstruct the oscillating velocity and
12 pressure $(\mathbf{u}_\varepsilon, p_\varepsilon)$ as a linear combination of the derivatives of their formal average $(\mathbf{u}_\varepsilon^*, p_\varepsilon^*)$ weighted
13 by suitable corrector tensors. The formal average $(\mathbf{u}_\varepsilon^*, p_\varepsilon^*)$ is itself the solution to a formal, infinite
14 order homogenized equation, whose truncation at any finite order is in general ill-posed. Inspired
15 by the variational truncation method of [53, 27], we derive, for any $K \in \mathbb{N}$, a well-posed model of
16 order $2K + 2$ which yields approximations of the original solutions with an error of order $O(\varepsilon^{K+3})$
17 in the L^2 norm. Furthermore, the error improves up to the order $O(\varepsilon^{2K+4})$ if a slight modification
18 of this model remains well-posed. Finally, we find asymptotics of all homogenized tensors in the low
19 volume fraction limit $\eta \rightarrow 0$ and in dimension $d \geq 3$. This allows us to obtain that our effective
20 equations converge coefficient-wise to either of the Brinkman or Darcy regimes which arise when η
21 is respectively equivalent, or greater than the critical scaling $\eta_{\text{crit}} \sim \varepsilon^{2/(d-2)}$.

22 **Key words.** Homogenization, higher order models, porous media, Stokes system, strange term.

23 **AMS subject classifications.** 35B27, 76M50, 35J30

24 **1. Introduction.** This article is concerned with the high order homogenization
25 of the Stokes system in a periodic porous medium. Let $D := (0, L)^d$ be a d -dimensional
26 box filled with periodic obstacles $\omega_\varepsilon := \varepsilon(\mathbb{Z}^d + \eta T) \cap D$ (the setting is illustrated on
27 Figure 1). The parameter ε denotes the size of the periodic cell, it is equal to $\varepsilon := L/N$
28 where $N \in \mathbb{N}$ is a large integer and L is the length of the box. The parameter η is the
29 scaling ratio between the radius $a_\varepsilon := \eta\varepsilon$ of the obstacles and the length ε of the cells.
30 The total fluid domain is denoted by $D_\varepsilon := D \setminus \overline{\omega_\varepsilon}$ and it is assumed to be connected.
31 $P = (0, 1)^d$ is the unit cell and $Y = P \setminus \eta T$ denotes its fluid component.

32 We consider $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H^1(D_\varepsilon, \mathbb{R}^d) \times L^2(D_\varepsilon)/\mathbb{R}$ the solution to the Stokes system

$$33 \quad (1.1) \quad \begin{cases} -\Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon = \mathbf{f} & \text{in } D_\varepsilon \\ \operatorname{div}(\mathbf{u}_\varepsilon) = 0 \\ \mathbf{u}_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon \\ \mathbf{u}_\varepsilon & \text{is } D\text{-periodic,} \end{cases}$$

34 where $\mathbf{f} \in C_{\text{per}}^\infty(D, \mathbb{R}^d)$ (and all its derivatives) is a smooth, D -periodic right hand-
35 side. The goal of this paper is to derive high order effective models for (1.1); i.e. a
36 family of well-posed partial differential equations posed in the homogeneous domain
37 D (without the holes) and whose solutions approximate the macroscopic behavior of
38 $(\mathbf{u}_\varepsilon, p_\varepsilon)$ at any desired order of accuracy in ε as $\varepsilon \rightarrow 0$.

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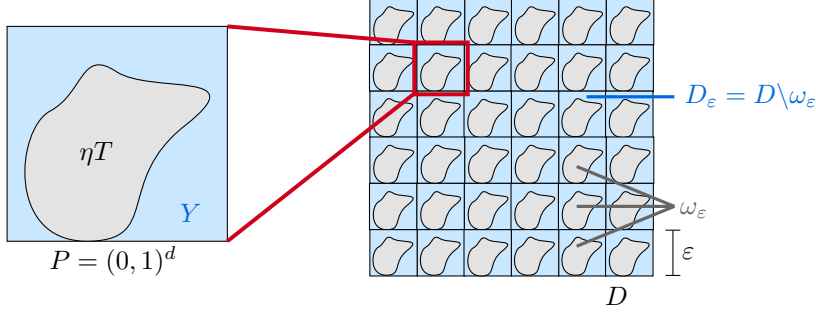


FIG. 1. The perforated domain $D_\varepsilon = D \setminus \omega_\varepsilon$ and the unit cell $Y = P \setminus (\eta T)$.

39 The literature [52, 47, 29, 5, 7, 4, 8] describes the occurrence of different asymp-
 40 totic regimes depending on how the size $a_\varepsilon = \eta\varepsilon$ of the holes compares to the critical
 41 size $\sigma_\varepsilon := \varepsilon^{d/(d-2)}$ in dimension $d \geq 3$ (if $d = 2$, then these regimes depend on
 42 how $\log(a_\varepsilon)$ compares to $-\varepsilon^{-2}$, see [7]). In loose mathematical terms, these can be
 43 summarized as follows (see e.g. [5, 7] for the precise statements):

- 44 • if $a_\varepsilon = o(\sigma_\varepsilon)$, then the holes have no effect and $(\mathbf{u}_\varepsilon, p_\varepsilon)$ converges as $\varepsilon \rightarrow 0$
 45 to the solution (\mathbf{u}, p) of the Stokes equation in the homogeneous domain D :

$$46 \quad (1.2) \quad \begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } D \\ \operatorname{div}(\mathbf{u}) = 0 \\ \mathbf{u} \text{ is } D\text{-periodic.} \end{cases}$$

- 47 • if $a_\varepsilon = c\sigma_\varepsilon$ for a constant $c > 0$, then $(\mathbf{u}_\varepsilon, p_\varepsilon)$ converges as $\varepsilon \rightarrow 0$ to the
 48 solution (\mathbf{u}, p) of the Brinkman equation

$$49 \quad (1.3) \quad \begin{cases} -\Delta \mathbf{u} + cF\mathbf{u} + \nabla p = \mathbf{f} & \text{in } D \\ \operatorname{div}(\mathbf{u}) = 0 \\ \mathbf{u} \text{ is } D\text{-periodic,} \end{cases}$$

50 where the so-called *strange term* $cF\mathbf{u}$ involves a symmetric positive definite
 51 $d \times d$ matrix F which can be computed by means of an exterior problem in
 52 $\mathbb{R}^d \setminus T$ (see [4] and section 5).

- 53 • if $\sigma_\varepsilon = o(a_\varepsilon)$ and $a_\varepsilon = \eta\varepsilon$ with $\eta \rightarrow 0$ as $\varepsilon \rightarrow 0$, then the holes are “large”
 54 and $(a_\varepsilon^{d-2}\varepsilon^{-d}\mathbf{u}_\varepsilon, p_\varepsilon)$ converges to the solution (\mathbf{u}, p) of the Darcy problem

$$55 \quad (1.4) \quad \begin{cases} F\mathbf{u} + \nabla p = \mathbf{f} & \text{in } D \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in } D \\ \mathbf{u} \text{ is } D\text{-periodic,} \end{cases}$$

56 where F is the same symmetric positive definite $d \times d$ matrix as in (1.3).

- 57 • if $a_\varepsilon = \eta\varepsilon$ with the ratio η fixed, then $(\varepsilon^{-2}\mathbf{u}_\varepsilon, p_\varepsilon)$ converges to the solution
 58 (\mathbf{u}, p) of the Darcy problem

$$59 \quad (1.5) \quad \begin{cases} M^0\mathbf{u} + \nabla p = \mathbf{f} & \text{in } D \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in } D \\ \mathbf{u} \text{ is } D\text{-periodic,} \end{cases}$$

60 where M^0 is another positive symmetric $d \times d$ matrix (which depends on η).
 61 Furthermore $M^0/|\log(\eta)| \rightarrow F$ if $d = 2$, and $M^0/\eta^{d-2} \rightarrow F$ (if $d \geq 3$) when
 62 $\eta \rightarrow 0$, so that there is a continuous transition from (1.5) to (1.4), see [6].

63 One of the long-term motivations driving this work is the need to lay down theo-
 64 retical material that would allow to optimize the design of fluid systems by homog-
 65 enization methods similar to those available in the context of mechanical structures
 66 [21, 20, 10, 50, 14]. To date, the Brinkman [24, 25, 30] and the Darcy models [56, 51]
 67 are commonly used by topology optimization algorithms in order to conveniently in-
 68 terpolate the physics of the fluid at intermediate “gray” regions featuring locally a
 69 mixture of fluid and solid. However, the above conclusions imply that these models
 70 are consistent only in specific ranges of obstacle sizes a_ε : the Brinkman model (1.3)
 71 is relevant when there are none or tiny obstacles, while the Darcy models (1.4) and
 72 (1.5) should be used at locations where the obstacles are large enough. The arising
 73 of these different regimes (1.2)–(1.5) is consequently a major obstacle towards the de-
 74 velopment of ‘de-homogenization’ methods [14, 37, 50, 39, 40] for the optimal design
 75 of fluid systems, which would enable to interpret “gray” designs as locally periodic
 76 “black and white” microstructures (featuring for instance many small tubes or thin
 77 plates).

78 It turns out that there is a continuous transition between these regimes which
 79 can be captured by higher order homogenized equations, which is the object of the
 80 present article. These higher order models are obtained by adding corrective terms
 81 scaled by increasing powers of ε to the Darcy equation (1.5); they yield more accurate
 82 approximations of $(\mathbf{u}_\varepsilon, p_\varepsilon)$ when ε is “not so small”. For a desired order $K \in \mathbb{N}$, the
 83 homogenized model of order $2K + 2$ reads

$$84 \quad (1.6) \quad \begin{cases} \sum_{k=0}^{2K+2} \varepsilon^{k-2} \mathbb{D}_K^k \cdot \nabla^k \mathbf{v}_{\varepsilon,K}^* + \nabla q_{\varepsilon,K}^* = \mathbf{f}, \\ \operatorname{div}(\mathbf{v}_{\varepsilon,K}^*) = 0, \\ \mathbf{v}_{\varepsilon,K}^* \text{ is } D\text{-periodic,} \end{cases}$$

85 where $(\mathbf{v}_{\varepsilon,K}^*, q_{\varepsilon,K}^*)$ is a high order homogenized approximation of $(\mathbf{u}_\varepsilon, p_\varepsilon)$. The coeffi-
 86 cient \mathbb{D}_K^k is a k -th order *matrix valued* tensor which can be computed by a procedure
 87 involving the resolution of cell problems; it makes $\mathbb{D}_K^k \cdot \nabla^k$ a differential operator of
 88 order k (the notation is defined in section 2 below). Finally, the high order equation
 89 (1.6) encompasses at least the Brinkman and the Darcy regimes in the sense that it
 90 converges coefficient-wise to either of (1.3) and (1.4) for the corresponding asymptotic
 91 regime of the scaling η (see Remarks 5.6 and 5.7) (the analysis of the subcritical case
 92 leading to the Stokes regime (1.2) requires more sophisticated arguments which are
 93 to be investigated in future works).

94 A rather striking feature of (1.6) is the arising of *odd* order differential operators
 95 (these vanish, however, in case the obstacle ηT is symmetric with respect to the cell
 96 axes; see Corollary 3.16). This fact is closely related to the vectorial nature of the
 97 Stokes system (1.1): the tensors \mathbb{D}_K^k are symmetric and antisymmetric valued matrices
 98 for respectively even and odd values of k . This property ensures that eventually,
 99 $\mathbb{D}_K^k \cdot \nabla^k$ is a symmetric operator for any $0 \leq k \leq 2K + 2$ (see Remark 3.12). To our
 100 knowledge, such terms have so far not been proposed in the literature seeking similar
 101 higher order corrections for the Stokes system, although these have been observed
 102 in other vectorial contexts [27, 28, 53]. Most of the available works have focused on
 103 situations with low regularity for \mathbf{f} , T and D (see [52, 5]), where the homogenization

104 process can be justified only for the approximation at the leading order in ε . Error
 105 bounds for higher order approximations of $(\mathbf{u}_\varepsilon, p_\varepsilon)$ (namely for the truncation of the
 106 ansatz (1.7) below) have been obtained in [46, 26], without relating these to effective
 107 models. A few additional works have sought corrector terms from physical modelling
 108 considerations [35, 18, 17], without considering odd order operators.

109 Our derivation is inspired from the works [19, 53, 15]; it is based on (non stan-
 110 dard) two-scale asymptotic expansions and formal operations on related power series
 111 which give rise to several families of tensors and homogenized equations at any order.
 112 We extend our previous works [34, 33] where we investigated the cases of the perfor-
 113 rated Poisson problem and of the perforated elasticity system. Expectedly, the major
 114 difficulty in extending the analysis to (1.6) is the treatment of the pressure variable
 115 p_ε and of the incompressibility constraint $\operatorname{div}(\mathbf{u}_\varepsilon) = 0$. Note that the D -periodicity
 116 assumption on \mathbf{f} and \mathbf{u}_ε is made in order to eliminate additional difficulties related
 117 to the arising of boundary layers (see [43, 22, 23, 11]).

118 The starting point of the method of two-scale expansions is to postulate an ansatz
 119 for the velocity and pressure solution $(\mathbf{u}_\varepsilon, p_\varepsilon)$:

$$120 \quad (1.7) \quad \mathbf{u}_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^{i+2} \mathbf{u}_i(x, x/\varepsilon), \quad p_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^i (p_i^*(x) + \varepsilon p_i(x, x/\varepsilon)), \quad x \in D_\varepsilon,$$

where the functions $\mathbf{u}_i(x, y)$ and $p_i(x, y)$ are P -periodic with respect to $y \in P$, and
 D -periodic with respect to $x \in D$. In (1.7), the oscillating function $p_i(x, y)$ is required
 to be of zero average with respect to y :

$$\int_Y p_i(x, y) dy = 0, \quad \forall i \geq 0.$$

121 The aim of the homogenization process is to obtain effective equations for the formal
 122 “infinite order” homogenized averages \mathbf{u}_ε^* and p_ε^* defined by

$$123 \quad (1.8) \quad \mathbf{u}_\varepsilon^*(x) := \sum_{i=0}^{+\infty} \varepsilon^{i+2} \int_Y \mathbf{u}_i(x, y) dy, \quad p_\varepsilon^*(x) := \sum_{i=0}^{+\infty} \varepsilon^i p_i^*(x), \quad x \in D.$$

124 In Proposition 3.7 below, we obtain that $(\mathbf{u}_\varepsilon^*, p_\varepsilon^*)$ solves the following formal “infinite-
 125 order” homogenized equation,

$$126 \quad (1.9) \quad \begin{cases} \sum_{k=0}^{+\infty} \varepsilon^{k-2} M^k \cdot \nabla^k \mathbf{u}_\varepsilon^* + \nabla p_\varepsilon^* = \mathbf{f}, \\ \operatorname{div}(\mathbf{u}_\varepsilon^*) = 0, \\ \mathbf{u}_\varepsilon^* \text{ is } D\text{-periodic,} \end{cases}$$

127 which involves a family of constant matrix-valued tensors $(M^k)_{k \in \mathbb{N}}$. Classically, trun-
 128 cating directly (1.9) yields, in general, an ill-posed model [12]. Several methods have
 129 been proposed to address this issue in order to obtain nonetheless well-posed higher
 130 order equations [16, 13, 1, 2, 15]. In our case, we adapt an idea from [53], whereby
 131 the coefficients \mathbb{D}_K^k are obtained thanks to a minimization principle (described in sec-
 132 tion 4) which makes indeed (1.6) well-posed. It is based on the existence of remarkable

133 identities which relate the oscillating solution $(\mathbf{u}_\varepsilon, p_\varepsilon)$ to its formal average $(\mathbf{u}_\varepsilon^*, p_\varepsilon^*)$:

$$134 \quad (1.10) \quad \begin{cases} \mathbf{u}_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^i N^i(x/\varepsilon) \cdot \nabla^i \mathbf{u}_\varepsilon^*(x) \\ p_\varepsilon(x) = p_\varepsilon^*(x) + \sum_{i=0}^{+\infty} \varepsilon^{i-1} \beta^i(x/\varepsilon) \cdot \nabla^i \mathbf{u}_\varepsilon^*(x), \end{cases} \quad \forall x \in D_\varepsilon,$$

135 where $(N^i(y))_{i \in \mathbb{N}}$ and $(\beta^i(y))_{i \in \mathbb{N}}$ are different families of respectively matrix valued
136 and vector valued P -periodic tensors (of order i). The ansatz (1.10) is substantially
137 different from (1.7); following [15], we call it “criminal” because the expansions of
138 (1.10) depend on \mathbf{u}_ε^* which is itself a formal power series in ε (eqn. (1.8)).

139 The order of accuracy at which the solution $(\mathbf{v}_{\varepsilon,K}^*, p_{\varepsilon,K}^*)$ yields an approximation
140 of the original solution $(\mathbf{u}_\varepsilon, p_\varepsilon)$ is determined by how many leading coefficients of (1.6)
141 and (1.9) coincide (Proposition 4.5). In Proposition 4.10, we show that $\mathbb{D}_K^k = M^k$ for
142 $0 \leq k \leq K$, which allows to infer error estimates of order $O(\varepsilon^{K+3})$ in the $L^2(D)$ norm.
143 It may seem disappointing that one needs to solve an equation of order $2K+2$ in order
144 to obtain approximations of order $O(\varepsilon^{K+3})$ “only”. This shortcoming is related to the
145 zero-divergence constraint: in the scalar and elasticity cases considered in [34, 33], it
146 turns out that $K+1$ extra coefficients coincide, namely $\mathbb{D}_K^k = M^k$ for $0 \leq k \leq 2K+1$,
147 which yields error estimates of order $O(\varepsilon^{2K+4})$. In the present context devoted to the
148 Stokes system (1.1), the equation obtained by substituting \mathbb{D}_K^k with M^k in (1.6) for
149 $K+1 \leq k \leq 2K+1$,

$$150 \quad (1.11) \quad \begin{cases} \varepsilon^{2K} \mathbb{D}_K^{2K+2} \cdot \nabla^{2K+2} \widehat{\mathbf{v}}_{\varepsilon,K}^* + \sum_{k=0}^{2K+1} \varepsilon^{k-2} M^k \cdot \nabla^k \mathbf{v}_{\varepsilon,K}^* + \nabla \widehat{q}_{\varepsilon,K}^* = \mathbf{f} \\ \operatorname{div}(\widehat{\mathbf{v}}_{\varepsilon,K}^*) = 0 \\ \widehat{\mathbf{v}}_{\varepsilon,K}^* \text{ is } D\text{-periodic,} \end{cases}$$

151 corresponds to applying the truncation method of [53] to the mixed variational for-
152 mulation rather than to the minimization problem associated with (1.1) (see Re-
153 mark 4.11). While the minimization principle ensures that (1.6) is well-posed, we do
154 not know whether this is the case for (1.11). However if it is, then Proposition 4.5
155 implies that (1.11) improves the approximation accuracy up to the order $O(\varepsilon^{2K+4})$.

156 The article outlines as follows. Notation conventions related to tensors and tech-
157 nical assumptions are exposed in section 2.

158 In section 3, we introduce cell problems and their solution tensors $(\mathcal{X}^k, \boldsymbol{\alpha}^k)$ which
159 allow to identify the functions \mathbf{u}_i , p_i^* and p_i in the ansatz (1.7). We show that the
160 formal average $(\mathbf{u}_\varepsilon^*, p_\varepsilon^*)$ solves the infinite order homogenized equation (1.9) involving
161 the tensors M^k . After defining the tensors $N^k(y)$ and $\beta^k(y)$, we derive the “criminal”
162 ansatz (1.10) expressing $(\mathbf{u}_\varepsilon, p_\varepsilon)$ in terms of p_ε^* and of the derivatives of \mathbf{u}_ε^* . Through-
163 out this section, a number of algebraic properties are stated for the various tensors
164 coming at play, such as the symmetry and the antisymmetry of the matrix valued
165 tensors M^k for respectively even and odd values of k , and the simplifications taking
166 place in case the obstacle ηT is symmetric with respect to the cell axes.

167 Section 4 details the truncation process of the infinite order equation (1.9) leading
168 to the well-posed model (1.6). We then provide an error analysis of the homogenized
169 approximations of $(\mathbf{u}_\varepsilon, p_\varepsilon)$ generated by our procedure: our main result is stated in

170 **Corollary 4.15** where we show that the solution $(\mathbf{v}_{\varepsilon,K}^*, q_{\varepsilon,K}^*)$ of (1.6) yield approxima-
 171 tions of $(\mathbf{u}_\varepsilon, p_\varepsilon)$ in the $L^2(D_\varepsilon)$ norm of order $K + 3$ and $K + 1$ for the velocity and
 172 the pressure respectively. We establish explicit formulas relating the coefficients \mathbb{D}_K^k
 173 to the coefficients M^k and we briefly discuss the improvement provided by (1.11) in
 174 case it is well-posed.

175 The last **section 5** investigates asymptotics of the tensors M^k in the low volume
 176 fraction limit where the scaling of the obstacle η converges to zero. Our main result is
 177 **Corollary 5.5** where we obtain the “coefficient-wise” convergence of the infinite order
 178 homogenized equation as well as the one of (1.6) towards either of the Brinkman or
 179 Darcy regimes (1.4) and (1.5) when η is respectively equivalent or greater than the
 180 critical size $\eta_{\text{crit}} \sim \varepsilon^{2/(d-2)}$, and towards the Stokes regime (1.3) for $\eta = o(\varepsilon^{2/(d-2)})$
 181 in the case $K = 0$. Although our error estimates for (1.6), are a priori not uniform
 182 in η , this suggests that our higher order model (1.6) has the potential to yield valid
 183 approximations in any regime of size of holes (at least for $K = 0$ or above the critical
 184 scale). Note that our analysis is unfortunately insufficient to establish the convergence
 185 of the high order coefficients $\varepsilon^{k-2}M^k$ with $k > 2$ towards 0 as $\eta \rightarrow 0$. Future works
 186 will investigate higher order asymptotics of the tensors M^k in the subcritical regime
 187 $\eta = o(\varepsilon^{2/(d-2)})$ which are required to establish or invalidate such a claim.

188 **2. Setting and notation conventions related to tensors.** In the sequel, we
 189 consider the following two classical assumptions for the distributions of the holes ω_ε
 190 (we recall the schematic of **Figure 1**), following [5]:

191 **(H1)** $Y = P \setminus (\eta T) \subset P$, as a subset of the unit torus (opposite matching faces
 192 of $(0, 1)^d$ are identified) is a smooth connected set with non-empty interior.

193 **(H2)** The fluid component $D_\varepsilon = D \setminus \omega_\varepsilon$ is a smooth connected set.

194 *Remark 2.1.* Assumption **(H1)** does not necessarily imply **(H2)**, see [3] for a coun-
 195 terexample. Assumption **(H1)** is not very restrictive and can easily be generalized to
 196 the case where the subset Y has m connected components with $m \in \mathbb{N}$ (see Appendix
 197 7.5.6 in [33]). Assumption **(H2)** is stronger, but is also more connected to physical
 198 applications. It forbids the existence of isolated fluid inclusions. Most of our deri-
 199 vations only assume **(H1)**. However, we rely on both assumptions **(H1)** and **(H2)** in
 200 order to obtain error bounds **section 4**, because we use some technical results of [5].

201 Below and further on, we consider scalar and vectorial functions such as

$$202 \quad (2.1) \quad \begin{array}{l} u : D \times P \rightarrow \mathbb{R} \\ (x, y) \mapsto u(x, y) \end{array}, \quad \begin{array}{l} \mathbf{u} : D \times P \rightarrow \mathbb{R}^d \\ (x, y) \mapsto \mathbf{u}(x, y) \end{array}$$

203 which are both D and P -periodic with respect to respectively the first and the second
 204 variable, and which vanish on the hole $D \times (\eta T)$. The arguments x and y of $u(x, y)$ are
 205 respectively called the “slow” and the “fast” or “oscillating” variable. With a small
 206 abuse of notation, the partial derivative with respect to the variable y_j (respectively
 207 x_j) is simply written ∂_j instead of ∂_{y_j} (respectively ∂_{x_j}) when the context is clear.

The star-“*”-symbol is used to indicate that a quantity is “macroscopic” in the
 sense that it does not depend on the fast variable x/ε ; e.g. $(\mathbf{v}_{\varepsilon,K}^*, q_{\varepsilon,K}^*)$ or $(\mathbf{u}_\varepsilon^*, p_\varepsilon^*)$ in
 (1.6) and (1.9). In the particular case where a two-variable quantity $u(x, y)$ is given
 such as (2.1), $u^*(x)$ always denotes the average of $y \mapsto u(x, y)$ with respect to the y
 variable:

$$u^*(x) := \int_P u(x, y) dy = \int_Y u(x, y) dy, \quad x \in D,$$

where the last equality is a consequence of u vanishing on $P \setminus Y = \overline{\eta T}$. When a function $\mathcal{X} : P \rightarrow \mathbb{R}$ depends only on the y variable, we find occasionally more convenient to write its cell average with the usual angle bracket symbols:

$$\langle \mathcal{X} \rangle := \int_P \mathcal{X}(y) dy.$$

208 In all what follows, unless otherwise specified, the Einstein summation convention
 209 over repeated *subscript* indices is assumed (but never on *superscript* indices). Vectors
 210 $\mathbf{b} \in \mathbb{R}^d$ are written in bold face notation.

211 The notation conventions used for tensor related operations are summarized in the
 212 nomenclature below. Some of them are not standard; they allow to avoid to system-
 213 atically write partial derivative indices (e.g. $1 \leq i_1 \dots i_k \leq d$) and to distinguish them
 214 from spatial indices (e.g. $1 \leq l, m \leq d$) associated with vector or matrix components.

215 Scalar, vector, and matrix valued tensors and their coordinates

216	\mathbf{b}	Vector of \mathbb{R}^d
217	$(b_j)_{1 \leq j \leq d}$	Coordinates of the vector \mathbf{b}
218	b^k	Scalar valued tensor of order k ($b_{i_1 \dots i_k}^k \in \mathbb{R}$ for $1 \leq i_1, \dots, i_k \leq d$)
219	\mathbf{b}^k	Vector valued tensor of order k ($\mathbf{b}_{i_1 \dots i_k}^k \in \mathbb{R}^d$ for $1 \leq i_1, \dots, i_k \leq d$)
220	B^k	Matrix valued tensor of order k ($B_{i_1 \dots i_k}^k \in \mathbb{R}^{d \times d}$ for $1 \leq i_1, \dots, i_k \leq d$)
221	$(b_j^k)_{1 \leq j \leq d}$	Coordinates of the vector valued tensor \mathbf{b}^k (b_j^k is a <i>scalar</i> tensor of order 222 k).
223	$(B_{lm}^k)_{1 \leq l, m \leq d}$	Coefficients of the matrix valued tensor B^k (B_{lm}^k is a <i>scalar</i> tensors of 224 order k).
225	$b_{i_1 \dots i_k, j}^k$	Coefficient of the vector valued tensor \mathbf{b}^k ($1 \leq i_1, \dots, i_k, j \leq d$)
226	$B_{i_1 \dots i_k, lm}^k$	Coefficients of the matrix valued tensor B^k ($1 \leq i_1, \dots, i_k, l, m \leq d$)

227 Tensor products

228 $b^p \otimes c^{k-p}$ Tensor product of scalar tensors b^p and c^{k-p} :

$$229 \quad (2.2) \quad (b^p \otimes c^{k-p})_{i_1 \dots i_k} := b_{i_1 \dots i_p}^p c_{i_{p+1} \dots i_k}^{k-p}.$$

230 $a^p \otimes \mathbf{b}^{k-p}$ Tensor product of a scalar tensors a^p and a vector valued tensor \mathbf{b}^{k-p} :

$$231 \quad (2.3) \quad (a^p \otimes \mathbf{b}^{k-p})_{i_1 \dots i_k} := a_{i_1 \dots i_p}^p \mathbf{b}_{i_{p+1} \dots i_k}^{k-p}.$$

232 $B^p \otimes C^{k-p}$ Tensor product of matrix valued tensors B^p and C^{k-p} :

$$233 \quad (2.4) \quad (B^p \otimes C^{k-p})_{i_1 \dots i_k, lm} := B_{i_1 \dots i_p, lj}^p C_{i_{p+1} \dots i_k, jm}^{k-p}.$$

234 Hence a matrix product is implicitly assumed in the notation $B^p \otimes C^{k-p}$.
 235 $B^p : C^{k-p}$ Tensor product and Frobenius product of matrix tensors B^p and C^{k-p} :

$$236 \quad (2.5) \quad (B^p : C^{k-p})_{i_1 \dots i_k} := B_{i_1 \dots i_p, lm}^p C_{i_{p+1} \dots i_k, lm}^{k-p}.$$

237 $\mathbf{b}^p \cdot \mathbf{c}^{k-p}$ Tensor product and inner product of vector valued tensors \mathbf{b}^p and \mathbf{c}^{k-p} :

$$238 \quad (2.6) \quad (\mathbf{b}^p \cdot \mathbf{c}^{k-p})_{i_1 \dots i_k} := b_{i_1 \dots i_p, m}^p c_{i_{p+1} \dots i_k, m}^{k-p}.$$

239 $B^p \cdot \mathbf{c}^{k-p}$ Tensor product of a matrix tensor B^p and a vector tensors \mathbf{c}^{k-p} :

240 (2.7) $(B^p \cdot \mathbf{c}^{k-p})_{i_1 \dots i_k, l} := B_{i_1 \dots i_p, lm}^p c_{i_{p+1} \dots i_k, m}^{k-p}$.

241 Hence a matrix-vector product is implicitly assumed in $B^p \cdot \mathbf{c}^{k-p}$.

242 **Contraction with partial derivatives**

243 $b^k \cdot \nabla^k$ Differential operator of order k associated with a scalar tensor b^k :

244 (2.8) $b^k \cdot \nabla^k := b_{i_1 \dots i_k}^k \partial_{i_1 \dots i_k}^k$.

245 $\mathbf{b}^k \cdot \nabla^k$ Differential operator of order k associated with a vector tensor \mathbf{b}^k : for
246 any smooth vector field $\mathbf{v} \in \mathcal{C}_{per}^\infty(D, \mathbb{R}^d)$,

247 (2.9) $\mathbf{b}^k \cdot \nabla^k \mathbf{v} = b_{i_1 \dots i_k, l}^k \partial_{i_1 \dots i_k}^k v_l$.

248 $B^k \cdot \nabla^k$ Differential operator of order k associated with a matrix valued tensor
249 B^k : for any smooth vector field $\mathbf{v} \in \mathcal{C}_{per}^\infty(D, \mathbb{R}^d)$,

250 (2.10) $(B^k \cdot \nabla^k \mathbf{v})_l = B_{i_1 \dots i_k, lm}^k \partial_{i_1 \dots i_k}^k v_m$.

251 **Special tensors**

252 $(e_j)_{1 \leq j \leq d}$ Vectors of the canonical basis of \mathbb{R}^d .

253 e_j Scalar valued tensor of order 1 given by $e_{j, i_1} := \delta_{i_1 j}$ (with $1 \leq j \leq d$).

254 δ_{ij} Kronecker symbol: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

I Identity tensor of order 2:

$$I_{i_1 i_2} = \delta_{i_1 i_2}.$$

255 The identity tensor is another notation for the Kronecker tensor and it
256 holds $I = e_j \otimes e_j$ with summation on the index $1 \leq j \leq d$.

J^{2k} Tensor of order $2k$ defined by:

$$J^{2k} := \overbrace{I \otimes I \otimes \dots \otimes I}^{k \text{ times}}.$$

257

258 With a small abuse of notation, we consider zeroth order tensors b^0 to be constants
259 (i.e. $b^0 \in \mathbb{R}$ if b^0 is scalar) and we still denote by $b^0 \otimes c^k := b^0 c^k$ the tensor product
260 with a k -th order tensor c^k . The same convention also applies to vector valued and
261 matrix valued tensors.

In all what follows, a k -th order tensor b^k (scalar, vector or matrix valued) truly makes sense when contracted with k partial derivatives, as in (2.8)–(2.10). Therefore all the tensors considered throughout this work are identified to their symmetrization:

$$b_{i_1 \dots i_k}^k \equiv \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} b_{i_{\sigma(1)} \dots i_{\sigma(k)}},$$

262 where \mathfrak{S}_k is the permutation group of order k . Consequently, the order in which the
263 (derivative) indices i_1, \dots, i_k are written in $b_{i_1 \dots i_k}^k$ does not matter.

264 Finally, in the whole work, we write C , C_K or $C_K(\mathbf{f})$ to denote universal constants
265 that do not depend on ε but whose values may change from lines to lines (and which
266 may depend on η or on the obstacle T).

267 *Remark 2.2.* In a limited number of places, the superscript or subscript indices
268 $p, q \in \mathbb{N}$ are used. Naturally, these are not to be confused with the pressure variables
269 p_ε or q_ε introduced in (1.1).

270 **3. Infinite order homogenized equation and criminal ansatz.** We start
 271 by identifying the two-scale structure of $(\mathbf{u}_\varepsilon, p_\varepsilon)$ which arise in the form of the ansatz
 272 (1.7). Because it helps emphasizing the arising of Cauchy products, we assume, *in*
 273 *this section only*, that the right-hand side \mathbf{f} can be formally decomposed into a power
 274 series in ε :

$$275 \quad (3.1) \quad \forall x \in D, \mathbf{f}(x) = \sum_{i=0}^{+\infty} \varepsilon^i \mathbf{f}^i(x).$$

276 **3.1. Identification of the “classical” ansatz: tensors $(\mathcal{X}^k, \alpha^k)$.** Inserting
 277 (1.7) into the Stokes system (1.1) yields the following cascade of equations:

$$278 \quad (3.2) \quad \left\{ \begin{array}{l} -\Delta_{yy} \mathbf{u}_{i+2} + \nabla_y p_{i+2} = \mathbf{f}_{i+2} - \nabla_x p_{i+2}^* - \nabla_x p_{i+1} + \Delta_{xy} \mathbf{u}_{i+1} + \Delta_{xx} \mathbf{u}_i, \\ \operatorname{div}_y(\mathbf{u}_{i+2}) = -\operatorname{div}_x(\mathbf{u}_{i+1}), \\ \mathbf{u}_{-2} = \mathbf{u}_{-1} = 0, p_{-1} = 0, \\ \mathbf{u}_i(x, \cdot) = 0 \text{ on } \partial(\eta T) \\ \mathbf{u}_i(x, \cdot) \text{ is } P\text{-periodic for any } x \in D, \\ \mathbf{u}_i(\cdot, y) \text{ is } D\text{-periodic for any } y \in P, \end{array} \right.$$

for any $i \geq -2$, where the operators $-\Delta_{yy}$, $-\Delta_{xy}$, $-\Delta_{yx}$ are defined by

$$-\Delta_{xx} = -\operatorname{div}_x(\nabla_x \cdot), \quad -\Delta_{xy} = -\operatorname{div}_x(\nabla_y \cdot) - \operatorname{div}_y(\nabla_x \cdot), \quad -\Delta_{yy} := -\operatorname{div}_y(\nabla_y \cdot).$$

279 In order to solve (3.2), we introduce a family of respectively vector valued tensors
 280 $(\mathcal{X}_j^k(y))_{1 \leq j \leq d}$ and scalar valued tensors $(\alpha_j^k(y))_{1 \leq j \leq d}$ defined by induction as the
 281 unique solutions in $H_{per}^1(Y, \mathbb{R}^d) \times L^2(Y)/\mathbb{R}$ to the following cell problems:

$$282 \quad (3.3) \quad \left\{ \begin{array}{l} -\Delta_{yy} \mathcal{X}_j^0 + \nabla_y \alpha_j^0 = \mathbf{e}_j \text{ in } Y, \\ \operatorname{div}_y(\mathcal{X}_j^0) = 0 \text{ in } Y \end{array} \right.$$

$$283 \quad (3.4) \quad \left\{ \begin{array}{l} -\Delta_{yy} \mathcal{X}_j^1 + \nabla_y \alpha_j^1 = (2\partial_l \mathcal{X}_j^0 - \alpha_j^0 \mathbf{e}_l) \otimes \mathbf{e}_l \text{ in } Y \\ \operatorname{div}_y(\mathcal{X}_j^1) = -(\mathcal{X}_j^0 - \langle \mathcal{X}_j^0 \rangle) \cdot \mathbf{e}_l \otimes \mathbf{e}_l \text{ in } Y, \end{array} \right.$$

$$284 \quad (3.5) \quad \left\{ \begin{array}{l} -\Delta_{yy} \mathcal{X}_j^{k+2} + \nabla_y \alpha_j^{k+2} = (2\partial_l \mathcal{X}_j^{k+1} - \alpha_j^{k+1} \mathbf{e}_l) \otimes \mathbf{e}_l + \mathcal{X}_j^k \otimes I \text{ in } Y \\ \operatorname{div}_y(\mathcal{X}_j^{k+2}) = -(\mathcal{X}_j^{k+1} - \langle \mathcal{X}_j^{k+1} \rangle) \cdot \mathbf{e}_l \otimes \mathbf{e}_l \text{ in } Y \end{array} \right. \quad \forall k \geq 0.$$

286 Equations (3.3)–(3.5) are supplemented with the following boundary conditions:

$$287 \quad (3.6) \quad \left\{ \begin{array}{l} \int_Y \alpha_j^k dy = 0 \\ \mathcal{X}_j^k = 0 \text{ on } \partial(\eta T) \\ (\mathcal{X}_j^k, \alpha_j^k) \text{ is } P\text{-periodic} \end{array} \right. \quad \forall k \geq 0.$$

288

Remark 3.1. In view of the notation conventions of section 2, the non bold symbols $\otimes \mathbf{e}_l$ and $\otimes I$ indicate the arising of extra partial derivatives indices. For instance, the first line of (3.5) must be understood as

$$-\Delta_{yy} \mathcal{X}_{j, i_1 \dots i_{k+2}}^{k+2} + \nabla_y \alpha_{j, i_1 \dots i_{k+2}}^{k+2} = 2\partial_{i_{k+2}} \mathcal{X}_{j, i_1 \dots i_{k+1}}^{k+1} - \alpha_{j, i_1 \dots i_{k+1}}^{k+1} \mathbf{e}_{i_{k+2}} + \mathcal{X}_{j, i_1 \dots i_k}^k \delta_{i_{k+1} i_{k+2}}.$$

We introduce the k -th order matrix valued tensors \mathcal{X}^k whose columns are the vector valued tensors (\mathcal{X}_j^k) :

$$(\mathcal{X}_{ij}^k(y))_{1 \leq i, j \leq d} := [\mathcal{X}_1^k(y) \quad \dots \quad \mathcal{X}_d^k(y)], \quad \forall y \in Y, \quad \forall k \geq 0.$$

We also denote by α^k the k -th order vector valued tensor whose coordinates are the scalar tensors α_j^k :

$$\alpha^k(y) := (\alpha_j^k(y))_{1 \leq j \leq d}, \quad \forall y \in Y, \quad \forall k \geq 0.$$

289 Following the conventions of [section 2](#), we use a star notation to denote the average
290 of respectively the tensor \mathcal{X}^k and of the vector fields \mathbf{u}_i :

$$291 \quad (3.7) \quad \mathcal{X}^{k*} := \int_Y \mathcal{X}^k(y) dy, \quad \forall k \geq 0, \quad \mathbf{u}_i^*(x) := \int_Y \mathbf{u}_i(x, y) dy, \quad \forall x \in D, \quad \forall i \geq 0.$$

292 The tensors \mathcal{X}^k and α^k enable to solve the cascade of equations [\(3.2\)](#):

293 **PROPOSITION 3.2.** *Assume [\(H1\)](#). The solutions $\mathbf{u}_i(x, y)$, $p_i(x, y)$ of the cascade*
294 *of equations [\(3.2\)](#) are given by*

$$295 \quad (3.8) \quad \begin{aligned} \mathbf{u}_i(x, y) &= \sum_{k=0}^i \mathcal{X}^k(y) \cdot \nabla^k (\mathbf{f}_{i-k}(x) - \nabla p_{i-k}^*(x)) \\ p_i(x, y) &= \sum_{k=0}^i \alpha^k(y) \cdot \nabla^k (\mathbf{f}_{i-k}(x) - \nabla p_{i-k}^*(x)), \end{aligned}$$

296 where the functions p_i^* are uniquely determined recursively as the solutions to the
297 following elliptic system: for any $i \geq 0$,

$$298 \quad (3.9) \quad \left\{ \begin{array}{l} -\operatorname{div}_x (\mathcal{X}^{0*} \nabla_x p_i^*) = -\operatorname{div}_x (\mathcal{X}^{0*} \mathbf{f}_i) \\ \quad - \sum_{k=1}^i \operatorname{div} (\mathcal{X}^{k*} \cdot \nabla^k (\mathbf{f}_{i-k} - \nabla_x p_{i-k}^*)) \text{ in } D_\varepsilon, \\ \int_D p_i^* dx = 0 \\ p_i^* \text{ is } D\text{-periodic.} \end{array} \right.$$

299 Recognizing Cauchy products, the identities [\(3.8\)](#) and [\(3.9\)](#) rewrite formally in terms
300 of equality of formal power series:

$$301 \quad (3.10) \quad \mathbf{u}_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^{i+2} \mathcal{X}^i(x/\varepsilon) \cdot \nabla^i (\mathbf{f}(x) - \nabla p_\varepsilon^*(x)),$$

$$302 \quad (3.11) \quad p_\varepsilon(x) = p_\varepsilon^*(x) + \sum_{i=0}^{+\infty} \varepsilon^{i+1} \alpha^i(x/\varepsilon) \cdot \nabla^i (\mathbf{f}(x) - \nabla p_\varepsilon^*(x)),$$

$$303 \quad (3.12) \quad \operatorname{div}(\mathbf{u}_\varepsilon^*(x)) = 0 \text{ where } \mathbf{u}_\varepsilon^*(x) = \sum_{i=0}^{+\infty} \varepsilon^{i+2} \mathcal{X}^{i*} \cdot \nabla^i (\mathbf{f}(x) - \nabla p_\varepsilon^*(x)).$$

305 *Proof.* The result is proved by induction. The case $i = -1$ is straightforward
306 thanks to the convention $\mathbf{u}_{-1} = p_{-1} = 0$. In this proof we use the short-hand

307 notation $\mathbf{h}_i(x) = \mathbf{f}_i(x) - \nabla p_i^*(x)$. Assuming (3.8) and (3.9) hold till rank $i + 1$ with
 308 $i \geq -2$, we compute, substituting (3.8) into (3.2):

$$309 \quad (3.13) \quad \begin{cases} (-\Delta_{yy}\mathbf{u}_{i+2} + \nabla_y p_{i+2})(x, y) \\ = h_{i+2,j}(x)\mathbf{e}_j + (2\partial_l \mathcal{X}_j^0(y) - \alpha_j^0(y)\mathbf{e}_l) \otimes \mathbf{e}_l \cdot \nabla h_{i+1,j}(x) \\ + \sum_{k=0}^i ((2\partial_l \mathcal{X}_j^{k+1}(y) - \alpha_j^{k+1}(y)\mathbf{e}_l) \otimes \mathbf{e}_l + \mathcal{X}_j^k(y) \otimes I) \cdot \nabla^{k+2} h_{i-k,j}(x) \\ \operatorname{div}_y(\mathbf{u}_{i+2})(x, y) = - \sum_{k=0}^{i+1} (\mathcal{X}_j^k(y) \cdot \mathbf{e}_l \otimes \mathbf{e}_l) \cdot \nabla^{k+1} h_{i+1-k,j}(x). \end{cases}$$

The system (3.13) admits a unique solution $(\mathbf{u}_{i+2}, p_{i+2})$ with $\int_Y p_{i+2}(x, y) dy = 0$ and only if the following compatibility condition (the so-called ‘‘Fredholm alternative’’) holds (for any $i \geq -1$):

$$\int_Y \operatorname{div}_y(\mathbf{u}_{i+2})(x, y) dy = - \sum_{k=0}^{i+1} [(\mathcal{X}_j^k) \cdot \mathbf{e}_l \otimes \mathbf{e}_l] \cdot \nabla^{k+1} h_{i+1-k,j}(x) = 0.$$

The above equation determines p_{i+1}^* given the values of p_k^* for $0 \leq k \leq i$:

$$(\langle \mathcal{X}_j^0 \rangle \cdot \mathbf{e}_l) \partial_l (f_{i+1,j} - \partial_j p_{i+1}^*) = - \sum_{k=1}^{i+1} [(\mathcal{X}_j^k) \cdot \mathbf{e}_l \otimes \mathbf{e}_l] \cdot \nabla^{k+1} (f_{i+1-k,j} - \partial_j p_{i+1-k}^*),$$

310 which is (3.9) at order $i + 1$. This identity allows to rewrite $\operatorname{div}_y(\mathbf{u}_{i+2})$ as

$$311 \quad (3.14) \quad \operatorname{div}_y(\mathbf{u}_{i+2})(x, y) = - \sum_{k=0}^{i+1} [(\mathcal{X}_j^k(y) - \langle \mathcal{X}_j^k \rangle) \cdot \mathbf{e}_l \otimes \mathbf{e}_l] \cdot \nabla^{k+1} h_{i+1-k,j}(x).$$

312 By linearity, (3.13) and (3.14) and the definitions of $(\mathcal{X}_j^k, \alpha_j^k)$ through the cell problems
 313 (3.3)–(3.5) imply the result at rank $i + 2$. \square

314 *Remark 3.3.* The truncation of the series (3.12) at first order yields the well-
 315 known Darcy’s law [52]. The next terms of the series have been obtained in [46, 26],
 316 at least up to the order $i = 1$.

317 *Remark 3.4.* The ansatz (3.10) is already non standard (when compared to (1.7))
 318 because it features p_ε^* which is a formal power series in ε (recall (1.8)).

319 The next proposition establishes the symmetry and antisymmetry of the matrices
 320 \mathcal{X}^{k*} (eqn. (3.7)) for respectively odd and even values of k . We note that similar
 321 identities have been found for the Poisson [34] or the wave equation [1].

322 **PROPOSITION 3.5.** *For any $k \geq 0$ and $0 \leq p \leq k$, $1 \leq i, j \leq d$, the following*
 323 *identity holds for the matrix valued tensor \mathcal{X}^{k*} :*

$$324 \quad (3.15) \quad \mathcal{X}_{ij}^{k*} = (-1)^p \int_Y ((-\Delta_{yy}\mathcal{X}_i^p + \nabla \alpha_i^p) \cdot \mathcal{X}_j^{k-p} + \nabla \alpha_j^{k-p} \cdot \mathcal{X}_i^p - \mathcal{X}_j^{k-p-1} \cdot \mathcal{X}_i^{p-1} \otimes I) dy$$

325 with $\mathcal{X}_i^{-1} = 0$ by convention. In particular, for any $k \geq 0$, \mathcal{X}^{2k*} and \mathcal{X}^{2k+1*} take
 326 values respectively in the set of $d \times d$ symmetric and antisymmetric matrices:

$$327 \quad (3.16) \quad \mathcal{X}_{ij}^{2k*} = (-1)^k \int_Y (\nabla \mathcal{X}_i^k : \nabla \mathcal{X}_j^k + \nabla \alpha_i^k \cdot \mathcal{X}_j^k + \nabla \alpha_j^k \cdot \mathcal{X}_i^k - \mathcal{X}_i^{k-1} \cdot \mathcal{X}_j^{k-1} \otimes I) dy$$

328

$$\begin{aligned}
\mathcal{X}_{ij}^{2k+1*} &= (-1)^k \int_Y (\boldsymbol{x}_i^k \cdot \nabla \boldsymbol{x}_j^k - \boldsymbol{x}_j^k \cdot \nabla \boldsymbol{x}_i^k + \alpha_i^k \boldsymbol{x}_j^k - \alpha_j^k \boldsymbol{x}_i^k) \cdot \boldsymbol{e}_l \otimes \boldsymbol{e}_l dy \\
&+ (-1)^k \int_Y (\boldsymbol{x}_j^{k-1} \cdot \boldsymbol{x}_j^k - \boldsymbol{x}_i^{k-1} \cdot \boldsymbol{x}_j^k) dy.
\end{aligned}
\tag{3.17}$$

Proof. The result holds for $p = 0$ because

$$\boldsymbol{x}_{ij}^{k*} = \int_Y \boldsymbol{x}_j^k \cdot \boldsymbol{e}_i dy = \int_Y \boldsymbol{x}_j^k \cdot (-\Delta_{yy} \boldsymbol{x}_i^0 + \nabla \alpha_i^0) dy.$$

Assuming now that (3.15) holds till rank p with $k > p \geq 0$, we prove the result at rank $p + 1$. We write, after an integration by parts and by using (3.3)–(3.5):

$$\begin{aligned}
\mathcal{X}_{ij}^{k*} &= (-1)^p \int_Y [-\boldsymbol{x}_i^p \cdot \Delta \boldsymbol{x}_j^{k-p} - \alpha_i^p \operatorname{div}(\boldsymbol{x}_j^{k-p}) - \alpha_j^{k-p} \operatorname{div}(\boldsymbol{x}_i^p) \\
&\quad - \boldsymbol{x}^{k-p-1} \cdot \boldsymbol{x}_i^{p-1} \otimes I] dy \\
&= (-1)^p \int_Y [(\boldsymbol{x}_i^p \cdot (2\partial_l \boldsymbol{x}_j^{k-p-1} - \alpha_j^{k-p-1} \boldsymbol{e}_l) \otimes \boldsymbol{e}_l + \boldsymbol{x}_j^{k-p-2} \otimes I - \nabla \alpha_j^{k-p}) \cdot \boldsymbol{x}_i^p \\
&\quad + \alpha_i^p \boldsymbol{x}_j^{k-p-1} \cdot \boldsymbol{e}_l \otimes \boldsymbol{e}_l + \alpha_j^{k-p} \boldsymbol{x}_i^{p-1} \cdot \boldsymbol{e}_l \otimes \boldsymbol{e}_l - \boldsymbol{x}_j^{k-p-1} \cdot \boldsymbol{x}_i^{p-1} \otimes I] dy \\
&= (-1)^p \int_Y [-\boldsymbol{x}_j^{k-p-1} \cdot ((2\partial_l \boldsymbol{x}_i^p - \alpha_i^p \boldsymbol{e}_l) \otimes \boldsymbol{e}_l + \boldsymbol{x}_i^{p-1} \otimes I) + \alpha_j^{k-p-1} \operatorname{div}(\boldsymbol{x}_i^{p+1}) \\
&\quad - \nabla \alpha_j^{k-p} \cdot \boldsymbol{x}_i^p - \alpha_j^{k-p} \operatorname{div}(\boldsymbol{x}_i^p) + \boldsymbol{x}_j^{k-p-2} \cdot \boldsymbol{x}_i^p \otimes I] dy \\
&= (-1)^p \int_Y [-\boldsymbol{x}_j^{k-p-1} \cdot (-\Delta_{yy} \boldsymbol{x}_i^{p+1} + \nabla \alpha_i^{p+1}) \\
&\quad - \nabla \alpha_j^{k-p-1} \cdot \boldsymbol{x}_i^{p+1} + \boldsymbol{x}_j^{k-p-2} \cdot \boldsymbol{x}_i^p \otimes I] dy,
\end{aligned}$$

whence (3.15) at rank $p + 1$. Finally, the expression (3.16) for \mathcal{X}_{ij}^{2k*} is obtained by setting $k \leftarrow 2k$ and $p \leftarrow k$ in (3.15). The expression for \mathcal{X}_{ij}^{2k+1*} is obtained by setting $k \leftarrow 2k + 1$ and $p \leftarrow k$ and performing an integration by parts. \square

3.2. Derivation of the infinite order homogenized equation and of the criminal ansatz. We now proceed on the derivation of the infinite order homogenized equation (1.9). Let us recall the classical positive definiteness of the Darcy tensor \mathcal{X}^{0*} .

COROLLARY 3.6. *Assume (H1). The matrix $\mathcal{X}^{0*} = (\mathcal{X}_{ij}^{0*})_{1 \leq i, j \leq d}$ (defined in (3.7)) is positive symmetric definite.*

Proof. See [52] or Corollary 7.8 in [33]. \square

Hence, the following definition of the tensors $(M^k)_{k \in \mathbb{N}}$ makes sense.

PROPOSITION 3.7. *Let M^k be the tensor of order k defined by induction as follows:*

$$\begin{cases} M^0 = (\mathcal{X}^{0*})^{-1} \\ M^k = -(\mathcal{X}^{0*})^{-1} \sum_{p=0}^{k-1} \mathcal{X}^{k-p*} \otimes M^p, \quad \forall k \geq 1. \end{cases}
\tag{3.18}$$

353 Then the source terms \mathbf{f}_i (eqn. (3.1)) can be expressed in terms of the averaged
 354 summands $\mathbf{u}_i^*(x)$ and $p_i^*(x)$ ((1.8) and (3.7)) through the following identity:

$$355 \quad (3.19) \quad \forall i \geq 0, \mathbf{f}_i(x) - \nabla p_i^*(x) = \sum_{k=0}^i M^k \cdot \nabla^k \mathbf{u}_{i-k}^*(x).$$

356 Recognizing a Cauchy product, (3.19) and (3.12) rewrite formally as the “infinite
 357 order” homogenized system (1.9) for the formal average $(\mathbf{u}_\varepsilon^*, p_\varepsilon^*)$ defined in (1.8).

358 *Proof.* The proof is identical to the one of Proposition 5 in [33], it amounts to
 359 average the first line of (3.8) with respect to y and to solve the resulting triangular
 360 system determining $\mathbf{f}_{i-k} - \nabla p_{i-k}^*$ in terms of \mathbf{u}_i^* . \square

361 The definition (3.18) essentially states that $\sum_{k=0}^{+\infty} \varepsilon^{k-2} M^k \cdot \nabla^k$ is the inverse of the
 362 formal power series $\sum_{k=0}^{+\infty} \varepsilon^{k+2} \mathcal{X}^{k*} \cdot \nabla^k$. In this spirit, it is even possible to write a
 363 fully explicit formula (see [34], Proposition 6 and Remark 2 for the proof):

364 PROPOSITION 3.8. For any $k \geq 1$, the tensor M^k is explicitly given by

$$365 \quad (3.20) \quad M^k = \sum_{p=1}^k (-1)^p \sum_{\substack{i_1 + \dots + i_p = k \\ 1 \leq i_1, \dots, i_p \leq k}} (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_1*} \otimes \dots \otimes (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_p*} \otimes (\mathcal{X}^{0*})^{-1}.$$

366 We now introduce matrix valued tensors N^k and vector valued tensors β^k which allow
 367 to obtain the “criminal ansatz” (1.10) expressing the velocity and pressure $(\mathbf{u}_\varepsilon, p_\varepsilon)$ in
 368 terms of their formal average $(\mathbf{u}_\varepsilon^*, p_\varepsilon^*)$.

PROPOSITION 3.9. Let N^k and β^k be respectively the k -th order matrix valued
 and vector valued tensors defined for any $k \in \mathbb{N}$ by

$$N^k(y) := \sum_{p=0}^k \mathcal{X}^{k-p}(y) \otimes M^p, \quad \beta^k(y) := \sum_{p=0}^k (-1)^p M^p \cdot \alpha^{k-p}(y), \quad \forall y \in Y.$$

369 Then the summands $\mathbf{u}_i(x, y)$ and $p_i(x, y)$ of (3.10) and (3.11) are given for any $i \geq 0$
 370 in terms of the averages \mathbf{u}_i^* (eqn. (3.7)) and p_i^* as follows:

$$371 \quad (3.21) \quad \mathbf{u}_i(x, y) = \sum_{k=0}^i N^k(y) \cdot \nabla^k \mathbf{u}_{i-k}^*(x), \quad p_i(x, y) = \sum_{k=0}^i \beta^k(y) \cdot \nabla^k \mathbf{u}_{i-k}^*(x).$$

372 Recognizing Cauchy products, the identities (3.21) can be rewritten formally as the
 373 “criminal ansatz” (1.10).

Proof. The result is obtained by substituting (3.19) into (3.8) which yields

$$\begin{aligned} \mathbf{u}_i(x, y) &= \sum_{p=0}^i \sum_{q=0}^{i-p} \mathcal{X}^p(y) \otimes M^q \cdot \nabla^{p+q} \mathbf{u}_{i-p-q}^*(x) \\ &= \sum_{k=0}^i \sum_{p=0}^k (\mathcal{X}^p(y) \otimes M^{p-k}) \cdot \nabla^k \mathbf{u}_{i-k}^*(x) \quad (\text{change of indices } k = p + q) \end{aligned}$$

from where the identity (3.21) for $\mathbf{u}_i(x, y)$ follows by inverting the summation. Simi-
 larly, we obtain

$$p_i(x, y) = \sum_{k=0}^i \sum_{p=0}^k ((M^{p-k})^T \cdot \alpha^p(y)) \cdot \nabla^k \mathbf{u}_{i-k}^*(x),$$

374 hence (3.21) by using $(M^{p-k})^T = (-1)^{p-k} M^{p-k}$ (see Corollary 3.11 below). \square

In what follows, we denote by $(\mathbf{N}_j^k)_{1 \leq j \leq d}$ and by $(\beta_j^k)_{1 \leq j \leq d}$ respectively the column vectors and the coefficients of $N^k(y)$ and $\beta^k(y)$:

$$\forall 1 \leq i, j \leq d, \mathbf{N}_j^k := N^k \mathbf{e}_j \text{ and } \beta_j^k := \beta^k \cdot \mathbf{e}_j.$$

375 In addition, the convention $\mathbf{N}_j^{-1} = 0$ is assumed. We shall in the sequel use several
376 times the following properties of $(\mathbf{N}_j^k, \beta_j^k)$ which are dual to those of $(\boldsymbol{\chi}_j^k, \alpha_j^k)$.

377 PROPOSITION 3.10. *The k -th order tensors N^k , $(\mathbf{N}_j^k)_{1 \leq j \leq d}$, β^k and $(\beta_j^k)_{1 \leq j \leq d}$*
378 *satisfy:*

379 (i) $\int_Y N^0(y) dy = I$ and $\int_Y N^k(y) dy = 0$ for any $k \geq 1$;

380 (ii) $\int_Y \beta^k(y) dy = 0$ for any $k \geq 0$;

381 (iii) For any $k \geq -2$ and $1 \leq j \leq d$,

$$382 \quad (3.22) \quad \begin{cases} -\Delta_{yy} \mathbf{N}_j^{k+2} + \nabla \beta_j^{k+2} = (2\partial_l \mathbf{N}_j^{k+1} - \beta_j^{k+1} \mathbf{e}_l) \otimes \mathbf{e}_l + \mathbf{N}_j^k \otimes I + M^{k+2} \mathbf{e}_j, \\ \operatorname{div}(\mathbf{N}_j^{k+2}) = -(\mathbf{N}_j^{k+1} - \langle \mathbf{N}_j^{k+1} \rangle) \cdot \mathbf{e}_l \otimes \mathbf{e}_l; \end{cases}$$

383 *Proof.* (i) and (ii) are straightforward consequences of (3.18).

384 (iii) is obtained by writing, for $k \geq 0$ (implicit summation on the repeated index
385 j assumed):

$$\begin{aligned} -\Delta_{yy} \mathbf{N}_j^{k+2} + \nabla \beta_j^{k+2} &= -\Delta_{yy} \left(\sum_{p=0}^{k+2} \boldsymbol{\chi}_i^{k+2-p}(y) \otimes M_{ij}^p \right) + \nabla \left(\sum_{p=0}^{k+2} \alpha_i^{k+2-p}(y) \otimes M_{ij}^p \right) \\ &= \sum_{p=0}^k \left[(2\partial_l \boldsymbol{\chi}_i^{k+1-p} - \alpha_i^{k+1-p} \mathbf{e}_l) \otimes \mathbf{e}_l + \boldsymbol{\chi}_i^{k-p} \otimes I \right] M_{ij}^p \\ &\quad + (2\partial_l \boldsymbol{\chi}_i^0 - \alpha_i^0 \mathbf{e}_l) M_{ij}^{k+1} + M_{ij}^{k+2} \mathbf{e}_i \\ &= (2\partial_l \mathbf{N}_j^{k+1} - \beta_j^{k+1} \mathbf{e}_l) \otimes \mathbf{e}_l + \mathbf{N}_j^k \otimes I + M^{k+2} \mathbf{e}_j. \\ \operatorname{div}(\mathbf{N}_j^{k+2}) &= \sum_{p=0}^{k+2} \operatorname{div}(\boldsymbol{\chi}_i^{k+2-p}) M_{ij}^p = - \sum_{p=0}^{k+1} M_{ij}^p (\boldsymbol{\chi}_i^{k+1-p} - \langle \boldsymbol{\chi}_i^{k+1-p} \rangle) \cdot \mathbf{e}_l \otimes \mathbf{e}_l. \end{aligned}$$

386 The proof is identical for $k = -1$ and $k = -2$. \square

387 The identity (3.22) allows to infer important properties characterizing the tensors M^k
388 which are similar to those of Proposition 3.5.

COROLLARY 3.11. *For any $1 \leq p \leq k-1$, it holds*

$$M_{ij}^k = (-1)^{p+1} \int_Y ((-\Delta_{yy} \mathbf{N}_i^p + \nabla \beta_i^p) \cdot \mathbf{N}_j^{k-p} + \nabla \beta_j^{k-p} \cdot \mathbf{N}_i^p - \mathbf{N}_i^{p-1} \cdot \mathbf{N}_j^{k-p-1} \otimes I) dy.$$

389 Consequently, for any $k \geq 0$,

- M^{2k} is a symmetric matrix valued tensor, and the following identities hold:

$$M_{ij}^0 = \int_Y \nabla \mathbf{N}_i^0 : \nabla \mathbf{N}_j^0 dy,$$

$$\forall k \geq 1, M_{ij}^{2k} = (-1)^{k+1} \int_Y (\nabla \mathbf{N}_i^k : \nabla \mathbf{N}_j^k + \nabla \beta_i^k \cdot \mathbf{N}_j^k + \nabla \beta_j^k \cdot \mathbf{N}_i^k - \mathbf{N}_i^{k-1} \cdot \mathbf{N}_j^{k-1} \otimes I) dy.$$

- M^{2k+1} is an antisymmetric matrix valued tensor and it holds:

$$M_{ij}^{2k+1} = (-1)^{k+1} \int_Y (\mathbf{N}_i^k \cdot \nabla \mathbf{N}_j^k - \mathbf{N}_j^k \cdot \nabla \mathbf{N}_i^k + \beta_i^k \mathbf{N}_j^k - \beta_j^k \mathbf{N}_i^k) \cdot \mathbf{e}_l \otimes \mathbf{e}_l dy$$

$$+ (-1)^{k+1} \int_Y (\mathbf{N}_j^{k-1} \cdot \mathbf{N}_i^k - \mathbf{N}_i^{k-1} \cdot \mathbf{N}_j^k) \otimes Idy.$$

Proof. The proof is very similar to the one of [Proposition 3.5](#) and is omitted, see also [Proposition 7.34](#) in [\[33\]](#). \square

Remark 3.12. The antisymmetry of odd order tensors M^{2k+1} ensures that the associated differential operators $\varepsilon^{2k-1} M^{2k+1} \cdot \nabla^{2k+1}$ arising in the “infinite order” homogenized equation [\(1.9\)](#) are symmetric. Indeed, the antisymmetry of M^{2k+1} “compensates” the one induced by odd order derivatives which makes $M^{2k+1} \cdot \nabla^{2k+1}$ be a symmetric operator: for two vector fields $\mathbf{u} := (u_i)_{1 \leq i \leq d}$, $\mathbf{v} = (v_i)_{1 \leq i \leq d}$, it holds

$$\int_Y \mathbf{v} \cdot M^{2k+1} \cdot \nabla^{2k+1} \mathbf{u} dy = \int_Y (M_{ij}^{2k+1} \cdot \nabla^{2k+1} u_j) v_i dy = - \int_Y (M_{ij}^{2k+1} \cdot \nabla^{2k+1} v_i) u_j dy$$

$$= \int_Y (M_{ji}^{2k+1} \cdot \nabla^{2k+1} v_i) u_j dy = \int_Y \mathbf{u} \cdot M^{2k+1} \cdot \nabla^{2k+1} \mathbf{v} dy.$$

Remark 3.13. It is not completely straightforward to exhibit an instance of hole ∂T and $k \in \mathbb{N}$ for which we can actually prove that M^{2k+1} is not zero. However simple numerical evidences tend to confirm this conjecture, see section 7.4.5 in [\[33\]](#) for an example featuring $M^1 \neq 0$ in the case of the elasticity system .

3.3. Simplifications for the tensors \mathcal{X}^{k*} and M^k in case of symmetries.

In the final part of this section, we examine how the symmetries of the obstacle ηT with respect to the cell axes reflect into the coefficients of the matrix valued tensors \mathcal{X}^{k*} and M^k . Our final result is stated in [Corollary 3.16](#), which implies that odd order tensors \mathcal{X}^{2k+1} and M^{2k+1} vanish in case ηT is symmetric with respect to the cell axes. It is based on the following elementary lemma:

LEMMA 3.14. *Let $S \in \mathbb{R}^{d \times d}$ an orthogonal symmetry, i.e. $S = S^T$ and $SS = I$. The following identities hold for any smooth vector field \mathcal{X} and scalar field α :*

$$(3.23) \quad -\Delta(S\mathcal{X} \circ S) + \nabla(\alpha \circ S) = S(-\Delta\mathcal{X} + \nabla\alpha) \circ S,$$

$$(3.24) \quad \operatorname{div}(S\mathcal{X} \circ S) = \operatorname{div}(\mathcal{X}) \circ S,$$

$$(3.25) \quad \partial_i(S\mathcal{X} \circ S) = S_{ij} S(\partial_j \mathcal{X}) \circ S.$$

Proof. The first two identities are obtained by writing

$$-\Delta(S\mathcal{X} \circ S) + \nabla(\alpha \circ S) = -S\partial_{ij}\mathcal{X} \circ SS_{il}S_{jl} + S(\nabla\alpha) \circ S$$

$$= -S(\Delta\mathcal{X} + \nabla\alpha) \circ S,$$

$$\operatorname{div}(S\mathcal{X} \circ S) = \operatorname{Tr}(\nabla(S\mathcal{X} \circ S)) = \operatorname{Tr}(S(\nabla\mathcal{X}) \circ SS) = \operatorname{Tr}((\nabla\mathcal{X}) \circ S) = \operatorname{div}(\mathcal{X}) \circ S.$$

Identity [\(3.25\)](#) is an elementary consequence of the chain rule. \square

PROPOSITION 3.15. *If the cell $Y = P \setminus (\eta T)$ is invariant with respect to a symmetry S , i.e. $S(Y) = Y$, then the following identity holds for the tensors $(\mathcal{X}_l^k, \alpha_l^k)$ (defined in [\(3.3\)](#)–[\(3.5\)](#)):*

$$(3.26) \quad S\mathcal{X}_{i_1 \dots i_k, l}^k \circ S = S_{i_1 j_1} \dots S_{i_k j_k} S_{lm} \mathcal{X}_{j_1 \dots j_k, m}^k,$$

- for $1 \leq m, l \leq d$ with $m \neq l$, the symmetry S^{lm} with respect to the diagonal axis $\mathbf{e}_l - \mathbf{e}_m$:

$$S^{lm} = I - \mathbf{e}_l \mathbf{e}_l^T - \mathbf{e}_m \mathbf{e}_m^T + \mathbf{e}_l \mathbf{e}_m^T + \mathbf{e}_m \mathbf{e}_l^T.$$

COROLLARY 3.16. 1. If the cell Y is symmetric with respect to all cell axes $(\mathbf{e}_l)_{1 \leq l \leq d}$, i.e. $S^l(Y) = Y$ for any $1 \leq l \leq d$, then

$$\mathcal{X}_{i_1 \dots i_k, pq}^{k*} = 0 \text{ and } M_{i_1 \dots i_k, pq}^k = 0$$

436 whenever any given integer $1 \leq l \leq d$ occurs an odd number of times in the
437 indices $i_1 \dots i_k, p, q$. In particular, this implies $\mathcal{X}^{2k+1*} = 0$ and $M^{2k+1} = 0$.

2. If the cell Y is symmetric with respect to all diagonal axes $\mathbf{e}_l - \mathbf{e}_m$, i.e. $S^{l,m}(Y) = Y$ for any $1 \leq l < m \leq d$, then for any permutation $\sigma \in \mathfrak{S}_d$,

$$\mathcal{X}_{\sigma(i_1) \dots \sigma(i_k), \sigma(p)\sigma(q)}^{k*} = \mathcal{X}_{i_1 \dots i_k, pq}^{k*} \text{ and } M_{\sigma(i_1) \dots \sigma(i_k), \sigma(p)\sigma(q)}^k = M_{i_1 \dots i_k, pq}^k.$$

438 *Proof.* The result is obtained by applying (3.28) and (3.29) to the particular
439 symmetries S^l and S^{lm} . See also Corollary 3 in [34]. \square

440 Let us illustrate how the previous properties translate for the tensors M^0 , M^2 and
441 M^4 :

- if the cell Y is symmetric with respect to all cell axes $(\mathbf{e}_l)_{1 \leq l \leq d}$, only the coefficients of the form $M_{i,i}^0$ are non zero. For M^2 , only

$$M_{ii,jj}^2, M_{ij,ij}^2, M_{ii,ii}^2$$

with $i \neq j$ are non zero. For M^4 , only the coefficients of the form

$$M_{ijj,kk}^4, M_{iijk,jk}^4, M_{iii,jj}^4, M_{ijj,ii}^4, M_{iij,ij}^4, M_{iii,ii}^4$$

442 are non zero with distinct integers i, j, k .

- If in addition the obstacle is symmetric with respect to all diagonal axes, then the values of the above coefficients do not depend on the choice of the distinct integers i, j, k . As a result, M^0 is proportional to the identity tensor, M^2 reduces to at most three coefficients (the material is said to be orthotropic), and M^4 reduces to at most 6 coefficients for $d \geq 3$, and to 4 coefficients for $d = 2$. For instance there are three constants α, β, γ such that $M^2 \cdot \nabla$ is the operator

$$M^2 \cdot \nabla \mathbf{v} = \alpha \Delta \mathbf{v} + \beta \nabla \operatorname{div}(\mathbf{v}) + \gamma \sum_{i=1}^d \partial_{ii} v_i \mathbf{e}_i.$$

443 **4. Homogenized equations of order $2K + 2$: tensors \mathbb{D}_K^k .** In this section,
444 we derive the well-posed high order homogenized system (1.6) and we justify the
445 homogenization process by means of quantitative error estimates.

446 The formal identities (1.10) lead us to introduce, for any order $K \in \mathbb{N}$, the
447 truncated ansatz $\mathbf{W}_{\varepsilon, K}(\mathbf{v})$ and $Q_{\varepsilon, K}(\mathbf{v}, \phi)$ for the reconstructed velocity and pressure:

$$448 \quad (4.1) \quad \mathbf{W}_{\varepsilon, K}(\mathbf{v})(x, y) := \sum_{k=0}^K \varepsilon^k N^k(y) \cdot \nabla^k \mathbf{v}(x), \quad x \in D, y \in Y$$

$$449 \quad (4.2) \quad Q_{\varepsilon, K}(\mathbf{v}, \phi)(x, y) := \phi(x) + \sum_{k=0}^K \varepsilon^{k-1} \beta^k(y) \cdot \nabla^k \mathbf{v}(x), \quad x \in D, y \in Y$$

450

451 for any $\mathbf{v} \in H^{K+1}(D, \mathbb{R}^d)$ and $\phi \in L^2(D)$ which are sought to approximate the
 452 homogenized averages \mathbf{u}_ε^* and p_ε^* respectively. Similarly we denote by $\widetilde{\mathbf{W}}_{\varepsilon, K}(\mathbf{v})$ and
 453 $\widetilde{Q}_{\varepsilon, K}(\mathbf{v}, \phi)$ the reconstructed *oscillating* functions defined for any $x \in D_\varepsilon$ by

$$454 \quad (4.3) \quad \widetilde{\mathbf{W}}_{\varepsilon, K}(\mathbf{v})(x) := \mathbf{W}_{\varepsilon, K}(\mathbf{v})(x, x/\varepsilon), \quad \widetilde{Q}_{\varepsilon, K}(\mathbf{v}, \phi)(x) := Q_{\varepsilon, K}(\mathbf{v}, \phi)(x, x/\varepsilon).$$

455

Most of the results of this section are consequences of the following observation:

456

LEMMA 4.1. *For any $K' \in \mathbb{N}$, $(\mathbf{v}, \phi) \in H^1(D, \mathbb{R}^d) \times L^2(D)$, the reconstructed
 457 velocity and pressure $(\widetilde{\mathbf{W}}_{\varepsilon, K'}(\mathbf{v}), \widetilde{Q}_{\varepsilon, K'}(\mathbf{v}, \phi))$ of (4.3) satisfy*

$$458 \quad (4.4) \quad \begin{aligned} & -\Delta \widetilde{\mathbf{W}}_{\varepsilon, K'}(\mathbf{v}) + \nabla \widetilde{Q}_{\varepsilon, K'}(\mathbf{v}, \phi) \\ & = \sum_{k=0}^{K'} \varepsilon^{k-2} M^k \nabla^k \mathbf{v} - \varepsilon^{K'-1} ((2\partial_l \mathbf{N}_j^{K'} - \beta_j^{K'} \mathbf{e}_l) \otimes \mathbf{e}_l) (\cdot/\varepsilon) \cdot \nabla^{K'+1} v_j \\ & \quad + (\mathbf{N}_j^{K'-1} \otimes I) (\cdot/\varepsilon) \cdot \nabla^{K'+1} v_j - \varepsilon^{K'} \mathbf{N}_j^{K'} (\cdot/\varepsilon) \otimes I \cdot \nabla^{K'+2} v_j, \end{aligned}$$

459

$$460 \quad (4.5) \quad \operatorname{div}(\widetilde{\mathbf{W}}_{\varepsilon, K'}(\mathbf{v})) = \operatorname{div}(\mathbf{v}) + \varepsilon^{K'} \mathbf{N}_j^{K'} (\cdot/\varepsilon) \cdot \mathbf{e}_l \otimes \mathbf{e}_l \cdot \nabla^{K'+1} v_j.$$

461

Proof. (4.4) and (4.5) are obtained by applying the Laplace and gradient opera-
 462 tors on (4.1) and (4.2) and by using the identity (3.22). \square

463

4.1. Sufficient conditions leading to error estimates. The purpose of this
 464 part is to demonstrate that a sequence of functions $(\mathbf{v}_\varepsilon^*, q_\varepsilon^*)_{\varepsilon>0}$ yields an approxima-
 465 tion of $(\mathbf{u}_\varepsilon, p_\varepsilon)$ at the order $O(\varepsilon^{K'})$ provided it solves the infinite order homogenized
 466 equation (1.9) up to a remainder of order $O(\varepsilon^{K'+1})$. The derivation of a *finite-order*
 467 homogenized equation such as (1.6) reduces then to determine $2K+2-K'$ tensors
 468 \mathbb{D}_K^k for $K'+1 \leq k \leq 2K+2$ such that the equation

$$469 \quad (4.6) \quad \sum_{k=K'+1}^{2K+2} \varepsilon^{k-2} \mathbb{D}_K^k \cdot \nabla^k \mathbf{v}_\varepsilon^* + \sum_{k=0}^{K'} \varepsilon^{k-2} M^k \cdot \nabla^k \mathbf{v}_\varepsilon^* + \nabla q_\varepsilon^* = \mathbf{f}$$

470 is well-posed. The proof is based on the next three technical results.

LEMMA 4.2. *There exists a constant C independent of $\varepsilon > 0$ such that for any
 $\mathbf{v} \in H^1(D_\varepsilon, \mathbb{R}^d)$ with $\mathbf{v} = 0$ on $\partial\omega_\varepsilon$, the following Poincaré inequality holds:*

$$\|\mathbf{v}\|_{L^2(D_\varepsilon, \mathbb{R}^d)} \leq C\varepsilon \|\nabla \mathbf{v}\|_{L^2(D, \mathbb{R}^{d \times d})}.$$

471

Proof. See e.g. [44] or the appendix of [49]. \square

472

The next lemma states the existence of a continuous right inverse for the divergence
 473 B_ε —so-called a Bogovskii's operator—with a bound explicit in ε on the uniform con-
 474 tinuity constant.

475

LEMMA 4.3. *Assume (H1) and (H2). Then there exists a linear operator $B_\varepsilon : L^2(D_\varepsilon) \rightarrow H^1(D_\varepsilon, \mathbb{R}^d)$ satisfying, for any $\phi \in L^2(D_\varepsilon)$ with $\int_{D_\varepsilon} \phi dx = 0$:*

477

(i) $\operatorname{div}(B_\varepsilon \phi) = \phi$ in D_ε ,

478

(ii) $B_\varepsilon \phi = 0$ on $\partial\omega_\varepsilon$ and $B_\varepsilon \phi$ is D -periodic,

479

(iii) $\|\nabla(B_\varepsilon \phi)\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} \leq C\varepsilon^{-1} \|\phi\|_{L^2(D_\varepsilon)}$, for a constant $C > 0$ independent of ϕ
 480 and ε .

481 *Proof.* See [32], Lemma 2.1, or [33], Lemma 7.9. \square

482 **COROLLARY 4.4.** Assume (H1) and (H2). For any $\mathbf{h} \in L^2(D_\varepsilon, \mathbb{R}^d)$ and $g \in$
 483 $L^2(D_\varepsilon)$ satisfying $\int_{D_\varepsilon} g dx = 0$, let $(\mathbf{v}, \phi) \in H^1(D_\varepsilon, \mathbb{R}^d) \times L^2(D_\varepsilon)$ be the unique solution
 484 to the Stokes problem

$$485 \quad (4.7) \quad \begin{cases} -\Delta \mathbf{v} + \nabla \phi = \mathbf{h} & \text{in } D_\varepsilon \\ \operatorname{div}(\mathbf{v}) = g & \text{in } D_\varepsilon \\ \int_{D_\varepsilon} \phi dx = 0 \\ \mathbf{v} = 0 & \text{on } \partial\omega_\varepsilon \\ \mathbf{v} \text{ is } D\text{-periodic.} \end{cases}$$

486 There exists a constant C independent of ε , \mathbf{h} and g such that

$$487 \quad (4.8) \quad \|\nabla \mathbf{v}\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} + \varepsilon \|\phi\|_{L^2(D_\varepsilon)} \leq C(\varepsilon \|\mathbf{h}\|_{L^2(D_\varepsilon, \mathbb{R}^d)} + \varepsilon^{-1} \|g\|_{L^2(D_\varepsilon)}),$$

Proof. We use the operator B_ε of Lemma 4.3 to lift the divergence of \mathbf{v} . Let us define the vector field $\mathbf{w} := \mathbf{v} - B_\varepsilon g \in H_{per}^1(D_\varepsilon, \mathbb{R}^d)$ which satisfies

$$\begin{cases} \operatorname{div}(\mathbf{w}) = 0 & \text{in } D_\varepsilon, \\ \mathbf{w} = 0 & \text{on } \partial\omega_\varepsilon. \end{cases}$$

After an integration by part, we obtain:

$$\begin{aligned} \|\nabla \mathbf{w}\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})}^2 &= \int_{D_\varepsilon} \mathbf{h} \cdot \mathbf{w} dx - \int_{D_\varepsilon} \nabla(B_\varepsilon g) : \nabla \mathbf{w} dx \\ &\leq \|\mathbf{h}\|_{L^2(D_\varepsilon, \mathbb{R}^d)} \|\mathbf{w}\|_{L^2(D_\varepsilon, \mathbb{R}^d)} + \|\nabla(B_\varepsilon g)\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} \|\nabla \mathbf{w}\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} \\ &\leq C(\varepsilon \|\mathbf{h}\|_{L^2(D_\varepsilon, \mathbb{R}^d)} + \|\nabla(B_\varepsilon g)\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})}) \|\nabla \mathbf{w}\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})}, \end{aligned}$$

where the last inequality is a consequence of Lemma 4.2. Therefore, simplifying by $\|\nabla \mathbf{w}\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})}$ and using the point (iii) of Lemma 4.3 yields

$$\begin{aligned} \|\nabla \mathbf{v}\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} &\leq \|\nabla \mathbf{w}\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} + \|\nabla(B_\varepsilon g)\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} \\ &\leq C(\varepsilon \|\mathbf{h}\|_{L^2(D_\varepsilon, \mathbb{R}^d)} + \varepsilon^{-1} \|g\|_{L^2(D_\varepsilon)}), \end{aligned}$$

488 which proves the first part of the bound (4.8) on $\nabla \mathbf{v}$. The bound on the pressure is
 489 then obtained by using $B_\varepsilon \phi$ as a test function: we write

$$\begin{aligned} 490 \quad \|\phi\|_{L^2(D_\varepsilon)}^2 &= \int_{D_\varepsilon} \phi \operatorname{div}(B_\varepsilon \phi) dx = - \int_{D_\varepsilon} \nabla \phi \cdot B_\varepsilon \phi dx \\ 491 \quad &= \int_{D_\varepsilon} (-\Delta \mathbf{v} - \mathbf{h}) \cdot B_\varepsilon \phi dx = \int_{D_\varepsilon} (\nabla \mathbf{v} \cdot \nabla(B_\varepsilon \phi) - \mathbf{h} \cdot B_\varepsilon \phi) dx \\ 492 \quad &\leq \|\nabla \mathbf{v}\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} \|\nabla(B_\varepsilon \phi)\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} + \|\mathbf{h}\|_{L^2(D_\varepsilon, \mathbb{R}^d)} \|B_\varepsilon \phi\|_{L^2(D_\varepsilon, \mathbb{R}^d)} \\ 493 \quad &\leq C(\varepsilon \|\mathbf{h}\|_{L^2(D_\varepsilon, \mathbb{R}^d)} + \varepsilon^{-1} \|g\|_{L^2(D_\varepsilon)}) \|\nabla(B_\varepsilon \phi)\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} \\ 494 \quad &\leq C\varepsilon^{-1} (\varepsilon \|\mathbf{h}\|_{L^2(D_\varepsilon, \mathbb{R}^d)} + \varepsilon^{-1} \|g\|_{L^2(D_\varepsilon)}) \|\phi\|_{L^2(D_\varepsilon)}, \end{aligned}$$

495 which concludes the proof. \square

497 We are now in position to state the main result of this section.

498 **PROPOSITION 4.5.** *Let $(\mathbf{v}_\varepsilon^*, q_\varepsilon^*)_{\varepsilon>0}$ be a sequence of functions of $H^1(D_\varepsilon, \mathbb{R}^d) \times$*
 499 *$L^2(D_\varepsilon)$, D -periodic, depending on ε (and possibly on K' and \mathbf{f}) satisfying the follow-*
 500 *ing conditions:*

501 1. $(\mathbf{v}_\varepsilon, q_\varepsilon)$ solves the infinite order homogenized equation (1.9) up to an error of
 502 order $O(\varepsilon^{K'+1})$:

$$503 \quad (4.9) \quad \left\| \sum_{k=0}^{K'} \varepsilon^{k-2} M^k \cdot \nabla^k \mathbf{v}_\varepsilon^* + \nabla q_\varepsilon^* - \mathbf{f} \right\|_{L^2(D, \mathbb{R}^d)} \leq C_{K'}(\mathbf{f}) \varepsilon^{K'+1}$$

504

$$505 \quad (4.10) \quad \operatorname{div}(\mathbf{v}_\varepsilon^*) = 0 \text{ in } D$$

506

$$507 \quad (4.11) \quad (\mathbf{v}_\varepsilon^*, p_\varepsilon^*) \text{ is } D\text{-periodic.}$$

508 2. For any $m \in \mathbb{N}$, there exists a constant C_m independent of ε such that

$$509 \quad (4.12) \quad \|\mathbf{v}_\varepsilon^*\|_{H^m(D, \mathbb{R}^d)} \leq C_m \varepsilon^2.$$

Then the reconstructed functions $(\widetilde{\mathbf{W}}_{\varepsilon, K'}(\mathbf{v}_\varepsilon^*), \widetilde{Q}_{\varepsilon, K'-1}(q_\varepsilon^*, \mathbf{v}_\varepsilon^*))$ (eqn. (4.1) and (4.2)) yield approximations of $(u_\varepsilon, p_\varepsilon)$ of order $O(K' + 2)$ in the $H^1(D_\varepsilon, \mathbb{R}^d)$ norm and $O(\varepsilon^{K'+3})$ in the $L^2(D_\varepsilon, \mathbb{R}^d)$ norm:

$$\left\| \nabla(\mathbf{u}_\varepsilon - \widetilde{\mathbf{W}}_{\varepsilon, K'}(\mathbf{v}_\varepsilon^*)) \right\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} + \varepsilon \left\| p_\varepsilon - \widetilde{Q}_{\varepsilon, K'-1}(q_\varepsilon^*, \mathbf{v}_\varepsilon^*) \right\|_{L^2(D_\varepsilon, \mathbb{R}^d)} \leq C_{K'}(\mathbf{f}) \varepsilon^{K'+2},$$

$$\left\| \mathbf{u}_\varepsilon - \widetilde{\mathbf{W}}_{\varepsilon, K'}(\mathbf{v}_\varepsilon^*) \right\|_{L^2(D_\varepsilon, \mathbb{R}^d)} \leq C_{K'}(\mathbf{f}) \varepsilon^{K'+3}.$$

Proof. According to Lemma 4.1 and (4.10), it holds

$$-\Delta \widetilde{\mathbf{W}}_{\varepsilon, K'+1}(\mathbf{v}_\varepsilon^*) + \nabla \widetilde{Q}_{\varepsilon, K'+1}(q_\varepsilon^*, \mathbf{v}_\varepsilon^*) = \sum_{k=0}^{K'} \varepsilon^{k-2} M^k \cdot \nabla^k \mathbf{v}_\varepsilon^* + \nabla q_\varepsilon^* + O_{L^2(D_\varepsilon, \mathbb{R}^d)}(\varepsilon^{K'+1})$$

$$\operatorname{div}(\widetilde{\mathbf{W}}_{\varepsilon, K'+1}) = O_{L^2(D_\varepsilon)}(\varepsilon^{K'+3}),$$

where we have used (4.12) to estimate the right-hand side terms. Applying now Corollary 4.4 to $(\mathbf{v}, \phi) \equiv (\mathbf{u}_\varepsilon - \widetilde{\mathbf{W}}_{\varepsilon, K'+1}(\mathbf{v}_\varepsilon^*), p_\varepsilon - \widetilde{Q}_{\varepsilon, K'+1}(q_\varepsilon^*, \mathbf{v}_\varepsilon^*))$ yields the error estimate

$$\left\| \nabla(\mathbf{u}_\varepsilon - \widetilde{\mathbf{W}}_{\varepsilon, K'+1}(\mathbf{v}_\varepsilon^*)) \right\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} + \varepsilon \left\| p_\varepsilon - \widetilde{Q}_{\varepsilon, K'+1}(q_\varepsilon^*, \mathbf{v}_\varepsilon^*) \right\|_{L^2(D_\varepsilon)} \leq C_{K'}(\mathbf{f}) \varepsilon^{K'+2}.$$

Finally, remarking that the highest order terms are already of order $O(\varepsilon^{K'+2})$, i.e.

$$\left\| \nabla(\widetilde{\mathbf{W}}_{\varepsilon, K'+1}(\mathbf{v}_\varepsilon^*) - \widetilde{\mathbf{W}}_{\varepsilon, K'}(\mathbf{v}_\varepsilon^*)) \right\|_{L^2(D_\varepsilon, \mathbb{R}^d)} \leq C_{K'} \varepsilon^{K'+2},$$

$$\varepsilon \left\| \widetilde{Q}_{\varepsilon, K'+1}(q_\varepsilon^*, \mathbf{v}_\varepsilon^*) - \widetilde{Q}_{\varepsilon, K'-1}(q_\varepsilon^*, \mathbf{v}_\varepsilon^*) \right\|_{L^2(D_\varepsilon)} \leq C_{K'} \varepsilon^{K'+2},$$

510 we obtain the result by using the triangle's inequality. \square

511 *Remark 4.6.* We need only $K - 1$ derivatives in the truncated criminal ansatz
 512 $Q_{\varepsilon, K-1}(\mathbf{v}_\varepsilon^*, q_\varepsilon^*)$ for the pressure (eqn. (4.2)), because the term of highest order has a
 513 norm of order ε^K while \mathbf{v}_ε^* is of order ε^2 by the assumption (4.12).

514 *Remark 4.7.* As a result of the scaling ε^{-1} in Corollary 4.4, we pay a factor ε^{-1}
 515 in the error induced by the non zero divergence constraint. However we are able to
 516 obtain the right order of ε in the error estimates of Proposition 4.5 thanks to the use
 517 of higher order terms of the ansatz (3.21) which are removed at the end of the proof.
 518 This strategy is quite classical in the truncation analysis of two-scale expansions, see
 519 e.g. [26, 11].

520 **4.2. Construction of a well-posed higher order effective models by a**
 521 **minimization principle.** We now derive the well-posed homogenized equation (1.6)
 522 of (finite) order $2K+2$ by following the variational method introduced by Smyshlyaev
 523 and Cherednychenko in [53] and used in the further works [28, 27, 34]. In the present
 524 context, of the Stokes system (1.1), we shall see that (1.6) can be obtained as (4.6)
 525 with $K' = K$, which yields error estimates of order $O(\varepsilon^{K+3})$ in the $L^2(D_\varepsilon)$ norm.

526 Recall that the velocity \mathbf{u}_ε solution to the Stokes system (1.1) is the unique
 527 minimizer of the constrained minimization problem

$$528 \quad (4.13) \quad \mathbf{u}_\varepsilon = \arg \min_{\mathbf{w} \in H^1(D_\varepsilon, \mathbb{R}^d)} J(\mathbf{w}, \mathbf{f}) := \int_D \left(\frac{1}{2} \nabla \mathbf{w} : \nabla \mathbf{w} - \mathbf{f} \cdot \mathbf{w} \right) dy$$

$$\text{s.t.} \quad \begin{cases} \operatorname{div}(\mathbf{w}) = 0 \text{ in } D_\varepsilon \\ \mathbf{w} = 0 \text{ on } \partial\omega_\varepsilon \\ \mathbf{w} \text{ is } D\text{-periodic.} \end{cases}$$

529 In the context of the homogenization of linearized elasticity, the main idea of the
 530 method of [53] is to restrict (4.13) to functions of the form $\mathbf{w} = \widetilde{\mathbf{W}}_{\varepsilon, K}(\mathbf{v})$ given by
 531 (4.1), where $\mathbf{v} \in H^{K+1}(D, \mathbb{R}^d)$ is an unknown function sought to approximate \mathbf{u}_ε^* . In
 532 the present setting, we consider the following approximation of (4.13):

$$533 \quad (4.14) \quad \min_{\mathbf{v} \in H^{K+1}(D, \mathbb{R}^d)} J(\widetilde{\mathbf{W}}_{\varepsilon, K}(\mathbf{v}), \mathbf{f})$$

$$\text{s.t.} \quad \begin{cases} \operatorname{div}(\mathbf{v}) = 0 \text{ in } D, \\ \mathbf{v} \text{ is } D\text{-periodic.} \end{cases}$$

534 Note that (4.14) is not exactly the restriction of (4.13) to such functions $\widetilde{\mathbf{W}}_{\varepsilon, K}(\mathbf{v})$
 535 because $\operatorname{div}(\widetilde{\mathbf{W}}_{\varepsilon, K}(\mathbf{v})) \neq 0$ (it is of order ε^K , see (4.5)). The next step of the process is
 536 to eliminate the oscillating variable x/ε in $J(\widetilde{\mathbf{W}}_{\varepsilon, K}(\mathbf{v}), \mathbf{f})$ so as to obtain an effective
 537 energy $J_K^*(\mathbf{v}, \mathbf{f}, \varepsilon) \simeq J(\widetilde{\mathbf{W}}_{\varepsilon, K}(\mathbf{v}), \mathbf{f})$ which does not involve oscillating functions. Such
 538 is achieved thanks to the classical lemma of two-scale convergence [9].

LEMMA 4.8. *Let ϕ be a $P = (0, 1)^d$ -periodic function and $f \in C_{per}^\infty(D)$ be a smooth D -periodic function. Then for any $p \in \mathbb{N}$, there exists a constant $C_p(f, \phi)$ independent of ε such that:*

$$\left| \int_D f(x) \phi(x/\varepsilon) dx - \int_D \int_P f(x) \phi(y) dy dx \right| \leq C_p(f, \phi) \varepsilon^p.$$

539 *Proof.* See Appendix C. of [53] or Lemma 7.3 in [33]. \square

Applying Lemma 4.8 to (4.14) in order to pass to the limit in the terms of $J(\widetilde{\mathbf{W}}_{\varepsilon, K}(\mathbf{v}), \mathbf{f})$ which depends on the oscillating variable x/ε , we obtain the existence of a functional J_K^* such that for any $\mathbf{v} \in C_{per}^\infty(D, \mathbb{R}^d)$, it holds

$$J(\widetilde{\mathbf{W}}_{\varepsilon, K}(\mathbf{v}), \mathbf{f}) = J_K^*(\mathbf{v}, \mathbf{f}, \varepsilon) + o(\varepsilon^p)$$

540 with $p \in \mathbb{N}$ arbitrarily large. The functional J_K^* is given explicitly by

$$541 \quad (4.15) \quad J_K^*(\mathbf{v}, \mathbf{f}, \varepsilon) := \int_D \int_P \frac{1}{2} \left\| (\nabla_x + \varepsilon^{-1} \nabla_y) \left(\mathbf{W}_{\varepsilon, K}(\mathbf{v})(x, y) \right) \right\|^2 dy dx - \int_D \mathbf{f} \cdot \mathbf{v} dx,$$

542 where we have used the point 1 of Proposition 3.10 to simplify the linear part of
 543 the energy. Replacing $J(\widetilde{\mathbf{W}}_{\varepsilon, K}(\mathbf{v}), \mathbf{f})$ by $J_K^*(\mathbf{v}, \mathbf{f}, \varepsilon)$ in (4.14) allows to obtain the
 544 homogenized equation (1.6) of order $2K+2$:

545 DEFINITION 4.9. For any $K \in \mathbb{N}$, we call homogenized equation of order $2K + 2$
 546 associated with the Stokes system (1.1) the Euler-Lagrange equation of the minimiza-
 547 tion problem

$$548 \quad (4.16) \quad \begin{array}{l} \min \\ \text{s.t.} \end{array} \begin{cases} J_K^*(\mathbf{v}, \mathbf{f}, \varepsilon) \\ \mathbf{v} \in H^{K+1}(D, \mathbb{R}^d), \\ \operatorname{div}(\mathbf{v}) = 0 \text{ in } D, \\ \mathbf{v} \text{ is } D\text{-periodic.} \end{cases}$$

549 This Euler-Lagrange equation can be written as (1.6) where the constant (matrix val-
 550 ued) tensors \mathbb{D}_K^k are inferred from (4.15) and where $(\mathbf{v}_{\varepsilon, K}^*, q_{\varepsilon, K}^*) \in H^{K+1}(D, \mathbb{R}^d) \times$
 551 $L^2(D)$ defines the higher order homogenized solution.

552 The next two propositions verify that (1.6) is indeed a “good” candidate effective
 553 model, by relating the coefficients \mathbb{D}_K^k to the tensors M^k (in view of (4.6)), and by
 554 establishing the well-posedness of (1.6).

555 PROPOSITION 4.10. The coefficients of the matrix valued tensor \mathbb{D}_K^k are explicitly
 556 given for any $1 \leq i, j \leq d$ by:

$$557 \quad (4.17) \quad \mathbb{D}_{K, ij}^k = \begin{cases} M^k & \text{if } 0 \leq k \leq K \\ M^k + \mathbb{A}_K^k & \text{if } K + 1 \leq k \leq 2K + 1 \\ (-1)^{K+1} \int_Y \mathbf{N}_i^K \cdot \mathbf{N}_j^K \otimes \operatorname{Id}_y & \text{if } k = 2K + 2. \end{cases}$$

558 where the matrix valued tensor \mathbb{A}_K^k is given for any $K + 1 \leq k \leq 2K + 1$ by

$$559 \quad (4.18) \quad \mathbb{A}_{K, ij}^k := (-1)^{K+1} \int_Y (\nabla \beta_j^{k-K-1} \cdot \mathbf{N}_i^{K+1} + (-1)^k \nabla \beta_i^{k-K-1} \cdot \mathbf{N}_j^{K+1}) \operatorname{d}y.$$

560 *Proof.* Let us denote by V_K the space

$$561 \quad (4.19) \quad V_K := \{\mathbf{v} \in H^{K+1}(D, \mathbb{R}^d) \mid \operatorname{div}(\mathbf{v}) = 0 \text{ and } \mathbf{v} \text{ is } D\text{-periodic}\}.$$

562 We identify the coefficients \mathbb{D}_K^k by computing the Euler-Lagrange equation associated
 563 with (4.15). For any $(\mathbf{v}, \mathbf{w}) \in V_K$, it holds, in a distributional sense:

$$564 \quad \int_D \int_P (\nabla_x + \varepsilon^{-1} \nabla_y) \mathbf{W}_{\varepsilon, K}(\mathbf{v}) : (\nabla_x + \varepsilon^{-1} \nabla_y) \mathbf{W}_{\varepsilon, K}(\mathbf{w}) \operatorname{d}x \operatorname{d}y \\ 565 \quad = \int_D \int_P [(-\Delta_{xx} - \varepsilon^{-1} \Delta_{xy} - \varepsilon^{-2} \Delta_{yy}) \mathbf{W}_{\varepsilon, K}(\mathbf{v}) \\ 566 \quad + (\nabla_x + \varepsilon^{-1} \nabla_y) Q_{\varepsilon, K}(\mathbf{v}, 0)] \cdot \mathbf{W}_{\varepsilon, K}(\mathbf{w}) \operatorname{d}x \operatorname{d}y \\ 567 \quad + \int_D \int_Y Q_{\varepsilon, K}(\mathbf{v}, 0) [(\operatorname{div}_x + \varepsilon^{-1} \operatorname{div}_y) \mathbf{W}_{\varepsilon, K}(\mathbf{w})] \operatorname{d}x \operatorname{d}y. \\ 568$$

569 By using (4.1) and the point (i) of Proposition 3.10, the above quantity is equal to

$$570 \quad \int_D \int_Y \left[\sum_{k=0}^K \varepsilon^{k-2} M^k \nabla^k \mathbf{v}(x) \right] \cdot \sum_{k=0}^K \varepsilon^k N^k(y) \cdot \nabla^k \mathbf{w}(x) \operatorname{d}x \operatorname{d}y \\ 571 \quad - \int_D \int_Y \varepsilon^{K-1} [(2\partial_l N_j^K(y) - \beta_j^K(y) e_l) \otimes e_l] \cdot \nabla^{K+1} v_j(x) \cdot \mathbf{W}_{\varepsilon, K}(\mathbf{w}) \operatorname{d}x \operatorname{d}y$$

$$\begin{aligned}
572 & - \int_D \int_Y [\varepsilon^{K-1} (\mathbf{N}_j^{K-1}(y) \otimes I) \cdot \nabla^{K+1} \mathbf{v}(x)] \cdot \mathbf{W}_{\varepsilon,K}(\mathbf{w}) dx dy \\
573 & - \int_D \int_Y [\varepsilon^K (\mathbf{N}_j^K(y) \otimes I) \cdot \nabla^{K+2} \mathbf{v}(x)] \cdot \mathbf{W}_{\varepsilon,K}(\mathbf{w}) dx dy \\
574 & + \int_D \int_Y \sum_{k=0}^K \varepsilon^{k+K-1} (\beta_j^k(y) \cdot \nabla^k v_j(x)) (\mathbf{N}_i^K(y) \cdot \mathbf{e}_l \otimes \mathbf{e}_l \cdot \nabla^{K+1} w_i(x)) dy dx \\
575 & = \int_D \left(\sum_{k=0}^K \varepsilon^{k-2} M^k \cdot \nabla^k \mathbf{v} + \sum_{k=K+1}^{2K+2} \varepsilon^{k-2} \mathbb{D}_K^k \cdot \nabla^k \mathbf{v} \right) \cdot \mathbf{w} dx \\
576 &
\end{aligned}$$

where we identify (by integration by parts) $\mathbb{D}_{K,ij}^{2K+2} := -(-1)^K \int_Y \mathbf{N}_i^K \cdot \mathbf{N}_j^K \otimes Idy$ as claimed. The coefficients of the tensor \mathbb{D}_K^k are given for $K+1 \leq k \leq 2K+1$ by

$$\begin{aligned}
\mathbb{D}_{K,ij}^k &= -(-1)^{k-K-1} \int_Y ((2\partial_l \mathbf{N}_j^K - \beta_j^K \mathbf{e}_l) \otimes \mathbf{e}_l + \mathbf{N}_j^{K-1} \otimes I) \cdot \mathbf{N}_i^{k-K-1} dy \\
& - (-1)^{k-K-2} \int_Y \mathbf{N}_j^K \cdot \mathbf{N}_i^{k-K-2} \otimes Idy + (-1)^{K+1} \int_Y \beta_j^{k-K-1} \mathbf{N}_i^K \cdot \mathbf{e}_l \otimes \mathbf{e}_l dy \\
&= -(-1)^{K+1} \int_Y (-1)^k (-\Delta_{yy} \mathbf{N}_j^{K+1} + \nabla \beta_j^{K+1} - M^{K+1} \mathbf{e}_j) \cdot \mathbf{N}_i^{k-K-1} dy \\
& + (-1)^{K+1} \int_Y [(-1)^k \mathbf{N}_j^K \cdot \mathbf{N}_i^{k-K-2} \otimes I + \beta_j^{k-K-1} \mathbf{N}_i^K \cdot \mathbf{e}_l \otimes \mathbf{e}_l] dy,
\end{aligned}$$

577 where we have used extensively [Proposition 3.10](#). We now distinguish two cases:

1. if $k = K+1$, then the above expression reads

$$\begin{aligned}
\mathbb{D}_{K,ij}^{K+1} &= (M^{K+1} \mathbf{e}_j) \cdot \mathbf{e}_i - \int_Y \mathbf{N}_j^{K+1} \cdot (-\Delta_{yy} \mathbf{N}_i^0 + \nabla \beta_i^0) dy + \int_Y \nabla \beta_i^0 \cdot \mathbf{N}_j^{K+1} \\
& + (-1)^{K+1} \int_Y \nabla \beta_j^{k-K-1} \cdot \mathbf{N}_i^{K+1} dy = M_{ij}^{K+1} + \mathbb{A}_{ij}^{K+1},
\end{aligned}$$

578 whence the result for $k = K+1$;

2. if $K+2 \leq k \leq 2K+1$, then we read instead

$$\begin{aligned}
\mathbb{D}_{K,ij}^k &= (-1)^{k+K} \int_Y ((-\Delta_{yy} \mathbf{N}_j^{K+1} + \nabla \beta_j^{K+1}) \cdot \mathbf{N}_i^{k-K-1} + \nabla \beta_i^{k-K-1} \cdot \mathbf{N}_j^{K+1}) dy \\
& - (-1)^{k+K} \int_Y \mathbf{N}_j^K \cdot \mathbf{N}_i^{k-K-2} \otimes Idy + \mathbb{A}_{ij}^k.
\end{aligned}$$

579 Applying finally [Corollary 3.11](#) with $p = K+1$, we obtain that the first two term of
580 the above equation are equal to $(-1)^k M_{ji}^k = M_{ij}^k$, which yields the final result. \square

581 *Remark 4.11.* In view of the proof of [Proposition 4.10](#), it is possible to show that
582 [\(1.11\)](#) (with the same definition for the leading coefficient \mathbb{D}_K^{2K+2}) is the strong form
583 of the following “mixed” variational formulation: find $(\widehat{\mathbf{v}}_{\varepsilon,K}^*, \widehat{q}_{\varepsilon,K}^*) \in V_K \times L^2(D)$ such
584 that for any $(\mathbf{w}, \phi) \in V_K \times L^2(D)$,

$$\begin{aligned}
585 & \\
586 & (4.20) \quad \int_D \int_Y [(\varepsilon^{-1} \nabla_y + \nabla_x) \mathbf{W}_{\varepsilon,K}(\widehat{\mathbf{v}}_{\varepsilon,K}^*)] : [(\varepsilon^{-1} \nabla_y + \nabla_x) \mathbf{W}_{\varepsilon,K}(\mathbf{w})] dy dx \\
587 & - \int_D \int_Y Q_{\varepsilon,K}(\widehat{\mathbf{v}}_{\varepsilon,K}^*, \widehat{q}_{\varepsilon,K}^*) [(\varepsilon^{-1} \operatorname{div}_y + \operatorname{div}_x) \mathbf{W}_{\varepsilon,K}(\mathbf{w})] dy dx
\end{aligned}$$

$$- \int_D \int_Y Q_{\varepsilon,K}(\mathbf{w}, \phi) [(\varepsilon^{-1} \operatorname{div}_y + \operatorname{div}_x) \mathbf{W}_{\varepsilon,K}(\widehat{\mathbf{v}}_{\varepsilon,K}^*)] dy dx = \int_D \mathbf{f} \cdot \mathbf{w} dx.$$

This result is quite surprising. Indeed, (4.20) is built from the truncated ansatz $(\mathbf{W}_{\varepsilon,K}(\widehat{\mathbf{v}}_{\varepsilon,K}^*), Q_{\varepsilon,K}(\widehat{\mathbf{v}}_{\varepsilon,K}^*, \widehat{q}_{\varepsilon,K}^*))$ which is expected to yield approximations of $(\mathbf{u}_\varepsilon, p_\varepsilon)$ at order $O(\varepsilon^{K+3})$ only (eqn. (4.1) and (4.2)). However, the strong form (1.11) turns out to exhibit $2K+1$ “correct” coefficients M^k . As a result, if (1.11) is well-posed, the reconstructed oscillating functions $\widetilde{\mathbf{W}}_{\varepsilon,2K+1}(\widehat{\mathbf{v}}_{\varepsilon,K}^*)$ and $\widetilde{Q}_{\varepsilon,2K}(\widehat{\mathbf{v}}_{\varepsilon,K}^*, \widehat{q}_{\varepsilon,K}^*)$ approximate \mathbf{u}_ε and p_ε with an error rate as good as $O(\varepsilon^{2K+4})$ in the $L^2(D_\varepsilon)$ norm (Proposition 4.5). This improvement (which had not been noticed in the original paper [53]) actually holds in the context of the Poisson or elasticity equations for which there is no difference between (1.6) and (1.11) (see [34, 33]). Unfortunately in the case of the Stokes system, we do not know whether the mixed formulation (4.20) with the “velocity-dependent” pressure $Q_{\varepsilon,K}(\mathbf{w}, \phi)$ yields a well-posed problem, hence our commitment to consider (1.6) instead of (1.11).

The leading tensor \mathbb{D}_K^{2K+2} is nonnegative according to (4.17). Under a rather unrestrictive additional non-degeneracy assumption, we obtain that the minimization principle (4.16) makes (1.6) be a well posed problem.

PROPOSITION 4.12. *Assume the dominant tensor $\mathbb{D}_K^{2K+2} = (-1)^{K+1} \mathbb{B}_K^{K+1, K+1}$ is non-degenerate, that is there exists a constant $\nu > 0$ such that for any constant vector tensor $\boldsymbol{\xi}^{K+1} = \xi_{i_1 \dots i_{K+1}, p}^{K+1} \in \mathbb{R}^{d^{K+1}} \times \mathbb{R}^d$ of order $K+1$, it holds*

$$(4.21) \quad \int_Y [(N^K \otimes e_l) \cdot \boldsymbol{\xi}^{K+1}] \cdot [(N^K \otimes e_l) \cdot \boldsymbol{\xi}^{K+1}] dy \geq \nu \xi_{i_1 \dots i_{K+1}}^{K+1} \cdot \xi_{i_1 \dots i_{K+1}}^{K+1}.$$

Then there exists a unique velocity and pressure couple $(\mathbf{v}_{\varepsilon,K}^*, q_{\varepsilon,K}^*) \in H^{K+1}(D, \mathbb{R}^d) \times L^2(D)/\mathbb{R}$ solving the higher order homogenized equation (1.6).

Proof. The proof relies on the positivity of the quadratic part of the energy $J_K^*(\mathbf{v}, \mathbf{f}, \varepsilon)$. By adapting the arguments of the proof of Proposition 12 in [34], we obtain indeed that the bilinear form associated with the energy (4.15) is coercive on the space V_K defined in (4.19). This is enough to apply standard theory for saddle point problems involving the zero divergence constraint (see e.g. the textbooks [55, 54, 38, 31]) which ensures the existence and uniqueness of a solution for (1.6). \square

Remark 4.13. The assumption (4.21) could fail for $K \geq 1$ in case the obstacle ηT is invariant along some of the directions \mathbf{e}_i of the cell P , however it is not restrictive. Indeed, since the leading order tensor \mathbb{D}_K^{2K+2} has no influence on the error estimates of Proposition 4.5, it is always possible to add to \mathbb{D}_K^{2K+2} a small non-negative tensor making it non-degenerate.

4.3. Error estimates: justification of the homogenization process. We conclude this section by stating error estimates holding for the solution $(\mathbf{v}_{\varepsilon,K}^*, q_{\varepsilon,K}^*)$ to the high order homogenized model (1.6). We know from Proposition 4.10 that $\mathbb{D}_K^k = M^k$ for $0 \leq k \leq K$, therefore the assumptions of Proposition 4.5 are satisfied provided we verify the uniform regularity estimate (4.12).

LEMMA 4.14. *The solution $(\mathbf{v}_{\varepsilon,K}^*, q_{\varepsilon,K}^*)$ of (1.6) is smooth and for any $m \in \mathbb{N}$, there exists a constant $C_m(\mathbf{f})$ depending only on m and \mathbf{f} such that*

$$\|\mathbf{v}_{\varepsilon,K}^*\|_{H^m(D, \mathbb{R}^d)} \leq C_m(\mathbf{f}) \varepsilon^2.$$

627 *Proof.* This result can be obtained by solving (1.6) explicitly with Fourier series
 628 in the periodic domain D and by adapting the proof of Lemma 5 in [34]. \square

629 Since we have verified that all the assumptions of Proposition 4.5 hold with $K' = K$,
 630 we are finally in position to state the following error bounds.

COROLLARY 4.15. *Let $(\mathbf{v}_{\varepsilon,K}^*, q_{\varepsilon,K}^*)$ be the unique solution to the high order homogenized equation (1.6). There exists a constant $C_K(\mathbf{f})$ independent of ε (but depending on K , \mathbf{f} , and a priori on the shape of the hole (ηT)) such that the following error estimates hold:*

$$\begin{aligned} \left\| \mathbf{u}_\varepsilon - \sum_{k=0}^K \varepsilon^k N^k(\cdot/\varepsilon) \cdot \nabla^k \mathbf{v}_{\varepsilon,K}^* \right\|_{L^2(D_\varepsilon, \mathbb{R}^d)} &\leq C_K(\mathbf{f}) \varepsilon^{K+3}, \\ \left\| \nabla \left(\mathbf{u}_\varepsilon - \sum_{k=0}^K \varepsilon^k N^k(\cdot/\varepsilon) \cdot \nabla^k \mathbf{v}_{\varepsilon,K}^* \right) \right\|_{L^2(D_\varepsilon, \mathbb{R}^{d \times d})} &\leq C_K(\mathbf{f}) \varepsilon^{K+2}, \\ \left\| p_\varepsilon - \left(q_{\varepsilon,K}^* + \sum_{k=0}^{K-1} \varepsilon^{k-1} \beta^k(\cdot/\varepsilon) \cdot \nabla^k \mathbf{v}_{\varepsilon,K}^* \right) \right\|_{L^2(D_\varepsilon)} &\leq C_K(\mathbf{f}) \varepsilon^{K+1}. \end{aligned}$$

631 *Remark 4.16.* As the reader may expect, error bounds with the same order of
 632 convergence hold for the truncation at order K of the “classical” ansatz (1.7), see
 633 [26, 46] up to the order $K = 1$, and in Proposition 7.37 of [33] at all orders.

634 **5. Low volume fraction limits when the scaling η of the obstacle vanishes.**
 635 In this section, we provide evidences that (1.6) is “well-behaved” in the sense
 636 that it has the potential to capture the homogenized regimes (1.2)–(1.4) in the low
 637 volume fraction limit where the size of the obstacles vanish. Our results supporting
 638 this claim are obtained by analyzing the asymptotics of the tensors \mathcal{X}^{k*} , M^k and
 639 \mathbb{D}_K^{2K+2} as the scaling ratio η converges to 0.

In this whole subsection, we assume for simplicity, that the space dimension is greater than 3:

$$d \geq 3.$$

640 Similar results are expected to hold in dimension $d = 2$ but would require a different
 641 treatment, as e.g. in [6, 41]. The hole ηT is assumed to be an non-empty open subset
 642 strictly included in the unit cell for any $\eta \leq 1$ (it does not touch the boundary):
 643 $\eta T \subset\subset P$.

644 Let us recall the definition of the Deny-Lions (or Beppo-Levi) space denoted by
 645 $\mathcal{D}^{1,2}(\mathbb{R}^d \setminus T, \mathbb{R}^d)$ (the reader is referred to [6, 4, 8] and also [48], p.59. for more details).

DEFINITION 5.1 (Deny-Lions space). *The Deny-Lions space $\mathcal{D}^{1,2}(\mathbb{R}^d \setminus T, \mathbb{R}^d)$ is the completion of the space of smooth vector fields by the L^2 norm of their gradients:*

$$\mathcal{D}^{1,2}(\mathbb{R}^d \setminus T, \mathbb{R}^d) := \overline{\mathcal{D}(\mathbb{R}^d \setminus T, \mathbb{R}^d)}^{\|\nabla \cdot\|_{L^2(\mathbb{R}^d \setminus T, \mathbb{R}^d)}},$$

646 where $\mathcal{D}(\mathbb{R}^d \setminus T, \mathbb{R}^d)$ is the space of compactly supported smooth vector fields. When
 647 $d \geq 3$, it admits the following characterization:

$$\begin{aligned} 648 &\mathcal{D}^{1,2}(\mathbb{R}^d \setminus T, \mathbb{R}^d) \\ 649 &= \{ \mathbf{v} \text{ measurable} \mid \|\mathbf{v}\|_{L^{2d/(d-2)}(\mathbb{R}^d \setminus T, \mathbb{R}^d)} < +\infty \text{ and } \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^d \setminus T, \mathbb{R}^{d \times d})} < +\infty \}. \end{aligned}$$

652 For any $1 \leq j \leq d$, we consider the unique solution (Ψ_j, σ_j) to the exterior Stokes
653 problem

$$654 \quad (5.1) \quad \left\{ \begin{array}{l} -\Delta \Psi_j + \nabla \sigma_j = 0 \text{ in } \mathbb{R}^d \setminus T \\ \operatorname{div}(\Psi_j) = 0 \text{ in } \mathbb{R}^d \setminus T \\ \Psi_j = 0 \text{ on } \partial T \\ \Psi_j \rightarrow \mathbf{e}_j \text{ at } \infty \\ \sigma_j \in L^2(\mathbb{R}^d \setminus T). \end{array} \right.$$

655 The convergence condition $\Psi_j \rightarrow \mathbf{e}_j$ at infinity must be understood in the sense that
656 $\Psi_j - \mathbf{e}_j$ belongs to $\mathcal{D}^{1,2}(\mathbb{R}^d \setminus T, \mathbb{R}^d)$. Similarly, the pressures $(\sigma_j)_{1 \leq j \leq d}$ are uniquely
657 determined by the condition $\sigma_j \in L^2(\mathbb{R}^d \setminus T)$ (see e.g. Lemma 1.1, article V. of [36]).
658 We denote by $F := (F_{ij})_{1 \leq i, j \leq d}$ the matrix collecting the drag force components:

$$659 \quad (5.2) \quad F_{ij} := \int_{\mathbb{R}^d \setminus T} \nabla \Psi_i : \nabla \Psi_j dx = - \int_{\partial T} \mathbf{e}_j \cdot (\nabla \Psi_i - \sigma_i I) \cdot \mathbf{n} ds,$$

660 where the normal \mathbf{n} is pointing *inward* T .

661 **5.1. Technical estimates in the growing periodic domain $\eta^{-1}P \setminus T$.** In all
662 this section, vector fields of the rescaled cell $\eta^{-1}P$ are indicated by a tilde $\tilde{\cdot}$ notation.
663 For a given vector field $\tilde{\mathbf{v}} \in L^2(\eta^{-1}P, \mathbb{R}^d)$, we denote by $\langle \tilde{\mathbf{v}} \rangle$ the average $\langle \tilde{\mathbf{v}} \rangle :=$
664 $\eta^d \int_{\eta^{-1}P} \tilde{\mathbf{v}}(y) dy$.

665 Let us recall that for any $\mathbf{v} \in H^1(P \setminus (\eta T), \mathbb{R}^d)$, if $\tilde{\mathbf{v}}$ is the rescaled function defined
666 by $\tilde{\mathbf{v}}(y) := \mathbf{v}(\eta y)$ in the rescaled cell $\eta^{-1}P \setminus T$, then the L^2 norms of \mathbf{v} , $\tilde{\mathbf{v}}$ and of their
667 gradients are related by the following identities:

$$668 \quad \|\mathbf{v}\|_{L^2(P \setminus (\eta T), \mathbb{R}^d)} = \eta^{d/2} \|\tilde{\mathbf{v}}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^d)}$$

$$669 \quad \|\nabla \mathbf{v}\|_{L^2(P \setminus (\eta T), \mathbb{R}^{d \times d})} = \eta^{d/2-1} \|\nabla \tilde{\mathbf{v}}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})}.$$

671 The asymptotic behaviors of the tensors \mathcal{X}^{k*}, M^k are obtained by following the
672 methodology of [6, 41, 34], which relies on several technical results stated in this part.

673 **LEMMA 5.2.** *Assume $d \geq 3$. There exists a constant $C > 0$ independent of $\eta > 0$
674 such that for any $\tilde{\mathbf{v}} \in H^1(\eta^{-1}P \setminus T, \mathbb{R}^d)$ which vanishes on the hole ∂T and which is
675 $\eta^{-1}P$ periodic, the following inequalities hold:*

$$676 \quad (5.3) \quad \|\tilde{\mathbf{v}}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^d)} \leq C \eta^{-d/2} \|\nabla \tilde{\mathbf{v}}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})},$$

$$677 \quad (5.4) \quad |\langle \tilde{\mathbf{v}} \rangle| \leq C \|\nabla \tilde{\mathbf{v}}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})},$$

$$678 \quad (5.5) \quad \|\tilde{\mathbf{v}} - \langle \tilde{\mathbf{v}} \rangle\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^d)} \leq C \eta^{-1} \|\nabla \tilde{\mathbf{v}}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})},$$

$$679 \quad (5.6) \quad \|\tilde{\mathbf{v}} - \langle \tilde{\mathbf{v}} \rangle\|_{L^{2d/(d-2)}(\eta^{-1}P \setminus T, \mathbb{R}^d)} \leq C \|\nabla \tilde{\mathbf{v}}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})}.$$

681 *Proof.* See [41, 45, 6, 42]. □

682 **LEMMA 5.3.** *Consider $\mathbf{h} \in L^2(\eta^{-1}P \setminus T, \mathbb{R}^d)$ and $g \in L^2(\eta^{-1}P \setminus T)$ a function
683 satisfying $\int_{\eta^{-1}P \setminus T} g dx = 0$. Let $(\mathbf{v}, \phi) \in H^1(\eta^{-1}P \setminus T, \mathbb{R}^d) \times L^2(\eta^{-1}P \setminus T)$ be the*

684 *unique solution to the following Stokes system:*

$$685 \quad (5.7) \quad \begin{cases} -\Delta \mathbf{v} + \nabla \phi = \mathbf{h} & \text{in } \eta^{-1}P \setminus T \\ \operatorname{div}(\mathbf{v}) = g & \text{in } \eta^{-1}P \setminus T \\ \int_{\eta^{-1}P \setminus T} \phi dx = 0 \\ \mathbf{v} = 0 & \text{on } \partial T \\ \mathbf{v} \text{ is } \eta^{-1}P\text{-periodic.} \end{cases}$$

686 *There exists a constant $C > 0$ independent of η , \mathbf{h} and g such that*

687

$$688 \quad (5.8) \quad \|\nabla \mathbf{v}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})} + \|\phi\|_{L^2(\eta^{-1}P \setminus T)} \\ 689 \quad \leq C(\eta^{-1}\|\mathbf{h} - \langle \mathbf{h} \rangle\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^d)} + \eta^{-d}|\langle \mathbf{h} \rangle| + \|g\|_{L^2(\eta^{-1}P \setminus T)}).$$

691 *Proof.* From Lemma 2.2.4 in [7], for any $\eta > 0$, there exists a linear ‘‘Bogovskii’s’’
692 operator $B_\eta : L^2(P \setminus (\eta T)) \rightarrow H^1(P \setminus (\eta T), \mathbb{R}^d)$ satisfying for any $\phi \in L^2(P \setminus (\eta T))$
693 such that $\int_{P \setminus (\eta T)} \phi dy = 0$:

694 (i) $\operatorname{div}(B_\eta \phi) = \phi$,

695 (ii) $B_\eta \phi = 0$ on $\partial(\eta T)$,

696 (iii) $B_\eta \phi$ is P -periodic,

697 (iv) $\|\nabla(B_\eta \phi)\|_{L^2(P \setminus (\eta T), \mathbb{R}^{d \times d})} \leq C\|\phi\|_{L^2(P \setminus (\eta T))}$ for a constant C independent of η
698 and ϕ .

For any $\tilde{\phi} \in L^2(\eta^{-1}P \setminus T)$ such that $\int_{\eta^{-1}P \setminus T} \tilde{\phi} dy = 0$, we define

$$\tilde{B}_\eta(\tilde{\phi}) := \eta^{-1}[B_\eta(\tilde{\phi}(\eta^{-1} \cdot))(\eta \cdot)].$$

699 The operator $\tilde{B}_\eta : L^2(\eta^{-1}P \setminus T) \rightarrow H^1(\eta^{-1}P \setminus T, \mathbb{R}^d)$ satisfies the following properties:

700 for any $\tilde{\phi} \in L^2(\eta^{-1}P \setminus T)$ such that $\int_{\eta^{-1}P \setminus T} \tilde{\phi} dx = 0$,

701 (i) $\operatorname{div}(\tilde{B}_\eta \tilde{\phi}) = \tilde{\phi}$ in $\eta^{-1}P \setminus T$,

702 (ii) $\tilde{B}_\eta \tilde{\phi} = 0$ on ∂T ,

703 (iii) $\tilde{B}_\eta \tilde{\phi}$ is $\eta^{-1}P$ -periodic,

704 (iv) $\|\nabla(\tilde{B}_\eta \tilde{\phi})\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})} \leq C\|\tilde{\phi}\|_{L^2(\eta^{-1}P \setminus T)}$ for a constant C independent of η
705 and $\tilde{\phi}$.

706 The proof follows then classically along the lines of Corollary 4.4. Upon an integration
707 by parts and by using Lemma 5.2, it is readily obtained with $\mathbf{w} := \mathbf{v} - \tilde{B}_\eta g$:

$$708 \quad \|\nabla \mathbf{w}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})}^2 = \int_{\eta^{-1}P \setminus T} \mathbf{h} \cdot \mathbf{w} dy \\ 709 \quad = \int_{\eta^{-1}P \setminus T} (\mathbf{h} - \langle \mathbf{h} \rangle) \cdot (\mathbf{w} - \langle \mathbf{w} \rangle) dy + \int_{\eta^{-1}P \setminus T} \langle \mathbf{h} \rangle \cdot \langle \mathbf{w} \rangle dy \\ 710 \quad \leq C(\|\mathbf{h} - \langle \mathbf{h} \rangle\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^d)} \|\mathbf{w} - \langle \mathbf{w} \rangle\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^d)} + \eta^{-d}|\langle \mathbf{h} \rangle| |\langle \mathbf{w} \rangle|) \\ 711 \quad (5.9) \quad \leq C(\eta^{-1}\|\mathbf{h} - \langle \mathbf{h} \rangle\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^d)} + \eta^{-d}|\langle \mathbf{h} \rangle|) \|\nabla \mathbf{w}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})}$$

for a constant $C > 0$ independent of η and \mathbf{h} . This implies

$$\|\nabla \mathbf{v}\|_{L^2(\eta^{-1}P \setminus T)} \leq \|\nabla \mathbf{w}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})} + \|\nabla(\tilde{B}_\eta g)\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})} \\ \leq C(\|\nabla \mathbf{w}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})} + \|g\|_{L^2(\eta^{-1}P \setminus T)}),$$

713 whence the bound on $\|\nabla \mathbf{v}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})}$ by using (5.9). The bound for the pressure
714 is obtained by writing

715

$$716 \quad \int_{\eta^{-1}P \setminus T} \phi^2 dx = \int_{\eta^{-1}P \setminus T} \phi \operatorname{div}(B_\eta \phi) dx$$

$$717 \quad = - \int_{\eta^{-1}P \setminus T} \nabla \phi \cdot B_\eta \phi dx = \int_{\eta^{-1}P \setminus T} (\nabla \mathbf{v} : \nabla(B_\eta \phi) - \mathbf{h} \cdot (B_\eta \phi)) dx,$$

718

719 from where (5.8) follows analogously. \square

720 **5.2. Asymptotic convergences of homogenized tensors in the low vol-**
721 **ume fraction limit $\eta \rightarrow 0$.** The asymptotics of the corrector tensors $(\mathcal{X}_j^0, \alpha_j^0)$ and
722 of \mathcal{X}^{0*} (defined in (3.3) and (3.7)) have been obtained in of Theorem 3.1 in [6]. The
723 following proposition extends this result to the whole family of tensors $(\mathcal{X}^k, \alpha_j^k)_{k \in \mathbb{N}}$
724 and $(\mathcal{X}^{k*})_{k \in \mathbb{N}}$.

PROPOSITION 5.4. *Assume $d \geq 3$. For any $k \geq 0$ and $1 \leq j \leq d$, denote by $(\tilde{\mathcal{X}}_j^{2k}, \tilde{\alpha}_j^{2k})$ and $(\tilde{\mathcal{X}}_j^{2k+1}, \tilde{\alpha}_j^{2k+1})$ the rescaled tensors in $\eta^{-1}P \setminus T$ defined by*

$$\begin{cases} \tilde{\mathcal{X}}_j^{2k}(x) := \eta^{(d-2)(k+1)} \mathcal{X}_j^{2k}(\eta x) & \tilde{\mathcal{X}}_j^{2k+1}(x) := \eta^{(d-2)(k+1)} \mathcal{X}_j^{2k+1}(\eta x) \\ \tilde{\alpha}_j^{2k}(x) := \eta^{(d-2)(k+1)-1} \alpha_j^{2k}(\eta x) & \tilde{\alpha}_j^{2k+1}(x) := \eta^{(d-2)(k+1)-1} \alpha_j^{2k+1}(\eta x) \end{cases}$$

725 for any $x \in \eta^{-1}P \setminus T$. Then:

1. there exists a constant C independent of $\eta > 0$ such that

$$\forall \eta > 0, \|\nabla \tilde{\mathcal{X}}_j^{2k}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})} + \|\tilde{\alpha}_j^{2k}\|_{L^2(\eta^{-1}P \setminus T)} \leq C,$$

$$\forall \eta > 0, \|\nabla \tilde{\mathcal{X}}_j^{2k+1}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})} + \|\tilde{\alpha}_j^{2k+1}\|_{L^2(\eta^{-1}P \setminus T)} \leq C;$$

726 2. the following convergences hold as $\eta \rightarrow 0$:

$$727 \quad (5.10) \quad (\tilde{\mathcal{X}}_i^{2k}, \tilde{\alpha}_i^{2k}) \rightharpoonup (c_{ij}^{2k} \Psi_j, c_{ij}^{2k} \sigma_j) \quad \text{weakly in } H_{loc}^1(\mathbb{R}^d \setminus T, \mathbb{R}^d) \times L_{loc}^2(\mathbb{R}^d \setminus T),$$

$$728 \quad (5.11) \quad (\tilde{\mathcal{X}}_i^{2k+1}, \tilde{\alpha}_i^{2k+1}) \rightharpoonup (0, 0) \quad \text{weakly in } H_{loc}^1(\mathbb{R}^d \setminus T, \mathbb{R}^d) \times L_{loc}^2(\mathbb{R}^d \setminus T),$$

$$729 \quad (5.12) \quad \mathcal{X}^{2k*} \sim \frac{1}{\eta^{(d-2)(k+1)}} c^{2k},$$

$$730 \quad (5.13) \quad \mathcal{X}^{2k+1*} = o\left(\frac{1}{\eta^{(d-2)(k+1)}}\right),$$

731

where c_{ij}^{2k} denotes the coefficients of the matrix valued tensor $c^{2k} := (c_{ij}^{2k})_{1 \leq i, j \leq d}$ of order $2k$ given by

$$c^{2k} := F^{-(k+1)} J^{2k} \quad \text{with } J^{2k} = \overbrace{I \otimes I \otimes \dots \otimes I}^{k \text{ times}}.$$

732 *Proof.* The result is proved by induction on k .

733 1. *Case $2k$ with $k = 0$.* The tensor $(\tilde{\mathcal{X}}_i^0, \tilde{\alpha}_i^0)$ satisfies

$$734 \quad (5.14) \quad \begin{cases} -\Delta \tilde{\mathcal{X}}_i^0 + \nabla \tilde{\alpha}_i^0 = \eta^d \mathbf{e}_i \text{ in } \mathbb{R}^d \setminus T \\ \operatorname{div}(\tilde{\mathcal{X}}_i^0) = 0 \text{ in } \mathbb{R}^d \setminus T, \end{cases}$$

as well as the other boundary conditions of (5.7). Lemma 5.3 implies then

$$\|\nabla \tilde{\mathcal{X}}_i^0\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})} + \|\tilde{\alpha}_i^0\|_{L^2(\eta^{-1}P \setminus T)} \leq C\eta^{-d}\eta^d|\langle \mathbf{e}_i \rangle| \leq C.$$

From (5.4), we also obtain that $\langle \tilde{\mathcal{X}}_i^0 \rangle$ is bounded. Hence, up to extracting a subsequence, there exists a constant matrix $c^0 := (c_{ij}^0)_{1 \leq i, j \leq d}$, and fields $(\widehat{\Psi}_i^0, \widehat{\sigma}_i^0)_{1 \leq i \leq d}$ such that

$$\langle \tilde{\mathcal{X}}_i^0 \rangle \cdot \mathbf{e}_j \rightarrow c_{ij}^0,$$

$$(\tilde{\mathcal{X}}_i^0, \tilde{\alpha}_i^0) \rightharpoonup (\widehat{\Psi}_i^0, \widehat{\sigma}_i^0) \text{ weakly in } H_{loc}^1(\eta^{-1}P \setminus T, \mathbb{R}^d) \times L_{loc}^2(\eta^{-1}P \setminus T).$$

Multiplying (5.14) by a compactly supported test function $\Phi \in C_c^\infty(\mathbb{R}^d \setminus T)$ and integrating by parts yields

$$\int_{\eta^{-1}P \setminus T} (\nabla \tilde{\mathcal{X}}_i^0 : \nabla \Phi - \tilde{\alpha}_i^0 \operatorname{div}(\Phi)) dx = \int_{\eta^{-1}P \setminus T} \eta^d \Phi \cdot \mathbf{e}_i dx.$$

Passing to the limit as $\eta \rightarrow 0$ implies then

$$\begin{cases} -\Delta \widehat{\Psi}_i^0 + \nabla \widehat{\sigma}_i^0 = 0 \text{ in } \mathbb{R}^d \setminus T \\ \operatorname{div}(\widehat{\Psi}_i^0) = 0 \text{ in } \mathbb{R}^d \setminus T \\ \widehat{\Psi}_i^0 = 0 \text{ on } \partial T. \end{cases}$$

By applying the point (5.6) of Lemma 5.2 and by using the lower semi-continuity of the Lebesgue space norms, we infer $(\widehat{\Phi}_i^0 - c_{ij}^0 \mathbf{e}_j, \widehat{\sigma}_i^0) \in \mathcal{D}^{1,2}(\mathbb{R}^d \setminus T) \times L^2(\mathbb{R}^d \setminus T)$ (see the proof of Theorem 3.1 in [6] for a detailed justification). By linearity, it is then necessary that $(\widehat{\Phi}_i^0, \widehat{\sigma}_i^0) = (c_{ij}^0 \Psi_j, c_{ij}^0 \sigma_j)$ where (Ψ_j, σ_j) are the solution to the exterior problem (5.1). In order to identify the coefficient c_{ij}^0 , we integrate (5.1) by parts against the test function $\Phi = \mathbf{e}_j$ then yields

$$0 = \eta^d \int_{\eta^{-1}P \setminus T} \delta_{ij} dx + \int_{\partial T} \mathbf{e}_j \cdot (\nabla \tilde{\mathcal{X}}_i^0 - \tilde{\alpha}_i^0 \mathbf{I}) \cdot \mathbf{n} dx.$$

Passing to the limit as $\eta \rightarrow 0$ by using the continuity of the drag force with respect to the weak convergence and (5.2) yields then

$$0 = \delta_{ij} + \int_{\partial T} \mathbf{e}_j \cdot (\nabla \widehat{\Phi}_i^0 - \widehat{\sigma}_i^0) \cdot \mathbf{n} dx = \delta_{ij} - c_{ip}^0 F_{pj}.$$

This implies $c^0 = F^{-1}$ as claimed and the convergence of the whole sequence by uniqueness of the limit. The asymptotic for \mathcal{X}^{0*} as $\eta \rightarrow 0$ follows by the change of variable $y = \eta x$:

$$\mathcal{X}_{ij}^{0*} = \mathbf{e}_i \cdot \int_{P \setminus (\eta T)} \mathcal{X}_j^0 dy = \eta^{2-d} \eta^d \mathbf{e}_i \cdot \int_{\eta^{-1}P \setminus T} \tilde{\mathcal{X}}_j^0 dy \sim \eta^{2-d} \langle \tilde{\mathcal{X}}_j^0 \rangle \cdot \mathbf{e}_i \sim \eta^{2-d} c_{ji}^0.$$

735 2. Case $2k + 1$ with $k = 0$. The tensor $(\tilde{\mathcal{X}}_i^1, \tilde{\alpha}_i^1)$ satisfies

$$736 \quad (5.15) \quad \begin{cases} -\Delta \tilde{\mathcal{X}}_i^1 + \nabla \tilde{\alpha}_i^1 = \eta(2\partial_l \tilde{\mathcal{X}}_i^0 - \tilde{\alpha}_j^0 \mathbf{e}_l) \otimes \mathbf{e}_l \text{ in } \eta^{-1}P \setminus T \\ \operatorname{div}(\tilde{\mathcal{X}}_i^1) = -\eta(\tilde{\mathcal{X}}_j^0 - \langle \tilde{\mathcal{X}}_j^0 \rangle) \cdot \mathbf{e}_l \otimes \mathbf{e}_l \text{ in } \eta^{-1}P \setminus T. \end{cases}$$

Applying [Lemma 5.3](#) and remarking that $\langle 2\partial_l \tilde{\mathcal{X}}_i^0 - \tilde{\alpha}_i^0 e_l \rangle = 0$, we obtain

$$\|\nabla \tilde{\mathcal{X}}_i^1\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})} + \|\tilde{\alpha}_i^1\|_{L^2(\eta^{-1}P \setminus T)} \leq C.$$

Integrating [\(5.15\)](#) by parts against a compactly supported test function $\Phi \in C^c(\mathbb{R}^d \setminus T)$ and passing to the limit as $\eta \rightarrow 0$, we obtain similarly the existence of a matrix valued tensor $c^1 := (c_{ij}^1)_{1 \leq i, j \leq d}$ (of order 1) such that, up to the extraction of a subsequence:

$$\langle \tilde{\mathcal{X}}_i^1 \rangle \cdot e_j \rightarrow c_{ij}^1,$$

$$(\tilde{\mathcal{X}}_i^1, \tilde{\alpha}_i^1) \rightharpoonup (c_{ij}^1 \Psi_j, c_{ij}^1 \sigma_j) \text{ weakly in } H_{loc}^1(\eta^{-1}P \setminus T, \mathbb{R}^d) \times L_{loc}^2(\eta^{-1}P \setminus T).$$

737 Integrating [\(5.15\)](#) by parts against the test function e_j and passing to the limit as
738 $\eta \rightarrow 0$ yields in this situation $0 = c_{ij}^1 F_{pj}$ whence $c^1 = 0$.

739 **3. General case.** Assuming that the result holds till rank k , the differential equations
740 satisfied by the rescaled tensors in $\eta^{-1}P \setminus T$ read:

$$741 \quad (5.16) \quad \begin{cases} -\Delta \tilde{\mathcal{X}}_i^{2k+2} + \nabla \tilde{\alpha}_i^{2k+2} = \eta^{d-1} (\partial_l \tilde{\mathcal{X}}_i^{2k+1} - \tilde{\alpha}_i^{2k+1} e_l) \otimes e_l + \eta^d \tilde{\mathcal{X}}_i^{2k} \otimes I \\ \operatorname{div}(\tilde{\mathcal{X}}_i^{2k+2}) = -\eta^{d-1} (\tilde{\mathcal{X}}_i^{2k+1} - \langle \tilde{\mathcal{X}}_i^{2k+1} \rangle) \cdot e_l \otimes e_l. \end{cases}$$

742

$$743 \quad (5.17) \quad \begin{cases} -\Delta \tilde{\mathcal{X}}_i^{2k+3} + \nabla \tilde{\alpha}_i^{2k+3} = \eta (\partial_l \tilde{\mathcal{X}}_i^{2k+2} - \tilde{\alpha}_i^{2k+2} e_l) \otimes e_l + \eta^d \tilde{\mathcal{X}}_i^{2k+1} \otimes I \\ \operatorname{div}(\tilde{\mathcal{X}}_i^{2k+3}) = -\eta (\tilde{\mathcal{X}}_i^{2k+2} - \langle \tilde{\mathcal{X}}_i^{2k+2} \rangle) \cdot e_l \otimes e_l. \end{cases}$$

Using [Lemma 5.2](#), [Lemma 5.3](#) and the point 1. of the proposition at rank k , we readily obtain

$$\|\nabla \tilde{\mathcal{X}}_i^{2k+2}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})} + \|\tilde{\alpha}_i^{2k+2}\|_{L^2(\eta^{-1}P \setminus T)} \leq C,$$

$$\|\nabla \tilde{\mathcal{X}}_i^{2k+3}\|_{L^2(\eta^{-1}P \setminus T, \mathbb{R}^{d \times d})} + \|\tilde{\alpha}_i^{2k+3}\|_{L^2(\eta^{-1}P \setminus T)} \leq C.$$

Repeating the above arguments, we obtain, up to the extraction of a subsequence, the existence of matrix valued tensors c^{2k+2} and c^{2k+3} such that

$$\langle \tilde{\mathcal{X}}_i^{2k+2} \rangle \cdot e_j \rightarrow c_{ij}^{2k+2} \text{ and } \langle \tilde{\mathcal{X}}_i^{2k+3} \rangle \cdot e_j \rightarrow c_{ij}^{2k+3},$$

$$(\tilde{\mathcal{X}}_i^{2k+2}, \tilde{\alpha}_i^{2k+2}) \rightharpoonup (c_{ij}^{2k+2} \Psi_j, c_{ij}^{2k+2} \sigma_j) \text{ weakly in } H_{loc}^1(\mathbb{R}^d \setminus T, \mathbb{R}^d) \times L_{loc}^2(\mathbb{R}^d \setminus T),$$

$$(\tilde{\mathcal{X}}_i^{2k+3}, \tilde{\alpha}_i^{2k+3}) \rightharpoonup (c_{ij}^{2k+3} \Psi_j, c_{ij}^{2k+3} \sigma_j) \text{ weakly in } H_{loc}^1(\mathbb{R}^d \setminus T, \mathbb{R}^d) \times L_{loc}^2(\mathbb{R}^d \setminus T).$$

The last step consists in integrating [\(5.16\)](#) and [\(5.17\)](#) by part against the test function e_j and to pass to the limit as $\eta \rightarrow 0$ in order to identify c_{ij}^{2k+2} and c_{ij}^{2k+3} . Performing this computation as above yields

$$0 = c_{ij}^{2k} \otimes I - c_{ip}^{2k+2} F_{pj},$$

$$0 = c_{ij}^{2k+1} \otimes I - c_{ip}^{2k+3} F_{pj}$$

744 from where we infer $c^{2k+2} = c^{2k} F^{-1} \otimes I$, $c^{2k+3} = c^{2k+1} F^{-1} \otimes I$, whence the result
745 (recall $c^1 = 0$ from the point 2. of the proof). \square

746 Using the identity of (3.20), we obtain the asymptotics for the coefficients M^k of the
747 infinite order homogenized equation (1.9).

748 **COROLLARY 5.5.** *Assume $d \geq 3$. The following convergences hold for the matrix*
749 *valued tensors M^k as $\eta \rightarrow 0$:*

$$750 \quad (5.18) \quad M^0 \sim \eta^{d-2} F,$$

$$751 \quad (5.19) \quad M^1 = o(\eta^{d-2}),$$

$$752 \quad (5.20) \quad M^2 \rightarrow -I,$$

$$753 \quad (5.21) \quad \forall k \geq 1, M^{2k} = o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right),$$

$$754 \quad (5.22) \quad \forall k \geq 1, M^{2k+1} = o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right).$$

755

Proof. We replace the asymptotics of (5.4) in the explicit formula for the tensors M^k given in (3.20). (5.18) is an immediate consequence of $M^0 = (\mathcal{X}^{0*})^{-1}$. The convergence (5.20) is obtained by writing, according to (3.20):

$$\begin{aligned} M^2 &= -(\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{2*} \otimes (\mathcal{X}^{0*})^{-1} + (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{1*} \otimes (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{1*} \otimes (\mathcal{X}^{0*})^{-1} \\ &= -\frac{\eta^{2(d-2)}}{\eta^{2(d-2)}} F \otimes c^2 \otimes F + o\left(\frac{\eta^{3(d-2)}}{\eta^{2(d-2)}}\right) = -(FF^{-2}F) \otimes I + o(\eta^{d-2}) \\ &= -I + o(\eta^{d-2}). \end{aligned}$$

756 For M^{2k+1} with $k \geq 0$, we use (3.20) and we observe that, for any $0 \leq p \leq 2k+1$
757 and indices $1 \leq i_1 \dots i_p \leq 2k+1$ such that $i_1 + \dots + i_p = 2k+1$, there exists at least
758 one odd index i_q with $1 \leq q \leq p$. Using (5.12) and (5.13), we arrive at

759

$$760 \quad (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_1*} \otimes \dots \otimes (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_p*} \otimes (\mathcal{X}^{0*})^{-1} \\ 761 \quad = o\left(\frac{\eta^{(p+1)(d-2)}}{\eta^{(p+\lfloor i_1/2 \rfloor + \dots + \lfloor i_p/2 \rfloor)(d-2)}}\right) = o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right),$$

762

763 which implies (5.19) and (5.22). For M^{2k} with $k > 1$, we separate the summands
764 of (3.20) into two categories. For a given p indices such that $1 \leq p \leq 2k$ and
765 $i_1 + \dots + i_p = 2k$, there are only two possibilities:

1. either there exists at least one odd index i_q , in that case the above reasoning implies as well

$$(\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_1*} \otimes \dots \otimes (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_p*} \otimes (\mathcal{X}^{0*})^{-1} = o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right).$$

766 2. or all indices $i_1 + \dots + i_p$ are even, in that case we may write, as $\eta \rightarrow 0$:

767

$$\begin{aligned} &(\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_1*} (\mathcal{X}^{0*})^{-1} \otimes \dots \otimes (\mathcal{X}^{0*})^{-1} \otimes \mathcal{X}^{i_p*} \otimes (\mathcal{X}^{0*})^{-1} \\ 768 \quad &\sim \frac{\eta^{(d-2)(p+1)}}{\eta^{(d-2)(p+k)}} F \otimes c^{i_1} \otimes \dots \otimes F \otimes c^{i_p} \otimes F \\ 769 \quad &\sim \frac{1}{\eta^{(d-2)(k-1)}} (FF^{-(i_1/2+1)}) \times \dots \times (FF^{-(i_p/2+1)}) F J^{2k} \\ 770 \quad &\sim \frac{1}{\eta^{(d-2)(k-1)}} F^{-(k-1)} \otimes J^{2k}. \end{aligned}$$

771

Note that in the latter case, the asymptotic does not depend on the choice of indices $i_1 + \dots + i_p = 2k$. Therefore, by isolating the terms featuring only even indices $i_1 := 2j_1, \dots, i_p := 2j_p$ in (3.20), we obtain finally

$$M^{2k} = \frac{1}{\eta^{(d-2)(k-1)}} F^{-(k-1)} \otimes J^{2k} \left(\sum_{p=1}^{2k} (-1)^p \sum_{\substack{2j_1 + \dots + 2j_p = 2k \\ 1 \leq j_1, \dots, j_p \leq k}} 1 \right) + o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right).$$

772 The asymptotic (5.21) follows from the fact that the summation over p in the above
773 expression is zero (see e.g. the end of the proof of Corollary 6 in [34]). \square

774 *Remark 5.6.* We have therefore obtained the following asymptotic estimates for
775 the coefficients $\varepsilon^{k-2} M^k$ of the infinite order homogenized equation (1.9) as $\eta \rightarrow 0$:

$$\begin{aligned} 776 \quad & \varepsilon^{-2} M^0 \sim (\eta^{d-2}/\varepsilon^2) F, \\ 777 \quad & \varepsilon^{-1} M^1 = o(\varepsilon(\eta^{d-2}/\varepsilon^2)), \\ 778 \quad & \varepsilon^0 M^2 \rightarrow -I, \\ 779 \quad & \varepsilon^{2k-2} M^{2k} = o\left((\varepsilon^2/\eta^{d-2})^{k-1}\right) \text{ for } k \geq 1, \\ 780 \quad & \varepsilon^{2k-1} M^{2k+1} = o\left(\varepsilon(\varepsilon^2/\eta^{d-2})^{k-1}\right) \text{ for } k \geq 1. \end{aligned}$$

782 These asymptotics bring into play the ratio ε^2/η^{d-2} and so the critical scaling $\eta_{\text{crit}} \sim$
783 $\varepsilon^{2/(d-2)}$. They imply thus the ‘‘coefficient-wise’’ convergence of (1.9) to the Brinkman
784 regime (1.3) at the critical rate $\eta \sim \varepsilon^{2/(d-2)}$, in which case $\varepsilon^{-2} M^0 \rightarrow F$ and
785 $\varepsilon^{k-2} M^k \rightarrow 0$ for any $k > 2$. Note that $\varepsilon^0 M^2 \rightarrow -I$ whatever the rate of conver-
786 gence at which $\eta \rightarrow 0$. The Darcy regimes (1.4) and (1.5) correspond to the situation
787 where $\eta^{d-2}/\varepsilon^2 \rightarrow +\infty$; in that case the zeroth order term $\varepsilon^{-2} M^0$ is dominant.

788 Finally, the leading coefficients of the Stokes regime (1.2) are retrieved for $\eta =$
789 $o(\varepsilon^{2/(d-2)})$, since in this case, $\varepsilon^{-2} M^0 \rightarrow 0$, $\varepsilon^{-1} M^1 \rightarrow 0$ and $\varepsilon^0 M^2 \rightarrow -I$. However
790 the present analysis is not sufficient to conclude that the coefficients $\varepsilon^{k-2} M^k$ of order
791 $k > 2$ converge to zero in the subcritical regime $\eta = o(\varepsilon^{2/(d-2)})$. Indeed $\varepsilon^{k-2} M^k$ is just
792 bounded by $(\varepsilon^2/\eta^{d-2})^k$, a quantity which can potentially blow up for too small values
793 of η . This matter is to be addressed in a future work through a more accurate analysis
794 of the rate of convergence of the coefficients \mathcal{X}^{2k*} and \mathcal{X}^{2k+1*} in the asymptotic (5.12)
795 and (5.13). To date, let us note that Jing obtained recently $\mathcal{X}^{0*} = F/\eta^{d-2} + O(1)$ in
796 the scalar case (proof of the Lemma 5.1 in [41]) by using layer potential techniques.

Remark 5.7. From the estimates of (5.4), we obtain that the coefficients (\mathbb{D}_K^k) of
(1.6) satisfy the same asymptotic convergences of Corollary 5.5. Indeed, by using the
definition (3.21) we find that there exists a constant $C > 0$ independent of η such
that for any $K \geq 0$, $K + 1 \leq k \leq 2K + 1$ and $1 \leq j \leq d$:

$$\begin{aligned} & \|\nabla \mathbf{N}_j^K\|_{L^2(Y, \mathbb{R}^{d \times d})} \leq C \eta^{-(d-2)\lfloor K/2 \rfloor} \eta^{d/2-1} \\ & \|\beta_j^{k-K-1}\|_{L^2(Y, \mathbb{R}^d)} \leq C \eta^{-(d-2)\lfloor \frac{k-K-1}{2} \rfloor + 1} \eta^{d/2}. \end{aligned}$$

Applying the inequality $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$, we obtain that the coefficients \mathbb{A}_{ij}^k of
Proposition 4.10 satisfy for $K + 1 \leq k \leq 2K + 1$,

$$|\mathbb{A}_{ij}^k| \leq C \eta^d \eta^{-\lfloor \frac{k-1}{2} \rfloor (d-2)} \leq C \eta^2 M_{ij}^k.$$

797 Hence the discrepancy induced by the coefficients A_{ij}^k is small and $\mathbb{D}_{K,ij}^k$ also satisfies
 798 (5.21) and (5.22) for $K + 1 \leq k \leq 2K + 1$. Similarly (see Corollary 7 of [34]), we may
 799 show that $\mathbb{D}_0^2 \rightarrow -I$. The remaining coefficients \mathbb{D}_K^k are equal to M^k .

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