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Jean-Yves Charbonnel

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PROJECTIVE DIMENSION AND COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA

JEAN-YVES CHARBONNEL

Abstract. The commuting variety of a reductive Lie algebra $\mathfrak{g}$ is the underlying variety of a well defined subscheme of $\mathfrak{g} \times \mathfrak{g}$. In this note, it is proved that this scheme is normal and Cohen-Macaulay. In particular, its ideal of definition is a prime ideal. As a matter of fact, this theorem results from a so called Property (**P**) for a simple Lie algebra. This property says that some cohomology complexes are exact.

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1. Introduction

In this note, the base field $\Bbbk$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, ℓ is its rank, and G is its adjoint group.

1.1. The dual of $\mathfrak{g}$ identifies with $\mathfrak{g}$ by a non degenerate symmetric bilinear form on $\mathfrak{g}$ extending the Killing form of the derived algebra of $\mathfrak{g}$. Denote by $(v,w) \mapsto \langle v,w\rangle$ this bilinear form and $I_{\mathfrak{g}}$ the ideal of $\Bbbk[\mathfrak{g}\times\mathfrak{g}]$ generated by the functions $(x,y) \mapsto \langle v,[x,y]\rangle$, $v \in \mathfrak{g}$. The commuting variety $\mathcal{C}(\mathfrak{g})$ of $\mathfrak{g}$ is the subvariety of elements (x,y) of $\mathfrak{g}\times\mathfrak{g}$ such that $[x,y] = 0$. It is the underlying variety of the subscheme $\mathcal{S}(\mathfrak{g})$ of $\mathfrak{g}\times\mathfrak{g}$ defined by $I_{\mathfrak{g}}$. It is a well known and long standing open question whether or not this scheme is reduced, that is $\mathcal{C}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g})$. According to Richardson [Ri79], $\mathcal{C}(\mathfrak{g})$ is irreducible and according to Popov [Po08, Theorem 1], the singular locus of $\mathcal{S}(\mathfrak{g})$ has codimension at least 2 in $\mathcal{C}(\mathfrak{g})$. Then, according to Serre's normality criterion, arises the question to know whether or not $\mathcal{C}(\mathfrak{g})$ is normal. There are many results about the commuting variety. A result of Dixmier [Di79] proves that $I_{\mathfrak{g}}$ contains all the elements of the radical of $I_{\mathfrak{g}}$, of degree 1 in the second variable. In [Ga-Gi06], Gan and Ginzburg prove that for $\mathfrak{g}$ simple of type A, the invariant elements under G of $I_{\mathfrak{g}}$ is a radical ideal of the algebra $\Bbbk[\mathfrak{g}\times\mathfrak{g}]^G$ of invariant elements of $\Bbbk[\mathfrak{g}\times\mathfrak{g}]$ under G. In [Gi12], Ginzburg proves that the normalization of $\mathcal{C}(\mathfrak{g})$ is Cohen-Macaulay.

1.2. **Main results.** According to the identification of $\mathfrak{g}$ and its dual, $\Bbbk[\mathfrak{g}\times\mathfrak{g}]$ is equal to the symmetric algebra $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$ of $\mathfrak{g}\times\mathfrak{g}$. The main result of this note is the following theorem:

Theorem 1.1. *The subscheme of $\mathfrak{g}\times\mathfrak{g}$ defined by $I_{\mathfrak{g}}$ is Cohen-Macaulay and normal. Furthermore, $I_{\mathfrak{g}}$ is a prime ideal of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$.*

According to Richardson's result and Popov's result, it suffices to prove that the scheme $\mathcal{S}(\mathfrak{g})$ is Cohen-Macaulay. The main idea of the proof in the theorem uses the main argument of the Dixmier's proof: for a finitely generated module M over $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$, $M = 0$ if the codimension of its support is at least $l+2$ with l the projective dimension of M (see Appendix B).

All the complexes considered in this note are localizations of submodules of the algebra $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathrm{S}(\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$. We introduce the characteristic submodule of $\mathfrak{g}$, denoted by $\mathrm{B}_{\mathfrak{g}}$. By definition, $\mathrm{B}_{\mathfrak{g}}$ is a submodule of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ and an element φ of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ is in $\mathrm{B}_{\mathfrak{g}}$ if and only if for all (x,y) in a dense subset of $\mathfrak{g}\times\mathfrak{g}$, $\varphi(x,y)$ is in the sum of the subspaces $\mathfrak{g}^{ax+by}$ with (a,b) in $\Bbbk^2\setminus\{0\}$ and $\mathfrak{g}^{ax+by}$ the centralizer of $ax+by$ in $\mathfrak{g}$. According to a Bolsinov's result, $\mathrm{B}_{\mathfrak{g}}$ is a free $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-module of rank $\mathrm{b}_{\mathfrak{g}}$, the dimension of the Borel subalgebras of $\mathfrak{g}$. Moreover, the orthogonal complement to $\mathrm{B}_{\mathfrak{g}}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ is a free $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-module of rank $\mathrm{b}_{\mathfrak{g}}-\ell$ and for φ in $\mathrm{B}_{\mathfrak{g}}$, $\langle\varphi(x,y),[x,y]\rangle = 0$ for all (x,y) in $\mathfrak{g}\times\mathfrak{g}$. Let d be the $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-derivation of the algebra $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$ such that for v in $\mathfrak{g}$, $\mathrm{d}v$ is the function on $\mathfrak{g}\times\mathfrak{g}$: $(x,y)\mapsto\langle v,[x,y]\rangle$. Then d is a structure of complex on $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$ and the ideal of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$ generated by $\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ is a subcomplex. The usual gradation of $\bigwedge(\mathfrak{g})$ induces a gradation of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$. Denote by

$C_\bullet(\mathfrak{g})$ the graded subcomplex of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$ such that $C_{i+\mathrm{b}_\mathfrak{g}}(\mathfrak{g}) := \mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge^i(\mathfrak{g})\wedge\bigwedge^{\mathrm{b}_\mathfrak{g}}(\mathrm{B}_\mathfrak{g})$. Then Theorem 1.1 is a consequence of the following theorem:

Theorem 1.2. *The complex $C_\bullet(\mathfrak{g})$ has no homology of degree bigger than $\mathrm{b}_\mathfrak{g}$ and $I_\mathfrak{g}$ is isomorphic to the space of boundaries of degree $\mathrm{b}_\mathfrak{g}$.*

By standard results of homological algebra (see appendix), Theorem 1.2 is a consequence of the key theorem of this note:

Theorem 1.3. *For $i = 1,\dots,\mathrm{b}_\mathfrak{g}-\ell$, $C_{i+\mathrm{b}_\mathfrak{g}}(\mathfrak{g})$ has projective dimension at most i.*

As a matter of fact, it is easy to see that the support of the homology of $C_\bullet(\mathfrak{g})$ is contained in $\mathcal{C}(\mathfrak{g})$. Then, by Theorem 1.3, $C_\bullet(\mathfrak{g})$ has no homology of degree bigger than $\mathrm{b}_\mathfrak{g}$ and $I_\mathfrak{g}$ has projective dimension at most $2(\mathrm{b}_\mathfrak{g}-\ell)-1$. So, by Auslander-Buchsbaum's theorem, $\mathcal{S}(\mathfrak{g})$ is Cohen-Macaulay.

For the proof of Theorem 1.3, we consider the algebra $\mathrm{S}(\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$ and the $\bigwedge(\mathfrak{g})$-derivation d such that

$$\mathrm{d}v\otimes a = v\wedge a \quad \text{with} \quad v\in\mathfrak{g}, \quad a\in\bigwedge(\mathfrak{g}).$$

Then $(\mathrm{S}(\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g}),\mathrm{d})$ is a complex and the usual gradation of $\bigwedge(\mathfrak{g})$ induces on this complex a structure of graded cohomology complex denoted by $D^\bullet(\mathfrak{g})$. For k nonnegative integer, denote by $D^\bullet_k(\mathfrak{g})$ the graded subcomplex of $D^\bullet(\mathfrak{g})$ such that

$$D^i_k(\mathfrak{g}) := \mathrm{S}^{k-i}(\mathfrak{g})\otimes_{\Bbbk}\bigwedge^i(\mathfrak{g})$$

and $D^\bullet_k(\mathfrak{g},\mathrm{B}_\mathfrak{g})$ the graded subcomplex of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}D^\bullet(\mathfrak{g})$ such that

$$D^{i+\mathrm{b}_\mathfrak{g}}_k(\mathfrak{g},\mathrm{B}_\mathfrak{g}) := \mathrm{S}^{k-i}(\mathfrak{g})\otimes_{\Bbbk}\bigwedge^i(\mathfrak{g})\wedge\bigwedge^{\mathrm{b}_\mathfrak{g}}(\mathrm{B}_\mathfrak{g}).$$

Definition 1.4. Let $n := \mathrm{b}_\mathfrak{g}-\ell$ and $j = 1,\dots,n$. We say that $\mathfrak{g}$ has Property $(\mathbf{P}_j)$ if $D^\bullet_k(\mathfrak{g},\mathrm{B}_\mathfrak{g})$ has no cohomology of degree different from $\mathrm{b}_\mathfrak{g}$ for $k = 1,\dots,j$.

We say that $\mathfrak{g}$ has Property $(\mathbf{P})$ if it has Property $(\mathbf{P}_n)$.

By an induction argument, Theorem 1.3 is a consequence of the following Theorem:

Theorem 1.5. *All simple Lie algebra has Property* $(\mathbf{P})$.

As a matter of fact, the proof of this theorem is the main part of this note.

1.3. **Sketch of proofs.** We suppose $\mathfrak{g}$ simple and we prove Theorem 1.5 by induction on the rank of $\mathfrak{g}$. For $k = 1,\dots,n$, we denote by S_k the support in $\mathfrak{g}\times\mathfrak{g}$ of the cohomology of $D^\bullet_k(\mathfrak{g},\mathrm{B}_\mathfrak{g})$ of degree different from $\mathrm{b}_\mathfrak{g}$. This subset of $\mathfrak{g}\times\mathfrak{g}$ is invariant under the diagonal action of G and the canonical action of $\mathrm{GL}_2(\Bbbk)$ since $\mathrm{B}_\mathfrak{g}$ is a relatively equivariant module under these actions. As a result, the image of S_k by the first projection

$$\mathfrak{g}\times\mathfrak{g} \xrightarrow{\varpi_1} \mathfrak{g}$$

is a G-invariant closed subset of $\mathfrak{g}$. In particular, if $\varpi_1(S_k)$ does not contain semisimple elements different from 0, $\varpi_1(S_k)$ is contained in the nilpotent cone $\mathfrak{N}_\mathfrak{g}$ of $\mathfrak{g}$ and S_k is contained in the so-called nilpotent bicone $\mathcal{N}_\mathfrak{g}$ of $\mathfrak{g}$. By definition, $\mathcal{N}_\mathfrak{g}$ is the subset of elements (x, y) of $\mathfrak{g}\times\mathfrak{g}$ such that the subspace of $\mathfrak{g}$, generated by x and y, is contained in $\mathfrak{N}_\mathfrak{g}$. By [CMo08, Theorem 1.2], $\mathcal{N}_\mathfrak{g}$ has codimension $\mathrm{b}_\mathfrak{g} + \ell$ in $\mathfrak{g} \times \mathfrak{g}$. So, when $\varpi_1(S_k)$ is contained in $\mathfrak{N}_\mathfrak{g}$ for $k = 1, \dots, j$, $\mathfrak{g}$ has Property $(\mathbf{P}_j)$ by Corollary B.3 and an induction argument on j.

As a result, the main step of the proof of Theorem 1.5 is the equality $\varpi_1(S_k) \cap \mathfrak{h} = \{0\}$ for $k = 1, \dots, n$. Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$. Let z be in $\mathfrak{h}$. Denote by $\mathfrak{g}^z$ the centralizer of z in $\mathfrak{g}$. The orbit of z under the Weyl group contains an element z' such that $\mathfrak{g}^{z'} + \mathfrak{b}$ is a parabolic subalgbera. So, we can suppose that $\mathfrak{p} := \mathfrak{g}^z + \mathfrak{b}$ is an algebra. Then $\mathfrak{l} := \mathfrak{g}^z$ is the reductive factor of $\mathfrak{p}$ containing $\mathfrak{h}$. Denote by $\mathfrak{d}$ the derived algebra of $\mathfrak{l}$ and $\mathfrak{l}_*$ the subset of elements x of $\mathfrak{l}$ such that $\mathfrak{g}^x$ is contained in $\mathfrak{l}$. Then $\mathfrak{l}_*$ is a principal open subset of $\mathfrak{l}$. Let $\mathfrak{d}_1, \dots, \mathfrak{d}_\mathrm{n}$ be the simple factors of $\mathfrak{d}$, $\mathfrak{z}$ the center of $\mathfrak{l}$, $\mathfrak{p}_{\pm,\mathrm{u}}$ the sum of root spaces with respect to $\mathfrak{h}$, not contained in $\mathfrak{l}$ and d the half dimension of $\mathfrak{p}_{\pm,\mathrm{u}}$. When z is regular, $\mathrm{n} = 0$. When n is positive, for $i = 1, \dots, \mathrm{n}$, denote by $\ell_{\mathfrak{d}_i}$ the rank of $\mathfrak{d}_i$, $\mathrm{b}_{\mathfrak{d}_i}$ the dimension of the Borel subalgebras of $\mathfrak{d}_i$ and set $n_i := \mathrm{b}_{\mathfrak{d}_i} - \ell_{\mathfrak{d}_i}$. For j nonnegative integer, set:

$$\mathbb{I}_j := \{(i_{-1}, \dots, i_\mathrm{n}) \in \mathbb{N}^{\mathrm{n}+2} \mid i_1 \leq n_1, \dots, i_\mathrm{n} \leq n_\mathrm{n},\ i_{-1} + \cdots + i_\mathrm{n} = j\},$$

and for $k = 1, \dots, n$ and $\iota = (i_{-1}, \dots, i_\mathrm{n})$ in $\mathbb{I}_k$, denote by $D^\bullet_{k,\iota,\#}(\mathfrak{g})$ the simple complex deduced from the multicomplex

$$D^\bullet_{i_{-1}}(\mathfrak{p}_{\pm,\mathrm{u}}) \otimes_\Bbbk D^\bullet_{i_0}(\mathfrak{z}) \otimes_\Bbbk D^\bullet_{i_1}(\mathfrak{d}_1) \otimes_\Bbbk \cdots \otimes_\Bbbk D^\bullet_{i_\mathrm{n}}(\mathfrak{d}_\mathrm{n})$$

and set:

$$D^\bullet_{k,\#}(\mathfrak{g}) := \bigoplus_{\iota \in \mathbb{I}_k} D^\bullet_{k,\iota,\#}(\mathfrak{g}).$$

Let B be the restriction of $\mathrm{B}_\mathfrak{g}$ to $\mathfrak{l}_*\times\mathfrak{g}$ and $\widetilde{\mathrm{B}}$ the $\Bbbk[G]\otimes_\Bbbk\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]$-submodule of $\Bbbk[G]\otimes_\Bbbk\Bbbk[\mathfrak{l}_* \times \mathfrak{g}]\otimes_\Bbbk\mathfrak{g}$ generated by the maps

$$(g, x, y) \longmapsto g.\varphi(x, y) \quad \text{with} \quad \varphi \in \mathrm{B}.$$

Then B and $\widetilde{\mathrm{B}}$ are free modules of rank $\mathrm{b}_\mathfrak{g}$. Denote by $D^\bullet_{k,\#}(\mathfrak{g}, \mathrm{B})$ and $D^\bullet_{k,\#}(\mathfrak{g}, \widetilde{\mathrm{B}})$ the graded subcomplexes of $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk D^\bullet(\mathfrak{g})$ and $\Bbbk[G] \otimes_\Bbbk \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk D^\bullet(\mathfrak{g})$,

$$D^\bullet_{k,\#}(\mathfrak{g}, \mathrm{B}) := D^\bullet_{k,\#}(\mathfrak{g})[-\mathrm{b}_\mathfrak{g}] \wedge \textstyle\bigwedge^{\mathrm{b}_\mathfrak{g}}(\mathrm{B}) \quad \text{and} \quad D^\bullet_{k,\#}(\mathfrak{g}, \widetilde{\mathrm{B}}) := D^\bullet_{k,\#}(\mathfrak{g})[-\mathrm{b}_\mathfrak{g}] \wedge \textstyle\bigwedge^{\mathrm{b}_\mathfrak{g}}(\widetilde{\mathrm{B}}).$$

An important step of the proof of Theorem 1.5 is the following proposition:

Proposition 1.6. *Suppose that the simple factors of* $\mathfrak{l}$ *have Property* $(\mathbf{P})$*. Then, for* $k = 1, \dots, n$*,* $D^\bullet_{k,\#}(\mathfrak{g}, \widetilde{\mathrm{B}})$ *has no cohomology of degree different from* $\mathrm{b}_\mathfrak{g}$*.*

When $\mathfrak{n} = 0$, z is regular and when $\mathfrak{n} = 1$, $\mathfrak{d}$ is simple. In this case, Proposition 1.6 is given by Proposition 12.12. When $\mathfrak{n} \geq 2$, Proposition 1.6 is given by Proposition 13.8. Denote by $D_k^\bullet(\mathfrak{g}, \widetilde{\mathrm{B}})$ the graded complex

$$D_k^\bullet(\mathfrak{g}, \widetilde{\mathrm{B}}) := D_k^\bullet(\mathfrak{g})[-\mathrm{b}_\mathfrak{g}] \wedge \bigwedge^{\mathrm{b}_\mathfrak{g}}(\widetilde{\mathrm{B}}).$$

Then, from Proposition 1.6, Proposition D.6 and [C20, Theorem 1.1], we deduce that $D_k^\bullet(\mathfrak{g}, \widetilde{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_\mathfrak{g}$. As a matter of fact, [C20, Theorem 1.1] is only true for simple Lie algebras. The complex $D_k^\bullet(\mathfrak{g}, \widetilde{\mathrm{B}})$ is a subcomplex of $\Bbbk[G] \otimes_\Bbbk \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk D_k^\bullet(\mathfrak{g})$ and the morphism

$$G \times \mathfrak{l}_* \times \mathfrak{g} \longrightarrow \mathfrak{g} \times \mathfrak{g}\ , \qquad (g, x, y) \longmapsto (g(x), g(y))$$

is a flat morphism whose image is the cartesian product of an open neighborhood of z and $\mathfrak{g}$, whence the following corollary:

Corollary 1.7. *Suppose that the simple factors of* $\mathfrak{l}$ *have Property* (**P**). *Then, for* $k = 1, \ldots, n$, z *is not in* $\varpi_1(S_k)$.

Corollary 1.7 is given by Proposition 6.5.

As a result, one of the main step to prove Theorem 1.5 is the proof of Proposition 1.6. For that purpose, denote by $\overline{\mathrm{B}}$ the restriction of B to $\mathfrak{l}_* \times \mathfrak{p}$ and $\overline{\mathrm{B}}_\mathfrak{l}$ the restriction of $\mathrm{B}_\mathfrak{l}$ to $\mathfrak{l}_* \times \mathfrak{l}$. Then $\overline{\mathrm{B}}$ and $\overline{\mathrm{B}}_\mathfrak{l}$ are free modules of rank $\mathrm{b}_\mathfrak{g}$ and $\mathrm{b}_\mathfrak{l}$ respectively. Denoting by $\mathfrak{p}_\mathrm{u}$ the nilpotent radical of $\mathfrak{p}$ and $\mathfrak{p}_{-,\mathrm{u}}$ the complement to $\mathfrak{p}_\mathrm{u}$ in $\mathfrak{p}_{\pm,\mathrm{u}}$, invariant under the adjoint action of $\mathfrak{h}$,

$$\mathfrak{g} = \mathfrak{p}_{-,\mathrm{u}} \oplus \mathfrak{p} \quad \text{and} \quad \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_\mathrm{u}$$

so that $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$ and $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$ are subalgebras of $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$. Let $\overline{\mathrm{B}}_+$ be the submodule of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk \mathfrak{g}$ generated by $\overline{\mathrm{B}}_\mathfrak{l}$ and $\mathfrak{p}_\mathrm{u}$. Then $\overline{\mathrm{B}}_+$ is a free module of rank $\mathrm{b}_\mathfrak{g}$ and $\overline{\mathrm{B}}$ is a submodule of $\overline{\mathrm{B}}_+$. For M free submodule of rank $\mathrm{b}_\mathfrak{g}$ of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk \mathfrak{g}$, denote by $D_{k,\#}^\bullet(\mathfrak{g}, M)$ the graded subcomplex of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk D^\bullet(\mathfrak{g})$,

$$D_{k,\#}^\bullet(\mathfrak{g}, M) := D_{k,\#}^\bullet(\mathfrak{g})[-\mathrm{b}_\mathfrak{g}] \wedge \bigwedge^{\mathrm{b}_\mathfrak{g}}(M).$$

By Property (**P**) for the simple factors of $\mathfrak{l}$, for $k = 1, \ldots, n - d$, the complex $D_{k,\#}^\bullet(\mathfrak{l})[-\mathrm{b}_\mathfrak{l}] \wedge \bigwedge^{\mathrm{b}_\mathfrak{l}}(\overline{\mathrm{B}}_\mathfrak{l})$ has no cohomology of degree different from $\mathrm{b}_\mathfrak{l}$ since $\mathfrak{l}_*$ is a principal open subset of $\mathfrak{l}$ and the center of $\mathfrak{l}$ is contained in $\overline{\mathrm{B}}_\mathfrak{l}$. Then, for $k = 1, \ldots, n$, $D_{k,\#}^\bullet(\mathfrak{g}, \overline{\mathrm{B}}_+)$ and $D_{k,\#}^\bullet(\mathfrak{g}, \overline{\mathrm{B}})$ have no cohomology of degree different from $\mathrm{b}_\mathfrak{g}$.

Let $\mathcal{O}$ be the local ring of G at the identity and $\hat{\mathcal{O}}$ the completion of $\mathcal{O}$ with respect to the $\mathfrak{m}$-adic topology with $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$. Let $\widehat{\mathrm{B}}$ be the $\hat{\mathcal{O}} \otimes_\Bbbk \Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$-submodule of $\hat{\mathcal{O}} \otimes_\Bbbk \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk \mathfrak{g}$ generated by $\widetilde{\mathrm{B}}$ and $D_{k,*}(\mathfrak{g}, \widehat{\mathrm{B}})$ the subcomplex of $\hat{\mathcal{O}} \otimes_\Bbbk \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk D(\mathfrak{g})$,

$$D_{k,*}(\mathfrak{g}, \widehat{\mathrm{B}}) := D_{k,\#}(\mathfrak{g}) \wedge \bigwedge(\mathfrak{p}_\mathrm{u}) \wedge \bigwedge^{\mathrm{b}_\mathfrak{g}}(\widehat{\mathrm{B}}).$$

The space $D_{k,*}(\mathfrak{g},\widehat{\mathrm{B}})$ is a module over the algebra

$$\hat{\mathcal{O}}\otimes_{\Bbbk}\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}\bigwedge(\mathfrak{p}_{\mathrm{u}}).$$

Denoting by J the ideal of definition of $\mathfrak{l}_*\times\mathfrak{p}$ in $\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]$, let $\hat{J}$ be the ideal of this algebra generated by $\mathfrak{m}$, J and $\mathfrak{p}_{\mathrm{u}}$. When $\mathrm{n}\leq 1$, the subspace $\hat{J}D_{k,*}(\mathfrak{g},\widehat{\mathrm{B}})$ is the kernel of the restriction morphism

$$D_{k,*}(\mathfrak{g},\widehat{\mathrm{B}})\longrightarrow D_{k,\#}(\mathfrak{g},\overline{\mathrm{B}})\ .$$

This important result comes from the invertibility of some square matrices (see Subsection 9.2). As a matter of fact, the powers of $\hat{J}$ induce a filtration of $D_{k,*}(\mathfrak{g},\widehat{\mathrm{B}})$ and the graded space associate to this filtration is isomorphic to

$$A\otimes_{\Bbbk[\mathfrak{l}_*\times\mathfrak{p}]}D_{k,\#}(\mathfrak{g},\overline{\mathrm{B}})\quad\text{with}\quad A=\bigoplus_{l\in\mathbb{N}}\hat{J}^l/\hat{J}^{l+1}.$$

Then, by the above result on the cohomology of $D_{k,\#}(\mathfrak{g},\overline{\mathrm{B}})$ and the acyclicity of the complex $D_k^\bullet(\mathfrak{p}_{\mathrm{u}})$, $D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$, whence Proposition 1.6 for the case $\mathrm{n}\leq 1$ since $\hat{\mathcal{O}}$ is a faithfully flat extension of $\mathcal{O}$ and $D_{k,\#}(\mathfrak{g},\widetilde{\mathrm{B}})$ is G-equivariant.

Suppose $\mathrm{n}\geq 2$. Denote by E the $\Bbbk[\mathfrak{l}_*\times\mathfrak{l}]$-submodule of $\Bbbk[\mathfrak{l}_*\times\mathfrak{l}]\otimes_{\Bbbk}\mathfrak{p}$ generated by $\mathfrak{p}_{\mathrm{u}}$ and the generators of the modules $\mathrm{B}_{\mathfrak{b}_j}$, $j=1,\dots,\mathrm{n}$ which are not in $\Bbbk[\mathfrak{l}]\otimes_{\Bbbk}\mathfrak{l}$. Let $\hat{J}_+$ be the ideal of the algebra

$$\hat{\mathcal{O}}\otimes_{\Bbbk}\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk[\mathfrak{l}_*\times\mathfrak{l}]}\bigwedge(\mathrm{E})$$

generated by $\mathfrak{m}$, J, E. Let $D_{k,*}(\mathfrak{g},\widehat{\mathrm{B}})$ be the subcomplex of $\hat{\mathcal{O}}\otimes_{\Bbbk}\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}D(\mathfrak{g})$,

$$D_{k,*}(\mathfrak{g},\widehat{\mathrm{B}}):=D_{k,\#}(\mathfrak{g})\wedge\bigwedge(\mathrm{E})\wedge\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\widehat{\mathrm{B}}).$$

Then $D_{k,*}(\mathfrak{g},\widehat{\mathrm{B}})$ is a module over the algebra

$$\hat{\mathcal{O}}\otimes_{\Bbbk}\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk[\mathfrak{l}_*\times\mathfrak{l}]}\bigwedge(\mathrm{E}).$$

The subspace $\hat{J}_+D_{k,*}(\mathfrak{g},\widehat{\mathrm{B}})$ is the kernel of the restriction morphism

$$D_{k,*}(\mathfrak{g},\widehat{\mathrm{B}})\longrightarrow D_{k,\#}(\mathfrak{g},\overline{\mathrm{B}})\ .$$

As a matter of fact, the powers of $\hat{J}_+$ induce a filtration of $D_{k,*}(\mathfrak{g},\widehat{\mathrm{B}})$ and the graded space associate to this filtration is isomorphic to

$$A\otimes_{\Bbbk[\mathfrak{l}_*\times\mathfrak{p}]}D_{k,\#}(\mathfrak{g},\overline{\mathrm{B}})\quad\text{with}\quad A=\bigoplus_{l\in\mathbb{N}}\hat{J}_+^l/\hat{J}_+^{l+1}.$$

Then, by the above result on the cohomology of $D_{k,\#}(\mathfrak{g},\overline{\mathrm{B}})$ and the acyclicity of the complex$D_{k,\#}^\bullet(\mathrm{E})$, $D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$, whence Proposition 1.6 for the case $\mathrm{n}\geq 2$ since $\hat{\mathcal{O}}$ is a faithfully flat extension of $\mathcal{O}$ and $D_{k,\#}(\mathfrak{g},\widetilde{\mathrm{B}})$ is G-equivariant.

1.4. **Notations.** • As usual $\Bbbk^* := \Bbbk \setminus \{0\}$. For m positive integer and for $i = (i_1, \dots, i_m)$ in $\mathbb{N}^m$, set:

$$|i| := i_1 + \cdots + i_m$$

and for d in $\mathbb{N}$, denote by $\mathbb{N}^m_d$ the subset of $\mathbb{N}^m$:

$$\mathbb{N}^m_d := \{i \in \mathbb{N}^m \mid |i| = d\}$$

The set $\mathbb{N}^m$ is ordered by the lexicographic order induced by the usual order of $\mathbb{N}$. As a result, all subset of $\mathbb{N}^m$ is well ordered.

• For V a module over a $\Bbbk$-algebra, its dual is denoted by V^* and its symmetric and exterior algebras are denoted by $\mathrm{S}(V)$ and $\bigwedge(V)$ respectively. For all integer i, $\mathrm{S}^i(V)$ and $\bigwedge^i(V)$ are the spaces of degree i of $\mathrm{S}(V)$ and $\bigwedge(V)$ with respect to the usual gradation. In particular, for i negative, $\mathrm{S}^i(V)$ and $\bigwedge^i(V)$ are equal to $\{0\}$. If E is a subset of V, the submodule of V generated by E is denoted by $\mathrm{span}(E)$. When V is a vector space over $\Bbbk$, the grassmannian of all d-dimensional subspaces of V is denoted by $\mathrm{Gr}_d(V)$.

• All topological terms refer to the Zariski topology. If Y is a subset of a topological space X, denote by $\overline{Y}$ the closure of Y in X. For Y an open subset of the algebraic variety X, Y is called *a big open subset* if the codimension of $X \setminus Y$ in X is at least 2. For Y a closed subset of an algebraic variety X, its dimension is the biggest dimension of its irreducible components and its codimension in X is the smallest codimension in X of its irreducible components. For X an algebraic variety, $\Bbbk[X]$ is the algebra of regular functions on X, $\mathcal{O}_X$ is its structural sheaf and for x in X, $\mathcal{O}_{X,x}$ is the local ring of X at x.

• All the complexes considered in this note are graded complexes over $\mathbb{Z}$ of vector spaces and their differentials are homogeneous of degree ± 1 and they are denoted by d. As usual, the gradation of the complex is denoted by $C_\bullet$ if the degree of d is -1 and $C^\bullet$ otherwise.

For E a graded space over $\mathbb{Z}$ and for i integer, $E[i]$ is the graded space over $\mathbb{Z}$ whose subspace of degree n is the subspace of degree $n + i$ of E.

• The dimension of the Borel subalgebras of $\mathfrak{g}$ is denoted by $\mathrm{b}_\mathfrak{g}$. Set $n := \mathrm{b}_\mathfrak{g} - \ell$ so that

$$\dim \mathfrak{g} = 2\mathrm{b}_\mathfrak{g} - \ell_\mathfrak{g} = 2n + \ell.$$

Denote by $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$.

• The dual $\mathfrak{g}^*$ of $\mathfrak{g}$ identifies with $\mathfrak{g}$ by a given non degenerate, invariant, symmetric bilinear form $\langle ., . \rangle$ on $\mathfrak{g} \times \mathfrak{g}$ extending the Killing form of $[\mathfrak{g}, \mathfrak{g}]$.

• Let $\mathcal{R}$ be the root system of $\mathfrak{h}$ in $\mathfrak{g}$ and $\mathcal{R}_+$ the positive root system of $\mathcal{R}$ defined by $\mathfrak{b}$. The Weyl group of $\mathcal{R}$ is denoted by $W(\mathcal{R})$ and the basis of $\mathcal{R}_+$ is denoted by Π. For α in $\mathcal{R}$, its coroot is denoted by H_α, the corresponding root subspace is denoted by $\mathfrak{g}^\alpha$ and a generator x_α of $\mathfrak{g}^\alpha$ is chosen so that $\langle x_\alpha, x_{-\alpha} \rangle = 1$.

• Let $\mathcal{P}(\mathcal{R})$ be the set of weights of the root system $\mathcal{R}$ and $\mathcal{P}_+(\mathcal{R})$ the subset of dominant weights with respect to $\mathcal{R}_+$. For λ in $\mathcal{P}_+(\mathcal{R})$, denote by V_λ a simple $\mathfrak{g}$-module of highest weight λ.

• Let e be the sum of the x_β's, β in Π, and h the element of $\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$ such that $\beta(h) = 2$ for all β in Π. Then there exists a unique f in $[\mathfrak{g}, \mathfrak{g}]$ such that (e, h, f) is a principal $\mathfrak{sl}_2$-triple. The one parameter subgroup of G generated by $\mathrm{ad}\, h$ is denoted by $t \mapsto \rho(t)$. The Borel subalgebra containing f is denoted by $\mathfrak{b}_-$ and its nilpotent radical is denoted by $\mathfrak{u}_-$. Let B and B_- be the normalizers of $\mathfrak{b}$ and $\mathfrak{b}_-$ in G, U and U_- the unipotent radicals of B and B_- respectively.

Lemma 1.8. *Let O be a principal open subset of $\mathfrak{h}$ and Σ an irreducible hypersurface of $O + \mathfrak{u}$. Suppose that Σ is invariant under the one-parameter subgroup $t \mapsto \rho(t)$ of G. Then O is contained in Σ or $\Sigma = \Sigma \cap \mathfrak{h} + \mathfrak{u}$.*

Proof. As O is a principal open subset of $\mathfrak{h}$, $\Bbbk[O+\mathfrak{u}]$ is a factorial ring. Hence Σ is the nullvariety in $O + \mathfrak{u}$ of an element of $\Bbbk[O + \mathfrak{u}]$. As a result, O is contained in Σ or $\Sigma \cap \mathfrak{h}$ is an hypersurface of O. Suppose that O is not contained in Σ. For (x, y) in $\mathfrak{h} \times \mathfrak{u}$,

$$\lim_{t\to 0} \rho(t).(x + y) = x.$$

Hence $\Sigma \cap \mathfrak{h}$ is the image of Σ by the canonical projection $O + \mathfrak{u} \longrightarrow O$ and Σ is contained in $\Sigma \cap \mathfrak{h} + \mathfrak{u}$. Moreover, $\Sigma \cap \mathfrak{h}$ is an irreducible hypersurface of O as the image of an irreducible subset. Then $\Sigma \cap \mathfrak{h} + \mathfrak{u}$ is an irreducible hypersurface of O, whence the lemma. □

• For α a positive root, let $|\alpha|$ be its height so that $\alpha(h) = 2|\alpha|$.

• For $x \in \mathfrak{g}$, denote by x_{s} its semisimple component, x_{n} its nilpotent component and $\mathfrak{g}^x$ its centralizer in $\mathfrak{g}$. The set of regular elements of $\mathfrak{g}$ is

$$\mathfrak{g}_{\mathrm{reg}} := \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = \ell\}.$$

We denote by $\mathfrak{g}_{\mathrm{reg,ss}}$ the set of regular semisimple elements of $\mathfrak{g}$. Then $\mathfrak{g}_{\mathrm{reg}}$ and $\mathfrak{g}_{\mathrm{reg,ss}}$ are G-invariant dense open subsets of $\mathfrak{g}$. According to [V72], $\mathfrak{g} \setminus \mathfrak{g}_{\mathrm{reg}}$ is equidimensional of codimension 3.

• Let $\mathfrak{h}_{\mathfrak{b}}$ be the subset of elements x of $\mathfrak{h}$ such that $\mathfrak{g}^x + \mathfrak{b}$ is a subalgebra of $\mathfrak{g}$. The orbits of $W(\mathcal{R})$ in $\mathfrak{h}$ have a nonempty intersection with $\mathfrak{h}_{\mathfrak{b}}$.

• Denote by $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ the algebra of $\mathfrak{g}$-invariant elements of $\mathrm{S}(\mathfrak{g})$. Let $p_1, \dots, p_\ell$ be homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of degree $d_1, \dots, d_\ell$ respectively. Choose the polynomials $p_1, \dots, p_\ell$ so that $d_1 \leq \cdots \leq d_\ell$. For $i = 1, \dots, \ell$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$, consider a shift of p_i in direction y: $p_i(x + ty)$ with $t \in \Bbbk$. Expanding $p_i(x + ty)$ as a polynomial in t, one obtains

$$p_i(x + ty) = \sum_{m=0}^{d_i} p_i^{(m)}(x, y) t^m, \qquad \forall (t, x, y) \in \Bbbk \times \mathfrak{g} \times \mathfrak{g} \tag{1}$$

where $y \mapsto (m!)p_i^{(m)}(x,y)$ is the derivative at x of p_i at the order m in the direction y. The elements $p_i^{(m)}$ defined by (1) are invariant elements of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{S}(\mathfrak{g})$ under the diagonal action of G in $\mathfrak{g} \times \mathfrak{g}$. Remark that $p_i^{(0)}(x,y) = p_i(x)$ while $p_i^{(d_i)}(x,y) = p_i(y)$ for all (x,y) in $\mathfrak{g} \times \mathfrak{g}$. Set:

$$I_0 := \{(i,m) \in \{1,\dots,\ell\} \times \mathbb{N} \mid 0 \leq m \leq d_i - 1\} \quad \text{and} \quad I_{*,0} := I_0 \cap \{1,\dots,\ell\} \times (\mathbb{N} \setminus \{0\}).$$

According to our notations, I_0 is totally ordered.

Remark 1.9. The family $\mathcal{P}_x := \{p_i^{(m)}(x,.);\ 1 \leq i \leq \ell, 0 \leq m \leq d_i\}$ for $x \in \mathfrak{g}$, is a Poisson-commutative family of $\mathrm{S}(\mathfrak{g})$ by Mishchenko-Fomenko [MF78]. One says that the family $\mathcal{P}_x$ is constructed by the *argument shift method*.

• Let $i \in \{1,\dots,\ell\}$. For x in $\mathfrak{g}$, denote by $\varepsilon_i(x)$ the element of $\mathfrak{g}$ given by

$$\langle \varepsilon_i(x), y \rangle = \frac{\mathrm{d}}{\mathrm{d}t} p_i(x+ty)\,|_{t=0}$$

for all y in $\mathfrak{g}$. Thereby, ε_i is an invariant element of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ under the canonical action of G. According to [Ko63, Theorem 9], for x in $\mathfrak{g}$, x is in $\mathfrak{g}_{\mathrm{reg}}$ if and only if $\varepsilon_1(x),\dots,\varepsilon_\ell(x)$ are linearly independent. In this case, $\varepsilon_1(x),\dots,\varepsilon_\ell(x)$ is a basis of $\mathfrak{g}^x$.

Denote by $\varepsilon_i^{(m)}$, for $0 \leq m \leq d_i - 1$, the elements of $\mathrm{S}(\mathfrak{g} \times \mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ defined by the equality:

$$\varepsilon_i(x+ty) = \sum_{m=0}^{d_i-1} \varepsilon_i^{(m)}(x,y)t^m, \qquad \forall (t,x,y) \in \Bbbk \times \mathfrak{g} \times \mathfrak{g} \tag{2}$$

and set:

$$V_{x,y} := \mathrm{span}(\{\varepsilon_i^{(m)}(x,y),\ (i,m) \in I_0\})$$

for (x,y) in $\mathfrak{g} \times \mathfrak{g}$.

• Let $\mathfrak{N}_{\mathfrak{g}}$ be the nilpotent cone of $\mathfrak{g}$. For (x,y) in $\mathfrak{g} \times \mathfrak{g}$, denote by $P_{x,y}$ the subspace of $\mathfrak{g}$ generated by x and y. Let $\mathcal{N}_{\mathfrak{g}}$ be the nilpotent bicone of $\mathfrak{g}$. By definition, $\mathcal{N}_{\mathfrak{g}}$ is the subset of elements (x,y) of $\mathfrak{g} \times \mathfrak{g}$ such that $P_{x,y}$ is contained in $\mathfrak{N}_{\mathfrak{g}}$. In particular, $\mathcal{N}_{\mathfrak{g}}$ is invariant under the diagonal action of G in $\mathfrak{g} \times \mathfrak{g}$ and the canonical action of $\mathrm{GL}_2(\Bbbk)$ in $\mathfrak{g} \times \mathfrak{g}$.

1.5. **Organization of the note.** In Section 2, the characteristic submodule $\mathrm{B}_{\mathfrak{g}}$ is introduced and some of its properties are given. In particular, its restrictions to parabolic subalgebras are considered. In Section 3, we prove that the main theorem and Theorem 1.2 results from Theorem 1.3. In Section 4, we prove that Theorem 1.3 results from Theorem 1.5 so that we suppose that $\mathfrak{g}$ is simple in the following sections. In Section 5, we consider the support S_k in $\mathfrak{g} \times \mathfrak{g}$ of the cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$ of the complex $D_k^\bullet(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})$ and we prove that under some hypothesis on $\varpi_1(S_k)$ the codimension of S_k in $\mathfrak{g} \times \mathfrak{g}$ is at least $k+2$ so that $\mathfrak{g}$ has Property $(\mathbf{P}_k)$ if it has Property $(\mathbf{P}_{k-1})$. In Section 6, we recall a result of flatness and prove that $\varpi_1(S_i)$ does not contain z for $i = 1,\dots,k$ if $D_{i,\#}(\mathfrak{g}, \widetilde{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$ for $i = 1,\dots,k$. In Section 7, we study the restriction $\overline{\mathrm{B}}$ of $\mathrm{B}_{\mathfrak{g}}$ to $\mathfrak{l}_* \times \mathfrak{p}$. In particular, $\overline{\mathrm{B}}$ has a

good decomposition with respect to the decomposition of $\mathfrak{p} = \mathfrak{l} + \mathfrak{p}_u$ (see Proposition 7.6). Some other results are given so that the decomposition is more precise. This decomposition introduces some functions in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$ and their restrictions to $\mathfrak{l}_* \times \mathfrak{p} \cap \mathfrak{h} \times \mathfrak{h}$ are considered in Section 8 (see Proposition 8.5). In Section 9, we study the generator ε of $\bigwedge^{b_{\mathfrak{g}}}(B)$. In particular, some of its coordinates in the canonical basis of $\bigwedge^{b_{\mathfrak{g}}}(\mathfrak{g})$ are considered. In Subsection 9.2, some square matrices whose the coefficients depend on these coordinates are considered and we prove that their determinant are different from 0 (see Corollary 9.4). In Section 10, Proposition 10.3 is a key result for proving Corollary 12.7 under the hypothesis $n \leq 1$ and in Section 11, Proposition 11.2 is a key result for proving Corollary 13.3 in the general case. In Section 12, we prove Proposition 1.6 for the case $n \leq 1$ and in Section 13, we prove Proposition 1.6 for the general case. Then we can complete the proof of Theorem 1.5.

The appendix has four sections. In Section A, we give some general results on some complexes. In Section B, we recall some well known results of cohomology. In Section C, we introduce some rational numbers and we prove that some of them are different from 0. In Section D, we prove the equalities of two $\mathfrak{g}$-submodules of a rational $\mathfrak{g}$-module under certain hypothesis. This result is used to prove Proposition 6.5.

2. Characteristic module

For (x, y) in $\mathfrak{g} \times \mathfrak{g}$, set:

$$V'_{x,y} = \sum_{(a,b)\in\Bbbk^2\setminus\{0\}} \mathfrak{g}^{ax+by}.$$

By definition, the characteristic module $B_{\mathfrak{g}}$ of $\mathfrak{g}$ is the submodule of elements φ of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ such that $\varphi(x, y)$ is in $V'_{x,y}$ for all (x, y) in a dense subset of $\mathfrak{g} \times \mathfrak{g}$. In this section, some properties of $B_{\mathfrak{g}}$ are given.

2.1. First properties of $B_{\mathfrak{g}}$.

Denote by $\Omega_{\mathfrak{g}}$ the subset of elements (x, y) of $\mathfrak{g} \times \mathfrak{g}$ such that $P_{x,y}$ has dimension 2 and $P_{x,y} \setminus \{0\}$ is contained in $\mathfrak{g}_{\mathrm{reg}}$. According to [CMo08, Corollary 10], $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$.

Proposition 2.1. *Let (x, y) be in $\mathfrak{g} \times \mathfrak{g}$ such that $P_{x,y} \cap \mathfrak{g}_{\mathrm{reg}}$ is not empty.*

(i) *Let O be an open subset of $\Bbbk^2$ such that $ax + by$ is in $\mathfrak{g}_{\mathrm{reg}}$ for all (a, b) in O. Then $V_{x,y}$ is the sum of the $\mathfrak{g}^{ax+by}$'s, $(a, b) \in O$.*

(ii) *The spaces $[x, V_{x,y}]$ and $[y, V_{x,y}]$ are equal.*

(iii) *The space $V_{x,y}$ has dimension at most $b_{\mathfrak{g}}$ and the equality holds if and only if (x, y) is in $\Omega_{\mathfrak{g}}$.*

(iv) *The space $[x, V_{x,y}]$ is orthogonal to $V_{x,y}$. Furthermore, (x, y) is in $\Omega_{\mathfrak{g}}$ if and only if $[x, V_{x,y}]$ is the orthogonal complement to $V_{x,y}$ in $\mathfrak{g}$.*

(v) *The space $V_{x,y}$ is contained in $V'_{x,y}$. Moreover, $V_{x,y} = V'_{x,y}$ if (x, y) is in $\Omega_{\mathfrak{g}}$.*

(vi) *For (i, m) in I_0, $\varepsilon_i^{(m)}$ is a G-equivariant map.*

Proof. (i) For z in $\mathfrak{g}_{\mathrm{reg}}$, $\varepsilon_1(z), \dots, \varepsilon_\ell(z)$ is a basis of $\mathfrak{g}^z$ by [Ko63, Theorem 9]. Hence $\mathfrak{g}^{ax+by}$ is contained in $V_{x,y}$ for all (a,b) in O since the maps $\varepsilon_1, \dots, \varepsilon_\ell$ are homogeneous. For pairwise different elements $t_{i,0}, \dots, t_{i,d_i-1}$, $i = 1, \dots, \ell$ of $\Bbbk$, $\varepsilon_i^{(m)}(x,y)$ is a linear combination of $\varepsilon_i(x + t_{i,j}y)$, $j = 0, \dots, d_i - 1$ for $m = 0, \dots, d_i - 1$. We can choose $t_{i,0}, \dots, t_{i,d_i-1}$ so that $(a_i, a_i t_{i,0}), \dots, (a_i, a_i t_{i,d_i-1}))$ are in O for some a_i in $\Bbbk^*$, whence the assertion since the maps $\varepsilon_1, \dots, \varepsilon_\ell$ are homogeneous.

(ii) Let O be an open subset of $\Bbbk^{*2}$ such that $ax + by$ is in $\mathfrak{g}_{\mathrm{reg}}$ for all (a,b) in O. For all (a,b) in O, $[x, \mathfrak{g}^{ax+by}] = [y, \mathfrak{g}^{ax+by}]$ since $[ax + by, \mathfrak{g}^{ax+by}] = 0$ and $ab \neq 0$, whence the assertion by (i).

(iii) According to [Bou02, Ch. V, §5, Proposition 3],

$$d_1 + \cdots + d_\ell = \mathrm{b}_{\mathfrak{g}}.$$

So $V_{x,y}$ has dimension at most $\mathrm{b}_{\mathfrak{g}}$. By [Bol91, Theorem 2.1], $V_{x,y}$ has dimension $\mathrm{b}_{\mathfrak{g}}$ if and only if (x,y) is in $\Omega_{\mathfrak{g}}$.

(iv) According to [Bol91, Theorem 2.1], $V_{x,y}$ is a totally isotropic subspace with respect to the skew bilinear form on $\mathfrak{g}$

$$(v,w) \longmapsto \langle ax + by, [v,w] \rangle$$

for all (a,b) in $\Bbbk^2$. As a result, by invariance of $\langle ., . \rangle$, $V_{x,y}$ is orthogonal to $[x, V_{x,y}]$. If (x,y) is in $\Omega_{\mathfrak{g}}$, $\mathfrak{g}^x$ has dimension ℓ and it is contained in $V_{x,y}$. Hence, by (iii),

$$\dim [x, V_{x,y}] = \mathrm{b}_{\mathfrak{g}} - \ell = \dim \mathfrak{g} - \dim V_{x,y}$$

so that $[x, V_{x,y}]$ is the orthogonal complement to $V_{x,y}$ in $\mathfrak{g}$. Conversely, if $[x, V_{x,y}]$ is the orhogonal complement to $V_{x,y}$ in $\mathfrak{g}$, then

$$\dim V_{x,y} + \dim [x, V_{x,y}] = \dim \mathfrak{g}.$$

Since $P_{x,y} \cap \mathfrak{g}_{\mathrm{reg}}$ is not empty, $\mathfrak{g}^{ax+by} \cap V_{x,y}$ has dimension ℓ for all (a,b) in a dense open subset of $\Bbbk^2$. By continuity, $\mathfrak{g}^x \cap V_{x,y}$ has dimension at least ℓ so that

$$2\dim V_{x,y} - \ell \geq \dim \mathfrak{g}.$$

Hence, by (iii), (x,y) is in $\Omega_{\mathfrak{g}}$.

(v) By (i), $V_{x,y} \subset V'_{x,y}$. Suppose that (x,y) is in $\Omega_{\mathfrak{g}}$. According to [Ko63, Theorem 9], for all (a,b) in $\Bbbk^2 \setminus \{0\}$, $\varepsilon_1(ax+by), \dots, \varepsilon_\ell(ax+by)$ is a basis of $\mathfrak{g}^{ax+by}$. Hence $\mathfrak{g}^{ax+by}$ is contained in $V_{x,y}$, whence the assertion.

(vi) Let i be in $\{1, \dots, \ell\}$. Since p_i is G-invariant, ε_i is a G-equivariant map. As a result, its 2-polarizations $\varepsilon_i^{(0)}, \dots, \varepsilon_i^{(d_i-1)}$ are G-equivariant under the diagonal action of G in $\mathfrak{g} \times \mathfrak{g}$. □

Theorem 2.2. (i) *The module* $\mathrm{B}_{\mathfrak{g}}$ *is a free module of rank* $\mathrm{b}_{\mathfrak{g}}$ *whose a basis is the sequence* $\varepsilon_i^{(m)}$, $(i,m) \in I_0$.

(ii) *For* φ *in* $\mathrm{S}(\mathfrak{g} \times \mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$, φ *is in* $\mathrm{B}_{\mathfrak{g}}$ *if and only if* $p\varphi \in \mathrm{B}_{\mathfrak{g}}$ *for some* p *in* $\mathrm{S}(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$.

(iii) *For all* φ *in* $\mathrm{B}_{\mathfrak{g}}$ *and for all* (x,y) *in* $\mathfrak{g} \times \mathfrak{g}$, $\varphi(x,y)$ *is orthogonal to* $[x,y]$.

Proof. (i) and (ii) According to Proposition 2.1(v), $\varepsilon_i^{(m)}$ is in $\mathrm{B}_{\mathfrak{g}}$ for all (i,m). Moreover, according to Proposition 2.1(iii), these elements are linearly independent over $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$. Let φ be an element of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ such that $p\varphi$ is in $\mathrm{B}_{\mathfrak{g}}$ for some p in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\setminus\{0\}$. Then $\varphi(x,y)$ is in $V_{x,y}$ for all (x,y) in a dense open subset of $\Omega_{\mathfrak{g}}$ by Proposition 2.1(v). According to Proposition 2.1(iii), the map

$$\Omega_{\mathfrak{g}} \longrightarrow \mathrm{Gr}_{\mathrm{b}_{\mathfrak{g}}}(\mathfrak{g}) \ , \qquad (x,y)\longmapsto V_{x,y}$$

is regular. So $\varphi(x,y)$ is in $V_{x,y}$ for all (x,y) in $\Omega_{\mathfrak{g}}$ and for some regular functions $a_{i,m}$, $(i,m)\in I_0$ on $\Omega_{\mathfrak{g}}$,

$$\varphi(x,y)=\sum_{(i,m)\in I_0} a_{i,m}(x,y)\varepsilon_i^{(m)}(x,y)$$

for all (x,y) in $\Omega_{\mathfrak{g}}$. Since $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g}\times\mathfrak{g}$ and $\mathfrak{g}\times\mathfrak{g}$ is normal, the $a_{i,m}$'s have a regular extension to $\mathfrak{g}\times\mathfrak{g}$. Hence φ is a linear combination of the $\varepsilon_i^{(m)}$'s with coefficients in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$. As a result, the sequence $\varepsilon_i^{(m)}$, $(i,m)\in I_0$ is a basis of the module $\mathrm{B}_{\mathfrak{g}}$ and $\mathrm{B}_{\mathfrak{g}}$ is the subset of elements φ of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ such that $p\varphi\in\mathrm{B}_{\mathfrak{g}}$ for some p in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\setminus\{0\}$.

(iii) Let φ be in $\mathrm{B}_{\mathfrak{g}}$. According to (i) and Proposition 2.1(iv), for all (x,y) in $\Omega_{\mathfrak{g}}$, $[x,\varphi(x,y)]$ is orthogonal to $V_{x,y}$. Then, since y is in $V_{x,y}$, $[x,\varphi(x,y)]$ is orthogonal to y and $\langle\varphi(x,y),[x,y]\rangle=0$, whence the assertion. □

2.2. **Orthogonal of $\mathrm{B}_{\mathfrak{g}}$.** Denote again by $\langle .,.\rangle$ the canonical extension of $\langle .,.\rangle$ to the module $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$.

Proposition 2.3. *Let $\mathrm{C}_{\mathfrak{g}}$ be the orthogonal complement to $\mathrm{B}_{\mathfrak{g}}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$.*

(i) *For φ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$, φ is in $\mathrm{C}_{\mathfrak{g}}$ if and only if $\varphi(x,y)$ is in $[x,V_{x,y}]$ for all (x,y) in a nonempty open subset of $\mathfrak{g}\times\mathfrak{g}$.*

(ii) *The module $\mathrm{C}_{\mathfrak{g}}$ is free of rank $\mathrm{b}_{\mathfrak{g}}-\ell$. Furthermore, the sequence of maps*

$$(x,y)\mapsto[x,\varepsilon_i^{(m)}(x,y)],\ (i,m)\in I_{*,0}$$

is a basis of $\mathrm{C}_{\mathfrak{g}}$.

(iii) *The orthogonal complement to $\mathrm{C}_{\mathfrak{g}}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ is equal to $\mathrm{B}_{\mathfrak{g}}$.*

Proof. (i) Let φ be in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$. If φ is in $\mathrm{C}_{\mathfrak{g}}$, then $\varphi(x,y)$ is orthogonal to $V_{x,y}$ for all (x,y) in $\Omega_{\mathfrak{g}}$. Then, according to Proposition 2.1(iv), $\varphi(x,y)$ is in $[x,V_{x,y}]$ for all (x,y) in $\Omega_{\mathfrak{g}}$. Conversely, suppose that $\varphi(x,y)$ is in $[x,V_{x,y}]$ for all (x,y) in a nonempty open subset O of $\mathfrak{g}\times\mathfrak{g}$. By Proposition 2.1(iv) again, for all (x,y) in $O\cap\Omega_{\mathfrak{g}}$, $\varphi(x,y)$ is orthogonal to $\varepsilon_i^{(m)}(x,y)$ for all (i,m) in I_0, whence the assertion by Theorem 2.2(i).

(ii) Let C be the submodule of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ generated by the maps

$$(x,y)\mapsto[x,\varepsilon_i^{(m)}(x,y)],\ (i,m)\in I_{*,0}.$$

According to (i), C is a submodule of $\mathrm{C}_{\mathfrak{g}}$. This module is free of rank $\mathrm{b}_{\mathfrak{g}} - \ell$ since $[x, V_{x,y}]$ has dimension $\mathrm{b}_{\mathfrak{g}} - \ell$ for all (x, y) in $\Omega_{\mathfrak{g}}$ by Proposition 2.1, (iii) and (iv). According to (i), for φ in $\mathrm{C}_{\mathfrak{g}}$, for all (x, y) in $\Omega_{\mathfrak{g}}$,

$$\varphi(x, y) = \sum_{(i,m)\in I_{*,0}} a_{i,m}(x, y)[x, \varepsilon_i^{(m)}(x, y)]$$

with the $a_{i,m}$'s regular on $\Omega_{\mathfrak{g}}$ and uniquely defined by this equality. Since $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$ and $\mathfrak{g} \times \mathfrak{g}$ is normal, the $a_{i,m}$'s have a regular extension to $\mathfrak{g} \times \mathfrak{g}$. As a result, φ is in C, whence the assertion.

(iii) Let φ be in the orthogonal complement to $\mathrm{C}_{\mathfrak{g}}$ in $\mathrm{S}(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$. According to (ii), for all (x, y) in $\Omega_{\mathfrak{g}}$, $\varphi(x, y)$ is orthogonal to $[x, V_{x,y}]$. Hence by Proposition 2.1(iv), $\varphi(x, y)$ is in $V_{x,y}$ for all (x, y) in $\Omega_{\mathfrak{g}}$. So, by Theorem 2.2, φ is in $\mathrm{B}_{\mathfrak{g}}$, whence the assertion. □

2.3. **Restriction to a parabolic subalgebra.** For $\mathfrak{a}$ subalgebra of $\mathfrak{g}$, set $\mathfrak{a}_{\mathrm{reg}} := \mathfrak{a} \cap \mathfrak{g}_{\mathrm{reg}}$.

Lemma 2.4. *Let $\mathfrak{a}$ be an algebraic subalgebra of $\mathfrak{g}$.*

(i) *Suppose that $\mathfrak{a}$ contains $\mathfrak{g}^x$ for all x in a dense open subset of $\mathfrak{a}$ and suppose that $\mathfrak{a}_{\mathrm{reg}}$ is not empty. Then $V_{x,y}$ is contained in $\mathfrak{a}$ for all (x, y) in $\mathfrak{a} \times \mathfrak{a}$.*

(ii) *Suppose that $\mathfrak{a}$ contains a Cartan subalgebra of $\mathfrak{g}$. Then $V_{x,y}$ is contained in $\mathfrak{a}$ for all (x, y) in $\mathfrak{a} \times \mathfrak{a}$.*

Proof. (i) By hypothesis, for all x in a dense open subset of $\mathfrak{a}$, x is a regular element and $\mathfrak{g}^x$ is contained in $\mathfrak{a}$. So by [Ko63, Theorem 9], $\varepsilon_1(x), \ldots, \varepsilon_\ell(x)$ are in $\mathfrak{a}$ for all x in a dense open subset of $\mathfrak{a}$. Then, so is it for all x in $\mathfrak{a}$ by continuity. As a result, for all (x, y) in $\mathfrak{a}\times\mathfrak{a}$, $\varepsilon_i^{(m)}(x, y)$, $(i, m) \in I_0$ is in $\mathfrak{a}$, whence the assertion.

(ii) Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{a}$. Since $\mathfrak{a}$ is an algebraic subalgebra of $\mathfrak{g}$, all semisimple element of $\mathfrak{a}$ is conjugate under the adjoint group of $\mathfrak{a}$ to an element of $\mathfrak{c}$. Hence for all x in $\mathfrak{g}_{\mathrm{reg,ss}} \cap \mathfrak{a}$, $\mathfrak{g}^x$ is contained in $\mathfrak{a}$, whence the assertion by (i) since $\mathfrak{g}_{\mathrm{reg,ss}} \cap \mathfrak{a}$ is a dense open subset of $\mathfrak{a}$. □

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. Denote by $\mathfrak{l}$ its reductive factor containing $\mathfrak{h}$, $\mathfrak{p}_{\mathrm{u}}$ its nilpotent radical and ϖ the canonical projection $\mathfrak{p} \longrightarrow \mathfrak{l}$.

Corollary 2.5. *For all (x, y) in $\mathfrak{p} \times \mathfrak{p}$, $V_{x,y}$ is contained in $\mathfrak{p}$. In particular, for all (x, y) in a dense open subset of $\mathfrak{b} \times \mathfrak{b}$, $V_{x,y} = \mathfrak{b}$.*

Proof. Since $\mathfrak{h}$ is contained in $\mathfrak{p}$, for all (x, y) in $\mathfrak{p} \times \mathfrak{p}$, $V_{x,y}$ is contained in $\mathfrak{p}$ by Lemma 2.4(ii). Since (h, e) is in $\Omega_{\mathfrak{g}}$, $\Omega_{\mathfrak{g}} \cap \mathfrak{b} \times \mathfrak{b}$ is a dense open subset of $\mathfrak{b} \times \mathfrak{b}$, whence the corollary by Proposition 2.1(iii). □

Let $\mathfrak{l}_{\mathrm{reg}}$ be the subset of regular elements of $\mathfrak{l}$ and $\Omega_{\mathfrak{l}}$ the subset of elements (x, y) of $\mathfrak{l} \times \mathfrak{l}$ such that $P_{x,y} \setminus \{0\}$ is contained in $\mathfrak{l}_{\mathrm{reg}}$. For (x, y) in $\mathfrak{l} \times \mathfrak{l}$, the image of $\mathrm{B}_{\mathfrak{l}}$ by the evaluation map at (x, y)

is denoted by $V^{\mathfrak{l}}_{x,y}$. Set:

$$R_{\mathfrak{p}} := \varpi^{-1}(\mathfrak{l}_{\mathrm{reg}}) \cap \mathfrak{g}_{\mathrm{reg}}.$$

Lemma 2.6. *Let $R'_{\mathfrak{p}}$ be the subset of elements x of $R_{\mathfrak{p}}$ such that $\mathfrak{g}^x \cap \mathfrak{p}_{\mathrm{u}} = \{0\}$.*

(i) *The sets $\mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{p}$ and $R_{\mathfrak{p}}$ are big open subsets of $\mathfrak{p}$.*

(ii) *For all x in $R_{\mathfrak{p}}$, $\varpi(\mathfrak{g}^x) = \mathfrak{l}^{\varpi(x)}$ if and only if $\mathfrak{g}^x \cap \mathfrak{p}_{\mathrm{u}} = \{0\}$.*

(iii) *The set $R'_{\mathfrak{p}}$ is a dense open subset of $\mathfrak{p}$.*

(iv) *For all (x, y) in $\mathfrak{p} \times \mathfrak{p}$, $V_{x,y}$ is contained in $V^{\mathfrak{l}}_{\varpi(x),\varpi(y)} + \mathfrak{p}_{\mathrm{u}}$.*

(v) *For all (x, y) in $R'_{\mathfrak{p}} \times \mathfrak{p}$, $\varpi(V_{x,y}) = V^{\mathfrak{l}}_{\varpi(x),\varpi(y)}$.*

Proof. (i) According to [V72], $\mathfrak{l}_{\mathrm{reg}}$ is a big open subset of $\mathfrak{l}$. Hence $\varpi^{-1}(\mathfrak{l}_{\mathrm{reg}})$ is a big open subset of $\mathfrak{p}$. As a result, it remains to prove that $\mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{p}$ is a big open subset of $\mathfrak{p}$. Suppose that $\mathfrak{p} \setminus \mathfrak{g}_{\mathrm{reg}}$ has an irreducible component Σ of codimension 1 in $\mathfrak{p}$. A contradiction is expected. As $\mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{p}$ is a cone invariant under B, Σ is a closed cone invariant under B. Since $\Bbbk[\mathfrak{p}]$ is a factorial ring, for some p in $\Bbbk[\mathfrak{p}]$, homogeneous and relatively invariant under B, the nullvariety of p in $\mathfrak{p}$ is equal to Σ. As a result, $\Sigma \cap \mathfrak{b}$ is an equidimensional closed cone of codimension 1 of $\mathfrak{b}$ since $\mathfrak{b} \cap \mathfrak{g}_{\mathrm{reg}}$ is not empty. So, by Lemma 1.8, $\Sigma = \Sigma \cap \mathfrak{h} + \mathfrak{u}$ and $\mathfrak{u}$ is contained in Σ since 0 is in $\Sigma \cap \mathfrak{h}$, whence a contradiction since $\mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{u}$ is not empty.

(ii) Let x be in $R_{\mathfrak{p}}$. By Lemma 2.4(ii), $\mathfrak{g}^x$ is contained in $\mathfrak{p}$. As ϖ is a surjective morphism of Lie algebra, $\varpi(\mathfrak{g}^x)$ is contained in $\mathfrak{l}^{\varpi(x)}$. Furthermore, $\dim \varpi(\mathfrak{g}^x) = \ell$ if and only if $\mathfrak{g}^x \cap \mathfrak{p}_{\mathrm{u}} = \{0\}$ since $\mathfrak{l}$ has rank ℓ.

(iii) For x regular semisimple in a Cartan subalgebra, contained in $\mathfrak{l}$, x is in $R'_{\mathfrak{p}}$ since the elements of $\mathfrak{g}^x$ are semisimple. So $R'_{\mathfrak{p}}$ is not empty. The map $x \mapsto \mathfrak{g}^x$ from $R_{\mathfrak{p}}$ to $\mathrm{Gr}_\ell(\mathfrak{g})$ is regular. So $R'_{\mathfrak{p}}$ is an open subset of $R_{\mathfrak{p}}$ and $\mathfrak{p}$ by (i).

(iv) Let $L_{\mathfrak{l}}$ be the submodule of elements φ of $\mathrm{S}(\mathfrak{l}) \otimes_{\Bbbk} \mathfrak{l}$ such that $[\varphi(x), x] = 0$ for all x in $\mathfrak{l}$. Then $L_{\mathfrak{l}}$ is a free module of rank ℓ according to [Di79]. Denote by $\varphi_1, \ldots, \varphi_\ell$ a basis of $L_{\mathfrak{l}}$. For x in $R_{\mathfrak{p}}$ and for $i = 1, \ldots, \ell$, $\varpi \circ \varepsilon_i(x)$ is in $\mathfrak{l}^{\varpi(x)}$. So there exists a unique element $(a_{i,1}(x), \ldots, a_{i,\ell}(x))$ of $\Bbbk^\ell$ such that

$$\varpi \circ \varepsilon_i(x) = a_{i,1}(x)\varphi_1 \circ \varpi(x) + \cdots + a_{i,\ell}(x)\varphi_\ell \circ \varpi(x).$$

The functions $a_{i,1}, \ldots, a_{i,\ell}$ so defined on $R_{\mathfrak{p}}$ have a regular extension to $\mathfrak{p}$ since they are regular, $\mathfrak{p}$ is normal and $R_{\mathfrak{p}}$ is a big open subset of $\mathfrak{p}$ by (i). As a result, for all (x, y) in $\mathfrak{p} \times \mathfrak{p}$ and for all (a, b) in $\Bbbk^2$, $\varpi \circ \varepsilon_i(ax + by)$ is a linear combination of the elements $\varphi_1(\varpi(ax + by)), \ldots, \varphi_\ell(\varpi(ax + by))$. Hence $\varpi(V_{x,y})$ is contained in $V^{\mathfrak{l}}_{\varpi(x),\varpi(y)}$ for all (x, y) in $\mathfrak{p} \times \mathfrak{p}$, whence the assertion.

(v) Let (x, y) be in $R'_{\mathfrak{p}} \times \mathfrak{p}$. By (iii), for all z in a dense open subset of $P_{x,y}$, z is in $R'_{\mathfrak{p}}$. So by (ii), $\mathfrak{l}^{\varpi(z)}$ is contained in $\varpi(V_{x,y})$ for all z in a dense open subset of $P_{x,y}$. As a result, according to Proposition 2.1(i), $V^{\mathfrak{l}}_{\varpi(x),\varpi(y)}$ is contained in $\varpi(V_{x,y})$, whence the assertion by (iv). □

Corollary 2.7. *For all (x, y) in $\Omega_{\mathfrak{g}} \cap \mathfrak{p} \times \mathfrak{p}$, $V_{x,y} = V^{\mathfrak{l}}_{\varpi(x),\varpi(y)} + \mathfrak{p}_{\mathrm{u}}$.*

Proof. As (h, e) is in $\mathfrak{p}\times\mathfrak{p}$, $\Omega_{\mathfrak{g}}\cap\mathfrak{p}\times\mathfrak{p}$ is a dense open subset of $\mathfrak{p}\times\mathfrak{p}$. Let (x, y) be in $\Omega_{\mathfrak{g}}\cap R'_{\mathfrak{p}}\times\mathfrak{p}$. By Lemma 2.6(v), $\varpi(V_{x,y}) = V^{\mathfrak{l}}_{\varpi(x),\varpi(y)}$. Furthermore, $\dim V_{x,y} = \mathrm{b}_{\mathfrak{g}}$ since (x, y) is in $\Omega_{\mathfrak{g}}$. Hence $\mathfrak{p}_{\mathfrak{u}}$ is contained in $V_{x,y}$ and $\dim V^{\mathfrak{l}}_{\varpi(x),\varpi(y)} = \mathrm{b}_{\mathfrak{l}}$ since $\mathrm{b}_{\mathfrak{g}} = \mathrm{b}_{\mathfrak{l}} + \dim \mathfrak{p}_{\mathfrak{u}}$. According to Lemma 2.4(ii), the map $(x, y) \mapsto V_{x,y}$ is a regular map from $\Omega_{\mathfrak{g}}\cap\mathfrak{p}\times\mathfrak{p}$ to $\mathrm{Gr}_{\mathrm{b}_{\mathfrak{g}}}(\mathfrak{p})$. So, for all (x, y) in $\Omega_{\mathfrak{g}}\cap\mathfrak{p}\times\mathfrak{p}$, $\mathfrak{p}_{\mathfrak{u}}$ is contained in $V_{x,y}$ and $\dim \varpi(V_{x,y}) = \mathrm{b}_{\mathfrak{l}}$, whence the assertion by Lemma 2.6(iv) since $V^{\mathfrak{l}}_{\varpi(x),\varpi(y)}$ has dimension at most $\mathrm{b}_{\mathfrak{l}}$. □

3. Proof of the main theorem

In this section, we prove that Theorem 1.1 and Theorem 1.2 results from Theorem 1.3. So we suppose that Theorem 1.3 is true for $\mathfrak{g}$. By Definition, $C_\bullet(\mathfrak{g})$ is the graded submodule of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$ such that $C_{i+\mathrm{b}_{\mathfrak{g}}}(\mathfrak{g}) := \bigwedge^i(\mathfrak{g})\wedge\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ for $i = 0, \dots, n$. Let d be the $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-derivation of the algebra $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$ such that for v in $\mathfrak{g}$, $\mathrm{d}v$ is the function on $\mathfrak{g}\times\mathfrak{g}$, $(x, y) \mapsto \langle v, [x, y]\rangle$.

Lemma 3.1. (i) *The graded module $C_\bullet(\mathfrak{g})$ is a graded subcomplex of* $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$.

(ii) *The ideal $I_{\mathfrak{g}}$ is isomorphic to the space of boundaries of degree* $\mathrm{b}_{\mathfrak{g}}$ *of $C_\bullet(\mathfrak{g})$.*

(iii) *The support of the homology of $C_\bullet(\mathfrak{g})$ is contained in* $\mathcal{C}(\mathfrak{g})$.

Proof. (i) Set:

$$\varepsilon := \wedge_{(i,m)\in I_0}\varepsilon_i^{(m)}, \tag{3}$$

where the order of the product is induced by the order of I_0. Then $C_\bullet(\mathfrak{g})$ is the ideal of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$ generated by ε since $\varepsilon_i^{(m)}$, $(i, m) \in I_0$ is a basis of $\mathrm{B}_{\mathfrak{g}}$ by Theorem 2.2(i). According to Theorem 2.2(iii), for (i, m) in I_0, $\varepsilon_i^{(m)}$ is a cycle of the complex $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$. Hence so is ε and $C_\bullet(\mathfrak{g})$ is a subcomplex of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$ as an ideal generated by a cycle.

(ii) As for v in $\mathfrak{g}$, $\mathrm{d}v$ is in $I_{\mathfrak{g}}$, $I_{\mathfrak{g}}\varepsilon$ is the space of boundaries of degree $\mathrm{b}_{\mathfrak{g}}$ of $C_\bullet(\mathfrak{g})$.

(iii) Let (x_0, y_0) be in $\mathfrak{g}\times\mathfrak{g}\setminus\mathcal{C}(\mathfrak{g})$ and v in $\mathfrak{g}$ such that $\langle v, [x_0, y_0]\rangle \neq 0$. For some affine open subset O of $\mathfrak{g}\times\mathfrak{g}$, containing (x_0, y_0), $\langle v, [x, y]\rangle \neq 0$ for all (x, y) in O. Then $\mathrm{d}v$ is an invertible element of $\Bbbk[O]$. For c a cycle of $\Bbbk[O]\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} C_\bullet(\mathfrak{g})$,

$$\mathrm{d}(v\wedge c) = (\mathrm{d}v)c$$

so that c is a boundary of $\Bbbk[O]\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} C_\bullet(\mathfrak{g})$, whence the assertion. □

Theorem 3.2. (i) *The complex $C_\bullet(\mathfrak{g})$ has no homology of degree bigger than* $\mathrm{b}_{\mathfrak{g}}$.

(ii) *The ideal $I_{\mathfrak{g}}$ has projective dimension* $2n - 1$.

(iii) *The algebra* $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})/I_{\mathfrak{g}}$ *is Cohen-Macaulay.*

(iv) *The projective dimension of the module* $\bigwedge^n(\mathfrak{g})\wedge\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ *is equal to n.*

Proof. (i) Let Z be the space of cycles of degree $\mathrm{b}_{\mathfrak{g}} + 1$ of $C_\bullet(\mathfrak{g})$, whence a graded subcomplex of $C_\bullet(\mathfrak{g})$,

$$0 \longrightarrow C_{2n+\ell}(\mathfrak{g}) \longrightarrow \cdots \longrightarrow C_{n+\ell+2}(\mathfrak{g}) \longrightarrow Z \longrightarrow 0\ .$$

According to Lemma 3.1(iii), the support of its homology is contained in $\mathcal{C}_{\mathfrak{g}}$. In particular, its codimension in $\mathfrak{g} \times \mathfrak{g}$ is at least

$$4n + 2\ell - (2n + 2\ell) = 2n = n + n - 1 + 1$$

According to Theorem 1.3, for $i = n + \ell + 2, \dots, 2n + \ell$, $C_{i+\mathrm{b}_{\mathfrak{g}}}(\mathfrak{g})$ has projective dimension at most n. Hence, by Corollary B.3, this complex is acyclic and Z has projective dimension at most $2n - 2$, whence the assertion.

(ii) and (iii) Since $\mathrm{B}_{\mathfrak{g}}$ is a free module of rank $\mathrm{b}_{\mathfrak{g}}$, $\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ is a free module of rank 1. By definition, the short sequence

$$0 \longrightarrow Z \longrightarrow \mathfrak{g} \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}}) \longrightarrow I_{\mathfrak{g}} \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}}) \longrightarrow 0$$

is exact, whence the short exact sequence

$$0 \longrightarrow Z \longrightarrow \mathfrak{g} \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}}) \longrightarrow I_{\mathfrak{g}} \longrightarrow 0 \ .$$

Moreover, by Theorem 1.3, $\mathfrak{g} \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ has projective dimension at most 1. Then, by (i) and Lemma B.5, $I_{\mathfrak{g}}$ has projective dimension at most $2n-1$. As a result the $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-module $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})/I_{\mathfrak{g}}$ has projective dimension at most $2n$. Then by Auslander-Buchsbaum's theorem [Bou98, §3, n°3, Théorème 1], the depth of the graded $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})$-module $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ is at least

$$4\mathrm{b}_{\mathfrak{g}} - 2\ell - 2n = 2\mathrm{b}_{\mathfrak{g}}$$

so that, according to [Bou98, §1, n°3, Proposition 4], the depth of the graded algebra $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})/I_{\mathfrak{g}}$ is at least $2\mathrm{b}_{\mathfrak{g}}$. In other words, $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})/I_{\mathfrak{g}}$ is Cohen-Macaulay since it has dimension $2\mathrm{b}_{\mathfrak{g}}$. Moreover, since the graded algebra $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ has depth $2\mathrm{b}_{\mathfrak{g}}$, the graded $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})$-module $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ has projective dimension $2n$. Hence $I_{\mathfrak{g}}$ has projective dimension $2n - 1$.

(iv) As $I_{\mathfrak{g}}$ has projective dimension $2n - 1$, $\bigwedge^{n}(\mathfrak{g}) \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ has projective dimension n by (i), Lemma B.5 and Theorem 1.3. □

Theorem 1.2 is given by Theorem 3.2(i) and Lemma 3.1(ii) and Theorem 1.1 is a corollary of Theorem 3.2.

Corollary 3.3. *The subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by $I_{\mathfrak{g}}$ is Cohen-Macaulay and normal. Furthermore, $I_{\mathfrak{g}}$ is a prime ideal.*

Proof. According to Theorem 3.2(iii), the subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by $I_{\mathfrak{g}}$ is Cohen-Macaulay. According to [Po08, Theorem 1], it is smooth in codimension 1. So by Serre's normality criterion [Bou98, §1, n°10, Théorème 4], it is normal. In particular, it is reduced and $I_{\mathfrak{g}}$ is radical. According to [Ri79], $\mathcal{C}(\mathfrak{g})$ is irreducible. Hence $I_{\mathfrak{g}}$ is a prime ideal. □

4. Proof of Theorem 1.3

In this section we prove that Theorem 1.3 results from Theorem 1.5. Moreover, we prove that $\mathfrak{sl}_2(\Bbbk)$ has Property (**P**). For k nonnegative integer, the complex $D_k^\bullet(\mathfrak{g}, \mathrm{B}_\mathfrak{g})$ is given by Definition A.1 with $V = \mathfrak{g}$ and $L = \mathrm{B}_\mathfrak{g}$ and for $i = 1, \dots, n$, $C_{i+\mathrm{b}_\mathfrak{g}}(\mathfrak{g}) := \bigwedge^i(\mathfrak{g}) \wedge \bigwedge^{\mathrm{b}_\mathfrak{g}}(\mathrm{B}_\mathfrak{g})$. The embedding of $\mathrm{S}^k(\mathrm{B}_\mathfrak{g}) \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \bigwedge^{\mathrm{b}_\mathfrak{g}}(\mathrm{B}_\mathfrak{g})$ in $D_k^{\mathrm{b}_\mathfrak{g}}(\mathfrak{g}, \mathrm{B}_\mathfrak{g})$ is an augmentation of $D_k^\bullet(\mathfrak{g}, \mathrm{B}_\mathfrak{g})$. Denote by $\overline{D}_k^\bullet(\mathfrak{g}, \mathrm{B}_\mathfrak{g})$ this augmented complex.

Proposition 4.1. *Let k be a nonnegative integer.*

(i) *The complex $\overline{D}_k^\bullet(\mathfrak{g}, \mathrm{B}_\mathfrak{g})$ has no cohomology of degree smaller than $\mathrm{b}_\mathfrak{g} + 1$.*

(ii) *For $k = 0, 1$, $\overline{D}_k^\bullet(\mathfrak{g}, \mathrm{B}_\mathfrak{g})$ is acyclic. In particular, $\mathfrak{sl}_2(\Bbbk)$ has Property (**P**).*

(iii) *If $\mathfrak{g}$ is simple and has Property (**P**), then for $i = 1, \dots, n$, $C_{i+\mathrm{b}_\mathfrak{g}}(\mathfrak{g})$ has projective dimension at most i.*

Proof. (i) By definition, $\overline{D}_k^\bullet(\mathfrak{g}, \mathrm{B}_\mathfrak{g})$ has no cohomology of degree smaller than $\mathrm{b}_\mathfrak{g}$. According to Theorem 2.2, $\mathrm{B}_\mathfrak{g}$ is a free module of rank $\mathrm{b}_\mathfrak{g}$. Since $\Omega_\mathfrak{g}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$ and $\mathfrak{g} \times \mathfrak{g}$ is normal, $\overline{D}_k^\bullet(\mathfrak{g}, \mathrm{B}_\mathfrak{g})$ has no cohomology of degree $\mathrm{b}_\mathfrak{g}$ for all positive integer k by Lemma A.4(iii).

(ii) By definition $\overline{D}_0^\bullet(\mathfrak{g}, \mathrm{B}_\mathfrak{g})$ is acyclic. As $\overline{D}_1^\bullet(\mathfrak{g}, \mathrm{B}_\mathfrak{g})$ has no cohomology of degree bigger than $\mathrm{b}_\mathfrak{g}$ by definition, $\overline{D}_1^\bullet(\mathfrak{g}, \mathrm{b}_\mathfrak{g})$ is acyclic by (i). For $\mathfrak{g} = \mathfrak{sl}_2(\Bbbk)$, $\mathrm{b}_\mathfrak{g} = 2$ and $\ell = 1$. Hence $\mathfrak{sl}_2(\Bbbk)$ has Property (**P**).

(iii) Prove the proposition by induction on i. By (i), $C_{1+\mathrm{b}_\mathfrak{g}}(\mathfrak{g})$ has projective dimension at most 1. Suppose that $C_{j+\mathrm{b}_\mathfrak{g}}(\mathfrak{g})$ has projective dimension at most j for $j < i$. By (i) and Property (**P**), the complex $\overline{D}_i^\bullet(\mathfrak{g}, \mathrm{B}_\mathfrak{g})$ is acyclic. Then, by induction hypothesis and Corollary B.4, $C_{i+\mathrm{b}_\mathfrak{g}}(\mathfrak{g})$ has projective dimension at most i. □

Corollary 4.2. *Suppose that all simple factors of $\mathfrak{g}$ have Property (**P**). Then, for $i = 1, \dots, n$, $C_{i+\mathrm{b}_\mathfrak{g}}(\mathfrak{g})$ has projective dimension at most i.*

Proof. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$ and $\mathfrak{d}$ the derived algebra of $\mathfrak{g}$. Denote by $\ell_\mathfrak{d}$ the rank of $\mathfrak{d}$. As $\mathfrak{z}$ is contained in $\mathrm{B}_\mathfrak{g}$, for $i = 1, \dots, n$, we have an isomorphism

$$\mathrm{S}(\mathfrak{z}) \otimes_\Bbbk \bigwedge^{\dim \mathfrak{z}}(\mathfrak{z}) \otimes_\Bbbk C_{i+\mathrm{b}_\mathfrak{d}}(\mathfrak{d}) \longrightarrow C_{i+\mathrm{b}_\mathfrak{g}}(\mathfrak{g}).$$

Hence the proposition for $\mathfrak{g}$ results from the proposition for $\mathfrak{g} = \mathfrak{d}$ since $\mathrm{b}_\mathfrak{d} - \ell_\mathfrak{d} = n$.

Denote by $\mathfrak{d}_1, \dots, \mathfrak{d}_\mathrm{n}$ the simple factors of $\mathfrak{g} = \mathfrak{d}$ and prove the proposition by induction on n. For $\mathrm{n} = 1$, the proposition results from the hypothesis by Proposition 4.1(iii). Suppose $\mathrm{n} \geq 2$ and the proposition true for $\mathrm{n} - 1$. Let $\mathfrak{a}$ be the direct product of $\mathfrak{d}_1, \dots, \mathfrak{d}_{\mathrm{n}-1}$. From the equalities:

$$\begin{aligned} \textstyle\bigwedge^{\mathrm{b}_\mathfrak{g}}(\mathrm{B}_\mathfrak{g}) &= \textstyle\bigwedge^{\mathrm{b}_\mathfrak{a}}(\mathrm{B}_\mathfrak{a}) \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}_\mathrm{n}}}(\mathrm{B}_{\mathfrak{d}_\mathrm{n}}) \\ \textstyle\bigwedge^{i}(\mathfrak{g}) &= \textstyle\bigoplus_{j=0}^{i} \bigwedge^{j}(\mathfrak{a}) \wedge \bigwedge^{i-j}(\mathfrak{d}_\mathrm{n}) \\ \mathrm{b}_\mathfrak{g} - \ell &= \mathrm{b}_\mathfrak{a} - \ell_\mathfrak{a} + \mathrm{b}_{\mathfrak{d}_\mathrm{n}} - \ell_{\mathfrak{d}_\mathrm{n}} \end{aligned}$$

we deduce an isomorphism

$$\bigoplus_{j=0}^{i} C_{j+\mathrm{b}_{\mathfrak{a}}}(\mathfrak{a}) \otimes_{\Bbbk} C_{i-j+\mathrm{b}_{\mathfrak{g}_{\mathrm{n}}}}(\mathfrak{d}_{\mathrm{n}}) \longrightarrow C_{i+\mathrm{b}_{\mathfrak{g}}}(\mathfrak{g})$$

for $i = 1, \dots, n$. By induction hypothesis, $C_{j+\mathrm{b}_{\mathfrak{a}}}(\mathfrak{a})$ has projective dimension at most j. By the hypothesis and Proposition 4.1(iii), $C_{i-j+\mathrm{b}_{\mathfrak{g}_{\mathrm{n}}}}(\mathfrak{d}_{\mathrm{n}})$ has projective dimension at most $i-j$. Hence $C_{i+\mathrm{b}_{\mathfrak{g}}}(\mathfrak{g})$ has projective dimension at most i. □

Let $\mathfrak{d}$, $\mathfrak{z}$, $\mathfrak{d}_1, \dots, \mathfrak{d}_{\mathrm{n}}$ be as in the proof of Corollary 4.2. Set:

$$n_1 := \mathrm{b}_{\mathfrak{d}_1} - \ell_{\mathfrak{d}_1}, \dots, n_{\mathrm{n}} := \mathrm{b}_{\mathfrak{d}_{\mathrm{n}}} - \ell_{\mathfrak{d}_{\mathrm{n}}}, \quad \mathbb{I}' := \{(i_0, \dots, i_{\mathrm{n}}) \in \mathbb{N}^{\mathrm{n}+1} \mid i_1 \leq n_1, \dots, i_{\mathrm{n}} \leq n_{\mathrm{n}}\},$$

$$\mathbb{I}'' := \mathbb{I}' \cap \{0\} \times \mathbb{N}^{\mathrm{n}}, \quad \mathbb{I}'_k := \mathbb{N}_k^{\mathrm{n}+1} \cap \mathbb{I}', \quad \mathbb{I}''_k := \mathbb{I}'_k \cap \mathbb{I}''$$

for k nonnegative integer. The sets $\mathbb{I}''$ and $\mathbb{I}''_k$ identify with subsets of $\mathbb{N}^{\mathrm{n}}$. For I subset of $\mathbb{I}'_k$ and $i = 0, \dots, k$, let I_i be the subset of elements ι of $\mathbb{I}''_i$ such that $(k-i, \iota)$ is in I. For $k = 0, \dots, n$ and I subset of $\mathbb{I}''_k$, denote by $D^{\bullet}_{k,I,\#}(\mathfrak{d})$ the simple complex deduced from the multicomplex

$$\bigoplus_{(j_1, \dots, j_{\mathrm{n}}) \in I} D^{\bullet}_{j_1}(\mathfrak{d}_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} D^{\bullet}_{j_{\mathrm{n}}}(\mathfrak{d}_{\mathrm{n}}).$$

Then $D^{\bullet}_{k,I,\#}(\mathfrak{d})$ is a graded subcomplex of $D^{\bullet}_k(\mathfrak{d})$. For $k = 0, \dots, n$ and I subset of $\mathbb{I}'_k$, the simple complex $D^{\bullet}_{k,I,\#}(\mathfrak{g})$ deduced from the double complex

$$\bigoplus_{i=0}^{k} D^{\bullet}_{k-i}(\mathfrak{z}) \otimes_{\Bbbk} D^{\bullet}_{i,I_i,\#}(\mathfrak{d})$$

is a graded subcomplex of $D^{\bullet}_k(\mathfrak{g})$ and

$$D^{\bullet}_{k,I,\#}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}}) := D^{\bullet}_{k,I,\#}(\mathfrak{g})[-\mathrm{b}_{\mathfrak{g}}] \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$$

is a graded subcomplex of $D^{\bullet}_k(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})$. For simplicity, we set:

$$D^{\bullet}_{k,\#}(\mathfrak{g}) := D^{\bullet}_{k,\mathbb{I}'_k,\#}(\mathfrak{g}) \quad \text{and} \quad D^{\bullet}_{k,\#}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}}) := D^{\bullet}_{k,\mathbb{I}'_k,\#}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}}).$$

Lemma 4.3. *Suppose that* $\mathfrak{d}_1, \dots, \mathfrak{d}_{\mathrm{n}}$ *have Property* (**P**). *Then for* $k = 0, \dots, n$ *and* I *subset of* $\mathbb{I}'_k$, $D^{\bullet}_{k,I,\#}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})$ *has no cohomology of degree different from* $\mathrm{b}_{\mathfrak{g}}$.

Proof. As $\mathrm{B}_{\mathfrak{d}}$ is the $\mathrm{S}(\mathfrak{d} \times \mathfrak{d})$-submodule of $\mathrm{S}(\mathfrak{d} \times \mathfrak{d}) \otimes_{\Bbbk} \mathfrak{d}$ generated by the direct sum

$$\mathrm{B}_{\mathfrak{d}_1} \oplus \cdots \oplus \mathrm{B}_{\mathfrak{d}_{\mathrm{n}}},$$

the $\mathrm{S}(\mathfrak{d} \times \mathfrak{d})$-module $\bigwedge^{\mathrm{b}_{\mathfrak{d}}}(\mathrm{B}_{\mathfrak{d}})$ is isomorphic to

$$\bigwedge^{\mathrm{b}_{\mathfrak{d}_1}}(\mathrm{B}_{\mathfrak{d}_1}) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \bigwedge^{\mathrm{b}_{\mathfrak{d}_{\mathrm{n}}}}(\mathrm{B}_{\mathfrak{d}_{\mathrm{n}}}).$$

As a result, for $j = 0, \dots, n$ and J subset of $\mathbb{I}''_j$, the complex $D_{j,J,\#}(\mathfrak{d}, \mathrm{B}_{\mathfrak{d}})$ is isomorphic to the simple complex deduced from the multicomplex

$$\bigoplus_{(i_1,\dots,\, i_n)\in J} D^\bullet_{i_1}(\mathfrak{d}_1, \mathrm{B}_{\mathfrak{d}_1}) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} D^\bullet_{i_n}(\mathfrak{d}_n, \mathrm{B}_{\mathfrak{d}_n}).$$

Then $D^\bullet_{j,J,\#}(\mathfrak{d}, \mathrm{B}_{\mathfrak{d}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{d}}$ since $\mathfrak{d}_1, \dots, \mathfrak{d}_n$ have Property (**P**). As $\mathfrak{z}$ is contained in $\mathrm{B}_{\mathfrak{g}}$,

$$D^\bullet_{k,I,\#}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}}) = \bigoplus_{i=0}^{k} \mathrm{S}^{k-i}(\mathfrak{z}) \otimes_{\Bbbk} D^\bullet_{i,I_i,\#}(\mathfrak{d}, \mathrm{B}_{\mathfrak{d}})[-\dim \mathfrak{z}] \wedge \bigwedge\nolimits^{\dim \mathfrak{z}}(\mathfrak{z}).$$

As a result, $D^\bullet_{k,I,\#}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$ since $\mathrm{b}_{\mathfrak{g}} = \mathrm{b}_{\mathfrak{d}} + \dim \mathfrak{z}$. □

For I subset of $\mathbb{I}'$ and $j = 0, \dots, \mathrm{n}$, let $I_{j,*}$ be the subset of elements i of I such that $i_j > 0$ and $I_{j,-}$ the image of $I_{j,*}$ by the map

$$\mathbb{N}^{\mathrm{n}+1} \longrightarrow \mathbb{N}^{\mathrm{n}+1}, \qquad i \longmapsto (i_0, \dots, i_{j-1}, i_j - 1, i_{j+1}, \dots, i_{\mathrm{n}}).$$

Denote by I_- the union of $I_{0,-}, \dots, I_{\mathrm{n},-}$. For $k = 0, \dots, n$, let $K^\bullet_{k,I}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})$ be the kernel of the morphism

$$\mathrm{S}(\mathfrak{g} \times \mathfrak{g}) \otimes_{\Bbbk} D^\bullet_{k,I,\#}(\mathfrak{g}) \longrightarrow D^\bullet_{k,I,\#}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})[\mathrm{b}_{\mathfrak{g}}], \qquad \varphi \longmapsto \varphi \wedge \varepsilon.$$

In particular, $K^\bullet_{k,I}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})$ is a graded subcomplex of $\mathrm{S}(\mathfrak{g} \times \mathfrak{g}) \otimes_{\Bbbk} D^\bullet_{k,I,\#}(\mathfrak{g})$.

Proposition 4.4. *Suppose that* $\mathfrak{d}_1, \dots, \mathfrak{d}_{\mathrm{n}}$ *have Property* (**P**). *Let* $k = 1, \dots, n$ *and* $I \subset \mathbb{I}'_k$. *Then* $K^\bullet_{k,I}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})$ *is equal to* $\mathrm{S}(\mathfrak{g} \times \mathfrak{g}) \otimes_{\Bbbk} D^\bullet_{k,I,\#}(\mathfrak{g}) \cap D^\bullet_{k-1,I_-,\#}(\mathfrak{g})[-1] \wedge \mathrm{B}_{\mathfrak{g}}$.

Proof. Since $\mathrm{S}(\mathfrak{g} \times \mathfrak{g}) \otimes_{\Bbbk} D^\bullet_{k,I,\#}(\mathfrak{g}) \cap D^\bullet_{k-1,I_-,\#}(\mathfrak{g})[-1] \wedge \mathrm{B}_{\mathfrak{g}}$ is clearly contained in $K^\bullet_{k,I}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})$, it is sufficient to prove that $K^\bullet_{k,I}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})$ is contained in $D^\bullet_{k-1,I_-,\#}(\mathfrak{g})[-1] \wedge \mathrm{B}_{\mathfrak{g}}$. Prove the assertion by induction on k. For $k = 1$, it is true by Proposition 4.1(i). Suppose $k > 1$ and $K^\bullet_{j,J}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})$ contained in $D^\bullet_{j-1,J_-,\#}(\mathfrak{g})[-1] \wedge \mathrm{B}_{\mathfrak{g}}$ for $j < k$ and $J \subset \mathbb{I}'_j$.

Let $j = 1, \dots, k-1$. For $\upsilon = (i_0, \dots, i_{\mathrm{n}})$ in $\mathbb{N}^{\mathrm{n}+1}_{k-j}$, set:

$$\begin{aligned} V_\upsilon &:= \mathrm{S}^{i_0}(\mathfrak{z}) \otimes_{\Bbbk} \mathrm{S}^{i_1}(\mathfrak{d}_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathrm{S}^{i_{\mathrm{n}}}(\mathfrak{d}_{\mathrm{n}}) \\ I_\upsilon &:= \{(l_0, \dots, l_{\mathrm{n}}) \in \mathbb{N}^{\mathrm{n}+1} \mid (i_0 + l_0, \dots, i_{\mathrm{n}} + l_{\mathrm{n}}) \in I\}. \end{aligned}$$

In particular, I_υ is contained in $\mathbb{I}'_j$ when it is not empty. Then

$$D^j_{k,I,\#}(\mathfrak{g}) = \bigoplus_{\upsilon \in \mathbb{N}^{\mathrm{n}+1}_{k-j}} V_\upsilon \otimes_{\Bbbk} D^j_{j,I_\upsilon,\#}(\mathfrak{g}).$$

So, by induction hypothesis, $K^j_{k,I}(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})$ is contained in $D^{j-1}_{k-1,I_-,\#}(\mathfrak{g}) \wedge \mathrm{B}_{\mathfrak{g}}$.

We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K^k_{k,I}(\mathfrak{g},\mathrm{B}_\mathfrak{g}) & \longrightarrow & \mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_\Bbbk D^k_{k,I,\#}(\mathfrak{g}) & \longrightarrow & D^{k+\mathrm{b}_\mathfrak{g}}_{k,I,\#}(\mathfrak{g},\mathrm{B}_\mathfrak{g}) & \longrightarrow & 0 \\
 & & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \\
0 & \longrightarrow & K^{k-1}_{k,I}(\mathfrak{g},\mathrm{B}_\mathfrak{g}) & \longrightarrow & \mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_\Bbbk D^{k-1}_{k,I,\#}(\mathfrak{g}) & \longrightarrow & D^{k-1+\mathrm{b}_\mathfrak{g}}_{k,I,\#}(\mathfrak{g},\mathrm{B}_\mathfrak{g}) & \longrightarrow & 0 \\
 & & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \\
0 & \longrightarrow & K^{k-2}_{k,I}(\mathfrak{g},\mathrm{B}_\mathfrak{g}) & \longrightarrow & \mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_\Bbbk D^{k-2}_{k,I,\#}(\mathfrak{g}) & \longrightarrow & D^{k-2+\mathrm{b}_\mathfrak{g}}_{k,I,\#}(\mathfrak{g},\mathrm{B}_\mathfrak{g}) & \longrightarrow & 0
\end{array}.$$

By definition, the rows are exact, by Lemma 4.3, the right column is exact and by Lemma A.2(ii), the middle column is exact. Denoting by δ the horizontal arrows, for a in $K^k_{k,I}(\mathfrak{g},\mathrm{B}_\mathfrak{g})$, $a = \mathrm{d}b$ for some b in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_\Bbbk D^{k-1}_{k,I,\#}(\mathfrak{g})$, $\delta b = \mathrm{d}c$ for c in $D^{k-2+\mathrm{b}_\mathfrak{g}}_{k,I,\#}(\mathfrak{g},\mathrm{B}_\mathfrak{g})$ and $c = \delta c'$ for some c' in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_\Bbbk D^{k-2}_{k,I,\#}(\mathfrak{g})$, whence $b - \mathrm{d}c' = \delta b'$ for some b' in $K^{k-1}_{k,I}(\mathfrak{g},\mathrm{B}_\mathfrak{g})$ and $a = \mathrm{d}b'$. As $\mathrm{d}D^{k-2}_{k-1,I_-,\#}(\mathfrak{g})\wedge\mathrm{B}_\mathfrak{g} = D^{k-1}_{k-1,I_-,\#}(\mathfrak{g})\wedge\mathrm{B}_\mathfrak{g}$, a is in $D^{k-1}_{k-1,I_-,\#}(\mathfrak{g})\wedge\mathrm{B}_\mathfrak{g}$ since $K^{k-1}_{k,I}(\mathfrak{g},\mathrm{B}_\mathfrak{g})$ is contained in $D^{k-2}_{k-1,I_-,\#}(\mathfrak{g})\wedge\mathrm{B}_\mathfrak{g}$, whence the proposition. □

As it remains to prove Theorem 1.5, we suppose $\mathfrak{g}$ simple of rank at least 2 in the following sections.

5. First step to the proof of Theorem 1.5

Let ϖ_1 be the first projection $\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$. For $k = 1,\dots,n$, denote by S_k the support in $\mathfrak{g}\times\mathfrak{g}$ of the cohomology of $D^\bullet_k(\mathfrak{g},\mathrm{B}_\mathfrak{g})$ of degree different from $\mathrm{b}_\mathfrak{g}$.

Lemma 5.1. *Let $k = 1,\dots,n$.*

(i) *The set S_k is a closed subset of $\mathfrak{g}\times\mathfrak{g}$ invariant under the actions of G and $\mathrm{GL}_2(\Bbbk)$ in $\mathfrak{g}\times\mathfrak{g}$.*

(ii) *The subset $\varpi_1(S_k)$ of $\mathfrak{g}$ is closed and G-invariant.*

(iii) *If $\varpi_1(S_k)\cap\mathfrak{h} = \{0\}$, then S_k has codimension at least $n+2$ in $\mathfrak{g}\times\mathfrak{g}$.*

Proof. (i) According to Proposition 2.1(vi), $\mathrm{B}_\mathfrak{g}$ is a free module generated by a basis of G-invariant elements. Moreover, by definition, $\mathrm{B}_\mathfrak{g}$ is invariant under the action of $\mathrm{GL}_2(\Bbbk)$. Hence $\bigwedge^{\mathrm{b}_\mathfrak{g}}(\mathrm{B}_\mathfrak{g})$ is generated by a $G\times\mathrm{GL}_2(\Bbbk)$-semi-invariant element. Then, as the differential of $D^\bullet(\mathfrak{g})$ is $G\times\mathrm{GL}_2(\Bbbk)$-equivariant, so is the differential of $D^\bullet_k(\mathfrak{g},\mathrm{B}_\mathfrak{g})$. Hence S_k is invariant under $G\times\mathrm{GL}_2(\Bbbk)$.

(ii) As S_k is invariant under $\Bbbk^*\times\Bbbk^*$, $\varpi_1(S_k)\times\{0\} = S_k\cap\mathfrak{g}\times\{0\}$ so that $\varpi_1(S_k)$ is a closed subset of $\mathfrak{g}$. As S_k is G-invariant so is $\varpi_1(S_k)$.

(iii) Suppose $\varpi_1(S_k)\cap\mathfrak{h} = \{0\}$. By (ii), $\varpi_1(S_k)$ is contained in the nilpotent cone $\mathfrak{N}_\mathfrak{g}$ of $\mathfrak{g}$. Then S_k is contained in the nilpotent bicone $\mathcal{N}_\mathfrak{g}$ since S_k is invariant under the action of $\mathrm{GL}_2(\Bbbk)$.

As result, S_k has codimension at least $n+2$ in $\mathfrak{g}\times\mathfrak{g}$ since $\mathcal{N}_{\mathfrak{g}}$ has dimension $3n$ by [CMo08, Theorem 1.2]. □

Proposition 5.2. *Let $k=2,\dots,n$. Suppose that $\mathfrak{g}$ has Property $(\mathbf{P}_{k-1})$ and $\varpi_1(S_k)\cap\mathfrak{h}=\{0\}$. Then $\mathfrak{g}$ has Property $(\mathbf{P}_k)$.*

Proof. According to Proposition 4.1(i), for $i=1,\dots,k-1$, the augmented complex $\overline{D}_i^{\bullet}(\mathfrak{g},\mathrm{B}_{\mathfrak{g}})$ is acyclic since $\mathfrak{g}$ has Property $(\mathbf{P}_{k-1})$. So, by Corollary B.4, for $i=1,\dots,k-1$, $\bigwedge^i(\mathfrak{g})\wedge\bigwedge^{\mathfrak{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ has projective dimension at most i. Again by Proposition 4.1(i), S_k is the support in $\mathfrak{g}\times\mathfrak{g}$ of the cohomology of $\overline{D}_k^{\bullet}(\mathfrak{g},\mathrm{B}_{\mathfrak{g}})$. By hypothesis and Lemma 5.1(iii), S_k has codimension at least $k+2$ in $\mathfrak{g}\times\mathfrak{g}$. So, by Corollary B.4, $\overline{D}_k^{\bullet}(\mathfrak{g},\mathrm{B}_{\mathfrak{g}})$ is acyclic, whence the proposition. □

Remark 5.3. By Proposition 4.1(ii), $\mathfrak{g}$ has Property $(\mathbf{P}_1)$. So, by Proposition 5.2 and an induction argument, $\mathfrak{g}$ has Property $(\mathbf{P})$ if for $i=2,\dots,n$, $\varpi_1(S_i)\cap\mathfrak{h}=\{0\}$.

6. At the neighborhood of a semisimple element

Let z be in $\mathfrak{h}_{\mathfrak{b}}\setminus\{0\}$ and $\mathfrak{p}:=\mathfrak{g}^z+\mathfrak{b}$. Then $\mathfrak{l}:=\mathfrak{g}^z$ is the reductive factor of $\mathfrak{p}$ containing $\mathfrak{h}$. Denoting by $\mathfrak{z}$ the center of $\mathfrak{l}$, $\mathfrak{d}$ the derived algebra of $\mathfrak{l}$ and δ the function on $\mathfrak{l}$,

$$x\longmapsto\delta(x)=\det(\operatorname{ad}x\big|_{[\mathfrak{z},\mathfrak{g}]}),$$

let $\mathfrak{l}_*$ be the complement in $\mathfrak{l}$ to the nullvariety of δ.

6.1. A smooth morphism.

Let θ be the morphism:

$$G\times\mathfrak{l}_*\times\mathfrak{g}\longrightarrow G(\mathfrak{l}_*)\times\mathfrak{g}\qquad(g,x,y)\longmapsto(g(x),g(y))\ .$$

Lemma 6.1. *Set $\Omega_*:=\Omega_{\mathfrak{g}}\cap\mathfrak{l}_*\times\mathfrak{g}$.*

(i) *The subset $G(\mathfrak{l}_*)$ of $\mathfrak{g}$ is open and θ is a faithfully flat morphism.*

(ii) *The subset Ω_* of $\mathfrak{l}_*\times\mathfrak{g}$ is a big open subset.*

Proof. (i) Since the map $(g,x)\mapsto g(x)$ from $G\times\mathfrak{l}_*$ to $\mathfrak{g}$ is a submersion, $G(\mathfrak{l}_*)$ is an open subset of $\mathfrak{g}$ and this map is a smooth surjective morphism from $G\times\mathfrak{l}_*$ to $G(\mathfrak{l}_*)$. As a result, θ is a faithfully flat morphism from $G\times\mathfrak{l}_*\times\mathfrak{g}$ onto the open subset $G(\mathfrak{l}_*)\times\mathfrak{g}$ of $\mathfrak{g}\times\mathfrak{g}$ since the endomorphism of $G\times\mathfrak{l}_*\times\mathfrak{g}$, $(g,x,y)\mapsto(g,x,g(y))$ is an isomorphism.

(ii) By (i), the fibers of θ are equidimensional of dimension $\dim\mathfrak{l}$. Hence Ω_* is a big open subset of $\mathfrak{l}_*\times\mathfrak{g}$ since $\Omega_{\mathfrak{g}}$ is a G-invariant big open subset of $\mathfrak{g}\times\mathfrak{g}$. □

For any $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-module M, denote by $\mathcal{M}$ the restriction to $G(\mathfrak{l}_*)\times\mathfrak{g}$ of the localization of M on $\mathfrak{g}\times\mathfrak{g}$ and $\overline{M}$ the space of global sections of $\theta^*(\mathcal{M})$.

Corollary 6.2. *Let M be a $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-module and N a submodule of M. The modules $\overline{M}$ and $\overline{N}$ are equal if and only if $G(\mathfrak{l}_*)\times\mathfrak{g}$ has an empty intersection with the support of M/N.*

Proof. Denote by $\mathcal{M}'$ the restriction to $G(\mathfrak{l}_*) \times \mathfrak{g}$ of the localization on $\mathfrak{g} \times \mathfrak{g}$ of M/N. As the localization functor is exact, we have a short exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}' \longrightarrow 0\ .$$

By Lemma 6.1(i), θ is a faithfully flat morphism from $G \times \mathfrak{l}_* \times \mathfrak{g}$ to $G(\mathfrak{l}_*) \times \mathfrak{g}$. Hence the short sequence

$$0 \longrightarrow \theta^*(\mathcal{N}) \longrightarrow \theta^*(\mathcal{M}) \longrightarrow \theta^*(\mathcal{M}') \longrightarrow 0$$

is exact and $\mathcal{M}' = 0$ if and only if $\theta^*(\mathcal{M}') = 0$. Moreover, $\theta^*(\mathcal{M}') = 0$ if and only if $\overline{M} = \overline{N}$ since $G \times \mathfrak{l}_* \times \mathfrak{g}$ is an affine variety. Hence $\overline{M} = \overline{N}$ if and only if the support of M/N in $\mathfrak{g} \times \mathfrak{g}$ has an empty intersection with $G(\mathfrak{l}_*) \times \mathfrak{g}$. □

6.2. As $\mathfrak{l}+\mathfrak{b}$ is a parabolic subalgebra of $\mathfrak{g}$, so is $\mathfrak{p}_- := \mathfrak{l}+\mathfrak{b}_-$. Let $\mathfrak{d}_1, \dots, \mathfrak{d}_\mathrm{n}$ be the simple factors of $\mathfrak{d}$. Denote by $\mathfrak{p}_{\pm,\mathrm{u}}$ the complement to $\mathfrak{l}$ in $\mathfrak{g}$, invariant under $\mathrm{ad}\,\mathfrak{h}$ and $\mathfrak{p}_{-,\mathrm{u}}$ the nilpotent radical of $\mathfrak{p}_-$. Set:

$$\mathfrak{l}_0 := \mathfrak{z} + \mathfrak{p}_{\pm,\mathrm{u}}, \quad \mathfrak{l}_{0,-} := \mathfrak{z} + \mathfrak{p}_{-,\mathrm{u}}, \quad n_1 := \mathrm{b}_{\mathfrak{d}_1} - \ell_{\mathfrak{d}_1}, \dots, n_\mathrm{n} := \mathrm{b}_{\mathfrak{d}_\mathrm{n}} - \ell_{\mathfrak{d}_\mathrm{n}},$$

$$\mathbb{I} := \{(i_{-1}, \dots, i_\mathrm{n}) \in \mathbb{N}^{\mathrm{n}+2} \mid i_1 \leq n_1, \dots, i_\mathrm{n} \leq n_\mathrm{n}\}, \quad \mathbb{I}_k := \mathbb{N}_k^{\mathrm{n}+2} \cap \mathbb{I}$$

for k nonnegative integer. The sets $\mathbb{I}'$ and $\mathbb{I}'_k$ of Section 4 identify with $\mathbb{I} \cap \{0\} \times \mathbb{N}^{\mathrm{n}+1}$ and $\mathbb{I}_k \cap \{0\} \times \mathbb{N}^{\mathrm{n}+1}$ respectively. According to the notations of Section 4, for $k = 1, \dots, n$, denote by $D^\bullet_{k,\#}(\mathfrak{g})$ the simple complex deduced from the double complexe

$$\bigoplus_{i=0}^{k} D^\bullet_{k-i}(\mathfrak{p}_{\pm,\mathrm{u}}) \otimes_{\Bbbk} D^\bullet_{i,\#}(\mathfrak{l}).$$

Then $D^\bullet_{k,\#}(\mathfrak{g})$ is a graded subcomplex of $D^\bullet_k(\mathfrak{g})$.

Denoting by $\mathcal{B}$ the localization on $\mathfrak{g} \times \mathfrak{g}$ of $\mathrm{B}_\mathfrak{g}$, let $\widetilde{\mathrm{B}}$ be the space of global section of $\theta^*(\mathcal{B})$. In particular, $\widetilde{\mathrm{B}}$ is a free submodule of rank $\mathrm{b}_\mathfrak{g}$ of $\Bbbk[G \times \mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \mathfrak{g}$. For $k = 1, \dots, n$, set:

$$\begin{aligned}
\mathrm{B} :=& \quad \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \mathrm{B}_\mathfrak{g} \\
\mathrm{C} :=& \quad \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \mathrm{C}_\mathfrak{g} \\
D^\bullet_{k,\#}(\mathfrak{g}, \widetilde{\mathrm{B}}) :=& \quad D^\bullet_{k,\#}(\mathfrak{g})[-\mathrm{b}_\mathfrak{g}] \wedge \textstyle\bigwedge^{\mathrm{b}_\mathfrak{g}}(\widetilde{\mathrm{B}}) \\
D^\bullet_{k,\#}(\mathfrak{g}, \mathrm{B}_\mathfrak{g}) :=& \quad D^\bullet_{k,\#}(\mathfrak{g})[-\mathrm{b}_\mathfrak{g}] \wedge \textstyle\bigwedge^{\mathrm{b}_\mathfrak{g}}(\mathrm{B}_\mathfrak{g}).
\end{aligned}$$

Lemma 6.3. *Let $k = 1, \dots, n$.*

(i) *The orthogonal complement to* B *in* $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \mathfrak{g}$ *is equal to* C *and the orthogonal complement to* C *in* $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \mathfrak{g}$ *is equal to* B.

(ii) *The set $\varpi_1(S_k)$ does not contain z if and only if $D^\bullet_k(\mathfrak{g}, \widetilde{\mathrm{B}})$ has no cohomology of degree different from* $\mathrm{b}_\mathfrak{g}$.

Proof. (i) By Proposition 2.3, C is orthogonal to B. Let φ and ψ be in $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \mathfrak{g}$ orthogonal to B and C respectively. For (x, y) in Ω_*, the orthogonal complement to $V_{x,y}$ is equal to $[x, V_{x,y}]$ by Proposition 2.1(iv). Then, for some regular functions $a_{i,m}$, $(i, m) \in I_{*,0}$ and $b_{i,m}$, $(i, m) \in I_0$,

$$\varphi(x, y) = \sum_{(i,m)\in I_{*,0}} a_{i,m}(x, y)[x, \varepsilon_i^{(m)}(x, y)] \quad \text{and} \qquad \psi(x, y) = \sum_{(i,m)\in I_0} b_{i,m}(x, y)\varepsilon_i^{(m)}(x, y)$$

for all (x, y) in Ω_*. By Lemma 6.1(ii), Ω_* is a big open subset of $\mathfrak{l}_* \times \mathfrak{g}$. So, the regular functions $a_{i,m}$ $(i, m) \in I_{*,0}$ and $b_{i,m}$ $(i, m) \in I_0$ have a regular extension to $\mathfrak{l}_* \times \mathfrak{g}$ since $\mathfrak{l}_* \times \mathfrak{g}$ is a normal variety. As a result, φ and ψ are in C and B respectively, whence the assertion.

(ii) For j integer, denote by Z_k^j and B_k^j the spaces of cocycles and coboundaries of degree j of $D_k^\bullet(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})$. By Lemma 6.1(i), from the short exact sequence

$$0 \longrightarrow Z_k^j \longrightarrow D_k^j(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}}) \longrightarrow B_k^{j+1} \longrightarrow 0$$

for j integer, we deduce the short exact sequence

$$0 \longrightarrow \overline{Z_k^j} \longrightarrow \overline{D_k^j(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})} \longrightarrow \overline{B_k^{j+1}} \longrightarrow 0$$

since $G \times \mathfrak{l}_* \times \mathfrak{g}$ is an affine variety. As $\widetilde{\mathrm{B}}$ is the space of global sections of $\theta^*(\mathcal{B})$, $D_k^j(\mathfrak{g}, \widetilde{\mathrm{B}}) = \overline{D_k^j(\mathfrak{g}, \mathrm{B}_{\mathfrak{g}})}$. Hence $\overline{Z_k^j}$ and $\overline{B_k^j}$ are the spaces of cocycles and coboundaries of degree j of $D_k^\bullet(\mathfrak{g}, \widetilde{\mathrm{B}})$. By Corollary 6.2, $z \notin \varpi_1(S_k)$ if and only $\overline{Z_k^j} = \overline{B_k^j}$ for $j \neq \mathrm{b}_{\mathfrak{g}}$, whence the assertion. □

Denote by $\mathcal{O}$ the localization at the identity of $\Bbbk[G]$.

Lemma 6.4. *Let $k = 1, \dots, n$. Suppose that for $i = 1, \dots, k$, $\mathcal{O} \otimes_{\Bbbk[G]} D_{i,\#}^\bullet(\mathfrak{g}, \widetilde{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$. Then, for $i = 1, \dots, k$ and φ in $\mathcal{O} \otimes_{\Bbbk[G]} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D_{i,\#}^i(\mathfrak{g})$*

$$\varphi \wedge \textstyle\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\widetilde{\mathrm{B}}) = \{0\} \Longrightarrow \varphi \in \mathcal{O} \otimes_{\Bbbk[G]} D_{i,\#}^{i-1}(\mathfrak{g}) \wedge \widetilde{\mathrm{B}}.$$

Proof. Let $\tilde{\varepsilon}$ be a generator of $\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\widetilde{\mathrm{B}})$. Prove the lemma by induction on i. By Proposition A.7, it is true for $i = 1$. Suppose that it is true for $i - 1$. For some ψ in $\mathcal{O} \otimes_{\Bbbk[G]} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D_{i,\#}^{i-1}(\mathfrak{g})$, $\varphi = \mathrm{d}\psi$. Then $\psi \wedge \tilde{\varepsilon}$ is a cocycle of degree $i - 1$ of $\mathcal{O} \otimes_{\Bbbk[G]} D_{i,\#}^\bullet(\mathfrak{g}, \widetilde{\mathrm{B}})$. So, by hypothesis, for some ψ' in $\mathcal{O} \otimes_{\Bbbk[G]} D_{i,\#}^{i-2}(\mathfrak{g})$, $\psi \wedge \tilde{\varepsilon} = \mathrm{d}\psi' \wedge \tilde{\varepsilon}$. Then, by induction hypothesis,

$$\psi - \mathrm{d}\psi' \in \mathcal{O} \otimes_{\Bbbk[G]} D_{i,\#}^{i-1}(\mathfrak{g}) \wedge \widetilde{\mathrm{B}},$$

whence the lemma. □

Proposition 6.5. *Let $k = 1, \dots, n$. Suppose that the following condition is satisfied: for $i = 1, \dots, k$, $\mathcal{O} \otimes_{\Bbbk[G]} D_{i,\#}^\bullet(\mathfrak{g}, \widetilde{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$.*

(i) *For $i = 1, \dots, k$, $D_i^\bullet(\mathfrak{g}, \widetilde{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$.*

(ii) *For $i = 1, \dots, k$, z is not in $\varpi_1(S_i)$.*

Proof. (i) We consider the action of G in $\Bbbk[G\times \mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}\mathfrak{g}$ given by

$$k.a\otimes v(g,x,y):=a(k^{-1}g,x,y)k.v.$$

This action has a natural extension to $\Bbbk[G\times \mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}D(\mathfrak{g})$ and $D_i^\bullet(\mathfrak{g},\widetilde{\mathrm{B}})$ is invariant under this action for $i=1,\dots,k$ since $\mathrm{B}_{\mathfrak{g}}$ is a G-invariant submodule of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ by Proposition 2.1(vi). This action induces an action of $\mathfrak{g}$ in $\mathcal{O}\otimes_{\Bbbk}\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}D(\mathfrak{g})$ and $\mathcal{O}\otimes_{\Bbbk[G]}D_i^\bullet(\mathfrak{g},\widetilde{\mathrm{B}})$ is invariant under this action.

Prove the assertion by induction on i. By Proposition 4.1(ii) and Lemma 6.3(ii), it is true for $i=1$. Suppose $i>1$ and the assertion true for the integers smaller than i. Set:

$$E':=\mathcal{O}\otimes_{\Bbbk}\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}D^i_{i,\#}(\mathfrak{g}),\quad N_1':=\mathcal{O}\otimes_{\Bbbk[G]}\textstyle\bigwedge^{i-1}(\mathfrak{g})\wedge\widetilde{\mathrm{B}}\quad\text{and}$$

$$N_2':=\{\varphi\in\mathcal{O}\otimes_{\Bbbk}\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}\textstyle\bigwedge^i(\mathfrak{g})\mid(g,x,y)\in G\times\mathfrak{l}_*\times\mathfrak{g}\Longrightarrow\varphi(g,x,y)\wedge\varepsilon(g(x),g(y))=0\}.$$

By the hypothesis, Proposition A.7 and Lemma 6.4, $N_2'\cap E'\subset N_1'\cap E'$. Moreover, N_1' and N_2' are $\mathfrak{g}$-submodules of $\mathcal{O}\otimes_{\Bbbk}\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}\bigwedge^i(\mathfrak{g})$.

Set:

$$M:=\Bbbk[G]\otimes_{\Bbbk}\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}\textstyle\bigwedge^i(\mathfrak{g}),\quad M':=\Bbbk[G]\otimes_{\Bbbk}\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}D^i_{i,\#}(\mathfrak{g}),\quad N_j:=N_j'\cap M$$

for $j=1,2$. Then M is a rational $\mathfrak{g}$-module since so is $\Bbbk[G]$, M' is a $\mathfrak{l}$-submodule of M, N_1 and N_2 are $\mathfrak{g}$-submodules of M such that N_1 is contained in N_2 and $N_2\cap M'$ is contained in $N_1\cap M'$. By [C20, Theorem 1.1], $D^i_{i,\#}(\mathfrak{g})$ generates the G-module of $\bigwedge^i(\mathfrak{g})$. So M' generates the G-module M since $\Bbbk[G]\otimes_{\Bbbk}\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]$ is a G-module. Then, by Proposition D.6, $N_1=N_2$.

For a in N_2', pa is in N_2 for some p in $\Bbbk[G]$ such that $p(1_{\mathfrak{g}})\neq 0$. Hence pa is in N_1 and a is in N_1'. As a result, $N_1'=N_2'$ since N_1' is contained in N_2'. Then, again by Proposition A.7 and the induction hypothesis, the complex $\mathcal{O}\otimes_{\Bbbk[G]}D_i^\bullet(\mathfrak{g},\widetilde{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$. As a result, the support in $G\times\mathfrak{l}_*\times\mathfrak{g}$ of the cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$ of $D_i^\bullet(\mathfrak{g},\widetilde{\mathrm{B}})$ does not contain $\{1_{\mathfrak{g}}\}\times\mathfrak{l}_*\times\mathfrak{g}$. As $D_i^\bullet(\mathfrak{g},\widetilde{\mathrm{B}})$ is a G-equivariant complex, this support is invariant under G. Hence it is empty and $D_i^\bullet(\mathfrak{g},\widetilde{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$.

(ii) By (i) and Lemma 6.3(ii), for $i=1,\dots,k$, z is not in $\varpi_1(S_i)$. □

Definition 6.6. For $k=1,\dots,n$ and z in $\mathfrak{h}_{\mathfrak{b}}$, we say that $\mathfrak{g}$ has Property $(\mathbf{P}_{z,k})$ if $D^\bullet_{i,\#}(\mathfrak{g},\widetilde{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$ for $i=1,\dots,k$.

Remark 6.7. Suppose that $\mathfrak{g}$ has Property $(\mathbf{P}_{z,n})$ for all z in $\mathfrak{h}_{\mathfrak{b}}\setminus\{0\}$. Then $\mathfrak{g}$ has Property $(\mathbf{P})$ by Proposition 6.5(ii) and Remark 5.3.

7. Restriction to a parabolic subalgebra

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$, $\mathfrak{l}$ the reductive factor of $\mathfrak{p}$ containing $\mathfrak{h}$, $\mathfrak{d}$ the derived algebra of $\mathfrak{l}$, $\mathfrak{z}$ the center of $\mathfrak{l}$ and P the normalizer of $\mathfrak{p}$ in G. Denote by $\mathcal{R}_{\mathfrak{l}}$ the set of

roots α such that $\mathfrak{g}^\alpha$ is contained in $\mathfrak{l}$ and set:

$$d_0 := \dim \mathfrak{z}, \qquad \mathfrak{p}_\mathfrak{u} := \bigoplus_{\alpha \in \mathcal{R}_+ \setminus \mathcal{R}_\mathfrak{l}} \mathfrak{g}^\alpha, \qquad d := \dim \mathfrak{p}_\mathfrak{u}$$

so that $\mathfrak{p}_\mathfrak{u}$ is the nilpotent radical of $\mathfrak{p}$. Let L be the centralizer of $\mathfrak{z}$ in G. According to [Ko63, §3.2, Lemma 5], L is connected. When $\mathfrak{b}$ is strictly contained in $\mathfrak{p}$, we denote by $\mathfrak{d}_1, \dots, \mathfrak{d}_\mathfrak{n}$ the simple factors of $\mathfrak{d}$. Let z be in $\mathfrak{z}$ such that $\beta(z) = 1$ for all β in $\Pi \setminus \mathcal{R}_\mathfrak{l}$ and $t \mapsto \zeta(t)$ the one parameter subgroup of G generated by $\mathrm{ad}\, z$.

Let $\mathfrak{l}_*$ be the open subset of $\mathfrak{l}$ as in Section 6. The usual gradation of $\Bbbk[\mathfrak{p}]$ induces a gradation of the polynomial algebra $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$ over $\Bbbk[\mathfrak{l}_*]$. Let ϖ be the canonical projection $\mathfrak{p} \longrightarrow \mathfrak{l}$ and set:

$$\overline{\mathrm{B}} := \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \mathrm{B}_\mathfrak{g}, \quad \overline{\mathrm{B}}_\mathfrak{l} := \Bbbk[\mathfrak{l}_* \times \mathfrak{l}] \otimes_{\mathrm{S}(\mathfrak{l}\times\mathfrak{l})} \mathrm{B}_\mathfrak{l}, \quad \overline{\mathrm{B}}_\mathfrak{u} := \overline{\mathrm{B}} \cap \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk \mathfrak{p}_\mathfrak{u}.$$

As (h, e) is in $\Omega_\mathfrak{g}$, $\Omega_\mathfrak{g} \cap \mathfrak{l}_* \times \mathfrak{p}$ is a dense open subset of $\mathfrak{l}_* \times \mathfrak{p}$ so that $\overline{\mathrm{B}}$ is a free submodule of rank $\mathrm{b}_\mathfrak{g}$ of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk \mathfrak{p}$ by Proposition 2.1(iii) and Corollary 2.7. Again by Proposition 2.1(iii), $\overline{\mathrm{B}}_\mathfrak{l}$ is a free $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$-module of rank $\mathrm{b}_\mathfrak{l}$. From the direct sum

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_\mathfrak{u},$$

we deduce the inclusions

$$\Bbbk[\mathfrak{l}] \subset \Bbbk[\mathfrak{p}] \quad \text{and} \quad \Bbbk[\mathfrak{l}_* \times \mathfrak{l}] \subset \Bbbk[\mathfrak{l}_* \times \mathfrak{p}].$$

In this section, for (i, m) in I_0, the restriction of $\varepsilon_i^{(m)}$ to $\mathfrak{l}_* \times \mathfrak{p}$ is again denoted by $\varepsilon_i^{(m)}$. According to this convention, ε is a generator of the $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-module $\bigwedge^{\mathrm{b}_\mathfrak{g}}(\overline{\mathrm{B}})$ (see Equality 3). As in Subsection 2.3, for (x, y) in $\mathfrak{l} \times \mathfrak{l}$, the image of $\mathrm{B}_\mathfrak{l}$ by the evaluation map at (x, y) is denoted by $V^\mathfrak{l}_{x,y}$.

7.1. **Elementary properties of the module $\overline{\mathrm{B}}$.** Denote by $\beta_1, \dots, \beta_{d_0}$ the simple roots in $\Pi \setminus \mathcal{R}_\mathfrak{l}$ and $h_1, \dots, h_{d_0}$ the basis of $\mathfrak{z}$ dual of $\beta_1, \dots, \beta_{d_0}$. Let $q_1, \dots, q_\ell$ be homogeneous generators of $\mathrm{S}(\mathfrak{l})^L$ and $d'_1, \dots, d'_\ell$ their respective degrees, chosen so that

(1) $d'_1 \leq \cdots \leq d'_\ell$,
(2) for $i = 1, \dots, \ell$, $q_i \in \mathrm{S}(\mathfrak{z}) \cup \mathrm{S}(\mathfrak{d}_1)^L \cup \cdots \cup \mathrm{S}(\mathfrak{d}_\mathfrak{n})^L$,
(3) for $i = 1, \dots, d_0$, $q_i = h_i$.

For $i = 1, \dots, \ell$, denote by η_i the differential of q_i.

Lemma 7.1. (i) *For $i = 1, \dots, \ell$, there exists a unique sequence $c_{i,j}$, $j = 1, \dots, \ell$ in $\Bbbk[\mathfrak{l}]$ such that*

$$\varepsilon_i(x) = \sum_{j=1}^{\ell} c_{i,j}(x)\eta_j(x)$$

for all x in $\mathfrak{l}$. Moreover, $c_{i,j}$ is invariant under L and homogeneous of degree $d_i - d'_j$.

(ii) *For x in $\mathfrak{l}_*$, the matrix*

$$(c_{i,j}(x),\ 1 \leq i, j \leq \ell)$$

is invertible.

Proof. (i) Let $i = 1, \ldots, \ell$. For x in $\mathfrak{l}_{\mathrm{reg}}$, $\eta_1(x), \ldots, \eta_\ell(x)$ is a basis of $\mathfrak{l}^x$ by [Ko63, Theorem 9]. By Lemma 2.4(i), for all x in $\mathfrak{l}$, $\varepsilon_i(x)$ is in $\mathfrak{l}^x$. Then there exists a unique sequence $c_{i,j},\ j = 1, \ldots, \ell$ in $\Bbbk[\mathfrak{l}_{\mathrm{reg}}]$ such that

$$\varepsilon_i(x) = \sum_{j=1}^{\ell} c_{i,j}(x)\eta_j(x)$$

for all x in $\mathfrak{l}_{\mathrm{reg}}$. By [V72], $\mathfrak{l}_{\mathrm{reg}}$ is a big open subset of $\mathfrak{l}$. Then $c_{i,1}, \ldots, c_{i,\ell}$ have a regular extension to $\mathfrak{l}$ since $\mathfrak{l}$ is normal, whence

$$\varepsilon_i(x) = \sum_{j=1}^{\ell} c_{i,j}(x)\eta_j(x)$$

for all x in $\mathfrak{l}$. As ε_i and η_j are invariant under L and homogeneous of degree $d_i - 1$ and $d'_j - 1$ respectively, $c_{i,j}$ is invariant under L and homogeneous of degree $d_i - d'_j$.

(ii) For x in $\mathfrak{l}_*$, $\mathfrak{g}^x$ is contained in $\mathfrak{l}$. Then $\mathfrak{l}_* \cap \mathfrak{l}_{\mathrm{reg}}$ is contained in $\mathfrak{g}_{\mathrm{reg}}$. As a result, for $i = 1, \ldots, \ell$, there exists a unique sequence $c'_{i,j},\ j = 1, \ldots, \ell$ in $\Bbbk[\mathfrak{l}_* \cap \mathfrak{l}_{\mathrm{reg}}]$ such that

$$\eta_i(x) = \sum_{j=1}^{\ell} c'_{i,j}(x)\varepsilon_j(x)$$

for all x in $\mathfrak{l}_* \cap \mathfrak{l}_{\mathrm{reg}}$ since $\varepsilon_1(x), \ldots, \varepsilon_\ell(x)$ is a basis of $\mathfrak{g}^x$ fo x in $\mathfrak{g}_{\mathrm{reg}}$ by [Ko63, Theorem 9]. As $\mathfrak{l}_* \cap \mathfrak{l}_{\mathrm{reg}}$ is a big open subset of $\mathfrak{l}_*$, $c'_{i,1}, \ldots, c'_{i,\ell}$ have a regular extension to $\mathfrak{l}_*$. Then, for $i = 1, \ldots, \ell$ and x in $\mathfrak{l}_*$,

$$\eta_i(x) = \sum_{j=1}^{\ell} c'_{i,j}(x)\varepsilon_j(x),$$

whence the assertion. □

For a homogeneous of degree d_a in $\Bbbk[\mathfrak{l}]$ and for $k = 0, \ldots, d_a$, denote by $a^{(k)}$ the 2-polarization of bidegree $(d_a - k, k)$ of a. Set:

$$I'_0 := \{(i, m) \in \{1, \ldots, \ell\} \times \mathbb{N} \mid 0 \leq m \leq d'_i - 1\}, \quad I_{\mathrm{u}} := I_0 \setminus I'_0.$$

Then $|I'_0| = \mathrm{b}_{\mathfrak{l}}$ and $|I_{\mathrm{u}}| = d$. For (i, m) in I'_0, let $\eta_i^{(m)}$ be the 2-polarization of η_i of bidegree $(d'_i - 1 - m, m)$. For (i, m) in I_0 and (j, m') in I'_0, set $c_{i,m,j,m'} := c_{i,j}^{(m-m')}$. In particular, $c_{i,m,j,m'} = 0$ if $m' > m$ or $m - m' > d_i - d'_j$ since $c_{i,j}$ is homogeneous of degree $d_i - d'_j$.

Lemma 7.2. (i) *For (i, m) in I_0,*

$$\varepsilon_i^{(m)}(x, y) = \sum_{(j,m') \in I'_0} c_{i,m,j,m'}(x, y)\eta_j^{(m')}(x, y)$$

for all (x, y) *in* $\mathfrak{l} \times \mathfrak{l}$.

(ii) *For* (i, m) *in* I_0 *and* (j, m') *in* I'_0*, the function* $c_{i,m,j,m'}$ *is invariant under the diagonal action of* L *in* $\mathfrak{l} \times \mathfrak{l}$.

(iii) *For* (x, y) *in* $\mathfrak{h} \times (\mathfrak{u} \cap \mathfrak{l})$ *and* (i, m) *in* I'_0,

$$\varepsilon_i^{(m)}(x, y) = \sum_{j=1}^{\ell} c_{i,j}(x)\eta_j^{(m)}(x, y).$$

(iv) *For* (i, m) *in* I_0 *and* y' *in* $\mathfrak{p}_{\mathfrak{u}}$,

$$\varpi \circ \varepsilon_i^{(m)}(x, y + y') = \varepsilon_i^{(m)}(x, y)$$

for all (x, y) *in* $\mathfrak{l} \times \mathfrak{l}$.

Proof. (i) By Lemma 7.1(i), for (x, y) in $\mathfrak{l} \times \mathfrak{l}$,

$$\varepsilon_i(x + sy) = \sum_{j=1}^{\ell} c_{i,j}(x + sy)\eta_j(x + sy)$$

for all s in $\Bbbk$, whence

$$\sum_{m=0}^{d_i-1} s^m \varepsilon_i^{(m)}(x, y) = \sum_{l=0}^{d_i-d'_j} \sum_{k=0}^{d'_j-1} s^{l+k} c_{i,j}^{(l)}(x, y)\eta_j^{(k)}(x, y) = \sum_{m=0}^{d_i-1} s^m \sum_{m'=0}^{\inf\{m,d'_j-1\}} c_{i,j}^{(m-m')}(x, y)\eta_j^{(m')}(x, y).$$

As a result,

$$\varepsilon_i^{(m)}(x, y) = \sum_{(j,m')\in I'_0} c_{i,m,j,m'}(x, y)\eta_j^{(m')}(x, y)$$

for all (x, y) in $\mathfrak{l} \times \mathfrak{l}$.

(ii) By Lemma 7.1(i), $c_{i,m,j,m'}$ is invariant under the diagonal action of L in $\mathfrak{l} \times \mathfrak{l}$.

(iii) Let (x, y) be in $\mathfrak{h} \times (\mathfrak{u} \cap \mathfrak{l})$. By (i),

$$\varepsilon_i^{(m)}(x, y) = \sum_{(j,m')\in I'_0} c_{i,m,j,m'}(x, y)\eta_j^{(m')}(x, y).$$

By (ii), for t in $\Bbbk^*$ and (j, m') in I'_0,

$$c_{i,m,j,m'}(x, y) = c_{i,m,j,m'}(x, \rho(t).y),$$

whence

$$c_{i,m,j,m'}(x, y) = c_{i,m,j,m'}(x, 0) \quad \text{since} \quad \lim_{t\to 0} \rho(t).y = 0.$$

As a result, $c_{i,m,j,m'}(x, y) = 0$ for $m \neq m'$ and $c_{i,m,j,m}(x, y) = c_{i,j}(x)$, whence the assertion.

(iv) According to Corollary 2.5, for all x in $\mathfrak{p}$, $\varepsilon_i(x)$ is in $\mathfrak{p}$. For x in $\mathfrak{l}$ and y in $\mathfrak{p}_{\mathfrak{u}}$,

$$\lim_{t\to 0} \zeta(t)(x + y) = x \quad \text{whence} \quad p_i(x + y) = p_i(x) \quad \text{and} \quad \langle \varepsilon_i(x), v\rangle = \langle \varepsilon_i(x + y), v\rangle$$

for all v in $\mathfrak{l}$. As a result, for all x in $\mathfrak{l}$ and for all y in $\mathfrak{p}_{\mathfrak{u}}$, $\varepsilon_i(x) - \varepsilon_i(x+y)$ is in $\mathfrak{p}_{\mathfrak{u}}$ since $\mathfrak{p}_{\mathfrak{u}}$ is the orthogonal complement to $\mathfrak{p}$ in $\mathfrak{g}$, whence the assertion. □

The order of I_0 induces an order of I_0' and we get a square matrix of order $|I_0'|$,

$$M_0 := (c_{i,m,j,m'},\ ((i,m),(j,m')) \in I_0' \times I_0'),$$

with coefficients in $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$.

Corollary 7.3. *For all* (x,y) *in* $\mathfrak{l}_* \times \mathfrak{l}$, $\det M_0(x,y) \neq 0$.

Proof. Set:

$$\varepsilon_0 := \wedge_{(i,m)\in I_0'} \varepsilon_i^{(m)} \quad \text{and} \quad \eta_0 := \wedge_{(j,m')\in I_0'} \eta_j^{(m')}.$$

In these equalities, the order of the products is induced by the order of I_0'. Let $\overline{\varepsilon_0}$ be the restriction of ε_0 to $\mathfrak{l}_* \times \mathfrak{l}$. By Lemma 7.2(i), $\overline{\varepsilon_0}$ is in $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}] \otimes_{\Bbbk} \bigwedge^{\mathfrak{b}_{\mathfrak{l}}}(\mathfrak{l})$ and

$$\overline{\varepsilon_0} = \det M_0 \eta_0.$$

As $\overline{\varepsilon_0}$ and η_0 are homogeneous of degree

$$\sum_{i=1}^{\ell} \frac{d_i'(d_i'-1)}{2},$$

$\det M_0$ is in $\Bbbk[\mathfrak{l}_*]$.

Denote by Σ the nullvariety of $\det M_0$ in $\mathfrak{l}_*$. Suppose that Σ is nonempty. A contradiction is expected. As the maps $\varepsilon_i^{(m)}$ and $\eta_i^{(m)}$ are homogeneous and invariant under L for all (i,m) in I_0', Σ is a closed cone of $\mathfrak{l}_*$, invariant under L.

Claim 7.4. For some x in $(\mathfrak{l}_* \cap \mathfrak{h})$, $x + \varpi(e)$ is in Σ.

Proof. [Proof of Claim 7.4] As $\Omega_{\mathfrak{l}} \cap \mathfrak{l}_* \times \mathfrak{l}$ contains $(h, \varpi(e))$, for all t in $\Bbbk$, $h + t\varpi(e)$ is a regular element of $\mathfrak{g}$ and $\mathfrak{l}$, whence a sequence $p_{i,j}$, $1 \leq i,j \leq \ell$ of polynomials such that

$$\eta_i(h + t\varpi(e)) = p_{i,1}(t)\varepsilon_1(h + t\varpi(e)) + \cdots + p_{i,\ell}(t)\varepsilon_\ell(h + t\varpi(e)$$

for all t in $\Bbbk$ and $i = 1, \ldots, \ell$. As a result, for all (i,m) in I_0', $\eta_i^{(m)}(h, \varpi(e))$ is a linear combination of $\varepsilon_j^{(m')}(h, \varpi(e)$. Hence $\mathfrak{h} \cap \mathfrak{l}_*$ is not contained in Σ. Let Σ' be an irreducible component of $\Sigma \cap \mathfrak{b}$. Then Σ' is an hypersurface of $\mathfrak{b}$ as an irreducible component of the nullvariety of $\det M_0$ in $\mathfrak{b} \cap \mathfrak{l}_*$. Then, by lemma 1.8,

$$\Sigma' = \Sigma' \cap \mathfrak{h} + \mathfrak{u} \cap \mathfrak{l}$$

since $\mathfrak{l}_* \cap \mathfrak{b} = \mathfrak{l}_* \cap \mathfrak{h} + \mathfrak{u} \cap \mathfrak{l}$ and Σ and Σ' are invariant under the one parameter subgroup $t \mapsto \rho(t)$ of G. As a result, for x in $\Sigma' \cap \mathfrak{h}$, $x + \varpi(e)$ is in Σ', whence the claim. □

From the equality

$$\lim_{t\to\infty} t^{-2}\rho(t).(x+\varpi(e)) = \varpi(e),$$

we deduce that $x+\varpi(e)$ is a regular element of $\mathfrak{l}$ since so is $\varpi(e)$. Hence $x+\varpi(e)$ is a regular element of $\mathfrak{g}$ since it is in $\mathfrak{l}_*$. Then, from the equalities

$$\lim_{t\to 0} \rho(t).(h+e-\varpi(e)) = h \quad \text{and}$$

$$\lim_{t\to\infty} t^{-2}\rho(t).(x+\varpi(e)+a(h+e-\varpi(e))) = \varpi(e)+a(e-\varpi(e)),$$

for a in $\Bbbk$, we deduce

$$h+e-\varpi(e) \in \mathfrak{g}_{\mathrm{reg}} \quad \text{and} \quad x+\varpi(e)+a(h+e-\varpi(e)) \in \mathfrak{g}_{\mathrm{reg}}$$

for all a in $\Bbbk^*$ since $\varpi(e)+a(e-\varpi(e))$ is a regular element of $\mathfrak{g}$. Hence

$$(x+\varpi(e), h+e-\varpi(e)) \in \Omega_{\mathfrak{g}}.$$

As a result, by Corollary 2.7 and Lemma 7.2(iv), the elements

$$\varepsilon_i^{(m)}(x+\varpi(e), h),\ (i,m) \in I_0'$$

are linearly independent, whence the contradiction. □

7.2. **Decomposition of $\overline{\mathrm{B}}$.** Let $\overline{\mathrm{B}}_0$ be the submodule of $\overline{\mathrm{B}}$ generated by $\varepsilon_i^{(m)}$, $(i,m) \in I_0'$. For (i,m) in I_0', denote by $\overline{\varepsilon_i^{(m)}}$ the restriction of $\varepsilon_i^{(m)}$ to $\mathfrak{l}_* \times \mathfrak{l}$. By Lemma 2.4(ii), $\overline{\varepsilon_i^{(m)}}$ is in $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}] \otimes_{\Bbbk} \mathfrak{l}$.

Lemma 7.5. (i) *The $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-modules $\overline{\mathrm{B}}$ and $\overline{\mathrm{B}}_0$ are free of rank $\mathrm{b}_{\mathfrak{g}}$ and $\mathrm{b}_{\mathfrak{l}}$ respectively.*

(ii) *The module $\overline{\mathrm{B}}_{\mathfrak{l}}$ is the image of $\overline{\mathrm{B}}_0$ by the restriction map from $\mathfrak{l}_* \times \mathfrak{p}$ to $\mathfrak{l}_* \times \mathfrak{l}$. In particular, $\overline{\varepsilon_i^{(m)}}$, $(i,m) \in I_0'$ is a basis of $\overline{\mathrm{B}}_{\mathfrak{l}}$.*

Proof. (i) As $\Omega_{\mathfrak{g}} \cap \mathfrak{l}_* \times \mathfrak{p}$ is not empty and $\overline{\mathrm{B}}$ is generated by $\varepsilon_i^{(m)}$, $(i,m) \in I_0$, the assertion results from Proposition 2.1(iii) since $|I_0| = \mathrm{b}_{\mathfrak{g}}$ and $|I_0'| = \mathrm{b}_{\mathfrak{l}}$.

(ii) By Lemma 7.2(i), the restriction of $\overline{\mathrm{B}}_0$ to $\mathfrak{l}_* \times \mathfrak{l}$ is contained in $\overline{\mathrm{B}}_{\mathfrak{l}}$. By Corollary 7.3, the matrix $M_0(x,y)$ is invertible for all (x,y) in $\mathfrak{l}_* \times \mathfrak{l}$. Then, for all (i,m) in I_0', $\eta_i^{(m)}$ is a linear combination with coefficients in $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$ of $\overline{\varepsilon_j^{(m')}}$, $(j,m') \in I_0'$, whence the assertion. □

Set:

$$\mathfrak{p}_{-,\mathrm{u}} := \bigoplus_{\alpha\in\mathcal{R}_+\setminus\mathcal{R}_{\mathfrak{l}}} \mathfrak{g}^{-\alpha} \quad \text{and} \quad \mathfrak{p}_- := \mathfrak{l} \oplus \mathfrak{p}_{-,\mathrm{u}}$$

so that $\mathfrak{p}_-$ is a parabolic subalgebra of $\mathfrak{g}$. Let J_- be the ideal of definition of $\mathfrak{l}_* \times \mathfrak{l}$ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$. As $\Bbbk[\mathfrak{p}]$ identifies with $\mathrm{S}(\mathfrak{p}_-)$, $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] = \Bbbk[\mathfrak{l}_*] \otimes_{\Bbbk} \mathrm{S}(\mathfrak{p}_-)$ and J_- is the ideal generated by $1 \otimes \mathfrak{p}_{-,\mathrm{u}}$.

Proposition 7.6. (i) *For* (i,m) *in* I_0, $\varepsilon_i^{(m)} - \overline{\varepsilon_i^{(m)}}$ *is in* $J_- \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}}$.

(ii) *For* (i,m) *in* I_{u}, *for a well defined sequence* $a_{i,m,j,m'}$, $(j,m') \in I_0'$ *in* $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]^L$,

$$\nu_i^{(m)} := \varepsilon_i^{(m)} - \sum_{(j,m')\in I_0'} a_{i,m,j,m'} \varepsilon_j^{(m')} \in J_- \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}}.$$

(iii) *The module* $\overline{\mathrm{B}}$ *is the direct sum of* $\overline{\mathrm{B}}_0$ *and* $\overline{\mathrm{B}}_{\mathrm{u}}$.

(iv) *The module* $\overline{\mathrm{B}}_{\mathrm{u}}$ *is free of rank* d *and* $\nu_i^{(m)}$, $(i,m) \in I_{\mathrm{u}}$ *is a basis of* $\overline{\mathrm{B}}_{\mathrm{u}}$.

Proof. (i) According to Lemma 7.2(iv), $\varepsilon_i^{(m)} - \overline{\varepsilon_i^{(m)}}$ is in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}}$. By Proposition 2.1(vi), $\varepsilon_i^{(m)}$ is invariant under the one parameter subgroup $t \mapsto \zeta(t)$. Then so is $\varepsilon_i^{(m)} - \overline{\varepsilon_i^{(m)}}$, whence the assertion since the elements of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}}$, invariant under $t \mapsto \zeta(t)$, are in $J_- \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}}$.

(ii) According to Corollary 2.7, $\overline{\varepsilon_i^{(m)}}$ is in $\overline{\mathrm{B}}_{\mathfrak{l}}$. Then, by Lemma 7.5(ii), for a well defined sequence $a_{i,m,j,m'}$, $(j,m') \in I_0'$ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$,

$$\overline{\varepsilon_i^{(m)}} = \sum_{(j,m')\in I_0'} a_{i,m,j,m'} \overline{\varepsilon_j^{(m')}}.$$

For all (j,m') in I_0, $\varepsilon_j^{(m')}$ is invariant under L by Proposition 2.1(vi). Hence $a_{i,m,j,m'}$ is invariant under L for all (j,m') in I_0'. Moreover, by (i), $\nu_i^{(m)}$ is in $J_- \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}}$.

(iii) By (ii), $\overline{\mathrm{B}}$ is the sum of $\overline{\mathrm{B}}_0$ and $\overline{\mathrm{B}}_{\mathrm{u}}$ since $\overline{\mathrm{B}}$ is generated by the $\varepsilon_i^{(m)}$'s. Suppose $\overline{\mathrm{B}}_0 \cap \overline{\mathrm{B}}_{\mathrm{u}} \neq \{0\}$. A contradiction is expected. Let φ be a non zero element of $\overline{\mathrm{B}}_0 \cap \overline{\mathrm{B}}_{\mathrm{u}}$. Then

$$\varphi = \sum_{(i,m)\in I_0'} \varphi_{i,m} \varepsilon_i^{(m)}$$

with $\varphi_{i,m}$ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$. As φ is in $\overline{\mathrm{B}}_{\mathrm{u}}$, $\varpi \circ \varphi = 0$. For some linearly independent homogeneous elements $r_1, \ldots, r_l$ of $\mathrm{S}(\mathfrak{p}_{-,\mathrm{u}})$,

$$\varphi_{i,m} = \sum_{j=1}^{l} r_j \varphi_{i,m,j}$$

with $\varphi_{i,m,1}, \ldots, \varphi_{i,m,l}$ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$ for all (i,m). By (i), denoting by $I_{\min}$ the set of indices j such that r_j has minimal degree and $\varphi_{i,m,j} \neq 0$ for some (i,m),

$$\sum_{j\in I_{\min}} r_j \Big(\sum_{(i,m)\in I_0'} \varphi_{i,m,j} \overline{\varepsilon_i^{(m)}} \Big) = 0$$

since φ is in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}}$. As a result, for j in $I_{\min}$, $\varphi_{i,m,j} = 0$ for all (i,m), whence the contradiction and the assertion.

(iv) By (iii), $\overline{\mathrm{B}}_{\mathrm{u}}$ has rank d since $\overline{\mathrm{B}}_0$ and $\overline{\mathrm{B}}$ have rank $\mathrm{b}_{\mathfrak{l}}$ and $\mathrm{b}_{\mathfrak{g}}$ respectively. Set:

$$\varepsilon := \wedge_{(i,m)\in I_0} \varepsilon_i^{(m)} \quad \text{and} \quad \nu := \wedge_{(i,m)\in I_{\mathrm{u}}} \nu_i^{(m)}.$$

In these equalities, the order of the products is induced by the order of I_0. By (ii),

$$\varepsilon_0 \wedge \nu = \pm\varepsilon,$$

with ε_0 as in the proof of Corollary 7.3. Again by (iii), $\bigwedge^{\mathfrak{b}_{\mathfrak{g}}}(\overline{\mathrm{B}})$ is isomorphic to $\bigwedge^{\mathfrak{b}_{\mathfrak{l}}}(\overline{\mathrm{B}}_0) \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]} \bigwedge^d(\overline{\mathrm{B}}_{\mathrm{u}})$. So, for φ in $\bigwedge^d(\overline{\mathrm{B}}_{\mathrm{u}})$, $\varepsilon_0 \wedge \varphi = a\varepsilon$ for some a in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$ since ε is a generator of $\bigwedge^{\mathfrak{b}_{\mathfrak{g}}}(\overline{\mathrm{B}})$. As a result,

$$\varepsilon_0 \wedge (\varphi \mp a\nu) = 0.$$

Denoting by $\overline{\varepsilon_0}$ the restriction of ε_0 to $\mathfrak{l}_* \times \mathfrak{l}$,

$$\varepsilon_0 - \overline{\varepsilon_0} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}} \wedge \bigwedge^{\mathfrak{b}_{\mathfrak{l}}-1}(\mathfrak{p})$$

by (i). Hence $\overline{\varepsilon_0} \wedge (\varphi \mp a\nu) = 0$ and $\varphi = \pm a\nu$. As a result, ν is a generator of $\bigwedge^d(\overline{\mathrm{B}}_{\mathrm{u}})$.

Let μ be in $\overline{\mathrm{B}}_{\mathrm{u}}$. Then

$$\mu = \sum_{(i,m)\in I_{\mathrm{u}}} c_{i,m}\nu_i^{(m)}$$

for some sequence $c_{i,m}$, $(i,m) \in I_{\mathrm{u}}$ in $\Bbbk(\mathfrak{l}_* \times \mathfrak{p})$ since $\overline{\mathrm{B}}_{\mathrm{u}}$ has rank d and $\nu_i^{(m)}$, $(i,m) \in I_{\mathrm{u}}$ are linearly independent over $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$. For (i,m) in I_{u}, set:

$$\nu_{i,m} := \wedge_{(j,m')\in I_{\mathrm{u}}\setminus\{(i,m)\}}\nu_j^{(m')}.$$

In this equality, the order of the product is induced by the order of I_0. As ν is a generator of $\bigwedge^d(\overline{\mathrm{B}}_{\mathrm{u}})$, for some $c'_{i,m}$ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$,

$$c'_{i,m}\nu = \mu \wedge \nu_{i,m} = \pm c_{i,m}\nu,$$

whence $c_{i,m}$ is in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$. As a result, $\overline{\mathrm{B}}_{\mathrm{u}}$ is a free module generated by $\nu_i^{(m)}$, $(i,m) \in I_{\mathrm{u}}$. Then, by (ii), $\overline{\mathrm{B}}_{\mathrm{u}}$ is contained in $J_- \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}}$. □

As $\mathfrak{l}_*$ is a principal open subset of $\mathfrak{l}$, containing $\mathfrak{h}_{\mathrm{reg}}$, $\mathfrak{h}_* := \mathfrak{l}_* \cap \mathfrak{h}$ is a nonempty principal open susbet of $\mathfrak{h}$.

Lemma 7.7. *Let (i,m) be in I_{u}.*

(i) *For (j,m') in I'_0, $a_{i,m,j,m'}$ is bihomogeneous of bidegree $(d_i - d_j + m' - m, m - m')$. In particular, $a_{i,m,j,m'} = 0$ when $m' > m$.*

(ii) *For (j,m') in I'_0, (x,x') in $\mathfrak{h}_* \times \mathfrak{h}$ and (y,y') in $\mathfrak{u} \times \mathfrak{u}$, $a_{i,m,j,m'}(x+y, x'+y') = a_{i,m,j,m'}(x,x')$ and $a_{i,m,j,m'}(x,y) = a_{i,m,j,m'}(x,0)$. In particular, $a_{i,m,j,m'}(x,y) = 0$ when $m \neq m'$.*

(iii) *For (j,m') in I_0 and (t,x,v) in $\Bbbk \times \mathfrak{g} \times \mathfrak{g}$,*

$$\varepsilon_j^{(m')}(x, tx+v) = \sum_{k=0}^{m'} \binom{d_j - 1 - k}{m' - k} t^{m'-k}\varepsilon_j^{(k)}(x,v).$$

Proof. (i) By Proposition 7.6(ii),

$$\varepsilon_i^{(m)} - \sum_{(j,m')\in I_0'} a_{i,m,j,m'}\varepsilon_j^{(m')} \in J_- \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}}.$$

As $\varepsilon_j^{(m')}$ is bihomogeneous of bidegree $(d_j - 1 - m', m')$ for all (j, m') in I_0, $a_{i,m,j,m'}$ is bihomogeneous of bidegree $(d_i - d_j + m' - m, m - m')$ for all (j, m') in I_0'. Again, by Proposition 7.6(ii), $a_{i,m,j,m'}$ is in $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$ so that the second degree of $a_{i,m,j,m'}$ is nonnegative. Hence $a_{i,m,j,m'} = 0$ when $m' > m$.

(ii) By Proposition 7.6(ii), $a_{i,m,j,m'}$ is invariant under the one parameter subgroup $t \mapsto \rho(t)$ of G, whence

$$a_{i,m,j,m'}(x+y, x'+y') = a_{i,m,j,m'}(x+\rho(t).y, x'+\rho(t).y) \quad \text{and} \quad a_{i,m,j,m'}(x, y) = a_{i,m,j,m'}(x, \rho(t).y), \quad \forall t \in \Bbbk^*.$$

As a result, $a_{i,m,j,m'}(x + y, x' + y) = a_{i,m,j,m'}(x, x')$ and $a_{i,m,j,m'}(x, y) = a_{i,m,j,m'}(x, 0)$ since

$$\lim_{t\to 0}\rho(t).y = \lim_{t\to 0}\rho(t).y' = 0.$$

Then, by (i), $a_{i,m,j,m'}(x, y) = 0$ when $m' \neq m$.

(iii) For s in $\Bbbk$,

$$\begin{aligned}
\varepsilon_j(x + s(tx + v)) &= \textstyle\sum_{k=0}^{d_j-1} s^k \varepsilon_j^{(k)}((1 + st)x, v) \\
&= \textstyle\sum_{k=0}^{d_j-1} s^k (1 + st)^{d_j-1-k} \varepsilon_i^{(k)}(x, v) \\
&= \textstyle\sum_{m'=0}^{d_j-1} s^{m'} \sum_{k=0}^{m'} \binom{d_j-1-k}{m'-k} t^{m'-k} \varepsilon_j^{(k)}(x, v),
\end{aligned}$$

whence the equality. □

For k, l positive integers such that $k \leq l$, set:

$$r(k, l) := \sum_{j=0}^{k-1} (-1)^j \binom{l}{j}.$$

Corollary 7.8. *Let (i, m) be in I_{u} and x in $\mathfrak{h}_*$.*

(i) *For $j = 0, \ldots, d_i' - 1$ and l in $\{1, \ldots, \ell\} \setminus \{i\}$,*

$$\binom{d_i - 1 - j}{m - j} - \sum_{m'=j}^{d_i'-1} \binom{d_i - 1 - j}{m' - j} a_{i,m,i,m'}(x, x) = 0 \quad \textit{and} \quad \sum_{m'=j}^{d_l'-1} \binom{d_l - 1 - j}{m' - j} a_{i,m,l,m'}(x, x) = 0.$$

(ii) *For $m' = 0, \ldots, d_i' - 1$,*

$$a_{i,m,i,m'}(x, x) = \sum_{j=0}^{m'} (-1)^{m'-j} \frac{(d_i - 1 - m')!}{(d_i - 1 - m)!(m - j)!(m' - j)!}.$$

(iii) *If* $d'_i \geq 2$ *then*

$$r(m, d_i - 1) - \sum_{m'=1}^{d'_i - 1} (-1)^{m-m'} r(m', d_i - 1) a_{i,m,i,m'}(x, x) \neq 0.$$

Proof. (i) Let (t, y) be in $\Bbbk \times (\mathfrak{u} \cap \mathfrak{l})$. By Proposition 7.6(ii),

$$\varepsilon_i^{(m)}(x, tx + y) = \sum_{(l', m') \in I'_0} a_{i,m,l',m'}(x, tx + y) \varepsilon_{l'}^{(m')}(x, tx + y)$$

since $\varepsilon_j^{(m')}(x, tx + y)$ is in $\mathfrak{l}$ for all (j, m') in I_0 by Lemma 2.4(i). Then, by Lemma 7.7,(ii) and (iii),

$$\sum_{j=0}^{m} \binom{d_i - 1 - j}{m - j} t^{m-j} \varepsilon_i^{(j)}(x, y) - \sum_{(l', m') \in I'_0} \sum_{j=0}^{m'} a_{i,m,l',m'}(x, x) \binom{d_{l'} - 1 - j}{m' - j} t^{m-j} \varepsilon_{l'}^{(j)}(x, y) = 0,$$

whence, for $j = 0, \ldots, m$,

$$\binom{d_i - 1 - j}{m - j} \varepsilon_i^{(j)}(x, y) - \sum_{d'_{l'} > j} \sum_{m'=j}^{d'_{l'} - 1} a_{i,m,l',m'}(x, x) \binom{d_{l'} - 1 - j}{m' - j} \varepsilon_{l'}^{(j)}(x, y) = 0.$$

For x in a dense open subset of $\mathfrak{h}_*$, $(x, \varpi(e))$ is in $\Omega_{\mathfrak{l}}$ since $\varpi(e)$ is a nilpotent element of a principal $\mathfrak{sl}_2$-triple of $\mathfrak{l}$. Then $\varepsilon_{l'}^{(m')}(x, \varpi(e))$, $(l', m') \in I'_0$ are linearly independent by Proposition 2.1(iii) and Lemma 7.5(ii). As a result, for $j = 0, \ldots, d'_i - 1$ and $j' = 0, \ldots, d'_l - 1$,

$$\binom{d_i - 1 - j}{m - j} - \sum_{m'=j}^{d'_i - 1} a_{i,m,i,m'}(x, x) \binom{d_i - 1 - j}{m' - j} = 0 \quad \text{and} \quad \sum_{m'=j'}^{d'_l - 1} a_{i,m,l,m'}(x, x) \binom{d_l - 1 - j'}{m' - j'} = 0.$$

(ii) For x in $\mathfrak{h}_*$ and $m' = 0, \ldots, d'_i - 1$, set:

$$\psi_{m'}(x) := a_{i,m,i,m'}(x, x) \frac{(d_i - 1 - m)!}{(d_i - 1 - m')!}.$$

By (i), the sequence $\psi_{m'}(x), m' = 0, \ldots, d'_i - 1$ is a solution of the linear system

$$\sum_{m'=j}^{d'_i - 1} \psi_{m'}(x) \frac{1}{(m' - j)!} = \frac{1}{(m - j)!}.$$

The inverse of the matrix of this system is equal to

$$(a_{m',j}, 0 \leq m', j \leq d'_i - 1) \quad \text{with} \quad a_{m',j} := \begin{cases} \frac{(-1)^{m'-j}}{(m'-j)!} & \text{if} \quad m' \geq j \\ 0 & \text{if} \quad m' < j \end{cases},$$

whence the equality of the assertion.

(iii) For e, k, l integers such that $2 \leq e \leq k \leq l$, set:

$$c(k,l) := \sum_{j=0}^{k} \frac{(-1)^j}{j!(l+j)!} \quad \text{and} \quad \psi(e,k,l) := r(k,l) - \sum_{j=1}^{e-1} (-1)^{k-j} c(j,k-j) r(j,l) \frac{(l-j)!}{(l-k)!}.$$

By (ii),

$$a_{i,m,i,m'}(x,x) = \frac{(d_i - 1 - m')!}{(d_i - 1 - m)!} c(m', m - m').$$

Then

$$\psi(d_i', m, d_i - 1) = r(m, d_i - 1) - \sum_{m'=1}^{d_i'-1} (-1)^{m-m'} r(m', d_i - 1) a_{i,m,i,m'}(x,x),$$

whence the assertion by Proposition C.3. □

7.3. **An invariance property.** In this subsection, $\varepsilon_i^{(m)}$, $(i,m) \in I_0$ is a sequence in $\mathrm{S}(\mathfrak{g} \times \mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$. Set:

$$\Gamma := (\mathfrak{b} \cap \mathfrak{l}_* \cap \mathfrak{g}_{\mathrm{reg,ss}}) \times (\mathfrak{b} \cap \mathfrak{l}).$$

Then Γ is an affine variety invariant under $B \cap L$ since $\mathfrak{l}_*$ and $\mathfrak{g}_{\mathrm{reg,ss}}$ are principal open subsets of $\mathfrak{l}$ and $\mathfrak{g}$ respectively, invariant under $B \cap L$. We identify Γ with the subset $\{1_G\} \times \Gamma$ of $G \times \Gamma$ by the map $x \mapsto (1_G, x)$. Let X be the quotient of $G \times \Gamma$ under the right action of $B \cap L$ given by $(g,x,y).k := (gk, k^{-1}(x), k^{-1}(y))$ and υ the quotient map. Denote by τ the canonical morphism

$$X \xrightarrow{\ \tau\ } \mathfrak{g} \times \mathfrak{g} \quad \text{such that} \quad \tau \circ \upsilon(g,x,y) = (g(x), g(y)).$$

Lemma 7.9. *Set* $\Omega_\Gamma := G.\tau^{-1}(\Omega_{\mathfrak{l}} \cap \Gamma)$ *and* $Y := X \setminus \Omega_\Gamma$.

(i) *The restriction of* υ *to* Γ *is a closed embedding.*

(ii) *The subset* Y *of* X *is closed and* G*-invariant.*

Proof. (i) Let (g,x,y) be in $G \times \Gamma$ such that $(g(x), g(y))$ is in Γ. As x is regular semisimple, for some b in $B \cap L$, $g(x) = b(x)$, whence $b^{-1}g$ is in $B \cap L$ since the centralizer of x in G is contained in $B \cap L$. As a result, $\tau^{-1}(\Gamma) = \upsilon(\Gamma)$, whence the assertion since the restriction of $\tau \circ \upsilon$ to Γ is the identity.

(ii) By (i), $\tau^{-1}(\Omega_{\mathfrak{l}} \cap \Gamma)$ is equal to $\upsilon(\Omega_{\mathfrak{l}} \cap \Gamma)$ since $\Omega_{\mathfrak{l}} \cap \Gamma$ is invariant under $B \cap L$. Hence $\Omega_\Gamma = G.\upsilon(\Omega_{\mathfrak{l}} \cap \Gamma)$. As a result, $Y = G.\upsilon(\Gamma \setminus \Omega_{\mathfrak{l}})$.

The variety X is a fiber bundle over $G/(B \cap L)$. Denote by σ the bundle projection

$$X \xrightarrow{\ \sigma\ } G/(B \cap L)\ .$$

Let O be a trivializing affine open subset of $G/(B \cap L)$. Then we have a commutative diagram,

$$\begin{tikzcd} \sigma^{-1}(O) \arrow[r, "\Phi"] \arrow[dr, "\sigma"'] & O \times \Gamma \arrow[d, "\mathrm{pr}_1"] \\ & O \end{tikzcd}$$

with Φ an isomorphism. As Y is G-invariant, $\Phi(Y \cap \sigma^{-1}(O)) = O \times (\Gamma \setminus \Omega_{\mathfrak{l}})$. Then Y is closed since $G/(B \cap L)$ has a cover by trivializing affine open subsets. □

We identify Γ with $\upsilon(\Gamma)$ by υ.

Lemma 7.10. *Let (i, m) be in $I_{\mathfrak{u}}$.*

(i) *The maps $\varepsilon_j^{(m')} \circ \tau$, $(j, m') \in I'_0$ are linearly independent over $\Bbbk(X)$.*

(ii) *For a well defined sequence $a'_{i,m,j,m'}$, $(j, m') \in I'_0$ in $\Bbbk[X \setminus Y]$,*

$$\varepsilon_i^{(m)} \circ \tau(x) = \sum_{(j,m') \in I'_0} a'_{i,m,j,m'}(x) \varepsilon_j^{(m')} \circ \tau(x)$$

for all x in $X \setminus Y$.

(iii) *For (j, m') in I'_0, the rational function $a'_{i,m,j,m'}$ is invariant under G.*

(iv) *For (j, m') in I'_0, $a_{i,m,j,m'}$ and $a'_{i,m,j,m'}$ have the same restrictions to $\Gamma \cap \Omega_{\mathfrak{l}}$.*

Proof. (i) For all x in $\Omega_{\mathfrak{l}} \cap \Gamma$, $\eta_j^{(m')}(x)$, $(j, m') \in I'_0$ are linearly independent. Then, for all x in $\Gamma \cap \Omega_{\mathfrak{l}}$, $\varepsilon_j^{(m')}(x)$, $(j, m') \in I'_0$ are linearly independent by Lemma 7.5(ii). As a result, the maps $\varepsilon_j^{(m')} \circ \tau$, $(j, m') \in I'_0$ are linearly independent over $\Bbbk(X)$ since $\varepsilon_j^{(m')}$, $(j, m') \in I'_0$ are G-equivariant by Proposition 2.1(vi).

(ii) Let ε_0 be as in the proof of Corollary 7.8. By Proposition 7.6(ii), the restriction to Γ of $\varepsilon_i^{(m)} \wedge \varepsilon_0$ is equal to 0. Then, by G-equivariance of the $\varepsilon_j^{(m')}$'s, $\varepsilon_i^{(m)} \wedge \varepsilon_0 \circ \tau$ is equal to 0. So, by (i), for a well defined sequence $a'_{i,m,j,m'}$, $(j, m') \in I'_0$ in $\Bbbk(X)$,

$$\varepsilon_i^{(m)} \circ \tau(x) = \sum_{(j,m') \in I'_0} a'_{i,m,j,m'}(x) \varepsilon_j^{(m')} \circ \tau(x)$$

for all x in a dense open subset of X. As for x in $X \setminus Y$, $\varepsilon_j^{(m')} \circ \tau(x)$, $(j, m') \in I'_0$ are linearly independent, the functions $a'_{i,m,j,m'}$, $(j, m') \in I'_0$ are regular on $X \setminus Y$, whence the assertion.

(iii) By unicity of the sequence $a'_{i,m,j,m'}$, $(j, m') \in I'_0$ and the G-equivariance of the maps $\varepsilon_k^{(l)}$, $(k, l) \in I_0$, the rational functions $a'_{i,m,j,m'}$, $(j, m') \in I'_0$ are G-invariant.

(iv) By (ii) and (iii), for all (x, y) in $\Gamma \cap \Omega_{\mathfrak{l}}$,

$$\varepsilon_i^{(m)}(x, y) = \sum_{(j,m') \in I'_0} a'_{i,m,j,m'}(x, y) \varepsilon_j^{(m')}(x, y).$$

By Proposition 7.6(ii), for all (x, y) in $\mathfrak{l}_* \times \mathfrak{l}$,

$$\varepsilon_i^{(m)}(x, y) = \sum_{(j,m') \in I'_0} a_{i,m,j,m'}(x, y) \varepsilon_j^{(m')}(x, y),$$

whence the assertion since $\varepsilon_j^{(m')}(x, y)$, $(j, m') \in I'_0$ are linearly independent for all (x, y) in $\Gamma \cap \Omega_{\mathfrak{l}}$ by Lemma 7.5(ii). □

Let $\mathfrak{h}_{**}$ be the open subset of $\mathfrak{h}$,

$$\mathfrak{h}_{**} := \bigcap_{g\in W(\mathcal{R})} g(\mathfrak{h}_*).$$

Then $\mathfrak{h}_{**}$ is a dense principal open subset of $\mathfrak{h}$, invariant under the Weyl group $W(\mathcal{R})$ of $\mathcal{R}$.

Proposition 7.11. *Let (i,m) be in I_{u}. For (j,m') in I'_0, the restriction of $a_{i,m,j,m'}$ to $\mathfrak{h}_{**}\times\mathfrak{h}$ is invariant under $W(\mathcal{R})$.*

Proof. Let π be the canonical projection $\mathfrak{b}\longrightarrow\mathfrak{h}$. According to Lemma 7.7(ii),

$$a_{i,m,j,m'}(x,y) = a_{i,m,j,m'}(\pi(x),\pi(y)) \qquad \forall (x,y)\in\Gamma.$$

Hence the function on $G\times\Gamma$,

$$(g,x,y)\longmapsto a_{i,m,j,m'}(x,y)$$

is constant on the $B\cap L$-orbits. As a result, $a_{i,m,j,m'}$ defines through the quotient a G-invariant regular function on X. Denote it by $\overline{a}_{i,m,j,m'}$. By Lemma 7.10(iii), $a'_{i,m,j,m'}$ is a G-invariant rational function on X. Then, by Lemma 7.10(iv), $a'_{i,m,j,m'} = \overline{a}_{i,m,j,m'}$ since $\overline{a}_{i,m,j,m'}$ is G-invariant. In particular, $a'_{i,m,j,m'}$ is regular on X and its restriction to $\Gamma\cap\mathfrak{h}_{**}\times\mathfrak{h}$ is invariant under $W(\mathcal{R})$. As a result, again by Lemma 7.10(iv), the restriction of $a_{i,m,j,m'}$ to $\mathfrak{h}_{**}\times\mathfrak{h}$ is invariant under $W(\mathcal{R})$ since $\Gamma\cap\mathfrak{h}_{**}\times\mathfrak{h}$ is dense in $\mathfrak{h}_{**}\times\mathfrak{h}$. □

8. Expansion along a parabolic subalgebra

Let $\mathfrak{p}$ be a parabolic subalgebra containing $\mathfrak{b}$ and

$$\mathfrak{l},\quad \mathfrak{p}_{\mathrm{u}},\quad \mathfrak{d},\quad \mathfrak{z},\quad \mathcal{R}_{\mathfrak{l}},\quad L,\quad d,\quad d_0,\quad \mathfrak{l}_*,\quad z,\quad \zeta,\quad I_0,\quad I'_0,\quad I_{\mathrm{u}}$$

as in Section 7. Set:

$$\mathfrak{p}_{-,\mathrm{u}} := \bigoplus_{\alpha\in\mathcal{R}_+\setminus\mathcal{R}_{\mathfrak{l}}} \quad\text{and}\quad \mathfrak{p}_- := \mathfrak{l}\oplus\mathfrak{p}_{-,\mathrm{u}}.$$

8.1. **Some results about the expansion along $\mathfrak{p}$.** Denote by $\alpha_1,\dots,\alpha_d$ the positive roots which are not in $\mathcal{R}_{\mathfrak{l}}$ and ordered so that $|\alpha_i|\le|\alpha_j|$ for $i\le j$. For $i=1,\dots,d$, set:

$$v_i := x_{\alpha_i},\quad w_i := x_{-\alpha_i}.$$

Then $v_1,\dots,v_d$ and $w_1,\dots,w_d$ are basis of $\mathfrak{p}_{\mathrm{u}}$ and $\mathfrak{p}_{-,\mathrm{u}}$ respectively. As usual, for $r:=(r_1,\dots,r_d)$ in $\mathbb{N}^d$,

$$v^r := v_1^{r_1}\cdots v_d^{r_d} \quad\text{and}\quad w^r := w_1^{r_1}\cdots w_d^{r_d}.$$

Lemma 8.1. *Let $i=1,\dots,\ell$.*

(i) *For all x in a dense open subset of $\mathfrak{h}$, $\varepsilon_i(x)$ is a regular element of $\mathfrak{g}$. In particular, for x in a dense open subset of $\mathfrak{l}_*$, $\varepsilon_i(x)$ has a non zero component on all simple factor of $\mathfrak{d}$.*

(ii) *For $k=1,\dots,d$, for some homogeneous element $p_{i,k}$ of degree d_i-2 in $\mathrm{S}(\mathfrak{h})$,*

$$\alpha_k\circ\varepsilon_i = \alpha_k p_{i,k}.$$

(iii) *For* $k = 1, \dots, d$ *and* (t, x) *in* $\Bbbk \times \mathfrak{h}$,

$$\varepsilon_i(x + tw_k) = \varepsilon_i(x) + tp_{i,k}(x)w_k.$$

Proof. (i) Suppose that $\varepsilon_i(x)$ is not regular for all x in $\mathfrak{h}$. A contradiction is expected. Then, for some root α, $\alpha \circ \varepsilon_i(x) = 0$ for all x in $\mathfrak{h}$. In particular, for x in $\mathfrak{h}$ and g in $W(\mathcal{R})$, $\alpha \circ \varepsilon_i(g(x)) = 0$. As ε_i is G-equivariant, $\varepsilon_i(x)$ is in the kernel of $g^{-1}(\alpha)$. As $\mathfrak{g}$ is simple, $\mathfrak{h}$ is a simple $W(\mathcal{R})$-module so that the orthogonal complement in $\mathfrak{h}$ to $g(\alpha)$, $g \in W(\mathcal{R})$ is equal to $\{0\}$, whence a contradiction since for x in $\mathfrak{h}_{\mathrm{reg}}$, $\varepsilon_i(x) \neq 0$ as an element of a basis of $\mathfrak{h}$ by [Ko63, Theorem 9].

As a result, for all x in a dense open subset of $\mathfrak{l}_*$, $\varepsilon_i(x)$ is a regular element of $\mathfrak{g}$ and $\mathfrak{l}$. Let x be in $\mathfrak{l}_*$ such that $\varepsilon_i(x)$ is a regular element of $\mathfrak{g}$ and $\mathfrak{l}$, and $\mathfrak{d}_1$ a simple factor of $\mathfrak{l}$. As $\mathfrak{g}^{\varepsilon_i(x)}$ is a commutative algebra, $\mathfrak{d}_1$ is not contained in $\mathfrak{g}^{\varepsilon_i(x)}$. Hence the component of $\varepsilon_i(x)$ on $\mathfrak{d}_1$ is different from 0 since $\mathfrak{l}^{\varepsilon_i(x)} = \mathfrak{g}^{\varepsilon_i(x)}$.

(ii) Let x be in $\mathfrak{h}$ such that $\alpha_k(x) = 0$. Then v_k and w_k are in $\mathfrak{g}^x$. As ε_i is a G equivariant map, $\varepsilon_i(x)$ is in the center of $\mathfrak{g}^x$ so that

$$0 = [\varepsilon_i(x), v_k] = \alpha_k \circ \varepsilon_i(x) v_k.$$

As a result, the nullvariety of α_k in $\mathfrak{h}$ is contained in the nullvariety of $\alpha_k \circ \varepsilon_i$. Hence α_k divides $\alpha_k \circ \varepsilon_i$ in $\mathrm{S}(\mathfrak{h})$ since $\mathrm{S}(\mathfrak{h})$ is a factorial ring and α_k is a prime element. As ε_i is homogeneous of degree $d_i - 1$, the quotient $\alpha_k \circ \varepsilon_i / \alpha_k$ is homogeneous of degree $d_i - 2$.

(iii) Let x be in $\mathfrak{h}$ such that $\alpha_k(x) \neq 0$. Then

$$\exp\left(\frac{t}{\alpha_k(x)} \operatorname{ad} w_k\right)(x) = x + tw_k.$$

As ε_i is G-equivariant,

$$\varepsilon_i(x + tw_k) = \exp\left(\frac{t}{\alpha_k(x)} \operatorname{ad} w_k\right) \circ \varepsilon_i(x),$$

whence

$$\varepsilon_i(x + tw_k) = \varepsilon_i(x) + tp_{i,k}(x)w_k$$

by (ii). □

For $i = 1, \dots, \ell$ and x in $\mathfrak{g}$, denote by $\varepsilon_{i,-}(x)$ the element of $\mathfrak{p}_{-,\mathrm{u}}$ such that $\varepsilon_i(x) - \varepsilon_{i,-}(x)$ is in $\mathfrak{p}$.

Lemma 8.2. *Let* $i = 1, \dots, \ell$.

(i) *For uniquely defined functions* $a_{i,j,k}$, $1 \leq j, k \leq d$ *in* $\Bbbk[\mathfrak{p}]$, *the polynomial map*

$$t \longmapsto \varepsilon_{i,-}(x + ty) - \sum_{1 \leq j,k \leq d} t\langle v_j, y\rangle a_{i,j,k}(x) w_k$$

is divisible by t^2 *in* $\Bbbk[t] \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}}$ *for all* (x, y) *in* $\mathfrak{p} \times \mathfrak{p}_{-,\mathrm{u}}$.

(ii) *For* $1 \leq j, k \leq d$, $a_{i,j,k}$ *is homogeneous of degree* $d_i - 2$.

(iii) *For* $1 \leq j, k \leq d$, *the function* $a_{i,j,k}$ *has weight* $\alpha_k - \alpha_j$ *with respect to the adjoint action of* $\mathfrak{h}$ *in* $\Bbbk[\mathfrak{p}]$.

Proof. (i) Let (x, y) be in $\mathfrak{p} \times \mathfrak{p}_{-,\mathrm{u}}$. According to Corollary 2.5, $\varepsilon_i(x)$ is in $\mathfrak{p}$. Moreover, ε_i is homogeneous of degree $d_i - 1$. Hence the polynomial map $t \mapsto \varepsilon_{i,-}(x + ty)$ has a unique expansion

$$\varepsilon_{i,-}(x + ty) = \sum_{k=1}^{d} \sum_{j=1}^{d_i-1} \sum_{r \in \mathbb{N}_j^d} t^{|r|} v^r(y) a_{i,r,k}(x) w_k$$

with the $a_{i,r,k}$ in $\Bbbk[\mathfrak{p}]$. For $j = 1, \ldots, d$ and r in $\mathbb{N}_1^d$ such that $r_l = 0$ for $l \neq j$, set $a_{i,j,k} := a_{i,r,k}$. Then the polynomial map

$$t \longmapsto \varepsilon_{i,-}(x + ty) - \sum_{1 \leq j,k \leq d} t\langle v_j, y\rangle a_{i,j,k}(x) w_k$$

is divisible by t^2 in $\Bbbk[t] \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}}$.

(ii) As ε_i is homogeneous of degree $d_i - 1$, so is $\varepsilon_{i,-}$ and for (s, x, y) in $\Bbbk^* \times \mathfrak{p} \times \mathfrak{p}_{-,u}$, the polynomial map

$$t \longmapsto \varepsilon_{i,-}(x + ty) - s^{1-d_i} \sum_{1 \leq j,k \leq d} t\langle v_j, sy\rangle a_{i,j,k}(sx) w_k$$

is divisible by t^2 in $\Bbbk[t] \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}}$. Hence the $a_{i,j,k}$'s are homogeneous of degree $d_i - 2$.

(iii) Let g be in H. As ε_i is a G-equivariant map and $\mathfrak{p}_{-,\mathrm{u}}$ is invariant under H, $\varepsilon_{i,-}$ is H-equivariant. Then, for (x, y, t) in $\mathfrak{p} \times \mathfrak{p}_{-,\mathrm{u}} \times \Bbbk$,

$$g.\varepsilon_{i,-}(x + ty) - \sum_{1 \leq j,k \leq d} t\langle v_j, y\rangle a_{i,j,k}(x) g.w_k = \varepsilon_{i,-}(g(x) + tg(y)) - \sum_{1 \leq j,k \leq d} t\langle v_j, y\rangle a_{i,j,k}(x) g.w_k.$$

By (i), the polynomial map

$$t \longmapsto \varepsilon_{i,-}(g(x) + tg(y)) - \sum_{1 \leq j,k \leq d} t\langle v_j, g(y)\rangle a_{i,j,k}(g(x)) w_k$$

is divisible by t^2 in $\Bbbk[t] \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}}$. Then

$$\langle v_j, g(y)\rangle a_{i,j,k}(g(x)) w_k = \langle v_j, y\rangle a_{i,j,k}(x) g.w_k$$

for all (x, y) in $\mathfrak{p} \times \mathfrak{p}_{\mathrm{u}}$. As a result, for $1 \leq j, k \leq d$ and x in $\mathfrak{h}$,

$$\alpha_j(x) a_{i,j,k} + [x, a_{i,j,k}] = \alpha_k(x) a_{i,j,k},$$

whence the assertion. □

Denote by $\mathrm{m}_1 < \cdots < \mathrm{m}_{d'}$ the strictly increasing sequence of the values of the map

$$\mathcal{R}_+ \setminus \mathcal{R}_{\mathfrak{l}} \longrightarrow \mathbb{N}\,, \qquad \alpha \longmapsto \alpha(z).$$

For $j = 1, \ldots, d'$, let I_j be the set of indices i such that $\alpha_i(z) = \mathrm{m}_j$ and set:

$$\mathfrak{p}_{-,\mathrm{u}}^{(j)} := \mathrm{span}(\{w_k \mid k \in I_j\}, \qquad \mathfrak{p}_{-,\mathrm{u},j} := \bigoplus_{k=j}^{d'} \mathfrak{p}_{-,\mathrm{u}}^{(k)}.$$

Corollary 8.3. *Let $j = 1, \dots, d'$ and $i = 1, \dots, \ell$.*

(i) *The space $\mathfrak{l} + \mathfrak{p}_{-,\mathrm{u},j}$ is a subalgebra of $\mathfrak{g}$ and $[\mathfrak{p}_{-,u,j}, \mathfrak{p}_{-,u,j'}]$ is contained in $\mathfrak{p}_{-,\mathrm{u},j''+1}$ with $j'' := \sup\{j, j'\}$.*

(ii) *For $j' > j$ and (k, k') in $I_j \times I_{j'}$, $a_{i,k,k'} = 0$, and for (k, k') in $I_j \times I_j$, $a_{i,k,k'}$ is in $\Bbbk[\mathfrak{l}]$.*

(iii) *Let $j = 1, \dots, d'$. For x in a dense open subset of $\mathfrak{h}$, the matrix $(a_{i,k,k'}(x),\ (k, k') \in I_j \times I_j)$ is diagonal and invertible for $i = 1, \dots, \ell$.*

Proof. (i) For $j_1, j_2 = 1, \dots, d'$ and (k, k') in $I_{j_1} \times I_{j_2}$,

$$[z, [w_k, w_{k'}]] = -(\mathrm{m}_{j_1} + \mathrm{m}_{j_2})[w_k, w_{k'}]$$

so that $[\mathfrak{p}_{-,\mathrm{u}}^{(j_1)}, \mathfrak{p}_{-,\mathrm{u}}^{(j_2)}]$ is contained in $\mathfrak{p}_{-,\mathrm{u},j_3+1}$ with $j_3 := \sup\{j_1, j_2\}$, whence the assertion since $[\mathfrak{l}, \mathfrak{p}_{-,\mathrm{u}}^{(j')}]$ is contained in $\mathfrak{p}_{-,\mathrm{u}}^{(j')}$ for all j'.

(ii) For $1 \le k, k' \le d$, the function $a_{i,k,k'}$ on $\mathfrak{p}$ has an expansion

$$a_{i,k,k'} = \sum_{u \in \mathbb{N}^d} w^u a_{u,i,k,k'} \quad \text{with} \quad a_{u,i,k,k'} \in \Bbbk[\mathfrak{l}].$$

For (s, x) in $\Bbbk^* \times \mathfrak{l}$ and

$$y := y_1 v_1 + \cdots + y_d v_d \in \mathfrak{p}_{\mathrm{u}},$$

$$\zeta(s).\varepsilon_i(x + y + t w_k) = \varepsilon_i(x + \sum_{l=1}^{d} s^{\alpha_l(z)} y_l v_l + t s^{-\alpha_k(z)} w_k)$$

since ε_i is G-equivariant and z is in $\mathfrak{z}$. Then, by Lemma 8.2(i), for k in I_j,

$$\sum_{k'=1}^{d} \sum_{u \in \mathbb{N}^d} s^{-\alpha_{k'}(z)} w^u(y) a_{u,i,k,k'}(x) w_{k'} = \sum_{k'=1}^{d} \sum_{u \in \mathbb{N}^d} s^{u_1\alpha_1(z) + \cdots + u_d\alpha_d(z) - \mathrm{m}_j} w^u(y) a_{u,i,k,k'}(x) w_{k'},$$

whence

$$a_{u,i,k,k'}(x) \neq 0 \Longrightarrow \mathrm{m}_j - \alpha_{k'}(z) = u_1\alpha_1(z) + \cdots + u_d\alpha_d(z).$$

For $j' > j$ and k' in $I_{j'}$, $\alpha_{k'}(z) > \mathrm{m}_j$. Hence $a_{u,i,k,k'} = 0$ for all u in $\mathbb{N}^d$ since the integers $\alpha_1(z), \dots, \alpha_d(z)$ are positive. Moreover, for k' in I_j, $a_{u,i,k,k'} = 0$ if u is different from 0, whence the assertion.

(iii) Let k be in I_j. By Lemma 8.1(iii),

$$a_{i,k,k'}(x) = \begin{cases} p_{i,k}(x) & \text{if} \quad k = k' \\ 0 & \text{if} \quad k \neq k'. \end{cases}.$$

By Lemma 8.1(i), for all x in a dense open subset of $\mathfrak{h}_{\mathrm{reg}}$, $p_{i,k}(x) \neq 0$ for $i = 1, \dots, \ell$. As a result, for such x, the matrix $(a_{i,k,k'}(x),\ (k, k')$ in $I_j \times I_j)$ is diagonal and invertible. □

8.2. **Some functions different from** 0**.** With the notations of Lemma 8.2(i), for $i = 1, \dots, \ell$ and $1 \leq j, k \leq d$, denote by $a^{(m)}_{i,j,k}$ the 2-polarization of $a_{i,j,k}$ of bidegree $(d_i - 2 - m, m)$ for $m = 0, \dots, d_i - 2$. For (i, m) in I_0, let $\psi_{i,m}$ be the map

$$\mathfrak{h} \times \mathfrak{h} \longrightarrow \mathrm{S}^{m+1}(\mathfrak{h}) \ , \qquad (x, y) \longmapsto \sum_{j=0}^{m} (-1)^{m-j} x^j y^{m-j} \varepsilon_i^{(j)}(x, y).$$

Lemma 8.4. *Let $k = 1, \dots, d$, $i = 1, \dots, \ell$ and $m = 0, \dots, d_i - 2$. For (x, y) in $\mathfrak{h} \times \mathfrak{h}$,*

$$\alpha_k(x)^{m+1} a^{(m)}_{i,k,k}(x, y) = \alpha_k^{m+1}(\psi_{i,m}(x, y)).$$

Proof. According to Lemma 8.1,(ii) and (iii),

$$\alpha_k(\varepsilon_i(x)) = \alpha_k(x) a_{i,k,k}(x) \quad \text{and} \quad a_{i,k,k}(x) = p_{i,k}(x)$$

for all x in $\mathfrak{h}$. As a result, for all (x, y) in $\mathfrak{h} \times \mathfrak{h}$,

$$\sum_{m=0}^{d_i-1} s^m \alpha_k \circ \varepsilon_i^{(m)}(x, y) = \alpha_k(x + sy) a_{i,k,k}(x + sy) = \sum_{m=0}^{d_i-2} s^m \alpha_k(x) a^{(m)}_{i,k,k}(x, y) + \sum_{m=1}^{d_i-1} s^m \alpha_k(y) a^{(m-1)}_{i,k,k}(x, y)$$

for all s in $\Bbbk$, whence

$$\alpha_k(x) a^{(m)}_{i,k,k}(x, y) + \alpha_k(y) a^{(m-1)}_{i,k,k}(x, y) = \alpha_k \circ \varepsilon_i^{(m)}(x, y)$$

for $m = 1, \dots, d_i - 2$. Then, by induction on m,

$$\alpha_k(x)^{m+1} a^{(m)}_{i,k,k}(x, y) = \sum_{j=0}^{m} (-1)^{m-j} \alpha_k(x)^j \alpha_k(y)^{m-j} \alpha_k \circ \varepsilon_i^{(j)}(x, y) = \alpha_k^{m+1}(\psi_{i,m}(x, y))$$

for $m = 0, \dots, d_i - 2$. □

For (i, m) in I_{u} and $1 \leq j, k \leq d$, set:

$$b^{(m-1)}_{i,j,k} := a^{(m-1)}_{i,j,k} - \sum_{(l,m') \in I'_{*,0}} a_{i,m,l,m'} a^{(m'-1)}_{l,j,k} \quad \text{with} \quad I'_{*,0} := I'_0 \setminus \{1, \dots, \ell\} \times \{0\}$$

and $a_{i,m,j,m'}$, $((i, m), (j, m')) \in I_{\mathrm{u}} \times I'_0$ as in Proposition 7.6. For (i, m) in I_{u}, denote by $\varphi_{i,m}$ the map

$$\mathfrak{h}_* \times \mathfrak{h} \xrightarrow{\varphi_{i,m}} \mathrm{S}^m(\mathfrak{h}) \ , \qquad (x, y) \longmapsto \psi_{i,m-1}(x, y) - \sum_{(j,m') \in I'_{*,0}} a_{i,m,j,m'}(x, y) x^{m-m'} \psi_{j,m'-1}(x, y).$$

Proposition 8.5. *Let (i, m) be in I_{u} and $k = 1, \dots, d$.*

(i) *For j in $\{1, \dots, d\} \setminus \{k\}$, the restriction of $b^{(m-1)}_{i,j,k}$ to $\mathfrak{h}_* \times \mathfrak{h}$ is equal to* 0.

(ii) *The restriction of $b^{(m-1)}_{i,k,k}$ to $\mathfrak{h}_* \times \mathfrak{h}$ is different from* 0.

Proof. (i) According to Lemma 8.2(iii), the functions $a^{(m')}_{l,j,k}$, $(l,m') \in I_0$ have weight $\alpha_k - \alpha_j$ with respect to the adjoint action of $\mathfrak{h}$ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$. As the functions $a_{i,m,l,m'}$, $(i,m) \in I_{\mathrm{u}}, (l,m') \in I'_0$ are invariant under L, the function $b^{(m-1)}_{i,j,k}$ has weight $\alpha_k - \alpha_j$. Then the restriction of $b^{(m-1)}_{i,j,k}$ to $\mathfrak{h}_* \times \mathfrak{h}$ is equal to 0 since $\alpha_j \neq \alpha_k$.

(ii) According to Lemma 8.4, for (x,y) in $\mathfrak{h}_* \times \mathfrak{h}$,

$$\alpha_k^m(x) b^{(m-1)}_{i,k,k}(x,y) = \alpha_k^m(\varphi_{i,m}(x,y)).$$

By definition, $\varphi_{i,m}(x,y)$ has an expansion

$$\begin{aligned}\varphi_{i,m}(x,y) = \quad & \textstyle\sum_{j=0}^{m-1}(-1)^j x^{m-1-j} y^j \varepsilon_i^{(m-1-j)}(x,y) - \\ & \textstyle\sum_{(l,m')\in I'_{*,0}} \sum_{j=0}^{m'-1}(-1)^j a_{i,m,l,m'}(x,y) x^{m-1-j} y^j \varepsilon_l^{(m'-1-j)}(x,y).\end{aligned}$$

Then $\varphi_{i,m}(x,x) = x^{m-1}\varphi'_{i,m}(x)$ with $\varphi'_{i,m}$ in $\Bbbk[\mathfrak{h}_*] \otimes_{\Bbbk} \mathfrak{h}$ defined by the following equality:

$$\begin{aligned}\varphi'_{i,m}(x) = \quad & \textstyle\sum_{j=0}^{m-1}(-1)^j \binom{d_i-1}{m-1-j} \varepsilon_i(x) - \\ & \textstyle\sum_{(l,m')\in I'_{*,0}} \sum_{j=0}^{m'-1}(-1)^j \binom{d_l-1}{m'-1-j} a_{i,m,l,m'}(x,x)\varepsilon_l(x)\end{aligned}$$

by Lemma 7.7(iii). Hence $\varphi'_{i,m}$ is in the free submodule of $\Bbbk[\mathfrak{h}_*] \otimes_{\Bbbk} \mathfrak{h}$ generated by $\varepsilon_1, \dots, \varepsilon_\ell$. Moreover, by Proposition 7.11, its restriction to $\mathfrak{h}_{**}$ is equivariant under $W(\mathcal{R})$ since so are $\varepsilon_1, \dots, \varepsilon_\ell$.

Denote by $\varphi'_{i,m,i}$ the coordinate of $\varphi'_{i,m}$ at ε_i. With the notations of Subsection 7.2,

$$(-1)^{m-1}\varphi'_{i,m,i}(x) = r(m, d_i - 1) - \sum_{m'=1}^{d'_i-1}(-1)^{m-m'} r(m', d_i - 1) a_{i,m,i,m'}(x,x)$$

for all x in $\mathfrak{h}_*$. Then, by Corollary 7.8(iii), $\varphi'_{i,m} \neq 0$. Suppose $\alpha_k \circ \varphi'_{i,m}(x) = 0$ for all x in $\mathfrak{h}_{**}$. A contradiction is expected. Then $g.\alpha_k \circ \varphi'_{i,m}(x) = 0$ for all (g,x) in $W(\mathcal{R}) \times \mathfrak{h}_{**}$. As $\mathfrak{g}$ is simple, so is the $W(\mathcal{R})$-module $\mathfrak{h}^*$. As a result, $\mathfrak{h}^*$ is generated by $g.\alpha_k$, $g \in W(\mathcal{R})$ and $\varphi'_{i,m}(x) = 0$ for all x in $\mathfrak{h}_{**}$, whence the contradiction. Then the function $x \mapsto \alpha_k^{m+1}(\varphi_{i,m}(x,x))$ on $\mathfrak{h}_*$ is different from 0 since $\alpha_k(x) \neq 0$ for x regular in $\mathfrak{h}$, whence the assertion. □

9. At the neighborood of a semi-simple element II

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. We then use the notations

$$\mathfrak{l}, \quad \mathfrak{p}_{\mathrm{u}}, \quad \mathfrak{d}, \quad \mathfrak{z}, \quad \mathcal{R}_{\mathfrak{l}}, \quad L, \quad d, \quad d_0, \quad \mathfrak{l}_*, \quad z, \quad \zeta, \quad I_0, \quad I'_0, \quad I_{\mathrm{u}}, \quad \mathfrak{p}_-, \quad \mathfrak{p}_{-,\mathrm{u}}$$

of Section 8. For (i,m) in I_0, the restriction of $\varepsilon_i^{(m)}$ to $\mathfrak{l}_* \times \mathfrak{g}$ is again denoted by $\varepsilon_i^{(m)}$. The usual gradation of $\Bbbk[\mathfrak{g}]$ induces a gradation of the polynomial algebra $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$ over $\Bbbk[\mathfrak{l}_*]$. From the direct sum

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{p}_{-,\mathrm{u}},$$

we deduce the inclusions

$$\Bbbk[\mathfrak{p}] \subset \Bbbk[\mathfrak{g}] \quad \text{and} \quad \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \subset \Bbbk[\mathfrak{l}_* \times \mathfrak{g}].$$

Moreover, $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$ is a subalgebra of $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$. Set $\mathrm{B} := \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \mathrm{B}_{\mathfrak{g}}$.

9.1. **Decomposition of** B. As $\Bbbk[\mathfrak{l}_*\times\mathfrak{p}]$ is contained in $\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]$, the sequence $a_{i,m,j,m'}$, $((i,m),(j,m')) \in I_{\mathrm{u}} \times I'_0$ of Proposition 7.6(ii) is contained in $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$. For (i,m) in I_{u}, let $\nu_i^{(m)}$ be the element of $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \mathfrak{g}$,

$$\nu_i^{(m)} := \varepsilon_i^{(m)} - \sum_{(j,m')\in I'_0} a_{i,m,j,m'} \varepsilon_j^{(m')}.$$

Denote by B_0 and B_{u} the submodules of B generated by the sequences $\varepsilon_i^{(m)}$, $(i,m) \in I'_0$ and $\nu_i^{(m)}$, $(i,m) \in I_{\mathrm{u}}$ respectively. Let J and J_- be the ideals of $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$ generated by $1\otimes\mathfrak{p}_{\mathrm{u}}$ and $1\otimes\mathfrak{p}_{-,\mathrm{u}}$ respectively.

Proposition 9.1. *For (i,m) in I_0, let $\overline{\varepsilon_i^{(m)}}$ be the restriction of $\varepsilon_i^{(m)}$ to $\mathfrak{l}_* \times \mathfrak{l}$.*

(i) *The modules B_0 and B_{u} are free of rank $\mathrm{b}_{\mathfrak{l}}$ and d respectively. Moreover* B *is the direct sum of B_0 and B_{u}.*

(ii) *The sequence $\overline{\varepsilon_i^{(m)}}$, $(i,m) \in I'_0$ is a basis of $\overline{\mathrm{B}}_{\mathfrak{l}}$.*

(iii) *For (i,m) in I_0,*

$$\varepsilon_i^{(m)} - \overline{\varepsilon_i^{(m)}} \in J \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}} + J_- \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}} + JJ_- \otimes_{\Bbbk} \mathfrak{l}.$$

(iv) *For $(i,m) \in I_{\mathrm{u}}$,*

$$\nu_i^{(m)} \in J \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}} + J_- \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}} + JJ_- \otimes_{\Bbbk} \mathfrak{l}.$$

Proof. (i) By Lemma 6.1(ii) and Theorem 2.2(i), B is a free module of rank $\mathrm{b}_{\mathfrak{g}}$ generated by $\varepsilon_i^{(m)}$, $(i,m) \in I_0$. Then B_0 and B_{u} are free of rank $\mathrm{b}_{\mathfrak{l}}$ and d respectively since $|I'_0| = \mathrm{b}_{\mathfrak{l}}$ and $|I_{\mathrm{u}}| = d$. Moreover, B_{u} is a submodule of B and B is the direct sum of B_0 and B_{u}.

(ii) The assertion results of Lemma 7.5(ii).

(iii) Let (i,m) be in I_0. As J is the ideal of definition of $\mathfrak{l}_* \times \mathfrak{p}$ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$,

$$\varepsilon_i^{(m)} - \overline{\varepsilon_i^{(m)}} \in J_- \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}} + J \otimes_{\Bbbk} \mathfrak{g}$$

by Proposition 7.6(i). Moreover, $\varepsilon_i^{(m)}$ and $\overline{\varepsilon_i^{(m)}}$ are invariant under L, whence the assertion since an element of $J\otimes_{\Bbbk}\mathfrak{g}$, invariant under the one parameter subgroup $t \mapsto \zeta(t)$, is in $J\otimes_{\Bbbk}\mathfrak{p}_{-,\mathrm{u}} + JJ_-\otimes_{\Bbbk}\mathfrak{l}$.

(iv) As $\nu_i^{(m)}$ is a linear combination with coefficients in $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$ of $\varepsilon_i^{(m)}$ and $\varepsilon_j^{(m')}$, $(j,m') \in I'_0$, the assertion results from (iii) and Proposition 7.6(ii). □

9.2. **Some expansions and invertible matrices.** Let $v_1, \dots, v_d, w_1, \dots, w_d$ be as in Subsection 8.1. Let $\mathfrak{I}$ be the union of $\{0\}$ and the set of strictly increasing sequences in $\{1, \dots, d\}$ and for $\iota = i_1 < \cdots < i_j$ in $\mathfrak{I}$, set:

$$\{\iota\} := \{i_1, \dots, i_j\}, \qquad |\iota| = j, \qquad v_\iota := v_{i_1} \wedge \cdots \wedge v_{i_j}, \qquad w_\iota := w_{i_1} \wedge \cdots \wedge w_{i_j}.$$

By definition, v_ι and w_ι are in $\bigwedge^{|\iota|}(\mathfrak{p}_{\mathrm{u}})$ and $\bigwedge^{|\iota|}(\mathfrak{p}_{-,\mathrm{u}})$. For $\iota = 0$, $|\iota| := 0$ and $v_\iota := w_\iota := 1$, and for $|\iota| = d$, $\mu_+ := v_\iota$ is a generator of $\bigwedge^d(\mathfrak{p}_{\mathrm{u}})$. For $j = 0, \dots, d$, set:

$$\mathfrak{I}_j := \{\iota \in \mathfrak{I} \mid |\iota| = j\} \quad \text{and} \quad \Lambda_j := \{(r, \kappa) \in \mathbb{N}^d \times \mathfrak{I} \mid |r| = |\kappa| = j\}.$$

Let $I_{*,0}$, $I'_{*,0}$, $I_{0,*}$ be the sets:

$$I_{*,0} := \{(i, m) \in I_0 \mid m > 0\}, \quad I'_{*,0} := I'_0 \cap I_{*,0}, \quad I_{0,*} := \{(i, m, k) \mid (i, m) \in I_{*,0},\ k \in \{1, \dots, m\}\}.$$

Denote by $I'_{0,*}$ and $I_{\mathrm{u},*}$ the inverse images of $I'_{*,0}$ and I_{u} by the canonical projection $I_{0,*} \longrightarrow I_{*,0}$. Set:

$$\varepsilon_0 := \wedge_{(i,m) \in I'_0} \varepsilon_i^{(m)}, \quad \varepsilon_{0,0} := \varepsilon_1 \wedge \cdots \wedge \varepsilon_\ell, \quad \varepsilon_{I'_{*,0}} := \wedge_{(i,m) \in I'_{*,0}} \varepsilon_i^{(m)}, \quad \nu_{\mathrm{u}} := \wedge_{(i,m) \in I_{\mathrm{u}}} \nu_i^{(m)}.$$

For some ϵ_* in $\{-1, +1\}$,

$$\varepsilon_0 = \epsilon_* \varepsilon_{0,0} \wedge \varepsilon_{I'_{*,0}}.$$

In these equalities, the order of the products is induced by the order of I_0. Then ε_0 and ν_{u} are generators of $\bigwedge^{\mathfrak{b}_{\mathfrak{l}}}(\mathrm{B}_0)$ and $\bigwedge^d(\mathrm{B}_{\mathrm{u}})$ respectively. Moreover, the restriction $\overline{\varepsilon_0}$ of ε_0 to $\mathfrak{l}_* \times \mathfrak{l}$ is a generator of $\bigwedge^{\mathfrak{b}_{\mathfrak{l}}}(\overline{\mathrm{B}}_{\mathfrak{l}})$ and $\varepsilon := \varepsilon_0 \wedge \nu_{\mathrm{u}}$ is a generator of $\bigwedge^{\mathfrak{b}_{\mathfrak{g}}}(\mathrm{B})$ by Proposition 9.1(i).

By Proposition 9.1(iii), for (x, y, y') in $\mathfrak{l}_* \times \mathfrak{p} \times \mathfrak{p}_{-,\mathrm{u}}$ and (i, m) in I'_0 and (j, l) in I_{u}, the polynomial maps

$$t \longmapsto \varepsilon_i^{(m)}(x, y + ty') \quad \text{and} \quad t \longmapsto \nu_j^{(l)}(x, y + ty')$$

have an expansion

$$\varepsilon_i^{(m)}(x, y + ty') = \varepsilon_i^{(m)}(x, y) + \sum_{k=1}^m t^k \omega_{i,m,k,-}(x, y, y') + \sum_{k=1}^m t^k \omega_{i,m,k,+}(x, y, y') + \sum_{k=1}^m t^k \omega_{i,m,k,0}(x, y, y')$$

$$\nu_j^{(l)}(x, y + ty') = \nu_j^{(l)}(x, y) + \sum_{k=1}^m t^k \omega_{j,l,k,-}(x, y, y') + \sum_{k=1}^m t^k \omega_{j,l,k,+}(x, y, y') + \sum_{k=1}^m t^k \omega_{j,l,k,0}(x, y, y')$$

with

$$\omega_{i,m,k,-} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathrm{S}^k(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}}, \quad \omega_{i,m,k,+} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathrm{S}^k(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}},$$
$$\omega_{i,m,k,0} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathrm{S}^k(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \mathfrak{l}$$

for (i, m) in I_0. For I subset of $I_{*,0}$, K subset of I_{u} and I' subset of $I_{0,*}$, set:

$$\varepsilon_I := \wedge_{(i,m) \in I} \varepsilon_i^{(m)}, \quad \nu_K := \wedge_{(i,m) \in K} \nu_i^{(m)}, \quad S_{I'} := \sum_{(i,m,k) \in I'} k,$$

$$\omega_{I',-} := \wedge_{(i,m,k) \in I'} \omega_{i,m,k,-}, \quad \omega_{I',+} := \wedge_{(i,m,k) \in I'} \omega_{i,m,k,+}, \quad \omega_{I',0} := \wedge_{(i,m,k) \in I'} \omega_{i,m,k,0}.$$

In these equalities, the order of the products are induced by the orders of I_0 and $I_{0,*}$. For K subset of $I_{0,*}$, denote by $K^\#$ the image of K by the projection $(i,m,k) \mapsto (i,m)$. When $S_K = |K|$, we identify K and $K^\#$.

For ι in $\mathfrak{I}$ and $j = 0,\dots,d$, set:

$$\varepsilon_\iota := v_\iota \wedge \varepsilon \quad \text{and} \quad \mathcal{K}_j := \{K \subset I_{\mathrm{u}} \mid |K| = d - j\}.$$

Lemma 9.2. *Let $j = 0,\dots,d$, ι in $\mathfrak{I}_j$ and (x,y,y') in $\mathfrak{l}_* \times \mathfrak{p} \times \mathfrak{p}_{-,\mathrm{u}}$. Denote by $c_\iota(x,y,y')$ the coefficient of t^j of the polynomial map $t \mapsto \varepsilon_\iota(x, y + ty')$.*

(i) *The polynomial map $t \mapsto \varepsilon_\iota(x, y + ty')$ is divisible by t^j in $\Bbbk[t] \otimes_{\Bbbk} \bigwedge^{\mathfrak{b}_{\mathfrak{g}}+j}(\mathfrak{g})$.*

(ii) *For a well defined map*

$$\mathcal{K}_j \xrightarrow{\quad \epsilon \quad} \{-1, 1\}\ ,$$

$$c_\iota(x,y,y') - c'_\iota(x,y,y') \in \bigoplus_{|\kappa|<j} w_\kappa \wedge \bigwedge^{j-|\kappa|}(\mathfrak{l}) \wedge \overline{\varepsilon_0}(x,y) \wedge \mu_+$$

with

$$c'_\iota(x,y,y') := \sum_{K \in \mathcal{K}_j} \epsilon(K)\, v_\iota \wedge \overline{\varepsilon_0}(x,y) \wedge \nu_K(x,y) \wedge \omega_{I_{\mathrm{u}}\setminus K,-}(x,y,y').$$

(iii) *For K in $\mathcal{K}_j$, for well defined functions $a_{r,\kappa,K,\iota}$, $(r,\kappa) \in \Lambda_j$ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$,*

$$\epsilon(K)\, v_\iota \wedge \overline{\varepsilon_0}(x,y) \wedge \nu_K(x,y) \wedge \omega_{I_{\mathrm{u}}\setminus K,-}(x,y,y') = \sum_{(r,\kappa)\in\Lambda_j} v^r(y') a_{r,\kappa,K,\iota}(x,y) w_\kappa \wedge \overline{\varepsilon_0}(x,y) \wedge \mu_+.$$

Proof. As $\varepsilon = \varepsilon_0 \wedge \nu_{\mathrm{u}}$, the cofficient of t^k of the polynomial functions $t \longmapsto \varepsilon(x, y + ty')$ is the sum of the value at (x,y,y') of products

$$\epsilon(I,K,I_-,K_-,I_+,K_+,I_\pm,K_\pm)\, \varepsilon_{0,0} \wedge \varepsilon_I \wedge \nu_K \wedge \omega_{I_-,-} \wedge \omega_{K_-,-} \wedge \omega_{I_+,+} \wedge \omega_{K_+,+} \wedge \omega_{I_\pm,0} \wedge \omega_{K_\pm,0}$$

with

$$I \subset I'_{*,0}, \quad K \subset I_{\mathrm{u}}, \quad I_- \subset I'_{0,*}, \quad K_- \subset I_{\mathrm{u},*}, \quad I_+ \subset I'_{0,*}, \quad K_+ \subset I_{\mathrm{u},*}, \quad I_\pm \subset I'_{0,*}, \quad K_\pm \subset I_{\mathrm{u},*},$$

$$\epsilon(I,K,I_-,K_-,I_+,K_+,I_\pm,K_\pm) \in \{-1,1\}$$

such that

$$I \cup I_-^\# \cup I_+^\# \cup I_\pm^\# = I \sqcup I_-^\# \sqcup I_+^\# \sqcup I_\pm^\#, \quad K \cup K_-^\# \cup K_+^\# \cup K_\pm^\# = K \sqcup K_-^\# \sqcup K_+^\# \sqcup K_\pm^\#,$$

$$|I| + |I_-^\#| + |I_+^\#| + |I_\pm^\#| = n - d, \quad |K| + |K_-^\#| + |K_+^\#| + |K_\pm^\#| = d,$$

$$S_{I_-} + S_{I_+} + S_{I_\pm} + S_{K_-} + S_{K_+} + S_{K_\pm} = k.$$

For $|K| + |I_+^\#| + |K_+^\#| > d - j$,

$$v_\iota \wedge \varepsilon_{0,0} \wedge \varepsilon_I \wedge \nu_K \wedge \omega_{I_-,-} \wedge \omega_{K_-,-} \wedge \omega_{I_+,+} \wedge \omega_{K_+,+} \wedge \omega_{I_\pm,0} \wedge \omega_{K_\pm,0} = 0$$

since $\nu_K(x,y)$ is in $\bigwedge^{|K|}(\mathfrak{p}_{\mathrm{u}})$, whence

$$|K| + |I_+^\#| + |K_+^\#| \le d - j, \quad j \le |K_-^\#| + |K_\pm^\#| - |I_+^\#|,$$

$$|I_-^\#| + |I_+^\#| + |I_\pm^\#| + |K_-^\#| + |K_+^\#| + |K_\pm^\#| \leq k$$

since

$$|I_-^\#| + |I_+^\#| + |I_\pm^\#| + |K_-^\#| + |K_+^\#| + |K_\pm^\#| \leq S_{I_-} + S_{I_+} + S_{I_\pm} + S_{K_-} + S_{K_+} + S_{K_\pm}.$$

As a result, the coefficient of t^k of the polynomial map $t \mapsto \varepsilon_\iota(x, y + ty')$ is equal to 0 when $k < j$ and for $k = j$, it is the sum of the value at (x, y, y') of the products

$$\epsilon(I, K, I_-, K_-, I_+, K_+, I_\pm, K_\pm)\, v_\iota \wedge \varepsilon_{0,0} \wedge \varepsilon_I \wedge \nu_K \wedge \omega_{I_-,-} \wedge \omega_{K_-,-} \wedge \omega_{I_\pm,-} \wedge \omega_{K_\pm,0}$$

with

$$|K| + |I_+^\#| + |K_+^\#| \leq d - j, \quad j \leq |K_-^\#| + |K_\pm^\#| - |I_+^\#|,$$
$$|I_-| = |I_-^\#| = S_{I_-}, \quad |I_+| = |I_+^\#| = S_{I_+}, \quad |I_\pm| = |I_\pm^\#| = S_{I_\pm},$$
$$|K_-| = |K_-^\#| = S_{K_-}, \quad |K_+| = |K_+^\#| = S_{K_+}, \quad |K_\pm| = |K_\pm^\#| = S_{K_\pm},$$

whence

$$|I_+| = |I_-| = |I_\pm| = |K_+| = 0, \quad I = I'_{*,0}, \quad |K| = d - j, \quad |K_-| + |K_\pm| = j.$$

For K in $\mathcal{K}_j$, set:

$$\epsilon(K) := \epsilon_* \epsilon(I'_{*,0}, K, \emptyset, I_{\mathrm{u}} \setminus K, \emptyset, \emptyset, \emptyset, \emptyset).$$

Then, by Proposition 9.1,

$$c_\iota(x, y, y') - \sum_{K \in \mathcal{K}_j} \epsilon(K)\, v_\iota \wedge \overline{\varepsilon_0}(x, y) \wedge \nu_K(x, y) \wedge \omega_{I_{\mathrm{u}} \setminus K,-}(x, y, y') \in \bigoplus_{|\kappa| < j} w_\kappa \wedge \bigwedge^{j-|\kappa|}(\mathfrak{l}) \wedge \overline{\varepsilon_0}(x, y) \wedge \mu_+$$

since

$$v_\iota \wedge \varepsilon_{0,0}(x) \wedge \varepsilon_{I'_{*,0}}(x, y) \wedge \nu_K = \epsilon_* v_\iota \wedge \overline{\varepsilon_0}(x, y) \wedge \nu_K(x, y)$$

and $|K_\pm| > 0$ when $K_- \subsetneq I_{\mathrm{u}} \setminus K$. Moreover, for K in $\mathcal{K}_j$, for well defined functions $a_{r,\kappa,K,\iota}$, $(r, \kappa) \in \Lambda_j$ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$,

$$\epsilon(K)\, v_\iota \wedge \overline{\varepsilon_0}(x, y) \wedge \nu_K(x, y) \wedge \omega_{I_{\mathrm{u}} \setminus K,-}(x, y, y') = \sum_{(r,\kappa) \in \Lambda_j} v^r(y') a_{r,\kappa,K,\iota}(x, y) w_\kappa \wedge \overline{\varepsilon_0}(x, y) \wedge \mu_+,$$

whence the lemma. □

For (i, m) in I_0 and $k = 1, \ldots, d$, denote by $b_{i,k}^{(m-1)}$ the coordinate of $\omega_{i,m,1,-}$ at w_k. Then $b_{i,k}^{(m-1)}$ has an expansion

$$b_{i,k}^{(m-1)} = \sum_{l=1}^{d} v_l \otimes b_{i,l,k}^{(m-1)}$$

with $b_{i,l,k}^{(m-1)}$ homogeneous of degree $m - 1$ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$. For $j = 1, \ldots, d$ and $r = (r_1, \ldots, r_d)$ in $\mathbb{N}_j^d$, define an increasing sequence $j_1 < \cdots < j_l$ by the following conditions:

- j_1 is the smallest integer such that $r_{j_1} \neq 0$,
- j_m is the smallest integer bigger than j_{m-1} such that $r_{j_m} \neq 0$.

and denote by $r'_1, \ldots, r'_l$ and $u_1, \ldots, u_j$ the sequences defined by the following conditions:

- $r'_1 = r_1$ and $r'_m = r'_{m-1} + r_{j_m}$,
- $u_m = j_1$ for $m = 1, \dots, r_1$,
- $u_m = j_k$ for $r'_{k-1} + 1 \le m \le r'_k$.

For (ι, κ) in $\mathfrak{I}_j \times \mathfrak{I}_j$ and K in $\mathcal{K}_j$, let $B_{r,\kappa,K}$ be the element of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$:

$$B_{r,\kappa,K} := \det(b^{(m_l-1)}_{i_l,u_l,k_{l'}},\ 1 \le l, l' \le j) \quad \text{with} \quad I_{\mathrm{u}} \setminus K = ((i_1, m_1) \prec \cdots \prec (i_j, m_j), \quad \kappa = k_1, \dots, k_j$$

and $a_{K,\iota}$ the function in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$ such that

$$v_\iota \wedge \overline{\varepsilon_0} \wedge \nu_K = a_{K,\iota} \overline{\varepsilon_0} \wedge \mu_+.$$

Lemma 9.3. *Let $j = 1, \dots, d$, ι in $\mathfrak{I}_j$, K in $\mathcal{K}_j$, $K_- := I_{\mathrm{u}} \setminus K$.*

(i) *The map $\omega_{K_-,-}$ has an expansion*

$$\omega_{K_-,-} = \sum_{r \in \mathbb{N}^d_j} \sum_{\kappa \in \mathfrak{I}_j} v^r B_{r,\kappa,K} w_\kappa.$$

(ii) *For $\kappa = k_1, \dots, k_j$ in $\mathfrak{I}_j$ and r in $\mathbb{N}^d_j$ such that $r_{k_l} = 1$ for $l = 1, \dots, j$, $B_{r,\kappa,K} \neq 0$.*

(iii) *For all (r, κ) in Λ_j,*

$$a_{r,\kappa,K,\iota} = (-1)^{j\mathfrak{b}_{\mathfrak{g}}} \epsilon(K)\, a_{K,\iota} B_{r,\kappa,K}.$$

Proof. (i) Let (x, y, y') be in $\mathfrak{l}_* \times \mathfrak{p} \times \mathfrak{p}_{-,\mathrm{u}}$. Denote by $(i_1, m_1), \dots, (i_j, m_j)$ the elements of K_-, ordered so that the sequence is increasing. For κ in $\mathfrak{I}_j$, let $\omega_{K_-,-,\kappa}$ be the coordinate of $\omega_{K_-,-}$ at w_κ. By definition, for $\kappa = k_1 < \cdots < k_j$,

$$\omega_{K_-,-,\kappa} := \det(b^{(m_l-1)}_{i_l,k_{l'}},\ 1 \le l, l' \le j) \quad \text{whence} \quad \omega_{K_-,-,\kappa} = \sum_{r \in \mathbb{N}^d_j} v^r B_{r,\kappa,K}.$$

(ii) Let (x, y, y') be in $\mathfrak{l}_* \times \mathfrak{p} \times \mathfrak{p}_{-,\mathrm{u}}$. By Lemma 8.1(i), for $l = 1, \dots, \ell$ and s in $\Bbbk$, the polynomial map

$$t \longmapsto \varepsilon_{l,-}(x + sy + sty') - \sum_{1 \le l_1, k \le d} st\langle v_{l_1}, y'\rangle a_{l,l_1,k}(x + sy) w_k$$

is divisible by t^2 in $\Bbbk[t] \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}}$. Hence the polynomial map

$$t \longmapsto \varepsilon^{(m')}_{l,-}(x, y + ty') - \sum_{1 \le l_1, k \le d} t\langle v_{l_1}, y'\rangle a^{(m'-1)}_{l,l_1,k}(x, y) w_k$$

is divisible by t^2 in $\Bbbk[t] \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}}$ for $m' = 0, \dots, d_l - 1$. By definition, for (i, m) in I_{u},

$$\nu^{(m)}_i = \varepsilon^{(m)}_i - \sum_{(l,m') \in I'_0} a_{i,m,l,m'} \varepsilon^{(m')}_l \quad \text{and} \quad \nu^{(m)}_{i,-} := \varepsilon^{(m)}_{i,-} - \sum_{(l,m') \in I'_0} a_{i,m,l,m'} \varepsilon^{(m')}_{l,-}$$

with $a_{i,m,l,m'}$ homogeneous and L-invariant element of $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$. So, the polynomial map

$$t \longmapsto \nu^{(m)}_{i,-}(x, y + ty') - \sum_{1 \le l_1, k \le d} t\langle v_{l_1}, y'\rangle (a^{(m-1)}_{i,l_1,k}(x, y) - \sum_{(l,m') \in I'_0} a_{i,m,l,m'}(x, y) a^{(m'-1)}_{l,l_1,k}(x, y)) w_k$$

is divisible by t^2 in $\Bbbk[t]\otimes_{\Bbbk}\mathfrak{p}_{-,\mathfrak{u}}$. As a result, for $u,u'=1,\dots,j$,

$$b^{(m_u-1)}_{i_u,k_u,k_{u'}}(x,y)=a^{(m_u-1)}_{i_u,k_u,k_{u'}}(x,y)-\sum_{(l,m')\in I'_0}a_{i_u,m_u,l,m'}(x,y)a^{(m'-1)}_{l,k_u,k_{u'}}(x,y)$$

so that $b^{(m_u-1)}_{i_u,k_u,k_{u'}}$ is the element of $\Bbbk[\mathfrak{l}_*\times\mathfrak{p}]$ defined in Subsection 8.2. By definition,

$$B_{r,\kappa,K}=\det(b^{(m_u-1)}_{i_u,k_u,k_{u'}},\ 1\le u,u'\le j).$$

According to Proposition 8.5, the restriction to $\mathfrak{h}_*\times\mathfrak{h}$ of the matrix

$$(b^{(m_u-1)}_{i_u,k_u,k_{u'}},\ 1\le u,u'\le j)$$

is diagonal and all its diagonal entries are different from 0, whence the assertion.

(iii) By (i),

$$v_\iota\wedge\overline{\varepsilon_0}\wedge\nu_K\wedge\omega_{K_-,-}=\sum_{(r,\kappa)\in\Lambda_j}v^rB_{r,\kappa,K}v_\iota\wedge\overline{\varepsilon_0}\wedge\nu_K\wedge w_\kappa=\sum_{(r,\kappa)\in\Lambda_j}v^rB_{r,\kappa,K}a_{K,\iota}\overline{\varepsilon_0}\wedge\mu_+\wedge w_\kappa,$$

whence the assertion by Lemma 9.2(iii). □

For $j=1,\dots,d$, (ι,κ) in $\mathfrak{I}_j\times\mathfrak{I}_j$, K in $\mathcal{K}_j$, set:

$$B_{\kappa,K}:=\sum_{r\in\mathbb{N}^d_j}v^rB_{r,\kappa,K},\quad a_{\kappa,\iota}:=\sum_{r\in\mathbb{N}^d_j}\sum_{K\in\mathcal{K}_j}v^ra_{r,\kappa,K,\iota},$$

$$P_j:=\det(a_{\kappa,\iota},\ (\iota,\kappa)\in\mathfrak{I}_j\times\mathfrak{I}_j),\quad Q_j:=\det(B_{\kappa,K},\ (\kappa,K)\in\mathfrak{I}_j\times\mathcal{K}_j).$$

Corollary 9.4. *Let $j=1,\dots,d$.*

(i) *For some (x,y,y') in $\mathfrak{l}_*\times\mathfrak{p}\times\mathfrak{p}_{-,\mathfrak{u}}$, $Q_j(x,y,y')\neq 0$.*

(ii) *For some (x,y,y') in $\mathfrak{l}_*\times\mathfrak{p}\times\mathfrak{p}_{-,\mathfrak{u}}$, $P_j(x,y,y')\neq 0$.*

Proof. (i) Prove the assertion by induction on j. By Lemma 9.3, the assertion is true for $j=1$. Suppose that it is true for $j-1$. By induction hypothesis and Lemma 9.3(ii), for all (x,y,y') in a dense open subset of $\mathfrak{l}_*\times\mathfrak{p}\times\mathfrak{p}_{-,\mathfrak{u}}$, $Q_{j-1}(x,y,y')\neq 0$ and $B_{\kappa,K}(x,y,y')\neq 0$ for $l=1,\dots,d$ and for all (κ,K) in $\mathfrak{I}_l\times\mathcal{K}_l$. Let (x,y,y') be in this open subset of $\mathfrak{l}_*\times\mathfrak{p}\times\mathfrak{p}_{-,\mathfrak{u}}$ and c_κ, $\kappa\in\mathfrak{I}_j$ in $\Bbbk$ such that

$$\sum_{\kappa\in\mathfrak{I}_j}c_\kappa B_{\kappa,K}(x,y,y')=0,\qquad\forall K\in\mathcal{K}_j.$$

For K in $\mathcal{K}_j$ and $l=1,\dots,j$, set:

$$K_-:=I_{\mathfrak{u}}\setminus K:=\{(i_1,m_1),\dots,(i_j,m_j)\}\quad\text{and}\quad K^{(l)}:=K\cup\{(i_l,m_l)\},$$

and for κ in $\mathfrak{I}_j$, denote by $\kappa^{(l)}$ the element of $\mathfrak{I}_{j-1}$ such that $\{\kappa^{(l)}\}=\{\kappa\}\setminus\{k_l\}$. Since

$$B_{\kappa,K}=\det(b^{(m_l-1)}_{i_l,k_{l'}},\ 1\le l,l'\le j),$$

for $l = 1, \dots, j$,

$$B_{\kappa,K} = \sum_{l'=1}^{j} (-1)^{l+l'} b_{i_l,k_{l'}}^{(m_l-1)} B_{\kappa^{(l')},K^{(l)}},$$

whence

$$\sum_{\kappa\in\mathfrak{I}_j} c_\kappa \sum_{l'=1}^{j} (-1)^{l'} b_{i_l,k_{l'}}^{(m_l-1)} B_{\kappa^{(l')},K^{(l)}}(x,y,y') = 0.$$

For (i,m) in I_{u}, let $\mathcal{K}_{j-1}(i,m)$ be the set of elements K of $\mathcal{K}_{j-1}$, containing (i,m), and for K in $\mathcal{K}_{j-1}(i,m)$ denote by $K^{i,m}$ the element $K \setminus \{(i,m)\}$ of $\mathcal{K}_j$. For $\kappa = k_1, \dots, k_{j-1}$ in $\mathfrak{I}_{j-1}$, denote by I_κ the subset of $\{1, \dots, j\}$ defined by the following conditions:

$$l \in I_\kappa \quad \text{if and only if} \quad \begin{cases} l = 1 & \text{and} \quad 2 \le k_1 \\ l = j & \text{and} \quad k_{j-1} \le d-1 \\ 1 < l < j & \text{and} \quad k_{l-1} + 2 \le k_l \end{cases}$$

For l in I_κ, let $a_{\kappa,l}$ and $b_{\kappa,l}$ be in $\{1, \dots, d\}$ defined by the following conditions:

$$a_{\kappa,l} := \begin{cases} 1 & \text{if} \quad l = 1 \\ k_{l-1}+1 & \text{if} \quad 2 \le l \end{cases} \quad \text{and} \quad b_{\kappa,l} := \begin{cases} d & \text{if} \quad l = j \\ k_{l+1}-1 & \text{if} \quad l \le j-1 \end{cases}$$

and for k in $[a_{\kappa,l}, b_{\kappa,l}]$, denote by $\kappa^{l,k}$ the element of $\mathfrak{I}_j$ such that $\{\kappa\}$ is contained in $\{\kappa^{l,k}\}$ and k is the l-th element of the sequence $\kappa^{k,l}$. Setting

$$\mathcal{E} := \{(\kappa,l,k) \in \mathfrak{I}_{j-1} \times \{1,\dots,j\} \times \{1,\dots,d\} \mid l \in I_\kappa \quad \text{and} \quad k \in [a_{\kappa,l}, b_{\kappa,l}]\},$$

the map

$$\mathcal{E} \longrightarrow \mathfrak{I}_j \times \{1,\dots,j\}\ , \qquad (\kappa,l,k) \longmapsto (\kappa^{l,k}, l)$$

is bijective.

For (i,m) in I_{u} and for κ in $\mathfrak{I}_{j-1}$, set:

$$c'_{\kappa,i,m} = \sum_{l\in I_\kappa} \sum_{k=a_{\kappa,l}}^{b_{\kappa,l}} (-1)^l c_{\kappa^{l,k}} b_{i,k}^{(m-1)}(x,y,y').$$

Then, for K in $\mathcal{K}_{j-1}$,

$$\sum_{\kappa\in\mathfrak{I}_{j-1}} c'_{\kappa,i,m} B_{\kappa,K}(x,y,y') = \sum_{(\kappa,l,k)\in\mathcal{E}} (-1)^l c_{\kappa^{l,k}} b_{i,k}^{(m-1)}(x,y,y') B_{\kappa,K}(x,y,y') =$$

$$\sum_{\kappa\in\mathfrak{I}_j} \sum_{l=1}^{j} (-1)^l c_\kappa b_{i,k_l}^{(m)}(x,y,y') B_{\kappa^{(l)},K}(x,y,y').$$

As a result, by the above equalities,

$$\sum_{\kappa\in\mathfrak{I}_{j-1}} c'_{\kappa,i,m} B_{\kappa,K}(x,y,y') = 0, \qquad \forall K \in \mathcal{K}_{j-1}(i,m).$$

For K in $\mathcal{K}_{j-1} \setminus \mathcal{K}_{j-1}(i,m)$ and κ in $\mathfrak{I}_j$,

$$\sum_{l=1}^{j}(-1)^l b_{i,k_l}^{(m-1)} B_{\kappa^{(l)},K} = 0$$

as the determinant of a matrix having two equal lines. Hence

$$\sum_{\kappa\in\mathfrak{I}_{j-1}} c'_{\kappa,i,m} B_{\kappa,K}(x,y,y') = 0, \qquad \forall K \in \mathcal{K}_{j-1}.$$

Since $Q_{j-1}(x,y,y') \neq 0$, for all κ in $\mathfrak{I}_{j-1}$ and for all (i,m) in I_{u}, $c'_{\kappa,i,m} = 0$.

Let κ be in $\mathfrak{I}_{j-1}$. Denote by N_κ the cardinality of the union

$$\bigcup_{l\in I_\kappa}[a_{\kappa,l}, b_{k,l}],$$

and κ_* the element of $\mathfrak{I}_{N_\kappa}$ equal to the ordered sequence of this union. As for K in $\mathcal{K}_{N_\kappa}$, $B_{\kappa_*,K}(x,y,y') \neq 0$, $c_{\kappa^{l,k}} = 0$ for all l in I_κ and all k in $[a_{\kappa,l}, b_{\kappa,l}]$ since $c'_{\kappa,i,m} = 0$ for all (i,m) in I_{u}. As a result, $c_\kappa = 0$ for all κ in $\mathfrak{I}_j$, whence $Q_j(x,y,y') \neq 0$.

(ii) As the $\nu_i^{(m)}$, $(i,m) \in I_{\mathrm{u}}$ are linearly independent over $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$, so are the ν_K, $K \in \mathcal{K}_j$. So, by (i), for all (x,y,y') in a dense open subset of $\mathfrak{l}_* \times \mathfrak{p} \times \mathfrak{p}_{-,\mathrm{u}}$, $Q_j(x,y,y') \neq 0$, $\overline{\varepsilon_0}(x,y) \neq 0$ and the $\nu_K(x,y)$, $K \in \mathcal{K}_j$ are linearly independent. Let (x,y,y') be in this open subset of $\mathfrak{l}_* \times \mathfrak{p} \times \mathfrak{p}_{-,\mathrm{u}}$ and c_κ, $\kappa \in \mathfrak{I}_j$ in $\Bbbk$ such that

$$\sum_{\kappa\in\mathfrak{I}_j} c_\kappa a_{\kappa,\iota}(x,y,y') = 0, \quad \forall \iota \in \mathfrak{I}_j.$$

By Lemma 9.3,(iii), for all (ι,κ) in $\mathfrak{I}_j \times \mathfrak{I}_j$,

$$a_{\kappa,\iota} = \sum_{K\in\mathcal{K}_j} (-1)^{j\mathfrak{b}_{\mathfrak{g}}} \epsilon(K)\, a_{K,\iota} B_{\kappa,K},$$

whence

$$\sum_{K\in\mathcal{K}_j} \epsilon(K)\, a_{K,\iota}(x,y) \sum_{\kappa\in\mathfrak{I}_j} c_\kappa B_{\kappa,K}(x,y,y') = 0, \qquad \forall \iota \in \mathfrak{I}_j.$$

Then, setting:

$$c'_K := \sum_{\kappa\in\mathfrak{I}_j} c_\kappa \epsilon(K)\, B_{\kappa,K}(x,y,y')$$

for K in $\mathcal{K}_j$,

$$0 = \sum_{K\in\mathcal{K}_j} c'_K a_{K,\iota}(x,y)\overline{\varepsilon_0}(x,y) \wedge \mu_+ = \sum_{K\in\mathcal{K}_j} c'_K v_\iota \wedge \overline{\varepsilon_0}(x,y) \wedge \nu_K(x,y), \qquad \forall \iota \in \mathfrak{I}_j.$$

As a result, for all ι in $\mathfrak{I}_j$,

$$\sum_{K\in\mathcal{K}_j} c'_K v_\iota \wedge \nu_K(x,y) = 0.$$

Since v_ι, $\iota \in \mathfrak{I}_j$ is a basis of $\bigwedge^j(\mathfrak{p}_{\mathrm{u}})$,

$$\sum_{K\in\mathcal{K}_j} c'_K \nu_K(x,y) = 0$$

so that

$$\sum_{\kappa\in\mathfrak{I}_j} c_\kappa \epsilon(K)\, B_{\kappa,K}(x,y,y') = 0$$

for all K in $\mathcal{K}_j$ since the $\nu_K(x,y)$, $K \in \mathcal{K}_j$ are linearly independent. As a result, the c_κ, $\kappa \in \mathfrak{I}_j$ are all equal to 0 since $Q_j(x,y,y') \neq 0$, whence $P_j(x,y,y') \neq 0$. □

9.3. **An other expansion.** Let $\mathfrak{d}_1, \dots, \mathfrak{d}_{\mathrm{n}}$ be the simple factors of $\mathfrak{d}$ when $\mathfrak{z}$ is strictly contained in $\mathfrak{l}$. Set:

$$n_1 := \mathrm{b}_{\mathfrak{d}_1} - \ell_{\mathfrak{d}_1},\ ,\dots,\ n_{\mathrm{n}} := \mathrm{b}_{\mathfrak{d}_{\mathrm{n}}} - \ell_{\mathfrak{d}_{\mathrm{n}}}$$

so that

$$n - d = n_1 + \cdots + n_{\mathrm{n}}.$$

As $\mathfrak{g}$ is the direct sum of $\mathfrak{z}$, $\mathfrak{d}_1, \dots, \mathfrak{d}_{\mathrm{n}}$, $\mathfrak{p}_{\mathrm{u}}$, $\mathfrak{p}_{-,\mathrm{u}}$, for $i = 1, \dots, \mathrm{n}$, $\Bbbk[\mathfrak{d}_i]$ is a subalgebra of $\Bbbk[\mathfrak{l}]$ and $\Bbbk[\mathfrak{g}]$ and $\Bbbk[\mathfrak{l}_* \times \mathfrak{d}_i]$ is a subalgebra of $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$. Set $\mathrm{B}_i := \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{d}_i\times\mathfrak{d}_i]} \mathrm{B}_{\mathfrak{d}_i}$. Then B_i is a free submodule of rank $\mathrm{b}_{\mathfrak{d}_i}$ of $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \mathfrak{d}_i$. According to Lemma 7.1(ii), for some $\lambda_{i,1}, \dots, \lambda_{i,n_i}$ in $\mathrm{B}_{\mathfrak{d}_i}$, the $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$-module B_i is contained in the submodule generated $\lambda_{i,1}, \dots, \lambda_{i,n_i}$ and $\varepsilon_1^{(0)}, \dots, \varepsilon_\ell^{(0)}$ so that $\mathrm{B}_i \wedge \varepsilon$ is generated by $\lambda_{i,1} \wedge \varepsilon, \dots, \lambda_{i,n_i} \wedge \varepsilon$.

Let $\mathfrak{J}$ be the union of $\{0\}$ and the set of strictly increasing sequences in

$$\{(i,j) \mid i = 1, \dots, \mathrm{n},\ j = 1, \dots, n_i\}.$$

For υ in $\mathfrak{J}$, $i = 1, \dots, \mathrm{n}$ and $j = 1, \dots, n-d$, set:

$$\{\upsilon\} := \{i_1, \dots, i_k\}, \quad |\upsilon| := k, \quad \lambda_\upsilon := \lambda_{i_1} \wedge \cdots \wedge \lambda_{i_k},$$

$$\mathfrak{J}^{(i)} := \{\upsilon \in \mathfrak{J} \mid \{\upsilon\} \subset \{(i,1), \dots, (i,n_i)\}\}, \quad \mathfrak{J}_j := \{\upsilon \in \mathfrak{J} \mid |\upsilon| = j\}, \quad \mathfrak{J}_j^{(i)} := \mathfrak{J}^{(i)} \cap \mathfrak{J}_j.$$

For $\upsilon = 0$, $|\upsilon| := 0$ and $\lambda_\upsilon = 1$. For υ in $\mathfrak{J}$ and $j = 0, \dots, n-d$, set:

$$\varepsilon_\upsilon := \lambda_\upsilon \wedge \varepsilon \quad \text{and} \quad \mathcal{J}_j := \{I \subset I_{*,0} \mid |I| = n - j \quad \text{and} \quad |I \cap I'_{*,0}| \geq n - d - j\}.$$

For (x,y,y') in $\mathfrak{l}_* \times \mathfrak{p} \times \mathfrak{p}_{-,\mathrm{u}}$ and (i,m) in I_0, the polynomial map

$$t \longmapsto \varepsilon_i^{(m)}(x, y + ty')$$

has an expansion

$$\varepsilon_i^{(m)}(x, y+ty') = \varepsilon_i^{(m)}(x,y) + \sum_{k=1}^m t^k \omega_{i,m,k,-}(x,y,y') + \sum_{k=1}^m t^k \omega_{i,m,k,+}(x,y,y') + \sum_{k=1}^m t^k \omega_{i,m,k,0}(x,y,y')$$

with

$$\omega_{i,m,k,-} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathrm{S}^k(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}}, \quad \omega_{i,m,k,+} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathrm{S}^k(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}},$$

$$\omega_{i,m,k,0} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathrm{S}^k(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \mathfrak{l}.$$

For I' subset of $I_{0,*}$, set:

$$\omega_{I',-} := \wedge_{(i,m,k)\in I'}\omega_{i,m,k,-}, \quad \omega_{I',+} := \wedge_{(i,m,k)\in I'}\omega_{i,m,k,+}, \quad \omega_{I',0} := \wedge_{(i,m,k)\in I'}\omega_{i,m,k,0}.$$

In these equalities, the order of the products are induced by the orders of I_0 and $I_{0,*}$.

Lemma 9.5. *Let $j = 0, \dots, n-d$, υ in $\mathfrak{J}_j$ and (x, y, y') in $\mathfrak{l}_* \times \mathfrak{p} \times \mathfrak{p}_{-,\mathrm{u}}$. Denote by $c_\upsilon(x, y, y')$ the coefficient of t^j of the polynomial map $t \mapsto \varepsilon_\upsilon(x, y + ty')$.*

(i) *The polynomial map $t \mapsto \varepsilon_\upsilon(x, y + ty')$ is divisible by t^j in $\Bbbk[t] \otimes_\Bbbk \bigwedge^{\mathrm{b}_\mathfrak{g}+j}(\mathfrak{g})$.*

(ii) *For a well defined map*

$$\mathcal{J}_j \xrightarrow{\quad\epsilon\quad} \{-1, 1\}\ ,$$

$$c_\upsilon(x, y, y') - c'_\upsilon(x, y, y') \in \bigoplus_{|\kappa|<j} w_\kappa \wedge \bigwedge^{j-|\kappa|}(\mathfrak{l}) \wedge \overline{\varepsilon_0}(x, y) \wedge \mu_+$$

with

$$c'_\upsilon(x, y, y') := \sum_{I\in\mathcal{J}_j} \epsilon(I)\, \lambda_\upsilon(x, y) \wedge \varepsilon_{0,0}(x) \wedge \varepsilon_I(x, y) \wedge \omega_{I'_{*,0}\setminus I,-}(x, y, y').$$

(iii) *For I in $\mathcal{J}_j$, for well defined functions $a_{r,\kappa,I,\upsilon}$, $(r, \kappa) \in \Lambda_j$ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$,*

$$\epsilon(I)\, \lambda_\upsilon(x, y) \wedge \varepsilon_{0,0}(x) \wedge \varepsilon_I(x, y) \wedge \omega_{I'_{*,0}\setminus I,-}(x, y, y') = \sum_{(r,\kappa)\in\Lambda_j} v^r(y') a_{r,\kappa,I,\upsilon}(x, y) w_\kappa \wedge \overline{\varepsilon_0}(x, y) \wedge \mu_+.$$

Proof. (i) As already observed in the proof of Lemma 9.2, the cofficient of t^k of the polynomial function $t \longmapsto \varepsilon(x, y + ty')$ is the sum of the value at (x, y, y') of products

$$\epsilon(I, I_-, I_+, I_\pm)\, \varepsilon_{0,0} \wedge \varepsilon_I \wedge \omega_{I_-,-} \wedge \omega_{I_+,+} \wedge \omega_{I_\pm,0}$$

with

$$I \subset I_{*,0}, \quad I_- \subset I_{0,*}, \quad I_+ \subset I_{0,*}, \quad I_\pm \subset I_{0,*}, \quad \epsilon(I, I_-, I_+, I_\pm) \in \{-1, 1\}$$

such that

$$I \cup I_-^\# \cup I_+^\# \cup I_\pm^\# = I \sqcup I_-^\# \sqcup I_+^\# \sqcup I_\pm^\#, \quad |I| + |I_-^\#| + |I_+^\#| + |I_\pm^\#| = n,$$
$$S_{I_-} + S_{I_+} + S_{I_\pm} = k.$$

According to Corollary 2.7,

$$\lambda_\upsilon \wedge \varepsilon_{0,0} \wedge \varepsilon_I \in \bigoplus_{m=0}^{d} \bigwedge^{|I|+j+\ell-m}(\overline{\mathrm{B}}_\mathfrak{l}) \wedge \bigwedge^m(\mathfrak{p}_\mathrm{u}).$$

Then

$$|I| + |I_+^\#| \le n - j \quad \text{whence} \quad j \le |I_-^\#| + |I_\pm^\#| \le k.$$

As a result, the coefficient of t^k of the polynomial map $t \mapsto \varepsilon_\upsilon(x, y + ty')$ is equal to 0 when $k < j$ and for $k = j$, it is the sum of the value at (x, y, y') of the products

$$\epsilon(I, I_-, I_+, I_\pm)\, \lambda_\upsilon \wedge \varepsilon_{0,0} \wedge \varepsilon_I \wedge \omega_{I_-,-} \wedge \omega_{I_\pm,-}$$

with

$$j = |I_-^\#| + |I_\pm^\#|, \quad I_+ = \emptyset, \quad |I_-| = |I_-^\#| = S_{I_-}, \quad |I_\pm| = |I_\pm^\#| = S_{I_\pm}.$$

So, in any case,

$$\lambda_\upsilon(x,y) \wedge \varepsilon_{0,0}(x) \wedge \varepsilon_I(x,y) \in \overline{\varepsilon_0}(x,y) \wedge \bigwedge^d(\mathfrak{p}_{\mathrm{u}}).$$

For I in $\mathcal{J}_j$, set:

$$\epsilon(I) := \epsilon_* \epsilon(I, I_{*,0} \setminus I, \emptyset, \emptyset).$$

Then

$$c_\upsilon(x,y,y') - \sum_{I \in \mathcal{J}_j} \epsilon(I)\, \lambda_\upsilon(x,y) \wedge \varepsilon_{0,0}(x) \wedge \varepsilon_I(x,y) \wedge \omega_{I_{*,0}\setminus I,-}(x,y) \in$$
$$\bigoplus_{|\kappa|<j} w_\kappa \wedge \bigwedge^{j-|\kappa|}(\mathfrak{l}) \wedge \overline{\varepsilon_0}(x,y) \wedge \mu_+.$$

since $|I_\pm| > 0$ when $|I_-| < j$. Moreover, for I in $\mathcal{J}_j$, for well defined functions $a_{r,\kappa,I,\upsilon}$, $(r,\kappa) \in \Lambda_j$ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$,

$$\epsilon(I)\, \lambda_\upsilon(x,y) \wedge \varepsilon_{0,0}(x) \wedge \varepsilon_I(x,y) \wedge \omega_{I_{*,0}\setminus I,-}(x,y) = \sum_{(r,\kappa)\in\Lambda_j} v^r(y') a_{r,\kappa,I,\upsilon}(x,y) w_\kappa \wedge \overline{\varepsilon_0}(x,y) \wedge \mu_+,$$

whence the lemma. □

10. Some spaces related to parabolic subalgebras

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. We then use the notations

$$\mathfrak{l}, \quad \mathfrak{p}_{\mathrm{u}}, \quad \mathfrak{d}, \quad \mathfrak{z}, \quad \mathcal{R}_{\mathfrak{l}}, \quad L, \quad d, \quad d_0, \quad \mathfrak{l}_*, \quad z, \quad \zeta, \quad I_0, \quad I_0', \quad I_{\mathrm{u}}, \quad \mathfrak{p}_-, \quad \mathfrak{p}_{-,\mathrm{u}}$$

of Section 8.

10.1. **An equivalence.** For φ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk \bigwedge(\mathfrak{g})$, φ has a unique expansion

$$\varphi = \sum_{\iota \in \mathfrak{I}} \varphi_\iota \wedge v_\iota \quad \text{with} \quad \varphi_\iota \in \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk \bigwedge(\mathfrak{p}_-)$$

and for ι in $\mathfrak{I}$, φ_ι has a unique expansion

$$\varphi_\iota = \sum_{\kappa \in \mathfrak{I}} \varphi_{\iota,\kappa} \wedge w_\kappa \quad \text{with} \quad \varphi_{\iota,\kappa} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk \bigwedge(\mathfrak{l}).$$

For all (ι,κ) in $\mathfrak{I} \times \mathfrak{I}$, $\varphi_{\iota,\kappa}$ has a unique expansion

$$\varphi_{\iota,\kappa} := \sum_{r \in \mathbb{N}^d} v^r \varphi_{r,\iota,\kappa} \quad \text{with} \quad \varphi_{r,\iota,\kappa} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk \bigwedge(\mathfrak{l}).$$

For l nonnegative integer, let $\mathcal{P}_l$ be the subspace of $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk \bigwedge(\mathfrak{g})$ defined by the following condition:

$$\varphi \in \mathcal{P}_l \Longleftrightarrow (|r| + |\iota| \neq l \Longrightarrow \varphi_{r,\iota,\kappa} = 0).$$

By Proposition 2.3 and Lemma 6.3(i), C is a free submodule of rank n of $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \mathfrak{g}$, generated by the maps

$$(x, y) \longmapsto [x, \varepsilon_i^{(m)}(x, y)], \ (i, m) \in I_{*,0}.$$

For ν in C, denote by ι_ν the $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$-derivation of the algebra $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \bigwedge(\mathfrak{g})$ such that $\iota_\nu(v) = \langle \nu, v \rangle$ for all v in $\mathfrak{g}$.

Lemma 10.1. *Let V be a subspace of $\mathfrak{g}$ such that V is contained in a complement to $V_{x,y}$ in $\mathfrak{g}$ for all (x, y) in a dense open subset of $\Omega_* \cap \mathfrak{l}_* \times \mathfrak{p}$.*

(i) *For v in $\mathfrak{p}_{\mathrm{u}}$ and w in $\mathfrak{p}_{-,\mathrm{u}}$, $\iota_\nu(v)$ and $\iota_\nu(w)$ are in J and J_- respectively for all ν in C.*

(ii) *For $k = 1, \dots, \dim V$ and ψ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \bigwedge^k(V)$, if $\iota_\nu(\psi) = 0$ for all ν in C, then $\psi = 0$.*

Proof. (i) By Proposition 9.1(iii), for all (i, m) in I_0,

$$[x, \varepsilon_i^{(m)}] \in \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \mathfrak{l} + J \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}} + J_- \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}},$$

whence the assertion.

(ii) Suppose that $\iota_\nu(\psi) = 0$ for all ν in C and prove the assertion by induction on $\dim V$. As V is contained in a complement to $V_{x,y}$ in $\mathfrak{g}$ for all (x, y) in a dense open subset of $\Omega_* \cap \mathfrak{l}_* \times \mathfrak{p}$, for v in V, $v = 0$ if and only if $\iota_\nu(v) = 0$ for all ν in C since $V_{x,y}$ is the orthogonal complement to $[x, V_{x,y}]$ for all (x, y) in Ω_* by Proposition 2.1(iv). As a result, the assertion is true for $\dim V = 1$. Suppose $\dim V > 1$ and the assertion true for the subspaces of V. Let V' be an hyperplane of V and v in $V \setminus V'$. Then

$$\psi = v \wedge \psi' + \psi_0$$

with ψ' in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \bigwedge^{k-1}(V')$ and ψ_0 in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \bigwedge^k(V')$. As $\iota_\nu(\psi) = 0$ for all ν in C,

$$\iota_\nu(\psi') = 0 \quad \text{and} \quad \iota_\nu(v)\psi' + \iota_\nu(\psi_0) = 0,$$

whence $\psi' = 0$ and $\psi_0 = 0$ by induction hypothesis, and $\psi = 0$. □

For l nonnegative integer, denote by M_l the subspace of elements φ of $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \bigwedge(\mathfrak{g})$ such that $\varphi \wedge \varepsilon$ is in $J^l \otimes_{\Bbbk} \bigwedge(\mathfrak{g})$. For $k = 1, \dots, n$, denote by $\mathcal{P}_{l,k}$ the intersection of $\mathcal{P}_l$ and $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \bigwedge^k(\mathfrak{g})$. For φ in M_l and ν in C, $\iota_\nu(\varphi)$ is in M_l since $\iota_\nu(\varepsilon) = 0$.

Lemma 10.2. *Let $k = 1, \dots, n$, l a positive integer and φ in $\mathcal{P}_{l,k}$.*

(i) *The element φ is in M_l.*

(ii) *If $\varphi_{r,\iota,\kappa} \wedge \overline{\varepsilon_0} = 0$ for all (r, ι, κ), then φ is in M_{l+1}.*

(iii) *Suppose that φ is in $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \bigwedge^k(\mathfrak{p})$. If φ is in M_{l+1} then $\varphi_{r,\iota,0} \wedge \overline{\varepsilon_0} = 0$ for all (r, ι).*

Proof. (i) Let (x, y, y') be in $\mathfrak{l}_* \times \mathfrak{p} \times \mathfrak{p}_{-,\mathrm{u}}$. According to lemma 9.2 and the notations of Subsection 9.2, for $j = 1, \dots, d$ and ι in $\mathfrak{I}_j$, the polynomial map

$$t \longmapsto v_\iota \wedge \varepsilon(x, y + ty')$$

is divisible by t^j in $\Bbbk[t]\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$ and the coefficient $c_\iota(x,y,y')$ of t^j satisfies the relation

$$c_\iota(x,y,y')-\sum_{\kappa'\in\mathfrak{I}_j}a_{\kappa',\iota}(x,y,y')w_{\kappa'}\wedge\overline{\varepsilon_0(x,y)}\wedge\mu_+\in$$

$$\bigoplus_{|\kappa'|<j}w_{\kappa'}\wedge\bigwedge^{j-|\kappa'|}(\mathfrak{l})\wedge\overline{\varepsilon_0}(x,y)\wedge\mu_+.$$

Hence the polynomial map

$$t\longmapsto\varphi(x,y+ty')\wedge\varepsilon(x,y+ty')$$

is divivisble by t^l in $\Bbbk[t]\otimes_{\Bbbk}\bigwedge^k(\mathfrak{g})$ since φ is in $\mathcal{P}_{l,k}$. As a result, φ is in M_l.

(ii) Suppose that $\varphi_{r,\iota,\kappa}\wedge\overline{\varepsilon_0}=0$ for all (r,ι,κ). By the above relation, the polynomial map

$$t\longmapsto\varphi(x,y+ty')\wedge\varepsilon(x,y+ty')$$

is divivisble by t^{l+1} in $\Bbbk[t]\otimes_{\Bbbk}\bigwedge^k(\mathfrak{g})$ for all (x,y,y') in $\mathfrak{l}_*\times\mathfrak{p}\times\mathfrak{p}_{-,\mathfrak{u}}$. Hence φ is in M_{l+1}.

(iii) As φ is in $\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}\bigwedge^k(\mathfrak{p})$, $\varphi_{r,\iota,\kappa}=0$ for $\kappa\neq 0$. Suppose $\varphi_{r,\iota,0}\wedge\overline{\varepsilon_0}\neq 0$ for some (r,ι). A contradiction is expected. Denote by $\lambda(\varphi)$ the biggest integer j such that $\varphi_{r,\iota,0}\wedge\overline{\varepsilon_0}\neq 0$ for some (r,ι) in $\mathbb{N}^d_{l-j}\times\mathfrak{I}_j$. By (ii), we can suppose $\varphi_{r,\iota,0}=0$ for $|\iota|>\lambda(\varphi)$. As the polynomial map

$$t\longmapsto\varphi(x,y+ty')\wedge\varepsilon(x,y+ty')$$

is divivisble by t^{l+1},

$$\sum_{\kappa\in\mathfrak{I}_{\lambda(\varphi)}}a_{\kappa,\iota}\sum_{r\in\mathbb{N}^d_{l-\lambda(\varphi)}}v^r\varphi_{r,\iota,0}\wedge\overline{\varepsilon_0}=0,\qquad\forall\iota\in\mathfrak{I}_{\lambda(\varphi)},$$

by maximality of $\lambda(\varphi)$. By Corollary 9.4(ii), for all (x,y,y') in a dense open subset of $\mathfrak{l}_*\times\mathfrak{p}\times\mathfrak{p}_{-,\mathfrak{u}}$, the matrix

$$(a_{\kappa,\iota}(x,y,y'),\ (\iota,\kappa)\in\mathfrak{I}_{\lambda(\varphi)}\times\mathfrak{I}_{\lambda(\varphi)})$$

is invertible. Hence $\varphi_{r,\iota,0}\wedge\overline{\varepsilon_0}=0$ for all (r,ι) in $\mathbb{N}^d_{l-\lambda(\varphi)}\times\mathfrak{I}_{\lambda(\varphi)}$, whence the contradiction. □

Proposition 10.3. *Let $k=1,\dots,n$, l a positive integer and φ in $\mathcal{P}_{l,k}$. Then φ is in M_{l+1} if and only if $\varphi_{r,\iota,\kappa}\wedge\overline{\varepsilon_0}=0$ for all (r,ι,κ).*

Proof. By Lemma 10.2(ii), the condition is sufficient. Suppose that φ is in M_{l+1}. By Lemma 10.1(i),

$$\sum_{(r,\iota,\kappa)}v^r\iota_\nu(\varphi_{r,\iota,\kappa})\wedge w_\kappa\wedge v_\iota+\sum_{(r,\iota,\kappa)}v^r\varphi_{r,\iota,\kappa}\wedge\iota_\nu(w_\kappa)\wedge v_\iota\in M_{l+1}$$

for ν in C since $\iota_\nu(\varepsilon)=0$. Prove the assertion by induction on k.

Suppose $k>1$, the assertion true for $k-1$ and the proposition not true for k. A contradiction is expected. Denote by j the biggest integer such that for some κ in $\mathfrak{I}_j$, $\varphi_{r,\iota,\kappa}\wedge\overline{\varepsilon_0}\neq 0$. By Lemma 10.2(ii), we can suppose

$$|\kappa|>j\Longrightarrow\varphi_{r,\iota,\kappa}=0.$$

By Lemma 10.2(iii), $j > 0$. By the above relation and the induction hypothesis,

$$\iota_\nu(\varphi_{r,\iota,\kappa}) \wedge \overline{\varepsilon_0} = 0, \qquad \forall(\nu, r, \iota, \kappa) \in \mathrm{C} \wedge \mathbb{N}^d \times \mathfrak{I} \times \mathfrak{I}_j$$

by maximality of j.

Let (r, ι, κ) be in $\mathbb{N}^d \times \mathfrak{I} \times \mathfrak{I}_j$. Set $k' := k - j - |\iota|$. Then $\iota_\nu(\varphi_{r,\iota,\kappa}) \wedge \varepsilon(x, y) = 0$ for all (ν, x, y) in $\mathrm{C} \times \mathfrak{l}_* \times \mathfrak{p}$ by Corollary 2.7. For some principal open subset O of $\Omega_* \cap \mathfrak{l}_* \times \mathfrak{p}$ and some subspace V of $\mathfrak{g}$, V is the complement to $V_{x,y}$ for all (x, y) in O. Then

$$\varphi_{r,\iota,\kappa}|_O = \varphi' + \varphi'' \quad \text{with} \quad \varphi' \in \Bbbk[O] \otimes_\Bbbk \bigwedge^{k'}(V) \quad \text{and} \quad \varphi''(x, y) \in \bigwedge^{k'-1}(\mathfrak{g}) \wedge V_{x,y}$$

for all (x, y) in O, whence $\iota_\nu(\varphi') = 0$ for all ν in C. As a result, $\varphi' = 0$ by Lemma 10.1(ii) and $\varphi_{r,\iota,\kappa} \wedge \overline{\varepsilon_0} = 0$ by Corollary 2.7 since $\varphi_{r,\iota,\kappa}$ is in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk \bigwedge^{k'}(\mathfrak{l})$ by definition, whence the contradiction.

For $k = 1$,

$$\varphi_{r,0,0} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk \mathfrak{l}, \quad \varphi_{r,0,j} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}], \quad \varphi_{r,j,0} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$$

for $j = 1, \dots, d$, whence

$$\iota_\nu(\varphi_{r,0,0} + \sum_{j=1}^{d} \varphi_{r,0,j} w_j) \wedge \varepsilon(x, y) = 0$$

for all (r, ν) in $\mathbb{N}_l^d \times \mathrm{C}$ and all (x, y) in $\mathfrak{l}_* \times \mathfrak{p}$ by Lemma 10.1(i). Then, arguing as above, by Lemma 10.1(ii), for all r in $\mathbb{N}_l^d$, $\varphi_{r,0,0} \wedge \overline{\varepsilon_0} = 0$ and $\varphi_{r,0,j} = 0$ for $j = 1, \dots, d$. As a result, by Lemma 10.2,(ii) and (iii), $\varphi_{r,j,0} = 0$ for all (r, j) in $\mathbb{N}_l^d \times \{1, \dots, d\}$, whence the proposition. □

10.2. **On the G-action.** Let $\mathcal{O}$ be the local ring of G at the identity and $\mathfrak{m}$ its maximal ideal. Denote by $\hat{\mathcal{O}}$ the completion of $\mathcal{O}$ for the $\mathfrak{m}$-adic topology and again by $\mathfrak{m}$ its maximal ideal. Let $z_1, \dots, z_{2n+\ell}$ be a system of coordinates of $\mathcal{O}$. Then $\hat{\mathcal{O}} = \Bbbk[[z_1, \dots, z_{2n+\ell}]]$. As usual for $s = (s_1, \dots, s_{2n+\ell})$ in $\mathbb{N}^{2n+\ell}$, set:

$$z^s := z_1^{s_1} \cdots z_{2n+\ell}^{s_{2n+\ell}}.$$

Let $\overline{J}$ be the ideal of $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk \bigwedge(\mathfrak{p}_\mathrm{u})$ generated by $J \otimes 1$ and $1 \otimes \mathfrak{p}_\mathrm{u}$ and $\hat{J}$ the ideal of $\hat{\mathcal{O}} \otimes_\Bbbk \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk \bigwedge(\mathfrak{p}_\mathrm{u})$ generated by $\mathfrak{m} \otimes 1$ and $1 \otimes \overline{J}$. Denote by $\widetilde{\mathrm{B}}$ and $\widehat{\mathrm{B}}$ the submodules of $\mathcal{O} \otimes_\Bbbk \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk \mathfrak{g}$ and $\hat{\mathcal{O}} \otimes_\Bbbk \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk \mathfrak{g}$ generated by the maps

$$(g, x, y) \longmapsto \varepsilon_i^{(m)}(g(x), g(y)), \ (i, m) \in I_0,$$

respectively. Then the map

$$G \times \mathfrak{l}_* \times \mathfrak{g} \xrightarrow{\hat{\varepsilon}} \bigwedge^{\mathfrak{b}_\mathfrak{g}}(\mathfrak{g}), \qquad (g, x, y) \longmapsto \wedge_{(i,m) \in I_0} \varepsilon_i^{(m)}(g(x), g(y))$$

is a generator of $\bigwedge^{\mathfrak{b}_\mathfrak{g}}(\widetilde{\mathrm{B}})$ and $\bigwedge^{\mathfrak{b}_\mathfrak{g}}(\widehat{\mathrm{B}})$. As usual, the order of the product is induced by the order of I_0.

Set:

$$\mathbb{N}_\mathfrak{p} := \mathbb{N}^{2n+\ell} \times \mathbb{N}^d \times \mathfrak{I}, \quad |(s, r, \iota)| := |s| + |r| + |\iota|,$$

$$\mathcal{L} := \{\varphi \in (\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \bigwedge(\mathfrak{g}))^{\mathbb{N}_{\mathfrak{p}}} \mid \exists N_\varphi \in \mathbb{N} \quad \text{such that} \quad |r| \geq N_\varphi \Longrightarrow \varphi_{s,r,\iota} = 0\},$$
$$\kappa(\varphi) := \sum_{(s,r,\iota)\in\mathbb{N}_{\mathfrak{p}}} z^s v^r \varphi_{s,r,\iota} \wedge v_\iota$$

for (s, r, ι) in $\mathbb{N}_{\mathfrak{p}}$ and φ in $\mathcal{L}$. For $k = 1, \ldots, n$, denote by $\mathcal{L}^{(k)}$ the subspace of elements φ of $\mathcal{L}$ such that

$$\kappa(\varphi) \in \hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \bigwedge^k(\mathfrak{g}).$$

and set:

$$\mathcal{L}_* := \{\varphi \in \mathcal{L} \mid s \neq 0 \Longrightarrow \varphi_{s,r,\iota} = 0\}, \quad \mathcal{L}_*^{(k)} = \mathcal{L}_* \cap \mathcal{L}^{(k)}, \quad \varphi_{r,\iota} := \varphi_{0,r,\iota}, \quad |(r,\iota)| := |(0,r,\iota)|$$

for φ in $\mathcal{L}_*$ and (r, ι) in $\mathbb{N}^d \times \mathfrak{I}$. For l nonnegative integer, denote by Δ_l the subset of elements (r, ι) of $\mathbb{N}^d \times \mathfrak{I}$ such that $|(r, \iota)| = l$.

Lemma 10.4. *Let $k = 1, \ldots, n$, l a positive integer and φ in $\mathcal{L}_*^{(k)}$ such that $\kappa(\varphi)$ is in M_{l+1}. If $\varphi_{r,\iota}$ is in $\overline{J}^{l-|(r,\iota)|} \bigwedge(\mathfrak{g})$ for all (r, ι), then $\varphi_{r,\iota}$ is in $M_{l+1-|(r,\iota)|}$ for all (r, ι).*

Proof. According to the hypothesis, $\varphi_{r,\iota} = 0$ for all (r, ι) such that $|(r, \iota)| > l$. For (r_1, ι_1) in $\mathbb{N}^d \times \mathfrak{I}$, denote by $\Gamma(r_1, \iota_1)$ the subset of elements (r, ι) of $\mathbb{N}^d \times \mathfrak{I}$ such that

$$r_1 - r \in \mathbb{N}^d \quad \text{and} \quad \{\iota\} \subset \{\iota_1\}.$$

For (r, ι) in $\Gamma(r_1, \iota_1)$, denote by $\iota_1 \setminus \iota$ the element of $\mathfrak{I}$ and $\epsilon(\iota_1, \iota)$ the element of $\{-1, +1\}$ such that

$$\{\iota_1 \setminus \iota\} = \{\iota_1\} \setminus \{\iota\} \quad \text{and} \quad v_{\iota_1\setminus\iota} \wedge v_\iota = \epsilon(\iota_1, \iota) v_{\iota_1}.$$

Let j be the smallest integer such that $\varphi_{r,\iota} \neq 0$ for some (r, ι) such that $|(r, \iota)| = j$ and i the biggest integer such that $\varphi_{r',\iota'} \neq 0$ for some (r', ι') such that $|(r', \iota')| = i$. Then $j \leq i \leq l$. Prove the lemma by induction on $(l - j, i)$. For $j = l$, by Proposition 10.3, for all (r, ι), $\varphi_{r,\iota} \wedge \overline{\varepsilon_0} = 0$. As a result, again by Proposition 10.3, $\varphi_{r,\iota}$ is in M_1. Suppose $j < l$ and the lemma true for all $(l - j', i')$ smaller than $(l - j, i)$.

(a) Suppose $j = i$. For (r, ι) in Δ_j, $\varphi_{r,\iota}$ has an expansion

$$\varphi_{r,\iota} = \sum_{(r',\iota')\in\Delta_1} v^{r'} \varphi_{r,\iota,r',\iota'} \wedge v_{\iota'} \quad \text{with} \quad \varphi_{r,\iota,r',\iota'} \in \overline{J}^{l-j-1} \bigwedge(\mathfrak{g}).$$

For (r_1, ι_1) in Δ_{j+1}, set:

$$\psi_{r_1,\iota_1} := \sum_{(r,\iota)\in\Gamma(r_1,\iota_1)} \epsilon(\iota_1, \iota) \varphi_{r,\iota,r_1-r,\iota_1\setminus\iota}.$$

Then

$$\sum_{(r,\iota)\in\mathbb{N}^d\times\mathfrak{I}} v^r \varphi_{r,\iota} \wedge v_\iota = \sum_{(r_1,\iota_1)\in\Delta_{j+1}} v^{r_1} \psi_{r_1,\iota_1} \wedge v_{\iota_1}.$$

By induction hypothesis, for all (r_1, ι_1) in Δ_{j+1}, ψ_{r_1,ι_1} is in M_{l-j}.

Prove by induction on $m := |\iota|$ that $\varphi_{r,\iota}$ is in M_{l+1-j}. For k nonnegative integer, set:

$$N_k := \{(r, r_*) \in \mathbb{N}_k^d \times \mathbb{N}_{k+1}^d \mid r_* - r \in \mathbb{N}^d\} \quad \text{and} \quad N_k(r_*) := \{r \in \mathbb{N}_k^d \mid (r, r_*) \in N_k\}$$

for r_* in $\mathbb{N}^d_{k+1}$.

Claim 10.5. Let a_{r,r_*}, $(r, r_*) \in N_k$ in $\Bbbk$ such that

$$\sum_{r \in N_k(r_*)} a_{r,r_*} = 0, \qquad \forall r_* \in \mathbb{N}^d_{k+1}.$$

Then $a_{r,r_*} = 0$ for all (r, r_*).

Proof. [Proof of Claim 10.5] Prove the claim by induction on k. For $k = 0$, there is nothing to prove. Suppose $k > 0$ and the claim true for $k-1$. Let r be in $\mathbb{N}^d_k$ such that the first component r_1 of r is positive. Denote by $N_{k,1}$ the subset of elements (x, x_*) of N_k such that the first component of x is positive. For x in $\mathbb{N}^d$, denote by $x^\#$ the element of $\mathbb{Z}^d$ such that

$$x^\#_1 := x_1 - 1, \qquad x^\#_s := x_s \quad \text{for} \quad s = 2, \dots, d.$$

Then the map

$$N_{k,1} \longrightarrow N_{k-1}\ , \qquad (x, x_*) \longmapsto (x^\#, x^\#_*)$$

is bijective. As a result, setting, $b_{x^\#,x^\#_*} := a_{x,x_*}$ for (x, x_*) in $N_{k,1}$,

$$\sum_{x \in N_{k-1}(x_*)} b_{x,x_*} = 0, \qquad \forall x_* \in \mathbb{N}^d_k.$$

So, by induction hypothesis, $a_{x,x_*} = 0$ for all (x, x_*) in $N_{k,1}$. Then, after permutation of the indices, $a_{r,r_*} = 0$ for all (r, r_*) in N_k since for some indice s, r_s is positive, whence the claim. □

By Claim 10.5, for (r, r_1) in N_j, $\varphi_{r,0,r_1-r,0}$ is in M_{l-j}. Hence $\varphi_{r,0}$ is in M_{l+1-j}. Suppose $m > 0$ and $\varphi_{r,\iota} \in M_{l+1-j}$ for all (r, ι) such that $|\iota| < m$. Then we can suppose $\varphi_{r,\iota} = 0$ for all (r, ι) such that $|\iota| < m$. Let ι be in $\mathfrak{I}_m$. For all r_1 in $\mathbb{N}^d_{j+1-m}$,

$$\sum_{(r,\iota) \in \Gamma(r_1,\iota)} \varphi_{r,\iota,r_1-r,\iota} \in M_{l-j}.$$

By Claim 10.5, $\varphi_{r,\iota,r_1-r,\iota}$ is in M_{l-j} for all (r, r_1) in N_{j-m}. Then $\varphi_{r,\iota}$ is in M_{l+1-j} for all (r, ι) such that $|\iota| = m$, whence the assertion in this case.

(b) Suppose $j < i$. For (r, ι) in $\mathbb{N}^d \times \mathfrak{I}$ such that $|(r, \iota)| \leq i$, $\varphi_{r,\iota}$ has an expansion

$$\varphi_{r,\iota} = \sum_{(r',\iota') \in \Delta_{i-|(r,\iota)|}} v^{r'} \varphi_{r,\iota,r',\iota'} \wedge v_{\iota'} \quad \text{with} \quad \varphi_{r,\iota,r',\iota'} \in \overline{J}^{l-i} \otimes_{\Bbbk} \bigwedge(\mathfrak{g}).$$

For (r_1, ι_1) in Δ_i, set:

$$\psi_{r_1,\iota_1} := \sum_{(r,\iota) \in \Gamma(r_1,\iota_1)} \epsilon(\iota_1, \iota) \varphi_{r,\iota,r_1-r,\iota_1 \setminus \iota}.$$

Then

$$\sum_{(r,\iota) \in \mathbb{N}^d \times \mathfrak{I}} v^r \varphi_{r,\iota} \wedge v_\iota = \sum_{(r_1,\iota_1) \in \Delta_i} v^{r_1} \psi_{r_1,\iota_1} \wedge v_{\iota_1}.$$

By (a), for all (r_1, ι_1) in Δ_i, ψ_{r_1,ι_1} is in M_{l+1-i} since ψ_{r_1,ι_1} is in $\overline{J}^{l-i} \bigwedge(\mathfrak{g})$.

Let Λ be the subset of elements (r,ι) of Δ_i such that $\varphi_{r,\iota}\neq 0$. For (r,ι) in Λ,

$$\Gamma(r,\iota)\setminus\{(r,\iota)\}\subset\bigcup_{(r',\iota')\in\Delta_i\setminus\{(r,\iota)\}}\Gamma(r',\iota'),$$

whence $\varphi_{r,\iota}=\psi_{r,\iota}$ is in M_{l+1-i}. Denote by φ' the element of $\mathcal{L}_*$ such that $\varphi'_{r,\iota}=\varphi_{r,\iota}$ for (r,ι) not in Λ and 0 otherwise. Denoting by i' the biggest integer such that $\varphi'_{r,\iota}\neq 0$ for some (r,ι) in $\Delta_{i'}$, i' is smaller than i. So, by induction hypothesis, $\varphi'_{r,\iota}$ is in $M_{l+1-|(r,\iota)|}$ since $\kappa(\varphi')$ is in M_{l+1}, whence the lemma. □

Proposition 10.6. *Let $k=1,\dots,n$, l a positive integer and φ in $\mathcal{L}^{(k)}$ such that $\kappa(\varphi)\wedge\hat{\varepsilon}$ is in $\hat{J}^{l+1}\bigwedge^k(\mathfrak{g})$. If $\varphi_{s,r,\iota}$ is in $\overline{J}^{l-|(s,r,\iota)|}\bigwedge^k(\mathfrak{g})$ for all (s,r,ι), then $\varphi_{s,r,\iota}$ is in $M_{l+1-|(s,r,\iota)|}$ for all (s,r,ι).*

Proof. By Proposition 2.1(vi), for (g,x,y) in $G\times\mathfrak{l}_*\times\mathfrak{g}$,

$$\hat{\varepsilon}(g(x),g(y))=g.\varepsilon(x,y).$$

Then

$$\hat{\varepsilon}-\varepsilon\in\mathfrak{m}\otimes_{\Bbbk}\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}\bigwedge^{\mathfrak{b}_{\mathfrak{g}}}(\mathfrak{g})\quad\text{and}\quad z^sv^r\varphi_{s,r,\iota}\wedge v_\iota\wedge(\varepsilon-\hat{\varepsilon})\in\hat{J}^{l+1}\bigwedge(\mathfrak{g}),$$

whence

$$\kappa(\varphi)\wedge\varepsilon\in\hat{J}^{l+1}\bigwedge(\mathfrak{g}).$$

As a result, from the equality

$$\hat{J}^{l+1}\bigwedge^k(\mathfrak{g})=\bigoplus_{i=0}^{l+1}\mathfrak{m}^i\otimes_{\Bbbk}\overline{J}^{l+1-i}\bigwedge^k(\mathfrak{g}),$$

we deduce

$$\sum_{s\in\mathbb{N}_i^{2n+\ell}}\sum_{(r,\iota)\in\mathfrak{J}\times\mathfrak{J}}\varphi_{s,r,\iota}\wedge\varepsilon\in\overline{J}^{l+1-i}\bigwedge^k(\mathfrak{g})$$

for $i=0,\dots,l+1$. Then, by Lemma 10.4, $\varphi_{s,r,\iota}$ is in $M_{l+1-|(s,r,\iota)|}$ for all (s,r,ι). □

11. Some spaces related to parabolic subalgebras II

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. We then use the notations of Section 10 and Subsection 9.3. Set:

$$\mathcal{Q}:=(\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}\bigwedge(\mathfrak{p}_-))^{\mathfrak{J}}$$

and denote by θ the map

$$\mathcal{Q}\xrightarrow{\ \theta\ }\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}\bigwedge(\mathfrak{p}_-)\ ,\qquad(\varphi_\upsilon,\ \upsilon\in\mathfrak{J})\longmapsto\sum_{\upsilon\in\mathfrak{J}}\varphi_\upsilon\wedge\lambda_\upsilon.$$

For υ in $\mathfrak{J}$, φ_υ has a unique expansion

$$\varphi_\upsilon=\sum_{\kappa\in\mathfrak{J}}\varphi_{\upsilon,\kappa}\wedge w_\kappa,\quad\text{with}\quad\varphi_{\upsilon,\kappa}\in\Bbbk[\mathfrak{l}_*\times\mathfrak{g}]\otimes_{\Bbbk}\bigwedge(\mathfrak{l}).$$

For l nonnegative integer, let $\mathcal{Q}_l$ be the subspace of $\mathcal{Q}$ defined by the following condition:

$$\varphi \in \mathcal{Q}_l \Longleftrightarrow (|r| + |\upsilon| \neq l \Longrightarrow \varphi_{r,\upsilon} = 0).$$

For $k = 1, \ldots, n$, denote by $\mathcal{Q}_{l,k}$ the subspace of elements φ of $\mathcal{Q}_l$ such that $\theta(\varphi)$ is in $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \bigwedge^k(\mathfrak{p}_-)$.

Lemma 11.1. *Let $k = 1, \ldots, n$, l a positive integer and φ in $\mathcal{Q}_{l,k}$.*

(i) *The element $\theta(\varphi)$ is in M_l.*

(ii) *If $\varphi_{r,\upsilon,\kappa} \wedge \overline{\varepsilon_0} = 0$ for all (r, υ, κ), then $\theta(\varphi)$ is in M_{l+1}.*

Proof. Let (x, y, y') be in $\mathfrak{l}_* \times \mathfrak{p} \times \mathfrak{p}_{-,\mathfrak{u}}$. According to Lemma 9.5, for $j = 1, \ldots, n-d$, υ in $\mathfrak{J}_j$, the polynomial map

$$t \longmapsto \lambda_\upsilon \wedge \varepsilon(x, y + ty')$$

is divisible by t^j in $\Bbbk[t] \otimes_{\Bbbk} \bigwedge(\mathfrak{g})$ and the coefficient $c_\upsilon(x, y, y')$ of t^j satisfies the relation

$$c_\upsilon(x, y, y') - \sum_{\kappa \in \mathfrak{J}_j} a_{\kappa,\upsilon}(x, y, y') w_\kappa \wedge \overline{\varepsilon_0(x, y)} \wedge \mu_+ \in$$

$$\bigoplus_{|\kappa| < j} w_\kappa \wedge \bigwedge^{j-|\kappa|}(\mathfrak{l}) \wedge \overline{\varepsilon_0}(x, y) \wedge \mu_+ \quad \text{with}$$

$$a_{\kappa,\upsilon} := \sum_{I \in \mathfrak{J}_j} \sum_{r \in \mathbb{N}_j^d} \upsilon^r a_{r,\kappa,I,\upsilon}.$$

Then the polynomial map

$$t \longmapsto \theta(\varphi) \wedge \varepsilon(x, y + ty')$$

is divisible by t^l in $\Bbbk[t] \otimes_{\Bbbk} \bigwedge(\mathfrak{g})$ since $\varphi_{r,\upsilon,\kappa} = 0$ when $|r| + |\upsilon| < l$. Hence $\theta(\varphi)$ is in M_l. Moreover, the coefficient of t^l of this polynomial map is equal to 0 if $\varphi_{r,\upsilon,\kappa} \wedge \overline{\varepsilon_0} = 0$ for all (r, υ, κ), whence the lemma. □

Proposition 11.2. *Let $k = 1, \ldots, n$, l a positive integer and φ in $\mathcal{Q}_{l,k}$. Then $\theta(\varphi)$ is in M_{l+1} if and only if $\varphi_{r,\upsilon,\kappa} \wedge \overline{\varepsilon_0} = 0$ for all (r, υ, κ).*

Proof. By Lemma 11.1(ii), the condition is sufficient. Suppose that $\theta(\varphi)$ is in M_{l+1}. For all ν in C,

$$\sum_{(r,\upsilon,\kappa)} \upsilon^r \iota_\nu(\varphi_{r,\upsilon,\kappa}) \wedge w_\kappa + \sum_{(r,\upsilon,\kappa)} \upsilon^r \varphi_{r,\upsilon,\kappa} \wedge \iota_\nu(w_\kappa) \in M_{l+1}$$

since $\iota_\nu(\varepsilon) = 0$. Prove the proposition by induction on k.

Suppose $k > 1$, the assertion true for $k - 1$ and the proposition not true for k. A contradiction is expected. Denote by j the biggest integer such that for some κ in $\mathfrak{J}_j$, $\varphi_{r,\upsilon,\kappa} \wedge \overline{\varepsilon_0} \neq 0$. By Lemma 11.1(ii), we can suppose

$$|\kappa| > j \Longrightarrow \varphi_{r,\upsilon,\kappa} = 0.$$

By the above relation and the induction hypothesis,

$$\iota_\nu(\varphi_{r,\upsilon,\kappa}) \wedge \overline{\varepsilon_0} = 0, \qquad \forall(\nu, r, \upsilon, \kappa) \in \mathrm{C} \wedge \mathbb{N}^d \times \mathfrak{J} \times \mathfrak{I}_j$$

by maximality of j. Then, arguing as in the proof of Proposition 10.3, $\varphi_{r,\upsilon,\kappa} \wedge \overline{\varepsilon_0} = 0$ for all (r, υ, κ) such that $|\kappa| = j$, whence the contradiction.

For $k = 1$,

$$\varphi_{r,0,0} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathfrak{l} \quad \text{and} \quad \varphi_{r,0,j} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}}$$

for $j = 1, \dots, d$, whence

$$(\varphi_{r,0,0} + \sum_{j=1}^{d} \varphi_{r,0,j} w_j) \wedge \varepsilon(x, y) = 0$$

for all r in $\mathbb{N}_l^d$ and all (x, y) in $\mathfrak{l}_* \times \mathfrak{p}$. Then, for all r in $\mathbb{N}_l^d$, $\varphi_{r,0,j} = 0$ for $j = 1, \dots, d$ and $\varphi_{r,0,0} \wedge \overline{\varepsilon_0} = 0$ since $\varepsilon(x, y)$ is colinear with $\overline{\varepsilon_0}(x, y) \wedge \mu_+$ for all (x, y) in $\mathfrak{l}_* \times \mathfrak{p}$, whence the proposition. □

Set $\mathbb{N}_{\mathfrak{p},*} := \mathbb{N}^d \times \mathfrak{J} \times \mathfrak{I}$.

Corollary 11.3. *Let $k = 1, \dots, n$, l a positive integer and φ in $(\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \bigwedge(\mathfrak{p}))^{\mathbb{N}_{\mathfrak{p},*}}$ such that*

$$\varphi_{r,\upsilon,\iota} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \bigwedge^{k-|\iota|}(\mathfrak{l}) \quad \textit{and} \quad |(r, \upsilon, \iota)| \neq l \Longrightarrow \varphi_{r,\upsilon,\iota} = 0.$$

Set:

$$\psi = \sum_{(r,\upsilon,\iota)} v^r \varphi_{r,\upsilon,\iota} \wedge \lambda_\upsilon \wedge v_\iota.$$

If ψ is in M_{l+1} then $\varphi_{r,\upsilon,\iota} \wedge \overline{\varepsilon_0} = 0$ for all (r, υ, ι).

Proof. For (r, ι) in $\mathbb{N}^d \times \mathfrak{I}$, set:

$$\psi_{r,\iota} = \sum_{\upsilon \in \mathfrak{J}} \varphi_{r,\upsilon,\iota} \wedge \lambda_\upsilon.$$

By hypothesis, for all (r, ι), $\psi_{r,\iota}$ is in $\overline{J}^{l-|(r,\iota)|} \bigwedge^{k-|\iota|}(\mathfrak{l})$. So, by Lemma 10.4, $\psi_{r,\iota}$ is in $M_{l+1-|(r,\iota)|}$. Hence, by Proposition 11.2, $\varphi_{r,\upsilon,\iota} \wedge \overline{\varepsilon_0} = 0$ for all (r, υ, ι). □

12. Induction. Case $\mathrm{n} \leq 1$

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. Then we use the notations of Section 9. In particular, $\mathfrak{l}$ is the reductive factor of $\mathfrak{p}$ containing $\mathfrak{h}$, $\mathfrak{d}$ is its derived algebra and n is the number of simple factors of $\mathfrak{d}$. We suppose that the simple factors of $\mathfrak{d}$ have Property (**P**). Let $\mathcal{O}$, $\hat{\mathcal{O}}$, $\mathfrak{m}$, $\overline{J}$, $\hat{J}$ be as in Subsection 10.2. Denote by J_1 the ideal of $\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$ generated by $\mathfrak{m}$ and J. In Subsections 12.2 and 12.3, we suppose $\mathrm{n} \leq 1$.

12.1. **Some complexes.** Recall that $\mathfrak{d}_1, \dots, \mathfrak{d}_n$ are the simple factors of $\mathfrak{l}$. Set: $\mathfrak{p}_{\pm,\mathrm{u}} := \mathfrak{p}_{\mathrm{u}} \oplus \mathfrak{p}_{-,\mathrm{u}}$. Let $\mathbb{I}$ and $\mathbb{I}_k$ be as in Subsection 6.2, $\mathbb{I}'$, $\mathbb{I}'_k$ as in Section 4. With the notations of Subsection 6.2, for $k = 0, \dots, n$, denote by $D^\bullet_{k,\#}(\mathfrak{g})$ and $D^\bullet_{k,\#}(\mathfrak{p}_-)$ the graded subcomplexes of $D^\bullet_k(\mathfrak{g})$ deduced from the double complexes

$$\bigoplus_{i=0}^{k} D^\bullet_{k-i}(\mathfrak{p}_{\pm,\mathrm{u}}) \otimes_{\Bbbk} D^\bullet_{i,\#}(\mathfrak{l}) \quad \text{and} \quad \bigoplus_{i=0}^{k} D^\bullet_{k-i}(\mathfrak{p}_{-,\mathrm{u}}) \otimes_{\Bbbk} D^\bullet_{i,\#}(\mathfrak{l})$$

respectively.

Let B, $\overline{\mathrm{B}}$, $\overline{\mathrm{B}}_{\mathfrak{l}}$, ε be as in Section 7, $\overline{\mathrm{B}}_0$, $\overline{\varepsilon_0}$ as in Subsection 7.2, $\widetilde{\mathrm{B}}$, $\widehat{\mathrm{B}}$, $\hat{\varepsilon}$ as in Subsection 10.2. For $k = 0, \dots, n$, denote by $D^\bullet_{k,\#}(\mathfrak{g}, \mathrm{B})$ the graded subcomplex of $D^\bullet_k(\mathfrak{g}, \mathrm{B})$:

$$D^\bullet_{k,\#}(\mathfrak{g}, \mathrm{B}) := D^\bullet_{k,\#}(\mathfrak{g})[-\mathrm{b}_{\mathfrak{g}}] \wedge \textstyle\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}).$$

The graded subcomplexes $D^\bullet_{k,\#}(\mathfrak{g}, \widetilde{\mathrm{B}})$ and $D^\bullet_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$ of $\hat{\mathcal{O}} \otimes_{\Bbbk} D^\bullet_k(\mathfrak{g})$ are equal to

$$D^\bullet_{k,\#}(\mathfrak{g})[-\mathrm{b}_{\mathfrak{g}}] \wedge \textstyle\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\widetilde{\mathrm{B}}) \quad \text{and} \quad D^\bullet_{k,\#}(\mathfrak{g})[-\mathrm{b}_{\mathfrak{g}}] \wedge \textstyle\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\widehat{\mathrm{B}})$$

respectively.

For $k = 0, \dots, n$ and $l = 0, \dots, n-d$, set:

$$\begin{aligned} \overline{\mathrm{B}}_+ &:= \quad \overline{\mathrm{B}}_0 \oplus \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}} \\ D^\bullet_{k,\#}(V, \overline{\mathrm{B}}_+) &:= \quad D^\bullet_{k,\#}(V)[-\mathrm{b}_{\mathfrak{g}}] \wedge \textstyle\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}}_+) \\ D^\bullet_{l,\#}(\mathfrak{l}, \overline{\mathrm{B}}_+) &:= \quad D^\bullet_{l,\#}(\mathfrak{l})[-\mathrm{b}_{\mathfrak{g}}] \wedge \textstyle\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}}_+) \end{aligned}$$

with $V = \mathfrak{p}_-$ or $V = \mathfrak{g}$. Then $\overline{\mathrm{B}}_+$ is a free module of rank $\mathrm{b}_{\mathfrak{g}}$ containing $\overline{\mathrm{B}}$, $D^\bullet_{k,\#}(V, \overline{\mathrm{B}}_+)$ is a graded subcomplex of $D^\bullet_k(V, \overline{\mathrm{B}}_+)$, $D^\bullet_{l,\#}(\mathfrak{l}, \overline{\mathrm{B}}_+)$ is a graded subcomplex of $D^\bullet_l(\mathfrak{l}, \overline{\mathrm{B}}_+)$.

Lemma 12.1. (i) *For $k = 0, \dots, n-d$, $D^\bullet_{k,\#}(\mathfrak{l}, \overline{\mathrm{B}}_+)$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$.*

(ii) *For $k = 0, \dots, n$, $D^\bullet_{k,\#}(\mathfrak{p}_-, \overline{\mathrm{B}}_{\mathfrak{l}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{l}}$.*

(iii) *For $k = 0, \dots, n$, $D^\bullet_{k,\#}(\mathfrak{p}_-, \overline{\mathrm{B}}_+)$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$.*

(iv) *For $k = 0, \dots, n$, $D^\bullet_{k,\#}(\mathfrak{g}, \overline{\mathrm{B}}_+)$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$.*

Proof. (i) According to Lemma 4.3, the complex $D^\bullet_{k,\#}(\mathfrak{l}, \mathrm{B}_{\mathfrak{l}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{l}}$ since $\mathfrak{d}_1, \dots, \mathfrak{d}_n$ have Property (**P**). By restriction to the principal open susbet $\mathfrak{l}_* \times \mathfrak{l}$ of $\mathfrak{l} \times \mathfrak{l}$, the subcomplex $D^\bullet_{k,\#}(\mathfrak{l}, \overline{\mathrm{B}}_{\mathfrak{l}})$ of $D^\bullet_k(\mathfrak{l}, \overline{\mathrm{B}}_{\mathfrak{l}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{l}}$. As $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] = \Bbbk[\mathfrak{p}_{\mathrm{u}}] \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$, $\overline{\mathrm{B}}_+$ is the direct sum of $\Bbbk[\mathfrak{p}_{\mathrm{u}}] \otimes_{\Bbbk} \overline{\mathrm{B}}_{\mathfrak{l}}$ and $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}}$ by Lemma 7.5(ii). As a result,

$$D^\bullet_{k,\#}(\mathfrak{l}, \overline{\mathrm{B}}_+) = \Bbbk[\mathfrak{p}_{\mathrm{u}}] \otimes_{\Bbbk} D^\bullet_{k,\#}(\mathfrak{l}, \overline{\mathrm{B}}_{\mathfrak{l}})[-d] \wedge \textstyle\bigwedge^{d}(\mathfrak{p}_{\mathrm{u}}),$$

whence an isomorphism of graded compexes

$$\textstyle\bigwedge^{d}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \Bbbk[\mathfrak{p}_{\mathrm{u}}] \otimes_{\Bbbk} D^\bullet_{k,\#}(\mathfrak{l}, \overline{\mathrm{B}}_{\mathfrak{l}})[-d] \longrightarrow D^\bullet_{k,\#}(\mathfrak{l}, \overline{\mathrm{B}}_+) \ ,$$

and the assertion.

(ii) and (iii) As $\mathfrak{p}_-$ is the direct sum of $\mathfrak{l}$ and $\mathfrak{p}_{-,\mathfrak{u}}$, $D^\bullet_{k,\#}(\mathfrak{p}_-,\overline{\mathrm{B}}_\mathfrak{l})$ and $D^\bullet_{k,\#}(\mathfrak{p}_-,\overline{\mathrm{B}}_+)$ are isomorphic to the simple complexes deduced from the double complexes:

$$\bigoplus_{j=\sup\{0,k-n+d\}}^{k} D^\bullet_j(\mathfrak{p}_{-,\mathfrak{u}})\otimes_\Bbbk D^\bullet_{k-j,\#}(\mathfrak{l},\overline{\mathrm{B}}_\mathfrak{l}) \quad \text{and} \quad \bigoplus_{j=\sup\{0,k-n+d\}}^{k} D^\bullet_j(\mathfrak{p}_{-,\mathfrak{u}})\otimes_\Bbbk D^\bullet_{k-j,\#}(\mathfrak{l},\overline{\mathrm{B}}_+)$$

respectively. By Lemma A.2(ii), for j positive integer, $D^\bullet_j(\mathfrak{p}_{-,\mathfrak{u}})$ is an acyclic complex, whence the assertion by (i) and its proof.

(iv) As $\mathfrak{p}_\mathfrak{u}$ is contained in $\overline{\mathrm{B}}_+$,

$$D^\bullet_{k,\#}(\mathfrak{g},\overline{\mathrm{B}}_+) = \bigoplus_{j=0}^{k} \mathrm{S}^{k-j}(\mathfrak{p}_\mathfrak{u})\otimes_\Bbbk D^\bullet_{k-j,\#}(\mathfrak{p}_-,\overline{\mathrm{B}}_+)$$

since

$$\bigoplus_{j=l'}^{l} \mathrm{S}^{l-j}(\mathfrak{p}_\mathfrak{u})\otimes_\Bbbk \mathrm{S}^{j-l'}(\mathfrak{p}_-) = \mathrm{S}^{l-l'}(\mathfrak{g})$$

for $0\le l'\le l$. So, by (iii), $D^\bullet_{k,\#}(\mathfrak{g},\overline{\mathrm{B}}_+)$ has no cohomology of degree different from $\mathrm{b}_\mathfrak{g}$. □

For $k=0,\dots,n$, denote by $D^\bullet_{k,\#}(\mathfrak{g},\overline{\mathrm{B}})$ the graded subcomplex of $\Bbbk[\mathfrak{l}_*\times\mathfrak{p}]\otimes_\Bbbk D^\bullet_k(\mathfrak{g})$,

$$D^\bullet_{k,\#}(\mathfrak{g},\overline{\mathrm{B}}) := D^\bullet_{k,\#}(\mathfrak{g})[-\mathrm{b}_\mathfrak{g}]\wedge\textstyle\bigwedge^{\mathrm{b}_\mathfrak{g}}(\overline{\mathrm{B}}).$$

Corollary 12.2. *For $k=0,\dots,n$, $D^\bullet_{k,\#}(\mathfrak{g},\overline{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_\mathfrak{g}$.*

Proof. As the modules $\overline{\mathrm{B}}_+$ and $\overline{\mathrm{B}}$ are free modules of rank $\mathrm{b}_\mathfrak{g}$ and $\overline{\mathrm{B}}$ is contained in $\overline{\mathrm{B}}_+$, for some p in $\Bbbk[\mathfrak{l}_*\times\mathfrak{p}]\setminus\{0\}$, $\bigwedge^{\mathrm{b}_\mathfrak{g}}(\overline{\mathrm{B}}) = p\bigwedge^{\mathrm{b}_\mathfrak{g}}(\overline{\mathrm{B}}_+)$. As a result, the map

$$\mathrm{S}(\mathfrak{g})\otimes_\Bbbk\textstyle\bigwedge(\mathfrak{g})\wedge\bigwedge^{\mathrm{b}_\mathfrak{g}}(\overline{\mathrm{B}}_+)\xrightarrow{\ \tau\ }\mathrm{S}(\mathfrak{g})\otimes_\Bbbk\bigwedge(\mathfrak{g})\wedge\bigwedge^{\mathrm{b}_\mathfrak{g}}(\overline{\mathrm{B}}),\qquad \varphi\longmapsto p\varphi$$

is an isomorphism of graded complexes such that

$$\tau(D^\bullet_{k,\#}(\mathfrak{g},\overline{\mathrm{B}}_+)) = D^\bullet_{k,\#}(\mathfrak{g},\overline{\mathrm{B}}).$$

So, by Lemma 12.1(iv), $D^\bullet_{k,\#}(\mathfrak{g},\overline{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_\mathfrak{g}$. □

For j,k integers such that $0\le j\le k\le n$, denote by $D^\bullet_{k,j,\#}(\mathfrak{g})$ the graded subspace of $D^\bullet_{k,\#}(\mathfrak{g})$,

$$D^\bullet_{k,j,\#}(\mathfrak{g}) := \mathrm{S}^j(\mathfrak{p}_\mathfrak{u})D^\bullet_{k-j,\#}(\mathfrak{p}_-)$$

and $D^\bullet_{k,j,\#}(\mathfrak{g},\overline{\mathrm{B}})$ the graded subspace of $D_{k,\#}(\mathfrak{g},\overline{\mathrm{B}})$,

$$D^\bullet_{k,j,\#}(\mathfrak{g},\overline{\mathrm{B}}) := D^\bullet_{k,j,\#}(\mathfrak{g})[-\mathrm{b}_\mathfrak{g}]\wedge\textstyle\bigwedge^{\mathrm{b}_\mathfrak{g}}(\overline{\mathrm{B}}).$$

In particular,

$$D_{k,0,\#}(\mathfrak{g},\overline{\mathrm{B}}) = D_{k,\#}(\mathfrak{p}_-,\overline{\mathrm{B}}) := D^\bullet_{k,\#}(\mathfrak{p}_-)[-\mathrm{b}_\mathfrak{g}]\wedge\textstyle\bigwedge^{\mathrm{b}_\mathfrak{g}}(\overline{\mathrm{B}}).$$

Since $\mathfrak{p}_{\mathrm{u}} \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}}) = \{0\}$, $D^{\bullet}_{k,j,\#}(\mathfrak{g}, \overline{\mathrm{B}})$ is a graded subcomplex of $D^{\bullet}_{k,\#}(\mathfrak{g}, \overline{\mathrm{B}})$.

Lemma 12.3. *Let $k = 1, \dots, n$.*

(i) *For φ in $D^{\mathrm{b}_{\mathfrak{g}}}_{k,\#}(\mathfrak{p}_-, \overline{\mathrm{B}})$, φ is a cocycle of $D^{\bullet}_{k,\#}(\mathfrak{g}, \overline{\mathrm{B}})$ if and only if φ is in $\mathrm{S}^k(\overline{\mathrm{B}}_{\mathfrak{l}}) \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}})$.*

(ii) *For $0 \leq j \leq k$ and φ in $D^{\mathrm{b}_{\mathfrak{g}}}_{k,j,\#}(\mathfrak{g}, \overline{\mathrm{B}})$, φ is a cocycle of $D^{\bullet}_{k,\#}(\mathfrak{g}, \overline{\mathrm{B}})$ if and only if*

$$\varphi \in \mathrm{S}^j(\mathfrak{p}_{\mathrm{u}})\mathrm{S}^{k-j}(\overline{\mathrm{B}}_{\mathfrak{l}}) \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}}).$$

Proof. Denote again by ε the restriction of ε to $\mathfrak{l}_* \times \mathfrak{p}$. Then ε is a generator of $\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}})$.

(i) As usual, for $r = (r_1, \dots, r_d)$ in $\mathbb{N}^d$, set:

$$w^r := w_1^{r_1} \cdots w_d^{r_d} \quad \text{and} \quad v^r := v_1^{r_1} \cdots v_d^{r_d}.$$

Let ψ be in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D^0_{k,\#}(\mathfrak{p}_-)$. Then ψ has an expansion

$$\psi = \sum_{j=0}^{k} \sum_{r \in \mathbb{N}^d_j} w^r \psi_r \quad \text{with} \quad \psi_r \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D^0_{k-|r|,\#}(\mathfrak{l}).$$

With the notations of Subsection 9.2, since $\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}})$ is a submodule of $\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}}_+)$, $\psi \wedge \varepsilon$ is a cocycle of $D^{\bullet}_{k,\#}(\mathfrak{g}, \overline{\mathrm{B}})$ if and only if $\psi \wedge \overline{\varepsilon_0} \wedge \mu_+$ is a cocycle of $D^{\bullet}_{k,\#}(\mathfrak{g}, \overline{\mathrm{B}}_+)$. In particular, the condition of the assertion is sufficient.

Suppose that $\psi \wedge \varepsilon$ is a cocycle of $D^{\bullet}_{k,\#}(\mathfrak{g}, \overline{\mathrm{B}})$. Denote by $\nu(\psi)$ the biggest element $(|r|, r)$ of $\mathbb{N} \times \mathbb{N}^d$ such that $\psi_r \neq 0$. Suppose that $\nu(\psi)$ is different from $(0, 0)$. A contradiction is expected. Let j be the smallest indice such that $r_j \neq 0$. Denote by $\tilde{r}$ the element of $\mathbb{N}^d_{|r|-1}$ such that $\tilde{r}_l = r_l$ for $l \neq j$ and $\mathfrak{R}_{r,\tilde{r}}$ the subset of elements r' of $\mathbb{N}^d_{|r|}$ such that $w^{\tilde{r}}$ divides $w^{r'}$. For r' in $\mathfrak{R}_{r,r'}$, let $k_{r'}$ be the indice such that $w^{r'} = w_{k_{r'}} w^{\tilde{r}}$. As $\psi \wedge \varepsilon$ is a cocycle, by maximality of $\nu(\psi)$ and minimality of j,

$$\sum_{r' \in \mathfrak{R}_{r,r'}} r'_{k_{r'}} w^{\tilde{r}} \psi_{r'} \otimes w_{k_{r'}} = 0.$$

In particular, $\psi_r = 0$, whence the contradition. As a result, $\psi = \psi_0$. As a matter of fact, $\psi = 0$ when $k > n - d$. Otherwise, $\psi \wedge \overline{\varepsilon_0}$ is a cocycle of $\Bbbk[\mathfrak{p}_{\mathrm{u}}] \otimes_{\Bbbk} D^{\bullet}_{k,\#}(\mathfrak{l}, \overline{\mathrm{B}}_{\mathfrak{l}})$. So, by Lemma A.4(iii), ψ is in $\Bbbk[\mathfrak{p}_{\mathrm{u}}] \otimes_{\Bbbk} \mathrm{S}^k(\overline{\mathrm{B}}_{\mathfrak{l}})$, whence the assertion.

(ii) Let ψ be in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D^0_{k,j,\#}(\mathfrak{g})$. Since $\psi \wedge \varepsilon$ is a cocycle if and only if $\psi \wedge \overline{\varepsilon_0} \wedge \mu_+$ is a cocycle of degree $\mathrm{b}_{\mathfrak{g}}$ of $D^{\bullet}_{k,\#}(\mathfrak{g}, \overline{\mathrm{B}}_+)$, the condition is sufficient. Suppose that $\psi \wedge \varepsilon$ is a cocycle of $D^{\bullet}_{k,\#}(\mathfrak{g}, \overline{\mathrm{B}})$. The element ψ has an expansion

$$\psi = \sum_{r \in \mathbb{N}^d_j} v^r \psi_r \quad \text{with} \quad \psi_r \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D^0_{k-j,\#}(\mathfrak{p}_-).$$

Since $\mathfrak{p}_{\mathrm{u}} \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}}) = \{0\}$, for all r, $\psi_r \wedge \varepsilon$ is a cocycle of $D_{k-j,\#}(\mathfrak{p}_-, \overline{\mathrm{B}})$, whence the assertion by (i). $\square$

12.2. **Quasi-regularity.** Let $k = 0, \dots, n$ and set:

$$D_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}}) := D_{k,\#}(\mathfrak{g}) \wedge \bigwedge^{\mathfrak{b}_{\mathfrak{g}}}(\widehat{\mathrm{B}}).$$

Let $\mathbb{N}_{\mathfrak{p}}$ be as in Subsection 10.2. For $j = 0, \dots, n$, set:

$$\mathcal{M}_j := \{\varphi \in (\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D(\mathfrak{g}))^{\mathbb{N}_{\mathfrak{p}}} \mid \exists N_\varphi \in \mathbb{N} \quad \text{such that} \quad |r| \geq N_\varphi \Longrightarrow \varphi_{s,r,\iota} = 0 \quad \text{and}$$
$$\varphi_{s,r,\iota} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D_{j-|\iota|,\#}(\mathfrak{g})\} \quad \text{and}$$
$$\kappa(\varphi) := \sum_{(s,r,\iota)\in\mathbb{N}'_{\mathfrak{p}}} z^s v^r \varphi_{s,r,\iota} \wedge v_\iota \wedge \hat{\varepsilon}$$

for φ in $\mathcal{M}_j$. For l nonnegative integer, denote by $\mathcal{M}_{j,l}$ the subspace of elements φ of $\mathcal{M}_j$ such that

$$|(s, r, \iota)| \neq l \Longrightarrow \varphi_{s,r,\iota} = 0$$

and $\mathcal{M}_{j,l,+}$ the sum of $\mathcal{M}_{j,i}$, $i = l, l+1, \dots$.

Lemma 12.4. *Let l be a positive integer. Suppose $k > 0$.*

(i) *For $m = 1, \dots, n-d$ and φ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D_{m,\#}(\mathfrak{l})$ such that $\varphi(x, y) \wedge \overline{\varepsilon_0}(x, y) = 0$ for all (x, y) in $\mathfrak{l}_* \times \mathfrak{p}$, φ is in $D_{m-1,\#}(\mathfrak{l}) \wedge \overline{\mathrm{B}}_{\mathfrak{l}}$.*

(ii) *For φ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D_{k,\#}(\mathfrak{p}_-)$, if the restriction of $\varphi \wedge \varepsilon$ to $\mathfrak{l}_* \times \mathfrak{p}$ is equal to 0, then*

$$\varphi \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} D_{k-1,\#}(\mathfrak{p}_-) \wedge \overline{\mathrm{B}}_{\mathfrak{l}}.$$

(iii) *For φ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D_{k,\#}(\mathfrak{g})$, the restriction of $\varphi \wedge \varepsilon$ to $\mathfrak{l}_* \times \mathfrak{p}$ is equal to 0 if and only if*

$$\varphi \in D_{k-1,\#}(\mathfrak{g}) \wedge \mathrm{B} + JD_{k,\#}(\mathfrak{g}) + \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D_{k-1,\#}(\mathfrak{g}) \wedge \mathfrak{p}_{\mathrm{u}}.$$

Moreover, in this case, $\varphi \wedge \hat{\varepsilon}$ is in $\kappa(\mathcal{M}_{k,1,+})$.

Proof. (i) As $\mathfrak{d} = \{0\}$ or $\mathfrak{d}$ has Property **P**, for some $\psi_1, \dots, \psi_s$ in $\overline{\mathrm{B}}_{\mathfrak{l}}$,

$$\varphi = \sum_{j=1}^{s} \varphi_j \wedge \psi_j \quad \text{with} \quad \varphi_j \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D_{m-1,\#}(\mathfrak{l})$$

for $j = 1, \dots, s$ by Proposition 4.4 since $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] = \Bbbk[\mathfrak{p}_{\mathrm{u}}] \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$, whence the assertion.

(ii) Let φ be in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D_{k,\#}(\mathfrak{p}_-)$ such that $\varphi(x, y) \wedge \varepsilon(x, y) = 0$ for all (x, y) in $\mathfrak{l}_* \times \mathfrak{p}$. The element φ has an expansion

$$\varphi = \sum_{\kappa\in\mathfrak{I}} \varphi_\kappa \wedge w_\kappa \quad \text{with} \quad \varphi_\kappa \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \bigoplus_{m=0}^{\inf\{n-d,k\}} \mathrm{S}^{k-m}(\mathfrak{p}_{-,\mathrm{u}}) \otimes_{\Bbbk} D_{m,\#}(\mathfrak{l}).$$

For (x, y) in $\mathfrak{l}_* \times \mathfrak{p}$,

$$\varepsilon(x, y) \in \overline{\varepsilon_0}(x, y) \wedge \bigwedge^{d}(\mathfrak{p}_{\mathrm{u}})$$

by Proposition 7.6(i), whence

$$\varphi_\kappa(x, y) \wedge \overline{\varepsilon_0}(x, y) = 0$$

for all κ in $\mathfrak{I}$ and all (x, y) in $\mathfrak{l}_* \times \mathfrak{p}$. So, by (i),

$$\varphi_\kappa \in \bigoplus_{m=1}^{\inf\{n-d,k\}} \mathrm{S}^{k-m}(\mathfrak{p}_{-,\mathrm{u}}) \otimes_{\Bbbk} D_{m-1,\#}(\mathfrak{l}) \wedge \overline{\mathrm{B}}_{\mathfrak{l}},$$

for all κ, whence the assertion.

(iii) Since $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] = \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] + J$, we can suppose that φ is in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D_{k,\#}(\mathfrak{g})$. As $\mathfrak{g}$ is the direct sum of $\mathfrak{p}_-$ and $\mathfrak{p}_{\mathrm{u}}$,

$$\bigwedge(\mathfrak{g}) = \bigwedge(\mathfrak{p}_-) \oplus \bigwedge(\mathfrak{g}) \wedge \mathfrak{p}_{\mathrm{u}} \quad \text{and} \quad D_{k,\#}(\mathfrak{g}) = \bigoplus_{m=0}^{k} \mathrm{S}^m(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} D_{k-m,\#}(\mathfrak{p}_-) \oplus D_{k-1,\#}(\mathfrak{g}) \wedge \mathfrak{p}_{\mathrm{u}}.$$

Then $\varphi = \varphi_1 + \varphi_2$ with

$$\varphi_1 \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} \bigoplus_{m=0}^{k} \mathrm{S}^m(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} D_{k-m,\#}(\mathfrak{p}_-) \quad \text{and} \quad \varphi_2 \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D_{k-1,\#}(\mathfrak{g}) \wedge \mathfrak{p}_{\mathrm{u}}.$$

As $\mathfrak{p}_{\mathrm{u}} \wedge \varepsilon(x, y) = \{0\}$ for all (x, y) in $\mathfrak{l}_* \times \mathfrak{p}$, the restriction of $\varphi_2 \wedge \varepsilon$ to $\mathfrak{l}_* \times \mathfrak{p}$ is equal to 0. Moreover, the restriction of $\varphi \wedge \varepsilon$ to $\mathfrak{l}_* \times \mathfrak{p}$ is equal to 0 if

$$\varphi \in D_{k-1,\#}(\mathfrak{g}) \wedge \mathrm{B} + JD_{k,\#}(\mathfrak{g}) + D_{k-1,\#}(\mathfrak{g}) \wedge \mathfrak{p}_{\mathrm{u}}.$$

Conversely, by (ii),

$$\varphi_1 \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D_{k-1,\#}(\mathfrak{p}_-) \wedge \overline{\mathrm{B}}_{\mathfrak{l}}.$$

By Proposition 9.1(ii),

$$\overline{\mathrm{B}}_{\mathfrak{l}} \subset \mathrm{B} + J \otimes_{\Bbbk} \mathfrak{p}_{-,\mathrm{u}} + J_- \otimes_{\Bbbk} \mathfrak{p}_{\mathrm{u}} + JJ_- \otimes_{\Bbbk} \mathfrak{l}.$$

Hence

$$\varphi_1 \in D_{k-1,\#}(\mathfrak{p}_-) \wedge \mathrm{B} + JD_{k,\#}(\mathfrak{g}) + D_{k-1,\#}(\mathfrak{g}) \wedge \mathfrak{p}_{\mathrm{u}},$$

since $\mathfrak{d}$ is simple or equal to $\{0\}$. Indeed, φ_1 has an expansion

$$\varphi_1 = \sum_{m=0}^{\inf\{k-1,n-d\}} \sum_{j=0}^{\inf\{m,n-d-1\}} \varphi_{1,m,j} \quad \text{with} \quad \varphi_{1,m,j} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} D_{k-m}(\mathfrak{p}_{-,\mathrm{u}}) \otimes_{\Bbbk} D_m^j(\mathfrak{l}) \wedge \overline{\mathrm{B}}_{\mathfrak{l}}.$$

For (i, m) in I_0, the map

$$G \times \mathfrak{l}_* \times \mathfrak{g} \longrightarrow \mathfrak{g}\,, \qquad (g, x, y) \longmapsto \varepsilon_i^{(m)}(g(x), g(y)) - \varepsilon_i^{(m)}(x, y)$$

is in $\mathfrak{m} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \mathfrak{g}$ since $\varepsilon_i^{(m)}$ is a G-equivariant by Proposition 2.1(vi), whence

$$\varphi_{1,m,j} \in D_{k-1,\#}(\mathfrak{g}) \wedge \widehat{\mathrm{B}} + J \otimes_{\Bbbk} D_{k,\#}(\mathfrak{g}) + \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D_{k-1,\#}(\mathfrak{g}) \wedge \mathfrak{p}_{\mathrm{u}} + \mathfrak{m} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D_{k,\#}(\mathfrak{g})$$

for all (m, j). As a result, $\varphi \wedge \hat{\varepsilon}$ is in $\kappa(\mathcal{M}_{k,1,+})$. □

Remark 12.5. Assertion (i) and Assertion (ii) are true when $\mathfrak{d}$ is not simple.

Proposition 12.6. *Let l be a positive integer and φ in $\mathcal{M}_{k,l}$. Then $\kappa(\varphi)$ is in $\kappa(\mathcal{M}_{k,l+1,+})$ if and only if $\varphi_{s,r,\iota} \wedge \hat{\varepsilon}$ is in $\kappa(\mathcal{M}_{k-|\iota|,1,+})$ for all (s,r,ι).*

Proof. The condition is clearly sufficient. Suppose that $\kappa(\varphi)$ is in $\kappa(\mathcal{M}_{k,l+1,+})$. As $\mathfrak{g}$ is the direct sum of $\mathfrak{p}_-$ and $\mathfrak{p}_\mathrm{u}$ and $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$ is the direct sum of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$ and J,

$$\varphi_{s,r,\iota} = \varphi'_{s,r,\iota} + \varphi''_{s,r,\iota} \quad \text{with}$$

$$\varphi'_{s,r,\iota} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk D_{k-|\iota|,\#}(\mathfrak{p}_-) \quad \text{and} \quad \varphi''_{s,r,\iota} \in J \otimes_\Bbbk D_{k-|\iota|,\#}(\mathfrak{g}) + \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk D_{k-|\iota|-1,\#}(\mathfrak{g}) \wedge \mathfrak{p}_\mathrm{u}$$

for all (s,r,ι). Setting $\varphi' := (\varphi'_{s,r,\iota},\ (s,r,\iota) \in \mathbb{N}_\mathfrak{p})$, φ' is in $\mathcal{M}_{k,l}$ and $\kappa(\varphi')$ is in $\kappa(\mathcal{M}_{k,l+1,+})$ since the condition is sufficient.

Let (s,r,ι) be in $\mathbb{N}_\mathfrak{p}$ such that $|(s,r,\iota)| = l$. By Proposition 10.6,

$$\varphi'_{s,r,\iota} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk D_{k,\#}(\mathfrak{p}_-) \cap \mathrm{S}(\mathfrak{g}) \otimes_\Bbbk M_1.$$

Then the restriction of $\varphi'_{s,r,\iota} \wedge \varepsilon$ to $\mathfrak{l}_* \times \mathfrak{p}$ is equal to 0. So, by Lemma 12.4(iii), the restriction of $\varphi_{s,r,\iota} \wedge \varepsilon$ to $\mathfrak{l}_* \times \mathfrak{p}$ is equal to 0 and $\varphi_{s,r,\iota} \wedge \hat{\varepsilon}$ is in $\kappa(\mathcal{M}_{k-|\iota|,1,+})$ for all (s,r,ι). □

For $l = 0,1,\ldots$, let F_l be the subspace $\kappa(\mathcal{M}_{k,l,+})$ of $D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$. Then the sequence F_l, $l = 0,1,\ldots$ is a decreasing filtration of $D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$. Denote by $\mathrm{gr}\, D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$ the associate graded space to this filtration and $\mathrm{gr}_l D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$ the subspace of degree l of $\mathrm{gr}\, D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$. For $j = 0,\ldots,k$, let $D^\bullet_{j,\#}(\mathfrak{g},\overline{\mathrm{B}})$ be the graded subcomplex of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk D^\bullet(\mathfrak{g})$,

$$D^\bullet_{j,\#}(\mathfrak{g},\overline{\mathrm{B}}) := D^\bullet_{j,\#}(\mathfrak{g})[-\mathrm{b}_\mathfrak{g}] \wedge \textstyle\bigwedge^{\mathrm{b}_\mathfrak{g}}(\overline{\mathrm{B}}),$$

and A the algebra

$$A := \Bbbk[z_1,\ldots,z_{2n+\ell}] \otimes_\Bbbk \mathrm{S}(\mathfrak{p}_\mathrm{u}) \otimes_\Bbbk \textstyle\bigwedge(\mathfrak{p}_\mathrm{u}).$$

This algebra has a bigradation $A^\bullet_\bullet$ such that

$$A^i := \Bbbk[z_1,\ldots,z_{2n+\ell}] \otimes_\Bbbk \mathrm{S}(\mathfrak{p}_\mathrm{u}) \otimes_\Bbbk \textstyle\bigwedge^i(\mathfrak{p}_\mathrm{u}) \quad \text{and}$$

$$A_l := \bigoplus_{(j_1,j_2,j_3)\in\mathbb{N}^3_l} \Bbbk[z_1,\ldots,z_{2n+\ell}]_{j_1} \otimes_\Bbbk \mathrm{S}^{j_2}(\mathfrak{p}_\mathrm{u}) \otimes_\Bbbk \textstyle\bigwedge^{j_3}(\mathfrak{p}_\mathrm{u})$$

with $\Bbbk[z_1,\ldots,z_{2n+\ell}]_j$ the space of homogeneous polynomials of degree j of $\Bbbk[z_1,\ldots,z_{2n+\ell}]$. Consider on $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk A \otimes_\Bbbk D(\mathfrak{g})$ the simple gradation deduced from the double gradation $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk A^\bullet \otimes_\Bbbk D^\bullet(\mathfrak{g})$. The trivial structure of complex on A and the structure of complex on $D(\mathfrak{g})$ induce a structure of graded complex on $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk A \otimes_\Bbbk D(\mathfrak{g})$. For $l = 0,1,\ldots$ and i nonnegative integer, set:

$$C^i_l = \bigoplus_{j=0}^{i} \bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-j}} \Bbbk[z_1,\ldots,z_{2n+\ell}]_{l_1} \otimes_\Bbbk \mathrm{S}^{l_2}(\mathfrak{p}_\mathrm{u}) \otimes_\Bbbk \textstyle\bigwedge^j(\mathfrak{p}_\mathrm{u}) \otimes_\Bbbk D^{i-j}_{k-j,\#}(\mathfrak{g},\overline{\mathrm{B}}).$$

Denoting by $C^\bullet_l$ the sum C^i_l, $i = 0,1,\ldots$, $C^\bullet_l$ is a graded subcomplex of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk A \otimes_\Bbbk D(\mathfrak{g})$.

Corollary 12.7. (i) *For $l = 0, 1, \ldots$, $F_l^\bullet$ is a graded subcomplex of the graded complex $D_{k,\#}^\bullet(\mathfrak{g}, \widehat{\mathrm{B}})$.*

(ii) *The graded complex $\mathrm{gr}_0 D_{k,\#}^\bullet(\mathfrak{g}, \widehat{\mathrm{B}})$ is isomorphic to the graded complex $D_{k,\#}^\bullet(\mathfrak{g}, \overline{\mathrm{B}})$.*

(iii) *For $l = 1, 2, \ldots$, the graded complex $\mathrm{gr}_l D_{k,\#}^\bullet(\mathfrak{g}, \widehat{\mathrm{B}})$ is isomorphic to $C_l^\bullet$.*

Proof. (i) For all l, $\kappa(\mathcal{M}_{k,l})$ is a graded subspace of $D_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$. For φ in $\mathcal{M}_{k,l}$,

$$\mathrm{d}\kappa(\varphi) = \sum_{(s,r,\iota)\in\mathbb{N}_{\mathfrak{p}}} z^s v^r \mathrm{d}\varphi_{s,r,\iota} \wedge v_\iota \wedge \hat{\varepsilon}.$$

Hence F_l is a graded subcomplex of $D_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$.

(ii) Let $r_{\mathfrak{g},\mathfrak{p}}$ be the quotient morphism

$$\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \xrightarrow{\quad r_{\mathfrak{g},\mathfrak{p}} \quad} \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \ .$$

Then $r_{\mathfrak{g},\mathfrak{p}}$ induces a morphism

$$\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D(\mathfrak{g}) \xrightarrow{\quad r_{\mathfrak{g},\mathfrak{p}} \quad} \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D(\mathfrak{g}) \ .$$

As $\overline{\mathrm{B}} = r_{\mathfrak{g},\mathfrak{p}}(\widehat{\mathrm{B}})$, we have a surjective morphism

$$D_{k,\#}^\bullet(\mathfrak{g}, \widehat{\mathrm{B}}) \xrightarrow{\quad r_{\mathfrak{g},\mathfrak{p}} \quad} D_{k,\#}^\bullet(\mathfrak{g}, \overline{\mathrm{B}}) \ .$$

and F_1 is its kernel by Proposition 12.6, whence the assertion.

(iii) Let l be a nonnegative integer. Denote by γ_l the morphism of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-modules

$$\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} A_l \otimes_{\Bbbk} D(\mathfrak{g}) \xrightarrow{\quad \gamma_l \quad} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D(\mathfrak{g}) \qquad \psi\otimes a\otimes v_\iota\otimes\varphi \longmapsto \psi a\otimes\varphi \wedge v_\iota$$

and $\mathcal{M}'_{k,l}$ the subspace of elements φ of $\mathcal{M}_{k,l}$ such that

$$\varphi_{s,r,\iota} \in \gamma_l(\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} A_l \otimes_{\Bbbk} D(\mathfrak{g})), \qquad \forall (s, r, \iota) \in \mathbb{N}_{\mathfrak{p}}.$$

Then $F_l = \kappa(\mathcal{M}'_{k,l}) + F_{l+1}$ since

$$\hat{J}^l = \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} A_l \oplus \hat{J}^{l+1}.$$

By Proposition 12.6, for φ in $\mathcal{M}'_{k,l}$, $\kappa(\varphi)$ is in F_{l+1} if and only if $r_{\mathfrak{g},\mathfrak{p}}(\varphi_{s,r,\iota} \wedge \hat{\varepsilon}) = 0$ for all (s, r, ι). Then, by (ii), the restriction of $r_{\mathfrak{g},\mathfrak{p}}$ to F_l defines through the quotient an isomorphism of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-modules,

$$\mathrm{gr}_l D_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}}) \longrightarrow C_l \ .$$

Moreover, this isomorphism is an isomorphism of graded complex. □

12.3. **Annulation of cohomology.** Let $k = 1, \dots, n$. For i integer, denote by Z^i and B^i the spaces of cocycles and coboundaries of degree i of $D^\bullet_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$. For $l = 1, \dots, n$, denote by $D^\bullet_{l,\mathrm{u}}$ the graded subcomplex of $D^\bullet_{l,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$,

$$D^\bullet_{l,\mathrm{u}} := D^\bullet_l(\mathfrak{p}_\mathrm{u})[-\mathrm{b}_\mathfrak{g}] \wedge \bigwedge^{\mathrm{b}_\mathfrak{g}}(\widehat{\mathrm{B}}).$$

Lemma 12.8. *Let $l = 1, \dots, n$. The morphism*

$$\hat{\mathcal{O}} \otimes_\Bbbk \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk D^\bullet_l(\mathfrak{p}_\mathrm{u})[-\mathrm{b}_\mathfrak{g}] \longrightarrow D^\bullet_{k,\mathrm{u}}\ , \qquad \varphi \longmapsto \varphi \wedge \hat{\varepsilon}$$

is an isomorphism of graded complexes. In particular, $D^\bullet_{l,\mathrm{u}}$ is acyclic.

Proof. Denote by $\tilde{D}^\bullet_{l,\mathrm{u}}$ the graded subcomplex of $D^\bullet_{l,\#}(\mathfrak{g}, \widetilde{\mathrm{B}})$,

$$\tilde{D}^\bullet_{l,\mathrm{u}} := D^\bullet_l(\mathfrak{p}_\mathrm{u})[-\mathrm{b}_\mathfrak{g}] \wedge \bigwedge^{\mathrm{b}_\mathfrak{g}}(\widetilde{\mathrm{B}}).$$

By Corollary 2.7, for all (x, y) in a dense open subset of $\mathfrak{l}_* \times \mathfrak{p}_-$, $V_{x,y}$ is contained in $\mathfrak{p}_-$. So, for (g, x, y) in a dense open subset of $G \times \mathfrak{l}_* \times \mathfrak{g}$, $V_{g(x),g(y)} \cap \mathfrak{p}_\mathrm{u} = \{0\}$. As a result, the morphism

$$\mathcal{O} \otimes_\Bbbk \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk D^\bullet_l(\mathfrak{p}_\mathrm{u})[-\mathrm{b}_\mathfrak{g}] \longrightarrow \tilde{D}^\bullet_{l,\mathrm{u}}\ , \qquad \varphi \longmapsto \varphi \wedge \hat{\varepsilon}$$

is an isomorphism of graded complexes since it is surjective. By Lemma A.2(ii), $D^\bullet_l(\mathfrak{p}_\mathrm{u})$ is an acyclic complex since l is positive. Hence so is $\tilde{D}^\bullet_{l,\mathrm{u}}$, whence the lemma since $\hat{\mathcal{O}}$ is a faithfully flat extension of $\mathcal{O}$. □

For j nonnegative integer, denote by $\mathrm{S}^j_\#(\overline{\mathrm{B}}_\mathfrak{l})$ the intersection of $\mathrm{S}^j(\overline{\mathrm{B}}_\mathfrak{l})$ and $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}] \otimes_\Bbbk D^0_{j,\#}(\mathfrak{l})$. For $i = \mathrm{b}_\mathfrak{g}, \dots, k + \mathrm{b}_\mathfrak{g}$ and l nonnegative integer, set:

$$K^i := \sum_{j=0}^{k-i+\mathrm{b}_\mathfrak{g}} \mathrm{S}^j_\#(\overline{\mathrm{B}}_\mathfrak{l}) D^i_{k-j,\mathrm{u}} \quad \text{and}$$

$$K^i_l = \bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_\mathfrak{g}}} \Bbbk[z_1, \dots, z_{2n+\ell}]_{l_2} \otimes_\Bbbk \mathrm{S}^{l_2}(\mathfrak{p}_\mathrm{u}) \otimes_\Bbbk \sum_{j=0}^{k-i+\mathrm{b}_\mathfrak{g}} \mathrm{S}^j_\#(\overline{\mathrm{B}}_\mathfrak{l}) D^{i-\mathrm{b}_\mathfrak{g}}_{k-j}(\mathfrak{p}_\mathrm{u}) \wedge \hat{\varepsilon}.$$

Lemma 12.9. *Let $i = \mathrm{b}_\mathfrak{g} + 1, \dots, k + \mathrm{b}_\mathfrak{g}$ and l a nonnegative integer.*
(i) *The subspace $Z^i \cap F^i_l$ of F^i_l is contained in $\mathrm{d}F^{i-1}_l + K^i + F^i_{l+1}$.*
(ii) *The space $K^i \cap F^i_l$ is contained in $K^i_l + F^i_{l+1}$.*
(iii) *The intersection of F^i_{l+1} and K^i is equal to $J_1^{l-i+\mathrm{b}_\mathfrak{g}+1} K^i$.*

Proof. (i) Let φ be in $Z^i \cap F^i_l$ and $\overline{\varphi}$ its image in $\mathrm{gr}_l D^i_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$. By Corollary 12.7, $\overline{\varphi}$ is a cocycle of degree i of the graded complex $C^\bullet_l$. By Corollary 12.2, for $j = 0, \dots, k$, the complex $D^\bullet_{k-j,\#}(\mathfrak{g}, \overline{\mathrm{B}})$

has no cohomolgy of degree different from $\mathfrak{b}_{\mathfrak{g}}$. Then, for some $\overline{\psi}$ in C_l^{i-1},

$$\overline{\varphi} - \mathrm{d}\overline{\psi} \in \bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathfrak{b}_{\mathfrak{g}}}} \Bbbk[z_1,\dots,z_{2n+\ell}]_{l_1} \otimes_{\Bbbk} \mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \bigwedge^{i-\mathfrak{b}_{\mathfrak{g}}}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} D^{\mathfrak{b}_{\mathfrak{g}}}_{k-i+\mathfrak{b}_{\mathfrak{g}},\#}(\mathfrak{g},\overline{\mathrm{B}}).$$

Then, by Lemma 12.3(ii),

$$\overline{\varphi} - \mathrm{d}\overline{\psi} \in \bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathfrak{b}_{\mathfrak{g}}}} \Bbbk[z_1,\dots,z_{2n+\ell}]_{l_1} \otimes_{\Bbbk} \mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \bigwedge^{i-\mathfrak{b}_{\mathfrak{g}}}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk}$$

$$\sum_{j=0}^{k-i+\mathfrak{b}_{\mathfrak{g}}} \mathrm{S}^j_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}}) D^0_{k-i+\mathfrak{b}_{\mathfrak{g}}-j}(\mathfrak{p}_{\mathrm{u}}) \wedge \bigwedge^{\mathfrak{b}_{\mathfrak{g}}}(\overline{\mathrm{B}}).$$

So, for a representative ψ of $\overline{\psi}$ in F_l^{i-1},

$$\varphi - \mathrm{d}\psi \in K^i + F^i_{l+1}.$$

(ii) By definition, K^i_l is contained in F^i_l. Prove by induction on l that $K^i \cap F^i_l$ is contained in $K^i_l + F^i_{l+1}$. For $l = 0$, there is nothing to prove. Suppose the assertion true for $l-1$. By definition,

$$K^i = \sum_{s=0}^{+\infty} K^i_s,$$

so by induction hypothesis,

$$K^i \cap F^i_{l-1} = \sum_{s=l-1}^{\infty} K^i_s.$$

The image of K^i_{l-1} in $\mathrm{gr}\, D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$ is contained in $\mathrm{gr}_{l-1} D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$. Then, by Corollary 12.7(ii), $K^i \cap F^i_l$ is contained in $K^i_l + F^i_{l+1}$.

(iii) By definition $J_1^{l-i+\mathfrak{b}_{\mathfrak{g}}+1} K^i$ is contained $F^i_{l+1} \cap K^i$. Prove the assertion by induction on l. By Corollary 12.7(iii), the quotient of K^i by $K^i \cap F^i_{1+i-\mathfrak{b}_{\mathfrak{g}}}$ is equal to the subspace of $\mathrm{gr}_{i-\mathfrak{b}_{\mathfrak{g}}} D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$,

$$\bigwedge^{i-\mathfrak{b}_{\mathfrak{g}}}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \sum_{j=0}^{k-i+\mathfrak{b}_{\mathfrak{g}}} \mathrm{S}^j_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}}) D^0_{k-i+\mathfrak{b}_{\mathfrak{g}}-j}(\mathfrak{p}_{\mathrm{u}}) \wedge \bigwedge^{\mathfrak{b}_{\mathfrak{g}}}(\overline{\mathrm{B}}).$$

Moreover, by Lemma 12.8, the quotient of K^i by $J_1 K^i$ is equal to

$$\sum_{j=0}^{k-i+\mathfrak{b}_{\mathfrak{g}}} \mathrm{S}^j_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}}) D^{i-\mathfrak{b}_{\mathfrak{g}}}_{k-j}(\mathfrak{p}_{\mathrm{u}}) \wedge \hat{\varepsilon}.$$

In particular, the $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-module K^i/J_1K^i is free. Again by Lemma 12.8, the three $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-modules,

$$\sum_{j=0}^{k-i+\mathfrak{b}_{\mathfrak{g}}} \mathrm{S}^j_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}}) D^{i-\mathfrak{b}_{\mathfrak{g}}}_{k-j}(\mathfrak{p}_{\mathrm{u}}) \wedge \hat{\varepsilon}, \qquad \sum_{j=0}^{k-i+\mathfrak{b}_{\mathfrak{g}}} \mathrm{S}^j_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}}) D^{i-\mathfrak{b}_{\mathfrak{g}}}_{k-j}(\mathfrak{p}_{\mathrm{u}}),$$

$$\bigwedge^{i-\mathrm{b}_{\mathfrak{g}}}(\mathfrak{p}_{\mathrm{u}})\otimes_{\Bbbk}\sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}}\mathrm{S}^{j}_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}})D^{0}_{k-i+\mathrm{b}_{\mathfrak{g}}-j}(\mathfrak{p}_{\mathrm{u}})\wedge\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}})$$

are three free modules of the same finite rank. Hence for some p in $\Bbbk[\mathfrak{l}_*\times\mathfrak{p}]\setminus\{0\}$, $pK^i\cap F^i_{i-\mathrm{b}_{\mathfrak{g}}+1}$ is contained J_1K^i. As a result, $K^i\cap F^i_{i-\mathrm{b}_{\mathfrak{g}}+1}$ is equal to J_1K^i since the $\Bbbk[\mathfrak{l}_*\times\mathfrak{p}]$-module $K^i/J_1K^i_1$ is torsion free.

Suppose $l>i-\mathrm{b}_{\mathfrak{g}}$ and the assertion true for $l-1$. By induction hypothesis, it remains to prove that $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}+1}K^i$ is the intersection of $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}}K^i$ and F^i_{l+1}. By Corollary 12.7(iii), the quotient of $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}}K^i$ by $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}}K^i\cap F^i_{l+1}$ is equal to the subspace of $\mathrm{gr}_l D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$,

$$\bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_{\mathfrak{g}}}}\Bbbk[z_1,\dots,z_{2n+\ell}]_{l_2}\otimes_{\Bbbk}\mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}})\otimes_{\Bbbk}\bigwedge^{i-\mathrm{b}_{\mathfrak{g}}}(\mathfrak{p}_{\mathrm{u}})\otimes_{\Bbbk}\sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}}\mathrm{S}^{j}_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}})D^{0}_{k-i+\mathrm{b}_{\mathfrak{g}}-j}(\mathfrak{p}_{\mathrm{u}})\wedge\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}}).$$

Moreover, by Lemma 12.8, the quotient of $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}}K^i$ by $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}+1}K^i$ is equal to

$$\bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_{\mathfrak{g}}}}\Bbbk[z_1,\dots,z_{2n+\ell}]_{l_2}\otimes_{\Bbbk}\mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}})\otimes_{\Bbbk}\sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}}\mathrm{S}^{j}_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}})D^{i-\mathrm{b}_{\mathfrak{g}}}_{k-j}(\mathfrak{p}_{\mathrm{u}})\wedge\hat{\varepsilon}.$$

In particular, the $\Bbbk[\mathfrak{l}_*\times\mathfrak{p}]$-module $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}}K^i/J_1^{l-i+\mathrm{b}_{\mathfrak{g}}+1}K^i$ is free. Again by Lemma 12.8, the three $\Bbbk[\mathfrak{l}_*\times\mathfrak{p}]$-modules

$$\bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_{\mathfrak{g}}}}\Bbbk[z_1,\dots,z_{2n+\ell}]_{l_2}\otimes_{\Bbbk}\mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}})\otimes_{\Bbbk}\bigwedge^{i-\mathrm{b}_{\mathfrak{g}}}(\mathfrak{p}_{\mathrm{u}})\otimes_{\Bbbk}\sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}}\mathrm{S}^{j}_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}})D^{0}_{k-i+\mathrm{b}_{\mathfrak{g}}-j}(\mathfrak{p}_{\mathrm{u}})\wedge\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}}),$$

$$\bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_{\mathfrak{g}}}}\Bbbk[z_1,\dots,z_{2n+\ell}]_{l_2}\otimes_{\Bbbk}\mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}})\otimes_{\Bbbk}\sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}}\mathrm{S}^{j}_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}})D^{i-\mathrm{b}_{\mathfrak{g}}}_{k-j}(\mathfrak{p}_{\mathrm{u}}),$$

$$\bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_{\mathfrak{g}}}}\Bbbk[z_1,\dots,z_{2n+\ell}]_{l_2}\otimes_{\Bbbk}\mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}})\otimes_{\Bbbk}\sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}}\mathrm{S}^{j}_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}})D^{i-\mathrm{b}_{\mathfrak{g}}}_{k-j}(\mathfrak{p}_{\mathrm{u}})\wedge\hat{\varepsilon},$$

are three free modules of the same finite rank. Hence for some p in $\Bbbk[\mathfrak{l}_*\times\mathfrak{p}]\setminus\{0\}$, $pK^i\cap F^i_{l+i-\mathrm{b}_{\mathfrak{g}}+1}$ is contained $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}+1}K^i$. As a result, $K^i\cap F^i_{i-\mathrm{b}_{\mathfrak{g}}+1}$ is equal to $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}+1}K^i$ since the $\Bbbk[\mathfrak{l}_*\times\mathfrak{p}]$-module $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}}K^i/J_1^{l-i+\mathrm{b}_{\mathfrak{g}}+1}K^i_1$ is torsion free. □

Let d_1 and d_2 be the morphisms from $K^\bullet_l$ to $F^\bullet_l$ such that

$$\mathrm{d}_1 a\otimes\omega\nu\wedge\hat{\varepsilon}=(-1)^{i-\mathrm{b}_{\mathfrak{g}}}a\otimes\nu(\mathrm{d}\omega)\wedge\hat{\varepsilon},\quad \mathrm{d}_2 a\otimes\omega\nu\wedge\hat{\varepsilon}=a\otimes\omega\mathrm{d}\nu\wedge\hat{\varepsilon}\quad\text{with}$$

$$a\in\bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_{\mathfrak{g}}}}\Bbbk[z_1,\dots,z_{2n+\ell}]_{l_2}\otimes_{\Bbbk}\mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}}),$$

$$\omega \in \mathrm{S}^j_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}}), \quad \nu \in D^{i-\mathrm{b}_{\mathfrak{g}}}_{k-j}(\mathfrak{p}_{\mathrm{u}}), \quad i = \mathrm{b}_{\mathfrak{g}}, \dots, k + \mathrm{b}_{\mathfrak{g}}, \quad j = 0, \dots, k - i + \mathrm{b}_{\mathfrak{g}}.$$

Lemma 12.10. *Let $i = \mathrm{b}_{\mathfrak{g}}, \dots, k + \mathrm{b}_{\mathfrak{g}}$ and l a nonnegative integer.*

(i) *For i nonnegative integer and $j > 0$,*

$$\mathrm{d}K^i_l \subset \mathrm{d}_2 K^i_l + F^{i+1}_{l+1}.$$

(ii) *If $i > \mathrm{b}_{\mathfrak{g}}$ then $Z^i \cap F^i_l$ is contained in $\mathrm{d}F^{i-1}_l + F^i_{l+1}$.*

Proof. (i) By definition $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2$. Let a, ω, ν be as in the above definition. Then

$$\mathrm{d}a{\otimes}\omega\nu \wedge \hat{\varepsilon} \in \mathrm{d}_2 a{\otimes}\omega \wedge \hat{\varepsilon} + a{\otimes}\mathrm{S}^{j-1}_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}}) D^{i-\mathrm{b}_{\mathfrak{g}}}_{k-j}(\mathfrak{p}_{\mathrm{u}}) \wedge \hat{\varepsilon} \wedge \overline{\mathrm{B}}_{\mathfrak{l}}.$$

By Corollary 12.7,

$$a{\otimes}\mathrm{S}^{j-1}_{\#}(\overline{\mathrm{B}}_{\mathfrak{l}}) D^{i-\mathrm{b}_{\mathfrak{g}}}_{k-j}(\mathfrak{p}_{\mathrm{u}}) \wedge \hat{\varepsilon} \wedge \overline{\mathrm{B}}_{\mathfrak{l}} \subset F^{i+1}_{l+1}$$

since the image of $\hat{\varepsilon}$ by the restriction morphism $r_{\mathfrak{g},\mathfrak{p}}$ is a generator of $\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}})$, whence the assertion.

(ii) Let φ be in $Z^i \cap F^i_l$. By Lemma 12.9(i), for some ψ in F^{i-1}_l,

$$\varphi - \mathrm{d}\psi \in K^i + F^i_{l+1}.$$

As a matter of fact, by Lemma 12.9(ii),

$$\varphi - \mathrm{d}\psi \in K^i_l + F^i_{l+1}.$$

By Lemma 12.9(iii) and Lemma 12.8, for $i' = i, i+1$, the sum $K^{i'}_l + F^{i'}_{l+1}$ is direct. Let φ_1 be the component of $\varphi - \mathrm{d}\psi$ on K^i_l. Then, by (i), $\mathrm{d}_2\varphi_1 = 0$ since $\mathrm{d}_2\varphi_1$ is in K^{i+1} and $\mathrm{d}F^i_{l+1}$ is contained in F^{i+1}_{l+1}. As a result, by Lemma 12.8, for some φ'_1 in K^{i-1}_l, $\varphi_1 = \mathrm{d}_2\varphi'_1$. Then, by (i),

$$\psi + \varphi'_1 \in F^{i-1}_l \quad \text{and} \quad \varphi - \mathrm{d}\psi - \mathrm{d}\varphi'_1 \in F^i_{l+1},$$

whence the assertion. □

Corollary 12.11. *Let $i = \mathrm{b}_{\mathfrak{g}} + 1, \dots, \mathrm{b}_{\mathfrak{g}} + k$.*

(i) *For all nonnegative integer l, Z^i is contained in $B^i + F^i_l$.*

(ii) *For some p in $\mathfrak{m} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_*]$, $(1+p)Z^i$ is contained in B^i.*

Proof. (i) By Lemma 12.10(ii), for l nonnegative integer,

$$Z^i \cap F^i_l \subset \mathrm{d}F^{i-1}_l + F^i_{l+1}.$$

Then, by induction on l, Z^i is contained in $B^i + F^i_l$.

(ii) The natural gradation of $\Bbbk[\mathfrak{g}]$ induces a gradation of $\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D(\mathfrak{g})$. As $\widehat{\mathrm{B}}$ is a graded submodule of $\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D(\mathfrak{g})$ so are $D^{\bullet}_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$, $D^{\bullet}_{k,\mathrm{u}}$, $F^{\bullet}_l$, $l = 0, 1, \dots$. Then Z^i and B^i are graded submodules of $D^i_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$ since the differential of $D^{\bullet}_k(\mathfrak{g}, \widehat{\mathrm{B}})$ is homogeneous of degree 0 with respect to this gradation.

Let l be a nonnegative integer. Denote by

$$\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}]^{(l)}, \quad D^{i,l}_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}}), \quad Z^{i,l}, \quad B^{i,l}$$

the subspaces of degree l of

$$\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}], \quad D^{i}_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}}), \quad Z^{i}, \quad B^{i}$$

respectively. In particular, these spaces are finitely generated $\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_*]$-modules. Then, by [MA86, Ch. 3, Theorem 8.9], for some p_l in $\mathfrak{m} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_*]$,

$$(1 + p_l) \bigcap_{j \in \mathbb{N}} (B^{i,l} + \mathfrak{m}^j D^{i,l}_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})) \subset B^{i,l}.$$

By (i),

$$Z^{i,l} \subset B^{i,l} + \mathfrak{m}^{j-l} \otimes_{\Bbbk} D^{i,l}_{k}(\mathfrak{g}, \widehat{\mathrm{B}})$$

for all integer j bigger than l since F^i_s is graded for all nonnegative integer s and J is generated by elements of positive degree. As a result,

$$(1 + p_l) Z^{i,l} \subset B^{i,l}.$$

Then, for some p in $\mathfrak{m} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_*]$,

$$(1 + p) Z^{i} \subset B^{i}$$

since Z^i is a finitely generated module over $\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$. □

Proposition 12.12. *For $k = 0, \dots, n$, $D^{\bullet}_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$ and $D^{\bullet}_{k,\#}(\mathfrak{g}, \widetilde{\mathrm{B}})$ have no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$.*

Proof. As $\hat{\mathcal{O}}$ is a faithfully flat extension of $\mathcal{O}$, it is sufficient to prove that $D^{\bullet}_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_{\mathfrak{g}}$. For $i < \mathrm{b}_{\mathfrak{g}}$ or $i > k + \mathrm{b}_{\mathfrak{g}}$, $D^{i}_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}}) = \{0\}$. By definition, $D^{\bullet}_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$ has no cohomology of degree $k + \mathrm{b}_{\mathfrak{g}}$. So, it is true for $k = 0, 1$. Let $k = 2, \dots, n$ and $i = \mathrm{b}_{\mathfrak{g}} + 1, \dots, k - 1 + \mathrm{b}_{\mathfrak{g}}$. Denote by T_i the support of Z^i/B^i in $\mathrm{Spec}(\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}])$. As Z^i/B^i is finitely generated, T_i is a closed subset of $\mathrm{Spec}(\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}])$. Since $\mathfrak{m}$ is contained in all maximal ideal of $\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$, T_i does not contain a maximal ideal by Corollary 12.11. Then T_i is empty and $Z^i = B^i$, whence the proposition. □

13. Induction. Case $\mathrm{n} \geq 2$

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. Then we use the notations of Section 9 and some notations of Section 12. In particular, $\mathfrak{l}$ is the reductive factor of $\mathfrak{p}$ containing $\mathfrak{h}$, $\mathfrak{d}$ is its derived algebra, n is the number of simple factors of $\mathfrak{d}$. We suppose that n is bigger than 1 and the simple factors $\mathfrak{d}_1, \dots, \mathfrak{d}_{\mathrm{n}}$ of $\mathfrak{d}$ have Property (**P**). For $i = 1, \dots, \mathrm{n}$, let E'_i be the submodule of $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}] \otimes_{\Bbbk} \mathfrak{l}$ generated by the elements $\lambda_{i,1}, \dots, \lambda_{i,n_i}$ defined in Subsection 9.3. Denote by E the

$\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]$-submodule of $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}] \otimes_{\Bbbk} \mathfrak{g}$ generated by $\mathfrak{p}_{\mathrm{u}}$ and $\mathrm{E}'_1, \dots, \mathrm{E}'_{\mathrm{n}}$. Let $\overline{J}_+$ be the ideal of the algebra $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} \bigwedge(\mathrm{E}')$ generated by $J \otimes 1$ and $1 \otimes \mathrm{E}$. Denote by $\tilde{J}_+$ and $\hat{J}_+$ the ideals of

$$\mathcal{O} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} \bigwedge(\mathrm{E}) \quad \text{and} \quad \hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} \bigwedge(\mathrm{E})$$

generated by $\mathfrak{m} \otimes 1$ and $\overline{J}_+$. Let J_1 be the ideal of $\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$ generated by $\mathfrak{m}$ and J.

13.1. **Quasi-regularity.** Let $\mathbb{I}$ be as in Subsection 6.2. According to notations of Subsection 6.2 and Section 4, for I subset of $\mathbb{I}_k$ and (j_1, j_2) in $\mathbb{N}^2$ such that $j_1 + j_2 \leq k$, let I_{j_1,j_2} be the subset of elements ι of $\mathbb{I}''_{k-j_1-j_2}$ such that (j_1, j_2, ι) is in I. Then, for I subset of $\mathbb{I}_k$ and $V = \mathfrak{p}_-$ or $\mathfrak{g}$, denote by $D_{k,I,\#}(V)$ the graded subcomplex of $D^\bullet_k(V)$ deduced from the triple complex

$$\bigoplus_{i=0}^{k} \bigoplus_{(j_1,j_2)\in\mathbb{N}^2_i} D^\bullet_{j_1}(V') \otimes_{\Bbbk} D^\bullet_{j_2}(\mathfrak{z}) \otimes_{\Bbbk} D_{i,I_{j_1,j_2},\#}(\mathfrak{d}),$$

with $V' = \mathfrak{p}_{-,\mathrm{u}}$ when $V = \mathfrak{p}_-$ and $V' = \mathfrak{p}_{\pm,\mathrm{u}}$ when $V = \mathfrak{g}$. For $j = (j_1, \dots, j_{\mathrm{n}})$ in $\mathbb{I}''_k$, set:

$$\mathbb{I}^{(j)} := \{(i_{-1}, \dots, i_{\mathrm{n}}) \in \mathbb{I} \mid i_1 \leq n_1 - j_1, \dots, i_{\mathrm{n}} \leq n_{\mathrm{n}} - j_{\mathrm{n}}\},$$

$$D^\bullet_{k,\#,j}(V) := D_{k-|j|,\mathbb{I}^{(j)},\#}(V)[-|j|] \wedge \bigwedge\nolimits^{j_1}(\mathrm{E}'_1) \wedge \cdots \wedge \bigwedge\nolimits^{j_{\mathrm{n}}}(\mathrm{E}'_{\mathrm{n}}),$$

$$D^\bullet_{k,\#,j,*}(V) := \bigwedge\nolimits^{j_1}(\mathrm{E}'_1) \wedge \cdots \wedge \bigwedge\nolimits^{j_{\mathrm{n}}}(\mathrm{E}'_{\mathrm{n}}) \otimes_{\Bbbk} D^\bullet_{k-|j|,\mathbb{I}^{(j)},\#}(V)[-|j|].$$

Then $D^\bullet_{k,\#,j}(V)$ is a graded subcomplex of $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}'] \otimes_{\Bbbk} D^\bullet_{k,\#}(V)$.

For $k = 0, \dots, n$ and j non negative integer, denote by $D^\bullet_{k,\#,j}(V)$ the graded submodule of $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}] \otimes_{\Bbbk} D_{k,\#}(V)$,

$$D^\bullet_{k,\#,j}(V) := \bigoplus_{(j_1,\dots,j_{\mathrm{n}})\in\mathbb{N}^{\mathrm{n}}_j} D^\bullet_{k,\#,(j_1,\dots,j_{\mathrm{n}})}(V)$$

and $D^\bullet_{k,\#,j,\times}(\mathfrak{g})$ the graded submodule of $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}] \otimes_{\Bbbk} \bigwedge^j(\mathrm{E}) \otimes_{\Bbbk} D^\bullet_{k-j,\#}(\mathfrak{g})$,

$$D^\bullet_{k,\#,j,\times}(\mathfrak{g}) := \bigoplus_{i=0}^{j} \bigoplus_{(j_1,\dots,j_{\mathrm{n}})\in\mathbb{N}^{\mathrm{n}}_{j-i}} \bigwedge\nolimits^i(\mathfrak{p}_{\mathrm{u}}) \wedge D^\bullet_{k-i,\#,(j_1,\dots,j_{\mathrm{n}}),\times}(\mathfrak{g}).$$

Let $k = 0, \dots, n$. Set:

$$\mathbb{N}_{\mathfrak{p},+} := \mathbb{N}^{2n+\ell} \times \mathbb{N}^d \times \mathfrak{J} \times \mathfrak{I}, \quad |(s, r, \upsilon, \iota)| := |s| + |r| + |\upsilon| + |\iota|,$$

for (s, r, υ, ι) in $\mathbb{N}_{\mathfrak{p},+}$. For $j = 0, \dots, n$, set:

$$\mathcal{M}_{j,*} := \{\varphi \in (\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D(\mathfrak{g}))^{\mathbb{N}_{\mathfrak{p},+}} \mid \exists N_\varphi \in \mathbb{N} \quad \text{such that} \quad |r| \geq N_\varphi \Longrightarrow \varphi_{s,r,\upsilon,\iota} = 0 \quad \text{and}$$

$$\varphi_{s,r,\upsilon,\iota} \wedge \lambda_\upsilon \in \Bbbk[\mathfrak{l}_* \times \mathfrak{l}] \otimes_{\Bbbk} D_{j-|\iota|,\#,|\upsilon|}(\mathfrak{g})\},$$

$$\kappa(\varphi) := \sum_{(s,r,\upsilon,\iota)\in\mathbb{N}'_{\mathfrak{p},+}} z^s \upsilon^r \varphi_{s,r,\upsilon,\iota} \wedge \lambda_\upsilon \wedge \upsilon_\iota \wedge \hat{\varepsilon}$$

for φ in $\mathcal{M}_{j,*}$. For l nonnegative integer, denote by $\mathcal{M}_{j,*,l}$ the subspace of elements φ of $\mathcal{M}_{j,*}$ such that

$$|(s,r,\upsilon,\iota)| \neq l \Longrightarrow \varphi_{s,r,\upsilon,\iota} = 0$$

and $\mathcal{M}_{j,*,l,+}$ the sum of $\mathcal{M}_{j,*,i}$, $i = l, l+1, \ldots$.

Recall that $\overline{\mathrm{B}}_{0,0}$ is the submodule of $\overline{\mathrm{B}}$ generated by $\varepsilon_1^{(0)}, \ldots, \varepsilon_\ell^{(0)}$.

Lemma 13.1. *For φ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk D_{k,\#}(\mathfrak{g})$, the restriction of $\varphi \wedge \varepsilon$ to $\mathfrak{l}_* \times \mathfrak{p}$ is equal to 0 if and only if*

$$\varphi \in \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} D_{k,\#,1}(\mathfrak{g}) + D_{k-1,\#}(\mathfrak{g}) \wedge \overline{\mathrm{B}}_{0,0} + JD_{k,\#}(\mathfrak{g}) + \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk D_{k-1,\#}(\mathfrak{g}) \wedge \mathfrak{p}_\mathrm{u}.$$

Moreover, in this case $\varphi \wedge \hat{\varepsilon}$ is in $\kappa(\mathcal{M}_{k,*,1,+})$.

Proof. Since $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] = \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] + J$, we can suppose that φ is in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk D_{k,\#}(\mathfrak{g})$. As $\mathfrak{g}$ is the direct sum of $\mathfrak{p}_-$ and $\mathfrak{p}_\mathrm{u}$,

$$\bigwedge(\mathfrak{g}) = \bigwedge(\mathfrak{p}_-) \oplus \bigwedge(\mathfrak{g}) \wedge \mathfrak{p}_\mathrm{u} \quad \text{and} \quad D_{k,\#}(\mathfrak{g}) = \bigoplus_{m=0}^{k} \mathrm{S}^m(\mathfrak{p}_\mathrm{u}) \otimes_\Bbbk D_{k-m,\#}(\mathfrak{p}_-) \oplus D_{k-1,\#}(\mathfrak{g}) \wedge \mathfrak{p}_\mathrm{u}.$$

Then $\varphi = \varphi_1 + \varphi_2$ with

$$\varphi_1 \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk \bigoplus_{m=0}^{k} \mathrm{S}^m(\mathfrak{p}_\mathrm{u}) \otimes_\Bbbk D_{k-m,\#}(\mathfrak{p}_-) \quad \text{and} \quad \varphi_2 \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk D_{k-1,\#}(\mathfrak{g}) \wedge \mathfrak{p}_\mathrm{u}.$$

By Lemma 12.4(ii) and Remark 12.5, the restriction of $\varphi_2 \wedge \varepsilon$ to $\mathfrak{l}_* \times \mathfrak{p}$ is equal to 0 and the condition is sufficient.

Conversely, suppose that the restriction of $\varphi \wedge \varepsilon$ to $\mathfrak{l}_* \times \mathfrak{p}$ is equal to 0. Since the condition is sufficient, the restrictions of $\varphi_2 \wedge \varepsilon$ and $\varphi_1 \wedge \varepsilon$ to $\mathfrak{l}_* \times \mathfrak{p}$ are equal to 0. By Proposition 4.4, setting:

$$\mathbb{I}''_{i,-} := \{(j_1, \ldots, j_\mathrm{n}) \in \mathbb{N}_i^\mathrm{n} \mid j_1 \leq n_1 - 1, \ldots, j_\mathrm{n} \leq n_\mathrm{n} - 1\},$$

and arguing as in the proof of Lemma 12.4(ii),

$$\varphi_1 = \varphi'_1 + \sum_{j=1}^{\ell} \varphi_{1,j} \wedge \varepsilon_j^{(0)} \quad \text{with} \quad \varphi'_1 \in \Bbbk[\mathfrak{p}_\mathrm{u}] \otimes_\Bbbk D_{k,\#,1}(\mathfrak{p}_-) \quad \text{and}$$

$$\varphi_{1,j} \in \bigoplus_{i=0}^{k} \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_\Bbbk D_{k,\mathbb{I}''_{i,-},\#}(\mathfrak{p}_-)$$

for $j = 1, \ldots, \ell$ since $\varepsilon_j(x)$ has a nonzero component on $\mathfrak{d}_l$ for $l = 1, \ldots, \mathrm{n}$ for all x in a dense open subset of $\mathfrak{l}_*$ by Lemma 8.1(i). As a result,

$$\varphi \in D_{k-1,\#}(\mathfrak{g}) \wedge \widehat{\mathrm{B}} + J \otimes_\Bbbk D_{k,\#}(\mathfrak{g}) + \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} D_{k,\#,1}(\mathfrak{g}) + \mathfrak{m} \otimes_\Bbbk \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_\Bbbk D_{k,\#}(\mathfrak{g})$$

since for $j = 1, \ldots, \ell$, the map

$$G \times \mathfrak{l}_* \times \mathfrak{g} \longrightarrow \mathfrak{g}\ , \qquad (g, x) \longmapsto \varepsilon_j(g(x)) - \varepsilon_j(x)$$

is in $\mathfrak{m} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_*] \otimes_{\Bbbk} \mathfrak{g}$, whence the lemma. □

Proposition 13.2. *Let l be a positive integer and φ in $\mathcal{M}_{k,*,l}$. Then $\kappa(\varphi)$ is in $\kappa(\mathcal{M}_{k,*,l+1,+})$ if and only if $\varphi_{s,r,\upsilon,\iota} \wedge \hat{\varepsilon}$ is in $\kappa(\mathcal{M}_{k-|\upsilon|-|\iota|,*,1,+})$ for all (s,r,υ,ι). Moreover, in this case,*

$$\varphi_{s,r,\upsilon,\iota} \wedge \lambda_\upsilon \wedge \hat{\varepsilon} \in \kappa(\mathcal{M}_{k-|\iota|,*,|\upsilon|+1,+})$$

for all (s,r,υ,ι).

Proof. The condition is clearly sufficient. Suppose that $\kappa(\varphi)$ is in $\kappa(\mathcal{M}_{k,*,l+1,+})$. As $\mathfrak{g}$ is the direct sum of $\mathfrak{p}_-$ and $\mathfrak{p}_{\mathrm{u}}$ and $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}]$ is the direct sum of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$ and J,

$$\varphi_{s,r,\upsilon,\iota} = \varphi'_{s,r,\upsilon,\iota} + \varphi''_{s,r,\upsilon,\iota} \quad \text{with} \quad \varphi'_{s,r,\upsilon,\iota} \wedge \lambda_\upsilon \in \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} D_{k-|\iota|,\#,|\upsilon|}(\mathfrak{p}'_-) \quad \text{and}$$
$$\varphi''_{s,r,\upsilon,\iota} \wedge \lambda_\upsilon \in J \otimes_{\Bbbk} D_{k-|\iota|,\#}(\mathfrak{g}) + \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D_{k-|\iota|-1,\#}(\mathfrak{g}) \wedge \mathfrak{p}_{\mathrm{u}}$$

for all (s,r,υ,ι). Setting $\varphi' := (\varphi'_{s,r,\upsilon,\iota},\ (s,r,\upsilon,\iota) \in \mathbb{N}_{\mathfrak{p},+})$, φ' is in $\mathcal{M}_{k,*,l}$ and $\kappa(\varphi')$ is in $\kappa(\mathcal{M}_{k,*,l+1,+})$ since the condition is sufficient.

For (s,r,ι) in $\mathbb{N}_{\mathfrak{p}}$, set:

$$\psi_{s,r,\iota} := \sum_{\upsilon \in \mathfrak{I}'} \varphi'_{s,r,\upsilon,\iota} \wedge \lambda_\upsilon \quad \text{and} \quad \psi := (\psi_{s,r,\iota},\ (s,r,\iota) \in \mathbb{N}_{\mathfrak{p}}).$$

Then

$$l < |(s,r,\iota)| \Longrightarrow \psi_{s,r,\iota} = 0 \quad \text{and} \quad \kappa(\psi) \wedge \hat{\varepsilon} \in \kappa(\mathcal{M}_{k,*,l+1,+}).$$

As a result, by Proposition 10.6, $\psi_{s,r,\iota}$ is in M_1 for all (s,r,ι). Then by Proposition 11.2, the restrictions of $\varphi'_{s,r,\upsilon,\iota} \wedge \varepsilon$ and $\varphi_{s,r,\upsilon,\iota} \wedge \varepsilon$ to $\mathfrak{l}_* \times \mathfrak{p}$ are equal to 0 since $\varepsilon(x,y)$ and $\overline{\varepsilon_0}(x,y) \wedge \mu_+$ are colinear for all (x,y) in $\mathfrak{l}_* \times \mathfrak{p}$ by Corollary 2.7. So, by Lemma 13.1(ii),

$$\varphi_{s,r,\upsilon,\iota} \wedge \hat{\varepsilon} \in \kappa(\mathcal{M}_{k-|\iota|-|\upsilon|,*,1,+}) \quad \text{and} \quad \varphi_{s,r,\upsilon,\iota} \wedge \lambda_\upsilon \wedge \hat{\varepsilon} \in \kappa(\mathcal{M}_{k-|\iota|,*,|\upsilon|+1,+})$$

for all (s,r,υ,ι). □

For $l = 0,1,\ldots$, let F_l be the subspace $\kappa(\mathcal{M}_{k,*,l,+})$ of $D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$. Then the sequence F_l, $l = 0,1,\ldots$ is a decreasing filtration of $D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$. Denote by $\operatorname{gr} D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$ the associate graded space to this filtration and $\operatorname{gr}_l D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$ the subspace of degree l of $\operatorname{gr} D_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$. For $j = 0,\ldots,k$ and $i = 0,\ldots,j$, let $D^\bullet_{j,\#,i,\times}(\mathfrak{g},\overline{\mathrm{B}})$ be the graded subcomplex of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} \bigwedge(\mathrm{E}) \otimes_{\Bbbk} D^\bullet(\mathfrak{g})$,

$$D^\bullet_{j,\#,i,\times}(\mathfrak{g},\overline{\mathrm{B}}) := D^\bullet_{j,\#,i,\times}(\mathfrak{g})[-\mathrm{b}_{\mathfrak{g}}] \wedge \textstyle\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\overline{\mathrm{B}}),$$

and A the algebra

$$A := \Bbbk[z_1,\ldots,z_{2n+\ell}] \otimes_{\Bbbk} \mathrm{S}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \textstyle\bigwedge(\mathrm{E}).$$

This algebra has a bigradation $A^\bullet_\bullet$ such that

$$A^i := \Bbbk[z_1,\ldots,z_{2n+\ell}] \otimes_{\Bbbk} \mathrm{S}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \textstyle\bigwedge^i(\mathrm{E}) \quad \text{and}$$
$$A_l := \bigoplus_{(j_1,j_2,j_3) \in \mathbb{N}_l^3} \Bbbk[z_1,\ldots,z_{2n+\ell}]_{j_1} \otimes_{\Bbbk} \mathrm{S}^{j_2}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \textstyle\bigwedge^{j_3}(\mathrm{E}).$$

Consider on $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} A \otimes_{\Bbbk} D(\mathfrak{g})$ the simple gradation deduced from the double gradation $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} A^\bullet \otimes_{\Bbbk} D^\bullet(\mathfrak{g})$. The trivial structure of complex on A and the structure of complex on $D(\mathfrak{g})$ induce a structure of graded complex on $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} A \otimes_{\Bbbk} D(\mathfrak{g})$. For $l = 0, 1, \ldots$ and i nonnegative integer, set:

$$C_l^i = \bigoplus_{j=0}^{i} \bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-j}} \Bbbk[z_1, \ldots, z_{2n+\ell}]_{l_1} \otimes_{\Bbbk} \mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} D^{i-j}_{k,\#,j,\times}(\mathfrak{g}, \overline{\mathrm{B}}).$$

Denoting by $C_l^\bullet$ the sum C_l^i, $i = 0, 1, \ldots$, $C_l^\bullet$ is a graded subcomplex of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} A \otimes_{\Bbbk} D(\mathfrak{g})$.

Corollary 13.3. (i) *For $l = 0, 1, \ldots$, $F_l^\bullet$ is a graded subcomplex of $D^\bullet_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$.*

(ii) *The graded complex $\mathrm{gr}_0 D^\bullet_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$ is isomorphic to the graded complex $D^\bullet_{k,\#}(\mathfrak{g}, \overline{\mathrm{B}})$.*

(iii) *For $l = 1, 2, \ldots$, the graded complex $\mathrm{gr}_l D^\bullet_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$ is isomorphic to $C_l^\bullet$.*

Proof. (i) The proof is the same as the proof of Corollary 12.7(i).

(ii) The proof is the same as the proof of Corollary 12.7(ii) and results from Proposition 13.2.

(iii) Let l be a nonnegative integer. Denote by γ_l the morphism of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-modules

$$\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} A_l \otimes_{\Bbbk} D(\mathfrak{g}) \xrightarrow{\gamma_l} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} D(\mathfrak{g}) \qquad \psi\otimes a\otimes\lambda_\upsilon \wedge v_\iota\otimes\varphi \longmapsto \psi a\otimes\varphi \wedge \lambda_\upsilon \wedge v_\iota$$

and $\mathcal{M}'_{k,l}$ the subspace of elements φ of $\mathcal{M}_{k,l}$ such that

$$\varphi_{s,r,\upsilon,\iota} \in \gamma_l(\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} A_l \otimes_{\Bbbk} D(\mathfrak{g})).$$

Then $F_l = \kappa(\mathcal{M}'_{k,l}) + F_{l+1}$ since

$$\hat{J}^l = \Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \otimes_{\Bbbk} A_l \oplus \hat{J}^{l+1}.$$

By Proposition 13.2, for φ in $\mathcal{M}'_{k,l}$, $\kappa(\varphi)$ is in F_{l+1} if and only if $r_{\mathfrak{g},\mathfrak{p}}(\varphi_{s,r,\upsilon,\iota} \wedge \hat{\varepsilon}) = 0$ for all (s, r, υ, ι). Then, by (ii), the restriction of $r_{\mathfrak{g},\mathfrak{p}}$ toF_l defines through the quotient an isomorphism of $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-modules,

$$\mathrm{gr}_l D_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}}) \longrightarrow C_l\ .$$

Moreover, this isomorphism is an isomorphism of graded complex. □

13.2. **Annulation of cohomology.** Let $k = 1, \ldots, n$. For i integer, denote by Z^i and B^i the spaces of cocycles and coboundaries of degree i of $D^\bullet_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$. For $j = 1, \ldots, n$ and i in $\mathbb{I}''_j$, denote by $D^\bullet_{j,\#}(\mathrm{E}')$ the graded the submodule of $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}] \otimes_{\Bbbk} D^\bullet_j(\mathfrak{l})$ deduced from the multigraded module

$$\bigoplus_{i\in\mathbb{I}''_j} D^\bullet_{i_1}(\mathrm{E}'_1) \wedge \cdots \wedge D^\bullet_{i_n}(\mathrm{E}'_n).$$

For $l = 1, \ldots, n$, let $D^\bullet_{l,\#}(\mathrm{E})$ be the graded submodule of $D^\bullet_l(\mathrm{E})$ deduced from the bigraded module

$$\bigoplus_{j=0}^{l} D_{l-j}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} D^\bullet_{j,\#}(\mathrm{E}').$$

Set:

$$\tilde{D}^{\bullet}_{l,\mathrm{u}} := D^{\bullet}_{l,\#}(\mathrm{E})[-\mathrm{b}_{\mathfrak{g}}] \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}) \quad \text{and} \quad D^{\bullet}_{l,\mathrm{u}} := D^{\bullet}_{l,\#}(\mathrm{E})[-\mathrm{b}_{\mathfrak{g}}] \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\widehat{\mathrm{B}}).$$

Then $D^{\bullet}_{l,\mathrm{u}}$ is a graded subcomplex of $D^{\bullet}_{l,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$.

Lemma 13.4. *Let $l = 1, \dots, n$.*

(i) *The morphisms*

$$\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} D^{\bullet}_l(\mathrm{E})[-\mathrm{b}_{\mathfrak{g}}] \xrightarrow{\theta_l} D^{\bullet}_l(\mathfrak{g}, \mathrm{B}) \ , \qquad \varphi \longmapsto \varphi \wedge \varepsilon \quad \textit{and}$$

$$\hat{\mathcal{O}} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} D^{\bullet}_l(\mathrm{E})[-\mathrm{b}_{\mathfrak{g}}] \xrightarrow{\hat{\theta}_l} D^{\bullet}_l(\mathfrak{g}, \widehat{\mathrm{B}}) \ , \qquad \varphi \longmapsto \varphi \wedge \hat{\varepsilon}$$

are injective.

(ii) *The complexes $D^{\bullet}_l(\mathrm{E})$, $\tilde{D}^{\bullet}_{l,\mathrm{u}}$, $D^{\bullet}_{l,\mathrm{u}}$ are acyclic.*

Proof. (i) For ψ in $\mathcal{M}_{l,*}$, set:

$$\kappa(\psi) := \sum_{(s,r,\upsilon,\iota) \in \mathbb{N}_{\mathrm{p},+}} z^s v^r \psi_{s,r,\upsilon,\iota} \wedge \lambda_\upsilon \wedge v_\iota.$$

Let $\mathcal{E}$ be the subset of elements ψ of $\mathcal{M}_{l,*}$ such that

$$s \neq 0 \Longrightarrow \psi_{s,r,\upsilon,\iota} = 0 \quad \text{and} \quad \psi_{s,r,\upsilon,\iota} \in \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} \mathrm{S}^l(\mathrm{E}).$$

For m nonnegative integer, denote by $\mathcal{E}_m$ the subspace of elements ψ of $\mathcal{E}$ such that

$$|(r, \upsilon, \iota)| < m \Longrightarrow \psi_{0,r,\upsilon,\iota} = 0.$$

For φ in $\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} D^{\bullet}_l(\mathrm{E})$, $\varphi = \kappa(\psi)$ for some ψ in $\mathcal{E}$. Denote by $\lambda(\varphi)$ the biggest integer m such that $\varphi = \kappa(\psi)$ for some ψ in $\mathcal{E}_m \setminus \mathcal{E}_{m+1}$.

Suppose that θ_l is not injective. A contradiction is expected. Let φ be in its kernel such that $\lambda(\varphi)$ is minimal. For some ψ in $\mathcal{E}_{\lambda(\varphi)}$, $\varphi = \kappa(\psi)$. By Corollary 11.3, for (r, υ, ι) such that $|(r, \upsilon, \iota)| = \lambda(\varphi)$,

$$\psi_{0,r,\upsilon,\iota} \in J\mathrm{S}(\mathrm{E}),$$

since $\mathrm{S}(\mathrm{E})$ is a free module, whence the contradiction since φ is not in $\mathcal{E}_{\lambda(\varphi)+1}$.

Suppose that $\hat{\theta}_l$ is not injective. A contradiction is expected. Let φ be a nonzero element of degree i of its kernel. The element φ has an expansion

$$\varphi = \sum_{s \in \mathbb{N}^{2n+\ell}} z^s \varphi_s \quad \text{with} \quad \varphi_s \in \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} \mathrm{S}^{l-i}(\mathrm{E}) \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} \bigwedge^i(\mathrm{E})$$

for all s. Denote by σ the smallest integer such that $\varphi_s \neq 0$ for some s in $\mathbb{N}^{2n+\ell}_\sigma$. By minimality of σ, $\varphi_s \wedge \varepsilon = 0$ for all s in $\mathbb{N}^{2n+\ell}_\sigma$ since $\hat{\varepsilon} - \varepsilon$ is in $\mathfrak{m} \otimes_{\Bbbk} \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk} \mathfrak{g}$ as already observed. Then, by the injectivity of θ_l, $\varphi_s = 0$ for all s in $\mathbb{N}^{2n+\ell}_\sigma$, whence the contradiction.

(ii) As E is a free module, $D^{\bullet}_l(\mathrm{E})$ is an acyclic complex by Lemma A.2(ii) since l is positive. Then $D^{\bullet}_{l,\#}(\mathrm{E})$ is acyclic since $D^{\bullet}_{l,\#}(\mathrm{E})$ is a direct factor of $D^{\bullet}_l(\mathrm{E})$. By (i), the restriction of θ_l to

$\Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} D^\bullet_{l,\#}(\mathrm{E})[-\mathrm{b}_\mathfrak{g}]$ is an isomorphism of graded complexes onto $\widetilde{D}^\bullet_{l,\mathrm{u}}$. Hence $\widetilde{D}^\bullet_{l,\mathrm{u}}$ is acyclic. Again by (i), the restriction of $\hat{\theta}_l$ to $\hat{\mathcal{O}} \otimes_\Bbbk \Bbbk[\mathfrak{l}_* \times \mathfrak{g}] \otimes_{\Bbbk[\mathfrak{l}_* \times \mathfrak{l}]} D^\bullet_{l,\#}(\mathrm{E})[-\mathrm{b}_\mathfrak{g}]$ is an isomorphism of graded complexes onto $D^\bullet_{l,\mathrm{u}}$. Hence $D^\bullet_{l,\mathrm{u}}$ is acyclic. □

For $j = 0, \dots, k$, denote by $D^\bullet_{k,j,\#}$ the intersection of $\Bbbk[\mathfrak{l}_* \times \mathfrak{l}] \otimes_\Bbbk D^\bullet_{k,\#}(\mathfrak{g})$ and $\mathrm{S}^j(\overline{\mathrm{B}}_{0,0}) D^\bullet_{k-j}(\mathrm{E})$. For $i = \mathrm{b}_\mathfrak{g}, \dots, k + \mathrm{b}_\mathfrak{g}$ and l nonnegative integer, set:

$$K^i := \sum_{j=0}^{k-i+\mathrm{b}_\mathfrak{g}} D^{i-\mathrm{b}_\mathfrak{g}}_{k,j,\#} \wedge \bigwedge\nolimits^{\mathrm{b}_\mathfrak{g}}(\widehat{\mathrm{B}}) \quad \text{and}$$

$$K^i_l = \bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_\mathfrak{g}}} \Bbbk[z_1, \dots, z_{2n+\ell}]_{l_2} \otimes_\Bbbk \mathrm{S}^{l_2}(\mathfrak{p}_\mathrm{u}) \otimes_\Bbbk \sum_{j=0}^{k-i+\mathrm{b}_\mathfrak{g}} D^{i-\mathrm{b}_\mathfrak{g}}_{k,j,\#} \wedge \hat{\varepsilon}.$$

Denote by $D^0_{k-j,i-\mathrm{b}_\mathfrak{g},\#,\times}$ the image of $D^{i-\mathrm{b}_\mathfrak{g}}_{k,j,\#} \wedge \hat{\varepsilon}$ by the quotient morphism $F_{i-\mathrm{b}_\mathfrak{g}} \longrightarrow \mathrm{gr}_{\,i-\mathrm{b}_\mathfrak{g}} D_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$.

Lemma 13.5. *Let $i = \mathrm{b}_\mathfrak{g} + 1, \dots, k + \mathrm{b}_\mathfrak{g}$ and l a nonnegative integer.*
(i) *The subspace $Z^i \cap F^i_l$ of F^i_l is contained in $\mathrm{d}F^{i-1}_l + K^i + F^i_{l+1}$.*
(ii) *The space $K^i \cap F^i_l$ is contained in $K^i_l + F^i_{l+1}$.*
(ii) *The intersection of F^i_{l+1} and K^i is equal to $J_1^{l-i+\mathrm{b}_\mathfrak{g}+1} K^i$.*

Proof. (i) Let φ be in $Z^i \cap F^i_l$ and $\overline{\varphi}$ its image in $\mathrm{gr}_{\,l} D^i_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$. By Corollary 13.3, $\overline{\varphi}$ is a cocycle of degree i of the graded complex $C^\bullet_l$. By Corollary 12.2, for $j = 0, \dots, k$, the complex $D^\bullet_{k-j,\#}(\mathfrak{g}, \overline{\mathrm{B}})$ has no cohomolgy of degree different from $\mathrm{b}_\mathfrak{g}$. Then, for some $\overline{\psi}$ in C^{i-1}_l,

$$\overline{\varphi} - \mathrm{d}\overline{\psi} \in \bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_\mathfrak{g}}} \Bbbk[z_1, \dots, z_{2n+\ell}]_{l_1} \otimes_\Bbbk \mathrm{S}^{l_2}(\mathfrak{p}_\mathrm{u}) \otimes_\Bbbk D^{\mathrm{b}_\mathfrak{g}}_{k,i-\mathrm{b}_\mathfrak{g},\#,\times}(\mathfrak{g}, \overline{\mathrm{B}}).$$

Then, by Lemma 12.3(ii),

$$\overline{\varphi} - \mathrm{d}\overline{\psi} \in \bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_\mathfrak{g}}} \Bbbk[z_1, \dots, z_{2n+\ell}]_{l_1} \otimes_\Bbbk \mathrm{S}^{l_2}(\mathfrak{p}_\mathrm{u}) \otimes_\Bbbk \sum_{j=0}^{k-i+\mathrm{b}_\mathfrak{g}} D_{k-j,i-\mathrm{b}_\mathfrak{g},\#,\times}.$$

So, for a representative ψ of $\overline{\psi}$ in F^{i-1}_l,

$$\varphi - \mathrm{d}\psi \in K^i + F^i_{l+1}.$$

(ii) The proof is analogous to the proof of Lemma 12.9(ii).

(iii) By definition $J_1^{l-i+\mathrm{b}_\mathfrak{g}+1} K^i$ is contained $F^i_{l+1} \cap K^i$. Prove the assertion by induction on l. By Corollary 13.3(iii), the quotient of K^i by $K^i \cap F^i_{1+i-\mathrm{b}_\mathfrak{g}}$ is equal to the subspace of $\mathrm{gr}_{\,i-\mathrm{b}_\mathfrak{g}} D_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$,

$$\sum_{j=0}^{k-i+\mathrm{b}_\mathfrak{g}} D^0_{k-j,i-\mathrm{b}_\mathfrak{g},\#,\times}.$$

Moreover, by Lemma 13.4(i), the quotient of K^i by J_1K^i is equal to

$$\sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}} D^{i-\mathrm{b}_{\mathfrak{g}}}_{k,j,\#} \wedge \hat{\varepsilon}.$$

In particular, the $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-module K^i/J_1K^i is free. Again by Lemma 13.4(i), the three $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-modules,

$$\sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}} D^{i-\mathrm{b}_{\mathfrak{g}}}_{k,j,\#} \wedge \hat{\varepsilon}, \qquad \sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}} D^{i-\mathrm{b}_{\mathfrak{g}}}_{k,j,\#},$$

$$\sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}} D^{0}_{k-j,i-\mathrm{b}_{\mathfrak{g}},\#,\times}$$

are three free modules of the same finite rank. Hence for some p in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \setminus \{0\}$, $pK^i \cap F^i_{i-\mathrm{b}_{\mathfrak{g}}+1}$ is contained J_1K^i. As a result, $K^i \cap F^i_{i-\mathrm{b}_{\mathfrak{g}}+1}$ is equal to J_1K^i since the $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-module $K^i/J_1K^i_1$ is torsion free.

Suppose $l > i - \mathrm{b}_{\mathfrak{g}}$ and the assertion true for $l-1$. By induction hypothesis, it remains to prove that $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}+1}K^i$ is the intersection of $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}}K^i$ and F^i_{l+1}. By Corollary 13.3(iii), the quotient of $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}}K^i$ by $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}}K^i \cap F^i_{l+1}$ is equal to the subspace of $\mathrm{gr}_l D_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$,

$$\bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_{\mathfrak{g}}}} \Bbbk[z_1,\dots,z_{2n+\ell}]_{l_2} \otimes_{\Bbbk} \mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}} D^{0}_{k-j,i-\mathrm{b}_{\mathfrak{g}},\#,\times}.$$

Moreover, by Lemma 12.8, the quotient of $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}}K^i$ by $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}+1}K^i$ is equal to

$$\bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_{\mathfrak{g}}}} \Bbbk[z_1,\dots,z_{2n+\ell}]_{l_2} \otimes_{\Bbbk} \mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}} D^{i-\mathrm{b}_{\mathfrak{g}}}_{k,j,\#} \wedge \hat{\varepsilon}.$$

In particular, the $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-module $J_1^{l-i+\mathrm{b}_{\mathfrak{g}}}K^i/J_1^{l-i+\mathrm{b}_{\mathfrak{g}}+1}K^i$ is free. Again by Lemma 12.8, the three $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-modules

$$\bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_{\mathfrak{g}}}} \Bbbk[z_1,\dots,z_{2n+\ell}]_{l_2} \otimes_{\Bbbk} \mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}} D^{0}_{k-j,i-\mathrm{b}_{\mathfrak{g}},\#,\times},$$

$$\bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_{\mathfrak{g}}}} \Bbbk[z_1,\dots,z_{2n+\ell}]_{l_2} \otimes_{\Bbbk} \mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}} D^{i-\mathrm{b}_{\mathfrak{g}}}_{k,j,\#},$$

$$\bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_{\mathfrak{g}}}} \Bbbk[z_1,\dots,z_{2n+\ell}]_{l_2} \otimes_{\Bbbk} \mathrm{S}^{l_2}(\mathfrak{p}_{\mathrm{u}}) \otimes_{\Bbbk} \sum_{j=0}^{k-i+\mathrm{b}_{\mathfrak{g}}} D^{i-\mathrm{b}_{\mathfrak{g}}}_{k,j,\#} \wedge \hat{\varepsilon},$$

are three free modules of the same finite rank. Hence for some p in $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}] \setminus \{0\}$, $pK^i \cap F^i_{l+i-\mathrm{b}_\mathfrak{g}+1}$ is contained $J_1^{l-i+\mathrm{b}_\mathfrak{g}+1}K^i$. As a result, $K^i \cap F^i_{i-\mathrm{b}_\mathfrak{g}+1}$ is equal to $J_1^{l-i+\mathrm{b}_\mathfrak{g}+1}K^i$ since the $\Bbbk[\mathfrak{l}_* \times \mathfrak{p}]$-module $J_1^{l-i+\mathrm{b}_\mathfrak{g}}K^i/J_1^{l-i+\mathrm{b}_\mathfrak{g}+1}K^i_1$ is torsion free. □

Let d_1 and d_2 be the morphisms from $K^\bullet_l$ to $F^\bullet_l$ such that

$$\mathrm{d}_1 a\otimes\omega\nu \wedge \hat{\varepsilon} = (-1)^{i-\mathrm{b}_\mathfrak{g}})a\otimes\nu(\mathrm{d}\omega) \wedge \hat{\varepsilon}, \quad \mathrm{d}_2 a\otimes\omega\nu \wedge \hat{\varepsilon} = a\otimes\omega\mathrm{d}\nu \wedge \hat{\varepsilon} \quad \text{with}$$

$$a \in \bigoplus_{(l_1,l_2)\in\mathbb{N}^2_{l-i+\mathrm{b}_\mathfrak{g}}} \Bbbk[z_1,\dots,z_{2n+\ell}]_{l_2} \otimes_\Bbbk \mathrm{S}^{l_2}(\mathfrak{p}_\mathrm{u}),$$

$$\omega \in \mathrm{S}^j(\overline{\mathrm{B}}_{0,0}), \quad \nu \in D^{i-\mathrm{b}_\mathfrak{g}}_{k-j}(\mathrm{E}), \quad a\otimes\omega\nu \in D^i_{k,j,\#}, \quad i = \mathrm{b}_\mathfrak{g},\dots,k+\mathrm{b}_\mathfrak{g}, \quad j = 0,\dots,k-i+\mathrm{b}_\mathfrak{g}.$$

Lemma 13.6. *Let $i = \mathrm{b}_\mathfrak{g},\dots,k+\mathrm{b}_\mathfrak{g}$ and l a nonnegative integer.*

(i) *For i nonnegative integer and $j > 0$, the subspace $\mathrm{d}K^i_l$ of $D^{i+1}_{k,\#}(\mathfrak{g},\widehat{\mathrm{B}})$ is contained in $\mathrm{d}_2K^i_l + F^{i+1}_{l+1}$.*

(ii) *If $i > \mathrm{b}_\mathfrak{g}$ then $Z^i \cap F^i_l$ is contained in $\mathrm{d}F^{i-1}_l + F^i_{l+1}$.*

Proof. (i) By definition $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2$. Let a, ω, ν be as in the definition of d_2. Then

$$\mathrm{d}a\otimes\omega\nu \wedge \hat{\varepsilon} \in \mathrm{d}_2 a\otimes\omega \wedge \hat{\varepsilon} + a\otimes D^{i-\mathrm{b}_\mathfrak{g}}_{k,j-1,\#} \wedge \hat{\varepsilon} \wedge \overline{\mathrm{B}}_{0,0}.$$

By Corollary 13.3, $a\otimes\nu\mathrm{d}\omega \wedge \hat{\varepsilon}$ is in F^{i+1}_{l+1} since the image of $\hat{\varepsilon}$ by the restriction morphism $r_{\mathfrak{g},\mathfrak{p}}$ is a generator of $\bigwedge^{\mathrm{b}_\mathfrak{g}}(\overline{\mathrm{B}})$ and $\overline{\mathrm{B}}_{0,0}$ is a submodule of $\overline{\mathrm{B}}$, whence the assertion.

(ii) Let φ be in $Z^i \cap F^i_{l+1}$. By Lemma 13.5(i), for some ψ in F^{i-1}_l,

$$\varphi - \mathrm{d}\psi \in K^i + F^i_{l+1}.$$

As a matter of fact, by Lemma 13.5(ii),

$$\varphi - \mathrm{d}\psi \in K^i_l + F^i_{l+1}.$$

By Lemma 13.5(iii) and Lemma 13.4(i), for $i' = i, i+1$, the sum $K^{i'}_l + F^{i'}_{l+1}$ is direct. Let φ_1 be the component of $\varphi - \mathrm{d}\psi$ on K^i_l. Then, by (i), $\mathrm{d}_2\varphi_1 = 0$ since $\mathrm{d}_2\varphi_1$ is in K^{i+1} and $\mathrm{d}F^i_{l+1}$ is contained in F^{i+1}_{l+1}. As a result, by Lemma 13.4(ii), for some φ'_1 in K^{i-1}_l, $\varphi_1 = \mathrm{d}_2\varphi'_1$. Then, by (i),

$$\psi + \varphi'_1 \in F^{i-1}_l \quad \text{and} \quad \varphi - \mathrm{d}\psi - \mathrm{d}\varphi'_1 \in F^i_{l+1},$$

whence the assertion. □

Arguing as in the proof of Corollary 12.11, we deduce the following corollary from Lemma 13.6(ii).

Corollary 13.7. *Let $i = \mathrm{b}_\mathfrak{g}+1,\dots,\mathrm{b}_\mathfrak{g}+k$ and l a natural integer.*

(i) *For all nonnegative integer l, Z^i is contained in $B^i + F^i_l$.*

(ii) *For some p in $\mathfrak{m}\otimes_\Bbbk \Bbbk[\mathfrak{l}_*]$, $(1+p)Z^i$ is contained in B^i.*

Arguing as in the proof of Proposition 12.12, we deduce the following proposition from Corollary 13.7.

Proposition 13.8. *For $k = 0, \dots, n$, $D^\bullet_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$ and $D^\bullet_{k,\#}(\mathfrak{g}, \widetilde{\mathrm{B}})$ have no cohomology of degree different from $\mathrm{b}_\mathfrak{g}$.*

13.3. **End of the proof of Theorem 1.5.** We can now complete the proof of Theorem 1.5.

We prove the theorem by induction on the dimension of $\mathfrak{g}$. By Proposition 4.1(ii), the theorem is true for $\ell = 1$. Suppose $\ell > 1$ and the theorem true for the simple algebras of rank smaller than ℓ. By Proposition 12.12, Proposition 13.8 and the induction hypothesis, for $k = 1, \dots, n$ and z in $\mathfrak{h}_\mathfrak{b} \setminus \{0\}$, $D^\bullet_{k,\#}(\mathfrak{g}, \widehat{\mathrm{B}})$ has no cohomology of degree different from $\mathrm{b}_\mathfrak{g}$. Then, by Proposition 6.5, $\varpi_1(S_k) \cap \mathfrak{h} = \{0\}$. As a result, by remark 6.7, $\mathfrak{g}$ has Property (**P**), whence the theorem.

Appendix

Appendix A. Some complexes

Let X be an affine irreducible variety. The canonical injection from V into $\bigwedge(V)$ has a unique extension as a derivation of the algebra $\mathrm{S}(V) \otimes_\Bbbk \bigwedge(V)$ which is equal to 0 on the subalgebra $1 \otimes_\Bbbk \bigwedge(V)$. Then $\mathrm{S}(V) \otimes_\Bbbk \bigwedge(V)$ is a graded cohomology complex whose gradation is induced by the natural gradation of $\bigwedge(V)$. We denote this complex by $D^\bullet(V)$ and its derivation by d.

A.1. **General facts.** For k nonnegative integer, set:

$$D^\bullet_k(V) := \bigoplus_{i=0}^{k} \mathrm{S}^{k-i}(V) \otimes_\Bbbk \bigwedge{}^i(V),$$

so that $D^\bullet_k(V)$ is a graded subcomplex of $D^\bullet(V)$.

Definition A.1. Let L be a free submodule of positive rank r of $\Bbbk[X] \otimes_\Bbbk V$. For k nonnegative integer, denote by $D^\bullet_k(V, L)$ the graded subcomplex of $\Bbbk[X] \otimes_\Bbbk D^\bullet(V)$:

$$D^\bullet_k(V, L) := D^\bullet_k(V, L)[-r] \wedge \bigwedge{}^r(L).$$

The restriction to $D^\bullet_k(V, L)$ of the derivation of $\Bbbk[X] \otimes_\Bbbk D^\bullet(V)$ is also denoted by d.

For W subspace of V, let $D^\bullet_k(W, L)$ be the graded subspace of $D^\bullet_k(V, L)$ such that

$$D^{i+r}_k(W, L) := \mathrm{S}^{k-i}(W) \otimes_\Bbbk \bigwedge{}^i(W) \wedge \bigwedge{}^r(L).$$

Then $D^\bullet_k(W, L)$ is a graded subcomplex of $D^\bullet_k(V, L)$.

The embedding of $\mathrm{S}^k(L) \otimes_\Bbbk \bigwedge{}^r(L)$ into $\mathrm{S}^k(V) \otimes_\Bbbk \bigwedge{}^r(L)$ is an augmentation of $D^\bullet_k(V, L)$. Denote by $\overline{D}^\bullet_k(V, L)$ this augmented complex. In particular, $\overline{D}^\bullet_0(V, L)$ is acyclic.

Lemma A.2. *Let k be a positive integer.*

(i) *The cohomology of $D^\bullet(V)$ is equal to $\Bbbk$.*

(ii) *The complex $D^\bullet_k(V)$ is acyclic.*

(iii) *For any subspace E of V, $\overline{D}^\bullet_k(V, E)$ is an acyclic complex.*

Proof. (i) We prove the statement by induction on the dimension of V. For V equal to zero, $D^\bullet(V)$ is equal to $\Bbbk$ and its differential is equal to 0. We suppose the statement true for any vector space whose dimension is strictly smaller than $\dim V$. Let W be an hyperplane of V and let v be in $V\backslash W$. Let a be a homogeneous cocycle of degree d of $D^\bullet(V)$. Then a has a unique expansion

$$a = v^m a_m + \cdots + a_0 \ ,$$

with $a_0, \ldots, a_m$ in $\mathrm{S}(W) \otimes_\Bbbk \bigwedge(V)$. If $d = 0$, then

$$m v^{m-1} a_m \otimes v + \cdots + a_1 \otimes v = 0$$

so that $a = a_0$ is in $\Bbbk$. Suppose $d > 0$. Then for $i = 0, \ldots, m$,

$$a_i = a_i' + a_i'' \wedge v,$$

with a_i' and a_i'' in $\mathrm{S}(W) \otimes_\Bbbk \bigwedge^d(W)$ and $\mathrm{S}(W) \otimes_\Bbbk \bigwedge^{d-1}(W)$ respectively. From the equality

$$0 = \sum_{i=0}^m v^i \mathrm{d}a_i' + \sum_{i=1}^m (-1)^d i v^{i-1} a_i' \wedge v + \sum_{i=0}^m v^i \mathrm{d}a_i'' \wedge v,$$

we deduce that $a_0', \ldots, a_m'$ are cocycles. So, by the induction hypothesis, for $i = 0, \ldots, m$, $a_i' = \mathrm{d}b_i$ for some b_i in $\mathrm{S}(W) \otimes_\Bbbk \bigwedge^{d-1}(W)$. Then

$$a - \mathrm{d}(\sum_{i=0}^m v^i b_i) = v^m a_m'' \wedge v + \sum_{i=0}^{m-1} v^i ((-1)^d (i+1) b_{i+1} + a_i'') \wedge v \ .$$

Hence a_m'' and $(-1)^d(i+1)b_{i+1} + a_i''$ are cocycles of degree $d-1$ for $i = 0, \ldots, m-1$. If $d = 1$,

$$a = \mathrm{d}(\sum_{i=0}^m v^i b_i + \frac{1}{m+1} v^{m+1} a_m'' + \sum_{i=0}^{m-1} \frac{1}{i+1} v^{i+1} ((-1)^d (i+1) b_{i+1} + a_i'')) \ .$$

For d bigger than 1, by induction hypothesis, a_m'' is the coboundary of an element c_m in $\mathrm{S}(W) \otimes_\Bbbk \bigwedge^{d-2}(W)$ and for $i = 0, \ldots, m-1$, $(-1)^d(i+1)b_{i+1} + a_i''$ is the coboundary of an element c_i in $\mathrm{S}(W) \otimes_\Bbbk \bigwedge^{d-2}(W)$ so that

$$a = \mathrm{d}(\sum_{i=0}^m v^i b_i + \sum_{i=0}^m v^i c_i \wedge v).$$

(ii) As $D^\bullet(V)$ is the direct sum of $D_i^\bullet(V)$, $i \in \mathbb{N}$, the assertion results from (i).

(iii) Let F be a complement to E in V and d the dimension of E. For $i = 0, \ldots, k$,

$$D_k^{i+d}(V, E) = \bigoplus_{j=0}^{k-i} \mathrm{S}^{k-i-j}(F) \otimes_\Bbbk \mathrm{S}^j(E) \otimes_\Bbbk \bigwedge^i(F) \wedge \bigwedge^d(E),$$

whence

$$\overline{D}_k^\bullet(V, E) = \bigoplus_{j=0}^{k} \mathrm{S}^j(E) \otimes_\Bbbk D_{k-j}^\bullet(F)[-d] \wedge \bigwedge^d(E).$$

By (ii), for $j < k$, $D^\bullet_{k-j}(F)$ is acyclic. As a result $D^\bullet_k(V, E)$ has no cohomology of degree different from d, whence the assertion since for v in V, $v \wedge \bigwedge^d(E) = \{0\}$ if and only if v is in E. □

For π automorphism of X, denote by $\pi^\#$ the automorphism of the algebra $\Bbbk[X] \otimes_\Bbbk \mathrm{S}(V) \otimes_\Bbbk \bigwedge(V)$ induced by the comorphism of π. Let L be a free submodule of rank r of $\Bbbk[X] \otimes_\Bbbk V$.

Lemma A.3. *Let k be a positive integer and π an automorphism of X.*

(i) *The restriction of $\pi^\#$ to $D^\bullet_k(V, L)$ is an isomorphism from $D^\bullet_k(V, L)$ onto $D^\bullet_k(V, \pi^\#(L))$.*

(ii) *For any positive integer j, the image by π^{-1} of the support in X of the cohomology of degree j of $D^\bullet_k(V, L)$ is the support in X of the cohomology of degree j of $D^\bullet_k(V, \pi^\#(L))$.*

Proof. (i) For i positive integer, $\pi^\#(D^i_k(V, L))$ is equal to $D^i_k(V, \pi^\#(L))$. Hence $\pi^\#(D^\bullet_k(V, L))$ is equal to $D^\bullet_k(V, \pi^\#(L))$. As $\pi^\#$ is an automorphism of the complex $\Bbbk[X] \otimes_\Bbbk D^\bullet(V)$, the restriction of $\pi^\#$ to $D^\bullet_k(V, L)$ is an isomorphism of the complex $D^\bullet_k(V, L)$ onto the complex $D^\bullet_k(V, \pi^\#(L))$.

(ii) Let j be a positive integer, J_j and $J_{j,\pi}$ the ideals of definition in $\Bbbk[X]$ of the supports of the cohomology of degree j of $D^\bullet_k(V, L)$ and $D^\bullet_k(V, \pi^\#(L))$. If a is a cocycle of degree j of $D^\bullet_k(V, L)$ and p is in J_j, for m sufficiently big positive integer, $p^m a$ is a coboundary of $D^\bullet_k(V, L)$. Hence by (i), $\pi^\#(p)^m \pi^\#(a)$ is a coboundary of $D^\bullet_k(V, \pi^\#(L))$. So $J_{j,\pi}$ contains $\pi^\#(J_j)$. By the same argument, J_j contains $\pi^\#(J_{j,\pi})$ since π is an automorphism. Hence $J_{j,\pi}$ is equal to $\pi^\#(J_j)$, whence the assertion. □

For any x in X, denote by $L(x)$ the image of L by the map $\varphi \mapsto \varphi(x)$.

Lemma A.4. *Let X' be the subset of elements x of X such that $L(x)$ has dimension r and $\mathcal{L}$ the localization of L on X.*

(i) *The subset X' of X is open and nonempty. Moreover, X' has a finite cover by affine open subsets Y which have the following property:*

- *there exists a subspace E of V which is a complement to $L(x)$ in V for all x in Y.*

(ii) *For all positive integer k, the support in X of the cohomology of $\overline{D}^\bullet_k(V, L)$ has an empty intersection with X'.*

(iii) *Suppose that X is normal and X' is a big open subset of X. Then $\overline{D}^\bullet_k(V, L)$ has no cohomology of degree r.*

Proof. (i) Let $\eta_1, \ldots, \eta_r$ be a basis of L. For all x in X, $L(x)$ is the subspace of V generated by $\eta_1(x), \ldots, \eta_r(x)$. Then X' is a nonempty open subset of X. Let x be in X'. Let E be a complement to $L(x)$ in V. Then, for all y in an open neighborhood Y_x of x in X, $L(y)$ has dimension r and E is a complement to $L(y)$ in V. In particular, Y_x is contained in X'.

(ii) Let k be a positive integer and Y an affine open subset of X' which satisfies the condition of (i). Denoting by L_Y the space of sections of $\mathcal{L}$ above Y, we have to prove that $\overline{D}^\bullet_k(V, L_Y)$ is acyclic.

Let x_0 be in Y. For x in Y, denote by $\tau(x)$ the linear automorphism of V such that $\tau(x)(v) = v$ for all v in E and for w in $L(x)$, $\tau(x)(w)$ is the element of $L(x_0)$ such that $w - \tau(x)(w)$ is in E. Let $\overline{\tau}$ be the automorphism of the algebra $\Bbbk[Y] \otimes_{\Bbbk} \mathrm{S}(V) \otimes_{\Bbbk} \bigwedge(V)$ such that $\overline{\tau}(\varphi)$ is the map

$$Y \longrightarrow \mathrm{S}(V) \otimes_{\Bbbk} \bigwedge(V) \ , \qquad x \longmapsto \tau(x)(\varphi(x)).$$

The images of L_Y and $\overline{D}_k^\bullet(V, L_Y)$ by $\overline{\tau}$ are equal to $\Bbbk[Y] \otimes_{\Bbbk} L_Y(x_0)$ and $\Bbbk[Y] \otimes_{\Bbbk} \overline{D}_k^\bullet(V, L_Y(x_0))$ respectively. Moreover, the map

$$\overline{D}_k^\bullet(V, L_Y) \xrightarrow{\ \overline{\tau}\ } \Bbbk[Y] \otimes_{\Bbbk} \overline{D}_k^\bullet(V, L_Y(x_0))$$

is an isomorphism of graded complexes. Hence by lemma A.2(iii), $D_k^\bullet(V, L_Y)$ is acyclic.

(iii) Let a be a cocycle of degree r of $\overline{D}_k^\bullet(V, L)$. By (ii), the restriction of a to Y is the image by d of a unique element φ of $\mathrm{S}^k(L_Y) \otimes_{\Bbbk[Y]} \bigwedge^r(L_Y)$. So, by (i), the restriction of a to X' is the image by d of a section above X' of the localization on X of $\mathrm{S}^k(L) \otimes_{\Bbbk[X]} \bigwedge^r(L)$. As L is a free module, $\mathrm{S}^k(L) \otimes_{\Bbbk[X]} \bigwedge^r(L)$ is a free module and any section above X' of its localization on X is the restriction to X' of an element of $\mathrm{S}^k(L) \otimes_{\Bbbk} \bigwedge^r(L)$ since X' is a big open subset of the normal variety X. Hence a is a coboundary. □

A.2. **Some equivalence.** Let L be a free submodule of rank r of $\Bbbk[X] \otimes_{\Bbbk} V$ such that $2r > \dim V$. For $k = 0, \dots, \dim V - r$, denote by $K_k^\bullet(V, L)$ the graded subcomplex of $\Bbbk[X] \otimes_{\Bbbk} D^\bullet(V)$ whose subspace of degree i is

$$K_k^i(V, L) := \bigwedge^i(V) \otimes_{\Bbbk} \mathrm{S}^{k-i}(L)$$

for $i = 0, \dots, k$. Denote by δ the restriction of d to $K_k^\bullet(V, L)$. The map

$$K_k^k(V, L) \xrightarrow{\ \theta_k\ } D_k^{k+r}(V, L) \ , \qquad \varphi \longmapsto \varphi \wedge \eta$$

with η a generator of $\bigwedge^r(L)$, is an augmentation of $K_k^\bullet(V, L)$. Denote by $\overline{K}_k^\bullet(V, L)$ the augemented complex so defined.

Lemma A.5. *Let X' be the subset of elements x of X such that $L(x)$ has dimension r.*

(i) *For $k = 1, \dots, \dim V - r$, the support of the cohomology of $\overline{K}_k^\bullet(V, L)$ is contained in $X \setminus X'$.*

(ii) *The complex $K_k^\bullet(V, L)$ has no cohomology of degree* 0.

Proof. (i) Let Y be an affine open subset of X' and E a subspace of V satisfying the condition of Lemma A.4(i) and set

$$L_Y := \Bbbk[Y] \otimes_{\Bbbk[X]} L.$$

The complex $K_k^\bullet(V, L_Y)$ is isomorphic to

$$\bigoplus_{j=0}^{k} \bigwedge^j(E) \otimes_{\Bbbk} D_{k-j}^\bullet(L_Y)[-j].$$

Then $K_k^\bullet(V, L_Y)$ has no cohomology of degree different from k by Lemma A.2(ii) since L_Y is a free module. Moreover, the space of cocycles of degree k of $\overline{K}_k^\bullet(V, L_Y)$ is equal to

$$\bigoplus_{j=1}^{k} \bigwedge^{k-j}(E) \wedge \bigwedge^{j}(L_Y).$$

Hence $\overline{K}_k^\bullet(V, L_Y)$ is acyclic and the support of the cohomology of $\overline{K}_k^\bullet(V, L)$ is contained in $X \setminus X'$ since X' has a cover by affine open subsets satisfying the condition of Lemma A.4(i).

(ii) By (i), $K_k^\bullet(V, L)$ has no cohomology of degree 0 since $\mathrm{S}^k(L)$ is a torsion free module. □

Denote by $\eta_1, \dots, \eta_r$ a basis of L. Let $\mathfrak{I}$ be the union of $\{0\}$ and the set of strictly increasing sequences in $\{1, \dots, r\}$ and for $\iota = i_1 < \cdots < i_j$ in $\mathfrak{I}$, set:

$$\{\iota\} := \{i_1, \dots, i_j\}, \qquad |\iota| = j, \qquad \eta_\iota := \eta_{i_1} \wedge \cdots \wedge \eta_{i_j}.$$

For $j = 0, \dots, r$ and $s = (s_1, \dots, s_r)$ in $\mathbb{N}^r$, set:

$$\mathfrak{I}_j := \{\iota \in \mathfrak{I} \mid |\iota| = j\} \quad \text{and} \quad \eta^s := \eta_1^{s_1} \cdots \eta_r^{s_r}.$$

For $k = 1, \dots, \dim V - r$ and $j = 1, \dots, k$, denote by $K_{k,j}^\bullet(V, L)$ the graded subcomplex of $K_k^\bullet(V, L)$ whose subspace of degree i is

$$K_{k,j}^i(V, L) := \bigwedge^{i-j}(V) \wedge \bigwedge^{j}(L) \otimes_{\Bbbk} \mathrm{S}^{k-j}(L),$$

Z_k^i and B_k^i the space of cocycles and coboundaries of degree i of $\overline{K}_k^\bullet(V, L)$ respectively.

Lemma A.6. *Let $k = 1, \dots, \dim V - r$. Suppose that $\bigwedge^{i-1}(V) \wedge L$ is the kernel of θ_i for $i = 1, \dots, k$.*

(i) *Suppose $2 \le k$. For $i = 1, \dots, k-1$, Z_k^i is contained in $B_k^i + K_{k,i}^i(V, L)$.*

(ii) *The complex $\overline{K}_k^\bullet(V, L)$ is acyclic.*

Proof. (i) For $j = 0, \dots, i$, set $Z_{k,j}^i := Z_k^i \cap K_{k,j}^i(V, L)$ and prove that $Z_{k,j}^i$ is contained in $B_k^i + Z_{k,j+1}^i$ for $j < i$. Let φ be in $Z_{k,j}^i$. Then φ has an expansion

$$\varphi = \sum_{s \in \mathbb{N}_{k-i}^r} \sum_{\iota \in \mathfrak{I}_j} \eta_\iota \wedge \varphi_{s,\iota} \otimes \eta^s \quad \text{with} \quad \varphi_{s,\iota} \in \Bbbk[X] \otimes_{\Bbbk} \bigwedge^{i-j}(V).$$

For s' in $\mathbb{N}_{k-i-1}^r$, set

$$I_{s'} := \{(s, l) \in \mathbb{N}_{k-i}^r \times \{1, \dots, r\} \mid \eta^s = \eta_l \eta^{s'}\}.$$

As φ is a cocycle,

$$\sum_{s' \in \mathbb{N}_{k-i-1}^r} \sum_{(s,l) \in I_{s'}} \sum_{\iota \in \mathfrak{I}_j} s_l \eta_l \wedge \eta_\iota \wedge \varphi_{s,\iota} \otimes \eta^{s'} = 0 \quad \text{whence} \quad \sum_{(s,l) \in I_{s'}} \sum_{\iota \in \mathfrak{I}_j} s_l \eta_l \wedge \eta_\iota \wedge \varphi_{s,\iota} = 0$$

for all s' in $\mathbb{N}_{k-i-1}^r$.

Let Y be an affine open subset of X' and E a subspace of V satisfying the condition of Lemma A.4(iii). For (s,ι) in $\mathbb{N}^r_{k-i}\times\mathfrak{I}_j$,

$$\varphi_{s,\iota}\,|_Y=\varphi'_{s,\iota}+\varphi''_{s,\iota}\quad\text{with}\quad \varphi'_{s,\iota}\in L_Y\wedge\bigwedge^{i-j-1}(V),\quad \varphi''_{s,\iota}\in\Bbbk[Y]\otimes_{\Bbbk}\bigwedge^{i-j}(E).$$

For (l,ι) in $\{1,\dots,r\}\times\mathfrak{I}_j$ such that $l\notin\{\iota\}$, denote by $\upsilon(l,\iota)$ the element of $\mathfrak{I}_{r-j-1}$ and $\epsilon(l,\iota)$ the element of $\{-1,1\}$ such that

$$\{l\}\cup\{\iota\}\cup\{\upsilon(l,\iota)\}=\{1,\dots,r\}\quad\text{and}\quad \eta_l\wedge\eta_\iota\wedge\eta_{\upsilon(l,\iota)}=\epsilon(l,\iota)\eta.$$

For s' in $\mathbb{N}^r_{k-i-1}$ and $l=1,\dots,r$, set:

$$I_{s',l}:=\{s\in\mathbb{N}^r_{k-i}\mid(s,l)\in I_{s'}\}.$$

For s' in $\mathbb{N}^r_{k-i-1}$ and for (l,ι) in $\{1,\dots,r\}\times\mathfrak{I}_j$ such that $l\notin\{\iota\}$, by the above equality, after multiplication by $\eta_{\upsilon(l,\iota)}$,

$$\sum_{s\in I_{s',l}}\epsilon(l,\iota)s_l\varphi''_{s,\iota}\wedge\eta=0\quad\text{whence}\quad\sum_{s\in I_{s',l}}\epsilon(l,\iota)s_l\varphi''_{s,\iota}=0.$$

Since for (s,l) in $I_{s'}$, $\eta^s=\eta_l\eta^{s'}$, $|I_{s',l}|=1$. Hence $\varphi''_{s,\iota}=0$ for all (s,ι) in $\mathbb{N}^r_{k-i}\times\mathfrak{I}_j$ such that $s_l\neq0$ for some $l\notin\{\iota\}$. As a result for such (s,l), $\varphi_{s,\iota}\wedge\eta$ is equal to 0 since so is its restriction to Y. So, by hypothesis, for such (s,ι), $\varphi_{s,\iota}$ is in $L\wedge\bigwedge^{i-j-1}(V)$ and

$$\eta_\iota\wedge\varphi_{s,\iota}\otimes\eta^s\in K^i_{k,j+1}(V,L).$$

Let (s,ι) be in $\mathbb{N}^r_{k-i}\times\mathfrak{I}_j$ such that

$$s_l\neq0\Longrightarrow l\in\{\iota\}.$$

Then, for l in $\{\iota\}$ such that $s_l\neq0$,

$$\pm\eta_\iota\otimes\eta^s=\mathrm{d}\frac{1}{s_l+1}\eta_{\iota'}\wedge\eta_l\eta^s\quad\text{and}\quad\pm\,\eta_\iota\wedge\varphi_{s,\iota}\otimes\eta^s=\mathrm{d}\frac{1}{s_l+1}\eta_{\iota'}\wedge\varphi_{s,\iota}\otimes\eta_l\eta^s\quad\text{with}\quad\{\iota'\}=\{\iota\}\setminus\{l\},$$

whence the assertion.

(ii) By hypothesis, $\overline{K}^\bullet_k(V,L)$ has no cohomology of degree k and by Lemma A.5(ii), it has no cohomology of degree 0. Let $i=1,\dots,k-1$. By (i),

$$Z^i_k\subset B^i_k+K^i_{k,i}(V,L)=B^i_k+\bigwedge^i(L)\otimes_{\Bbbk[X]}\mathrm{S}^{k-i}(L).$$

As L is a free module, $K^\bullet_{k,i}(V,L)$ has no cohomology of degree i by Lemma A.2(ii). Hence $Z^i_k=B^i_k$. □

Proposition A.7. *Let $k=1,\dots,\dim V-r$.*

(i) *The complex $\overline{K}^\bullet_k(V,L)$ is acyclic if $D^\bullet_k(V,L)$ has no cohomology of degree different from r.*

(ii) *The complex $D^\bullet_k(V,L)$ has no cohomolgy of degree different from r if $\overline{K}^\bullet_i(V,L)$ is acyclic for $i=1,\dots,k$.*

(iii) *Suppose that* $\bigwedge^{i-1}(V) \wedge L$ *is the kernel of* θ_i *for* $i = 1, \dots, k$. *Then* $D_k^\bullet(V, L)$ *has no cohomology of degree different from* r.

Proof. (i) and (ii) For $i = 1, \dots, k$, denote by δ the restriction to $K_i^\bullet(V, L)$ of the derivation d of $\Bbbk[X] \otimes_\Bbbk D^\bullet(V)$. Setting

$$E_k^{i,j} := \begin{cases} \mathrm{S}^{k-i}(V) \otimes_\Bbbk \bigwedge^j(V) \otimes_\Bbbk \mathrm{S}^{i-j}(L) & \text{if} \quad j \le i \le k \\ \mathrm{S}^{k-j}(V) \otimes_\Bbbk \bigwedge^j(V) \wedge \bigwedge^r(L) & \text{if} \quad i = j-1 < k \\ 0 & \text{otherwise} \end{cases},$$

we have the equalities

$$E_k^{j-1,j} = D_k^j(V, L), \quad E_k^{i,j} = D_{k-i+j}^j(V) \otimes_\Bbbk \mathrm{S}^{i-j}(L) = \mathrm{S}^{k-i}(V) \otimes_\Bbbk K_i^j(V, L)$$

for $j \le i \le k$. Denoting again by δ the map

$$E_k^{j,j} \xrightarrow{\ \delta\ } E_k^{j-1,j}, \qquad \varphi \longmapsto \varphi \wedge \eta,$$

we have the double complex

$$\begin{tikzcd} & E_k^{i,j+1} & \\ \arrow[r, "\mathrm{d}"] & E_k^{i,j} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & E_k^{i+1,j+1} \\ & \arrow[u, "\delta"] & \end{tikzcd}.$$

Along a line, $i - j$ is constant and a line corresponding to a nonnegative constant is acyclic by Lemma A.2(ii). Then $\overline{K}_k^\bullet(V, L)$ has no cohomology of positive degree if $D_k^\bullet(V, L)$ has no cohomology of degree different from r since the maps

$$E_k^{j,j} \xrightarrow{\ \delta\ } E_k^{j-1,j} \quad \text{and} \quad E_k^{k-1,j} \xrightarrow{\ \mathrm{d}\ } E_k^{k,j}$$

are surjective, whence Assertion (i) by Lemma A.5(ii). As $E_k^{i,\bullet} = \mathrm{S}^{k-i}(V) \otimes_\Bbbk K_i^\bullet(V, L)$, $D_k^\bullet(V, L)$ has no cohomology of degree different from r if $\overline{K}_i^\bullet(V, L)$ is acyclic for $i = 1, \dots, k$.

(iii) By Lemma A.6(ii), $\overline{K}_i^\bullet(V, L)$ is acyclic for $i = 1, \dots, k$, whence the assertion by (ii). □

Appendix B. Projective dimension and cohomology

Recall in this section classical results. Let X be a Cohen-Macaulay irreducible affine algebraic variety and S a closed subset of codimension p of X. Let $P_\bullet$ be a complex of finitely generated projective $\Bbbk[X]$-modules whose length l is finite and let ε be an augmentation morphism of $P_\bullet$ whose image is R, whence an augmented complex of $\Bbbk[X]$-modules,

$$0 \longrightarrow P_l \longrightarrow P_{l-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\ \varepsilon\ } R \longrightarrow 0\ .$$

Denote by $\mathcal{P}_\bullet$, $\mathcal{R}$, $\mathcal{K}_0$ the localizations on X of $P_\bullet$, R, the kernel of ε respectively and denote by $\mathcal{K}_i$ the kernel of the morphism $\mathcal{P}_i \longrightarrow \mathcal{P}_{i-1}$ for i positive integer.

Lemma B.1. *Suppose that S contains the support of the homology of the augmented complex $P_\bullet$.*

(i) *For all positive integer $i < p-1$ and for all projective $\mathcal{O}_X$-module $\mathcal{P}$, $\mathrm{H}^i(X \setminus S, \mathcal{P})$ is equal to* 0.

(ii) *For all nonnegative integer $j \leq l$ and for all positive integer $i < p-j$, the cohomology group $\mathrm{H}^i(X \setminus S, \mathcal{K}_{l-j})$ is equal to zero.*

Proof. (i) Let $i < p-1$ be a positive integer. Since the functor $\mathrm{H}^i(X \setminus S, \bullet)$ commutes with the direct sum, it suffices to prove $\mathrm{H}^i(X \setminus S, \mathcal{O}_X) = 0$. Since S is a closed subset of X, we have the relative cohomology long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i_S(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^i(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^i(X \setminus S, \mathcal{O}_X) \longrightarrow \mathrm{H}^{i+1}_S(X, \mathcal{O}_X) \longrightarrow \cdots .$$

Since X is affine, $\mathrm{H}^i(X, \mathcal{O}_X)$ is equal to zero and $\mathrm{H}^i(X \setminus S, \mathcal{O}_X)$ is isomorphic to $\mathrm{H}^{i+1}_S(X, \mathcal{O}_X)$. Since X is Cohen-Macaulay, the codimension p of S in X is equal to the depth of its ideal of definition in $\Bbbk[X]$ [MA86, Ch. 6, Theorem 17.4]. Hence, according to [Gro67, Theorem 3.8], $\mathrm{H}^{i+1}_S(X, \mathcal{O}_X)$ and $\mathrm{H}^i(X \setminus S, \mathcal{O}_X)$ are equal to 0 since $i+1 < p$.

(ii) Let j be a nonnegative integer. Since S contains the support of the homology of the complex $P_\bullet$, for all nonnegative integer j, we have the short exact sequence of $\mathcal{O}_{X\setminus S}$-modules

$$0 \longrightarrow \mathcal{K}_{j+1}|_{X\setminus S} \longrightarrow \mathcal{P}_{j+1}|_{X\setminus S} \longrightarrow \mathcal{K}_j|_{X\setminus S} \longrightarrow 0$$

whence the long exact sequence of cohomology

$$\cdots \longrightarrow \mathrm{H}^i(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow \mathrm{H}^i(X \setminus S, \mathcal{K}_j) \longrightarrow \mathrm{H}^{i+1}(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow \mathrm{H}^{i+1}(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow \cdots .$$

Then, by (i), for $0 < i < p-2$, the cohomology groups $\mathrm{H}^i(X \setminus S, \mathcal{K}_j)$ and $\mathrm{H}^{i+1}(X \setminus S, \mathcal{K}_{j+1})$ are isomorphic since P_{j+1} is a projective module. Since $\mathcal{P}_i = 0$ for $i > l$, $\mathcal{K}_{l-1}$ and $\mathcal{P}_l$ have isomorphic restrictions to $X \setminus S$. In particular, by (i), for $0 < i < p-1$, $\mathrm{H}^i(X \setminus S, \mathcal{K}_{l-1})$ equal zero. Then, by induction on j, for $0 < i < p-j$, $\mathrm{H}^i(X \setminus S, \mathcal{K}_{l-j})$ is equal to zero. □

Proposition B.2. *Let R' be a $\Bbbk[X]$-module containing R. Suppose that the following conditions are satisfied:*

(1) *p is at least $l+2$,*
(2) *X is normal,*
(3) *S contains the support of the homology of the augmented complex $P_\bullet$.*

(i) *The complex $P_\bullet$ is a projective resolution of R of length l.*

(ii) *Suppose that R' is torsion free and that S contains the support in X of R'/R. Then $R' = R$.*

Proof. (i) Let j be a positive integer. We have to prove that $\mathrm{H}^0(X, \mathcal{K}_j)$ is the image of P_{j+1}. By Condition (3), the short sequence of $\mathcal{O}_{X\setminus S}$-modules

$$0 \longrightarrow \mathcal{K}_{j+1}\big|_{X\setminus S} \longrightarrow \mathcal{P}_{j+1}\,|_{X\setminus S} \longrightarrow \mathcal{K}_j\big|_{X\setminus S} \longrightarrow 0$$

is exact, whence the cohomology long exact sequence

$$0 \longrightarrow \mathrm{H}^0(X\setminus S, \mathcal{K}_{j+1}) \longrightarrow \mathrm{H}^0(X\setminus S, \mathcal{P}_{j+1}) \longrightarrow \mathrm{H}^0(X\setminus S, \mathcal{K}_j) \longrightarrow \mathrm{H}^1(X\setminus S, \mathcal{K}_{j+1}) \longrightarrow \cdots .$$

By Lemma B.1(ii), $\mathrm{H}^1(X\setminus S, \mathcal{K}_{j+1})$ equals 0 since $1 < p - l + j + 1$, whence the short exact sequence

$$0 \longrightarrow \mathrm{H}^0(X\setminus S, \mathcal{K}_{j+1}) \longrightarrow \mathrm{H}^0(X\setminus S, \mathcal{P}_{j+1}) \longrightarrow \mathrm{H}^0(X\setminus S, \mathcal{K}_j) \longrightarrow 0 .$$

As the codimension of S in X is at least 2 and X is irreducible and normal, the restriction morphism from P_{j+1} to $\mathrm{H}^0(X\setminus S, \mathcal{P}_{j+1})$ is an isomorphism. Let φ be in $\mathrm{H}^0(X, \mathcal{K}_j)$. Then there exists an element ψ of P_{j+1} whose image ψ' in $\mathrm{H}^0(X, \mathcal{K}_j)$ has the same restriction to $X\setminus S$ as φ. Since P_j is a projective module and X is irreducible, P_j is torsion free. Then $\varphi = \psi'$ since $\varphi - \psi'$ is a torsion element of P_j, whence the assertion.

(ii) Let $\mathcal{R}'$ be the localization of R' on X. Arguing as in (i), since S contains the support of R'/R and $1 < p - l$, the short sequence

$$0 \longrightarrow \mathrm{H}^0(X\setminus S, \mathcal{K}_0) \longrightarrow \mathrm{H}^0(X\setminus S, \mathcal{P}_0) \longrightarrow \mathrm{H}^0(X\setminus S, \mathcal{R}') \longrightarrow 0$$

is exact. Moreover, the restriction morphism from P_0 to $\mathrm{H}^0(X/S, \mathcal{P}_0)$ is an isomorphism since the codimension of S in X is at least 2 and X is irreductible and normal. Let φ be in R'. Then for some ψ in P_0, $\varphi - \varepsilon(\psi)$ is a torsion element of R'. So $\varphi = \varepsilon(\psi)$ since R' is torsion free, whence the assertion. □

Corollary B.3. *Let $C_\bullet$ be a homology complex of finitely generated $\Bbbk[X]$-modules whose length l is finite and positive. For $j = 0, \dots, l$, denote by Z_j the space of cycles of degree j of $C_\bullet$. Suppose that the following conditions are satisfied:*

(1) *S contains the support of the homology of the complex $C_\bullet$,*
(2) *for all i, C_i is a submodule of a free module,*
(3) *for $i = 1, \dots, l$, C_i has projective dimension at most d,*
(4) *X is normal and $l + d \leq p - 1$.*

Then $C_\bullet$ is acyclic and for $j = 0, \dots, l$, Z_j has projective dimension at most $l + d - j - 1$.

Proof. Prove by induction on $l - j$ that the complex

$$0 \longrightarrow C_l \longrightarrow \cdots \longrightarrow C_{j+1} \longrightarrow Z_j \longrightarrow 0$$

is acyclic and Z_j has projective dimension at most $l+d-j-1$. For $j = l$, Z_j is equal to zero since C_l is torsion free by Condition (2) and Z_l is a submodule of C_l, supported by S by Condition

(1). Suppose $j \leq l-1$ and the statement true for $j+1$. By Condition (3), C_{j+1} has a projective resolution $P_\bullet$ whose length is at most d and whose terms are finitely generated. By induction hypothesis, Z_{j+1} has a projective resolution $Q_\bullet$ whose length is at most $l+d-j-2$ and whose terms are finitely generated, whence an augmented complex $R_\bullet$ of projective modules whose length is $l+d-j-1$,

$$0 \longrightarrow Q_{l+d-j-2} \oplus P_{l+d-j-1} \longrightarrow \cdots \longrightarrow Q_0 \oplus P_1 \longrightarrow P_0 \longrightarrow Z_j \longrightarrow 0\ .$$

Denoting by d the differentials of $Q_\bullet$ and $P_\bullet$, the restriction to $Q_i \oplus P_{i+1}$ of the differential of $R_\bullet$ is the map

$$(x, y) \mapsto (\mathrm{d}x, \mathrm{d}y + (-1)^i \delta(x)),$$

with δ the map which results from the injection of Z_{j+1} into C_{j+1}. Since $P_\bullet$ and $Q_\bullet$ are projective resolutions, the complex $R_\bullet$ is a complex of projective modules having no homology of positive degree. Hence the support of the homology of the augmented complex $R_\bullet$ is contained in S by Condition (1). Then, by Proposition B.2 and Condition (4), $R_\bullet$ is a projective resolution of Z_j of length $l+d-j-1$ since Z_j is a submodule of a free module by Condition (2), whence the corollary since $Z_0 = C_0$ by definition. □

Corollary B.4. *Let*

$$0 \longrightarrow E_{-1} \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_l \longrightarrow 0$$

be a complex of finitely generated $\mathbb{C}[X]$-modules. Suppose that the following conditions are satisfied:

(1) *E_{-1} is projective and for $i = 0, \ldots, l-1$, E_i has projective dimension at most i,*
(2) *S contains the support of the cohomology of this complex,*
(3) *for $i = 0, \ldots, l$, E_i is a submodule of a free module,*
(4) *X is normal and $p \geq l+2$.*

Then the complex is acyclic and E_l has projective dimension at most l.

Proof. Prove the corollary by induction on l. For $l = 0$, by Conditions (2), (3), (4), the arrow $E_{-1} \longrightarrow E_0$ is an isomorphism. Suppose the corollary true for the integers smaller than l. Let Z_{l-1} be the kernel of the arrow $E_{l-1} \longrightarrow E_l$, whence the two complexes

$$0 \longrightarrow E_{-1} \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_{l-2} \longrightarrow Z_{l-1} \longrightarrow 0$$

$$0 \longrightarrow Z_{l-1} \longrightarrow E_{l-1} \longrightarrow E_l \longrightarrow 0\ .$$

By Condition (2), the support of the cohomology of these two complexes is contained in S. Then, by induction hypothesis, the first complex is acyclic and Z_{l-1} has projective dimension at most $l-1$. As a result, arguing as in the proof of Corollary B.3, we have a complex of $\Bbbk[X]$-modules

$$0 \longrightarrow P_l \longrightarrow P_{l-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow E_l \longrightarrow 0$$

such that $P_0, \dots, P_l$ are projective, the image of P_0 in E_l is the image of the arrow $E_{l-1} \longrightarrow E_l$ and the support of its homology is contained in S. Then, by Condition (4) and Proposition B.2, it is acyclic so that E_l has projective dimension at most l and the complex

$$0 \longrightarrow E_{-1} \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_l \longrightarrow 0$$

is acyclic. □

Let

$$0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0$$

be a short exact sequence of $\Bbbk[X]$-modules.

Lemma B.5. *Suppose that for $i = 0, 1, 2$, M_i has a finite projective dimension d_i. Then we have the inequalities*

$$d_2 \leq \sup\{d_0 + 1, d_1\} \quad \textit{and} \quad d_0 \leq \sup\{d_2 - 1, d_1\}.$$

Proof. Let N be a $\Bbbk[X]$-module. We have to prove $\mathrm{Ext}^j(M_2, N) = 0$ for j bigger than $\{d_0 + 1, d_1\}$ and $\mathrm{Ext}^j(M_1, N) = 0$ for j bigger than $\{d_2 - 1, d_1\}$. From the short exact sequence, we deduce the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^j(M_1, N) \longrightarrow \mathrm{Ext}^j(M_0, N) \longrightarrow \mathrm{Ext}^{j+1}(M_2, N) \longrightarrow \mathrm{Ext}^{j+1}(M_1, N) \longrightarrow \cdots.$$

For $j + 1 > \sup\{d_0 + 1, d_1\}$, $\mathrm{Ext}^{j+1}(M_1, N) = 0$ and $\mathrm{Ext}^j(M_0, N) = 0$, whence $\mathrm{Ext}^{j+1}(M_2, N) = 0$. For $j > \sup\{d_2 - 1, d_1\}$, $\mathrm{Ext}^j(M_1, N) = 0$ and $\mathrm{Ext}^{j+1}(M_2, N) = 0$, whence $\mathrm{Ext}^j(M_0, N) = 0$. □

Appendix C. Some computations

For k, l positive integers such that $k \leq l$, set:

$$r(k, l) := \sum_{j=0}^{k-1} (-1)^j \binom{l}{j}$$

and for k, l nonnegative integers, set:

$$c(k, l) := \sum_{j=0}^{k} \frac{(-1)^j}{j!(l + j)!}.$$

For e, k, l integers such that $2 \leq e \leq k \leq l$, set:

$$\psi(e, k, l) := r(k, l) - \sum_{j=1}^{e-1} (-1)^{k-j} c(j, k - j) r(j, l) \frac{(l - j)!}{(l - k)!}.$$

Let p_k, $k = 0, \dots$ be the sequence of polynomials defined by the induction relations:

$$p_0 := 1, \quad p_1(x) := x, \quad p_k(x) := k(x - k)p_{k-1}(x) + (-1)^k.$$

Lemma C.1. *Let k, l be positive integers.*

(i) *For $k \leq l$,*

$$r(k,l) := \frac{(-1)^{k-1}(l-1)!}{(k-1)!(l-k)!}.$$

(ii) *The rational number $c(k,l)$ is equal to $p_k(l)/(k!(l+k)!)$.*

(iii) *For $k \geq 2$, $p_k(l)$ is in $k\mathbb{Z} \pm 1$.*

Proof. (i) Prove the assertion by induction on k. For $k = 1$, $r(k,l) = 1$. Suppose $k > 1$ and the assertion true for $k-1$. Then

$$r(k,l) = \frac{(-1)^{k-2}(l-1)!}{(k-2)!(l-k+1)!} + \frac{(-1)^{k-1}l!}{(k-1)!(l-k+1)!} = \frac{(-1)^{k-2}(l-1)!}{(k-1)!(l-k+1)!}(k-1-l),$$

whence the assertion.

(ii) Prove the assertion by induction on k. For $k = 1$, $c(k,l) = l/(l+1)!$. Suppose $k > 1$ and the assertion true for $k-1$. Then

$$c(k,l) = \frac{p_{k-1}(l)}{(k-1)!(l-k+1)!} + \frac{(-1)^k}{k!(l+k)!} = \frac{1}{k!(l+k)!}(k(l-k)p_{k-1}(l) + (-1)^k),$$

whence the assertion.

(iii) Prove the assertion by induction on k. For $k = 2$, there is nothing to prove. Suppose $k > 2$ and the assertion true for $k-1$. From the equality

$$p_k(l) = k(l-k)p_{k-1}(l) + (-1)^k$$

and the induction hypothesis, we deduce the assertion. □

For e, k positive integers such that $2 \leq e \leq k$, set:

$$\varphi(e,k) := k - \sum_{j=1}^{e-1} \frac{j}{(j!)^2} p_j(k-j).$$

Corollary C.2. *Let e, k, l be positive integers such that $2 \leq e \leq k \leq l$. Then*

$$(-1)^{k-1}\frac{k!(l-k)!}{(l-1)!}\psi(e,k,l) = \varphi(e,k).$$

Proof. By Lemma C.1,

$$\psi(e,k,l) = \frac{(-1)^{k-1}(l-1)!}{(k-1)!(l-k)!} - \sum_{j=1}^{e-1} \frac{p_j(k-j)}{j!k!}\frac{(-1)^{k-1}(l-1)!}{(j-1)!(l-j)!}\frac{(l-j)!}{(l-k)!} =$$

$$\frac{(-1)^{k-1}(l-1)!}{(k-1)!(l-k)!} - \sum_{j=1}^{e-1} \frac{(-1)^{k-1}j(l-1)!p_j(k-j)}{(j!)^2k!(l-k)!},$$

whence

$$(1)^{k-1}\frac{k!(l-k)!}{(l-1)!}\psi(e,k,l) = k - \sum_{j=1}^{e-1}\frac{j}{(j!)^2}p_j(k-j).$$

□

Proposition C.3. *Let e, k, l be positive integers such that $2 \le e \le k \le l$. Then $\psi(e,k,l) \neq 0$*

Proof. By Corollary C.2, it is equivalent to prove $\varphi(e,k) \neq 0$. By definition,

$$\varphi(2,k) = k - k + 1 \quad \text{and} \quad \varphi(3,k) = \varphi(2,k) - \frac{1}{2}p_2(k-2) \in \mathbb{Z} \pm \frac{1}{2},$$

whence the proposition for $e = 2$ and $e = 3$. Suppose $e \ge 4$ and prove by induction on e,

$$((e-2)!)^2\varphi(e,k) \in \mathbb{Z} \pm \frac{1}{e-1}.$$

From the equality

$$\varphi(4,k) = \varphi(3,k) - \frac{3}{(3!)^2}p_3(k-3),$$

we deduce the assertion by Lemma C.1(iii) and the relation for $\varphi(3,k)$. Suppose $e > 4$ and the assertion true for $e-1$. By definition,

$$\varphi(e,k) = \varphi(e-1,k) - \frac{e-1}{((e-1)!)^2}p_{e-1}(k-e+1) \quad \text{whence}$$

$$((e-2)!)^2\varphi(e,k) = ((e-2)!)^2\varphi(e-1,k) + \frac{1}{e-1}p_{e-1}(k-e+1).$$

By induction hypothesis,

$$((e-2)!)^2\varphi(e-1,k) \in \mathbb{Z},$$

and by Lemma C.1(i),

$$\frac{1}{e-1}p_{e-1}(k-e+1) \in \mathbb{Z} \pm \frac{1}{e-1},$$

whence the assertion and the proposition. □

Appendix D. Some remarks about representations

In this section, $\mathfrak{g}$ is a semisimple Lie algebra, $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, containing $\mathfrak{b}$, $\mathfrak{l}$ is the reductive factor of $\mathfrak{p}$, containing $\mathfrak{h}$, and $\mathfrak{d}$ is the derived algebra of $\mathfrak{l}$. Let $\mathcal{R}_{\mathfrak{l}}$ the set of roots α such that $\mathfrak{g}^\alpha$ is contained in $\mathfrak{l}$ and $\mathcal{R}_{\mathfrak{l},+}$ the intersection of $\mathcal{R}_{\mathfrak{l}}$ and $\mathcal{R}_+$. Denote by $\mathcal{P}_\#$ the subset of elements of $\mathcal{P}(\mathcal{R})$ whose restriction to $\mathfrak{h} \cap \mathfrak{d}$ is a dominant weight of the root system $\mathcal{R}_{\mathfrak{l}}$ with respect to the positive root system $\mathcal{R}_{\mathfrak{l},+}$.

Let M be a rational $\mathfrak{g}$-module. For λ in $\mathcal{P}_+(\mathcal{R})$, denote by M_λ the isotypic component of type V_λ of the $\mathfrak{g}$-module M. Let $\mathcal{P}_M$ be the subset of dominant weights λ such that $M_\lambda \neq 0$.

Lemma D.1. *The space M is the direct sum of M_λ, $\lambda \in \mathcal{P}_M$.*

Proof. As M is a rational $\mathfrak{g}$-module, M is a union of $\mathfrak{g}$-modules of finite dimension. In particular, all simple $\mathfrak{g}$-module contained in M has finite dimension. Hence M_λ, $\lambda \in \mathcal{P}_M$ is the set of isotypic components of M. Moreover, M is the direct sum of M_λ, $\lambda \in \mathcal{P}_M$. □

For λ in $\mathcal{P}_M$, denote by ϖ_λ the canonical projection $M \longrightarrow M_\lambda$. Let N_1 and N_2 be two $\mathfrak{g}$-submodules of M such that N_1 is contained in N_2. For λ in $\mathcal{P}_M$, denote by $M_{\lambda,+}$ the subspace of highest weight vectors of M_λ. For the trivial action of G on $M_{\lambda,+}$, $V_\lambda \otimes_{\Bbbk} M_{\lambda,+}$ is a $\mathfrak{g}$-module. For $j = 1, 2$, set $N_{j,\lambda,+} := N_j \cap M_{\lambda,+}$ and for v in $M_{\lambda,+}$, denote by M_v the $\mathfrak{g}$-submodule of M_λ generated by v.

Lemma D.2. *Let λ be in $\mathcal{P}_M$.*

(i) *For $j = 1, 2$, N_j is the direct sum of $N_i \cap M_\gamma$, $\gamma \in \mathcal{P}_M$.*

(ii) *For v in $M_{\lambda,+}$, the $\mathfrak{g}$-modules V_λ and M_v are isomorphic.*

(iii) *There exists a basis v_i, $i \in I_\lambda$ of $M_{\lambda,+}$ satisfying the following condition: for some subsets $I_{\lambda,1}$ and $I_{\lambda,2}$ of I_λ, $I_{\lambda,1} \subset I_{\lambda,2}$, v_i, $i \in I_{\lambda,1}$ and v_i, $i \in I_{\lambda,2}$ are basis of $N_{1,\lambda,+}$ and $N_{2,\lambda,+}$ respectively.*

(iv) *For i in I_λ, denote by $\tau_{\lambda,i}$ an isomorphism of $\mathfrak{g}$-modules $V_\lambda \longrightarrow M_{v_i}$. Then the linear map*

$$V_\lambda \otimes_{\Bbbk} M_{\lambda,+} \xrightarrow{\tau_\lambda} M_\lambda\ , \qquad v \otimes v_i \longmapsto \tau_{\lambda,i}(v)$$

is an isomorphism of $\mathfrak{g}$-modules such that $\tau_\lambda(V_\lambda \otimes_{\Bbbk} N_{j,\lambda,+}) = N_j \cap M_\lambda$ for $j = 1, 2$.

Proof. (i) As N_j is a $\mathfrak{g}$-submodule of M, it is rational. So, by Lemma D.1(i), N_j is the direct sum of its isotypic components, whence the assertion since an isotypic component of N_j is contained in the isotypic component of M of the same type.

(ii) As v is in $M_{\lambda,+}$, M_v is a module of highest weight λ and the space of highest weight vectors in M_v is generated by v. Hence M_v is simple and isomorphic to V_λ.

(iii) is straightforward. Moreover, if $N_j \cap M_\lambda = \{0\}$ then $I_{\lambda,j}$ is empty.

(iv) By (ii), the isomorphisms $\tau_{\lambda,i}$ does exist. As v_i, $i \in I_\lambda$ is a basis of $M_{\lambda,+}$, M_λ is the direct sum of the subspaces M_{v_i}, $i \in I_\lambda$. Hence τ_λ is an isomorphism of $\mathfrak{g}$-modules. Moreover, for $j = 1, 2$, for i in I_λ, $\tau_\lambda(V_\lambda \otimes v_i)$ is contained in N_j if and only if i is in $I_{\lambda,j}$, whence the assertion since N_j is a $\mathfrak{g}$-module. □

Let M' be a $\mathfrak{l}$-submodule of M. For μ in $\mathcal{P}_\#$, denote by V'_μ a simple $\mathfrak{l}$-module of highest weight μ and M'_μ the isotypic component of type V'_μ of M'. Denote by $\mathcal{P}_{M'}$ the subset of elements μ of $\mathcal{P}_\#$ such that $M'_\mu \neq \{0\}$ and $\mathcal{P}_{M,M'}$ the subset of elements (λ, μ) of $\mathcal{P}_M \times \mathcal{P}_{M'}$ such that $\varpi_\lambda(M'_\mu) \neq \{0\}$.

Lemma D.3. (i) *The space M' is the direct sum of M'_μ, $\mu \in \mathcal{P}_{M'}$.*

(ii) *For (λ, μ) in $\mathcal{P}_{M,M'}$, V'_μ is isomorphic to a $\mathfrak{l}$-submodule of V_λ.*

Proof. (i) As M' is a $\mathfrak{l}$-submodule of the rational $\mathfrak{g}$-module M, M' is a rational $\mathfrak{l}$-module, whence the assertion by Lemma D.1(i).

(iii) Let (λ,μ) in $\mathcal{P}_{M,M'}$ and V_0 a simple $\mathfrak{l}$-module contained in $\varpi_\lambda(M'_\mu)$. According to Lemma D.2,(ii) and (iii),

$$M_\lambda = \bigoplus_{i\in I_\lambda} M_{v_i}.$$

For i in I_λ, denote by π_i the projection

$$M_\lambda \xrightarrow{\pi_i} M_{v_i}$$

corresponding to this decomposition. For some i, the restriction to V_0 is different from 0. As V_0 is a simple $\mathfrak{l}$-module, this restriction is an embedding of V_0 into M_{v_i}, whence the assertion since M_{v_i} is isomorphic to V_λ. □

For λ in $\mathcal{P}_M$, denote by $V^{\mathfrak{l}}_\lambda$ the subspace of elements of V_λ, annihilated by $\mathfrak{u}\cap\mathfrak{l}$, and for (λ,μ) in $\mathcal{P}_{M,M'}$, let $V^{\mathfrak{l}}_{\lambda,\mu}$ be the subspace of weight μ of $V^{\mathfrak{l}}_\lambda$.

Lemma D.4. *Let (λ,μ) be in $\mathcal{P}_{M,M'}$.*

(i) *There exists an isomorphism of $\mathfrak{l}$-modules*

$$V'_\mu\otimes_{\Bbbk} V^{\mathfrak{l}}_{\lambda,\mu} \xrightarrow{\tau_{\mu,\lambda}} \mathrm{U}(\mathfrak{l}).V^{\mathfrak{l}}_{\lambda,\mu}\ .$$

(ii) *For a well defined subspace $E_{\lambda,\mu}$ of $V^{\mathfrak{l}}_{\lambda,\mu}\otimes_{\Bbbk} M_{\lambda,+}$,*

$$\varpi_\lambda(M'_\mu) = \tau_\lambda\circ(\tau_{\mu,\lambda}\otimes\mathrm{id}_{M_{\lambda,+}})(V'_\mu\otimes_{\Bbbk} E_{\lambda,\mu}).$$

(iii) *For $j=1,2$, let $E_{\lambda,\mu,j}$ be the intersection of $E_{\lambda,\mu}$ and $V'_\mu\otimes_{\Bbbk} N_{j,\lambda,+}$. Then*

$$\varpi_\lambda(N_j\cap M'_\mu) = \tau_\lambda\circ(\tau_{\mu,\lambda}\otimes\mathrm{id}_{M_{\lambda,+}})(V'_\mu\otimes_{\Bbbk} E_{\lambda,\mu,j}).$$

Proof. (i) Let $w_1,\dots,w_m$ be a basis of $V^{\mathfrak{l}}_{\lambda,\mu}$. For $i=1,\dots,m$, denote by V'_i the $\mathfrak{l}$-submodule of V_λ generated by w_i. As w_i is weight vector of weight μ of $V^{\mathfrak{l}}_{\lambda,\mu}$, V'_i is a module of highest weight μ and the space of highest weight vectors in V'_i is generated by w_i so that V'_i is a simple module isomorphic to V'_μ. Moreover, $\mathrm{U}(\mathfrak{l}).V^{\mathfrak{l}}_{\lambda,\mu}$ is the direct sum of V'_i, $i=1,\dots,m$ since $w_1,\dots,w_m$ is a basis of $V^{\mathfrak{l}}_{\lambda,\mu}$, whence an isomorphism

$$V'_\mu\otimes_{\Bbbk} V^{\mathfrak{l}}_{\lambda,\mu} \xrightarrow{\tau_{\mu,\lambda}} \mathrm{U}(\mathfrak{l}).V^{\mathfrak{l}}_{\lambda,\mu}\ .$$

(ii) For v in $\tau_\lambda^{-1}(\varpi_\lambda(M'_\mu))$, v has an expansion

$$v=\sum_{i\in I_\lambda} v'_i\otimes v_i$$

with v'_i, $i\in I_\lambda$ in V_λ. As $\tau_\lambda(v)$ is in $\varpi_\lambda(M'_\mu)$, for i in I_λ, $u.v'_i$ is in $V^{\mathfrak{l}}_{\lambda,\mu}$ for some u in $\mathrm{U}(\mathfrak{u}\cap\mathfrak{l})$. As a result, $\tau_\lambda^{-1}(\varpi_\lambda(M'_\mu))$ is a subspace of $\mathrm{U}(\mathfrak{l}).V^{\mathfrak{l}}_{\lambda,\mu}\otimes_{\Bbbk} M_{\lambda,+}$, whence the assertion by (i).

(iii) Let v be in $E_{\lambda,\mu}$. By Lemma D.2(ii), $\tau_\lambda\circ(\tau_{\lambda,\mu}\otimes\mathrm{id}_{M_{\lambda,+}})(v)$ is in N_j if and only if $\tau_{\lambda,\mu}\otimes\mathrm{id}_{M_{\lambda,+}}(v)$ is in $V_\lambda\otimes_{\Bbbk} N_{j,\lambda,+}$. Then, by (ii), $\tau_\lambda\circ(\tau_{\lambda,\mu}\otimes\mathrm{id}_{M_{\lambda,+}})(v)$ is in $\varpi_\lambda(N_j\cap M'_\mu)$ if and only if v is in $E_{\lambda,\mu,j}$. □

For λ in $\mathcal{P}_M$, let θ_λ be the linear map

$$V_\lambda^* \otimes_{\Bbbk} V_\lambda \xrightarrow{\theta_\lambda} \Bbbk\ , \quad v'\otimes v \longmapsto \langle v', v\rangle$$

given by the duality. The kernel K_λ of θ_λ is a G-submodule of $V_\lambda^* \otimes_{\Bbbk} V_\lambda$ so that

$$V_\lambda^* \otimes_{\Bbbk} V_\lambda \otimes_{\Bbbk} M_{\lambda,+} = M_{\lambda,+} \oplus K_\lambda \otimes_{\Bbbk} M_{\lambda,+}.$$

Corollary D.5. *Suppose that M' generates the $\mathfrak{g}$-module M. Let λ be in $\mathcal{P}_M$. Then*

$$M_{\lambda,+} = \theta_\lambda\otimes\mathrm{id}_{M_{\lambda,+}}(\bigoplus_{\mu\in\mathcal{P}_{M,M',\lambda}} V_\lambda^* \otimes_{\Bbbk} E_{\lambda,\mu}) \quad \textit{with} \quad \mathcal{P}_{M,M',\lambda} := \{\mu \in \mathcal{P}_{M'} \mid (\lambda,\mu) \in \mathcal{P}_{M,M'}\}$$

and for $j = 1, 2$,

$$N_{j,\lambda,+} = \theta_\lambda\otimes\mathrm{id}_{M_{\lambda,+}}(\bigoplus_{\mu\in\mathcal{P}_{M,M',\lambda}} V_\lambda^* \otimes_{\Bbbk} E_{\lambda,\mu,j}).$$

Proof. As $E_{\lambda,\mu}$ is contained in $V^{\mathfrak{l}}_{\lambda,\mu} \otimes_{\Bbbk} M_{\lambda,+}$, the sums

$$\sum_{\mu\in\mathcal{P}_{V,V',\lambda}} E_{\lambda,\mu} \quad \text{and} \quad \sum_{\mu\in\mathcal{P}_{V,V',\lambda}} M_{\lambda,+} \cap V_\lambda^* \otimes_{\Bbbk} E_{\lambda,\mu}$$

are direct. By the hypothesis, M_λ is the $\mathfrak{g}$-submodule of M generated by $\varpi_\lambda(M')$. The map $\theta_\lambda\otimes\mathrm{id}_{M_{\lambda,+}}$ is a morphism of $\mathfrak{g}$-modules for the trivial action of $\mathfrak{g}$ in $M_{\lambda,+}$. Hence

$$\theta_\lambda\otimes\mathrm{id}_{M_{\lambda,+}}(\bigoplus_{\mu\in\mathcal{P}_{M,M',\lambda}} V_\lambda^* \otimes_{\Bbbk} \tau_\lambda^{-1}(\varpi_\lambda(M'_\mu))) = M_{\lambda,+}$$

by Lemma D.3(i). Then, by Lemma D.4(ii),

$$\theta_\lambda\otimes\mathrm{id}_{M_{\lambda,+}}(\bigoplus_{\mu\in\mathcal{P}_{M,M',\lambda}} V_\lambda^* \otimes_{\Bbbk} E_{\lambda,\mu}) = M_{\lambda,+},$$

and by Lemma D.4(iii),

$$\theta_\lambda\otimes\mathrm{id}_{M_{\lambda,+}}(\bigoplus_{\mu\in\mathcal{P}_{M,M',\lambda}} V_\lambda^* \otimes_{\Bbbk} E_{\lambda,\mu,j}) = N_{j,\lambda,+}.$$

□

Proposition D.6. *Suppose that the following conditions are satisfied:*

(1) *M' generates the G-module M,*
(2) *$N_2 \cap M'$ is contained in $N_1 \cap M'$.*

Then $N_1 = N_2$.

Proof. By Lemma D.3(i), for $j = 1, 2$, $N_j \cap M'$ is the direct sum of $N_j \cap M'_\mu$ since $N_j \cap M'$ is a $\mathfrak{l}$-submodule of M'. So, by Condition (2), for all (λ, μ) in $\mathcal{P}_{M,M'}$, $E_{\lambda,\mu,2}$ is contained in $E_{\lambda,\mu,1}$. Then, by Condition (1) and Corollary D.5, $N_{2,\lambda,+}$ is contained in $N_{1,\lambda,+}$ for λ in $\mathcal{P}_M$. As a result, by Lemma D.2(iii), $N_1 \cap M_\lambda = N_2 \cap M_\lambda$ for all λ in $\mathcal{P}_M$ since N_1 is contained in N_2, whence the proposition by Lemma D.2(i). □

References

[Bol91] A.V. Bolsinov, *Commutative families of functions related to consistent Poisson brackets*, Acta Applicandae Mathematicae, **24** (1991), **n°1**, p. 253–274.

[Bou02] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6. Translated from the 1968 French original by Andrew Pressley*, Springer-Verlag, Berlin (2002).

[Bou98] N. Bourbaki, *Algèbre commutative, Chapitre 10, Éléments de mathématiques*, Masson (1998), Paris.

[Bru] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics **n°39**, Cambridge University Press, Cambridge (1996).

[CMo08] J.-Y. Charbonnel and A. Moreau, *Nilpotent bicone and characteristic submodule of a reductive Lie algebra*, Tranformation Groups, **14**, (2008), p. 319–360.

[C20] J.-Y. Charbonnel, *On some subspaces of the exterior algebra of a simple algebra*, arXiv 2006-08994.

[Di74] J. Dixmier, *Algèbres enveloppantes*, Gauthier-Villars (1974).

[Di79] J. Dixmier, *Champs de vecteurs adjoints sur les groupes et algèbres de Lie semi-simples*, Journal für die reine und angewandte Mathematik, Band. **309** (1979), p. 183–190.

[Ga-Gi06] W. L. Gan, V. Ginzburg, *Almost-commuting variety, $\mathcal{D}$-modules, and Cherednik algebras.*, International Mathematics Research Papers, **2**, (2006), p. 1–54.

[Gi12] V. Ginzburg, *Isospectral commuting variety, the Harish-Chandra $\mathcal{D}$-module, and principal nilpotent pairs*, Duke Mathematical Journal, **161**, (2012), p. 2023–2111.

[Gro67] A. Grothendieck, *Local cohomology*, Lecture Notes in Mathematics **n°41** (1967), Springer-Verlag, Berlin, Heidelberg, New York.

[H77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **n°52** (1977), Springer-Verlag, Berlin Heidelberg New York.

[HuWi97] G. Huneke and R. Wiegand, *Tensor products of modules, Rigidity and Local cohomology*, Mathematica Scandinavica, **81**, (1997), p. 161–183.

[Ko63] B. Kostant, *Lie group representations on polynomial rings*, American Journal of Mathematics **85** (1963), p. 327–404.

[MA86] H. Matsumura, *Commutative ring theory* Cambridge studies in advanced mathematics **n°8** (1986), Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney.

[MF78] A.S. Mishchenko and A.T. Fomenko, *Euler equations on Lie groups*, Math. USSR-Izv. **12** (1978), p. 371–389.

[Mu88] D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Mathematics **n°1358** (1988), Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo.

[Po08] V.L. Popov, *Irregular and singular loci of commuting varieties*, Transformation Groups **13** (2008), p. 819–837.

[Po08] V.L. Popov and E. B. Vinberg, *Invariant Theory, in: Algebraic Geometry IV*, Encyclopaedia of MathematicalSciences **n°55** (1994), Springer-Verlag, Berlin, p.123–284.

[Ri79] R. W. Richardson, *Commuting varieties of semisimple Lie algebras and algebraic groups*, Compositio Mathematica **38** (1979), p. 311–322.

[Sh94] I.R. Shafarevich, *Basic algebraic geometry 2*, Springer-Verlag (1994), Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong-Kong, Barcelona, Budapest.

[V72] F.D. Veldkamp, *The center of the universal enveloping algebra of a Lie algebra in characteristic p*, Annales Scientifiques de L'École Normale Supérieure **5** (1972) , p. 217–240.

Jean-Yves Charbonnel, Université de Paris - CNRS, Institut de Mathématiques de Jussieu - Paris Rive Gauche, UMR 7586, Groupes, représentations et géométrie, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France

E-mail address: jean-yves.charbonnel@imj-prg.fr