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# On the automorphisms of the Drinfel'd double of a Borel Lie subalgebra

Michaël Bulois and Nicolas Ressayre

June 22, 2020

## Abstract

Let  $\mathfrak{g}$  be a complex simple Lie algebra with Borel subalgebra  $\mathfrak{b}$ . Consider the semidirect product  $I\mathfrak{b} = \mathfrak{b} \ltimes \mathfrak{b}^*$ , where the dual  $\mathfrak{b}^*$  of  $\mathfrak{b}$ , is equipped with the coadjoint action of  $\mathfrak{b}$  and is considered as an abelian ideal of  $I\mathfrak{b}$ . We describe the automorphism group  $\text{Aut}(I\mathfrak{b})$  of the Lie algebra  $I\mathfrak{b}$ . In particular we prove that it contains the automorphism group of the extended Dynkin diagram of  $\mathfrak{g}$ . In type  $A_n$ , the dihedral subgroup was recently proved to be contained in  $\text{Aut}(I\mathfrak{b})$  by Dror Bar-Natan and Roland Van Der Veen in [Bv20] (where  $I\mathfrak{b}$  is denoted by  $I\mathfrak{u}_n$ ). Their construction is handmade and they ask for an explanation: this note fully answers the question.

## 1 Introduction

Given any complex Lie algebra  $\mathfrak{a}$ , one can form its “inhomogeneous version”  $I\mathfrak{a} := \mathfrak{a} \ltimes \mathfrak{a}^*$ . It is the semidirect product of  $\mathfrak{a}$  with its dual  $\mathfrak{a}^*$  where  $\mathfrak{a}^*$  is considered as an abelian ideal and  $\mathfrak{a}$  acts on  $\mathfrak{a}^*$  via the coadjoint action.

As mentioned in [Bv20], for applications in knot theory and representation theory, the most important case is when  $\mathfrak{a} = \mathfrak{b}$  is the Borel subalgebra of some simple Lie algebra  $\mathfrak{g}$ . It is precisely the situation studied here. In addition to [Bv20], several examples of these algebras appear with variations in the literature. In [NW93], Nappi-Wittney use the case when  $\mathfrak{g} = \mathfrak{sl}_2$  in conformal field theory. Several authors also consider  $\mathfrak{b} \ltimes \mathfrak{n}^*$  where  $\mathfrak{n}$  is the derived subalgebra of  $\mathfrak{b}$ . It is the quotient of  $I\mathfrak{b}$  by its center. In [KZJ07], Knutson and Zinn-Justin meet this algebra for  $\mathfrak{g} = \mathfrak{gl}_n$  in the associative setting, see below. In [Fei12, Fei11], Feigin uses  $\mathfrak{b} \ltimes \mathfrak{n}^*$  in order to study degenerate flag varieties for  $\mathfrak{g} = \mathfrak{sl}_n$ . For a general semisimple Lie algebra  $\mathfrak{g}$ , in [PY12], Panyushev and Yakimova study the invariants of  $\mathfrak{b} \ltimes \mathfrak{n}^*$  under the action of their adjoint group. Finally, in [PY13, Pho20], similar considerations are studied replacing  $\mathfrak{b}$  by an arbitrary parabolic subalgebra of  $\mathfrak{g}$ .

The aim of this note is to give a new interpretation of  $I\mathfrak{b}$  in the language of Kac-Moody algebras and to completely describe the automorphism group of  $I\mathfrak{b}$ .

Before describing this group, we introduce some notation. Let  $r$  denote the rank of  $\mathfrak{g}$  and  $G$  the adjoint group with Lie algebra  $\mathfrak{g}$ . Let  $B$  be the Borel subgroup of  $G$  with  $\mathfrak{b}$  as


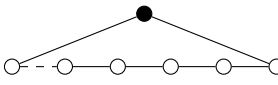
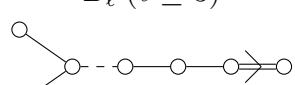

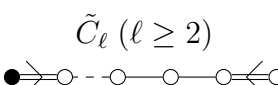
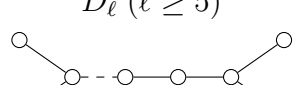
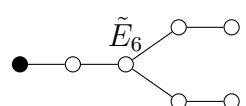
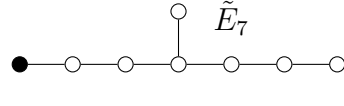
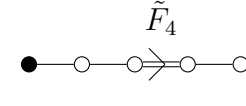
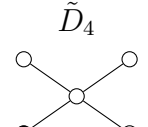
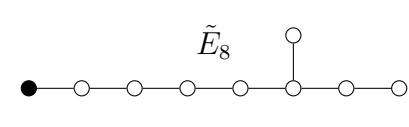
$\tilde{A}_1$  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z}$	$\tilde{A}_\ell (\ell \geq 2)$  $\text{Aut}(\tilde{\mathcal{D}}) = D_{(\ell+1)}$	$\tilde{B}_\ell (\ell \geq 3)$  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z}$
$\tilde{G}_2$  $\text{Aut}(\tilde{\mathcal{D}})$ is trivial	$\tilde{C}_\ell (\ell \geq 2)$  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z}$	$\tilde{D}_\ell (\ell \geq 5)$  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}/4\mathbb{Z}$
$\tilde{E}_6$  $\text{Aut}(\tilde{\mathcal{D}}) = \mathfrak{S}_3$	$\tilde{E}_7$  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z}$	$\tilde{F}_4$  $\text{Aut}(\tilde{\mathcal{D}})$ is trivial
$\tilde{D}_4$  $\text{Aut}(\tilde{\mathcal{D}}) = \mathfrak{S}_4$	$\tilde{E}_8$  $\text{Aut}(\tilde{\mathcal{D}})$ is trivial	

Figure 1: Extended Dynkin diagrams and their automorphisms

Lie algebra. Consider two abelian additive groups: the quotient  $\mathfrak{g}/\mathfrak{b}$  and the space  $\mathcal{M}_r(\mathbb{C})$  of square matrices.

An important ingredient is the extended Dynkin diagram of  $\mathfrak{g}$ . On Figure 1, these diagrams and their automorphisms are shortly recalled.

**Theorem 1.** *The neutral component  $\text{Aut}(I\mathfrak{b})^\circ$  of the automorphism group  $\text{Aut}(I\mathfrak{b})$  of the Lie algebra  $I\mathfrak{b}$  decomposes as*

$$\mathbb{C}^* \ltimes \left( (B \ltimes \mathfrak{g}/\mathfrak{b}) \times \mathcal{M}_r(\mathbb{C}) \right).$$

*The group of components  $\text{Aut}(I\mathfrak{b})/\text{Aut}(I\mathfrak{b})^\circ$  is isomorphic to the automorphism group of the affine Dynkin diagram of  $\mathfrak{g}$  and can be lifted as a subgroup of  $\text{Aut}(I\mathfrak{b})$ .*

The details of how these subgroups act on  $I\mathfrak{b}$  are given in Section 3. Section 4 explains how the semidirect products are formed.

One of the amazing facts is that the extended Dynkin diagram of  $\mathfrak{g}$  plays a crucial role in  $\text{Aut}(I\mathfrak{b})$ . On one hand, we explain this by constructing the extended Cartan matrix of  $\mathfrak{g}$  in terms of  $I\mathfrak{b}$  in Section 3.1. On the other hand, this diagram is the Dynkin diagram of the

untwisted affine Lie algebra constructed from the loop algebra of  $\mathfrak{g}$ . A second explanation is given by Theorem 2 that realizes  $I\mathfrak{b}$  as a subquotient of the affine Lie algebra associated to  $\mathfrak{g}$ .

In [KZJ07], Knutson and Zinn-Justin defined a degeneration  $\bullet$  of the standard associative product on  $\mathcal{M}_n(\mathbb{C})$ . Let  $\mathfrak{b}$  denote the set of upper triangular matrices. Identifying the vector space  $\mathcal{M}_n(\mathbb{C})$  with  $\mathfrak{b} \times \mathcal{M}_n(\mathbb{C})/\mathfrak{b}$  in a natural way one gets

$$(R, L) \bullet (V, M) = (RV, RM + LV),$$

for any  $R, V \in \mathfrak{b}$  and  $L, M \in \mathcal{M}_n(\mathbb{C})/\mathfrak{b}$ . The Lie algebra of the group  $(\mathcal{M}_n(\mathbb{C}), \bullet)^\times$  of invertible elements of this algebra is  $\mathfrak{b} \times \mathcal{M}_n(\mathbb{C})/\mathfrak{b}$ , where the product is defined similarly with that of  $I\mathfrak{b}$ . Note also that a cyclic automorphism (corresponding in our setting with the cyclic automorphism of the affine Dynkin diagram of type  $A_{n-1}$  and with the unexpected cyclic automorphism of [Bv20]) appears in [KZJ07]. Moreover [KZJ07, Proposition 2], which realizes  $(\mathcal{M}_n(\mathbb{C}), \bullet)$  as a subquotient of  $\mathcal{M}_n(\mathbb{C}[t])$ , is similar with our Theorem 2.

**Motivation and story of this work.** In [Bv20], the authors constructed an “unexpected” cyclic automorphism of  $I\mathfrak{b}$  when  $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ . The first version of this work was an explanation for this automorphism by using affine Lie algebras. Simultaneously with this first version, A. Knutson mentioned to Bar-Natan his earlier work [KZJ07] with Zinn-Justin.

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## 2 The Lie algebras $I\mathfrak{b}$ , $\mathfrak{g}_+^\epsilon$ and $\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$

### 2.1 Definitions of $I\mathfrak{b}$ and $\mathfrak{g}_+^\epsilon$

Let  $\mathfrak{g}$  be a complex simple Lie algebra with Lie bracket denoted by  $[\ , \ ]$ . Fix a Borel subalgebra  $\mathfrak{b}$  of  $\mathfrak{g}$  and a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{b}$ . Let  $\mathfrak{b}^-$  be the opposite Borel subalgebra of  $\mathfrak{b}$  containing  $\mathfrak{h}$ . Set  $\mathcal{V} = \mathfrak{b} \oplus \mathfrak{b}^-$  viewed as a vector space. In this section, we define the Lie bracket  $[\ , \ ]_\epsilon$  on  $\mathcal{V}$  depending on the complex parameter  $\epsilon$ , interpolating between  $I\mathfrak{b}$  and the direct product  $\mathfrak{g} \oplus \mathfrak{h}$ .

Let  $\mathfrak{n}$  and  $\mathfrak{n}^-$  denote the derived subalgebras of  $\mathfrak{b}$  and  $\mathfrak{b}^-$  respectively. Fix  $\epsilon \in \mathbb{C}$ . Define the skew-symmetric bilinear bracket  $[\ , \ ]_\epsilon$  on  $\mathcal{V}$  by

$$\begin{aligned} [x, x']_\epsilon &= [x, x'] & \forall x, x' \in \mathfrak{b} \\ [y, y']_\epsilon &= \epsilon[y, y'] & \forall y, y' \in \mathfrak{b}^- \\ [x, y]_\epsilon &= (\epsilon X + \epsilon \frac{H}{2}, \frac{H}{2} + Y) & \forall x \in \mathfrak{b} \ y \in \mathfrak{b}^- \quad \text{where } [x, y] = X + H + Y \in \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^- \end{aligned}$$

Then  $[\cdot, \cdot]_\epsilon$  satisfies Jacobi identity (see below for a proof). Endowed with this Lie bracket,  $\mathcal{V}$  is denoted by  $\mathfrak{g}_+^\epsilon$ . Assume, for a moment that  $\epsilon$  is nonzero. The invertible linear map

$$\begin{aligned} \varphi_\epsilon : \mathfrak{b} \oplus \mathfrak{b}^- &\longrightarrow \mathfrak{b} \oplus \mathfrak{b}^- \\ (x, y) &\longmapsto (x, \epsilon y) \quad \text{for any } x \in \mathfrak{b}, y \in \mathfrak{b}^- \end{aligned}$$

allows to interpret  $\mathfrak{g}_+^\epsilon$  as an Inönü-Wigner contraction [IW53] of  $\mathfrak{g}_+^1$ . Indeed, for any nonzero  $\epsilon$ , we have

$$[X, Y]_\epsilon = \varphi_\epsilon^{-1}([\varphi_\epsilon(X), \varphi_\epsilon(Y)]_1) \quad \forall X, Y \in \mathcal{V}. \quad (1)$$

We now describe  $\mathfrak{g}_+^1$ . Using the triangular decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-, \quad (2)$$

one defines the injective linear map

$$\begin{aligned} \iota_{\mathfrak{g}}^1 : \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^- &\longrightarrow \mathfrak{g}_+^1 \\ (\xi, \alpha, \zeta) &\longmapsto (\xi + \frac{\alpha}{2}, \frac{\alpha}{2} + \zeta) \end{aligned}$$

and checks that it is a Lie algebra homomorphism. Moreover, the image of

$$\begin{aligned} \iota_{\mathfrak{h}}^1 : \mathfrak{h} &\longrightarrow \mathfrak{g}_+^1 \\ \alpha &\longmapsto (-\alpha, \alpha) \end{aligned}$$

is the center of  $\mathfrak{g}_+^1$  and

$$\mathfrak{g}_+^1 = \iota_{\mathfrak{g}}^1(\mathfrak{g}) \oplus \iota_{\mathfrak{h}}^1(\mathfrak{h}). \quad (3)$$

Observe that we never used the Jacobi identity for  $[\cdot, \cdot]_1$  to prove the isomorphism (3). Hence we can deduce from it that  $[\cdot, \cdot]_1$  satisfies the Jacobi identity. Then, the expression (1) with  $\varphi_\epsilon$  of  $[\cdot, \cdot]_\epsilon$  from  $[\cdot, \cdot]_1$  implies that  $[\cdot, \cdot]_\epsilon$  satisfies the Jacobi identity for any nonzero  $\epsilon$ . Since this property is closed on the space of bilinear maps, it is satisfied by  $[\cdot, \cdot]_0$  too.

Consider now  $I\mathfrak{b}$  with its Lie bracket  $[\cdot, \cdot]_{I\mathfrak{b}}$  defined by:  $\mathfrak{b}^*$  is an abelian ideal on which  $\mathfrak{b}$  acts by the coadjoint action. Denote by  $\kappa : \mathfrak{g} \longrightarrow \mathfrak{g}^*$  the Killing form on  $\mathfrak{g}$ . Since the orthogonal of  $\mathfrak{b}$  with respect to  $\kappa$  is  $\mathfrak{n}$ ,  $\mathfrak{b}^*$  identifies with  $\mathfrak{g}/\mathfrak{n}$  as a  $\mathfrak{b}$ -module. Identify  $\mathfrak{g}/\mathfrak{n}$  with  $\mathfrak{b}^-$  in a canonical way (that is by  $y \in \mathfrak{b}^- \longmapsto y + \mathfrak{n}$ ) and denote by  $\pi : \mathfrak{g} \longrightarrow \mathfrak{b}^-$  the quotient map. Then  $I\mathfrak{b} = \mathfrak{b} \oplus \mathfrak{b}^*$  identifies with  $\mathfrak{b} \oplus \mathfrak{b}^- = \mathcal{V}$ . Let  $[\cdot, \cdot]_I$  denote the Lie bracket transported to  $\mathcal{V}$  from  $[\cdot, \cdot]_{I\mathfrak{b}}$ . Let  $x, x' \in \mathfrak{b}$  and  $y, y' \in \mathfrak{b}^-$  and decompose  $[x, y'] - [x', y]$  as  $X + H + Y$  with respect to  $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ . Then

$$[(x, y), (x', y')]_I = ([x, x'], H + Y). \quad (4)$$

We now describe  $\mathfrak{g}_+^0$ . The Lie bracket  $[\cdot, \cdot]_0$  on  $\mathcal{V} = \mathfrak{g}_+^0$  is given by

$$[(x, y), (x', y')]_0 = ([x, x'], \frac{H}{2} + Y). \quad (5)$$

Comparing (4) and (5), one gets that the following linear map  $\eta$  is a Lie algebra isomorphism:

$$\begin{aligned} \eta : \mathcal{V} = \mathfrak{b} \oplus (\mathfrak{h} \oplus \mathfrak{n}^-) &\longrightarrow \mathfrak{b} \oplus \mathfrak{b}^* = I\mathfrak{b} \\ (x, h, y) &\longmapsto (x, \kappa(2h + y, \square)). \end{aligned}$$

## 2.2 The affine Kac-Moody Lie algebra

The affine Kac-Moody Lie algebra  $\mathfrak{g}^{\text{KM}}$  is constructed from the semisimple Lie algebra  $\mathfrak{g}$ . We refer to [Kum02, Chapters I and XIII] for the basic properties of  $\mathfrak{g}^{\text{KM}}$ . Denote by  $\mathfrak{z}(\mathfrak{g}^{\text{KM}})$  the one dimensional center of  $\mathfrak{g}^{\text{KM}}$ . Consider the Borel subalgebra  $\mathfrak{b}^{\text{KM}}$  of  $\mathfrak{g}^{\text{KM}}$  and its derived subalgebra  $\mathfrak{n}^{\text{KM}}$ . By killing the semi-direct product and the central extension from the construction of  $\mathfrak{g}^{\text{KM}}$ , one gets

$$\begin{aligned}\tilde{\mathfrak{g}} &:= [\mathfrak{g}^{\text{KM}}, \mathfrak{g}^{\text{KM}}] / \mathfrak{z}(\mathfrak{g}^{\text{KM}}) \\ &\cong \mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g},\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathfrak{b}} &:= (\mathfrak{b}^{\text{KM}} \cap [\mathfrak{g}^{\text{KM}}, \mathfrak{g}^{\text{KM}}]) / \mathfrak{z}(\mathfrak{g}^{\text{KM}}) \subset \tilde{\mathfrak{g}} \\ \tilde{\mathfrak{n}} &:= (\mathfrak{n}^{\text{KM}} \cap [\mathfrak{g}^{\text{KM}}, \mathfrak{g}^{\text{KM}}]) / \mathfrak{z}(\mathfrak{g}^{\text{KM}}) = [\tilde{\mathfrak{b}}, \tilde{\mathfrak{b}}].\end{aligned}$$

Identify  $\mathfrak{g}$  with the subspace  $\mathbb{C} \otimes \mathfrak{g} \subset \tilde{\mathfrak{g}}$ . Note that  $\mathfrak{g}^{\text{KM}} / \mathfrak{z}(\mathfrak{g}^{\text{KM}}) = \tilde{\mathfrak{g}} + \mathbb{C}d$  where  $d$  acts as the derivation with respect to  $t$ .

We consider the set of (positive) roots  $\Phi^{(+)}$  (resp.  $\tilde{\Phi}^{(+)}$ ) of  $\mathfrak{g}$  (resp.  $\tilde{\mathfrak{g}}$ ) and the set of simple roots  $\Delta$  (resp.  $\tilde{\Delta}$ ) with respect to  $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$  (resp.  $\mathfrak{h} + \mathbb{C}d + \mathfrak{z}(\mathfrak{g}^{\text{KM}}) \subset \mathfrak{b}^{\text{KM}} \subset \mathfrak{g}^{\text{KM}}$ ). We recall the following classical facts:

$$\mathfrak{n}^{\text{KM}} \cong \tilde{\mathfrak{n}} = \bigoplus_{\alpha \in \tilde{\Phi}^+} \tilde{\mathfrak{g}}_{\alpha}$$

where  $\tilde{\mathfrak{g}}_{\alpha} \cong \mathfrak{g}_{\alpha}^{\text{KM}}$  is the root space associated to  $\alpha$ . Moreover,  $\tilde{\mathfrak{n}}$  is generated, as a Lie algebra by the subspaces  $(\tilde{\mathfrak{g}}_{\alpha})_{\alpha \in \tilde{\Delta}}$ . The identification of  $\Delta$  with  $\{\alpha \in \tilde{\Delta} \mid \alpha(d) = 0\}$  yields the above-described embedding  $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ . Denoting by  $\delta$  the indivisible positive imaginary root in  $\tilde{\Phi}$ , we have

$$\begin{aligned}\tilde{\Phi} &= \{n\delta + \alpha \mid \alpha \in \Delta \cup \{0\}, n \in \mathbb{Z}\} \setminus \{0\} \\ \tilde{\Delta} &= \Delta \cup \{\alpha_0 + \delta\}\end{aligned}$$

where  $-\alpha_0$  is the highest root of  $\Phi$ .

Finally, the extended Dynkin diagram can be reconstructed from the combinatorics of  $\tilde{\Delta}$  in  $\tilde{\Phi}$ . Indeed, the nodes correspond to the elements of  $\tilde{\Delta}$  and the non-diagonal entries  $a_{\alpha, \beta}$  of the generalized Cartan matrix (encoding the arrows of the diagram) are  $a_{\alpha, \beta} = -\max\{n \in \mathbb{N} \mid \beta + n\alpha \in \tilde{\Phi}\}$  by Serre relations.

We list in Figure 1 the extended Dynkin diagram  $\tilde{\mathcal{D}}_{\mathfrak{g}}$  in each simple type. The black node corresponds to the simple root  $\alpha_0 + \delta$ . We also provide the automorphism group of  $\tilde{\mathcal{D}}_{\mathfrak{g}}$ . Note that by the definition of  $\mathfrak{g}^{\text{KM}}$  given in [Kum02, §1.1], any  $\theta \in \text{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}})$  provides an automorphism  $\theta^{\text{KM}} \in \text{Aut}(\mathfrak{g}^{\text{KM}})$  stabilizing both  $\mathfrak{h} + \mathbb{C}d + \mathfrak{z}(\mathfrak{g}^{\text{KM}})$  and  $\mathfrak{b}^{\text{KM}}$  and permuting  $\tilde{\Delta}$ <sup>1</sup> as  $\theta$  does. Since  $\mathfrak{z}(\mathfrak{g}^{\text{KM}})$  and  $[\mathfrak{g}^{\text{KM}}, \mathfrak{g}^{\text{KM}}]$  are characteristic in  $\mathfrak{g}^{\text{KM}}$ , this yields an automorphism  $\tilde{\theta} \in \text{Aut}(\tilde{\mathfrak{g}})$ .

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<sup>1</sup>and even permuting the set of generators  $e_{\alpha}$ ,  $\alpha \in \tilde{\Delta}$

## 2.3 Realisation of $\mathfrak{g}_+^\epsilon$

The Lie algebras  $\tilde{\mathfrak{b}}$  and  $\tilde{\mathfrak{n}}$  decompose as

$$\begin{aligned}\tilde{\mathfrak{b}} &= \mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{n}^-, \\ \tilde{\mathfrak{n}} &= \mathbb{C}[t]\mathfrak{n} \oplus t\mathbb{C}[t]\mathfrak{b}^-.\end{aligned}$$

Moreover,  $(t - \epsilon)\tilde{\mathfrak{n}}$  is an ideal of  $\tilde{\mathfrak{b}}$ , and  $\tilde{\mathfrak{b}}/((t - \epsilon)\tilde{\mathfrak{n}})$  is a Lie algebra.

**Theorem 2.** *Let  $\epsilon \in \mathbb{C}$ . The Lie algebras  $\mathfrak{g}_+^\epsilon$  and  $\tilde{\mathfrak{b}}/(t - \epsilon)\tilde{\mathfrak{n}}$  are isomorphic.*

*Proof.* From Section 2.1, we have  $\mathfrak{g}_+^1 \stackrel{v.s.}{=} \mathfrak{b} \oplus \mathfrak{b}^-$ . Elements of  $\mathfrak{g}_+^1$  will be written as couples with respect to this decomposition.

Set  $\widetilde{\mathfrak{g}}_+^1 := \mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g}_+^1$  and extend  $\iota_{\mathfrak{g}}^1$  to an injective  $\mathbb{C}[t^{\pm 1}]$ -linear map  $\widetilde{\mathfrak{g}} \rightarrow \widetilde{\mathfrak{g}}_+^1$ . Consider the subspace  $\mathfrak{w} := \mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{b}^-$  that is a Lie subalgebra of  $\widetilde{\mathfrak{g}}_+^1$ . The Inönü-Wigner contraction on  $\mathfrak{g}_+^1$  with respect to the decomposition  $\mathfrak{b} \oplus \mathfrak{b}^-$  gives rise to  $\mathfrak{g}_+^\epsilon$  ( $\epsilon \in \mathbb{C}$ ). In particular, the linear map

$$\begin{aligned}\mathfrak{g}_+^\epsilon &\longrightarrow \mathfrak{w}/(t - \epsilon)\mathfrak{w} \\ (x, y) &\longmapsto x + ty + (t - \epsilon)\mathfrak{w} \quad \text{for any } x \in \mathfrak{b} \text{ and } y \in \mathfrak{b}^-.\end{aligned}\tag{6}$$

is a Lie algebra isomorphism

Set  $\mathfrak{b}_0^- := \iota_{\mathfrak{g}}^1(\mathfrak{b}^-) = \{(h, h) | h \in \mathfrak{h}\} \oplus \mathfrak{n}^-$ . Observe that  $t\mathfrak{b}_0^-$  is contained in  $\mathfrak{w}$ . Indeed, for any  $h \in \mathfrak{h}$ , the element  $t(h, h) = t(h, 0) + t(0, h)$  belongs to  $\mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{b}^-$ . In particular, one gets a linear map induced by the inclusions of  $\mathfrak{b}$  and  $t\mathfrak{b}_0^-$  in  $\mathfrak{w}$ :

$$\mathfrak{b} \oplus t\mathfrak{b}_0^- \longrightarrow \mathfrak{w}.$$

One can easily check that it induces a linear isomorphism  $\mathfrak{b} \oplus t\mathfrak{b}_0^- \longrightarrow \mathfrak{w}/(t - \epsilon)\mathfrak{w}$ . Setting  $\tilde{\mathfrak{b}}_{\mathfrak{w}} := \langle \mathfrak{b} \oplus t\mathfrak{b}_0^- \rangle_{Lie} \subset \mathfrak{w}$ , we thus get a Lie algebra isomorphism.

$$\tilde{\mathfrak{b}}_{\mathfrak{w}}/((t - \epsilon)\mathfrak{w} \cap \tilde{\mathfrak{b}}_{\mathfrak{w}}) \longrightarrow \mathfrak{w}/(t - \epsilon)\mathfrak{w}.\tag{7}$$

Since,  $\mathfrak{b} = \{(h, 0) | h \in \mathfrak{h}\} \oplus \iota_{\mathfrak{g}}^1(\mathfrak{n})$  and  $\langle \iota_{\mathfrak{g}}^1(\mathfrak{n}) \oplus \iota_{\mathfrak{g}}^1(t\mathfrak{b}^-) \rangle_{Lie} = \iota_{\mathfrak{g}}^1(\langle \mathfrak{n} \oplus t\mathfrak{b}^- \rangle_{Lie}) = \iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$ , we have

$$\tilde{\mathfrak{b}}_{\mathfrak{w}} = \{(h, 0) | h \in \mathfrak{h}\} \oplus \iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}}) \cong \iota_{\mathfrak{g}}^1(\tilde{\mathfrak{b}}) \cong \tilde{\mathfrak{b}},\tag{8}$$

the middle Lie isomorphism being the identity on  $\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$  and sending  $(h, 0)$  to  $\frac{1}{2}(h, h)$  for each  $h \in \mathfrak{h}$ . Moreover,  $(t - \epsilon)\mathfrak{w} \cap \tilde{\mathfrak{b}}_{\mathfrak{w}} = (t - \epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$ . Indeed,  $(t - \epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$  is contained in  $(t - \epsilon)\mathfrak{w} \cap \tilde{\mathfrak{b}}_{\mathfrak{w}}$ , and  $\mathfrak{b} \oplus t\mathfrak{b}_0^-$  is complementary to  $(t - \epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$  in  $\tilde{\mathfrak{b}}_{\mathfrak{w}}$ .

We finally get the desired Lie isomorphism

$$\tilde{\mathfrak{b}}/(t - \epsilon)\tilde{\mathfrak{n}} \stackrel{(8)}{\cong} \tilde{\mathfrak{b}}_{\mathfrak{w}}/((t - \epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})) \stackrel{(7)}{\cong} \mathfrak{w}/(t - \epsilon)\mathfrak{w} \stackrel{(6)}{\cong} \mathfrak{g}_+^\epsilon$$

□

In addition, we can make explicit the isomorphism of Theorem 2:

$$\begin{aligned} \gamma_\epsilon : \quad \mathfrak{g}_+^\epsilon &\xrightarrow{\cong} \tilde{\mathfrak{b}}/(t-\epsilon)\tilde{\mathfrak{n}} \\ x &\mapsto x && \text{if } x \in \mathfrak{n} \\ y &\mapsto ty && \text{if } y \in \mathfrak{n}^- \\ (a, b) &\mapsto (a - \epsilon b) + 2tb && \text{if } a, b \in \mathfrak{h} \end{aligned}$$

and its inverse map is induced by

$$\begin{aligned} \theta : \quad \tilde{\mathfrak{b}} &\longrightarrow \mathcal{V} \\ Px &\mapsto P(\epsilon)x && \text{if } x \in \mathfrak{n} \\ tRy &\mapsto R(\epsilon)y && \text{if } y \in \mathfrak{n}^- \\ Qh &\mapsto \left( \frac{Q(\epsilon)+Q(0)}{2}h, \frac{Q(\epsilon)-Q(0)}{2\epsilon}h \right) && \text{if } h \in \mathfrak{h} (\epsilon \neq 0) \\ &&& (Q(0)h, \frac{1}{2}Q'(0)h) && \text{if } h \in \mathfrak{h} (\epsilon = 0) \end{aligned}$$

Note that, in order to prove Theorem 2, we could alternatively have checked directly that  $\theta$  is a Lie algebra homomorphism from  $\tilde{\mathfrak{b}}$  onto  $\mathfrak{g}_+^\epsilon$  with Kernel  $(t-\epsilon)\tilde{\mathfrak{n}}$ .

### 3 Some subgroups of $\text{Aut}(I\mathfrak{b})$

#### 3.1 The roots of $I\mathfrak{b}$

From Sections 2.1 and 2.3, we can interpret the algebra  $I\mathfrak{b}$  in the Kac-Moody world via the isomorphism

$$\begin{aligned} I\mathfrak{b} &\longrightarrow \tilde{\mathfrak{b}}/t\tilde{\mathfrak{n}} \\ (x, y) &\mapsto x + ty \quad \left( \begin{array}{l} x \in \mathfrak{b}, \\ y \in \mathfrak{b}^- \cong \mathfrak{g}/\mathfrak{n} \cong \mathfrak{b}^* \end{array} \right) \end{aligned}$$

From now on, this identification will be made systematically. In particular, we write  $I\mathfrak{b} = \mathfrak{b} \oplus t\mathfrak{b}^-$ . We first describe some basic properties of  $I\mathfrak{b}$  in this language.

**Lemma 3.** *1. The subalgebra  $\mathfrak{c} := \mathfrak{h} \oplus t\mathfrak{h}$  is a Cartan subalgebra of  $I\mathfrak{b}$ . Namely,  $\mathfrak{c}$  is abelian and equal to its normalizer.*

*2. Under the action of  $\mathfrak{c}$ ,  $I\mathfrak{b}$  decomposes as*

$$I\mathfrak{b} = \mathfrak{c} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^-} t\mathfrak{g}_\alpha.$$

*For  $\alpha \in \Phi^+$ ,  $\mathfrak{c}$  acts on  $\mathfrak{g}_\alpha$  with the weight  $(\alpha, 0) \in \mathfrak{h}^* \times t\mathfrak{h}^*$ . It acts on  $t\mathfrak{g}_\alpha$  with the weight  $(\alpha, 0) \in \mathfrak{h}^* \times t\mathfrak{h}^*$ , if  $\alpha \in \Phi^-$ . Here, we identified  $\mathfrak{c}^*$  with  $\mathfrak{h}^* \times t\mathfrak{h}^*$  in a natural way.*

*3. The set of ad-nilpotent elements of  $I\mathfrak{b}$  is  $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}} = \mathfrak{n} \oplus t\mathfrak{b}^-$ .*

*4. The centre of  $I\mathfrak{b}$  is  $\mathfrak{z}(I\mathfrak{b}) = t\mathfrak{h}$ .*



5. The derived subalgebra of  $I\mathfrak{b}$  is  $[I\mathfrak{b}; I\mathfrak{b}] = \tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ .

*Proof.* 1-2) The fact that  $\mathfrak{c}$  is abelian is clear from the definition of  $\tilde{\mathfrak{g}}$ . The decomposition in eigenspaces also. The action of  $t\mathfrak{h}$  is zero since it sends  $\tilde{\mathfrak{n}}$  to  $t\tilde{\mathfrak{n}}$  that vanishes itself in  $I\mathfrak{b}$ . The decomposition of  $I\mathfrak{b}$  in weight spaces under the action of  $\mathfrak{c}$  follows. Then this decomposition also implies that  $N_{I\mathfrak{b}}(\mathfrak{c}) = \mathfrak{c}$ .

3) The elements of  $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$  are clearly ad-nilpotent. From 2), an element with non-zero component in  $\mathfrak{h}$  is not ad-nilpotent.

4) Since it acts as 0 on  $\tilde{\mathfrak{n}}$  and on  $\mathfrak{h}$ , we have  $t\mathfrak{h} \subset \mathfrak{z}(I\mathfrak{b})$ . The decomposition in weight spaces implies the converse inclusion.

5) The inclusion  $[I\mathfrak{b}, I\mathfrak{b}] \subset \tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$  is clear. On the other hand we deduce from the weight space decomposition that the subspaces  $(\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \tilde{\Delta}}$  belong to  $[I\mathfrak{b}, I\mathfrak{b}]$ . Since they generate  $\tilde{\mathfrak{n}}$  in  $\tilde{\mathfrak{g}}$ , the result follows.  $\square$

From Lemma 3 (2), the set of nonzero weights  $\Phi(I\mathfrak{b})$  of  $\mathfrak{c}$  acting on  $I\mathfrak{b}$  identifies with  $\tilde{\Phi}$ . It is also useful to embed  $\Phi(I\mathfrak{b})$  in  $\tilde{\Phi}$  by

$$\begin{aligned} \varphi : \quad \Phi(I\mathfrak{b}) &\longrightarrow \tilde{\Phi} \\ \alpha \in \Phi^+ &\longmapsto \alpha \\ \alpha \in \Phi^- &\longmapsto \delta + \alpha \end{aligned}$$

Indeed, the weight space  $(I\mathfrak{b})_\alpha$  identifies with  $\tilde{\mathfrak{g}}_{\varphi(\alpha)}$ , for any  $\alpha \in \Phi(I\mathfrak{b})$ . In particular, for  $\alpha, \beta \in \tilde{\Phi} \cup \{0\}$ , we have  $[I\mathfrak{b}_{\varphi^{-1}(\alpha)}, I\mathfrak{b}_{\varphi^{-1}(\beta)}] \subset I\mathfrak{b}_{\varphi^{-1}(\alpha+\beta)}$  with equality when  $\alpha, \beta, \alpha + \beta \notin \{0, \delta\}$ . Set also  $\Delta(I\mathfrak{b}) = \varphi^{-1}(\tilde{\Delta}) = \Delta \cup \{-\alpha_0\}$ .

**Lemma 4.** 1. The derived subalgebra of  $I\mathfrak{b}^{(1)} := [I\mathfrak{b}, I\mathfrak{b}]$  is

$$I\mathfrak{b}^{(2)} = \bigoplus_{\alpha \in \Phi(I\mathfrak{b}) \setminus \Delta(I\mathfrak{b})} (I\mathfrak{b})_\alpha$$

2. Assume that  $\mathfrak{g}$  is not  $\mathfrak{sl}_2$ . For  $\alpha, \beta \in \Delta(I\mathfrak{b})$  ( $\alpha \neq \beta$ ), the corresponding entry of the generalized Cartan Matrix of  $\mathfrak{g}^{\text{KM}}$  is given by

$$a_{\alpha, \beta} = -\max\{n \in \mathbb{N} \mid \beta + n\alpha \in \Phi(I\mathfrak{b})\}.$$

*Proof.* 1) Recall that  $\tilde{\mathfrak{n}}$  is generated as a Lie algebra by the  $(\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \tilde{\Delta}}$ . Thus, for weight reasons, the  $(\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \tilde{\Phi} \setminus \tilde{\Delta}}$  are root spaces included in  $[\tilde{\mathfrak{n}}, \tilde{\mathfrak{n}}]$ . Since  $\tilde{\Delta}$  is a linearly independent family, they are in fact the only root spaces not contained in  $[\tilde{\mathfrak{n}}, \tilde{\mathfrak{n}}]$ . Quotienting, this yields  $\bigoplus_{\alpha \in \Phi(I\mathfrak{b}) \setminus \Delta(I\mathfrak{b})} (I\mathfrak{b})_\alpha = I\mathfrak{b}^{(2)}$ .

2) Recall that the statement is valid if we replace  $\Phi(I\mathfrak{b})$  by  $\tilde{\Phi}$ , see Section 2.2. It is thus sufficient to show that

$$\beta + n\alpha \in \tilde{\Phi} \Rightarrow \beta + n\alpha \in \Phi(I\mathfrak{b}).$$

When  $\alpha, \beta \in \Delta$ , the statement is clear since  $\Phi^+ \subset \Phi(I\mathfrak{b})$ .

If  $\beta = \delta + \alpha_0$ , then  $\beta + n\alpha \in \tilde{\Phi}$  means that  $\alpha_0 + n\alpha \in \Phi$ . Since  $\alpha_0 + n\alpha$  has elements of  $-\Delta$

in its support, it has to lie in  $\Phi^-$ . Thus  $\beta + n\alpha \in \Phi(I\mathfrak{b})$ .

If  $\alpha = \delta + \alpha_0$ , then  $\beta + n\alpha \in \tilde{\Phi}$  means that  $\beta + n\alpha_0 \in \Phi$ . For height reasons, we must have  $n \in \{0, 1\}$ . Then,  $\beta + n\alpha \in \Phi(I\mathfrak{b})$ .  $\square$

**Remark.** One can observe that the first assertion of Lemma 4 is similar with

$$[\mathfrak{n}, \mathfrak{n}] = \bigoplus_{\alpha \in \Phi^+ \setminus \Delta} \mathfrak{b}_\alpha.$$

### 3.2 The adjoint subgroup of $\text{Aut}(I\mathfrak{b})$

Let  $G$  be the adjoint group with Lie algebra  $\mathfrak{g}$ . Let  $T$  and  $B$  be the subgroups of  $G$  with Lie algebras  $\mathfrak{h}$  and  $\mathfrak{b}$ . Consider now  $\mathfrak{b}^- \cong \mathfrak{g}/\mathfrak{n}$  equipped with the addition as an abelian algebraic group. The adjoint action of  $B$  on  $\mathfrak{g}$  stabilizes  $\mathfrak{n}$  and induces a linear action on  $\mathfrak{b}^- \cong \mathfrak{g}/\mathfrak{n}$  by group isomorphisms. We can perform the semidirect product:

$$IB := B \ltimes \mathfrak{b}^-.$$

By construction the Lie algebra of  $IB$  identifies with  $I\mathfrak{b}$ . The adjoint action of  $IB$  on  $I\mathfrak{b}$  is given by

$$\begin{aligned} IB \times I\mathfrak{b} &\longrightarrow I\mathfrak{b} \\ ((b, f), x + ty) &\longmapsto b \cdot x + t(b \cdot y + [f, x]) \quad \text{for } b \in B, f, x \in \mathfrak{b} \text{ and } y \in \mathfrak{b}^-, \end{aligned} \quad (9)$$

where  $\cdot$  denotes the  $B$  action on  $\mathfrak{b}$  and on  $\mathfrak{b}^-$ . It induces a morphism

$$\text{Ad} : IB \longrightarrow \text{Aut}(I\mathfrak{b})$$

with Kernel  $Z(IB) \cong (1, \mathfrak{h})$ . In particular, one gets:

**Lemma 5.** *The image  $\text{Ad}(IB)$  is isomorphic with  $B \ltimes \mathfrak{g}/\mathfrak{b}$ .*

Note also that  $\text{Ad}(IB) = H \ltimes (N \ltimes \mathfrak{g}/\mathfrak{b})$  where  $N$  and  $H$  are the connected subgroups of  $B$  with respective Lie algebras  $\mathfrak{n}$  and  $\mathfrak{h}$ . Since  $\mathfrak{n} + t\mathfrak{b}^-$  is the set of ad-nilpotent elements of  $I\mathfrak{b}$ , we get the following result from (9).

**Lemma 6.** *1. The group of elementary automorphisms  $\text{Aut}_e(I\mathfrak{b}) = \exp(\mathfrak{n} + t\mathfrak{b}^-)$  coincides with  $N \ltimes \mathfrak{g}/\mathfrak{b}$ .*

*2.  $\text{Ad}(IB) = \exp(I\mathfrak{b})$*

### 3.3 An unipotent subgroup of $\text{Aut}(I\mathfrak{b})$

**Lemma 7.** *The following map is an injective group homomorphism*

$$\begin{aligned} \text{Hom}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}], \mathfrak{z}(I\mathfrak{b})) &\longrightarrow \text{Aut}(I\mathfrak{b}) \\ u &\longmapsto \left( \begin{array}{ccc} \bar{u} : I\mathfrak{b} &\longrightarrow & I\mathfrak{b} \\ X &\longmapsto & X + u(X) \end{array} \right). \end{aligned}$$

We denote by  $U \subset \text{Aut}(I\mathfrak{b})$  the image of this map. From Lemma 3, we have  $U \cong \mathcal{M}_r(\mathbb{C})$  where  $r = \dim \mathfrak{h}$ .

*Proof.* Let  $X, Y \in I\mathfrak{b}$ . On one hand, we have

$$[\bar{u}(X), \bar{u}(Y)] = [X + u(X), Y + u(Y)] = [X, Y],$$

since the image of  $u$  is contained in the center. On the other hand,

$$\bar{u}([X, Y]) = [X, Y],$$

since  $u$  vanishes on the derived subalgebra. It follows that  $\bar{u}$  is a Lie algebra isomorphism.

Since  $\mathfrak{z}$  is contained in  $[I\mathfrak{b}, I\mathfrak{b}]$  the map of the lemma is a group homomorphism from  $(\text{Hom}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}], \mathfrak{z}), +)$  to  $(\text{Aut}(I\mathfrak{b}), \circ)$ .  $\square$

Note that the existence of this group of automorphisms is quite general. Indeed, the only useful property of  $I\mathfrak{b}$  in this proof is  $\mathfrak{z}(I\mathfrak{b}) \cap [I\mathfrak{b}, I\mathfrak{b}] = \{0\}$ .

### 3.4 The loop subgroup

**Lemma 8.** *The following map is an injective group homomorphism*

$$\begin{aligned} \mathbb{C}^* &\longrightarrow \text{Aut}(I\mathfrak{b}) \\ \tau &\longmapsto \left( \begin{array}{ccc} \delta_\tau : I\mathfrak{b} &\longrightarrow & I\mathfrak{b} \\ x &\longmapsto & x \quad \text{if } x \in \mathfrak{b} \\ ty &\longmapsto & \tau ty \quad \text{if } y \in \mathfrak{b}^- \end{array} \right). \end{aligned}$$

We denote by  $D \subset \text{Aut}(I\mathfrak{b})$  the image of this map

*Proof.* It is a straightforward check on  $\mathfrak{b} \ltimes t\mathfrak{b}^-$  that the  $\delta_\tau$  are automorphisms of  $I\mathfrak{b}$ .  $\square$

**Remark.** The map  $\delta_\tau$  corresponds with the variable changing  $t \mapsto \tau t$  in the  $\mathbb{C}[t]$ -Lie algebra  $\tilde{\mathfrak{b}}/t\tilde{\mathfrak{n}}$ . Moreover, the Lie algebra of  $D$  acts on  $I\mathfrak{b}$  like  $\mathbb{C}d$  where  $d$  is the derivation involved in the definition of  $\mathfrak{g}^{KM}$ .

### 3.5 Automorphisms stabilizing the Cartan subalgebra

For any  $\alpha \in \Delta(I\mathfrak{b})$ , fix a nonzero element  $X_\alpha$  in the corresponding root space  $(I\mathfrak{b})_\alpha$ . Set

$$G := \left\{ \theta \in \text{Aut}(I\mathfrak{b}) \mid \begin{array}{l} \theta(\mathfrak{h}) \subset \mathfrak{h} \\ \theta(\{X_\alpha : \alpha \in \Delta(I\mathfrak{b})\}) = \{X_\alpha : \alpha \in \Delta(I\mathfrak{b})\} \end{array} \right\}.$$

Note that, since  $\mathfrak{c}$  is the sum of  $\mathfrak{h}$  with  $\mathfrak{z}(I\mathfrak{b})$  and since the center is characteristic, the elements of  $G$  also stabilize  $\mathfrak{c}$ .

**Proposition 9.** *The group  $G$  is isomorphic to the automorphism group of the affine Dynkin diagram of  $\mathfrak{g}$ .*

*Proof.* The group  $G$  stabilizes  $\mathfrak{c}$  and hence the set of weights of  $\mathfrak{c}$  acting on  $I\mathfrak{b}$ . This yields an action of  $G$  on  $\Phi(I\mathfrak{b})$ .

Moreover,  $I\mathfrak{b}^{(1)} = [I\mathfrak{b}, I\mathfrak{b}]$  and  $I\mathfrak{b}^{(2)} = [I\mathfrak{b}^{(1)}, I\mathfrak{b}^{(1)}]$  are characteristic and stabilized by  $G$ . Now, Lemma 4 implies that  $G$  stabilizes  $\Phi(I\mathfrak{b}) \setminus \Delta(I\mathfrak{b})$  and hence  $\Delta(I\mathfrak{b})$ . Moreover, by Lemma 4 (2), we have for  $g \in G$  and  $\alpha, \beta \in \Delta(I\mathfrak{b})$ :

$$\begin{aligned} a_{\alpha, \beta} &= -\max\{n \mid (\text{ad } X_\alpha)^n(X_\beta) \neq 0\} \\ &= -\max\{n \mid g((\text{ad } X_\alpha)^n(X_\beta)) \neq 0\} \\ &= -\max\{n \mid (\text{ad } X_{g(\alpha)})^n(X_{g(\beta)}) \neq 0\} = a_{g(\alpha), g(\beta)}. \end{aligned}$$

Hence  $g$  actually induces an automorphism of the extended Dynkin diagram<sup>2</sup> and we thus obtain a group homomorphism

$$\Theta : G \rightarrow \text{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}}).$$

We claim that  $\Theta$  is surjective. Indeed, fix a group automorphism  $\theta$  of  $\tilde{\mathcal{D}}_{\mathfrak{g}}$ . As seen in Section 2.2, there exists  $\tilde{\theta} \in \text{Aut}(\tilde{\mathfrak{g}})$  which stabilizes both  $\mathfrak{h}$  and  $\tilde{\mathfrak{b}}$  and which permutes  $\Delta(I\mathfrak{b})$  as  $\theta$  does. Then it stabilizes  $\tilde{\mathfrak{n}} = [\tilde{\mathfrak{b}}, \tilde{\mathfrak{b}}]$  and thus induces the desired element of  $\text{Aut}(\tilde{\mathfrak{b}}/\tilde{\mathfrak{t}}\tilde{\mathfrak{n}})$ .

We now prove that  $\Theta$  is injective. Let  $\theta$  in its Kernel. By the definition of the group  $G$ ,  $\theta$  stabilizes  $\mathfrak{h}$ . Since the restrictions of the elements of  $\Delta(I\mathfrak{b})$  span  $\mathfrak{h}^*$ , the restriction of  $\theta$  to  $\mathfrak{h}$  has to be the identity. In particular,  $\theta$  acts trivially on  $\Phi(I\mathfrak{b})$  and stabilizes each root space  $(I\mathfrak{b})_\alpha$  for  $\alpha \in \Phi(I\mathfrak{b})$ . But  $\theta$  stabilizes the set  $\{X_\alpha : \alpha \in \Delta(I\mathfrak{b})\}$ . Hence  $\theta$  acts trivially on each  $\tilde{\mathfrak{g}}_\alpha$  for  $\alpha \in \Delta(I\mathfrak{b})$ . Since  $\tilde{\mathfrak{n}}$  is generated by the  $(\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \Delta(I\mathfrak{b})}$ , the restriction of  $\theta$  to  $\tilde{\mathfrak{n}}/\tilde{\mathfrak{t}}\tilde{\mathfrak{n}}$  is the identity map. Finally,  $\theta$  is trivial and  $\Theta$  is injective.  $\square$

**Remark.** [Bv20, Theorem 2] is the construction of an explicit order  $n$  automorphism of  $\mathfrak{gl}_{n+}^\epsilon$ . Theorem 2 and the above proof of the surjectivity of  $\Theta$  also show the existence of such an automorphism for  $\mathfrak{sl}_{n+}^\epsilon$  with nonzero  $\epsilon$ . Hence we just got both an explanation and an extension (to any simple  $\mathfrak{g}$ ) of the Bar-Natan-van der Veen's Theorem 2.

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<sup>2</sup>If  $\mathfrak{g}$  is  $\mathfrak{sl}_2$ , Lemma 4 (2) does not apply. However, any permutation of  $\tilde{\Delta}$  is an automorphism of the extended Dynkin diagram in this case.

## 4 Description of $\text{Aut}(I\mathfrak{b})$

In this section, we describe the structure of

$$\text{Aut}(I\mathfrak{b}) = \{g \in \text{GL}(I\mathfrak{b}) : \forall X, Y \in I\mathfrak{b} \quad g([X, Y]) = [g(X), g(Y)]\}$$

in terms of the subgroups  $U$ ,  $\text{Ad}(IB)$ ,  $D$  and  $G$  introduced in Section 3.

Observe that  $\text{Aut}(I\mathfrak{b})$  is a Zariski closed subgroup of the linear group  $\text{GL}(I\mathfrak{b})$ .

**Theorem 10.** *We have the following decomposition*

$$\text{Aut}(I\mathfrak{b}) = G \ltimes (D \ltimes (\text{Ad}(IB) \times U)).$$

In particular, the neutral component is  $\text{Aut}(I\mathfrak{b})^\circ = D \ltimes (\text{Ad}(IB) \times U)$  and  $G \cong \text{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}})$  can be seen as the component group of  $\text{Aut}(I\mathfrak{b})$ .

The result is a consequence of the lemmas provided below. Indeed, by Lemma 12, the four subgroups generate  $\text{Aut}(I\mathfrak{b})$ . By Lemma 11, the subgroup generated by  $U$  and  $\text{Ad}(IB)$  is a direct product  $U \times \text{Ad}(IB)$ . Then the structure of  $\text{Aut}(I\mathfrak{b})$  follows from lemma 13.

Since  $D$ ,  $\text{Ad}(IB)$  and  $U$  are connected and  $G$  is discrete,  $\text{Aut}(I\mathfrak{b}) = \bigsqcup_{g \in G} gD\text{Ad}(IB)U$  is a finite disjoint union of irreducible subsets of the same dimension. They are thus the irreducible components of  $\text{Aut}(I\mathfrak{b})$  and the remaining statements of Theorem 10 follow.

**Lemma 11.** *The subgroups  $U$  and  $\text{Ad}(IB)$  are normal in  $\text{Aut}(I\mathfrak{b})$ . Moreover,  $U \cap \text{Ad}(IB) = \{\text{Id}\}$ .*

*Proof.* Recall that  $\text{Ad}(IB)$  is generated by the exponentials of  $\text{ad}(x)$  with  $x \in I\mathfrak{b}$ . Then for any  $\theta \in \text{Aut}(I\mathfrak{b})$ ,

$$\theta \text{Ad}(IB) \theta^{-1} = \theta \exp(I\mathfrak{b}) \theta^{-1} = \exp(\theta(I\mathfrak{b})) = \exp(I\mathfrak{b}) = \text{Ad}(IB).$$

Take now  $u \in \text{Hom}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}], \mathfrak{z}(I\mathfrak{b}))$  so that  $\bar{u} \in U$ . Since  $[I\mathfrak{b}, I\mathfrak{b}]$  (resp.  $\mathfrak{z}(I\mathfrak{b})$ ) is a characteristic subspace of  $I\mathfrak{b}$ , it is stabilized by  $\theta^{-1}$  (resp.  $\theta$ ). Hence  $\theta u \theta^{-1}$  vanishes on  $[I\mathfrak{b}, I\mathfrak{b}]$  and takes values in  $\mathfrak{z}$ . So  $\theta \bar{u} \theta^{-1}$  is an element of  $U$ .

Let  $(b, g) \in IB$  and  $h \in \mathfrak{h}$ . Then  $\text{Ad}(b, g)(h) = b \cdot h + t[g, h] \subset h + (\mathfrak{n} \oplus t\mathfrak{n}^-)$ . As a consequence, whenever  $\text{Ad}(b, g)(\mathfrak{h}) \subset \mathfrak{h} + \mathfrak{z}$  we have  $\text{Ad}(b, g)(\mathfrak{h}) \subset \mathfrak{h}$ . Hence  $\text{Ad}(IB) \cap U = \{\text{Id}\}$ .  $\square$

**Lemma 12.** *We have  $\text{Aut}(I\mathfrak{b}) = GD\text{Ad}(IB)U$ .*

*Proof.* Let  $\theta \in \text{Aut}(I\mathfrak{b})$ . Since the two Cartan subalgebras  $\mathfrak{c}$  and  $\theta(\mathfrak{c})$  are Ad-conjugate (see [Bou75, §3, n° 3, th. 2]), there exists  $\theta_1 \in \text{Ad}(IB)\theta$  which stabilizes  $\mathfrak{c}$ .

Then  $\theta_1(\mathfrak{h})$  is complementary to the center  $t\mathfrak{h} = \theta_1(t\mathfrak{h})$  in  $\mathfrak{c}$ . Thus, there exists  $\theta_2 \in U\theta_1$  such that  $\theta_2$  stabilizes  $\mathfrak{h}$ .

Since  $\theta_2$  stabilizes  $\mathfrak{c}$ , it acts on  $\Phi(I\mathfrak{b})$ . Arguing as in the proof of Proposition 9, we show that it stabilizes  $\Delta(I\mathfrak{b})$  and that the induced permutation is actually an automorphism of

the extended Dynkin diagram. Thus there exists  $\theta_3 \in G\theta_2$  with the additional property that the induced permutation on  $\Delta(I\mathfrak{b})$  and thus on  $\Phi(I\mathfrak{b})$  are trivial. Then  $\theta_3$  acts on each  $(I\mathfrak{b})_\alpha$  for  $\alpha \in \Delta(I\mathfrak{b})$ .

Since  $\Delta$  is a basis of  $\mathfrak{h}^*$ , one can find  $h \in H \subset B \subset IB$  such that  $\text{Ad}(h) \circ \theta_3$  acts trivially on each  $(I\mathfrak{b})_\alpha$  for  $\alpha \in \Delta$ . Moreover,  $D$  acts trivially on these roots spaces and with weight 1 on  $(I\mathfrak{b})_{\alpha_0}$ . This yields  $\theta_4 \in D\text{Ad}(H)G\text{UAd}(IB)\theta$  which acts trivially on  $\mathfrak{h}$  and on each  $(I\mathfrak{b})_\alpha$ ,  $\alpha \in \Delta(I\mathfrak{b})$ .

Recall now that  $\tilde{\mathfrak{n}}/\tilde{\mathfrak{t}}\tilde{\mathfrak{n}}$  is generated by the spaces  $((I\mathfrak{b})_\alpha)_{\alpha \in \Delta(I\mathfrak{b})}$ . Since  $\theta_4$  acts trivially on  $\tilde{\mathfrak{n}}$  and on  $\mathfrak{h}$ , it has to be trivial. As a consequence,  $\theta \in \text{Ad}(IB)U\text{GAd}(H)D = G\text{DAd}(IB)U$ , the last equality following from Lemma 11.  $\square$

**Lemma 13.** *The intersections  $D \cap (\text{Ad}(IB) \times U)$  and  $G \cap (D \times (\text{Ad}(IB) \times U))$  are the trivial group  $\{\text{Id}\}$ . Moreover,  $(D \times (\text{Ad}(IB) \times U))$  is normal in  $\text{Aut}(I\mathfrak{b})$ .*

*Proof.* Let  $\tau \in \mathbb{C}^*$ ,  $b \in B$ ,  $f \in \mathfrak{g}/\mathfrak{n}$  and  $u \in \text{Hom}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}], \mathfrak{z}(I\mathfrak{b}))$  such that the elements associated  $\delta_\tau \in D$ ,  $(b, f) \in IB$  and  $\tilde{u} \in U$  (see Section 3) satisfy  $\delta_\tau = \text{Ad}(b, f) \circ \tilde{u}$ . For  $b' \in \mathfrak{b}$ , we have

$$b' = \delta_\tau(b') = (\text{Ad}(b, f) \circ \tilde{u})(b') = \text{Ad}(b, f)(b' + u(b')) = b \cdot b' + (b \cdot u(b') + t([f, b'])).$$

In particular,  $b \cdot b' = b'$  and, whenever  $b' \in \mathfrak{n}$ ,  $[f, b'] = 0$  in  $\mathfrak{g}/\mathfrak{n}$ . So  $b \in B$  centralizes  $\mathfrak{b}$  and  $\text{ad}_{\mathfrak{g}} f$  normalizes  $\mathfrak{n}$ . As a consequence,  $b = 1_B$ ,  $f$  is 0 in  $\mathfrak{g}/\mathfrak{b}$  and  $u = 0$ . Thus the only element of  $D \cap (\text{Ad}(IB) \times U)$  is the trivial one.

Since  $[I\mathfrak{b}, I\mathfrak{b}]$  is characteristic in  $I\mathfrak{b}$ , we have a natural group morphism  $p : \text{Aut}(I\mathfrak{b}) \rightarrow \text{Aut}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}])$ . From the description of  $[I\mathfrak{b}, I\mathfrak{b}]$  in Lemma 3, it is straightforward that  $D$ ,  $\text{Ad}(IB)$  and  $U$  are included in  $\text{Ker}(p)$  while  $p|_G$  is injective. From Lemma 12, we then deduce that  $D \times (\text{Ad}(IB) \times U) = \text{Ker}(p)$  and the desired properties follow.  $\square$

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