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Jonathan Chappelon, Jorge Luis Ramírez Alfonsín

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# THE SQUARE FROBENIUS NUMBER

JONATHAN CHAPPELON AND JORGE LUIS RAMÍREZ ALFONSÍN

ABSTRACT. Let  $S = \langle s_1, \dots, s_n \rangle$  be a numerical semigroup generated by the relatively prime positive integers  $s_1, \dots, s_n$ . Let  $k \geq 2$  be an integer. In this paper, we consider the following  $k$ -power variant of the Frobenius number of  $S$  defined as

$${}^k g(S) := \text{the largest } k\text{-power integer not belonging to } S.$$

We investigate the case  $k = 2$ . We give an upper bound for  ${}^2 g(S_A)$  for an infinity family of semigroups  $S_A$  generated by *arithmetic progressions*. The latter turns out to be the exact value of  ${}^2 g(\langle s_1, s_2 \rangle)$  under certain conditions. We present an exact formula for  ${}^2 g(\langle s_1, s_1 + d \rangle)$  when  $d = 3, 4$  and  $5$ , study  ${}^2 g(\langle s_1, s_1 + 1 \rangle)$  and  ${}^2 g(\langle s_1, s_1 + 2 \rangle)$  and put forward two relevant conjectures. We finally discuss some related questions.

## 1. INTRODUCTION

Let  $s_1, \dots, s_n$  be relatively prime positive integers. Let

$$S = \langle s_1, \dots, s_n \rangle = \left\{ \sum_{i=1}^n x_i s_i \mid x_i \text{ integer, } x_i \geq 0 \right\}$$

be the numerical semigroup generated by  $s_1, \dots, s_n$ . The largest integer which is not an element of  $S$ , denoted by  $g(S)$  or  $g(\langle s_1, \dots, s_n \rangle)$ , is called the *Frobenius number* of  $S$ . It is well known that  $g(\langle s_1, s_2 \rangle) = s_1 s_2 - s_1 - s_2$ . However, calculate  $g(S)$  is a difficult problem in general. In [1] was shown that computing  $g(S)$  is *NP-hard*. We refer the reader to [2] for an extensive literature on the Frobenius number.

The non-negative integers not in  $S$  are called the *gaps* of  $S$ . The number of gaps of  $S$ , denoted by  $N(S)$  (that is,  $N(S) = \#(\mathbb{N} \setminus S)$ ) is called the *genus* of  $S$ . We recall that the *multiplicity* of  $S$  is the smallest positive element belonging to  $S$ .

Given a particular (arithmetical, number theoretical, etc.) *Property P*, one might consider the following two *P-type functions* :

$${}^P R(S) := \text{the largest integer having property } P \text{ not belonging to } S$$

and

$${}^P r(S) := \text{the smallest integer having property } P \text{ belonging to } S.$$

Notice that the multiplicity and the Frobenius number are *P-type functions* where  $P$  is the property of being a positive integer<sup>1</sup>.

In this spirit, we consider the following  $k$ -power variant of  $g(S)$ . Let  $k \geq 2$  be an integer, we define

$${}^k g(S) := \text{the largest } k\text{-power integer not belonging to } S.$$

We call  ${}^k g(S)$  the *k-power Frobenius number* of  $S$ .

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<sup>1</sup>*P-type functions* were introduced by the second author (often mentioned during his lectures) with the hope to understand better certain properties  $P$  in terms of linear forms.

In this paper we investigate the 2-power Frobenius number, we call it the *square Frobenius number*.

In Section 2, we study the square Frobenius number of semigroups  $S_A$  generated by *arithmetic progressions*. We give an upper bound for  ${}^2g(S_A)$  for an infinity family (Theorem 2.4) which turns out to be the exact value when the arithmetic progression consists of two generators (Corollary 2.5).

In Section 3, we present exact formulas for  ${}^2g(\langle a, a+3 \rangle)$  where  $a \geq 3$  is an integer not divisible by 3 (Theorem 3.1), for  ${}^2g(\langle a, a+4 \rangle)$  where  $a \geq 3$  is an odd integer (Theorem 3.2) and for  ${}^2g(\langle a, a+5 \rangle)$  where  $a \geq 2$  is an integer not divisible by 5 (Theorem 3.3).

In Sections 4 and 5, we turn our attention to the cases  $\langle a, a+1 \rangle$  where  $a \geq 2$  and  $\langle a, a+2 \rangle$  where  $a \geq 3$  is an odd integer. We present formulas for the corresponding square Frobenius number in the case when neither of the generators are square integers (Propositions 4.1 and 5.1). We also put forward two conjectures on the values of  ${}^2g(\langle a, a+1 \rangle)$  and  ${}^2g(\langle a, a+2 \rangle)$  in the case when one of the generators is a square integer (Conjectures 4.2 and 5.2). The conjectured values have an unexpected close connection with a known recursive sequence (Equation (10)) and in which  $\sqrt{2}$  and  $\sqrt{3}$  (strangely) appear. A number of computer experiments support our conjectures.

Finally, Section 6 contains some concluding remarks.

## 2. ARITHMETIC PROGRESSION

Let  $a, d$  and  $k$  be positive integers. In this section we consider a semigroup  $S_A$  generated by an *arithmetic progression*, that is, a sequence of relatively prime positive integers  $a, a+d, \dots, a+kd$ . Since the generators are relatively prime, we have, in particular,  $\gcd(a, d) = 1$ . The following result is given in [3]; for a short proof see [4, Lemma 1].

**Lemma 2.1.** *We have  $M \in \langle a, a+d, \dots, a+kd \rangle$  if and only if there exist integers  $x, y$  such that*

$$M = ax + dy \text{ with } 0 \leq y \leq kx.$$

**Proposition 2.2.** *Let  $i$  be an integer and let  $0 \leq \lambda_i \leq d-1$  be the integer such that  $\lambda_i a + i^2 \equiv 0 \pmod{d}$ . Then,*

$$(a-i)^2 \in S_A \iff (i+kd)^2 \leq \left( \left( \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor + k \right) d - \lambda_i \right) (a+kd).$$

*Proof.* Let  $i \in \mathbb{Z}$  and let  $\lambda_i \in \{0, 1, \dots, d-1\}$  such that  $\lambda_i a + i^2 \equiv 0 \pmod{d}$ . We have that

$$\begin{aligned} (a-i)^2 &= (a-2i)a + i^2 \\ &= (a-2i-\lambda_i)a + \frac{i^2 + \lambda_i a}{d}d \\ &= \left( a-2i-\lambda_i + \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor d \right) a + \left( \frac{i^2 + \lambda_i a}{d} - \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor \right) d. \end{aligned}$$

We thus have, by Lemma 2.1, that  $(a - i)^2 \in S_A$  if and only if

$$\begin{aligned}
 & \frac{i^2 + \lambda_i a}{d} - \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor a \leq k \left( a - 2i - \lambda_i + \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor d \right) \\
 \iff & \frac{i^2 + \lambda_i a}{d} \leq k(a - 2i - \lambda_i) + \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor (a + kd) \\
 \iff & i^2 + \lambda_i a \leq kd(a - 2i - \lambda_i) + \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor d(a + kd) \\
 \iff & i^2 + 2ikd \leq kda - \lambda_i(a + kd) + \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor d(a + kd) \\
 \iff & i^2 + 2ikd + k^2 d^2 \leq kd(a + kd) - \lambda_i(a + kd) + \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor d(a + kd) \\
 \iff & (i + kd)^2 \leq \left( \left( \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor + k \right) d - \lambda_i \right) (a + kd).
 \end{aligned}$$

□

**Definition 2.3.** Let  $a$  and  $d$  be relatively prime integers with  $d \geq 3$  and let

$$\lambda^* = \max_{0 \leq i \leq d-1} \{ \lambda_i \in \{0, 1, \dots, d-1\} \mid \lambda_i a + i^2 \equiv 0 \pmod{d} \}.$$

Let  $\{\alpha_1, \dots, \alpha_n\} \subseteq \{0, 1, \dots, d-1\}$  such that  $\lambda_{\alpha_j} = \lambda^*$  and take  $\alpha_{n+1} = d + \alpha_1$ . Let  $1 \leq j \leq n$  be the unique index such that

$$(\mu d + \alpha_j)^2 \leq (kd - \lambda^*)(a + kd) < (\mu d + \alpha_{j+1})^2,$$

for some integer  $\mu \geq 0$ .

Let us consider the integer function  $h(a, d, k)$  defined as

$$h(a, d, k) := (a - ((\mu - k)d + \alpha_{j+1}))^2.$$

**Theorem 2.4.** Let  $a$  and  $d$  be relatively prime integers with  $d \geq 3$ . If  $a + kd > 4k\lambda^*d^3$ , then

$${}^2g(S_A) \leq h(a, d, k).$$

*Proof.* It is known [3] that

$$g(S_A) = \left( \left\lfloor \frac{a-2}{k} \right\rfloor + 1 \right) a + (d-1)(a-1) - 1.$$

Since  $a^2 > (\lfloor \frac{a-2}{k} \rfloor + 1)a$ ,  $2akd > (d-1)(a-1)$  and  $(kd)^2 > 0$  then

$$g(S_A) < a^2 + 2kda + (kd)^2 = (a - (-kd))^2.$$

Therefore, it is enough to show that  $(a - i)^2 \in S$  for all  $-kd \leq i < (\mu - k)d + \alpha_{j+1}$ .

We have two cases.

*Case 1.*  $-kd \leq i \leq (\mu - k)d + \alpha_j$ .

We have that

$$\begin{aligned}
 (i + kd)^2 & \leq (\mu d + \alpha_j)^2 && (\text{since } i \leq (\mu - k)d + \alpha_j) \\
 & \leq (kd - \lambda^*)(a + kd) && (\text{by hypothesis}) \\
 & \leq \left( \left( \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor + k \right) d - \lambda_i \right) (a + kd) && (\text{since } \lambda^* \geq \lambda_i \text{ and } \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor > 0)
 \end{aligned}$$

Therefore, by Proposition 2.2, we obtain that  $(a - i)^2 \in S_A$ .

Case 2.  $(\mu - k)d + \alpha_j < i < (\mu - k)d + \alpha_{j+1}$ .

In this case we have that  $\alpha_j < i \pmod{d} < \alpha_{j+1}$  implying that  $\lambda_i \leq \lambda^* - 1$  and thus

$$(1) \quad (kd - \lambda_i)(a + kd) \geq (kd - \lambda^*)(a + kd) + (a + kd).$$

Moreover,

$$(2) \quad \begin{aligned} (i + kd)^2 &< (\mu d + \alpha_{j+1})^2 \\ &= ((\mu d + \alpha_j) + (\alpha_{j+1} - \alpha_j))^2 \\ &= (\mu d + \alpha_j)^2 + (\alpha_{j+1} - \alpha_j)(2(\mu d + \alpha_j) + (\alpha_{j+1} - \alpha_j)) \\ &= (\mu d + \alpha_j)^2 + (\alpha_{j+1} - \alpha_j)(2\mu d + \alpha_j + \alpha_{j+1}). \end{aligned}$$

Now, for each  $j \in \{1, \dots, n-1\}$  we have

$$(3) \quad \alpha_{j+1} - \alpha_j < \alpha_{j+1} \leq \alpha_n < d \text{ and } \alpha_j + \alpha_{j+1} < 2\alpha_n < 2d.$$

Moreover, for  $j = n$ , we know that  $\alpha_j = \alpha_n = d - \alpha_1$  and  $\alpha_{j+1} = \alpha_{n+1} = d + \alpha_1$  implying that

$$(4) \quad \alpha_{j+1} - \alpha_j = 2\alpha_1 \leq d \text{ and } \alpha_j + \alpha_{j+1} = 2d.$$

Therefore, combining (2), (3) and (4) we obtain

$$(5) \quad (i + kd)^2 < (\mu d + \alpha_j)^2 + d(2\mu d + 2d) = (\mu d + \alpha_j)^2 + 2d^2(\mu + 1)$$

for any  $j \in \{1, \dots, n\}$ .

Since

$$(\mu d + \alpha_j)^2 \stackrel{\text{(hypothesis)}}{\leq} (kd - \lambda^*)(a + kd) \stackrel{(1)}{\leq} (kd - \lambda_i)(a + kd) - (a + kd)$$

then

$$(6) \quad (i + kd)^2 < (kd - \lambda_i)(a + kd) + 2d^2(\mu + 1) - (a + kd).$$

We claim that  $2d^2(\mu + 1) \leq a + kd$ . We have two subcases

Subcase *i*) For  $j \in \{1, \dots, n-1\}$ . Since  $(\mu d + \alpha_j)^2 \leq (kd - \lambda^*)(a + kd) < (\mu d + \alpha_{j+1})^2$ ,  $\alpha_j \geq 0$  and  $\alpha_{j+1} \leq \alpha_n < d$ , we obtain that

$$\mu = \left\lfloor \frac{\sqrt{(kd - \lambda^*)(a + kd)}}{d} \right\rfloor.$$

Moreover, since  $a + kd > 4(kd - \lambda^*)d^2$ , it follows that

$$(7) \quad \mu > 2(kd - \lambda^*).$$

Therefore

$$(kd - \lambda^*)(a + kd) \geq (\mu d + \alpha_j)^2 \stackrel{(\alpha_j \geq 1)}{>} \mu^2 d^2 > (\mu^2 - 1)d^2 = (\mu - 1)(\mu + 1)d^2 \geq 2(kd - \lambda^*)(\mu + 1)d^2.$$

Obtaining the desired inequality  $a + kd > 2(\mu + 1)d^2$  for  $j \in \{1, 2, \dots, n-1\}$ .

Subcase *ii*) For  $j = n$ . From  $\alpha_j = \alpha_n = d - \alpha_1$  and  $\alpha_{j+1} = d + \alpha_1$ , we obtain

$$\left\lfloor \frac{\sqrt{(kd - \lambda^*)(a + kd)}}{d} \right\rfloor \in \{\mu, \mu + 1\}.$$

Since

$$((\mu + 1)d - \alpha_1)^2 \leq (kd - \lambda^*)(a + kd) < ((\mu + 1)d + \alpha_1)^2,$$

then

$$\mu + 1 > 2(kd - \lambda^*).$$

We thus have

$$\begin{aligned} (kd - \lambda^*)(a + kd) &\geq ((\mu + 1)d - \alpha_1)^2 \geq \left( (\mu + 1)d - \frac{d}{2} \right)^2 = \left( \mu^2 + \mu + \frac{1}{4} \right) d^2 \\ &> \mu(\mu + 1)d^2 \geq 2(kd - \lambda^*)(\mu + 1)d^2. \end{aligned}$$

Obtaining the desired inequality  $a + kd > 2(\mu + 1)d^2$ , for  $j = n$ .

Finally, for any  $j \in \{1, \dots, n\}$ , we obtain

$$(i + kd)^2 < (kd - \lambda_i)(a + kd) + 2d^2(\mu + 1) - (a + kd) \leq (kd - \lambda_i)(a + kd)$$

and we deduce that  $(a - i)^2 \in S_A$  from Proposition 2.2, in this case. □

We believe that the upper bound  $h(a, d, k)$  of  ${}^2g(S_A)$  given in Theorem 2.4 is actually an equality. We are able to establish the latter in the case when  $k = 1$  for any  $d \geq 3$ .

**Corollary 2.5.** *Let  $a$  and  $d$  be relatively prime integers with  $d \geq 3$ . If  $a + d > 4\lambda^*d^3$ , then*

$${}^2g(\langle a, a + d \rangle) = h(a, d, 1).$$

*Proof.* By Theorem 2.4, we have  ${}^2g(\langle a, a + d \rangle) \leq (a - ((\mu - 1)d + \alpha_{j+1}))^2$ . It is thus enough to show that  $(a - ((\mu - 1)d + \alpha_{j+1}))^2 \notin \langle a, a + d \rangle$ .

Let  $i = (\mu - 1)d + \alpha_{j+1}$ . We have

$$\begin{aligned} i^2 &= ((\mu - 1)d + \alpha_{j+1})^2 \\ &= ((\mu d + \alpha_j) - (d + \alpha_j - \alpha_{j+1}))^2 \\ &= (\mu d + \alpha_j)^2 - (d + \alpha_j - \alpha_{j+1})(2(\mu d + \alpha_j) - (d + \alpha_j - \alpha_{j+1})) \\ &= (\mu d + \alpha_j)^2 - (d + \alpha_j - \alpha_{j+1})((2\mu - 1)d + \alpha_j + \alpha_{j+1}). \end{aligned}$$

Since  $d + \alpha_j - \alpha_{j+1} \geq 2$  and  $(2\mu - 1)d + \alpha_j + \alpha_{j+1} > (2\mu - 1)d$ , it follows that

$$i^2 < (\mu d + \alpha_j)^2 - 2(2\mu - 1)d \leq (d - \lambda^*)(a + d) - 2(2\mu - 1)d.$$

Since  $a + d > 4\lambda^*d^3$  then  $a + d > (d - \lambda^*)d^2$ .

We already know that  $\mu > 2(d - \lambda^*)$  (see Equation (7)). Moreover, since  $\mu \geq 2$ , we have

$$d - \lambda^* < \frac{1}{2}\mu < 2(2\mu - 1).$$

Obtaining

$$i^2 < (d - \lambda^*)(a + d) - (d - \lambda^*)d = (d - \lambda^*)a$$

and

$$\frac{i^2 + \lambda_i a}{ad} = \frac{i^2 + \lambda^* a}{ad} < \frac{(d - \lambda^*)a + \lambda^* a}{ad} = 1.$$

We may thus deduce that

$$\left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor = 0.$$

Finally,

$$(i + d)^2 = (\mu d + \alpha_{j+1})^2 > (d - \lambda^*)(a + d) = \left( \left( \left\lfloor \frac{i^2 + \lambda a}{ad} \right\rfloor + 1 \right) d - \lambda_i \right) (a + d)$$

we deduce that  $(a - i)^2 \notin \langle a, a + d \rangle$  from Proposition 2.2, as desired. □

Unfortunately, the value of  ${}^2g(\langle a, a+d \rangle)$  given in the above corollary does not hold in general (if the condition  $a+d > 4\lambda^*d^3$  is not satisfied). However, as we will see below, the number of values of  $a$  not holding the equality  ${}^2g(\langle a, a+d \rangle) = h(a, d, 1)$  is finite for each fixed  $d$ .

### 3. FORMULAS FOR $\langle a, a+d \rangle$ WITH SMALL $d \geq 3$

In this section, we investigate the value of  ${}^2g(\langle a, a+d \rangle)$  when  $d$  is small.

For any positive integer  $d \geq 3$ , we may define the set  $E(d)$  to be the set of integers  $a$  coprime to  $d$  not holding the equality of Corollary 2.5, that is,

$$E(d) := \{a \in \mathbb{N} \setminus \{0, 1\} \mid \gcd(a, d) = 1 \text{ and } {}^2g(\langle a, a+d \rangle) \neq h(a, d, 1)\}.$$

Since  $\lambda^* \leq d-1$  then, from Corollary 2.5, we obtain that  $E(d) \subset [2, 4(d-1)d^3 - d] \cap \mathbb{N}$ . We completely determine the set  $E(d)$  for a few values of  $d \geq 3$  by computer calculations, see Table 1.

$d$	$ E(d) $	$E(d)$
3	0	$\emptyset$
4	0	$\emptyset$
5	5	$\{2, 4, 13, 27, 32\}$
6	0	$\emptyset$
7	10	$\{2, 3, 4, 9, 16, 18, 19, 23, 30, 114\}$
8	5	$\{5, 9, 21, 45, 77\}$
9	5	$\{2, 4, 7, 8, 16\}$
10	14	$\{3, 9, 13, 23, 27, 33, 43, 123, 133, 143, 153, 163, 333, 343\}$
11	14	$\{2, 3, 4, 5, 7, 8, 9, 14, 16, 18, 25, 36, 38, 47\}$
12	9	$\{13, 19, 25, 31, 67, 79, 139, 151, 235\}$

TABLE 1.  $E(d)$  for the first values of  $d \geq 3$ .

The exact values of  ${}^2g(\langle a, a+d \rangle)$  when  $a \in E(d)$ , for  $d \in \{3, \dots, 12\}$ , are given in Appendix A.

For each value  $d \in \{3, \dots, 12\}$ , it can be presented an explicit formula for  ${}^2g(\langle a, a+d \rangle)$  excluding the values given in Table 1. The latter can be done by using (essentially) the same arguments as those applied in the proofs of Theorem 2.4 and Corollary 2.5. We present the proof for the case  $d = 3$ .

**Theorem 3.1.** *Let  $a \geq 3$  be an integer not divisible by 3 and let  $S = \langle a, a+3 \rangle$ . Then,*

$${}^2g(S) = \begin{cases} (a - (3b - 1))^2 & \text{if either } (3b + 1)^2 \leq a + 3 < (3b + 2)^2 \text{ and } a \equiv 1 \pmod{3} \\ & \text{or } (3b + 1)^2 \leq 2(a + 3) < (3b + 2)^2 \text{ and } a \equiv 2 \pmod{3}, \\ (a - (3b + 1))^2 & \text{if either } (3b + 2)^2 \leq a + 3 < (3b + 4)^2 \text{ and } a \equiv 1 \pmod{3} \\ & \text{or } (3b + 2)^2 \leq 2(a + 3) < (3b + 4)^2 \text{ and } a \equiv 2 \pmod{3}. \end{cases}$$

*Proof.* Since  $g(S) = (a-1)(a+2) - 1 = a^2 + a - 3 < (a+1)^2$  then

$${}^2g(S) \leq (a-1)^2.$$

By Proposition 2.2, we know that

$$(8) \quad (a-i)^2 \in S \iff (i+3)^2 \leq \left(3 \left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right) (a+3),$$

where  $\lambda_i \in \{0, 1, 2\}$  such that  $\lambda_i a + i^2 \equiv 0 \pmod{3}$ , that is,

$$\lambda_i = \begin{cases} 0 & \text{if } i \equiv 0 \pmod{3} \text{ and } a \equiv 1, 2 \pmod{3}, \\ 1 & \text{if } i \equiv 1, 2 \pmod{3} \text{ and } a \equiv 2 \pmod{3}, \\ 2 & \text{if } i \equiv 1, 2 \pmod{3} \text{ and } a \equiv 1 \pmod{3}. \end{cases}$$

We have four cases.

*Case 1.* Suppose that  $a \equiv 1 \pmod{3}$  with  $(3b+1)^2 \leq a+3 < (3b+2)^2$ .  
If  $i \leq 3b-2$  then

$$(i+3)^2 = (3b+1)^2 \leq a+3 \leq \left(3 \left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right) (a+3),$$

Obtaining, by Equation (8), that  $(a-i)^2 \in S$ .

If  $i = 3b-1$  then

$$i^2 = (3b-1)^2 = 9b^2 - 6b + 1 \stackrel{b \geq 1}{<} 9b^2 + 6b - 2 = (3b+1)^2 - 3 \leq a,$$

obtaining that

$$0 \leq \frac{i^2 + \lambda_i a}{3a} = \frac{i^2 + 2a}{3a} < 1 \quad (\text{since } 3b-1 \equiv 2 \pmod{3})$$

and thus

$$\left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor = 0.$$

Moreover, since

$$\left(3 \left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right) (a+3) = a+3 < (3b+2)^2 = (i+3)^2,$$

therefore, by Equation (8), we have that  $(a-i)^2 \notin S$ .

*Case 2.* Suppose that  $a \equiv 1 \pmod{3}$  with  $(3b+2)^2 \leq a+3 < (3b+4)^2$ .  
If  $i \leq 3b-1$ , then

$$(i+3)^2 = (3b+2)^2 \leq a+3 \leq \left(3 \left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right) (a+3),$$

Obtaining, by Equation (8), that  $(a-i)^2 \in S$ .

If  $i = 3b$  then

$$(i+3)^2 = (3b+3)^2 \leq 3(3b+2)^2 \leq 3(a+3) \leq \left(3 \left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right) (a+3).$$

Obtaining, by Equation (8), that  $(a-i)^2 \in S$ .

If  $i = 3b+1$  then

$$i^2 = (3b+1)^2 = 9b^2 + 6b + 1 \stackrel{b \geq 1}{<} 9b^2 + 12b + 1 = (3b+2)^2 - 3 \leq a,$$

obtaining that

$$0 \leq \frac{i^2 + \lambda_i a}{3a} = \frac{i^2 + 2a}{3a} < 1 \quad (\text{since } 3b+1 \equiv 1 \pmod{3})$$

and thus

$$\left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor = 0.$$

Moreover, since

$$\left(3 \left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor + 3 - \lambda_i\right) (a+3) = a+3 < (3b+4)^2 = (i+3)^2,$$



therefore, by Equation (8), we have that  $(a - i)^2 \notin S$ .

*Case 3.* Suppose that  $a \equiv 2 \pmod{3}$  with  $(3b + 1)^2 \leq 2(a + 3) < (3b + 2)^2$ .  
If  $i \leq 3b - 2$  then

$$(i + 3)^2 = (3b + 1)^2 \leq 2(a + 3) \leq \left( 3 \left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor + 3 - \lambda_i \right) (a + 3),$$

Obtaining, by Equation (8), that  $(a - i)^2 \in S$ .

If  $i = 3b - 1$  then

$$i^2 = (3b - 1)^2 = 9b^2 - 6b + 1 \stackrel{b \geq 1}{<} 9b^2 + 6b - 5 = (3b + 1)^2 - 6 \leq 2a,$$

obtaining that

$$0 \leq \frac{i^2 + \lambda_i a}{3a} = \frac{i^2 + a}{3a} < 1 \quad (\text{since } 3b - 1 \equiv 2 \pmod{3})$$

and thus

$$\left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor = 0.$$

Moreover, since

$$\left( 3 \left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor + 3 - \lambda_i \right) (a + 3) = 2(a + 3) < (3b + 2)^2 = (i + 3)^2,$$

therefore, by Equation (8), we have that  $(a - i)^2 \notin S$ .

*Case 4.* Suppose that  $a \equiv 2 \pmod{3}$  with  $(3b + 2)^2 \leq 2(a + 3) < (3b + 4)^2$ .  
If  $i \leq 3b - 1$  then

$$(i + 3)^2 = (3b + 2)^2 \leq 2(a + 3) \leq \left( 3 \left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor + 3 - \lambda_i \right) (a + 3),$$

Obtaining, by Equation (8), that  $(a - i)^2 \in S$ .

If  $i = 3b$  then  $a^2 \in S$  when  $b = 0$  and

$$(i + 3)^2 = (3b + 3)^2 \stackrel{b \geq 1}{<} \frac{3}{2}(3b + 2)^2 \leq 3(a + 3) \leq \left( 3 \left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor + 3 - \lambda_i \right) (a + 3),$$

when  $b \geq 1$ . Therefore, by Equation (8), we have  $(a - i)^2 \in S$ .

If  $i = 3b + 1$  then

$$i^2 = (3b + 1)^2 = 9b^2 + 6b + 1 \stackrel{b \geq 1}{<} 9b^2 + 12b - 2 = (3b + 2)^2 - 6 \leq 2a,$$

when  $b \geq 1$  and clearly  $i^2 = 1 < 2a$  when  $b = 0$ , obtaining that

$$0 \leq \frac{i^2 + \lambda_i a}{3a} = \frac{i^2 + a}{3a} < 1$$

and

$$\left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor = 0.$$

Moreover, since

$$\left( 3 \left\lfloor \frac{i^2 + \lambda_i a}{3a} \right\rfloor + 3 - \lambda_i \right) (a + 3) = 2(a + 3) < (3b + 4)^2 = (i + 3)^2,$$

therefore, by Equation(8), we have that  $(a - i)^2 \notin S$ .

□

The proofs of the following two theorems are completely analogue to that of Theorem 3.1 with a larger number of cases to be analyzed (in each case, the appropriate inequality is obtained in order to apply Proposition 2.2).

**Theorem 3.2.** *Let  $a \geq 3$  be an odd integer and let  $S = \langle a, a + 4 \rangle$ . Then,*

$${}^2g(S) = \begin{cases} (a - (4b - 1))^2 & \text{if either } (4b + 1)^2 \leq a + 4 < (4b + 3)^2 \text{ and } a \equiv 1 \pmod{4} \\ & \text{or } (4b + 1)^2 \leq 3(a + 4) < (4b + 3)^2 \text{ and } a \equiv 3 \pmod{4}, \\ (a - (4b + 1))^2 & \text{if either } (4b + 3)^2 \leq a + 4 < (4b + 5)^2 \text{ and } a \equiv 1 \pmod{4} \\ & \text{or } (4b + 3)^2 \leq 3(a + 4) < (4b + 5)^2 \text{ and } a \equiv 3 \pmod{4}. \end{cases}$$

**Theorem 3.3.** *Let  $a \geq 2$  be an integer not divisible by 5 and let  $S = \langle a, a + 5 \rangle$ . Then,*

$${}^2g(S) = \begin{cases} (a - (5b - 2))^2 & \text{if either } (5b + 2)^2 \leq a + 5 < (5b + 3)^2 \text{ and } a \equiv 4 \pmod{5} \\ & \text{or } (5b + 2)^2 \leq 2(a + 5) < (5b + 3)^2 \text{ and } a \equiv 2 \pmod{5}, \\ (a - (5b - 1))^2 & \text{if either } (5b + 1)^2 \leq a + 5 < (5b + 4)^2 \text{ and } a \equiv 1 \pmod{5} \\ & \text{or } (5b + 1)^2 \leq 2(a + 5) < (5b + 4)^2 \text{ and } a \equiv 3 \pmod{5}, a \neq 13, \\ (a - (5b + 1))^2 & \text{if either } (5b + 4)^2 \leq a + 5 < (5b + 6)^2 \text{ and } a \equiv 1 \pmod{5} \\ & \text{or } (5b + 4)^2 \leq 2(a + 5) < (5b + 6)^2 \text{ and } a \equiv 3 \pmod{5}, \\ (a - (5b + 2))^2 & \text{if either } (5b + 3)^2 \leq a + 5 < (5b + 7)^2 \text{ and } a \equiv 4 \pmod{5}, a \neq 4 \\ & \text{or } (5b + 3)^2 \leq 2(a + 5) < (5b + 7)^2 \text{ and } a \equiv 2 \pmod{5}, a \neq 2, 27, 32. \end{cases}$$

#### 4. STUDY OF $\langle a, a + 1 \rangle$

We investigate the square Frobenius number of  $\langle a, a + 1 \rangle$  with  $a \geq 2$ . We first study the case when neither  $a$  nor  $a + 1$  is a square integer.

**Proposition 4.1.** *Let  $a$  be a positive integer such that  $b^2 < a < a + 1 < (b + 1)^2$  for some integer  $b \geq 1$ . Then,*

$${}^2g(\langle a, a + 1 \rangle) = (a - b)^2.$$

*Proof.* Since  $g(\langle a, a + 1 \rangle) = a^2 - a - 1$  then

$$(a - 1)^2 \leq g(\langle a, a + 1 \rangle) < a^2.$$

We thus have that  ${}^2g(\langle a, a + 1 \rangle) < a^2$ . We shall show that  $(a - i)^2 \in \langle a, a + 1 \rangle$  for  $i \in \{1, 2, \dots, b - 1\}$ .

We first observe that

$$(9) \quad (a - i)^2 = a^2 - 2ai + i^2 = (a - 2i)a + i^2 = (a - 2i - i^2)a + i^2(a + 1).$$

for any integer  $i$ .

Since for any  $i \in \{1, 2, \dots, b - 1\}$  we have

$$a - 2i - i^2 = a - i(i + 2) \geq a - (b - 1)(b + 1) = a - b^2 + 1 > 0$$

and

$$i^2 > 0$$

then, by (9), we deduce that  $(a - i)^2 \in \langle a, a + 1 \rangle$  for any  $i \in \{1, 2, \dots, b - 1\}$ .

Finally, since  $a + 1 < (b + 1)^2$  (implying that  $a - 2b - b^2 < 0$ ) and  $0 < b^2 < a$  then we may deduce, from (9), that  $(a - b)^2 \notin \langle a, a + 1 \rangle$ .  $\square$

Let  $(u_n)_{n \geq 1}$  be the recursive sequence defined by  $u_1 = 1$ ,  $u_2 = 2$ ,  $u_3 = 3$  and

$$(10) \quad u_{2n} = u_{2n-1} + u_{2n-2} \quad \text{and} \quad u_{2n+1} = u_{2n} + u_{2n-2} \quad \text{for all } n \geq 2.$$

The first few values of  $(u_n)_{n \geq 1}$  are

$$1, 2, 3, 5, 7, 12, 17, 29, 41, 70, 99, 169, 239, 408, 577, 985, \dots$$

This sequence also appear in a number of other context, for instance, in connection with the *Farey fractions*, see [5].

We pose the following conjecture in the case when either  $a$  or  $a + 1$  is an square integer.

**Conjecture 4.2.** *Let  $(u_n)_{n \geq 1}$  be the recursive sequence given in (10).*

*If  $a = b^2$  for some integer  $b \geq 1$  then*

$${}^2g(\langle a, a + 1 \rangle) = \begin{cases} (a - \lfloor b\sqrt{2} \rfloor)^2 & \text{if } b \notin \bigcup_{n \geq 0} \{u_{4n+1}, u_{4n+2}\}, \\ (a - \lfloor b\sqrt{3} \rfloor)^2 & \text{if } b \in \bigcup_{n \geq 0} \{u_{4n+1}, u_{4n+2}\}. \end{cases}$$

*If  $a + 1 = b^2$  for some integer  $b \geq 1$  then*

$${}^2g(\langle a, a + 1 \rangle) = \begin{cases} (a - \lfloor b\sqrt{2} \rfloor)^2 & \text{if } b \notin \bigcup_{n \geq 1} \{u_{4n-1}, u_{4n}\}, \\ (a - \lfloor b\sqrt{3} \rfloor)^2 & \text{if } b \in \bigcup_{n \geq 1} \{u_{4n}, u_{4n+3}\}, \\ 2^2 & \text{if } b = u_3 = 3. \end{cases}$$

The formulas of Conjecture 4.2 have been verified by computer for all integers  $a \geq 2$  up to  $10^6$ .

## 5. STUDY OF $\langle a, a + 2 \rangle$

We investigate the square Frobenius number of  $\langle a, a + 2 \rangle$  with  $a \geq 3$  odd. We first study the case when neither  $a$  nor  $a + 2$  is a square integer.

**Proposition 5.1.** *Let  $a \geq 3$  be an odd integer such that  $(2b + 1)^2 < a < a + 2 < (2b + 3)^2$  for some integer  $b \geq 1$ . Then,*

$${}^2g(\langle a, a + 2 \rangle) = (a - (2b + 1))^2.$$

*Proof.* Since  $g(\langle a, a + 2 \rangle) = (a - 1)(a + 1) - 1 = a^2 - 2$  then

$$(a - 1)^2 < g(\langle a, a + 2 \rangle) < a^2.$$

We thus have that  ${}^2g(\langle a, a + 2 \rangle) < a^2$ . We shall show that  $(a - i)^2 \in \langle a, a + 2 \rangle$  for  $i \in \{1, 2, \dots, 2b\}$ .

We first observe that for any integer  $i$ , we have

$$(11) \quad (a - 2i)^2 = a^2 - 4ai + 4i^2 = (a - 4i)a + 4i^2 = (a - 4i - 2i^2)a + 2i^2(a + 2).$$

Since for any  $i \in \{1, 2, \dots, b\}$  we have

$$a - 4i - 2i^2 = a - 2i(i + 2) \geq a - 2i(2i + 1) > a - (2i + 1)^2 \geq a - (2b + 1)^2 > 0$$

and

$$2i^2 > 0$$

then, by (11), it follows that  $(a - 2i)^2 \in \langle a, a + 2 \rangle$  for any  $i \in \{1, 2, \dots, b\}$ .

Moreover, for any integer  $i$ , we have

$$\begin{aligned}
 (12) \quad (a - (2i + 1))^2 &= a^2 - 2a(2i + 1) + (2i + 1)^2 = (a - 2(2i + 1))a + (2i + 1)^2 \\
 &= (a - 4i - 3)a + (2i + 1)^2 + a \\
 &= \left( a - 4i - 3 - \frac{(2i+1)^2+a}{2} \right) a + \frac{(2i+1)^2+a}{2}(a + 2) \\
 &= \frac{a-4i^2-12i-7}{2}a + \frac{(2i+1)^2+a}{2}(a + 2) \\
 &= \frac{a+2-(2i+3)^2}{2}a + \frac{(2i+1)^2+a}{2}(a + 2).
 \end{aligned}$$

Since, for any  $i \in \{0, 1, \dots, b - 1\}$  we have

$$\frac{a + 2 - (2i + 3)^2}{2} \geq \frac{a + 2 - (2b + 1)^2}{2} > 0$$

and

$$\frac{(2i + 1)^2 + a}{2} > 0$$

then it follows, from (12), that  $(a - (2i + 1))^2 \in \langle a, a + 2 \rangle$ , for any  $i \in \{0, 1, \dots, b - 1\}$ .

Finally, since

$$0 < \frac{(2b + 1)^2 + a}{2} < a$$

and

$$\frac{a + 2 - (2b + 3)^2}{2} < 0,$$

then we have, from (12), that  $(a - (2b + 1))^2 \notin \langle a, a + 2 \rangle$ .  $\square$

We pose the following conjecture in the case when either  $a$  or  $a + 2$  is an square integer.

**Conjecture 5.2.** *Let  $(u_n)_{n \geq 1}$  be the recursive sequence given in (10).*

*If  $a = (2b + 1)^2$  for some integer  $b \geq 1$  then*

$${}^2g(\langle a, a + 2 \rangle) = \begin{cases} \left( a - 2 \left\lfloor \frac{(2b+1)\sqrt{2}}{2} \right\rfloor \right)^2 & \text{if } (2b + 1) \notin \bigcup_{n \geq 1} \{u_{4n+1}\}, \\ \left( a - \lfloor (2b + 1)\sqrt{3} \rfloor \right)^2 & \text{if } (2b + 1) \in \bigcup_{n \geq 2} \{u_{4n+1}\}, \\ 38^2 & \text{if } 2b + 1 = u_5 = 7. \end{cases}$$

*If  $a + 2 = (2b + 1)^2$  for some integer  $b \geq 1$  then*

$${}^2g(\langle a, a + 2 \rangle) = \begin{cases} \left( a - 2 \left\lfloor \frac{(2b+1)\sqrt{2}}{2} \right\rfloor \right)^2 & \text{if } (2b + 1) \notin \bigcup_{n \geq 0} \{u_{4n+3}\}, \\ \left( a - \lfloor (2b + 1)\sqrt{3} \rfloor \right)^2 & \text{if } (2b + 1) \in \bigcup_{n \geq 0} \{u_{4n+3}\}. \end{cases}$$

The formulas of Conjecture 5.2 have been verified by computer for all odd integers  $a \geq 3$  up to  $10^6$ .

## 6. CONCLUDING REMARKS

In the process of investigating square Frobenius numbers different problems arose. In the same flavor as  ${}^k g(S)$ , we may consider the  $P$ -type function  ${}_k r(S)$  defined as,

$${}_k r(S) := \text{the smallest } k\text{-power integer belonging to } S.$$

It is clear that

$$(13) \quad s \leqslant {}_k r(S) \leqslant s^k$$

where  $s$  is the multiplicity of  $S$ .

**Theorem 6.1.** *Let  $S_A = \langle a, a + d, \dots, a + kd \rangle$  where  $a, d, k$  are positive integers with  $\gcd(a, d) = 1$ . If  $\lceil \sqrt{a} \rceil \leqslant d \leqslant \frac{ak}{1+2k}$  then*

$${}_2 r(S_A) \leqslant (a - d)^2.$$

*Proof.* We shall use the characterization given in Proposition 2.2 with  $i = d$ . In this case  $\lambda_d = 0$  and thus

$$\left( \left( \left\lfloor \frac{d^2 + 0a}{ad} \right\rfloor + k \right) d - 0 \right) (a + kd) = \left( \left( \left\lfloor \frac{d}{a} \right\rfloor + k \right) d - 0 \right) (a + kd) = akd + (kd)^2.$$

Thus,

$$(d + kd)^2 \leqslant akd + (kd)^2 \iff d^2 + 2kd \leqslant akd \iff d \leqslant \frac{ak}{1 + 2k}.$$

Therefore, by Proposition 2.2,  $(a - d)^2 \in S_A$ .  $\square$

**Problem 1.** *Let  $k \geqslant 2$  be an integer and let  $S$  be a numerical semigroup. Investigate the computational complexity to determine  ${}^k g(S)$  and/or  ${}_k r(S)$ .*

Or more ambitious,

**Question 1.** *Let  $k \geqslant 2$  be an integer. Is there a closed formula for  ${}^k g(S)$  and/or  ${}_k r(S)$  for any semigroup  $S$ ?*

Maybe a first step on this direction might be the following.

**Problem 2.** *Give a formula for  ${}^2 g(\langle F_i, F_j \rangle)$  and/or  ${}_2 r(\langle F_i, F_j \rangle)$  with  $\gcd(F_i, F_j) = 1$  where  $F_k$  denotes the  $k^{\text{th}}$  Fibonacci number. What about  ${}^2 g(\langle a^2, b^2 \rangle)$  where  $a$  and  $b$  are relatively prime integers? We clearly have that  ${}_2 r(\langle a^2, b^2 \rangle) = a^2$  for  $1 \leqslant a < b$ .*

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APPENDIX A. COMPLEMENT TO FORMULAS FOR  $\langle a, a + d \rangle$  WITH SMALL  $d \geq 3$ 

In Tabular 2, we compare the exact values of  ${}^2g(\langle a, a + d \rangle)$  and the formula  $h(a, d, 1)$ , when  $a \in E(d)$  for  $d \in \{3, \dots, 12\}$ .

$d$	$a$	${}^2g(\langle a, a + d \rangle)$	$h(a, d, 1)$	$d$	$a$	${}^2g(\langle a, a + d \rangle)$	$h(a, d, 1)$
5	2	1	0	10	43	$40^2$	$39^2$
5	4	1	$2^2$	10	123	$110^2$	$109^2$
5	13	$10^2$	$9^2$	10	133	$120^2$	$119^2$
5	27	$21^2$	$20^2$	10	143	$130^2$	$129^2$
5	32	$26^2$	$25^2$	10	153	$140^2$	$139^2$
7	2	1	$5^2$	10	163	$150^2$	$149^2$
7	3	$2^2$	0	10	333	$310^2$	$309^2$
7	4	$5^2$	$6^2$	10	343	$320^2$	$319^2$
7	9	$7^2$	$6^2$	11	2	$3^2$	$4^2$
7	16	$14^2$	$13^2$	11	3	$5^2$	$9^2$
7	18	$17^2$	$16^2$	11	4	$5^2$	$6^2$
7	19	$14^2$	$13^2$	11	5	$7^2$	$9^2$
7	23	$21^2$	$20^2$	11	7	$4^2$	$2^2$
7	30	$28^2$	$27^2$	11	8	$7^2$	$12^2$
7	114	$105^2$	$104^2$	11	9	$8^2$	$12^2$
8	5	$4^2$	$3^2$	11	14	$13^2$	$19^2$
8	9	$10^2$	$12^2$	11	16	$14^2$	$20^2$
8	21	$16^2$	$15^2$	11	18	$15^2$	$13^2$
8	45	$36^2$	$35^2$	11	25	$22^2$	$20^2$
8	77	$64^2$	$63^2$	11	36	$33^2$	$31^2$
9	2	$3^2$	$4^2$	11	38	$36^2$	$34^2$
9	4	$3^2$	$6^2$	11	47	$44^2$	$42^2$
9	7	$6^2$	$11^2$	12	13	$14^2$	$18^2$
9	8	$6^2$	$4^2$	12	19	$17^2$	$16^2$
9	16	$9^2$	$12^2$	12	25	$26^2$	$30^2$
10	3	$2^2$	$7^2$	12	31	$29^2$	$28^2$
10	9	$7^2$	$12^2$	12	67	$59^2$	$58^2$
10	13	$10^2$	$9^2$	12	79	$71^2$	$70^2$
10	23	$20^2$	$19^2$	12	139	$125^2$	$124^2$
10	27	$26^2$	$25^2$	12	151	$137^2$	$136^2$
10	33	$30^2$	$29^2$	12	235	$215^2$	$214^2$

TABLE 2.  ${}^2g(\langle a, a + d \rangle)$  and  $h(a, d, 1)$  when  $a \in E(d)$  for  $d \in \{3, \dots, 12\}$ 

IMAG, UNIV. MONTPELLIER, CNRS, MONTPELLIER, FRANCE

*Email address:* jonathan.chappelton@umontpellier.fr

IMAG, UNIV. MONTPELLIER, CNRS, MONTPELLIER, FRANCE AND UMI2924 - JEAN-CHRISTOPHE YOCOZ, CNRS-IMPA

*Email address:* jorge.ramirez-alfonsin@umontpellier.fr