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On the automorphisms of the Drinfel'd double of a Borel Lie subalgebra

Michaël Bulois and Nicolas Ressayre

June 21, 2020

Abstract

Let \mathfrak{g} be a complex simple Lie algebra with Borel subalgebra \mathfrak{b} . Consider the semidirect-product $I\mathfrak{b} = \mathfrak{b} \ltimes \mathfrak{b}^*$, where \mathfrak{b}^* , the dual of \mathfrak{b} , is equipped with the coadjoint action of \mathfrak{b} and is considered as an abelian ideal of $I\mathfrak{b}$. Given an automorphism γ the extended Dynkin diagram of \mathfrak{g} , we construct an automorphism of $I\mathfrak{b}$ of the same order.

In type A, these automorphisms were recently constructed by Dror Bar-Natan and Roland Van Der Veen in [Bv20] (where $I\mathfrak{b}$ is denoted by $I\mathfrak{u}_n$). Their construction is by hand and they ask for an explanation: this note fully answers the question.

1 Introduction

Given any Lie algebra \mathfrak{a} , one can form its “inhomogeneous version” $I\mathfrak{a} := \mathfrak{a} \ltimes \mathfrak{a}^*$. It is the semidirect product of \mathfrak{a} with its dual \mathfrak{a}^* where \mathfrak{a}^* is considered as an abelian Lie algebra and \mathfrak{a} acts on \mathfrak{a}^* via the coadjoint action.

Here we consider the case when $\mathfrak{a} = \mathfrak{b}$ is the Borel subalgebra of some complex simple Lie algebra \mathfrak{g} . In this case, \mathfrak{b}^* can be understood differently. Indeed, let \mathfrak{n} be the derived subalgebra of \mathfrak{b} and \mathfrak{b}^- be the opposite Borel subalgebra of \mathfrak{b} (relatively to some fixed Cartan subalgebra). Now, \mathfrak{b}^- identifies with $\mathfrak{g}/\mathfrak{n}$ and hence inherits from a structure of a \mathfrak{b} -module. Then \mathfrak{b}^- identifies with \mathfrak{b}^* as a \mathfrak{b} -module.

Consider now the affine Kac-Moody Lie algebra \mathfrak{g}^{KM} , its center $Z(\mathfrak{g}^{\text{KM}})$ and its Borel subalgebra \mathfrak{b}^{KM} . Let \mathfrak{n}^{KM} denote the derived subalgebra of \mathfrak{b}^{KM} . By killing the semidirect-product and the central extension from the construction of \mathfrak{g}^{KM} , one gets

$$\begin{aligned} \tilde{\mathfrak{g}} &:= [\mathfrak{g}^{\text{KM}}, \mathfrak{g}^{\text{KM}}]/Z(\mathfrak{g}^{\text{KM}}) \\ &\cong \mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g}, \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathfrak{b}} &:= (\mathfrak{b}^{\text{KM}} \cap [\mathfrak{g}^{\text{KM}}, \mathfrak{g}^{\text{KM}}])/Z(\mathfrak{g}^{\text{KM}}) \subset \tilde{\mathfrak{g}} \\ \tilde{\mathfrak{n}} &:= (\mathfrak{n}^{\text{KM}} \cap [\mathfrak{g}^{\text{KM}}, \mathfrak{g}^{\text{KM}}])/Z(\mathfrak{g}^{\text{KM}}) = [\tilde{\mathfrak{b}}, \tilde{\mathfrak{b}}]. \end{aligned}$$

We identify \mathfrak{g} with the subspace $\mathbb{C} \otimes \mathfrak{g} \subset \tilde{\mathfrak{g}}$.

Theorem 1. *The map*

$$\begin{aligned} I\mathfrak{b} &\longrightarrow \tilde{\mathfrak{b}}/(t\tilde{\mathfrak{n}}) \\ \xi + \zeta &\longmapsto \xi + t\zeta, \end{aligned}$$

with $\xi \in \mathfrak{b}$ and $\zeta \in \mathfrak{b}^-$ is a well defined isomorphism of Lie algebras.

Consider now an automorphism γ of the extended Dynkin diagram $\tilde{\mathcal{D}}_{\mathfrak{g}}$ of \mathfrak{g} . Since \mathfrak{g}^{KM} is canonically defined from its Dynkin diagram, γ induces an automorphism (still denoted by γ) of \mathfrak{g}^{KM} . Moreover, by construction γ preserves both the Cartan subalgebra of \mathfrak{g}^{KM} , its Borel subalgebra \mathfrak{b}^{KM} and \mathfrak{n}^{KM} . Thus it induces an isomorphism $\bar{\gamma}$ of $\tilde{\mathfrak{b}}/(t\tilde{\mathfrak{n}})$.

Corollary 2. *The above construction induces an injective morphism from $\text{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}})$ to $\text{Aut}(I\mathfrak{b})$.*

At the end of the paper, a table sums up the groups $\text{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}})$ for \mathfrak{g} simple.

Comments. This note was first sent to the authors of [Bv20]. We thank Dror Bar-Natan and Roland van der Veen for their prompt answers. In doing so, we learned that they were already informed by A. Knutson that very similar connexion was made in [KZJ07, Sections 2.1 and 2.3], at least in type A.

2 Proof

The proofs are simple checkings. First, it follows from the constuction of the affine Kac-Moody algebra that we have the following decompositions

$$\begin{aligned} \tilde{\mathfrak{b}} &= \mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{n}^- \\ \tilde{\mathfrak{n}} &= \mathbb{C}[t]\mathfrak{n} \oplus t\mathbb{C}[t]\mathfrak{b}^-, \end{aligned}$$

where \mathfrak{n}^- is the derived subalgebra of \mathfrak{b}^- . Then, it is obvious that $t\tilde{\mathfrak{n}}$ is an ideal of $\tilde{\mathfrak{b}}$ and that $\tilde{\mathfrak{b}}/(t\tilde{\mathfrak{n}})$ is a Lie algebra. These decompositions imply that the linear map from Theorem 1 is well defined.

Consider the quotient map $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{b} \simeq \mathfrak{b}^-$.

Let $\xi, \xi' \in \mathfrak{b}$ and $\zeta, \zeta' \in \mathfrak{b}^-$. Then, computing in $\tilde{\mathfrak{b}}$, we get:

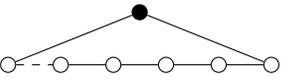
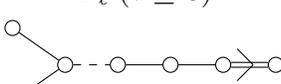
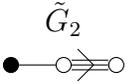
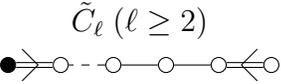
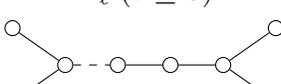
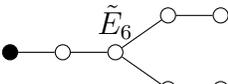
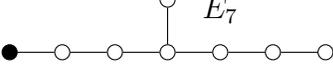
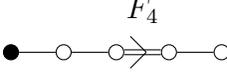
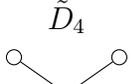
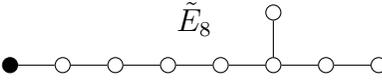
$$[\xi + t\zeta, \xi' + t\zeta'] = [\xi, \xi'] + t([\zeta, \xi'] + [\zeta', \xi]) + t^2[\zeta, \zeta'].$$

Its class modulo $t\tilde{\mathfrak{n}}$ is

$$[\xi, \xi'] + t\pi([\zeta, \xi'] + [\zeta', \xi]).$$

This implies that the map is a Lie algebra isomorphism.

Finally, a diagram automorphism γ is characterized by the induced permutation of the root spaces of \mathfrak{g}^{KM} associated to simple roots. Since these root spaces survive in $I\mathfrak{b}$, the injectivity statement in the corollary is clear.

\tilde{A}_1  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z}$	$\tilde{A}_\ell (\ell \geq 2)$  $\text{Aut}(\tilde{\mathcal{D}}) = D_{(\ell+1)}$	$\tilde{B}_\ell (\ell \geq 3)$  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z}$
\tilde{G}_2  $\text{Aut}(\tilde{\mathcal{D}})$ is trivial	$\tilde{C}_\ell (\ell \geq 2)$  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z}$	$\tilde{D}_\ell (\ell \geq 5)$  $\text{Aut}(\tilde{\mathcal{D}}) = (\mathbb{Z}/2\mathbb{Z})^3$
\tilde{E}_6  $\text{Aut}(\tilde{\mathcal{D}}) = \mathfrak{S}_3$	\tilde{E}_7  $\text{Aut}(\tilde{\mathcal{D}}) = \mathbb{Z}/2\mathbb{Z}$	\tilde{F}_4  $\text{Aut}(\tilde{\mathcal{D}})$ is trivial
\tilde{D}_4  $\text{Aut}(\tilde{\mathcal{D}}) = \mathfrak{S}_4$	\tilde{E}_8  $\text{Aut}(\tilde{\mathcal{D}})$ is trivial	

References

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- [KZJ07] Allen Knutson and Paul Zinn-Justin. A scheme related to the Brauer loop model. *Adv. Math.*, 214(1):40–77, 2007.

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