Projective dimension and commuting variety of a reductive Lie algebra
Jean-Yves Charbonnel

To cite this version:
Jean-Yves Charbonnel. Projective dimension and commuting variety of a reductive Lie algebra. 2020. hal-02877905

HAL Id: hal-02877905
https://hal.science/hal-02877905
Preprint submitted on 22 Jun 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
PROJECTIVE DIMENSION AND COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA

JEAN-YVES CHARBONNEL

Abstract. The commuting variety of a reductive Lie algebra \( \mathfrak{g} \) is the underlying variety of a well defined subscheme of \( \mathfrak{g} \times \mathfrak{g} \). In this note, it is proved that this scheme is normal and Cohen-Macaulay. In particular, its ideal of definition is a prime ideal. As a matter of fact, this theorem results from a so called Property (P) for a simple Lie algebra. This property says that some cohomology complexes are exact.

Contents

1. Introduction .................................................. 2
2. Characteristic module ......................................... 10
3. Proof of the main theorem .................................... 15
4. Proof of Theorem 1.3 ........................................... 17
5. First step to the proof of Theorem 1.5 ....................... 20
6. At the neighborhood of a semisimple element ............. 21
7. Restriction to a parabolic subalgebra ....................... 24
8. Expansion along a parabolic subalgebra .................... 36
9. At the neighborhood of a semi-simple element II .......... 41
10. Some spaces related to parabolic subalgebras ............. 52
11. Some spaces related to parabolic subalgebras II ......... 58
12. Induction. Case \( n \leq 1 \) .................................... 60
13. Induction. Case \( n \geq 2 \) .................................... 72
Appendix .......................................................... 81
Appendix A. Some complexes .................................. 81
Appendix B. Projective dimension and cohomology .......... 87
Appendix C. Some computations ................................ 91
Appendix D. Some remarks about representations .......... 93
References .......................................................... 97
1. Introduction

In this note, the base field $\mathbb{k}$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, $\ell$ is its rank, and $G$ is its adjoint group.

1.1. The dual of $\mathfrak{g}$ identifies with $\mathfrak{g}$ by a non degenerate symmetric bilinear form on $\mathfrak{g}$ extending the Killing form of the derived algebra of $\mathfrak{g}$. Denote by $(v, w) \mapsto \langle v, w \rangle$ this bilinear form and $I_\mathfrak{g}$ the ideal of $\mathbb{k}[\mathfrak{g} \times \mathfrak{g}]$ generated by the functions $(x, y) \mapsto \langle v, [x, y] \rangle$, $v \in \mathfrak{g}$. The commuting variety $\mathcal{C}(\mathfrak{g})$ of $\mathfrak{g}$ is the subvariety of elements $(x, y)$ of $\mathfrak{g} \times \mathfrak{g}$ such that $[x, y] = 0$. It is the underlying variety of the subscheme $S(\mathfrak{g})$ of $\mathfrak{g} \times \mathfrak{g}$ defined by $I_\mathfrak{g}$. It is a well known and long standing open question whether or not this scheme is reduced, that is $\mathcal{C}(\mathfrak{g}) = S(\mathfrak{g})$. According to Richardson [Ri79], $\mathcal{C}(\mathfrak{g})$ is irreducible and according to Popov [Po08, Theorem 1], the singular locus of $S(\mathfrak{g})$ has codimension at least 2 in $\mathcal{C}(\mathfrak{g})$. Then, according to Serre’s normality criterion, arises the question to know whether or not $\mathcal{C}(\mathfrak{g})$ is normal. There are many results about the commuting variety. A result of Dixmier [Di79] proves that $I_\mathfrak{g}$ contains all the elements of the radical of $I_\mathfrak{g}$, of degree 1 in the second variable. In [Ga-Gi06], Gan and Ginzburg prove that for $\mathfrak{g}$ simple of type A, the invariant elements under $G$ of $I_\mathfrak{g}$ is a radical ideal of the algebra $\mathbb{k}[\mathfrak{g} \times \mathfrak{g}]^G$ of invariant elements of $\mathbb{k}[\mathfrak{g} \times \mathfrak{g}]$ under $G$. In [Gi12], Ginzburg proves that the normalization of $\mathcal{C}(\mathfrak{g})$ is Cohen-Macaulay.

1.2. Main results. According to the identification of $\mathfrak{g}$ and its dual, $\mathbb{k}[\mathfrak{g} \times \mathfrak{g}]$ is equal to the symmetric algebra $S(\mathfrak{g} \times \mathfrak{g})$ of $\mathfrak{g} \times \mathfrak{g}$. The main result of this note is the following theorem:

**Theorem 1.1.** The subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by $I_\mathfrak{g}$ is Cohen-Macaulay and normal. Furthermore, $I_\mathfrak{g}$ is a prime ideal of $S(\mathfrak{g} \times \mathfrak{g})$.

According to Richardson’s result and Popov’s result, it suffices to prove that the scheme $S(\mathfrak{g})$ is Cohen-Macaulay. The main idea of the proof in the theorem uses the main argument of the Dixmier’s proof: for a finitely generated module $M$ over $S(\mathfrak{g} \times \mathfrak{g})$, $M = 0$ if the codimension of its support is at least $l + 2$ with $l$ the projective dimension of $M$ (see Appendix B).

All the complexes considered in this note are localizations of submodules of the algebra $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{k} S(\mathfrak{g}) \otimes_\mathbb{k} \wedge(\mathfrak{g})$. We introduce the characteristic submodule of $\mathfrak{g}$, denoted by $B_\mathfrak{g}$. By definition, $B_\mathfrak{g}$ is a submodule of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{k} \mathfrak{g}$ and an element $\varphi$ of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{k} \mathfrak{g}$ is in $B_\mathfrak{g}$ if and only if for all $(x, y)$ in a dense subset of $\mathfrak{g} \times \mathfrak{g}$, $\varphi(x, y)$ is in the sum of the subspaces $\mathfrak{g}^{ax+by}$ with $(a, b)$ in $\mathbb{k}^2 \setminus \{0\}$ and $\mathfrak{g}^{ax+by}$ the centralizer of $ax + by$ in $\mathfrak{g}$. According to a Bolsinov’s result, $B_\mathfrak{g}$ is a free $S(\mathfrak{g} \times \mathfrak{g})$-module of rank $b_\mathfrak{g}$, the dimension of the Borel subalgebras of $\mathfrak{g}$. Moreover, the orthogonal complement to $B_\mathfrak{g}$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{k} \mathfrak{g}$ is a free $S(\mathfrak{g} \times \mathfrak{g})$-module of rank $b_\mathfrak{g} - \ell$ and for $\varphi$ in $B_\mathfrak{g}$, $\langle \varphi(x, y), [x, y] \rangle = 0$ for all $(x, y)$ in $\mathfrak{g} \times \mathfrak{g}$. Let $d$ be the $S(\mathfrak{g} \times \mathfrak{g})$-derivation of the algebra $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{k} \wedge(\mathfrak{g})$ such that for $v$ in $\mathfrak{g}$, $dv$ is the function on $\mathfrak{g} \times \mathfrak{g}$: $(x, y) \mapsto \langle v, [x, y] \rangle$. Then $d$ is a structure of complex on $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{k} \wedge(\mathfrak{g})$ and the ideal of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{k} \wedge(\mathfrak{g})$ generated by $B_\mathfrak{g}^\ell(B_\mathfrak{g})$ is a subcomplex. The usual gradation of $\wedge(\mathfrak{g})$ induces a gradation of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{k} \wedge(\mathfrak{g})$. Denote by
COMMUTING VARIETY

C∗(g) the graded subcomplex of S(g × g)⊗k ∧ (g) such that C∗i+b∗(g) := S(g × g)⊗k ∧ i(g)∧ ∧ b∗(Bg).

Then Theorem 1.1 is a consequence of the following theorem:

**Theorem 1.2.** The complex C∗(g) has no homology of degree bigger than b∗ and I∗ is isomorphic to the space of boundaries of degree b∗.

By standard results of homological algebra (see appendix), Theorem 1.2 is a consequence of the key theorem of this note:

**Theorem 1.3.** For i = 1, ..., b∗ − ℓ, C∗i+b*(g) has projective dimension at most i.

As a matter of fact, it is easy to see that the support of the homology of C•(g) is contained in C(g). Then, by Theorem 1.3, C•(g) has no homology of degree bigger than b∗ and I∗ has projective dimension at most 2(b∗ − ℓ) − 1. So, by Auslander-Buchsbaum’s theorem, S(g) is Cohen-Macaulay.

For the proof of Theorem 1.3, we consider the algebra S(g)⊗k ∧ (g) and the ∧ (g)-derivation d such that

du⊗a = v ∧ a with v ∈ g, a ∈ ∧ (g).

Then (S(g)⊗k ∧ (g), d) is a complex and the usual gradation of ∧ (g) induces on this complex a structure of graded cohomology complex denoted by D*(g). For k nonnegative integer, denote by D∗k(g) the graded subcomplex of D*(g) such that

D∗k(g) := Sk−i(g)⊗k ∧ i(g)

and D∗k(g, Bg) the graded subcomplex of S(g × g)⊗k D*(g) such that

D∗k,i+b∗(g, Bg) := Sk−i(g)⊗k ∧ i(g)∧ ∧ b∗(Bg).

**Definition 1.4.** Let n := b∗ − ℓ and j = 1, ..., n. We say that g has Property (Pj) if D∗k(g, Bg) has no cohomology of degree different from b∗ for k = 1, ..., j.

We say that g has Property (P) if it has Property (Pn).

By an induction argument, Theorem 1.3 is a consequence of the following Theorem:

**Theorem 1.5.** All simple Lie algebra has Property (P).

As a matter of fact, the proof of this theorem is the main part of this note.

1.3. **Sketch of proofs.** We suppose g simple and we prove Theorem 1.5 by induction on the rank of g. For k = 1, ..., n, we denote by Sk the support in g × g of the cohomology of D∗k(g, Bg) of degree different from b∗. This subset of g × g is invariant under the diagonal action of G and the canonical action of GL2(k) since Bg is a relatively equivariant module under these actions. As a result, the image of Sk by the first projection

\[ g \times g \xrightarrow{\pi_1} g \]
is a $G$-invariant closed subset of $\mathfrak{g}$. In particular, if $\sigma_1(S_k)$ does not contain semisimple elements different from 0, $\sigma_1(S_k)$ is contained in the nilpotent cone $\mathcal{N}_\mathfrak{g}$ of $\mathfrak{g}$ and $S_k$ is contained in the so-called nilpotent bicone $\mathcal{N}_\mathfrak{g}$ of $\mathfrak{g}$. By definition, $\mathcal{N}_\mathfrak{g}$ is the subset of elements $(x, y)$ of $\mathfrak{g} \times \mathfrak{g}$ such that the subspace of $\mathfrak{g}$, generated by $x$ and $y$, is contained in $\mathcal{R}_\mathfrak{g}$. By [CMo08, Theorem 1.2], $\mathcal{N}_\mathfrak{g}$ has codimension $b_\mathfrak{g} + \ell$ in $\mathfrak{g} \times \mathfrak{g}$. So, when $\sigma_1(S_k)$ is contained in $\mathcal{R}_\mathfrak{g}$ for $k = 1, \ldots, j$, $\mathfrak{g}$ has Property (P) by Corollary B.3 and an induction argument on $j$.

As a result, the main step of the proof of Theorem 1.5 is the equality $\sigma_1(S_k) \cap \mathfrak{b} = \{0\}$ for $k = 1, \ldots, n$. Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$. Let $z$ be in $\mathfrak{b}$. Denote by $\mathfrak{g}^z$ the centralizer of $z$ in $\mathfrak{g}$. The orbit of $z$ under the Weyl group contains an element $\mathfrak{g}^z + \mathfrak{h}$ is a parabolic subalgebra. So, we can suppose that $\mathfrak{p} := \mathfrak{g}^z + \mathfrak{h}$ is an algebra. Then $\mathfrak{l} := \mathfrak{g}^z$ is the reductive factor of $\mathfrak{p}$ containing $\mathfrak{h}$. Denote by $\mathfrak{d}$ the derived algebra of $\mathfrak{l}$ and $\mathfrak{l}$, the subset of elements $x$ of $\mathfrak{l}$ such that $\mathfrak{g}^x$ is contained in $\mathfrak{l}$. Then $\mathfrak{l}$, is a principal open subset of $\mathfrak{l}$. Let $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ be the simple factors of $\mathfrak{d}$, $\mathfrak{z}$ the center of $\mathfrak{l}$, $\mathfrak{p}_{\pm,\mathfrak{u}}$ the sum of root spaces with respect to $\mathfrak{h}$, not contained in $\mathfrak{l}$ and $d$ the half dimension of $\mathfrak{p}_{\pm,\mathfrak{u}}$. When $z$ is regular, $n = 0$. When $n$ is positive, for $i = 1, \ldots, n$, denote by $\mathfrak{d}_i$ the rank of $\mathfrak{d}_i$, $b_{\mathfrak{d}_i}$ the dimension of the Borel subalgebras of $\mathfrak{d}_i$ and set $n_i := b_{\mathfrak{d}_i} - b_{\mathfrak{d}_i}$. For $j$ nonnegative integer, set:

$$\mathbb{I}_j := \{(i_1, \ldots, i_n) \in \mathbb{N}_n^2 \mid i_1 \leq n_1, \ldots, i_n \leq n_n, i_1 + \cdots + i_n = j\},$$

and for $k = 1, \ldots, n$ and $\tau = (i_1, \ldots, i_n)$ in $\mathbb{I}_k$, denote by $D^*_k$ the simple complex deduced from the multicomplex

$$D^*_{i-1}(\mathfrak{p}_{\pm,\mathfrak{u}}) \otimes_k D^*_0(\mathfrak{g}) \otimes_k D^*_1(\mathfrak{d}_1) \otimes_k \cdots \otimes_k D^*_n(\mathfrak{d}_n)$$

and set:

$$D^*_k := \bigoplus_{\tau \in \mathbb{I}_k} D^*_k.$$  

Let $B$ be the restriction of $B_\mathfrak{g}$ to $\mathfrak{l} \times \mathfrak{g}$ and $\mathcal{B}$ the $\mathbb{K}[\mathfrak{g}] \otimes_\mathbb{K} \mathbb{K}[\mathfrak{l} \times \mathfrak{g}]$-submodule of $\mathbb{K}[\mathfrak{g}] \otimes_\mathbb{K} \mathbb{K}[\mathfrak{l} \times \mathfrak{g}] \otimes_\mathbb{K} \mathfrak{g}$ generated by the maps

$$(g, x, y) \mapsto g.\varphi(x, y) \quad \text{with} \quad \varphi \in B.$$  

Then $B$ and $\mathcal{B}$ are free modules of rank $b_\mathfrak{g}$. Denote by $D^*_k\mathcal{B}$ and $D^*_k\mathcal{B}$ the graded subcomplexes of $\mathbb{K}[\mathfrak{g}] \otimes_\mathbb{K} \mathfrak{D}^*(\mathfrak{g})$ and $\mathbb{K}[\mathfrak{g}] \otimes_\mathbb{K} \mathfrak{D}^*(\mathfrak{g})$, 

$$D^*_k\mathcal{B} := D^*_k(\mathfrak{g})[\mathfrak{b}_\mathfrak{g}] \wedge \mathfrak{b}_\mathfrak{g}(\mathcal{B}) \quad \text{and} \quad D^*_k\mathcal{B} := D^*_k(\mathfrak{g})[\mathfrak{b}_\mathfrak{g}] \wedge \mathfrak{b}_\mathfrak{g}(\mathcal{B})$$

An important step of the proof of Theorem 1.5 is the following proposition:

**Proposition 1.6.** Suppose that the simple factors of $\mathfrak{l}$ have Property (P). Then, for $k = 1, \ldots, n$, $D^*_k\mathcal{B}$ has no cohomology of degree different from $b_\mathfrak{g}$. 
When $n = 0$, $z$ is regular and when $n = 1$, $\mathfrak{b}$ is simple. In this case, Proposition 1.6 is given by Proposition 12.12. When $n \geq 2$, Proposition 1.6 is given by Proposition 13.8. Denote by $D^\bullet_k(g, \tilde{B})$ the graded complex

$$D^\bullet_k(g, \tilde{B}) := D^\bullet_k(g)[-b_\mathfrak{g}] \wedge \Lambda^{b_\mathfrak{b}}(\tilde{B}).$$

Then, from Proposition 1.6, Proposition D.6 and [C20, Theorem 1.1], we deduce that $D^\bullet_k(g, \tilde{B})$ has no cohomology of degree different from $b_\mathfrak{g}$. As a matter of fact, [C20, Theorem 1.1] is only true for simple Lie algebras. The complex $D^\bullet_k(g, \tilde{B})$ is a subcomplex of $\tilde{k}[G] \otimes_k \tilde{k}[l \times g] \otimes_k D^\bullet_k(g)$ and the morphism

$$G \times l, \times g \longrightarrow g \times g,$$

$$(g, x, y) \longmapsto (g(x), g(y))$$

is a flat morphism whose image is the cartesian product of an open neighborhood of $z$ and $\mathfrak{g}$, whence the following corollary:

**Corollary 1.7.** Suppose that the simple factors of $l$ have Property (P). Then, for $k = 1, \ldots, n$, $z$ is not in $\sigma_1(S_k)$.

Corollary 1.7 is given by Proposition 6.5.

As a result, one of the main steps to prove Theorem 1.5 is the proof of Proposition 1.6. For that purpose, denote by $\tilde{B}$ the restriction of $B$ to $l \times p$ and $\overline{B}_1$ the restriction of $B_1$ to $l_1 \times l$. Then $\overline{B}$ and $\overline{B}_1$ are free modules of rank $b_\mathfrak{b}$ and $b_1$, respectively. Denoting by $p_u$ the nilpotent radical of $p$ and $p_{-u}$ the complement to $p_u$ in $p_{\pm u}$, invariant under the adjoint action of $\mathfrak{b}$,

$$g = p_{-u} \oplus p \quad \text{and} \quad p = l \oplus p_u$$

so that $k[l_1 \times l]$ and $k[l_1 \times p]$ are subalgebras of $k[l \times g]$. Let $\overline{B}_+\mathfrak{b}$ be the submodule of $k[l_1 \times p] \otimes_k \mathfrak{g}$ generated by $\overline{B}_1$ and $p_u$. Then $\overline{B}_+\mathfrak{b}$ is a free module of rank $b_\mathfrak{b}$ and $\overline{B}$ is a submodule of $\overline{B}_+\mathfrak{b}$. For $M$ a free submodule of rank $b_\mathfrak{b}$ of $k[l_1 \times p] \otimes_k \mathfrak{g}$, denote by $D_{k,\#}^\bullet(g, M)$ the graded submodule of $k[l_1 \times p] \otimes_k D^\bullet(g)$,

$$D_{k,\#}^\bullet(g, M) := D_{k,\#}^\bullet(g)[-b_\mathfrak{b}] \wedge \Lambda^{b_\mathfrak{b}}(M).$$

By Property (P) for the simple factors of $l$, for $k = 1, \ldots, n - d$, the complex $D_{k,\#}^\bullet(l)[-b_1] \wedge \Lambda^{b_1}(\overline{B}_1)$ has no cohomology of degree different from $b_1$ since $l$ is a principal open subset of $l$ and the center of $l$ is contained in $\overline{B}_1$. Then, for $k = 1, \ldots, n$, $D_{k,\#}^\bullet(g, \overline{B}_1)$ and $D_{k,\#}^\bullet(g, \overline{B})$ have no cohomology of degree different from $b_\mathfrak{g}$.

Let $\hat{O}$ be the local ring of $G$ at the identity and $\hat{\mathfrak{b}}$ the completion of $\hat{O}$ with respect to the $m$-adic topology with $m$ the maximal ideal of $\hat{O}$. Let $\tilde{B} = \hat{O} \otimes_k k[l_1 \times g]$-submodule of $\hat{O} \otimes_k k[l \times g] \otimes_k \mathfrak{g}$ generated by $\tilde{B}$ and $D_{k,\#}^\bullet(g, \tilde{B})$ the subcomplex of $\hat{O} \otimes_k k[l \times g] \otimes_k D^\bullet(g)$,

$$D_{k,\#}^\bullet(g, \tilde{B}) := D_{k,\#}^\bullet(g) \wedge \Lambda/(p_u) \wedge \Lambda^{b_\mathfrak{b}}(\overline{B}).$$
The space $D_{k,*}(\mathfrak{g}, \widehat{B})$ is a module over the algebra

$$\hat{\mathcal{O}} \otimes_{k} \mathbb{k}[\mathfrak{l} \times \mathfrak{g}] \otimes_{k}(p_{u}).$$

Denoting by $J$ the ideal of definition of $\mathfrak{l} \times p$ in $\mathbb{k}[\mathfrak{l} \times \mathfrak{g}]$, let $\hat{J}$ be the ideal of this algebra generated by $\mathfrak{m}$, $J$ and $p_{u}$. When $n \leq 1$, the subspace $\hat{J}D_{k,*}(\mathfrak{g}, \widehat{B})$ is the kernel of the restriction morphism

$$D_{k,*}(\mathfrak{g}, \widehat{B}) \to D_{k,#}(\mathfrak{g}, \widehat{B}).$$

This important result comes from the invertibility of some square matrices (see Subsection 9.2). As a matter of fact, the powers of $\hat{J}$ induce a filtration of $D_{k,*}(\mathfrak{g}, \widehat{B})$ and the graded space associate to this filtration is isomorphic to

$$A \otimes_{\mathbb{k}[\mathfrak{l} \times p]} D_{k,#}(\mathfrak{g}, \widehat{B}) \quad \text{with} \quad A = \bigoplus_{l \in \mathbb{N}} \hat{J}_{l}^{l}/\hat{J}_{l+1}^{l+1}.$$

Then, by the above result on the cohomology of $D_{k,#}(\mathfrak{g}, \widehat{B})$ and the acyclicity of the complex $D_{k,*}^{*}(p_{u})$, $D_{k,#}(\mathfrak{g}, \widehat{B})$ has no cohomology of degree different from $b_{g}$, whence Proposition 1.6 for the case $n \leq 1$ since $\hat{\mathcal{O}}$ is a faithfully flat extension of $\mathcal{O}$ and $D_{k,#}(\mathfrak{g}, \widehat{B})$ is $G$-equivariant.

Suppose $n \geq 2$. Denote by $E$ the $\mathbb{k}[\mathfrak{l} \times \mathfrak{l}]$-submodule of $\mathbb{k}[\mathfrak{l} \times \mathfrak{l}] \otimes_{\mathbb{k}} p$ generated by $p_{u}$ and the generators of the modules $B_{\mathfrak{j}}$, $\mathfrak{j} = 1, \ldots, n$ which are not in $\mathbb{k}[\mathfrak{l}] \otimes_{\mathbb{k}} \mathfrak{l}$. Let $\hat{J}_{+}$ be the ideal of the algebra

$$\hat{\mathcal{O}} \otimes_{\mathbb{k}} \mathbb{k}[\mathfrak{l} \times \mathfrak{g}] \otimes_{\mathbb{k}[\mathfrak{l} \times \mathfrak{l}]} \wedge(E)$$

generated by $\mathfrak{m}$, $J$, $E$. Let $D_{k,*}(\mathfrak{g}, \widehat{B})$ be the subcomplex of $\hat{\mathcal{O}} \otimes_{\mathbb{k}} \mathbb{k}[\mathfrak{l} \times \mathfrak{g}] \otimes_{\mathbb{k}} D(\mathfrak{g})$.

$$D_{k,*}(\mathfrak{g}, \widehat{B}) := D_{k,#}(\mathfrak{g}) \wedge \wedge(E) \wedge \wedge_{b_{g}}(\widehat{B}).$$

Then $D_{k,*}(\mathfrak{g}, \widehat{B})$ is a module over the algebra

$$\hat{\mathcal{O}} \otimes_{\mathbb{k}} \mathbb{k}[\mathfrak{l} \times \mathfrak{g}] \otimes_{\mathbb{k}[\mathfrak{l} \times \mathfrak{l}]} \wedge(E).$$

The subspace $\hat{J}_{+}D_{k,*}(\mathfrak{g}, \widehat{B})$ is the kernel of the restriction morphism

$$D_{k,*}(\mathfrak{g}, \widehat{B}) \to D_{k,#}(\mathfrak{g}, \widehat{B}).$$

As a matter of fact, the powers of $\hat{J}_{+}$ induce a filtration of $D_{k,*}(\mathfrak{g}, \widehat{B})$ and the graded space associate to this filtration is isomorphic to

$$A \otimes_{\mathbb{k}[\mathfrak{l} \times p]} D_{k,#}(\mathfrak{g}, \widehat{B}) \quad \text{with} \quad A = \bigoplus_{l \in \mathbb{N}} \hat{J}_{+}^{l}/\hat{J}_{+}^{l+1}.$$

Then, by the above result on the cohomology of $D_{k,#}(\mathfrak{g}, \widehat{B})$ and the acyclicity of the complex $D_{k,*}^{*}(E)$, $D_{k,#}(\mathfrak{g}, \widehat{B})$ has no cohomology of degree different from $b_{g}$, whence Proposition 1.6 for the case $n \geq 2$ since $\hat{\mathcal{O}}$ is a faithfully flat extension of $\mathcal{O}$ and $D_{k,#}(\mathfrak{g}, \widehat{B})$ is $G$-equivariant.
1.4. Notations. • As usual $\mathbb{k}^* := \mathbb{k} \setminus \{0\}$. For $m$ positive integer and for $i = (i_1, \ldots, i_m)$ in $\mathbb{N}^m$, set:

$$|i| := i_1 + \cdots + i_m$$

and for $d$ in $\mathbb{N}$, denote by $\mathbb{N}_d^m$ the subset of $\mathbb{N}^m$:

$$\mathbb{N}_d^m := \{ i \in \mathbb{N}^m | |i| = d \}$$

The set $\mathbb{N}^m$ is ordered by the lexicographic order induced by the usual order of $\mathbb{N}$. As a result, all subset of $\mathbb{N}^m$ is well ordered.

• For $V$ a module over a $\mathbb{k}$-algebra, its dual is denoted by $V^*$ and its symmetric and exterior algebras are denoted by $S(V)$ and $\wedge(V)$ respectively. For all integer $i$, $S^i(V)$ and $\wedge^i(V)$ are the spaces of degree $i$ of $S(V)$ and $\wedge(V)$ with respect to the usual gradation. In particular, for $i$ negative, $S^i(V)$ and $\wedge^i(V)$ are equal to $\{0\}$. If $E$ is a subset of $V$, the submodule of $V$ generated by $E$ is denoted by $\text{span}(E)$. When $V$ is a vector space over $\mathbb{k}$, the grassmannian of all $d$-dimensional subspaces of $V$ is denoted by $\text{Gr}_d(V)$.

• All topological terms refer to the Zariski topology. If $Y$ is a subset of a topological space $X$, denote by $\overline{Y}$ the closure of $Y$ in $X$. For $Y$ an open subset of the algebraic variety $X$, $Y$ is called a big open subset if the codimension of $X \setminus Y$ in $X$ is at least 2. For $Y$ a closed subset of an algebraic variety $X$, its dimension is the biggest dimension of its irreducible components and its codimension in $X$ is the smallest codimension in $X$ of its irreducible components. For $X$ an algebraic variety, $\mathbb{k}[X]$ is the algebra of regular functions on $X$, $\mathcal{O}_X$ is its structural sheaf and for $x$ in $X$, $\mathcal{O}_{X,x}$ is the local ring of $X$ at $x$.

• All the complexes considered in this note are graded complexes over $\mathbb{Z}$ of vector spaces and their differentials are homogeneous of degree $\pm 1$ and they are denoted by $d$. As usual, the gradation of the complex is denoted by $C_*$ if the degree of $d$ is $-1$ and $C^*$ otherwise.

For $E$ a graded space over $\mathbb{Z}$ and for $i$ integer, $E[i]$ is the graded space over $\mathbb{Z}$ whose subspace of degree $n$ is the subspace of degree $n + i$ of $E$.

• The dimension of the Borel subalgebras of $\mathfrak{g}$ is denoted by $b_\mathfrak{g}$. Set $n := b_\mathfrak{g} - \ell$ so that

$$\dim \mathfrak{g} = 2b_\mathfrak{g} - \ell_\mathfrak{g} = 2n + \ell.$$ 

Denote by $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$.

• The dual $\mathfrak{g}^*$ of $\mathfrak{g}$ identifies with $\mathfrak{g}$ by a given non degenerate, invariant, symmetric bilinear form $\langle ., . \rangle$ on $\mathfrak{g} \times \mathfrak{g}$ extending the Killing form of $[\mathfrak{g}, \mathfrak{g}]$.

• Let $\mathcal{R}$ be the root system of $\mathfrak{h}$ in $\mathfrak{g}$ and $\mathcal{R}_+$ the positive root system of $\mathcal{R}$ defined by $\mathfrak{b}$. The Weyl group of $\mathcal{R}$ is denoted by $W(\mathcal{R})$ and the basis of $\mathcal{R}_+$ is denoted by $\Pi$. For $\alpha$ in $\mathcal{R}$, its coroot is denoted by $H_\alpha$, the corresponding root subspace is denoted by $\mathfrak{g}^\alpha$ and a generator $x_\alpha$ of $\mathfrak{g}^\alpha$ is chosen so that $\langle x_\alpha, x_{-\alpha} \rangle = 1$. 
• Let $\mathcal{P}(\mathcal{R})$ be the set of weights of the root system $\mathcal{R}$ and $\mathcal{P}_+(\mathcal{R})$ the subset of dominant weights with respect to $\mathcal{R}_+$. For $\lambda$ in $\mathcal{P}_+(\mathcal{R})$, denote by $V_\lambda$ a simple $\mathfrak{g}$-module of highest weight $\lambda$.

• Let $e$ be the sum of the $x_\beta$’s, $\beta$ in $\Pi$, and $h$ the element of $\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$ such that $\beta(h) = 2$ for all $\beta$ in $\Pi$. Then there exists a unique $f$ in $[\mathfrak{g}, \mathfrak{g}]$ such that $(e, h, f)$ is a principal $\mathfrak{sl}_2$-triple. The one parameter subgroup of $G$ generated by $\text{ad}h$ is denoted by $t \mapsto \rho(t)$. The Borel subalgebra containing $f$ is denoted by $\mathfrak{b}_-$ and its nilpotent radical is denoted by $\mathfrak{u}_-$. Let $B$ and $B_-$ be the normalizers of $\mathfrak{b}$ and $\mathfrak{b}_-$ in $G$, $U$ and $U_-$ the unipotent radicals of $B$ and $B_-$ respectively.

**Lemma 1.8.** Let $O$ be a principal open subset of $\mathfrak{h}$ and $\Sigma$ an irreducible hypersurface of $O + u$. Suppose that $\Sigma$ is invariant under the one-parameter subgroup $t \mapsto \rho(t)$ of $G$. Then $O$ is contained in $\Sigma$ or $\Sigma = \Sigma \cap \mathfrak{h} + u$.

*Proof.* As $O$ is a principal open subset of $\mathfrak{h}$, $\mathbb{K}[O + u]$ is a factorial ring. Hence $\Sigma$ is the nullvariety in $O + u$ of an element of $\mathbb{K}[O + u]$. As a result, $O$ is contained in $\Sigma$ or $\Sigma \cap \mathfrak{h}$ is an hypersurface of $O$. Suppose that $O$ is not contained in $\Sigma$. For $(x, y)$ in $\mathfrak{h} \times u$,

$$\lim_{t \to 0} \rho(t)(x + y) = x.$$  

Hence $\Sigma \cap \mathfrak{h}$ is the image of $\Sigma$ by the canonical projection $O + u \longrightarrow O$ and $\Sigma$ is contained in $\Sigma \cap \mathfrak{h} + u$. Moreover, $\Sigma \cap \mathfrak{h}$ is an irreducible hypersurface of $O$ as the image of an irreducible subset. Then $\Sigma \cap \mathfrak{h} + u$ is an irreducible hypersurface of $O$, whence the lemma. \(\square\)

• Let $\alpha$ a positive root, let $|\alpha|$ be its height so that $\alpha(h) = 2|\alpha|$.

• For $x \in \mathfrak{g}$, denote by $x_s$ its semisimple component, $x_n$ its nilpotent component and $\mathfrak{g}^x$ its centralizer in $\mathfrak{g}$. The set of regular elements of $\mathfrak{g}$ is

$$\mathfrak{g}_{\text{reg}} := \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = \ell\}.$$  

We denote by $\mathfrak{g}_{\text{reg, ss}}$ the set of regular semisimple elements of $\mathfrak{g}$. Then $\mathfrak{g}_{\text{reg}}$ and $\mathfrak{g}_{\text{reg, ss}}$ are $G$-invariant dense open subsets of $\mathfrak{g}$. According to [V72], $\mathfrak{g} \setminus \mathfrak{g}_{\text{reg}}$ is equidimensional of codimension 3.

• Let $\mathfrak{b}_0$ be the subset of elements $x$ of $\mathfrak{h}$ such that $\mathfrak{g}^x + \mathfrak{b}$ is a subalgebra of $\mathfrak{g}$. The orbits of $W(\mathcal{R})$ in $\mathfrak{h}$ have a nonempty intersection with $\mathfrak{b}_0$.

• Denote by $S(\mathfrak{g})^\mathfrak{b}$ the algebra of $\mathfrak{g}$-invariant elements of $S(\mathfrak{g})$. Let $p_1, \ldots, p_\ell$ be homogeneous generators of $S(\mathfrak{g})^\mathfrak{b}$ of degree $d_1, \ldots, d_\ell$ respectively. Choose the polynomials $p_1, \ldots, p_\ell$ so that $d_1 \leq \cdots \leq d_\ell$. For $i = 1, \ldots, \ell$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$, consider a shift of $p_i$ in direction $y$: $p_i(x + ty)$ with $t \in \mathbb{K}$. Expanding $p_i(x + ty)$ as a polynomial in $t$, one obtains

$$p_i(x + ty) = \sum_{m=0}^{d_i} p_i^{(m)}(x, y)t^m, \quad \forall (t, x, y) \in \mathbb{K} \times \mathfrak{g} \times \mathfrak{g} (1)$$
where \( y \mapsto (m!)p_i^{(m)}(x, y) \) is the derivative at \( x \) of \( p_i \) at the order \( m \) in the direction \( y \). The elements \( p_i^{(m)} \) defined by (1) are invariant elements of \( S(\mathfrak{g}) \otimes_k S(\mathfrak{g}) \) under the diagonal action of \( G \) in \( \mathfrak{g} \times \mathfrak{g} \).

Remark that \( p_i^{(0)}(x, y) = p_i(x) \) while \( p_i^{(d_i)}(x, y) = p_i(y) \) for all \((x, y)\) in \( \mathfrak{g} \times \mathfrak{g} \). Set:

\[
I_0 := \{(i, m) \in \{1, \ldots, \ell\} \times \mathbb{N} \mid 0 \leq m \leq d_i - 1\} \quad \text{and} \quad I_{\epsilon, 0} := I_0 \cap \{1, \ldots, \ell\} \times (\mathbb{N} \setminus \{0\}).
\]

According to our notations, \( I_0 \) is totally ordered.

**Remark 1.9.** The family \( \mathcal{P}_x := \{p_i^{(m)}(x, \cdot) \mid 1 \leq i \leq \ell, 0 \leq m \leq d_i\} \) for \( x \in \mathfrak{g} \), is a Poisson-commutative family of \( S(\mathfrak{g}) \) by Mishchenko-Fomenko [MF78]. One says that the family \( \mathcal{P}_x \) is constructed by the argument shift method.

- Let \( i \in \{1, \ldots, \ell\} \). For \( x \in \mathfrak{g} \), denote by \( \varepsilon_i(x) \) the element of \( \mathfrak{g} \) given by

\[
\langle \varepsilon_i(x), y \rangle = \frac{d}{dt} p_i(x + ty) \big|_{t=0}
\]

for all \( y \) in \( \mathfrak{g} \). Thereby, \( \varepsilon_i \) is an invariant element of \( S(\mathfrak{g}) \otimes_k \mathfrak{g} \) under the canonical action of \( G \). According to [Ko63, Theorem 9], for \( x \in \mathfrak{g} \), \( x \) is in \( \mathfrak{g}_{\text{reg}} \) if and only if \( \varepsilon_1(x), \ldots, \varepsilon_{\ell}(x) \) are linearly independent. In this case, \( \varepsilon_1(x), \ldots, \varepsilon_{\ell}(x) \) is a basis of \( \mathfrak{g}^\times \).

Denote by \( \varepsilon_i^{(m)} \), for \( 0 \leq m \leq d_i - 1 \), the elements of \( S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \mathfrak{g} \) defined by the equality:

\[
\varepsilon_i(x + ty) = \sum_{m=0}^{d_i-1} \varepsilon_i^{(m)}(x, y)t^m, \quad \forall (t, x, y) \in \mathbb{k} \times \mathfrak{g} \times \mathfrak{g}
\]

and set:

\[
V_{x,y} := \text{span}(\{\varepsilon_i^{(m)}(x, y), (i, m) \in I_0\})
\]

for \((x, y)\) in \( \mathfrak{g} \times \mathfrak{g} \).

- Let \( \mathfrak{N}_\mathfrak{g} \) be the nilpotent cone of \( \mathfrak{g} \). For \((x, y)\) in \( \mathfrak{g} \times \mathfrak{g} \), denote by \( P_{x,y} \) the subspace of \( \mathfrak{g} \) generated by \( x \) and \( y \). Let \( \mathcal{N}_\mathfrak{g} \) be the nilpotent bicone of \( \mathfrak{g} \). By definition, \( \mathcal{N}_\mathfrak{g} \) is the subset of elements \((x, y)\) of \( \mathfrak{g} \times \mathfrak{g} \) such that \( P_{x,y} \) is contained in \( \mathfrak{N}_\mathfrak{g} \). In particular, \( \mathcal{N}_\mathfrak{g} \) is invariant under the diagonal action of \( G \) in \( \mathfrak{g} \times \mathfrak{g} \) and the canonical action of \( \text{GL}_2(\mathbb{k}) \) in \( \mathfrak{g} \times \mathfrak{g} \).

**1.5. Organization of the note.** In Section 2, the characteristic submodule \( B_0 \) is introduced and some of its properties are given. In particular, its restrictions to parabolic subalgebras are considered. In Section 3, we prove that the main theorem and Theorem 1.2 results from Theorem 1.3. In Section 4, we prove that Theorem 1.3 results from Theorem 1.5 so that we suppose that \( \mathfrak{g} \) is simple in the following sections. In Section 5, we consider the support \( S_k \) in \( \mathfrak{g} \times \mathfrak{g} \) of the cohomology of degree different from \( b_0 \) of the complex \( D^*_k(\mathfrak{g}, B_0) \) and we prove that under some hypothesis on \( \sigma_1(S_k) \) the codimension of \( S_k \) in \( \mathfrak{g} \times \mathfrak{g} \) is at least \( k + 2 \) so that \( \mathfrak{g} \) has Property \( (P_k) \) if it has Property \( (P_{k-1}) \). In Section 6, we recall a result of flatness and prove that \( \sigma_i(S_1) \) does not contain \( z \) for \( i = 1, \ldots, k \) if \( D_{b, x}(\mathfrak{g}, \overline{B}) \) has no cohomology of degree different from \( b_0 \) for \( i = 1, \ldots, k \). In Section 7, we study the restriction \( \overline{B} \) of \( B_0 \) to \( l \times \mathfrak{p} \). In particular, \( \overline{B} \) has a
good decomposition with respect to the decomposition of \( p = I + p_u \) (see Proposition 7.6). Some other results are given so that the decomposition is more precise. This decomposition introduces some functions in \( \mathbb{k}[I, \times p] \) and their restrictions to \( I, \times p \cap h \times h \) are considered in Section 8 (see Proposition 8.5). In Section 9, we study the generator \( \varepsilon \) of \( \bigwedge^{b_h}(B) \). In particular, some of its coordinates in the canonical basis of \( \bigwedge^{b_h}(g) \) are considered. In Subsection 9.2, some square matrices whose the coefficients depend on these coordinates are considered and we prove that their determinant are different from 0 (see Corollary 9.4). In Section 10, Proposition 10.3 is a key result for proving Corollary 12.7 under the hypothesis \( n \leq 1 \) and in Section 11, Proposition 11.2 is a key result for proving Corollary 13.3 in the general case. In Section 12, we prove Proposition 1.6 for the case \( n \leq 1 \) and in Section 13, we prove Proposition 1.6 for the general case. Then we can complete the proof of Theorem 1.5.

The appendix has four sections. In Section A, we give some general results on some complexes. In Section B, we recall some well known results of cohomology. In Section C, we introduce some rational numbers and we prove that some of them are different from 0. In Section D, we prove the equalities of two \( g \)-submodules of a rational \( g \)-module under certain hypothesis. This result is used to prove Proposition 6.5.

2. Characteristic module

For \((x, y)\) in \( g \times g\), set:

\[
V'_{x,y} = \sum_{(a,b) \in \mathbb{k}^2 \setminus \{0\}} g^{ax+by}.
\]

By definition, the characteristic module \( B_g \) of \( g \) is the submodule of elements \( \varphi \) of \( S(g \times g) \otimes_{\mathbb{k}} g \) such that \( \varphi(x, y) \) is in \( V'_{x,y} \) for all \((x, y)\) in a dense subset of \( g \times g\). In this section, some properties of \( B_g \) are given.

2.1. First properties of \( B_g \). Denote by \( \Omega_g \) the subset of elements \((x, y)\) of \( g \times g\) such that \( P_{x,y} \) has dimension 2 and \( P_{x,y} \setminus \{0\} \) is contained in \( \mathfrak{g}_{\text{reg}} \). According to [CM08, Corollary 10], \( \Omega_g \) is a big open subset of \( g \times g\).

**Proposition 2.1.** Let \((x, y)\) be in \( g \times g\) such that \( P_{x,y} \cap \mathfrak{g}_{\text{reg}} \) is not empty.

(i) Let \( O \) be an open subset of \( \mathbb{k}^2 \) such that \( ax + by \) is in \( \mathfrak{g}_{\text{reg}} \) for all \((a, b)\) in \( O \). Then \( V_{x,y} \) is the sum of the \( g^{ax+by} \)'s, \((a, b) \in O\).

(ii) The spaces \([x, V_{x,y}]\) and \([y, V_{x,y}]\) are equal.

(iii) The space \( V_{x,y} \) has dimension at most \( b_h \) and the equality holds if and only if \((x, y)\) is in \( \Omega_g \).

(iv) The space \([x, V_{x,y}]\) is orthogonal to \( V_{x,y} \). Furthermore, \((x, y)\) is in \( \Omega_g \) if and only if \([x, V_{x,y}]\) is the orthogonal complement to \( V_{x,y} \) in \( g \).

(v) The space \( V_{x,y} \) is contained in \( V'_{x,y} \). Moreover, \( V_{x,y} = V'_{x,y} \) if \((x, y)\) is in \( \Omega_g \).

(vi) For \((i, m)\) in \( I_0 \), \( \varepsilon_i^{(m)} \) is a \( G \)-equivariant map.
Proof. (i) For \( z \) in \( \mathfrak{g}_{\text{reg}} \), \( \epsilon_1(z), \ldots, \epsilon_{\ell}(z) \) is a basis of \( \mathfrak{g}^z \) by [Ko63, Theorem 9]. Hence \( \mathfrak{g}^{ax+by} \) is contained in \( V_{x,y} \) for all \((a, b)\) in \( O \) since the maps \( \epsilon_1, \ldots, \epsilon_{\ell} \) are homogeneous. For pairwise different elements \( t_1, t_2, \ldots, t_{d_i} \), \( i = 1, \ldots, \ell \) of \( \mathbb{K} \), \( \epsilon^{(m)}_i(x, y) \) is a linear combination of \( \epsilon_i(x + t_{i,j}), \quad j = 0, \ldots, d_i - 1 \) for \( m = 0, \ldots, d_i - 1 \). We can choose \( t_1, t_2, \ldots, t_{d_i} \) so that \((a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_{i+d_i})\) are in \( O \) for some \( a_i \) in \( \mathbb{K}^* \), whence the assertion since the maps \( \epsilon_1, \ldots, \epsilon_{\ell} \) are homogeneous.

(ii) Let \( O \) be an open subset of \( \mathbb{K}^2 \) such that \( ax + by \) is in \( \mathfrak{g}_{\text{reg}} \) for all \((a, b)\) in \( O \). For all \((a, b)\) in \( O \), \([x, g^{ax+by}] = [y, g^{ax+by}] \) since \([ax + by, g^{ax+by}] = 0 \) and \( ab \neq 0 \), whence the assertion by (i).

(iii) According to [Bou02, Ch. V, §5, Proposition 3],
\[
d_1 + \cdots + d_\ell = b_0.
\]
So \( V_{x,y} \) has dimension at most \( b_0 \). By [Bol91, Theorem 2.1], \( V_{x,y} \) has dimension \( b_0 \) if and only if \((x, y)\) is in \( \Omega_\mathfrak{g} \).

(iv) According to [Bol91, Theorem 2.1], \( V_{x,y} \) is a totally isotropic subspace with respect to the skew bilinear form on \( \mathfrak{g} \)
\[
(v, w) \longmapsto \langle ax + by, [v, w] \rangle
\]
for all \((a, b)\) in \( \mathbb{K}^2 \). As a result, by invariance of \( \langle \cdot, \cdot \rangle \), \( V_{x,y} \) is orthogonal to \([x, V_{x,y}] \). If \((x, y)\) is in \( \Omega_\mathfrak{g} \), \( \mathfrak{g} \) has dimension \( \ell \) and it is contained in \( V_{x,y} \). Hence, by (iii),
\[
\dim [x, V_{x,y}] = b_0 - \ell = \dim \mathfrak{g} - \dim V_{x,y}
\]
so that \([x, V_{x,y}] \) is the orthogonal complement to \( V_{x,y} \) in \( \mathfrak{g} \). Conversely, if \([x, V_{x,y}] \) is the orthogonal complement to \( V_{x,y} \) in \( \mathfrak{g} \), then
\[
\dim V_{x,y} + \dim [x, V_{x,y}] = \dim \mathfrak{g}.
\]
Since \( P_{x,y} \cap \mathfrak{g}_{\text{reg}} \) is not empty, \( \mathfrak{g}^{ax+by} \cap V_{x,y} \) has dimension \( \ell \) for all \((a, b)\) in a dense open subset of \( \mathbb{K}^2 \). By continuity, \( \mathfrak{g}^{ax+by} \cap V_{x,y} \) has dimension at least \( \ell \) so that
\[
2\dim V_{x,y} - \ell \geq \dim \mathfrak{g}.
\]
Hence, by (iii), \((x, y)\) is in \( \Omega_\mathfrak{g} \).

(v) By (i), \( V_{x,y} \subset V_{x,y}^* \). Suppose that \((x, y)\) is in \( \Omega_\mathfrak{g} \). According to [Ko63, Theorem 9], for all \((a, b)\) in \( \mathbb{K}^2 \setminus \{0\} \), \( \epsilon_1(ax + by), \ldots, \epsilon_{\ell}(ax + by) \) is a basis of \( \mathfrak{g}^{ax+by} \). Hence \( \mathfrak{g}^{ax+by} \) is contained in \( V_{x,y} \), whence the assertion.

(vi) Let \( i \) be in \( \{1, \ldots, \ell\} \). Since \( p_i \) is \( G \)-invariant, \( \epsilon_i \) is a \( G \)-equivariant map. As a result, its 2-polarizations \( \epsilon_i^{(0)}, \ldots, \epsilon_i^{(d_i)} \) are \( G \)-equivariant under the diagonal action of \( G \) in \( \mathfrak{g} \times \mathfrak{g} \).

\[ \square \]

Theorem 2.2. (i) The module \( B_{b_0} \) is a free module of rank \( b_0 \) whose a basis is the sequence \( \epsilon_i^{(m)}, \ (i, m) \in I_0 \).

(ii) For \( \varphi \) in \( S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g} \), \( \varphi \) is in \( B_0 \) if and only if \( p\varphi \in B_{b_0} \) for some \( p \) in \( S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\} \).

(iii) For all \( \varphi \) in \( B_{b_0} \) and for all \((x, y)\) in \( \mathfrak{g} \times \mathfrak{g} \), \( \varphi(x, y) \) is orthogonal to \([x, y]\).
Proposition 2.3. According to Proposition 2.1(v), $e^{(m)}_i$ is in $B_\mathfrak{g}$ for all $(i, m)$. Moreover, according to Proposition 2.1(iii), these elements are linearly independent over $S(\mathfrak{g} \times \mathfrak{g})$. Let $\varphi$ be an element of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \mathfrak{g}$ such that $p\varphi$ is in $B_\mathfrak{g}$ for some $p$ in $S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$. Then $\varphi(x, y)$ is in $V_{x,y}$ for all $(x, y)$ in a dense open subset of $\Omega_\mathfrak{g}$ by Proposition 2.1(v). According to Proposition 2.1(iii), the map

$$
\Omega_\mathfrak{g} \twoheadrightarrow \text{Gr}_{B_\mathfrak{g}}(\mathfrak{g}), \quad (x, y) \mapsto V_{x,y}
$$

is regular. So $\varphi(x, y)$ is in $V_{x,y}$ for all $(x, y)$ in $\Omega_\mathfrak{g}$ and for some regular functions $a_{i,m}$, $(i, m) \in I_0$ on $\Omega_\mathfrak{g}$,

$$
\varphi(x, y) = \sum_{(i,m) \in I_0} a_{i,m}(x, y)e^{(m)}_i(x, y)
$$

for all $(x, y)$ in $\Omega_\mathfrak{g}$. Since $\Omega_\mathfrak{g}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$ and $\mathfrak{g} \times \mathfrak{g}$ is normal, the $a_{i,m}$'s have a regular extension to $\mathfrak{g} \times \mathfrak{g}$. Hence $\varphi$ is a linear combination of the $e^{(m)}_i$'s with coefficients in $S(\mathfrak{g} \times \mathfrak{g})$. As a result, the sequence $e^{(m)}_i(i, m) \in I_0$ is a basis of the module $B_\mathfrak{g}$ and $B_\mathfrak{g}$ is the subset of elements $\varphi$ of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \mathfrak{g}$ such that $p\varphi \in B_\mathfrak{g}$ for some $p$ in $S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$.

(iii) Let $\varphi$ be in $B_\mathfrak{g}$. According to (i) and Proposition 2.1(iv), for all $(x, y)$ in $\Omega_\mathfrak{g}$, $[x, \varphi(x, y)]$ is orthogonal to $V_{x,y}$. Then, since $y$ is in $V_{x,y}$, $[x, \varphi(x, y)]$ is orthogonal to $y$ and $\langle \varphi(x, y), [x, y] \rangle = 0$, whence the assertion. $\square$

2.2. Orthogonal of $B_\mathfrak{g}$. Denote again by $\langle \cdot, \cdot \rangle$ the canonical extension of $\langle \cdot, \cdot \rangle$ to the module $S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \mathfrak{g}$.

Proposition 2.3. Let $C_\mathfrak{g}$ be the orthogonal complement to $B_\mathfrak{g}$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \mathfrak{g}$.

(i) For $\varphi$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \mathfrak{g}$, $\varphi$ is in $C_\mathfrak{g}$ if and only if $\varphi(x, y)$ is in $[x, V_{x,y}]$ for all $(x, y)$ in a nonempty open subset of $\mathfrak{g} \times \mathfrak{g}$.

(ii) The module $C_\mathfrak{g}$ is free of rank $b_\mathfrak{g} - \ell$. Furthermore, the sequence of maps

$$(x, y) \mapsto [x, e^{(m)}_i(x, y)], \ (i, m) \in I_{*,0}$$

is a basis of $C_\mathfrak{g}$.

(iii) The orthogonal complement to $C_\mathfrak{g}$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \mathfrak{g}$ is equal to $B_\mathfrak{g}$.

Proof. (i) Let $\varphi$ be in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \mathfrak{g}$. If $\varphi$ is in $C_\mathfrak{g}$, then $\varphi(x, y)$ is orthogonal to $V_{x,y}$ for all $(x, y)$ in $\Omega_\mathfrak{g}$. Then, according to Proposition 2.1(iv), $\varphi(x, y)$ is in $[x, V_{x,y}]$ for all $(x, y)$ in $\Omega_\mathfrak{g}$. Conversely, suppose that $\varphi(x, y)$ is in $[x, V_{x,y}]$ for all $(x, y)$ in a nonempty open subset $O$ of $\mathfrak{g} \times \mathfrak{g}$. By Proposition 2.1(iv) again, for all $(x, y)$ in $O \cap \Omega_\mathfrak{g}$, $\varphi(x, y)$ is orthogonal to $e^{(m)}_i(x, y)$ for all $(i, m)$ in $I_0$, whence the assertion by Theorem 2.2(i).

(ii) Let $C$ be the submodule of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \mathfrak{g}$ generated by the maps

$$(x, y) \mapsto [x, e^{(m)}_i(x, y)], \ (i, m) \in I_{*,0}.$$
According to (i), C is a submodule of Cg. This module is free of rank b_g – ℓ since [x, V_{x,y}] has dimension b_g – ℓ for all (x, y) in Ω_g by Proposition 2.1, (iii) and (iv). According to (i), for φ in Cg, for all (x, y) in Ω_g,

φ(x, y) = \sum_{(i,m) \in I_0} a_{i,m}(x, y)[x, e_i^{(m)}(x, y)]

with the a_{i,m}’s regular on Ω_g and uniquely defined by this equality. Since Ω_g is a big open subset of g x g and g x g is normal, the a_{i,m}’s have a regular extension to g x g. As a result, φ is in C, whence the assertion.

(iii) Let φ be in the orthogonal complement to Cg in S(g \times g) \otimes_k g. According to (ii), for all (x, y) in Ω_g, φ(x, y) is orthogonal to [x, V_{x,y}]. Hence by Proposition 2.1(iv), φ(x, y) is in V_{x,y} for all (x, y) in Ω_g. So, by Theorem 2.2, φ is in B_g, whence the assertion. □

2.3. Restriction to a parabolic subalgebra. For a subalgebra of g, set a_{reg} := a \cap g_{reg}.

Lemma 2.4. Let a be an algebraic subalgebra of g.

(i) Suppose that a contains g^x for all x in a dense open subset of a and suppose that a_{reg} is not empty. Then V_{x,y} is contained in a for all (x, y) in a x a.

(ii) Suppose that a contains a Cartan subalgebra of g. Then V_{x,y} is contained in a for all (x, y) in a x a.

Proof. (i) By hypothesis, for all x in a dense open subset of a, x is a regular element and g^x is contained in a. So by [Ko63, Theorem 9], ε_1(x), ..., ε_ℓ(x) are in a for all x in a dense open subset of a. Then, so is it for all x in a by continuity. As a result, for all (x, y) in a x a, ε_i^{(m)}(x, y), (i, m) \in I_0 is in a, whence the assertion.

(ii) Let c be a Cartan subalgebra of g contained in a. Since a is an algebraic subalgebra of g, all semisimple element of a is conjugate under the adjoint group of a to an element of c. Hence for all x in g_{reg,ss} \cap a, g^x is contained in a, whence the assertion by (i) since g_{reg,ss} \cap a is a dense open subset of a. □

Let p be a parabolic subalgebra of g containing b. Denote by l its reductive factor containing b, \nu_u its nilpotent radical and σ the canonical projection p → l.

Corollary 2.5. For all (x, y) in p x p, V_{x,y} is contained in p. In particular, for all (x, y) in a dense open subset of b x b, V_{x,y} = b.

Proof. Since b is contained in p, for all (x, y) in p x p, V_{x,y} is contained in p by Lemma 2.4(ii). Since (h, e) is in Ω_g, Ω_g \cap b x b is a dense open subset of b x b, whence the corollary by Proposition 2.1(iii). □

Let l_{reg} be the subset of regular elements of l and Ω_l the subset of elements (x, y) of l x l such that P_{x,y} \setminus \{0\} is contained in l_{reg}. For (x, y) in l x l, the image of B_l by the evaluation map at (x, y)
is denoted by $V^l_{x,y}$. Set:

$$R_p := \varpi^{-1}(1_{\text{reg}}) \cap \mathfrak{g}_{\text{reg}}.$$  

**Lemma 2.6.** Let $R'_p$ be the subset of elements $x$ of $R_p$ such that $\mathfrak{g}^x \cap \mathfrak{p}_u = \{0\}$.

(i) The sets $\mathfrak{g}_{\text{reg}} \cap \mathfrak{p}$ and $R_p$ are big open subsets of $\mathfrak{p}$.
(ii) For all $x$ in $R_p$, $\varpi(\mathfrak{g}^x) = I^{\varpi(x)}$ if and only if $\mathfrak{g}^x \cap \mathfrak{p}_u = \{0\}$.
(iii) The set $R'_p$ is a dense open subset of $\mathfrak{p}$.
(iv) For all $(x, y)$ in $\mathfrak{p} \times \mathfrak{p}$, $V_{x,y}$ is contained in $V^l_{\varpi(x),\varpi(y)} + \mathfrak{p}_u$.
(v) For all $(x, y)$ in $R'_p \times \mathfrak{p}$, $\varpi(V_{x,y}) = V^l_{\varpi(x),\varpi(y)}$.

**Proof.** (i) According to [V72], $1_{\text{reg}}$ is a big open subset of $\mathfrak{l}$. Hence $\varpi^{-1}(1_{\text{reg}})$ is a big open subset of $\mathfrak{p}$. As a result, it remains to prove that $\mathfrak{g}_{\text{reg}} \cap \mathfrak{p}$ is a big open subset of $\mathfrak{p}$. Suppose that $\mathfrak{p} \setminus \mathfrak{g}_{\text{reg}}$ has an irreducible component $\Sigma$ of codimension 1 in $\mathfrak{p}$. A contradiction is expected. As $\mathfrak{g}_{\text{reg}} \cap \mathfrak{p}$ is a cone invariant under $B$, $\Sigma$ is a closed cone invariant under $B$. Since $k[\mathfrak{p}]$ is a factorial ring, for some $p$ in $k[\mathfrak{p}]$, homogeneous and relatively invariant under $B$, the nullvariety of $p$ in $\mathfrak{p}$ is equal to $\Sigma$. As a result, $\Sigma \cap \mathfrak{b}$ is an equidimensional closed cone of codimension 1 of $\mathfrak{b}$ since $\mathfrak{b} \cap \mathfrak{g}_{\text{reg}}$ is not empty. So, by Lemma 1.8, $\Sigma = \Sigma \cap \mathfrak{b} + \mathfrak{u}$ and $\mathfrak{u}$ is contained in $\Sigma$ since 0 is in $\Sigma \cap \mathfrak{b}$, whence a contradiction since $\mathfrak{g}_{\text{reg}} \cap \mathfrak{u}$ is not empty.

(ii) Let $x$ be in $R_p$. By Lemma 2.4(ii), $\mathfrak{g}^x$ is contained in $\mathfrak{p}$. As $\varpi$ is a surjective morphism of Lie algebra, $\varpi(\mathfrak{g}^x)$ is contained in $I^{\varpi(x)}$. Furthermore, dim $\varpi(\mathfrak{g}^x) = \ell$ if and only if $\mathfrak{g}^x \cap \mathfrak{p}_u = \{0\}$ since $\mathfrak{l}$ has rank $\ell$.

(iii) For $x$ regular semisimple in a Cartan subalgebra, contained in $\mathfrak{l}$, $x$ is in $R'_p$ since the elements of $\mathfrak{g}^x$ are semisimple. So $R'_p$ is not empty. The map $x \mapsto \mathfrak{g}^x$ from $R_p$ to $\text{Gr}_x(\mathfrak{g})$ is regular. So $R'_p$ is an open subset of $R_p$ and by (i).

(iv) Let $L_i$ be the submodule of elements $\varphi$ of $S(\mathfrak{l}) \otimes_k \mathfrak{l}$ such that $[\varphi(x), x] = 0$ for all $x$ in $\mathfrak{l}$. Then $L_i$ is a free module of rank $\ell$ according to [Di79]. Denote by $\varphi_1, \ldots, \varphi_\ell$ a basis of $L_i$. For $x$ in $R_p$ and for $i = 1, \ldots, \ell$, $\varpi \varphi_i(x)$ is in $I^{\varpi(x)}$. So there exists a unique element $(a_{i,1}(x), \ldots, a_{i,\ell}(x))$ of $k^\ell$ such that

$$\varpi \varphi_i(x) = a_{i,1}(x)\varphi_1 \circ \varpi(x) + \cdots + a_{i,\ell}(x)\varphi_\ell \circ \varpi(x).$$

The functions $a_{i,1}, \ldots, a_{i,\ell}$ so defined on $R_p$ have a regular extension to $\mathfrak{p}$ since they are regular, $\mathfrak{p}$ is normal and $R_p$ is a big open subset of $\mathfrak{p}$ by (i). As a result, for all $(x, y)$ in $\mathfrak{p} \times \mathfrak{p}$ and for all $(a, b)$ in $k^2$, $\varpi \varphi_i(ax + by)$ is a linear combination of the elements $\varphi_1(\varpi(ax + by)), \ldots, \varphi_\ell(\varpi(ax + by))$. Hence $\varpi(V_{x,y})$ is contained in $V^l_{\varpi(x),\varpi(y)}$ for all $(x, y)$ in $\mathfrak{p} \times \mathfrak{p}$, whence the assertion.

(v) Let $(x, y)$ be in $R'_p \times \mathfrak{p}$. By (iii), for all $z$ in a dense open subset of $P_{x,y}$, $z$ is in $R'_p$. So by (ii), $I^{\varpi(z)}$ is contained in $\varpi(V_{x,y})$ for all $z$ in a dense open subset of $P_{x,y}$. As a result, according to Proposition 2.1(i), $V^l_{\varpi(x),\varpi(y)}$ is contained in $\varpi(V_{x,y})$, whence the assertion by (iv).  

**Corollary 2.7.** For all $(x, y)$ in $\Omega_\mathfrak{g} \cap \mathfrak{p} \times \mathfrak{p}$, $V_{x,y} = V^l_{\varpi(x),\varpi(y)} + \mathfrak{p}_u$. 

Proof. As \((h, e)\) is in \(p \times p\), \(\Omega_3 \cap p \times p\) is a dense open subset of \(p \times p\). Let \((x, y)\) be in \(\Omega_3 \cap R'_p \times p\). By Lemma 2.6(v), \(\varpi(V_{x,y}) = V^l_{(x),\varpi(y)}\). Furthermore, \(\dim V_{x,y} = b_3\) since \((x, y)\) is in \(\Omega_3\). Hence \(p_v\) is contained in \(V_{x,y}\) and \(\dim V^l_{(x),\varpi(y)} = b_3\) since \(b_3 = b_l + \dim p_v\). According to Lemma 2.4(ii), the map \((x, y) \mapsto V_{x,y}\) is a regular map from \(\Omega_3 \cap p \times p\) to \(\text{Gr}_{b_3}(p)\). So, for all \((x, y)\) in \(\Omega_3 \cap p \times p\), \(p_v\) is contained in \(V_{x,y}\) and \(\dim \varpi(V_{x,y}) = b_l\), whence the assertion by Lemma 2.6(iv) since \(V^l_{(x),\varpi(y)}\) has dimension at most \(b_l\).

3. Proof of the main theorem

In this section, we prove that Theorem 1.1 and Theorem 1.2 results from Theorem 1.3. So we suppose that Theorem 1.3 is true for \(g\). By Definition, \(C_\bullet(g)\) is the graded submodule of \(S(g \times g) \otimes_k \wedge(g)\) such that \(C_{i+b_3}(g) := \wedge^i(g) \wedge \wedge^{b_3}(B_g)\) for \(i = 0, \ldots, n\). Let \(d\) be the \(S(g \times g)\)-derivation of the algebra \(S(g \times g) \otimes_k \wedge(g)\) such that for \(v \in g\), \(dv\) is the function on \(g \times g\), \((x, y) \mapsto \langle v, [x, y]\rangle\).

**Lemma 3.1.** (i) The graded module \(C_\bullet(g)\) is a graded subcomplex of \(S(g \times g) \otimes_k \wedge(g)\).

(ii) The ideal \(I_\delta\) is isomorphic to the space of boundaries of degree \(b_\delta\) of \(C_\bullet(g)\).

(iii) The support of the homology of \(C_\bullet(g)\) is contained in \(\partial(g)\).

**Proof.** (i) Set:

\[
\varepsilon := \wedge_{(i, m) \in I_0} \varepsilon_i^{(m)},
\]

where the order of the product is induced by the order of \(I_0\). Then \(C_\bullet(g)\) is the ideal of \(S(g \times g) \otimes_k \wedge(g)\) generated by \(\varepsilon\) since \(\varepsilon_i^{(m)}\), \((i, m) \in I_0\) is a basis of \(B_g\) by Theorem 2.2(i). According to Theorem 2.2(ii), for \((i, m)\) in \(I_0\), \(\varepsilon_i^{(m)}\) is a cycle of the complex \(S(g \times g) \otimes_k \wedge(g)\). Hence so is \(\varepsilon\) and \(C_\bullet(g)\) is a subcomplex of \(S(g \times g) \otimes_k \wedge(g)\) as an ideal generated by a cycle.

(ii) As for \(v \in g\), \(dv\) is in \(I_\delta\), \(I_\delta \varepsilon\) is the space of boundaries of degree \(b_\delta\) of \(C_\bullet(g)\).

(iii) Let \((x_0, y_0)\) be in \(g \times g \setminus \partial(g)\) and \(v \in g\) such that \(\langle v, [x_0, y_0}\rangle \neq 0\). For some affine open subset \(O\) of \(g \times g\), containing \((x_0, y_0)\), \(\langle v, [x, y]\rangle \neq 0\) for all \((x, y)\) in \(O\). Then \(dv\) is an invertible element of \(k[O]\). For \(c\) a cycle of \(k[O] \otimes_{S(g \times g)} C_\bullet(g)\),

\[
d(v \wedge c) = (dv)c
\]

so that \(c\) is a boundary of \(k[O] \otimes_{S(g \times g)} C_\bullet(g)\), whence the assertion. \(\square\)

**Theorem 3.2.** (i) The complex \(C_\bullet(g)\) has no homology of degree bigger than \(b_\delta\).

(ii) The ideal \(I_\delta\) has projective dimension \(2n-1\).

(iii) The algebra \(S(g \times g)/I_\delta\) is Cohen-Macaulay.

(iv) The projective dimension of the module \(\wedge^n(g) \wedge \wedge^{b_\delta}(B_g)\) is equal to \(n\).

**Proof.** (i) Let \(Z\) be the space of cycles of degree \(b_\delta + 1\) of \(C_\bullet(g)\), whence a graded subcomplex of \(C_\bullet(g)\),

\[
0 \longrightarrow C_{2n+\ell}(g) \longrightarrow \cdots \longrightarrow C_{n+\ell+2}(g) \longrightarrow Z \longrightarrow 0.
\]
According to Lemma 3.1(iii), the support of its homology is contained in $C_g$. In particular, its codimension in $g \times g$ is at least

$$4n + 2\ell - (2n + 2\ell) = 2n = n + n - 1 + 1$$

According to Theorem 1.3, for $i = n + \ell + 2, \ldots, 2n + \ell$, $C_{i+\ell}(g)$ has projective dimension at most $n$. Hence, by Corollary B.3, this complex is acyclic and $Z$ has projective dimension at most $2n - 2$, whence the assertion.

(ii) and (iii) Since $B_g$ is a free module of rank $b_g$, $\bigwedge^{b_g}(B_g)$ is a free module of rank 1. By definition, the short sequence

$$0 \to Z \to g \wedge \bigwedge^{b_g}(B_g) \to I_g \bigwedge^{b_g}(B_g) \to 0$$

is exact, whence the short exact sequence

$$0 \to Z \to g \wedge \bigwedge^{b_g}(B_g) \to I_g \to 0.$$ 

Moreover, by Theorem 1.3, $g \wedge \bigwedge^{b_g}(B_g)$ has projective dimension at most 1. Then, by (i) and Lemma B.5, $I_g$ has projective dimension at most $2n - 1$. As a result the $S(g \times g)$-module $S(g \times g)/I_g$ has projective dimension at most $2n$. Then by Auslander-Buchsbaum’s theorem [Bou98, §3, n°3, Théorème 1], the depth of the graded $S(g \times g)$-module $S(g \times g)/I_g$ is at least

$$4b_g - 2\ell - 2n = 2b_g$$

so that, according to [Bou98, §1, n°3, Proposition 4], the depth of the graded algebra $S(g \times g)/I_g$ is at least $2b_g$. In other words, $S(g \times g)/I_g$ is Cohen-Macaulay since it has dimension $2b_g$. Moreover, since the graded algebra $S(g \times g)/I_g$ has depth $2b_g$, the graded $S(g \times g)$-module $S(g \times g)/I_g$ has projective dimension $2n$. Hence $I_g$ is Cohen-Macaulay since it has dimension $2n$. Hence $I_g$ has projective dimension $2n - 1$.

(iv) As $I_g$ has projective dimension $2n - 1$, $\wedge^n(g) \wedge \bigwedge^{b_g}(B_g)$ has projective dimension $n$ by (i), Lemma B.5 and Theorem 1.3.

Theorem 1.2 is given by Theorem 3.2(i) and Lemma 3.1(ii) and Theorem 1.1 is a corollary of Theorem 3.2.

**Corollary 3.3.** The subscheme of $g \times g$ defined by $I_g$ is Cohen-Macaulay and normal. Furthermore, $I_g$ is a prime ideal.

**Proof.** According to Theorem 3.2(iii), the subscheme of $g \times g$ defined by $I_g$ is Cohen-Macaulay. According to [Po08, Theorem 1], it is smooth in codimension 1. So by Serre’s normality criterion [Bou98, §1, n°10, Théorème 4], it is normal. In particular, it is reduced and $I_g$ is radical. According to [Ri79], $\mathcal{C}(g)$ is irreducible. Hence $I_g$ is a prime ideal. □
4. Proof of Theorem 1.3

In this section we prove that Theorem 1.3 results from Theorem 1.5. Moreover, we prove that \(\mathfrak{s}_{2}(\mathfrak{k})\) has Property \((P)\). For \(k\) nonnegative integer, the complex \(D^{*}_{k}(g, B_{g})\) is given by Definition A.1 with \(V = g\) and \(L = B_{g}\) and for \(i = 1, \ldots, n\), \(C_{i+b_{g}}(g) := \bigwedge^{i}(g) \wedge \bigwedge^{b_{g}}(B_{g})\). The embedding of \(S^{k}(B_{g}) \otimes_{S(g \times g)} \bigwedge^{b_{g}}(B_{g})\) in \(D^{b_{g}}_{k}(g, B_{g})\) is an augmentation of \(D^{*}_{k}(g, B_{g})\). Denote by \(\overrightarrow{D}^{*}_{k}(g, B_{g})\) this augmented complex.

**Proposition 4.1.** Let \(k\) be a nonnegative integer:

(i) The complex \(\overrightarrow{D}^{*}_{k}(g, B_{g})\) has no cohomology of degree smaller than \(b_{g} + 1\).

(ii) For \(k = 0, 1\), \(\overrightarrow{D}^{*}_{k}(g, B_{g})\) is acyclic. In particular, \(\mathfrak{s}_{2}(\mathfrak{k})\) has Property \((P)\).

(iii) If \(g\) is simple and has Property \((P)\), then for \(i = 1, \ldots, n\), \(C_{i+b_{g}}(g)\) has projective dimension at most \(i\).

*Proof.* (i) By definition, \(\overrightarrow{D}^{*}_{k}(g, B_{g})\) has no cohomology of degree smaller than \(b_{g}\). According to Theorem 2.2, \(B_{g}\) is a free module of rank \(b_{g}\). Since \(\Omega_{g}\) is a big open subset of \(g \times g\) and \(g \times g\) is normal, \(\overrightarrow{D}^{*}_{k}(g, B_{g})\) has no cohomology of degree \(b_{g}\) for all positive integer \(k\) by Lemma A.4(iii).

(ii) By definition \(\overrightarrow{D}^{*}_{0}(g, B_{g})\) is acyclic. As \(\overrightarrow{D}^{*}_{1}(g, B_{g})\) has no cohomology of degree bigger than \(b_{g}\) by definition, \(\overrightarrow{D}^{*}_{1}(g, b_{g})\) is acyclic by (i). For \(g = \mathfrak{s}_{2}(\mathfrak{k})\), \(b_{g} = 2\) and \(\ell = 1\). Hence \(\mathfrak{s}_{2}(\mathfrak{k})\) has Property \((P)\).

(iii) Prove the proposition by induction on \(i\). By (i), \(C_{1+b_{g}}(g)\) has projective dimension at most 1. Suppose that \(C_{j+b_{g}}(g)\) has projective dimension at most \(j\) for \(j < i\). By (i) and Property \((P)\), the complex \(\overrightarrow{D}^{*}_{i}(g, B_{g})\) is acyclic. Then, by induction hypothesis and Corollary B.4, \(C_{i+b_{g}}(g)\) has projective dimension at most \(i\). \(\square\)

**Corollary 4.2.** Suppose that all simple factors of \(g\) have Property \((P)\). Then, for \(i = 1, \ldots, n\), \(C_{i+b_{g}}(g)\) has projective dimension at most \(i\).

*Proof.* Let \(\mathfrak{z}\) be the center of \(g\) and \(\mathfrak{d}\) the derived algebra of \(g\). Denote by \(\ell_{\mathfrak{d}}\) the rank of \(\mathfrak{d}\). As \(\mathfrak{z}\) is contained in \(B_{g}\), for \(i = 1, \ldots, n\), we have an isomorphism

\[
S(\mathfrak{z}) \otimes_{k} \bigwedge^{\dim_{k}(\mathfrak{z})} \otimes_{k} C_{i+b_{g}}(\mathfrak{d}) \rightarrow C_{i+b_{g}}(g).
\]

Hence the proposition for \(g\) results from the proposition for \(g = \mathfrak{d}\) since \(b_{g} - \ell_{\mathfrak{d}} = n\).

Denote by \(\mathfrak{d}_{1}, \ldots, \mathfrak{d}_{n}\) the simple factors of \(g = \mathfrak{d}\) and prove the proposition by induction on \(n\). For \(n = 1\), the proposition results from the hypothesis by Proposition 4.1(iii). Suppose \(n \geq 2\) and the proposition true for \(n - 1\). Let \(a\) be the direct product of \(\mathfrak{d}_{1}, \ldots, \mathfrak{d}_{n-1}\). From the equalities:

\[
\bigwedge^{b_{g}}(B_{g}) = \bigwedge^{b_{g}}(B_{a}) \wedge \bigwedge^{b_{a}}(B_{a})
\]

\[
\bigwedge^{t}(g) = \bigoplus_{j=0}^{t} \bigwedge^{j}(\mathfrak{a}) \wedge \bigwedge^{\ell-j}(\mathfrak{d}_{a})
\]

\[
b_{g} - \ell = b_{a} - \ell_{a} + b_{b_{a}} - \ell_{b_{a}}
\]
we deduce an isomorphism

\[ \bigoplus_{i=0}^i C_{j+b_i}(a) \otimes_k C_{i-j+b_{i+n}}(b_n) \cong C_{i+b_0}(g) \]

for \( i = 1, \ldots, n \). By induction hypothesis, \( C_{j+b_i}(a) \) has projective dimension at most \( j \). By the hypothesis and Proposition 4.1(iii), \( C_{i-j+b_{i+m}}(b_n) \) has projective dimension at most \( i-j \). Hence \( C_{i+b_0}(g) \) has projective dimension at most \( i \). \( \square \)

Let \( b, b_1, \ldots, b_n \) be as in the proof of Corollary 4.2. Set:

\[ n_1 := b_0 - \ell_{b_1}, \ldots, n_n := b_n - \ell_{b_n}, \quad \mathcal{I}' := \{(i_0, \ldots, i_n) \in \mathbb{N}^{n+1} | i_1 \leq n_1, \ldots, i_n \leq n_n\}, \]

\[ \mathcal{I}'' := \mathcal{I}' \cap \{0\} \times \mathbb{N}^n, \quad \mathcal{I}_k := \mathbb{N}^{n+1} \cap \mathcal{I}', \quad \mathcal{I}_k'' := \mathcal{I}_k \cap \mathcal{I}'' \]

for \( k \) nonnegative integer. The sets \( \mathcal{I}'' \) and \( \mathcal{I}_k'' \) identify with subsets of \( \mathbb{N}^n \). For \( I \) subset of \( \mathcal{I}_k' \) and \( i = 0, \ldots, k \), let \( I_i \) be the subset of elements \( i \) of \( \mathcal{I}_k'' \) such that \( (k-i, i) \) is in \( I \). For \( k = 0, \ldots, n \) and \( I \) subset of \( \mathcal{I}_k'' \), denote by \( D_{k,l,h}^*(b) \) the simple complex deduced from the multicomplex

\[ \bigoplus_{(j_1, \ldots, j_n) \in \mathcal{I}'} D_{j_1}^*(b_1) \otimes_k \cdots \otimes_k D_{j_n}^*(b_n). \]

Then \( D_{k,l,h}^*(b) \) is a graded subcomplex of \( D_{k}^*(b) \). For \( k = 0, \ldots, n \) and \( I \) subset of \( \mathcal{I}_k'' \), the simple complex \( D_{k,l,h}^*(g) \) deduced from the double complex

\[ \bigoplus_{i=0}^k D_{k,i}^*(g) \otimes_k D_{l,h}^*(b) \]

is a graded subcomplex of \( D_{k}^*(g) \) and

\[ D_{k,l,h}^*(g, B_0) := D_{k,l,h}^*(g)[-b_0] \wedge \wedge_{b_0}^0(B_0) \]

is a graded subcomplex of \( D_{k}^*(g, B_0) \). For simplicity, we set:

\[ D_{k,0}^*(g) := D_{k,0,0}^*(g) \quad \text{and} \quad D_{k,h}^*(g, B_0) := D_{k,h,0}^*(g, B_0). \]

**Lemma 4.3.** Suppose that \( b_1, \ldots, b_n \) have Property (P). Then for \( k = 0, \ldots, n \) and \( I \) subset of \( \mathcal{I}_k'' \), \( D_{k,l,h}^*(g, B_0) \) has no cohomology of degree different from \( b_0 \).

**Proof.** As \( B_0 \) is the \( S(b \times b) \)-submodule of \( S(b \times b) \otimes_k b \) generated by the direct sum

\[ B_{b_1} \oplus \cdots \oplus B_{b_n}, \]

the \( S(b \times b) \)-module \( \wedge_{b_0}^0(B_0) \) is isomorphic to

\[ \wedge_{b_0}^0(B_{b_1}) \otimes_k \cdots \otimes_k \wedge_{b_0}^0(B_{b_n}). \]
As a result, for \( j = 0, \ldots, n \) and \( J \) subset of \( \mathbb{I}_I^* \), the complex \( D_{j, I, \#}(B, B_g) \) is isomorphic to the simple complex deduced from the multicomplex

\[
\bigoplus_{(i_1, \ldots, i_n) \in J} D^*_{i_1}(B_{i_1}, B_{g_1}) \otimes \cdots \otimes D^*_{i_n}(B_{i_n}, B_{g_n}).
\]

Then \( D^*_{j, I, \#}(B, B_g) \) has no cohomology of degree different from \( b_0 \) since \( b_1, \ldots, b_n \) have Property (P). As \( \mathfrak{s} \) is contained in \( B_g \),

\[
D^*_{k, I, \#}(B, B_g) = \bigoplus_{i=0}^k S^{k-i}(\mathfrak{s}) \otimes \mathbb{C} D^*_{i, I, \#}(B, B_g) [-\dim \mathfrak{s}] \wedge \dim \mathfrak{s}.
\]

As a result, \( D^*_{j, I, \#}(B, B_g) \) has no cohomology of degree different from \( b_0 \) since \( b_0 = b_0 + \dim \mathfrak{s} \). \( \square \)

For \( I \) subset of \( \mathbb{I}_I^* \) and \( j = 0, \ldots, n \), let \( I_{j, *}, \) be the subset of elements \( i \) of \( I \) such that \( i_j > 0 \) and \( I_{j, -} \) the image of \( I_{j, *} \) by the map

\[
\mathbb{I}_I^{n+1} \xrightarrow{i \mapsto (i_0, \ldots, i_{j-1}, i_j - 1, i_{j+1}, \ldots, i_n)}. \]

Denote by \( I_{-} \) the union of \( I_{0, -}, \ldots, I_{n, -} \). For \( k = 0, \ldots, n \), let \( K^*_{k, I}(B, B_g) \) be the kernel of the morphism

\[
S(\mathfrak{s} \times \mathfrak{s}) \otimes \mathbb{C} D^*_{k, I, \#}(\mathfrak{s}) \xrightarrow{\mathbb{C} \otimes \mathbb{C}} D^*_{k, I, \#}(B, B_g)[b_0], \quad \varphi \mapsto \varphi \wedge \varepsilon.
\]

In particular, \( K^*_{k, I}(B, B_g) \) is a graded subcomplex of \( S(\mathfrak{s} \times \mathfrak{s}) \otimes \mathbb{C} D^*_{k, I, \#}(\mathfrak{s}) \).

**Proposition 4.4.** Suppose that \( \mathfrak{d}_1, \ldots, \mathfrak{d}_n \) have Property (P). Let \( k = 1, \ldots, n \) and \( I \subset \mathbb{I}_k^* \). Then \( K^*_{k, I}(B, B_g) \) is equal to \( S(\mathfrak{s} \times \mathfrak{s}) \otimes \mathbb{C} D^*_{k, I, \#}(\mathfrak{s}) \cap D^*_{k-1, I, \#}(\mathfrak{s})[-1] \wedge B_g \).

**Proof.** Since \( S(\mathfrak{s} \times \mathfrak{s}) \otimes \mathbb{C} D^*_{k, I, \#}(\mathfrak{s}) \cap D^*_{k-1, I, \#}(\mathfrak{s})[-1] \wedge B_g \) is clearly contained in \( K^*_{k, I}(B, B_g) \), it is sufficient to prove that \( K^*_{k, I}(B, B_g) \) is contained in \( D^*_{k-1, I, \#}(\mathfrak{s}) \wedge B_g \). Prove the assertion by induction on \( k \). For \( k = 1 \), it is true by Proposition 4.1(i). Suppose \( k > 1 \) and \( K^*_{k-1, I}(B, B_g) \) contained in \( D^*_{k-1, I, \#}(\mathfrak{s})[-1] \wedge B_g \) for \( j < k \) and \( J \subset \mathbb{I}_j^* \).

Let \( j = 1, \ldots, k-1 \). For \( \nu = (i_0, \ldots, i_n) \) in \( \mathbb{I}_k^{n+1} \), set:

\[
V_{\nu} := S^\nu(\mathfrak{s}) \otimes \mathbb{C} S^{i_1}(\mathfrak{d}_1) \otimes \cdots \otimes \mathbb{C} S^{i_n}(\mathfrak{d}_n)
\]

\[
I_{\nu} := \{(i_0, \ldots, i_n) \in \mathbb{I}_k^{n+1} | (i_0 + l_0, \ldots, i_n + l_n) \in I\}.
\]

In particular, \( I_{\nu} \) is contained in \( \mathbb{I}_j^* \) when it is not empty. Then

\[
D^j_{k, I, \#}(\mathfrak{s}) = \bigoplus_{\nu \in \mathbb{I}_k^{n+1}} V_{\nu} \otimes D^j_{k, I, \#}(\mathfrak{s}).
\]

So, by induction hypothesis, \( K^j_{k, I}(B, B_g) \) is contained in \( D^j_{k-1, I, \#}(\mathfrak{s}) \wedge B_g \).
We have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & K_{k,l}^k(g, B_g) & \rightarrow & S(g \times g) \otimes_k D_{k,l,\#}^k(g) & \rightarrow & D_{k,l,\#}^{k+b_0}(g, B_g) & \rightarrow & 0 \\
\downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \rightarrow & K_{k,l}^{k-1}(g, B_g) & \rightarrow & S(g \times g) \otimes_k D_{k,l,\#}^{k-1}(g) & \rightarrow & D_{k,l,\#}^{k-1+b_0}(g, B_g) & \rightarrow & 0 \\
\downarrow d & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \rightarrow & K_{k,l}^{k-2}(g, B_g) & \rightarrow & S(g \times g) \otimes_k D_{k,l,\#}^{k-2}(g) & \rightarrow & D_{k,l,\#}^{k-2+b_0}(g, B_g) & \rightarrow & 0
\end{array}
\]

By definition, the rows are exact, by Lemma 4.3, the right column is exact and by Lemma A.2(ii), the middle column is exact. Denoting by \(\delta\) the horizontal arrows, for \(a \in K_{k,l}^k(g, B_g)\), \(a = db\) for some \(b\) in \(S(g \times g) \otimes_k D_{k,l,\#}^{k-1}(g)\), \(\delta b = dc\) for \(c\) in \(D_{k,l,\#}^1(g, B_g)\) and \(c = \delta c'\) for some \(c'\) in \(S(g \times g) \otimes_k D_{k,l,\#}^{k-1}(g)\), whence \(b - dc' = \delta b'\) for some \(b'\) in \(K_{k,l}^{k-1}(g, B_g)\) and \(a = db'\). As \(dD_{k-1,l,\#}^{k-2}(g) \land B_g = D_{k-1,l,\#}^{k-1}(g) \land B_g\), \(a\) is in \(D_{k-1,l,\#}^{k-1}(g) \land B_g\) since \(K_{k,l}^{k-1}(g, B_g)\) is contained in \(D_{k-1,l,\#}^{k-2}(g) \land B_g\), whence the proposition.

As it remains to prove Theorem 1.5, we suppose \(g\) simple of rank at least 2 in the following sections.

5. First step to the proof of Theorem 1.5

Let \(\varpi_1\) be the first projection \(g \times g \rightarrow g\). For \(k = 1, \ldots, n\), denote by \(S_k\) the support in \(g \times g\) of the cohomology of \(D_k^*(g, B_g)\) of degree different from \(b_0\).

Lemma 5.1. Let \(k = 1, \ldots, n\).

(i) The set \(S_k\) is a closed subset of \(g \times g\) invariant under the actions of \(G\) and \(GL_2(k)\) in \(g \times g\).

(ii) The subset \(\varpi_1(S_k)\) of \(g\) is closed and \(G\)-invariant.

(iii) If \(\varpi_1(S_k) \cap h = \{0\}\), then \(S_k\) has codimension at least \(n + 2\) in \(g \times g\).

Proof. (i) According to Proposition 2.1(vi), \(B_g\) is a free module generated by a basis of \(G\)-invariant elements. Moreover, by definition, \(B_g\) is invariant under the action of \(GL_2(k)\). Hence \(\bigwedge^{b_0}(B_g)\) is generated by a \(G\times GL_2(k)\)-semi-invariant element. Then, as the differential of \(D^*(g)\) is \(G\times GL_2(k)\)-equivariant, so is the differential of \(D_k^*(g, B_g)\). Hence \(S_k\) is invariant under \(G\times GL_2(k)\).

(ii) As \(S_k\) is invariant under \(k^* \times k^*\), \(\varpi_1(S_k) \times \{0\} = S_k \cap g \times \{0\}\) so that \(\varpi_1(S_k)\) is a closed subset of \(g\). As \(S_k\) is \(G\)-invariant so is \(\varpi_1(S_k)\).

(iii) Suppose \(\varpi_1(S_k) \cap h = \{0\}\). By (ii), \(\varpi_1(S_k)\) is contained in the nilpotent cone \(N_0\) of \(g\). Then \(S_k\) is contained in the nilpotent bicone \(N_0\) since \(S_k\) is invariant under the action of \(GL_2(k)\).
As result, $S_k$ has codimension at least $n + 2$ in $\mathfrak{g} \times \mathfrak{g}$ since $N_\mathfrak{g}$ has dimension $3n$ by [CMo08, Theorem 1.2]. □

**Proposition 5.2.** Let $k = 2, \ldots, n$. Suppose that $\mathfrak{g}$ has Property $(P_{k-1})$ and $\varpi_1(S_k) \cap \mathfrak{h} = \{0\}$. Then $\mathfrak{g}$ has Property $(P_k)$.

**Proof.** According to Proposition 4.1(i), for $i = 1, \ldots, k - 1$, the augmented complex $\overline{D}_i^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{B}_\mathfrak{g})$ is acyclic since $\mathfrak{g}$ has Property $(P_{k-1})$. So, by Corollary B.4, for $i = 1, \ldots, k - 1$, $\wedge^i(\mathfrak{g}) \wedge \wedge^b_\mathfrak{g}(\mathfrak{B}_\mathfrak{g})$ has projective dimension at most $i$. Again by Proposition 4.1(i), $S_k$ is the support in $\mathfrak{g} \times \mathfrak{g}$ of the cohomology of $\overline{D}_k^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{B}_\mathfrak{g})$. By hypothesis and Lemma 5.1(iii), $S_k$ has codimension at least $k + 2$ in $\mathfrak{g} \times \mathfrak{g}$. So, by Corollary B.4, $\overline{D}_k^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{B}_\mathfrak{g})$ is acyclic, whence the proposition. □

**Remark 5.3.** By Proposition 4.1(ii), $\mathfrak{g}$ has Property $(P_1)$. So, by Proposition 5.2 and an induction argument, $\mathfrak{g}$ has Property $(P)$ if for $i = 2, \ldots, n$, $\varpi_1(S_i) \cap \mathfrak{h} = \{0\}$.

6. **At the neighborhood of a semisimple element**

Let $\mathfrak{z}$ be in $\mathfrak{b}_\mathfrak{g} \setminus \{0\}$ and $\mathfrak{p} := \mathfrak{g}^\mathfrak{z} + \mathfrak{b}$. Then $\mathfrak{l} := \mathfrak{g}^\mathfrak{z}$ is the reductive factor of $\mathfrak{p}$ containing $\mathfrak{b}$. Denoting by $\mathfrak{z}$ the center of $\mathfrak{l}$, $\mathfrak{d}$ the derived algebra of $\mathfrak{l}$ and $\delta$ the function on $\mathfrak{l}$,

$$x \mapsto \delta(x) = \det(\text{ad} x|_{[\mathfrak{x}, \mathfrak{d}]}) ,$$

let $\mathfrak{l}_S$ be the complement in $\mathfrak{l}$ to the nullvariety of $\delta$.

6.1. **A smooth morphism.** Let $\theta$ be the morphism:

$$G \times \mathfrak{l}_S \times \mathfrak{g} \rightarrow G(\mathfrak{l}_S) \times \mathfrak{g} \quad (g, x, y) \mapsto (g(x), g(y)) .$$

**Lemma 6.1.** Set $\Omega_\mathfrak{g} := \Omega_\mathfrak{g} \cap \mathfrak{l}_S \times \mathfrak{g}$.

(i) The subset $G(\mathfrak{l}_S)$ of $\mathfrak{g}$ is open and $\theta$ is a faithfully flat morphism.

(ii) The subset $\Omega_\mathfrak{g}$ of $\mathfrak{l}_S \times \mathfrak{g}$ is a big open subset.

**Proof.** (i) Since the map $(g, x) \mapsto g(x)$ from $G \times \mathfrak{l}_S$ to $\mathfrak{g}$ is a submersion, $G(\mathfrak{l}_S)$ is an open subset of $\mathfrak{g}$ and this map is a smooth surjective morphism from $G \times \mathfrak{l}_S$ to $G(\mathfrak{l}_S)$. As a result, $\theta$ is a faithfully flat morphism from $G \times \mathfrak{l}_S$ onto the open subset $G(\mathfrak{l}_S) \times \mathfrak{g}$ of $\mathfrak{g} \times \mathfrak{g}$ since the endomorphism of $G \times \mathfrak{l}_S \times \mathfrak{g}$, $(g, x, y) \mapsto (g(x), g(y))$ is an isomorphism.

(ii) By (i), the fibers of $\theta$ are equidimensional of dimension $\dim \mathfrak{l}$. Hence $\Omega_\mathfrak{g}$ is a big open subset of $\mathfrak{l}_S \times \mathfrak{g}$ since $\Omega_\mathfrak{g}$ is a $G$-invariant big open subset of $\mathfrak{g} \times \mathfrak{g}$.

For any $\mathfrak{g} \times \mathfrak{g}$-module $M$, denote by $\mathcal{M}$ the restriction to $G(\mathfrak{l}_S) \times \mathfrak{g}$ of the localization of $M$ on $\mathfrak{g} \times \mathfrak{g}$ and $\overline{M}$ the space of global sections of $\theta^*(\mathcal{M})$.

**Corollary 6.2.** Let $M$ be a $\mathfrak{g} \times \mathfrak{g}$-module and $N$ a submodule of $M$. The modules $\overline{M}$ and $\overline{N}$ are equal if and only if $G(\mathfrak{l}_S) \times \mathfrak{g}$ has an empty intersection with the support of $M/N$. 
Proof. Denote by $M'$ the restriction to $G(l_*) \times g$ of the localization on $g \times g$ of $M/N$. As the localization functor is exact, we have a short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M' \longrightarrow 0.$$ 

By Lemma 6.1(i), $\theta$ is a faithfully flat morphism from $G \times l_* \times g$ to $G(l_*) \times g$. Hence the short sequence

$$0 \longrightarrow \theta^*(N) \longrightarrow \theta^*(M) \longrightarrow \theta^*(M') \longrightarrow 0$$

is exact and $M' = 0$ if and only if $\theta^*(M') = 0$. Moreover, $\theta^*(M') = 0$ if and only if $\overline{M} = \overline{N}$ since $G \times l_* \times g$ is an affine variety. Hence $\overline{M} = \overline{N}$ if and only if the support of $M/N$ in $g \times g$ has an empty intersection with $G(l_*) \times g$.

6.2. As $l + b$ is a parabolic subalgebra of $g$, so is $p_- := l + b_-$. Let $b_1, \ldots, b_n$ be the simple factors of $b$. Denote by $p_{z,u}$ the complement to $l$ in $g$, invariant under $\text{ad} b$ and $p_{z,u}$ the nilpotent radical of $p_-$. Set:

$$l_0 := z + p_{z,u}, \quad l_{0,-} := z + p_{z,u}, \quad n_1 := b_{b_1} - \ell_{b_1}, \ldots, n_n := b_{b_n} - \ell_{b_n},$$

$$I := \{(i_1, \ldots, i_n) \in \mathbb{N}^{n+2} \mid i_1 \leq n_1, \ldots, i_n \leq n_n\}, \quad \mathbb{I}_k := \mathbb{N}^{n+2} \cap I$$

for $k$ nonnegative integer. The sets $I'$ and $\mathbb{I}_k'$ of Section 4 identify with $I \cap \{0\} \times \mathbb{N}^{n+1}$ and $\mathbb{I}_k \cap \{0\} \times \mathbb{N}^{n+1}$ respectively. According to the notations of Section 4, for $k = 1, \ldots, n$, denote by $D_{k,\#}^*(g)$ the simple complex deduced from the double complex

$$\bigoplus_{i=0}^k D_{k-i}^*(p_{z,u}) \otimes_k D_{i,\#}^*(l).$$

Then $D_{k,\#}^*(g)$ is a graded subcomplex of $D_{k}^*(g)$.

Denoting by $\mathcal{B}$ the localization on $g \times g$ of $B$, let $\overline{\mathcal{B}}$ be the space of global section of $\theta^*(\mathcal{B})$. In particular, $B$ is a free submodule of rank $b_0$ of $\mathbb{K}[G \times l_* \times g] \otimes_k g$. For $k = 1, \ldots, n$, set:

$$B := \mathbb{K}[l_* \times g] \otimes_{\mathbb{K}(g \times g)} B,$$

$$C := \mathbb{K}[l_* \times g] \otimes_{\mathbb{K}(g \times g)} C,$$

$$D_{k,\#}^*(g, \mathcal{B}) := D_{k,\#}^*(g)[\ldots] \wedge \wedge^{b_0} \mathcal{B},$$

$$D_{k,\#}^*(g, B) := D_{k,\#}^*(g)[\ldots] \wedge \wedge^{b_0} B.$$

Lemma 6.3. Let $k = 1, \ldots, n$.

(i) The orthogonal complement to $B$ in $\mathbb{K}[l_* \times g] \otimes_k g$ is equal to $C$ and the orthogonal complement to $C$ in $\mathbb{K}[l_* \times g] \otimes_k g$ is equal to $B$.

(ii) The set $\sigma_1(S_k)$ does not contain $z$ if and only if $D_{k,\#}^*(g, \mathcal{B})$ has no cohomology of degree different from $b_0$. 


Lemma 6.4. Let $k$ be in $\mathbb{k}[I_+ \times \mathfrak{g}] \otimes \mathbb{k} \mathfrak{g}$ orthogonal to $B$ and $C$ respectively. For $(x, y)$ in $\Omega$, the orthogonal complement to $V_{x, g}$ is equal to $[x, V_{x, g}]$ by Proposition 2.1(iv). Then, for some regular functions $a_{i,m}, (i, m) \in I_{+0}$ and $b_{i,m}, (i, m) \in I_0$, $\varphi(x, y) = \sum_{(i, m) \in I_{+0}} a_{i,m}(x, y)[x, \varepsilon_i^{(m)}(x, y)]$ and $\psi(x, y) = \sum_{(i, m) \in I_0} b_{i,m}(x, y)\varepsilon_i^{(m)}(x, y)$ for all $(x, y)$ in $\Omega$. By Lemma 6.1(ii), $\Omega$ is a big open subset of $I_+ \times \mathfrak{g}$. So, the regular functions $a_{i,m}, (i, m) \in I_{+0}$ and $b_{i,m}, (i, m) \in I_0$ have a regular extension to $I_+ \times \mathfrak{g}$ since $I_+ \times \mathfrak{g}$ is a normal variety. As a result, $\varphi$ and $\psi$ are in $C$ and $B$ respectively, whence the assertion.

(ii) For $j$ integer, denote by $Z^j_k$ and $B^j_k$ the spaces of cocycles and coboundaries of degree $j$ of $D^j_k(\mathfrak{g}, B)$. By Lemma 6.1(i), the short exact sequence $\begin{CD} 0 @>>> Z^j_k @>>> D^j_k(\mathfrak{g}, B) @>>> B^{j+1}_k @>>> 0 \end{CD}$ for $j$ integer, we deduce the short exact sequence $\begin{CD} 0 @>>> \overline{Z}^j_k @>>> \overline{D}^j_k(\mathfrak{g}, B) @>>> \overline{B}^{j+1}_k @>>> 0 \end{CD}$ since $G \times I_+ \times \mathfrak{g}$ is an affine variety. As $\overline{B}$ is the space of global sections of $\Theta^*(\mathbb{P})$, $D^j_k(\mathfrak{g}, \overline{B}) = \overline{D}^j_k(\mathfrak{g}, \overline{B})$. Hence $Z^j_k$ and $B^j_k$ are the spaces of cocycles and coboundaries of degree $j$ of $D^j_k(\mathfrak{g}, B)$. By Corollary 6.2, $z \notin \sigma_1(S)$ if and only $Z^j_k = B^j_k$ for $j \neq b$, whence the assertion. \( \square \)

Denote by $\mathcal{O}$ the localization at the identity of $\mathbb{k}[G]$.

Lemma 6.4. Let $k = 1, \ldots, n$. Suppose that for $i = 1, \ldots, k$, $\mathcal{O} \otimes_{\mathbb{k}[G]} D^i_{*, k}(\mathfrak{g}, \overline{B})$ has no cohomology of degree different from $b$. Then, for $i = 1, \ldots, k$ and $\varphi$ in $\mathcal{O} \otimes_{\mathbb{k}[G]} \mathbb{k}[I_+ \times \mathfrak{g}] \otimes \mathbb{k} D^i_{l, k}(\mathfrak{g})$ $\varphi \wedge \wedge^{b}(\overline{B}) = [0] \implies \varphi \in \mathcal{O} \otimes_{\mathbb{k}[G]} D^i_{l-1, k}(\mathfrak{g}) \wedge \overline{B}$.

Proof. Let $\tilde{\varepsilon}$ be a generator of $\wedge^{b}(\overline{B})$. Prove the lemma by induction on $i$. By Proposition A.7, it is true for $i = 1$. Suppose that it is true for $i - 1$. For some $\psi$ in $\mathcal{O} \otimes_{\mathbb{k}[G]} \mathbb{k}[I_+ \times \mathfrak{g}] \otimes \mathbb{k} D^{i-1}_{l, k}(\mathfrak{g})$, $\varphi = d\psi$. Then $\varphi \wedge \tilde{\varepsilon}$ is a cocycle of degree $i - 1$ of $\mathcal{O} \otimes_{\mathbb{k}[G]} D^i_{l, k}(\mathfrak{g}, \overline{B})$. So, by hypothesis, for some $\psi'$ in $\mathcal{O} \otimes_{\mathbb{k}[G]} D^{i-2}_{l, k}(\mathfrak{g})$, $\psi \wedge \tilde{\varepsilon} = d\psi' \wedge \tilde{\varepsilon}$. Then, by induction hypothesis, $\psi - d\psi' \in \mathcal{O} \otimes_{\mathbb{k}[G]} D^{i-2}_{l, k}(\mathfrak{g}) \wedge \overline{B}$, whence the lemma. \( \square \)

Proposition 6.5. Let $k = 1, \ldots, n$. Suppose that the following condition is satisfied: for $i = 1, \ldots, k$, $\mathcal{O} \otimes_{\mathbb{k}[G]} D^i_{*, k}(\mathfrak{g}, \overline{B})$ has no cohomology of degree different from $b$. 

(i) For $i = 1, \ldots, k$, $D^i_{*}(\mathfrak{g}, \overline{B})$ has no cohomology of degree different from $b$. 

(ii) For $i = 1, \ldots, k$, $z$ is not in $\sigma_1(S)$.
Proof. (i) We consider the action of $G$ in $\mathbb{k}[G \times \mathfrak{l} \times \mathfrak{g}] \otimes_\mathbb{k} \mathfrak{g}$ given by

$$k.a\otimes (g, x, y) := a(k^{-1}g, x, y)k.v.$$  

This action has a natural extension to $\mathbb{k}[G \times \mathfrak{l} \times \mathfrak{g}] \otimes_\mathbb{k} D(\mathfrak{g})$ and $D_i^\ast(\mathfrak{g}, \overline{B})$ is invariant under this action for $i = 1, \ldots, k$ since $B_\mathfrak{g}$ is a $G$-invariant submodule of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{k} \mathfrak{g}$ by Proposition 2.1(vi). This action induces an action of $\mathfrak{g}$ in $\mathfrak{g} \otimes_\mathbb{k} \mathfrak{k} \otimes_\mathbb{k} D(\mathfrak{g})$ and $\mathfrak{g} \otimes_\mathbb{k} D_i^\ast(\mathfrak{g}, \overline{B})$ is invariant under this action.

Prove the assertion by induction on $i$. By Proposition 4.1(ii) and Lemma 6.3(ii), it is true for $i = 1$. Suppose $i > 1$ and the assertion true for the integers smaller than $i$. Set:

$$E' := \mathfrak{g} \otimes_\mathbb{k} \mathfrak{k} \otimes_\mathbb{k} D_i^\ast(\mathfrak{g}), \quad N_i' := \mathfrak{g} \otimes_\mathbb{k} \mathfrak{k} \otimes_\mathbb{k} D_i^\ast(\mathfrak{g}), \quad N_j := N_j' \cap M$$

for $j = 1, 2$. Then $M$ is a rational $\mathfrak{g}$-module since so is $\mathbb{k}[G]$, $M'$ is a $l$-submodule of $M$, $N_1$ and $N_2$ are $\mathfrak{g}$-submodules of $M$ such that $N_1$ is contained in $N_2$ and $N_2 \cap M'$ is contained in $N_1 \cap M'$. By [C20, Theorem 1.1], $D_i^\ast(\mathfrak{g})$ generates the $G$-module of $\mathfrak{g}$. So $M'$ generates the $G$-module $M$ since $\mathbb{k}[G] \otimes_\mathbb{k} \mathfrak{k} \otimes_\mathbb{k} \mathfrak{k} \otimes_\mathbb{k} \mathfrak{g}$ is a $G$-module. Then, by Proposition D.6, $N_1 = N_2$.

For $a$ in $N_2'$, $pa$ is in $N_2$ for some $p$ in $\mathbb{k}[G]$ such that $p(1_\mathfrak{g}) \neq 0$. Hence $pa$ is in $N_1$ and $a$ is in $N_1'$. As a result, $N_1' = N_2'$ since $N_1'$ is contained in $N_2'$. Then, again by Proposition A.7 and the induction hypothesis, the complex $\mathfrak{g} \otimes_\mathbb{k} D_i^\ast(\mathfrak{g}, \overline{B})$ has no cohomology of degree different from $b_\mathfrak{g}$. As a result, the support in $G \times \mathfrak{l} \times \mathfrak{g}$ of the cohomology of degree different from $\mathfrak{b}_\mathfrak{g}$ of $D_i^\ast(\mathfrak{g}, \overline{B})$ does not contain $\{1_\mathfrak{g}\} \times \mathfrak{g}$ of $\mathfrak{g}$. As $D_i^\ast(\mathfrak{g}, \overline{B})$ is a $G$-equivariant complex, this support is invariant under $G$. Hence it is empty and $D_i^\ast(\mathfrak{g}, \overline{B})$ has no cohomology of degree different from $b_\mathfrak{g}$.

(ii) By (i) and Lemma 6.3(ii), for $i = 1, \ldots, k$, $z$ is not in $\sigma_1(S_i)$. □

Definition 6.6. For $k = 1, \ldots, n$ and $z$ in $\mathfrak{b}_\mathfrak{g}$, we say that $\mathfrak{g}$ has Property (P$_{z,k}$) if $D_i^\ast(\mathfrak{g}, \overline{B})$ has no cohomology of degree different from $b_\mathfrak{g}$ for $i = 1, \ldots, k$.

Remark 6.7. Suppose that $\mathfrak{g}$ has Property (P$_{z,n}$) for all $z$ in $\mathfrak{b}_\mathfrak{g} \setminus \{0\}$. Then $\mathfrak{g}$ has Property (P) by Proposition 6.5(ii) and Remark 5.3.

7. Restriction to a parabolic subalgebra

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$, $l$ the reductive factor of $\mathfrak{p}$ containing $\mathfrak{b}$, $\mathfrak{d}$ the derived algebra of $l$, $\mathfrak{z}$ the center of $l$ and $P$ the normalizer of $\mathfrak{p}$ in $G$. Denote by $\mathcal{R}_l$ the set of
roots $\alpha$ such that $g^\alpha$ is contained in $\frak{l}$ and set:

$$d_0 := \dim_\frak{k} \frak{z}, \quad p_u := \bigoplus_{\alpha \in \frak{R}_u \cap \frak{R}_l} g^\alpha, \quad d := \dim p_u$$

so that $p_u$ is the nilpotent radical of $p$. Let $L$ be the centralizer of $\frak{z}$ in $G$. According to [Ko63, §3.2, Lemma 5], $L$ is connected. When $b$ is strictly contained in $p$, we denote by $d_1, \ldots, d_n$ the simple factors of $b$. Let $z$ be in $\frak{z}$ such that $\beta(z) = 1$ for all $\beta$ in $\Pi \setminus \frak{R}_l$ and $t \mapsto \zeta(t)$ the one parameter subgroup of $G$ generated by $ad z$.

Let $l_0$ be the open subset of $l$ as in Section 6. The usual gradation of $\frak{k}[p]$ induces a gradation of the polynomial algebra $\frak{k}[l \times p]$ over $\frak{k}[l_0]$. Let $\pi$ be the canonical projection $p \longrightarrow l$ and set:

$$\frak{B} := \frak{k}[l \times p] \otimes_{\frak{k}(\frak{g} \otimes \frak{g})} \frak{B}_0, \quad \frak{B}_l := \frak{k}[l \times l] \otimes_{\frak{k}(\frak{g} \otimes \frak{g})} \frak{B}_1, \quad \frak{B}_u := \frak{B} \cap \frak{k}[l \times p] \otimes_{\frak{k}} p_u.$$  

As $(h, e)$ is in $\Omega_0$, $\Omega_0 \cap l \times p$ is a dense open subset of $l \times p$ so that $\frak{B}$ is a free submodule of rank $b_0$ of $\frak{k}[l \times p] \otimes_{\frak{k}} p$ by Proposition 2.1(iii) and Corollary 2.7. Again by Proposition 2.1(iii), $\frak{B}_l$ is a free $\frak{k}[l \times l]$-module of rank $b_1$. From the direct sum

$$p = l \oplus p_u,$$

we deduce the inclusions

$$\frak{k}[l] \subset \frak{k}[p] \quad \text{and} \quad \frak{k}[l \times l] \subset \frak{k}[l \times p].$$

In this section, for $(i, m)$ in $I_0$, the restriction of $e_i^{(m)}$ to $l \times p$ is again denoted by $e_i^{(m)}$. According to this convention, $e$ is a generator of the $\frak{k}[l \times p]$-module $\wedge^{b_1}(\frak{B})$ (see Equality 3). As in Subsection 2.3, for $(x, y)$ in $l \times l$, the image of $\frak{B}_l$ by the evaluation map at $(x, y)$ is denoted by $\mathcal{V}^l_{x, y}$.

7.1. **Elementary properties of the module $\frak{B}$**. Denote by $\beta_1, \ldots, \beta_{b_0}$ the simple roots in $\Pi \setminus \frak{R}_l$ and $h_1, \ldots, h_{b_0}$ the basis of $\frak{z}$ dual of $\beta_1, \ldots, \beta_{b_0}$. Let $q_1, \ldots, q_\ell$ be homogeneous generators of $S(l)^L$ and $d_1', \ldots, d_\ell'$ their respective degrees, chosen so that

1. $d_1' \leq \cdots \leq d_\ell'$,
2. for $i = 1, \ldots, \ell$, $q_i \in S(\frak{z}) \cup S(b_1)^L \cup \cdots \cup S(b_0)^L$,
3. for $i = 1, \ldots, d_0$, $q_i = h_i$.

For $i = 1, \ldots, \ell$, denote by $\eta_i$ the differential of $q_i$.

**Lemma 7.1.** (i) For $i = 1, \ldots, \ell$, there exists a unique sequence $c_{i,j}$, $j = 1, \ldots, \ell$ in $\frak{k}[l]$ such that

$$e_i(x) = \sum_{j=1}^\ell c_{i,j}(x)\eta_j(x)$$

for all $x$ in $l$. Moreover, $c_{i,j}$ is invariant under $L$ and homogeneous of degree $d_i - d_j$.
(ii) For \( x \) in \( l_s \), the matrix
\[
(c_{i,j}(x), \ 1 \leq i, j \leq \ell)
\]
is invertible.

**Proof.** (i) Let \( i = 1, \ldots, \ell \). For \( x \) in \( l_{\text{reg}} \), \( \eta_1(x), \ldots, \eta_{\ell}(x) \) is a basis of \( l^* \) by [Ko63, Theorem 9]. By Lemma 2.4(i), for all \( x \) in \( l \), \( \epsilon_i(x) \) is in \( l^* \). Then there exists a unique sequence \( c_{i,j}, \ j = 1, \ldots, \ell \) in \( \kappa[l_{\text{reg}}] \) such that
\[
\epsilon_i(x) = \sum_{j=1}^{\ell} c_{i,j}(x) \eta_j(x)
\]
for all \( x \) in \( l_{\text{reg}} \). By [V72], \( l_{\text{reg}} \) is a big open subset of \( l \). Then \( c_{i,1}, \ldots, c_{i,\ell} \) have a regular extension to \( l \) since \( l \) is normal, whence
\[
\epsilon_i(x) = \sum_{j=1}^{\ell} c_{i,j}(x) \eta_j(x)
\]
for all \( x \) in \( l \). As \( \epsilon_i \) and \( \eta_j \) are invariant under \( L \) and homogeneous of degree \( d_i - 1 \) and \( d'_j - 1 \) respectively, \( c_{i,j} \) is invariant under \( L \) and homogeneous of degree \( d_i - d'_j \).

(ii) For \( x \) in \( l_s \), \( g^x \) is contained in \( l \). Then \( l_s \cap l_{\text{reg}} \) is contained in \( g_{\text{reg}} \). As a result, for \( i = 1, \ldots, \ell \), there exists a unique sequence \( c'_{i,j}, \ j = 1, \ldots, \ell \) in \( \kappa[l_s \cap l_{\text{reg}}] \) such that
\[
\eta_i(x) = \sum_{j=1}^{\ell} c'_{i,j}(x) \epsilon_j(x)
\]
for all \( x \) in \( l_s \cap l_{\text{reg}} \) since \( \epsilon_i(x), \ldots, \epsilon_{\ell}(x) \) is a basis of \( g^x \) for \( x \) in \( g_{\text{reg}} \) by [Ko63, Theorem 9]. As \( l_s \cap l_{\text{reg}} \) is a big open subset of \( l_s \), \( c'_{i,1}, \ldots, c'_{i,\ell} \) have a regular extension to \( l_s \). Then, for \( i = 1, \ldots, \ell \) and \( x \) in \( l_s \),
\[
\eta_i(x) = \sum_{j=1}^{\ell} c'_{i,j}(x) \epsilon_j(x),
\]
whence the assertion. \( \square \)

For a homogeneous of degree \( d_a \) in \( \kappa[l] \) and for \( k = 0, \ldots, d_a \), denote by \( a^{(k)} \) the 2-polarization of bidegree \( (d_a - k, k) \) of \( a \). Set:
\[
I'_0 := \{(i,m) \in \{1, \ldots, \ell\} \times \mathbb{N} \mid 0 \leq m \leq d'_i - 1\}, \quad I_a := I_0 \setminus I'_0.
\]
Then \( |I'_0| = b_l \) and \( |I_a| = d \). For \((i,m)\) in \( I'_0 \), let \( \eta^{(m)}_i \) be the 2-polarization of \( \eta_i \) of bidegree \( (d'_i - 1 - m, m) \). For \((i,m)\) in \( I_0 \) and \((j,m')\) in \( I'_0 \), set \( c_{i,m,j,m'} := c_{i,j}^{(m-m')} \). In particular, \( c_{i,m,j,m'} = 0 \) if \( m' > m \) or \( m - m' > d_i - d'_j \) since \( c_{i,j} \) is homogeneous of degree \( d_i - d'_j \).

**Lemma 7.2.** (i) For \((i,m)\) in \( I_0 \),
\[
\epsilon^{(m)}_i(x,y) = \sum_{(j,m') \in I'_0} c_{i,m,j,m'}(x,y) \eta^{(m')}_{j}(x,y)
\]
for all \((x, y)\) in \(l \times l\).

(ii) For \((i, m)\) in \(I_0\) and \((j, m')\) in \(I_0'\), the function \(c_{i,m,j,m'}\) is invariant under the diagonal action of \(L\) in \(l \times l\).

(iii) For \((x, y)\) in \(h \times (u \cap l)\) and \((i, m)\) in \(I_0'\),

\[
\varepsilon_i^{(m)}(x, y) = \sum_{j=1}^{\ell} c_{i,j}(x) \eta_j^{(m)}(x, y).
\]

(iv) For \((i, m)\) in \(I_0\) and \(y'\) in \(p_u\),

\[
\pi^{\ast}\varepsilon_i^{(m)}(x, y + y') = \varepsilon_i^{(m)}(x, y)
\]

for all \((x, y)\) in \(l \times l\).

**Proof.** (i) By Lemma 7.1(i), for \((x, y)\) in \(l \times l\),

\[
\varepsilon_i(x + sy) = \sum_{j=1}^{\ell} c_{i,j}(x + sy) \eta_j(x + sy)
\]

for all \(s\) in \(k\), whence

\[
\sum_{m=0}^{d_i-1} \sum_{k=0}^{d_j-1} s^k \varepsilon_i^{(m)}(x, y) = \sum_{m=0}^{d_i-1} \sum_{k=0}^{d_j-1} s^k c_{i,j}^{(l)}(x, y) \eta_j^{(m)}(x, y) = \sum_{m=0}^{d_i-1} \sum_{m'=0}^{\inf(m, d_j'-1)} c_{i,j}^{(m-m')} (x, y) \eta_j^{(m')} (x, y).
\]

As a result,

\[
\varepsilon_i^{(m)}(x, y) = \sum_{(j, m') \in I_0'} c_{i,m,j,m'}(x, y) \eta_j^{(m')}(x, y)
\]

for all \((x, y)\) in \(l \times l\).

(ii) By Lemma 7.1(i), \(c_{i,m,j,m'}\) is invariant under the diagonal action of \(L\) in \(l \times l\).

(iii) Let \((x, y)\) be in \(h \times (u \cap l)\). By (i),

\[
\varepsilon_i^{(m)}(x, y) = \sum_{(j, m') \in I_0} c_{i,m,j,m'}(x, y) \eta_j^{(m')}(x, y).
\]

By (ii), for \(t\) in \(k^*\) and \((j, m')\) in \(I_0'\),

\[
c_{i,m,j,m'}(x, y) = c_{i,m,j,m'}(x, p(t)\cdot y),
\]

whence

\[
c_{i,m,j,m'}(x, y) = c_{i,m,j,m'}(x, 0) \text{ since } \lim_{t \to 0} p(t)\cdot y = 0.
\]

As a result, \(c_{i,m,j,m'}(x, y) = 0\) for \(m \neq m'\) and \(c_{i,m,j,m}(x, y) = c_{i,j}(x)\), whence the assertion.

(iv) According to Corollary 2.5, for all \(x\) in \(p\), \(\varepsilon_i(x)\) is in \(p\). For \(x\) in \(l\) and \(y\) in \(p_u\),

\[
\lim_{t \to 0} \zeta(t)(x + y) = x \text{ whence } p_i(x + y) = p_i(x) \text{ and } \langle \varepsilon_j(x), v \rangle = \langle \varepsilon_j(x + y), v \rangle
\]
for all \( v \) in \( l \). As a result, for all \( x \) in \( l \) and for all \( y \) in \( p_u \), \( \varepsilon_i(x) - \varepsilon_i(x + y) \) is in \( p_u \) since \( p_u \) is the orthogonal complement to \( p \) in \( g \), whence the assertion.

The order of \( I_0 \) induces an order of \( I_0' \) and we get a square matrix of order \(|I_0'|\),

\[
M_0 := (c_{i,m,j,m'}, ((i,m),(j,m'))) \in I_0' \times I_0',
\]

with coefficients in \( \mathbb{k}[l \times l] \).

**Corollary 7.3.** For all \((x,y)\) in \( l_+ \times l \), \( \det M_0(x,y) \neq 0 \).

**Proof.** Set:

\[
\varepsilon_0 := \wedge_{(i,m) \in I_0'} \varepsilon_i^{(m)} \quad \text{and} \quad \eta_0 := \wedge_{(j,m') \in I_0'} \eta_j^{(m')}.
\]

In these equalities, the order of the products is induced by the order of \( I_0' \). Let \( \overline{\varepsilon_0} \) be the restriction of \( \varepsilon_0 \) to \( l_+ \times l \). By Lemma 7.2(i), \( \overline{\varepsilon_0} \) is in \( \mathbb{k}[l_+ \times l] \otimes \mathbb{k} \wedge^b (l) \) and

\[
\overline{\varepsilon_0} = \det M_0 \eta_0.
\]

As \( \overline{\varepsilon_0} \) and \( \eta_0 \) are homogeneous of degree

\[
\sum_{j=1}^\ell d'_j(d'_j - 1)\frac{2}{2},
\]

\( \det M_0 \) is in \( \mathbb{k}[l_+] \).

Denote by \( \Sigma \) the nullvariety of \( \det M_0 \) in \( l_+ \). Suppose that \( \Sigma \) is nonempty. A contradiction is expected. As the maps \( \varepsilon_i^{(m)} \) and \( \eta_j^{(m')} \) are homogeneous and invariant under \( L \) for all \((i,m)\) in \( I_0' \), \( \Sigma \) is a closed cone of \( l_+ \), invariant under \( L \).

**Claim 7.4.** For some \( x \) in \((l_+ \cap b) \), \( x + \varpi(e) \) is in \( \Sigma \).

**Proof.** [Proof of Claim 7.4] As \( \Omega_l \cap l \times l \) contains \((h, \varpi(e))\), for all \( t \) in \( \mathbb{k} \), \( h + t\varpi(e) \) is a regular element of \( g \) and \( l \), whence a sequence \( p_{i,j}, 1 \leq i, j \leq \ell \) of polynomials such that

\[
\eta_i(h + t\varpi(e)) = p_{i,1}(t)\varepsilon_1(h + t\varpi(e)) + \cdots + p_{i,\ell}(t)\varepsilon_\ell(h + t\varpi(e))
\]

for all \( t \) in \( \mathbb{k} \) and \( i = 1, \ldots, \ell \). As a result, for all \((i,m)\) in \( I_0' \), \( \eta_i^{(m)}(h, \varpi(e)) \) is a linear combination of \( \varepsilon_j^{(m')}(h, \varpi(e)) \). Hence \( b \cap l \) is not contained in \( \Sigma \). Let \( \Sigma' \) be an irreducible component of \( \Sigma \cap b \). Then \( \Sigma' \) is an hypersurface of \( b \) as an irreducible component of the nullvariety of \( \det M_0 \) in \( b \cap l_+ \). Then, by lemma 1.8,

\[
\Sigma' = \Sigma' \cap b + u \cap l
\]

since \( l_+ \cap b = l_+ \cap b + u \cap l \) and \( \Sigma \) and \( \Sigma' \) are invariant under the one parameter subgroup \( t \mapsto \rho(t) \) of \( G \). As a result, for \( x \) in \( \Sigma' \cap b \), \( x + \varpi(e) \) is in \( \Sigma' \), whence the claim. \( \square \)
From the equality
\[ \lim_{t \to \infty} t^{-2} \rho(t) \cdot (x + \mathfrak{w}(e)) = \mathfrak{w}(e), \]
we deduce that \( x + \mathfrak{w}(e) \) is a regular element of \( \mathfrak{l} \) since so is \( \mathfrak{w}(e) \). Hence \( x + \mathfrak{w}(e) \) is a regular element of \( \mathfrak{g} \) since it is in \( \mathfrak{l}_e \). Then, from the equalities
\[ \lim_{t \to \infty} t^{-2} \rho(t) \cdot (h + e - \mathfrak{w}(e)) = h \quad \text{and} \]
\[ \lim_{t \to \infty} t^{-2} \rho(t) \cdot (x + \mathfrak{w}(e) + a(h + e - \mathfrak{w}(e))) = \mathfrak{w}(e) + a(e - \mathfrak{w}(e)), \]
for a in \( \mathbb{k} \), we deduce
\[ h + e - \mathfrak{w}(e) \in \mathfrak{g}_{\text{reg}} \quad \text{and} \quad x + \mathfrak{w}(e) + a(h + e - \mathfrak{w}(e)) \in \mathfrak{g}_{\text{reg}} \]
for all a in \( \mathbb{k}^* \) since \( \mathfrak{w}(e) + a(e - \mathfrak{w}(e)) \) is a regular element of \( \mathfrak{g} \). Hence
\[ (x + \mathfrak{w}(e), h + e - \mathfrak{w}(e)) \in \Omega_{\mathfrak{h}}. \]
As a result, by Corollary 2.7 and Lemma 7.2(iv), the elements
\[ \mathcal{E}_i^{(m)}(x + \mathfrak{w}(e), h), (i, m) \in I'_0 \]
are linearly independent, whence the contradiction. \( \square \)

7.2. **Decomposition of \( \mathcal{B} \).** Let \( \mathcal{B}_0 \) be the submodule of \( \mathcal{B} \) generated by \( \mathcal{E}_i^{(m)}, (i, m) \in I'_0 \). For \( (i, m) \) in \( I'_0 \), denote by \( \mathcal{E}_i^{(m)} \) the restriction of \( \mathcal{E}_i^{(m)} \) to \( \mathfrak{l}, \mathfrak{x}_l \). By Lemma 2.4(ii), \( \mathcal{E}_i^{(m)} \) is in \( \mathbb{k}[\mathfrak{l}, \times \mathfrak{l}] \otimes_{\mathbb{k}} \mathfrak{l} \).

**Lemma 7.5.** (i) The \( \mathbb{k}[\mathfrak{l}, \times \mathfrak{p}] \)-modules \( \mathcal{B} \) and \( \mathcal{B}_0 \) are free of rank \( b_0 \) and \( b_1 \) respectively.

(ii) The module \( \mathcal{B}_0 \) is the image of \( \mathcal{B}_0 \) by the restriction map from \( \mathfrak{l}, \mathfrak{x}_l \) to \( \mathfrak{l}, \mathfrak{x}_l \). In particular, \( \mathcal{E}_i^{(m)}, (i, m) \in I'_0 \) is a basis of \( \mathcal{B}_0 \).

**Proof.** (i) As \( \Omega_{\mathfrak{h}} \cap \mathfrak{l} \times \mathfrak{p} \) is not empty and \( \mathcal{B} \) is generated by \( \mathcal{E}_i^{(m)}, (i, m) \in I_0 \), the assertion results from Proposition 2.1(iii) since \( |I_0| = b_0 \) and \( |I'_0| = b_1 \).

(ii) By Lemma 7.2(i), the restriction of \( \mathcal{B}_0 \) to \( \mathfrak{l} \times \mathfrak{l} \) is contained in \( \mathcal{B}_l \). By Corollary 7.3, the matrix \( M_0(x, y) \) is invertible for all \( (x, y) \) in \( \mathfrak{l}, \times \mathfrak{l} \). Then, for all \( (i, m) \) in \( I'_0 \), \( \eta_i^{(m)} \) is a linear combination with coefficients in \( \mathbb{k}[\mathfrak{l}, \times \mathfrak{l}] \) of \( \mathcal{E}_j^{(m')} \), \( (j, m') \in I'_0 \), whence the assertion. \( \square \)

Set:
\[ \mathfrak{p}_{-u} := \bigoplus_{a \in \mathbb{R} \cup \{0\}} \mathfrak{g}^{-a} \quad \text{and} \quad \mathfrak{p}_- := \mathfrak{l} \oplus \mathfrak{p}_{-u} \]
so that \( \mathfrak{p}_- \) is a parabolic subalgebra of \( \mathfrak{g} \). Let \( J_- \) be the ideal of definition of \( \mathfrak{l} \times \mathfrak{l} \) in \( \mathbb{k}[\mathfrak{l}, \times \mathfrak{p}] \). As \( \mathbb{k}[\mathfrak{p}] \) identifies with \( \mathbb{S}(\mathfrak{p}_-), \mathbb{k}[\mathfrak{l}, \times \mathfrak{p}] = \mathbb{k}[\mathfrak{l}_e] \otimes_{\mathbb{k}} \mathbb{S}(\mathfrak{p}_-) \) and \( J_- \) is the ideal generated by \( 1 \oplus \mathfrak{p}_{-u} \).
Proposition 7.6.  (i) For $(i, m)$ in $I_0$, $\varepsilon_i^{(m)} - \overline{\varepsilon_i^{(m)}}$ is in $J_- \otimes_k v_u$.

(ii) For $(i, m)$ in $I_u$, for a well defined sequence $a_{i,m,j,m'}$, $(j, m') \in I'_0$ in $\mathbb{k}[l \times l]$, 

$$v_i^{(m)} := \varepsilon_i^{(m)} - \sum_{(j,m') \in I'_0} a_{i,m,j,m'} \varepsilon_j^{(m')} \in J_- \otimes_k v_u.$$ 

(iii) The module $\overline{B}$ is the direct sum of $\overline{B}_0$ and $\overline{B}_u$.

(iv) The module $\overline{B}_u$ is free of rank $d$ and $v_i^{(m)}$, $(i, m) \in I_u$ is a basis of $\overline{B}_u$.

Proof.  (i) According to Lemma 7.2(iv), $\varepsilon_i^{(m)} - \overline{\varepsilon_i^{(m)}}$ is in $\mathbb{k}[l \times p] \otimes_k v_u$. By Proposition 2.1(vi), $\varepsilon_i^{(m)}$ is invariant under the one parameter subgroup $t \mapsto \zeta(t)$. Then so is $\varepsilon_i^{(m)} - \overline{\varepsilon_i^{(m)}}$, whence the assertion since the elements of $\mathbb{k}[l \times p] \otimes_k v_u$, invariant under $t \mapsto \zeta(t)$, are in $J_- \otimes_k v_u$.

(ii) According to Corollary 2.7, $\varepsilon_i^{(m)}$ is in $\overline{B}_i$. Then, by Lemma 7.5(ii), for a well defined sequence $a_{i,m,j,m'}$, $(j, m') \in I'_0$ in $\mathbb{k}[l \times l]$, 

$$\overline{\varepsilon_i^{(m)}} = \sum_{(j,m') \in I'_0} a_{i,m,j,m'} \overline{\varepsilon_j^{(m')}}.$$ 

For all $(j, m')$ in $I_0$, $\varepsilon_j^{(m')}$ is invariant under $L$ by Proposition 2.1(vi). Hence $a_{i,m,j,m'}$ is invariant under $L$ for all $(j, m')$ in $I'_0$. Moreover, by (i), $\overline{v_i^{(m)}}$ is in $J_- \otimes_k v_u$.

(iii) By (ii), $\overline{B}$ is the sum of $\overline{B}_0$ and $\overline{B}_u$ since $\overline{B}$ is generated by the $\varepsilon_i^{(m)}$'s. Suppose $\overline{B}_0 \cap \overline{B}_u \neq \{0\}$. A contradiction is expected. Let $\varphi$ be a non zero element of $\overline{B}_0 \cap \overline{B}_u$. Then 

$$\varphi = \sum_{(i,m) \in I'_0} \varphi_{i,m} \varepsilon_i^{(m)}$$ 

with $\varphi_{i,m}$ in $\mathbb{k}[l \times p]$. As $\varphi$ is in $\overline{B}_u$, $\varphi \circ \varphi = 0$. For some linearly independent homogeneous elements $r_1, \ldots, r_l$ of $S(v_{-u})$, 

$$\varphi_{i,m} = \sum_{j=1}^l r_j \varphi_{i,m,j}$$ 

with $\varphi_{i,m,1}, \ldots, \varphi_{i,m,l}$ in $\mathbb{k}[l \times l]$ for all $(i, m)$. By (i), denoting by $I_{\min}$ the set of indices $j$ such that $r_j$ has minimal degree and $\varphi_{i,m,j} \neq 0$ for some $(i, m)$, 

$$\sum_{j \in I_{\min}} r_j \left( \sum_{(i,m) \in I'_0} \varphi_{i,m,j} \overline{\varepsilon_i^{(m)}} \right) = 0$$ 

since $\varphi$ is in $\mathbb{k}[l \times p] \otimes_k v_u$. As a result, for $j$ in $I_{\min}$, $\varphi_{i,m,j} = 0$ for all $(i, m)$, whence the contradiction and the assertion.

(iv) By (iii), $\overline{B}_u$ has rank $d$ since $\overline{B}_0$ and $\overline{B}$ have rank $b_1$ and $b_0$ respectively. Set: 

$$\varepsilon := \land_{(i,m) \in I_0} \varepsilon_i^{(m)} \quad \text{and} \quad \nu := \land_{(i,m) \in I_u} \nu_i^{(m)}.$$
In these equalities, the order of the products is induced by the order of $I_0$. By (ii),

$$\varepsilon_0 \wedge \nu = \pm \varepsilon,$$

with $\varepsilon_0$ as in the proof of Corollary 7.3. Again by (iii), $\wedge^b_u(\mathcal{B})$ is isomorphic to $\wedge^b_u(\mathcal{B}_0) \otimes_k [l \times p]$ \& $\wedge^d_u(\mathcal{B}_u)$. So, for $\varphi$ in $\wedge^d_u(\mathcal{B}_u)$, $\varepsilon_0 \wedge \varphi = a\varepsilon$ for some $a$ in $k[l \times p]$ since $\varepsilon$ is a generator of $\wedge^b_u(\mathcal{B})$.

As a result,

$$\varepsilon_0 \wedge (\varphi \mp av) = 0.$$

Denoting by $\varepsilon_0^\prime$ the restriction of $\varepsilon_0$ to $l_\ast \times 1$,

$$\varepsilon_0 - \varepsilon_0^\prime \in k[l_\ast \times p] \otimes_k \nu \wedge \wedge^b_{l-1}(p)$$

by (i). Hence $\varepsilon_0^\prime \wedge (\varphi \mp av) = 0$ and $\varphi = \pm av$. As a result, $\nu$ is a generator of $\wedge^d_u(\mathcal{B}_u)$.

Let $\mu$ be in $\mathcal{B}_u$. Then

$$\mu = \sum_{(i,m) \in I_u} c_{i,m} \nu^{(m)}_i$$

for some sequence $c_{i,m}$, $(i, m) \in I_u$ in $k(l_\ast \times p)$ since $\mathcal{B}_u$ has rank $d$ and $\nu^{(m)}_i$, $(i, m) \in I_u$ are linearly independent over $k[l_\ast \times p]$. For $(i, m)$ in $I_u$, set:

$$\nu_{i,m} := \wedge^d((j,m') \in I_u \setminus ((i,m)), \nu^{(m')}_{j})$$

In this equality, the order of the product is induced by the order of $I_0$. As $\nu$ is a generator of $\wedge^d_u(\mathcal{B}_u)$, for some $c'_{i,m}$ in $k[l_\ast \times p]$,

$$c'_{i,m} \nu = \mu \wedge \nu_{i,m} = \pm c_{i,m} \nu,$$

whence $c_{i,m}$ is in $k[l_\ast \times p]$. As a result, $\mathcal{B}_u$ is a free module generated by $\nu^{(m)}_i$, $(i, m) \in I_u$. Then, by (ii), $\mathcal{B}_u$ is contained in $J_{\ast} \otimes_k \nu_u$. \hfill $\Box$

As $l_\ast$ is a principal open subset of $l$, containing $h_{\text{reg}}$, $h_\ast := l_\ast \cap h$ is a nonempty principal open subset of $h$.

**Lemma 7.7.** Let $(i, m)$ be in $I_u$.

(i) For $(j, m')$ in $I'_{0}$, $a_{i,m,j,m'}$ is bihomogeneous of bidegree $(d_i - d_j + m' - m, m - m')$. In particular, $a_{i,m,j,m'} = 0$ when $m' > m$.

(ii) For $(j, m')$ in $I'_{0}$, $(x, x')$ in $h_\ast \times h$ and $(y, y')$ in $u \times u$, $a_{i,m,j,m'}(x + y, x' + y') = a_{i,m,j,m'}(x, x')$ and $a_{i,m,j,m'}(x, y) = a_{i,m,j,m'}(x, 0)$. In particular, $a_{i,m,j,m'}(x, y) = 0$ when $m \neq m'$.

(iii) For $(j, m')$ in $I_0$ and $(t, x, v)$ in $k \times g \times g$,

$$e_j^{(m')}[(tx + v) = \sum_{k=0}^{m'} \binom{d_j - 1 - k}{m' - k} e_j^{(k)}(x, v).$$
Corollary 7.8. Let $(i, m)$ be in $I_0$ and $x$ in $b_*$.

(i) For $j = 0, \ldots, d'_i - 1$ and $l$ in $\{1, \ldots, \ell\} \setminus \{i\}$,

\[
\left( \frac{d_i - 1 - j}{m - j} \right) - \sum_{m' = j}^{d_i - 1 - j} \binom{d_i - 1 - j}{m' - j} a_{i,m,m'}(x, x) = 0 \quad \text{and} \quad \sum_{m' = j}^{d_i - 1} \binom{d_i - 1 - j}{m' - j} a_{i,m,m'}(x, x) = 0.
\]

(ii) For $m' = 0, \ldots, d'_i - 1$,

\[
a_{i,m,m'}(x, x) = \sum_{j=0}^{m'} (-1)^{m'-j} \frac{(d_i - 1 - m')!}{(d_i - 1 - j)! (m - j)!(m' - j)!}.
\]
(iii) If \(d'_i \geq 2\) then

\[
r(m, d_i - 1) - \sum_{m' = 1}^{d'_i - 1} (-1)^{m-m'} r(m', d_i - 1) a_{i,m,i,m'}(x,x) \neq 0.
\]

**Proof.** (i) Let \((t, y)\) be in \(\mathbb{k} \times (u \cap l)\). By Proposition 7.6(ii),

\[
\ell^{(m)}_x(t, y + x) = \sum_{(l', m') \in I_0} a_{i,m',l'}(x, tx + y) \ell^{(m')}_{l'}(x, tx + y)
\]

since \(\ell^{(m')}_{l'}(x, tx + y)\) is in \(l\) for all \((j, m')\) in \(I_0\) by Lemma 2.4(i). Then, by Lemma 7.7, (ii) and (iii),

\[
\sum_{j=0}^{m} \binom{d_i - 1 - j}{m-j} \ell^{(j)}_{x}(x, y) - \sum_{j=0}^{d'_i-1} \sum_{j' = j}^{d'_i-1} a_{i,m,l',m'}(x, x) \binom{d_i - 1 - j}{m-j} \ell^{(j)}_{l'}(x, y) = 0,
\]

whence, for \(j = 0, \ldots, m,\)

\[
\binom{d_i - 1 - j}{m-j} \ell^{(j)}_{x}(x, y) - \sum_{j' = j}^{d'_i-1} a_{i,m,l',m'}(x, x) \binom{d_i - 1 - j}{m-j} \ell^{(j)}_{l'}(x, y) = 0.
\]

For \(x\) in a dense open subset of \(h_+\), \((x, \Psi(e))\) is in \(\Omega_i\) since \(\Psi(e)\) is a nilpotent element of a principal \(\mathfrak{sl}_2\)-triple of \(l\). Then \(\ell^{(m')}_{l'}(x, \Psi(e))\), \((l', m') \in I_0\) are linearly independent by Proposition 2.1(iii) and Lemma 7.5(ii). As a result, for \(j = 0, \ldots, d'_i - 1\) and \(j' = 0, \ldots, d'_i - 1,\)

\[
\binom{d_i - 1 - j}{m-j} - \sum_{j' = j}^{d'_i-1} a_{i,m,i,m'}(x, x) \binom{d_i - 1 - j}{m-j} = 0 \text{ and } \sum_{j' = j}^{d'_i-1} a_{i,m,i,m'}(x, x) \binom{d_i - 1 - j}{m-j} = 0.
\]

(ii) For \(x\) in \(h_+\) and \(m' = 0, \ldots, d'_i - 1,\) set:

\[
\psi_{m'}(x) := a_{i,m,i,m'}(x, x) \frac{(d_i - 1 - m)!}{(d_i - 1 - m')!}.
\]

By (i), the sequence \(\psi_{m'}(x), m' = 0, \ldots, d'_i - 1\) is a solution of the linear system

\[
\sum_{m' = j}^{d'_i-1} \psi_{m'}(x) \frac{1}{(m' - j)!} = \frac{1}{(m-j)!}.
\]

The inverse of the matrix of this system is equal to

\[
(a_{m', j}, 0 \leq m', j \leq d'_i - 1) \quad \text{with} \quad a_{m', j} := \begin{cases} (-1)^{m'-j} & \text{if } m' \geq j, \\ 0 & \text{if } m' < j, \end{cases}
\]

whence the equality of the assertion.
(iii) For $e, k, l$ integers such that $2 \leq e \leq k \leq l$, set:

$$c(k, l) := \sum_{j=0}^{k} \frac{(-1)^j}{j!(l+j)!} \quad \text{and} \quad \psi(e, k, l) := r(k, l) - \sum_{j=1}^{e-1} \frac{(-1)^{k-j} c(j, k-j) r(j, l)(l-j)!}{(l-k)!},$$

By (ii),

$$a_{i,m,m'}(x, x) = \frac{(d_i - 1 - m')!}{(d_i - 1 - m)!} c(m', m - m').$$

Then

$$\psi(d_i', m, d_i - 1) = r(m, d_i - 1) - \sum_{m' = d_i - 1}^{d_i - 1} (-1)^{m-m'} r(m', d_i - 1) a_{i,m,m'}(x, x),$$

whence the assertion by Proposition C.3.

\[ \square \]

7.3. An invariance property. In this subsection, $e^{(m)}_i$, $(i, m) \in I_0$ is a sequence in $S(g \times g) \otimes k g$. Set:

$$\Gamma := (b \cap \lambda_s \cap g_{\text{reg,ss}}) \times (b \cap \lambda).$$

Then $\Gamma$ is an affine variety invariant under $B \cap L$ since $\lambda_s$ and $g_{\text{reg,ss}}$ are principal open subsets of $\lambda$ and $g$ respectively, invariant under $B \cap L$. We identify $\Gamma$ with the subset $\{1_G\} \times \Gamma$ of $G \times \Gamma$ by the map $x \mapsto (1_G, x)$. Let $X$ be the quotient of $G \times \Gamma$ under the right action of $B \cap L$ given by $(g, x, y).k := (gk, k^{-1}(x), k^{-1}(y))$ and $\nu$ the quotient map. Denote by $\tau$ the canonical morphism

$$X \longrightarrow \Gamma \times g \quad \text{such that} \quad \tau \circ \nu(g, x, y) = (g(x), g(y)).$$

**Lemma 7.9.** Set $\Omega := G \tau^{-1}(\Omega \cap \Gamma)$ and $Y := X \setminus \Omega$.

(i) The restriction of $\nu$ to $\Gamma$ is a closed embedding.

(ii) The subset $Y$ of $X$ is closed and $G$-invariant.

**Proof.** (i) Let $(g, x, y)$ be in $G \times \Gamma$ such that $(g(x), g(y))$ is in $\Gamma$. As $x$ is regular semisimple, for some $b$ in $B \cap L$, $g(x) = b(x)$, whence $b^{-1}g$ is in $B \cap L$ since the centralizer of $x$ in $G$ is contained in $B \cap L$. As a result, $\tau^{-1}(\Gamma) = \nu(\Gamma)$, whence the assertion since the restriction of $\tau \circ \nu$ to $\Gamma$ is the identity.

(ii) By (i), $\tau^{-1}(\Omega \cap \Gamma)$ is equal to $\nu(\Omega \cap \Gamma)$ since $\Omega \cap \Gamma$ is invariant under $B \cap L$. Hence $\Omega = G.\nu(\Omega \cap \Gamma \setminus \Omega)$. As a result, $Y = G.\nu(\Gamma \setminus \Omega)$.

The variety $X$ is a fiber bundle over $G/(B \cap L)$. Denote by $\sigma$ the bundle projection

$$X \longrightarrow G/(B \cap L).$$

Let $O$ be a trivializing affine open subset of $G/(B \cap L)$. Then we have a commutative diagram,

$$\sigma^{-1}(O) \xrightarrow{\Phi} O \times \Gamma \xrightarrow{pr_1} O.$$
with $\Phi$ an isomorphism. As $Y$ is $G$-invariant, $\Phi(Y \cap \sigma^{-1}(O)) = O \times (\Gamma \setminus \Omega)$. Then $Y$ is closed since $G/(B \cap L)$ has a cover by trivializing affine open subsets.

We identify $\Gamma$ with $\nu(\Gamma)$ by $\nu$.

**Lemma 7.10.** Let $(i, m)$ be in $I_0$.

(i) The maps $\varepsilon_j^{(m') \circ \tau}$, $(j, m') \in I_0'$ are linearly independent over $\mathbb{k}(X)$.

(ii) For a well defined sequence $a'_{i,m,j,m'}$, $(j, m') \in I_0'$ in $\mathbb{k}[X \setminus Y]$,

$$\varepsilon_i^{(m)} \circ \tau(x) = \sum_{(j,m') \in I_0'} a'_{i,m,j,m'}(x) \varepsilon_j^{(m')} \circ \tau(x)$$

for all $x$ in $X \setminus Y$.

(iii) For $(j, m')$ in $I_0'$, the rational function $a'_{i,m,j,m'}$ is invariant under $G$.

(iv) For $(j, m')$ in $I_0'$, $a_{i,m,j,m'}$ and $a'_{i,m,j,m'}$ have the same restrictions to $\Gamma \cap \Omega_i$.

**Proof.** (i) For all $x$ in $\Omega_i \cap \Gamma$, $\eta_j^{(m')}(x)$, $(j, m') \in I_0'$ are linearly independent. Then, for all $x$ in $\Gamma \cap \Omega_i$, $\varepsilon_j^{(m')}(x)$, $(j, m') \in I_0'$ are linearly independent by Lemma 7.5(ii). As a result, the maps $\varepsilon_j^{(m')} \circ \tau$, $(j, m') \in I_0'$ are linearly independent over $\mathbb{k}(X)$ since $\varepsilon_j^{(m)}$, $(j, m') \in I_0'$ are $G$-equivariant by Proposition 2.1(vi).

(ii) Let $\varepsilon_0$ be as in the proof of Corollary 7.8. By Proposition 7.6(ii), the restriction to $\Gamma$ of $\varepsilon_i^{(m)} \wedge \varepsilon_0$ is equal to 0. Then, by $G$-equivariance of the $\varepsilon_j^{(m')}$s, $\varepsilon_i^{(m)} \wedge \varepsilon_0 \circ \tau$ is equal to 0. So, by (i), for a well defined sequence $a'_{i,m,j,m'}$, $(j, m') \in I_0'$ in $\mathbb{k}(X)$,

$$\varepsilon_i^{(m)} \circ \tau(x) = \sum_{(j,m') \in I_0'} a'_{i,m,j,m'}(x) \varepsilon_j^{(m')} \circ \tau(x)$$

for all $x$ in a dense open subset of $X$. As for $x$ in $X \setminus Y$, $\varepsilon_j^{(m')} \circ \tau(x)$, $(j, m') \in I_0'$ are linearly independent, the functions $a'_{i,m,j,m'}$, $(j, m') \in I_0'$ are regular on $X \setminus Y$, whence the assertion.

(iii) By unicity of the sequence $a'_{i,m,j,m'}$, $(j, m') \in I_0'$ and the $G$-equivariance of the maps $\varepsilon_k^{(l)}$, $(k, l) \in I_0$, the rational functions $a'_{i,m,j,m'}$, $(j, m') \in I_0'$ are $G$-invariant.

(iv) By (ii) and (iii), for all $(x, y)$ in $\Gamma \cap \Omega_i$,

$$\varepsilon_i^{(m)}(x, y) = \sum_{(j,m') \in I_0'} a'_{i,m,j,m'}(x, y) \varepsilon_j^{(m')}(x, y).$$

By Proposition 7.6(ii), for all $(x, y)$ in $l_\times l$,

$$\varepsilon_i^{(m)}(x, y) = \sum_{(j,m') \in I_0'} a_{i,m,j,m'}(x, y) \varepsilon_j^{(m')}(x, y),$$

whence the assertion since $\varepsilon_j^{(m')}(x, y)$, $(j, m') \in I_0'$ are linearly independent for all $(x, y)$ in $\Gamma \cap \Omega_i$ by Lemma 7.5(ii).
Let $\mathfrak{h}_{ss}$ be the open subset of $\mathfrak{h}$,

$$\mathfrak{h}_{ss} := \bigcap_{g \in W(\mathcal{R})} g(\mathfrak{h}).$$

Then $\mathfrak{h}_{ss}$ is a dense principal open subset of $\mathfrak{h}$, invariant under the Weyl group $W(\mathcal{R})$ of $\mathcal{R}$.

**Proposition 7.11.** Let $(i, m)$ be in $I_u$. For $(j, m')$ in $I'_v$, the restriction of $a_{i,m,j,m'}$ to $\mathfrak{h}_{ss} \times \mathfrak{h}$ is invariant under $W(\mathcal{R})$.

**Proof.** Let $\pi$ be the canonical projection $\mathfrak{b} \longrightarrow \mathfrak{b}$. According to Lemma 7.7(ii),

$$a_{i,m,j,m'}(x, y) = a_{i,m,j,m'}(\pi(x), \pi(y)) \quad \forall (x, y) \in \Gamma.$$

Hence the function on $G \times \Gamma$,

$$(g, x, y) \mapsto a_{i,m,j,m'}(x, y)$$

is constant on the $B \cap L$-orbits. As a result, $a_{i,m,j,m'}$ defines through the quotient a $G$-invariant regular function on $X$. Denote it by $\overline{a}_{i,m,j,m'}$. By Lemma 7.10(iii), $a'_{i,m,j,m'}$ is a $G$-invariant rational function on $X$. Then, by Lemma 7.10(iv), $a'_{i,m,j,m'} = \overline{a}_{i,m,j,m'}$ since $\overline{a}_{i,m,j,m'}$ is $G$-invariant. In particular, $a'_{i,m,j,m'}$ is regular on $X$ and its restriction to $\Gamma \cap \mathfrak{h}_{ss} \times \mathfrak{h}$ is invariant under $W(\mathcal{R})$. As a result, again by Lemma 7.10(iv), the restriction of $a_{i,m,j,m'}$ to $\mathfrak{h}_{ss} \times \mathfrak{h}$ is invariant under $W(\mathcal{R})$ since $\Gamma \cap \mathfrak{h}_{ss} \times \mathfrak{h}$ is dense in $\mathfrak{h}_{ss} \times \mathfrak{h}$. \hfill $\square$

### 8. Expansion along a parabolic subalgebra

Let $\mathfrak{p}$ be a parabolic subalgebra containing $\mathfrak{b}$ and

$$\mathfrak{l}, \mathfrak{p}_u, \mathfrak{d}, \mathfrak{z}, \mathcal{R}_l, L, d, d_0, \mathfrak{l}_s, z, \zeta, I_0, I'_0, I_u$$

as in Section 7. Set:

$$p_{-u} := \bigoplus_{\alpha \in \mathcal{R}_s \setminus \mathcal{R}_l} \mathfrak{p} \quad \text{and} \quad p_- := \mathfrak{l} \oplus p_{-u}.$$

#### 8.1. Some results about the expansion along $\mathfrak{p}$

Denote by $\alpha_1, \ldots, \alpha_d$ the positive roots which are not in $\mathcal{R}_l$ and ordered so that $|\alpha_i| \leq |\alpha_j|$ for $i \leq j$. For $i = 1, \ldots, d$, set:

$$v_i := x_{\alpha_i}, \quad w_i := x_{-\alpha_i}.$$

Then $v_1, \ldots, v_d$ and $w_1, \ldots, w_d$ are basis of $\mathfrak{p}_u$ and $\mathfrak{p}_{-u}$ respectively. As usual, for $r := (r_1, \ldots, r_d)$ in $\mathbb{N}^d$,

$$v^r := v_1^{r_1} \cdots v_d^{r_d} \quad \text{and} \quad w^r := w_1^{r_1} \cdots w_d^{r_d}.$$

**Lemma 8.1.** Let $i = 1, \ldots, \ell$.

(i) For all $x$ in a dense open subset of $\mathfrak{h}$, $\varepsilon_i(x)$ is a regular element of $\mathfrak{g}$. In particular, for $x$ in a dense open subset of $\mathfrak{l}$, $\varepsilon_i(x)$ has a non zero component on all simple factor of $\mathfrak{d}$.

(ii) For $k = 1, \ldots, d$, for some homogeneous element $p_{i,k}$ of degree $d_i - 2$ in $S(\mathfrak{h})$,

$$\alpha_k \varepsilon_i = \alpha_k p_{i,k}.$$
(iii) For $k = 1, \ldots, d$ and $(t, x)$ in $k \times \mathfrak{h}$,

$$
\epsilon_i(x + tw_k) = \epsilon_i(x) + tp_{i,k}(x)w_k.
$$

**Proof.** (i) Suppose that $\epsilon_i(x)$ is not regular for all $x$ in $\mathfrak{h}$. A contradiction is expected. Then, for some root $\alpha$, $\alpha \circ \epsilon_i(x) = 0$ for all $x$ in $\mathfrak{h}$. In particular, for $x$ in $\mathfrak{h}$ and $g$ in $W(\mathcal{R})$, $\alpha \circ \epsilon_i(g(x)) = 0$. As $\epsilon_i$ is $G$-equivariant, $\epsilon_i(x)$ is in the kernel of $g^{-1}(\alpha)$. As $\mathfrak{g}$ is simple, $\mathfrak{h}$ is a simple $W(\mathcal{R})$-module so that the orthogonal complement in $\mathfrak{g}$ to $g(\alpha)$, $g \in W(\mathcal{R})$ is equal to $\{0\}$, whence a contradiction since for $x$ in $\mathfrak{h}_{\text{reg}}$, $\epsilon_i(x) \neq 0$ as an element of a basis of $\mathfrak{h}$ by [Ko63, Theorem 9].

As a result, for all $x$ in a dense open subset of $\mathfrak{h}$, $\epsilon_i(x)$ is a regular element of $\mathfrak{g}$ and $\mathfrak{l}$. Let $x$ be in $\mathfrak{l}$, such that $\epsilon_i(x)$ is a regular element of $\mathfrak{g}$ and $\mathfrak{l}$, and $\mathfrak{d}_l$ a simple factor of $\mathfrak{l}$. As $\mathfrak{g}^{\epsilon_i(x)}$ is a commutative algebra, $\mathfrak{d}_l$ is not contained in $\mathfrak{g}^{\epsilon_i(x)}$. Hence the component of $\epsilon_i(x)$ on $\mathfrak{d}_l$ is different from 0 since $t^{\epsilon_i(x)} = g^{\epsilon_i(x)}$.

(ii) Let $x$ be in $\mathfrak{h}$ such that $\alpha_k(x) = 0$. Then $v_k$ and $w_k$ are in $\mathfrak{g}^{\epsilon_i}$. As $\epsilon_i$ is a $G$-equivariant map, $\epsilon_i(x)$ is in the center of $\mathfrak{g}^{\epsilon_i}$ so that

$$
0 = [\epsilon_i(x), v_k] = \alpha_k \circ \epsilon_i(x)v_k.
$$

As a result, the nullvariety of $\alpha_k$ in $\mathfrak{h}$ is contained in the nullvariety of $\alpha_k \circ \epsilon_i$. Hence $\alpha_k$ divides $\alpha_k \circ \epsilon_i$ in $S(\mathfrak{h})$ since $S(\mathfrak{h})$ is a factorial ring and $\alpha_k$ is a prime element. As $\epsilon_i$ is homogeneous of degree $d_i - 1$, the quotient $\alpha_k \circ \epsilon_i / \alpha_k$ is homogeneous of degree $d_i - 2$.

(iii) Let $x$ be in $\mathfrak{h}$ such that $\alpha_k(x) \neq 0$. Then

$$
\exp(\frac{t}{\alpha_k(x)} \text{ad} w_k)(x) = x + tw_k.
$$

As $\epsilon_i$ is $G$-equivariant,

$$
\epsilon_i(x + tw_k) = \exp(\frac{t}{\alpha_k(x)} \text{ad} w_k) \circ \epsilon_i(x),
$$

whence

$$
\epsilon_i(x + tw_k) = \epsilon_i(x) + tp_{i,k}(x)w_k
$$

by (ii). \qed

For $i = 1, \ldots, \ell$ and $x$ in $\mathfrak{g}$, denote by $\epsilon_{i, -}(x)$ the element of $\mathfrak{p}_{-\mathfrak{u}}$ such that $\epsilon_{i}(x) - \epsilon_{i, -}(x)$ is in $\mathfrak{p}$.

**Lemma 8.2.** Let $i = 1, \ldots, \ell$.

(i) For uniquely defined functions $a_{i,j,k}$, $1 \leq j, k \leq \ell$ in $k[\mathfrak{p}]$, the polynomial map

$$
t \mapsto \epsilon_{i, -}(x + ty) - \sum_{1 \leq j, k \leq d} t(v_j, y)a_{i,j,k}(x)w_k
$$

is divisible by $t^2$ in $k[t] \otimes_k \mathfrak{p}_{-\mathfrak{u}}$ for all $(x, y)$ in $\mathfrak{p} \times \mathfrak{p}_{-\mathfrak{u}}$.

(ii) For $1 \leq j, k \leq \ell$, $a_{i,j,k}$ is homogeneous of degree $d_i - 2$.

(iii) For $1 \leq j, k \leq \ell$, the function $a_{i,j,k}$ has weight $\alpha_k - \alpha_j$ with respect to the adjoint action of $\mathfrak{h}$ in $k[\mathfrak{p}]$. 

Proof. (i) Let \((x, y)\) be in \(p \times p_{-u}\). According to Corollary 2.5, \(\varepsilon_i(x)\) is in \(p\). Moreover, \(\varepsilon_i\) is homogeneous of degree \(d_i - 1\). Hence the polynomial map \(t \mapsto \varepsilon_{i,-}(x + ty)\) has a unique expansion

\[
\varepsilon_{i,-}(x + ty) = \sum_{k=1}^{d_i} \sum_{j=1}^{d_i - 1} \sum_{r \in S^d_j} t^r t'(y) a_{i,r,k}(x) w_k
\]

with the \(a_{i,r,k}\) in \(\mathbb{k}[p]\). For \(j = 1, \ldots, d\) and \(r\) in \(\mathbb{N}_1^d\) such that \(r_i = 0\) for \(i \not\sim j\), set \(a_{i,j,k} := a_{i,r,k}\). Then the polynomial map

\[
t \mapsto \varepsilon_{i,-}(x + ty) - \sum_{1 \leq j \leq d} t(v_j, y) a_{i,j,k}(x) w_k
\]

is divisible by \(t^2\) in \(\mathbb{k}[t] \otimes_{\mathbb{k}} p_{-u}\).

(ii) As \(\varepsilon_j\) is homogeneous of degree \(d_i - 1\), so is \(\varepsilon_{i,-}\) and for \((s, x, y)\) in \(\mathbb{k}^* \times p \times p_{-u}\), the polynomial map

\[
t \mapsto \varepsilon_{i,-}(x + ty) - s^{1-d_i} \sum_{1 \leq j \leq d} t(v_j, sy) a_{i,j,k}(sx) w_k
\]

is divisible by \(t^2\) in \(\mathbb{k}[t] \otimes_{\mathbb{k}} p_{-u}\). Hence the \(a_{i,j,k}\)'s are homogeneous of degree \(d_i - 2\).

(iii) Let \(g\) be in \(H\). As \(\varepsilon_i\) is a \(G\)-equivariant map and \(p_{-u}\) is invariant under \(H\), \(\varepsilon_{i,-}\) is \(H\)-equivariant. Then, for \((x, y, t)\) in \(p \times p_{-u} \times \mathbb{k}\),

\[
g \cdot \varepsilon_{i,-}(x + ty) - \sum_{1 \leq j \leq d} t(v_j, y) a_{i,j,k}(x) g.w_k = \varepsilon_{i,-}(g(x) + tg(y)) - \sum_{1 \leq j \leq d} t(v_j, y) a_{i,j,k}(g(x)) g.w_k.
\]

By (i), the polynomial map

\[
t \mapsto \varepsilon_{i,-}(g(x) + tg(y)) - \sum_{1 \leq j \leq d} t(v_j, g(y)) a_{i,j,k}(g(x)) w_k
\]

is divisible by \(t^2\) in \(\mathbb{k}[t] \otimes_{\mathbb{k}} p_{-u}\). Then

\[
\langle v_j, g(y) \rangle a_{i,j,k}(g(x)) w_k = \langle v_j, y \rangle a_{i,j,k}(x) g.w_k
\]

for all \((x, y)\) in \(p \times p_{-u}\). As a result, for \(1 \leq j, k \leq d\) and \(x\) in \(\mathfrak{h}\),

\[
\alpha_j(x) a_{i,j,k} + [x, a_{i,j,k}] = \alpha_k(x) a_{i,j,k},
\]

whence the assertion. \(\square\)

Denote by \(m_1 < \cdots < m_d\) the strictly increasing sequence of the values of the map

\[
\mathcal{R}_+ \setminus \mathcal{R}_1 \longrightarrow \mathbb{N}, \quad \alpha \mapsto \alpha(z).
\]

For \(j = 1, \ldots, d'\), let \(I_j\) be the set of indices \(i\) such that \(\alpha_i(z) = m_j\) and set:

\[
p^{(j)}_{-u} := \text{span}\{w_k \mid k \in I_j\}, \quad p_{-u,j} := \bigoplus_{k=j}^{d'} p^{(k)}_{-u}.
\]
Corollary 8.3. Let $j = 1, \ldots, d'$ and $i = 1, \ldots, \ell$.

(i) The space $\mathfrak{l} + \mathfrak{p}_{-u,j}$ is a subalgebra of $\mathfrak{g}$ and $[\mathfrak{p}_{-u,j}, \mathfrak{p}_{-u,j'}]$ is contained in $\mathfrak{p}_{-u,j''}$ with $j'' := \sup \{j, j'\}$.

(ii) For $j' > j$ and $(k, k')$ in $I_j \times I_{j'}$, $a_{i,k,k'} = 0$, and for $(k, k')$ in $I_j \times I_{j'}$, $a_{i,k,k'}$ is in $\mathbb{k}[I]$. 

(iii) Let $j = 1, \ldots, d'$. For $x$ in a dense open subset of $\mathfrak{h}$, the matrix $(a_{i,k,k'}(x), (k, k') \in I_j \times I_{j'})$ is diagonal and invertible for $i = 1, \ldots, \ell$.

Proof. (i) For $j_1, j_2 = 1, \ldots, d'$ and $(k, k')$ in $I_{j_1} \times I_{j_2}$,

$$[z, [w_k, w_{k'}]] = -(m_{j_1} + m_{j_2})[w_k, w_{k'}]$$

so that $[\mathfrak{p}_{u,j_1}, \mathfrak{p}_{u,j_2}]$ is contained in $\mathfrak{p}_{-u,j_3}$ with $j_3 := \sup \{j_1, j_2\}$, whence the assertion since $[\mathfrak{l}, \mathfrak{p}_{u,j}]$ is contained in $\mathfrak{p}_{-u,j}$ for all $j'$.

(ii) For $1 \leq k, k' \leq d$, the function $a_{i,k,k'}$ on $\mathfrak{p}$ has an expansion

$$a_{i,k,k'} = \sum_{u \in \mathbb{N}^d} u^\alpha a_{u,i,k,k'} \quad \text{with} \quad a_{u,i,k,k'} \in \mathbb{k}[I].$$

For $(s, x)$ in $\mathbb{k}^\times \times I$ and $y := y_1v_1 + \cdots + y_dv_d \in \mathfrak{p}_u$,

$$\zeta(s, x + y \alpha(z) = \epsilon_i(x + \sum_{j=1}^d s^{\alpha_j(z)} y_j + s - 1) w_k)$$

since $\epsilon_i$ is $G$-equivariant and $z$ is in $\mathfrak{z}$. Then, by Lemma 8.2(i), for $k$ in $I_j$,

$$\sum_{k'=1}^d \sum_{u \in \mathbb{N}^d} s^{-\alpha_k(z)} w^u(y) a_{u,i,k,k'}(x) w_{k'} = \sum_{k'=1}^d \sum_{u \in \mathbb{N}^d} u_1 \alpha_1(z) + \cdots + u_d \alpha_d(z) - m_j w^u(y) a_{u,i,k,k'}(x) w_{k'},$$

whence

$$a_{u,i,k,k'}(x) \neq 0 \iff m_j - \alpha_k(z) = u_1 \alpha_1(z) + \cdots + u_d \alpha_d(z).$$

For $j' > j$ and $k'$ in $I_{j'}$, $\alpha_k(z) > m_j$. Hence $a_{u,i,k,k'} = 0$ for all $u$ in $\mathbb{N}^d$ since the integers $\alpha_1(z), \ldots, \alpha_d(z)$ are positive. Moreover, for $k'$ in $I_{j'}$, $a_{u,i,k,k'} = 0$ if $u$ is different from 0, whence the assertion.

(iii) Let $k$ be in $I_j$. By Lemma 8.1(iii),

$$a_{i,k,k'}(x) = \begin{cases} p_{i,k}(x) & \text{if } k = k' \\ 0 & \text{if } k \neq k'. \end{cases}$$

By Lemma 8.1(i), for all $x$ in a dense open subset of $\mathfrak{b}_{\text{reg}}$, $p_{i,k}(x) \neq 0$ for $i = 1, \ldots, \ell$. As a result, for such $x$, the matrix $(a_{i,k,k'}(x), (k, k') \in I_j \times I_{j'})$ is diagonal and invertible.
Lemma 8.4. Let \( k = 1, \ldots, d, i = 1, \ldots, \ell \) and \( m = 0, \ldots, d_i - 2 \). For \( (x, y) \) in \( \mathfrak{h} \times \mathfrak{h} \),
\[
\alpha_k(x)^m a_{i,k,k}^{(m)}(x, y) = \alpha_k^{m+1}(\psi_{i,m}(x, y)).
\]

Proof. According to Lemma 8.1.(ii) and (iii),
\[
\alpha_k(\mathfrak{c}_i(x)) = \alpha_k(x) a_{i,k,k}(x) \quad \text{and} \quad a_{i,k,k}(x) = p_{i,k}(x)
\]
for all \( x \) in \( \mathfrak{h} \). As a result, for all \( (x, y) \) in \( \mathfrak{h} \times \mathfrak{h} \),
\[
\sum_{m=0}^{d_i-1} s^m \alpha_k \mathfrak{c}_i^{(m)}(x, y) = \alpha_k(x + sy) a_{i,k,k}(x + sy) = \sum_{m=0}^{d_i-2} s^m \alpha_k(x) a_{i,k,k}^{(m)}(x, y) + \sum_{m=1}^{d_i-1} s^m \alpha_k(y) a_{i,k,k}^{(m-1)}(x, y)
\]
for all \( s \) in \( \mathfrak{k} \), whence
\[
\alpha_k(x) a_{i,k,k}^{(m)}(x, y) + \alpha_k(y) a_{i,k,k}^{(m-1)}(x, y) = \alpha_k \mathfrak{c}_i^{(m)}(x, y)
\]
for \( m = 1, \ldots, d_i - 2 \). Then, by induction on \( m \),
\[
\alpha_k(x)^m a_{i,k,k}^{(m)}(x, y) = \sum_{j=0}^{m} (-1)^{m-j} \alpha_k(x)^j \alpha_k(y)^{m-j} \mathfrak{c}_i^{(j)}(x, y) = \alpha_k^{m+1}(\psi_{i,m}(x, y))
\]
for \( m = 0, \ldots, d_i - 2 \).

For \( (i, m) \) in \( I_o \) and \( 1 \leq j, k \leq d \), set:
\[
b_i^{(m-1)} := a_{i,k,k}^{(m-1)} - \sum_{(l,m') \in I_o} a_{i,m,l,m'}^{(m'-1)} \quad \text{with} \quad I_o := I_o' \setminus \{1, \ldots, \ell\} \times \{0\}
\]
and \( a_{i,m,j,m'} \), \( ((i, m), (j, m')) \in I_u \times I_o' \) as in Proposition 7.6. For \( (i, m) \) in \( I_u \), denote by \( \varphi_{i,m} \) the map
\[
\varphi_{i,m} : \mathfrak{h} \times \mathfrak{h} \longrightarrow S^m(\mathfrak{h}) , \quad (x, y) \longmapsto \psi_{i,m-1}(x, y) - \sum_{(j,m') \in I_o} a_{i,m,j,m'}(x, y) x^{m-m'} \psi_{j,m'-1}(x, y).
\]

Proposition 8.5. Let \( (i, m) \) be in \( I_u \) and \( k = 1, \ldots, d \).

(i) For \( j \) in \( \{1, \ldots, d\} \setminus \{k\} \), the restriction of \( b_i^{(m-1)} \) to \( \mathfrak{h}_s \times \mathfrak{h} \) is equal to 0.

(ii) The restriction of \( b_i^{(m-1)} \) to \( \mathfrak{h}_s \times \mathfrak{h} \) is different from 0.
Proof. (i) According to Lemma 8.2(iii), the functions $a^{(m)}_{i,j,k}$, $(l, m') \in I_0$ have weight $\alpha_k - \alpha_j$ with respect to the adjoint action of $\mathfrak{h}$ in $\mathbb{K}[l \times \mathfrak{p}]$. As the functions $a_{i,m,l,m'}$, $(i, m) \in I_0$, $(l, m') \in I_0'$ are invariant under $L$, the function $b^{(m-1)}_{i,j,k}$ has weight $\alpha_k - \alpha_j$. Then the restriction of $b^{(m-1)}_{i,j,k}$ to $\mathfrak{h}_* \times \mathfrak{h}$ is equal to 0 since $\alpha_j \neq \alpha_k$.

(ii) According to Lemma 8.4, for $(x, y)$ in $\mathfrak{h}_* \times \mathfrak{h}$,

$$a^m_k(x)b^{(m-1)}_{i,j,k}(x, y) = a^m_k(\varphi_{i,m}(x, y)).$$

By definition, $\varphi_{i,m}(x, y)$ has an expansion

$$\varphi_{i,m}(x, y) = \sum_{j=0}^{m-1} (-1)^j x^{m-1-j} y^j \varepsilon_j^{(m-1)}(x, y) - \sum_{(l,m') \in I_0'} \sum_{j=0}^{m'-1} (-1)^j a_{i,m,l,m'}(x, y)x^{m-1-j} y^j \varepsilon_j^{(m-1)}(x, y).$$

Then $\varphi_{i,m}(x, y) = x^{m-1}\varphi'_{i,m}(x)$ with $\varphi'_{i,m}$ in $\mathbb{K}[\mathfrak{h}_*] \otimes_{\mathbb{K}} \mathfrak{h}$ defined by the following equality:

$$\varphi'_{i,m}(x) = \sum_{j=0}^{m-1} (-1)^j \left( \frac{d_{i-1}}{m-1-j} \right) \varepsilon_j(x) - \sum_{(l,m') \in I'_0} \sum_{j=0}^{m'-1} (-1)^j \left( \frac{d_{i-1}}{m'-1-j} \right) a_{i,m,l,m'}(x, y) \varepsilon_j(x)$$

by Lemma 7.7(iii). Hence $\varphi'_{i,m}$ is in the free submodule of $\mathbb{K}[\mathfrak{h}_*] \otimes_{\mathbb{K}} \mathfrak{h}$ generated by $\varepsilon_1, \ldots, \varepsilon_l$. Moreover, by Proposition 7.11, its restriction to $\mathfrak{h}_*$ is equivariant under $W(\mathcal{R})$ since so are $\varepsilon_1, \ldots, \varepsilon_l$.

Denote by $\varphi'_{i,m,i}$ the coordinate of $\varphi'_{i,m}$ at $\varepsilon_i$. With the notations of Subsection 7.2,

$$(-1)^{m-1} \varphi'_{i,m,i}(x) = r(m, d_i - 1) - \sum_{m'=1}^{d_i-1} (-1)^{m-m'} r(m', d_i - 1) a_{i,m,l,m'}(x, x)$$

for all $x$ in $\mathfrak{h}_*$. Then, by Corollary 7.8(iii), $\varphi'_{i,m} \neq 0$. Suppose $\alpha_k \varphi'_{i,m}(x) = 0$ for all $x$ in $\mathfrak{h}_*$. A contradiction is expected. Then $g.\alpha_k \varphi'_{i,m}(x) = 0$ for all $(g, x)$ in $W(\mathcal{R}) \times \mathfrak{h}_*$. As $g$ is simple, so is the $W(\mathcal{R})$-module $\mathfrak{h}_*$. As a result, $\mathfrak{h}_*$ is generated by $g.\alpha_k$, $g \in W(\mathcal{R})$ and $\varphi'_{i,m}(x) = 0$ for all $x$ in $\mathfrak{h}_*$, whence the contradiction. Then the function $x \mapsto \alpha_k^{m+1}(\varphi_{i,m}(x, x))$ on $\mathfrak{h}_*$ is different from 0 since $\alpha_k(x) \neq 0$ for $x$ regular in $\mathfrak{h}$, whence the assertion. \hfill $\square$

9. At the neighborhood of a semi-simple element II

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. We then use the notations

$$1, \ p_u, \ \mathfrak{d}, \ \mathfrak{z}, \ \mathcal{R}_t, \ L, \ d, \ d_0, \ l, \ z, \ \zeta, \ I_0, \ I_0', \ I_u, \ p_-, \ p_{-u}$$

of Section 8. For $(i, m)$ in $I_0$, the restriction of $\varepsilon_i^{(m)}$ to $1 \times \mathfrak{g}$ is again denoted by $\varepsilon_i^{(m)}$. The usual gradation of $\mathbb{K}[\mathfrak{g}]$ induces a gradation of the polynomial algebra $\mathbb{K}[1 \times \mathfrak{g}]$ over $\mathbb{K}[1, \zeta]$. From the direct sum

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{p}_{-u},$$
we deduce the inclusions
\[ \mathbb{k}[p] \subset \mathbb{k}[g] \quad \text{and} \quad \mathbb{k}[I \times p] \subset \mathbb{k}[I \times g]. \]

Moreover, \( \mathbb{k}[I \times I] \) is a subalgebra of \( \mathbb{k}[I \times g] \). Set \( B := \mathbb{k}[I \times g] \otimes_{\mathbb{k}[g]} B_g \).

### 9.1. Decomposition of \( B \)

As \( \mathbb{k}[I \times p] \) is contained in \( \mathbb{k}[I \times g] \), the sequence \( a_{i,m,j,m'} \), \( ((i, m), (j, m')) \in I_u \times I'_0 \) of Proposition 7.6(ii) is contained in \( \mathbb{k}[I \times g] \). For \((i, m) \in I_u\), let \( \nu_i^{(m)} \) be the element of \( \mathbb{k}[I \times g] \otimes_{\mathbb{k}} g \).

\[
\nu_i^{(m)} := \varepsilon_i^{(m)} - \sum_{(j,m') \in I'_0} a_{i,m,j,m'} \varepsilon_j^{(m')}.
\]

Denote by \( B_0 \) and \( B_u \) the submodules of \( B \) generated by the sequences \( \varepsilon_i^{(m)} \), \((i, m) \in I'_0\) and \( \nu_i^{(m)} \), \((i, m) \in I_u \) respectively. Let \( J \) and \( J_\lambda \) be the ideals of \( \mathbb{k}[I \times g] \) generated by \( 1 \otimes p_u \) and \( 1 \otimes p_{-u} \) respectively.

**Proposition 9.1.** For \((i, m) \in I_0\), let \( \overline{\varepsilon_i^{(m)}} \) be the restriction of \( \varepsilon_i^{(m)} \) to \( I \times I \).

(i) The modules \( B_0 \) and \( B_u \) are free of rank \( b_1 \) and \( d \) respectively. Moreover \( B \) is the direct sum of \( B_0 \) and \( B_u \).

(ii) The sequence \( \overline{\varepsilon_i^{(m)}} \), \((i, m) \in I'_0\) is a basis of \( \overline{B_u} \).

(iii) For \((i, m) \in I_0\),

\[
\varepsilon_i^{(m)} - \overline{\varepsilon_i^{(m)}} \in J \otimes_{\mathbb{k}} p_{-u} + J_\lambda \otimes_{\mathbb{k}} p_u + JJ_\lambda \otimes_{\mathbb{k}} I.
\]

(iv) For \((i, m) \in I_u\),

\[
\nu_i^{(m)} \in J \otimes_{\mathbb{k}} p_{-u} + J_\lambda \otimes_{\mathbb{k}} p_u + JJ_\lambda \otimes_{\mathbb{k}} I.
\]

**Proof.**

(i) By Lemma 6.1(ii) and Theorem 2.2(i), \( B \) is a free module of rank \( b_0 \) generated by \( \varepsilon_i^{(m)} \), \((i, m) \in I_0\). Then \( B_0 \) and \( B_u \) are free of rank \( b_1 \) and \( d \) respectively since \( |I'_0| = b_1 \) and \( |I_u| = d \).

Moreover, \( B_u \) is a submodule of \( B \) and \( B \) is the direct sum of \( B_0 \) and \( B_u \).

(ii) The assertions results of Lemma 7.5(ii).

(iii) Let \((i, m) \) be in \( I_0 \). As \( J \) is the ideal of definition of \( I \times p \) in \( \mathbb{k}[I \times g] \),

\[
\varepsilon_i^{(m)} - \overline{\varepsilon_i^{(m)}} \in J_\lambda \otimes_{\mathbb{k}} p_u + J \otimes_{\mathbb{k}} g
\]

by Proposition 7.6(i). Moreover, \( \varepsilon_i^{(m)} \) and \( \overline{\varepsilon_i^{(m)}} \) are invariant under \( L \), whence the assertion since an element of \( J \otimes_{\mathbb{k}} g \), invariant under the one parameter subgroup \( t \mapsto \xi(t) \), is in \( J \otimes_{\mathbb{k}} p_{-u} + JJ_\lambda \otimes_{\mathbb{k}} I \).

(iv) As \( \nu_i^{(m)} \) is a linear combination with coefficients in \( \mathbb{k}[I \times I] \) of \( \varepsilon_i^{(m)} \) and \( \varepsilon_j^{(m')} \), \((j, m') \in I'_0\), the assertion results from (iii) and Proposition 7.6(ii). \( \square \)
9.2. Some expansions and invertible matrices. Let $v_1, \ldots, v_d, w_1, \ldots, w_d$ be as in Subsection 8.1. Let $\mathcal{S}$ be the union of $\{0\}$ and the set of strictly increasing sequences in $\{1, \ldots, d\}$ and for $\iota = i_1 < \cdots < i_j$ in $\mathcal{S}$, set:

$$\{\iota\} := \{i_1, \ldots, i_j\}, \quad |\iota| = j, \quad v_\iota := v_{i_1} \wedge \cdots \wedge v_{i_j}, \quad w_\iota := w_{i_1} \wedge \cdots \wedge w_{i_j}. $$

By definition, $v_\iota$ and $w_\iota$ are in $\bigwedge^{|\iota|}(p_u)$ and $\bigwedge^{|\iota|}(p_{-u})$. For $i = 0$, $|\iota| := 0$ and $v_i := w_i := 1$, and for $|\iota| = d$, $\mu_\iota := v_\iota$ is a generator of $\bigwedge^d(p_u)$. For $j = 0, \ldots, d$, set:

$$\mathfrak{S}_j := \{\iota \in \mathcal{S} \mid |\iota| = j\} \quad \text{and} \quad \Lambda_j := \{(r, \kappa) \in \mathbb{N}^d \times \mathfrak{S} \mid |r| = |\kappa| = j\}.$$

Let $I_{*, 0}$, $I'_{*, 0}$, $I_{0,*}$ be the sets:

$$I_{*, 0} := \{(i, m) \in I_0 \mid m > 0\}, \quad I'_{*, 0} := I'_0 \cap I_{*, 0}, \quad I_{0,*} := \{(i, m, k) \mid (i, m) \in I_{*, 0}, k \in \{1, \ldots, m\}\}.$$

Denote by $I'_{*, 0}$ and $I_{0,*}$ the inverse images of $I'_{*, 0}$ and $I_{0,*}$ by the canonical projection $I_{0,*} \longrightarrow I_{*, 0}$. Set:

$$\epsilon_0 := \wedge_{(i, m) \in I_0} \omega_i^{(m)}, \quad \epsilon_{0, 0} := \wedge_1 \cdots \wedge \epsilon_\iota, \quad \epsilon_{I'_0} := \wedge_{(i, m) \in I'_{*, 0}} \omega_i^{(m)} , \quad \nu_u := \wedge_{(i, m) \in I_u} \omega_i^{(m)}.$$

For some $\epsilon_\iota$ in $\{-1, +1\}$,

$$\epsilon_0 = \epsilon_\iota \epsilon_{0, 0} \land \epsilon_{I'_0}.$$

In these equalities, the order of the products is induced by the order of $I_0$. Then $\epsilon_0$ and $\nu_u$ are generators of $\bigwedge^b(B_0)$ and $\bigwedge^d(B_u)$ respectively. Moreover, the restriction $\overline{\epsilon_0}$ of $\epsilon_0$ to $l \times l$ is a generator of $\bigwedge^b(B)$ and $\epsilon := \overline{\epsilon_0} \land \nu_u$ is a generator of $\bigwedge^b(B)$ by Proposition 9.1(i).

By Proposition 9.1(iii), for $(x, y, y')$ in $l \times p \times p_{-u}$ and $(i, m)$ in $I'_0$ and $(j, l)$ in $I_u$, the polynomial maps

$$t \longmapsto \omega_i^{(m)}(x, y + ty') \quad \text{and} \quad t \longmapsto \omega_j^{(l)}(x, y + ty')$$

have an expansion

$$\omega_i^{(m)}(x, y + ty') = \omega_i^{(m)}(x, y) + \sum_{k=1}^m t^k \omega_{i, j, k, -}(x, y, y') + \sum_{k=1}^m t^k \omega_{i, j, k, +}(x, y, y') + \sum_{k=1}^m t^k \omega_{i, j, k, 0}(x, y, y')$$

$$\omega_j^{(l)}(x, y + ty') = \omega_j^{(l)}(x, y) + \sum_{k=1}^m t^k \omega_{j, k, +}(x, y, y') + \sum_{k=1}^m t^k \omega_{j, k, -}(x, y, y') + \sum_{k=1}^m t^k \omega_{j, k, 0}(x, y, y')$$

with

$$\omega_{i, j, k, -} \in \mathbb{k}[l \times p] \otimes \mathbb{k} S^k(p_u) \otimes \mathbb{k} p_{-u}, \quad \omega_{i, j, k, +} \in \mathbb{k}[l \times p] \otimes \mathbb{k} S^k(p_u) \otimes \mathbb{k} p_u,$$

$$\omega_{i, j, k, 0} \in \mathbb{k}[l \times p] \otimes \mathbb{k} S^k(p_u) \otimes \mathbb{k} l$$

for $(i, m)$ in $I_0$. For $I$ subset of $I_{*, 0}$, $K$ subset of $I_u$ and $I'$ subset of $I_{0,*}$, set:

$$\epsilon_I := \wedge_{(i, m) \in I} \omega_i^{(m)} , \quad \nu_K := \wedge_{(i, m) \in K} \omega_i^{(m)} , \quad S_I := \sum_{(i, m, k) \in I'} k,$$

$$\omega_{I', -} := \wedge_{(i, m, k) \in I'} \omega_{i, m, k, -}, \quad \omega_{I', +} := \wedge_{(i, m, k) \in I'} \omega_{i, m, k, +}, \quad \omega_{I', 0} := \wedge_{(i, m, k) \in I'} \omega_{i, m, k, 0}. $$
In these equalities, the order of the products are induced by the orders of $I_0$ and $I_{0,*}$. For $K$ subset of $I_{0,*}$, denote by $K^\#$ the image of $K$ by the projection $(i,m,k) \mapsto (i,m)$. When $S_K = |K|$, we identify $K$ and $K^\#$.

For $i$ in $\mathcal{S}$ and $j = 0, \ldots, d$, set:

$$\varepsilon_i := v_i \land \varepsilon \quad \text{and} \quad \mathcal{K}_j := \{K \subset I_0 \mid |K| = d - j\}.$$ 

**Lemma 9.2.** Let $j = 0, \ldots, d$, $i$ in $\mathcal{S}_j$ and $(x,y,y')$ in $I_0 \times p \times p_{-u}$. Denote by $c_i(x,y,y')$ the coefficient of $t^i$ of the polynomial map $t \mapsto \varepsilon_i(x,y + ty')$.

(i) The polynomial map $t \mapsto \varepsilon_i(x,y + ty')$ is divisible by $t^i$ in $\mathbb{k}[t] \otimes \Lambda_{1/2}$.

(ii) For a well defined map

$$\mathcal{K}_j \xrightarrow{\varepsilon} \{-1, 1\},$$

$$c_i(x,y,y') - c_i'(x,y,y') \in \bigoplus_{|i| < j} w_i \land \Lambda^{|-|i|}(I) \land \overline{\varepsilon}_0(x,y) \land \mu_+$$

with

$$c_i'(x,y,y') := \sum_{K \in \mathcal{K}_j} \varepsilon(K)v_i \land \overline{\varepsilon}_0(x,y) \land v_K(x,y) \land \omega_{I_i \setminus K}(x,y,y').$$

(iii) For $K$ in $\mathcal{K}_j$, for well defined functions $a_{r,k,K,i}$, $(r,k) \in \Lambda_j$ in $\mathbb{k}[I_0 \times p]$

$$\varepsilon(K)v_i \land \overline{\varepsilon}_0(x,y) \land v_K(x,y) \land \omega_{I_i \setminus K}(x,y,y') = \sum_{(r,k) \in \Lambda_j} v'(y')a_{r,k,K,i}(x,y)w_r \land \overline{\varepsilon}_0(x,y) \land \mu_+.$$ 

**Proof.** As $\varepsilon = \varepsilon_0 \land v_u$, the coefficient of $t^k$ of the polynomial functions $t \mapsto \varepsilon(x,y + ty')$ is the sum of the value at $(x,y,y')$ of products

$$\varepsilon(I, K, I_+, K_+, I_0, K_0) \varepsilon_{0,0} \land \varepsilon_I \land v_K \land \omega_{I_{-n}} \land \omega_{I_{+n}} \land \omega_{I_+} \land \omega_{I_{-n}} \land \omega_{I_0} \land \omega_{I_{0,n}}$$

with

$$I \subset I'_{+0}, \quad K \subset I_0, \quad I_0 \subset I'_0, \quad K_+ \subset I_{0,+}, \quad I_\pm \subset I_{0,\pm}, \quad I_0 \subset I_{0,0}, \quad K_\pm \subset I_{0,\pm}, \quad K_+ \subset I_{0,+}, \quad K_{-} \subset I_{0,-}, \quad I_{0,0} \subset I_{0,0}$$

such that

$$I \cup I_0^\# \cup I_+^\# \cup I_-^\# = I \cup I_0^\# \cup I_+^\# \cup I_-^\# \cup K^\# = I \cup K^\# \cup K_0^\# \cup K_+^\# \cup K_-^\#,$$

$$|I| + |I_0^\#| + |I_+^\#| + |I_-^\#| = n - d, \quad |K| + |K_0^\#| + |K_+^\#| + |K_-^\#| = d,$$

$$S_{I_0} + S_{I_0} + S_{I_+} + S_{I_-} + S_{K_+} + S_{K_-} = k.$$ 

For $|K| + |K_0^\#| + |K_+^\#| > d - j$,

$$v_i \land \varepsilon_{0,0} \land \varepsilon_I \land v_K \land \omega_{I_{-n}} \land \omega_{I_{+n}} \land \omega_{I_+} \land \omega_{I_{-n}} \land \omega_{I_0} \land \omega_{I_{0,n}} = 0$$

since $v_K(x,y)$ is in $\Lambda^{|K|}(p_u)$, whence

$$|K| + |I_0^\#| + |K_0^\#| \leq d - j, \quad j \leq |K_+^\#| + |K_-^\#| - |I_+^\#|. $$
\[ |I_k^0| + |I_k^1| + |I_k^2| + |K_k^0| + |K_k^1| \leq k \]

since
\[ |I_k^0| + |I_k^1| + |I_k^2| + |K_k^0| + |K_k^1| \leq S_L + S_{I_k} + S_{K_k} + S_{K_{\pm}}. \]

As a result, the coefficient of \( t^k \) of the polynomial map \( t \mapsto \varepsilon(x, y + ty') \) is equal to 0 when \( k < j \) and for \( k = j \), it is the sum of the value of the product at \((x, y, y')\) of the products
\[ \varepsilon(I, K, I_-, K_+, I_+, K_+, I_{\pm}, K_{\pm}) v_i \land \varepsilon_{0,0} \land \varepsilon_I \land \nu_K \land \omega_{I_-,} \land \omega_{K_-,} \land \omega_{I_+} \land \omega_{K_+} \]

with
\[ |K| + |I_k^0| + |I_k^1| \leq d - j, \quad j \leq |K_k^0| + |K_k^1| - |I_k^1|, \]
\[ |I_-| = |I_k^0| = S_L, \quad |I_+| = |I_k^1| = S_{I_k}, \quad |I_{\pm}| = |I_k^2| = S_{I_k}, \]
\[ |K_-| = |K_k^0| = S_{K_k}, \quad |K_+| = |K_k^1| = S_{K_k}, \quad |K_{\pm}| = |K_k^2| = S_{K_k}, \]

whence
\[ |I_+| = |I_-| = |I_{\pm}| = |K_{\pm}| = 0, \quad I = I_{0,0}, \quad |K| = d - j, \quad |K_-| + |K_{\pm}| = j. \]

For \( K \in \mathcal{K}_j \), set:
\[ \varepsilon(K) := \varepsilon, \varepsilon(K, \emptyset, I_u \setminus K, 0, 0, 0, 0). \]

Then, by Proposition 9.1,
\[ c_t(x, y, y') = \sum_{K \in \mathcal{K}_j} \varepsilon(K) v_i \land \varepsilon_0(x, y) \land \nu_K(x, y) \land \omega_{I_u \setminus K} (x, y, y') \in \bigoplus_{|k| < j} w_k \land \bigwedge_{j - |k|} (1) \land \varepsilon_0(x, y) \land \mu_+ \]

since
\[ v_i \land \varepsilon_{0,0}(x) \land \varepsilon_{I,0}(x, y) \land \nu_K = \varepsilon, v_i \land \varepsilon_0(x, y) \land \nu_K(x, y) \]

and \(|K_{\pm}| > 0\) when \( K_+ \subseteq I_u \setminus K \). Moreover, for \( K \in \mathcal{K}_j \), for well defined functions \( a_{r, K, k}, (r, k) \in \Lambda_j \) in \( \mathbb{K}[1, \times, p] \),
\[ \varepsilon(K) v_i \land \varepsilon_0(x, y) \land \nu_K(x, y) \land \omega_{I_u \setminus K} (x, y, y') = \sum_{(r, k) \in \Lambda_j} v(r') a_{r, K, k}(x, y) w_k \land \varepsilon_0(x, y) \land \mu_+, \]

whence the lemma.

For \((i, m)\) in \(I_0\) and \(k = 1, \ldots, d\), denote by \( b_{i,k}^{(m-1)} \) the coordinate of \( \omega_{i,m,1,-} \) at \( w_k \). Then \( b_{i,k}^{(m-1)} \) has an expansion
\[ b_{i,k}^{(m-1)} = \sum_{l=1}^d v_l \otimes b_{i,l,k}^{(m-1)} \]

with \( b_{i,l,k}^{(m-1)} \) homogeneous of degree \( m - 1 \) in \( \mathbb{K}[1, \times, p] \). For \( j = 1, \ldots, d \) and \( r = (r_1, \ldots, r_d) \) in \( \mathbb{N}^d \),

define an increasing sequence \( j_1 < \cdots < j_l \) by the following conditions:

- \( j_1 \) is the smallest integer such that \( r_{j_1} \neq 0 \),
- \( j_m \) is the smallest integer bigger than \( j_{m-1} \) such that \( r_{j_m} \neq 0 \).

and denote by \( r_1', \ldots, r_l' \) and \( u_1, \ldots, u_j \) the sequences defined by the following conditions:
Lemma 9.3. Let $j$ be divisible by $\kappa$ and $1 \leq m \leq r_j$.

(i) Let $(i, \kappa)$ be in $\mathcal{J}_j$, $(i)$ let $(i_m, \kappa)$ be in $\mathcal{K}$, for $m = 1, \ldots, r_j$.

(ii) Let $(i, \kappa)$ be in $\mathcal{J}_j$, $(i)$ let $(i_m, \kappa)$ be in $\mathcal{K}$.

(iii) For all $(r, \kappa)$ in $\Lambda$, $\omega_{e, K, d} = (-1)^{\delta} e(K) a_{K, d} B_{e, K, d}$.

Proof. (i) Let $(x, y, y')$ be in $I_u \times \mathcal{P} \times \mathcal{P}_u$. Denote by $(i_1, m_1), \ldots, (i_j, m_j)$ the elements of $\mathcal{K}$, ordered so that the sequence is increasing. For $\kappa$ in $\mathcal{J}_j$, let $\omega_{e, K, d}$ be the coordinate of $\omega_{e, K, d}$ at $w_k$. By definition, for $\kappa = k_1 < \cdots < k_j$,

$$\omega_{e, K, d} := \det(b_{i, m, \kappa}^{(m-1)}, 1 \leq l, l' \leq j) \quad \text{whence} \quad \omega_{e, K, d} = \sum_{r \in \mathbb{N}_j^d} v^r B_{e, K, d}.$$ 

(ii) Let $(x, y, y')$ be in $I_u \times \mathcal{P} \times \mathcal{P}_u$. By Lemma 8.1(i), for $l = 1, \ldots, \ell$ and $s$ in $\kappa$, the polynomial map

$$t \mapsto e_{l, s}(x + sy + sty' - \sum_{1 \leq l, k \leq d} st(v_{l, s, y'}) a_{l, s, k}(x + sy) w_k$$

is divisible by $t^2$ in $\mathcal{K}$. Hence the polynomial map

$$t \mapsto e_{l, s}^{(m')}(x, y + ty' - \sum_{1 \leq l, k \leq d} t(v_{l, s, y'}) a_{l, s, k}^{(m'-1)}(x, y) w_k$$

is divisible by $t^2$ in $\mathcal{K}$ for $m' = 0, \ldots, d_l - 1$. By definition, for $(i, m)$ in $I_u$,

$$v_i^{(m)} = e_i^{(m)} - \sum_{(l, m') \in I_0} a_{i, m, l, m'} e_i^{(m)} \quad \text{and} \quad v_i^{(m)} := e_i^{(m)} - \sum_{(l, m') \in I_0} a_{i, m, l, m'} e_i^{(m)}$$

with $a_{i, m, l, m'}$ homogeneous and $L$-invariant element of $\mathcal{K}$. So, the polynomial map

$$t \mapsto v_{l, s}^{(m)}(x, y + ty' - \sum_{1 \leq l, k \leq d} t(v_{l, s, y'}) a_{l, s, k}^{(m'-1)}(x, y) - \sum_{(l, m') \in I_0} a_{i, m, l, m'}(x, y) a_{l, s, k}^{(m'-1)}(x, y)) w_k$$
is divisible by $t^2$ in $\mathbb{k}[t] \otimes_{\mathbb{k}} \mathfrak{p}_{-u}$. As a result, for $u, u' = 1, \ldots, j$,
\[
\begin{align*}
b_{i_u k_u k_{u'}}^{(m_u-1)}(x, y) &= a_{i_u k_u k_{u'}}^{(m_u-1)}(x, y) - \sum_{(l m') \in l_0'} a_{i_u m_u l m'}(x, y) a_{i_u k_u k_{u'}}^{(m_u-1)}(x, y) \\
\end{align*}
\]
so that $b_{i_u k_u k_{u'}}^{(m_u-1)}$ is the element of $\mathbb{k}[l_1 \times \mathfrak{p}]$ defined in Subsection 8.2. By definition,
\[
B_r \kappa K = \det (b_{i_u k_u k_{u'}}^{(m_u-1)}, 1 \leq u, u' \leq j).
\]
According to Proposition 8.5, the restriction to $\mathfrak{h}_r \times \mathfrak{h}_s$ of the matrix
\[
(b_{i_u k_u k_{u'}}^{(m_u-1)}, 1 \leq u, u' \leq j)
\]
is diagonal and all its diagonal entries are different from 0, whence the assertion.

(iii) By (i),
\[
\begin{align*}
v_i \wedge \mathcal{E}_0 \wedge \nu_K \wedge \omega_{K,-} &= \sum_{(r, \kappa) \in \Lambda_j} v' B_r \kappa K v_i \wedge \mathcal{E}_0 \wedge \nu_K \wedge \omega_{K}, \\
&= \sum_{(r, \kappa) \in \Lambda_j} v' B_r \kappa K a_{k_1 \kappa} \mathcal{E}_0 \wedge \mu_+ \wedge \omega_{K},
\end{align*}
\]
whence the assertion by Lemma 9.2(iii).

For $j = 1, \ldots, d$, $(u, \kappa)$ in $\mathfrak{H}_j \times \mathfrak{S}_j$, $K$ in $\mathfrak{K}_j$, set:
\[
B_{\kappa, K} := \sum_{r \in \mathfrak{H}_j} v' B_{r, \kappa, K}, \quad a_{k_1 \kappa} := \sum_{r \in \mathfrak{H}_j} \sum_{K \in \mathfrak{S}_j} v' a_{r, \kappa, K},
\]
\[
P_j := \det (a_{k_1 \kappa}, (u, \kappa) \in \mathfrak{H}_j \times \mathfrak{S}_j), \quad Q_j := \det (B_{\kappa, K}, (\kappa, K) \in \mathfrak{H}_j \times \mathfrak{K}_j).
\]

**Corollary 9.4.** Let $j = 1, \ldots, d$.

(i) For some $(x, y, y')$ in $l_1 \times \mathfrak{p} \times \mathfrak{p}_{-u}$, $Q_j(x, y, y') \neq 0$.

(ii) For some $(x, y, y')$ in $l_1 \times \mathfrak{p} \times \mathfrak{p}_{-u}$, $P_j(x, y, y') \neq 0$.

**Proof.** (i) Prove the assertion by induction on $j$. By Lemma 9.3, the assertion is true for $j = 1$. Suppose that it is true for $j - 1$. By induction hypothesis and Lemma 9.3(ii), for all $(x, y, y')$ in a dense open subset of $l_1 \times \mathfrak{p} \times \mathfrak{p}_{-u}$, $Q_{j-1}(x, y, y') \neq 0$ and $B_{k, x, k, y, y'} \neq 0$ for $l = 1, \ldots, d$ and for all $(\kappa, K)$ in $\mathfrak{H}_j \times \mathfrak{K}_j$. Let $(x, y, y')$ be in this open subset of $l_1 \times \mathfrak{p} \times \mathfrak{p}_{-u}$ and $c_{k, \kappa} \in \mathfrak{K}_j$ in $\mathbb{k}$ such that
\[
\sum_{k \in \mathfrak{K}_j} c_{k} B_{k, x, k, y, y'} = 0, \quad \forall K \in \mathfrak{K}_j.
\]

For $K$ in $\mathfrak{K}_j$ and $l = 1, \ldots, j$, set:
\[
K_l := I_0 \setminus K := \{(i_1, m_1), \ldots, (i_j, m_j)\} \quad \text{and} \quad K^{(l)} := K \cup \{(i_l, m_l)\},
\]
and for $k$ in $\mathfrak{S}_j$, denote by $k^{(l)}$ the element of $\mathfrak{S}_{j-1}$ such that $\{k^{(l)}\} = \{k\} \setminus \{k_l\}$. Since
\[
B_{k, K} = \det (b_{i_u k_{u'}}^{(m_u-1)}, 1 \leq l, l' \leq j),
\]
for \( l = 1, \ldots, j \),
\[
B_{k,K} = \sum_{\ell=1}^{j} (-1)^{\ell+j} b_{\ell,k_{\ell}}^{(m-1)} B_{k^{(\ell)},K^{(\ell)}},
\]
whence
\[
\sum_{k \in \Gamma_{j}} c_{k} \sum_{\ell=1}^{j} (-1)^{\ell} b_{\ell,k_{\ell}}^{(m-1)} B_{k^{(\ell)},K^{(\ell)}}(x, y, y') = 0.
\]

For \((i, m)\) in \(I_{a}\), let \(\mathcal{K}_{j-1}(i, m)\) be the set of elements \(K\) of \(\mathcal{K}_{j-1}\), containing \((i, m)\), and for \(K\) in \(\mathcal{K}_{j-1}(i, m)\) denote by \(K^{i,m}\) the element \(K \setminus \{(i, m)\}\) of \(\mathcal{K}_{j}\). For \(\kappa = k_{1}, \ldots, k_{j-1}\) in \(\mathcal{J}_{j-1}\), denote by \(I_{\kappa}\) the subset of \(\{1, \ldots, j\}\) defined by the following conditions:
\[
l \in I_{\kappa} \quad \text{if and only if} \quad \begin{cases} 
  l = 1 & \text{and} \quad 2 \leq k_{l} \\
  l = j & \text{and} \quad k_{j-1} \leq d - 1 \\
  1 < l < j & \text{and} \quad k_{l-1} + 2 \leq k_{l}
\end{cases}
\]
For \(l\) in \(I_{\kappa}\), let \(a_{\kappa,l}\) and \(b_{\kappa,l}\) be in \(\{1, \ldots, d\}\) defined by the following conditions:
\[
a_{\kappa,l} := \begin{cases} 
  1 & \text{if} \quad l = 1 \\
  k_{l-1} + 1 & \text{if} \quad 2 \leq l
\end{cases} \quad \text{and} \quad b_{\kappa,l} := \begin{cases} 
  d & \text{if} \quad l = j \\
  k_{l+1} - 1 & \text{if} \quad l \leq j - 1
\end{cases}
\]
and for \(k\) in \([a_{\kappa,l}, b_{\kappa,l}]\), denote by \(\kappa^{l,k}\) the element of \(\mathcal{J}_{j}\) such that \(\{\kappa\}\) is contained in \(\{\kappa^{l,k}\}\) and \(k\) is the \(l\)-th element of the sequence \(\kappa^{l,d}\). Setting
\[
\mathcal{E} := \{(\kappa, l, k) \in \mathcal{J}_{j-1} \times \{1, \ldots, j\} \times \{1, \ldots, d\} \mid l \in I_{\kappa} \quad \text{and} \quad k \in [a_{\kappa,l}, b_{\kappa,l}]\},
\]
the map
\[
\mathcal{E} \xrightarrow{\sim} \mathcal{J}_{j} \times \{1, \ldots, j\}, \quad (\kappa, l, k) \mapsto (\kappa^{l,k}, l)
\]
is bijective.

For \((i, m)\) in \(I_{a}\) and for \(\kappa\) in \(\mathcal{J}_{j-1}\), set:
\[
c'_{\kappa,i,m} = \sum_{l \in I_{\kappa}} \sum_{k = a_{\kappa,l}}^{b_{\kappa,l}} (-1)^{l} c_{\kappa,l,k} b_{l,k}^{(m-1)}(x, y, y').
\]
Then, for \(K\) in \(\mathcal{K}_{j-1}\),
\[
\sum_{\kappa \in \mathcal{J}_{j-1}} c'_{\kappa,i,m} B_{k,K}(x, y, y') = \sum_{(\kappa, l, k) \in \mathcal{E}} (-1)^{l} c_{\kappa,l,k} b_{l,k}^{(m-1)}(x, y, y') B_{k,K}(x, y, y') =
\]
\[
\sum_{\kappa \in \mathcal{J}_{j-1}} \sum_{l=1}^{j} (-1)^{l} c_{\kappa,l,k} b_{l,k}^{(m)}(x, y, y') B_{k^{(l)},K}(x, y, y').
\]
As a result, by the above equalities,
\[
\sum_{\kappa \in \mathcal{J}_{j-1}} c'_{\kappa,i,m} B_{k,K}(x, y, y') = 0, \quad \forall K \in \mathcal{K}_{j-1}(i, m).
\]
For $K$ in $\mathcal{K}_{j-1} \setminus \mathcal{K}_{j-1}(i,m)$ and $\kappa$ in $\mathfrak{S}_j$,

$$\sum_{j} (-1)^{j} b^{(m-1)}_{j,i,j} B_{\kappa,i,j,K} = 0$$

as the determinant of a matrix having two equal lines. Hence

$$\sum_{\kappa \in \mathfrak{S}_j} c'_{\kappa,i,m} B_{\kappa,K}(x, y, y') = 0, \quad \forall K \in \mathcal{K}_{j-1}.$$

Since $Q_{j-1}(x, y, y') \neq 0$, for all $\kappa$ in $\mathfrak{S}_{j-1}$ and for all $(i,m)$ in $I_u$, $c'_{\kappa,i,m} = 0$.

Let $\kappa$ be in $\mathfrak{S}_{j-1}$. Denote by $N_\kappa$ the cardinality of the union

$$\bigcup_{l \in I_u} [a_{\kappa,l}, b_{\kappa,l}],$$

and $\kappa$, the element of $\mathfrak{S}_{N_\kappa}$ equal to the ordered sequence of this union. As for $K$ in $\mathcal{K}_{N_\kappa}$, $B_{\kappa,K}(x, y, y') \neq 0$, for all $l$ in $I_u$ and all $\kappa$ in $[a_{\kappa,l}, b_{\kappa,l}]$ since $c'_{\kappa,i,m} = 0$ for all $(i,m)$ in $I_u$. As a result, $c_\kappa = 0$ for all $\kappa$ in $\mathfrak{S}_j$, whence $Q_{j}(x, y, y') \neq 0$.

(ii) As the $u_{i,m}^{(l)}$, $(i,m)$ in $I_u$ are linearly independent over $\mathbb{K}[l, x, y]$, so are the $v_{K}$, $K \in \mathcal{K}_j$. So, by (i), for all $(x, y, y')$ in a dense open subset of $l, x, y$, $Q_{j}(x, y, y') \neq 0$, $v_{0}(x, y) \neq 0$ and the $v_{K}(x, y)$, $K \in \mathcal{K}_j$ are linearly independent. Let $(x, y, y')$ be in this open subset of $l, x, y$ and $c_{\kappa}, \kappa \in \mathfrak{S}_j$ in $\mathbb{K}$ such that

$$\sum_{\kappa \in \mathfrak{S}_j} c_\kappa a_{\kappa,l}(x, y, y') = 0, \quad \forall l \in \mathfrak{S}_j.$$

By Lemma 9.3.(iii), for all $(l, \kappa)$ in $\mathfrak{S}_j \times \mathfrak{S}_j$,

$$a_{\kappa,l} = \sum_{K \in \mathcal{K}_j} (-1)^{b_{0}} e(K) a_{K_{,l}} B_{\kappa,K},$$

whence

$$\sum_{K \in \mathcal{K}_j} e(K) a_{K_{,l}}(x, y) \sum_{\kappa \in \mathfrak{S}_j} c_\kappa B_{\kappa,K}(x, y, y') = 0, \quad \forall l \in \mathfrak{S}_j.$$

Then, setting:

$$c'_{K} := \sum_{\kappa \in \mathfrak{S}_j} c_\kappa e(K) B_{\kappa,K}(x, y, y')$$

for $K$ in $\mathcal{K}_j$,

$$0 = \sum_{K \in \mathcal{K}_j} c'_{K} a_{K_{,l}}(x, y) e_{0}(x, y) \wedge \mu_+ = \sum_{K \in \mathcal{K}_j} c'_{K} v_{l} \wedge e_{0}(x, y) \wedge v_{K}(x, y), \quad \forall l \in \mathfrak{S}_j.$$

As a result, for all $l$ in $\mathfrak{S}_j$,

$$\sum_{K \in \mathcal{K}_j} c'_{K} v_{l} \wedge v_{K}(x, y) = 0.$$
Since \( v, \ i \in \mathcal{S}_j \) is a basis of \( \wedge^j(p_u) \),
\[
\sum_{k \in \mathcal{S}_j} c'_K v_K(x, y) = 0
\]
so that
\[
\sum_{k \in \mathcal{S}_j} c_k \varepsilon(K) B_{k,K}(x, y, y') = 0
\]
for all \( K \in \mathcal{S}_j \) since the \( v_K(x, y) \) are linearly independent. As a result, the \( c_k, \ k \in \mathcal{S}_j \) are all equal to 0 since \( Q_j(x, y, y') \neq 0 \), whence \( P_j(x, y, y') \neq 0 \).

\section{An other expansion.} Let \( d_1, \ldots, d_n \) be the simple factors of \( d \) when \( \mathfrak{z} \) is strictly contained in \( \mathfrak{l} \). Set:
\[
n_1 := b_{b_1} - \ell_{b_1}, \ldots, n_n := b_{b_n} - \ell_{b_n}
\]
so that
\[
n - d = n_1 + \cdots + n_n.
\]
As \( g \) is the direct sum of \( \mathfrak{z}, \ d_1, \ldots, d_n, \ p_u, \ p_{-u}, \) for \( i = 1, \ldots, n, \ \mathbb{k}[b_i] \) is a subalgebra of \( \mathbb{k}[l] \) and \( \mathbb{k}[g] \) and \( \mathbb{k}[l \times b_i] \) is a subalgebra of \( \mathbb{k}[l \times g] \). Set \( B_i := \mathbb{k}[l \times g] \otimes_{\mathbb{k}[b_i \times b_i]} B_{b_i} \). Then \( B_i \) is a free submodule of rank \( b_{b_i} \) of \( \mathbb{k}[l \times g] \otimes \mathfrak{d} \). According to Lemma 7.1(ii), for some \( \lambda_{i,1}, \ldots, \lambda_{i,n_i} \) in \( B_{b_i} \), the \( \mathbb{k}[l \times g] \)-module \( B_i \) is contained in the submodule generated \( \lambda_{i,1}, \ldots, \lambda_{i,n_i} \) and \( \varepsilon_1^{(0)}, \ldots, \varepsilon_i^{(0)} \) so that \( B_i \) is a basis of \( j_i \).

Let \( \mathfrak{z} \) be the union of \( \{0\} \) and the set of strictly increasing sequences in
\[
\{(i, j) \mid i = 1, \ldots, n, \ j = 1, \ldots, n_i\}.
\]
For \( \nu \) in \( \mathfrak{z}, \ i = 1, \ldots, n \) and \( j = 1, \ldots, n - d \), set:
\[
\nu := \{i_1, \ldots, i_k\}, \quad |\nu| := k, \quad \lambda_{\nu} := \lambda_{i_1} \wedge \cdots \wedge \lambda_{i_k},
\]
\[
\mathfrak{z}^{(i)} := \{\nu \in \mathfrak{z} \mid \nu \subset \{(i, 1), \ldots, (i, n_i)\}\}, \quad \mathfrak{z}_j := \{\nu \in \mathfrak{z} \mid |\nu| = j\}, \quad \mathfrak{z}_j^{(i)} := \mathfrak{z}^{(i)} \cap \mathfrak{z}_j.
\]
For \( \nu = 0, |\nu| = 0 \) and \( \lambda_{\nu} = 1 \). For \( \nu \) in \( \mathfrak{z} \) and \( j = 0, \ldots, n - d \), set:
\[
\varepsilon_{\nu} := \lambda_{\nu} \wedge \varepsilon \quad \text{and} \quad \mathfrak{z}_j := \{I \subset I_{y,0} \mid |I| = n - j \quad \text{and} \quad |I \cap I_{y,0}| \geq n - d - j\}.
\]
For \( (x, y, y') \) in \( l \times p \times p_{-u} \) and \( (i, m) \) in \( I_0 \), the polynomial map
\[
t \mapsto \varepsilon_i^{(m)}(x, y + ty')
\]
has an expansion
\[
\varepsilon_i^{(m)}(x, y + ty') = \varepsilon_i^{(m)}(x, y) + \sum_{k=1}^{m} \ell^k \omega_{i,m,k,-}(x, y, y') + \sum_{k=1}^{m} \ell^k \omega_{i,m,k,+}(x, y, y') + \sum_{k=1}^{m} \ell^k \omega_{i,m,k,0}(x, y, y')
\]
with
\[
\omega_{i,m,k,-} \in \mathbb{k}[l \times p] \otimes_{\mathbb{k}} S^k(p_u) \otimes_{\mathbb{k}} p_{-u}, \quad \omega_{i,m,k,+} \in \mathbb{k}[l \times p] \otimes_{\mathbb{k}} S^k(p_u) \otimes_{\mathbb{k}} p_u,
\]
\[
\omega_{i,m,k,0} \in \mathbb{k}[l \times p] \otimes_{\mathbb{k}} S^k(p_u) \otimes_{\mathbb{k}} 1.
\]
For $I'$ subset of $I_{0,\ast}$, set:

$$\omega_{I',-} := \land_{(m,k)\in I'} \omega_{I,m,k,-}, \quad \omega_{I',+} := \land_{(m,k)\in I'} \omega_{I,m,k,+}, \quad \omega_{I',0} := \land_{(m,k)\in I'} \omega_{I,m,k,0}.$$ 

In these equalities, the order of the products are induced by the orders of $I_0$ and $I_{0,\ast}$.

**Lemma 9.5.** Let $j = 0, \ldots, n-d, \nu$ in $\mathfrak{z} J$ and $(x, y, y')$ in $L_0 \times p \times p_{-\nu}$. Denote by $c_\nu(x, y, y')$ the coefficient of $t^j$ of the polynomial map $t \mapsto \epsilon_\nu(x, y + ty')$.

(i) The polynomial map $t \mapsto \epsilon_\nu(x, y + ty')$ is divisible by $t^j$ in $\mathbb{K} t \otimes_\mathbb{K} \land^{{\text{b}}+j}(q)$.

(ii) For a well defined map

$$\mathcal{J} \quad \epsilon \quad \{ -1, 1 \},$$

$$c_\nu(x, y, y') - c'_\nu(x, y, y') \in \bigoplus_{|k| < j} w_k \land |j-k|(l) \land \overline{\epsilon_0}(x, y) \land \mu_+$$

with

$$c'_\nu(x, y, y') := \sum_{I \in \mathcal{J}} \epsilon(l) \lambda_\nu(x, y) \land \epsilon_{0,0}(x) \land \epsilon_l(x, y) \land \omega_{I^0}(-)(x, y, y').$$

(iii) For $I$ in $\mathcal{J}$, for well defined functions $a_{r,k,l,u}$, $(r, k) \in \Lambda_j$ in $\mathbb{K}[L_0 \times p]$,

$$\epsilon(l) \lambda_\nu(x, y) \land \epsilon_{0,0}(x) \land \epsilon_l(x, y) \land \omega_{I^0}(-)(x, y, y') = \sum_{(r,k) \in \Lambda_j} v'(y') a_{r,k,l,u}(x, y)w_k \land \overline{\epsilon_0}(x, y) \land \mu_+.$$ 

**Proof.** (i) As already observed in the proof of Lemma 9.2, the coefficient of $t^k$ of the polynomial function $t \mapsto \epsilon(x, y + ty')$ is the sum of the value at $(x, y, y')$ of products

$$\epsilon(I, I_-, I_+, I_0) \lambda_\nu \land \epsilon_{0,0} \land \epsilon_l \land \omega_{I,\ast} \land \omega_{I,-} \land \omega_{I,-} \land \omega_{I,0}$$

with

$$I \subset I_{0,0}, \quad I_- \subset I_{0,\ast}, \quad I_+ \subset I_{0,\ast}, \quad I_0 \subset I_{0,\ast}, \quad \epsilon(I, I_-, I_+, I_0) \in \{-1, 1\}$$

such that

$$I \cup I_- \cup I_+ \cup I_0 = I \cup I_- \cup I_+ \cup I_0, \quad |I| + |I_-| + |I_+| + |I_0| = n,$$

$$S_{I_\ast} + S_{I_+} + S_{I_0} = k.$$ 

According to Corollary 2.7,

$$\lambda_\nu \land \epsilon_{0,0} \land \epsilon_l \in \bigoplus_{m=0}^d \land |l+j-l-m|(B_{I}) \land \land^m(p_u).$$

Then

$$|l| + |I_-| \leq n - j \quad \text{whence} \quad j \leq |I_-| + |I_+| \leq k.$$ 

As a result, the coefficient of $t^k$ of the polynomial map $t \mapsto \epsilon_\nu(x, y + ty')$ is equal to 0 when $k < j$ and for $k = j$, it is the sum of the value at $(x, y, y')$ of the products

$$\epsilon(I, I_-, I_+, I_0) \lambda_\nu \land \epsilon_{0,0} \land \epsilon_l \land \omega_{I,\ast} \land \omega_{I,-} \land \omega_{I,-} \land \omega_{I,0}.$$
with
\[ j = |I^\#_+| + |I^\#_|, \quad I_+ = \emptyset, \quad |I^-| = |I^\#_| = S_{I^-}, \quad |I_\pm| = |I^\#_\pm| = S_{I_\pm}. \]
So, in any case,
\[ \lambda_\nu(x, y) \land \varphi_0(x) \land \varphi_I(x, y) \in \mathfrak{H}(x, y) \land \wedge^d(p_\nu). \]
For \( I \in \mathcal{G}_j \), set:
\[ \epsilon(I) := \epsilon_x(I, I_{\nu, 0} \setminus I, \emptyset, \emptyset). \]
Then
\[ c_\nu(x, y, y') - \sum_{I \in \mathcal{G}_j} \epsilon(I) \lambda_\nu(x, y) \land \varphi_0(x) \land \varphi_I(x, y) \land \omega_{l, 0}\setminus j(x, y) \in \bigoplus_{|k| < j} w_\kappa \land \wedge^{j-|k|}(I) \land \mathfrak{H}(x, y) \land \mu_\kappa. \]
since \( |I_\pm| > 0 \) when \( |I_-| < j \). Moreover, for \( I \) in \( \mathcal{G}_j \), for well defined functions \( a_{r, \kappa, l, u} \), \( (r, \kappa) \in \Lambda_j \) in \( \mathbb{K}[L \times p] \),
\[ \epsilon(I) \lambda_\nu(x, y) \land \varphi_0(x) \land \varphi_I(x, y) \land \omega_{l, 0}\setminus j(x, y) = \sum_{(r, \kappa) \in \Lambda_j} u'(y') a_{r, \kappa, l, u}(x, y) w_\kappa \land \mathfrak{H}(x, y) \land \mu_\kappa, \]
whence the lemma. \( \Box \)

10. Some spaces related to parabolic subalgebras

Let \( \mathfrak{p} \) be a parabolic subalgebra of \( \mathfrak{g} \) containing \( \mathfrak{b} \). We then use the notations
\[ \mathfrak{l}, \mathfrak{p}_u, \mathfrak{d}, \mathfrak{z}, \mathfrak{R}, L, d, d_0, \mathfrak{I}, \mathfrak{z}, \zeta, I_0, I'_0, I_u, \mathfrak{p}_-, \mathfrak{p}_-u \]
of Section 8.

10.1. An equivalence. For \( \varphi \) in \( \mathbb{K}[L \times \mathfrak{g}] \otimes_{\mathbb{K}} \wedge (\mathfrak{g}) \), \( \varphi \) has a unique expansion
\[ \varphi = \sum_{i \in \mathcal{I}} \varphi_i \land v_i \quad \text{with} \quad \varphi_i \in \mathbb{K}[I \times \mathfrak{g}] \otimes_{\mathbb{K}} \wedge (\mathfrak{p}_-) \]
and for \( \iota \) in \( \mathcal{I} \), \( \varphi_i \) has a unique expansion
\[ \varphi_i = \sum_{\kappa \in \mathcal{I}} \varphi_{i, \kappa} \land w_\kappa \quad \text{with} \quad \varphi_{i, \kappa} \in \mathbb{K}[I \times \mathfrak{g}] \otimes_{\mathbb{K}} \wedge (\iota). \]
For all \( (\iota, \kappa) \) in \( \mathcal{I} \times \mathcal{I} \), \( \varphi_{i, \kappa} \) has a unique expansion
\[ \varphi_{i, \kappa} := \sum_{r \in \mathbb{N}^{d-}} u' \varphi_{r, i, \kappa} \quad \text{with} \quad \varphi_{r, i, \kappa} \in \mathbb{K}[I \times \mathfrak{p}] \otimes_{\mathbb{K}} \wedge (\iota). \]

For \( l \) nonnegative integer, let \( \mathcal{P}_l \) be the subspace of \( \mathbb{K}[L \times \mathfrak{g}] \otimes_{\mathbb{K}} \wedge (\mathfrak{g}) \) defined by the following condition:
\[ \varphi \in \mathcal{P}_l \iff (|r| + |d| \neq l \implies \varphi_{r, i, \kappa} = 0). \]
By Proposition 2.3 and Lemma 6.3(i), C is a free submodule of rank \(n\) of \(\mathbb{k}[\mathfrak{l} \times \mathfrak{g}] \otimes_{\mathbb{k}} \mathfrak{g}\), generated by the maps

\[
(x, y) \mapsto [x, \epsilon^{(m)}(x, y)], \quad (i, m) \in I_{\epsilon, 0}.
\]

For \(v\) in \(C\), denote by \(\iota_v\) the \(\mathbb{k}[\mathfrak{l} \times \mathfrak{g}]\)-derivation of the algebra \(\mathbb{k}[\mathfrak{l} \times \mathfrak{g}] \otimes_{\mathbb{k}} \mathfrak{g}\) such that \(\iota_v(v) = \langle v, v \rangle\) for all \(v\) in \(\mathfrak{g}\).

Lemma 10.1. Let \(V\) be a subspace of \(\mathfrak{g}\) such that \(V\) is contained in a complement to \(V_{x,y}\) in \(\mathfrak{g}\) for all \((x, y)\) in a dense open subset of \(\Omega - \mathfrak{l} \times \mathfrak{p}\).

(i) For \(v\) in \(\mathfrak{p}_u\) and \(w\) in \(\mathfrak{p}_{-u}\), \(\iota_v(v)\) and \(\iota_v(w)\) are in \(J\) and \(J_\epsilon\) respectively for all \(v\) in \(C\).

(ii) For \(k = 1, \ldots, \dim V\) and \(\psi\) in \(\mathbb{k}[\mathfrak{l} \times \mathfrak{p}] \otimes_{\mathbb{k}} \mathfrak{g}\) such that \(\iota_v(\psi) = 0\) for all \(v\) in \(C\), then \(\psi = 0\).

Proof. (i) By Proposition 9.1(iii), for all \((i, m)\) in \(I_\epsilon\),

\[
[x, \epsilon^{(m)}(x, y)] \in \mathbb{k}[\mathfrak{l} \times \mathfrak{g}] \otimes_{\mathbb{k}} 1 + J \otimes_{\mathbb{k}} \mathfrak{p}_{-u} + J_\epsilon \otimes_{\mathbb{k}} \mathfrak{p}_u,
\]

whence the assertion.

(ii) Suppose that \(\iota_v(\psi) = 0\) for all \(v\) in \(C\) and prove the assertion by induction on \(\dim V\). As \(V\) is contained in a complement to \(V_{x,y}\) in \(\mathfrak{g}\) for all \((x, y)\) in a dense open subset of \(\Omega - \mathfrak{l} \times \mathfrak{p}\), for \(v\) in \(V\), \(v = 0\) if and only if \(\iota_v(v) = 0\) for all \(v\) in \(C\) since \(V_{x,y}\) is the orthogonal complement to \([x, V_{x,y}]\) for all \((x, y)\) in \(\Omega\), by Proposition 2.1(iv). As a result, the assertion is true for \(\dim V = 1\). Suppose \(\dim V > 1\) and the assertion true for the subspaces of \(V\). Let \(V'\) be an hyperplane of \(V\) and \(v\) in \(V \setminus V'\). Then

\[
\psi = v \wedge \psi' + \psi_0
\]

with \(\psi'\) in \(\mathbb{k}[\mathfrak{l} \times \mathfrak{p}] \otimes_{\mathbb{k}} \mathfrak{g}\) and \(\psi_0\) in \(\mathbb{k}[\mathfrak{l} \times \mathfrak{p}] \otimes_{\mathbb{k}} \mathfrak{g}\). As \(\iota_v(\psi) = 0\) for all \(v\) in \(C\),

\[
\iota_v(\psi') = 0 \quad \text{and} \quad \iota_v(v)\psi' + \iota_v(\psi_0) = 0,
\]

whence \(\psi' = 0\) and \(\psi_0 = 0\) by induction hypothesis, and \(\psi = 0\). \(\square\)

For \(l\) nonnegative integer, denote by \(M_l\) the subspace of elements \(\varphi\) of \(\mathbb{k}[\mathfrak{l} \times \mathfrak{g}] \otimes_{\mathbb{k}} \mathfrak{g}\) such that \(\varphi \wedge \varepsilon\) is in \(J^l \otimes_{\mathbb{k}} \mathfrak{g}\). For \(k = 1, \ldots, n\), denote by \(\mathcal{P}_{l,k}\) the intersection of \(\mathcal{P}_l\) and \(\mathbb{k}[\mathfrak{l} \times \mathfrak{g}] \otimes_{\mathbb{k}} \mathfrak{g}\). For \(\varphi\) in \(M_l\) and \(v\) in \(C\), \(\iota_v(\varphi)\) is in \(M_l\) since \(\iota_v(\varepsilon) = 0\).

Lemma 10.2. Let \(k = 1, \ldots, n\), \(l\) a positive integer and \(\varphi\) in \(\mathcal{P}_{l,k}\).

(i) The element \(\varphi\) is in \(M_l\).

(ii) If \(\varphi_{r,t,k} \wedge \mathfrak{p}_0 = 0\) for all \((r, t, k)\), then \(\varphi\) is in \(M_{l+1}\).

(iii) Suppose that \(\varphi\) is in \(\mathbb{k}[\mathfrak{l} \times \mathfrak{g}] \otimes_{\mathbb{k}} \mathfrak{g}\). If \(\varphi\) is in \(M_{l+1}\) then \(\varphi_{r,t,0} \wedge \mathfrak{p}_0 = 0\) for all \((r, t)\).

Proof. (i) Let \((x, y, y')\) be in \(\mathfrak{l} \times \mathfrak{p} \times \mathfrak{p}_{-u}\). According to Lemma 9.2 and the notations of Subsection 9.2, for \(j = 1, \ldots, d\) and \(t\) in \(\mathfrak{Z}_j\), the polynomial map

\[
t \mapsto v_j \wedge \varepsilon(x, y + ty')
\]
is divisible by \( t^j \) in \( \mathbb{K}[t] \otimes \mathbb{K} \setminus (g) \) and the coefficient \( c_i(x, y, y') \) of \( t^j \) satisfies the relation
\[
c_i(x, y, y') = \sum_{k' \in \mathbb{O}_y} a_{k',i}(x, y, y') w_{k'} \wedge \varepsilon_0(x, y) \wedge \mu_+ \in \\
\bigoplus_{|k'| < j} w_{k'} \wedge \bigwedge^{j-|k'|}(1) \wedge \varepsilon_0(x, y) \wedge \mu_+.
\]
Hence the polynomial map
\[
t \mapsto \varphi(x, y + ty') \wedge \varepsilon(x, y + ty')
\]
is divisible by \( t^j \) in \( \mathbb{K}[t] \otimes \mathbb{K} \setminus (g) \) since \( \varphi \) is in \( \mathcal{P}_{l,k} \). As a result, \( \varphi \) is in \( M_l \).

(ii) Suppose that \( \varphi_{r,l,k} \wedge \varepsilon_0 = 0 \) for all \((r, l, \kappa)\). By the above relation, the polynomial map

\[
t \mapsto \varphi(x, y + ty') \wedge \varepsilon(x, y + ty')
\]
is divisible by \( t^{l+1} \) in \( \mathbb{K}[t] \otimes \mathbb{K} \setminus (g) \) for all \((x, y, y')\) in \( l \times p \times p_{-u} \). Hence \( \varphi \) is in \( M_{l+1} \).

(iii) As \( \varphi \) is in \( \mathbb{K}[l, g] \otimes \mathbb{K} \setminus (p) \), \( \varphi_{r,l,k} = 0 \) for \( k \not= 0 \). Suppose \( \varphi_{r,l,0} \wedge \varepsilon_0 \not= 0 \) for some \((r, l)\). A contradiction is expected. Denote by \( \lambda(\varphi) \) the biggest integer \( j \) such that \( \varphi_{r,l,0} \wedge \varepsilon_0 \not= 0 \) for some \((r, l)\) in \( \mathbb{N}_{p_{-u}} \times \mathcal{I}_j \). By (ii), we can suppose \( \varphi_{r,l,0} = 0 \) for \(|\nu| > \lambda(\varphi)\). As the polynomial map

\[
t \mapsto \varphi(x, y + ty') \wedge \varepsilon(x, y + ty')
\]
is divisible by \( t^{l+1} \),
\[
\sum_{k \in \mathcal{J}_{\lambda(\varphi)}} a_{k,l} \sum_{r \in \mathbb{N}_{p_{-u}}} v^r \varphi_{r,l,0} \wedge \varepsilon_0 = 0, \quad \forall t \in \mathcal{J}_{\lambda(\varphi)}, \]
by maximality of \( \lambda(\varphi) \). By Corollary 9.4(ii), for all \((x, y, y')\) in a dense open subset of \( l \times p \times p_{-u} \), the matrix
\[
(a_{k,l}(x, y, y'), \ (t, \kappa) \in \mathcal{J}_{\lambda(\varphi)} \times \mathcal{I}_{\lambda(\varphi)})
\]
is invertible. Hence \( \varphi_{r,l,0} \wedge \varepsilon_0 = 0 \) for all \((r, l)\) in \( \mathbb{N}_{p_{-u}} \times \mathcal{J}_{\lambda(\varphi)} \), whence the contradiction.

\[\square\]

**Proposition 10.3.** Let \( k = 1, \ldots, n, l \) a positive integer and \( \varphi \in \mathcal{P}_{l,k} \). Then \( \varphi \) is in \( M_{l+1} \) if and only if \( \varphi_{r,l,k} \wedge \varepsilon_0 = 0 \) for all \((r, l, \kappa)\).

**Proof.** By Lemma 10.2(ii), the condition is sufficient. Suppose that \( \varphi \) is in \( M_{l+1} \). By Lemma 10.1(i),
\[
\sum_{(r,l,k)} v^r \lambda_\nu(\varphi_{r,l,k}) \wedge w_k \wedge v_i + \sum_{(r,l,k)} v^r \varphi_{r,l,k} \wedge \lambda_\nu(w_k) \wedge v_i \in M_{l+1}
\]
for \( \nu \in C \) since \( \lambda_\nu(\varphi) = 0 \). Prove the assertion by induction on \( k \).

Suppose \( k > 1 \), the assertion true for \( k - 1 \) and the proposition not true for \( k \). A contradiction is expected. Denote by \( j \) the biggest integer such that for some \( \kappa \in \mathcal{I}_j \), \( \varphi_{r,l,k} \wedge \varepsilon_0 \not= 0 \). By Lemma 10.2(ii), we can suppose
\[
|\kappa| > j \implies \varphi_{r,l,k} = 0.
\]
By Lemma 10.2(iii), \( j > 0 \). By the above relation and the induction hypothesis,
\[
\nu_r(\varphi_{r,d,k}) \wedge \varepsilon_0 = 0, \quad \forall (\nu, r, \iota, \kappa) \in C \wedge \mathbb{N}_d \times \mathfrak{I} \times \mathfrak{I}_j
\]
by maximality of \( j \).

Let \((r, \iota, \kappa)\) be in \( \mathbb{N}_d \times \mathfrak{I} \times \mathfrak{I}_j \). Set \( k' := k - j - |l| \). Then \( \nu_r(\varphi_{r,d,k}) \wedge \varepsilon(x, y) = 0 \) for all \((\nu, x, y)\) in \( C \times l \times p \) by Corollary 2.7. For some principal open subset \( O \) of \( \Omega \), \( \cap l \times p \) and some subspace \( V \) of \( g, V \) is the complement to \( V_{x,y} \) for all \((x, y)\) in \( O \). Then
\[
\varphi_{r,d,k} \mid_O = \varphi' + \varphi'' \quad \text{with} \quad \varphi' \in k[O] \otimes_k \mathcal{K}'(V) \quad \text{and} \quad \varphi''(x, y) \in \mathcal{K}'^{-1}(g) \wedge V_{x,y}
\]
for all \((x, y)\) in \( O \), whence \( \nu_r(\varphi') = 0 \) for all \( \nu \) in \( C \). As a result, \( \varphi' = 0 \) by Lemma 10.1(ii) and \( \varphi_{r,d,k} \wedge \varepsilon_0 = 0 \) by Corollary 2.7 since \( \varphi_{r,d,k} \) is in \( k[l \times p] \otimes_k \mathcal{K}' \) by definition, whence the contradiction.

For \( k = 1 \),
\[
\varphi_{r,0,0} \in k[l \times p] \otimes_k 1, \quad \varphi_{r,0,j} \in k[l \times p], \quad \varphi_{r,j,0} \in k[l \times p]
\]
for \( j = 1, \ldots, d \), whence
\[
\nu_r(\varphi_{r,0,0} + \sum_{j=1}^d \varphi_{r,0,j} w_j) \wedge \varepsilon(x, y) = 0
\]
for all \((r, \nu)\) in \( \mathbb{N}_d \times C \) and all \((x, y)\) in \( l \times p \) by Lemma 10.1(i). Then, arguing as above, by Lemma 10.1(ii), for all \( r \) in \( \mathbb{N}_d \), \( \varphi_{r,0,0} \wedge \varepsilon_0 = 0 \) and \( \varphi_{r,0,j} = 0 \) for \( j = 1, \ldots, d \). As a result, by Lemma 10.2(ii) and (iii), \( \varphi_{r,j,0} = 0 \) for all \((r, j)\) in \( \mathbb{N}_d \times \{1, \ldots, d\} \), whence the proposition. \( \square \)

10.2. **On the \( G \)-action.** Let \( O \) be the local ring of \( G \) at the identity and \( m \) its maximal ideal. Denote by \( \hat{O} \) the completion of \( O \) for the \( m \)-adic topology and again by \( m \) its maximal ideal. Let \( z_1, \ldots, z_{2n+\ell} \) be a system of coordinates of \( O \). Then \( \hat{O} = k[[z_1, \ldots, z_{2n+\ell}]] \). As usual for \( s = (s_1, \ldots, s_{2n+\ell}) \) in \( \mathbb{N}^{2n+\ell} \), set:
\[
z^s := z_1^{s_1} \cdots z_{2n+\ell}^{s_{2n+\ell}}.
\]
Let \( \mathfrak{I} \) be the ideal of \( k[l \times g] \otimes_k \land (p_u) \) generated by \( \mathfrak{I} \otimes 1 \) and \( 1 \otimes p_u \) and \( \mathfrak{J} \) the ideal of \( \hat{O} \otimes_k k[l \times g] \otimes_k \land (p_u) \) generated by \( m \otimes 1 \) and \( 1 \otimes \mathfrak{I} \). Denote by \( \hat{B} \) and \( \mathfrak{B} \) the submodules of \( \hat{O} \otimes_k k[l \times g] \otimes_k g \) and \( \hat{O} \otimes_k k[l \times g] \otimes_k g \) generated by the maps
\[
(g, x, y) \mapsto e^{(m)}_i(g(x), g(y)), \quad (i, m) \in I_0,
\]
respectively. Then the map
\[
G \times l \times g \longrightarrow \bigwedge^{b_i} g, \quad (g, x, y) \mapsto \bigwedge_{(i,m) \in I_0} e^{(m)}_i(g(x), g(y))
\]
is a generator of \( \bigwedge^{b_i}(\hat{B}) \) and \( \bigwedge^{b_i}(\mathfrak{B}) \). As usual, the order of the product is induced by the order of \( I_0 \).

Set:
\[
\mathbb{N}_p := \mathbb{N}^{2n+\ell} \times \mathbb{N}_d \times \mathfrak{I}, \quad |(s, r, \iota)| := |s| + |r| + |l|,
\]
\[ \mathcal{L} := \{ \varphi \in (\mathbb{K}[l, g] \otimes \mathbb{K})^N \mid \exists N_{\varphi} \in \mathbb{N} \text{ such that } |r| \geq N_{\varphi} \implies \varphi_{s, r, t} = 0 \}, \]

\[ \kappa(\varphi) := \sum_{(s, r, t) \in \mathbb{N}_p} z^t \varphi_{s, r, t} \wedge v_i \]

for \((s, r, t) \in \mathbb{N}_p \) and \( \varphi \in \mathcal{L} \). For \( k = 1, \ldots, n \), denote by \( \mathcal{L}^{(k)} \) the subspace of elements \( \varphi \) of \( \mathcal{L} \) such that \( \kappa(\varphi) \in \hat{O} \otimes \mathbb{K}[l, g] \otimes \mathbb{K}^k \).

and set:

\[ \mathcal{L}_s := \{ \varphi \in \mathcal{L} \mid s \neq 0 \implies \varphi_{s, r, t} = 0 \}, \quad \mathcal{L}_s^{(k)} = \mathcal{L}_s \cap \mathcal{L}^{(k)}, \quad \varphi_{r, t} := \varphi_{0, r, t}, \quad |(r, t)| := |(0, r, t)| \]

for \( \varphi \) in \( \mathcal{L}_s \) and \((r, t) \in \mathbb{N}^d \times \mathbb{N} \). For \( l \) nonnegative integer, denote by \( \Delta_l \) the subset of elements \((r, t) \) of \( \mathbb{N}^d \times \mathbb{N} \) such that \( |(r, t)| = l \).

**Lemma 10.4.** Let \( k = 1, \ldots, n \), \( l \) a positive integer and \( \varphi \) in \( \mathcal{L}_s^{(k)} \) such that \( \kappa(\varphi) \) is in \( M_{l+1} \). If \( \varphi_{r, t} \) is in \( \mathcal{F}^{[l]} \cap (g) \) for all \((r, t) \), then \( \varphi_{r, t} \) is in \( M_{l+1-|(r, t)|} \) for all \((r, t) \).

**Proof.** According to the hypothesis, \( \varphi_{r, t} = 0 \) for all \((r, t) \) such that \( |(r, t)| > l \). For \((r, t) \) in \( \mathbb{N}^d \times \mathbb{N} \), denote by \( \Gamma(r_1, t_1) \) the subset of elements \((r, t) \) of \( \mathbb{N}^d \times \mathbb{N} \) such that

\[ r_1 - r \in \mathbb{N}^d \quad \text{and} \quad \{t\} \subset \{t_1\}. \]

For \((r, t) \) in \( \Gamma(r_1, t_1) \), denote by \( t_1 \setminus t \) the element of \( \mathbb{N} \) and \( e(t_1, t) \) the element of \(-1, +1 \) such that \( |t_1 \setminus t| = |t_1 \setminus t| \) and \( v_{t_1 \setminus t} \wedge v_i = e(t_1, t)v_i \).

Let \( j \) be the smallest integer such that \( \varphi_{r, t} \neq 0 \) for some \((r, t) \) such that \(|(r, t)| = j \) and \( i \) the biggest integer such that \( \varphi_{r', t'} \neq 0 \) for some \((r', t') \) such that \(|(r', t')| = i \). Then \( j \leq i \leq l \). Prove the lemma by induction on \((l - j, i) \). For \( j = l \), by Proposition 10.3, for all \((r, t) \), \( \varphi_{r, t} \wedge e_i = 0 \). As a result, again by Proposition 10.3, \( \varphi_{r, t} \) is in \( M_1 \). Suppose \( j < l \) and the lemma true for all \((l - j', i') \) smaller than \((l - j, i) \).

(a) Suppose \( j = i \). For \((r, t) \) in \( \Delta_j \), \( \varphi_{r, t} \) has an expansion

\[ \varphi_{r, t} = \sum_{(r', t') \in \Delta_1} v_{t,r} \varphi_{r, r', t'} \wedge v_i \quad \text{with} \quad \varphi_{r, r', t'} \in \mathcal{F}^{l-1} \cap (g). \]

For \((r_1, t_1) \) in \( \Delta_{j+1} \), set:

\[ \psi_{r_1, t_1} := \sum_{(r, t) \in \Gamma(r_1, t_1)} e(t_1, t)\varphi_{r, r_1-r, t_1-t}. \]

Then

\[ \sum_{(r, t) \in \mathbb{N}^d \times \mathbb{N}} v_r \varphi_{r, t} \wedge v_i = \sum_{(r, t) \in \mathbb{N}^d \times \mathbb{N}} \varphi_{r_1, t_1} \wedge v_i. \]

By induction hypothesis, for all \((r_1, t_1) \) in \( \Delta_{j+1} \), \( \psi_{r_1, t_1} \) is in \( M_{l-j} \).

Prove by induction on \( m := |t| \) that \( \varphi_{r, t} \) is in \( M_{l+1-j} \). For \( k \) nonnegative integer, set:

\[ N_k := \{ (r, r_*) \in \mathbb{N}^d \times \mathbb{N}^d_+ \mid r_* - r \in \mathbb{N}^d \} \quad \text{and} \quad N_k := \{ r \in \mathbb{N}^d_+ \mid (r, r_*) \in N_k \} \]
for $r_*$ in $\mathbb{N}^d_{k+1}$.

**Claim 10.5.** Let $a_{r,r_*}, (r, r_*) \in N_k$ in $\mathcal{K}$ such that
\[ \sum_{r \in N_k(r_*)} a_{r,r_*} = 0, \quad \forall r_* \in \mathbb{N}^d_{k+1}. \]

Then $a_{r,r_*} = 0$ for all $(r, r_*)$.

**Proof.** [Proof of Claim 10.5] Prove the claim by induction on $k$. For $k = 0$, there is nothing to prove. Suppose $k > 0$ and the claim true for $k - 1$. Let $r$ be in $\mathbb{N}^d_k$ such that the first component $r_1$ of $r$ is positive. Denote by $N_{k,1}$ the subset of elements $(x, x_*)$ of $N_k$ such that the first component of $x$ is positive. For $x$ in $\mathbb{N}^d$, denote by $x^\#$ the element of $\mathbb{Z}^d$ such that
\[ x^\# := x_1 - 1, \quad x^\#_s := x_s \quad \text{for} \quad s = 2, \ldots, d. \]

Then the map
\[ N_{k,1} \longrightarrow N_{k-1}, \quad (x, x_*) \mapsto (x^\#, x^\#_*) \]
is bijective. As a result, setting, $b_{x^\#, x^\#_*} := a_{x,x_*}$ for $(x, x_*)$ in $N_{k,1}$,
\[ \sum_{x \in N_{k,1}(x_*)} b_{x,x_*} = 0, \quad \forall x_* \in \mathbb{N}^d_k. \]

So, by induction hypothesis, $a_{x,x_*} = 0$ for all $(x, x_*)$ in $N_{k,1}$. Then, after permutation of the indices, $a_{r,r_*} = 0$ for all $(r, r_*)$ in $N_k$ since for some indice $s$, $r_s$ is positive, whence the claim. $\square$

By Claim 10.5, for $(r, r_1)$ in $N_j$, $\varphi_{r_0,r_1-r_0}$ is in $M_{l-j}$. Hence $\varphi_{r,0}$ is in $M_{l+1-j}$. Suppose $m > 0$ and $\varphi_{r,l} \in M_{l+1-j}$ for all $(r, l)$ such that $|l| < m$. Then we can suppose $\varphi_{r,l} = 0$ for all $(r, l)$ such that $|l| < m$. Let $\iota$ be in $\mathbb{Z}_m$. For all $r_1$ in $\mathbb{N}^d_{j+1-m}$,
\[ \sum_{(r_1) \in \Gamma(r_1, l)} \varphi_{r,l,r_1-r_1} \in M_{l-j}. \]

By Claim 10.5, $\varphi_{r,l,r_1-r_1}$ is in $M_{l-j}$ for all $(r, r_1)$ in $N_{j-m}$. Then $\varphi_{r,l}$ is in $M_{l+1-j}$ for all $(r, l)$ such that $|l| = m$, whence the assertion in this case.

(b) Suppose $j < i$. For $(r, l)$ in $\mathbb{N}^d \times \mathbb{Z}$ such that $|(r, l)| \leq i$, $\varphi_{r,l}$ has an expansion
\[ \varphi_{r,l} = \sum_{(r', l') \in \Delta_{i-|r|}} u^l \varphi_{r,r',l'} \wedge v_{l'}, \quad \text{with} \quad \varphi_{r,r',l'} \in \mathcal{J}^{-i}_{\mathcal{K}} \otimes (g). \]

For $(r_1, l_1)$ in $\Delta_{i}$, set:
\[ \psi_{r_1,l_1} := \sum_{(r_1) \in \Gamma(r_1, l_1)} e(t_1, l) \varphi_{r_1,r_1-r_1_l_1}. \]

Then
\[ \sum_{(r_1) \in \mathbb{N}^d \times \mathbb{Z}} u^l \varphi_{r_1,l_1} \wedge v_{l_1} = \sum_{(r_1, l_1) \in \Delta_{i}} u^l \psi_{r_1,l_1} \wedge v_{l_1}. \]

By (a), for all $(r_1, l_1)$ in $\Delta_{i}$, $\psi_{r_1,l_1}$ is in $M_{l+1-i}$ since $\psi_{r_1,l_1}$ is in $\mathcal{J}^{-i}_{\mathcal{K}} \otimes (g)$. 


Let $\Lambda$ be the subset of elements $(r, i)$ of $\Delta_i$ such that $\varphi_{r,i} \neq 0$. For $(r, i)$ in $\Lambda$, 
$$
\Gamma(r, i) \setminus \{(r, i)\} \subset \bigcup_{(r', i') \in \Delta_i \setminus \{(r, i)\}} \Gamma(r', i'),
$$
whence $\varphi_{r,i} = \psi_{r,i}$ is in $M_{i+1-i}$. Denote by $\varphi'$ the element of $\mathcal{L}_s$ such that $\varphi'_{r,i} = \varphi_{r,i}$ for $(r, i)$ not in $\Lambda$ and 0 otherwise. Denoting by $i'$ the biggest integer such that $\varphi'_{r,i} \neq 0$ for some $(r, i)$ in $\Delta_i$, $i'$ is smaller than $i$. So, by induction hypothesis, $\varphi'_{r,i}$ is in $M_{i+1-(r,i)}$ since $\kappa(\varphi')$ is in $M_{i+1}$, whence the lemma. 

**Proposition 10.6.** Let $k = 1, \ldots, n, l$ a positive integer and $\varphi$ in $\mathcal{L}^{(k)}$ such that $\kappa(\varphi) \wedge \hat{\varepsilon}$ is in $\tilde{j}^{l+1} \wedge^k(g)$. If $\varphi_{s,r,i}$ is in $\tilde{j}^{-[(s,r,i)]} \wedge^k(g)$ for all $(s, r, i)$, then $\varphi_{s,r,i}$ is in $M_{i+1-(s,r,i)}$ for all $(s, r, i)$.

**Proof.** By Proposition 2.1(vi), for $(g, x, y)$ in $G \times I_s \times g$,
$$
\hat{\varepsilon}(g(x), g(y)) = g.\varepsilon(x, y).
$$
Then
$$
\hat{\varepsilon} - \varepsilon \in m \otimes_k [I_s \times g] \otimes_k \wedge^h(g) \quad \text{and} \quad z^i u' \varphi_{s,r,i} \wedge \nu_i \wedge (\varepsilon - \hat{\varepsilon}) \in \tilde{j}^{l+1} \wedge (g),
$$
whence
$$
\kappa(\varphi) \wedge \varepsilon \in \tilde{j}^{l+1} \wedge (g).
$$
As a result, from the equality
$$
\tilde{j}^{l+1} \wedge^k(g) = \bigoplus_{i=0}^{\ell+1} m_i \otimes_k \tilde{j}^{l+1-i} \wedge^k(g),
$$
we deduce
$$
\sum_{s \in [L]^{[s,r,i]}} \sum_{(r,i) \in \mathfrak{Z} \setminus \mathfrak{Z}_0} \varphi_{s,r,i} \wedge \varepsilon \in \tilde{j}^{l+1-i} \wedge^k(g),
$$
for $i = 0, \ldots, l + 1$. Then, by Lemma 10.4, $\varphi_{s,r,i}$ is in $M_{i+1-(s,r,i)}$ for all $(s, r, i)$. 

11. **Some spaces related to parabolic subalgebras II**

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. We then use the notations of Section 10 and Subsection 9.3. Set:
$$
\mathcal{Q} := ([I_s \times g] \otimes_k \wedge (\mathfrak{p}_-))^3.
$$
and denote by $\theta$ the map
$$
\mathcal{Q} \xrightarrow{\theta} [I_s \times g] \otimes_k \wedge (\mathfrak{p}_-), \quad (\varphi_{\nu}, \nu \in \mathfrak{Z}) \longmapsto \sum_{\nu \in \mathfrak{Z}} \varphi_{\nu} \wedge \lambda_{\nu}.
$$
For $\nu$ in $\mathfrak{Z}$, $\varphi_{\nu}$ has a unique expansion
$$
\varphi_{\nu} = \sum_{k \in \mathfrak{Z}} \varphi_{\nu,k} \wedge w_{k}, \quad \text{with} \quad \varphi_{\nu,k} \in [I_s \times g] \otimes_k \wedge (l).
$$
For $l$ nonnegative integer, let $\Omega_l$ be the subspace of $\Omega$ defined by the following condition:

$$\varphi \in \Omega_l \iff (|r| + |u| \neq l \iff \varphi_{r,u} = 0).$$

For $k = 1, \ldots, n$, denote by $\Omega_{l,k}$ the subspace of elements $\varphi$ of $\Omega_l$ such that $\theta(\varphi)$ is in $\mathbb{k}[l \times g] \otimes_k \wedge^k(p_-).

**Lemma 11.1.** Let $k = 1, \ldots, n$, $l$ a positive integer and $\varphi$ in $\Omega_{l,k}$.

(i) The element $\theta(\varphi)$ is in $M_l$.

(ii) If $\varphi_{r,u,k} \wedge \overline{e}_0 = 0$ for all $(r, u, \kappa)$, then $\theta(\varphi)$ is in $M_{l+1}$.

**Proof.** Let $(x, y, y')$ be in $\mathbb{l} \times \mathbb{p} \times \mathbb{p}_-$. According to Lemma 9.5, for $j = 1, \ldots, n - d$, $u$ in $\mathcal{J}_j$, the polynomial map

$$t \mapsto \lambda_u \wedge \varepsilon(x, y + ty')$$

is divisible by $t^l$ in $\mathbb{k}[t] \otimes_k \wedge (g)$ and the coefficient $c_{r,u}(x, y, y')$ of $t^l$ satisfies the relation

$$c_{r,u}(x, y, y') = \sum_{k \in \mathcal{J}_j} a_{r,u}(x, y, y') w_k \wedge \overline{e}_0(x, y) \wedge \mu_+ \in \bigoplus_{|l| < j} w_k \wedge \wedge^{|-|\kappa|}(l) \wedge \overline{e}_0(x, y) \wedge \mu_+ \quad \text{with}$$

$$a_{r,u} := \sum_{l \in \mathcal{J}_j} \sum_{r \in \mathbb{N}^l} t^r a_{r,u,k}.\,$$

Then the polynomial map

$$t \mapsto \theta(\varphi) \wedge \varepsilon(x, y + ty')$$

is divisible by $t^l$ in $\mathbb{k}[t] \otimes_k \wedge (g)$ since $\varphi_{r,u,k} = 0$ when $|r| + |u| < l$. Hence $\theta(\varphi)$ is in $M_l$. Moreover, the coefficient of $t^l$ of this polynomial map is equal to 0 if $\varphi_{r,u,k} \wedge \overline{e}_0 = 0$ for all $(r, u, \kappa)$, whence the lemma.

**Proposition 11.2.** Let $k = 1, \ldots, n$, $l$ a positive integer and $\varphi$ in $\Omega_{l,k}$. Then $\theta(\varphi)$ is in $M_{l+1}$ if and only if $\varphi_{r,u,k} \wedge \overline{e}_0 = 0$ for all $(r, u, \kappa)$.

**Proof.** By Lemma 11.1(ii), the condition is sufficient. Suppose that $\theta(\varphi)$ is in $M_{l+1}$. For all $\nu$ in $\mathbb{C}$,

$$\sum_{(r,u,k)} t^r \nu_r(\varphi_{r,u,k}) \wedge w_k + \sum_{(r,u,k)} t^r \nu_r(\varphi_{r,u,k}) \wedge w_k \in M_{l+1}$$

since $t_r(\varepsilon) = 0$. Prove the proposition by induction on $k$.

Suppose $k > 1$, the assertion true for $k - 1$ and the proposition not true for $k$. A contradiction is expected. Denote by $j$ the biggest integer such that for some $\kappa$ in $\mathcal{J}_j$, $\varphi_{r,u,k} \wedge \overline{e}_0 \neq 0$. By Lemma 11.1(ii), we can suppose

$$|\kappa| > j \implies \varphi_{r,u,k} = 0.$$

By the above relation and the induction hypothesis,
\[ \iota_r(\varphi_{r,\nu,\kappa}) \wedge \overline{e_0} = 0, \quad \forall (n, r, \nu, \kappa) \in C \wedge \mathbb{N}^d \times \mathfrak{A} \times \mathfrak{A}_j \]
by maximality of $j$. Then, arguing as in the proof of Proposition 10.3, \( \varphi_{r,\nu,\kappa} \wedge \overline{e_0} = 0 \) for all $(r, \nu, \kappa)$ such that $|\kappa| = j$, whence the contradiction.

For $k = 1$,
\[ \varphi_{r,0,0} \in \mathbb{k}[I \times \mathfrak{p}] \otimes \mathfrak{k} \quad \text{and} \quad \varphi_{r,0,j} \in \mathbb{k}[I \times \mathfrak{p}] \otimes \mathfrak{k} \quad \text{for } j = 1, \ldots, d, \]
whence
\[ (\varphi_{r,0,0} + \sum_{j=1}^{d} \varphi_{r,0,j}w_j) \wedge \varepsilon(x, y) = 0 \]
for all $r$ in $\mathbb{N}^d_j$ and all $(x, y)$ in $l \times \mathfrak{p}$. Then, for all $r$ in $\mathbb{N}^d_j$, $\varphi_{r,0,j} = 0$ for $j = 1, \ldots, d$ and $\varphi_{r,0,0} \wedge \overline{e_0} = 0$ since $\varepsilon(x, y)$ is colinear with $\overline{e_0}(x, y) \wedge \mu_+$ for all $(x, y)$ in $l \times \mathfrak{p}$, whence the proposition.

Set $\mathbb{N}_{p,*} := \mathbb{N}^d \times \mathfrak{A} \times \mathfrak{A}$.

**Corollary 11.3.** Let $k = 1, \ldots, n$, $l$ a positive integer and $\varphi$ in $(\mathbb{k}[I \times \mathfrak{p}] \otimes \mathfrak{k} \wedge (\mathfrak{p}))^{\mathbb{N}_{p,*}}$ such that
\[ \varphi_{r,\nu,\kappa} \in \mathbb{k}[I \times \mathfrak{p}] \otimes \mathfrak{k} \wedge |k-\lambda| \quad \text{and} \quad |(r, \nu, \kappa)| \neq l \implies \varphi_{r,\nu,\kappa} = 0. \]
Set:
\[ \psi = \sum_{(r,\nu,\kappa)} \varphi_{r,\nu,\kappa} \wedge \lambda_\nu \wedge \nu_\kappa. \]
If $\psi$ is in $M_{l+1}$ then $\varphi_{r,\nu,\kappa} \wedge \overline{e_0} = 0$ for all $(r, \nu, \kappa)$.

**Proof.** For $(r, \nu)$ in $\mathbb{N}^d \times \mathfrak{A}$, set:
\[ \psi_{r,\nu} = \sum_{\kappa \in \mathfrak{A}} \varphi_{r,\nu,\kappa} \wedge \lambda_{\nu}. \]
By hypothesis, for all $(r, \nu)$, $\psi_{r,\nu}$ is in $\overline{J}_{\nu-\lambda_{(r,\nu)}}\wedge k-\lambda_{(r,\nu)}$. So, by Lemma 10.4, $\psi_{r,\nu}$ is in $M_{l+1-\lambda_{(r,\nu)}}$. Hence, by Proposition 11.2, $\varphi_{r,\nu,\kappa} \wedge \overline{e_0} = 0$ for all $(r, \nu, \kappa)$.

12. Induction. Case $n \leq 1$

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. Then we use the notations of Section 9. In particular, $l$ is the reductive factor of $\mathfrak{p}$ containing $\mathfrak{b}$, $\mathfrak{d}$ is its derived algebra and $n$ is the number of simple factors of $\mathfrak{d}$. We suppose that the simple factors of $\mathfrak{d}$ have Property (P). Let $\mathfrak{O}, \hat{\mathfrak{O}}, \mathfrak{m}, \mathfrak{J}, \hat{\mathfrak{J}}$ be as in Subsection 10.2. Denote by $J_1$ the ideal of $\hat{\mathfrak{O}} \otimes \mathbb{k}[I \times \mathfrak{g}]$ generated by $\mathfrak{m}$ and $J$. In Subsections 12.2 and 12.3, we suppose $n \leq 1$. 
12.1. Some complexes. Recall that $d_1, \ldots, d_n$ are the simple factors of $l$. Set: $p_{±, u} := p_u \oplus p_{-u}$.

Let $I$ and $I_k$ be as in Subsection 6.2, $I', I'_k$ as in Section 4. With the notations of Subsection 6.2, for $k = 0, \ldots, n$, denote by $D^*_{k, #}(g)$ and $D^*_{k, #}(p_-)$ the graded subcomplexes of $D_k^*(g)$ deduced from the double complexes

$$\bigoplus_{i=0}^k D^*_{k-i}(p_{±, u}) \otimes_k D^*_{i, #}(l)$$

$$\bigoplus_{i=0}^k D^*_{k-i}(p_{-u}) \otimes_k D^*_{i, #}(l)$$

respectively.

Let $B, \overline{B}, \overline{B}_1, e$ be as in Section 7, $\overline{B}_0, e_0$ as in Subsection 7.2, $\overline{B}, \overline{B}, \hat{e}$ as in Subsection 10.2. For $k = 0, \ldots, n$, denote by $D^*_{k, #}(g, B)$ the graded subcomplex of $D_k^*(g, B)$:

$$D^*_{k, #}(g, B) := D_k^*(g)[−b_g] \wedge^b_{k}(B).$$

The graded subcomplexes $D^*_{k, #}(g, \overline{B})$ and $D^*_{k, #}(g, \hat{B})$ of $D^*_{k, #}(g)$ are equal to

$$D^*_{k, #}(g)[−b_g] \wedge^b_{k}(B)$$

$$D^*_{k, #}(g)[−b_g] \wedge^b_{k}(\hat{B})$$

respectively.

For $k = 0, \ldots, n$ and $l = 0, \ldots, n − d$, set:

$$\overline{B}_+ := \overline{B}_0 \oplus \mathbb{k}[l, x] \otimes_k p_u$$

$$D^*_{k, #}(V, \overline{B}_+) := D^*_{k, #}(V)[−b_g] \wedge^b_{k}(\overline{B}_+)$$

$$D^*_{l, #}(l, \overline{B}_+) := D^*_{l, #}(l)[−b_g] \wedge^b_{k}(\overline{B}_+)$$

with $V = p_-$ or $V = g$. Then $\overline{B}_+$ is a free module of rank $b_g$ containing $\overline{B}$, $D^*_{k, #}(V, \overline{B}_+)$ is a graded subcomplex of $D^*_{k}(V, \overline{B}_+)$, $D^*_{l, #}(l, \overline{B}_+)$ is a graded subcomplex of $D^*_{l}(l, \overline{B}_+)$.  

Lemma 12.1. (i) For $k = 0, \ldots, n − d$, $D^*_{k, #}(l, \overline{B}_+)$ has no cohomology of degree different from $b_g$.

(ii) For $k = 0, \ldots, n$, $D^*_{k, #}(p_-, \overline{B}_1)$ has no cohomology of degree different from $b_l$.

(iii) For $k = 0, \ldots, n$, $D^*_{k, #}(p_-, \overline{B}_+)$ has no cohomology of degree different from $b_l$.

(iv) For $k = 0, \ldots, n$, $D^*_{k, #}(g, \overline{B}_1)$ has no cohomology of degree different from $b_l$.

Proof. (i) According to Lemma 4.3, the complex $D^*_{k, #}(l, B_1)$ has no cohomology of degree different from $b_l$ since $d_1, \ldots, d_n$ have Property (P). By restriction to the principal open subset $l$, $x$, $l \times l$ of $l \times l$, the subcomplex $D^*_{k, #}(l, \overline{B}_1)$ of $D_k^*(l, \overline{B}_1)$ has no cohomology of degree different from $b_l$. As $\mathbb{k}[l, x] = \mathbb{k}[p_u] \otimes \mathbb{k}[l, x]$, $\overline{B}_+$ is the direct sum of $\mathbb{k}[p_u] \otimes \overline{B}_1$ and $\mathbb{k}[l, x] \otimes p_u$ by Lemma 7.5(ii).

As a result,

$$D^*_{k, #}(l, \overline{B}_+) = \mathbb{k}[p_u] \otimes \mathbb{k}[l, \overline{B}_1][−d] \wedge^d(p_u),$$

whence an isomorphism of graded complexes

$$\wedge^d(p_u) \otimes \mathbb{k}[p_u] \otimes \mathbb{k} D^*_{k, #}(l, \overline{B}_1)[−d]$$

$$D^*_{k, #}(l, \overline{B}_+) ,$$
and the assertion.

(ii) and (iii) As \( p_+ \) is the direct sum of \( I \) and \( p_-, \) \( D^*_{k,\#}(p_-, \overline{B}_I) \) and \( D^*_{k,\#}(p_-, \overline{B}_+) \) are isomorphic to the simple complexes deduced from the double complexes:

\[
\bigoplus_{j=\sup(0,k-n+d)}^k D^*_j(p_-, \overline{B}_I) \otimes_k D^*_{k-j,\#}(1, \overline{B}_I) \quad \text{and} \quad \bigoplus_{j=\sup(0,k-n+d)}^k D^*_j(p_-, \overline{B}_+) \otimes_k D^*_{k-j,\#}(1, \overline{B}_+)
\]

respectively. By Lemma \( \text{A.2}(ii), \) for \( j \) positive integer, \( D^*_j(p_-, \overline{B}) \) is an acyclic complex, whence the assertion by (i) and its proof.

(iv) As \( p_u \) is contained in \( \overline{B}_+, \)

\[
D^*_{k,\#}(p_u, \overline{B}_+) = \bigoplus_{j=0}^k S^{k-j}(p_u) \otimes_k D^*_{k-j,\#}(p_-, \overline{B}_+)
\]

since

\[
\bigoplus_{j=l'}^{l} S^{l-j}(p_u) \otimes_k S^{j-l'}(p_-) = S^{l-l'}(g)
\]

for \( 0 \leq l' \leq l. \) So, by (iii), \( D^*_{k,\#}(p_u, \overline{B}_+) \) has no cohomology of degree different from \( b_\#. \)

\[ \square \]

For \( k = 0, \ldots, n, \) denote by \( D^*_{k,\#}(g, \overline{B}) \) the graded subcomplex of \( \mathbb{Z}[I \times p] \otimes_k D^*_g, \)

\[
D^*_{k,\#}(g, \overline{B}) := D^*_{k,\#}(g)[-b_\#] \wedge^{b_\#}(\overline{B}).
\]

**Corollary 12.2.** For \( k = 0, \ldots, n, \) \( D^*_{k,\#}(g, \overline{B}) \) has no cohomology of degree different from \( b_\#. \)

**Proof.** As the modules \( \overline{B}_+ \) and \( \overline{B} \) are free modules of rank \( b_\# \) and \( \overline{B} \) is contained in \( \overline{B}_+, \) for some \( p \) in \( \mathbb{Z}[I \times p] \setminus \{0\}, \) \( \wedge^{b_\#}(\overline{B}) = p \wedge^{b_\#}(\overline{B}_+). \) As a result, the map

\[
S(g) \otimes_k \wedge^{b_\#}(\overline{B}_+) \xrightarrow{\tau} S(g) \otimes_k \wedge^{b_\#}(\overline{B}), \quad \varphi \mapsto p\varphi
\]

is an isomorphism of graded complexes such that

\[
\tau(D^*_{k,\#}(g, \overline{B}_+)) = D^*_{k,\#}(g, \overline{B}).
\]

So, by Lemma 12.1(iv), \( D^*_{k,\#}(g, \overline{B}) \) has no cohomology of degree different from \( b_\#. \)

\[ \square \]

For \( j, k \) integers such that \( 0 \leq j \leq k \leq n, \) denote by \( D^*_{k,j,\#}(g) \) the graded subspace of \( D^*_{k,\#}(g), \)

\[
D^*_{k,j,\#}(g) := S^j(p_u)D^*_{k-j,\#}(p_-)
\]

and \( D^*_{k,j,\#}(g, \overline{B}) \) the graded subspace of \( D_{k,\#}(g, \overline{B}), \)

\[
D^*_{k,j,\#}(g, \overline{B}) := D^*_{k,j,\#}(g)[-b_\#] \wedge^{b_\#}(\overline{B}).
\]

In particular,

\[
D_{k,0,\#}(g, \overline{B}) = D_{k,\#}(p_-, \overline{B}) := D^*_{k,\#}(p_-)[-b_\#] \wedge^{b_\#}(\overline{B}).
\]
Since \( p_u \wedge \bigwedge^{b_k(\mathcal{B})} = \{0\} \), \( D^*_{k,j,#}(\mathfrak{g}, \mathcal{B}) \) is a graded subcomplex of \( D^*_{k,#}(\mathfrak{g}, \mathcal{B}) \).

**Lemma 12.3.** Let \( k = 1, \ldots, n \).

(i) For \( \varphi \in D^{b_k(\mathfrak{p} \times \mathfrak{p})}(\mathfrak{g}, \mathcal{B}) \), \( \varphi \) is a cocycle of \( D^*_{k,#}(\mathfrak{g}, \mathcal{B}) \) if and only if \( \varphi \) is in \( S^k(\mathcal{B}_l) \otimes_{k[x]} \bigwedge^{b_k(\mathcal{B})} \).

(ii) For \( 0 \leq j \leq k \) and \( \varphi \in D^{b_k(\mathfrak{g}, \mathcal{B})}(\mathfrak{g}, \mathcal{B}) \), \( \varphi \) is a cocycle of \( D^*_{k,#}(\mathfrak{g}, \mathcal{B}) \) if and only if

\[
\varphi \in S^j(p_u)S^{k-j}(\mathcal{B}_l) \otimes_{k[x]} \bigwedge^{b_k(\mathcal{B})}.
\]

**Proof.** Denote again by \( \varepsilon \) the restriction of \( \varepsilon \) to \( \mathfrak{l} \times \mathfrak{p} \). Then \( \varepsilon \) is a generator of \( \bigwedge^{b_k(\mathcal{B})} \).

(i) As usual, for \( r = (r_1, \ldots, r_d) \) in \( \mathbb{N}^d \), set:

\[
u' := w'^{r_1} \cdots w'^{r_d} \quad \text{and} \quad v' := v^{r_1} \cdots v^{r_d}.
\]

Let \( \psi \) be in \( \mathbb{k}[\mathfrak{l} \times \mathfrak{p}] \otimes_{\mathbb{k}} D^0_{k,#}(\mathfrak{p} \times \mathfrak{p}) \). Then \( \psi \) has an expansion

\[
\psi = \sum_{j=0}^k \sum_{r \in \mathbb{N}^d} w^r \psi_r, \quad \text{with} \quad \psi_r \in \mathbb{k}[\mathfrak{l} \times \mathfrak{p}] \otimes_{\mathbb{k}} D^0_{k-l,#}(\mathfrak{l}).
\]

With the notations of Subsection 9.2, since \( \bigwedge^{b_k(\mathcal{B})} \) is a submodule of \( \bigwedge^{b_k(\mathcal{B}_l)} \), \( \psi \wedge \varepsilon \) is a cocycle of \( D^*_{k,#}(\mathfrak{g}, \mathcal{B}) \) if and only if \( \psi \wedge \varepsilon_0 \wedge \mu_+ \) is a cocycle of \( D^*_{k,#}(\mathfrak{g}, \mathcal{B}_l) \). In particular, the condition of the assertion is sufficient.

Suppose that \( \psi \wedge \varepsilon \) is a cocycle of \( D^*_{k,#}(\mathfrak{g}, \mathcal{B}) \). Denote by \( v(\psi) \) the biggest element \((|r|, r)\) of \( \mathbb{N} \times \mathbb{N}^d \) such that \( \psi_r \neq 0 \). Suppose that \( v(\psi) \) is different from \((0, 0)\). A contradiction is expected.

Let \( j \) be the smallest indice such that \( r_j \neq 0 \). Denote by \( \tilde{r} \) the element of \( \mathbb{N}^{d-1} \) such that \( \tilde{r}_l = r_l \) for \( l \neq j \) and \( \mathfrak{R}_{r_j} \) the subset of elements \( r' \) of \( \mathbb{N}^{d-1} \) such that \( w^r \) divides \( w^{r'} \). For \( r' \) in \( \mathfrak{R}_{r_j} \), let \( k_r \) be the indice such that \( w^{r'} = w_{k_r} w^r \). As \( \psi \wedge \varepsilon \) is a cocycle, by maximality of \( v(\psi) \) and minimality of \( j \),

\[
\sum_{r' \in \mathfrak{R}_{r_j}} r' \psi_r w_{k_r} w^r = 0.
\]

In particular, \( \psi_r = 0 \), whence the contradiction. As a result, \( \psi = \psi_0 \). As a matter of fact, \( \psi = 0 \) when \( k > n - d \). Otherwise, \( \psi \wedge \varepsilon_0 \) is a cocycle of \( \mathbb{k}[\mathfrak{p}_u] \otimes_{\mathbb{k}} D^*_{k,#}(\mathfrak{l}, \mathcal{B}_l) \). So, by Lemma A.4(iii), \( \psi \) is in \( \mathbb{k}[\mathfrak{p}_u] \otimes_{\mathbb{k}} S^k(\mathcal{B}_l) \), whence the assertion.

(ii) Let \( \psi \) be in \( \mathbb{k}[\mathfrak{l} \times \mathfrak{p}] \otimes_{\mathbb{k}} D^0_{k,j,#}(\mathfrak{g}) \). Since \( \psi \wedge \varepsilon \) is a cocycle if and only if \( \psi \wedge \varepsilon_0 \wedge \mu_+ \) is a cocycle of degree \( b_0 \) of \( D^*_{k,#}(\mathfrak{g}, \mathcal{B}_l) \), the condition is sufficient. Suppose that \( \psi \wedge \varepsilon \) is a cocycle of \( D^*_{k,#}(\mathfrak{g}, \mathcal{B}) \). The element \( \psi \) has an expansion

\[
\psi = \sum_{r \in \mathbb{N}^d} v^r \psi_r, \quad \text{with} \quad \psi_r \in \mathbb{k}[\mathfrak{l} \times \mathfrak{p}] \otimes_{\mathbb{k}} D^0_{k-l,#}(\mathfrak{p} \times \mathfrak{p}) \).
\]

Since \( p_u \wedge \bigwedge^{b_k(\mathcal{B})} = \{0\} \), for all \( r, \psi_r \wedge \varepsilon \) is a cocycle of \( D_{k-j,#}(\mathfrak{p} \times \mathfrak{p}, \mathcal{B}) \), whence the assertion by (i). \( \square \)
12.2. **Quasi-regularity.** Let \( k = 0, \ldots, n \) and set:

\[
D_{k,#}(\mathcal{B}) := D_{k,#}(\mathcal{B}) \wedge \wedge_{k}(\mathcal{B}).
\]

Let \( \mathbb{N} \) be as in Subsection 10.2. For \( j = 0, \ldots, n \), set:

\[
\mathcal{M}_j := \{ \varphi \in (\mathbb{k}[l, \times p] \otimes \mathbb{k} D_{l}(\mathcal{B}))^{\mathbb{N}_{p}} \mid \exists \mathbb{N}_{\varphi} \in \mathbb{N} \text{ such that } |r| \geq \mathbb{N}_{\varphi} \implies \varphi_{x,r,t} = 0 \text{ and } \}
\]

\[
\varphi_{x,r,t} \in \mathbb{k}[l, \times p] \otimes \mathbb{k} D_{j-\{l\},#}(\mathcal{B}) \text{ and } \kappa(\varphi) := \sum_{(s,t,r) \in \mathcal{N}_{p}} z^{s} \psi \varphi_{x,r,t} \wedge v_{l} \wedge \hat{e}
\]

for \( \varphi \in \mathcal{M}_j \). For \( l \) nonnegative integer, denote by \( \mathcal{M}_{ji} \) the subspace of elements \( \varphi \) of \( \mathcal{M}_j \) such that

\[
|(s, r, t)| \neq l \implies \varphi_{x,r,t} = 0
\]

and \( \mathcal{M}_{ji} \), the sum of \( \mathcal{M}_{ji} \), \( i = l, l+1, \ldots \).

**Lemma 12.4.** Let \( l \) be a positive integer. Suppose \( k > 0 \).

(i) For \( m = 1, \ldots, n-d \) and \( \varphi \in \mathbb{k}[l, \times p] \otimes \mathbb{k} D_{m,#}(l) \) such that \( \varphi(x, y) \wedge \bar{e}_0(x, y) = 0 \) for all \( (x, y) \) in \( l \times p \), \( \varphi \) is in \( D_{m-1,#}(l) \wedge \overline{\mathcal{B}}_{l} \).

(ii) For \( \varphi \in \mathbb{k}[l, \times p] \otimes \mathbb{k} D_{k,#}(p_{-}) \), if the restriction of \( \varphi \wedge \epsilon \) to \( l \times p \) is equal to 0, then

\[
\varphi \in \mathbb{k}[l, \times p] \otimes \mathbb{k}(l) \wedge \mathcal{M}_l \wedge \mathcal{B}_{l}.
\]

(iii) For \( \varphi \in \mathbb{k}[l, \times g] \otimes \mathbb{k} D_{k,#}(g) \), the restriction of \( \varphi \wedge \epsilon \) to \( l \times p \) is equal to 0 if and only if

\[
\varphi \in D_{k-1,#}(g) \wedge \mathcal{B} + JD_{k,#}(g) + \mathbb{k}[l, \times g] \otimes \mathbb{k} D_{k-1,#}(g) \wedge p_{u}.
\]

Moreover, in this case, \( \varphi \wedge \hat{e} \) is in \( \kappa \mathcal{M}_{k,1,d} \).

**Proof.** (i) As \( d = \{0\} \) or \( d \) has Property \( P \), for some \( \psi_{1}, \ldots, \psi_{s} \) in \( \overline{\mathcal{B}}_{l} \),

\[
\varphi = \sum_{j=1}^{s} \varphi_{j} \wedge \psi_{j} \quad \text{with} \quad \varphi_{j} \in \mathbb{k}[l, \times p] \otimes \mathbb{k} D_{m-1,#}(l)
\]

for \( j = 1, \ldots, s \) by Proposition 4.4 since \( \mathbb{k}[l, \times p] = \mathbb{k}[p_{u}] \otimes \mathbb{k}[l, \times l] \), whence the assertion.

(ii) Let \( \varphi \) be in \( \mathbb{k}[l, \times p] \otimes \mathbb{k} D_{k,#}(p_{-}) \) such that \( \varphi(x, y) \wedge \epsilon(x, y) = 0 \) for all \( (x, y) \) in \( l \times p \). The element \( \varphi \) has an expansion

\[
\varphi = \sum_{k \in \mathbb{K}} \varphi_{k} \wedge w_{k} \quad \text{with} \quad \varphi_{k} \in \mathbb{k}[l, \times p] \otimes \mathbb{k} \bigoplus_{m=0}^{\inf(n-d,k)} S^{k-m}(p_{-,u}) \otimes \mathbb{k} D_{m,#}(l).
\]

For \( (x, y) \) in \( l \times p \),

\[
\epsilon(x, y) \in \bar{e}_0(x, y) \wedge \wedge_{d}(p_{u})
\]

by Proposition 7.6(i), whence

\[
\varphi_{k}(x, y) \wedge \bar{e}_0(x, y) = 0
\]
for all \( \kappa \) in \( \mathfrak{S} \) and all \((x, y)\) in \( I, \times p \). So, by (i),
\[
\varphi_\kappa \in \bigoplus_{m=1}^{\inf(n-d, k)} S^{k-m}(p_{-u}) \otimes_\kappa D_{m-1, \#}(l) \wedge \mathbf{B}_l,
\]
for all \( \kappa \), whence the assertion.

(iii) Since \( k[I, \times g] = k[I, \times p] + J \), we can suppose that \( \varphi \) is in \( k[I, \times p] \otimes_\kappa D_{k, \#}(g) \). As \( g \) is the direct sum of \( p_- \) and \( p_u \),
\[
\bigwedge(\varphi) = \bigwedge(p_-) \oplus \bigwedge(g) \wedge p_u \quad \text{and} \quad D_{k, \#}(g) = \bigoplus_{m=0}^{k} S^m(p_u) \otimes_\kappa D_{k-m, \#}(p_-) \oplus D_{k-1, \#}(g) \wedge p_u.
\]
Then \( \varphi = \varphi_1 + \varphi_2 \) with
\[
\varphi_1 \in k[I, \times p] \otimes_\kappa \bigoplus_{m=0}^{k} S^m(p_u) \otimes_\kappa D_{k-m, \#}(p_-) \quad \text{and} \quad \varphi_2 \in k[I, \times p] \otimes_\kappa D_{k-1, \#}(g) \wedge p_u.
\]
As \( p_u \wedge \varepsilon(x, y) = \{0\} \) for all \((x, y)\) in \( I, \times p \), the restriction of \( \varphi_2 \wedge \varepsilon \) to \( I, \times p \) is equal to 0. Moreover, the restriction of \( \varphi \wedge \varepsilon \) to \( I, \times p \) is equal to 0 if
\[
\varphi \in D_{k-1, \#}(g) \wedge B + JD_{k, \#}(g) + D_{k-1, \#}(g) \wedge p_u.
\]
Conversely, by (ii),
\[
\varphi_1 \in k[I, \times p] \otimes_\kappa D_{k-1, \#}(p_-) \wedge \mathbf{B}_l.
\]
By Proposition 9.1(ii),
\[
\mathbf{B}_l \subset B + J \otimes_\kappa p_{-u} + J_- \otimes_\kappa p_u + JJ_- \otimes_\kappa I.
\]
Hence
\[
\varphi_1 \in D_{k-1, \#}(p_-) \wedge B + JD_{k, \#}(g) + D_{k-1, \#}(g) \wedge p_u,
\]
since \( B \) is simple or equal to \( \{0\} \). Indeed, \( \varphi_1 \) has an expansion
\[
\varphi_1 = \sum_{m=0}^{\inf(k-1, n-d)} \sum_{n=0}^{\inf(m, n-d-1)} \varphi_{1,m,j} \quad \text{with} \quad \varphi_{1,m,j} \in k[I, \times p] \otimes_\kappa D_{k-m}(p_{-u}) \otimes_\kappa D_{m}(l) \wedge \mathbf{B}_l.
\]
For \((i, m)\) in \( I_0 \), the map
\[
G \times I_+ \times g \longrightarrow \mathbf{g}, \quad (g, x, y) \mapsto \varepsilon^{(m)}_i(g(x), g(y)) - \varepsilon^{(m)}_i(x, y)
\]
is in \( \mathfrak{m} \otimes_\kappa k[I, \times g] \otimes_\kappa g \) since \( \varepsilon^{(m)}_i \) is a \( G \)-equivariant by Proposition 2.1(vi), whence
\[
\varphi_{1,m,j} \in D_{k-1, \#}(g) \wedge \mathbf{B} + J \otimes_\kappa D_{k, \#}(g) + k[I, \times g] \otimes_\kappa D_{k-1, \#}(g) \wedge p_u + m \otimes_\kappa k[I, \times g] \otimes_\kappa D_{k, \#}(g)
\]
for all \((m, j)\). As a result, \( \varphi \wedge \varepsilon \) is in \( \kappa(N_{k,1,+}) \).

Remark 12.5. Assertion (i) and Assertion (ii) are true when \( \mathfrak{d} \) is not simple.
Proposition 12.6. Let \( l \) be a positive integer and \( \varphi \) in \( M_{k,l} \). Then \( \kappa(\varphi) \) is in \( \kappa(M_{k,l+1}) \) if and only if \( \varphi_{s,r,d} \wedge \hat{e} \) is in \( \kappa(M_{k-|d|,l+1}) \) for all \( (s,r,t) \).

Proof. The condition is clearly sufficient. Suppose that \( \kappa(\varphi) \) is in \( \kappa(M_{k,l+1}) \). As \( g \) is the direct sum of \( p_- \) and \( p_u \) and \( \mathbb{k}[l \times g] \) is the direct sum of \( \mathbb{k}[l \times p] \) and \( J \),

\[
\varphi_{s,r,d} = \varphi'_{s,r,d} + \varphi''_{s,r,d}
\]

with \( \varphi'_{s,r,d} \in \mathbb{k}[l \times p] \otimes \mathbb{k} D_{k-|d|,\#}(p_-) \) and \( \varphi''_{s,r,d} \in J \otimes \mathbb{k} D_{k-|d|,\#}(g) + \mathbb{k}[l \times g] \otimes \mathbb{k} D_{k-|d|,\#}(g) \wedge p_u \)

for all \( (s,r,t) \). Setting \( \varphi' := (\varphi'_{s,r,d}, (s,r,t) \in \mathbb{N}_p) \), \( \varphi' \) is in \( M_{k,l} \) and \( \kappa(\varphi') \) is in \( \kappa(M_{k,l+1}) \) since the condition is sufficient.

Let \( (s,r,t) \) be in \( \mathbb{N}_p \) such that \( |(s,r,t)| = l \). By Proposition 10.6,

\[
\varphi'_{s,r,d} \in \mathbb{k}[l \times g] \otimes \mathbb{k} D_{k,\#}(p_-) \cap S(g \otimes \mathbb{k} M_1).
\]

Then the restriction of \( \varphi'_{s,r,d} \wedge \varphi \) to \( l \times p \) is equal to 0. So, by Lemma 12.4(iii), the restriction of \( \varphi_{s,r,d} \wedge \varphi \) to \( l \times p \) is equal to 0 and \( \varphi_{s,r,d} \wedge \hat{e} \) is in \( \kappa(M_{k-|d|,l+1}) \) for all \( (s,r,t) \). \( \square \)

For \( l = 0, 1, \ldots \), let \( F_l \) be the subspace \( \kappa(M_{k,l+1}) \) of \( D_{k,\#}(\tilde{g}, \tilde{B}) \). Then the sequence \( F_l \), \( l = 0, 1, \ldots \) is a decreasing filtration of \( D_{k,\#}(\tilde{g}, \tilde{B}) \). Denote by \( \text{gr} D_{k,\#}(\tilde{g}, \tilde{B}) \) the associate graded space to this filtration and \( \text{gr}_{D_{k,\#}(\tilde{g}, \tilde{B})} \) the subspace of degree \( l \) of \( \text{gr} D_{k,\#}(\tilde{g}, \tilde{B}) \). For \( j = 0, \ldots, k \), let \( D^*_{j,\#}(\tilde{g}, \tilde{B}) \) be the graded subcomplex of \( \mathbb{k}[l \times p] \otimes \mathbb{k} D^*_{(g)} \),

\[
D^*_{j,\#}(\tilde{g}, \tilde{B}) := D^*_{j,\#}(g)[-b_0] \wedge b_3(\tilde{B}),
\]

and \( A \) the algebra

\[
A := \mathbb{k}[z_1, \ldots, z_{2n+1}] \otimes \mathbb{k} S(p_u) \otimes \mathbb{k} (p_u).
\]

This algebra has a bigradation \( A^* \) such that

\[
A^i := \mathbb{k}[z_1, \ldots, z_{2n+1}] \otimes \mathbb{k} S(p_u) \otimes \mathbb{k} \wedge^i (p_u)
\]

and

\[
A_l := \bigoplus_{(j_1,j_2,j_3) \in \mathbb{N}^3} \mathbb{k}[z_1, \ldots, z_{2n+1}]_{j_1} \otimes \mathbb{k} S^{j_2}(p_u) \otimes \mathbb{k} \wedge^j (p_u)
\]

with \( \mathbb{k}[z_1, \ldots, z_{2n+1}]_j \) the space of homogeneous polynomials of degree \( j \) of \( \mathbb{k}[z_1, \ldots, z_{2n+1}] \).

Consider on \( \mathbb{k}[l \times p] \otimes \mathbb{k} A \otimes \mathbb{k} D(g) \) the simple gradation deduced from the double gradation \( \mathbb{k}[l \times p] \otimes \mathbb{k} A^* \otimes \mathbb{k} D^*_{(g)} \). The trivial structure of complex on \( A \) and the structure of complex on \( D(g) \) induce a structure of graded complex on \( \mathbb{k}[l \times p] \otimes \mathbb{k} A \otimes \mathbb{k} D(g) \). For \( l = 0, 1, \ldots \) and \( i \) nonnegative integer, set:

\[
C^i_l = \bigoplus_{j=0}^i \bigoplus_{(l_1,l_2) \in \mathbb{N}^2} \mathbb{k}[z_1, \ldots, z_{2n+1}]_{l_1} \otimes \mathbb{k} S^{l_2}(p_u) \otimes \mathbb{k} \wedge^i (p_u) \otimes \mathbb{k} D_{k-j,\#}^{|d|(g, \tilde{B})}.
\]

Denoting by \( C^i_l \) the sum \( C^i_l, i = 0, 1, \ldots \), \( C^i_l \) is a graded subcomplex of \( \mathbb{k}[l \times p] \otimes \mathbb{k} A \otimes \mathbb{k} D(g) \).
Corollary 12.7. (i) For \( l = 0, 1, \ldots \), \( F_i^* \) is a graded subcomplex of the graded complex \( D_{k,#}^*(g, \hat{B}) \).
(ii) The graded complex \( \text{gr}_0 D_{k,#}^*(g, \hat{B}) \) is isomorphic to the graded complex \( D_{k,#}^*(g, \bar{B}) \).
(iii) For \( l = 1, 2, \ldots \), the graded complex \( \text{gr}_l D_{k,#}^*(g, \hat{B}) \) is isomorphic to \( C_l^* \).

Proof. (i) For all \( l, \kappa(M_{k,l}) \) is a graded subspace of \( D_{k,#}(g, \hat{B}) \). For \( \varphi \) in \( M_{k,l} \),
\[
\text{d} \kappa(\varphi) = \sum_{(s,r,d) \in \mathbb{N}_p} z^i v^d \varphi_{s,r,d} \wedge v_i \wedge \hat{e}.
\]
Hence \( F_i \) is a graded subcomplex of \( D_{k,#}(g, \hat{B}) \).
(ii) Let \( r_{\hat{g},p} \) be the quotient morphism
\[
\hat{\mathcal{O}} \otimes_k \mathbb{K}[l, \times \mathbb{G}] \xrightarrow{r_{\hat{g},p}} \mathbb{K}[l, \times \mathbb{P}].
\]
Then \( r_{\hat{g},p} \) induces a morphism
\[
\hat{\mathcal{O}} \otimes_k \mathbb{K}[l, \times \mathbb{G}] \otimes_k D(g) \xrightarrow{r_{\hat{g},p}} \mathbb{K}[l, \times \mathbb{P}] \otimes_k D(g).
\]
As \( \bar{B} = r_{\hat{g},p}(\hat{B}) \), we have a surjective morphism
\[
D_{k,#}^*(g, \hat{B}) \xrightarrow{r_{\hat{g},p}} D_{k,#}^*(g, \bar{B}).
\]
and \( F_1 \) is its kernel by Proposition 12.6, whence the assertion.
(iii) Let \( l \) be a nonnegative integer. Denote by \( \gamma_l \) the morphism of \( \mathbb{K}[l, \times \mathbb{P}] \)-modules
\[
\mathbb{K}[l, \times \mathbb{P}] \otimes_k A_l \otimes_k D(g) \xrightarrow{\gamma_l} \mathbb{K}[l, \times \mathbb{G}] \otimes_k D(g)
\]
and \( \mathcal{M}^\prime_{k,l} \) the subspace of elements \( \varphi \) of \( \mathcal{M}_{k,l} \) such that
\[
\varphi_{s,r,d} \in \gamma_l(\mathbb{K}[l, \times \mathbb{P}] \otimes_k A_l \otimes_k D(g)), \quad \forall (s, r, i) \in \mathbb{N}_p.
\]
Then \( F_l = \kappa(M^\prime_{k,l}) + F_{l+1} \) since
\[
\hat{f}_l = \mathbb{K}[l, \times \mathbb{P}] \otimes_k A_l \oplus \hat{f}^{l+1}.
\]
By Proposition 12.6, for \( \varphi \) in \( \mathcal{M}^\prime_{k,l} \), \( \kappa(\varphi) \) is in \( F_{l+1} \) if and only if \( r_{\hat{g},p}(\varphi_{s,r,d} \wedge \hat{e}) = 0 \) for all \( (s, r, i) \).
Then, by (ii), the restriction of \( r_{\hat{g},p} \) to \( F_l \) defines through the quotient an isomorphism of \( \mathbb{K}[l, \times \mathbb{P}] \)-modules,
\[
\text{gr}_l D_{k,#}(g, \hat{B}) \longrightarrow C_l^*.
\]
Moreover, this isomorphism is an isomorphism of graded complex. \( \square \)
12.3. **Annulation of cohomology.** Let \( k = 1, \ldots, n \). For \( i \) integer, denote by \( Z^i \) and \( B^i \) the spaces of cocycles and coboundaries of degree \( i \) of \( D_{k,#}^*(g, \overline{B}) \). For \( l = 1, \ldots, n \), denote by \( D_{l,u}^* \) the graded subcomplex of \( D_{l,#}^*(g, \overline{B}) \),

\[
D_{l,u}^* := D_l^*(p_u)[-b_g] \wedge b_\hat{\mathcal{B}}.
\]

**Lemma 12.8.** Let \( l = 1, \ldots, n \). The morphism

\[
\hat{\mathcal{O}} \otimes_\mathbb{K} [1 \times g] \otimes_\mathbb{K} D_l^*(p_u)[-b_g] \xrightarrow{\varphi \mapsto \varphi \wedge \hat{\varepsilon}} D_{l,u}^* \tag{1.12}
\]

is an isomorphism of graded complexes. In particular, \( D_{l,u}^* \) is acyclic.

**Proof.** Denote by \( \hat{D}_{l,u}^* \) the graded subcomplex of \( D_{l,#}^*(g, \overline{B}) \),

\[
\hat{D}_{l,u}^* := D_l^*(p_u)[-b_g] \wedge b_\hat{\mathcal{B}}.
\]

By Corollary 2.7, for all \((x, y)\) in a dense open subset of \( l \times p_\mathcal{B} \), \( V_{x,y} \) is contained in \( p_\mathcal{B} \). So, for \((g, x, y)\) in a dense open subset of \( G \times l \times g \), \( V_{g(x), g(y)} \cap p_u = \emptyset \). As a result, the morphism

\[
\hat{\mathcal{O}} \otimes_\mathbb{K} [1 \times g] \otimes_\mathbb{K} D_l^*(p_u)[-b_g] \xrightarrow{\varphi \mapsto \varphi \wedge \hat{\varepsilon}} \hat{D}_{l,u}^* \tag{1.12}
\]

is an isomorphism of graded complexes since it is surjective. By Lemma A.2(ii), \( D_l^*(p_u) \) is an acyclic complex since \( l \) is positive. Hence so is \( \hat{D}_{l,u}^* \), whence the lemma since \( \hat{\mathcal{O}} \) is a faithfully flat extension of \( \mathcal{O} \).

For \( j \) nonnegative integer, denote by \( S_j^\mathcal{O}(\overline{B}) \) the intersection of \( S^j(\overline{B}) \) and \( \mathcal{O}[1 \times 1] \otimes_\mathbb{K} D_{j,#}^0(\mathcal{I}) \).

For \( i = b_g, \ldots, k + b_g \) and \( l \) nonnegative integer, set:

\[
K^i := \bigoplus_{j=0}^{k-i+b_g} S_j^\mathcal{O}(\overline{B}) D_{k-j,u}^i \quad \text{and} \quad K^i = \bigoplus_{(l_1, l_2) \in \mathbb{Z}_{l_1+b_g}^2 \cap \mathcal{I}} \mathbb{K}[z_1, \ldots, z_{2n+l_2}] \otimes_\mathbb{K} S_{l_2}^\mathcal{O}(p_u) \otimes_\mathbb{K} \sum_{j=0}^{k-i+b_g} S_{j}^\mathcal{O}(\overline{B}) D_{k-j}^{l-b_g}(p_u) \wedge \hat{\varepsilon}.
\]

**Lemma 12.9.** Let \( i = b_g + 1, \ldots, k + b_g \) and \( l \) a nonnegative integer.

(i) The subspace \( Z^i \cap F^i_1 \) of \( F^i_1 \) is contained in \( dF^i_{l-1} + K^i + F^i_{l+1} \).

(ii) The space \( K^i \cap F^i_1 \) is contained in \( K^i + F^i_{l+1} \).

(iii) The intersection of \( F^i_{l+1} \) and \( K^i \) is equal to \( J^i_{l-1+b_g} + K^i \).

**Proof.** (i) Let \( \varphi \) be in \( Z^i \cap F^i_1 \) and \( \overline{\varphi} \) its image in \( \text{gr}_l D_{k,#}^*(g, \overline{B}) \). By Corollary 12.7, \( \overline{\varphi} \) is a cocycle of degree \( i \) of the graded complex \( C^*_l \). By Corollary 12.2, for \( j = 0, \ldots, k \), the complex \( D_{k-j,#}^*(g, \overline{B}) \)
has no cohomology of degree different from $b_\alpha$. Then, for some $\tilde{\psi}$ in $C^{i-1}_l$,

$$\tilde{\varphi} - d\tilde{\psi} \in \bigoplus_{(l_1, l_2) \in \mathbb{N}^2} \mathbb{k}[z_1, \ldots, z_{2n+\ell}]_l \otimes_k S^2_z(p_u) \otimes_k \wedge^{i-b_\alpha}(p_u) \otimes_k D^b_{k-i+b_\alpha, \#}(\mathfrak{g}, \mathfrak{B}).$$

Then, by Lemma 12.3(ii),

$$\tilde{\varphi} - d\tilde{\psi} \in \bigoplus_{(l_1, l_2) \in \mathbb{N}^2} \mathbb{k}[z_1, \ldots, z_{2n+\ell}]_l \otimes_k S^2_z(p_u) \otimes_k \wedge^{i-b_\alpha}(p_u) \otimes_k \sum_{j=0}^{k-i-b_\alpha} S^j_\#(\mathfrak{B}) D^0_{k-i+b_\alpha-j}(p_u) \wedge b_\alpha(\mathfrak{B}).$$

So, for a representative $\psi$ of $\tilde{\psi}$ in $F^{i-1}_l$,

$$\varphi - d\psi \in K^i + F^i_{l+1}.$$  

(ii) By definition, $K^i_l$ is contained in $F^i_l$. Prove by induction on $l$ that $K^i_l \cap F^i_{l+1}$ is contained in $K^i_{l+1} + F^i_{l+1}$. For $l = 0$, there is nothing to prove. Suppose the assertion true for $l - 1$. By definition,

$$K^i_l = \sum_{s=0}^{+\infty} K^i_s,$$

so by induction hypothesis,

$$K^i_l \cap F^i_{l-1} = \sum_{s=l-1}^{+\infty} K^i_s.$$  

The image of $K^i_{l-1}$ in $\text{gr} D_{k, \#}(\mathfrak{g}, \mathfrak{B})$ is contained in $\text{gr}_{l-1} D_{k, \#}(\mathfrak{g}, \mathfrak{B})$. Then, by Corollary 12.7(ii), $K^i \cap F^i_{l-1}$ is contained in $K^i_{l-1} + F^i_{l-1}$.

(iii) By definition $F^{i-1}_{l+1} \cap K^i_l$ is contained $F^i_{l+1} \cap K^i_l$. Prove the assertion by induction on $l$. By Corollary 12.7(iii), the quotient of $K^i_l$ by $K^i \cap F^i_{l+1}$ is equal to the subspace of $\text{gr}_{i-b_\alpha} D_{k, \#}(\mathfrak{g}, \mathfrak{B})$,

$$\wedge^{i-b_\alpha}(p_u) \otimes_k \sum_{j=0}^{k-i-b_\alpha} S^j_\#(\mathfrak{B}) D^0_{k-i+b_\alpha-j}(p_u) \wedge b_\alpha(\mathfrak{B}).$$

Moreover, by Lemma 12.8, the quotient of $K^i_l$ by $J^i_1 K^i_l$ is equal to

$$\sum_{j=0}^{k-i-b_\alpha} S^j_\#(\mathfrak{B}) D^0_{k-j}(p_u) \wedge \mathfrak{g}.$$  

In particular, the $\mathbb{k}[\mathfrak{g} \times \mathfrak{p}]$-module $K^i / J^i_1 K^i$ is free. Again by Lemma 12.8, the three $\mathbb{k}[\mathfrak{g} \times \mathfrak{p}]$-modules,

$$\sum_{j=0}^{k-i-b_\alpha} S^j_\#(\mathfrak{B}) D^0_{k-j}(p_u) \wedge \mathfrak{g}, \quad \sum_{j=0}^{k-i-b_\alpha} S^j_\#(\mathfrak{B}) D^0_{k-j}(p_u),$$
are three free modules of the same finite rank. Hence for some \( p \) in \( \mathbb{k}[l \times p] \setminus \{0\} \), \( pK^i \cap F^i_{l-b_3+1} \) is contained \( J_1K^i \). As a result, \( K^i \cap F^i_{l-b_3+1} \) is equal to \( J_1K^i \) since the \( \mathbb{k}[l \times p] \)-module \( K^i / J_1K^i \) is torsion free.

Suppose \( l > i - b_3 \) and the assertion true for \( l - 1 \). By induction hypothesis, it remains to prove that \( J^i_{l-i+b_3+1}K^i \) is the intersection of \( J^i_{l-i+b_3}K^i \) and \( F^i_{l+1} \). By Corollary 12.7(iii), the quotient of \( J^i_{l-i+b_3}K^i \) by \( J^i_{l-i+b_3}K^i \cap F^i_{l+1} \) is equal to the subspace of \( \text{gr}_1D_{k,\mathbb{k}}(\hat{\mathbb{g}}, \hat{\mathbb{B}}) \).

Moreover, by Lemma 12.8, the quotient of \( J^i_{l-i+b_3}K^i \) by \( J^i_{l-i+b_3+1}K^i \) is equal to

\[
\bigoplus_{(l_1, l_2) \in \mathbb{N}_{l-i+b_3}} \mathbb{k}[z_1, \ldots, z_{2n+\ell}]_{I_2} \otimes \mathbb{k} S^{l_2}(p_u) \otimes \mathbb{k} \bigoplus_{j=0}^{k-i-b_3} S_j^k(\mathbb{B})D^0_{k-i-b_3-j}(p_u) \wedge \mathbb{k}^{b_3}(\mathbb{B}).
\]

In particular, the \( \mathbb{k}[l \times p] \)-module \( J^i_{l-i+b_3}K^i / J^i_{l-i+b_3+1}K^i \) is free. Again by Lemma 12.8, the three \( \mathbb{k}[l \times p] \)-modules

\[
\bigoplus_{(l_1, l_2) \in \mathbb{N}_{l-i+b_3}} \mathbb{k}[z_1, \ldots, z_{2n+\ell}]_{I_2} \otimes \mathbb{k} S^{l_2}(p_u) \otimes \mathbb{k} \bigoplus_{j=0}^{k-i-b_3} S_j^k(\mathbb{B})D^0_{k-i-b_3-j}(p_u) \wedge \mathbb{k}^{b_3}(\mathbb{B}),
\]

\[
\bigoplus_{(l_1, l_2) \in \mathbb{N}_{l-i+b_3+1}} \mathbb{k}[z_1, \ldots, z_{2n+\ell}]_{I_2} \otimes \mathbb{k} S^{l_2}(p_u) \otimes \mathbb{k} \bigoplus_{j=0}^{k-i-b_3} S_j^k(\mathbb{B})D^0_{k-i-b_3-j}(p_u),
\]

\[
\bigoplus_{(l_1, l_2) \in \mathbb{N}_{l-i+b_3}} \mathbb{k}[z_1, \ldots, z_{2n+\ell}]_{I_2} \otimes \mathbb{k} S^{l_2}(p_u) \otimes \mathbb{k} \bigoplus_{j=0}^{k-i-b_3} S_j^k(\mathbb{B})D^0_{k-i-b_3-j}(p_u) \wedge \mathbb{k},
\]

are three free modules of the same finite rank. Hence for some \( p \) in \( \mathbb{k}[l \times p] \setminus \{0\} \), \( pK^i \cap F^i_{l-i+b_3+1} \) is contained \( J^i_1K^i \). As a result, \( K^i \cap F^i_{l-b_3+1} \) is equal to \( J^i_{l-i+b_3+1}K^i \) since the \( \mathbb{k}[l \times p] \)-module \( J^i_{l-i+b_3}K^i / J^i_{l-i+b_3+1}K^i \) is torsion free.

Let \( d_1 \) and \( d_2 \) be the morphisms from \( K^i_1 \) to \( F^i_1 \) such that

\[
d_1a \otimes \omega \wedge \hat{\varepsilon} = (-1)^{i-b_3} a \otimes (d\omega) \wedge \hat{\varepsilon}, \quad d_2 a \otimes \omega \wedge \hat{\varepsilon} = a \otimes d\omega \wedge \hat{\varepsilon}
\]

with

\[
a \in \bigoplus_{(l_1, l_2) \in \mathbb{N}_{l-i+b_3}} \mathbb{k}[z_1, \ldots, z_{2n+\ell}]_{I_2} \otimes \mathbb{k} S^{l_2}(p_u),
\]

\[
\]
\[ \omega \in S^j_n(\overline{B}), \quad \nu \in D^{i-b_j}_k(p_u), \quad i = b_k, \ldots, k + b_k, \quad j = 0, \ldots, k - i + b_k. \]

**Lemma 12.10.** Let \( i = b_k, \ldots, k + b_k \) and \( l \) a nonnegative integer.

(i) For \( i \) nonnegative integer and \( j > 0 \),
\[ dK^i_j \subset d_2K^i_j + F^{i+1}_{l+1}. \]

(ii) If \( i > b_k \) then \( Z^i \cap F^i_j \) is contained in \( dF^{i-1}_{l+1} + F^i_j \).

**Proof.** (i) By definition \( d = d_1 + d_2 \). Let \( a, \omega, \nu \) be as in the above definition. Then
\[ da\omega \nu \wedge \hat{e} \in d_2a\omega \wedge \hat{e} + a\wedge S^j_i(\overline{B}_l)D^{i-b_j}_k(p_u) \wedge \hat{e} \land \overline{B}_l. \]

By Corollary 12.7,
\[ a\wedge S^j_i(\overline{B}_l)D^{i-b_j}_k(p_u) \wedge \hat{e} \land \overline{B}_l \subset F^{i+1}_{l+1} \]

since the image of \( \hat{e} \) by the restriction morphism \( r_{a,p} \) is a generator of \( \wedge^b_k(\overline{B}) \), whence the assertion.

(ii) Let \( \varphi \) be in \( Z^i \cap F^i_j \). By Lemma 12.9(i), for some \( \psi \) in \( F^{i-1}_j \),
\[ \varphi - d\psi \in K^i_j + F^i_{l+1}. \]

As a matter of fact, by Lemma 12.9(ii),
\[ \varphi - d\psi \in K^i_j + F^i_{l+1}. \]

By Lemma 12.9(iii) and Lemma 12.8, for \( i' = i, i + 1 \), the sum \( K^{i'}_l + F^{i'}_{l+1} \) is direct. Let \( \varphi_1 \) be the component of \( \varphi - d\psi \) on \( K^i_j \). Then, by (i), \( d_2\varphi_1 = 0 \) since \( d_2\varphi_1 \) is in \( K^{i+1}_{l+1} \) and \( dF^i_{l+1} \) is contained in \( F^{i+1}_{l+1} \). As a result, by Lemma 12.8, for some \( \varphi'_1 \) in \( K^{i-1}_{l+1} \), \( \varphi_1 = d_2\varphi'_1 \). Then, by (i),
\[ \psi + \varphi'_1 \in F^{i-1}_j \quad \text{and} \quad \varphi - d\psi - d\varphi'_1 \in F^i_j, \]

whence the assertion. \( \square \)

**Corollary 12.11.** Let \( i = b_k + 1, \ldots, b_k + k \).

(i) For all nonnegative integer \( l \), \( Z^i \) is contained in \( B^i + F^i_j \).

(ii) For some \( p \) in \( m \otimes_k k[1_\times g] \), \( (1 + p)Z^i \) is contained in \( B^i \).

**Proof.** (i) By Lemma 12.10(ii), for \( l \) nonnegative integer,
\[ Z^i \cap F^i_j \subset dF^{i-1}_{l+1} + F^i_{l+1}. \]

Then, by induction on \( l \), \( Z^i \) is contained in \( B^i + F^i_j \).

(ii) The natural gradation of \( k[g] \) induces a gradation of \( \hat{O} \otimes_k k[1_\times g] \otimes_k D(g) \). As \( \hat{B} \) is a graded submodule of \( \hat{O} \otimes_k k[1_\times g] \otimes_k D(g) \) so are \( D^i_{k,a}(g, \hat{B}), D^i_{k,a} F^i_{l+1}, l = 0, 1, \ldots \). Then \( Z^i \) and \( B^i \) are graded submodules of \( D^i_{k,a}(g, \hat{B}) \) since the differential of \( D^i_{k,a}(g, \hat{B}) \) is homogeneous of degree 0 with respect to this gradation.
Let \( l \) be a nonnegative integer. Denote by
\[
\hat{\mathcal{O}} \otimes_k \mathbb{k}[l_\ast \times \mathfrak{g}]^{(l)}, \quad D^{i,l}_{k,\#}(\hat{\mathfrak{g}}, \hat{\mathfrak{B}}), \quad Z^{i,l}, \quad B^{i,l}
\]
the subspaces of degree \( l \) of
\[
\hat{\mathcal{O}} \otimes_k \mathbb{k}[l_\ast \times \mathfrak{g}], \quad D^{i,l}_{k,\#}(\hat{\mathfrak{g}}, \hat{\mathfrak{B}}), \quad Z^i, \quad B^i
\]
respectively. In particular, these spaces are finitely generated \( \hat{\mathcal{O}} \otimes_k \mathbb{k}[l_\ast] \)-modules. Then, by [MA86, Ch. 3, Theorem 8.9], for some \( p_j \) in \( \mathfrak{m} \otimes_k \mathbb{k}[l_\ast] \),
\[
(1 + p_j) \bigcap_{j \in \mathbb{N}} (B^{i,j} + \mathfrak{m}^{j,l} \otimes_k D^{i,l}_{k,\#}(\hat{\mathfrak{g}}, \hat{\mathfrak{B}})) \subset B^{i,l}.
\]
By (i),
\[
Z^{i,l} \subset B^{i,l} + \mathfrak{m}^{i,l} \otimes_k D^{i,l}_{k,\#}(\hat{\mathfrak{g}}, \hat{\mathfrak{B}})
\]
for all integer \( j \) bigger than \( l \) since \( F^i_s \) is graded for all nonnegative integer \( s \) and \( J \) is generated by elements of positive degree. As a result,
\[
(1 + p_j)Z^{i,l} \subset B^{i,l}.
\]
Then, for some \( p \) in \( \mathfrak{m} \otimes_k \mathbb{k}[l_\ast] \),
\[
(1 + p)Z^i \subset B^i
\]
since \( Z^i \) is a finitely generated module over \( \hat{\mathcal{O}} \otimes_k \mathbb{k}[l_\ast \times \mathfrak{g}] \).

\[\square\]

**Proposition 12.12.** For \( k = 0, \ldots, n \), \( D^{\ast}_{k,\#}(\hat{\mathfrak{g}}, \hat{\mathfrak{B}}) \) and \( D^{\ast}_{k,\#}(\hat{\mathfrak{g}}, \hat{\mathfrak{B}}) \) have no cohomology of degree different from \( b_0 \).

**Proof.** As \( \hat{\mathcal{O}} \) is a faithfully flat extension of \( \mathcal{O} \), it is sufficient to prove that \( D^{\ast}_{k,\#}(\hat{\mathfrak{g}}, \hat{\mathfrak{B}}) \) has no cohomology of degree different from \( b_0 \). For \( i < b_0 \) or \( i > k + b_0 \), \( D^{I}_{k,\#}(\hat{\mathfrak{g}}, \hat{\mathfrak{B}}) = [0] \). By definition, \( D^{\ast}_{k,\#}(\hat{\mathfrak{g}}, \hat{\mathfrak{B}}) \) has no cohomology of degree \( k + b_0 \). So, it is true for \( k = 0, 1 \). Let \( k = 2, \ldots, n \) and \( i = b_0 + 1, \ldots, k - 1 + b_0 \). Denote by \( T_i \) the support of \( Z^i/B^i \) in Spec(\( \hat{\mathcal{O}} \otimes_k \mathbb{k}[l_\ast \times \mathfrak{g}] \)). As \( Z^i/B^i \) is finitely generated, \( T_i \) is a closed subset of Spec(\( \hat{\mathcal{O}} \otimes_k \mathbb{k}[l_\ast \times \mathfrak{g}] \)). Since \( \mathfrak{m} \) is contained in all maximal ideal of \( \hat{\mathcal{O}} \otimes_k \mathbb{k}[l_\ast \times \mathfrak{g}] \), \( T_i \) does not contain a maximal ideal by Corollary 12.11. Then \( T_i \) is empty and \( Z^i = B^i \), whence the proposition. \[\square\]

13. Induction. Case \( n \geq 2 \)

Let \( \mathfrak{p} \) be a parabolic subalgebra of \( \mathfrak{g} \) containing \( \mathfrak{b} \). Then we use the notations of Section 9 and some notations of Section 12. In particular, \( \mathfrak{l} \) is the reductive factor of \( \mathfrak{p} \) containing \( \mathfrak{b} \), \( \mathfrak{d} \) is its derived algebra, \( n \) is the number of simple factors of \( \mathfrak{d} \). We suppose that \( n \) is bigger than 1 and the simple factors \( \mathfrak{d}_1, \ldots, \mathfrak{d}_n \) of \( \mathfrak{d} \) have Property (P). For \( i = 1, \ldots, n \), let \( E'_i \) be the submodule of \( \mathbb{k}[l_\ast \times \mathfrak{l}] \otimes_k \mathfrak{l} \) generated by the elements \( \lambda_{i,1}, \ldots, \lambda_{i,n} \), defined in Subsection 9.3. Denote by \( E \) the
$\mathbb{k}[l_1 \times l]$-submodule of $\mathbb{k}[l_1 \times l] \otimes \mathfrak{g}$ generated by $p_n$ and $E_1', \ldots, E_n'$. Let $\bar{J}_+$ be the ideal of the algebra $\mathbb{k}[l_1 \times g] \otimes_{\mathbb{k}[l_1 \times l]} (E')$ generated by $J \otimes 1$ and $1 \otimes E$. Denote by $\bar{J}_+$ and $\bar{J}_-$ the ideals of $\mathbb{k} \otimes \mathbb{k}[l_1 \times g] \otimes_{\mathbb{k}[l_1 \times l]} (E)$ generated by $m \otimes 1$ and $\bar{J}_+$. Let $J_1$ be the ideal of $\mathbb{k} \otimes \mathbb{k}[l_1 \times g]$ generated by $m$ and $J$.

13.1. **Quasi-regularity.** Let $I$ be as in Subsection 6.2. According to notations of Subsection 6.2 and Section 4, for $I$ subset of $I_k$ and $(j_1, j_2)$ in $\mathbb{N}^2$ such that $j_1 + j_2 \leq k$, let $I_{j_1, j_2}$ be the subset of elements $i$ of $\mathbb{N}^n_{k-j_1-j_2}$ such that $(j_1, j_2, i)$ is in $I$. Then, for $I$ subset of $I_k$ and $V = V_-$ or $g$, denote by $D_{I, I, \#}(V)$ the graded subcomplex of $D_k^*(V)$ deduced from the triple complex

$$\bigoplus_{i=0}^{k} \bigoplus_{(j_1, j_2) \in \mathbb{N}^2} D^*_j(V') \otimes_{\mathbb{k}} D^*_j(\mathbb{R}) \otimes_{\mathbb{k}} D_{I, I, j_1, j_2, \#}(V),$$

with $V' = V_- \wedge V$ when $V = V_-$ and $V' = V_+ \wedge V$ when $V = g$. For $j = (j_1, \ldots, j_n)$ in $\mathbb{N}^n_k$, set:

$$\mathbb{N}^j := \{(i_1, \ldots, i_n) \in \mathbb{I} \mid i_1 \leq n_1 - j_1, \ldots, i_n \leq n_n - j_n\},$$

$$D^*_{I, I, j}(V) := D_{I-\mathbb{N}^j \otimes \mathbb{R}^j}(V)[-j] \wedge \mathbb{i}^j(E'_1) \wedge \cdots \wedge \mathbb{i}^j(E'_n),$$

$$D^*_{I, I, j, \#}(V) := \mathbb{i}^j(E'_1) \wedge \cdots \wedge \mathbb{i}^j(E'_n) \otimes_{\mathbb{k}} D^*_{I-\mathbb{N}^j \otimes \mathbb{R}^j}(V)[-j].$$

Then $D^*_{I, I, j}(V)$ is a graded subcomplex of $\mathbb{k}[l_1 \times l'] \otimes_{\mathbb{k}} D^*_{I, I, j}(V)$.

For $k = 0, \ldots, n$ and $j$ non negative integer, denote by $D^*_k(V)$ the graded submodule of $\mathbb{k}[l_1 \times l] \otimes D_{I, I, \#}(V)$,

$$D^*_k(V) := \bigoplus_{(j_1, \ldots, j_n) \in \mathbb{N}^n_k} D^*_k(j_1, \ldots, j_n)(V),$$

and $D^*_k(j_1, \ldots, j_n)$ the graded submodule of $\mathbb{k}[l_1 \times l] \otimes \mathbb{i}^j(E) \otimes D^*_k(j_1, \ldots, j_n)$,

$$D^*_k(j_1, \ldots, j_n) := \bigoplus_{i=0}^{j} \bigoplus_{(j_1, \ldots, j_n) \in \mathbb{N}^n_k} \wedge \mathbb{i}^j(p_n) \wedge D^*_k-i, i(j_1, \ldots, j_n)(\mathfrak{g}).$$

Let $k = 0, \ldots, n$. Set:

$$\mathbb{N}_p := \mathbb{N}^{2n+\ell} \times \mathbb{N}^{\ell} \times \mathfrak{g} \times \mathfrak{g}, \quad |(s, r, u, t)| := |s| + |r| + |u| + |t|,$$

for $(s, r, u, t)$ in $\mathbb{N}_{p, \#}$. For $j = 0, \ldots, n$, set:

$$\mathcal{M}_{j, s} := \{ \varphi \in (\mathbb{k}[l_1 \times g] \otimes \mathbb{k} D(\mathfrak{g}))^{\mathbb{N}_p} \mid \exists N_{\varphi} \in \mathbb{N} \text{ such that } |r| \geq N_{\varphi} \implies \varphi_{s, r, v, t} = 0 \}$$

and

$$\varphi_{s, r, v, t} \wedge \lambda_u \in \mathbb{k}[l_1 \times l] \otimes \mathbb{k} D_{j-I, j, \#}(V),$$

$$\kappa(\varphi) := \sum_{(s, r, u, v, t) \in \mathbb{N}^{2n+\ell}_{p, \#}} z^i v^j \varphi_{s, r, v, t} \wedge \lambda_u \wedge v \wedge \hat{e}.$$
for $\varphi$ in $M_{j,\ast}$. For $l$ nonnegative integer, denote by $M_{j,\ast,l}$ the subspace of elements $\varphi$ of $M_{j,\ast}$ such that

$$|(s, r, u, i)| \neq l \implies \varphi_{s,r,u,i} = 0$$

and $M_{j,\ast,l+}$ the sum of $M_{j,\ast,i}$, $i = l, l + 1, \ldots$

Moreover, in this case $\varphi$ and arguing as in the proof of Lemma 74.

Recall that $\overline{B}_{0,0}$ is the submodule of $\overline{B}$ generated by $\varepsilon_{1}^{(0)}, \ldots, \varepsilon_{\ell}^{(0)}$.

**Lemma 13.1.** For $\varphi$ in $\mathbb{k}[l, x] \otimes_{\mathbb{k}} D_{k,\#}(g)$, the restriction of $\varphi \wedge \varepsilon$ to $l \times p$ is equal to 0 if and only if

$$\varphi \in \mathbb{k}[l, x] \otimes_{\mathbb{k}} D_{k,\#}(g) + D_{k-1,\#}(g) \wedge \overline{B}_{0,0} + JD_{k,\#}(g) + \mathbb{k}[l, x] \otimes_{\mathbb{k}} D_{k-1,\#}(g) \wedge p_u.$$ 

Moreover, in this case $\varphi \wedge \varepsilon$ is in $\mathbb{k}(M_{k,\ast,1,+})$.

**Proof.** Since $\mathbb{k}[l, x] = \mathbb{k}[l, x] + J$, we can suppose that $\varphi$ is in $\mathbb{k}[l, x] \otimes_{\mathbb{k}} D_{k,\#}(g)$. As $g$ is the direct sum of $p_\ast$ and $p_u$,

$$\wedge(g) = \wedge(p_\ast) \oplus \wedge(g) \wedge p_u \quad \text{and} \quad D_{k,\#}(g) = \bigoplus_{m=0}^{k} S^m(p_u) \otimes_{\mathbb{k}} D_{k-m,\#}(p_\ast) \oplus D_{k-1,\#}(g) \wedge p_u.$$ 

Then $\varphi = \varphi_1 + \varphi_2$ with

$$\varphi_1 \in \mathbb{k}[l, x] \otimes_{\mathbb{k}} \bigoplus_{m=0}^{k} S^m(p_u) \otimes_{\mathbb{k}} D_{k-m,\#}(p_\ast) \quad \text{and} \quad \varphi_2 \in \mathbb{k}[l, x] \otimes_{\mathbb{k}} D_{k-1,\#}(g) \wedge p_u.$$ 

By Lemma 12.4(ii) and Remark 12.5, the restriction of $\varphi_2 \wedge \varepsilon$ to $l \times p$ is equal to 0 and the condition is sufficient.

Conversely, suppose that the restriction of $\varphi \wedge \varepsilon$ to $l \times p$ is equal to 0. Since the condition is sufficient, the restrictions of $\varphi_2 \wedge \varepsilon$ and $\varphi_1 \wedge \varepsilon$ to $l \times p$ are equal to 0. By Proposition 4.4, setting:

$$\mathbb{N}_{n,\ast}^{n} := \{(j_1, \ldots, j_n) \in \mathbb{N}_n \mid j_1 \leq n_1 - 1, \ldots, j_n \leq n_n - 1\},$$

and arguing as in the proof of Lemma 12.4(ii),

$$\varphi_1 = \varphi_1' + \sum_{j=1}^{\ell} \varphi_{1,j} \wedge \varepsilon_j^{(0)} \quad \text{with} \quad \varphi_1' \in \mathbb{k}[p_u] \otimes_{\mathbb{k}} D_{k,\#}(p_\ast) \quad \text{and}$$

$$\varphi_{1,j} \in \bigoplus_{i=0}^{k} \mathbb{k}[l, x] \otimes_{\mathbb{k}} D_{k-1,\ast}(p_\ast)$$

for $j = 1, \ldots, \ell$ since $\varepsilon_j(x)$ has a nonzero component on $\mathcal{D}_i$ for $i = 1, \ldots, n$ for all $x$ in a dense open subset of $l$, by Lemma 8.1(i). As a result,

$$\varphi \in D_{k-1,\#}(g) \wedge \overline{B} + J \otimes_{\mathbb{k}} D_{k,\#}(g) + \mathbb{k}[l, x] \otimes_{\mathbb{k}[l, x]} D_{k,\#}(g) + m \otimes_{\mathbb{k}} \mathbb{k}[l, x] \otimes_{\mathbb{k}} D_{k,\#}(g)$$

since for $j = 1, \ldots, \ell$, the map

$$G \times l \times g \longrightarrow g, \quad (g, x) \mapsto \varepsilon_j(g(x)) - \varepsilon_j(x)$$
is in $m \otimes_k k[l_i] \otimes_k g$, whence the lemma.

**Proposition 13.2.** Let $l$ be a positive integer and $\varphi$ in $M_{k,*,l}$. Then $\kappa(\varphi)$ is in $\kappa(M_{k,*,l+1,+})$ if and only if $\varphi_{s,r,u,i} \wedge \hat{e}$ is in $\kappa(M_{k,|s|-|l|,|r|,|u|,|i|,1,+})$ for all $(s, r, u, i)$. Moreover, in this case,

$$\varphi_{s,r,u,i} \wedge \lambda_u \wedge \hat{e} \in \kappa(M_{k,|s|-|l|,|r|,|u|,|i|,1,+})$$

for all $(s, r, u, i)$.

**Proof.** The condition is clearly sufficient. Suppose that $\kappa(\varphi)$ is in $\kappa(M_{k,*,l+1,+})$. As $g$ is the direct sum of $p_-$ and $p_+$ and $k[l \times g]$ is the direct sum of $k[l \times p]$ and $J$,

$$\varphi_{s,r,u,i} = \varphi'_{s,r,u,i} + \varphi''_{s,r,u,i}$$

with $\varphi'_{s,r,u,i} \wedge \lambda_u \in k[l \times p] \otimes_k D_{k,-|l|,|b|}(p')$ and

$$\varphi''_{s,r,u,i} \wedge \lambda_u \in J \otimes_k D_{k,-|l|,|b|}(g) + k[l \times g] \otimes_k D_{k,-|l|,|b|}(g) \wedge p_u$$

for all $(s, r, u, i)$. Setting $\varphi' := (\varphi'_{s,r,u,i}, (s, r, u, i) \in \mathbb{N}_p)$, $\varphi'$ is in $M_{k,*,l}$ and $\kappa(\varphi')$ is in $\kappa(M_{k,*,l+1,+})$ since the condition is sufficient.

For $(s, r, i)$ in $\mathbb{N}_p$, set:

$$\psi_{s,r,i} := \sum_{i \in I'} \varphi'_{s,r,u,i} \wedge \lambda_u$$

and $\psi := (\psi_{s,r,i}, (s, r, i) \in \mathbb{N}_p)$.

Then

$$l < |(s, r, i)| \implies \psi_{s,r,i} = 0 \quad \text{and} \quad \kappa(\psi) \wedge \hat{e} \in \kappa(M_{k,*,l+1,+}).$$

As a result, by Proposition 10.6, $\psi_{s,r,i}$ is in $M_1$ for all $(s, r, i)$. Then by Proposition 11.2, the restrictions of $\varphi'_{s,r,u,i} \wedge \varepsilon$ and $\varphi_{s,r,u,i} \wedge \varepsilon$ to $l \times p$ are equal to 0 since $e(x, y)$ and $g(x, y) \wedge \mu_+$ are colinear for all $(x, y)$ in $l \times p$ by Corollary 2.7. So, by Lemma 13.1(ii),

$$\varphi_{s,r,u,i} \wedge \hat{e} \in \kappa(M_{k,|s|-|l|,|r|,|u|,|i|,1,+})$$

and $\varphi_{s,r,u,i} \wedge \lambda_u \wedge \hat{e} \in \kappa(M_{k,|s|-|l|,|r|,|u|,|i|,1,+})$

for all $(s, r, u, i)$.

For $l = 0, 1, \ldots$, let $F_l$ be the subspace $\kappa(M_{k,*,l,+})$ of $D_{k,\#}(g, \overline{B})$. Then the sequence $F_l$, $l = 0, 1, \ldots$ is a decreasing filtration of $D_{k,\#}(g, \overline{B})$. Denote by $\gr D_{k,\#}(g, \overline{B})$ the associate graded space to this filtration and $\gr_{l} D_{k,\#}(g, \overline{B})$ the subspace of degree $l$ of $\gr D_{k,\#}(g, \overline{B})$. For $j = 0, \ldots, k$ and $i = 0, \ldots, j$, let $D^*_{j,\#,(g, \overline{B})}$ be the graded subcomplex of $k[l \times p] \otimes_{k[l \times l]} (E) \otimes_k D^*(g)$,

$$D^*_{j,\#,(g, \overline{B})} := D^*_{j,\#,(g) [-b_\varphi]} \wedge b_\varphi \overline{B},$$

and $A$ the algebra

$$A := k[z_1, \ldots, z_{2n+\ell}] \otimes_k S(p_\varphi) \otimes_k \wedge (E).$$

This algebra has a bigradation $A^*_{ij}$ such that

$$A^' := k[z_1, \ldots, z_{2n+\ell}] \otimes_k S(p_\varphi) \otimes_k \wedge (E) \quad \text{and}$$

$$A^i := \bigoplus_{(j_1, j_2, j_3) \in \mathbb{N}^3} \mathbb{R}[z_1, \ldots, z_{2n+\ell}]_{j_1} \otimes \mathbb{R}[S(p_\varphi)] \otimes_k \wedge_{j_3}(E).$$
Consider on $\mathbb{k}[l \times p] \otimes_{\mathbb{k}[l \times l]} A \otimes_{\mathbb{k}} D(\mathfrak{g})$ the simple gradation deduced from the double gradation $\mathbb{k}[l \times p] \otimes_{\mathbb{k}[l \times l]} A^* \otimes_{\mathbb{k}} D^*(\mathfrak{g})$. The trivial structure of complex on $A$ and the structure of complex on $D(\mathfrak{g})$ induce a structure of graded complex on $\mathbb{k}[l \times p] \otimes_{\mathbb{k}[l \times l]} A \otimes_{\mathbb{k}} D(\mathfrak{g})$. For $l = 0, 1, \ldots$ and $i$ nonnegative integer, set:

$$C^i_l = \bigoplus_{j=0}^i \bigoplus_{(l_1, l_2) \in \mathbb{Z}^2_{-j}} \mathbb{k}[z_1, \ldots, z_{2n+l}]_{l_1} \otimes_{\mathbb{k}} S^J_z(p_n) \otimes_{\mathbb{k}} D^{i-j}_{k,\#;j,\#}(g, \widehat{B}).$$

Denoting by $C^i_l$ the sum $C^i_l$, $i = 0, 1, \ldots$, $C^i_l$ is a graded subcomplex of $\mathbb{k}[l \times p] \otimes_{\mathbb{k}[l \times l]} A \otimes_{\mathbb{k}} D(\mathfrak{g})$.

**Corollary 13.3.** (i) For $l = 0, 1, \ldots$, $F^* _l$ is a graded subcomplex of $D^* _{k,\#}(g, \widehat{B})$.

(ii) The graded complex $\text{gr}_l D^* _{k,\#}(g, \widehat{B})$ is isomorphic to the graded complex $D^* _{k,\#}(g, \widehat{B})$.

(iii) For $l = 1, 2, \ldots$, the graded complex $\text{gr}_l D^* _{k,\#}(g, \widehat{B})$ is isomorphic to $C^i_l$.

**Proof.** (i) The proof is the same as the proof of Corollary 12.7(i).

(ii) The proof is the same as the proof of Corollary 12.7(ii) and results from Proposition 13.2.

(iii) Let $l$ be a nonnegative integer. Denote by $\gamma_l$ the morphism of $\mathbb{k}[l \times p]$-modules

$$\mathbb{k}[l \times p] \otimes_{\mathbb{k}} A_I \otimes_{\mathbb{k}} D(\mathfrak{g}) \xrightarrow{\gamma_l} \mathbb{k}[l \times p] \otimes_{\mathbb{k}} D(\mathfrak{g})$$

and $M_{k,l}$ the subspace of elements $\varphi$ of $M_{k,l}$ such that

$$\varphi_{s,r,u,t} \in \gamma_l(\mathbb{k}[l \times p] \otimes_{\mathbb{k}} A_I \otimes_{\mathbb{k}} D(\mathfrak{g})).$$

Then $F_l = \kappa(M_{k,l}') + F_{l+1}$ since

$$j^l = \mathbb{k}[l \times p] \otimes_{\mathbb{k}} A_I \otimes j^{l+1}.$$

By Proposition 13.2, for $\varphi$ in $M_{k,l}'$, $\kappa(\varphi)$ is in $F_{l+1}$ if and only if $r_{g,p}(\varphi_{s,r,u,t} \wedge \hat{\varepsilon}) = 0$ for all $(s, r, u, t)$. Then, by (ii), the restriction of $r_{g,p}$ to $F_l$ defines through the quotient an isomorphism of $\mathbb{k}[l \times p]$-modules,

$$\text{gr}_l D_{k,\#}(g, \widehat{B}) \cong C_l.$$

Moreover, this isomorphism is an isomorphism of graded complex. \hfill $\Box$

13.2. **Annulation of cohomology.** Let $k = 1, \ldots, n$. For $i$ integer, denote by $Z^i$ and $B^i$ the spaces of cocycles and coboundaries of degree $i$ of $D^* _{k,\#}(g, \widehat{B})$. For $j = 1, \ldots, n$ and $i$ in $\mathbb{Z}^n$, denote by $D^* _{j,\#}(E')$ the graded the submodule of $\mathbb{k}[l \times l] \otimes_{\mathbb{k}} D^*_j(l)$ deduced from the multigraded module

$$\bigoplus_{j \in \mathbb{Z}^n} D^* _{j,\#}(E') \wedge \cdots \wedge D^* _{j,\#}(E').$$

For $l = 1, \ldots, n$, let $D^* _{l,\#}(E)$ be the graded submodule of $D^*_l(E)$ deduced from the bigraded module

$$\bigoplus_{j \in \mathbb{Z}^n} D_{l-j}(p_n) \otimes_{\mathbb{k}} D^* _{j,\#}(E').$$
Set: 
\[ \tilde{D}^*_l := D^*_l(E)[-b_0] \quad \text{and} \quad D^*_l := D^*_l(E)[-b_2] \quad \text{and} \quad D^*_l := D^*_l(E)[-b_3] \] 
Then \( D^*_l \) is a graded subcomplex of \( D^*_l(E) \).

**Lemma 13.4.** Let \( l = 1, \ldots, n \).

(i) The morphisms

\[ \kappa[I \times g] \otimes_{\varphi[I \times t]} D^*_l(E)[-b_0] \xrightarrow{\theta_l} D^*_l(E) \quad \varphi \mapsto \varphi \wedge \varepsilon \] 

\[ \hat{\theta} \otimes_{\kappa[I \times g]} D^*_l(E)[-b_0] \xrightarrow{\hat{\theta}_l} D^*_l(E) \quad \varphi \mapsto \varphi \wedge \hat{\varepsilon} \] 

are injective.

(ii) The complexes \( D^*_l(E), \tilde{D}^*_l, D^*_l \) are acyclic.

**Proof.** (i) For \( \psi \) in \( M_{l,s} \), set:

\[ \kappa(\psi) := \sum_{(s,r,v,t) \in N_{n+1}} z^s \psi_{s,r,u,v} \wedge \lambda_v \wedge v_t. \]

Let \( \mathcal{E} \) be the subset of elements \( \psi \) of \( M_{l,s} \) such that

\[ s \neq 0 \implies \psi_{s,r,u,v} = 0 \quad \text{and} \quad \psi_{s,r,u,v} \in \kappa[I \times g] \otimes_{\varphi[I \times t]} S^l(E). \]

For \( m \) nonnegative integer, denote by \( \mathcal{E}_m \) the subspace of elements \( \psi \) of \( E \) such that

\[ |(s,r,u,v)| < m \implies \psi_{0,r,u,v} = 0. \]

For \( \varphi \) in \( \kappa[I \times g] \otimes_{\varphi[I \times t]} D^*_l(E), \varphi = \kappa(\psi) \) for some \( \psi \) in \( \mathcal{E} \). Denote by \( \lambda(\varphi) \) the biggest integer \( m \) such that \( \varphi = \kappa(\psi) \) for some \( \psi \) in \( \mathcal{E}_m \setminus \mathcal{E}_{m+1} \).

Suppose that \( \theta_l \) is not injective. A contradiction is expected. Let \( \varphi \) be in its kernel such that \( \lambda(\varphi) \) is minimal. For some \( \psi \) in \( \mathcal{E}_{\lambda(\varphi)}, \varphi = \kappa(\psi) \). By Corollary 11.3, for \( (r,v,t) \) such that \(|(s,r,u,v)| = \lambda(\varphi), \psi_{0,r,u,v} \in JS(E), \)

since \( S(E) \) is a free module, whence the contradiction since \( \varphi \) is not in \( \mathcal{E}_{\lambda(\varphi)+1} \).

Suppose that \( \hat{\theta}_l \) is not injective. A contradiction is expected. Let \( \varphi \) be a nonzero element of degree \( i \) of its kernel. The element \( \varphi \) has an expansion

\[ \varphi = \sum_{s \in \mathbb{N}^{2n+l}} z^s \varphi_s \quad \text{with} \quad \varphi_s \in \kappa[I \times g] \otimes_{\varphi[I \times t]} S^{i-1}(E) \otimes_{\varphi[I \times t]} \wedge^i(E) \]

for all \( s \). Denote by \( \sigma \) the smallest integer such that \( \varphi_s \neq 0 \) for some \( s \) in \( \mathbb{N}^{2n+l} \). By minimality of \( \sigma, \varphi_s \wedge \varepsilon = 0 \) for all \( s \) in \( \mathbb{N}^{2n+l} \) and \( \varepsilon = 0 \) is in \( \varphi[I \times g] \otimes_{\varphi[I \times t]} \kappa[I \times g] \) as already observed. Then, by the injectivity of \( \theta_l \), \( \varphi_s = 0 \) for all \( s \) in \( \mathbb{N}^{2n+l} \), whence the contradiction.

(ii) As \( E \) is a free module, \( D^*_l(E) \) is an acyclic complex by Lemma A.2(ii) since \( l \) is positive. Then \( D^*_l(E) \) is acyclic since \( D^*_l(E) \) is a direct factor of \( D^*_l(E) \). By (i), the restriction of \( \theta_l \) to
\[ \mathbb{k}[l \times g] \otimes_{\mathbb{k}[l \times l]} D^\bullet_{l,a}(E)[-b_g] \] is an isomorphism of graded complexes onto \( \tilde{D}^\bullet_{l,a} \). Hence \( \tilde{D}^\bullet_{l,a} \) is acyclic. Again by (i), the restriction of \( \hat{\theta}_l \) to \( \hat{\theta}_l \otimes_{\mathbb{k}[l \times g]} \mathbb{k}[l \times l] D^\bullet_{l,a}(E)[-b_g] \) is an isomorphism of graded complexes onto \( D^\bullet_{l,a} \). Hence \( D^\bullet_{l,a} \) is acyclic. \( \square \)

For \( j = 0, \ldots, k \), denote by \( D^\bullet_{k,j} \) the intersection of \( \mathbb{k}[l \times l] \otimes_{\mathbb{k}} D^\bullet_{k,\#}(g) \) and \( S^l(\overline{B},0)D^\bullet_{k-j}(E) \). For \( i = b_g, \ldots, k + b_g \) and \( l \) nonnegative integer, set:

\[ K^i := \bigoplus_{j=0}^{k-i+b_g} D_{k,j}^{i-b_g} \wedge \wedge^j (\overline{B}) \quad \text{and} \quad K^i_1 = \bigoplus_{(l_1,l_2) \in H^2_{l-i+b_0}} \mathbb{k}[z_1, \ldots, z_{2n+l}]_{l_1} \otimes_{\mathbb{k}} S^{[2]} (p_u) \otimes_{\mathbb{k}} \sum_{j=0}^{k-i+b_g} D_{k,j}^{i-b_g} \wedge \overline{\mathcal{E}}. \]

Denote by \( D^0_{k-j-b_g,\# \times} \) the image of \( D^i_{k-b_g} \wedge \overline{\mathcal{E}} \) by the quotient morphism \( F_{i-b_g} \longrightarrow \text{gr}_{i-b_g} D_{k,\#}(g, \overline{B}) \).

**Lemma 13.5.** Let \( i = b_g + 1, \ldots, k + b_g \) and \( l \) a nonnegative integer.

(i) The subspace \( Z^i \cap F^i \) of \( F^i \) is contained in \( dF^{i-1} + K^i + F^i_{i+1} \).

(ii) The space \( K^i \cap F^i \) is contained in \( K^i + F^i_{i+1} \).

(iii) The intersection of \( F^{i-1} \) and \( K^i \) is equal to \( J^1_{l-i+b_g+1} K^i \).

**Proof.** (i) Let \( \varphi \) be in \( Z^i \cap F^i \) and \( \overline{\varphi} \) its image in \( \text{gr}_l D^\bullet_{k,\#}(g, \overline{B}) \). By Corollary 13.3, \( \overline{\varphi} \) is a cocycle of degree \( i \) of the graded complex \( C^\bullet_{l} \). By Corollary 12.2, for \( j = 0, \ldots, k \), the complex \( D^\bullet_{k-j,\#}(g, \overline{B}) \) has no cohomology of degree different from \( b_g \). Then, for some \( \overline{\psi} \) in \( C^\bullet_{l} \),

\[ \overline{\varphi} - d\overline{\psi} \in \bigoplus_{(l_1,l_2) \in H^2_{l-i+b_0}} \mathbb{k}[z_1, \ldots, z_{2n+l}]_{l_1} \otimes_{\mathbb{k}} S^{[2]} (p_u) \otimes_{\mathbb{k}} D^b_{k-j-b_0,\# \times}(g, \overline{B}). \]

Then, by Lemma 13.2(ii),

\[ \overline{\varphi} - d\overline{\psi} \in \bigoplus_{(l_1,l_2) \in H^2_{l-i+b_0}} \mathbb{k}[z_1, \ldots, z_{2n+l}]_{l_1} \otimes_{\mathbb{k}} S^{[2]} (p_u) \otimes_{\mathbb{k}} \sum_{j=0}^{k-i+b_g} D_{k-j-b_g,\# \times}(g, \overline{B}). \]

So, for a representative \( \psi \) of \( \overline{\psi} \) in \( F^{i-1}_{l} \),

\[ \varphi - d\psi \in K^i + F^i_{i+1}. \]

(ii) The proof is analogous to the proof of Lemma 12.9(ii).

(iii) By definition \( J^1_{l-i+b_g+1} K^i \) is contained \( F^i_{i+1} \cap K^i \). Prove the assertion by induction on \( l \). By Corollary 13.3(iii), the quotient of \( K^i \) by \( K^i \cap F^i_{i+1} \) is equal to the subspace of \( \text{gr}_{i-b_g} D^\bullet_{k,\#}(g, \overline{B}) \),

\[ \sum_{j=0}^{k-i+b_g} D^0_{k-j-b_g,\# \times}. \]
Moreover, by Lemma 13.4(i), the quotient of $K^i$ by $J_1K^i$ is equal to
\[ \sum_{j=0}^{k-i-b_0} D_{i,j,\#}^{k-i-b_0} \wedge \hat{e}. \]

In particular, the $\mathbb{k}[l, \times p]$-module $K^i/J_1K^i$ is free. Again by Lemma 13.4(i), the three $\mathbb{k}[l, \times p]$-modules,
\[ \sum_{j=0}^{k-i+b_0} D_{i,j,\#}^{k-i-b_0} \wedge \hat{e}, \quad \sum_{j=0}^{k-i+b_0} D_{i,j,\#}^{k-i-b_0}, \quad \sum_{j=0}^{k-i+b_0} D_{k-j,\#}^{k-i-b_0,\#}, \]
are three free modules of the same finite rank. Hence for some $p$ in $\mathbb{k}[l, \times p] \setminus \{0\}$, $pK^i \cap F_{i-b_0+1}^i$ is contained $J_1K^i$. As a result, $K^i \cap F_{i-b_0+1}^i$ is equal to $J_1K^i$ since the $\mathbb{k}[l, \times p]$-module $K^i/J_1K^i$ is torsion free.

Suppose $l > i - b_0$ and the assertion true for $l - 1$. By induction hypothesis, it remains to prove that $J_1^{l-i+b_0} K^i$ is the intersection of $J_1^{l-i+b_0} K^i$ and $F_{l-1}^i$. By Corollary 13.3(iii), the quotient of $J_1^{l-i+b_0} K^i$ by $J_1^{l-i+b_0} K^i \cap F_{l-1}^i$ is equal to the subspace of $\text{gr}_1D_{k,\#}^j(\mathfrak{g}, \mathfrak{B})$,
\[ \bigoplus_{(l_1, l_2) \in \mathbb{N}^2_{l-i+b_0}} \mathbb{k}[z_1, \ldots, z_{2n+l}]_l \otimes_{\mathbb{k}} S^{l_2}(p_u) \otimes_{\mathbb{k}} \sum_{j=0}^{k-i+b_0} D_{k,j,\#}^{k-i-b_0,\#}. \]

Moreover, by Lemma 12.8, the quotient of $J_1^{l-i+b_0} K^i$ by $J_1^{l-i+b_0+1} K^i$ is equal to
\[ \bigoplus_{(l_1, l_2) \in \mathbb{N}^2_{l-i+b_0}} \mathbb{k}[z_1, \ldots, z_{2n+l}]_l \otimes_{\mathbb{k}} S^{l_2}(p_u) \otimes_{\mathbb{k}} \sum_{j=0}^{k-i+b_0} D_{k,j,\#}^{k-i-b_0} \wedge \hat{e}. \]

In particular, the $\mathbb{k}[l, \times p]$-module $J_1^{l-i+b_0} K^i/J_1^{l-i+b_0+1} K^i$ is free. Again by Lemma 12.8, the three $\mathbb{k}[l, \times p]$-modules
\[ \bigoplus_{(l_1, l_2) \in \mathbb{N}^2_{l-i+b_0}} \mathbb{k}[z_1, \ldots, z_{2n+l}]_l \otimes_{\mathbb{k}} S^{l_2}(p_u) \otimes_{\mathbb{k}} \sum_{j=0}^{k-i+b_0} D_{k,j,\#}^{k-i-b_0,\#}, \]
\[ \bigoplus_{(l_1, l_2) \in \mathbb{N}^2_{l-i+b_0}} \mathbb{k}[z_1, \ldots, z_{2n+l}]_l \otimes_{\mathbb{k}} S^{l_2}(p_u) \otimes_{\mathbb{k}} \sum_{j=0}^{k-i+b_0} D_{k,j,\#}^{k-i-b_0}, \]
\[ \bigoplus_{(l_1, l_2) \in \mathbb{N}^2_{l-i+b_0}} \mathbb{k}[z_1, \ldots, z_{2n+l}]_l \otimes_{\mathbb{k}} S^{l_2}(p_u) \otimes_{\mathbb{k}} \sum_{j=0}^{k-i+b_0} D_{k,j,\#}^{k-i-b_0} \wedge \hat{e}, \]
are three free modules of the same finite rank. Hence for some \( p \) in \( \mathbb{k}[l, \times p] \setminus \{0\} \), \( pK_i^j \cap F_{i+j-b_j+1}^i \) is contained \( J_1^{l-i+b_j+1} K_i^j \). As a result, \( K_i^j \cap F_{i+j-b_j+1}^i \) is equal to \( J_1^{l-i+b_j+1} K_i^j \) since the \( \mathbb{k}[l, \times p] \)-module \( J_1^{l-i+b_j+1} \mathbb{K}/J_1^{l-i+b_j} K_i^j \) is torsion free.

Let \( d_1 \) and \( d_2 \) be the morphisms from \( K_i^j \) to \( F_{i+j}^{i+j} \) such that

\[
d_1 a \otimes \omega \wedge \hat{\omega} = (-1)^{l-b_j} a \otimes (d\omega) \wedge \hat{\omega}, \quad d_2 a \otimes \omega \wedge \hat{\omega} = a \otimes d\omega \wedge \hat{\omega}
\]

with

\[
a \in \bigoplus_{(l_1, l_2) \in \mathbb{B}_l^{i+j-b_j}} \mathbb{k}[z_1, \ldots, z_{2n+\ell}]_{l_2} \otimes_{\mathbb{k}} S_{l_1}^{i+j-b_j} (p_0),
\]

\[
\omega \in S_l^j (\overline{B}_{0,0}), \quad \nu \in D_{k-j}^l (E), \quad a \otimes \nu \in D_{k-j}^l (E) \quad \text{for all nonnegative integer } i, j, k, b, b_g, b_a,
\]

By Corollary 13.3, \( a \otimes d\omega \wedge \hat{\omega} \) is in \( F_{i+j-1}^i \) since the image of \( \hat{\omega} \) by the restriction morphism \( r_{g,0} \) is a generator of \( \wedge^{b_i} (\overline{B}) \) and \( \overline{B}_{0,0} \) is a submodule of \( \overline{B} \), whence the assertion.

(ii) Let \( \varphi \) be in \( Z_i^j \cap F_{i+j}^{i+j} \). By Lemma 13.5(i), for some \( \psi \) in \( F_{i+j}^{i+j-1} \),

\[
\varphi - d\psi \in K_i^j + F_{i+j}^{i+j-1}.
\]

As a matter of fact, by Lemma 13.5(ii),

\[
\varphi - d\psi \in K_i^j + F_{i+j}^{i+j-1}.
\]

By Lemma 13.5(iii) and Lemma 13.4(i), for \( i' = i, i+1 \), the sum \( K_i^{i'} + F_{i+j}^{i+j'} \) is direct. Let \( \varphi_1 \) be the component of \( \varphi - d\psi \) on \( K_i^j \). Then, by (i), \( d_2 \varphi_1 = 0 \) since \( d_2 \varphi_1 \) is in \( K_i^{i+j} \) and \( dF_{i+j}^{i+j-1} \) is contained in \( K_i^{i+j} \). As a result, by Lemma 13.4(ii), for some \( \varphi_1' \) in \( K_i^{i+j-1} \), \( \varphi_1 = d_2 \varphi_1' \). Then, by (i),

\[
\psi + \varphi_1' \in F_{i+j}^{i+j-1} \quad \text{and} \quad \varphi - d\psi - d\varphi_1' \in F_{i+j}^{i+j-1},
\]

whence the assertion.

Arguing as in the proof of Corollary 12.11, we deduce the following corollary from Lemma 13.6(ii).

**Corollary 13.7.** Let \( i = b_g + 1, \ldots, b_g + k \) and \( l \) a natural integer.

(i) For all nonnegative integer \( l \), \( Z_i^j \) is contained in \( \mathbb{B}^l + F_{i+j}^i \).

(ii) For some \( p \) in \( m \otimes \mathbb{k}[l] \), \( (1 + p)Z_i^j \) is contained in \( \mathbb{B}^l \).

Arguing as in the proof of Proposition 12.12, we deduce the following proposition from Corollary 13.7.
Proposition 13.8. For \( k = 0, \ldots, n \), \( D_{k,#}^*(g, \widehat{B}) \) and \( D_{k,*}(g, \widehat{B}) \) have no cohomology of degree different from \( b_0 \).

13.3. End of the proof of Theorem 1.5. We can now complete the proof of Theorem 1.5.

We prove the theorem by induction on the dimension of \( g \). By Proposition 4.1(ii), the theorem is true for \( \ell = 1 \). Suppose \( \ell > 1 \) and the theorem true for the simple algebras of rank smaller than \( \ell \). By Proposition 12.12, Proposition 13.8 and the induction hypothesis, for \( k = 1, \ldots, n \) and \( z \) in \( b_0 \setminus \{0\} \), \( D_{k,#}^*(g, \widehat{B}) \) has no cohomology of degree different from \( b_0 \). Then, by Proposition 6.5, \( \mathfrak{m}_1(S_k) \cap b = \{0\} \). As a result, by remark 6.7, \( g \) has Property (P), whence the theorem.

Appendix

Appendix A. Some complexes

Let \( X \) be an affine irreducible variety. The canonical injection from \( V \) into \( \wedge(V) \) has a unique extension as a derivation of the algebra \( S(V) \otimes_k \wedge(V) \) which is equal to 0 on the subalgebra \( 1 \otimes_k \wedge(V) \). Then \( S(V) \otimes_k \wedge(V) \) is a graded cohomology complex whose gradation is induced by the natural gradation of \( \wedge(V) \). We denote this complex by \( D^*(V) \) and its derivation by \( d \).

A.1. General facts. For \( k \) nonnegative integer, set:

\[
D^*_k(V) := \bigoplus_{i=0}^{k} S^{k-i}(V) \otimes_k \wedge^i(V),
\]
so that \( D^*_k(V) \) is a graded subcomplex of \( D^*(V) \).

Definition A.1. Let \( L \) be a free submodule of positive rank \( r \) of \( k[X] \otimes_k V \). For \( k \) nonnegative integer, denote by \( D^*_k(V, L) \) the graded subcomplex of \( k[X] \otimes_k D^*(V) \):

\[
D^*_k(V, L) := D^*_k(V, L)[-r] \wedge \wedge^r(L).
\]

The restriction to \( D^*_k(V, L) \) of the derivation of \( k[X] \otimes_k D^*(V) \) is also denoted by \( d \).

For \( W \) subspace of \( V \), let \( D^*_k(W, L) \) be the graded subspace of \( D^*_k(V, L) \) such that

\[
D^*_k(W, L) := S^{k-i}(W) \otimes_k \wedge^i(W) \wedge \wedge^r(L).
\]

Then \( D^*_k(W, L) \) is a graded subcomplex of \( D^*_k(V, L) \).

The embedding of \( S^k(L) \otimes_k \wedge^r(L) \) into \( S^k(V) \otimes_k \wedge^r(L) \) is an augmentation of \( D^*_k(V, L) \). Denote by \( \overline{D}_k(V, L) \) this augmented complex. In particular, \( \overline{D}_0(V, L) \) is acyclic.

Lemma A.2. Let \( k \) be a positive integer.

(i) The cohomology of \( D^*(V) \) is equal to \( k \).

(ii) The complex \( D^*_k(V) \) is acyclic.

(iii) For any subspace \( E \) of \( V \), \( \overline{D}_k(V, E) \) is an acyclic complex.
Proof. (i) We prove the statement by induction on the dimension of $V$. For $V$ equal to zero, $D^*(V)$ is equal to $\mathbb{K}$ and its differential is equal to 0. We suppose the statement true for any vector space whose dimension is strictly smaller than $\dim V$. Let $W$ be an hyperplane of $V$ and let $v$ be in $V \setminus W$. Let $a$ be a homogeneous cocycle of degree $d$ of $D^*(V)$. Then $a$ has a unique expansion

$$a = v^m a_m + \cdots + a_0,$$

with $a_0, \ldots, a_m$ in $S(W) \otimes_\mathbb{K} \wedge \langle V \rangle$. If $d = 0$, then

$$m v^{m-1} a_m \otimes v + \cdots + a_1 \otimes v = 0$$

so that $a = a_0$ is in $\mathbb{K}$. Suppose $d > 0$. Then for $i = 0, \ldots, m$,

$$a_i = a'_i + a''_i \wedge v,$$

with $a'_i$ and $a''_i$ in $S(W) \otimes_\mathbb{K} \wedge^d (W)$ and $S(W) \otimes_\mathbb{K} \wedge^{d-1} (W)$ respectively. From the equality

$$0 = \sum_{i=0}^m v^i d a'_i + \sum_{i=1}^m (-1)^i v^{i-1} a'_i \wedge v + \sum_{i=0}^m v^i d a''_i \wedge v,$$

we deduce that $a'_0, \ldots, a'_m$ are cocycles. So, by the induction hypothesis, for $i = 0, \ldots, m, a'_i = db_i$ for some $b_i$ in $S(W) \otimes_\mathbb{K} \wedge^{d-1} (W)$. Then

$$a - \text{d}(\sum_{i=0}^m v^i b_i) = v^m a''_m \wedge v + \sum_{i=0}^{m-1} v^i ((-1)^i (i+1) b_{i+1} + a''_i) \wedge v.$$

Hence $a''_m$ and $(-1)^i (i+1) b_{i+1} + a''_i$ are cocycles of degree $d - 1$ for $i = 0, \ldots, m - 1$. If $d = 1$,

$$a = d(\sum_{i=0}^m v^i b_i + \frac{1}{m+1} v^{m+1} a''_m + \sum_{i=0}^{m-1} \frac{1}{i+1} v^{i+1} (-1)^i (i+1) b_{i+1} + a''_i)).$$

For $d$ bigger than 1, by induction hypothesis, $a''_m$ is the coboundary of an element $c_m$ in $S(W) \otimes_\mathbb{K} \wedge^{d-2} (W)$ and for $i = 0, \ldots, m - 1$, $(-1)^i (i + 1) b_{i+1} + a''_i$ is the coboundary of an element $c_i$ in $S(W) \otimes_\mathbb{K} \wedge^{d-2} (W)$ so that

$$a = d(\sum_{i=0}^m v^i b_i + \sum_{i=0}^m v^i c_i \wedge v).$$

(ii) As $D^*(V)$ is the direct sum of $D^*_i(V), i \in \mathbb{N}$, the assertion results from (i).

(iii) Let $F$ be a complement to $E$ in $V$ and $d$ the dimension of $E$. For $i = 0, \ldots, k$,

$$D^{i+d}_k (V, E) = \bigoplus_{j=0}^{k-i} S^{k-i-j}(F) \otimes_\mathbb{K} S^j(E) \otimes_\mathbb{K} \wedge^i (F) \wedge \wedge^d (E),$$

whence

$$\overline{D}_k (V, E) = \bigoplus_{j=0}^k S^j(E) \otimes_\mathbb{K} D^*_{k-j}(F)[-d] \wedge \wedge^d (E).$$
By (ii), for $j < k$, $D^*_k(V, E)$ is acyclic. As a result $D^*_k(V, E)$ has no cohomology of degree different from $d$, whence the assertion since for $v$ in $V$, $v \wedge \wedge^d(E) = \{0\}$ if and only if $v$ is in $E$. 

For $\pi$ automorphism of $X$, denote by $\pi^#$ the automorphism of the algebra $k[X] \otimes_k S(V) \otimes E \wedge(V)$ induced by the comorphism of $\pi$. Let $L$ be a free submodule of rank $r$ of $k[X] \otimes_k V$.

**Lemma A.3.** Let $k$ be a positive integer and $\pi$ an automorphism of $X$.

(i) The restriction of $\pi^#$ to $D^*_k(V, L)$ is an isomorphism from $D^*_k(V, L)$ onto $D^*_k(V, \pi^#(L))$.

(ii) For any positive integer $j$, the image by $\pi^{-1}$ of the support in $X$ of the cohomology of degree $j$ of $D^*_k(V, L)$ is the support in $X$ of the cohomology of degree $j$ of $D^*_k(V, \pi^#(L))$.

**Proof.** (i) For $i$ positive integer, $\pi^#(D^*_k(V, L))$ is equal to $D^*_k(V, \pi^#(L))$. Hence $\pi^#(D^*_k(V, L))$ is equal to $D^*_k(V, \pi^#(L))$. As $\pi^#$ is an automorphism of the complex $k[X] \otimes_k D^*(V)$, the restriction of $\pi^#$ to $D^*_k(V, L)$ is an isomorphism of the complex $D^*_k(V, \pi^#(L))$ onto the complex $D^*_k(V, \pi^#(L))$.

(ii) Let $j$ be a positive integer, $J_j$ and $J_{j, \pi}$ the ideals of definition in $k[X]$ of the supports of the cohomology of degree $j$ of $D^*_k(V, L)$ and $D^*_k(V, \pi^#(L))$. If $a$ is a cocycle of degree $j$ of $D^*_k(V, L)$ and $p$ is in $J_j$, for $m$ sufficiently big positive integer, $p^m a$ is a coboundary of $D^*_k(V, L)$. Hence by (i), $\pi^#(p)^m \pi^#(a)$ is a coboundary of $D^*_k(V, \pi^#(L))$. So $J_{j, \pi}$ contains $\pi^#(J_j)$. By the same argument, $J_j$ contains $\pi^#(J_{j, \pi})$ since $\pi$ is an automorphism. Hence $J_{j, \pi}$ is equal to $\pi^#(J_j)$, whence the assertion.

For any $x$ in $X$, denote by $L(x)$ the image of $L$ by the map $\varphi \mapsto \varphi(x)$.

**Lemma A.4.** Let $X'$ be the subset of elements $x$ of $X$ such that $L(x)$ has dimension $r$ and $L$ the localization of $L$ on $X$.

(i) The subset $X'$ of $X$ is open and nonempty. Moreover, $X'$ has a finite cover by affine open subsets $Y$ which have the following property:

- there exists a subspace $E$ of $V$ which is a complement to $L(x)$ in $V$ for all $x$ in $Y$.

(ii) For all positive integer $k$, the support in $X$ of the cohomology of $\overline{D^*_k}(V, L)$ has an empty intersection with $X'$.

(iii) Suppose that $X$ is normal and $X'$ is a big open subset of $X$. Then $\overline{D^*_k}(V, L)$ has no cohomology of degree $r$.

**Proof.** (i) Let $\eta_1, \ldots, \eta_r$ be a basis of $L$. For all $x$ in $X$, $L(x)$ is the subspace of $V$ generated by $\eta_1(x), \ldots, \eta_r(x)$. Then $X'$ is a nonempty open subset of $X$. Let $x$ be in $X'$. Let $E$ be a complement to $L(x)$ in $V$. Then, for all $y$ in an open neighborhood $Y_x$ of $x$ in $X$, $L(y)$ has dimension $r$ and $E$ is a complement to $L(y)$ in $V$. In particular, $Y_x$ is contained in $X'$.

(ii) Let $k$ be a positive integer and $Y$ an affine open subset of $X'$ which satisfies the condition of (i). Denoting by $L_y$ the space of sections of $L$ above $Y$, we have to prove that $\overline{D^*_k}(V, L_y)$ is acyclic.
Let $x_0$ be in $Y$. For $x$ in $Y$, denote by $\tau(x)$ the linear automorphism of $V$ such that $\tau(x)(v) = v$ for all $v$ in $E$ and for $w$ in $L(x)$, $\tau(x)(w)$ is the element of $L(x_0)$ such that $w - \tau(x)(w)$ is in $E$. Let $\overline{\tau}$ be the automorphism of the algebra $\mathbb{k}[Y] \otimes_\mathbb{k} S(V) \otimes_\mathbb{k} \wedge(V)$ such that $\overline{\tau}(\varphi)$ is the map

$$Y \xrightarrow{\overline{\tau}} S(V) \otimes_\mathbb{k} \wedge(V), \quad x \mapsto \tau(x)(\varphi(x)).$$

The images of $L_Y$ and $\overline{D}_k(V, L_Y)$ by $\overline{\tau}$ are equal to $\mathbb{k}[Y] \otimes_\mathbb{k} L_Y(x_0)$ and $\mathbb{k}[Y] \otimes_\mathbb{k} \overline{D}_k(V, L_Y(x_0))$ respectively. Moreover, the map

$$\overline{D}_k(V, L_Y) \xrightarrow{\overline{\tau}} \mathbb{k}[Y] \otimes_\mathbb{k} \overline{D}_k(V, L_Y(x_0))$$

is an isomorphism of graded complexes. Hence by lemma A.2(iii), $D_k^\bullet(V, L_Y)$ is acyclic.

(iii) Let $a$ be a cocycle of degree $r$ of $\overline{D}_1(V, L)$. By (ii), the restriction of $a$ to $Y$ is the image by $d$ of a unique element $\varphi$ of $S^k(L_Y) \otimes_\mathbb{k}[Y] \wedge(L_Y)$. So, by (i), the restriction of $a$ to $X'$ is the image by $d$ of a section above $X'$ of the localization on $X$ of $S^k(L) \otimes_\mathbb{k}[X] \wedge(L)$. As $L$ is a free module, $S^k(L) \otimes_\mathbb{k}[X] \wedge(L)$ is a free module and any section above $X'$ of its localization on $X$ is the restriction to $X'$ of an element of $S^k(L) \otimes_\mathbb{k} \wedge(L)$ since $X'$ is a big open subset of the normal variety $X$. Hence $a$ is a coboundary. 

\[84\]

A.2. Some equivalence. Let $L$ be a free submodule of rank $r$ of $\mathbb{k}[X] \otimes_\mathbb{k} V$ such that $2r > \dim V$. For $k = 0, \ldots, \dim V - r$, denote by $K^\bullet_k(V, L)$ the graded subcomplex of $\mathbb{k}[X] \otimes_\mathbb{k} D^\bullet(V)$ whose subspace of degree $i$ is

$$K^i_k(V, L) := \wedge^i(V) \otimes_\mathbb{k} S^{k-i}(L)$$

for $i = 0, \ldots, k$. Denote by $\delta$ the restriction of $d$ to $K^\bullet_k(V, L)$. The map

$$K^i_k(V, L) \xrightarrow{\delta_k} D^{k+r}_k(V, L), \quad \varphi \mapsto \varphi \wedge \eta$$

with $\eta$ a generator of $\wedge^r(L)$, is an augmentation of $K^\bullet_k(V, L)$. Denote by $\overline{K}^\bullet_k(V, L)$ the aughtmented complex so defined.

\[\text{Lemma A.5.}\] Let $X'$ be the subset of elements $x$ of $X$ such that $L(x)$ has dimension $r$.

(i) For $k = 1, \ldots, \dim V - r$, the support of the cohomology of $\overline{K}^\bullet_k(V, L)$ is contained in $X \setminus X'$.

(ii) The complex $K^\bullet_k(V, L)$ has no cohomology of degree 0.

\[\text{Proof.}\] (i) Let $Y$ be an affine open subset of $X'$ and $E$ a subspace of $V$ satisfying the condition of Lemma A.4(i) and set

$$L_Y := \mathbb{k}[Y] \otimes_\mathbb{k}[X] L.$$ 

The complex $K^\bullet_k(V, L_Y)$ is isomorphic to

$$\bigoplus_{j=0}^k \wedge^j(E) \otimes_\mathbb{k} D^\bullet_{k-j}(L_Y)[-j].$$
Then $K^*_k(V, L_Y)$ has no cohomology of degree different from $k$ by Lemma A.2(ii) since $L_Y$ is a free module. Moreover, the space of cocycles of degree $k$ of $K^*_k(V, L_Y)$ is equal to

$$
\bigoplus_{j=1}^{k} \wedge^{k-j}(E) \wedge \wedge^j(L_Y).
$$

Hence $K^*_k(V, L_Y)$ is acyclic and the support of the cohomology of $K^*_k(V, L)$ is contained in $X \setminus X'$ since $X'$ has a cover by affine open subsets satisfying the condition of Lemma A.4(i).

(ii) By (i), $K^*_k(V, L)$ has no cohomology of degree 0 since $S^k(L)$ is a torsion free module.

Denote by $\eta_1, \ldots, \eta_r$ a basis of $L$. Let $\mathcal{A}$ be the union of $\{0\}$ and the set of strictly increasing sequences in $\{1, \ldots, r\}$ and for $\ell = i_1 < \cdots < i_j$ in $\mathcal{A}$, set:

$$
\{\ell\} := \{i_1, \ldots, i_j\}, \quad |\ell| = j, \quad \eta_\ell := \eta_{i_1} \wedge \cdots \wedge \eta_{i_j}.
$$

For $j = 0, \ldots, r$ and $s = (s_1, \ldots, s_r)$ in $\mathbb{N}^r$, set:

$$
\mathcal{A}_j := \{\ell \in \mathcal{A} \mid |\ell| = j\} \quad \text{and} \quad \eta^s := \eta_1^{s_1} \cdots \eta_r^{s_r}.
$$

For $k = 1, \ldots, \dim V - r$ and $j = 1, \ldots, k$, denote by $K^*_{k,j}(V, L)$ the graded subcomplex of $K^*_k(V, L)$ whose subspace of degree $i$ is

$$
K^*_{k,j}(V, L) := \wedge^{i-j}(V) \wedge \wedge^j(L) \otimes_k S^{k-j}(L),
$$

$Z^i_k$ and $B^i_k$ the space of cocycles and coboundaries of degree $i$ of $K^*_k(V, L)$ respectively.

**Lemma A.6.** Let $k = 1, \ldots, \dim V - r$. Suppose that $\wedge^{i-1}(V) \wedge L$ is the kernel of $\theta_i$ for $i = 1, \ldots, k$.

(i) Suppose $2 \leq k$. For $i = 1, \ldots, k - 1$, $Z^i_k$ is contained in $B^i_k + K^*_{k,i}(V, L)$.

(ii) The complex $K^*_k(V, L)$ is acyclic.

**Proof.** (i) For $j = 0, \ldots, i$, set $Z^i_{k,j} := Z^i_k \cap K^*_{k,j}(V, L)$ and prove that $Z^i_{k,j}$ is contained in $B^i_k + Z^i_{k,j+1}$ for $j < i$. Let $\varphi$ be in $Z^i_{k,j}$. Then $\varphi$ has an expansion

$$
\varphi = \sum_{s \in \mathcal{A}_i} \sum_{l \in \mathcal{A}_j} \eta_s \wedge \varphi_{s, l} \otimes \eta^l \quad \text{with} \quad \varphi_{s, l} \in \mathbb{k}[X] \otimes_k \wedge^{i-j}(V).
$$

For $s'$ in $\mathbb{N}^r_{k-i-1}$, set

$$
I_{s'} := \{(s, l) \in \mathbb{N}^r_{k-i} \times \{1, \ldots, r\} \mid \eta^s = \eta^s \eta^{s'}\}.
$$

As $\varphi$ is a cocycle,

$$
\sum_{s' \in \mathbb{N}^r_{k-i-1}} \sum_{(s, l) \in I_{s'}} \sum_{t \in \mathcal{A}_j} s_l \eta_l \wedge \eta_t \wedge \varphi_{s, t} \otimes \eta^{s'} = 0 \quad \text{whence} \quad \sum_{(s, l) \in I_{s'}} \sum_{t \in \mathcal{A}_j} s_l \eta_l \wedge \eta_t \wedge \varphi_{s, t} = 0
$$

for all $s'$ in $\mathbb{N}^r_{k-i-1}$.
Let $Y$ be an affine open subset of $X'$ and $E$ a subspace of $V$ satisfying the condition of Lemma A.4(iii). For $(s, t)$ in $N^r_{k-1} \times \mathfrak{S}_j$,
\[ \varphi_{s,t}|_Y = \varphi'_{s,t} + \varphi''_{s,t} \quad \text{with} \quad \varphi'_{s,t} \in L_Y \wedge \wedge^{i-j-1}(V), \quad \varphi''_{s,t} \in \mathbb{k}[Y] \otimes \mathbb{k} \wedge^{i-j}(E). \]
For $(l, t)$ in $\{1, \ldots, r\} \times \mathfrak{S}_j$ such that $l \notin \{t\}$, denote by $\nu(l, t)$ the element of $\mathfrak{S}_{r-j-1}$ and $\epsilon(l, t)$ the element of $\{-1, 1\}$ such that
\[ \{l\} \cup \{t\} \cup \{\nu(l, t)\} = \{1, \ldots, r\} \quad \text{and} \quad \eta_l \wedge \eta_t \wedge \eta_{\nu(l,t)} = \epsilon(l, t)\eta. \]
For $s'$ in $N^r_{k-1}$ and $l = 1, \ldots, r$, set:
\[ I_{l,s'} := \{ s \in N^r_{k-1} \mid (s, l) \in I_{s'} \}. \]
For $s'$ in $N^r_{k-1}$ and for $(l, t)$ in $\{1, \ldots, r\} \times \mathfrak{S}_j$ such that $l \notin \{t\}$, by the above equality, after multiplication by $\eta_{\nu(l,t)}$,
\[ \sum_{s \in I_{l,s'}} \epsilon(l, t)s_i \varphi''_{s,t} \wedge \eta = 0 \quad \text{whence} \quad \sum_{s \in I_{l,s'}} \epsilon(l, t)s_i \varphi''_{s,t} = 0. \]
Since for $(s, l)$ in $I_{s'}$, $\eta^i = \eta \eta''^i$, $|I_{s', l}| = 1$. Hence $\varphi''_{s,t} = 0$ for all $(s, t)$ in $N^r_{k-1} \times \mathfrak{S}_j$ such that $s_t \neq 0$ for some $l \notin \{t\}$. As a result for such $(s, l)$, $\varphi_{s,t} \wedge \eta$ is equal to 0 since so is its restriction to $Y$. So, by hypothesis, for such $(s, t)$, $\varphi_{s,t}$ is in $L \wedge \wedge^{i-j-1}(V)$ and
\[ \eta_l \wedge \varphi_{s,t} \otimes \eta^l \in K^i_{k,j-1}(V, L). \]
Let $(s, t)$ be in $N^r_{k-1} \times \mathfrak{S}_j$ such that
\[ s_t \neq 0 \implies l \in \{t\}. \]
Then, for $l$ in $\{t\}$ such that $s_t \neq 0$,
\[ \pm \eta \otimes \eta^t = \frac{1}{s_t + 1} \eta_l \wedge \eta \eta^t \quad \text{and} \quad \pm \eta \wedge \varphi_{s,t} \otimes \eta^t = \frac{1}{s_t + 1} \eta_l \wedge \varphi_{s,t} \otimes \eta \eta^t \quad \text{with} \quad \{t'\} = \{t\} \setminus \{l\}, \]
whence the assertion.

(ii) By hypothesis, $\overline{K}_k(V, L)$ has no cohomology of degree $k$ and by Lemma A.5(ii), it has no cohomology of degree $0$. Let $i = 1, \ldots, k - 1$. By (i),
\[ Z^i_k \subset B^i_k + K^i_{k,i}(V, L) = B^i_k + \bigwedge^i(L) \otimes \mathbb{k}[X] S^{k-i}(L). \]
As $L$ is a free module, $K^*_{k,i}(V, L)$ has no cohomology of degree $i$ by Lemma A.2(ii). Hence $Z^i_k = B^i_k$. \hfill \Box

**Proposition A.7.** Let $k = 1, \ldots, \dim V - r$.

(i) The complex $\overline{K}_k(V, L)$ is acyclic if $D^i_k(V, L)$ has no cohomology of degree different from $r$.

(ii) The complex $D^i_k(V, L)$ has no cohomology of degree different from $r$ if $\overline{K}_k(V, L)$ is acyclic for $i = 1, \ldots, k$. \hfill \Box
(iii) Suppose that $\bigwedge^{i-1}(V) \wedge L$ is the kernel of $\theta_i$ for $i = 1, \ldots, k$. Then $D^*_k(V, L)$ has no cohomology of degree different from $r$.

**Proof.** (i) and (ii) For $i = 1, \ldots, k$, denote by $\delta$ the restriction to $K^*_i(V, L)$ of the derivation $d$ of $k[X] \otimes_k D^*(V)$. Setting

$$E^{i,j}_k := \begin{cases} S^{k-i}(V) \otimes_k \bigwedge^j(V) \otimes_k S^{i-j}(L) & \text{if } j \leq i \leq k \\ S^{k-j}(V) \otimes_k \bigwedge^i(V) \wedge \bigwedge^r(L) & \text{if } i = j-1 < k \\ 0 & \text{otherwise} \end{cases}$$

we have the equalities

$$E^{i,j-1}_k = D^i_k(V, L), \quad E^{i,j}_k = D^i_{k-i+j}(V) \otimes_k S^{i-j}(L) = S^{k-i}(V) \otimes_k K^i_j(V, L)$$

for $j \leq i \leq k$. Denoting again by $\delta$ the map

$$E^{i,j}_k \xrightarrow{\delta} E^{i,j-1}_k, \quad \varphi \mapsto \varphi \wedge \eta,$$

we have the double complex

$$\begin{array}{c}
E^{i,j+1}_k \\
\delta \downarrow \\
\downarrow d \quad \downarrow d \\
E^{i,j}_k \\
\delta \downarrow \\
E^{i+1,j+1}_k
\end{array}$$

Along a line, $i - j$ is constant and a line corresponding to a nonnegative constant is acyclic by Lemma A.2(ii). Then $K^*_i(V, L)$ has no cohomology of positive degree if $D^*_k(V, L)$ has no cohomology of degree different from $r$ since the maps

$$E^{i,j}_k \xrightarrow{\delta} E^{i,j-1}_k \quad \text{and} \quad E^{k-i,j}_k \xrightarrow{d} E^{k,j}_k$$

are surjective, whence Assertion (i) by Lemma A.5(ii). As $E^{i,*}_k = S^{k-i}(V) \otimes_k K^*_i(V, L)$, $D^*_k(V, L)$ has no cohomology of degree different from $r$ if $K^*_i(V, L)$ is acyclic for $i = 1, \ldots, k$.

(iii) By Lemma A.6(ii), $K^*_i(V, L)$ is acyclic for $i = 1, \ldots, k$, whence the assertion by (ii). □

**Appendix B. Projective dimension and cohomology**

Recall in this section classical results. Let $X$ be a Cohen-Macaulay irreducible affine algebraic variety and $S$ a closed subset of codimension $p$ of $X$. Let $P_\bullet$ be a complex of finitely generated projective $\mathbb{k}[X]$-modules whose length $l$ is finite and let $e$ be an augmentation morphism of $P_\bullet$ whose image is $R$, whence an augmented complex of $\mathbb{k}[X]$-modules,

$$0 \rightarrow P_l \rightarrow P_{l-1} \rightarrow \cdots \rightarrow P_0 \rightarrow e R \rightarrow 0.$$
Denote by $\mathcal{P}_\bullet$, $\mathcal{R}$, $\mathcal{K}_0$ the localizations on $X$ of $P_\bullet$, $R$, the kernel of $\varepsilon$ respectively and denote by $\mathcal{K}_1$ the kernel of the morphism $\mathcal{P}_i \longrightarrow \mathcal{P}_{i-1}$ for $i$ positive integer.

**Lemma B.1.** Suppose that $S$ contains the support of the homology of the augmented complex $P_\bullet$.

(i) For all positive integer $i < p - 1$ and for all projective $\mathcal{O}_X$-module $\mathcal{P}$, $H^i(X \setminus S, \mathcal{P})$ is equal to zero.

(ii) For all nonnegative integer $j \leq l$ and for all positive integer $i < p - j$, the cohomology group $H^i(X \setminus S, \mathcal{K}_{l-j})$ is equal to zero.

**Proof.** (i) Let $i < p - 1$ be a positive integer. Since the functor $H^i(X \setminus S, \bullet)$ commutes with the direct sum, it suffices to prove $H^i(X \setminus S, \mathcal{O}_X) = 0$. Since $S$ is a closed subset of $X$, we have the relative cohomology long exact sequence

$$
\cdots \longrightarrow H^i_S(X, \mathcal{O}_X) \longrightarrow H^i(X, \mathcal{O}_X) \longrightarrow H^i(X \setminus S, \mathcal{O}_X) \longrightarrow H^{i+1}_S(X, \mathcal{O}_X) \longrightarrow \cdots.
$$

Since $X$ is affine, $H^i(X, \mathcal{O}_X)$ is equal to zero and $H^i(X \setminus S, \mathcal{O}_X)$ is isomorphic to $H^{i+1}_S(X, \mathcal{O}_X)$. Since $X$ is Cohen-Macaulay, the codimension $p$ of $S$ in $X$ is equal to the depth of its ideal of definition in $\mathbb{k}[X]$ [MA86, Ch. 6, Theorem 17.4]. Hence, according to [Gro67, Theorem 3.8], $H^{i+1}_S(X, \mathcal{O}_X)$ and $H^i(X \setminus S, \mathcal{O}_X)$ are equal to 0 since $i + 1 < p$.

(ii) Let $j$ be a nonnegative integer. Since $S$ contains the support of the homology of the complex $P_\bullet$, for all nonnegative integer $j$, we have the short exact sequence of $\mathcal{O}_{X\setminus S}$-modules

$$
0 \longrightarrow \mathcal{K}_{j+1}|_{X\setminus S} \longrightarrow \mathcal{P}_{j+1}|_{X\setminus S} \longrightarrow \mathcal{K}_j|_{X\setminus S} \longrightarrow 0
$$

whence the long exact sequence of cohomology

$$
\cdots \longrightarrow H^i(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow H^i(X \setminus S, \mathcal{K}_j) \longrightarrow H^{i+1}(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow H^{i+1}(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow \cdots.
$$

Then, by (i), for $0 < i < p - 2$, the cohomology groups $H^i(X \setminus S, \mathcal{K}_j)$ and $H^{i+1}(X \setminus S, \mathcal{K}_{j+1})$ are isomorphic since $P_{j+1}$ is a projective module. Since $\mathcal{P}_i = 0$ for $i > l$, $\mathcal{K}_{l-1}$ and $\mathcal{P}_l$ have isomorphic restrictions to $X \setminus S$. In particular, by (i), for $0 < i < p - 1$, $H^i(X \setminus S, \mathcal{K}_{l-1})$ equal zero. Then, by induction on $j$, for $0 < i < p - j$, $H^i(X \setminus S, \mathcal{K}_{l-j})$ is equal to zero. $\square$

**Proposition B.2.** Let $R'$ be a $\mathbb{k}[X]$-module containing $R$. Suppose that the following conditions are satisfied:

1. $p$ is at least $l + 2$,
2. $X$ is normal,
3. $S$ contains the support of the homology of the augmented complex $P_\bullet$.

(i) The complex $P_\bullet$ is a projective resolution of $R$ of length $l$.

(ii) Suppose that $R'$ is torsion free and that $S$ contains the support in $X$ of $R'/R$. Then $R' = R$. 

Proof. (i) Let $j$ be a positive integer. We have to prove that $H^0(X, \mathcal{K}_j)$ is the image of $P_{j+1}$. By Condition (3), the short sequence of $O_{X/S}$-modules

$$
0 \longrightarrow \mathcal{K}_{j+1} |_{X/S} \longrightarrow P_{j+1} |_{X/S} \longrightarrow \mathcal{K}_j |_{X/S} \longrightarrow 0
$$

is exact, whence the cohomology long exact sequence

$$
0 \longrightarrow H^0(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow H^0(X \setminus S, P_{j+1}) \longrightarrow H^0(X \setminus S, \mathcal{K}_j) \longrightarrow H^1(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow \cdots .
$$

By Lemma B.1(ii), $H^1(X \setminus S, \mathcal{K}_{j+1})$ equals 0 since $1 < p - l + j + 1$, whence the short exact sequence

$$
0 \longrightarrow H^0(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow H^0(X \setminus S, P_{j+1}) \longrightarrow H^0(X \setminus S, \mathcal{K}_j) \longrightarrow 0 .
$$

As the codimension of $S$ in $X$ is at least 2 and $X$ is irreducible and normal, the restriction morphism from $P_{j+1}$ to $H^0(X \setminus S, P_{j+1})$ is an isomorphism. Let $\varphi$ be in $H^0(X, \mathcal{K}_j)$. Then there exists an element $\psi$ of $P_{j+1}$ whose image $\psi'$ in $H^0(X, \mathcal{K}_j)$ has the same restriction to $X \setminus S$ as $\varphi$. Since $P_j$ is a projective module and $X$ is irreducible, $P_j$ is torsion free. Then $\varphi = \psi'$ since $\varphi - \psi'$ is a torsion element of $P_j$, whence the assertion.

(ii) Let $R'$ be the localization of $R'$ on $X$. Arguing as in (i), since $S$ contains the support of $R'/R$ and $1 < p - l$, the short sequence

$$
0 \longrightarrow H^0(X \setminus S, \mathcal{K}_0) \longrightarrow H^0(X \setminus S, P_0) \longrightarrow H^0(X \setminus S, R') \longrightarrow 0
$$

is exact. Moreover, the restriction morphism from $P_0$ to $H^0(X/S, P_0)$ is an isomorphism since the codimension of $S$ in $X$ is at least 2 and $X$ is irreducible and normal. Let $\varphi$ be in $R'$. Then for some $\psi$ in $P_0$, $\varphi - \varepsilon(\psi)$ is a torsion element of $R'$. So $\varphi = \varepsilon(\psi)$ since $R'$ is torsion free, whence the assertion. \qed

Corollary B.3. Let $C_\bullet$ be a homology complex of finitely generated $\mathbb{Z}[X]$-modules whose length $l$ is finite and positive. For $j = 0, \ldots, l$, denote by $Z_j$ the space of cycles of degree $j$ of $C_\bullet$. Suppose that the following conditions are satisfied:

1. $S$ contains the support of the homology of the complex $C_\bullet$,
2. for all $i$, $C_i$ is a submodule of a free module,
3. for $i = 1, \ldots, l$, $C_i$ has projective dimension at most $d$,
4. $X$ is normal and $l + d \leq p - 1$.

Then $C_\bullet$ is acyclic and for $j = 0, \ldots, l$, $Z_j$ has projective dimension at most $l + d - j - 1$.

Proof. Prove by induction on $l - j$ that the complex

$$
0 \longrightarrow C_i \longrightarrow \cdots \longrightarrow C_{j+1} \longrightarrow Z_j \longrightarrow 0
$$

is acyclic and $Z_j$ has projective dimension at most $l + d - j - 1$. For $j = l$, $Z_j$ is equal to zero since $C_l$ is torsion free by Condition (2) and $Z_l$ is a submodule of $C_l$, supported by $S$ by Condition
(1). Suppose $j \leq l - 1$ and the statement true for $j + 1$. By Condition (3), $C_{j+1}$ has a projective resolution $P_\bullet$ whose length is at most $d$ and whose terms are finitely generated. By induction hypothesis, $Z_{j+1}$ has a projective resolution $Q_\bullet$ whose length is at most $l + d - j - 2$ and whose terms are finitely generated, whence an augmented complex $R_\bullet$ of projective modules whose length is $l + d - j - 1$,

$$0 \to Q_{l+d-j-2} \oplus P_{l+d-j-1} \to \cdots \to Q_0 \oplus P_1 \to P_0 \to Z_j \to 0.$$

Denoting by $d$ the differentials of $Q_\bullet$ and $P_\bullet$, the restriction to $Q_i \oplus P_{i+1}$ of the differential of $R_\bullet$ is the map

$$(x, y) \mapsto (dx, dy + (-1)^i \delta(x)),$$

with $\delta$ the map which results from the injection of $Z_{j+1}$ into $C_{j+1}$. Since $P_\bullet$ and $Q_\bullet$ are projective resolutions, the complex $R_\bullet$ is a complex of projective modules having no homology of positive degree. Hence the support of the homology of the augmented complex $R_\bullet$ is contained in $S$ by Condition (1). Then, by Proposition B.2 and Condition (4), $R_\bullet$ is a projective resolution of $Z_j$ of length $l + d - j - 1$ since $Z_j$ is a submodule of a free module by Condition (2), whence the corollary since $Z_0 = C_0$ by definition. □

**Corollary B.4.** Let

$$0 \to E_{-1} \to E_0 \to \cdots \to E_i \to 0$$

be a complex of finitely generated $\mathbb{C}[X]$-modules. Suppose that the following conditions are satisfied:

1. $E_{-1}$ is projective and for $i = 0, \ldots, l - 1$, $E_i$ has projective dimension at most $i$,
2. $S$ contains the support of the cohomology of this complex,
3. for $i = 0, \ldots, l$, $E_i$ is a submodule of a free module,
4. $X$ is normal and $p \geq l + 2$.

Then the complex is acyclic and $E_l$ has projective dimension at most $l$.

**Proof.** Prove the corollary by induction on $l$. For $l = 0$, by Conditions (2), (3), (4), the arrow $E_{-1} \to E_0$ is an isomorphism. Suppose the corollary true for the integers smaller than $l$. Let $Z_{l-1}$ be the kernel of the arrow $E_{l-1} \to E_l$, whence the two complexes

$$0 \to E_{-1} \to E_0 \to \cdots \to E_{l-2} \to Z_{l-1} \to 0$$

$$0 \to Z_{l-1} \to E_{l-1} \to E_l \to 0.$$

By Condition (2), the support of the cohomology of these two complexes is contained in $S$. Then, by induction hypothesis, the first complex is acyclic and $Z_{l-1}$ has projective dimension at most $l - 1$. As a result, arguing as in the proof of Corollary B.3, we have a complex of $\mathbb{C}[X]$-modules

$$0 \to P_1 \to P_{l-1} \to \cdots \to P_0 \to E_l \to 0.$$
such that $P_0, \ldots, P_l$ are projective, the image of $P_0$ in $E_i$ is the image of the arrow $E_{i-1} \longrightarrow E_i$ and the support of its homology is contained in $S$. Then, by Condition (4) and Proposition B.2, it is acyclic so that $E_i$ has projective dimension at most $l$ and the complex

$$0 \longrightarrow E_{-1} \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_l \longrightarrow 0$$

is acyclic. □

Let

$$0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0$$

be a short exact sequence of $\mathbb{k}[X]$-modules.

**Lemma B.5.** Suppose that for $i = 0, 1, 2$, $M_i$ has a finite projective dimension $d_i$. Then we have the inequalities

$$d_2 \leq \sup\{d_0 + 1, d_1\} \quad \text{and} \quad d_0 \leq \sup\{d_2 - 1, d_1\}.$$

**Proof.** Let $N$ be a $\mathbb{k}[X]$-module. We have to prove $\text{Ext}^j(M_2, N) = 0$ for $j$ bigger than $\{d_0 + 1, d_1\}$ and $\text{Ext}^j(M_1, N) = 0$ for $j$ bigger than $\{d_2 - 1, d_1\}$. From the short exact sequence, we deduce the long exact sequence

$$\cdots \longrightarrow \text{Ext}^j(M_1, N) \longrightarrow \text{Ext}^j(M_0, N) \longrightarrow \text{Ext}^{j+1}(M_2, N) \longrightarrow \text{Ext}^{j+1}(M_1, N) \longrightarrow \cdots.$$

For $j + 1 > \sup\{d_0 + 1, d_1\}$, $\text{Ext}^{j+1}(M_1, N) = 0$ and $\text{Ext}^j(M_0, N) = 0$, whence $\text{Ext}^{j+1}(M_2, N) = 0$. For $j > \sup\{d_2 - 1, d_1\}$, $\text{Ext}^j(M_1, N) = 0$ and $\text{Ext}^{j+1}(M_2, N) = 0$, whence $\text{Ext}^j(M_0, N) = 0$. □

**Appendix C. Some computations**

For $k, l$ positive integers such that $k \leq l$, set:

$$r(k, l) := \sum_{j=0}^{k-1} (-1)^j \binom{l}{j}$$

and for $k, l$ nonnegative integers, set:

$$c(k, l) := \sum_{j=0}^{k} \frac{(-1)^j}{j!(l+j)!}.$$

For $e, k, l$ integers such that $2 \leq e \leq k \leq l$, set:

$$\psi(e, k, l) := r(k, l) - \sum_{j=1}^{e-1} (-1)^{k-j} c(j, k-j) r(j, l) \frac{(l-j)!}{(l-k)!}.$$

Let $p_k, k = 0, \ldots$ be the sequence of polynomials defined by the induction relations:

$$p_0 := 1, \quad p_1(x) := x, \quad p_k(x) := k(x - k) p_{k-1}(x) + (-1)^k.$$
Lemma C.1. Let \( k, l \) be positive integers.

(i) For \( k \leq l \),
\[
r(k, l) := \frac{(-1)^{k-1} (l-1)!}{(k-1)!(l-k)!}.
\]

(ii) The rational number \( c(k, l) \) is equal to \( p_k(l)/(k!(l+k)!). \)

(iii) For \( k \geq 2 \), \( p_k(l) \) is in \( k\mathbb{Z} \pm 1 \).

Proof. (i) Prove the assertion by induction on \( k \). For \( k = 1 \), \( r(k, l) = 1 \). Suppose \( k > 1 \) and the assertion true for \( k-1 \). Then
\[
r(k, l) = \frac{(-1)^{k-2} (l-1)!}{(k-2)!(l-k+1)!} + \frac{(-1)^{k-1} l!}{(k-1)!(l-k+1)!} = \frac{(-1)^{k-2} (l-1)!}{(k-1)!(l-k+1)!} (k-1-l),
\]
whence the assertion.

(ii) Prove the assertion by induction on \( k \). For \( k = 1 \), \( c(k, l) = l/(l+1)! \). Suppose \( k > 1 \) and the assertion true for \( k-1 \). Then
\[
c(k, l) = \frac{p_{k-1}(l)}{(k-1)!(l-k+1)!} + \frac{(-1)^k}{k!(l+k)!} = \frac{1}{k!(l+k)!} (k(l-k)p_{k-1}(l) + (-1)^k),
\]
whence the assertion.

(iii) Prove the assertion by induction on \( k \). For \( k = 2 \), there is nothing to prove. Suppose \( k > 2 \) and the assertion true for \( k-1 \). From the equality
\[
p_k(l) = k(l-k)p_{k-1}(l) + (-1)^k
\]
and the induction hypothesis, we deduce the assertion. \( \square \)

For \( e, k \) positive integers such that \( 2 \leq e \leq k \), set:
\[
\varphi(e, k) := k - \sum_{j=1}^{e-1} \frac{j}{(j!)^2} p_j(k-j).
\]

Corollary C.2. Let \( e, k, l \) be positive integers such that \( 2 \leq e \leq k \leq l \). Then
\[
(-1)^{k-1} \frac{k!(l-k)!}{(l-1)!} \psi(e, k, l) = \varphi(e, k).
\]

Proof. By Lemma C.1,
\[
\psi(e, k, l) = \frac{(-1)^{k-1} (l-1)!}{(k-1)!(l-k)!} - \sum_{j=1}^{e-1} \frac{p_j(k-j) (-1)^{k-1} (l-1)! (l-j)!}{j!(l-j)! (l-1)! (l-k)!} = \frac{(-1)^{k-1} (l-1)!}{(k-1)!(l-k)!} - \sum_{j=1}^{e-1} \frac{(-1)^{k-1} j(l-1)! p_j(k-j)}{(j!)^2 k!(l-k)!},
\]
whence
\[(1)^{k-1} \frac{k!(l-k)!}{(l-1)!} \psi(e, k, l) = k - \sum_{j=1}^{e-1} \frac{j}{j!} p_j(k - j).\]

\[\square\]

**Proposition C.3.** Let \(e, k, l\) be positive integers such that \(2 \leq e \leq k \leq l\). Then \(\psi(e, k, l) \neq 0\)

**Proof.** By Corollary C.2, it is equivalent to prove \(\varphi(e, k) \neq 0\). By definition,
\[\varphi(2, k) = k - k + 1\quad \text{and} \quad \varphi(3, k) = \varphi(2, k) - \frac{1}{2} p_2(k - 2) \in \mathbb{Z} \pm \frac{1}{2},\]
whence the proposition for \(e = 2\) and \(e = 3\). Suppose \(e \geq 4\) and prove by induction on \(e\),
\[((e - 2)!)^2 \varphi(e, k) \in \mathbb{Z} \pm \frac{1}{e - 1}.\]

From the equality
\[\varphi(4, k) = \varphi(3, k) - \frac{3}{(3!)^2} p_3(k - 3),\]
we deduce the assertion by Lemma C.1(iii) and the relation for \(\varphi(3, k)\). Suppose \(e > 4\) and the assertion true for \(e - 1\). By definition,
\[\varphi(e, k) = \varphi(e - 1, k) - \frac{e - 1}{((e - 1)!)^2} p_{e-1}(k - e + 1)\quad \text{whence}\]
\[((e - 2)!)^2 \varphi(e, k) = ((e - 2)!)^2 \varphi(e - 1, k) + \frac{1}{e - 1} p_{e-1}(k - e + 1).\]

By induction hypothesis,
\[((e - 2)!)^2 \varphi(e - 1, k) \in \mathbb{Z},\]
and by Lemma C.1(i),
\[\frac{1}{e - 1} p_{e-1}(k - e + 1) \in \mathbb{Z} \pm \frac{1}{e - 1},\]
whence the assertion and the proposition. \[\square\]

**Appendix D. Some remarks about representations**

In this section, \(\mathfrak{g}\) is a semisimple Lie algebra, \(\mathfrak{p}\) is a parabolic subalgebra of \(\mathfrak{g}\), containing \(\mathfrak{b}, \mathfrak{l}\) is the reductive factor of \(\mathfrak{p}\), containing \(\mathfrak{h}\), and \(\mathfrak{d}\) is the derived algebra of \(\mathfrak{l}\). Let \(\mathcal{R}_l\) the set of roots \(\alpha\) such that \(\mathfrak{g}^{\alpha}\) is contained in \(\mathfrak{l}\) and \(\mathcal{R}_{l,+}\) the intersection of \(\mathcal{R}_l\) and \(\mathcal{R}_+\). Denote by \(\mathcal{P}_\mathfrak{g}\) the subset of elements of \(\mathcal{P}(\mathfrak{R})\) whose restriction to \(\mathfrak{h} \cap \mathfrak{d}\) is a dominant weight of the root system \(\mathcal{R}_l\) with respect to the positive root system \(\mathcal{R}_{l,+}\).

Let \(M\) be a rational \(\mathfrak{g}\)-module. For \(\lambda\) in \(\mathcal{P}_+ (\mathcal{R})\), denote by \(M_\lambda\) the isotypic component of type \(V_\lambda\) of the \(\mathfrak{g}\)-module \(M\). Let \(\mathcal{P}_M\) be the subset of dominant weights \(\lambda\) such that \(M_\lambda \neq 0\).

**Lemma D.1.** The space \(M\) is the direct sum of \(M_\lambda\), \(\lambda \in \mathcal{P}_M\).
Proof. As $M$ is a rational $g$-module, $M$ is a union of $g$-modules of finite dimension. In particular, all simple $g$-module contained in $M$ has finite dimension. Hence $M_\lambda$, $\lambda \in \mathcal{P}_M$ is the set of isotypic components of $M$. Moreover, $M$ is the direct sum of $M_\lambda$, $\lambda \in \mathcal{P}_M$. □

For $\lambda$ in $\mathcal{P}_M$, denote by $\varpi_\lambda$ the canonical projection $M \longrightarrow M_\lambda$. Let $N_1$ and $N_2$ be two $g$-submodules of $M$ such that $N_1$ is contained in $N_2$. For $\lambda$ in $\mathcal{P}_M$, denote by $M_{\lambda,+}$ the subspace of highest weight vectors of $M_\lambda$. For the trivial action of $G$ on $M_\lambda$, $V_\lambda \otimes_k M_{\lambda,+}$ is a $g$-module. For $j = 1, 2$, set $N_{j,\lambda,+} := N_j \cap M_{\lambda,+}$ and for $v$ in $M_{\lambda,+}$, denote by $M_v$ the $g$-submodule of $M_\lambda$ generated by $v$.

Lemma D.2. Let $\lambda$ be in $\mathcal{P}_M$.

(i) For $j = 1, 2$, $N_j$ is the direct sum of $N_1 \cap M_\gamma$, $\gamma \in \mathcal{P}_M$.

(ii) For $v$ in $M_{\lambda,+}$, the $g$-modules $V_\lambda$ and $M_v$ are isomorphic.

(iii) There exists a basis $v_i$, $i \in I_A$ of $M_{\lambda,+}$ satisfying the following condition: for some subsets $I_{A,1}$ and $I_{A,2}$ of $I_A$, $I_{A,1} \subset I_{A,2}$, $v_i$, $i \in I_{A,1}$ and $v_i$, $i \in I_{A,2}$ are basis of $N_{1,\lambda,+}$ and $N_{2,\lambda,+}$ respectively.

(iv) For $i$ in $I_A$, denote by $\tau_{\lambda,i}$ an isomorphism of $g$-modules $V_\lambda \longrightarrow M_{v_i}$. Then the linear map

\[ V_\lambda \otimes_k M_{\lambda,+} \xrightarrow{\tau_{\lambda,i}} M_\lambda, \quad v \otimes v_i \longmapsto \tau_{\lambda,i}(v) \]

is an isomorphism of $g$-modules such that $\tau_{\lambda}(V_\lambda \otimes_k N_{j,\lambda,+}) = N_j \cap M_\lambda$ for $j = 1, 2$.

Proof. (i) As $N_j$ is a $g$-submodule of $M$, it is rational. So, by Lemma D.1(i), $N_j$ is the direct sum of its isotypic components, whence the assertion since an isotypic component of $N_j$ is contained in the isotypic component of $M$ of the same type.

(ii) As $v$ is in $M_{\lambda,+}$, $M_v$ is a module of highest weight $\lambda$ and the space of highest weight vectors in $M_v$ is generated by $v$. Hence $M_v$ is simple and isomorphic to $V_\lambda$.

(iii) is straightforward. Moreover, if $N_j \cap M_\lambda = \{0\}$ then $I_{A,j}$ is empty.

(iv) By (ii), the isomorphisms $\tau_{\lambda,i}$ does exist. As $v_i$, $i \in I_A$ is a basis of $M_{\lambda,+}$, $M_\lambda$ is the direct sum of the subspaces $M_{v_i}$, $i \in I_A$. Hence $\tau_{\lambda,i}$ is an isomorphism of $g$-modules. Moreover, for $j = 1, 2$, for $i$ in $I_A$, $\tau_{\lambda,i}(V_\lambda \otimes v_i)$ is contained in $N_j$ if and only if $i$ is in $I_{A,j}$, whence the assertion since $N_j$ is a $g$-module. □

Let $M'$ be a $l$-submodule of $M$. For $\mu$ in $\mathcal{P}_g$, denote by $V'_\mu$ a simple $l$-module of highest weight $\mu$ and $M'_\mu$ the isotypic component of type $V'_\mu$ of $M'$. Denote by $\mathcal{P}_M'$ the subset of elements $\mu$ of $\mathcal{P}_g$ such that $M'_\mu \neq \{0\}$ and $\mathcal{P}_{M,M'}$ the subset of elements $(\lambda, \mu)$ of $\mathcal{P}_M \times \mathcal{P}_M'$ such that $\varpi_\lambda(M'_\mu) \neq \{0\}$.

Lemma D.3. (i) The space $M'$ is the direct sum of $M'_\mu$, $\mu \in \mathcal{P}_M'$.

(ii) For $(\lambda, \mu)$ in $\mathcal{P}_{M,M'}$, $V'_\mu$ is isomorphic to a $l$-submodule of $V_\lambda$.

Proof. (i) As $M'$ is a $l$-submodule of the rational $g$-module $M$, $M'$ is a rational $l$-module, whence the assertion by Lemma D.1(i).
(iii) Let \((\lambda, \mu)\) in \(P_{M', M}\) and \(V_0\) a simple \(l\)-module contained in \(\varpi_{\lambda}(M'_\mu)\). According to Lemma D.2,(ii) and (iii),
\[ M_\lambda = \bigoplus_{i \in I_\lambda} M_{v_i}. \]
For \(i\) in \(I_\lambda\), denote by \(\pi_i\) the projection
\[ M_\lambda \xrightarrow{\pi_i} M_{v_i} \]
corresponding to this decomposition. For some \(i\), the restriction to \(V_0\) is different from 0. As \(V_0\) is a simple \(l\)-module, this restriction is an embedding of \(V_0\) into \(M_{v_i}\), whence the assertion since \(M_{v_i}\) is isomorphic to \(V_\lambda\).

For \(\lambda\) in \(P_M\), denote by \(V^l_\lambda\) the subspace of elements of \(V_\lambda\), annihilated by \(u \cap l\), and for \((\lambda, \mu)\) in \(P_{M', M}\), let \(V^l_{\lambda, \mu}\) be the subspace of weight \(\mu\) of \(V^l_\lambda\).

**Lemma D.4.** Let \((\lambda, \mu)\) be in \(P_{M, M'}\).

(i) There exists an isomorphism of \(l\)-modules
\[ V'_\mu \otimes_k V^l_{\lambda, \mu} \xrightarrow{\tau_{\mu, \lambda}} U(l).V^l_{\lambda, \mu}. \]

(ii) For a well defined subspace \(E_{\lambda, \mu}\) of \(V^l_{\lambda, \mu} \otimes_k M_{\lambda, +}\),
\[ \varpi_{\lambda}(M'_\mu) = \tau_{\mu, \lambda}(\tau_{\mu, \lambda} \otimes \text{id}_{M_{\lambda, +}})(V'_\mu \otimes_k E_{\lambda, \mu}). \]

(iii) For \(j = 1, 2\), let \(E_{\lambda, \mu, j}\) be the intersection of \(E_{\lambda, \mu}\) and \(V'_\mu \otimes_k N_{j, \lambda, +}\). Then
\[ \varpi_{\lambda}(N_j \cap M'_\mu) = \tau_{\mu, \lambda}(\tau_{\mu, \lambda} \otimes \text{id}_{M_{\lambda, +}})(V'_\mu \otimes_k E_{\lambda, \mu, j}). \]

**Proof.** (i) Let \(w_1, \ldots, w_m\) be a basis of \(V^l_{\lambda, \mu}\). For \(i = 1, \ldots, m\), denote by \(V'_i\) the \(l\)-submodule of \(V_\lambda\) generated by \(w_i\). As \(w_i\) is weight vector of weight \(\mu\) of \(V^l_{\lambda, \mu}\), \(V'_i\) is a module of highest weight \(\mu\) and the space of highest weight vectors in \(V'_i\) is generated by \(w_i\) so that \(V'_i\) is a simple module isomorphic to \(V'_\mu\). Moreover, \(U(l).V^l_{\lambda, \mu}\) is the direct sum of \(V'_i\), \(i = 1, \ldots, m\) since \(w_1, \ldots, w_m\) is a basis of \(V^l_{\lambda, \mu}\), whence an isomorphism
\[ V'_\mu \otimes_k V^l_{\lambda, \mu} \xrightarrow{\tau_{\mu, \lambda}} U(l).V^l_{\lambda, \mu}. \]

(ii) For \(v\) in \(\tau_{\lambda, \mu}^{-1}(\varpi_{\lambda}(M'_\mu))\), \(v\) has an expansion
\[ v = \sum_{i \in I_\lambda} u'_i \otimes v_i \]
with \(u'_i, i \in I_\lambda\) in \(V_\lambda\). As \(\varpi_{\lambda}(v)\) is \(\varpi_{\lambda}(M'_\mu)\), for \(i \in I_\lambda\), \(u \cdot v_i\) is in \(V^l_{\lambda, \mu}\) for some \(u\) in \(U(u \cap l)\). As a result, \(\tau_{\lambda, \mu}^{-1}(\varpi_{\lambda}(M'_\mu))\) is a subspace of \(U(l).V^l_{\lambda, \mu} \otimes_k M_{\lambda, +}\), whence the assertion by (i).

(iii) Let \(v\) be in \(E_{\lambda, \mu}\). By Lemma D.2(ii), \(\tau_{\lambda, \mu}(\tau_{\lambda, \mu} \otimes \text{id}_{M_{\lambda, +}})(v)\) is in \(N_j\) if and only if \(\tau_{\lambda, \mu} \otimes \text{id}_{M_{\lambda, +}}(v)\) is in \(V_\lambda \otimes_k N_{j, \lambda, +}\). Then, by (ii), \(\tau_{\lambda, \mu}(\tau_{\lambda, \mu} \otimes \text{id}_{M_{\lambda, +}})(v)\) is in \(\varpi_{\lambda}(N_j \cap M'_\mu)\) if and only if \(v\) is in \(E_{\lambda, \mu, j}\). \(\square\)
For \( \lambda \) in \( \mathcal{P}_M \), let \( \theta_\lambda \) be the linear map
\[
V_\lambda^* \otimes_k V_\lambda \xrightarrow{\theta_\lambda} k, \quad v' \otimes v \mapsto \langle v', v \rangle
\]
given by the duality. The kernel \( K_\lambda \) of \( \theta_\lambda \) is a \( G \)-submodule of \( V_\lambda^* \otimes_k V_\lambda \) so that
\[
V_\lambda^* \otimes_k V_\lambda \otimes_k M_{\lambda,+} = M_{\lambda,+} \oplus K_\lambda \otimes_k M_{\lambda,+}.
\]

**Corollary D.5.** Suppose that \( M' \) generates the \( g \)-module \( M \). Let \( \lambda \) be in \( \mathcal{P}_M \). Then
\[
M_{\lambda,+} = \theta_\lambda \otimes \text{id}_{M_{\lambda,+}} ( \bigoplus_{\mu \in \mathcal{P}_{M',\lambda}} V_\lambda^* \otimes_k E_{\lambda,\mu}) \quad \text{with} \quad \mathcal{P}_{M,M',\lambda} := \{ \mu \in \mathcal{P}_{M'} \mid (\lambda, \mu) \in \mathcal{P}_{M,M'} \}
\]
and for \( j = 1, 2 \),
\[
N_{j,\lambda,+} = \theta_\lambda \otimes \text{id}_{M_{\lambda,+}} ( \bigoplus_{\mu \in \mathcal{P}_{M',\lambda}} V_\lambda^* \otimes_k E_{\lambda,\mu,j}).
\]

**Proof.** As \( E_{\lambda,\mu} \) is contained in \( V_\lambda^* \otimes_k M_{\lambda,+} \), the sums
\[
\sum_{\mu \in \mathcal{P}_{M',\lambda}} E_{\lambda,\mu} \quad \text{and} \quad \sum_{\mu \in \mathcal{P}_{M',\lambda}} M_{\lambda,+} \cap V_\lambda^* \otimes_k E_{\lambda,\mu}
\]
are direct. By the hypothesis, \( M_1 \) is the \( g \)-submodule of \( M \) generated by \( \sigma_1(M') \). The map \( \theta_\lambda \otimes \text{id}_{M_{\lambda,+}} \) is a morphism of \( g \)-modules for the trivial action of \( g \) in \( M_{\lambda,+} \). Hence
\[
\theta_\lambda \otimes \text{id}_{M_{\lambda,+}} ( \bigoplus_{\mu \in \mathcal{P}_{M',\lambda}} V_\lambda^* \otimes_k E_{\lambda,\mu}) = M_{\lambda,+}
\]
by Lemma D.3(i). Then, by Lemma D.4(ii),
\[
\theta_\lambda \otimes \text{id}_{M_{\lambda,+}} ( \bigoplus_{\mu \in \mathcal{P}_{M',\lambda}} V_\lambda^* \otimes_k E_{\lambda,\mu}) = M_{\lambda,+},
\]
and by Lemma D.4(iii),
\[
\theta_\lambda \otimes \text{id}_{M_{\lambda,+}} ( \bigoplus_{\mu \in \mathcal{P}_{M',\lambda}} V_\lambda^* \otimes_k E_{\lambda,\mu,j}) = N_{j,\lambda,+}.
\]

**Proposition D.6.** Suppose that the following conditions are satisfied:

1. \( M' \) generates the \( G \)-module \( M \).
2. \( N_2 \cap M' \) is contained in \( N_1 \cap M' \).

Then \( N_1 = N_2 \).

**Proof.** By Lemma D.3(i), for \( j = 1, 2 \), \( N_j \cap M' \) is the direct sum of \( N_j \cap M'_\mu \) since \( N_j \cap M' \) is a \( \lambda \)-submodule of \( M' \). So, by Condition (2), for all \( (\lambda, \mu) \in \mathcal{P}_{M,M'} \), \( E_{\lambda,\mu,2} \) is contained in \( E_{\lambda,\mu,1} \).

Then, by Condition (1) and Corollary D.5, \( N_{2,\lambda,+} \) is contained in \( N_{1,\lambda,+} \) for \( \lambda \) in \( \mathcal{P}_M \). As a result, by Lemma D.2(iii), \( N_1 \cap M_\lambda = N_2 \cap M_\lambda \) for all \( \lambda \) in \( \mathcal{P}_M \) since \( N_1 \) is contained in \( N_2 \), whence the proposition by Lemma D.2(i). \( \square \)
References


Jean-Yves Charbonnel, Université de Paris - CNRS, Institut de Mathématiques de Jussieu - Paris Rive Gauche, UMR 7586, Groupes, représentations et géométrie, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France

E-mail address: jean-yves.charbonnel@imj-prg.fr