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On Some Special Substructures of Refined Neutrosophic Rings

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Abstract

The objective of this article is to define and study the concepts of refined neutrosophic AH-ideal and AHS-ideal in refined neutrosophic rings. We investigate the elementary properties of these concepts.

Keywords: Refined neutrosophic ring, Refined neutrosophic AH-ideal, Refined neutrosophic AHS-Ideal, Refined AHS-homomorphism.

1. Introduction

Neutrosophy as a new branch of philosophy can be applied into the algebraic systems, which leads to a better comprehension and evolution of these systems. The notion of neutrosophic groups and rings was defined by Kandasamy and Smarandache in [10], and studied widely in [4, 5, 7, 8]. Studies were carried out on neutrosophic rings and neutrosophic hyper rings. See [1, 3, 4, 6].

Refined neutrosophic rings were defined and studied carefully in [2, 3], where special substructures such as refined neutrosophic subrings and refined neutrosophic ideals are defined. Many interesting results were proved. In [1] concepts as AH-ideal and AHS-ideal were defined and studied as interesting substructures of neutrosophic ring. Some related concepts such as weak principal, maximal, and prime AH-ideals were introduced. These concepts have many properties which are similar to classical case of rings. In this paper, we try to define concepts such as AH-ideal and AHS-ideal in refined neutrosophic ring with some related concepts such as weak prime, principal and maximal refined neutrosophic AH-ideals. Also, we introduce the notion of refined AHS-homomorphism in similar way to AHS-homomorphism defined in [1].

Motivation

This paper is the continuation of the work began in the paper entitled "On Some Special Substructures of Neutrosophic Rings and Their properties".

2. Preliminaries

Definition 2.1: [5]

Let (R, +, ×) be a ring then R(I) = {a+bi; a, b ∈ R} is called the neutrosophic ring where I is a neutrosophic element with the condition I² = I.

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Remark 2.2: [2]
The element I can be split into two indeterminacies $I_1, I_2$ with conditions:

\[ I_1^2 = I_1, I_2^2 = I_2, I_1I_2 = I_2I_1 = I_1. \]

Definition 2.3: [2]
If $X$ is a set then $X(I_1, I_2) = \{(a, bl_1, cl_2); a, b, c \in X\}$ is called the refined neutrosophic set generated by $X, I_1, I_2.$

Definition 2.4: [2]
Let $(R, +, \times)$ be a ring, $(R(I_1, I_2), +, \times)$ is called a refined neutrosophic ring generated by $R, I_1, I_2.$

Definition 2.5: [2]
Let $(R(I_1, I_2), +, \times)$ be a refined neutrosophic ring; it is called commutative if

\[ x \times y = y \times x, \forall x, y \in R(I_1, I_2). \]

Theorem 2.6: [2]
Let $(R(I_1, I_2), +, \times)$ be a refined neutrosophic ring then it is a ring.

Definition 2.7: [3]
Let $(R(I_1, I_2), +, \times)$ be a refined neutrosophic ring and $J$ be a nonempty subset of $R(I_1, I_2)$ then $J$ is called a neutrosophic refined ideal if:

(a) $J$ is a refined neutrosophic subring of $R(I_1, I_2)$.

(b) For every $x \in J$ and $r \in R(I_1, I_2)$ then $x \times r \in R(I_1, I_2)$.

Definition 2.8: [1]
Let $R(I)$ be a neutrosophic ring and $P = P_0 + P_1I = \{a_0 + a_1I; a_0 \in P_0, a_1 \in P_1\}.$

(a) We say that $P$ is an AH-ideal if $P_0, P_1$ are ideals in the ring $R.$

(b) We say that $P$ is an AHS-ideal if $P_0 = P_1.$

Definition 2.9: [1]
Let $R(I), T(J)$ be two neutrosophic rings and the map $f: R(I) \rightarrow T(J)$; we say that $f$ is a neutrosophic AHS-homomorphism:

Restriction of the map $f$ on $R$ is a ring homomorphism from $R$ to $T,$ i.e $f_R: R \rightarrow T$ is homomorphism and

\[ f(\alpha + bl) = f_R(\alpha) + f_R(b)J. \]

We say that $R(I), T(J)$ are AHS-isomorphic neutrosophic rings if there is a neutrosophic AHS-homomorphism

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\( f: R(I) \to T(J) \) which is a bijective map; i.e ( \( R \cong T \) ), we say that \( f \) is a neutrosophic AHS-isomorphism.

**Definition 2.10:**

Let \( R(I) \) be a neutrosophic ring and \( P = P_0 + P_1 I \) be an AH-ideal, we define the AH-factor as:

\[
R(I)/P = R/P_0 + R/P_1 I.
\]

**Theorem 2.11:**

Let \( R(I) \) be a neutrosophic ring and \( P = P_0 + P_1 I \) be an AH-ideal then \( R(I)/P \) is a ring.

**Theorem 2.12:**

Let \( R(I), T(J) \) be two neutrosophic rings and \( f: R(I) \to T(J) \) is a neutrosophic ring AHS-homomorphism, let \( P = P_0 + P_1 I \) be an AH-ideal of \( R(I) \) and \( Q = Q_0 + Q_1 J \) be an AH-ideal of \( T(J) \), we have:

(a) \( f(P) \) is an AH-ideal of \( f(R(I)) \).
(b) \( f^{-1}(Q) \) is an AH-ideal of \( R(I) \).
(c) If \( P \) is an AHS-ideal of \( R(I) \), \( f(P) \) is an AHS-ideal of \( f(R(I)) \).
(d) \( AH - ker f = ker f_R + ker f_R I \) is an AHS-ideal; \( f_R \) is the restriction of \( f \) on the ring \( R \).
(e) The AH-factor \( R(I)/ker f \) is AHS-isomorphic to \( f(R(I)) \).

**Definition 2.13:**

Let \( R(I) \) be a neutrosophic commutative ring and \( P = P_0 + P_1 I \) be an AH-ideal, we say that:

(a) \( P \) is a weak prime AH-ideal if \( P_0, P_1 \) are prime ideals in \( R \).
(b) \( P \) is a weak maximal AH-ideal if \( P_0, P_1 \) are maximal ideals in \( R \).
(c) \( P \) is a weak principal AH-ideal if \( P_0, P_1 \) are principal ideals in \( R \).

3. Main concepts and discussion

**Definition 3.1:**

Let \((R, +, \times), (T, +, \times)\) be a refined neutrosophic ring, and \( P_0, P_1, P_2 \) be three ideals in the ring \( R \) then the set \( P = (P_0, P_1 I, P_2 I_2) = \{(a, b I_1, c I_2); a \in P_0, b \in P_1, c \in P_2\} \) is called a refined neutrosophic AH-ideal.

If \( P_0 = P_1 = P_2 \) then \( P \) is called a refined neutrosophic AHS-ideal.

**Definition 3.2:**

Let \( (R, +, \times), (T, +, \times) \) be two rings and \( f_R: R \to T \) is a homomorphism:

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The map \( f: R(I_1, I_2) \to T(I_1, I_2); f(x, y 
\times I_1, zI_2) = (f_R(x), f_R(y)I_1, f_R(z)I_2) \) is called a refined AHS-homomorphism.

It is easy to see that for all \( x, y \in R(I_1, I_2) \), we have \( f(x + y) = f(x) + f(y), f(x \times y) = f(x) \times f(y) \).

**Example 3.3:**

Suppose that \( R = (Z_{30^0}^+, \times), T = (Z_{30^0}^+, \times) \) are two rings and \( f_R: R \to T; f(a) = 5a \) is homomorphism, the related refined AHS-homomorphism can be defined:

\[
f: R(I_1, I_2) \to T(I_1, I_2); f(x, y)I_1, zI_2) = (5x, 5yI_1, 5zI_2).
\]

The previous example shows that refined AHS-homomorphism is not a refined neutrosophic homomorphism in general because:

\( f(I_1) \neq I_1 \)

**Definition 3.4:**

(a) Let \( f: R(I_1, I_2) \to T(I_1, I_2) \) be a refined AHS-homomorphism, we define refined AH-Kernel of \( f \) by:

\[
AH - Ker f = \{ (a,b,c)I_1; a,b,c \in Ker f_R \} = (Ker f_R, Ker f_R I_1, Ker f_R I_2).
\]

(b) Let \( S = (S_0, S_1I_1, S_2I_2) \) be a subset of \( R(I_1, I_2) \), then:

\[
f(S) = (f_R(S), f_R(S_1)I_1, f_R(S_2)I_2) = (f_R(a_0), f_R(a_1)I_1, f_R(a_2)I_2); a_1 \in S_1 \).
\]

(c) Let \( S = (S_0, S_1I_1, S_2I_2) \) be a subset of \( T(I_1, I_2) \), then:

\[
f^{-1}(S) = (f_R^{-1}(S_0), f_R^{-1}(S_1)I_1, f_R^{-1}(S_2)I_2).
\]

**Definition 3.5:**

Let \( f: R(I_1, I_2) \to T(I_1, I_2) \) be a refined AHS-homomorphism we say that \( f \) is a refined AHS-isomorphism if it is a bijective map, \( R(I_1, I_2), T(I_1, I_2) \) are called AHS-isomorphic refined neutrosophic rings.

It is easy to see that restriction \( f_R \) will be an isomorphism between \( R, T \).

**Theorem 3.6:**

Let \( f: R(I_1, I_2) \to T(I_1, I_2) \) be a refined AHS-homomorphism we have:

(a) \( AH - Ker f \) is a refined neutrosophic AH-ideal of \( R(I_1, I_2) \).

(b) If \( P \) is a refined neutrosophic AH-ideal of \( R(I_1, I_2) \), \( f(P) \) is a refined neutrosophic AH-ideal of \( T(I_1, I_2) \).

(c) If \( P \) is a refined neutrosophic AH-ideal of \( R(I_1, I_2) \), \( f(P) \) is a refined neutrosophic AH-ideal of \( T(I_1, I_2) \).

Proof:

(a) Since \( Ker f_R \) is an ideal of \( RAH - ker f = (Ker f_R, Ker f_R I_1, Ker f_R I_2) \) is a refined neutrosophic AH-ideal of \( R(I_1, I_2) \).

(b) Suppose that \( P = (P_0, P_1I_1, P_2I_2) \) is a refined neutrosophic AH-ideal of \( R(I_1, I_2) \). Since \( f_R(P_0) \) is an ideal of \( T, f(P) = (f_R(P_0), f_R(P_1)I_1, f_R(P_2)I_2) \) is a refined neutrosophic AH-ideal.

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(c) The proof is similar to (b).

**Definition 3.7:**

Let \((R(I_1, I_2), +, \times)\) be a refined neutrosophic ring and \(P = (P_0, P_1 I_1, P_2 I_2)\) be a refined neutrosophic AH-ideal then:

(a) We say that \(P\) is a weak prime refined neutrosophic AH-ideal if \(P_i; i \in \{0, 1, 2\}\) are prime ideals in \(R\).

(b) We say that \(P\) is a weak maximal refined neutrosophic AH-ideal if \(P_i; i \in \{0, 1, 2\}\) are maximal ideals in \(R\).

(c) We say that \(P\) is a weak principal refined neutrosophic AH-ideal if \(P_i; i \in \{0, 1, 2\}\) are principal ideals in \(R\).

(d) We define the refined neutrosophic AH-factor as:

\[
R(I_1, I_2)/P = (R/P_0, R/P_1 I_1, R/P_2 I_2) = \left\{ ([x_0 + P_0], [x_1 + P_1] I_1, [x_2 + P_2] I_2); \ x_0, x_1, x_2 \in R \right\}.
\]

**Theorem 3.8:**

Let \(f: R(I_1, I_2) \rightarrow T(I_1, I_2)\) be a refined AHS-homomorphism and \(P = (P_0, P_1 I_1, P_2 I_2)\) be a refined neutrosophic AH-ideal of \(R(I_1, I_2)\), let \(Q=(Q_0, Q_1 I_1, Q_2 I_2) \neq T(I_1, I_2)\) be a refined neutrosophic AH-ideal of \(T(I_1, I_2)\), assume that \(\text{Ker} f \subseteq P_i \neq R\) then:

(a) \(P\) is a weak prime refined neutrosophic AH-ideal of \(R(I_1, I_2)\) if and only if \(f(P)\) is a weak prime refined neutrosophic AH-ideal in \(f(R(I_1, I_2))\).

(b) \(P\) is a weak maximal AH-ideal of \(R(I_1, I_2)\) if and only if \(f(P)\) is a weak maximal in \(f(R(I_1, I_2))\).

(c) \(Q\) is a weak prime AH-ideal of \(T(I_1, I_2)\) if and only if \(f^{-1}(Q)\) is a weak prime in \(R(I_1, I_2)\).

(d) \(Q\) is a weak maximal AH-ideal of \(T(I_1, I_2)\) if and only if \(f^{-1}(Q)\) is a weak maximal in \(R(I_1, I_2)\).

Proof:

The proof is similar to the Theorem 3.15 in [1].

It is easy to see that conditions (a), (b) are still true if \(P\) is an AHS-ideal and conditions (c), (d) are still true if \(Q\) is an AHS-ideal.

**Theorem 3.9:**

The refined neutrosophic AH-factor \(R(I_1, I_2)/P\) is a ring with respect to the following operations:

Let \(x = (x_0 + P_0, (x_1 + P_1) I_1, (x_2 + P_2) I_2), y = (y_0 + P_0, (y_1 + P_1) I_1, (y_2 + P_2) I_2)\), be two arbitrary elements in \(R(I_1, I_2)\) then:

\[
x + y = ([x_0 + y_0] + P_0, ([x_1 + y_1] + P_1) I_1, ([x_2 + y_2] + P_2) I_2).
\]

\[
x \times y = ([x_0 \times y_0] + P_0, ([x_1 \times y_1] + P_1) I_1, ([x_2 \times y_2] + P_2) I_2).
\]

Proof:

The proof is similar to the Theorem 3.9 in [1].

**Example 3.10:**

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Let \( R = (\mathbb{Z}, +, \times) \) be two rings, and \( f \) be the refined neutrosophic \( AHS \)-homomorphism defined in Example 3.3, we have the following:

(a) \( P_0 = \{0,2,4\}, P_1 = \{0,3\} \) are two ideals in \( \mathbb{Z} \) thus \( P = (P_0, P_1, I_1, I_2) \) is a refined neutrosophic \( AHS \)-ideal of \( R(1_1, I_2) \).

\[
(f(P_0), f(P_1), I_1, f(P_2)I_2) = \{(0,0,0), (0,0,5I_2)\}
\]
is a refined neutrosophic \( AHS \)-ideal in \( T(I_1, I_2) \).

(c) \( Q_0 = \{0,2,4,6,8\} \) is a maximal ideal in \( \mathbb{Z} \) and \( f^{-1}(Q_0) = \{0,2,4\} \), so \( Q = (Q_0, Q_0I_1, Q_0I_2) \) is a weak maximal refined neutrosophic \( AHS \)-ideal in \( T(I_1, I_2) \), we have \( f^{-1}(Q) = \{0,2,4\}, \{0,2,4\}I_1, \{0,2,4\}I_2 \) is a weak maximal refined neutrosophic \( AHS \)-ideal in \( R(I_1, I_2) \).

**Example 3.11:**

(a) In the ring \( (\mathbb{Z}, +, \times) \), \( P = \langle 3 \rangle, Q = \langle 2 \rangle \) are two prime and maximal ideals, thus \( M = (P_0, Q_0I_1, Q_0I_2) = \langle 3a, 2bI_1, 2cI_2 \rangle \); \( a, b, c \in \mathbb{Z} \) is a weak maximal/prime refined neutrosophic \( AHS \)-homomorphism is

\[
\begin{align*}
\rho: \mathbb{Z}(I_1, I_2) &\rightarrow \mathbb{Z}_0(I_1, I_2); \\
\rho(a, bI_1, cI_2) &\equiv (a \mod 6, (b \mod 6)I_1, (c \mod 6)I_2), \\
\mathbb{A}H - \ker(\rho) &\equiv (6Z, 6ZI_1, 6ZI_2) \subseteq M
\end{align*}
\]

since \( 6Z \leq P_0, Q_0 \).

(b) The map \( f_2: Z \rightarrow Z_0; f(a) = a \mod 6 \) is a homomorphism so the related refined neutrosophic \( AHS \)-homomorphism is

\[
\begin{align*}
\rho: \mathbb{Z}(I_1, I_2) &\rightarrow \mathbb{Z}_0(I_1, I_2); \\
\rho(a, bI_1, cI_2) &\equiv (a \mod 6, (b \mod 6)I_1, (c \mod 6)I_2), \\
\mathbb{A}H - \ker(\rho) &\equiv (6Z, 6ZI_1, 6ZI_2) \subseteq M
\end{align*}
\]

(c) \( f(M) = \{(0,3), \{0,2,4\}I_1, \{0,2,4\}I_2 \} \) is a weak maximal/prime refined neutrosophic \( AHS \)-ideal of \( \mathbb{Z}_0(I_1, I_2) \).

**Definition 3.12:**

A refined neutrosophic ring \( R(1_1, I_2) \) is called weak principal refined neutrosophic \( AHS \)-ring if every refined neutrosophic \( AHS \)-ideal is weak principal.

**Theorem 3.13:**

Let \( R \) be a principal ideal ring then \( R(1_1, I_2) \) is weak principal refined neutrosophic \( AHS \)-ring.

**Proof:**

Let \( P = (P_0, P_1I_1, P_2I_2) \) be a refined neutrosophic \( AHS \)-ideal of \( R(1_1, I_2) \). Since \( P_1 \) are ideals in \( R \) and then principal, this implies that \( P \) is a weak refined neutrosophic \( AHS \)-ideal; thus \( R(1_1, I_2) \) must be weak principal refined neutrosophic \( AHS \)-ring.

**Example 3.14:**

The ring \( (\mathbb{Z}, +, \times) \) is principal ideals ring; thus \( Z(I_1, I_2) \) is weak principal refined neutrosophic \( AHS \)-ring.

**Definition 3.15:**

Let \( (R(I_1, I_2), +, \times) \) be a refined neutrosophic ring and \( P = (P_0, P_1I_1, P_2I_2) \), \( Q = (Q_0, Q_1I_1, Q_2I_2) \) be two refined neutrosophic \( AHS \)-ideals of \( R(I_1, I_2) \), then we define:

(a) \( P \cap Q = (P_0 \cap Q_0, [P_1 \cap Q_1]I_1, [P_2 \cap Q_2]I_2) \).

(b) \( P + Q = (P_0 + Q_0, [P_1 + Q_1]I_1, [P_2 + Q_2]I_2) \).

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(c) \( P \times Q = (P_0 \times Q_0, [P_1 \times Q_1]_1, [P_2 \times Q_2]_2) \).

**Theorem 3.16:**

Let \((R(I_1, I_2), +, \times)\) be a refined neutrosophic ring and \(P = (P_0, P_1, P_2)\), \(Q = (Q_0, Q_1, Q_2)\) be two refined neutrosophic AHS-ideals of \(R(I_1, I_2)\), then:

\[ P \cap Q, P + Q, P \times Q \text{ are refined neutrosophic AHS-ideals of } R(I_1, I_2). \]

Proof:

As a result of Theorem 2.5 in[1], we have \(P_i + Q_i, P_i \cap Q_i, P_i \times Q_i\) are ideals of \(R\), thus the proof is complete.

**Remark 3.17:**

Theorem 3.16 is still true if \(P\) and \(Q\) are refined neutrosophic AHS-ideals.

**Example 3.18:**

Let \(R(I_1, I_2) = Z_4(I_1, I_2)\) and \(Q = \{0, 4\}, S = \{0, 2, 4, 6\}\) be two principal ideals in \(R\), then:

(a) \(P = (S, Q_1, S_1)\) is a refined neutrosophic AHS-ideal of \(R(I_1, I_2)\), the related refined neutrosophic AHS-factor is:

\[ R(I_1, I_2)/P = (R/S, R/Q_1, R/S_1) = (\{S, 1+S\}, \{Q, 1+Q, 2+Q, 3+Q\}I_1, \{S, 1+S\}I_2). \]

To clarify addition and multiplication on \(R(I_1, I_2)/P\) we take:

\[ x = (1 + S, 1 + Q)I_1, S_1), y = (S, 2 + Q)I_1, (1 + S)I_2), \text{ we have:} \]

\[ x + y = ([1 + 0] + S, ([1 + 2] + Q)I_1, ([0 + 1] + S)I_2) = (1 + S, (3 + Q)I_1, (1 + S)I_2). \]

\[ x \times y = ([1 \times 0] + S, ([1 \times 2] + Q)I_1, ([0 \times 1] + S)I_2) = (S, (2 + Q)I_1, S_2). \]

**Conclusion**

In this article we defined concepts of refined neutrosophic AHS-ideal/ AHS-ideal in a refined neutrosophic ring. We studied some of elementary properties of these concepts. Also, notions as weak maximal, prime and principal refined neutrosophic AHS-ideal and refined AHS-homomorphisms were introduced and checked.

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