



**HAL**  
open science

## Arbitrage and equilibrium in economies with short-selling and ambiguity

Thai Ha-Huy, Cuong Le Van, Cuong Tran-Viet

► **To cite this version:**

Thai Ha-Huy, Cuong Le Van, Cuong Tran-Viet. Arbitrage and equilibrium in economies with short-selling and ambiguity. *Journal of Mathematical Economics*, 2018, 76, pp.95-100. 10.1016/j.jmateco.2018.01.004 . hal-02877948

**HAL Id: hal-02877948**

**<https://hal.science/hal-02877948>**

Submitted on 19 Jun 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Arbitrage and equilibrium in economies with short-selling and ambiguity \*

Thai Ha-Huy<sup>†</sup>, Cuong Le Van<sup>‡</sup>, Cuong Tran-Viet<sup>§</sup>

August 23, 2017

## Abstract

We consider a model with a finite number of states of nature where short sells are allowed. We present a notion of no-arbitrage price weaker than the one of Werner [23] that we call no-arbitrage price. We prove that in the case of separable risk averse utility functions, the existence of one common weak no-arbitrage price is equivalent to the existence of equilibrium.

**Keywords:** asset market equilibrium, individually rational attainable allocations, individually rational utility set, no-arbitrage prices, no-arbitrage condition.

**JEL Classification:** C62, D50, D81, D84, G1.

## 1 Introduction

Equilibrium conditions on financial markets differ with the ones on good market when short-selling is accepted. This assumption makes useless traditional techniques using fixed point theory. In the finite dimension case, there is a huge literature on well-known conditions called "no arbitrage conditions". These conditions in general imply the compactness of the allocations set or the utilities set. We can classify them in three main categories.

---

\*The authors are grateful to two referees and the associate editor for their thoughtful remarks, questions and advices which have helped them to improve the article. Thai Ha-Huy would like to thank the MME-DII Labex for supports.

<sup>†</sup>EPEE, University of Evry-val-d'Essonne. E-mail: thai.hahuy@univ-evry.fr

<sup>‡</sup>IPAG Business School, Paris School of Economics, CNRS, TIMAS. E-mail: Cuong.Le-Van@univ-paris1.fr

<sup>§</sup>University of Paris 1 Panthéon Sorbonne. Email: Viet-Cuong.Tran@univ-paris1.fr

The first category is based on conditions on net trade, for example Hart [17], Page [19], Nielsen [18], Page and Wooders [20], Allouch [1], Page, Wooders and Monteiro [21]. We define *Individual arbitrage opportunity* as the set of directions along which the agent wants to trade with infinite quantities. In the case the agents disagree too much, some agents can make an arbitrage which is "an opportunity with a mutually compatible set of net trades which are utility non-decreasing and, at most, costless to make" (Hart [17]). Taking the fact that this opportunity can be repeated indefinitely, equilibrium may not exist. The Weak-no-market-arbitrage WNMA requires that that all mutually compatible net trades which are non-decreasing be useless. Page [19] proposes the no-unbounded-arbitrage NUBA, a situation in which there is no group of agents can make mutually compatible, unbounded and utility increasing trades. In 2000, Page, Wooders and Monteiro [21], introduce *Inconsequential arbitrage* condition to ensure the existence for an equilibrium.

The second category is based on conditions on prices, for example Green [13], Grandmont [11], [12], Hammond [16] and Werner [23]. These authors define No-arbitrage price as element in the strictly positive dual of the set of useful vectors. If the intersection of No-arbitrage price cones of all agents is non empty (existence of No-arbitrage-price-system NAPS), then there exists a general equilibrium.

The third category includes authors, like Brown and Werner [2], Dana, Le Van, Magnien [5], who assume the compactness of attainable utility set to ensure the existence of equilibrium.

Obviously, we can wonder whether we can have equivalent conditions between the existence of general equilibrium and these no arbitrage conditions when the utility functions are not strictly concave. Unfortunately, the answer is no. In this paper, we give an example in which NAPS and NUBA, even WNMA conditions are violated, but a general equilibrium does exist (Subsection 3.3).

In 2010, Dana and Le Van [6], by considering the relationships between the agents beliefs and risk when there is ambiguity, propose to use the set of derivatives of the utility function as no-arbitrage prices set. By using this 'trick', they give a description of weak no-arbitrage prices and useful vectors. Furthermore, they give an equivalence between non-emptiness of the intersection of interiors of no-arbitrage price cones and NUBA condition, or non-emptiness of the intersection of relative interiors of no-arbitrage prices cones and WNMA condition. Hence, if this intersection is non empty, existence of a general equilibrium is ensured.

In this paper, we reconsider the equilibrium theory of assets with short-selling when there is risk and ambiguity. The agents have maximin expected utility functions. They are not only risk averse but also ambiguity averse. We

suppose the set of beliefs of each agent is polyhedral, i.e. it is the convex hull of a finite number of strictly positive probabilities. In particular when the agent has a linear utility with one prior, our no-arbitrage condition implies existence of equilibrium while the one by Dana and Le Van [6] cannot. Moreover, we give an example where NUBA, WNMA, no-arbitrage la Dana and Le Van are not satisfied. However, there exists an equilibrium because our condition holds.

Using the notion proposed by Dana and Le Van [6], that we call weak no-arbitrage prices, we prove the equivalence between existence of a general equilibrium and non-emptiness of the intersection of no-arbitrage prices cones. It is easy to prove that any equilibrium price is a weak no-arbitrage price. The difficulty is to prove the converse.

To the best of our knowledge, when the utility functions are not strictly concave, such a result does not exist in the literature<sup>1</sup>.

Our paper is organized as follows. In Section 2 we recall several well-known conditions (No unbounded arbitrage - NUBA, or Weak no market arbitrage - WNMA, or No arbitrage price system - NAPS) for the existence of an equilibrium in a general framework. In Section 3, we consider an economy in which any agent  $i$  has a maximin expected utility function. As in Gilboa and Schmeidler [14], each agent faces a set of subjective probabilities. Here we assume that this set is the convex hull of a finite number of strictly positive probabilities. We introduce the cone of common weak no-arbitrage prices and state the equivalence between existence of equilibrium and existence of a weak no-arbitrage price common to all the agents. In Subsection 3.3, we give an example of economy which does not satisfy either NUBA or WNMA or NAPS and however has an equilibrium since it satisfies our no-arbitrage condition.

## 2 Existence of equilibrium: the general case

We use Hart's model where the market is complete, the choice of portfolio is equivalent to the choice of wealth. Consider an unbounded exchange economy  $\mathcal{E}$  with  $S$  states of nature and  $m$  agents. Since short-sales are allowed, we suppose the consumption set  $X_i$  of agent  $i$  is  $\mathbb{R}^S$ , for any  $i$ . Agent  $i$  has an endowment  $e^i \in \mathbb{R}^S$  and a concave utility function  $U^i$  from  $\mathbb{R}^S$  into  $\mathbb{R}$ . We assume that for any  $i$ ,  $U^i$  is strictly increasing and concave.

For a subset  $X \subset \mathbb{R}^S$ , denote  $\text{int}X$  the interior of  $X$ ,  $X^0$  the polar of  $X$  where  $X^0 = \{p \in \mathbb{R}^S \mid p \cdot x \leq 0, \forall x \in X\}$  and  $X^{00} = (X^0)^0$ . It is well known that if  $X$  is closed, convex and contains the origin then  $X^{00} = X$ .

For  $x \in \mathbb{R}^S$ , agent  $i$ 's weak preferred set at  $x$  is

$$\widehat{P}^i(x) = \{y \in \mathbb{R}^S \mid U^i(y) \geq U^i(x)\}.$$

---

<sup>1</sup>This model is equivalent to a model where any agent has a finite number of priors

Let  $R^i(x)$  be recession cone of  $\widehat{P}^i(x)$  (see Rockaffellar [22]). The set  $R^i(x)$  is called the set of *useful vectors* for  $U^i$  is given as

$$R^i(x) = \{w \in \mathbb{R}^S \mid U^i(x + \lambda w) \geq U^i(x), \text{ for all } \lambda \geq 0\}.$$

It is easy to check that  $R^i(x)$  is a closed convex cone.

The linear space of  $i$  is defined by

$$L^i(x) = \{w \in \mathbb{R}^S \mid U^i(x + \lambda w) = U^i(x), \text{ for all } \lambda \in \mathbb{R}\} = R^i(x) \cap -R^i(x).$$

Elements in  $L^i(x)$  will be called *useless vectors* at  $x$ . From the concavity of function  $U^i$ ,  $R^i(x)$  and  $L^i(x)$  do not depend on  $x$ . Let us set  $R^i = R^i(e^i)$ ,  $L^i = L^i(e^i)$ . Denote by  $L^{i\perp}$  the orthogonal space of  $L^i$ .

Since  $U^i$  is strictly increasing, we have  $\mathbb{R}_{++}^2 \subset R^i$ .

Let us first recall the *no-unbounded-arbitrage condition* denoted from now on by NUBA introduced by Page [19] (see also Page-Wooders [20]), which requires non-existence of an unbounded set of mutually compatible net trades that are utility non decreasing.

**Definition 1** *The economy satisfies the NUBA condition if  $\sum_{i=1}^m w^i = 0$  and  $w^i \in R^i$  for all  $i$  implies  $w^i = 0$  for all  $i$ .*

A weaker condition, called the *weak-no-market-arbitrage condition* (WNMA), is introduced by Hart [17]. This condition requires that all mutually compatible net trades which are utility non-decreasing be useless. More precisely,

**Definition 2** *The economy satisfies the WNMA condition if  $\sum_{i=1}^m w^i = 0$  and  $w^i \in R^i$  for all  $i$  implies  $w^i \in L^i$  for all  $i$ .*

We recall the definitions of the attainable allocations set and the individually rational utility set.

**Definition 3** *1. The individually rational attainable allocations set  $\mathcal{A}$  is defined by*

$$\mathcal{A} = \{(x^1, x^2, \dots, x^m) \in (\mathbb{R}^S)^m \mid \sum_{i=1}^m x^i = \sum_{i=1}^m e^i \text{ and } U^i(x^i) \geq U^i(e^i) \text{ for all } i\}.$$

*2. The individually rational utility set  $\mathcal{U}$  is defined by*

$$\mathcal{U} = \{(v^1, v^2, \dots, v^m) \in \mathbb{R}^m \mid \exists x \in \mathcal{A} \text{ s.t. } U^i(e^i) \leq v^i \leq U^i(x^i) \text{ for all } i\}.$$

In [21], Page, Wooders and Monteiro presented the notion of *Inconsequential arbitrage*, which will be widely used in this paper.

**Definition 4** *The economy satisfies Inconsequential arbitrage condition if for any  $(w^1, w^2, \dots, w^m)$  with  $w^i \in \mathbb{R}^i$  for all  $i$ ,  $\sum_{i=1}^m w^i = 0$  and  $(w^1, w^2, \dots, w^m)$  is the limit of  $\lambda_n(x^1(n), x^2(n), \dots, x^m(n))$  with  $(x^1(n), x^2(n), \dots, x^m(n)) \in \mathcal{A}$  and  $\lambda_n$  converges to zero when  $n$  tends to infinity, there exists  $\epsilon > 0$  such that for  $n$  sufficiently big we have  $U^i(x^i(n) - \epsilon w^i) \geq U^i(x^i(n))$ .*

**Definition 5** *An equilibrium is a list  $((x^{i*})_{i=1, \dots, m}, p^*)$  such that  $p^* \in \mathbb{R}_+^S \setminus \{0\}$  and*

(a) *For any  $i$ ,  $U^i(x) > U^i(x^{i*}) \Rightarrow p^* \cdot x > p^* \cdot e^i$ .*

(b)  $\sum_{i=1}^m x^{i*} = \sum_{i=1}^m e^i$ .

**Definition 6** *An quasi equilibrium is a list  $((x^{i*})_{i=1, \dots, m}, p^*)$  such that  $p^* \in \mathbb{R}_+^S \setminus \{0\}$  and*

(a) *For any  $i$ ,  $U^i(x) > U^i(x^{i*}) \Rightarrow p^* \cdot x \geq p^* \cdot e^i$ .*

(b)  $\sum_{i=1}^m x^{i*} = \sum_{i=1}^m e^i$ .

We recall the following property:

$$\text{NUBA} \Rightarrow \text{WNMA} \Rightarrow \text{Inconsequential arbitrage} \Rightarrow \mathcal{U} \text{ is compact.}$$

**Theorem 1** *If  $\mathcal{U}$  is compact then there exists an equilibrium.*

**Proof:** See Dana, Le Van and Magnien [7] for the existence of quasi-equilibrium. Since short sales are allowed, in our model quasi equilibrium is also equilibrium. For this result, see Florenzano [9]. ■

### 3 Existence of equilibrium with maximin expected utility

We consider now the economy in which any agent  $i$  has a maximin expected utility function. Define  $\Delta = \{\pi \in \mathbb{R}_+^S \text{ such that } \sum_{s=1}^S \pi_s = 1\}$ . The agents are ambiguous in the probabilities of the outcomes and they are averse at ambiguity. As in Gilboa and Schmeidler [14], each agent faces a set of subjective probabilities  $\Delta^i \subset \Delta = \{\pi \in \mathbb{R}_+^S \text{ such that } \sum_{s=1}^S \pi_s = 1\}$  and their utility functions take the form

$$U^i(x) = \inf_{\pi \in \Delta^i} \sum_{s=1}^m \pi_s u^i(x_s),$$

where  $u^i : \mathbb{R} \rightarrow \mathbb{R}$  is a concave, strictly increasing, differentiable function <sup>2</sup>. Observe that the function  $U^i$  is concave<sup>3</sup>. Since taking infimum on  $\Delta^i$  is equiv-

<sup>2</sup>For the sake of simplicity, we assume the differentiability. The results do not change for general case with sub-differentials, but the calculus become tedious.

<sup>3</sup>The infimum of a family of concave functions is also concave. See Rockafellar [22].

alent to taking infimum on its convex hull, without loss of generality, suppose that  $\Delta^i$  is convex.

### 3.1 Characterization of useful vectors

The following lemma characterizes the useful vectors set of agent  $i$ .

Denote <sup>4</sup>

$$\begin{aligned} a^i &= \inf_{z \in \mathbb{R}} u^{i'}(z) = u^{i'}(+\infty) \\ b^i &= \sup_{z \in \mathbb{R}} u^{i'}(z) = u^{i'}(-\infty).^5 \end{aligned}$$

**Lemma 1** *The vector  $w \in \mathbb{R}^S$  is useful for agent  $i$  if and only if for any  $x \in \mathbb{R}^S$ , any  $\pi \in \Delta^i$  we have*

$$\sum_{s=1}^S \pi_s u^{i'}(x_s) w_s \geq 0.$$

**Proof:** See Proposition 2 in Dana and Le Van [6]. ■

For each vector  $w \in \mathbb{R}^S$ , define  $S_+(w) = \{s \text{ such that } w_s > 0\}$ , and  $S_-(w) = \{s \text{ such that } w_s < 0\}$ . The following proposition is a direct consequence of Lemma 1.

**Proposition 1** *The vector  $w$  is useful for agent  $i$ , if and only if, for any  $\pi \in \Delta^i$  we have*

$$a^i \sum_{s \in S_+(w)} \pi_s w_s + b^i \sum_{s \in S_-(w)} \pi_s w_s \geq 0.$$

**Proof:** See Appendix. ■

**Corollary 1** *If  $a^i = 0$  or  $b^i = \infty$ , then  $R^i = \mathbb{R}_+^S$ .*

### 3.2 Weak No-Arbitrage prices and existence of equilibrium

Following Dana and Le Van [6], we define the set of weak no-arbitrage prices.

$P^i = \{p \in \mathbb{R}^S \text{ such that } \exists \lambda > 0, x \in \mathbb{R}^S, \pi \in \Delta^i, \text{ satisfying } p_s = \lambda \pi_s u^{i'}(x_s) \forall s = 1, 2, \dots, S\}$ .

**Lemma 2** *For all  $i$ ,  $P^i$  is a convex cone.*

**Proof:** See Dana and Le Van [6]. ■

We have two cases.

*Case 1:* For all  $i$  except at most one agent, for any  $z \in \mathbb{R}$ , either  $a^i < u^{i'}(z)$  or  $b^i > u^{i'}(z)$ .

---

<sup>4</sup>We rule out the case  $a^i = b^i = 0$  which is not interesting. The utility function is constant in this case

**Proposition 2** *Suppose that for all  $i$  except at most one agent, we have either  $a^i < u^i(z)$ ,  $\forall z \in \mathbb{R}$ , or  $b^i > u^i(z)$ ,  $\forall z \in \mathbb{R}$ , then we have:*

$$\bigcap_{i=1}^m P^i \neq \emptyset \Leftrightarrow \text{NUBA condition} \Leftrightarrow \mathcal{U} \text{ is compact} \Leftrightarrow \text{there exists equilibrium.}$$

**Proof:** See Appendix. ■

*Case 2:* Now we consider the case where for some  $i$ , the utility function  $u^i$  becomes affine when the consumption is large enough, i.e. there exists  $\bar{z}^i$  such that

$$u^i(z) = a^i \text{ for } z \geq \bar{z}^i \text{ and } u^i(x) = b^i \text{ for } z \leq -\bar{z}^i.$$

**Lemma 3** *Fix  $x \in \mathbb{R}^S$ . Then  $\partial U^i(x)$  is the set*

$$Q = \{p : p = (\pi_1 u^i(x_1), \pi_2 u^i(x_2), \dots, \pi_S u^i(x_S))\}$$

where  $\pi \in \Delta^i$  satisfies  $U^i(x) = \sum_{s=1}^S \pi_s u^i(x_s)$ .

**Proof:** See Appendix. ■

We now state our main result.

**Theorem 2** *Suppose that each probabilities set  $\Delta^i$  is a convex hull of the set of  $M^i$  points, i.e.  $\Delta^i = \text{conv}\{\pi_0^i, \pi_1^i, \dots, \pi_{M^i}^i\}$ . Then:*

$$\bigcap_{i=1}^m P^i \neq \emptyset \Leftrightarrow \mathcal{U} \text{ is compact} \Leftrightarrow \text{there exists a general equilibrium.}$$

**Proof:** See Appendix. ■

We can give here an intuition about the hypothesis of Theorem 2. The ambiguity problems comes from the lack of information of economic agents. By imposing the assumption that  $\Delta^i$  is convex hull of a set of finite number of priors, we want to prove that in the case the agents dispose "not-too-bad" information, or the ambiguous situation is not extreme, then the existence of general equilibrium is equivalent to the existence of an weak no-arbitrage price, which is defined as a vector of marginal utilities. This results confirms the intuition that the prices are defined by marginal utilities.

The following Corollary is the direct consequence of Theorem 2.

**Corollary 2** *If for any  $i$ ,  $\Delta^i$  is a singleton, i.e  $\Delta^i = \{\pi^i\}$ , then  $P^i = \{p \in \mathbb{R}^S \text{ such that there exists } \lambda > 0, x \in \mathbb{R}^S : p_s = \lambda \pi_s^i u^i(x_s), \text{ for any } 1 \leq s \leq S\}$ .*

*We also have*

$$\bigcap_{i=1}^m P^i \neq \emptyset \Leftrightarrow \mathcal{U} \text{ is compact} \Leftrightarrow \text{there exists general equilibrium.}$$



### 3.3 Example

We present here an example in which the weak no-arbitrage prices cones are closed, their intersection is non empty, the model does not satisfy NAPS, NUBA or WNMA conditions, but there exists an equilibrium.

We consider an economy with two agents, the number of states is  $S = 2$ . The belief of agent 1 is represented by the probability  $\pi_1^1 = \pi_2^1 = \frac{1}{2}$ . The belief of agent 2 is  $\pi_1^2 = \frac{1}{3}, \pi_2^2 = \frac{2}{3}$ . Their endowments are 0. Their utility functions are defined as follows:

$$u^1(x) = \begin{cases} \ln(x) & \text{if } x \in [1/3, 1/2] \\ 2x - 1 - \ln 2 & \text{if } x \geq 1/2 \\ 3x - 1 - \ln 3 & \text{if } x \leq 1/3 \end{cases}$$

$$u^2(x) = \begin{cases} \ln(x) & \text{if } x \in [1/3, 4/9] \\ \frac{9}{4}x - 1 + \ln(\frac{4}{9}) & \text{if } x \geq \frac{4}{9} \\ 3x - 1 - \ln 3 & \text{if } x \leq \frac{1}{3} \end{cases}$$

We have  $u^{1'}(+\infty) = 2, u^{1'}(-\infty) = 3$  and  $u^{2'}(+\infty) = \frac{9}{4}, u^{2'}(-\infty) = 3$ . Therefore, the cone of no-arbitrage prices of agent 1 is  $P^1 = \{\lambda(\zeta_1, \zeta_2)\}_{\lambda > 0}$  with  $1 \leq \zeta_1 \leq \frac{3}{2}, 1 \leq \zeta_2 \leq \frac{3}{2}$ . The one of agent 2 is  $P^2 = \{\lambda(\zeta_1, \zeta_2)\}_{\lambda > 0}$  with  $\frac{3}{4} \leq \zeta_1 \leq 1, \frac{3}{2} \leq \zeta_2 \leq 2$ .

The set of common weak no-arbitrage prices is the intersection of the two cones  $P^1 \cap P^2 = \{\lambda(1, \frac{3}{2})\}_{\lambda > 0}$ . If  $S^1, S^2$  are the interiors of  $P^1$  and  $P^2$ , then  $S^1 \cap S^2 = \emptyset$ .

Our economy does not satisfies either NUBA or WNMA conditions. Indeed, consider the useful vector  $w^1 = (1, -\frac{2}{3})$  of agent 1, the useful vector  $w^2 = (-1, \frac{2}{3})$  of agent 2. We obtain that  $w^1 + w^2 = 0$ . That means NUBA is not satisfied. But  $-w^1, -w^2$  are not useful vectors, hence are not in the linearity spaces. That means WNMA does not hold. However, from our main Theorem 2, an equilibrium exists in this model. Calculus give one equilibrium allocation as  $x_1^{*1} = 1, x_2^{*1} = -2/3, x_1^{*2} = -1, x_2^{*2} = 2/3$ , with equilibrium prices  $p_1^* = 1, p_2^* = 3/2$ .

## 4 Appendix

### 4.1 Proof of Proposition 1

Suppose that  $w$  is a useful vector of agent  $i$ . Using Lemma 1, and letting  $x_s$  converge to  $+\infty$  when  $s \in S_+(w)$ , and  $x_s$  converge to  $-\infty$  when  $s \in S_-(w)$ , we obtain the inequality.

Now, we prove the converse. For any  $x \in \mathbb{R}^S$  we have  $a^i \leq u^i(x_s) \leq b^i$ . This implies  $\sum_{s=1}^S \pi_s u^i(x_s) w_s \geq a^i \sum_{s \in S_+(w)} \pi_s w_s + b^i \sum_{s \in S_-(w)} \pi_s w_s \geq 0$ . From Lemma 1, the vector  $w$  is useful for agent  $i$ . ■

## 4.2 Proof of Proposition 2

Suppose that for any  $i \neq i_0$ , we have either  $a^i < u^{i'}(z) \forall z$ , or  $b^i > u^{i'}(z), \forall z$ . Using Dana and Le Van [6], for any  $i \neq i_0$ ,  $P^i$  is open. We can hence chose  $p \in \bigcap_{i \neq i_0} P^i$ , and  $p \in \text{int} P^{i_0}$ . For details, see Rockafellar [22]. Applying Dana and Le Van [6], the Proposition is proved. ■

## 4.3 Proof of Lemma 3

First observe  $Q \subset \partial U^i(x)$ . Conversely, from Clarke [4],  $\partial U^i(x)$  is the convex hull of the derivatives  $(\tilde{\pi}_1 u^{i'}(x_1), \tilde{\pi}_2 u^{i'}(x_2), \dots, \tilde{\pi}_S u^{i'}(x_S))$ , the probabilities  $(\tilde{\pi})$  satisfy  $U^i(x) = \sum_{s=1}^S \tilde{\pi}_s u^i(x_s)$ . Hence  $\partial U^i(x) \subset Q$ .<sup>6</sup> ■

## 4.4 Proof of Theorem 2

Define  $\tilde{\Delta}^i = \{\pi_0^i, \pi_1^i, \pi_2^i, \dots, \pi_{M^i}^i\}$ . Observe that actually  $U^i(x) = \min_{\pi \in \tilde{\Delta}^i} \sum_{s=1}^S \pi_s u^i(x_s)$ .

Denote by  $\mathcal{I}$  the set of agents  $i$  such that there exists  $z^i: u^i(z) = a^i$  for any  $z > z^i$  and  $u^i(z) = b^i$  for any  $z < -z^i$ . For  $i \in \mathcal{I}$ , the weak no-arbitrage prices cone  $P^i$  is closed. For  $i \notin \mathcal{I}$ , the weak no-arbitrage prices cone  $P^i$  is open.

(a) Suppose that there exists general equilibrium,  $(p^*, x^*)$ . Using the same arguments in Ha-Huy and Le Van [15], we have  $\bigcap_i P^i \neq \emptyset$ .

(b) We prove the converse. Take  $\bar{p} \in \bigcap_i P^i$ . For each  $i$ , there exists  $\lambda_i > 0$ ,  $\bar{x}^i \in \mathbb{R}^S$  and  $\pi_0^i \in \Delta^i$  such that  $p_s = \lambda_i \pi_{0,s}^i u^{i'}(\bar{x}_s^i)$ , for any  $i$ , any  $s$ .

Firstly, we prove that  $\mathcal{U}$  is bounded. Indeed, for all  $(x^1, \dots, x^m) \in \mathcal{A}$ :

$$\begin{aligned} \sum_{i=1}^m \lambda_i U^i(x^i) - \sum_{i=1}^m \lambda_i \sum_{s=1}^S \pi_{0,s}^i u^{i'}(\bar{x}_s^i) &\leq \sum_{i=1}^m \lambda_i \sum_{s=1}^S \pi_{0,s}^i u^i(x_s^i) - \sum_{i=1}^m \lambda_i \sum_{s=1}^S \pi_{0,s}^i u^i(\bar{x}_s^i) \\ &= \bar{p} \cdot \left( \bar{e} - \sum_{i=1}^m \bar{x}^i \right). \end{aligned}$$

Hence for all  $i$  we have:

$$\lambda_i U^i(e^i) \leq \lambda_i U^i(x^i) \leq \bar{p} \cdot \left( \bar{e} - \sum_{j=1}^m \bar{x}^j \right) + \sum_{j=1}^m \lambda_j \sum_{s=1}^S \pi_{0,s}^j u^{j'}(\bar{x}_s^j) - \sum_{j \neq i} \lambda_j U^j(e^j).$$

Summing up, we have proved that  $\mathcal{U}$  is bounded. Now we prove that  $\mathcal{U}$  is closed.

We will construct a sequence of economies  $\{\mathcal{E}_t\}_{t \geq 0}$ . For each economy  $\mathcal{E}_t$ , there are  $m$  agents, each agent has a utility function

$$U_t^i(x) = \min_{\Delta_t^i} \sum_{s=1}^m \pi_s u^i(x_s^i) = \min_{\Delta_t^i} \sum_{s=1}^m \pi_s u^i(x_s^i),$$

<sup>6</sup>A more simplified version of Clarke's one, which can be used for the case of utility functions in theorem 2, can be found in [10].

where the probabilities set  $\Delta_t^i$  is the convex hull of  $\tilde{\Delta}_t^i$  which has  $M_t^i + 1$  points: the probability  $\pi_0^i$  used in the definition of  $\bar{p}$  and  $M_t^i$  other points.

Let  $T(\mathcal{E}_t) = \sum_{i=1}^m M_t^i$ .

Observe that for any opportunity of arbitrage  $(w^1, w^2, \dots, w^m)$ , i.e.  $\sum_{i=1}^m w^i = 0$  with  $w^i \in R^i$ , we have  $\bar{p} \cdot \sum_{i=1}^m w^i = 0$ . Since  $p \cdot w^i \geq 0$ , this implies  $\bar{p} \cdot w^i = 0$  for any  $i$ .

Moreover, we have

$$0 = \bar{p} \cdot w^i \geq a^i \sum_{s \in S_+(w^i)} \pi_{0,s}^i w_s^i + b^i \sum_{s \in S_-(w^i)} \pi_{0,s}^i w_s^i \geq 0,$$

which implies

$$a^i \sum_{s \in S_+(w^i)} \pi_{0,s}^i w_s^i + b^i \sum_{s \in S_-(w^i)} \pi_{0,s}^i w_s^i = 0.$$

But actually, if  $\sum_{i=1}^m w^i = 0$  with  $w^i \in R^i$ , then for any  $i \notin \mathcal{I}$ , we have  $w^i = 0$ . Indeed, in the contrary case, since for  $i \notin \mathcal{I}$ ,  $P^i$  is open, we have  $\bar{p} \cdot w^i > 0$ , a contradiction. Hence if  $a^i \sum_{s \in S_+(w^i)} \pi_{0,s}^i w_s^i + b^i \sum_{s \in S_-(w^i)} \pi_{0,s}^i w_s^i > 0$ , then  $i \in \mathcal{I}$ .

For each economy  $\mathcal{E}_t = \{U_t^i, \Delta_t^i, \tilde{\Delta}_t^i\}_{i=1}^m$ , denote

$$\mathcal{A}_t = \left\{ (x^1, x^2, \dots, x^m) \in (\mathbb{R}^S)^m \mid \sum_{i=1}^m x^i = \bar{e} \text{ and } U^i(e^i) \leq U_t^i(x^i) \right\}.$$

$$\mathcal{U}_t = \{ (v^1, v^2, \dots, v^m) \in \mathbb{R}^m \mid \exists x \in \mathcal{A}_t \text{ such that } U^i(e^i) \leq v^i \leq U_t^i(x^i) \}.$$

We denote also  $R_t^i$  the set of useful vectors of  $U_t^i$ , and  $W_t$  the set

$$W_t = \left\{ (w^1, w^2, \dots, w^m) \in R_t^1 \times R_t^2 \times \dots \times R_t^m \mid \sum_{i=1}^m w^i = 0 \right\}.$$

We denote by  $P_t^i$  the set of *weak-no-arbitrage-prices* of agent  $i$  in the economy  $\mathcal{E}_t$ . In this sequence  $(\mathcal{E}_t)$  we will have

$$\forall t, U_t^i \leq U_{t+1}^i \Rightarrow R_t^i \subseteq R_{t+1}^i \Rightarrow P_{t+1}^i \subseteq P_t^i.$$

However, we still have:  $\forall t, P_t^i$  is open if  $i \notin \mathcal{I}$ , and  $P_t^i$  is closed if  $i \in \mathcal{I}$ .

We now start the construction of our economies  $\mathcal{E}_t$ .

Our initial economy is  $\mathcal{E}_0$ :  $\Delta_0^i = \Delta^i$ ,  $M_0^i = M^i$ ,  $\tilde{\Delta}_0^i = \tilde{\Delta}^i$ , and utility function  $U_0^i(x^i) = U^i(x^i)$  for all  $i$ , for all  $x^i \in \mathbb{R}^S$ . If for all  $(w^1, \dots, w^m) \in W_0$ , for all  $\pi^i \in \Delta^i$  we have:

$$a^i \sum_{s \in S_+(w^i)} \pi_s^i w_s^i + b^i \sum_{s \in S_-(w^i)} \pi_s^i w_s^i = 0$$

then we stop.

Suppose that there exists  $(w^1, w^2, \dots, w^m) \in W_0$  such that:

$$a^{i_0} \sum_{s \in S_+(w^{i_0})} \tilde{\pi}_s^{i_0} w_s^{i_0} + b^{i_0} \sum_{s \in S_-(w^{i_0})} \tilde{\pi}_s^{i_0} w_s^{i_0} > 0$$

for an  $i_0$ , and a probability  $\tilde{\pi}^{i_0} \in \tilde{\Delta}_0^{i_0}$ .

Firstly, recall that  $i_0 \in \mathcal{I}$ . We define the  $\mathcal{E}_1$  as an economy with  $m$  agents, where agent  $i$  with  $i \neq i_0$  has the set of probabilities  $\Delta_1^i = \Delta_0^i$ , a utility function  $U_1^i(x^i) = U_0^i(x^i)$ , for all  $x^i \in \mathbb{R}^S$ . For agent  $i_0$ , we define  $\Delta_1^{i_0}$  as the convex hull of  $\tilde{\Delta}_0^{i_0} \setminus \{\tilde{\pi}^{i_0}\}$ . The new utility function is defined as:

$$\begin{aligned} U_1^{i_0}(x^{i_0}) &= \inf_{\pi \in \Delta_1^{i_0}} \sum_{s=1}^S \pi_s u^{i_0}(x_s^{i_0}) \\ &= \min_{\pi \in \Delta_1^{i_0}} \sum_{s=1}^S \pi_s u^{i_0}(x_s^{i_0}) \\ &= \min_{\pi \in \tilde{\Delta}_1^{i_0}} \sum_{s=1}^S \pi_s u^{i_0}(x_s^{i_0}). \end{aligned}$$

Observe that  $T(\mathcal{E}_1) = T(\mathcal{E}_0) - 1$ .

We will prove that for this new economy,  $\mathcal{U}_1 = \mathcal{U}_0$ .

Observe that for  $i \neq i_0$ ,  $U_0^i = U_1^i$ , and  $U_0^{i_0} \leq U_1^{i_0}$ . Hence  $\mathcal{U}_0 \subset \mathcal{U}_1$ . We also have  $P_1^i = P_0^i$  for  $i \neq i_0$  and  $P_1^{i_0} \subseteq P_0^{i_0}$ . Moreover,  $P_1^i$  is open if  $i \notin \mathcal{I}$  and  $P_1^i$  is closed if  $i \in \mathcal{I}$ .

We also observe that  $\bigcap_i P_1^i \neq \emptyset$ . Indeed,  $\pi_0^i \in \Delta_1^i$ , then  $\bar{p} \in P_1^i$  for all  $i$ .  $\mathcal{U}_1$  is then bounded. Denote  $C > 0$  this upper bound.

Suppose that  $(v^1, v^2, \dots, v^m) \in \mathcal{U}_1$ . There exists  $(x^1, x^2, \dots, x^m)$  such that  $\sum_i x^i = \bar{e}$  and  $U^i(e^i) \leq v^i \leq U_1^i(x^i)$  for all  $1 \leq i \leq m$ .

Since  $i_0 \in \mathcal{I}$ , there exists  $\bar{z}^{i_0} > 0$  such that  $u^{i_0'}(z) = a^{i_0}$  for  $z > \bar{z}^{i_0}$  and  $u^{i_0'}(z) = b^{i_0}$  for  $z < -\bar{z}^{i_0}$ .

Take  $\lambda > 0$  large enough such that  $x_s^{i_0} + \lambda w_s^{i_0} > \bar{z}^{i_0}$ ,  $\forall s \in S_+(w^{i_0})$  and  $x_s^{i_0} + \lambda w_s^{i_0} < -\bar{z}^{i_0}$ ,  $\forall s \in S_-(w^{i_0})$ . Observe that for  $z > \bar{z}^{i_0}$  we have  $u^{i_0}(z) = u^{i_0}(\bar{z}^{i_0}) + a^{i_0}(z - \bar{z}^{i_0})$ , and for  $z < -\bar{z}^{i_0}$  we have  $u^{i_0}(z) = u^{i_0}(\bar{z}^{i_0}) + b^{i_0}(z - \bar{z}^{i_0})$ .

Then:

$$\begin{aligned} & \sum_{s=1}^S \tilde{\pi}_s^{i_0} u^{i_0}(x_s^{i_0} + \lambda w_s^{i_0}) - \sum_{s=1}^S \tilde{\pi}_s^{i_0} u^{i_0}(x_s^{i_0}) \\ &= \sum_{s \in S_+(w^{i_0})} \tilde{\pi}_s^{i_0} [u^{i_0}(\bar{z}^{i_0}) + a^{i_0}(x_s^{i_0} + \lambda w_s^{i_0} - \bar{z}^{i_0})] \\ &+ \sum_{s \in S_-(w^{i_0})} \tilde{\pi}_s^{i_0} [u^{i_0}(\bar{z}^{i_0}) + b^{i_0}(x_s^{i_0} + \lambda w_s^{i_0} - \bar{z}^{i_0})]. \end{aligned}$$

And we have:

$$\sum_{s=1}^S \tilde{\pi}_s^{i_0} u^{i_0}(x_s^{i_0} + \lambda w_s^{i_0}) = Z + \lambda \left[ a^{i_0} \sum_{s \in S_+(w^{i_0})} \tilde{\pi}_s^{i_0} w_s^{i_0} + b^{i_0} \sum_{s \in S_-(w^{i_0})} \pi_s^{i_0} w_s^{i_0} \right], \quad (1)$$

where  $Z$  is a constant:

$$\begin{aligned} Z &= \sum_{s=1}^S \tilde{\pi}_s^{i_0} u^{i_0}(x_s^{i_0}) + \sum_{s \in S_+(w^{i_0})} \tilde{\pi}_s^{i_0} [u^{i_0}(\bar{z}^{i_0}) + a^{i_0}(x_s^{i_0} - \bar{z}^{i_0})] \\ &+ \sum_{s \in S_-(w^{i_0})} \tilde{\pi}_s^{i_0} [u^{i_0}(\bar{z}^{i_0}) + b^{i_0}(x_s^{i_0} - \bar{z}^{i_0})]. \end{aligned}$$

Since  $a^{i_0} \sum_{s \in S_+(w^{i_0})} \tilde{\pi}_s^{i_0} w_s^{i_0} + b^{i_0} \sum_{s \in S_-(w^{i_0})} \tilde{\pi}_s^{i_0} w_s^{i_0} > 0$ , if we let  $\lambda$  tends to infinity, the right hand term of (1) converges to infinity. Then for  $\lambda > 0$  large enough,  $\sum_{s=1}^S \tilde{\pi}_s^{i_0} u^{i_0}(x_s^{i_0} + \lambda w_s^{i_0}) > C$ , the upper bound of  $\mathcal{U}_1$ . Hence we have

$$\min_{\tilde{\Delta}_0^{i_0} \setminus \{\tilde{\pi}^{i_0}\}} \sum_{s=1}^S \pi_s u^{i_0}(x_s^{i_0} + \lambda w_s^{i_0}) = \min_{\pi \in \tilde{\Delta}_0^{i_0}} \sum_{s=1}^S \pi_s u^{i_0}(x_s^{i_0} + \lambda w_s^{i_0})$$

and then

$$\begin{aligned} U_0^{i_0}(x^{i_0} + \lambda w^{i_0}) &= \min_{\pi \in \tilde{\Delta}_0^{i_0}} \sum_{s=1}^S \pi_s u^{i_0}(x_s^{i_0} + \lambda w_s^{i_0}) \\ &= \min_{\tilde{\Delta}_0^{i_0} \setminus \{\tilde{\pi}^{i_0}\}} \sum_{s=1}^S \pi_s^{i_0} u^{i_0}(x_s^{i_0} + \lambda w_s^{i_0}) \\ &= \min_{\tilde{\Delta}_1^{i_0}} \sum_{s=1}^S \pi_s^{i_0} u^{i_0}(x_s^{i_0} + \lambda w_s^{i_0}) \\ &= U_1^{i_0}(x^{i_0} + \lambda w^{i_0}). \end{aligned}$$

So for  $\lambda > 0$  large enough, we have for all  $1 \leq i \leq m$ :

$$U^i(e^i) \leq v^i \leq U_1^i(x^i) \leq U_1^i(x^i + \lambda w^i) = U_0^i(x^i + \lambda w^i).$$

Observe that  $\sum_i (x^i + \lambda w^i) = \sum_i e^i$ . Hence  $\mathcal{U}_1 \subset \mathcal{U}_0$ .

Now we are ready to finish the proof. The economy  $\mathcal{E}_1$  has  $\mathcal{U}_1 = \mathcal{U}_0$ . Since  $\pi_0^i \in \Delta_1^i$  for all  $i$ , we have  $\bigcap_i P_1^i \neq \emptyset$ . This implies  $\mathcal{U}_1$  is bounded. We recall that  $T(\mathcal{E}_1) = T(\mathcal{E}_0) - 1$ . If in the economy  $\mathcal{E}_1$ , for any  $\sum_i w^i = 0$  with  $w^i \in R_1^i$ , we have

$$a^i \sum_{s \in S_+(w^i)} \pi_s^i w_s^i + b^i \sum_{s \in S_-(w^i)} \pi_s^i w_s^i = 0,$$

then we stop.

If there exists  $w \in W_1$  such that there exist  $i_1$  and  $\tilde{\pi}^{i_1} \in \Delta_1^{i_1}$  satisfying

$$a^{i_1} \sum_{s \in S_+(w^{i_1})} \tilde{\pi}_s^{i_1} w_s^{i_1} + b^{i_1} \sum_{s \in S_-(w^{i_1})} \pi_s^{i_1} w_s^{i_1} > 0,$$

then  $i_1 \in \mathcal{I}$  and by the same argument as for the previous step, we can construct a new economy  $\mathcal{E}_2$ , and so on. After each step, the number of probabilities which determine the ambiguity set of the agents diminishes by 1:  $T(\mathcal{E}_{t+1}) = T(\mathcal{E}_t) - 1$ . Since there exists a finite number of probabilities which determine  $\Delta^i$ , the process has to stop at some step  $T$ . The utility sets of  $T + 1$  economies are equal  $\mathcal{U}_0 = \mathcal{U}_1 = \dots = \mathcal{U}_T$ . The economy  $\mathcal{E}_T$  satisfies the property: for all  $w \in W_T$ , for all  $\pi^i \in \Delta_T^i$  we have

$$a^i \sum_{s \in S_+(w^i)} \pi_s^i w_s^i + b^i \sum_{s \in S_-(w^i)} \pi_s^i w_s^i = 0.$$

Now we will prove that  $\mathcal{E}_T$  satisfies *Inconsequential arbitrage* condition. Suppose there exist  $(w^1, \dots, w^m) \in W_T$ , a sequence  $\{(x_n^1, \dots, x_n^m)\} \subset \mathcal{A}$ , and another positive sequence  $\lambda_n \rightarrow 0$  such that  $\lim_n \lambda_n x_n^i = w^i$ . Observe if  $w_s > 0$  then  $x_{n,s}^i \rightarrow +\infty$  and if  $w_s < 0$  then  $x_{n,s}^i \rightarrow -\infty$ . Take  $\epsilon > 0$  arbitrarily.

For  $n$  large enough, for any  $i \in \mathcal{I}$ , we have for any  $s$ , either  $w_s^i = 0$ , or  $x_{n,s}^i - \epsilon w_s^i > \bar{z}^i$  if  $s \in S_+(w^i)$ , and  $x_{n,s}^i - \epsilon w_s^i < -\bar{z}^i$  if  $s \in S_-(w^i)$ .

For every  $n$ , there exists  $\pi_n^i \in \Delta_T^i$  such that  $U_T^i(x_n^i) = \sum_s \pi_{n,s}^i u^i(x_{n,s}^i)$ . And we have for  $n$  large enough:

$$\begin{aligned} U_T^i(x_n^i - \epsilon w^i) - U_T^i(x_n^i) &\geq \sum_{s=1}^S \pi_{n,s}^i u^i(x_{n,s}^i - \epsilon w_s^i) - \sum_{s=1}^S \pi_{n,s}^i u^i(x_{n,s}^i) \\ &\geq \sum_{s=1}^m \pi_{n,s}^i (-\epsilon w_s^i) u^i(x_{n,s}^i - \epsilon w_s^i) \\ &= -\epsilon [a^i \sum_{s \in S_+(w^i)} \pi_{n,s}^i w_s^i + b^i \sum_{s \in S_-(w^i)} \pi_{n,s}^i w_s^i] \\ &= 0. \end{aligned}$$

For  $i \notin \mathcal{I}$ , since  $w^i = 0$ , we have  $U_T^i(x^i - \epsilon w^i) = U_T^i(x^i)$ .

The economy  $\mathcal{E}_T$  satisfies *Inconsequential arbitrage* condition and hence  $\mathcal{U}_T$  is compact. This implies  $\mathcal{U} = \mathcal{U}_T$  is also compact. From Theorem 1, our initial economy has an equilibrium. ■

## References

- [1] N. Allouch, C. Le Van and F.H. Page (2002): The geometry of arbitrage and the existence of competitive equilibrium. *Journal of Mathematical Economics* **38**, 373-391.

- [2] D. Brown and J. Werner (1995): Existence theorems in the classical asset pricing model. *Econometrica* **59**, 1169-1174.
- [3] D. Brown. and J. Werner (1995): Arbitrage and existence of equilibrium in infinite asset markets. *Review of Economic Studies*, **62**, 101-114.
- [4] F.H. Clarke (1993): Optimization and nonsmooth analysis. *John Wileys and Sons*.
- [5] R.A Dana, and C. Le Van (1996): Asset Equilibria in  $L^p$  Spaces: A Duality Approach. *Journal of Mathematical Economics*, **25**, 263-280.
- [6] R.A. Dana and C. Le Van (2010): Overlapping risk adjusted sets of priors and the existence of efficient allocations and equilibria with short-selling. *Journal of Economic Theory*, **145**, 2186-2202.
- [7] R.A. Dana, C. Le Van and F. Magnien (1997): General Equilibrium in asset markets with or without short-selling. *Journal of Mathematical Analysis and Applications* **206**, 567-588.
- [8] R.A. Dana, C. Le Van and F. Magnien (1999): On the different notions of arbitrage and existence of equilibrium. *Journal of Economic Theory* **86**, 169-193.
- [9] M. Geistdorfer-Florenzano (1982): The Gale-Nikaido-Debreu Lemma and the Existence of Transitive Equilibrium With or Without the Free Disposal Assumption. *Journal of Mathematical Economics* **9**, 113-134.
- [10] M. Geistdorfer-Florenzano, C. Le Van: Finite dimensional convexity and optimization. *Springer*.
- [11] J.M. Grandmont (1972): Continuity properties of a von Neumann-Morgenstern utility. *Journal of Economic Theory*, **5**, 45-57.
- [12] J.M. Grandmont (1977): Temporary general equilibrium theory. *Econometrica* **45**, 535-572.
- [13] J.R. Green (1973): Temporary general equilibrium in a sequential trading model with spot and futures transactions. *Econometrica* **41**, 1103-1124.
- [14] I. Gilboa, D. Schmeidler (1989): Maximin expected utility with non-unique prior. *Journal of Mathematical Economics* **18**, 141-153.
- [15] T. Ha-Huy, C. Le Van (2017): Existence of equilibrium on asset markets with a countably infinite number of states. *Journal of Mathematical Economics*, forthcoming.

- [16] P. Hammond (1983): Overlapping expectations and Hart's conditions for equilibrium in a securities models. *Journal of Economic Theory* **31**, 170-175.
- [17] O. Hart (1974): On the Existence of an Equilibrium in a Securities Model. *Journal of Economic Theory* **9**, 293-311.
- [18] L.T. Nielsen (1989): Asset market equilibrium with short-selling. *Review of Economic Studies*, **41**, 392-469.
- [19] F.H. Page (1987): On equilibrium in Hart's securities exchange model. *Journal of Economic Theory* **41**, 392-404.
- [20] F.H. Page, and M.H. Wooders, (1996): A necessary and sufficient condition for compactness of individually rational and feasible outcomes and existence of an equilibrium. *Economics Letters* **52**, 153-162.
- [21] F.H. Page, M.H. Wooders and P.K. Monteiro (2000): Inconsequential arbitrage. *Journal of Mathematical Economics* **34**, 439-469.
- [22] J.T. Rockafellar (1970): Convex Analysis. *Princeton University Press*, Princeton, New Jersey.
- [23] J. Werner (1987): Arbitrage and the Existence of Competitive Equilibrium. *Econometrica* **55**, 1403-1418.