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On the automorphisms of the Drinfel'd double of a Borel Lie subalgebra

Michaël Bulois and Nicolas Ressayre

June 22, 2020

Abstract

Let \mathfrak{g} be a complex simple Lie algebra with Borel subalgebra \mathfrak{b} . Consider the semidirect product $I\mathfrak{b} = \mathfrak{b} \ltimes \mathfrak{b}^*$, where the dual \mathfrak{b}^* of \mathfrak{b} , is equipped with the coadjoint action of \mathfrak{b} and is considered as an abelian ideal of $I\mathfrak{b}$. We describe the automorphism group $\text{Aut}(I\mathfrak{b})$ of the Lie algebra $I\mathfrak{b}$. In particular we prove that it contains the automorphism group of the extended Dynkin diagram of \mathfrak{g} . In type A_n , the dihedral subgroup was recently proved to be contained in $\text{Aut}(I\mathfrak{b})$ by Dror Bar-Natan and Roland Van Der Veen in [Bv20] (where $I\mathfrak{b}$ is denoted by $I\mathfrak{u}_n$). Their construction is handmade and they ask for an explanation: this note fully answers the question.

1 Introduction

Given any complex Lie algebra \mathfrak{a} , one can form its “inhomogeneous version” $I\mathfrak{a} := \mathfrak{a} \ltimes \mathfrak{a}^*$. It is the semidirect product of \mathfrak{a} with its dual \mathfrak{a}^* where \mathfrak{a}^* is considered as an abelian ideal and \mathfrak{a} acts on \mathfrak{a}^* via the coadjoint action.

As mentioned in [Bv20], for applications in knot theory and representation theory, the most important case is when $\mathfrak{a} = \mathfrak{b}$ is the Borel subalgebra of some simple Lie algebra \mathfrak{g} . It is precisely the situation studied here. In addition to [Bv20], several examples of these algebras appear with variations in the literature. In [NW93], Nappi-Wittney use the case when $\mathfrak{g} = \mathfrak{sl}_2$ in conformal field theory. Several authors also consider $\mathfrak{b} \ltimes \mathfrak{n}^*$ where \mathfrak{n} is the derived subalgebra of \mathfrak{b} . It is the quotient of $I\mathfrak{b}$ by its center. In [KZJ07], Knutson and Zinn-Justin meet this algebra for $\mathfrak{g} = \mathfrak{gl}_n$ in the associative setting, see below. In [Fei12, Fei11], Feigin uses $\mathfrak{b} \ltimes \mathfrak{n}^*$ in order to study degenerate flag varieties for $\mathfrak{g} = \mathfrak{sl}_n$. For a general semisimple Lie algebra \mathfrak{g} , in [PY12], Panyushev and Yakimova study the invariants of $\mathfrak{b} \ltimes \mathfrak{n}^*$ under the action of their adjoint group. Finally, in [PY13, Pho20], similar considerations are studied replacing \mathfrak{b} by an arbitrary parabolic subalgebra of \mathfrak{g} .

The aim of this note is to give a new interpretation of $I\mathfrak{b}$ in the language of Kac-Moody algebras and to completely describe the automorphism group of $I\mathfrak{b}$.

Before describing this group, we introduce some notation. Let r denote the rank of \mathfrak{g} and G the adjoint group with Lie algebra \mathfrak{g} . Let B be the Borel subgroup of G with \mathfrak{b} as


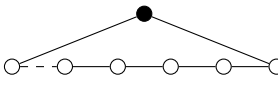
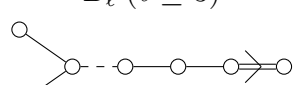

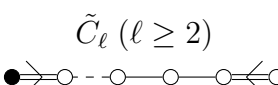
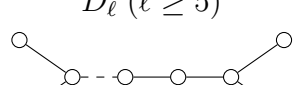
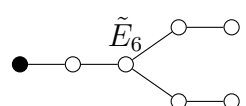
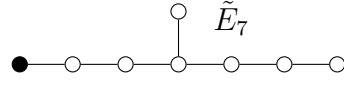
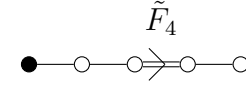
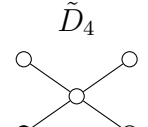
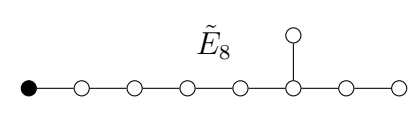
\tilde{A}_1  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z}$	$\tilde{A}_\ell (\ell \geq 2)$  $\text{Aut}(\tilde{\mathcal{D}}) = D_{(\ell+1)}$	$\tilde{B}_\ell (\ell \geq 3)$  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z}$
\tilde{G}_2  $\text{Aut}(\tilde{\mathcal{D}})$ is trivial	$\tilde{C}_\ell (\ell \geq 2)$  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z}$	$\tilde{D}_\ell (\ell \geq 5)$  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$
\tilde{E}_6  $\text{Aut}(\tilde{\mathcal{D}}) = \mathfrak{S}_3$	\tilde{E}_7  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z}$	\tilde{F}_4  $\text{Aut}(\tilde{\mathcal{D}})$ is trivial
\tilde{D}_4  $\text{Aut}(\tilde{\mathcal{D}}) = \mathfrak{S}_4$	\tilde{E}_8  $\text{Aut}(\tilde{\mathcal{D}})$ is trivial	

Figure 1: Extended Dynkin diagrams and their automorphisms

Lie algebra. Consider two abelian additive groups: the quotient $\mathfrak{g}/\mathfrak{b}$ and the space $\mathcal{M}_r(\mathbb{C})$ of square matrices.

An important ingredient is the extended Dynkin diagram of \mathfrak{g} . On Figure 1, these diagrams and their automorphisms are shortly recalled.

Theorem 1. *The neutral component $\text{Aut}(I\mathfrak{b})^\circ$ of the automorphism group $\text{Aut}(I\mathfrak{b})$ of the Lie algebra $I\mathfrak{b}$ decomposes as*

$$\mathbb{C}^* \times \left((B \times \mathfrak{g}/\mathfrak{b}) \times \mathcal{M}_r(\mathbb{C}) \right).$$

The group of components $\text{Aut}(I\mathfrak{b})/\text{Aut}(I\mathfrak{b})^\circ$ is isomorphic to the automorphism group of the affine Dynkin diagram of \mathfrak{g} and can be lifted as a subgroup of $\text{Aut}(I\mathfrak{b})$.

The details of how these subgroups act on $I\mathfrak{b}$ are given in Section 3. Section 4 explains how the semidirect products are formed.

One of the amazing facts is that the extended Dynkin diagram of \mathfrak{g} plays a crucial role in $\text{Aut}(I\mathfrak{b})$. On one hand, we explain this by constructing the extended Cartan matrix of \mathfrak{g} in terms of $I\mathfrak{b}$ in Section 3.1. On the other hand, this diagram is the Dynkin diagram of the

untwisted affine Lie algebra constructed from the loop algebra of \mathfrak{g} . A second explanation is given by Theorem 2 that realizes $I\mathfrak{b}$ as a subquotient of the affine Lie algebra associated to \mathfrak{g} .

In [KZJ07], Knutson and Zinn-Justin defined a degeneration \bullet of the standard associative product on $\mathcal{M}_n(\mathbb{C})$. Let \mathfrak{b} denote the set of upper triangular matrices. Identifying the vector space $\mathcal{M}_n(\mathbb{C})$ with $\mathfrak{b} \times \mathcal{M}_n(\mathbb{C})/\mathfrak{b}$ in a natural way one gets

$$(R, L) \bullet (V, M) = (RV, RM + LV),$$

for any $R, V \in \mathfrak{b}$ and $L, M \in \mathcal{M}_n(\mathbb{C})/\mathfrak{b}$. The Lie algebra of the group $(\mathcal{M}_n(\mathbb{C}), \bullet)^\times$ of invertible elements of this algebra is $\mathfrak{b} \times \mathcal{M}_n(\mathbb{C})/\mathfrak{b}$, where the product is defined similarly with that of $I\mathfrak{b}$. Note also that a cyclic automorphism (corresponding in our setting with the cyclic automorphism of the affine Dynkin diagram of type A_{n-1} and with the unexpected cyclic automorphism of [Bv20]) appears in [KZJ07]. Moreover [KZJ07, Proposition 2], which realizes $(\mathcal{M}_n(\mathbb{C}), \bullet)$ as a subquotient of $\mathcal{M}_n(\mathbb{C}[t])$, is similar with our Theorem 2.

Motivation and story of this work. In [Bv20], the authors constructed an “unexpected” cyclic automorphism of $I\mathfrak{b}$ when $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. The first version of this work was an explanation for this automorphism by using affine Lie algebras. Simultaneously with this first version, A. Knutson mentioned to Bar-Natan his earlier work [KZJ07] with Zinn-Justin.

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2 The Lie algebras $I\mathfrak{b}$, \mathfrak{g}_+^ϵ and $\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$

2.1 Definitions of $I\mathfrak{b}$ and \mathfrak{g}_+^ϵ

Let \mathfrak{g} be a complex simple Lie algebra with Lie bracket denoted by $[\cdot, \cdot]$. Fix a Borel subalgebra \mathfrak{b} of \mathfrak{g} and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Let \mathfrak{b}^- be the opposite Borel subalgebra of \mathfrak{b} containing \mathfrak{h} . Set $\mathcal{V} = \mathfrak{b} \oplus \mathfrak{b}^-$ viewed as a vector space. In this section, we define the Lie bracket $[\cdot, \cdot]_\epsilon$ on \mathcal{V} depending on the complex parameter ϵ , interpolating between $I\mathfrak{b}$ and the direct product $\mathfrak{g} \oplus \mathfrak{h}$.

Let \mathfrak{n} and \mathfrak{n}^- denote the derived subalgebras of \mathfrak{b} and \mathfrak{b}^- respectively. Fix $\epsilon \in \mathbb{C}$. Define the skew-symmetric bilinear bracket $[\cdot, \cdot]_\epsilon$ on \mathcal{V} by

$$\begin{aligned} [x, x']_\epsilon &= [x, x'] & \forall x, x' \in \mathfrak{b} \\ [y, y']_\epsilon &= \epsilon[y, y'] & \forall y, y' \in \mathfrak{b}^- \\ [x, y]_\epsilon &= (\epsilon X + \epsilon \frac{H}{2}, \frac{H}{2} + Y) & \forall x \in \mathfrak{b} \ y \in \mathfrak{b}^- \quad \text{where } [x, y] = X + H + Y \in \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^- \end{aligned}$$

Then $[\cdot, \cdot]_\epsilon$ satisfies Jacobi identity (see below for a proof). Endowed with this Lie bracket, \mathcal{V} is denoted by \mathfrak{g}_+^ϵ . Assume, for a moment that ϵ is nonzero. The invertible linear map

$$\begin{aligned} \varphi_\epsilon : \mathfrak{b} \oplus \mathfrak{b}^- &\longrightarrow \mathfrak{b} \oplus \mathfrak{b}^- \\ (x, y) &\longmapsto (x, \epsilon y) \quad \text{for any } x \in \mathfrak{b}, y \in \mathfrak{b}^- \end{aligned}$$

allows to interpret \mathfrak{g}_+^ϵ as an Inönü-Wigner contraction [IW53] of \mathfrak{g}_+^1 . Indeed, for any nonzero ϵ , we have

$$[X, Y]_\epsilon = \varphi_\epsilon^{-1}([\varphi_\epsilon(X), \varphi_\epsilon(Y)]_1) \quad \forall X, Y \in \mathcal{V}. \quad (1)$$

We now describe \mathfrak{g}_+^1 . Using the triangular decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-, \quad (2)$$

one defines the injective linear map

$$\begin{aligned} \iota_{\mathfrak{g}}^1 : \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^- &\longrightarrow \mathfrak{g}_+^1 \\ (\xi, \alpha, \zeta) &\longmapsto (\xi + \frac{\alpha}{2}, \frac{\alpha}{2} + \zeta) \end{aligned}$$

and checks that it is a Lie algebra homomorphism. Moreover, the image of

$$\begin{aligned} \iota_{\mathfrak{h}}^1 : \mathfrak{h} &\longrightarrow \mathfrak{g}_+^1 \\ \alpha &\longmapsto (-\alpha, \alpha) \end{aligned}$$

is the center of \mathfrak{g}_+^1 and

$$\mathfrak{g}_+^1 = \iota_{\mathfrak{g}}^1(\mathfrak{g}) \oplus \iota_{\mathfrak{h}}^1(\mathfrak{h}). \quad (3)$$

Observe that we never used the Jacobi identity for $[\cdot, \cdot]_1$ to prove the isomorphism (3). Hence we can deduce from it that $[\cdot, \cdot]_1$ satisfies the Jacobi identity. Then, the expression (1) with φ_ϵ of $[\cdot, \cdot]_\epsilon$ from $[\cdot, \cdot]_1$ implies that $[\cdot, \cdot]_\epsilon$ satisfies the Jacobi identity for any nonzero ϵ . Since this property is closed on the space of bilinear maps, it is satisfied by $[\cdot, \cdot]_0$ too.

Consider now $I\mathfrak{b}$ with its Lie bracket $[\cdot, \cdot]_{I\mathfrak{b}}$ defined by: \mathfrak{b}^* is an abelian ideal on which \mathfrak{b} acts by the coadjoint action. Denote by $\kappa : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ the Killing form on \mathfrak{g} . Since the orthogonal of \mathfrak{b} with respect to κ is \mathfrak{n} , \mathfrak{b}^* identifies with $\mathfrak{g}/\mathfrak{n}$ as a \mathfrak{b} -module. Identify $\mathfrak{g}/\mathfrak{n}$ with \mathfrak{b}^- in a canonical way (that is by $y \in \mathfrak{b}^- \longmapsto y + \mathfrak{n}$) and denote by $\pi : \mathfrak{g} \longrightarrow \mathfrak{b}^-$ the quotient map. Then $I\mathfrak{b} = \mathfrak{b} \oplus \mathfrak{b}^*$ identifies with $\mathfrak{b} \oplus \mathfrak{b}^- = \mathcal{V}$. Let $[\cdot, \cdot]_I$ denote the Lie bracket transported to \mathcal{V} from $[\cdot, \cdot]_{I\mathfrak{b}}$. Let $x, x' \in \mathfrak{b}$ and $y, y' \in \mathfrak{b}^-$ and decompose $[x, y'] - [x', y]$ as $X + H + Y$ with respect to $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Then

$$[(x, y), (x', y')]_I = ([x, x'], H + Y). \quad (4)$$

We now describe \mathfrak{g}_+^0 . The Lie bracket $[\cdot, \cdot]_0$ on $\mathcal{V} = \mathfrak{g}_+^0$ is given by

$$[(x, y), (x', y')]_0 = ([x, x'], \frac{H}{2} + Y). \quad (5)$$

Comparing (4) and (5), one gets that the following linear map η is a Lie algebra isomorphism:

$$\begin{aligned} \eta : \mathcal{V} = \mathfrak{b} \oplus (\mathfrak{h} \oplus \mathfrak{n}^-) &\longrightarrow \mathfrak{b} \oplus \mathfrak{b}^* = I\mathfrak{b} \\ (x, h, y) &\longmapsto (x, \kappa(2h + y, \square)). \end{aligned}$$

2.2 The affine Kac-Moody Lie algebra

The affine Kac-Moody Lie algebra \mathfrak{g}^{KM} is constructed from the semisimple Lie algebra \mathfrak{g} . We refer to [Kum02, Chapters I and XIII] for the basic properties of \mathfrak{g}^{KM} . Denote by $\mathfrak{z}(\mathfrak{g}^{\text{KM}})$ the one dimensional center of \mathfrak{g}^{KM} . Consider the Borel subalgebra \mathfrak{b}^{KM} of \mathfrak{g}^{KM} and its derived subalgebra \mathfrak{n}^{KM} . By killing the semi-direct product and the central extension from the construction of \mathfrak{g}^{KM} , one gets

$$\begin{aligned}\tilde{\mathfrak{g}} &:= [\mathfrak{g}^{\text{KM}}, \mathfrak{g}^{\text{KM}}] / \mathfrak{z}(\mathfrak{g}^{\text{KM}}) \\ &\cong \mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g},\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathfrak{b}} &:= (\mathfrak{b}^{\text{KM}} \cap [\mathfrak{g}^{\text{KM}}, \mathfrak{g}^{\text{KM}}]) / \mathfrak{z}(\mathfrak{g}^{\text{KM}}) \subset \tilde{\mathfrak{g}} \\ \tilde{\mathfrak{n}} &:= (\mathfrak{n}^{\text{KM}} \cap [\mathfrak{g}^{\text{KM}}, \mathfrak{g}^{\text{KM}}]) / \mathfrak{z}(\mathfrak{g}^{\text{KM}}) = [\tilde{\mathfrak{b}}, \tilde{\mathfrak{b}}].\end{aligned}$$

Identify \mathfrak{g} with the subspace $\mathbb{C} \otimes \mathfrak{g} \subset \tilde{\mathfrak{g}}$. Note that $\mathfrak{g}^{\text{KM}} / \mathfrak{z}(\mathfrak{g}^{\text{KM}}) = \tilde{\mathfrak{g}} + \mathbb{C}d$ where d acts as the derivation with respect to t .

We consider the set of (positive) roots $\Phi^{(+)}$ (resp. $\tilde{\Phi}^{(+)}$) of \mathfrak{g} (resp. $\tilde{\mathfrak{g}}$) and the set of simple roots Δ (resp. $\tilde{\Delta}$) with respect to $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ (resp. $\mathfrak{h} + \mathbb{C}d + \mathfrak{z}(\mathfrak{g}^{\text{KM}}) \subset \mathfrak{b}^{\text{KM}} \subset \mathfrak{g}^{\text{KM}}$). We recall the following classical facts:

$$\mathfrak{n}^{\text{KM}} \cong \tilde{\mathfrak{n}} = \bigoplus_{\alpha \in \tilde{\Phi}^+} \tilde{\mathfrak{g}}_{\alpha}$$

where $\tilde{\mathfrak{g}}_{\alpha} \cong \mathfrak{g}_{\alpha}^{\text{KM}}$ is the root space associated to α . Moreover, $\tilde{\mathfrak{n}}$ is generated, as a Lie algebra by the subspaces $(\tilde{\mathfrak{g}}_{\alpha})_{\alpha \in \tilde{\Delta}}$. The identification of Δ with $\{\alpha \in \tilde{\Delta} \mid \alpha(d) = 0\}$ yields the above-described embedding $\mathfrak{g} \subset \tilde{\mathfrak{g}}$. Denoting by δ the indivisible positive imaginary root in $\tilde{\Phi}$, we have

$$\begin{aligned}\tilde{\Phi} &= \{n\delta + \alpha \mid \alpha \in \Delta \cup \{0\}, n \in \mathbb{Z}\} \setminus \{0\} \\ \tilde{\Delta} &= \Delta \cup \{\alpha_0 + \delta\}\end{aligned}$$

where $-\alpha_0$ is the highest root of Φ .

Finally, the extended Dynkin diagram can be reconstructed from the combinatorics of $\tilde{\Delta}$ in $\tilde{\Phi}$. Indeed, the nodes correspond to the elements of $\tilde{\Delta}$ and the non-diagonal entries $a_{\alpha, \beta}$ of the generalized Cartan matrix (encoding the arrows of the diagram) are $a_{\alpha, \beta} = -\max\{n \in \mathbb{N} \mid \beta + n\alpha \in \tilde{\Phi}\}$ by Serre relations.

We list in Figure 1 the extended Dynkin diagram $\tilde{\mathcal{D}}_{\mathfrak{g}}$ in each simple type. The black node corresponds to the simple root $\alpha_0 + \delta$. We also provide the automorphism group of $\tilde{\mathcal{D}}_{\mathfrak{g}}$. Note that by the definition of \mathfrak{g}^{KM} given in [Kum02, §1.1], any $\theta \in \text{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}})$ provides an automorphism $\theta^{\text{KM}} \in \text{Aut}(\mathfrak{g}^{\text{KM}})$ stabilizing both $\mathfrak{h} + \mathbb{C}d + \mathfrak{z}(\mathfrak{g}^{\text{KM}})$ and \mathfrak{b}^{KM} and permuting $\tilde{\Delta}$ ¹ as θ does. Since $\mathfrak{z}(\mathfrak{g}^{\text{KM}})$ and $[\mathfrak{g}^{\text{KM}}, \mathfrak{g}^{\text{KM}}]$ are characteristic in \mathfrak{g}^{KM} , this yields an automorphism $\tilde{\theta} \in \text{Aut}(\tilde{\mathfrak{g}})$.

¹and even permuting the set of generators e_{α} , $\alpha \in \tilde{\Delta}$

2.3 Realisation of \mathfrak{g}_+^ϵ

The Lie algebras $\tilde{\mathfrak{b}}$ and $\tilde{\mathfrak{n}}$ decompose as

$$\begin{aligned}\tilde{\mathfrak{b}} &= \mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{n}^-, \\ \tilde{\mathfrak{n}} &= \mathbb{C}[t]\mathfrak{n} \oplus t\mathbb{C}[t]\mathfrak{b}^-.\end{aligned}$$

Moreover, $(t - \epsilon)\tilde{\mathfrak{n}}$ is an ideal of $\tilde{\mathfrak{b}}$, and $\tilde{\mathfrak{b}}/((t - \epsilon)\tilde{\mathfrak{n}})$ is a Lie algebra.

Theorem 2. *Let $\epsilon \in \mathbb{C}$. The Lie algebras \mathfrak{g}_+^ϵ and $\tilde{\mathfrak{b}}/((t - \epsilon)\tilde{\mathfrak{n}})$ are isomorphic.*

Proof. From Section 2.1, we have $\mathfrak{g}_+^1 \stackrel{v.s.}{=} \mathfrak{b} \oplus \mathfrak{b}^-$. Elements of \mathfrak{g}_+^1 will be written as couples with respect to this decomposition.

Set $\tilde{\mathfrak{g}}_+^1 := \mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g}_+^1$ and extend $\iota_{\mathfrak{g}}^1$ to an injective $\mathbb{C}[t^{\pm 1}]$ -linear map $\tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}_+^1$. Consider the subspace $\mathfrak{w} := \mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{b}^-$ that is a Lie subalgebra of $\tilde{\mathfrak{g}}_+^1$. The Inönü-Wigner contraction on \mathfrak{g}_+^1 with respect to the decomposition $\mathfrak{b} \oplus \mathfrak{b}^-$ gives rise to \mathfrak{g}_+^ϵ ($\epsilon \in \mathbb{C}$). In particular, the linear map

$$\begin{aligned}\mathfrak{g}_+^\epsilon &\longrightarrow \mathfrak{w}/(t - \epsilon)\mathfrak{w} \\ (x, y) &\longmapsto x + ty + (t - \epsilon)\mathfrak{w} \quad \text{for any } x \in \mathfrak{b} \text{ and } y \in \mathfrak{b}^-.\end{aligned}\tag{6}$$

is a Lie algebra isomorphism

Set $\mathfrak{b}_0^- := \iota_{\mathfrak{g}}^1(\mathfrak{b}^-) = \{(h, h) | h \in \mathfrak{h}\} \oplus \mathfrak{n}^-$. Observe that $t\mathfrak{b}_0^-$ is contained in \mathfrak{w} . Indeed, for any $h \in \mathfrak{h}$, the element $t(h, h) = t(h, 0) + t(0, h)$ belongs to $\mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{b}^-$. In particular, one gets a linear map induced by the inclusions of \mathfrak{b} and $t\mathfrak{b}_0^-$ in \mathfrak{w} :

$$\mathfrak{b} \oplus t\mathfrak{b}_0^- \longrightarrow \mathfrak{w}.$$

One can easily check that it induces a linear isomorphism $\mathfrak{b} \oplus t\mathfrak{b}_0^- \longrightarrow \mathfrak{w}/(t - \epsilon)\mathfrak{w}$. Setting $\tilde{\mathfrak{b}}_{\mathfrak{w}} := \langle \mathfrak{b} \oplus t\mathfrak{b}_0^- \rangle_{Lie} \subset \mathfrak{w}$, we thus get a Lie algebra isomorphism.

$$\tilde{\mathfrak{b}}_{\mathfrak{w}}/((t - \epsilon)\mathfrak{w} \cap \tilde{\mathfrak{b}}_{\mathfrak{w}}) \longrightarrow \mathfrak{w}/(t - \epsilon)\mathfrak{w}.\tag{7}$$

Since, $\mathfrak{b} = \{(h, 0) | h \in \mathfrak{h}\} \oplus \iota_{\mathfrak{g}}^1(\mathfrak{n})$ and $\langle \iota_{\mathfrak{g}}^1(\mathfrak{n}) \oplus \iota_{\mathfrak{g}}^1(t\mathfrak{b}^-) \rangle_{Lie} = \iota_{\mathfrak{g}}^1(\langle \mathfrak{n} \oplus t\mathfrak{b}^- \rangle_{Lie}) = \iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$, we have

$$\tilde{\mathfrak{b}}_{\mathfrak{w}} = \{(h, 0) | h \in \mathfrak{h}\} \oplus \iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}}) \cong \iota_{\mathfrak{g}}^1(\tilde{\mathfrak{b}}) \cong \tilde{\mathfrak{b}},\tag{8}$$

the middle Lie isomorphism being the identity on $\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$ and sending $(h, 0)$ to $\frac{1}{2}(h, h)$ for each $h \in \mathfrak{h}$. Moreover, $(t - \epsilon)\mathfrak{w} \cap \tilde{\mathfrak{b}}_{\mathfrak{w}} = (t - \epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$. Indeed, $(t - \epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$ is contained in $(t - \epsilon)\mathfrak{w} \cap \tilde{\mathfrak{b}}_{\mathfrak{w}}$, and $\mathfrak{b} \oplus t\mathfrak{b}_0^-$ is complementary to $(t - \epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$ in $\tilde{\mathfrak{b}}_{\mathfrak{w}}$.

We finally get the desired Lie isomorphism

$$\tilde{\mathfrak{b}}/((t - \epsilon)\tilde{\mathfrak{n}}) \stackrel{(8)}{\cong} \tilde{\mathfrak{b}}_{\mathfrak{w}}/((t - \epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})) \stackrel{(7)}{\cong} \mathfrak{w}/(t - \epsilon)\mathfrak{w} \stackrel{(6)}{\cong} \mathfrak{g}_+^\epsilon$$

□

In addition, we can make explicit the isomorphism of Theorem 2:

$$\begin{aligned} \gamma_\epsilon : \quad \mathfrak{g}_+^\epsilon &\xrightarrow{\cong} \tilde{\mathfrak{b}}/(t-\epsilon)\tilde{\mathfrak{n}} \\ x &\longmapsto x && \text{if } x \in \mathfrak{n} \\ y &\longmapsto ty && \text{if } y \in \mathfrak{n}^- \\ (a, b) &\longmapsto (a - \epsilon b) + 2tb && \text{if } a, b \in \mathfrak{h} \end{aligned}$$

and its inverse map is induced by

$$\begin{aligned} \theta : \quad \tilde{\mathfrak{b}} &\longrightarrow \mathcal{V} \\ Px &\longmapsto P(\epsilon)x && \text{if } x \in \mathfrak{n} \\ tRy &\longmapsto R(\epsilon)y && \text{if } y \in \mathfrak{n}^- \\ Qh &\longmapsto \left(\frac{Q(\epsilon)+Q(0)}{2}h, \frac{Q(\epsilon)-Q(0)}{2\epsilon}h \right) && \text{if } h \in \mathfrak{h} (\epsilon \neq 0) \\ &&& (Q(0)h, \frac{1}{2}Q'(0)h) && \text{if } h \in \mathfrak{h} (\epsilon = 0) \end{aligned}$$

Note that, in order to prove Theorem 2, we could alternatively have checked directly that θ is a Lie algebra homomorphism from $\tilde{\mathfrak{b}}$ onto \mathfrak{g}_+^ϵ with Kernel $(t-\epsilon)\tilde{\mathfrak{n}}$.

3 Some subgroups of $\text{Aut}(I\mathfrak{b})$

3.1 The roots of $I\mathfrak{b}$

From Sections 2.1 and 2.3, we can interpret the algebra $I\mathfrak{b}$ in the Kac-Moody world via the isomorphism

$$\begin{aligned} I\mathfrak{b} &\longrightarrow \tilde{\mathfrak{b}}/t\tilde{\mathfrak{n}} \\ (x, y) &\longmapsto x + ty \quad \left(\begin{array}{l} x \in \mathfrak{b}, \\ y \in \mathfrak{b}^- \cong \mathfrak{g}/\mathfrak{n} \cong \mathfrak{b}^* \end{array} \right) \end{aligned}$$

From now on, this identification will be made systematically. In particular, we write $I\mathfrak{b} = \mathfrak{b} \oplus t\mathfrak{b}^-$. We first describe some basic properties of $I\mathfrak{b}$ in this language.

Lemma 3. *1. The subalgebra $\mathfrak{c} := \mathfrak{h} \oplus t\mathfrak{h}$ is a Cartan subalgebra of $I\mathfrak{b}$. Namely, \mathfrak{c} is abelian and equal to its normalizer.*

2. Under the action of \mathfrak{c} , $I\mathfrak{b}$ decomposes as

$$I\mathfrak{b} = \mathfrak{c} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^-} t\mathfrak{g}_\alpha.$$

For $\alpha \in \Phi^+$, \mathfrak{c} acts on \mathfrak{g}_α with the weight $(\alpha, 0) \in \mathfrak{h}^ \times t\mathfrak{h}^*$. It acts on $t\mathfrak{g}_\alpha$ with the weight $(\alpha, 0) \in \mathfrak{h}^* \times t\mathfrak{h}^*$, if $\alpha \in \Phi^-$. Here, we identified \mathfrak{c}^* with $\mathfrak{h}^* \times t\mathfrak{h}^*$ in a natural way.*

3. The set of ad-nilpotent elements of $I\mathfrak{b}$ is $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}} = \mathfrak{n} \oplus t\mathfrak{b}^-$.

4. The centre of $I\mathfrak{b}$ is $\mathfrak{z}(I\mathfrak{b}) = t\mathfrak{h}$.

5. The derived subalgebra of $I\mathfrak{b}$ is $[I\mathfrak{b}; I\mathfrak{b}] = \tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$.

Proof. 1-2) The fact that \mathfrak{c} is abelian is clear from the definition of $\tilde{\mathfrak{g}}$. The decomposition in eigenspaces also. The action of $t\mathfrak{h}$ is zero since it sends $\tilde{\mathfrak{n}}$ to $t\tilde{\mathfrak{n}}$ that vanishes itself in $I\mathfrak{b}$. The decomposition of $I\mathfrak{b}$ in weight spaces under the action of \mathfrak{c} follows. Then this decomposition also implies that $N_{I\mathfrak{b}}(\mathfrak{c}) = \mathfrak{c}$.

3) The elements of $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ are clearly ad-nilpotent. From 2), an element with non-zero component in \mathfrak{h} is not ad-nilpotent.

4) Since it acts as 0 on $\tilde{\mathfrak{n}}$ and on \mathfrak{h} , we have $t\mathfrak{h} \subset \mathfrak{z}(I\mathfrak{b})$. The decomposition in weight spaces implies the converse inclusion.

5) The inclusion $[I\mathfrak{b}, I\mathfrak{b}] \subset \tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ is clear. On the other hand we deduce from the weight space decomposition that the subspaces $(\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \tilde{\Delta}}$ belong to $[I\mathfrak{b}, I\mathfrak{b}]$. Since they generate $\tilde{\mathfrak{n}}$ in $\tilde{\mathfrak{g}}$, the result follows. \square

From Lemma 3 (2), the set of nonzero weights $\Phi(I\mathfrak{b})$ of \mathfrak{c} acting on $I\mathfrak{b}$ identifies with $\tilde{\Phi}$. It is also useful to embed $\Phi(I\mathfrak{b})$ in $\tilde{\Phi}$ by

$$\begin{aligned} \varphi : \quad \Phi(I\mathfrak{b}) &\longrightarrow \tilde{\Phi} \\ \alpha \in \Phi^+ &\longmapsto \alpha \\ \alpha \in \Phi^- &\longmapsto \delta + \alpha \end{aligned}$$

Indeed, the weight space $(I\mathfrak{b})_\alpha$ identifies with $\tilde{\mathfrak{g}}_{\varphi(\alpha)}$, for any $\alpha \in \Phi(I\mathfrak{b})$. In particular, for $\alpha, \beta \in \tilde{\Phi} \cup \{0\}$, we have $[I\mathfrak{b}_{\varphi^{-1}(\alpha)}, I\mathfrak{b}_{\varphi^{-1}(\beta)}] \subset I\mathfrak{b}_{\varphi^{-1}(\alpha+\beta)}$ with equality when $\alpha, \beta, \alpha + \beta \notin \{0, \delta\}$. Set also $\Delta(I\mathfrak{b}) = \varphi^{-1}(\tilde{\Delta}) = \Delta \cup \{-\alpha_0\}$.

Lemma 4. 1. The derived subalgebra of $I\mathfrak{b}^{(1)} := [I\mathfrak{b}, I\mathfrak{b}]$ is

$$I\mathfrak{b}^{(2)} = \bigoplus_{\alpha \in \Phi(I\mathfrak{b}) \setminus \Delta(I\mathfrak{b})} (I\mathfrak{b})_\alpha$$

2. Assume that \mathfrak{g} is not \mathfrak{sl}_2 . For $\alpha, \beta \in \Delta(I\mathfrak{b})$ ($\alpha \neq \beta$), the corresponding entry of the generalized Cartan Matrix of \mathfrak{g}^{KM} is given by

$$a_{\alpha, \beta} = -\max\{n \in \mathbb{N} \mid \beta + n\alpha \in \Phi(I\mathfrak{b})\}.$$

Proof. 1) Recall that $\tilde{\mathfrak{n}}$ is generated as a Lie algebra by the $(\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \tilde{\Delta}}$. Thus, for weight reasons, the $(\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \tilde{\Phi} \setminus \tilde{\Delta}}$ are root spaces included in $[\tilde{\mathfrak{n}}, \tilde{\mathfrak{n}}]$. Since $\tilde{\Delta}$ is a linearly independent family, they are in fact the only root spaces not contained in $[\tilde{\mathfrak{n}}, \tilde{\mathfrak{n}}]$. Quotienting, this yields $\bigoplus_{\alpha \in \Phi(I\mathfrak{b}) \setminus \Delta(I\mathfrak{b})} (I\mathfrak{b})_\alpha = I\mathfrak{b}^{(2)}$.

2) Recall that the statement is valid if we replace $\Phi(I\mathfrak{b})$ by $\tilde{\Phi}$, see Section 2.2. It is thus sufficient to show that

$$\beta + n\alpha \in \tilde{\Phi} \Rightarrow \beta + n\alpha \in \Phi(I\mathfrak{b}).$$

When $\alpha, \beta \in \Delta$, the statement is clear since $\Phi^+ \subset \Phi(I\mathfrak{b})$.

If $\beta = \delta + \alpha_0$, then $\beta + n\alpha \in \tilde{\Phi}$ means that $\alpha_0 + n\alpha \in \Phi$. Since $\alpha_0 + n\alpha$ has elements of $-\Delta$

in its support, it has to lie in Φ^- . Thus $\beta + n\alpha \in \Phi(I\mathfrak{b})$.

If $\alpha = \delta + \alpha_0$, then $\beta + n\alpha \in \tilde{\Phi}$ means that $\beta + n\alpha_0 \in \Phi$. For height reasons, we must have $n \in \{0, 1\}$. Then, $\beta + n\alpha \in \Phi(I\mathfrak{b})$. \square

Remark. One can observe that the first assertion of Lemma 4 is similar with

$$[\mathfrak{n}, \mathfrak{n}] = \bigoplus_{\alpha \in \Phi^+ \setminus \Delta} \mathfrak{b}_\alpha.$$

3.2 The adjoint subgroup of $\text{Aut}(I\mathfrak{b})$

Let G be the adjoint group with Lie algebra \mathfrak{g} . Let T and B be the subgroups of G with Lie algebras \mathfrak{h} and \mathfrak{b} . Consider now $\mathfrak{b}^- \cong \mathfrak{g}/\mathfrak{n}$ equipped with the addition as an abelian algebraic group. The adjoint action of B on \mathfrak{g} stabilizes \mathfrak{n} and induces a linear action on $\mathfrak{b}^- \cong \mathfrak{g}/\mathfrak{n}$ by group isomorphisms. We can perform the semidirect product:

$$IB := B \ltimes \mathfrak{b}^-.$$

By construction the Lie algebra of IB identifies with $I\mathfrak{b}$. The adjoint action of IB on $I\mathfrak{b}$ is given by

$$\begin{aligned} IB \times I\mathfrak{b} &\longrightarrow I\mathfrak{b} \\ ((b, f), x + ty) &\longmapsto b \cdot x + t(b \cdot y + [f, x]) \quad \text{for } b \in B, f, x \in \mathfrak{b} \text{ and } y \in \mathfrak{b}^-, \end{aligned} \quad (9)$$

where \cdot denotes the B action on \mathfrak{b} and on \mathfrak{b}^- . It induces a morphism

$$\text{Ad} : IB \longrightarrow \text{Aut}(I\mathfrak{b})$$

with Kernel $Z(IB) \cong (1, \mathfrak{h})$. In particular, one gets:

Lemma 5. *The image $\text{Ad}(IB)$ is isomorphic with $B \ltimes \mathfrak{g}/\mathfrak{b}$.*

Note also that $\text{Ad}(IB) = H \ltimes (N \ltimes \mathfrak{g}/\mathfrak{b})$ where N and H are the connected subgroups of B with respective Lie algebras \mathfrak{n} and \mathfrak{h} . Since $\mathfrak{n} + t\mathfrak{b}^-$ is the set of ad-nilpotent elements of $I\mathfrak{b}$, we get the following result from (9).

Lemma 6. *1. The group of elementary automorphisms $\text{Aut}_e(I\mathfrak{b}) = \exp(\mathfrak{n} + t\mathfrak{b}^-)$ coincides with $N \ltimes \mathfrak{g}/\mathfrak{b}$.*

2. $\text{Ad}(IB) = \exp(I\mathfrak{b})$

3.3 An unipotent subgroup of $\text{Aut}(I\mathfrak{b})$

Lemma 7. *The following map is an injective group homomorphism*

$$\begin{aligned} \text{Hom}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}], \mathfrak{z}(I\mathfrak{b})) &\longrightarrow \text{Aut}(I\mathfrak{b}) \\ u &\longmapsto \left(\begin{array}{ccc} \bar{u} : I\mathfrak{b} &\longrightarrow & I\mathfrak{b} \\ X &\longmapsto & X + u(X) \end{array} \right). \end{aligned}$$

We denote by $U \subset \text{Aut}(I\mathfrak{b})$ the image of this map. From Lemma 3, we have $U \cong \mathcal{M}_r(\mathbb{C})$ where $r = \dim \mathfrak{h}$.

Proof. Let $X, Y \in I\mathfrak{b}$. On one hand, we have

$$[\bar{u}(X), \bar{u}(Y)] = [X + u(X), Y + u(Y)] = [X, Y],$$

since the image of u is contained in the center. On the other hand,

$$\bar{u}([X, Y]) = [X, Y],$$

since u vanishes on the derived subalgebra. It follows that \bar{u} is a Lie algebra isomorphism.

Since \mathfrak{z} is contained in $[I\mathfrak{b}, I\mathfrak{b}]$ the map of the lemma is a group homomorphism from $(\text{Hom}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}], \mathfrak{z}), +)$ to $(\text{Aut}(I\mathfrak{b}), \circ)$. \square

Note that the existence of this group of automorphisms is quite general. Indeed, the only useful property of $I\mathfrak{b}$ in this proof is $\mathfrak{z}(I\mathfrak{b}) \cap [I\mathfrak{b}, I\mathfrak{b}] = \{0\}$.

3.4 The loop subgroup

Lemma 8. *The following map is an injective group homomorphism*

$$\begin{aligned} \mathbb{C}^* &\longrightarrow \text{Aut}(I\mathfrak{b}) \\ \tau &\longmapsto \left(\begin{array}{ccc} \delta_\tau : I\mathfrak{b} &\longrightarrow & I\mathfrak{b} \\ x &\longmapsto & x \quad \text{if } x \in \mathfrak{b} \\ ty &\longmapsto & \tau ty \quad \text{if } y \in \mathfrak{b}^- \end{array} \right). \end{aligned}$$

We denote by $D \subset \text{Aut}(I\mathfrak{b})$ the image of this map

Proof. It is a straightforward check on $\mathfrak{b} \ltimes t\mathfrak{b}^-$ that the δ_τ are automorphisms of $I\mathfrak{b}$. \square

Remark. The map δ_τ corresponds with the variable changing $t \mapsto \tau t$ in the $\mathbb{C}[t]$ -Lie algebra $\tilde{\mathfrak{b}}/t\tilde{\mathfrak{n}}$. Moreover, the Lie algebra of D acts on $I\mathfrak{b}$ like $\mathbb{C}d$ where d is the derivation involved in the definition of \mathfrak{g}^{KM} .

3.5 Automorphisms stabilizing the Cartan subalgebra

For any $\alpha \in \Delta(I\mathfrak{b})$, fix a nonzero element X_α in the corresponding root space $(I\mathfrak{b})_\alpha$. Set

$$G := \left\{ \theta \in \text{Aut}(I\mathfrak{b}) \mid \begin{array}{l} \theta(\mathfrak{h}) \subset \mathfrak{h} \\ \theta(\{X_\alpha : \alpha \in \Delta(I\mathfrak{b})\}) = \{X_\alpha : \alpha \in \Delta(I\mathfrak{b})\} \end{array} \right\}.$$

Note that, since \mathfrak{c} is the sum of \mathfrak{h} with $\mathfrak{z}(I\mathfrak{b})$ and since the center is characteristic, the elements of G also stabilize \mathfrak{c} .

Proposition 9. *The group G is isomorphic to the automorphism group of the affine Dynkin diagram of \mathfrak{g} .*

Proof. The group G stabilizes \mathfrak{c} and hence the set of weights of \mathfrak{c} acting on $I\mathfrak{b}$. This yields an action of G on $\Phi(I\mathfrak{b})$.

Moreover, $I\mathfrak{b}^{(1)} = [I\mathfrak{b}, I\mathfrak{b}]$ and $I\mathfrak{b}^{(2)} = [I\mathfrak{b}^{(1)}, I\mathfrak{b}^{(1)}]$ are characteristic and stabilized by G . Now, Lemma 4 implies that G stabilizes $\Phi(I\mathfrak{b}) \setminus \Delta(I\mathfrak{b})$ and hence $\Delta(I\mathfrak{b})$. Moreover, by Lemma 4 (2), we have for $g \in G$ and $\alpha, \beta \in \Delta(I\mathfrak{b})$:

$$\begin{aligned} a_{\alpha, \beta} &= -\max\{n \mid (\text{ad } X_\alpha)^n(X_\beta) \neq 0\} \\ &= -\max\{n \mid g((\text{ad } X_\alpha)^n(X_\beta)) \neq 0\} \\ &= -\max\{n \mid (\text{ad } X_{g(\alpha)})^n(X_{g(\beta)}) \neq 0\} = a_{g(\alpha), g(\beta)}. \end{aligned}$$

Hence g actually induces an automorphism of the extended Dynkin diagram² and we thus obtain a group homomorphism

$$\Theta : G \rightarrow \text{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}}).$$

We claim that Θ is surjective. Indeed, fix a group automorphism θ of $\tilde{\mathcal{D}}_{\mathfrak{g}}$. As seen in Section 2.2, there exists $\tilde{\theta} \in \text{Aut}(\tilde{\mathfrak{g}})$ which stabilizes both \mathfrak{h} and $\tilde{\mathfrak{b}}$ and which permutes $\Delta(I\mathfrak{b})$ as θ does. Then it stabilizes $\tilde{\mathfrak{n}} = [\tilde{\mathfrak{b}}, \tilde{\mathfrak{b}}]$ and thus induces the desired element of $\text{Aut}(\tilde{\mathfrak{b}}/\tilde{\mathfrak{t}\tilde{\mathfrak{n}}})$.

We now prove that Θ is injective. Let θ in its Kernel. By the definition of the group G , θ stabilizes \mathfrak{h} . Since the restrictions of the elements of $\Delta(I\mathfrak{b})$ span \mathfrak{h}^* , the restriction of θ to \mathfrak{h} has to be the identity. In particular, θ acts trivially on $\Phi(I\mathfrak{b})$ and stabilizes each root space $(I\mathfrak{b})_\alpha$ for $\alpha \in \Phi(I\mathfrak{b})$. But θ stabilizes the set $\{X_\alpha : \alpha \in \Delta(I\mathfrak{b})\}$. Hence θ acts trivially on each $\tilde{\mathfrak{g}}_\alpha$ for $\alpha \in \Delta(I\mathfrak{b})$. Since $\tilde{\mathfrak{n}}$ is generated by the $(\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \Delta(I\mathfrak{b})}$, the restriction of θ to $\tilde{\mathfrak{n}}/\tilde{\mathfrak{t}\tilde{\mathfrak{n}}}$ is the identity map. Finally, θ is trivial and Θ is injective. \square

Remark. [Bv20, Theorem 2] is the construction of an explicit order n automorphism of $\mathfrak{gl}_{n+}^\epsilon$. Theorem 2 and the above proof of the surjectivity of Θ also show the existence of such an automorphism for $\mathfrak{sl}_{n+}^\epsilon$ with nonzero ϵ . Hence we just got both an explanation and an extension (to any simple \mathfrak{g}) of the Bar-Natan-van der Veen's Theorem 2.

²If \mathfrak{g} is \mathfrak{sl}_2 , Lemma 4 (2) does not apply. However, any permutation of $\tilde{\Delta}$ is an automorphism of the extended Dynkin diagram in this case.

4 Description of $\text{Aut}(I\mathfrak{b})$

In this section, we describe the structure of

$$\text{Aut}(I\mathfrak{b}) = \{g \in \text{GL}(I\mathfrak{b}) : \forall X, Y \in I\mathfrak{b} \quad g([X, Y]) = [g(X), g(Y)]\}$$

in terms of the subgroups U , $\text{Ad}(IB)$, D and G introduced in Section 3.

Observe that $\text{Aut}(I\mathfrak{b})$ is a Zariski closed subgroup of the linear group $\text{GL}(I\mathfrak{b})$.

Theorem 10. *We have the following decomposition*

$$\text{Aut}(I\mathfrak{b}) = G \rtimes (D \rtimes (\text{Ad}(IB) \times U)).$$

In particular, the neutral component is $\text{Aut}(I\mathfrak{b})^\circ = D \rtimes (\text{Ad}(IB) \times U)$ and $G \cong \text{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}})$ can be seen as the component group of $\text{Aut}(I\mathfrak{b})$.

The result is a consequence of the lemmas provided below. Indeed, by Lemma 12, the four subgroups generate $\text{Aut}(I\mathfrak{b})$. By Lemma 11, the subgroup generated by U and $\text{Ad}(IB)$ is a direct product $U \times \text{Ad}(IB)$. Then the structure of $\text{Aut}(I\mathfrak{b})$ follows from lemma 13.

Since D , $\text{Ad}(IB)$ and U are connected and G is discrete, $\text{Aut}(I\mathfrak{b}) = \bigsqcup_{g \in G} gD\text{Ad}(IB)U$ is a finite disjoint union of irreducible subsets of the same dimension. They are thus the irreducible components of $\text{Aut}(I\mathfrak{b})$ and the remaining statements of Theorem 10 follow.

Lemma 11. *The subgroups U and $\text{Ad}(IB)$ are normal in $\text{Aut}(I\mathfrak{b})$. Moreover, $U \cap \text{Ad}(IB) = \{\text{Id}\}$.*

Proof. Recall that $\text{Ad}(IB)$ is generated by the exponentials of $\text{ad}(x)$ with $x \in I\mathfrak{b}$. Then for any $\theta \in \text{Aut}(I\mathfrak{b})$,

$$\theta \text{Ad}(IB) \theta^{-1} = \theta \exp(I\mathfrak{b}) \theta^{-1} = \exp(\theta(I\mathfrak{b})) = \exp(I\mathfrak{b}) = \text{Ad}(IB).$$

Take now $u \in \text{Hom}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}], \mathfrak{z}(I\mathfrak{b}))$ so that $\bar{u} \in U$. Since $[I\mathfrak{b}, I\mathfrak{b}]$ (resp. $\mathfrak{z}(I\mathfrak{b})$) is a characteristic subspace of $I\mathfrak{b}$, it is stabilized by θ^{-1} (resp. θ). Hence $\theta u \theta^{-1}$ vanishes on $[I\mathfrak{b}, I\mathfrak{b}]$ and takes values in \mathfrak{z} . So $\theta \bar{u} \theta^{-1}$ is an element of U .

Let $(b, g) \in IB$ and $h \in \mathfrak{h}$. Then $\text{Ad}(b, g)(h) = b \cdot h + t[g, h] \subset h + (\mathfrak{n} \oplus t\mathfrak{n}^-)$. As a consequence, whenever $\text{Ad}(b, g)(\mathfrak{h}) \subset \mathfrak{h} + \mathfrak{z}$ we have $\text{Ad}(b, g)(\mathfrak{h}) \subset \mathfrak{h}$. Hence $\text{Ad}(IB) \cap U = \{\text{Id}\}$. \square

Lemma 12. *We have $\text{Aut}(I\mathfrak{b}) = GD\text{Ad}(IB)U$.*

Proof. Let $\theta \in \text{Aut}(I\mathfrak{b})$. Since the two Cartan subalgebras \mathfrak{c} and $\theta(\mathfrak{c})$ are Ad-conjugate (see [Bou75, §3, n° 3, th. 2]), there exists $\theta_1 \in \text{Ad}(IB)\theta$ which stabilizes \mathfrak{c} .

Then $\theta_1(\mathfrak{h})$ is complementary to the center $t\mathfrak{h} = \theta_1(t\mathfrak{h})$ in \mathfrak{c} . Thus, there exists $\theta_2 \in U\theta_1$ such that θ_2 stabilizes \mathfrak{h} .

Since θ_2 stabilizes \mathfrak{c} , it acts on $\Phi(I\mathfrak{b})$. Arguing as in the proof of Proposition 9, we show that it stabilizes $\Delta(I\mathfrak{b})$ and that the induced permutation is actually an automorphism of

the extended Dynkin diagram. Thus there exists $\theta_3 \in G\theta_2$ with the additional property that the induced permutation on $\Delta(I\mathfrak{b})$ and thus on $\Phi(I\mathfrak{b})$ are trivial. Then θ_3 acts on each $(I\mathfrak{b})_\alpha$ for $\alpha \in \Delta(I\mathfrak{b})$.

Since Δ is a basis of \mathfrak{h}^* , one can find $h \in H \subset B \subset IB$ such that $\text{Ad}(h) \circ \theta_3$ acts trivially on each $(I\mathfrak{b})_\alpha$ for $\alpha \in \Delta$. Moreover, D acts trivially on these roots spaces and with weight 1 on $(I\mathfrak{b})_{\alpha_0}$. This yields $\theta_4 \in D\text{Ad}(H)G\text{UAd}(IB)\theta$ which acts trivially on \mathfrak{h} and on each $(I\mathfrak{b})_\alpha$, $\alpha \in \Delta(I\mathfrak{b})$.

Recall now that $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ is generated by the spaces $((I\mathfrak{b})_\alpha)_{\alpha \in \Delta(I\mathfrak{b})}$. Since θ_4 acts trivially on $\tilde{\mathfrak{n}}$ and on \mathfrak{h} , it has to be trivial. As a consequence, $\theta \in \text{Ad}(IB)U\text{GAd}(H)D = G\text{DAd}(IB)U$, the last equality following from Lemma 11. \square

Lemma 13. *The intersections $D \cap (\text{Ad}(IB) \times U)$ and $G \cap (D \times (\text{Ad}(IB) \times U))$ are the trivial group $\{\text{Id}\}$. Moreover, $(D \times (\text{Ad}(IB) \times U))$ is normal in $\text{Aut}(I\mathfrak{b})$.*

Proof. Let $\tau \in \mathbb{C}^*$, $b \in B$, $f \in \mathfrak{g}/\mathfrak{n}$ and $u \in \text{Hom}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}], \mathfrak{z}(I\mathfrak{b}))$ such that the elements associated $\delta_\tau \in D$, $(b, f) \in IB$ and $\tilde{u} \in U$ (see Section 3) satisfy $\delta_\tau = \text{Ad}(b, f) \circ \tilde{u}$. For $b' \in \mathfrak{b}$, we have

$$b' = \delta_\tau(b') = (\text{Ad}(b, f) \circ \tilde{u})(b') = \text{Ad}(b, f)(b' + u(b')) = b \cdot b' + (b \cdot u(b') + t([f, b'])).$$

In particular, $b \cdot b' = b'$ and, whenever $b' \in \mathfrak{n}$, $[f, b'] = 0$ in $\mathfrak{g}/\mathfrak{n}$. So $b \in B$ centralizes \mathfrak{b} and $\text{ad}_{\mathfrak{g}} f$ normalizes \mathfrak{n} . As a consequence, $b = 1_B$, f is 0 in $\mathfrak{g}/\mathfrak{b}$ and $u = 0$. Thus the only element of $D \cap (\text{Ad}(IB) \times U)$ is the trivial one.

Since $[I\mathfrak{b}, I\mathfrak{b}]$ is characteristic in $I\mathfrak{b}$, we have a natural group morphism $p : \text{Aut}(I\mathfrak{b}) \rightarrow \text{Aut}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}])$. From the description of $[I\mathfrak{b}, I\mathfrak{b}]$ in Lemma 3, it is straightforward that D , $\text{Ad}(IB)$ and U are included in $\text{Ker}(p)$ while $p|_G$ is injective. From Lemma 12, we then deduce that $D \times (\text{Ad}(IB) \times U) = \text{Ker}(p)$ and the desired properties follow. \square

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