Existence of equilibrium on asset markets with a countably infinite number of states

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Abstract

We consider a model with a countably infinite number of states of nature. The agents have equivalent probability beliefs and von Neumann - Morgenstern utilities. The No-Arbitrage Prices in this this paper are, up to a scalar, the marginal utilities. We introduce the *Beliefs Strong Equivalence* and the *No Half Line Condition of the same type* conditions. Under these conditions, the No Arbitrage price condition is sufficient for the existence of an equilibrium when the commodity space is $l^p, 1 \leq p < +\infty$. This No Arbitrage condition is necessary and sufficient for the existence of equilibrium when the total endowment is in l^∞. Moreover, it is equivalent to the compactness of the individually rational utility set.

Keywords: beliefs strong equivalence, asset market equilibrium, individually rational attainable allocations, individually rational utility set, no-arbitrage prices, no-arbitrage condition.

JEL Classification: C62, D50, D81,D84,G1.

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1 Introduction

In finite dimensional markets with short-selling, conditions on agents' utilities ensuring the existence of equilibria are by now well understood. They can be interpreted as no-arbitrage conditions. In an uncertainty setting, where agents have different beliefs and different risk aversions, as originally shown by Hart (1974), the no-arbitrage conditions may be interpreted as compatibility of agent's risk adjusted beliefs.

There is a huge literature on sufficient and necessary conditions for the existence of equilibria.

In finite dimension, there are three categories of proofs of existence of equilibrium. The proofs consist in finding conditions called no-arbitrage conditions which, at the end, conduct to the compactness of $\mathcal{U}$.

In the *first category*, we can cite Green [12], Grandmont [11], Hammond [13], Werner [22], Allouch et al. [1], and Dana and Le Van [8] . They impose the existence of a no-arbitrage price. This condition implies the existence of equilibrium.

In the *second category*, see e.g. Page [19], Nielsen [18], Page and Wooders [20], Dana et al. [10], Page et al. [21]). They impose a condition on useful net trades.

The *third category* assumes directly the set $\mathcal{U}$ is compact, see Dana et al. [10].

In infinite dimension asset markets, the no-arbitrage conditions used for finite dimension do not imply existence of equilibrium. The standard assumption is to assume that the individually rational utility set is compact (see e.g. Brown and Werner [3], Dana and Le Van [7], Dana et al [9], Dana and Le Van [8].

In this paper, we consider a model with a countably infinite number of states of nature, a finite number of agents and von Neumann - Morgenstern utilities with different expectations.

More precisely, we consider a model where the utility of agent i is

$$U^i(x^i) = \sum_{s=1}^{\infty} \pi^i_s u^i(x^i_s)$$

where π^i is her belief and x^i is her consumption. The commodity space is $l^p(\pi)$ with $p \in \{1, \dots, +\infty\}$.[1]

[1] We use the model proposed by Hart [14]. Investors are interested only in their wealth in the second period. We suppose, as in Cheng [4], Brown and Werner [3], Le Van and Truong Xuan [17], that the market is complete with an asset system $(r^1, r^2, \dots, r^k, \dots)$. For each portfolio $z = (z^1, z^2, \dots, z^k, \dots)$, the wealth of the investor when state s occurs is

$$w_s = \sum_k z^k r^k_s.$$

When the number of states is finite, say K states, following Werner [22], one can introduce for any agent i the set of useful vectors W^i to obtain the set of no-arbitrage prices denoted by S^i, which are defined as the set of vectors p which satisfy $p \cdot w > 0$ for any $w \in W^i \setminus \{0\}$. We say that the no-arbitrage condition holds if $\bigcap_i S^i \neq \emptyset$. When the utility functions are strictly concave, strictly increasing, this condition is necessary and sufficient for the existence of equilibrium.

Dana and Le Van [8] introduce, for every agent i, the convex cone P^i generated by the vectors $\{\pi^i_s u^{i\prime}(x^i_s)\}_{s=1,\ldots,K}$ where $x^i \in \mathbb{R}^K$ and $u^{i\prime}(+\infty) < u^{i\prime}(x^i_s) < u^{i\prime}(-\infty), \forall s, \forall i$. The no-arbitrage cone S^i is proved to be the interior of the cone P^i.

In this paper, we weaken the concept of no-arbitrage prices *à la* Dana and Le Van [8]. We define no-arbitrage prices p for agent i as follows: for any state s,

$$p_s = \lambda_i \pi^i_s u^{i\prime}(x^i_s)$$

where $\lambda_i > 0, x^i \in l^\infty(\pi)$ and

$$u^{i\prime}(+\infty) \leq u^{i\prime}(x^i_s) \leq u^{i\prime}(-\infty).$$

We say that the no-arbitrage condition (NA) holds if :

$$\lambda_i \pi^i_s u^{i\prime}(x^i_s) = \lambda_j \pi^j_s u^{j\prime}(x^j_s), \forall i, \forall j, \forall s$$

and if for any i,

$$\begin{aligned}
&1. \forall x, u^{i\prime}(x) < u^{i\prime}(-\infty) \Rightarrow \sup_s u^{i\prime}(x^i_s) < u^{i\prime}(-\infty),\\
&2. \forall x, u^{i\prime}(x) > u^{i\prime}(+\infty) \Rightarrow \inf_s u^{i\prime}(x^i_s) > u^{i\prime}(+\infty).
\end{aligned}$$

When the number of states is finite, condition (NA) ensures existence of equilibrium. When the number of states is infinite, in general, this condition only ensures the boundedness of the individually rational utility set. However, under a condition we call *Beliefs Strong Equivalence* and *No Half Line conditions of the same type*, no-arbitrage condition (NA) is sufficient for the existence of an equilibrium when the commodity space is $l^p(\pi), 1 \leq p < +\infty$. This no-arbitrage condition is necessary and sufficient when the total endowment is in $l^\infty(\pi)$ or when the consumption space is of finite dimension.

Until now, to our knowledge, for infinite dimension asset markets, there exists

Her expected utility is:

$$V(w^1, w^2, \ldots, w^s, \ldots) = \sum_s \pi_s u(w_s).$$

Since the market is complete, the choice of portfolio is equivalent to the choice of wealth. As in Hart's pioneer paper, we consider the expected utility function on wealth.

no paper where no-arbitrage condition on prices implies existence of an equilibrium and may be equivalent to the existence of equilibrium.
We summarize our contribution in this paper.

- We introduce the Beliefs Strong Equivalence condition for the beliefs of the agent.
- A new notion of no half line is also defined: No Halfline Condition of the Same Type.
- We prove that no-arbitrage condition implies the compactness of the Individually Rational Utility Set and hence Existence of Equilibrium. The Individually Rational Feasible Sets may be non compact.
- This no-arbitrage condition is necessary and sufficient for the existence of equilibrium when the total endowment is in $l^\infty(\pi)$ or when the consumption space is of finite dimension. Again, the Individually Rational Feasible Sets may be non compact.

The paper is organized as follows. In Section 2, we present the model, define the individually rational allocations set, the individually rational utility set and the equilibrium. Section 3 gives the definitions of the useful vectors of the agents, of Half Lines, No Half Line Condition, while No-Arbitrage prices are defined in Section 4. The assumptions are set in Section 5. We introduce there Beliefs Strong Equivalence, No Half Line of the Same Type. In Section 6, we introduce a no-arbitrage condition and show that this no-arbitrage condition is sufficient for the existence of and equilibrium. It becomes necessary and sufficient when the total endowment is in $l^\infty(\pi)$. Our results are discussed in a section entitled Comments, Section 7. We devote a section for an extensive review of literature on the relationship between no-arbitrage conditions and existence of equilibrium on finite dimension assets markets and infinite dimension assets as well (Section 8). In Section 9 which is the conclusion, we explain our contribution to the problem of the existence of equilibrium on assets markets with an infinitely countable number of states of nature. Sections 10, 11 and 12 are devoted to the proofs .

2 The model

There are m agents indexed by $i = 1, \ldots, m$. The consumption set of agent i is $X^i = l^p(\pi)$ with $p \in \{1, 2, \ldots, +\infty\}$ and agent i has an endowment $e^i \in l^p(\pi)$.
We assume that for each agent i
(1) her belief in state s is $\pi^i_s \geq 0$, and $\sum_{s=1}^{\infty} \pi^i_s = 1$.

(2) there exists a concave, strictly increasing, differentiable [2] function $u^i : \mathbb{R} \to \mathbb{R}$, such that, for any i, the function

$$U^i(x^i) = \sum_{s=1}^{\infty} \pi_s^i u^i(x_s^i)$$

is real-valued for any $x^i \in X^i$. Her utility function is represented by this function U^i. [3]
Let us denote by π the mean probability $\frac{1}{m}\sum_i \pi^i$. This allows us to use $l^p(\pi), 1 \leq p \leq +\infty$ as space of consumption goods[4].

Definition 1 *1. The individually rational attainable allocations set $\mathcal{A}$ is defined by*

$$\mathcal{A} = \{(x^i) \in (l^p(\pi))^m \mid \sum_{i=1}^m x^i = \sum_{i=1}^m e^i \ \text{ and } \ U^i(x^i) \geq U^i(e^i) \text{ for all } \ i\}.$$

2. The individually rational utility set $\mathcal{U}$ is defined by

$$\mathcal{U} = \{(v^1, v^2, ..., v^m) \in \mathbb{R}^m \mid \exists x \in \mathcal{A} \ \text{ s.t } \ U^i(e^i) \leq v^i \leq U^i(x^i) \ \text{ for all } i\}.$$

Definition 2 *An equilibrium is a list $\left((x^{i*})_{i=1,...,m}, p^*\right)$ such that $x^{i*} \in X^i$ for every i and $p^* \in l_+^q(\pi) \setminus \{0\}$ and*
(a) For any i, $U^i(x) > U^i(x^{i*}) \Rightarrow \sum_{s=1}^{\infty} p_s^* x_s > \sum_{s=1}^{\infty} p_s^* e_s^i$,
(b) $\sum_{i=1}^m x^{i*} = \sum_{i=1}^m e^i$.

3 Useful vectors

Define

$$\begin{aligned} a^i &= \inf_x u^{i\prime}(x) = u^{i\prime}(+\infty) \\ b^i &= \sup_x u^{i\prime}(x) = u^{i\prime}(-\infty). \end{aligned}$$

Definition 3 *Following Werner [22]), when $X^i = l^p(\pi)$, we say that a vector $w \in l^p(\pi) \setminus \{0\}$ is*

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[2] To simplify the proof of our result we assume differentiability. One can use subdifferentials of a concave function.

[3] From Le Van [16], the derivative of U^i is in $l^q(\pi)$ with $1/p + 1/q = 1$. Hence for any $w \in l^p(\pi)$, the sum $\sum_{s=1}^{\infty} \pi_s^i u^{i\prime}(x_s) w_s$ converges in $\mathbb{R}$.

[4] The space $l^p(\pi)$ for $p < +\infty$, is the space of infinite sequences $(x_1, \ldots, x_s, \ldots)$ such that $\sum_{s=1}^{\infty} \pi_s |x_s|^p < +\infty$. We are interested by the moments up to order p of the assets. The moments measure the dispersion of an asset around its mean value. When $p = +\infty$, $l^{\infty}(\pi)$ is the space of assets which are uniformly bounded above and below, over the states of nature.

- *useful for agent i if it satisfies* $\forall x \in l^p(\pi), \forall \lambda \geq 0, \sum_s \pi_s^i u^i(x_s + \lambda w_s) \geq \sum_s \pi_s^i u^i(x_s)$,
- *a half line for agent i, if it satisfies:*
  i) $\forall x \in l^p(\pi), \forall \lambda \geq 0, \sum_s \pi_s^i u^i(x_s + \lambda w_s) \geq \sum_s \pi_s^i u^i(x_s)$ *(i.e. it is useful),*
  ii) *and there exists* $z \in l^p(\pi)$ *such that* $\forall \lambda \geq 0, \sum_s \pi_s^i u^i(z_s + \lambda w_s) = \sum_s \pi_s^i u^i(z_s)$.

The following characterization of a useful vector is an easy extension to a countably infinite number of states of a result in Dana and Le Van [8].

Lemma 1 *A vector* $w \in l^p(\pi) \setminus \{0\}$ *is useful for agent i if, and only if,*

$$\forall x \in l^p(\pi), \sum_s \pi_s^i u^{i\prime}(x_s) w_s \geq 0. \tag{1}$$

Remark 1
- *If $a^i = 0$, then w is useful for agent i iff $w > 0$.*
- *if $b^i = +\infty$, then w is useful for agent i iff $w > 0$.*

For a vector $w \in l^p(\pi)$, let $S_+ = \{s : w_s \geq 0\}, S_- = \{s : w_s < 0\}$.

Lemma 2 *A vector* $w \in l^p(\pi) \setminus \{0\}$ *is a half line for agent i if, and only if, it is useful and if there exists* $z \in l^p(\pi)$ *such that*

$$\sum_{s \in S_+} \pi_s^i u^{i\prime}(z_s) w_s + \sum_{s \in S_-} \pi_s^i u^{i\prime}(z_s) w_s = 0. \tag{2}$$

Proof: See Appendix 1. ■

Remark 2
- *If $a^i = 0$ then agent i has no half line.*
- *If $b^i = +\infty$ then agent i has no half line.*

4 No-arbitrage prices

For any $i = 1, \ldots, m$, let P^i denote the cone

$$\begin{aligned} P^i &= \left\{ \left(\lambda \pi_s^i u^{i\prime}(x_s)\right)_s, x \in l^p(\pi), \lambda > 0, a^i < \inf_s u^{i\prime}(x_s^i) < \sup_s u^{i\prime}(x_s^i) < b^i \right\} \\ &= \left\{ \left(\lambda \pi_s^i u^{i\prime}(x_s)\right)_s, x \in l^\infty(\pi), \lambda > 0 \right\}. \end{aligned}$$

The cone P^i is open for the $l^\infty(\pi)$ topology. Hence, we obtain the following result, the proof of which is easy.

Lemma 3 *For any non-null useful vector w one has*

$$\sum_s p_s w_s > 0, \forall p \in P^i. \tag{3}$$

Define, as in finite dimension, the cone of no-arbitrage prices for agent i by

$$S^i = \left\{ p \in l^q(\pi) : \forall w \text{ non-null useful }, \sum_s p_s w_s > 0 \right\}$$

where $1/p + 1/q = 1$. Obviously, $P^i \subseteq S^i$. When the number of states S is finite then $P^i = S^i$.

Lemma 4 *1. If $\forall x, u^{i\prime}(x) > a^i$ then agent i has no half line.*

2. If $\forall x, u^{i\prime}(x) < b^i$ then agent i has no half line.

Proof: See Appendix 1. ■

5 The Assumptions

We add the following assumptions:
A0 (*Beliefs Strong Equivalence*)[5]: For all pair of agents (i, j), there exists s^i_j such that

$$\frac{\pi^i_{s^i_j}}{\pi^j_{s^i_j}} = \inf_s \frac{\pi^i_s}{\pi^j_s}.$$

Under **A0**, without loss of generality, one can assume that $\pi^i_s > 0$ for any i, any s. In this paper, we always suppose that condition **A0** is satisfied and $\pi^i_s > 0$ for any i, any s.
A1 (*The agents have the same type of No Half Line*).
1. Either any agent i satisfies $0 \le a^i < u^{i\prime}(x)$, $\forall x \in \mathbb{R}$,
2. Or any agent i satisfies $u^{i\prime}(x) < b^i \le +\infty$, $\forall x \in \mathbb{R}$.

Remark 3 *We recall that probabilities π^i, π^j are equivalent if there exists $h^{ij} > 0$ such that*

$$\forall s, \pi^i_s h^{ij} \le \pi^j_s \le \frac{\pi^i_s}{h^{ij}}.$$

*Obviously, our assumption **A0** implies equivalence between beliefs. It is stronger since for any pair (i, j), there exists an index s^i_j such that $h^{ij} = \frac{\pi^i_{s^i_j}}{\pi^j_{s^i_j}}$.*

[5]We observe that when all agents have the same belief as in Cheng [4], then **A0** is satisfied.

*We can interpret **A0** as follows. When we assume the beliefs are equivalent, we may suppose the agents have this information, i.e., they know that for any couple (i, j) of agents, their relative beliefs $\left(\frac{\pi_s^i}{\pi_s^j}\right)_s$ are uniformly bounded above. Assumption **A0** supposes they have a bit more information. For any couple (i, j) of agents, their relative beliefs $\left(\frac{\pi_s^i}{\pi_s^j}\right)_s$ are uniformly bounded by their relative belief for some state s_j^i. But they may not know this state.*

Remark 4

- *If the function u^i is strictly concave for any i, then the agents satisfy **A1** because $u^{i\prime}$ is strictly decreasing and we cannot have x with $u^{i\prime}(x) = a^i$ or $u^{i\prime}(x) = b^i$. Very often it is assumed that the reward utility functions are strictly concave to get single valued demand correspondences. In this case, in finite dimension, the cone P^i always exists.*
- *If the utility is not strictly concave, we are not sure that P^i exists. However, under **A1** which is weaker than strict concavity, the cone P^i is non empty (see Proposition 1).*

6 The Main Results

In this section, we prove our main results: (i) No-arbitrage condition (**NA**), which is given below, implies existence of equilibrium when $1 \leq p < +\infty$. (ii) If $e \in l^\infty(\pi)$ then No-arbitrage condition (**NA**) is equivalent to the existence of equilibrium.

We mimic Dana and Le Van [8] for a model with S states of nature, we restrict our No-arbitrage prices in the cone $P = \bigcap_{i=1}^m P^i$ where

$$P^i = \left\{p \in l^\infty(\pi) : p_s = \lambda \pi_s^i u^{i\prime}(\overline{x}_s^i) \text{ for all } s, \lambda > 0\right\}.$$

In this case $\overline{x}^i \in l^\infty(\pi)$.
A first no arbitrage condition (which coincides with the standard Werner no-arbitrage condition, in finite dimension) may be
(**NA0**) $\bigcap_{i=1}^m P^i \neq \emptyset$.

Actually, our No-arbitrage condition is:
(**NA**) There exists $x \in l^p(\pi)$ such that

$$(a)\ \lambda^1 \pi_s^1 u^{1\prime}(x_s^1) = \lambda^2 \pi_s^2 u^{2\prime}(x_s^2) = \ldots = \lambda^m \pi_s^m u^{m\prime}(x_s^m),\ \forall s$$

and $\forall i$,

$$(b) \text{ if } \forall x, u^{i\prime}(x) < b^i \text{ then } \sup_s u^{i\prime}(x_s^i) < b^i,$$

$$(c) \text{ if } \forall x, u^{i\prime}(x) > a^i \text{ then } \inf_s u^{i\prime}(x_s^i) > a^i.$$

A price p which satisfies (a), (b) and (c) will be called, in this paper, No-arbitrage price.

Remark 5 *If the agents have the same belief, i.e.* $\pi^i = \pi^j = \pi, \forall i, \forall j$, *and if the utilities functions are strictly concave, then* ***(NA)*** *is satisfied. Indeed, in this case the agents satisfy the half-line of the same type (see Remark 4). Define* $\lambda^1 = 1, \lambda^i = \frac{u^{1\prime}(0)}{u^{i\prime}(0)}, \forall i > 1$. *Take* $x_s^i = 0, \forall i, \forall s$. *Then we have* $\lambda^1 \pi_s^1 u^{1\prime}(x_s^1) = \lambda^2 \pi_s^2 u^{2\prime}(x_s^2) = \ldots = \lambda^m \pi_s^m u^{m\prime}(x_s^m), \forall s$.

To prove the existence of equilibrium we prove that the individually rational utility set $\mathcal{U}$ is compact. First, we impose conditions **(NA)** and **A1** which are sufficient to obtain the boundedness of $\mathcal{U}$. We then add condition **A0** and get the closedness of $\mathcal{U}$.

Proposition 1 *Assume* ***A1****. Then* ***(NA)*** $\Leftrightarrow$ ***(NA0)****. Assume* ***(NA)****. Then there exists* $C > 0$ *such that* $\|x\|_{l^1(\pi)} \leq C$ *for any* x *in* $\mathcal{A}$ *and* $\mathcal{U}$ *is bounded.*

Proof: See Appendix 2. ■

The following proposition is crucial and its proof is the most difficult of this paper.

Proposition 2 *Assume* ***A0****,* ***A1****. If* ***(NA)*** *holds, then* $\mathcal{U}$ *is compact.*

Proof: See Appendix 2. ■

The following theorems which are the main results of the paper follow Propositions 1 and 2 and Dana, Le Van and Magnien [9].

Theorem 1 *Assume* ***A0****,* ***A1****. Suppose that* (***NA***) *holds, then there exists an equilibrium for all* $1 \leq p < +\infty$.

Theorem 2 *Let* $e = \sum_{i=1}^m e^i$. *Assume* ***A0****,* ***A1****,* $1 \leq p \leq +\infty$, *and* $e \in l^\infty(\pi)$. *Then*

$$\textbf{\textit{NA}} \textit{ holds } \Leftrightarrow \mathcal{U} \textit{ is compact } \Leftrightarrow \textit{ there exists an equilibrium.}$$

Proof: See Appendix 3. ■

7 Comments on our assumptions and our results

1. Assumption (**NA**) deserves an explanation. In [15], we have an example with two agents, where (**NA0**) holds and we have no equilibrium because the agents do not have No Half-Line of the same line property. In our paper we introduce condition **A1** (No Half Line of the same type). In this case, it is necessary to precise that for any i,

$$\text{if } \forall x, u^{i\prime}(x) < b^i \text{ then } \sup_s u^{i\prime}(x^i_s) < b^i,$$
$$\text{if } \forall x, u^{i\prime}(x) > a^i \text{ then } \inf_s u^{i\prime}(x^i_s) > a^i,$$

   to obtain (**NA0**). More precisely, without **A1**, (**NA0**) and (**NA**) are not equivalent.

2. One can check that, in finite dimension, (**NA**) implies condition **(NUBA)** of Page [19] or condition *Positive semi-independence* of Nielsen [18].

3. Proposition 1 states that the set $\mathcal{A}$ is bounded in $l^1(\pi)$. In [15], the authors prove that $\mathcal{A}$ is $l^1(\pi)$ compact when $b^i = +\infty$ for any i. Hence $\mathcal{U}$ is compact since $l^\infty(\pi) \subset l^p(\pi) \subset l^1(\pi)$. Here we do not rule out the case with $b^i < +\infty$. In this case the set $\mathcal{A}$ is not necessary compact.

4. Some remarks:
   (i) Since $P^i \subseteq S^i$, we have $\bigcap_i P^i \neq \emptyset \Rightarrow \bigcap_i S^i \neq \emptyset$. In finite dimension, this condition implies the compactness of $\mathcal{A}$. In infinite dimension, it implies the boundedness of $\mathcal{U}$.
   (ii) In the von Neumann-Morgenstern model, if condition $\bigcap_i P^i \neq \emptyset$ implies the set $\mathcal{U}$ is bounded, we cannot prove that condition $\bigcap_i S^i \neq \emptyset$ gives the same result. Until now, there is no paper with a model in infinite dimension where condition $\bigcap_i S^i \neq \emptyset$ implies the boundedness of $\mathcal{U}$.

5. In Theorem 2, the equivalence between no-arbitrage condition and the existence of equilibrium requires a slightly stronger condition which is the total endowment is in $l^\infty(\pi)$. In particular, if the commodity space is $l^\infty(\pi)$, this condition is automatically satisfied.

   Theorem 2 obviously holds also for the finite number of states case.

   The no-arbitrage condition implies the compactness of the individualy rational utility sets. The individually rational feesible sets may not be compact.

6. In [15], if $a^i = 0, \forall i$, or $b^i = +\infty, \forall i$ and if the beliefs are equivalent, there exists an equilibrium. One can see immediately that the conditions

$a^i = 0, \forall i$, or $b^i = +\infty, \forall i$ imply our conditions **A1**. We will check that they also satisfy (**NA**). Since the beliefs are equivalent, there exists $h > 0$, such that $\frac{1}{h} \leq \frac{\pi_s^i}{\pi_s^j} \leq h$, for all i, j, s.
Consider the case where $a^i = 0, \forall i$. Take $x_s^1 = 0, \forall s$. Fix $\lambda^1 > 0$. Take $\bar{b}^j \in (0, b^j)$ for $j > 1$. Take λ^j which satisfies $\lambda^j \geq \frac{\lambda^1 h u^{1\prime}(0)}{\bar{b}^j}$. Define x_s^j by $u^{j\prime}(x_s^j) = \frac{\lambda^1 \pi_s^1 u^{1\prime}(0)}{\lambda^j \pi_s^j}$. Let $\bar{a}^j = \frac{\lambda^1 u^{1\prime}(0)}{\lambda^j h}$. We have

$$\forall j > 1, \lambda^1 \pi_s^1 u^{1\prime}(0) = \lambda^j \pi_s^j u^{j\prime}(x_s^j),$$

and

$$\forall j > 1, 0 < \bar{a}^j \leq u^{j\prime}(x_s^j) \leq \bar{b}^j < b^j.$$

The last inequalities show that $(x_s^j)_s \in l^\infty(\pi), \ \forall j$.
The same method applies when $b^j = \infty, \forall j$.
We have proved that (**NA**) holds.
However, [15] does not assume **A0** which is stronger than the equivalence of the beliefs. In our paper, **A0** is required since a^i may be non null and b^i may be finite.

7. The No Half Line of the same type condition **A1** is crucial for the proof of the existence of equilibrium. In [15], we give an example of economy with two agents who do not satisfy **A1**. They have no half line but not of the same type. Agent 1 satisfies the statement 2 of Lemma 4 while agent 2 satisfies the statement 1 of the same lemma. The economy exhibits a no-arbitrage condition and it has no equilibrium.

8. If the utility functions u^i are strictly concave and if the agents have the same belief, then from Remarks 4 and 5 we have an equilibrium. Cheng [4] in a model with a continuum number of states assumes the agents have the same belief and the utility functions are strictly concave. If we put his model in a model with a countably infinite number of states, we will get an equilibrium.

9. Le Van and Truong Xuan [17] generalize the model of Cheng. The utility functions are $\int_0^1 v^i(x_s, s)\mu(ds)$. Le Van and Truong Xuan give an example where $v^i(x, s) = u^i(x)h^i(s), \forall i, \forall s$ and v^i is concave, increasing, differentiable,
(a) h^i is continuous, $h^i(s) > 0$ for all s. In this case, for any i, j, there exists s_0 such that $\frac{h^i(s_0)}{h^j(s_0)} = \min_s \frac{h^i(s)}{h^j(s)}$.
They also assume that
(b) for any i, any a there exist $b > b'$ such that $u^{i\prime}(b) < u^{i\prime}(a) < u^{i\prime}(b')$ and
(c) $\forall i, u^{i\prime}(+\infty) = 0, u^{i\prime}(-\infty) = +\infty$.

We will show that if we put their model, with the assumptions of their example, in a model with an infinitely countable number of states, we have an equilibrium.

Indeed

- (a) implies that **A0** is satisfied.
- (b) implies that the agents satisfy No Half-Line of the same type.
- (c) implies **NA** is satisfied. To see that take $x_s^1 = 0, \forall s$. For any $i > 1$, any s we have $0 < \frac{\pi_s^1}{\pi_s^i} u^{1\prime}(0) < +\infty$. There exists x_s^i which satisfies $u^{i\prime}(x_s^i) = \frac{\pi_s^1}{\pi_s^i} u^{1\prime}(0)$. Finally, we get $\forall s, \forall i > 1$, $\pi_s^1 u^{1\prime}(x_s^1) = \pi_s^i u^{i\prime}(x_s^i)$.

8 Review of literature on existence of equilibrium on assets markets

We will present a review of literature on the no-arbitrage condition and its role on the existence of equilibrium on complete financial markets. [6] The papers we mention use the two-period model proposed by Hart. Investors are interested only in their wealth in the second period. There are k assets and S states of nature in the second period. We suppose at state s, the returns are given by a system $(r_s^1, r_s^2, \dots, r_s^k, \dots)$. For each portfolio $z = (z^1, z^2, \dots, z^k, \dots)$, the wealth of the investor when state s occurs is

$$w_s = \sum_k z^k r_s^k.$$

Given the belief $(\pi_1, \dots, \pi_S)$, the expected utility of the agent is:

$$V(w^1, w^2, \dots, w^s, \dots) = \sum_{s=1}^{S} \pi_s u(w_s).$$

Since the market is complete, the choice of portfolio is equivalent to the choice of wealth. As in Hart's pioneer paper, we consider the expected utility function on wealth.
We distinguish two cases: (i) The number of states is finite, (ii) the number of states in infinite.
The contribution of our paper will be discussed after this review of literature.
We suppose the economy has m agents with utility functions $u^i, i = 1, \dots, m$ which are concave, continuous.

[6]We focus on the case where the preferences are transitive and represented by utility functions which are concave. For the case of non transitive, non convex and not necessary continuous preferences, the reader can refer to Won and Yannelis [23].

8.1 No arbitrage conditions and equilibrium on finite dimension asset markets

For simplicity we assume that the consumption sets are $\mathbb{R}^S$. Let us recall the definitions of the following sets.

- The individually rational attainable allocations set $\mathcal{A}$ is defined by

$$\mathcal{A} = \{(x^i) \in (l^p(\pi))^m \mid \sum_{i=1}^{m} x^i = \sum_{i=1}^{m} e^i \text{ and } U^i(x^i) \geq U^i(e^i) \text{ for all } i\}.$$

- The individually rational utility set $\mathcal{U}$ is defined by

$$\mathcal{U} = \{(v^1, v^2, ..., v^m) \in \mathbb{R}^m \mid \exists x \in \mathcal{A} \text{ s.t } U^i(e^i) \leq v^i \leq U^i(x^i) \text{ for all } i\}.$$

- A vector $w \in \mathbb{R}^S$ is useful for agent i if $u^i(x + \lambda w) \geq u^i(x), \forall x, \forall \lambda \geq 0$. The set of useful vectors of agent i is denoted by W^i.

- A vector $w \in \mathbb{R}^S$ is useless for agent i if $u^i(x + \lambda w) = u^i(x) = u^i(x - \lambda w), \forall x, \forall \lambda \geq 0$. The set of useless vectors of agent i is denoted by L^i.

- A vector $w \in \mathbb{R}^S$ is a half-line for agent i if there exists x such that $u^i(x + \lambda w) = u^i(x), \forall \lambda \geq 0$.
An economy has no half-line if no agent of this economy has a half-line.

- A vector $p \in \mathbb{R}^S$ is a no-arbitrage price for agent i if $p \cdot w \geq 0$ for any $w \in W^i$. We denote by S^i the set of no-arbitrage prices for agent i.

To prove the existence of an equilibrium a sufficient condition is the compact ness of the set $\mathcal{U}$. Actually, since the utility functions are continuous, if $\mathcal{A}$ is compact then $\mathcal{U}$ is also compact. There are three categories of proofs of existence of equilibrium. The proofs consist in finding conditions called no-arbitrage conditions which, at the end, conduct to the compactness of $\mathcal{U}$.
In the first category, we can cite Green [12], Grandmont [11], Hammond [13], Werner [22], Allouch et al. [1], and Dana and Le Van [8] . They impose the existence of a no-arbitrage price. This condition implies the existence of equilibrium.
In the second category, see e.g. Page ([19]), Nielsen ([18]), Page and Wooders [20], Dana et alii [10], Page et alii[21]).
The third category assumes directly the set $\mathcal{U}$ is compact, see Dana et al. [10].

The results of the authors of the *first category* on the existence by imposing conditions on *prices* can be presented as follows.
Assume the following no-arbitrage condition:

NA0: $\bigcap_{i=1}^m S^i \neq \emptyset$.
This condition says there exists a price which is a common no-arbitrage price for all the agents. We have the following result:

$$\textbf{NA0} \text{ holds} \implies \mathcal{A} \text{ is compact} \implies \mathcal{U} \text{ is compact} \implies \text{ Existence of equilibrium.}$$

However, when the set of useless vectors of agent i is not reduced to $\{0\}$, the set S^i is empty. Werner [22] introduces a weaker no-arbitrage condition:
NA1: $W^i \setminus L^i \neq \emptyset, \forall i$ and there exists a vector p which satisfies $p \cdot w > 0$ for any $w \in W^i \setminus L^i$.
Allouch et al. [1] introduce a more general set of no-arbitrage prices

$$\begin{aligned} \tilde{S}^i &= \left\{p : p \cdot w > 0 \text{ if } w \in W^i \setminus L^i\right\} \\ &= L^{i\perp} \text{ if } W^i = L^i. \end{aligned}$$

Their no-arbitrage condition is:
NA2: $\bigcap_i \tilde{S}^i \neq \emptyset$.
Condition **NA2** is weaker than **NA1**. It coincides with **NA1** if $W^i \setminus L^i \neq \emptyset, \forall i$. However

$$\textbf{NA1} \text{ holds} \implies \mathcal{U} \text{ is compact} \implies \text{ Existence of equilibrium.}$$

and also

$$\textbf{NA2} \text{ holds} \implies \mathcal{U} \text{ is compact} \implies \text{ Existence of equilibrium.}$$

Dana and Le Van reconsider the equilibrium theory of assets with short-selling when there is risk and ambiguity. The variational preferences are used. They observe that the equilibrium prices equal, up to a scalar, the marginal utilities generated by the equilibrium allocations of the agents. They therefore consider prices which equal, up to a scalar, the marginal utilities of the allocations of the agents. For an agent i, they define P^i as the cone generated by the marginal utility of her allocations.
Their condition of non-arbitrage is
NA3: $\bigcap_i \text{int} P^i \neq \emptyset$.
The result is

$$\textbf{NA3} \text{ holds} \implies \mathcal{A} \text{ is compact} \implies \mathcal{U} \text{ is compact} \implies \text{ Existence of equilibrium.}$$

Observe that these no-arbitrage conditions are sufficient for the existence of an equilibrium. They become necessary if the economy has no half-line.

The *second category* of papers make use of net trades. A list $(w^1, \ldots, w^m) \in \mathbb{R}^{S \times m}$ is a net trade if $\sum_{i=1}^m w^i = 0$. Page [19] and Nielsen [18] introduce two

equivalent notions, No Unbounded Arbitrage **NUBA** by Page [19] and Positive Semi-Independence by Nielsen [18].
(NUBA) If $(w^1, \dots, w^m) \in \mathbb{R}^{S \times m}$ is a net trade which satifies $w^i \in W^i$ for all i, then $w^i = 0$.
We can easily prove that $\bigcap_i S^i \neq \emptyset \Rightarrow$ **(NUBA)**. Page and Wooders [20] show that, if $L^i = \{0\}$ for all i, then $\bigcap_i S^i \neq \emptyset \Leftrightarrow$ **(NUBA)**.
Page, Wooders and Monteiro [21] propose the notion of *Inconsequential arbitrage*. Here is the definition of *Inconsequential arbitrage*.
The economy satisfies *Inconsequential arbitrage condition* (**IC**) if for any net trade $(w^1, w^2, \dots, w^m)$ with $w_i \in W^i$ for all i, and $(w^1, w^2, \dots, w^m)$ is the limit of $\lambda_n(x^1(n), x^2(n), \dots, x^m(n))$ with $(x^1(n), x^2(n), \dots, x^m(n)) \in \mathcal{A}$ and λ_n converges to zero when n tends to infinity, then there exists $\epsilon > 0$ such that for n sufficiently big we have $U^i(x^i(n) - \epsilon w^i) \geq U^i(x^i(n))$.
Under this condition, Page, Wooders and Monteiro prove that the set $\mathcal{U}$ is compact.
In Allouch et al. [1], we find this result

(NUBA) $\Rightarrow$ **(NA2)** $\Rightarrow$ **(IC)** $\Rightarrow \mathcal{U}$ is compact $\Rightarrow$ Existence of an equilibrium.

These conditions are equivalent if the economy has no half-line.

Since the condition stating that $\mathcal{U}$ is compact seems the "weakest" one, the *third category* of papers assumes it directly. See e.g. Dana et al. [10] for a finite dimension model. This condition is sufficient but not necessary for the existence of an equilibrium.

8.2 No arbitrage conditions and equilibrium on infinite dimension asset markets

The proofs that the no-arbitrage conditions given above imply the compactness either of $\mathcal{A}$ or of $\mathcal{U}$ use the property that the unit-sphere is compact, in finite dimension. In infinite dimension this property does not hold anymore. Brown and Werner [3], Dana and Le Van [7], Dana et al. [9] in infinite dimension models assume directly the compactness of $\mathcal{U}$. Chichilnisky and Heal [5] in L^2, assume that $\mathcal{A}$ is norm-bounded. Since this one is closed, it is weakly compact in L^2. The utility functions being norm-continuous and concave, are weakly upper semi-continuous. Hence $\mathcal{U}$ is compact.

9 Conclusion: Our contribution

1. We follow Dana and Le Van [8] and use the marginal utilities of the agents as potential no-arbitrage prices. This idea is very natural since at the optimum prices equal marginal utilities.

2. We can observe that in infinite dimension the papers mentioned above do not provide any prices no-arbitrage condition. In this paper we give a prices no-arbitrage condition which coincides with the well-known condition $\bigcap_i S^i \neq \emptyset$ when the number of states S is finite. As we wrote above, in finite dimension, this condition implies the compactness of $\mathcal{A}$. The proof of this result is based on the property that the unit sphere is compact in finite dimension. In infinite dimension, this property could not hold. Our no-arbitrage condition implies that $\mathcal{U}$ is bounded and the set $\mathcal{A}$ is bounded in $l^1(\pi)$.

3. The proofs to obtain the compactness of $\mathcal{U}$ when we impose **(NA2)** or **Inconsequential Arbitrage** use also the fact that the unit sphere is compact in finite dimension.

4. Therefore, to prove the compactness of $\mathcal{U}$ we have to find another way of proof. That is the one we propose in the present paper. This proof is an adaptation of the proofs in Cheng [4], Le Van and Truong Xuan [17], Daher et al. [6] and Ha-Huy, Le Van and Nguyen [15].[7] Our proof of the closedness of the utility set requires additional assumptions on the behavior of the consumers: (a) The beliefs of the agents satisfy Assumption **A0**. Thanks to **A0**, $\mathcal{A}$ is bounded in $l^1(\pi)$ and this result is crucial for our proof. (b) The reward functions of the agents satisfy *No Half-Line of the Same Type*. This assumption is also crucial (see section 7, point 7).

5. Summing up, in this paper, where the number of states is infinitely countable and the agents are risk averse, we propose a no-arbitrage condition on the prices restricted to the cone of the marginal utilities of the allocations and prove that, under the conditions we denote by *Beliefs Strong Equivalence* and the *No Half Line Condition of the same type*, our no-arbitrage price condition is sufficient for the existence of an equilibrium when the commodity space is $l^p, 1 \leq p < +\infty$. This no-arbitrage condition is necessary and sufficient for the existence of equilibrium when the total endowment is in l^∞. Moreover, it is equivalent to the compactness of the individually rational utility set.[8]

[7] We cannot use the proof in Cheng [4] because the author assumes the marginal utilities of the allocations cannot be linear when the allocations become large. We cannot use the one in Le Van and Truong Xuan [17] or Daher et al. [6] because one of their assumptions is not satisfied in our case (H4 in [17], S4 in [6]). We cannot use Ha-Huy et al. [15] since the authors assume $b^i = +\infty$ for any i, or $a^i = 0$ for any i. In our paper these quantities may be finite.

[8] Our paper covers the results in Cheng [4], Le Van and Truong Xuan [17] if we put their models in an infinitely countable setting, and also Ha-Huy et al. [15]. We are not aware of the existence of another paper in infinite dimension which introduces explicitly a prices no-arbitrage condition which is sufficient for the existence of an equilibrium and which may become also necessary.

10 Appendix 1

Proof of Lemma 2

- Assume w is a half line for agent i. Then it is useful. There exists $z \in l^p(\pi)$ which satisfies
$$\forall \lambda > 0, \ \sum_s \pi_s^i u^i(z_s + \lambda w_s) = \sum_s \pi_s^i u^i(z_s).$$
Take the derivative with respect to $\lambda > 0$. [9] We get $\sum_s \pi_s^i u^{i\prime}(z_s) w_s = 0, \ \forall \lambda > 0$. Hence (2).

- Conversely, let w be useful and satisfy (2) for some $z \in l^p(\pi)$. This is equivalent to
$$\frac{d}{d\lambda}\left(\sum_s \pi_s^i u^i(z_s + \lambda w_s)\right) = 0, \forall \lambda > 0.$$
Hence $\sum_s \pi_s^i u^i(z_s + \lambda w_s) = C, \forall \lambda > 0$. Let $\lambda \to 0$. We get $\sum_s \pi_s^i u^i(z_s) = C$. Hence
$$\sum_s \pi_s^i u^i(z_s + \lambda w_s) = \sum_s \pi_s^i u^i(z_s), \ \forall \lambda \geq 0.$$
That proves w is a half line.

■

Proof of Lemma 4
If $b^i = +\infty$ or $a^i = 0$ then agent i has no half line (Remark 2).
We will assume $a^i > 0$ and $b^i < +\infty$. Since
$$\sum_{s=1}^{\infty} \pi_s^i u'(x_s) \lambda w_s \geq \sum_s \pi_s^i u^i(x_s + \lambda w_s) - \sum_s \pi_s^i u^i(x_s) \geq 0,$$
we have for any half line direction $w \neq 0$
$$\sum_{s \in S_+} \pi_s^i u^{i\prime}(x_s) w_s + \sum_{s \in S_-} \pi_s^i u^{i\prime}(x_s) w_s \geq 0.$$
For $s \in S_+$ let $x_s \to +\infty$ and for $s \in S_-$ let $x_s \to -\infty$ we obtain
$$a^i \sum_{s \in S^1} \pi_s^i w_s + b^i \sum_{s \in S_-} \pi_s^i w_s \geq 0.$$

[9] We can take the derivative since the function u^i is well defined in $l^p(\pi)$. Its derivative is in $l^q(\pi)$ with $1/p + 1/q = 1$. See Le Van [16].

Suppose that for some z we have

$$\sum_{s\in S_+} \pi_s^i u^{i\prime}(z_s)w_s + \sum_{s\in S_-} \pi_s^i u^{i\prime}(z_s)w_s = 0. \tag{4}$$

We have that $S_+ \neq \emptyset$, $S_- \neq \emptyset$. Indeed, if $S_+ = \emptyset$ then we have from (4) a contradiction

$$0 = \sum_{s\in S_-} \pi_s^i u^{i\prime}(z_s)w_s < 0.$$

Similarly, if $S_- = \emptyset$, we have another contradiction

$$0 = \sum_{s\in S_+} \pi_s^i u^{i\prime}(z_s)w_s > 0.$$

Consider statement 1. If $w_s > 0$, then $u^{i\prime}(z_s)w_s > a^i w_s$ and if $w_s < 0$ we have $u^{i\prime}(z_s) \geq b^i w_s$. Hence if $w \neq 0$ we have:

$$0 = \sum_{s\in S_+} \pi_s^i u^{i\prime}(z_s)w_s + \sum_{s\in S_-} \pi_s^i u^{i\prime}(z_s)w_s > a^i \sum_{s\in S_+} \pi_s^i w_s + b^i \sum_{s\in S_-} \pi_s^i w_s \geq 0.$$

We arrive to a contradiction.
Consider statement 2. If $w_s > 0$, then $u^{i\prime}(z_s)w_s \geq a^i w_s$ and if $w_s < 0$ we have $u^{i\prime}(z_s) > b^i w_s$. Hence if $w \neq 0$ we have:

$$0 = \sum_{s\in S_+} \pi_s^i u^{i\prime}(z_s)w_s + \sum_{s\in S_-} \pi_s^i u^{i\prime}(z_s)w_s > a^i \sum_{s\in S_+} \pi_s^i w_s + b^i \sum_{s\in S_-} \pi_s^i w_s \geq 0.$$

We arrive to another contradiction. ■

11 Appendix 2

Proof of Proposition 1
1. Consider the case where every agent i satisfies $\forall x,\ u^{i\prime}(x) < b^i$. In this case there exists (z_s^i) which satisfies

$$\forall i, \forall s, a^i < \inf_s u^{i\prime}(z_s^i) < \sup_s u^{i\prime}(z_s^i) < b^i$$

Indeed, there exists $\eta > 0$ such that $u^{i\prime}(z_s^i) = u^{i\prime}(x_s^i)(1+\eta) < b^i, \forall i, \forall s$. Use the proof of proposition 1 in [15] to obtain that $\mathcal{U}$ is bounded.
2. Now consider the case where every agent i satisfies $\forall x,\ u^{i\prime}(x) > a^i$. In this case we can find $\eta' > 0$ small enough such that for all i

$$a^i < \inf_s u^{i\prime}(z_s^i) < \sup_s u^{i\prime}(z_s^i) < b^i$$

with $u^{i\prime}(z_s^i) =: u^{i\prime}(x_s^i)(1-\eta'), \forall s, \forall i$. Now (z_s^i) plays the role of (x_s^i) while (x_s^i) plays the role of (z_s^i) in point 1 with $1+\eta = \frac{1}{1-\eta'}$. ■

Lemma 5 *Suppose that $\mathcal{A}$ is bounded and $(v^1, v^2, \dots, v^m)$ is in the closure of $\mathcal{U}$. Suppose that there exists a sequence $\{x(n)\}_n \subset \mathcal{A}$ such that there exists i such that* $\lim_n U^i(x^i(n)) > v^i$, *and for all $j \neq i$,* $\lim_n U^j(x^j(n)) \geq v^j$. *Then* $(v^1, v^2, \dots, v^m) \in \mathcal{U}$.

Proof: See the proof of Claim 5, page 37, in [15]. ■

Proof of Proposition 2

Suppose that $\{x^i(n)\}_n \subset \mathcal{A}$ such that $\lim_n x^i(n) = v^i$. We prove that $(v^1, \dots, v^m) \in \mathcal{U}$. By Proposition 1, there exists $C > 0$ such that for all n, $\|x^i(n)\|_{l^1(\pi)} < C$, or $|x^i_s(n)| < C/\pi^i_s$ for all s.

For any i, any j, pick one s^i_j which satisfies:

$$\frac{\pi^j_{s^i_j}}{\pi^i_{s^j_i}} = \min_s \frac{\pi^j_s}{\pi^i_s}.$$

Take $\underline{M}$ such that for any i, j:

$$\underline{M} > \frac{2C}{\pi^i_{s^j_i}}.$$

Case 1. For any i, for any x we have $a^i < u^{i\prime}(x)$.

Fix $0 < \delta < 1$ such that $u^{i\prime}(\underline{M}) > \frac{a^i}{\delta}$ for all i. Let $\overline{M}$ satisfy: for any i:

$$0 < u^i(x) < (1 + a^i)x \text{ for all } x > \frac{(m-1)\overline{M} + |\overline{e}_s|}{1-\delta}$$

$$u^{i\prime}(\underline{M}) > \frac{u^{i\prime}(\overline{M})}{\delta} \text{ for all } i.$$

Let E^i_n denote the set

$$E^i_n = \left\{ s \mid x^i_s(n) > \frac{(m-1)\overline{M} + |\overline{e}_s|}{1-\delta} \right\}.$$

Observe that for all j, $s^j_i \notin E^i_n$ since $C > \sum_s \pi^i_s |x^i_s|$.

We consider two sub-cases:

1. $\exists\, i$: $\limsup_n \sum_{s \in E^i_n} \pi^i_s x^i_s(n) > 0$.

2. $\forall\, i$: $\lim_n \sum_{s \in E^i_n} \pi^i_s x^i_s(n) = 0$.

Consider the first sub-case. By Proposition 1, $\mathcal{A}$ is bounded in $l^1(\pi)$. We can suppose, without loss generality, that $\sum_{s=n}^{\infty} \pi^i_s |x^i_s(n)| \to c^i > 0$ when $n \to \infty$. This implies $\lim_n \sum_{s=n}^{\infty} \pi^i_s x^{i+}_s(n) - \lim_n \sum_{s=n}^{\infty} \pi^i_s x^{i-}_s(n) = c^i$. The limits of these two sums exist because $x^i \in l^1(\pi)$. We know that $\sum_{j \neq i} x^i_s(n) = \overline{e}_s - x^i_s(n)$. So, for every s, $\exists j$ such that $x^j_s(n) \leq -\frac{x^i_s(n) - |\overline{e}_s|}{m-1}$. Since there is a finite number of agents $j \neq i$, we can assume that, for simplicity, there exist i and j which satisfy two properties:

1. $\exists\ E_n^i \subset \mathbb{N} \cap \{s \geq n\}$, $x_s^i > 0$ for all $s \in E_n^i$ and

$$\lim_n \sum_{s \in E_n^i} \pi_s^i x_s^i(n) = c^i > 0.$$

2. For all $s \in E_n^i$

$$x_s^j(n) \leq -\frac{x_s^i(n) - |\overline{e}_s|}{m-1}.$$

To simplify the notation, denote $s_0 = s_i^j$.

Define

$$a = \delta \frac{\lim_{n\to\infty} \sum_{s \in E_n^i} \pi_s^i x_s^i(n)}{(m-1)\pi_{s_0}^i}.$$

Obviously $0 < a < C/\pi_{s_0}^i$.

We define the sequence $\{y^k(n)\}_n$ by

$$\begin{array}{rcl}
y_s^i(n) & = & x_s^i(n) - \dfrac{x_s^i(n) - |\overline{e}_s|}{m-1} + \overline{M} \text{ if } s \in E_n^i \\
y_s^i(n) & = & x_s^i(n) + a \text{ if } s = s_0 \\
y_s^i(n) & = & x_s^i(n) \text{ if } s \notin E_n^i \cup \{s_0\}.
\end{array}$$

and $\{y^j(n)\}_n$ by

$$\begin{array}{rcl}
y_s^j(n) & = & x_s^j(n) + \dfrac{x_s^j(n) - |\overline{e}_s|}{m-1} - \overline{M} \text{ if } s \in E_n^i \\
y_s^j(n) & = & x_s^j(n) - a \text{ if } s = s_0 \\
y_s^j(n) & = & x_s^j(n) \text{ if } s \notin E_n^i \cup \{s_0\}
\end{array}$$

and $y^k(n) = x^k(n)$ for all $k \neq i, j$.

Denote

$$d = \frac{\pi_{s_0}^j}{\pi_{s_0}^i} = \min_s \frac{\pi_s^j}{\pi_s^i}.$$

For all s, $\pi_s^j \geq d\pi_s^i$. We have :

$$U^i(y^i(n)) - U^i(x^i(n)) = \pi_{s_0}^i \left[u^i(y_{s_0}^i(n)) - u^i(x_{s_0}^i(n))\right] + \sum_{s \in E_n^i} \pi_s^i [u^i(y_s^i(n)) - u^i(x_s^i(n))]$$

$$\begin{array}{rl}
\geq & a\pi_{s_0}^i u^{i\prime}(x_{s_0}^i(n) + a) - \displaystyle\sum_{s \in E_n^i} \pi_s^i u^{i\prime}(x_s^i(n) - \frac{x_s^i(n) - |\overline{e}_s|}{m-1} + \overline{M}) \left[\frac{x_s^i(n) - |\overline{e}_s|}{m-1} - \overline{M}\right] \\
\geq & au^{i\prime}(\underline{M})\pi_{s_0}^i - u^{i\prime}(\overline{M}) \displaystyle\sum_{s \in E_n^i} \pi_s^i \left[\frac{x_s^i(n) - |\overline{e}_s|}{m-1} - \overline{M}\right] \\
\geq & au^{i\prime}(\underline{M})\pi_{s_0}^i - u^{i\prime}(\overline{M}) \displaystyle\sum_{s \in E_n^i} \pi_s^i \frac{x_s^i(n)}{m-1} + u^{i\prime}(\overline{M}) \sum_{s \in E_n^i} \pi_s^i \left[\frac{|\overline{e}_s|}{m-1} + \overline{M}\right]
\end{array}$$

$$
\begin{aligned}
&\geq \quad \pi^i_{s_0}\left[au^{i\prime}(\underline{M}) - \frac{u^{i\prime}(\overline{M})}{\delta}\frac{\delta\sum_{s\in E^i_n}\pi^i_s x^i_s(n)}{(m-1)\pi^i_{s_0}}\right] + u^{i\prime}(\overline{M})\sum_{s\in E^i_n}\pi^i_s\left[\frac{|\overline{e}_s|}{m-1}+\overline{M}\right]\\
&> 0 \quad \text{by letting } n\to\infty
\end{aligned}
$$

because $u^{i\prime}(\underline{M}) > u^{i\prime}(\overline{M})/\delta$.

Also:

$$
\begin{aligned}
U^j(y^j(n)) - U^j(x^j(n)) &\geq -au^{j\prime}(-\underline{M})\pi^j_{s_0} + u^{j\prime}(-\overline{M})\sum_{s\in E^i_n}\pi^j_s\frac{\delta x^i_s(n)}{m-1}\\
&= -au^{j\prime}(-\underline{M})\pi^j_{s_0} + u^{j\prime}(-\overline{M})\sum_{s\in E^i_n}\pi^j_s\frac{\delta x^i_s(n)}{m-1}\\
&= -au^{j\prime}(-\underline{M})d\pi^i_{s_0} + u^{j\prime}(-\overline{M})\sum_{s\in E^i_n}\pi^j_s\frac{\delta x^i_s(n)}{m-1}\\
&\geq -au^{j\prime}(-\underline{M})d\pi^i_{s_0} + u^{j\prime}(-\overline{M})\sum_{s\in E^i_n}d\pi^i_s\frac{\delta x^i_s(n)}{m-1}\\
&= d\left[-au^{j\prime}(-\underline{M})\pi^i_{s_0} + u^{j\prime}(-\overline{M})\delta\frac{\sum_{s\in E^i_n}\pi^i_s x^i_s(n)}{m-1}\right]\\
&\geq 0 \text{ by letting } n \text{ go to infinity.}
\end{aligned}
$$

because $u^{j\prime}(-\overline{M}) \geq u^{j\prime}(-\underline{M})$.

Applying Lemma 5, we then have $(v^1, v^2, \ldots, v^m) \in \mathcal{U}$. The set $\mathcal{U}$ is compact.

Consider the second sub-case, $\forall\ i$: $\lim_n \sum_{s\in E^i_n}\pi^i_s x^i_s(n) = 0$.

Since $\lim_n \sum_{s\in E^i_n}\pi^i_s x^i_s(n) = 0$ for all i, observe that $\lim_n \min_s\{s\in E^i_n\} = \infty$. Hence for n big enough $0 < u^i(x^i_s(n)) < (1+a^i)x^i_s(n)$ with $s\in E^i_n$, then $\lim_n \sum_{s\in E^i_n}\pi^i_s u^i(x^i_s(n)) = 0$, $\forall i$.

We will construct a sequence satisfying the properties:

1. $\liminf_n U^i(y^i(n)) = v^i$.
2. There exists d_1 and d_2 such that for all i, n, s, we have $|y^i_s(n)| \leq |d_1| + |d_2||e_s|$.

Fix i. For all $s\in E^i_n$, $\sum_{j\neq i} x^j_s(n) = \overline{e}_s - x^i_s(n) < 0$. So, $0 \leq \sum_{j\neq i} x^{j+}_s(n) < \sum_{j\neq i} x^{j-}_s(n)$. Then there exists a sequence $0 \leq z^j_s(n) \leq x^{j-}_s(n)$ such that $\sum_{j\neq i} z^j_s(n) = \sum_{j\neq i} x^{j-}_s(n) - \sum_{j\neq i} x^{j+}_s(n) = x^i_s(n) - \overline{e}_s$. We define the sequence

$\{y^i(n)\}_n$

$$\begin{array}{rcl} y^i_s(n) & = & \overline{e}_s \text{ if } s \in E^i_n \\ y^i_s(n) & = & x^i_s(n) \text{ if } s \notin E^i_n \\ y^j_s(n) & = & x^j_s(n) + z^j_s(n) \text{ if } s \in E^i_n \\ y^j_s(n) & = & x^j_s(n) \text{ if } s \notin E^i_n \end{array}$$

and we have:

$$U^i(y^i(n)) - U^i(x^i(n)) = \sum_{s \in E^i_n} \pi^i_s u^i(\overline{e}_s) - \sum_{s \in E^i_n} \pi^i_s u^i(x^i_s(n)) \to 0$$

because $\min_s\{s \in E^i_n\} \to \infty$ and $\sum_{s \in E^i_n} \pi^i_s u^i(x^i_s(n)) \to 0$.

For all $j \neq i$ and for all s, $y^j_s(n) \geq x^j_s(n)$, we have

$$U^j(y^j(n)) - U^j(x^j(n)) \geq 0$$

And, for all k, $\liminf_n U^k(y^k) \geq v^k$.

We can check that $\sum_{k=1}^m y^k(n) = \overline{e}$. We have also two observations.

1. $y^i_s(n) < \frac{(m-1)\overline{M}+|e_s|}{1-\delta}$ for all s.
2. if $x^j_s(n) > 0$, $x^{j-}(n) = 0$, this implies $z^j_s(n) = 0$, and $y^j_s(n) = x^j_s(n)$.

Fix $j \neq i$. By the similar arguments as above, we construct a new sequence $\{y'^k(n)\}_n$ in $\mathcal{A}$ which satisfies for all s, all n: $y'^j(n) < \frac{(m-1)\overline{M}+|e_s|}{1-\delta}$. By the second observation given above, $y'^i_s(n) < \overline{M}(m-1)/\delta + |e_s|$ for all s. For $k \neq i, j$, $\liminf_n U^k(y'^k(n)) = v^k$.

We continue for the agents different from i, j. And finally, by induction, we obtain a sequence $\{y^i(n))\}_n \subset \mathcal{A}$ satisfying: $\forall i, \forall n, \forall s$, $y^i_s(n) < \frac{(m-1)\overline{M}+|\overline{e}_s|}{1-\delta}$.

Fix an i. We have:

$$\begin{array}{rcl} \dfrac{(m-1)\overline{M}+|\overline{e}_s|}{1-\delta} > y^i_s(n) & = & \overline{e}_s - \displaystyle\sum_{j \neq i} y^j_s(n) \\ & > & \overline{e}_s - \displaystyle\sum_{j \neq i} \frac{(m-1)\overline{M}+|\overline{e}_s|}{1-\delta} \\ & > & -\dfrac{(m-1)^2\overline{M}}{1-\delta} - |\overline{e}_s| - \dfrac{(m-1)|\overline{e}_s|}{1-\delta}. \end{array}$$

So there exist d_1 and d_2 such that for all i, n, s, we have

$$|y^i_s(n)| < d_1 + d_2|\overline{e}_s|.$$

Moreover, this sequence satisfies $\liminf_n U^i(y^i(n)) \geq v^i$ for all i.

We know that for all $p \geq 1$, there exists $C_p > 0$ such that $(a+b)^p \leq C_p(a^p + b^p)$ for all pair of real numbers $a, b \geq 0$.

Since the sequence $\{y^i(n)\}_n$ belongs to a compact set of product topology, we can assume that $y^i(n)$ converges to y^i.

Consider the case $1 \leq p < +\infty$. Fix $\epsilon > 0$. Firstly, observe that for all $n, m \geq 1$, for all N:

$$\begin{aligned}
\sum_{s=N}^{m} \pi_s^i |y_s^i(n) - y_s^i(m)|^p &\leq C_p \sum_{s=N}^{\infty} \pi_s^i (|y_s^i(n)|^p + |y_s^i(m)|^p) \\
&\leq C_p \sum_{s=N}^{\infty} 2^p \pi_s^i \left[(d_1 + d_2|\overline{e}_s|)^p\right] \\
&\leq C_p^2 2^p \sum_{s=N}^{\infty} \pi_s^i (d_1^p + d_2^p |\overline{e}_s|^p)
\end{aligned}$$

which tends to 0 when N converges to infinity.

Fix $N > 0$ such that $\sum_{s=N}^{m} \pi_s^i |y_s^i(n) - y_s^i(m)| < \frac{\epsilon^p}{2}$ for all n, m. Since $y_s^i(n)$ converges to y_s^i, for n, m big enough we have

$$\sum_{s=1}^{N} \pi_s^i |y_s^i(n) - y_s^i(m)| < \frac{\epsilon^p}{2}.$$

So, for n, m big enough, we have $\|y^i(n) - y^i(m)\|_{l^p(\pi)} < \epsilon$. This implies the sequence $\{y_n^i\}$ is a Cauchy sequence. Hence $y^i \in l^p(\pi)$. Easily, we can prove that $U^i(y^i) \geq v^i$ for all i, from the upper-continuity of U^i on compact set of $l^1(\pi)$.

Consider the case $p = \infty$. Since $\overline{e} \in l^\infty(\pi)$, there exists $M > 0$ such that for all i, n, s, $|y_s^i(n)| < M$. This implies $|y_s^i| < M$ for all i, s. So, for all i, $y^i \in l^\infty(\pi)$.

We can easily prove that $U^i(y^i) \geq \liminf_n U^i(y^i(n)) \geq v^i$ for all i.

Thus, $(v^1, v^2, \ldots, v^m) \in \mathcal{U}$. The set $\mathcal{U}$ is compact.

Case 2. For any i, for any x we have $u^{i\prime}(x) < b^i$.

Fix $0 < \delta < 1$ such that $u^{j\prime}(-\underline{M}) < \frac{b^j}{\delta}$ for all j. Let $\overline{M}$ satisfy: for any i, j:

$$0 < u^i(x) < (1 + a^i)x \text{ for all } x > \frac{(m-1)\overline{M} + |\overline{e}_s|}{1 - \delta},$$

$$u^{j\prime}(-\underline{M}) < \frac{u^{j\prime}(-\overline{M})}{\delta} \text{ for all } j.$$

Let E_n^i denote the set

$$E_n^i = \left\{ s \mid x_s^i(n) > \frac{(m-1)\overline{M} + |\overline{e}_s|}{1 - \delta} \right\}.$$

Observe that for all j, $s_i^j \notin E_n^i$ since $C > \sum_s \pi_s^i |x_s^i|$.

As in Case 1, we consider two sub-cases:

1. $\exists\ i$: $\limsup_n \sum_{s \in E_n^i} \pi_s^i x_s^i(n) > 0$.

2. $\forall\ i$: $\lim_n \sum_{s \in E_n^i} \pi_s^i x_s^i(n) = 0$.

In the second sub-case, using the arguments of Case 1, we have the compactness of $\mathcal{U}$. We have to consider only the first sub-case. Using the same arguments as in Case 1, we can prove the existence of i, j such that:

1. $\exists\ E_n^i \subset \mathbb{N} \cap \{s \geq n\}$, $x_s^i > 0$ for all $s \in E_n^i$ and
$$\lim_n \sum_{s \in E_n^i} \pi_s^i x_s^i(n) = c^i > 0.$$

2. For all $s \in E_n^i$
$$x_s^j(n) \leq -\frac{x_s^i(n) - |\overline{e}_s|}{m-1}.$$

To simplify the notation, denote $s_0 = s_i^j$.

Define
$$a = \frac{\lim_{n \to \infty} \sum_{s \in E_n^i} \pi_s^i x_s^i(n)}{(m-1)\pi_{s_0}^i}.$$

Obviously $0 < a < C/\pi_{s_0}^i$.

We define the sequence $\{y^k(n)\}_n$ by

$$\begin{aligned}
y_s^i(n) &= x_s^i(n) - \frac{x_s^i(n) - |\overline{e}_s|}{m-1} + \overline{M} \text{ if } s \in E_n^i \\
y_s^i(n) &= x_s^i(n) + a \text{ if } s = s_0 \\
y_s^i(n) &= x_s^i(n) \text{ if } s \notin E_n^i \cup \{s_0\}.
\end{aligned}$$

and $\{y^j(n)\}_n$ by

$$\begin{aligned}
y_s^j(n) &= x_s^j(n) + \frac{x_s^j(n) - |\overline{e}_s|}{m-1} - \overline{M} \text{ if } s \in E_n^i \\
y_s^j(n) &= x_s^j(n) - a \text{ if } s = s_0 \\
y_s^j(n) &= x_s^j(n) \text{ if } s \notin E_n^i \cup \{s_0\}
\end{aligned}$$

and $y^k(n) = x^k(n)$ for all $k \neq i, j$.

Denote
$$d = \frac{\pi_{s_0}^j}{\pi_{s_0}^i} = \min_s \frac{\pi_s^j}{\pi_s^i}.$$

For all s, $\pi^j_s \geq d\pi^i_s$. Using exactly the same calculus of Case 1, we get

$$U^i(y^i(n)) - U^i(x^i(n)) = \pi^i_{s_0}\left[u^i(y^i_{s_0}(n)) - u^i(x^i_{s_0}(n))\right] + \sum_{s\in E^i_n} \pi^i_s[u^i(y^i_s(n)) - u^i(x^i_s(n))]$$

$$\begin{aligned}
&\geq \quad \pi^i_{s_0}\left[au^{i\prime}(\underline{M}) - u^{i\prime}(\overline{M})\frac{\sum_{s\in E^i_n}\pi^i_s x^i_s(n)}{(m-1)\pi^i_{s_0}}\right] + u^{i\prime}(\overline{M})\sum_{s\in E^i_n}\pi^i_s\left[\frac{|\overline{e}_s|}{m-1} + \overline{M}\right] \\
&\geq 0 \quad \text{by letting } n \to \infty,
\end{aligned}$$

because $u^{i\prime}(\underline{M}) \geq u^{i\prime}(\overline{M})$.

Also:

$$\begin{aligned}
U^j(y^j(n)) - U^j(x^j(n)) &\geq -au^{j\prime}(-\underline{M})\pi^j_{s_0} + u^{j\prime}(-\overline{M})\sum_{s\in E^i_n}\pi^j_s\frac{\delta x^i_s(n)}{m-1} \\
&= d\left[-au^{j\prime}(-\underline{M})\pi^i_{s_0} + u^{j\prime}(-\overline{M})\delta\frac{\sum_{s\in E^i_n}\pi^i_s x^i_s(n)}{m-1}\right] \\
&> 0 \text{ by letting } n \text{ go to infinity.}
\end{aligned}$$

because $\delta u^{j\prime}(-\overline{M}) > u^{j\prime}(-\underline{M})$.

Applying Lemma 5, we then have $(v^1, v^2, \ldots, v^m) \in \mathcal{U}$. We have proved that $\mathcal{U}$ is compact.

■

12 Appendix 3

Proof of Theorem 2 We have just to prove the converse: if there exists an equilibrium then (**NA**) holds, and hence $\mathcal{U}$ is compact.

The equilibrium allocation (x^{i*}) solves the problem:

$$\max \sum_{s=1}^{\infty}\pi^i_s u^i(x^i_s)$$
$$\text{s.t. } \sum_{s=1}^{\infty} p^*_s x^i_s = \sum_{s=1}^{\infty} p^*_s e^i_s$$

From Theorem V.3.1, page 91, in Arrow-Hurwicz-Uzawa [2], for any i, there exists ζ_i s.t.

$$\sum_{s=1}^{\infty}\pi^i_s u^i(x^{*i}_s) - \zeta_i\sum_{s=1}^{\infty}p^*_s x^{*i}_s \geq \sum_{s=1}^{\infty}\pi^i_s u^i(x_s) - \zeta_i\sum_{s=1}^{\infty}p^*_s x_s$$

for any $x \in X^i$. Hence $\pi^i_s u^{i\prime}(x^{*i}_s) = \zeta_i p^*_s, \forall i$. Since u^i is strictly increasing $\zeta_i > 0$. Let $\lambda_i = \frac{1}{\zeta_i}$, $p_s = \lambda_i \pi^i_s u^{i\prime}(x^{i*}_s)$ for all i, s. We have for all i, j, s, $\lambda_i \pi^i_s u^{i\prime}(x^{*i}_s) = \lambda_j \pi^j_s u^{j\prime}(x^{*j}_s)$.

Consider the case $a^i < u^{i\prime}(x)$ for all i, x. We will prove that $\inf_s u^{i\prime}(x^{*i}_s) > a^i$ for all i. We always have

$$\frac{\pi^i_s}{\pi^j_s} = \frac{\lambda_j u^{j\prime}(x^{*j}_s)}{\lambda_i u^{i\prime}(x^{*i}_s)} < \frac{\lambda_j b^j}{\lambda_i a^i}. \tag{5}$$

If the claim is wrong, then there exists i and subsequence s_k such that $x^{*i}_{s_k}$ converges to $+\infty$. Since $e \in l^\infty(\pi)$, there exists j such that $x^{*j}_{s_k}$ converges to $-\infty$. In this case since we have **A0**

$$\max_s \frac{\pi^i_s}{\pi^j_s} = \sup_s \frac{\pi^i_s}{\pi^j_s} = \frac{\lambda_j b^j}{\lambda_i a^i},$$

contradicting (5).

The same arguments apply when $u^{i\prime}(x) < b^i$, for all i, x. We have prove that condition **(NA)** is satisfied. ■

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