



HAL
open science

A Kesten-Stigum type theorem for a supercritical multi-type branching process in a random environment

Ion Grama, Quansheng Liu, Erwan Pin

► **To cite this version:**

Ion Grama, Quansheng Liu, Erwan Pin. A Kesten-Stigum type theorem for a supercritical multi-type branching process in a random environment. 2019. hal-02878026

HAL Id: hal-02878026

<https://hal.science/hal-02878026>

Preprint submitted on 22 Jun 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A Kesten-Stigum type theorem for a supercritical multi-type branching process in a random environment

Grama Ion, Liu Quansheng and Pin Erwan

Université de Bretagne-Sud

LMBA UMR CNRS 6205

Vannes, France.

e-mail: ion.grama@univ-ubs.fr

quansheng.liu@univ-ubs.fr

erwan.pin@univ-ubs.fr

Abstract: Consider a supercritical d -type branching process $Z_n^i = (Z_n^i(1), \dots, Z_n^i(d))$, $n \geq 0$, in an i.i.d. environment $\xi = (\xi_0, \xi_1, \dots)$, starting with one particle of type i , whose offspring distributions of generation n depend on the environment ξ_n at time n . In the deterministic environment case, the famous Kesten-Stigum (1966) theorem states essentially that, if the mean matrix of the offspring distribution has spectral radius $\rho > 1$, then for all $i, j = 1, \dots, d$, almost surely $\lim_{n \rightarrow \infty} \frac{Z_n^i(j)}{\rho^n}$ exists and is finite; moreover, the limit variables are non-degenerate if and only if $\mathbb{E}(Z_1^i(j) \log^+ Z_1^i(j)) < +\infty$ for all i and j . The extension to the random environment case with $d = 1$ has been done by Athreya and Karlin (1971) and Tanny (1988). Extending the Kesten-Stigum theorem to the random environment case with $d > 1$ is a long-standing problem. The main objective of this paper is to resolve this delicate problem. In particular, under simple conditions, we prove that for any $1 \leq i, j \leq d$, as $n \rightarrow +\infty$, $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j) \rightarrow W^i$ in probability, where W^i is a non-negative random variable, $\mathbb{E}_\xi Z_n^i(j)$ is the conditional expectation of $Z_n^i(j)$ given the environment ξ , which diverges to ∞ with geometric rate in the sense that $\frac{1}{n} \log \mathbb{E}_\xi Z_n^i(j) \rightarrow \gamma > 0$ almost surely, γ being the Lyapunov exponent of the mean matrices of the offspring distributions; moreover W^i are non-degenerate for all i if and only if $\mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)} \right) < +\infty$ for all i and j , where $M_0(i, j)$ is the conditioned mean of the number of children of type j produced by a particle of type i at time 0, given the environment ξ . The key idea of the proof is the introduction of a non-negative martingale (W_n^i) which converges a.s. to W^i , and which reduces to the well-known fundamental martingale in the deterministic environment case. In addition, we prove that the direction $Z_n^i/\|Z_n^i\|$ converges in law conditioned on the explosion event $\{\|Z_n^i\| \rightarrow +\infty\}$. The case of stationary and ergodic environment is also considered. Our results open ways to prove important properties

such as central limit theorems with convergence rate and large deviation asymptotics.

MSC 2010 subject classifications: Primary 60J80, 60K37; secondary 60J85.

Keywords and phrases: Multi-type branching processes, random environment, Kesten-Stigum theorem, products of random matrices, martingale.

1. Introduction

Branching processes are rapidly developing areas of the theory of random processes. Their importance is mainly due to the large spectrum of applications in many fields including biology, chemistry, population dynamics, nuclear physics, etc. See for example the classical books by Harris [16] and Athreya and Ney [3]. The introduction of a random environment by Smith and Wilkinson [31] and Athreya and Karlin [1] brought an important advancement in the theory and applications of branching processes. The role of random environment has been by now well understood in the case of single type branching processes, for which a number of important properties have been established, see for example the recent book by Kersting and Vatutin [22]. For multi-type branching processes in random environments (MBPRE's), recent progress has been made for the critical and subcritical cases: see for example Peigné, Le Page and Pham [28], Vatutin and Dyakonova [36], and Vatutin and Wachtel [37], who studied the convergence rate of the survival probability; for the supercritical case, we have not found recent work in the literature, and we feel that too few results are known.

For a supercritical multi-type branching process (MBP), the fundamental problem is the description of the population size at time n . Let us recall the famous Kesten-Stigum's theorem [24] established in the deterministic environment case, which tells us exactly when the population size grows at an exponential rate. Consider a MBP $Z_n = (Z_n(1), \dots, Z_n(d))$, $n \geq 0$, where $Z_n(j)$ denotes the number of particles of types j at time n , Z_0 represents the initial population. Denote by M the (non-random) matrix of means of the offspring distributions, which is assumed to be primitive in the sense that there exists $k \geq 1$ such that $M^k > 0$. Let ρ be the spectral radius of the mean matrix M , and let $u = (u(1), \dots, u(d))$ and $v = (v(1), \dots, v(d))$ be respectively associated positive right and left eigenvectors with the normalization $\|u\| = 1$ and $\langle v, u \rangle = 1$, where $\|\cdot\|$ denotes the L_1 -norm and $\langle \cdot, \cdot \rangle$ the scalar product. Assume that $\rho > 1$, which means that the branching process is in the supercritical regime. Denote by $(Z_n^i)_{n \geq 0}$ the branching process $(Z_n)_{n \geq 0}$ starting with one initial particle of type i , that is when $Z_0 = e_i$,

where e_i is the unit vector whose i -th component is 1. Kesten and Stigum [24] showed that, for any $1 \leq i, j \leq d$, as $n \rightarrow +\infty$,

$$\frac{Z_n^i(j)}{\rho^n v(j)} \rightarrow W^i u(i) \quad \text{a.s.}, \quad (1.1)$$

where W^i is a non-negative random variable which is non-degenerate for all i if and only if $\mathbb{E}(Z_1^i(j) \log^+ Z_1^i(j)) < +\infty$ for all i and j , and when it is non-degenerate, $\mathbb{E}W^i = 1$. The proof of (1.1) is based on the fundamental non-negative martingale

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, u \rangle}{\rho^n u(i)}, \quad n \geq 1, \quad (1.2)$$

which converges a.s. to W^i .

Due to the importance of the Kesten-Stigum theorem and of the fundamental martingale (W_n^i) , a challenging problem is to find the corresponding results for the random environment case. For the single type process, this problem was considered at the very beginning of the study of the topic in the fundamental work of Athreya and Karlin [2] (1971). In [2] it was found that for a single type branching process $(Z_n)_{n \geq 0}$ in a stationary and ergodic random environment, the sequence

$$W_0 = 1, \quad W_n = \frac{Z_n}{m_0 \cdots m_{n-1}}, \quad n \geq 1, \quad (1.3)$$

with m_k denoting the conditioned mean of the offspring distribution at time k given the environment, constitutes a martingale, and that, in the supercritical case where $\mathbb{E} \log m_0 > 0$, the limit variable $W = \lim_{n \rightarrow \infty} W_n$ is non-degenerate if

$$\mathbb{E} \left(\frac{Z_1}{m_0} \log^+ Z_1 \right) < +\infty. \quad (1.4)$$

In case of an independent and identically distributed (i.i.d.) environment, this condition was proved to be also necessary for the non degeneracy of W by Tanny [35] (1988). Notice that when $\mathbb{E}|\log m_0| < \infty$, the moment condition (1.4) is equivalent to

$$\mathbb{E} \left(\frac{Z_1}{m_0} \log^+ \frac{Z_1}{m_0} \right) < +\infty.$$

For a multi-type branching process in random environment $Z_n = (Z_n(1), \dots, Z_n(d))$, $n \geq 0$, the situation is much more delicate. In fact, extending

the Kesten-Stigum theorem to this case is a long-standing problem. The only result that we found in the literature about the subject is a theorem by Cohn [6], which we briefly recall below. For $n \geq 0$, denote by M_n the matrix of the conditioned means of the offspring distribution of n -th generation given the environment: the (i, j) -th entry of M_n is

$$M_n(i, j) = \mathbb{E}_\xi[Z_{n+1}(j) \mid Z_n = e_i],$$

where \mathbb{E}_ξ denotes the conditional expectation given the environment ξ . Let $M_{0,n} = M_0 \cdots M_n$ be the product matrix. Assume that each entry of M_n is bounded a.s. from below and above by two positive constants, and that all the conditional second moments of the offspring distributions given the environment are bounded a.s. by a constant. We suppose that the MBPRE is in the supercritical regime, which means that

$$\gamma := \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\| > 0, \tag{1.5}$$

where $\|M_{0,n-1}\|$ is the L_1 -norm of the matrix $M_{0,n-1}$. This definition of the supercriticality agrees with that in the deterministic environment case, since in this case $\log \rho = \gamma$. Assume also the integrability condition $\mathbb{E}|\log \sum_{i=1}^d (1 - \mathbb{P}(\|Z_1^i\| = 0))| < \infty$. Under these conditions Cohn ([6], 1989) proved that for each $j = 1, \dots, d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \rightarrow W^i \quad \text{in } L^2, \tag{1.6}$$

where W^i is a non degenerate random variable satisfying $\mathbb{E}W^i = 1$. This result is already very interesting. However, it only gives sufficient conditions which are not necessary for $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$ to converge to a non degenerate random variable. Moreover, the sequence $(\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)})_{n \geq 0}$ in general is not a martingale; it turns out very useful to find the martingale which corresponds the fundamental martingale known in the constant environment case.

Our objective in this paper is to obtain a full extension of the Kesten-Stigum result (1.1) for a supercritical MBPRE $Z_n = (Z_n(1), \dots, Z_n(d))$, $n \geq 0$. For simplicity, let us consider the case where the Furstengerg-Kesten condition H4 (see Section 2) is satisfied, and where the environment is i.i.d. Assume the supercritical condition $\gamma > 0$. For $n, k \geq 0$, let $\rho_{n,n+k}$ be the spectral radius of the product matrix $M_{n,n+k} = M_n \cdots M_{n+k}$, and let $U_{n,n+k}$ and $V_{n,n+k}$ be respectively the associated non-negative right and left eigenvectors with the normalization $\|U_{n,n+k}\| = 1$ and $\langle V_{n,n+k}, U_{n,n+k} \rangle = 1$. Set $U_{n,\infty} = \lim_{k \rightarrow \infty} U_{n,n+k}$, where the limit exists a.s. according to a result of

Hennion [17]. Then we have the following analog of Kesten-Stigum's result (1.1) which describes the asymptotic behaviour of the coordinate $Z_n^i(j)$: for any $1 \leq i, j \leq d$, as $n \rightarrow +\infty$,

$$\frac{Z_n^i(j)}{\rho_{0,n-1} V_{0,n-1}(j)} \rightarrow W^i U_{0,\infty}(i) \quad \text{in probability,} \quad (1.7)$$

where $U_{0,\infty}(i) \in (0, 1)$, W^i is a non-negative random variable such that W^i is non-degenerate for all i if and only if

$$\mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i, j)} \log^+ \frac{Z_1^i(j)}{M_0(i, j)} \right) < +\infty \quad (1.8)$$

for all i and j , and when it is non-degenerate, $\mathbb{E}W^i = 1$. A result similar to (1.6) is also proved : in Theorem 2.11 we establish (under conditions weaker than those supposed by Cohn [6]) that for all $1 \leq i, j \leq d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \rightarrow W^i \quad \text{in probability,} \quad (1.9)$$

where W^i is the same variable as in (1.7). As it has been just noted, the condition (1.8) is sufficient and necessary for W^i to be non-degenerate.

The asymptotic behavior of the direction of the vector Z_n^i is also of interest. We show that the unit vector $Z_n^i/\|Z_n^i\|$ converges in law conditioned on the explosion event $\{\|Z_n^i\| \rightarrow +\infty\}$. This extends the corresponding result of Kurtz, Lyons, Pemantle and Perez [26] established for the deterministic environment case.

The key idea of the proof is the introduction of a non-negative martingale (W_n^i) which converges a.s. to W^i , and which reduces to the well-known fundamental martingale in the deterministic environment case and in the single-type random environment case. Since this is the key difficulty let us explain our construction in details. The straightforward way for a generalization of (1.2) would be replacing ρ^n and the right eigenvector u by the eigenvalues $\rho_{0,n-1}$ and the corresponding right eigenvectors $U_{0,n-1}$ of the matrix $M_{0,n-1}$; unfortunately, this does not lead to a martingale. Our definition is based on the analog of the Perron-Frobenius theorem for products of random matrices which has been established in Hennion [17]. From the results of [17, Theorem 1], the sequence of unit vectors $(U_{n,\infty})$ satisfies

$$M_n U_{n+1,\infty} = \lambda_n U_{n,\infty},$$

where $\lambda_n = \|M_n U_{n+1,\infty}\|$, $n \geq 0$, is a stationary and ergodic sequence. Iterating the last relation leads to the identity

$$M_{0,n-1} U_{n,\infty} = \lambda_{0,n-1} U_{0,\infty},$$

with $\lambda_{0,n} = \lambda_0 \cdots \lambda_n$. This allows us to associate with the branching process (Z_n^i) the positive martingale

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, U_{n,\infty} \rangle}{\lambda_{0,n-1} U_{0,\infty}(i)}, \quad n \geq 1. \quad (1.10)$$

When the environment is deterministic, the matrices M_n , $n \geq 0$, are identical to a single deterministic matrix, say M . In this case we have $U_{n,\infty} = U_{n,n+k} = u$, where u is the unit right eigenvector of M associated with the spectral radius ρ , and $\lambda_{0,n-1} = \rho^n$, so that $W_n^i = \frac{\langle Z_n^i, u \rangle}{\rho^n u(i)}$, which shows that the martingale (1.10) coincides with the martingale (1.2). For a single-type branching process in random environment, we have $U_{n,\infty} = 1$, $\lambda_{0,n-1} = m_0 \cdots m_{n-1}$, so that (1.10) coincides with (1.3).

In fact, in the paper we will establish more general results for stationary and ergodic environment without assuming the Furstengerg-Kesten condition. We refer the reader to Section 2 for details.

We mention that the results of this paper open ways to prove important properties such as central limit theorems with convergence rate and large deviation asymptotics, similar to those obtained in [4, 13]. See the preprints [14, 15].

The outline is as follows. In Section 2, we introduce the necessary notation and present the main results. In Section 3, we give some preliminaries for products of positive random matrices. The fundamental martingale (W_n^i) is constructed in Section 4; the non-degeneracy of its limit is considered in Sections 5-7. Section 8 is devoted to the convergence of the direction of Z_n . In Section 9 we study the convergence in probability of the normalized component $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$; its a.s. convergence is considered in Section 10. Section 11 is an appendix in which we prove several implications among the conditions used in the statements of the main results.

2. Background and main results

2.1. Notation and preliminary statements

We start this section by fixing some notation. For an integer $d \geq 1$ let \mathbb{R}^d be the d -dimensional space of vectors with real coordinates. For $1 \leq i \leq d$ denote by e_i the d -dimensional vector with 1 in the i -th place and 0 elsewhere. $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$ stands for the vector with all coordinates equal to 1. For any $x, y \in \mathbb{R}^d$, let

$$\langle x, y \rangle := \sum_{i=1}^d x(i) y(i) \quad \text{and} \quad \|x\| := \sum_{i=1}^d |x(i)|$$

be the scalar product and the L^1 norm in \mathbb{R}^d . The operator norm of a matrix $M = (M(i, j))_{1 \leq i, j \leq d} \in \mathcal{M}_d(\mathbb{R})$ is given by

$$\|M\| := \sup_{\|x\|=1} \|Mx\|.$$

For a matrix or a vector X , we write $X > 0$ to mean that each entry of X is strictly positive. The set of non-negative integers is denoted $\mathbb{N} = \{0, 1, \dots\}$. The symbol C denotes positive constants. The indicator of an event A is denoted by $\mathbf{1}_A$. The symbol $\xrightarrow{d(\mathbb{P})}$ denotes the convergence in distribution under \mathbb{P} , while $\xrightarrow{\mathbb{P}}$ means the convergence in probability \mathbb{P} .

Let $\xi = (\xi_n)_{n \geq 0}$ be a stationary and ergodic sequence of random variables with values in an abstract space \mathbb{X} . Each realization of ξ_n is associated with d probability distributions on \mathbb{N}^d characterized by their probability generating functions

$$f_n^r(s) = \sum_{k_1, \dots, k_d=0}^{\infty} p_{k_1, \dots, k_d}^r(\xi_n) s_1^{k_1} \cdots s_d^{k_d}, \quad s = (s_1, \dots, s_d) \in [0, 1]^d,$$

$1 \leq r \leq d$. A d -type branching process $Z_n = (Z_n(1), \dots, Z_n(d))$, $n \geq 0$, in the random environment ξ is a process with values in \mathbb{N}^d such that $Z_0 \in \mathbb{N}^d$ is fixed, and for all $n \geq 0$,

$$Z_{n+1} = \sum_{r=1}^d \sum_{l=1}^{Z_n(r)} N_{l,n}^r, \quad (2.1)$$

where $Z_n(j)$ represents the number of particles of type j of some population in generation n ; conditioned on the environment ξ , the random vectors $N_{l,n}^r = (N_{l,n}^r(1), \dots, N_{l,n}^r(d))$, with $N_{l,n}^r(j)$ denoting the offspring of type j at time $n+1$ of the l -th particle of type r in the generation n , are independent for $l \geq 1, n \geq 0, 1 \leq r \leq d$; each $N_{l,n}^r$ has the same probability generating function f_n^r for $l \geq 1$. In the sequel, when the branching process $(Z_n)_{n \geq 0}$ starts with one initial particle of type i , i.e. when $Z_0 = e_i$, we will write $(Z_n^i)_{n \geq 0}$ instead of $(Z_n)_{n \geq 0}$.

Let \mathbb{P}_ξ be the additional probability under which the process (Z_n) is defined given the environment ξ . The total probability \mathbb{P} can be formulated as $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx) \tau(d\xi)$, where τ denotes the law of the environment sequence ξ . The probability \mathbb{P}_ξ is usually called quenched law, while the total probability \mathbb{P} is called annealed law. The quenched law \mathbb{P}_ξ can be considered as the conditional law of \mathbb{P} given the environment ξ . The expectation with respect to \mathbb{P}_ξ and \mathbb{P} will be denoted respectively by \mathbb{E}_ξ and \mathbb{E} .

According to the definition of the model, under \mathbb{P}_ξ , the random vectors $N_{l,n}^r = (N_{l,n}^r(1), \dots, N_{l,n}^r(d))$ are independent and have the same probability generating function f_n^r :

$$f_n^r(s) = \mathbb{E}_\xi \left(\prod_{j=1}^d s_j^{N_{l,n}^r(j)} \right), \quad s = (s_1, \dots, s_d) \in [0, 1]^d.$$

Set for brevity $N_n^r := N_{1,n}^r$, and let $f_n = (f_n^1, \dots, f_n^d)$. Then f_n^r is the generating function of $N_n^r = (N_n^r(1), \dots, N_n^r(d))$ under \mathbb{P}_ξ .

We now introduce the sequence of matrices $(M_n)_{n \in \mathbb{N}}$ of conditional means given the environment, which will play a central role in our developments. For all $n \geq 0$, set

$$M_n = M_n(\xi_n) := \left(\frac{\partial f_n^i}{\partial s_j}(\mathbf{1}) \right)_{1 \leq i, j \leq d},$$

i.e., for any $1 \leq i, j \leq d$, the (i, j) -th entry of the matrix M_n is

$$M_n(i, j) = \frac{\partial f_n^i}{\partial s_j}(\mathbf{1}) = \mathbb{E}_\xi [Z_{n+1}(j) | Z_n = e_i],$$

which represents the conditioned mean of the number of children of type j produced by a particle of type i at time n , and which are supposed to be finite. Here and hereafter, for a d -dimensional probability generating function f , $\frac{\partial f}{\partial s_j}(\mathbf{1})$ denotes the left derivative at $\mathbf{1}$ of f with respect to s_j .

The matrix M_n depends only on ξ_n and the sequence of the matrices $(M_n)_{n \geq 0}$ is stationary and ergodic. Let $0 \leq k \leq n$. For the product of the matrices M_k, \dots, M_n it is convenient to use the notation

$$M_{k,n} := M_k \cdots M_n = \left(\frac{\partial f_k^i \circ f_{k+1} \circ \cdots \circ f_n}{\partial s_j}(\mathbf{1}) \right)_{1 \leq i, j \leq d},$$

where

$$\frac{\partial f_k^i \circ f_{k+1} \circ \cdots \circ f_n}{\partial s_j}(\mathbf{1}) = \mathbb{E}_\xi [Z_{n+1}(j) | Z_k = e_i].$$

In particular, with $k = 0$, we have for all $1 \leq i, j \leq d$,

$$\mathbb{E}_\xi Z_{n+1}^i(j) = M_{0,n}(i, j). \tag{2.2}$$

Denote by \mathcal{S} the semigroup of matrices of $\mathcal{M}_d(\mathbb{R})$ with positive entries which are allowable in the sense that every row and column contains a strictly positive element, and by \mathcal{S}^0 the subset of the matrices with strictly positive entries. Following Hennion [17], we shall assume that the matrices M_n satisfy the condition

H1. The matrix M_0 belongs to the semigroup \mathcal{S} \mathbb{P} -a.s. and

$$\mathbb{P}\left(\bigcup_{n \geq 0} \{M_{0,n} \in \mathcal{S}^0\}\right) > 0.$$

This means that with positive probability, there is n such that the product matrix $M_{0,n}$ is strictly positive.

Obviously if $G \in \mathcal{S}$ and $G^0 \in \mathcal{S}^0$ then $G^0 G \in \mathcal{S}^0$. Let θ_n be the least k such that $M_{n,n+k} \in \mathcal{S}^0$:

$$\theta_n := \inf \left\{ k \geq 0 : M_{n,n+k} \in \mathcal{S}^0 \right\},$$

with the convention that $\inf \emptyset = +\infty$. According to Lemma 3.1 in [17], under condition **H1**, we have $\theta_n < +\infty$ \mathbb{P} -a.s. for all $n \geq 0$.

We shall relate the branching process $(Z_n^i)_{n \geq 0}$ to a martingale which is the key point in our study. Our construction is based on the extension of the Perron-Frobenius theorem of Hennion [17] for products of random matrices. Recall that, under condition **H1**, for any $n, k \geq 0$, the product $M_{n,n+k}$ belongs to \mathcal{S} \mathbb{P} -a.s. Let $\rho_{n,n+k}$ be the spectral radius of $M_{n,n+k}$. By the classical Perron-Frobenius theorem (see e.g. [3]), $\rho_{n,n+k}$ is a strictly positive eigenvalue of $M_{n,n+k}$, associated to positive right and left eigenvectors $U_{n,n+k}$ and $V_{n,n+k}$, respectively, with the normalizations $\|U_{n,n+k}\| = 1$ and $\langle V_{n,n+k}, U_{n,n+k} \rangle = 1$.

The following propositions collect some results established by Hennion in [17, Lemma 3.3 and Theorem 1], which provide an analog of the Perron-Frobenius theorem for products of random matrices.

Proposition 2.1. Assume condition **H1**. For all $n \geq 0$, the following assertions hold :

1. for all $1 \leq i, j \leq d$:

$$\lim_{k \rightarrow +\infty} \frac{M_{n,n+k}(i, j)}{\rho_{n,n+k} U_{n,n+k}(i) V_{n,n+k}(j)} \mathbf{1}_{\{\theta_n \leq k\}} = 1 \quad \mathbb{P}\text{-a.s.}, \quad (2.3)$$

or equivalently

$$\lim_{k \rightarrow +\infty} \left(\frac{M_{n,n+k-1}}{\rho_{n,n+k-1}} - U_{n,n+k-1} V_{n,n+k-1}^T \right) = 0 \quad \mathbb{P}\text{-a.s.};$$

2. the sequence $(U_{n,n+k})_{k \geq 0}$ converges \mathbb{P} -a.s. to a random unit vector, say $U_{n,\infty} > 0$:

$$U_{n,n+k} \xrightarrow[k \rightarrow +\infty]{} U_{n,\infty};$$

3. the sequence $(V_{n,n+k}/\|V_{n,n+k}\|)_{k \geq 0}$ converges in law to a random unit vector, say $\bar{V}_{0,\infty} > 0$:

$$\frac{V_{n,n+k}}{\|V_{n,n+k}\|} \xrightarrow[k \rightarrow +\infty]{d(\mathbb{P})} \bar{V}_{0,\infty};$$

4. the scalars

$$\lambda_n := \lambda_n(\xi) = \|M_n U_{n+1,\infty}\| \tag{2.4}$$

are strictly positive and satisfy the relation

$$M_n U_{n+1,\infty} = \lambda_n U_{n,\infty}. \tag{2.5}$$

The sequence (λ_n) will play an important role in the following. The numbers λ_n will be called pseudo spectral radii of the products of random matrices. Notice that λ_n behaves as the spectral radius ρ_n which satisfies

$$M_n U_{n,n} = \rho_n U_{n,n}; \tag{2.6}$$

the point is that in (2.5), there is a shift of time in the vector $U_{n+1,\infty}$ appearing on the left side, which permits to iterate the formula leading to

$$M_{n,n+k} U_{n+k+1,\infty} = \lambda_{n,n+k} U_{n,\infty}, \tag{2.7}$$

where

$$\lambda_{n,n+k} = \prod_{r=n}^{n+k} \lambda_r, \quad \text{for } n, k \geq 0.$$

This shows that the relation (2.5) is stable for products of random matrices, while the corresponding relation (2.6) for the spectral radius does not have this stability. Notice that by (2.7)

$$\lambda_{n,n+k} = \|M_{n,n+k} U_{n+k+1,\infty}\|. \tag{2.8}$$

Let T be the shift operator of the environment sequence:

$$T\xi = (\xi_1, \xi_2, \dots) \quad \text{if} \quad \xi = (\xi_0, \xi_1, \dots),$$

and let T^n be its n -fold iteration. Note that the vector $U_{n,\infty}$ and the scalar λ_n depend on the whole sequence $T^n \xi = (\xi_n, \xi_{n+1}, \dots)$. Since the random environment $\xi = (\xi_n)_{n \geq 0}$ is a stationary ergodic sequence, from (2.4) it follows that $(\lambda_n)_{n \geq 0}$ is also a stationary ergodic sequence.

We complement Proposition 2.1 by establishing a relation between the product sequence $\lambda_{0,n-1}$ and the spectral radius $\rho_{0,n-1}$, which will be useful in the proof of the main results of the paper. For its proof, see Section 3.

Proposition 2.2. Assume condition **H1**. For all $n \geq 0$ and $1 \leq j \leq d$,

$$\lambda_n = \lim_{k \rightarrow +\infty} \frac{\rho_{n,n+k} V_{n,n+k}(j)}{\rho_{n+1,n+k} V_{n+1,n+k}(j)} \mathbf{1}_{\{\theta_{n+1} \leq k\}} \quad \mathbb{P}\text{-a.s.} \quad (2.9)$$

and

$$\lim_{n \rightarrow +\infty} \frac{\lambda_{0,n-1}}{\rho_{0,n-1} \langle V_{0,n-1}, U_{n,\infty} \rangle} = 1 \quad \mathbb{P}\text{-a.s.} \quad (2.10)$$

2.2. Main results

We first introduce the martingale related to MBPRE. Our definition is quite different from the one for a MBP with deterministic environment. However, we shall see below that in the case of deterministic environment it comes to the same. Consider the following filtration: $\mathcal{F}_0 = \sigma(\xi)$ and, for $n \geq 1$,

$$\mathcal{F}_n = \sigma\left(\xi, N_{l,k}^r, 0 \leq k \leq n-1, 1 \leq r \leq d, l \geq 1\right).$$

Define the process $(W_n^i)_{n \geq 0}$: for all $1 \leq i \leq d$, set

$$W_0^i := 1, \quad W_n^i := \frac{\langle Z_n^i, U_{n,\infty} \rangle}{\lambda_{0,n-1} U_{0,\infty}(i)}, \quad n \geq 1. \quad (2.11)$$

Our first theorem states that $(W_n^i)_{n \geq 0}$ is a non-negative martingale.

Theorem 2.3. Assume condition **H1**. For all $1 \leq i \leq d$ the sequence $(W_n^i)_{n \geq 0}$ is a non-negative martingale w.r.t. $(\mathcal{F}_n)_{n \geq 0}$ under the laws \mathbb{P}_ξ and \mathbb{P} , and hence converges \mathbb{P} -a.s. to a random variable $W^i \geq 0$ which satisfies $\mathbb{E}_\xi W^i \leq 1$ \mathbb{P} -a.s.

We next give a functional equation satisfied by the quenched Laplace transform $\phi_\xi^i(t) = \mathbb{E}_\xi e^{-tW^i}$, $t \geq 0$, $1 \leq i \leq d$. For a similar result in the deterministic environment we refer to Theorem 2, p.192 in [3].

Theorem 2.4. Assume condition **H1**. Then for each $1 \leq i \leq d$, the quenched Laplace transform ϕ_ξ^i of W^i satisfies

$$\phi_\xi^i(t) = f_0^i \left(\phi_{T\xi}^1 \left(t \frac{U_{1,\infty}(1)}{\lambda_0 U_{0,\infty}(1)} \right), \dots, \phi_{T\xi}^d \left(t \frac{U_{1,\infty}(d)}{\lambda_0 U_{0,\infty}(d)} \right) \right), \quad t \geq 0. \quad (2.12)$$

We now introduce a condition under which we can define the Lyapunov exponent γ of the sequence of random matrices $(M_n)_{n \geq 0}$.

H2. *The random matrix M_0 satisfies the moment condition*

$$\mathbb{E} \log^+ \|M_0\| < +\infty.$$

By the sub-additivity lemma, under **H2**, the limit

$$\gamma := \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\|$$

exists and is equal to the quantity $\inf_{k \geq 1} \frac{1}{k} \mathbb{E} \log \|M_{0,k-1}\|$. Moreover, the following strong law of large numbers has been established [12] :

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|M_{0,n-1}\| = \gamma \quad \mathbb{P}\text{-a.s.} \quad (2.13)$$

The Lyapunov exponent γ allows to introduce the following classification of MBPRE's. We say that a MBPRE is subcritical if $\gamma < 0$, critical if $\gamma = 0$, and supercritical if $\gamma > 0$. It is easy to see that our classification coincides with the standard classification of a MBP in a deterministic environment and with that of the uni-type BPPE.

All over the rest of the paper we shall focus only on the supercritical regime where $\gamma > 0$, which by (2.13) implies that

$$\lim_{n \rightarrow +\infty} \|M_{0,n}\| = +\infty \quad \mathbb{P}\text{-a.s.}$$

Using an extension of Birkhoff's theorem and (2.13), we obtain the following strong law of large numbers for the product sequence $\lambda_{0,n-1}$, and a new expression of γ (see Section 3.2).

Proposition 2.5. *Assume conditions **H1** and **H2**. Then the expectation $\mathbb{E} \log \lambda_0$ is well defined with value in $\mathbb{R} \cup \{-\infty\}$, and*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \lambda_{0,n-1} = \mathbb{E} \log \lambda_0 = \gamma \quad \mathbb{P}\text{-a.s.}$$

From Proposition 2.5 it is clear that, under the conditions **H1** and **H2**, the classification stated above can be reformulated in terms of the quantity $\mathbb{E} \log \lambda_0$.

We then investigate the non-degeneracy of the limits W^i , $1 \leq i \leq d$. Our first result gives a sufficient condition for non-degeneracy of W^i , $1 \leq i \leq d$ under condition **H1**. To state the result we need to introduce the following condition.

H3. *There exists a constant $C > 1$ such that, for all $1 \leq i \leq d$, \mathbb{P} -a.s.*

$$\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1,\infty} \rangle}{\lambda_n U_{n,\infty}(i)} \mathbf{1}_{\{\langle N_n^i, U_{n+1,\infty} \rangle \geq C^n\}} \right) < +\infty.$$

Let $q^i(\xi)$ be the probability of extinction of the process $(Z_n^i)_{n \geq 0}$:

$$q^i(\xi) := \mathbb{P}_\xi \left(\lim_{n \rightarrow +\infty} \|Z_n^i\| = 0 \right).$$

Theorem 2.6. *Assume conditions **H1**, **H2** and $\gamma > 0$. Then **H3** is a sufficient condition for W^i , $1 \leq i \leq d$ to be non-degenerate, that is,*

$$\mathbb{P}_\xi(W^i > 0) > 0, \quad \mathbb{P}\text{-a.s.}, \quad 1 \leq i \leq d. \quad (2.14)$$

Furthermore, when W^i , $1 \leq i \leq d$ are non-degenerate, then

$$\mathbb{E}_\xi W^i = 1 \quad \mathbb{P}\text{-a.s.}, \quad (2.15)$$

and

$$\mathbb{P}_\xi(W^i = 0) = q^i(\xi) \quad \mathbb{P}\text{-a.s.} \quad (2.16)$$

We will see that the sufficient condition **H3** can be replaced by a condition of type $EX \log^+ X < \infty$: see Remark 2.7. To obtain a necessary and sufficient condition for the non-degeneracy of W^i , $1 \leq i \leq d$, we need the following condition introduced by Furstenberg and Kesten [12]:

H4. *There exists a constant $D > 1$ such that \mathbb{P} -a.s.,*

$$1 \leq \frac{\max_{1 \leq i, j \leq d} M_0(i, j)}{\min_{1 \leq i, j \leq d} M_0(i, j)} \leq D.$$

Under condition **H4** we have $M_0 \in \mathcal{S}^0$ \mathbb{P} -a.s., so that condition **H1** is satisfied, and $\theta_n = 1$ \mathbb{P} -a.s. for any $n \geq 0$.

The following conditions, which are stronger than **H3**, will also be used.

H5. *There exists a constant $C > 1$ such that, for all $1 \leq i, j \leq d$, \mathbb{P} -a.s.,*

$$\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i, j)} \mathbf{1}_{\left\{ \frac{N_n^i(j)}{M_n(i, j)} \geq C^n \right\}} \right) < +\infty.$$

H6. *For all $1 \leq i \leq d$,*

$$\mathbb{E} \left(\frac{\langle Z_1^i, U_{1, \infty} \rangle}{\lambda_0 U_{0, \infty}(i)} \log^+ \langle Z_1^i, U_{1, \infty} \rangle \right) < +\infty.$$

H7. *For all $1 \leq i, j \leq d$,*

$$\mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i, j)} \log^+ \frac{Z_1^i(j)}{M_0(i, j)} \right) < +\infty.$$

Remark 2.7. In Theorem 2.6, condition **H3** can be replaced by each of the conditions **H5**, **H6** and **H7**. This can be seen by the following implications which will be proved in Appendix 11:

1. under **H1, H2**, we have: **H7** \Rightarrow **H5** \Rightarrow **H3**, **H7** \Rightarrow **H6** \Rightarrow **H3**;
2. under **H4, H2**, we have: **H5** \Leftrightarrow **H3**; **H7** \Leftrightarrow **H6**;
3. under **H4, H2** and when the environment is i.i.d., we have:
H3 \Leftrightarrow **H5** \Leftrightarrow **H6** \Leftrightarrow **H7**.

The following assertion is a consequence of Theorem 2.6. Let

$$E^i = \left\{ \lim_{n \rightarrow +\infty} \|Z_n^i\| = +\infty \right\}$$

be the explosion event on which the branching process explodes, starting with one particle of type i , $1 \leq i \leq d$.

Corollary 2.8. Assume conditions **H1**, **H2** and $\gamma > 0$. Assume also that one of the conditions **H3**, **H5**, **H6** or **H7** holds. Then for all $1 \leq i \leq d$ we have $q^i(\xi) < 1$ \mathbb{P} -a.s. and

$$\mathbb{P}_\xi(E^i) = 1 - q^i(\xi) \quad \mathbb{P}\text{-a.s.} \quad (2.17)$$

Moreover, on the explosion event E^i we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|Z_n^i\| = \gamma \quad \mathbb{P}\text{-a.s.} \quad (2.18)$$

Kaplan [21, Theorem 1] proved (2.17) under stronger conditions: he assumed that all the conditional means (given the environment) are bounded a.s. from below and above by two positive constants, and that all the conditional second moments of the offspring distributions are bounded a.s. by a constant. Tanny [34, Theorem 1] obtained (2.17) and (2.18) under different conditions.

The following theorem gives a necessary and sufficient condition for the non-degeneracy of W^i , $1 \leq i \leq d$, under the Furstenberg - Kesten condition **H4**. Notice that when the environment is i.i.d., the condition that we obtain is of the form $\mathbb{E}X \log^+ X < \infty$ as in the classic Kesten-Stigum theorem [24] established for the deterministic environment case. In this case our result coincides with the corresponding one of Kesten-Stigum [24].

Theorem 2.9. Assume conditions **H2**, **H4** and $\gamma > 0$. Then condition **H5** is necessary and sufficient for W^i , $1 \leq i \leq d$, to be non-degenerate (in the sense of (2.14)); this condition is equivalent to **H7** when the environment $(\xi_n)_{n \geq 0}$ is i.i.d. Furthermore, when W^i , $1 \leq i \leq d$, are non-degenerate, then (2.15) and (2.16) hold.

We finally present our results about the asymptotic behavior of the branching process (Z_n) . All these results will be stated for an i.i.d. environment under the Furstenberg-Kesten condition **H4**.

Under conditions **H2** and **H4**, Furstenberg and Kesten established in [12] a strong law of large numbers for all the components of the product of random matrices $M_{0,n-1}$: for all $1 \leq i, j \leq d$,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log M_{0,n-1}(i, j) = \gamma \quad \mathbb{P}\text{-a.s.} \quad (2.19)$$

Let $\mathbb{P}_{E^i} := \mathbb{P}(\cdot | E^i)$ be the probability conditioned on E^i , when $\mathbb{P}(E^i) > 0$. The next result compares the direction of the vector Z_n with that of the left eigenvector $V_{0,n-1}$ of the matrix $M_{0,n-1}$ and provides its limit law.

Theorem 2.10. *Assume conditions **H2**, **H4** and $\gamma > 0$. Assume additionally that the random environment sequence $\xi = (\xi_0, \xi_1, \dots)$ is i.i.d. Then, for all $1 \leq i \leq d$ such that $\mathbb{P}(E^i) > 0$, we have*

$$\left\| \frac{Z_n^i}{\|Z_n^i\|} - \frac{V_{0,n-1}}{\|V_{0,n-1}\|} \right\| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}_{E^i}} 0; \quad (2.20)$$

moreover, conditional on the event E^i , the sequence $(Z_n^i / \|Z_n^i\|)_{n \geq 0}$ converges in law to $\bar{V}_{0,\infty}$:

$$\frac{Z_n^i}{\|Z_n^i\|} \xrightarrow[n \rightarrow +\infty]{d(\mathbb{P}_{E^i})} \bar{V}_{0,\infty}. \quad (2.21)$$

From Theorem 2.10 and from the convergence of the martingale $(W_n^i)_{n \geq 0}$ we deduce the asymptotic behaviour of the components $Z_n^i(j)$, under two different normalizations $\mathbb{E}_\xi Z_n^i(j)$ and $\rho_{0,n-1} V_{0,n-1}(j)$. Recall that by (2.2), $\mathbb{E}_\xi Z_n^i(j) = M_{0,n-1}(i, j)$, and by (2.3) and **H4**, it holds that $\rho_{0,n-1} V_{0,n-1}(j) \sim \frac{M_{0,n-1}(i,j)}{U_{0,\infty}(i)}$ with $U_{0,\infty}(i) > 0$, as $n \rightarrow +\infty$.

Theorem 2.11. *Assume the conditions of Theorem 2.10. Then, for all $1 \leq i, j \leq d$,*

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i, j)} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} W^i \quad (2.22)$$

and

$$\frac{Z_n^i(j)}{\rho_{0,n-1} V_{0,n-1}(j)} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} W^i U_{0,\infty}(i). \quad (2.23)$$

Moreover, the limit variables W^i , $1 \leq i \leq d$, are non-degenerate (in the sense of (2.14)) if and only if **H7** holds; when **H7** holds, we have (2.15) and (2.16).

Under stronger assumptions than those used in Theorem 2.11, namely that the entries of the mean matrices M_n and those of the corresponding Hessian matrices are bounded, Cohn [6] proved that the convergence in (2.22) can be reinforced to the L^2 -convergence. Our result (2.23) can be compared to the well-known Kesten-Stigum theorem [24, Theorem 1] established in the deterministic environment case. In fact, when the environment is deterministic, (2.23) reduces to Kesten-Stigum's result (1.1), but with the a.s. convergence therein replaced by the convergence in probability. We will give below a sufficient condition to have the a.s. convergence in (2.23).

From Theorem 2.11 we get the following corollary which gives the asymptotic behaviour of the norm $\|Z_n^i\|$ of Z_n^i .

Corollary 2.12. *Assume the conditions of Theorem 2.10. Then for all $1 \leq i \leq d$,*

$$\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} = \frac{\|Z_n^i\|}{\|M_{0,n-1}(i, \cdot)\|} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} W^i$$

and

$$\frac{\|Z_n^i\|}{\rho_{0,n-1}\|V_{0,n-1}\|} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} W^i U_{0,\infty}(i).$$

Our last result states a sufficient condition to get the a.s. convergence instead of the convergence in probability in Theorems 2.10, 2.11 and Corollary 2.12.

Theorem 2.13. *Assume conditions H2, H4 and $\gamma > 0$. Assume additionally that the random environment sequence $\xi = (\xi_0, \xi_1, \dots)$ is i.i.d. Assume also that for some $p > 1$,*

$$\max_{1 \leq i, j \leq d} \mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i, j)} \right)^p < +\infty \quad \text{and} \quad \mathbb{E} \|M_0\|^{1-p} < +\infty. \quad (2.24)$$

Then the following assertions hold :

1. W^i , $1 \leq i \leq d$ are non-degenerate, and (2.15) and (2.16) hold.
2. For all $1 \leq i \leq d$, \mathbb{P} -a.s. on the event E^i ,

$$\left\| \frac{Z_n^i}{\|Z_n^i\|} - \frac{V_{0,n-1}}{\|V_{0,n-1}\|} \right\| \xrightarrow[n \rightarrow +\infty]{} 0. \quad (2.25)$$

3. For all $1 \leq i, j \leq d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i, j)} \xrightarrow[n \rightarrow +\infty]{} W^i \quad \mathbb{P}\text{-a.s.}, \quad (2.26)$$

$$\frac{Z_n^i(j)}{\rho_{0,n-1}V_{0,n-1}(j)} \xrightarrow[n \rightarrow +\infty]{} W^i U_{0,\infty}(i) \quad \mathbb{P}\text{-a.s.} \quad (2.27)$$

4. For all $1 \leq i \leq d$,

$$\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} = \frac{\|Z_n^i\|}{\|M_{0,n-1}(i, \cdot)\|} \xrightarrow[n \rightarrow +\infty]{} W^i \quad \mathbb{P}\text{-a.s.}, \quad (2.28)$$

$$\frac{\|Z_n^i\|}{\rho_{0,n-1}\|V_{0,n-1}\|} \xrightarrow[n \rightarrow +\infty]{} W^i U_{0,\infty}(i) \quad \mathbb{P}\text{-a.s.} \quad (2.29)$$

Under assumptions stronger than those of Theorem 2.13, one can show the L^p -convergence (with $p > 1$) instead of the a.s. convergence stated above, with an exponential rate. However this task is outside the scope of the present paper and will be done in a forthcoming work.

3. Asymptotic properties of the pseudo spectral radii for products of positive random matrices

In this section we prove Propositions 2.2 and 2.5 on the asymptotic properties of the sequence of pseudo spectral radii (λ_n) related to the products of positive random matrices (M_n) .

3.1. Proof of Proposition 2.2

From (2.3), it holds that for all $n \geq 1$ and $1 \leq j \leq d$,

$$\begin{aligned} M_n U_{n+1,\infty} &= \lim_{k \rightarrow +\infty} M_n U_{n+1,n+k} \\ &= \lim_{k \rightarrow +\infty} M_n \frac{M_{n+1,n+k}(\cdot, j)}{\rho_{n+1,n+k} V_{n+1,n+k}(j)} \mathbf{1}_{\{\theta_{n+1} \leq k\}} \\ &= \lim_{k \rightarrow +\infty} \frac{\rho_{n,n+k} V_{n,n+k}(j)}{\rho_{n+1,n+k} V_{n+1,n+k}(j)} \mathbf{1}_{\{\theta_{n+1} \leq k\}} \\ &\quad \times \lim_{k \rightarrow +\infty} \frac{M_{n,n+k}(\cdot, j)}{\rho_{n,n+k} V_{n,n+k}(j)} \mathbf{1}_{\{\theta_n \leq k\}} \\ &= \lim_{k \rightarrow +\infty} \frac{\rho_{n,n+k} V_{n,n+k}(j)}{\rho_{n+1,n+k} V_{n+1,n+k}(j)} \mathbf{1}_{\{\theta_{n+1} \leq k\}} U_{n,\infty}. \end{aligned}$$

Combining this with (2.5), the relation (2.9) follows. Now we prove (2.10). From (2.7) we get that for all $n \geq 1$,

$$V_{0,n-1}^T M_{0,n-1} U_{n,\infty} = \lambda_{0,n-1} V_{0,n-1}^T U_{0,\infty},$$

so

$$\frac{\lambda_{0,n-1}}{\rho_{0,n-1} \langle V_{0,n-1}, U_{n,\infty} \rangle} = \frac{1}{\langle V_{0,n-1}, U_{0,\infty} \rangle}. \quad (3.1)$$

Moreover, by Proposition 2.1 we know that $U_{0,n-1} \xrightarrow[n \rightarrow +\infty]{} U_{0,\infty} > 0$ \mathbb{P} -a.s., so there exist two random variables $A > 0$ and $N_0 \geq 1$ such that \mathbb{P} -a.s. for all $n \geq N_0$,

$$0 < A \leq \min_{1 \leq i \leq d} U_{0,n-1}(i) \leq 1.$$

Since for all $n \geq 1$ we have $\langle U_{0,n-1}, V_{0,n-1} \rangle = 1$, it follows that for all $n \geq N_0$,

$$1 \leq \|V_{0,n-1}\| \leq \frac{1}{A}. \quad (3.2)$$

Consequently, for all $n \geq N_0$,

$$|\langle V_{0,n-1}, U_{0,\infty} \rangle - 1| = |\langle V_{0,n-1}, U_{0,\infty} - U_{0,n-1} \rangle| \leq \frac{1}{A} \|U_{0,\infty} - U_{0,n-1}\|,$$

so that

$$\langle V_{0,n-1}, U_{0,\infty} \rangle \xrightarrow[n \rightarrow +\infty]{} 1 \quad \mathbb{P}\text{-a.s.} \quad (3.3)$$

Combining (3.1) and (3.3) gives (2.10), which ends the proof of Proposition 2.2.

3.2. Proof of Proposition 2.5

By (2.4) we have

$$\lambda_0 = \|M_0 U_{1,\infty}\| \leq \|M_0\| \quad \mathbb{P}\text{-a.s.}$$

Using condition **H2**, it follows that $\mathbb{E} \log^+ \lambda_0 < +\infty$, so that $\mathbb{E} \log \lambda_0$ is well defined with value in $\mathbb{R} \cup \{-\infty\}$. Recall that $(\lambda_n)_{n \geq 0}$ is a stationary ergodic sequence of random variables. Applying an extension of the Birkhoff Ergodic Theorem [18, Theorem 1] we deduce that, \mathbb{P} -a.s.,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \lambda_{0,n-1} = \mathbb{E} \log \lambda_0. \quad (3.4)$$

Moreover, from (2.8) we see that for all $n \geq 1$, \mathbb{P} -a.s.,

$$\frac{1}{n} \log \min_{1 \leq j \leq d} U_{n,\infty}(j) + \frac{1}{n} \log \|M_{0,n-1}\| \leq \frac{1}{n} \log \lambda_{0,n-1} \leq \frac{1}{n} \log \|M_{0,n-1}\|. \quad (3.5)$$

Since $\left(\log \min_{1 \leq j \leq d} U_{n,\infty}(j) \right)_{n \geq 0}$ is a stationary sequence of random variables, by Slutsky's lemma it follows that

$$\frac{1}{n} \log \min_{1 \leq j \leq d} U_{n,\infty}(j) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0.$$

Combining this with the law of large numbers (2.13) and letting $n \rightarrow +\infty$ in (3.5), we obtain that

$$\frac{1}{n} \log \lambda_{0,n-1} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} \gamma. \quad (3.6)$$

By (3.4), (3.6) and the uniqueness of the limit in probability, it holds that $\mathbb{E} \log \lambda_0 = \gamma$. This concludes the proof of Proposition 2.5.

4. The fundamental martingale (W_n^i)

In this section we prove that (W_n^i) is a martingale, and that the quenched Laplace transform of its limit variable satisfies a functional equation that we make precise.

4.1. Proof of Theorem 2.3

Clearly $(W_n^i)_{n \geq 0}$ is adapted to $(\mathcal{F}_n)_{n \geq 0}$, and using (2.7) we have for all $n \geq 0$ and $1 \leq i \leq d$,

$$\mathbb{E}_\xi W_n^i = \frac{\mathbb{E}_\xi \langle Z_n^i, U_{n,\infty} \rangle}{\lambda_{0,n-1} U_{0,\infty}(i)} = \frac{\langle M_{0,n-1}(i, \cdot), U_{n,\infty} \rangle}{\lambda_{0,n-1} U_{0,\infty}(i)} = 1 \quad \mathbb{P}\text{-a.s.}$$

Moreover, we know that $\mathbb{E}_\xi [Z_{n+1}^i | \mathcal{F}_n] = M_n^T Z_n^i$, so we obtain that for all $n \geq 0$ and $1 \leq i \leq d$,

$$\mathbb{E}_\xi [W_{n+1}^i | \mathcal{F}_n] = \frac{\langle M_n^T Z_n^i, U_{n+1,\infty} \rangle}{\lambda_{0,n} U_{0,\infty}(i)} = \frac{\langle Z_n^i, M_n U_{n+1,\infty} \rangle}{\lambda_{0,n} U_{0,\infty}(i)}.$$

Then applying (2.5) we get that for all $n \geq 0$,

$$\mathbb{E}_\xi \left[W_{n+1}^i | \mathcal{F}_n \right] = \frac{\langle Z_n^i, \lambda_n U_{n,\infty} \rangle}{\lambda_{0,n} U_{0,\infty}(i)} = \frac{\langle Z_n^i, U_{n,\infty} \rangle}{\lambda_{0,n-1} U_{0,\infty}(i)} = W_n^i.$$

This proves that the sequence $(W_n^i)_{n \geq 0}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n \geq 0}$ under the law \mathbb{P}_ξ . The argument is similar under the law \mathbb{P} . By Fatou's Lemma we have $\mathbb{E}_\xi W^i \leq 1$ \mathbb{P} -a.s. This ends the proof of Theorem 2.3.

4.2. Proof of Theorem 2.4

Conditioned on the environment ξ , the random vectors $Z_{l,n,k}^r = (Z_{l,n,k}^r(1), \dots, Z_{l,n,k}^r(d))$, with $Z_{l,n,k}^r(j)$ denoting the offspring of type j at time $n+k$ of the l -th particle of type r in the generation n , are independent and have the same probability generating function $f_n^r \circ f_{n+1} \circ \dots \circ f_{n+k-1}$. By iterating (2.1), it is easy to see that the process $(Z_n)_{n \geq 0}$ satisfies the relation

$$Z_{n+k} = \sum_{r=1}^d \sum_{l=1}^{Z_n(r)} Z_{l,n,k}^r, \quad n \geq 0, k \geq 1. \quad (4.1)$$

From (4.1) and (2.11) we get that for all $n \geq 0$ and $1 \leq i \leq d$,

$$\begin{aligned} W_{n+1}^i &= \sum_{r=1}^d \sum_{l=1}^{Z_1^i(r)} \frac{\langle Z_{l,1,n}^r, U_{n+1,\infty} \rangle}{\lambda_{0,n} U_{0,\infty}(i)} \\ &= \sum_{r=1}^d \frac{U_{1,\infty}(r)}{\lambda_0 U_{0,\infty}(i)} \sum_{l=1}^{Z_1^i(r)} W_{l,1,n}^r, \end{aligned} \quad (4.2)$$

where

$$W_{l,1,n}^r := \frac{\langle Z_{l,1,n}^r, U_{n+1,\infty} \rangle}{\lambda_{1,n} U_{1,\infty}(r)}.$$

Clearly $(Z_{l,1,n}^r)_{n \geq 0}$ is a supercritical MBPRE in the random environment $T\xi$, and $(W_{l,1,n}^r)_{n \geq 0}$ is its associated martingale which converges to a random variable denoted $W_{l,1}^r$. Moreover, when ξ is given, the random variables $W_{l,1}^r$, $l \geq 0$, are independent of each other and independent of Z_1^i under \mathbb{P}_ξ , with a common distribution such that

$$\mathbb{P}_\xi \left(W_{l,1}^r \in \cdot \right) = \mathbb{P}_{T\xi} \left(W^r \in \cdot \right).$$

Letting $n \rightarrow +\infty$ in (4.2) it follows that for all $1 \leq i \leq d$,

$$W^i = \sum_{r=1}^d \frac{U_{1,\infty}(r)}{\lambda_0 U_{0,\infty}(i)} \sum_{l=1}^{Z_1^i(r)} W_{l,1}^r.$$

Taking the Laplace transform and using the independence under \mathbb{P}_ξ of the random variables $W_{l,1}^r$ and $Z_1^i(r)$ for $l \geq 0$ and $1 \leq r \leq d$, we get that for all $1 \leq i \leq d$ and $t \geq 0$,

$$\begin{aligned} \phi_\xi^i(t) &= \mathbb{E}_\xi \left[\prod_{r=1}^d \prod_{l=1}^{Z_1^i(r)} \mathbb{E}_\xi \left[e^{\frac{-tU_{1,\infty}(r)}{\lambda_0 U_{0,\infty}(i)} W_{l,1}^r} \right] \right] \\ &= \mathbb{E}_\xi \left[\prod_{r=1}^d \left(\phi_{T\xi} \left(t \frac{U_{1,\infty}(r)}{\lambda_0 U_{0,\infty}(i)} \right) \right)^{Z_1^i(r)} \right] \\ &= f_0^i \left(\phi_{T\xi}^1 \left(t \frac{U_{1,\infty}(1)}{\lambda_0 U_{0,\infty}(i)} \right), \dots, \phi_{T\xi}^d \left(t \frac{U_{1,\infty}(d)}{\lambda_0 U_{0,\infty}(i)} \right) \right), \end{aligned}$$

which is the desired equation.

5. Proof of Theorem 2.6

In this section we prove Theorem 2.6 about the non degeneracy of the limit variables W^i . We shall adapt the proof of Lyons, Pemantle and Peres in [29], which first consists to interpret a branching process as a random tree. Let \mathcal{T} be the set of (colored) trees and denote by $\mathbb{T}^i \in \mathcal{T}$ the random tree associated to the MBRE $(Z_n^i)_{n \geq 0}$. In fact a multi-type branching process can be identified naturally as a random colored-tree (type i is considered as color i), which is a subset of $\cup_{n=1}^\infty \{1, \dots, d\}^n \times \cup_{n=0}^\infty \mathbb{N}^{*n}$ with $\mathbb{N}^{*0} = \{\emptyset\}$. The initial particle \emptyset of type i is denoted (i, \emptyset) ; a particle of type i of generation n is denoted by (i, u) with $u \in \mathbb{N}^{*n}$ a sequence of length n ; its k -th child of type j is denoted (ij, uk) , which is linked with its ancestor (i, u) .

We write $t \stackrel{n}{=} t'$ for $n \geq 0$ and $t, t' \in \mathcal{T}$, if t and t' coincide up to height n . It is known that this defines a relation of equivalence. The associated equivalence classes generate the σ -fields \mathcal{G}_n , which form a filtration on \mathcal{T} . For any $s \in t$ and $t \in \mathcal{T}$, denote by $y(s) \in \mathbb{N}^d$ the number of children, by $\text{gen}(s)$ the generation and by $\text{type}(s)$ the type of the particle s . The distribution of \mathbb{T}^i is characterized by

$$\mathbb{P}_\xi \left(\mathbb{T}^i \stackrel{n}{=} t \right) = \prod_{s \in t, \text{gen}(s) < n} \mathbb{P}_\xi \left(N_{\text{gen}(s)}^{\text{type}(s)} = y(s) \right), \quad (5.1)$$

for any $n \geq 0$ and $t \in \mathcal{T}$; it is well defined by the Kolmogorov extension theorem.

We shall construct an auxiliary random tree \mathbb{T}_*^i called "size-biased tree", for all $1 \leq i \leq d$. At time 0, we start with one initial particle of type i , labeled $(D_0, L_0) := (i, 1)$, which forms the generation 0 of the tree \mathbb{T}_*^i . In the following, the environment ξ is fixed, and the notion of independence is conditioned on ξ . We generate d independent random vectors $Y_0^j \in \mathbb{N}^d$, $1 \leq j \leq d$, such that

$$\mathbb{P}_\xi \left(Y_0^j = y \right) = \frac{\langle y, U_{1,\infty} \rangle}{\lambda_0 U_{0,\infty}(j)} \mathbb{P}_\xi \left(N_0^j = y \right), \quad y \in \mathbb{N}^d.$$

The above formula defines a probability due to the fact that $\mathbb{E}_\xi W_1^i = 1$. Let $Y_0 := Y_0^{D_0} = Y_0^i$ be the number of children of the initial particle $(D_0, L_0) = (i, 1)$. They constitute the particles of the first generation of the tree \mathbb{T}_*^i . At time $n = 1$, we pick at random one particle of type D_1 of the first generation, labeled (D_1, L_1) , with the distribution

$$\mathbb{P}_\xi \left((D_1, L_1) = (j, l) \mid Y_0 \right) = \frac{U_{1,\infty}(j)}{\langle Y_0, U_{1,\infty} \rangle}, \quad 1 \leq l \leq Y_0(j), 1 \leq j \leq d.$$

The l -th particle of type j of the first generation, except for the particle (D_1, L_1) , produces its descendants of the next generations according to a random tree $\mathbb{T}_1^j(l)$ (which forms the subtree of \mathbb{T}_*^i starting from this particle), with distribution

$$\mathbb{P}_\xi \left(\mathbb{T}_1^j(l) \in \cdot \right) = \mathbb{P}_{T\xi} \left(\mathbb{T}^j \in \cdot \right), \quad (j, l) \neq (D_1, L_1),$$

$1 \leq l \leq Y_0(j)$, $1 \leq j \leq d$; the random trees $\mathbb{T}_1^j(l)$, $1 \leq j \leq d, l \geq 1$, are independent of each other. Moreover, we generate independent random vectors $Y_1^j \in \mathbb{N}^d$, $1 \leq j \leq d$, which are also independent of (D_1, L_1) and independent of the trees $\mathbb{T}_1^j(l)$, with distributions

$$\mathbb{P}_\xi \left(Y_1^j \in \cdot \right) = \mathbb{P}_{T\xi} \left(Y_0^j \in \cdot \right).$$

The particle (D_1, L_1) of the first generation produces its children of the next generation according to

$$Y_1 := Y_1^{D_1} = \sum_{j=1}^d Y_1^j \mathbf{1}_{\{D_1=j\}},$$

namely, $Y_1(j)$ is the number of children of type j generated by the particle (D_1, L_1) . We then proceed in the same way. Assume that at time $n \geq 2$, we

have defined all the particles of generation n , and all the genealogical trees of the particles of generation n except for the direct children of (D_{n-1}, L_{n-1}) . We pick at random one particle of type D_n of the generation n , labeled (D_n, L_n) , with the distribution

$$\mathbb{P}_\xi \left((D_n, L_n) = (j, l) \mid Y_{n-1} \right) = \frac{U_{n,\infty}(j)}{\langle Y_{n-1}, U_{n,\infty} \rangle}, \quad 1 \leq l \leq Y_{n-1}(j), 1 \leq j \leq d,$$

where $Y_{n-1} = (Y_{n-1}(1), \dots, Y_{n-1}(d))$, with $Y_{n-1}(j)$ denoting the number of children of type j of the particle (D_{n-1}, L_{n-1}) . The l -th particle of type j of the children of (D_{n-1}, L_{n-1}) , except for the particle (D_n, L_n) , produces its descendants of the next generations according to a random tree $\mathbb{T}_n^j(l)$ (which forms the subtree of \mathbb{T}_*^i starting from this particle), with distribution

$$\mathbb{P}_\xi \left(\mathbb{T}_n^j(l) \in \cdot \right) = \mathbb{P}_{T^n \xi} \left(\mathbb{T}^j \in \cdot \right), \quad (j, l) \neq (D_n, L_n),$$

$1 \leq l \leq Y_{n-1}(j)$, $1 \leq j \leq d$; these trees $\mathbb{T}_n^j(l)$, $1 \leq j \leq d, l \geq 1$, are independent of each other. Moreover, we generate independent random vectors $Y_n^j \in \mathbb{N}^d$, $1 \leq j \leq d$, which are independent of the past, also independent of (D_n, L_n) and independent of the trees $\mathbb{T}_n^j(l)$, with distributions

$$\mathbb{P}_\xi \left(Y_n^j \in \cdot \right) = \mathbb{P}_{T^n \xi} \left(Y_0^j \in \cdot \right).$$

The particle (D_n, L_n) of the generation n produces its children of the next generation according to

$$Y_n := Y_n^{D_n} = \sum_{j=1}^d Y_n^j \mathbf{1}_{\{D_n=j\}},$$

namely, $Y_n(j)$ is the number of children of type j generated by the particle (D_n, L_n) . Therefore, by recurrence on n , we have defined the random tree \mathbb{T}_*^i .

For all $n \geq 0$, denote by Δ_n^i the distinguished path in \mathbb{T}_*^i formed by the particles (D_k, L_k) , $k \leq n$, which is identified to the last particle of the path. We show by induction that

$$\mathbb{P}_\xi \left(\mathbb{T}_*^i \stackrel{n}{=} t, \Delta_n^i = \sigma_j \right) = \frac{U_{n,\infty}(j)}{\lambda_{0,n-1} U_{0,\infty}(i)} \mathbb{P}_\xi \left(\mathbb{T}^i \stackrel{n}{=} t \right), \quad (5.2)$$

for all $n \geq 1$, $t \in \mathcal{T}$ a tree of height at least n , and $\sigma_j \in t$ a particle of type j in generation n . For $n = 1$ we have

$$\begin{aligned} \mathbb{P}_\xi \left(\mathbb{T}_*^i \stackrel{1}{=} t, \Delta_1^i = \sigma_j \right) &= \frac{U_{1,\infty}(j)}{\langle y, U_{1,\infty} \rangle} \frac{\langle y, U_{1,\infty} \rangle}{\lambda_0 U_{0,\infty}(i)} \mathbb{P}_\xi \left(N_0^i = y \right) \\ &= \frac{U_{1,\infty}(j)}{\lambda_0 U_{0,\infty}(i)} \mathbb{P}_\xi \left(\mathbb{T}^i \stackrel{1}{=} t \right), \end{aligned}$$

where y is the number of children of the initial particle in generation 0 in t . Now assume that (5.2) is true for some $n \geq 1$. Let $t \in \mathcal{T}$ be a tree of height at least $n + 1$, $\sigma_j \in t$ a particle of type j in generation $n + 1$, $\tilde{\sigma}_r \in t$ his ancestor of type r in generation n . Then using (5.1) and the notation introduced before (5.1), we have

$$\begin{aligned}
 & \mathbb{P}_\xi \left(\mathbb{T}_*^i \stackrel{n+1}{=} t, \Delta_{n+1}^i = \sigma_j \right) \\
 &= \mathbb{P}_\xi \left(\mathbb{T}_*^i \stackrel{n}{=} t, \Delta_n^i = \tilde{\sigma}_r \right) \mathbb{P}_\xi \left((D_{n+1}, L_{n+1}) = \sigma_j \right) \times \\
 & \quad \mathbb{P}_\xi \left(Y_n^r = y(\tilde{\sigma}_r) \right) \prod_{s \in t, s \neq \tilde{\sigma}_r, \text{gen}(s)=n} \mathbb{P}_\xi \left(N_n^{\text{type}(s)} = y(s) \right) \\
 &= \mathbb{P}_\xi \left(\mathbb{T}_*^i \stackrel{n}{=} t, \Delta_n^i = \tilde{\sigma}_r \right) \frac{U_{n+1, \infty}(j)}{\langle y(\tilde{\sigma}_r), U_{n+1, \infty} \rangle} \times \\
 & \quad \frac{\langle y(\tilde{\sigma}_r), U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(r)} \mathbb{P}_\xi \left(N_n^r = y(\tilde{\sigma}_r) \right) \prod_{s \in t, s \neq \tilde{\sigma}_r, \text{gen}(s)=n} \mathbb{P}_\xi \left(N_n^{\text{type}(s)} = y(s) \right) \\
 &= \frac{U_{n, \infty}(r)}{\lambda_{0, n-1} U_{0, \infty}(i)} \mathbb{P}_\xi \left(\mathbb{T}_*^i \stackrel{n}{=} t \right) \frac{U_{n+1, \infty}(j)}{\lambda_n U_{n, \infty}(r)} \prod_{s \in t, \text{gen}(s)=n} \mathbb{P}_\xi \left(N_n^{\text{type}(s)} = y(s) \right) \\
 &= \frac{U_{n+1, \infty}(j)}{\lambda_{0, n} U_{0, \infty}(i)} \mathbb{P}_\xi \left(\mathbb{T}_*^i \stackrel{n+1}{=} t \right).
 \end{aligned}$$

Hence (5.2) remains true for $n + 1$. By induction (5.2) holds for all $n \geq 1$. Summing over σ_j in (5.2), we see that for any $n \geq 1$ and $t \in \mathcal{T}$,

$$\mathbb{P}_\xi \left(\mathbb{T}_*^i \stackrel{n}{=} t \right) = \frac{\langle z_n(t), U_{n, \infty} \rangle}{\lambda_{0, n-1} U_{0, \infty}(i)} \mathbb{P}_\xi \left(\mathbb{T}_*^i \stackrel{n}{=} t \right), \quad (5.3)$$

where $z_n(t)$ is the vector counting the number of particles in t at generation n , of types $j = 1, \dots, d$.

By abuse of notation we denote by \mathbb{P}^i and \mathbb{P}_ξ^i respectively the annealed and quenched distributions of the tree $\mathbb{T}^i \in \mathcal{T}$. The annealed and quenched laws of biased tree $\mathbb{T}^i \in \mathcal{T}$ are denoted by \mathbb{P}_*^i and $\mathbb{P}_{*\xi}^i$, and defined according to

$$\mathbb{P}_*^i(\mathbb{T}^i \in \cdot) := \mathbb{P}^i(\mathbb{T}_*^i \in \cdot), \quad \mathbb{P}_{*\xi}^i(\mathbb{T}^i \in \cdot) := \mathbb{P}_\xi^i(\mathbb{T}_*^i \in \cdot). \quad (5.4)$$

By $\mathbb{P}_*^i|_{\mathcal{G}_n}$ and $\mathbb{P}_\xi^i|_{\mathcal{G}_n}$ we denote the restrictions of the respective laws to the σ -field \mathcal{G}_n . Then by (5.3) we obtain that for all $n \geq 0$,

$$d\mathbb{P}_*^i|_{\mathcal{G}_n} = W_n^i d\mathbb{P}_\xi^i|_{\mathcal{G}_n},$$

which means that $\mathbb{P}_*^i|_{\mathcal{G}_n}$ has the density W_n^i with respect to $\mathbb{P}^i|_{\mathcal{G}_n}$. However, \mathbb{P}_*^i is not necessarily absolutely continuous with respect to \mathbb{P}^i . Define

$$W^i := \limsup_{n \rightarrow +\infty} W_n^i.$$

Then according to Theorem 5.3.3 in [8] we have the following two equivalences :

$$\begin{cases} W^i = +\infty & \mathbb{P}_*^i\text{-a.s.} & \Leftrightarrow & W^i = 0 & \mathbb{P}^i\text{-a.s.}; \\ W^i < +\infty & \mathbb{P}_*^i\text{-a.s.} & \Leftrightarrow & \mathbb{E}W^i = 1. \end{cases} \quad (5.5)$$

Now we prove that the condition **H3** is sufficient for the random variable W^i to be finite \mathbb{P}_*^i -a.s., which will conclude the proof of Theorem 2.6 by (5.5). Assume **H3**. So there exists $C > 0$ such that for all $1 \leq j \leq d$, \mathbb{P} -a.s.,

$$\begin{aligned} \sum_{n=0}^{+\infty} \mathbb{P}_\xi \left(\frac{\log^+ \langle Y_n^j, U_{n+1, \infty} \rangle}{n} \geq C \right) \\ = \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^j, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(j)} \mathbf{1}_{\{\langle N_n^j, U_{n+1, \infty} \rangle \geq e^{Cn}\}} \right) < \infty. \end{aligned}$$

Since the random variables $\langle Y_n^j, U_{n+1, \infty} \rangle$, $n \geq 0$, are independent under \mathbb{P}_ξ , by the Borel-Cantelli lemma we deduce that for all $1 \leq j \leq d$, \mathbb{P}_ξ -a.s.,

$$\limsup_{n \rightarrow +\infty} \frac{\log^+ \langle Y_n^j, U_{n+1, \infty} \rangle}{n} \leq C. \quad (5.6)$$

Moreover $(\log^+ \langle Y_n^j, U_{n+1, \infty} \rangle)_{n \geq 0}$ is a non negative stationary and ergodic stochastic process, hence by a result of Tanny [32, Theorem 1] we know that $\limsup_{n \rightarrow +\infty} \log^+ \langle Y_n^j, U_{n+1, \infty} \rangle / n$ is either 0 \mathbb{P} -a.s. or $+\infty$ \mathbb{P} -a.s. Therefore, by (5.6) it follows that for all $1 \leq j \leq d$,

$$\lim_{n \rightarrow +\infty} \frac{\log^+ \langle Y_n^j, U_{n+1, \infty} \rangle}{n} = 0 \quad \mathbb{P}^i\text{-a.s.} \quad (5.7)$$

Since $\log^+ \langle Y_n, U_{n+1, \infty} \rangle = \sum_{j=1}^d \log^+ \langle Y_n^j, U_{n+1, \infty} \rangle \mathbf{1}_{\{D_n=j\}}$, this implies that

$$\lim_{n \rightarrow +\infty} \frac{\log^+ \langle Y_n, U_{n+1, \infty} \rangle}{n} = 0 \quad \mathbb{P}^i\text{-a.s.}$$

Furthermore, using Proposition 2.5, as $n \rightarrow +\infty$ we have

$$\log \lambda_{0, n-1} \sim \gamma n \quad \mathbb{P}\text{-a.s.}$$

The last two assertions imply that

$$\sum_{n=0}^{+\infty} \frac{\langle Y_n, U_{n+1, \infty} \rangle}{\lambda_{0, n-1} U_{0, \infty}(i)} < +\infty \quad \mathbb{P}^i\text{-a.s.} \quad (5.8)$$

For all $1 \leq i \leq d$ and $n \geq 0$, let $Z_{*n}^i \in \mathbb{N}^d$ be the vector whose j -th component is the number of particles in \mathbb{T}_*^i at generation n , of type j . For all $0 \leq k \leq n$, let $Z_{*k, n}$ be the vector whose j -th component is the number of the particles in \mathbb{T}_*^i at generation n , of type j , which have as ancestor one of the children of (D_k, L_k) , except (D_{k+1}, L_{k+1}) . Then the processes $\{Z_{*k, n}, n \geq k\}$, $k \geq 0$, are independent (under \mathbb{P}_ξ), with environment $T^{k+1}\xi$ and initial state $Z_{*k, k} = Y_k - e_{D_{k+1}}$. So, for all $n \geq 0$, we have the decomposition

$$Z_{*n}^i = e_{D_n} + \sum_{k=1}^n Z_{*k, n} \quad \mathbb{P}^i\text{-a.s.}$$

Set $\mathcal{Y} = \{Y_n^j, D_n : n \geq 0, 1 \leq j \leq d\}$. Then by Lemma 2.2, for $n \geq 1$, \mathbb{P}^i -a.s.,

$$\begin{aligned} & \mathbb{E}_\xi[\langle Z_{*n}^i - e_{D_n}, U_{n, \infty} \rangle | \mathcal{Y}, Z_{*k, n-1}, k \leq n-1] \\ &= \sum_{k=1}^n \langle \mathbb{E}_\xi[Z_{*k, n} | \mathcal{Y}, Z_{*k, n-1}, k \leq n-1], U_{n, \infty} \rangle \\ &= \sum_{k=1}^{n-1} \langle M_{n-1}^T Z_{*k, n-1}, U_{n, \infty} \rangle + \langle Y_{n-1} - e_{D_n}, U_{n, \infty} \rangle \\ &= \lambda_{n-1} \langle Z_{*n-1}^i - e_{D_{n-1}}, U_{n-1, \infty} \rangle + \langle Y_{n-1} - e_{D_n}, U_{n, \infty} \rangle. \end{aligned}$$

Consequently, conditioned on \mathcal{Y} and on the environment ξ , the process

$$A_n := \frac{\langle Z_{*n}^i - e_{D_n}, U_{n, \infty} \rangle}{\lambda_{0, n-1} U_{0, \infty}(i)} - \sum_{k=0}^{n-1} \frac{\langle Y_k - e_{D_{k+1}}, U_{k+1, \infty} \rangle}{\lambda_{0, k-1} U_{0, \infty}(i)}, \quad n \geq 0, \quad (5.9)$$

is a martingale w.r.t. $\sigma(\xi, \mathcal{Y}, \{Z_{*k, n}, k \leq n\})$, $n \geq 0$, under the law \mathbb{P}_ξ^i . Notice that A_n is bounded from below by the opposite of the series (5.8) which converges a.s., so this martingale converges \mathbb{P}^i -a.s. to a finite limit. From (5.9) and using the a.s. convergence of A_n and of the series (5.8), together with the fact that $\frac{\langle e_{D_n}, U_{n, \infty} \rangle}{\lambda_{0, n-1} U_{0, \infty}(i)} \rightarrow 0$ a.s., we deduce that

$$\lim_{n \rightarrow +\infty} \frac{\langle Z_{*n}^i, U_{n, \infty} \rangle}{\lambda_{0, n-1} U_{0, \infty}(i)} \quad \text{exists and is finite } \mathbb{P}^i\text{-a.s.}$$

Therefore, by the definition of \mathbb{P}_*^i (see (5.4)),

$$W^i = \limsup_{n \rightarrow +\infty} \frac{\langle Z_n^i, U_{n,\infty} \rangle}{\lambda_{0,n-1} U_{0,\infty}(i)} < +\infty \quad \mathbb{P}_*^i\text{-a.s.}$$

So applying (5.5) we see that $\mathbb{E}W^i = 1$, or equivalently $\mathbb{E}_\xi W^i = 1$ \mathbb{P} -a.s., which implies that W^i is non-degenerate.

Finally, if we denote by

$$\bar{q}^i(\xi) := \mathbb{P}_\xi(W^i = 0), \quad 1 \leq i \leq d,$$

then by letting $t \rightarrow +\infty$ in (2.12) we see that

$$\bar{q}(\xi) = f_0(\bar{q}(T\xi)), \quad (5.10)$$

where $\bar{q}(\xi) = (\bar{q}^1(\xi), \dots, \bar{q}^d(\xi))$. Clearly,

$$\left\{ \|Z_n^i\| \xrightarrow[n \rightarrow +\infty]{} 0 \right\} \subset \{W^i = 0\}. \quad (5.11)$$

So, if W^i is non-degenerate, then we have $q^i(\xi) \leq \bar{q}^i(\xi) < 1$ \mathbb{P} -a.s. Hence, using [21, Proposition 3.1] we deduce from (5.10) that $\bar{q}(\xi) = q(\xi)$ \mathbb{P} -a.s. This concludes the proof of Theorem 2.6.

6. Proof of Corollary 2.8

By Theorem 2.6, we know that

$$\mathbb{P}_\xi(W^i = 0) = q^i(\xi) < 1.$$

So from (5.11) we conclude that for all $1 \leq i \leq d$,

$$\left\{ \|Z_n^i\| \xrightarrow[n \rightarrow +\infty]{} 0 \right\} = \{W^i = 0\} \quad \mathbb{P}\text{-a.s.} \quad (6.1)$$

By the definition of W_n^i (cf. (2.11)), we obtain that for all $1 \leq i \leq d$ and $n \geq 0$,

$$W_n^i \leq \frac{\|Z_n^i\|}{\lambda_{0,n-1} U_{0,\infty}(i)}. \quad (6.2)$$

Using Proposition 2.5, it follows from (6.2) that, \mathbb{P} -a.s. on the event $\{W^i > 0\}$,

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \log \|Z_n^i\| \geq \gamma. \quad (6.3)$$

For any $\varepsilon > 0$, all $n \geq 0$ and $1 \leq i \leq d$, we have

$$\mathbb{P} \left(\|Z_n^i\| \geq e^{\varepsilon n} \|M_{0,n-1}\| \right) \leq \mathbb{E} \left(\frac{\mathbb{E}_\xi \|Z_n^i\|}{\|M_{0,n-1}\|} e^{-\varepsilon n} \right) \leq e^{-\varepsilon n}.$$

It follows that $\sum_{n \geq 1} \mathbb{P}(\|Z_n^i\| \geq e^{\varepsilon n} \|M_{0,n-1}\|) < +\infty$. Applying the Borel-Cantelli lemma, we deduce that

$$\mathbb{P} \left(\|Z_n^i\| \geq e^{\varepsilon n} \|M_{0,n-1}\| \quad \text{i.o.} \right) = 0,$$

where i.o. means infinitely often. Combining this with (2.13) we get that for all $1 \leq i \leq d$, \mathbb{P} -a.s.,

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log \|Z_n^i\| \leq \varepsilon + \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|M_{0,n-1}\| = \varepsilon + \gamma.$$

Letting $\varepsilon \rightarrow 0$ and using (6.1) and (6.3), we see that \mathbb{P} -a.s. on the explosion event E^i , $\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|Z_n^i\| = \gamma$. This ends the proof of Corollary 2.8.

7. Proof of Theorem 2.9

7.1. Auxiliary results

We will need the following preliminary lemma.

Lemma 7.1. *Assume condition H4. Then :*

1. for all $n, k \geq 0$ and $1 \leq i, j, r \leq d$, \mathbb{P} -a.s.,

$$\frac{1}{D} \leq \frac{M_{n,n+k}(i, j)}{M_{n,n+k}(i, r)} \leq D \quad \text{and} \quad \frac{1}{D} \leq \frac{M_{n,n+k}(i, j)}{M_{n,n+k}(r, j)} \leq D; \quad (7.1)$$

2. for all $n \geq 0$ and $1 \leq i \leq d$, \mathbb{P} -a.s.,

$$\frac{1}{dD} \leq U_{n,\infty}(i) \leq 1; \quad (7.2)$$

3. for all $n, k \geq 0$ and $1 \leq i, j \leq d$, \mathbb{P} -a.s.,

$$\frac{1}{dD^2} \leq \frac{M_{n,n+k}(i, j) U_{n+k+1,\infty}(j)}{\lambda_{n,n+k} U_{n,\infty}(i)} \leq 1. \quad (7.3)$$

Proof. For $k = 0$, (7.1) is a direct consequence of condition H4 and the fact that the sequence $(M_n)_{n \geq 0}$ is stationary. Moreover, for all $n \geq 0$, $k \geq 1$ and $1 \leq i, j, r \leq d$, we have

$$\frac{M_{n,n+k}(i, j)}{M_{n,n+k}(i, r)} = \sum_{l=1}^d \frac{M_{n,n+k-1}(i, l) M_{n+k}(l, r)}{\sum_{s=1}^d M_{n,n+k-1}(i, s) M_{n+k}(s, r)} \frac{M_{n+k}(l, j)}{M_{n+k}(l, r)}.$$

We note that

$$\left(\frac{M_{n,n+k-1}(i, l)M_{n+k}(l, r)}{\sum_{s=1}^d M_{n,n+k-1}(i, s)M_{n+k}(s, r)} \right)_{1 \leq i, l \leq d}$$

is a positive stochastic matrix. Therefore we get the first inequalities in (7.1) : for all $n \geq 0$, $k \geq 1$ and $1 \leq i, j, r \leq d$, \mathbb{P} -a.s.,

$$\frac{1}{D} \leq \min_{1 \leq l \leq d} \frac{M_{n+k}(l, j)}{M_{n+k}(l, r)} \leq \frac{M_{n,n+k}(i, j)}{M_{n,n+k}(i, r)} \leq \max_{1 \leq l \leq d} \frac{M_{n+k}(l, j)}{M_{n+k}(l, r)} \leq D.$$

A similar argument gives the second inequality in (7.1). So the proof of (7.1) is complete.

By (2.3) and (7.1) we get that for all $n \geq 0$ and $1 \leq i, j \leq d$, \mathbb{P} -a.s.,

$$\frac{U_{n,\infty}(i)}{U_{n,\infty}(j)} = \lim_{k \rightarrow +\infty} \frac{M_{n,n+k}(i, i)}{M_{n,n+k}(j, i)} \geq \frac{1}{D}.$$

Since $\|U_{n,\infty}\| = 1$, this implies (7.2).

Using (2.7), it is clear that for all $n, k \geq 0$,

$$\left(\frac{M_{n,n+k}(i, j)U_{n+k+1,\infty}(j)}{\lambda_{n,n+k}U_{n,\infty}(i)} \right)_{1 \leq i, j \leq d}$$

is a positive stochastic matrix. Then, applying (2.8), (7.2) and (7.1), it follows that for all $n, k \geq 0$ and $1 \leq i, j \leq d$, \mathbb{P} -a.s.,

$$\begin{aligned} \frac{M_{n,n+k}(i, j)U_{n+k+1,\infty}(j)}{\lambda_{n,n+k}U_{n,\infty}(i)} &= \frac{M_{n,n+k}(i, j)U_{n+k+1,\infty}(j)}{\sum_{r=1}^d M_{n,n+k}(i, r)U_{n+k+1,\infty}(r)} \\ &\geq \frac{1}{dD} \left(\sum_{r=1}^d \frac{M_{n,n+k}(i, r)U_{n+k+1,\infty}(r)}{M_{n,n+k}(i, j)} \right)^{-1} \\ &\geq \frac{1}{dD^2} \left(\sum_{r=1}^d U_{n+k+1,\infty}(r) \right)^{-1} = \frac{1}{dD^2}. \end{aligned} \quad (7.4)$$

This ends the proof of (7.3). \square

7.2. Proof of Theorem 2.9

Notice that the conclusion of Theorem 2.9 for an i.i.d. environment follows from that for a stationary and ergodic environment and the fact that the conditions H5 and H7 are equivalent in the i.i.d. case (cf. Lemma 11.1). So

we need only to prove Theorem 2.9 when the environment is stationary and ergodic.

By Theorem 2.6 and Lemma 11.1 we know that **H5** is sufficient for the non-degeneracy of all the W^i , $1 \leq i \leq d$.

We now prove that if **H5** fails, then each W^i is degenerate. Assume that **H5** fails. Then **H3** fails, since **H3** \Leftrightarrow **H5**, which means that for all $C > 0$,

$$\mathbb{P} \left(\max_{1 \leq r \leq d} \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^r, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(r)} \mathbf{1}_{\{\langle N_n^r, U_{n+1, \infty} \rangle \geq Cn\}} \right) = +\infty \right) > 0. \quad (7.5)$$

We keep the notation of the proof of Theorem 2.6. By the definition of the tree \mathbb{T}_*^i , for $n \geq 0$,

$$Z_{*n}^i \geq Y_{n-1} \quad \mathbb{P}^i\text{-a.s.} \quad (7.6)$$

Let $(\mathcal{F}_{*n})_{n \geq 0}$ be the filtration defined by $\mathcal{F}_{*0} = \sigma(\xi)$, and for $n \geq 1$,

$$\mathcal{F}_{*n} = \sigma(\xi, N_{l,k}^r, Y_k^r, D_k, 0 \leq k \leq n-1, 1 \leq r \leq d, l \geq 1).$$

By the conditional Borel-Cantelli lemma [8, Theorem 5.3.2] we get that for all $C > 0$,

$$\begin{aligned} & \left\{ \log^+ \langle Y_n, U_{n+1, \infty} \rangle \geq Cn \quad \text{i.o.} \right\} \\ &= \left\{ \sum_{n=1}^{+\infty} \mathbb{P} \left(\log^+ \langle Y_n, U_{n+1, \infty} \rangle \geq Cn \mid \mathcal{F}_{*n} \right) = +\infty \right\}. \end{aligned} \quad (7.7)$$

By the independence under \mathbb{P}_ξ between $\{Y_n^r : 1 \leq r \leq d\}$ and \mathcal{F}_{*n} , we have

$$\begin{aligned} & \sum_{n=1}^{+\infty} \mathbb{P} \left(\log^+ \langle Y_n, U_{n+1, \infty} \rangle \geq Cn \mid \mathcal{F}_{*n} \right) \\ &= \sum_{n=1}^{+\infty} \mathbb{P}_\xi \left(\sum_{r=1}^d \log^+ \langle Y_n^r, U_{n+1, \infty} \rangle \mathbf{1}_{\{D_n=r\}} \geq Cn \mid \mathcal{F}_{*n} \right) \\ &= \sum_{n=1}^{+\infty} \sum_{r=1}^d \mathbb{P}_\xi \left(\log^+ \langle Y_n^r, U_{n+1, \infty} \rangle \geq Cn \right) \mathbb{P}_\xi \left(D_n = r \mid \mathcal{F}_{*n} \right). \end{aligned} \quad (7.8)$$

For any $n \geq 1$, under \mathbb{P}_ξ , D_n is independent of the family $\{N_{l,k}^r, Y_k^r\}$ with $0 \leq k \leq n-1, 1 \leq r \leq d$ and $l \geq 1$. Therefore, for $1 \leq r \leq d$ and $n \geq 1$,

$$\mathbb{P}_\xi(D_n = r \mid \mathcal{F}_{*n}) = \mathbb{P}_\xi(D_n = r \mid D_k, 1 \leq k \leq n-1)$$

Moreover by the construction of (D_n) , for all $n \geq 1$ and $(j_1, \dots, j_{n-1}, r) \in \{1, \dots, d\}^n$,

$$\begin{aligned} \mathbb{P}_\xi(D_n = r | D_{n-1} = j_{n-1}, \dots, D_1 = j_1, D_0 = i) &= \mathbb{P}_\xi(D_n = r | D_{n-1} = j_{n-1}) \\ &= \sum_{y \in \mathbb{N}^d} \frac{y(r)U_{n,\infty}(r)}{\langle y, U_{n,\infty} \rangle} \mathbb{P}_\xi(Y_{n-1}^{j_{n-1}} = y) \\ &= \sum_{y \in \mathbb{N}^d} \frac{y(r)U_{n,\infty}(r)}{\lambda_{n-1}U_{n-1,\infty}(j_{n-1})} \mathbb{P}_\xi(N_{n-1}^{j_{n-1}} = y) \\ &= \frac{M_{n-1}(j_{n-1}, r)U_{n,\infty}(r)}{\lambda_{n-1}U_{n,\infty}(j_{n-1})}. \end{aligned}$$

This implies that for all $1 \leq r \leq d$ and $n \geq 1$, \mathbb{P} -a.s.,

$$\mathbb{P}_\xi(D_n = r | \mathcal{F}_{*n}) = \frac{M_{n-1}(D_{n-1}, r)U_{n,\infty}(r)}{\lambda_{n-1}U_{n,\infty}(D_{n-1})}.$$

Then, using (7.3), it follows that for all $1 \leq r \leq d$ and $n \geq 1$, \mathbb{P} -a.s.,

$$\mathbb{P}_\xi(D_n = r | \mathcal{F}_{*n}) \geq \frac{1}{dD^2}. \quad (7.9)$$

Combining equality (7.8) with inequalities (7.9) and (7.3), we get that for all $C > 0$, \mathbb{P} -a.s.,

$$\begin{aligned} &\sum_{n=1}^{+\infty} \mathbb{P}(\log^+ \langle Y_n, U_{n+1,\infty} \rangle \geq Cn | \mathcal{F}_{*n}) \\ &\geq \frac{1}{dD^2} \sum_{n=1}^{+\infty} \sum_{r=1}^d \mathbb{P}_\xi(\log^+ \langle Y_n^r, U_{n+1,\infty} \rangle \geq Cn) \end{aligned} \quad (7.10)$$

By the definition of Y_n^r , for all $C > 0$,

$$\begin{aligned} &\mathbb{P}_\xi(\log^+ \langle Y_n^r, U_{n+1,\infty} \rangle \geq Cn) \\ &= \mathbb{E}_\xi \left(\frac{\langle N_n^r, U_{n+1,\infty} \rangle}{\lambda_n U_{n,\infty}(r)} \mathbf{1}_{\{\log^+ \langle N_n^r, U_{n+1,\infty} \rangle \geq Cn\}} \right). \end{aligned}$$

Using this together with (7.5) and (7.10), we deduce that for all $C > 0$,

$$\mathbb{P} \left(\sum_{n=1}^{+\infty} \sum_{r=1}^d \mathbb{P}_\xi(\log^+ \langle Y_n^r, U_{n+1,\infty} \rangle \geq Cn) = +\infty \right) > 0. \quad (7.11)$$

Since $(\log^+ \langle Y_n^r, U_{n+1, \infty} \rangle)_{n \geq 0}$ is a non negative stationary and ergodic sequence, by [32, Theorem 1] of Tanny we know that

$$\limsup_{n \rightarrow +\infty} \frac{\log^+ \langle Y_n^r, U_{n+1, \infty} \rangle}{n} \text{ is either } 0 \text{ } \mathbb{P}\text{-a.s. or } +\infty \text{ } \mathbb{P}\text{-a.s.}$$

As $(Y_n^r, U_{n+1, \infty}), n \geq 1$, are independent under \mathbb{P}_ξ , by Borel-Cantelli lemma this implies that for all $C > 0$ and $1 \leq r \leq d$, either

$$\sum_{n=1}^{+\infty} \mathbb{P}_\xi \left(\log^+ \langle Y_n^r, U_{n+1, \infty} \rangle \geq Cn \right) < +\infty \text{ } \mathbb{P}\text{-a.s.},$$

or

$$\sum_{n=1}^{+\infty} \mathbb{P}_\xi \left(\log^+ \langle Y_n^r, U_{n+1, \infty} \rangle \geq Cn \right) = +\infty \text{ } \mathbb{P}\text{-a.s.}$$

This statement remains valid while $\mathbb{P}_\xi (\log^+ \langle Y_n^r, U_{n+1, \infty} \rangle \geq Cn)$ is replaced by $\sum_{r=1}^d \mathbb{P}_\xi (\log^+ \langle Y_n^r, U_{n+1, \infty} \rangle \geq Cn)$. Therefore from (7.11) we obtain that for all $C > 0$, \mathbb{P} -a.s.,

$$\sum_{n=1}^{+\infty} \sum_{r=1}^d \mathbb{P}_\xi \left(\log^+ \langle Y_n^r, U_{n+1, \infty} \rangle \geq Cn \right) = +\infty. \quad (7.12)$$

Combining (7.7), (7.10) and (7.12), we deduce that

$$\limsup_{n \rightarrow +\infty} \frac{\log^+ \langle Y_n, U_{n+1, \infty} \rangle}{n} = +\infty \text{ } \mathbb{P}\text{-a.s.}$$

It follows from (7.6) that, \mathbb{P}^i -a.s.,

$$\limsup_{n \rightarrow +\infty} \frac{\log^+ \langle Z_{*n}^i, U_{n, \infty} \rangle}{n} \geq \limsup_{n \rightarrow +\infty} \frac{\log^+ \langle Y_{n-1}, U_{n, \infty} \rangle}{n} = +\infty. \quad (7.13)$$

By Proposition 2.5 we have $\log \lambda_{0, n-1} \sim \gamma n$ \mathbb{P} -a.s. as $n \rightarrow +\infty$. So we get from (7.13) that

$$\limsup_{n \rightarrow +\infty} \frac{\langle Z_{*n}^i, U_{n, \infty} \rangle}{\lambda_{0, n-1} U_{0, \infty}(i)} = +\infty \text{ } \mathbb{P}^i\text{-a.s.},$$

or equivalently

$$W^i = \limsup_{n \rightarrow +\infty} \frac{\langle Z_n^i, U_{n, \infty} \rangle}{\lambda_{0, n-1} U_{0, \infty}(i)} = +\infty \text{ } \mathbb{P}_*^i\text{-a.s.}$$

By (5.5) we conclude that $W^i = 0$ \mathbb{P}^i -a.s. , for any $1 \leq i \leq d$. This ends the proof of Theorem 2.9.

8. Proof of Theorem 2.10

8.1. Auxiliary results

We need additional results on the products of the mean matrices (M_n) . Set

$$\delta = \frac{D^2 - 1}{D^2 + 1} \in (0, 1).$$

The following Lemma was proved by Kesten and Spitzer in [23]. It gives a uniform convergence in (2.3) with an exponential rate, under the condition **H4** of Furstenberg and Kesten [12].

Lemma 8.1. *Assume condition **H4**. Then there exists a constant $C > 0$ such that for all $1 \leq i, j \leq d$ and $k \geq 0$,*

$$\sup_{n \geq 0} \left| \frac{M_{n,n+k}(i, j)}{\rho_{n,n+k} U_{n,n+k}(i) V_{n,n+k}(j)} - 1 \right| \leq C \delta^k \quad \mathbb{P}\text{-a.s.} \quad (8.1)$$

The next result establishes a uniform convergence with an exponential rate for the left and right eigenvectors $U_{n,n+k}$ and $V_{n,n+k}/\|V_{n,n+k}\|$, as $k \rightarrow \infty$.

Lemma 8.2. *Assume condition **H4**. Then there exists a constant $C > 0$ such that for all $k \geq 0$,*

$$\sup_{n \geq 0} \|U_{n,n+k} - U_{n,\infty}\| \leq C \delta^k \quad \mathbb{P}\text{-a.s.} \quad (8.2)$$

and

$$\sup_{n \geq 0} \left\| \frac{V_{0,n+k}}{\|V_{0,n+k}\|} - \frac{V_{n,n+k}}{\|V_{n,n+k}\|} \right\| \leq C \delta^k \quad \mathbb{P}\text{-a.s.} \quad (8.3)$$

Proof. We only prove (8.3), since one can obtain (8.2) by similar arguments. Let $C > 0$ be as in Lemma 8.1. Denote by $k_0 \geq 0$ such that $C \delta^{k_0} < 1$. By Lemma 8.1, for all $n \geq 0$, $k \geq k_0$ and $1 \leq i, j \leq d$ we have

$$\frac{1 - C \delta^k V_{n,n+k}(j)}{1 + C \delta^k \|V_{n,n+k}\|} \leq \frac{M_{n,n+k}(i, j)}{\|M_{n,n+k}(i, \cdot)\|} \leq \frac{1 + C \delta^k V_{n,n+k}(j)}{1 - C \delta^k \|V_{n,n+k}\|} \quad \mathbb{P}\text{-a.s.}$$

From this and the fact that $\frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|} \leq 1$, we deduce that for all $1 \leq i \leq d$ and $k \geq k_0$,

$$\sup_{n \geq 0} \left| \frac{M_{n,n+k}(i, j)}{\|M_{n,n+k}(i, \cdot)\|} - \frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|} \right| \leq \frac{2C}{1 - C \delta^{k_0}} \delta^k \quad \mathbb{P}\text{-a.s.}$$

Therefore we get that for all $1 \leq i \leq d$ and $k \geq k_0$,

$$\sup_{n \geq 0} \left| \frac{M_{n,n+k}(i, j)}{\|M_{n,n+k}(i, \cdot)\|} - \frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|} \right| \leq C_1 \delta^k \quad \mathbb{P}\text{-a.s.}, \quad (8.4)$$

with $C_1 = 2C/(1 - C\delta^{k_0})$. From (8.4) it follows that for all $n \geq 0$, $k \geq k_0$ and $1 \leq j \leq d$, \mathbb{P} -a.s.,

$$\begin{aligned} & \left| \frac{V_{0,n+k}(j)}{\|V_{0,n+k}\|} - \frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|} \right| \\ & \leq \left| \frac{M_{0,n+k}(j, j)}{\|M_{0,n+k}(j, \cdot)\|} - \frac{M_{n,n+k}(j, j)}{\|M_{n,n+k}(j, \cdot)\|} \right| + C_1 \delta^{n+k} + C_1 \delta^k \\ & \leq \left| \sum_{i=1}^d M_{0,n-1}(j, i) \frac{M_{n,n+k}(i, j)}{\|M_{0,n+k}(j, \cdot)\|} - \frac{M_{n,n+k}(j, j)}{\|M_{n,n+k}(j, \cdot)\|} \right| + 2C_1 \delta^k \\ & = \left| \sum_{i=1}^d M_{0,n-1}(j, i) \frac{\|M_{n,n+k}(i, \cdot)\|}{\|M_{0,n+k}(j, \cdot)\|} \left(\frac{M_{n,n+k}(i, j)}{\|M_{n,n+k}(i, \cdot)\|} - \frac{M_{n,n+k}(j, j)}{\|M_{n,n+k}(j, \cdot)\|} \right) \right| \\ & \quad + 2C_1 \delta^k, \end{aligned}$$

So we obtain that for all $n \geq 0$, $k \geq k_0$ and $1 \leq j \leq d$, \mathbb{P} -a.s.,

$$\begin{aligned} & \left| \frac{V_{0,n+k}(j)}{\|V_{0,n+k}\|} - \frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|} \right| \\ & \leq \sum_{i=1}^d M_{0,n-1}(j, i) \frac{\|M_{n,n+k}(i, \cdot)\|}{\|M_{0,n+k}(j, \cdot)\|} \left| \frac{M_{n,n+k}(i, j)}{\|M_{n,n+k}(i, \cdot)\|} - \frac{M_{n,n+k}(j, j)}{\|M_{n,n+k}(j, \cdot)\|} \right| \\ & \quad + 2C_1 \delta^k \\ & \leq \max_{1 \leq r \leq d} \left| \frac{M_{n,n+k}(r, j)}{\|M_{n,n+k}(r, \cdot)\|} - \frac{M_{n,n+k}(j, j)}{\|M_{n,n+k}(j, \cdot)\|} \right| + 2C_1 \delta^k \\ & \leq 4C_1 \delta^k, \end{aligned}$$

where the last step holds by (8.4). Hence (8.3) holds for all $k \geq k_0$ and $C = 4C_1$. For $k < k_0$, since $\frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|} \leq 1$, we have $\left| \frac{V_{0,n+k}(j)}{\|V_{0,n+k}\|} - \frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|} \right| \leq 1 \leq \delta^{-k_0} \delta^k$. Therefore (8.3) holds for all $k \geq 0$ and $C = \max(4C_1, \delta^{-k_0})$. \square

The next assertion shows that conditioned on the explosion event $E^i = \{\|Z_n^i\| \rightarrow +\infty\}$, each component $Z_n^i(j)$ of Z_n^i tends to $+\infty$ in probability.

Proposition 8.3. *Assume conditions H2, H4, and $\gamma > 0$. Then, for all $1 \leq i, j \leq d$ such that $\mathbb{P}(E^i) > 0$, we have*

$$Z_n^i(j) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}^{E^i}} +\infty.$$

Proof. Clearly, it suffices to prove that for all $1 \leq i, j \leq d$ and $K \geq 0$,

$$\mathbb{P}(Z_n^i(j) \geq K, E^i) \xrightarrow{n \rightarrow +\infty} \mathbb{P}(E^i), \quad (8.5)$$

Set $K_1, K_2 \geq 0$. By (4.1), for $n, k \geq 1$ and $1 \leq i, j \leq d$, \mathbb{P} -a.s., we have

$$\begin{aligned} & \mathbb{P}_\xi \left(Z_{n+k}^i(j) \leq K_1, \|Z_n^i\| \geq K_2 \right) \\ &= \mathbb{P}_\xi \left(\sum_{r=1}^d \sum_{l=1}^{Z_n^i(r)} Z_{l,n,k}^r(j) \leq K_1, \|Z_n^i\| \geq K_2 \right) \\ &\leq \mathbb{P}_\xi \left(Z_{l,n,k}^r(j) \leq K_1, \|Z_n^i\| \geq K_2, 1 \leq r \leq d, 1 \leq l \leq Z_n^i(r) \right) \\ &= \mathbb{E}_\xi \left[\mathbb{P}_{T^n \xi} \left(Z_k^1(j) \leq K_1 \right)^{Z_n^i(1)} \cdots \mathbb{P}_{T^n \xi} \left(Z_k^d(j) \leq K_1 \right)^{Z_n^i(d)} \mathbf{1}_{\{\|Z_n^i\| \geq K_2\}} \right]. \end{aligned}$$

It follows that, \mathbb{P} -a.s.,

$$\mathbb{P}_\xi \left(Z_{n+k}^i(j) \leq K_1, \|Z_n^i\| \geq K_2 \right) \leq \left(\max_{1 \leq r \leq d} \mathbb{P}_{T^n \xi} \left(Z_k^r(j) \leq K_1 \right) \right)^{K_2}.$$

This together with the fact that

$$\limsup_{n \rightarrow +\infty} \mathbb{P}\{E^i, \|Z_n^i\| < K_2\} \leq \mathbb{P}(\limsup_{n \rightarrow +\infty} \{E^i, \|Z_n^i\| < K_2\}) = 0,$$

implies that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \mathbb{P}(Z_n^i(j) \leq K_1, E^i) &\leq \limsup_{n \rightarrow +\infty} \mathbb{P} \left(Z_{n+k}^i(j) \leq K_1, \|Z_n^i\| \geq K_2 \right) \\ &\leq \mathbb{E} \left(\max_{1 \leq r \leq d} \mathbb{P}_\xi \left(Z_k^r(j) \leq K_1 \right) \right)^{K_2}. \end{aligned}$$

Letting $K_2 \rightarrow +\infty$, it follows that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \mathbb{P}(Z_n^i(j) \leq K_1, E^i) &\leq \mathbb{P} \left(\max_{1 \leq r \leq d} \mathbb{P}_\xi \left(Z_k^r(j) \leq K_1 \right) = 1 \right) \\ &\leq \sum_{r=1}^d \mathbb{P}(\mathbb{P}_\xi(Z_k^r(j) \leq K_1) = 1). \end{aligned} \quad (8.6)$$

By (2.19) we know that

$$\mathbb{E}_\xi Z_k^r(j) = M_{0,k-1}(r, j) \xrightarrow{k \rightarrow +\infty} +\infty \quad \mathbb{P}\text{-a.s.}, \quad (8.7)$$

which implies that for all $K_1 \geq 0$,

$$\mathbb{P}(\mathbb{P}_\xi(Z_k^r(j) \leq K_1) = 1) \leq \mathbb{P}(\mathbb{E}_\xi Z_k^r(j) \leq K_1) \xrightarrow{k \rightarrow +\infty} 0.$$

Therefore from (8.6), we conclude that for all $K_1 \geq 0$,

$$\mathbb{P}(Z_n^i(j) \leq K_1, E^i) \xrightarrow{n \rightarrow +\infty} 0,$$

which implies (8.5) and ends the proof of Proposition 8.3. \square

8.2. Proof of Theorem 2.10

Let $1 \leq i \leq d$. For all $n, k \geq 0$ set

$$\bar{Z}_n^i = \frac{Z_n^i}{\|Z_n^i\|} \quad \text{and} \quad \bar{V}_{n,n+k} = \frac{V_{n,n+k}}{\|V_{n,n+k}\|}.$$

From (4.1) and on the event E^i , for any $n, k \geq 1$ we have

$$\begin{aligned} & \frac{1}{\rho_{n,n+k-1} \|V_{n,n+k-1}\|} \left\| \frac{Z_{n+k}^i}{\|Z_{n+k}^i\|} - M_{n,n+k-1}^T \bar{Z}_n^i \right\| \\ & \leq \frac{1}{\rho_{n,n+k-1} \|V_{n,n+k-1}\|} \frac{1}{\|Z_n^i\|} \sum_{j=1}^d \sum_{r=1}^d \left| \sum_{l=1}^{Z_n^i(r)} (Z_{l,n,k}^r(j) - M_{n,n+k-1}(r, j)) \right| \\ & = \sum_{j=1}^d \sum_{r=1}^d \frac{M_{n,n+k-1}(r, j)}{\rho_{n,n+k-1} \|V_{n,n+k-1}\|} \frac{1}{\|Z_n^i\|} \left| \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r, j)} - 1 \right) \right|. \end{aligned} \quad (8.8)$$

By the weak law of large numbers and Proposition 8.3 we get that for all $1 \leq r, j \leq d$,

$$\frac{1}{\|Z_n^i\|} \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r, j)} - 1 \right) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}_{E^i}} 0. \quad (8.9)$$

Let $C > 0$ be sufficiently large such that (8.1) and (8.3) hold. By (8.1), for any $1 \leq r, j \leq d$, $n \geq 0$ and $k \geq 1$, \mathbb{P} -a.s.,

$$\begin{aligned} \frac{M_{n,n+k-1}(r, j)}{\rho_{n,n+k-1} \|V_{n,n+k-1}\|} & \leq (1 + C\delta^k) \frac{U_{n,n+k-1}(r) V_{n,n+k-1}(j)}{\|V_{n,n+k-1}\|} \\ & \leq 1 + C\delta^k. \end{aligned}$$

Combining this with (8.8) and (8.9), we deduce that for all $k \geq 1$,

$$\frac{1}{\rho_{n,n+k-1} \|V_{n,n+k-1}\|} \left\| \frac{Z_{n+k}^i}{\|Z_{n+k}^i\|} - M_{n,n+k-1}^T \bar{Z}_n^i \right\| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}_{E^i}} 0. \quad (8.10)$$

Moreover, by Lemma 8.1 we get that that for any $n, k \geq 1$,

$$\begin{aligned}
 & \left\| \frac{M_{n,n+k-1}^T}{\rho_{n,n+k-1} \|V_{n,n+k-1}\|} \bar{Z}_n^i - \langle \bar{Z}_n^i, U_{n,n+k-1} \rangle \bar{V}_{n,n+k-1} \right\| \\
 & \leq \sum_{r=1}^d \sum_{j=1}^d \left| \frac{M_{n,n+k-1}(r,j)}{\rho_{n,n+k-1} \|V_{n,n+k-1}\|} \bar{Z}_n^i(r) - U_{n,n+k-1}(r) \bar{V}_{n,n+k-1}(j) \bar{Z}_n^i(r) \right| \\
 & \leq \sum_{r=1}^d \sum_{j=1}^d U_{n,n+k-1}(r) \bar{V}_{n,n+k-1}(j) \bar{Z}_n^i(r) \times \\
 & \quad \left| \frac{M_{n,n+k-1}(r,j)}{\rho_{n,n+k-1} U_{n,n+k-1}(r) V_{n,n+k-1}(j)} - 1 \right| \\
 & \leq \max_{1 \leq r, j \leq d} \left| \frac{M_{n,n+k-1}(r,j)}{\rho_{n,n+k-1} U_{n,n+k-1}(r) V_{n,n+k-1}(j)} - 1 \right| \\
 & \leq C\delta^k. \tag{8.11}
 \end{aligned}$$

Let $k_0 \in \mathbb{N}$ be large enough such that $C\delta^{k_0-1} \leq 1/(dD)$. Then, combining (7.2) and (8.2), we see that for all $1 \leq r \leq d$, $n \geq 1$ and $k \geq k_0$,

$$U_{n,n+k-1}(r) \geq \frac{1}{2dD} \quad \mathbb{P}\text{-a.s.}$$

It follows that for all $n \geq 1$ and $k \geq k_0$,

$$\langle \bar{Z}_n^i, U_{n,n+k-1} \rangle \geq \frac{1}{2dD} \quad \mathbb{P}\text{-a.s.} \tag{8.12}$$

Let $\varepsilon > 0$. Let $k_1 \in \mathbb{N}$ be such that $2dDC\delta^{k_1} \leq \varepsilon/8$ and $k_1 \geq k_0$. For all $n \geq 0$ and $k \geq k_1$, set

$$Y_{n,k}^i = \frac{\|Z_{n+k}^i\|}{\rho_{n,n+k-1} \|V_{n,n+k-1}\| \langle \bar{Z}_n^i, U_{n,n+k-1} \rangle \|Z_n^i\|},$$

which is well defined on the explosion event E^i . Notice that

$$\begin{aligned}
 Y_{n,k}^i \bar{Z}_{n+k}^i - \bar{V}_{n,n+k-1} &= \frac{1}{\langle \bar{Z}_n^i, U_{n,n+k-1} \rangle} \times \\
 & \left[\frac{Z_{n+k}^i}{\rho_{n,n+k-1} \|V_{n,n+k-1}\| \|Z_n^i\|} - \langle \bar{Z}_n^i, U_{n,n+k-1} \rangle \bar{V}_{n,n+k-1} \right].
 \end{aligned}$$

Therefore, combining the relations (8.10) and (8.11), together with (8.12), we obtain that for all $k \geq k_1$,

$$\mathbb{P}_{E^i} \left(\|Y_{n,k}^i \bar{Z}_{n+k}^i - \bar{V}_{n,n+k-1}\| > \frac{\varepsilon}{4} \right) \xrightarrow{n \rightarrow +\infty} 0. \tag{8.13}$$

Applying (8.13) and the triangle inequality we have that for all $k \geq k_1$,

$$\begin{aligned}
 & \limsup_{n \rightarrow +\infty} \mathbb{P}_{E^i} \left(|Y_{n,k}^i - 1| > \frac{\varepsilon}{4} \right) \\
 &= \limsup_{n \rightarrow +\infty} \mathbb{P}_{E^i} \left(\left| \|Y_{n,k}^i \bar{Z}_{n+k}^i\| - \|\bar{V}_{n,n+k-1}\| \right| > \frac{\varepsilon}{4} \right) \\
 &\leq \limsup_{n \rightarrow +\infty} \mathbb{P}_{E^i} \left(\|Y_{n,k}^i \bar{Z}_{n+k}^i - \bar{V}_{n,n+k-1}\| > \frac{\varepsilon}{4} \right) \\
 &= 0.
 \end{aligned} \tag{8.14}$$

Combining (8.13) with (8.14), we obtain that for all $k \geq k_1$,

$$\begin{aligned}
 & \mathbb{P}_{E^i} \left(\|\bar{Z}_{n+k}^i - \bar{V}_{n,n+k-1}\| > \frac{\varepsilon}{2} \right) \\
 &\leq \mathbb{P}_{E^i} \left(\|Y_{n,k}^i \bar{Z}_{n+k}^i - \bar{V}_{n,n+k-1}\| + |Y_{n,k}^i - 1| > \frac{\varepsilon}{2} \right) \\
 &\leq \mathbb{P}_{E^i} \left(\|Y_{n,k}^i \bar{Z}_{n+k}^i - \bar{V}_{n,n+k-1}\| > \frac{\varepsilon}{4} \right) + \mathbb{P}_{E^i} \left(|Y_{n,k}^i - 1| > \frac{\varepsilon}{4} \right) \\
 &\xrightarrow{n \rightarrow +\infty} 0.
 \end{aligned} \tag{8.15}$$

Notice that for any $k_2 \geq 0$,

$$\begin{aligned}
 & \limsup_{n \rightarrow +\infty} \mathbb{P}_{E^i} \left(\|\bar{Z}_n^i - \bar{V}_{0,n-1}\| > \varepsilon \right) \\
 &\leq \limsup_{n \rightarrow +\infty} \mathbb{P}_{E^i} \left(\|\bar{Z}_{n+k_2}^i - \bar{V}_{n,n+k_2-1}\| > \frac{\varepsilon}{2} \right) \\
 &\quad + \limsup_{n \rightarrow +\infty} \mathbb{P}_{E^i} \left(\|\bar{V}_{0,n+k_2-1} - \bar{V}_{n,n+k_2-1}\| > \frac{\varepsilon}{2} \right)
 \end{aligned}$$

Let $k_2 \geq k_1$ be such that $C\delta^{k_2-1} \leq \varepsilon/2$. Then by (8.3), the second term in the right hand side is 0. The first one is also 0 by (8.15). Hence

$$\limsup_{n \rightarrow +\infty} \mathbb{P}_{E^i} \left(\|\bar{Z}_n^i - \bar{V}_{0,n-1}\| > \varepsilon \right) = 0.$$

This proves (2.20).

Since $(\bar{V}_{0,n})_{n \geq 0}$ converges in law to $\bar{V}_{0,\infty}$ (see Proposition 2.1), from (2.20) we obtain directly the convergence in law of $(\bar{Z}_n^i)_{n \geq 0}$ to $\bar{V}_{0,\infty}$. This concludes the proof of Theorem 2.10.

9. Proof of Theorem 2.11 and Corollary 2.12

Proof of Theorem 2.11. By (7.3), for all $1 \leq i \leq d$ and $n \geq 1$, \mathbb{P} -a.s., we have

$$\begin{aligned} W_n^i &= \sum_{j=1}^d \frac{M_{0,n-1}(i,j)U_{n,\infty}(j)}{\lambda_{0,n-1}U_{0,\infty}(i)} \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \\ &\geq \frac{1}{dD^2} \sum_{j=1}^d \frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}. \end{aligned}$$

Consequently we get that $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j) \rightarrow 0$ \mathbb{P} -a.s. on the event $\{W^i = 0\}$, for all $1 \leq i, j \leq d$. Now we investigate on the event $\{W^i > 0\}$. By (6.2) and Proposition 2.5 it holds that $\|Z_n^i\| \rightarrow +\infty$ \mathbb{P} -a.s. on $\{W^i > 0\}$. Moreover, using (2.3) and Proposition 2.10 we have that for all $1 \leq i, j \leq d$, as $n \rightarrow +\infty$, \mathbb{P} -a.s.,

$$\begin{aligned} \frac{1}{W_n^i} \frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} &= \frac{U_{0,\infty}(i)\lambda_{0,n-1}}{M_{0,n-1}(i,j)} \frac{Z_n^i(j)}{\langle Z_n^i, U_{n,\infty} \rangle} \\ &\sim \frac{\langle V_{0,n-1}, U_{n,\infty} \rangle}{V_{0,n-1}(j)} \frac{Z_n^i(j)}{\langle Z_n^i, U_{n,\infty} \rangle} \\ &\sim \frac{\|V_{0,n-1}\|}{V_{0,n-1}(j)} \frac{Z_n^i(j)}{\|Z_n^i\|} \sum_{r=1}^d \frac{Z_n^i(r)U_{n,\infty}(r)}{\langle Z_n^i, U_{n,\infty} \rangle} \frac{V_{0,n-1}(r)}{\|V_{0,n-1}\|} \frac{\|Z_n^i\|}{Z_n^i(r)}. \quad (9.1) \end{aligned}$$

Applying Theorem 2.10 it follows that for all $1 \leq i, j \leq d$,

$$\left| \frac{Z_n^i(j)}{\|Z_n^i\|} - \frac{V_{0,n-1}(j)}{\|V_{0,n-1}\|} \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}_{E^i}} 0.$$

Since $(V_{0,n}/\|V_{0,n}\|)_{n \geq 0}$ converges in law to $\bar{V}_{0,\infty}$ with $\bar{V}_{0,\infty} > 0$ \mathbb{P} -a.s., this implies that

$$\left| \frac{\|V_{0,n-1}\|}{V_{0,n-1}(j)} \frac{Z_n^i(j)}{\|Z_n^i\|} - 1 \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}_{E^i}} 0. \quad (9.2)$$

Combining (9.1) and (9.2), we deduce that for all $1 \leq i, j \leq d$,

$$\frac{1}{W_n^i} \frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}_{E^i}} 1.$$

It follows that

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}_{E^i}} W^i,$$

which concludes the proof of (2.23). From (2.23) and (2.3), we deduce (2.22) : for all $1 \leq i, j \leq d$,

$$\frac{Z_n^i(j)}{\rho_{0,n-1} V_{0,n-1}(j)} = \frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} U_{0,n-1}(i) \frac{M_{0,n-1}(i, j)}{\rho_{0,n-1} U_{0,n-1}(i) V_{0,n-1}(j)} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} W^i U_{0,\infty}(i).$$

This concludes the proof of Theorem 2.11. □

Proof of Corollary 2.12. Notice that, for all $1 \leq i \leq d$ and $n \geq 0$,

$$\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} - W^i = \sum_{j=1}^d \frac{M_{0,n-1}(i, j)}{\|M_{0,n-1}(i, \cdot)\|} \left(\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} - W^i \right).$$

Then, letting $n \rightarrow +\infty$ and using Theorem 2.11, we get the first convergence in Corollary 2.12. Combining this with (2.3), we get the second convergence, and we conclude the proof of Corollary 2.12. □

10. Proof of Theorem 2.13

We need an auxiliary result to prove Theorem 2.13.

Lemma 10.1. *Let $(X_k)_{k \in \mathbb{N}^*}$ be a sequence of i.i.d. random centered variables. Then for all $n \in \mathbb{N}^*$ and $p > 1$:*

$$\mathbb{E} \left| \sum_{k=1}^n X_k \right|^p \leq \begin{cases} (B_p)^p \mathbb{E} |X_k|^p n, & \text{if } 1 < p \leq 2, \\ (B_p)^p \mathbb{E} |X_k|^p n^{\frac{p}{2}}, & \text{if } p > 2, \end{cases}$$

where $B_p = 2 \min \left\{ k^{1/2} : k \in \mathbb{N}, k \geq \frac{p}{2} \right\}$.

This result is a direct consequence of the Marcinkiewicz-Zygmund inequality, see [5, Theorem 1.5].

Proof of Theorem 2.13. Notice that the condition (2.24) implies H7. Therefore, using Theorem 2.9, we deduce that W^i , $1 \leq i \leq d$ are non-degenerate, and (2.15) and (2.16) hold. Now we shall prove the a.s. convergence (2.25)-(2.29). For that, it is sufficient to show that the convergence in probability in (8.9) can be reinforced to a.s. convergence. Indeed, if we prove that for all $1 \leq i, r, j \leq d$ and $k \geq 1$, \mathbb{P} -a.s. on the event E^i ,

$$\frac{1}{\|Z_n^i\|} \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r, j)} - 1 \right) \xrightarrow[n \rightarrow +\infty]{} 0, \quad (10.1)$$

then all the convergences in probability in the proofs of Theorems 2.10, 2.11 and Corollary 2.12 can be reinforced to a.s. convergences. Now we shall prove (10.1), which is equivalent to the following statement: for all $1 \leq i, r, j \leq d$, $k \geq 1$ and $0 \leq b < k$, \mathbb{P} -a.s. on E^i ,

$$\frac{1}{\|Z_{kn+b}^i\|} \sum_{l=1}^{Z_{kn+b}^{i(r)}} \left(\frac{Z_{l, kn+b, k}^r(j)}{M_{kn+b, k(n+1)+b-1}(r, j)} - 1 \right) \xrightarrow{n \rightarrow +\infty} 0. \quad (10.2)$$

Let $(\tilde{\mathcal{F}}_n)_{n \geq 0}$ be the filtration defined by: $\tilde{\mathcal{F}}_0 = \{\emptyset\}$ and, for $n \geq 1$,

$$\tilde{\mathcal{F}}_n = \sigma \left(\xi_s, N_{l,s}^r, 0 \leq s \leq n-1, 1 \leq r \leq d, l \geq 1 \right).$$

Applying the conditional Borel-Cantelli lemma [8, Theorem 5.3.2], we see that (10.2) holds if and only if for all $1 \leq i, r, j \leq d$, $k \geq 1$, $0 \leq b < k$, and $C > 0$, \mathbb{P} -a.s. on the event E^i ,

$$\begin{aligned} \sum_{n=1}^{+\infty} \mathbb{P} \left(\left| \sum_{l=1}^{Z_{kn+b}^{i(r)}} \left(\frac{Z_{l, kn+b, k}^r(j)}{M_{kn+b, k(n+1)+b-1}(r, j)} - 1 \right) \right| \right. \\ \left. \geq C \|Z_{kn+b}^i\| \middle| \tilde{\mathcal{F}}_{kn+b} \right) < +\infty. \quad (10.3) \end{aligned}$$

We can always assume that condition (2.24) holds for some $1 < p \leq 2$. Since the environment sequence $(\xi_n)_{n \geq 0}$ is i.i.d., $Z_{l, n, k}^r(j)/M_{n, n+k-1}(r, j)$ is independent of $\tilde{\mathcal{F}}_n$ for all $1 \leq i, r, j \leq d$ and $n, k, l \geq 1$. Therefore, using Tchebychev's inequality and Lemma 10.1, the series in (10.3) can be bounded as follows :

$$\begin{aligned} & \sum_{n=1}^{+\infty} \mathbb{P} \left(\left| \sum_{l=1}^{Z_{kn+b}^{i(r)}} \left(\frac{Z_{l, kn+b, k}^r(j)}{M_{kn+b, k(n+1)+b-1}(r, j)} - 1 \right) \right| \geq C \|Z_{kn+b}^i\| \middle| \tilde{\mathcal{F}}_{kn+b} \right) \\ & \leq \sum_{n=1}^{+\infty} \frac{1}{C^p \|Z_{kn+b}^i\|^p} \mathbb{E} \left(\left| \sum_{l=1}^{Z_{kn+b}^{i(r)}} \left(\frac{Z_{l, kn+b, k}^r(j)}{M_{kn+b, k(n+1)+b-1}(r, j)} - 1 \right) \right|^p \middle| \tilde{\mathcal{F}}_{kn+b} \right) \\ & \leq \sum_{n=1}^{+\infty} \left(\frac{B_p^p}{C^p} \frac{Z_{kn+b}^{i(r)}}{\|Z_{kn+b}^i\|^p} \mathbb{E} \left| \frac{Z_k^r(j)}{M_{0, k-1}(r, j)} - 1 \right|^p \right). \end{aligned}$$

The last series converges provided that \mathbb{P} -a.s. on E^i ,

$$\mathbb{E} \left| \frac{Z_k^r(j)}{M_{0, k-1}(r, j)} - 1 \right|^p \sum_{n=1}^{+\infty} \|Z_{kn+b}^i\|^{1-p} < +\infty. \quad (10.4)$$

Therefore (10.3) holds if (10.4) is satisfied for all $1 \leq i, r, j \leq d$, $k \geq 1$ and $0 \leq b < k$.

It remains to prove (10.4), which is done below. By Corollary 2.8, we know that for all $1 \leq i \leq d$, \mathbb{P} -a.s. on E^i ,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|Z_n^i\| = \gamma > 0.$$

Therefore we deduce that, \mathbb{P} -a.s. on E^i ,

$$\sum_{n=1}^{+\infty} \|Z_n^i\|^{1-p} < +\infty. \quad (10.5)$$

Now using (4.1) and the inequality $(x+y)^p \leq 2^{p-1}(x^p+y^p)$, $x, y \geq 0$, for all $1 \leq i, j \leq d$ and $k \geq 1$, we have

$$\begin{aligned} & \mathbb{E} \left(\frac{Z_{k+1}^i(j)}{M_{0,k}(i,j)} \right)^p \\ &= \mathbb{E} \left(\sum_{r=1}^d \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} \sum_{l=1}^{Z_1^i(r)} \left| \frac{Z_{l,1,k}^r(j)}{M_{1,k}(r,j)} - 1 + 1 \right| \right)^p \\ &\leq \mathbb{E} \left(\sum_{r=1}^d \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} \sum_{l=1}^{Z_1^i(r)} \left| \frac{Z_{l,1,k}^r(j)}{M_{1,k}(r,j)} - 1 \right| + \sum_{r=1}^d \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} Z_1^i(r) \right)^p \\ &\leq 2^{p-1} \mathbb{E} \left(\sum_{r=1}^d \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} \sum_{l=1}^{Z_1^i(r)} \left| \frac{Z_{l,1,k}^r(j)}{M_{1,k}(r,j)} - 1 \right| \right)^p \\ &\quad + 2^{p-1} \mathbb{E} \left(\sum_{r=1}^d \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} Z_1^i(r) \right)^p. \end{aligned} \quad (10.6)$$

By the convexity of the function $x \mapsto x^p$ on \mathbb{R}_+ we get that, for all $1 \leq i, j \leq d$ and $k \geq 1$,

$$\begin{aligned} \mathbb{E} \left(\sum_{r=1}^d \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} Z_1^i(r) \right)^p &= \mathbb{E} \left(\sum_{r=1}^d \frac{M_0(i,r)M_{1,k}(r,j)}{M_{0,k}(i,j)} \frac{Z_1^i(r)}{M_0(i,r)} \right)^p \\ &\leq \mathbb{E} \left(\sum_{r=1}^d \frac{M_0(i,r)M_{1,k}(r,j)}{M_{0,k}(i,j)} \left(\frac{Z_1^i(r)}{M_0(i,r)} \right)^p \right) \\ &\leq \sum_{r=1}^d \mathbb{E} \left(\frac{Z_1^i(r)}{M_0(r,j)} \right)^p. \end{aligned} \quad (10.7)$$

Using again the convexity of $x \mapsto x^p$ on \mathbb{R}_+ , together with Lemma 10.1, we obtain that for all $1 \leq i, j \leq d$ and $k \geq 1$,

$$\begin{aligned}
 & \mathbb{E} \left(\sum_{r=1}^d \frac{M_{1,k}(r, j)}{M_{0,k}(i, j)} \sum_{l=1}^{Z_1^i(r)} \left| \frac{Z_{l,1,k}^r(j)}{M_{1,k}(r, j)} - 1 \right| \right)^p \\
 = & \mathbb{E} \left(\sum_{r=1}^d \frac{M_0(i, r)M_{1,k}(r, j)}{M_{0,k}(i, j)} \frac{1}{M_0(i, r)} \sum_{l=1}^{Z_1^i(r)} \left| \frac{Z_{l,1,k}^r(j)}{M_{1,k}(r, j)} - 1 \right| \right)^p \\
 \leq & \mathbb{E} \left(\sum_{r=1}^d \frac{M_0(i, r)M_{1,k}(r, j)}{M_{0,k}(i, j)M_0(i, r)^p} \mathbb{E}_\xi \left[\left(\sum_{l=1}^{Z_1^i(r)} \left| \frac{Z_{l,1,k}^r(j)}{M_{1,k}(r, j)} - 1 \right| \right)^p \middle| Z_1^i(r) \right] \right) \\
 \leq & B_p^p \mathbb{E} \left(\sum_{r=1}^d \frac{Z_1^i(r)}{M_0(i, r)^p} \mathbb{E}_\xi \left| \frac{Z_k^r(j)}{M_{0,k-1}(r, j)} - 1 \right|^p \right) \\
 = & B_p^p \sum_{r=1}^d \mathbb{E} M_0(i, r)^{1-p} \mathbb{E} \left| \frac{Z_k^r(j)}{M_{0,k-1}(r, j)} - 1 \right|^p. \tag{10.8}
 \end{aligned}$$

Combining the relations (10.6)-(10.8), we obtain that for all $1 \leq i, j \leq d$ and $k \geq 1$,

$$\begin{aligned}
 \mathbb{E} \left(\frac{Z_{k+1}^i(j)}{M_{0,k}(i, j)} \right)^p & \leq 2^{p-1} B_p^p \sum_{r=1}^d \mathbb{E} M_0(i, r)^{1-p} \mathbb{E} \left| \frac{Z_k^r(j)}{M_{0,k-1}(r, j)} - 1 \right|^p \\
 & \quad + 2^{p-1} \sum_{r=1}^d \mathbb{E} \left(\frac{Z_1^i(r)}{M_0(r, j)} \right)^p. \tag{10.9}
 \end{aligned}$$

By (7.1), for all $1 \leq i, j \leq d$, \mathbb{P} -a.s., it holds

$$M_0(i, j) \geq \frac{1}{dD} \|M_0(\cdot, j)\| \geq \frac{1}{dD^2} \|M_0\|.$$

Combining this with (10.9), we get that for all $1 \leq j \leq d$ and $k \geq 0$,

$$\begin{aligned}
 \mathbb{E} \left(\frac{Z_{k+1}^i(j)}{M_{0,k}(i, j)} \right)^p & \leq 2^{p-1} B_p^p (dD^2)^{p-1} \mathbb{E} \|M_0\|^{1-p} \sum_{r=1}^d \mathbb{E} \left| \frac{Z_k^r(j)}{M_{0,k-1}(r, j)} - 1 \right|^p \\
 & \quad + 2^{p-1} \sum_{r=1}^d \mathbb{E} \left(\frac{Z_1^i(r)}{M_0(r, j)} \right)^p.
 \end{aligned}$$

Using the condition (2.24), by induction on k we conclude that for all $1 \leq i, j \leq d$ and $k \geq 1$,

$$\mathbb{E} \left(\frac{Z_k^i(j)}{M_{0,k-1}(i, j)} \right)^p < +\infty. \tag{10.10}$$

Putting together (10.5) and (10.10), we obtain (10.4), which implies (10.3), and ends the proof of Theorem 2.13. \square

11. Appendix

In this section we prove the several implications among the conditions **H3**-**H7**.

Lemma 11.1. *Assume conditions **H1** and **H2**. Then the following implications hold :*

$$\mathbf{H7} \Rightarrow \mathbf{H5} \Rightarrow \mathbf{H3}, \quad \text{and} \quad \mathbf{H7} \Rightarrow \mathbf{H6} \Rightarrow \mathbf{H3}.$$

If additionally condition **H4** holds, then we have the equivalences

$$\mathbf{H5} \Leftrightarrow \mathbf{H3} \quad \text{and} \quad \mathbf{H7} \Leftrightarrow \mathbf{H6}.$$

Moreover, when the environment (ξ_n) is i.i.d. and **H4** holds, then

$$\mathbf{H3} \Leftrightarrow \mathbf{H5} \Leftrightarrow \mathbf{H6} \Leftrightarrow \mathbf{H7}.$$

Proof of Lemma 11.1. We first prove that **H5** \Rightarrow **H3**. For all $C > 1$, $K > 1$ and $1 \leq i \leq d$, \mathbb{P} -a.s., we have

$$\begin{aligned} & \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \mathbf{1}_{\{\langle N_n^i, U_{n+1, \infty} \rangle \geq C^n\}} \right) \\ &= \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \mathbf{1}_{\{\langle N_n^i, U_{n+1, \infty} \rangle \geq C^n\}} \mathbf{1}_{\{\lambda_n U_{n, \infty}(i) < K^n\}} \right) \\ & \quad + \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \mathbf{1}_{\{\langle N_n^i, U_{n+1, \infty} \rangle \geq C^n\}} \mathbf{1}_{\{\lambda_n U_{n, \infty}(i) \geq K^n\}} \right) \\ &\leq \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \mathbf{1}_{\left\{ \frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \geq (CK)^n \right\}} \right) \\ & \quad + \sum_{n=0}^{+\infty} \mathbf{1}_{\{\lambda_n U_{n, \infty}(i) \geq K^n\}}. \end{aligned} \tag{11.1}$$

First, by **H2**, for all $K > 1$ and $1 \leq i \leq d$, it holds that

$$\begin{aligned} \mathbb{E} \left[\sum_{n=0}^{+\infty} \mathbf{1}_{\{\lambda_n U_{n, \infty}(i) \geq K^n\}} \right] &= \mathbb{E} \left[\sum_{n=0}^{+\infty} \mathbf{1}_{\{\log^+(\lambda_n U_{n, \infty}(i)) \geq n \log K\}} \right] \\ &\leq \mathbb{E} \left[\frac{\log^+(\lambda_0 U_{0, \infty}(i))}{\log(K)} + 1 \right] \\ &\leq \frac{\mathbb{E} \log^+ \|M_0\|}{\log(K)} + 1 < +\infty. \end{aligned} \tag{11.2}$$

By (2.5), we have $\sum_{j=1}^d \frac{M_n(i,j)U_{n+1,\infty}(j)}{\lambda_n U_{n,\infty}(i)} = 1$, so that the summands are bounded by 1. Therefore, for all $C > 1$ and $1 \leq i \leq d$, \mathbb{P} -a.s.,

$$\begin{aligned}
 & \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1,\infty} \rangle}{\lambda_n U_{n,\infty}(i)} \mathbf{1}_{\left\{ \frac{\langle N_n^i, U_{n+1,\infty} \rangle}{\lambda_n U_{n,\infty}(i)} \geq C^n \right\}} \right) \\
 &= \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\sum_{j=1}^d \frac{M_n(i,j)U_{n+1,\infty}(j)}{\lambda_n U_{n,\infty}(i)} \frac{N_n^i(j)}{M_n(i,j)} \times \right. \\
 & \quad \left. \mathbf{1}_{\left\{ \sum_{r=1}^d \frac{M_n(i,r)U_{n+1,\infty}(r)}{\lambda_n U_{n,\infty}(i)} \frac{N_n^i(r)}{M_n(i,r)} \geq C^n \right\}} \right) \\
 &\leq \sum_{n=0}^{+\infty} \sum_{j=1}^d \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i,j)} \mathbf{1}_{\left\{ \max_{1 \leq r \leq d} \left(\frac{N_n^i(r)}{M_n(i,r)} \right) \geq C^n \right\}} \right) \\
 &\leq \sum_{n=0}^{+\infty} \sum_{j=1}^d \sum_{r=1}^d \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i,j)} \mathbf{1}_{\left\{ \frac{N_n^i(r)}{M_n(i,r)} \geq C^n \right\}} \right). \tag{11.3}
 \end{aligned}$$

By a symmetry argument, for all $C > 1$ and $1 \leq j, r \leq d$, \mathbb{P} -a.s., we have

$$\begin{aligned}
 & \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i,j)} \mathbf{1}_{\left\{ \frac{N_n^i(r)}{M_n(i,r)} \geq C^n \right\}} \right) \\
 &= \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i,j)} \mathbf{1}_{\left\{ \frac{N_n^i(r)}{M_n(i,r)} \geq C^n \right\}} \mathbf{1}_{\left\{ \frac{N_n^i(j)}{M_n(i,j)} \geq \frac{N_n^i(r)}{M_n(i,r)} \right\}} \right) \\
 & \quad + \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i,j)} \mathbf{1}_{\left\{ \frac{N_n^i(r)}{M_n(i,r)} \geq C^n \right\}} \mathbf{1}_{\left\{ \frac{N_n^i(j)}{M_n(i,j)} < \frac{N_n^i(r)}{M_n(i,r)} \right\}} \right) \\
 &\leq \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i,j)} \mathbf{1}_{\left\{ \frac{N_n^i(j)}{M_n(i,j)} \geq C^n \right\}} \right) + \mathbb{E}_\xi \left(\frac{N_n^i(r)}{M_n(i,r)} \mathbf{1}_{\left\{ \frac{N_n^i(r)}{M_n(i,r)} \geq C^n \right\}} \right). \tag{11.4}
 \end{aligned}$$

Combining the inequalities (11.1)-(11.4), this shows that **H5** \Rightarrow **H3**.

We next prove that **H7** \Rightarrow **H5** and **H6** \Rightarrow **H3**. Since the sequence of the environments (ξ_n) is stationary, for all $C > 1$ and $1 \leq i, j \leq d$ we have

$$\begin{aligned}
 & \mathbb{E} \left(\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i,j)} \mathbf{1}_{\left\{ \frac{N_n^i(j)}{M_n(i,j)} \geq C^n \right\}} \right) \right) \\
 & \quad = \sum_{n=0}^{+\infty} \mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i,j)} \mathbf{1}_{\left\{ \log^+ \left(\frac{Z_1^i(j)}{M_0(i,j)} \right) \geq n \log C \right\}} \right).
 \end{aligned}$$

Therefore, we deduce that

$$\frac{B(i,j)}{\log C} \leq \mathbb{E} \left(\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i,j)} \mathbf{1}_{\left\{ \frac{N_n^i(j)}{M_n(i,j)} \geq C^n \right\}} \right) \right) \leq \frac{B(i,j)}{\log C} + 1, \tag{11.5}$$

with

$$B(i, j) := \mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i, j)} \log^+ \left(\frac{Z_1^i(j)}{M_0(i, j)} \right) \right).$$

The implication **H7** \Rightarrow **H5** follows. The implication **H6** \Rightarrow **H3** can be obtained by a similar argument.

We now prove that **H7** \Rightarrow **H6**. By the convexity of the function $x \mapsto x \log^+ x$ on \mathbb{R}^+ we obtain that for all $1 \leq i \leq d$,

$$\begin{aligned} & \mathbb{E} \left(\frac{\langle Z_1^i, U_{1, \infty} \rangle}{\lambda_0 U_{0, \infty}(i)} \log^+ \langle Z_1^i, U_{1, \infty} \rangle \right) \\ & \leq \mathbb{E} \left(\frac{\langle Z_1^i, U_{1, \infty} \rangle}{\lambda_0 U_{0, \infty}(i)} \log^+ \left(\frac{\langle Z_1^i, U_{1, \infty} \rangle}{\lambda_0 U_{0, \infty}(i)} \right) \right) + \mathbb{E} \log^+ (\lambda_0 U_{0, \infty}(i)) \\ & \leq \mathbb{E} \left(\sum_{j=1}^d \frac{M_0(i, j) U_{1, \infty}(j)}{\lambda_0 U_{0, \infty}(i)} \frac{Z_1^i(j)}{M_0(i, j)} \log^+ \left(\frac{Z_1^i(j)}{M_0(i, j)} \right) \right) + \mathbb{E} \log^+ \|M_0\| \\ & \leq \sum_{j=1}^d \mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i, j)} \log^+ \left(\frac{Z_1^i(j)}{M_0(i, j)} \right) \right) + \mathbb{E} \log^+ \|M_0\|. \end{aligned}$$

Using **H2**, this proves that **H7** \Rightarrow **H6**.

From now assume additionally the Furstenberg-Kesten condition **H4**. Then **H1** holds, so from the conclusions above we see that **H5** \Rightarrow **H3** and **H7** \Rightarrow **H6**. We will prove below the inverse implications **H3** \Rightarrow **H5** and **H7** \Rightarrow **H6**.

We first prove that **H3** \Rightarrow **H5**. For all $1 < K < C$ and $1 \leq i \leq d$, \mathbb{P} -a.s., we have

$$\begin{aligned} & \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \mathbf{1}_{\left\{ \frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \geq C^n \right\}} \right) \\ & = \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \mathbf{1}_{\left\{ \frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \geq C^n \right\}} \mathbf{1}_{\{\langle N_n^i, U_{n+1, \infty} \rangle > K^n\}} \right) \\ & \quad + \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \mathbf{1}_{\left\{ \frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \geq C^n \right\}} \mathbf{1}_{\{\langle N_n^i, U_{n+1, \infty} \rangle \leq K^n\}} \right) \\ & \leq \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \mathbf{1}_{\{\langle N_n^i, U_{n+1, \infty} \rangle \geq (CK)^n\}} \right) \\ & \quad + \sum_{n=0}^{+\infty} \mathbf{1}_{\{\lambda_n U_{n, \infty}(i) \leq (\frac{K}{C})^n\}}. \end{aligned} \tag{11.6}$$

Notice that, by (7.3),

$$\frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} = \sum_{j=1}^d \frac{M_n(i, j) U_{n+1, \infty}(j)}{\lambda_n U_{n, \infty}(i)} \frac{N_n^i(j)}{M_n(i, j)} \geq \frac{1}{dD^2} \sum_{j=1}^d \frac{N_n^i(j)}{M_n(i, j)}. \quad (11.7)$$

Therefore we get that for all $C > 0$ and $1 \leq i \leq d$, \mathbb{P} -a.s.,

$$\begin{aligned} \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \mathbf{1}_{\left\{ \frac{\langle N_n^i, U_{n+1, \infty} \rangle}{\lambda_n U_{n, \infty}(i)} \geq C^n \right\}} \right) \\ \geq \frac{1}{dD^2} \sum_{j=1}^d \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i, j)} \mathbf{1}_{\left\{ \frac{N_n^i(j)}{M_n(i, j)} \geq dD^2 C^n \right\}} \right). \end{aligned} \quad (11.8)$$

Moreover, for $1 < K < C$ and $1 \leq i \leq d$, it holds that

$$\begin{aligned} \mathbb{E} \left[\sum_{n=0}^{+\infty} \mathbf{1}_{\left\{ \lambda_n U_{n, \infty}(i) \leq \left(\frac{K}{C}\right)^n \right\}} \right] &= \mathbb{E} \left[\sum_{n=0}^{+\infty} \mathbf{1}_{\left\{ \log(\lambda_0 U_{0, \infty}(i)) \leq n \log\left(\frac{K}{C}\right) \right\}} \right] \\ &\leq \mathbb{E} \left[\sum_{n=0}^{+\infty} \mathbf{1}_{\left\{ |\log(\lambda_0 U_{0, \infty}(i))| \geq n \log\left(\frac{C}{K}\right) \right\}} \right] \\ &\leq \mathbb{E} \left[\frac{|\log(\lambda_0 U_{0, \infty}(i))|}{\log(C/K)} + 1 \right]. \end{aligned} \quad (11.9)$$

By (7.2) and Proposition (2.5), we get that

$$\begin{aligned} \mathbb{E} |\log(\lambda_0 U_{0, \infty}(i))| &\leq \mathbb{E} |\log \lambda_0| + \mathbb{E} |\log U_{0, \infty}(i)| \\ &\leq \mathbb{E} |\log \lambda_0| + \log(dD) < +\infty. \end{aligned} \quad (11.10)$$

From (11.9) and (11.10) we deduce that for $1 < K < C$ and $1 \leq i \leq d$,

$$\mathbb{E} \left[\sum_{n=0}^{+\infty} \mathbf{1}_{\left\{ \lambda_n U_{n, \infty}(i) \leq \left(\frac{K}{C}\right)^n \right\}} \right] \leq \frac{1}{\log(C/K)} \mathbb{E} |\log(\lambda_0 U_{0, \infty}(i))| + 1 < +\infty. \quad (11.11)$$

Combining the inequalities (11.6)-(11.11), we obtain the implication **H3** \Rightarrow **H5**.

It remains to prove that **H7**⇒**H6**. By (7.3), for all $1 \leq i, j \leq d$ we have

$$\begin{aligned}
 & \mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)\right) \\
 & \leq \mathbb{E}\left(dD^2 \frac{U_{1,\infty}(j)Z_1^i(j)}{\lambda U_{0,\infty}(i)} \log^+\left(dD^2 \frac{U_{1,\infty}(j)Z_1^i(j)}{\lambda U_{0,\infty}(i)}\right)\right) \\
 & \leq \mathbb{E}\left(dD^2 \frac{\langle Z_1^i, U_{1,\infty} \rangle}{\lambda_0 U_{0,\infty}(i)} \log^+\left(dD^2 \frac{\langle Z_1^i, U_{1,\infty} \rangle}{\lambda_0 U_{0,\infty}(i)}\right)\right) \\
 & \leq dD^2 \left(\mathbb{E}\left(\frac{\langle Z_1^i, U_{1,\infty} \rangle}{\lambda_0 U_{0,\infty}(i)} \log^+\langle Z_1^i, U_{1,\infty} \rangle\right) + \mathbb{E} \log^+\left(\frac{dD^2}{\lambda_0 U_{0,\infty}(i)}\right) \right) \\
 & \leq dD^2 \left(\mathbb{E}\left(\frac{\langle Z_1^i, U_{1,\infty} \rangle}{\lambda_0 U_{0,\infty}(i)} \log^+\langle Z_1^i, U_{1,\infty} \rangle\right) + \mathbb{E}\left|\log\left(\frac{\lambda_0 U_{0,\infty}(i)}{dD^2}\right)\right| \right).
 \end{aligned}$$

This, together with (11.10), proves that **H7**⇒**H6**.

Finally, in addition to the condition **H4**, we suppose that the environment ξ is i.i.d. Using the implications proved above, to show that all the conditions **H3-H7** are equivalent, it suffices to prove that **H5**⇔**H7**. Let us prove this below. Since (ξ_n) is i.i.d., for all $C > 0$ and $1 \leq i, j \leq d$ the random variables

$$\mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i,j)} \mathbf{1}_{\left\{ \frac{N_n^i(j)}{M_n(i,j)} \geq C^n \right\}} \right), \quad n \geq 0,$$

are independent, and bounded by 1. By the Kolmogorov's Three Series Theorem, we deduce that the condition **H5** holds if and only if for all $C > 0$ and $1 \leq i, j \leq d$,

$$\mathbb{E} \left(\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i,j)} \mathbf{1}_{\left\{ \frac{N_n^i(j)}{M_n(i,j)} \geq C^n \right\}} \right) \right) < +\infty.$$

Combining this with (11.5), it follows that **H5**⇔**H7**. This completes the proof of Lemma 11.1. \square

References

- [1] Athreya K.B., Karlin S., On branching processes with random environments I:Extinction probabilities, Ann. Math. Stat., 42(5), 1499-1520, 1971.
- [2] Athreya K.B., Karlin S., Branching Processes with random environments II:Limit theorems, Ann. Math. Stat., 42(6), 1843-1858, 1971.

- [3] Athreya K.B., Ney P.E., *Branching Processes*, Springer, New York, 1972.
- [4] Bansaye V., Berestycki J., Large deviations for branching processes in random environment, *Markov Process. Related Fields*, 15(4), 493-524, 2009.
- [5] Chow Y.S., Teicher H., *Probability Theory : Independence, Interchangeability, Martingales*, Springer Science and Business Media, 2012.
- [6] Cohn H., On the growth of the multitype supercritical branching process in a random environment, *Ann. Proba.*, 17(3), 1118-1123, 1989.
- [7] Cornfeld I.P., Fomin S.V., Sinai Y., *Ergodic Theory*, Springer, New York, 1982.
- [8] Durrett R., *Probability : Theory and Examples*, The Wadsworth, Pacific Grove, CA, 1991.
- [9] Dyakonova E. E., Asymptotics behaviour of the probability of non-extinction for a multi-type branching process in a random environment. *Discrete Math. Appl.* 9 (2), 119-136, 1999.
- [10] Dyakonova E. E., Multitype Galton-Watson branching processes in a Markov random environment, *Theory Probab. Appl.* 56 (3), 508-517, 2012.
- [11] Dyakonova E. E., Limit theorem for a multitype critical branching process evolving in a random medium, *Discrete Math. Appl.* 25 (3), 137-147, 2015.
- [12] Furstenberg H., Kesten H., Products of random matrices, *Ann. Math. Stat.*, 31(2), 457-469, 1960.
- [13] Grama I., Liu Q., Miqueu E., Berry-Esseen's bound and Cramér's large deviation expansion for a supercritical branching process in a random environment, *Stochastic Process. Appl.*, 127, 1255-1281, 2017.
- [14] Grama I., Liu Q., Pin E. Berry-Esseen's bound and harmonic moments for supercritical multi-type branching processes in random environments. Preprint.
- [15] Grama I., Liu Q., Pin E. Cramér type moderate deviation expansion for supercritical multi-type branching processes in random environments. Preprint.
- [16] Harris T.E., *The Theory of Branching Processes*, Berlin:Springer, 1963.
- [17] Hennion H., Limit Theorems for Products of positive Random Matrices, *Ann. Proba.*, 25(4), 1545-1587, 1997.
- [18] Hess C., Seri R., Choirat C., Ergodic theorems for extended real-valued random variables, *Stochastic Process. Appl.*, 120, 1908-1919, 2010.
- [19] Hong W., Liu M., Vatutin V., Limit theorems for supercritical MBPRE with linear fractional offspring distributions, *Markov Proc. Rel. Fields*,

25, 1-31, 2019.

- [20] Huang C., Liu Q., Convergence in L_p and its exponential rate for a branching process in a random environment, *Electron. J. Probab.*, 19(104), 1-22, 2014.
- [21] Kaplan N., Some results about multidimensional branching processes with random environments, *Ann. Proba.*, 2(3), 441-455, 1974.
- [22] Kersting G., Vatutin V., *Discrete time branching processes in random environment*, Wiley-ISTE, 2017.
- [23] Kesten H., Spitzer F., Convergence in Distribution of products of Random Matrices, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 67, 363-386, 1984.
- [24] Kesten H., Stigum B.P., A Limit Theorem for Multidimensional Galton-Watson Processes, *Ann. Math. Stat.*, 37(5), 1211-1223, 1966.
- [25] Kingman J.F.C, Subadditive ergodic theory, *Ann. Probab.*, 1, 883-899, 1973.
- [26] Kurtz T., Lyons R., Pemantle R., Peres Y., A conceptual proof of the Kesten-Stigum Theorem for multi-type branching processes, in *Classical and Modern Branching Processes*, 181-185. Ed. K. B. Athreya, P. Jagers, Springer, 1997.
- [27] Kyprianou A.E., Sani A.R., Martingale convergence and the functional equation in the multi-type branching random walk, *Bernoulli*, 7(4), 593-604, 2001.
- [28] Le Page E., Peigné M., Pham C., The survival probability of a critical multi-type branching process in i.i.d. random environment, *Ann. Probab.* 46(5), 2946-2972, 2018.
- [29] Lyons R., Pemantle R., Peres Y., Conceptual Proofs of L Log L Criteria for Mean Behavior of Branching Processes, *Ann. Proba.*, 23(3), 1125-1138, 1995.
- [30] Seneta E., *Non-negative Matrices and Markov Chains*, Springer, New York, 1981.
- [31] Smith W. L., Wilkinson W., On branching processes in random environments. *Ann. Math. Statist.*, 40(3): 814-827, 1969.
- [32] Tanny D., A Zero-One Law for Stationary Sequences, *Z. Wahrsch. Verw., Gebiete* 30, 139-148, 1974.
- [33] Tanny D., Normalizing constants for branching processes in random environments, *Stochastic Process. Appl.*, 6, 201-211, 1978.
- [34] Tanny D., On multitype branching processes in a random environment, *Adv. Appl. Probab.*, 13(3), 464-497, 1981.
- [35] Tanny D., A Necessary and sufficient condition for a branching process in a random environment to grow like the product of its means,

Stochastic Process. Appl., 28, 123–139, 1988.

- [36] Vatutin, V. A., Dyakonova, E. E., Multitype branching processes in a random environment: nonextinction probability in the critical case. Theory Probab. Appl. 62 (4), 506-521, 2018.
- [37] Vatutin, V.A.; Wachtel, V. Multi-type subcritical branching processes in a random environment. Adv. Appl. Probab. 50 (A), 281-289, 2018.