

Développement "ABCD" pour les schémas de Boltzmann sur réseau et application aux équations de Navier Stokes compressibles

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vendredi 19 juin 2020

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Macroscopique, mésoscopique, microscopique

trois échelles possibles pour représenter la nature

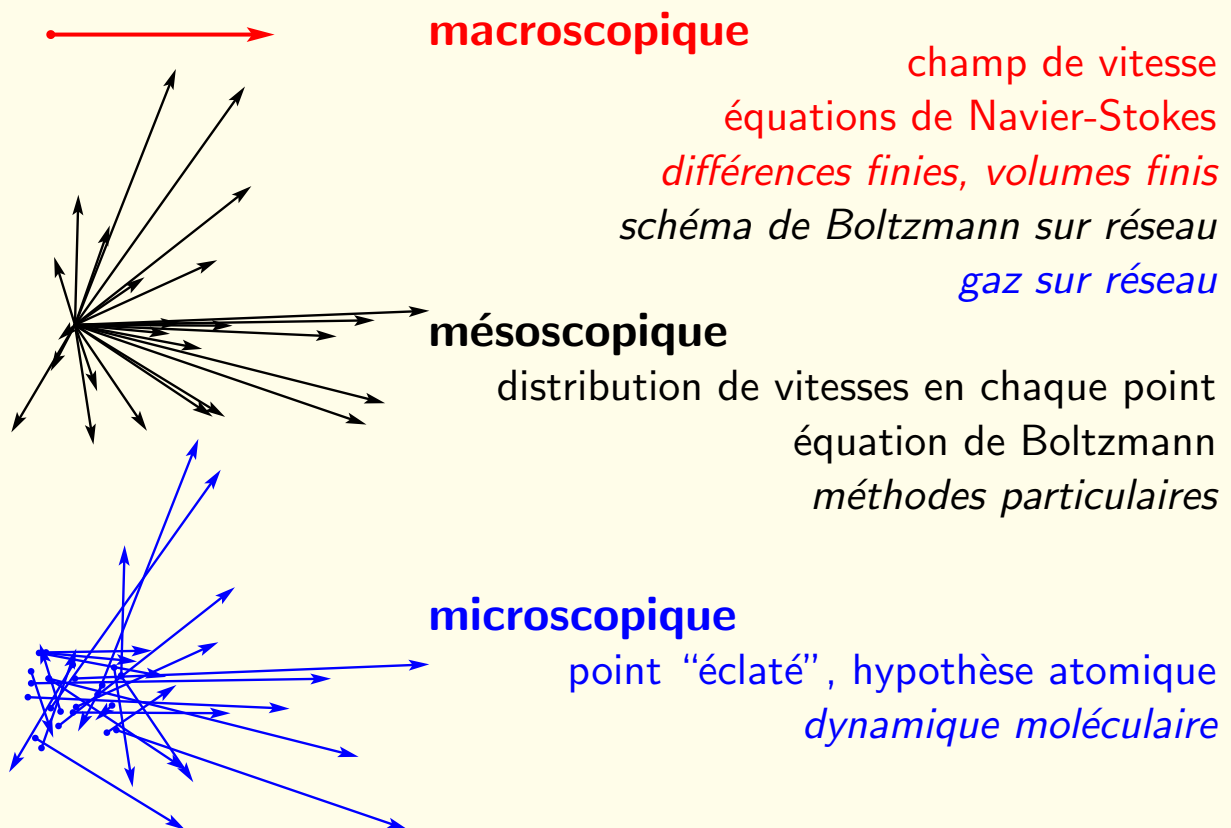
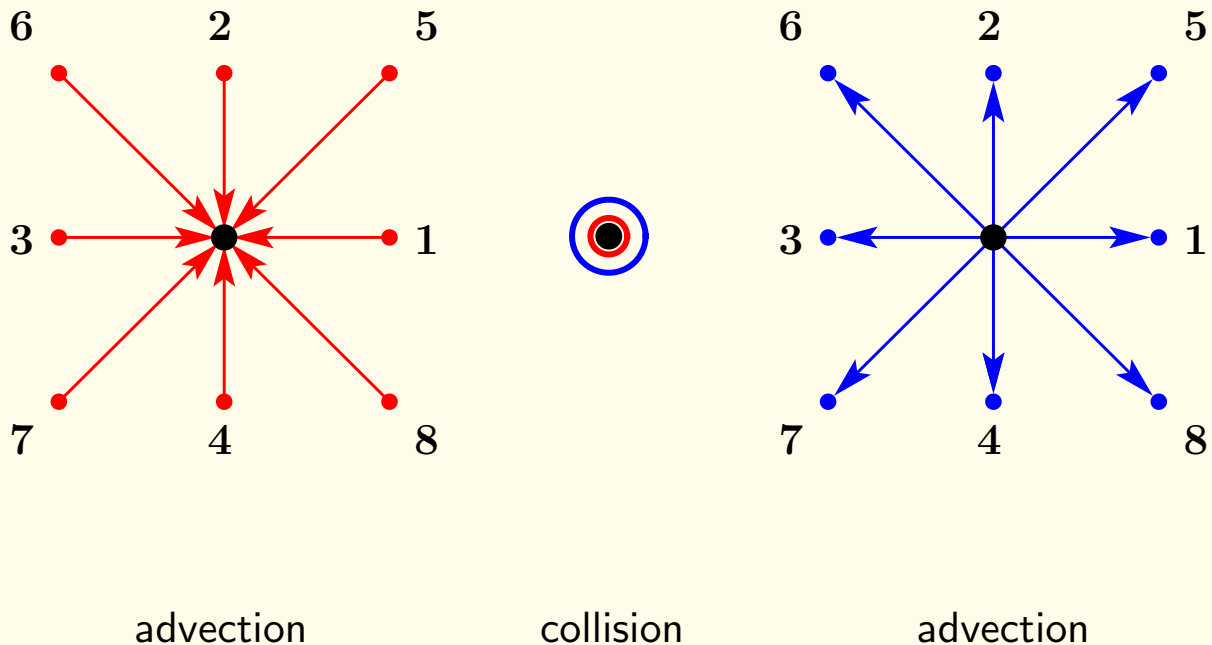


Schéma de Boltzmann sur réseau

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Outlook

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- Boltzmann equation with discrete velocities
 - Momentum-velocity operator
 - Chapman-Enskog expansion
- Multiple Relaxation Times Lattice Boltzmann scheme
 - Exact exponential expansion
 - "ABCD" method
- Isothermic Navier-Stokes in two and three space dimensions
 - D2Q9 and D2Q13
 - D3Q19, D3Q27 and D3Q33
- Navier-Stokes-Fourier in two space dimensions
 - D2Q13, D2Q17 and D2V17
- Conclusion

Boltzmann equation with discrete velocities

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space \mathbb{R}^d of dimension d

finite set of q discrete velocities $v_j \in \mathcal{V}$

with components v_j^α for $1 \leq \alpha \leq d$

unknowns of the Boltzmann equation

particle densities f_j

functions of space x , time t and discrete velocities v_j

$$f_j = f_j(x, t), \quad x \in \mathbb{R}^d, \quad t \geq 0, \quad 0 \leq j < q$$

collision vector $\mathbb{R}^q \ni f \mapsto Q(f) \in \mathbb{R}^q$

small parameter $\varepsilon > 0$

Knudsen number in the context of gas dynamics

Boltzmann equation with discrete velocities

$$\frac{\partial f_j}{\partial t} + \sum_{1 \leq \alpha \leq d} v_j^\alpha \frac{\partial f_j}{\partial x_\alpha} = \frac{1}{\varepsilon} Q_j(f), \quad 0 \leq j < q$$

Carleman (1957), Gross (1960), Broadwell (1964),
Gatignol (1970), ...

Moments

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constant invertible matrix M ("d'Humières matrix", 1992)

vector of moments $m \in \mathbb{R}^q$:

$$m_k \equiv \sum_{0 \leq j < q} M_{kj} f_j, \quad 0 \leq k < q$$

N conservation laws ($1 \leq N < q$)

the N first moments of the collision kernel are equal to zero:

$$\sum_{0 \leq j < q} M_{kj} Q_j(f) = 0, \quad \forall f \in \mathbb{R}^q, \quad 0 \leq k < N$$

divide the vector of moments into two families: $m \equiv \begin{pmatrix} W \\ Y \end{pmatrix}$

N conservation laws for macroscopic variables W

$$\frac{\partial W_k}{\partial t} + \sum_{1 \leq \alpha \leq d} \sum_{0 \leq j < q} M_{kj} v_j^\alpha \frac{\partial f_j}{\partial x_\alpha} = 0, \quad 0 \leq k < N$$

$q - N$ nonconserved moments or microscopic variables Y

Equilibrium states

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equilibrium states f^{eq} defined by the conditions $Q(f^{\text{eq}}) = 0$
 characterized with a regular vector field $\Phi : \mathbb{R}^N \longrightarrow \mathbb{R}^{q-N}$

$$f^{\text{eq}} = M^{-1} \begin{pmatrix} W \\ \Phi(W) \end{pmatrix}$$

the vector field $W \longmapsto Y^{\text{eq}} \equiv \Phi(W)$ defines the equilibrium states

hypothesis : the jacobian matrix $dQ(f^{\text{eq}})$ at equilibrium

is diagonalizable with real eigenvalues and real eigenvectors

there exists a diagonal matrix $Z = \text{diag} \left(\frac{1}{\tau_1}, \dots, \frac{1}{\tau_{q-N}} \right)$
 of order $q - N$, $\tau_\ell > 0$ such that

$$M dQ(f^{\text{eq}}) M^{-1} = - \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}$$

the jacobian operator $dQ(f^{\text{eq}})$

admits the matrix M^{-1} as a matrix of eigenvectors

the associated eigenvalues are all non positive

Momentum-velocity operator

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momentum-velocity operator matrix

$$\Lambda = M \text{diag} \left(\sum_{1 \leq \alpha \leq d} v^\alpha \partial_\alpha \right) M^{-1}$$

$q \times q$ operator matrix

composed by first-order space differential operators

advection operator seen in the space of moments

block decomposition of the momentum-velocity operator matrix

associated to the decomposition $m = \begin{pmatrix} W \\ Y \end{pmatrix}$

$$\Lambda \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

A : $N \times N$ first order space operator matrix

B : $N \times (q - N)$ first order space operator

C : $(q - N) \times N$ first order space operator

D : $(q - N) \times (q - N)$ first order operator

Boltmann-BGK system with discrete velocities

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Bhatnagar-Gross-Krook (1954) type hypothesis

the state f is close to equilibrium f^{eq}

collision kernel Q approximated by its first order expansion around the equilibrium state:

$$Q(f) \simeq Q(f^{\text{eq}}) + dQ(f^{\text{eq}}).(f - f^{\text{eq}})$$

condition $Q(f^{\text{eq}}) = 0$

Boltmann-BGK system with discrete velocities

$$\frac{\partial f_j}{\partial t} + \sum_{1 \leq \alpha \leq d} v_j^\alpha \frac{\partial f_j}{\partial x_\alpha} = \frac{1}{\varepsilon} dQ_j(f^{\text{eq}}).(f - f^{\text{eq}}), \quad 0 \leq j < q$$

f^{eq} is a function of the particle distribution f

$$W_k \equiv \sum_{0 \leq j < q} M_{kj} f_j, \quad 0 \leq k < N, \quad f^{\text{eq}} = M^{-1} \begin{pmatrix} W \\ \Phi(W) \end{pmatrix}$$

system of two coupled partial differential equations

$$\frac{\partial W}{\partial t} + A W + B Y = 0$$

$$\frac{\partial Y}{\partial t} + C W + D Y = -\frac{1}{\varepsilon} Z (Y - \Phi(W))$$

Chapman-Enskog expansion

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$$\frac{\partial W}{\partial t} + A W + B Y = 0, \quad \frac{\partial Y}{\partial t} + C W + D Y = -\frac{1}{\varepsilon} Z (Y - \Phi(W))$$

As ε tends to zero, the conserved moments W

satisfy the asymptotic system of partial differential equations

$$\frac{\partial W}{\partial t} + \Gamma_1(W) + \varepsilon \Gamma_2(W) = O(\varepsilon^2)$$

microscopic moments:

$$Y = \Phi(W) + \varepsilon Z^{-1} \Psi_1(W) + O(\varepsilon^2)$$

interlaced relations

$$\Gamma_1(W) = A W + B \Phi(W)$$

$$\Psi_1(W) = d\Phi(W).\Gamma_1(W) - (C W + D \Phi(W))$$

$$\Gamma_2(W) = B Z^{-1} \Psi_1(W)$$

reformulation of a classical expansion
see e.g. Gatignol (1975)

Chapman-Enskog expansion (ii)

inject the representation $Y = \Phi(W) + \varepsilon Z^{-1} \Psi_1(W) + O(\varepsilon^2)$

inside the equation $\frac{\partial W}{\partial t} + A W + B Y = 0$

$$\text{then } \frac{\partial W}{\partial t} + A W + B (\Phi(W) + \varepsilon Z^{-1} \Psi_1(W)) = O(\varepsilon^2)$$

after rearranging the terms,

$$\frac{\partial W}{\partial t} + A W + B \Phi(W) + \varepsilon B Z^{-1} \Psi_1(W) = O(\varepsilon^2)$$

$$\text{then } \Gamma_1(W) = A W + B \Phi(W) \text{ and } \Gamma_2(W) = B Z^{-1} \Psi_1(W)$$

write the equation $\frac{\partial Y}{\partial t} + C W + D Y = -\frac{1}{\varepsilon} Z (Y - \Phi(W))$

under the form $Y = \Phi - \varepsilon Z^{-1} (\partial_t Y + C W + D Y)$

observe that $-\partial_t Y = dY \cdot (-\partial_t W) = d\Phi(W) \cdot \Gamma_1 + O(\varepsilon)$

report this expression in the value of Y :

$$\begin{aligned} Y &= \Phi + \varepsilon Z^{-1} (-\partial_t Y - C W - D Y) \\ &= \Phi + \varepsilon Z^{-1} (d\Phi(W) \cdot \Gamma_1 - C W - D \Phi) + O(\varepsilon^2) \end{aligned}$$

$$\text{and } \Psi_1(W) = d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi) \quad \square$$

Regular cartesian lattice \mathcal{L}

vertices x separated by distances

that are simple expressions

of the space step Δx

discrete time t :

integer multiple of a time step $\Delta t > 0$

unknowns of a lattice Boltzmann scheme

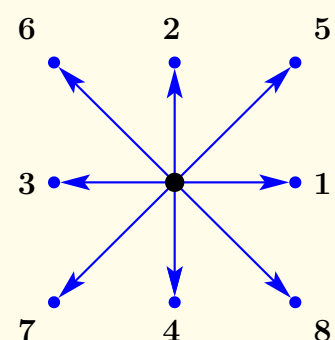
with q discrete velocities:

particle densities f_j

functions of discrete space x , discrete time t

and discrete velocities v_j for $0 \leq j < q$:

$$f_j = f_j(x, t), \quad x \in \mathcal{L}, \quad t = n \Delta t, \quad n \in \mathbb{N}, \quad v_j \in \mathcal{V}$$



The discrete velocity set \mathcal{V} does not depend on space or time.

simple Lie-Trotter splitting for the

Boltzmann-BGK system with discrete velocities

$$\frac{\partial f_j}{\partial t} + \sum_{1 \leq \alpha \leq d} v_j^\alpha \frac{\partial f_j}{\partial x_\alpha} = \frac{1}{\varepsilon} dQ_j(f^{\text{eq}}) \cdot (f - f^{\text{eq}}), \quad 0 \leq j < q$$

relaxation step $f(t) \rightarrow R(\Delta t) f(t) \equiv f^*$

for the approximation of the system of differential equations

$$\frac{\partial f}{\partial t} = \frac{1}{\varepsilon} dQ(f^{\text{eq}}) \cdot (f - f^{\text{eq}})$$

transport step $f^* \rightarrow T(\Delta t) f^* \equiv f(t + \Delta t)$

for the free transport

$$\frac{\partial f_j}{\partial t} + \sum_{1 \leq \alpha \leq d} v_j^\alpha \frac{\partial f_j}{\partial x_\alpha} = 0, \quad 0 \leq j < q$$

Relaxation step

$$\frac{\partial W}{\partial t} + A W + B Y = 0, \quad \frac{\partial Y}{\partial t} + C W + D Y = -\frac{1}{\varepsilon} Z (Y - \Phi(W))$$

relaxation scheme simply described in the moment representation

$$\frac{\partial W}{\partial t} = 0, \quad \frac{\partial Y}{\partial t} = -\frac{1}{\varepsilon} Z (Y - \Phi(W))$$

vector of moments m transformed into a new vector $m^* \equiv \begin{pmatrix} W^* \\ Y^* \end{pmatrix}$

conserved moments are not modified

during this relaxation step: $W^* = W$

explicit first order Euler scheme for the microscopic moments:

$$\frac{Y^* - Y}{\Delta t} = -\frac{1}{\varepsilon} Z (Y - \Phi(W))$$

$$\text{id est } Y^* = Y + S (\Phi(W) - Y) \text{ with } S = \frac{\Delta t}{\varepsilon} Z$$

relaxation matrix $S = \text{diag}(s_1, \dots, s_{q-N})$

fixed invertible square $(q - N) \times (q - N)$ matrix S

time step Δt is if the order of the Knudsen number ε

particle distribution f^* after relaxation: $f^* = M^{-1} m^*$

Necessary conditions for stability

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$$Y^* = Y + S(\Phi(W) - Y)$$

we deduce

$$Y^* - \Phi(W) = (I - S)(Y - \Phi(W))$$

natural stability condition: $\|I - S\| \leq 1$

id est

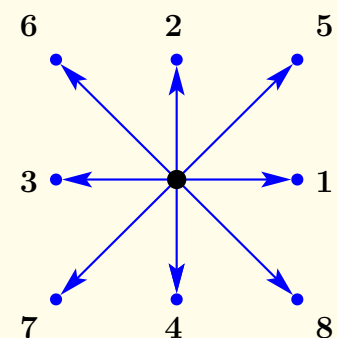
$$0 \leq s_j \leq 2, \quad 0 \leq j < q - N$$

over-relaxation $1 < s_j < 2$

widely used in the community of lattice Boltzmann schemes
to take into account **very low viscosities**
in fluid mechanics

Advection step

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second step of advection

method of characteristics applied

in the very particular case when it is **exact**

in one time step each particle distribution f_j

exactly transported from the node $x - v_j \Delta t$ of the lattice \mathcal{L}
to the vertex $x \in \mathcal{L}$

the advection step occurs with a Courant-Friedrichs-Lewy number

$cfl \equiv 1$ for all discrete velocities v_j

Lie-Trotter splitting scheme

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iteration of the lattice Boltzmann scheme

$$f(t + \Delta t) = T(\Delta t) R(\Delta t) f(t)$$

for each component:

$$f_j(x, t + \Delta t) = f_j^*(x - v_j \Delta t, t), \quad v_j \in \mathcal{V}, \quad x \in \mathcal{L}^0$$

very general framework

“BGK version”: $M = I$ and $Z = \frac{1}{\tau} I$.

usual “MRT” or “TRT” lattice Boltzmann schemes

two particle distributions

Alexander-Chen-Sterling (1993)

Guo-Zheng-Shi (2007), *etc.*

vectorial schemes developed by Graille (2014)

variant of entropy schemes (FD 2014)

in this case, non injective numbering $j \mapsto v_j$
of the velocities

Second order accurate Strang splitting scheme ?

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Lie-Trotter splitting scheme $f(t + \Delta t) = T(\Delta t) R(\Delta t) f(t)$

Strang splitting

$$f(t + \Delta t) = R\left(\frac{\Delta t}{2}\right) T(\Delta t) R\left(\frac{\Delta t}{2}\right) f(t)$$

with Dellar (2013), introduce $\tilde{f}(t) \equiv R\left(\frac{\Delta t}{2}\right)^{-1} f(t)$

Strang scheme: $\tilde{f}(t + \Delta t) = T(\Delta t) R\left(\frac{\Delta t}{2}\right) R\left(\frac{\Delta t}{2}\right) \tilde{f}(t)$

scheme identical to the Lie-Trotter scheme

the product $R\left(\frac{\Delta t}{2}\right) R\left(\frac{\Delta t}{2}\right)$ plays the role of $R(\Delta t)$.

- first possibility

divide the time step Δt by a factor 2

in the relation $Y^* = Y + S(\Phi(W) - Y)$

$$\bar{Y} = Y + \frac{1}{2} S(\Phi - Y), \quad Y^* = \bar{Y} + \frac{1}{2} S(\Phi - \bar{Y})$$

then $Y^* = (1 - S + \frac{1}{4} S^2) Y + (S - \frac{1}{4} S^2) \Phi(W)$

over-relaxation no more possible

because $s_j - \frac{1}{4} s_j^2 \leq 1$ if $s_j \in \mathbb{R}$

not a popular scheme!

Second order accurate Strang splitting scheme ? (ii) 19

- second possibility
enforce the relation $R(\frac{\Delta t}{2}) = \sqrt{R(\Delta t)}$
 $\bar{Y} = \sqrt{I-S} Y + (I - \sqrt{I-S}) \Phi(W)$
correct only if $s_j \leq 1$, over-relaxation excluded
 - exchange the roles of advection and relaxation ?
 $f(t + \Delta t) = T(\frac{\Delta t}{2}) R(\Delta t) T(\frac{\Delta t}{2}) f(t)$
or more elaborated splitting (Helluy *et al.*, 2019)
the transport step nomore at a Courant number equal to unity
interpolation procedure like in the Lax-Wendroff scheme
costly in terms of numerical viscosity
semi-Lagrangian method ?
complex to implement!
- Lie-Trotter splitting scheme $f(t + \Delta t) = T(\Delta t) R(\Delta t) f(t) !$

Multiple Relaxation Times Lattice Boltzmann scheme 20

Two steps for one time iteration

(i) Nonlinear relaxation

the particle distribution f is modified locally
into a new distribution f^*

(ii) Linear advection

method of characteristics when it is exact !

Compact description of the lattice Boltzmann scheme:

$$f_j(x, t + \Delta t) = f_j^*(x - v_j \Delta t, t), \quad v_j \in \mathcal{V}, \quad x \in \mathcal{L}^0.$$

d'Humières matrix M for moments: $m_k \equiv M_{kl} f_l$, $m \equiv \begin{pmatrix} W \\ Y \end{pmatrix}$

Conserved variables W ; after relaxation, $W^* = W$

Non-Conserved moments Y

equilibrium value: $Y^{\text{eq}} = \Phi(W)$

after relaxation: $Y^* = Y + S(Y^{\text{eq}} - Y)$, $m^* = \begin{pmatrix} W \\ Y^* \end{pmatrix}$

Compact form of the lattice Boltzmann scheme

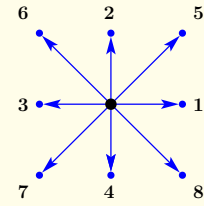
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Momentum-velocity operator matrix

$$\Lambda \equiv M \operatorname{diag}\left(\sum_{1 \leq \alpha \leq d} v^\alpha \partial_\alpha\right) M^{-1}$$

$d = \text{space dimension}$

block decomposition $\Lambda \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

example of Λ operator matrix for the isothermal D2Q9 scheme

| | | | | | | | | |
|-----------------------------------|----------------------------------|-----------------------------------|----------------------------------|---------------------------|------------------------|---------------------------|--------------------------|--------------------------|
| 0 | ∂_x | ∂_y | 0 | 0 | 0 | 0 | 0 | 0 |
| $\frac{2\lambda^2}{3} \partial_x$ | 0 | 0 | $\frac{1}{6} \partial_x$ | $\frac{1}{2} \partial_x$ | ∂_y | 0 | 0 | 0 |
| $\frac{2\lambda^2}{3} \partial_y$ | 0 | 0 | $\frac{1}{6} \partial_y$ | $-\frac{1}{2} \partial_y$ | ∂_x | 0 | 0 | 0 |
| 0 | $\lambda^2 \partial_x$ | $\lambda^2 \partial_y$ | 0 | 0 | 0 | ∂_x | ∂_y | 0 |
| 0 | $\frac{\lambda^2}{3} \partial_x$ | $-\frac{\lambda^2}{3} \partial_y$ | 0 | 0 | 0 | $-\frac{1}{3} \partial_x$ | $\frac{1}{3} \partial_y$ | 0 |
| 0 | $\frac{\lambda^2}{3} \partial_y$ | $-\frac{\lambda^2}{3} \partial_x$ | 0 | 0 | 0 | $-\frac{1}{3} \partial_y$ | $\frac{1}{3} \partial_x$ | 0 |
| 0 | 0 | 0 | $\frac{\lambda^2}{3} \partial_x$ | $-\lambda^2 \partial_x$ | $\lambda^2 \partial_y$ | 0 | 0 | $\frac{1}{3} \partial_x$ |
| 0 | 0 | 0 | $\frac{\lambda^2}{3} \partial_y$ | $\lambda^2 \partial_y$ | $\lambda^2 \partial_x$ | 0 | 0 | $\frac{1}{3} \partial_y$ |
| 0 | 0 | 0 | 0 | 0 | 0 | $\lambda^2 \partial_x$ | $\lambda^2 \partial_y$ | 0 |

Then $m(x, t + \Delta t) = \exp(-\Delta t \Lambda) m^*(x, t)$

Exact exponential expansion

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$$\begin{aligned}
 m_k(x, t + \Delta t) &= \sum_j M_{kj} f_j^*(x - v_j \Delta t, t) \\
 &= \sum_{j\ell} M_{kj} (M^{-1})_{j\ell} m_\ell^*(x - v_j \Delta t, t) \\
 &= \sum_{j\ell} M_{kj} (M^{-1})_{j\ell} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\Delta t \sum_{\alpha} v_j^\alpha \partial_\alpha\right)^n m_\ell^*(x, t) \\
 &= \sum_{\ell} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_j M_{kj} \left(-\Delta t \sum_{\alpha} v_j^\alpha \partial_\alpha\right)^n (M^{-1})_{j\ell} m_\ell^*(x, t) \\
 &= \sum_{\ell} \left[\sum_{n=0}^{\infty} \frac{1}{n!} (-\Delta t \Lambda)_{k\ell}^n \right] m_\ell^*(x, t) \\
 &= \sum_{\ell} \exp(-\Delta t \Lambda)_{k\ell} m_\ell^*(x, t) \\
 &= \left(\exp(-\Delta t \Lambda) m^*(x, t) \right)_k
 \end{aligned}$$

Exact exponential expansion (ii)

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Important elementary remark to simplify the final result

block decomposition $\Lambda \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$$\Lambda^2 = \Lambda \Lambda \equiv \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

with $\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$$A_2 = A^2 + B C$$

$$B_2 = A B + B D$$

$$C_2 = C A + D C$$

$$D_2 = C B + D^2$$

“ABCD” method

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Asymptotic hypothesis: emerging partial differential equations

$$\partial_t W + \Gamma_1 + \Delta t \Gamma_2 = O(\Delta t^2)$$

Γ_j : vector obtained after j space derivations
of the conserved moments W
and the equilibrium vector $\Phi(W)$.

Non-Conserved moments:

$$Y = \Phi(W) + \Delta t S^{-1} \Psi_1 + O(\Delta t^2)$$

General nonlinear “Vilnius” algorithm at second order

$$\Gamma_1 = A W + B \Phi(W)$$

$$\Psi_1 = d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi(W))$$

$$\Gamma_2 = B \Sigma \Psi_1 \quad \text{with } \Sigma \text{ the Hénon matrix: } \Sigma \equiv S^{-1} - \frac{1}{2} I$$

"ABCD" method (ii)

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Non-Conserved moments

Hénon matrix: $\Sigma \equiv S^{-1} - \frac{1}{2} I$

$$Y = \Phi(W) + \Delta t S^{-1} \Psi_1 + O(\Delta t^2)$$

then

$$\begin{aligned} Y^* &= Y + S(\Phi - Y) \\ &= Y - (\Delta t \Psi_1 + O(\Delta t^2)) \end{aligned}$$

$$\begin{aligned} &= \Phi(W) + \left(\Sigma + \frac{1}{2} I\right) (\Delta t \Psi_1 + O(\Delta t^2)) - (\Delta t \Psi_1 + O(\Delta t^2)) \\ &= \Phi(W) + \left(\Sigma - \frac{1}{2} I\right) \Delta t \Psi_1 + O(\Delta t^2) \end{aligned}$$

and

$$\begin{aligned} Y &= \Phi(W) + \left(\Sigma + \frac{1}{2} I\right) \Delta t \Psi_1 + O(\Delta t^2) \\ Y^* &= \Phi(W) + \left(\Sigma - \frac{1}{2} I\right) \Delta t \Psi_1 + O(\Delta t^2) \end{aligned}$$

"ABCD" method (iii)

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Zero-order expansion

Expand one iteration of the scheme:

$$m(x, t + \Delta t) = \exp(-\Delta t \Lambda) m^*(x, t), \quad m \equiv \begin{pmatrix} W \\ Y \end{pmatrix}$$

$$\begin{aligned} m + \Delta t \partial_t m + \frac{1}{2} \Delta t^2 \partial_t^2 m + O(\Delta t^3) &= \\ &= m^* - \Delta t \Lambda m^* + \frac{1}{2} \Delta t^2 \Lambda^2 m^* + O(\Delta t^3) \end{aligned}$$

We replace the vector m by its two components W and Y

$$\begin{aligned} W + \Delta t \partial_t W + \frac{1}{2} \Delta t^2 \partial_t^2 W + O(\Delta t^3) &= \\ &= W - \Delta t (A W + B Y^*) + \frac{1}{2} \Delta t^2 (A_2 W + B_2 Y^*) + O(\Delta t^3) \end{aligned}$$

$$\begin{aligned} Y + \Delta t \partial_t Y + \frac{1}{2} \Delta t^2 \partial_t^2 Y + O(\Delta t^3) &= \\ &= Y^* - \Delta t (C W + D Y^*) + \frac{1}{2} \Delta t^2 (C_2 W + D_2 Y^*) + O(\Delta t^3) \end{aligned}$$

At zero-order: $Y - Y^* = O(\Delta t)$ and $Y^* \equiv Y + S(\Phi(W) - Y)$

The matrix S is supposed fixed

$$\text{then } Y = \Phi(W) + O(\Delta t) \text{ and } Y^* = \Phi(W) + O(\Delta t)$$

"ABCD" method (iv)

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First-order expansion

$$Y^* = \Phi(W) + O(\Delta t)$$

Second-order partial differential equations:

$$\begin{aligned} \partial_t W + \frac{1}{2} \Delta t \partial_t^2 W + O(\Delta t^2) &= \\ &= -(A W + B Y^*) + \frac{1}{2} \Delta t (A_2 W + B_2 Y^*) + O(\Delta t^2) \end{aligned}$$

at first-order:

$$\partial_t W + O(\Delta t) = -(A W + B \Phi(W)) + O(\Delta t)$$

$$\text{then } \partial_t W + \Gamma_1 = O(\Delta t) \quad \text{with } \Gamma_1 = A W + B \Phi(W)$$

$$\text{then } \partial_t Y = d\Phi(W) \cdot \partial_t W + O(\Delta t) = d\Phi(W) \cdot (-\Gamma_1) + O(\Delta t)$$

$$\partial_t Y = -d\Phi(W) \cdot \Gamma_1 + O(\Delta t)$$

$$Y - Y^* = -\Delta t \partial_t Y - \Delta t (C W + D Y^*) + O(\Delta t^2)$$

$$\text{then } S(Y - \Phi) = \Delta t (d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi)) + O(\Delta t^2)$$

$$Y = \Phi + \Delta t S^{-1} (d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi)) + O(\Delta t^2)$$

$$\text{and } \Psi_1 = d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi)$$

"ABCD" method (v)

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Second-order expansion

$$\partial_t W = -\Gamma_1 + O(\Delta t) = -(A W + B \Phi(W)) + O(\Delta t)$$

$$\text{Then } \partial_t^2 W = -\partial_t (\Gamma_1 + O(\Delta t)) = -d\Gamma_1 \cdot \partial_t W + O(\Delta t)$$

$$= d\Gamma_1 \cdot \Gamma_1 + O(\Delta t)$$

$$= A \Gamma_1 + B d\Phi \cdot \Gamma_1 + O(\Delta t)$$

Second-order partial differential equations:

$$\begin{aligned} \partial_t W + \frac{1}{2} \Delta t \partial_t^2 W + O(\Delta t^2) &= \\ &= -(A W + B Y^*) + \frac{1}{2} \Delta t (A_2 W + B_2 Y^*) + O(\Delta t^2) \end{aligned}$$

$$\begin{aligned} \partial_t W &= -A W - B (\Phi + (\Sigma - \frac{1}{2}I) \Delta t \Psi_1) - \frac{1}{2} \Delta t (A \Gamma_1 + B d\Phi \cdot \Gamma_1) \\ &\quad + \frac{1}{2} \Delta t ((A^2 + B C) W + (A B + B D) \Phi) + O(\Delta t^2) \end{aligned}$$

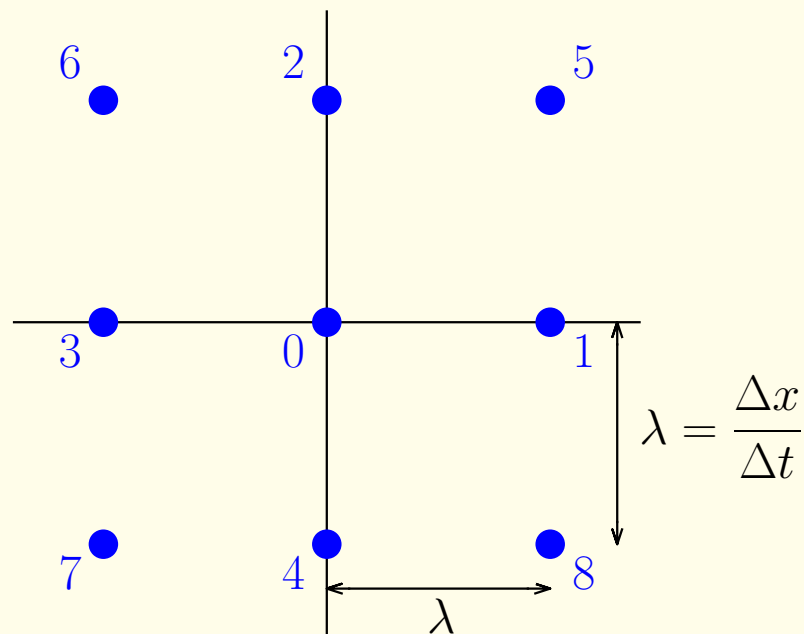
$$= -A W - B \Phi - \Delta t (B \Sigma \Psi_1 + \frac{1}{2} B (d\Phi \cdot \Gamma_1 - C W - D \Phi))$$

$$- \frac{1}{2} (A (A W + B \Phi)) - \frac{1}{2} B d\Phi \cdot \Gamma_1 + \frac{1}{2} (A^2 + B C) W$$

$$+ \frac{1}{2} (A B + B D) \Phi + O(\Delta t^2)$$

$$\partial_t W = -\Gamma_1 - \Delta t B \Sigma \Psi_1 + O(\Delta t^2)$$

$$\text{and } \Gamma_2 = B \Sigma \Psi_1$$



$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \lambda & 0 & -\lambda & 0 & \lambda & -\lambda & -\lambda & \lambda \\ 0 & 0 & \lambda & 0 & -\lambda & \lambda & \lambda & -\lambda & -\lambda \\ -4\lambda^2 & -\lambda^2 & -\lambda^2 & -\lambda^2 & -\lambda^2 & 2\lambda^2 & 2\lambda^2 & 2\lambda^2 & 2\lambda^2 \\ 0 & \lambda^2 & -\lambda^2 & \lambda^2 & -\lambda^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & -\lambda^2 & \lambda^2 & -\lambda^2 \\ 0 & -2\lambda^3 & 0 & 2\lambda^3 & 0 & \lambda^3 & -\lambda^3 & -\lambda^3 & \lambda^3 \\ 0 & 0 & -2\lambda^3 & 0 & 2\lambda^3 & \lambda^3 & \lambda^3 & -\lambda^3 & -\lambda^3 \\ 4\lambda^4 & -2\lambda^4 & -2\lambda^4 & -2\lambda^4 & -2\lambda^4 & \lambda^4 & \lambda^4 & \lambda^4 & \lambda^4 \end{bmatrix} \begin{matrix} \rho \\ J_x \\ J_y \\ \varepsilon \\ xx \\ xy \\ q_x \\ q_y \\ h \end{matrix}$$

the lines of this invertible matrix are chosen orthogonal

Momentum-velocity operator matrix $\Lambda \equiv M \text{diag}\left(\sum_{\alpha} v^{\alpha} \partial_{\alpha}\right) M^{-1}$
 $1 \leq \alpha \leq 2 = \text{space dimension}$

Block decomposition $\Lambda \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

Operator matrix Λ for the isothermal D2Q9 scheme

| ρ | J_x | J_y | ε | xx | xy | q_x | q_y | h | |
|-----------------------------------|----------------------------------|-----------------------------------|----------------------------------|---------------------------|------------------------|---------------------------|--------------------------|--------------------------|---------------|
| 0 | ∂_x | ∂_y | 0 | 0 | 0 | 0 | 0 | 0 | ρ |
| $\frac{2\lambda^2}{3} \partial_x$ | 0 | 0 | $\frac{1}{6} \partial_x$ | $\frac{1}{2} \partial_x$ | ∂_y | 0 | 0 | 0 | J_x |
| $\frac{2\lambda^2}{3} \partial_y$ | 0 | 0 | $\frac{1}{6} \partial_y$ | $-\frac{1}{2} \partial_y$ | ∂_x | 0 | 0 | 0 | J_y |
| 0 | $\lambda^2 \partial_x$ | $\lambda^2 \partial_y$ | 0 | 0 | 0 | ∂_x | ∂_y | 0 | ε |
| 0 | $\frac{\lambda^2}{3} \partial_x$ | $-\frac{\lambda^2}{3} \partial_y$ | 0 | 0 | 0 | $-\frac{1}{3} \partial_x$ | $\frac{1}{3} \partial_y$ | 0 | xx |
| 0 | $\frac{\lambda^2}{3} \partial_y$ | $-\frac{\lambda^2}{3} \partial_x$ | 0 | 0 | 0 | $-\frac{1}{3} \partial_y$ | $\frac{1}{3} \partial_x$ | 0 | xy |
| 0 | 0 | 0 | $\frac{\lambda^2}{3} \partial_x$ | $-\lambda^2 \partial_x$ | $\lambda^2 \partial_y$ | 0 | 0 | $\frac{1}{3} \partial_x$ | q_x |
| 0 | 0 | 0 | $\frac{\lambda^2}{3} \partial_y$ | $\lambda^2 \partial_y$ | $\lambda^2 \partial_x$ | 0 | 0 | $\frac{1}{3} \partial_y$ | q_y |
| 0 | 0 | 0 | 0 | 0 | 0 | $\lambda^2 \partial_x$ | $\lambda^2 \partial_y$ | 0 | h |

First order partial differential equations

We have at first order $\Gamma_1 = A W + B \Phi(W)$

$$\Gamma_1 = \begin{pmatrix} \partial_x J_x + \partial_y J_y \\ \frac{2}{3} \lambda^2 \partial_x \rho + \frac{1}{6} \partial_x \Phi_{\varepsilon} + \frac{1}{2} \partial_x \Phi_{xx} + \partial_y \Phi_{xy} \\ \frac{2}{3} \lambda^2 \partial_y \rho + \frac{1}{6} \partial_y \Phi_{\varepsilon} - \frac{1}{2} \partial_y \Phi_{xx} + \partial_x \Phi_{xy} \end{pmatrix}$$

First order terms of the Navier-Stokes equations

$$\partial_t \rho + \partial_x J_x + \partial_y J_y = \dots$$

$$\partial_t J_x + \partial_x \left(\frac{J_x^2}{\rho} + p \right) + \partial_y \left(\frac{J_x J_y}{\rho} \right) = \dots$$

$$\partial_t J_y + \partial_x \left(\frac{J_x J_y}{\rho} \right) + \partial_y \left(\frac{J_y^2}{\rho} + p \right) = \dots$$

We identify the two expressions ($J_x \equiv \rho u$, $J_y \equiv \rho v$)

$$\frac{2}{3} \lambda^2 \rho + \frac{1}{6} \Phi_{\varepsilon} + \frac{1}{2} \Phi_{xx} = \rho u^2 + p$$

$$\Phi_{xy} = \rho u v$$

$$\frac{2}{3} \lambda^2 \rho + \frac{1}{6} \Phi_{\varepsilon} - \frac{1}{2} \Phi_{xx} = \rho v^2 + p$$

Equilibrium function of second order moments

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Equivalent partial differential equations at first order

The Euler equations of gas dynamics are recovered by the D2Q9 fluid lattice Boltzmann scheme first order if and only if the second order moments ε , Φ_{xx} and Φ_{xy} have equilibrium values that satisfy

$$\begin{cases} \Phi_\varepsilon &= 6p - 4\lambda^2\rho + 3\rho(u^2 + v^2) \\ \Phi_{xx} &= \rho(u^2 - v^2) \\ \Phi_{xy} &= \rho uv. \end{cases}$$

Just solve the linear system

$$\begin{cases} \frac{2}{3}\lambda^2\rho + \frac{1}{6}\Phi_\varepsilon + \frac{1}{2}\Phi_{xx} &= \rho u^2 + p \\ \Phi_{xy} &= \rho uv \\ \frac{2}{3}\lambda^2\rho + \frac{1}{6}\Phi_\varepsilon - \frac{1}{2}\Phi_{xx} &= \rho v^2 + p. \end{cases}$$

D2Q9 : towards second order equations

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Φ : vector of moments at equilibrium

$$\Phi = (\Phi_\varepsilon, \Phi_{xx}, \Phi_{xy}, \Phi_{qx}, \Phi_{qy}, \Phi_h)^t$$

conservative first order fluxes

$$\Gamma_1 = \begin{bmatrix} \partial_x J_x + \partial_y J_y \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) \\ \partial_x(\rho uv) + \partial_x(\rho v^2 + p) \end{bmatrix}$$

$$\Psi_1 = d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi(W)) \in \mathbb{R}^6$$

viscous fluxes $-\Delta t \Gamma_2 = -\Delta t B \Sigma \Psi_1 =$ physical fluxes ?

$$= \begin{bmatrix} 0 \\ \partial_j \tau_{xj} \equiv \partial_x(2\mu \partial_x u + (\zeta - \mu)(\partial_x u + \partial_y v)) + \partial_y(\mu(\partial_x v + \partial_y u)) \\ \partial_j \tau_{yj} \equiv \partial_x(\mu(\partial_x v + \partial_y u)) + \partial_y((\zeta - \mu)(\partial_x u + \partial_y v) + 2\mu \partial_y v) \end{bmatrix}$$

D2Q9 : towards second order equations (ii)

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Do not forget that $\Gamma_2 = B \Sigma \Psi_1$

The Henon matrix Σ is a diagonal matrix
for this basic version of the MRT scheme

For this specific D2Q9 scheme, $B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} \partial_x & \frac{1}{2} \partial_x & \partial_y & 0 & 0 & 0 \\ \frac{1}{6} \partial_y & -\frac{1}{2} \partial_y & \partial_x & 0 & 0 & 0 \end{pmatrix}$

Only the 3 first components of the vector

$$\Psi_1 \equiv d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi(W))$$

have a role for the second order equations

The 3 first components of $d\Phi(W) \cdot \Gamma_1$ are composed
with derivatives of Φ_ε , Φ_{xx} and Φ_{xy}
and the known first order flux Γ_1

We just have to express these known expressions in terms
of the gradients $\partial_x \rho$, $\partial_y \rho$, $\partial_x u$, $\partial_y u$, $\partial_x v$ and $\partial_y v$
of the non-conservative variables ρ , u and v

D2Q9 : towards second order equations (iii)

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Φ : vector of moments at equilibrium

$$\Phi = (\Phi_\varepsilon, \Phi_{xx}, \Phi_{xy}, \Phi_{qx}, \Phi_{qy}, \Phi_h)^t$$

Three first components of the vector

$$\Psi_1 \equiv d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi(W))$$

3 first lines of matrix $C = \begin{pmatrix} 0 & \lambda^2 \partial_x & \lambda^2 \partial_y \\ 0 & \frac{\lambda^2}{3} \partial_x & -\frac{\lambda^2}{3} \partial_y \\ 0 & \frac{\lambda^2}{3} \partial_y & -\frac{\lambda^2}{3} \partial_x \end{pmatrix}$

3 first lines of matrix $D = \begin{pmatrix} 0 & 0 & 0 & \partial_x & \partial_y & 0 \\ 0 & 0 & 0 & -\frac{1}{3} \partial_x & \frac{1}{3} \partial_y & 0 \\ 0 & 0 & 0 & -\frac{1}{3} \partial_y & \frac{1}{3} \partial_x & 0 \end{pmatrix}$

The last moment at equilibrium Φ_h has no role

for the partial differential equations at second order

Is it possible to find two functions $\Phi_{qx} = \Phi_{qx}(\rho, u, v)$ and
 $\Phi_{qy} = \Phi_{qy}(\rho, u, v)$ in order to "fit" Navier Stokes ?

D2Q9 : towards second order equations (iv)

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Try to avoid the unphysical $\partial_x \rho$ and $\partial_y \rho$
from the second order fluxes :

system of 8 independent equations (*a priori* $2 \times 3 \times 2$)

No solution :- (

Discrepancy reduced to third order terms in velocity

if $\rho(\rho) = \frac{\lambda^2}{3} \rho$: isothermal fluid with $c_s = \frac{\lambda}{\sqrt{3}}$

and moreover if

$$\Phi_{qx} = -\rho \lambda^2 u + 3 \rho (u^2 + v^2) u$$

$$\Phi_{qy} = -\rho \lambda^2 v + 3 \rho (u^2 + v^2) v$$

What is the final discrepancy with this specific choices ?

D2Q9 : towards second order equations (v)

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Given vector Φ of moments at equilibrium

$$\Phi_\varepsilon = 3 \rho (u^2 + v^2) - 2 \lambda^2 \rho, \quad \Phi_{xx} = \rho (u^2 - v^2), \quad \Phi_{xy} = \rho u v$$

$$\Phi_{qx} = -\rho \lambda^2 u + 3 \rho (u^2 + v^2) u, \quad \Phi_{qy} = -\rho \lambda^2 v + 3 \rho (u^2 + v^2) v$$

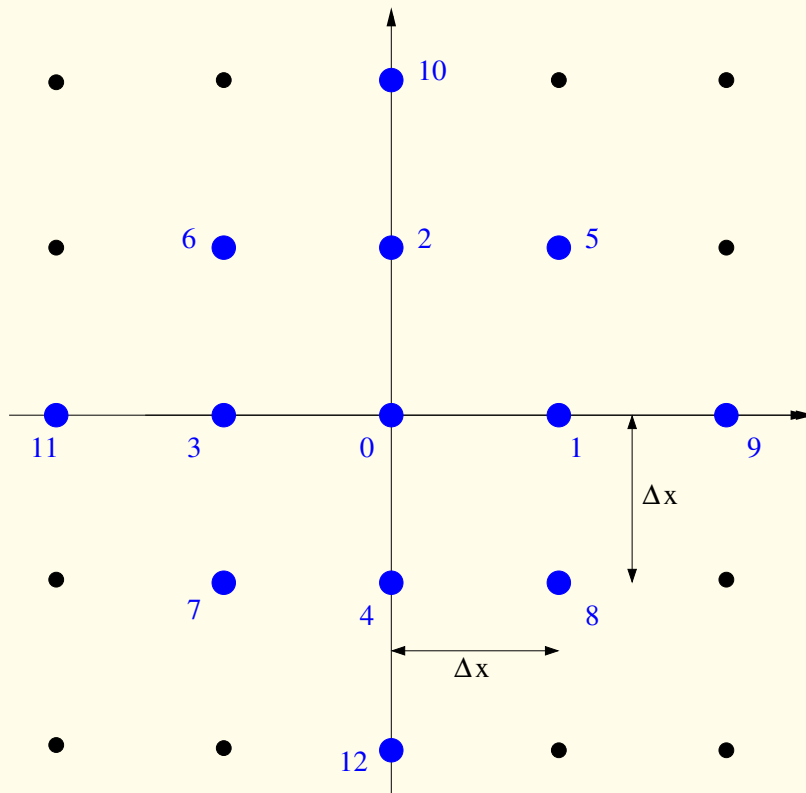
Hénon matrix $\Sigma = \text{diag}(\sigma_e, \sigma_x, \sigma_x, *, * *)$

With $\mu = \frac{\lambda}{3} \rho \sigma_x \Delta x$ and $\zeta = \frac{\lambda}{3} \rho \sigma_e \Delta x$, we have

$$-\Delta t \Gamma_2 =$$

$$\begin{pmatrix} 0 \\ \partial_j \tau_{xj} \equiv \partial_x (2 \mu \partial_x u + (\zeta - \mu) (\partial_x u + \partial_y v)) + \partial_y (\mu (\partial_x v + \partial_y u)) \\ \partial_j \tau_{yj} \equiv \partial_x (\mu (\partial_x v + \partial_y u)) + \partial_y ((\zeta - \mu) (\partial_x u + \partial_y v) + 2 \mu \partial_y v) \end{pmatrix} \\ + \sigma_x \Delta t \partial_x \begin{pmatrix} 0 \\ u^3 \partial_x \rho - v^3 \partial_y \rho + 3 \rho (u^2 \partial_x u - v^2 \partial_y v) \\ -v^3 \partial_x \rho - u^3 \partial_y \rho - 3 \rho (u^2 \partial_y u + v^2 \partial_x v) \end{pmatrix} \\ + \sigma_x \Delta t \partial_y \begin{pmatrix} 0 \\ -v^3 \partial_x \rho - u^3 \partial_y \rho - 3 \rho (u^2 \partial_y u + v^2 \partial_x v) \\ -u^3 \partial_x \rho + v^3 \partial_y \rho + 3 \rho (-u^2 \partial_x u + v^2 \partial_y v) \end{pmatrix}$$

D2Q13



| | |
|---------------|---|
| ρ | 0 |
| J_x | 1 |
| J_y | 1 |
| ε | 2 |
| xx | 2 |
| xy | 2 |
| q_x | 3 |
| q_y | 3 |
| r_x | 5 |
| r_y | 5 |
| h | 4 |
| x_{xe} | 4 |
| h_2 | 6 |

D2Q13 (ii)

Polynomials for the matrix of moments : $M_{kj} = p_k(u = v_j^x, v = v_j^y)$

| | |
|---------------|--|
| ρ | 1 |
| J_x, J_y | u, v |
| ε | $-28 + 13(u^2 + v^2)$ |
| xx, xy | $u^2 - v^2, uv$ |
| q_x, q_y | $u(-3 + (u^2 + v^2)), v(-3 + (u^2 + v^2))$ |
| r_x | $u \left(\frac{101}{6} - (u^2 + v^2) \right) \left(\frac{63}{4} - \frac{35}{12}(u^2 + v^2) \right)$ |
| r_y | $v \left(\frac{101}{6} - (u^2 + v^2) \right) \left(\frac{63}{4} - \frac{35}{12}(u^2 + v^2) \right)$ |
| h | $140 - (u^2 + v^2) \left(\frac{361}{2} - \frac{77}{2}(u^2 + v^2) \right)$ |
| xx_e | $(u^2 - v^2) \left(-\frac{65}{12} + \frac{17}{12}(u^2 + v^2) \right)$ |
| h_2 | $-12 + (u^2 + v^2) \left(\frac{581}{12} - (u^2 + v^2) \left(\frac{273}{8} - \frac{137}{24}(u^2 + v^2) \right) \right)$ |

D2Q13: momentum-velocity operator matrix Λ

$$\Lambda = \begin{bmatrix} \rho & J_x & J_y & \varepsilon & xx & xy & q_x & q_y & r_x & r_y & h & x_{xe} & h_2 \\ 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * \partial_x & 0 & 0 & * \partial_x & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * \partial_y & 0 & 0 & * \partial_y & * \partial_y & * \partial_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & 0 \\ 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & * \partial_x & * \partial_y & * \partial_x & * \partial_y & 0 & 0 & 0 \\ 0 & * \partial_y & * \partial_x & 0 & 0 & 0 & * \partial_y & * \partial_x & * \partial_y & * \partial_x & 0 & 0 & 0 \\ 0 & 0 & 0 & * \partial_x & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & * \partial_x & * \partial_x & 0 \\ 0 & 0 & 0 & * \partial_y & * \partial_y & * \partial_x & 0 & 0 & 0 & 0 & * \partial_y & * \partial_y & 0 \\ 0 & 0 & 0 & 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & * \partial_x & * \partial_x & * \partial_x \\ 0 & 0 & 0 & 0 & * \partial_y & * \partial_x & 0 & 0 & 0 & 0 & * \partial_y & * \partial_y & * \partial_y \\ 0 & 0 & 0 & 0 & 0 & 0 & * \partial_x & * \partial_y & * \partial_x & * \partial_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * \partial_x & * \partial_y & * \partial_x & * \partial_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \partial_x & * \partial_y & 0 & 0 & 0 \end{bmatrix}$$

D2Q13: first order equations

$$\begin{aligned} \partial_t \rho + \partial_x J_x + \partial_y J_y &= 0 & \partial_t \rho + \partial_x J_x + \partial_y J_y &= 0 \\ \partial_t J_x + \partial_x \left(\frac{14}{13} \lambda^2 \rho + \frac{1}{26} \Phi_\varepsilon + \frac{1}{2} \Phi_{xx} \right) + \partial_y \Phi_{xy} &= 0 & \partial_t J_x + \partial_x (\rho u^2 + p) + \partial_y (\rho u v) &= 0 \\ \partial_t J_y + \partial_x \Phi_{xy} + \partial_y \left(\frac{14}{13} \lambda^2 \rho + \frac{1}{26} \Phi_\varepsilon - \frac{1}{2} \Phi_{xx} \right) &= 0 & \partial_t J_y + \partial_x (\rho u v) + \partial_x (\rho v^2 + p) &= 0 \end{aligned}$$

then $\frac{14}{13} \lambda^2 \rho + \frac{1}{26} \Phi_\varepsilon + \frac{1}{2} \Phi_{xx} = \rho u^2 + p,$
 $\frac{14}{13} \lambda^2 \rho + \frac{1}{26} \Phi_\varepsilon - \frac{1}{2} \Phi_{xx} = \rho v^2 + p$
 and $\Phi_{xx} = \rho (u^2 - v^2), \Phi_{xy} = \rho u v$
 $\frac{28}{13} \lambda^2 \rho + \frac{1}{13} \Phi_\varepsilon = \rho (u^2 + v^2) + 2p$
 then $\Phi_\varepsilon = 13 \rho (u^2 + v^2) + 26 p - 28 \lambda^2 \rho$

The equilibrium values of the moments of **degree two** are fixed with the first order **Euler** equations

D2Q13 : second order equations

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Φ : vector of moments at equilibrium

$$\Phi = (\Phi_\varepsilon, \Phi_{xx}, \Phi_{xy}, \Phi_{qx}, \Phi_{qy}, \Phi_{rx}, \Phi_{ry}, \Phi_h, \Phi_{xxe}, \Phi_{h2})^t$$

conservative first order fluxes

$$\Gamma_1 = \begin{bmatrix} \partial_x J_x + \partial_y J_y \\ \partial_x(\rho u^2 + p) + \partial_y(\rho u v) \\ \partial_x(\rho u v) + \partial_y(\rho v^2 + p) \end{bmatrix}$$

$$\Psi_1 = d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi(W)) \in \mathbb{R}^6$$

viscous fluxes $-\Delta t \Gamma_2 = -\Delta t B \Sigma \Psi_1 =$ physical fluxes ?

$$= \begin{bmatrix} 0 \\ \partial_j \tau_{xj} \equiv \partial_x(2\mu \partial_x u + (\zeta - \mu)(\partial_x u + \partial_y v)) + \partial_y(\mu(\partial_x v + \partial_y u)) \\ \partial_j \tau_{yj} \equiv \partial_x(\mu(\partial_x v + \partial_y u)) + \partial_y((\zeta - \mu)(\partial_x u + \partial_y v) + 2\mu \partial_y v) \end{bmatrix}$$

D2Q13 : second order equations (ii)

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We have $\Gamma_2 = B \Sigma \Psi_1$

Henon matrix : $\Sigma = \text{diag}(\sigma_e, \sigma_x, \sigma_x, \dots)$

For the D2Q13 scheme,

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{26} \partial_x & \frac{1}{2} \partial_x & \partial_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{26} \partial_y & -\frac{1}{2} \partial_y & \partial_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Only the **3 first components** of the vector

$$\Psi_1 \equiv d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi(W))$$

have a role for the second order equations

The 3 first components of $d\Phi(W) \cdot \Gamma_1$ are composed

with derivatives of Φ_ε , Φ_{xx} and Φ_{xy}

and the known first order flux Γ_1

We express these known expressions in terms

of the gradients $\partial_x \rho$, $\partial_y \rho$, $\partial_x u$, $\partial_y u$, $\partial_x v$ and $\partial_y v$

of the non-conservative variables ρ , u and v

D2Q13 : second order equations (iii)

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Φ : vector of moments at equilibrium

$$\Phi = (\Phi_\varepsilon, \Phi_{xx}, \Phi_{xy}, \Phi_{qx}, \Phi_{qy}, \Phi_{rx}, \Phi_{ry}, \Phi_h, \Phi_{xxe}, \Phi_{h2})^t$$

Three first components of the vector

$$\Psi_1 \equiv d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi(W))$$

3 first lines of matrix $C = \begin{pmatrix} 0 & 11 \lambda^2 \partial_x & 11 \lambda^2 \partial_y \\ 0 & \frac{17 \lambda^2}{7} \partial_x & -\frac{17 \lambda^2}{7} \partial_y \\ 0 & \frac{2 \lambda^2}{7} \partial_y & \frac{2 \lambda^2}{7} \partial_x \end{pmatrix}$

3 first lines of matrix D :

$$\begin{pmatrix} 0 & 0 & 0 & 13 \partial_x & 13 \partial_y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{5} \partial_x & -\frac{7}{5} \partial_y & \frac{12}{35} \partial_x & -\frac{12}{35} \partial_y & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{5} \partial_y & -\frac{1}{5} \partial_x & -\frac{6}{35} \partial_y & -\frac{6}{35} \partial_x & 0 & 0 & 0 \end{pmatrix}$$

The 3 last moment at equilibrium Φ_h , Φ_{xxe} and Φ_{h2} have no role for the partial differential equations at second order

D2Q13 : second order equations (iv)

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After identification, the $3 \times 2 \times 2 \times 4 = 48$ equations can be solved

We find :

$$p = \lambda^2 c_s^2 \rho \quad \text{and the parameter } c_s \text{ is not constrained}$$

$$\Phi_{qx} = \rho (u^2 + v^2 + 4 \lambda^2 c_s^2 - 3 \lambda^2) u$$

$$\Phi_{qy} = \rho (u^2 + v^2 + 4 \lambda^2 c_s^2 - 3 \lambda^2) v$$

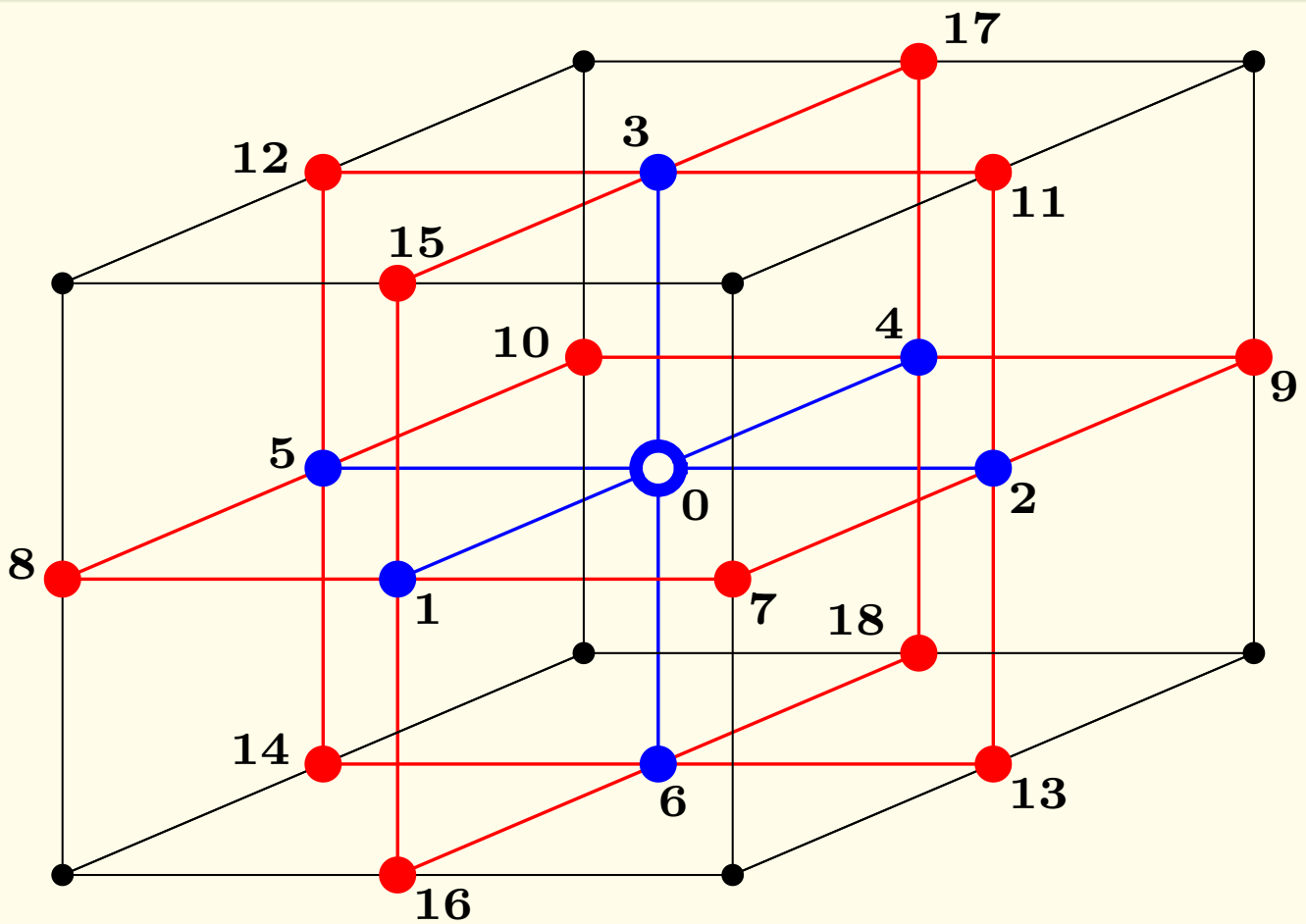
$$\Phi_{rx} = \rho \left(-\frac{7}{6} \lambda^2 u^2 - 7 \lambda^2 v^2 - \frac{21}{2} \lambda^4 c_s^2 + \frac{31}{6} \lambda^4 \right) u$$

$$\Phi_{ry} = \rho \left(-7 \lambda^2 u^2 - \frac{7}{6} \lambda^2 v^2 - \frac{21}{2} \lambda^4 c_s^2 + \frac{31}{6} \lambda^4 \right) v$$

$$\mu = \rho \sigma_x \lambda c_s^2 \Delta x$$

$$\zeta = \rho \sigma_e \lambda c_s^2 \Delta x$$

ok compared to a pure human algebraic calculus (april 2015)



$$M_{kj} = p_k(u = v_j^x, v = v_j^y, w = v_j^z)$$

| | |
|-----------------|--|
| ρ | 1 |
| j_x, j_y, j_z | u, v, w |
| ϵ | $19(u^2 + v^2 + w^2) - 30$ |
| xx, ww | $2u^2 - v^2 - w^2, v^2 - w^2$ |
| xy, yz, zx | uv, vw, wu |
| q_x, q_y, q_z | $u p_2, v p_2, w p_2$ |
| $x yz$ | $u(v^2 - w^2)$ |
| $y zx$ | $v(w^2 - u^2)$ |
| $z xy$ | $w(u^2 - v^2)$ |
| hh | $\frac{21}{2}(u^2 + v^2 + w^2)^2 - \frac{53}{2}(u^2 + v^2 + w^2) + 12$ |
| xx_e, ww_e | $(2u^2 - v^2 - w^2) q_2, (v^2 - w^2) q_2$ |

$$p_2 \equiv -9 + 5(u^2 + v^2 + w^2)$$

$$q_2 \equiv 3(u^2 + v^2 + w^2) - 5$$

D3Q19 : non null elements of the Λ matrix

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$$\begin{bmatrix}
 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 * & 0 & 0 & 0 & * & * & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 * & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 * & 0 & 0 & 0 & * & * & * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \\
 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & * & * & 0 & 0 & 0 \\
 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0 \\
 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & * & * & 0 & 0 & 0 & 0 \\
 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & * & * & 0 & 0 & 0 \\
 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & * & 0 & * & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & * & * & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 \\
 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
 0 & 0 & 0 & 0 & * & * & * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
 0 & 0 & 0 & 0 & 0 & * & * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & * & * & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0
 \end{bmatrix}$$

D3Q19 : recovering first order isothermal equations

50

isothermal flow : $p \equiv c_s^2 \rho$

$$\Phi_\varepsilon = 19 \rho (u^2 + v^2 + w^2) - 30 \rho \lambda^2 + 57 \rho c_s^2$$

$$\Phi_{xx} = 2 \rho u^2 - \rho v^2 - \rho w^2$$

$$\Phi_{ww} = \rho v^2 - \rho w^2$$

$$\Phi_{xy} = \rho u v$$

$$\Phi_{yz} = \rho v w$$

$$\Phi_{zx} = \rho w u$$

Then

$$\partial_t \rho + \text{div}(\rho \mathbf{u}) = O(\Delta x)$$

$$\partial_t(\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = O(\Delta x)$$

the value of the sound velocity is not imposed at this step

D3Q19 : recovering second order equations...

51

at first order relative to velocity...

In particular, the relation $c_s = \frac{\lambda}{\sqrt{3}}$ is mandatoryA total of $3 \times 3 \times 3 \times 4 = 108$ equations to solve to identify the second order terms of the Navier Stokes equations

3 equations for momentum

3 conservation terms per equation : $\partial_x[**]$, $\partial_y[**]$ and $\partial_z[**]$ 4 nonconserved variables ρ , u , v and w 3 partial derivatives ∂_x , ∂_y and ∂_z per variable

Expressions that concentrate the error to high order velocity terms

$$\Phi_{qx} = 5 \rho u (u^2 + v^2 + w^2) - \frac{2}{3} \rho u \lambda^2$$

$$\Phi_{qy} = 5 \rho v (u^2 + v^2 + w^2) - \frac{2}{3} \rho v \lambda^2$$

$$\Phi_{qz} = 5 \rho w (u^2 + v^2 + w^2) - \frac{2}{3} \rho w \lambda^2$$

$$\Phi_{x\ yz} = \rho u (v^2 - w^2), \quad \Phi_{y\ zx} = \rho v (w^2 - u^2)$$

$$\Phi_{z\ xy} = \rho w (u^2 - v^2)$$

A total of 61 equations remain unsolved

D3Q19 : recovering second order equations... (ii)

52

relaxation of second order moments

$$\begin{aligned} \varepsilon^* &= \varepsilon + s_\varepsilon (\Phi_\varepsilon - \varepsilon), & xx^* &= xx + s_x (\Phi_{xx} - xx) \\ ww^* &= ww + s_x (\Phi_{ww} - ww), & xy^* &= xy + s_x (\Phi_{xy} - xy) \\ yz^* &= yz + s_x (\Phi_{yz} - yz), & zx^* &= zx + s_x (\Phi_{zx} - zx) \end{aligned}$$

Hénon relations : $\sigma_x \equiv \frac{1}{s_x} - \frac{1}{2}$, $\sigma_e \equiv \frac{1}{s_e} - \frac{1}{2}$

$$\text{shear viscosity } \mu = \frac{1}{3} \rho \sigma_x \Delta t (\lambda^2 + O(|\mathbf{u}|^2))$$

$$\text{bulk viscosity } \zeta = \frac{2}{9} \rho \sigma_e \Delta t (\lambda^2 + O(|\mathbf{u}|^2))$$

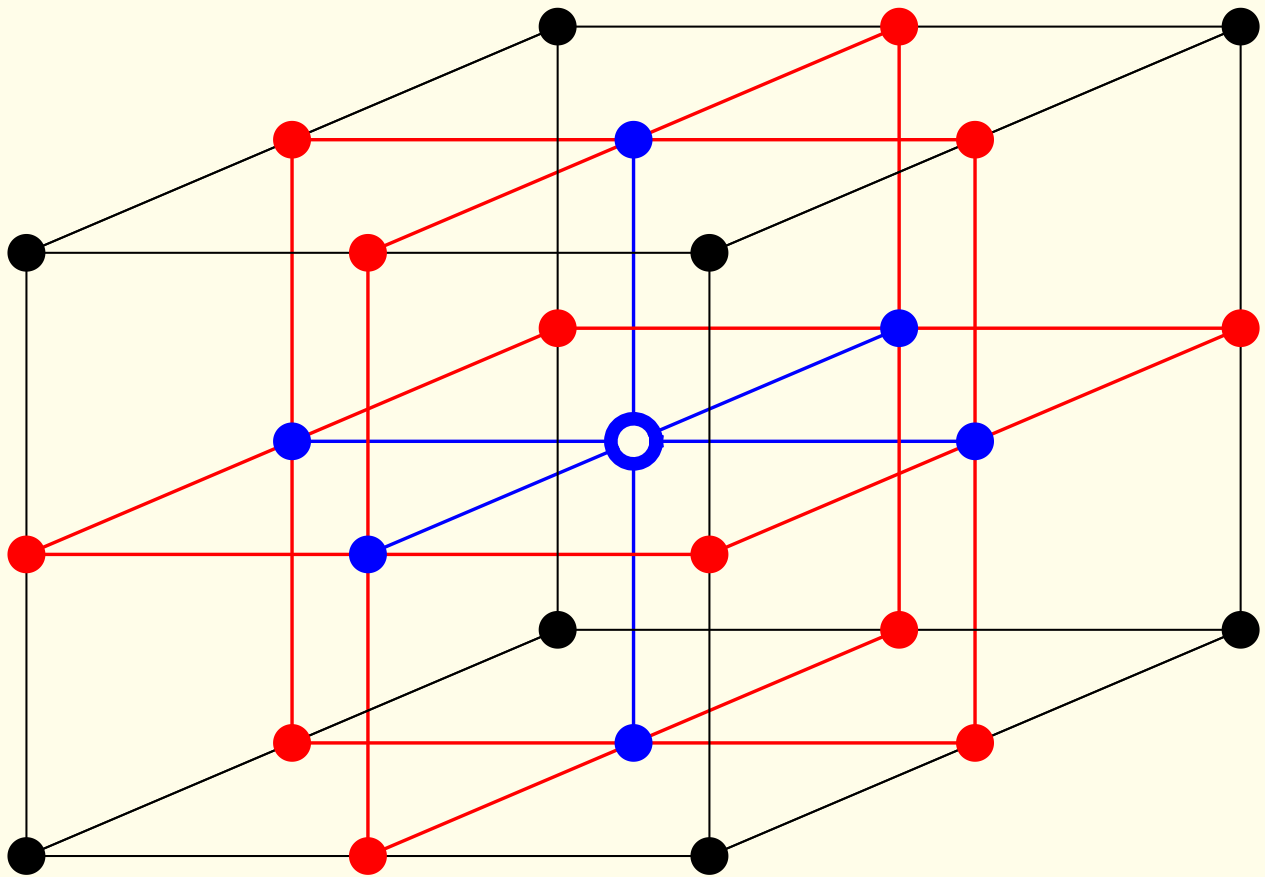
tensor of viscosities

$$\begin{aligned} \tau_{xx} &= 2 \mu \partial_x u + \left(\zeta - \frac{2}{3} \mu\right) \text{div } \mathbf{u} \\ \tau_{yy} &= 2 \mu \partial_y v + \left(\zeta - \frac{2}{3} \mu\right) \text{div } \mathbf{u}, & \tau_{zz} &= 2 \mu \partial_z w + \left(\zeta - \frac{2}{3} \mu\right) \text{div } \mathbf{u} \\ \tau_{xy} &= \mu (\partial_x v + \partial_y u), & \tau_{yz} &= \mu (\partial_y w + \partial_z v), & \tau_{zx} &= \mu (\partial_z u + \partial_x w) \end{aligned}$$

Approximative isothermal Navier-Stokes equations at second order

$$\partial_t \rho + \text{div}(\rho \mathbf{u}) = O(\Delta x^2)$$

$$\partial_t(\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - \Delta t \text{div } \tau = \Delta x O(|\mathbf{u}|^3) + O(\Delta x^2)$$



D3Q27 : Polynomials for the matrix of moments

$$M_{kj} = p_k(u = v_j^x, v = v_j^y, w = v_j^z)$$

| | |
|-----------------|-------------------------------|
| ρ | 1 |
| j_x, j_y, j_z | u, v, w |
| ϵ | $(u^2 + v^2 + w^2) - 2$ |
| xx, ww | $2u^2 - v^2 - w^2, v^2 - w^2$ |
| xy, yz, zx | uv, vw, wu |
| q_x, q_y, q_z | $u p_2, v p_2, w p_2$ |
| $x yz$ | $u(v^2 - w^2)$ |
| $y zx$ | $v(w^2 - u^2)$ |
| $z xy$ | $w(u^2 - v^2)$ |
| xyz | $u v w$ |

$$p_2 \equiv -7 + 3(u^2 + v^2 + w^2)$$

D3Q27 : recovering first order isothermal equations

57

isothermal flow : $p \equiv c_s^2 \rho$

$$\Phi_\epsilon = \rho(u^2 + v^2 + w^2) - 2\rho\lambda^2 + 3\rho c_s^2$$

$$\Phi_{xx} = 2\rho u^2 - \rho v^2 - \rho w^2$$

$$\Phi_{ww} = \rho v^2 - \rho w^2$$

$$\Phi_{xy} = \rho uv$$

$$\Phi_{yz} = \rho vw$$

$$\Phi_{zx} = \rho wu$$

Then

$$\partial_t \rho + \text{div}(\rho \mathbf{u}) = O(\Delta x)$$

$$\partial_t(\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = O(\Delta x)$$

the value of the sound velocity is not imposed at this step
essentially analogous to the D3Q19 case...

D3Q27 : recovering second order equations...

58

The relation $c_s = \frac{\lambda}{\sqrt{3}}$ is now imposed

A total of $3 \times 3 \times 3 \times 4 = 108$ equations to solve to identify

Nonisotropic expressions for high order velocity terms errors

$$\Phi_{qx} = 3\rho u(u^2 + v^2 + w^2) - 2\rho u\lambda^2$$

$$\Phi_{qy} = 3\rho v(u^2 + v^2 + w^2) - 2\rho v\lambda^2$$

$$\Phi_{qz} = 3\rho w(u^2 + v^2 + w^2) - 2\rho w\lambda^2$$

$$\Phi_{x\ yz} = \rho u(v^2 - w^2) - \rho u^3$$

$$\Phi_{y\ zx} = \rho v(w^2 - u^2) + \rho v^3$$

$$\Phi_{z\ xy} = \rho w(u^2 - v^2) - \rho w^3$$

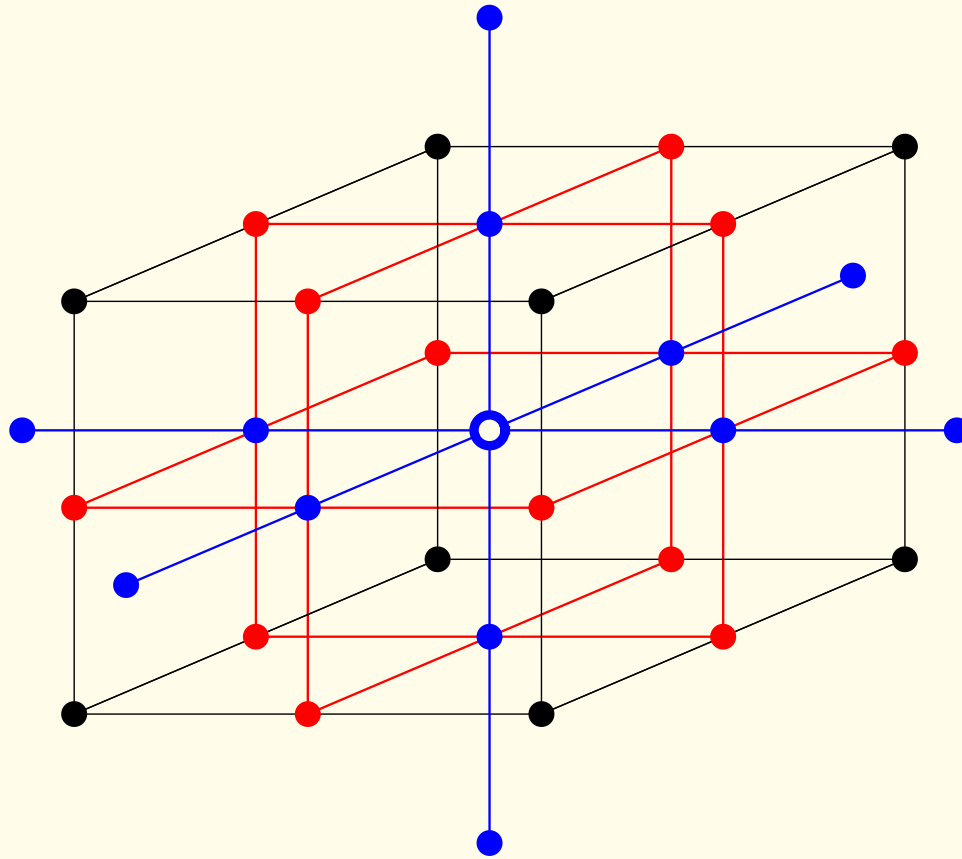
Approximative isothermal Navier-Stokes equations at second order

$$\partial_t \rho + \text{div}(\rho \mathbf{u}) = O(\Delta x^2)$$

$$\partial_t(\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - \Delta t \text{div} \tau = \Delta x O(|\mathbf{u}|^3) + O(\Delta x^2)$$

A total of 33 equations remain unsolved

(45 with the isotropic version!)



D3Q33 : Polynomials for the matrix of moments

$$M_{kj} = p_k(u = v_j^x, v = v_j^y, w = v_j^z)$$

| | |
|-----------------|-------------------------------|
| ρ | 1 |
| j_x, j_y, j_z | u, v, w |
| ϵ | $11(u^2 + v^2 + w^2) - 26$ |
| xx, ww | $2u^2 - v^2 - w^2, v^2 - w^2$ |
| xy, yz, zx | uv, vw, wu |
| q_x, q_y, q_z | $u p_2, v p_2, w p_2$ |
| $x yz$ | $u(v^2 - w^2)$ |
| $y zx$ | $v(w^2 - u^2)$ |
| $z xy$ | $w(u^2 - v^2)$ |
| xyz | $u v w$ |
| r_x, r_y, r_z | $u p_4, v p_4, w p_4$ |
| t_x, t_y, t_z | $u p_6, v p_6, w p_6$ |

$$p_2 \equiv -37 + 23(u^2 + v^2 + w^2)$$

$$p_4 \equiv 125 - 104(u^2 + v^2 + w^2) + 19(u^2 + v^2 + w^2)^2$$

$$p_6 \equiv -47 - \frac{206}{3}(u^2 + v^2 + w^2) - \frac{59}{2}(u^2 + v^2 + w^2)^2 + \frac{23}{6}(u^2 + v^2 + w^2)^3$$

D3Q33 : equilibrium value to recover first order

63

isothermal flow : $p \equiv c_s^2 \rho$
 the **sound velocity** c_s is *a priori* **not imposed**

$$\Phi_\varepsilon = 11 \rho (u^2 + v^2 + w^2) - 26 \rho \lambda^2 + 33 \rho c_s^2$$

$$\Phi_{xx} = 2 \rho u^2 - \rho v^2 - \rho w^2$$

$$\Phi_{ww} = \rho v^2 - \rho w^2$$

$$\Phi_{xy} = \rho u v$$

$$\Phi_{yz} = \rho v w$$

$$\Phi_{zx} = \rho w u$$

Then
$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) &= O(\Delta x) \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p &= O(\Delta x) \end{aligned}$$

D3Q33 : second order partial differential equations

64

The **108 equations** are completely **solved !**

Algebraic nonlinear expressions for high order moments

$$\Phi_{qx} = 13 \rho u (u^2 + v^2 + w^2) - 37 \rho u \lambda^2 + 65 \rho u c_s^2$$

$$\Phi_{qy} = 13 \rho v (u^2 + v^2 + w^2) - 37 \rho v \lambda^2 + 65 \rho v c_s^2$$

$$\Phi_{qz} = 13 \rho w (u^2 + v^2 + w^2) - 37 \rho w \lambda^2 + 65 \rho w c_s^2$$

$$\Phi_{x\ yz} = \rho u (v^2 - w^2)$$

$$\Phi_{y\ zx} = \rho v (w^2 - u^2)$$

$$\Phi_{z\ xy} = \rho w (u^2 - v^2)$$

$$\Phi_{xyz} = \rho u v w$$

$$\begin{aligned} \Phi_{rx} + \frac{38}{13} \frac{1}{\lambda^2} \Phi_{tx} &= -\frac{161}{39} \rho u^3 \lambda^2 - \frac{345}{13} \rho u (v^2 - w^2) \\ &\quad + \frac{1265}{39} \rho u \lambda^4 - \frac{2553}{39} \rho u \lambda^2 c_s^2 \end{aligned}$$

$$\begin{aligned} \Phi_{ry} + \frac{38}{13} \frac{1}{\lambda^2} \Phi_{ty} &= -\frac{161}{39} \rho v^3 \lambda^2 - \frac{345}{13} \rho v (w^2 - u^2) \\ &\quad + \frac{1265}{39} \rho v \lambda^4 - \frac{2553}{39} \rho v \lambda^2 c_s^2 \end{aligned}$$

$$\begin{aligned} \Phi_{rz} + \frac{38}{13} \frac{1}{\lambda^2} \Phi_{tz} &= -\frac{161}{39} \rho w^3 \lambda^2 - \frac{345}{13} \rho w (u^2 - v^2) \\ &\quad + \frac{1265}{39} \rho w \lambda^4 - \frac{2553}{39} \rho w \lambda^2 c_s^2 \end{aligned}$$

D3Q33 : second order partial differential equations (ii) 65

relaxation of second order moments

$$\begin{aligned}\varepsilon^* &= \varepsilon + s_\varepsilon (\Phi_\varepsilon - \varepsilon), & xx^* &= xx + s_x (\Phi_{xx} - xx) \\ ww^* &= ww + s_x (\Phi_{ww} - ww), & xy^* &= xy + s_x (\Phi_{xy} - xy) \\ yz^* &= yz + s_x (\Phi_{yz} - yz), & zx^* &= zx + s_x (\Phi_{zx} - zx)\end{aligned}$$

Hénon relations : $\sigma_x \equiv \frac{1}{s_x} - \frac{1}{2}$, $\sigma_e \equiv \frac{1}{s_e} - \frac{1}{2}$

shear viscosity $\mu = \rho c_s^2 \lambda \sigma_x \Delta x$

bulk viscosity $\zeta = \frac{2}{3} \rho c_s^2 \lambda \sigma_e \Delta x$

tensor of viscosities

$$\tau_{xx} = 2\mu \partial_x u + \left(\zeta - \frac{2}{3}\mu\right) \operatorname{div} \mathbf{u}$$

$$\tau_{yy} = 2\mu \partial_y v + \left(\zeta - \frac{2}{3}\mu\right) \operatorname{div} \mathbf{u}$$

$$\tau_{zz} = 2\mu \partial_z w + \left(\zeta - \frac{2}{3}\mu\right) \operatorname{div} \mathbf{u}$$

$$\tau_{xy} = \mu (\partial_x v + \partial_y u), \quad \tau_{yz} = \mu (\partial_y w + \partial_z v), \quad \tau_{zx} = \mu (\partial_z u + \partial_x w)$$

Isothermal Navier-Stokes equations satisfied at second order :-)

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = \mathcal{O}(\Delta x^2)$$

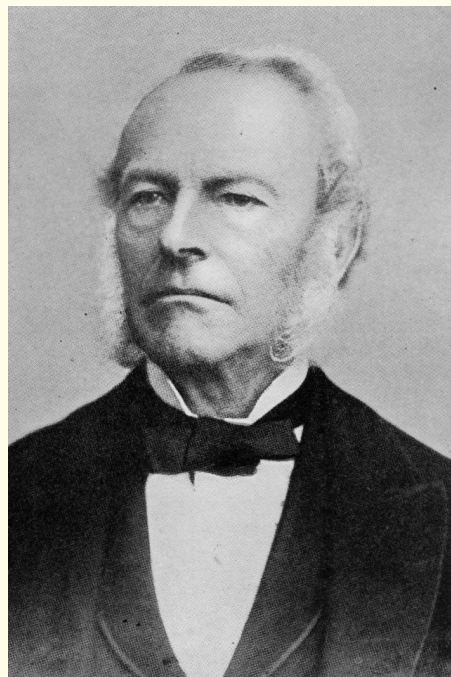
$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - \Delta t \operatorname{div} \tau = \mathcal{O}(\Delta x^2)$$

Navier - Stokes - Fourier

66



Claude Henri Navier
(1785 - 1836)



George Stokes
(1819 - 1903)



Joseph Fourier
(1768 - 1830)

Navier - Stokes - Fourier equations

... in one space dimension

conserved variables $\rho, J \equiv \rho u, E = \frac{1}{2} \rho u^2 + \rho e$

polytropic perfect gas $p = (\gamma - 1) \rho e, e = c_v T, \gamma = \frac{c_p}{c_v}$

Prandtl number $Pr = \frac{\mu c_p}{\kappa}$

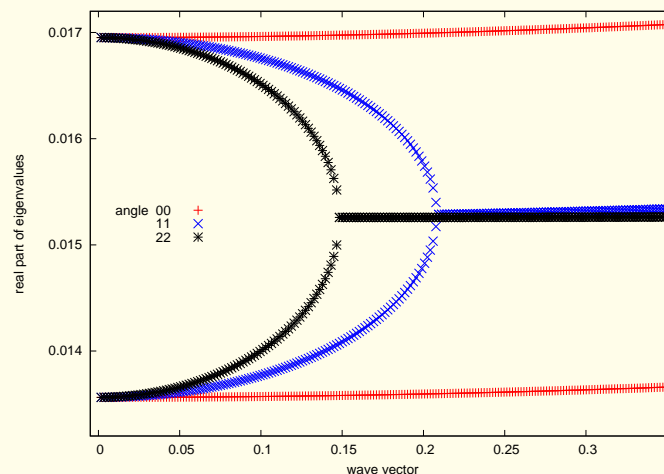
mass conservation $\partial_t \rho + \partial_x(\rho u) = 0$

momentum conservation $\partial_t(\rho u) + \partial_x(\rho u^2 + p) - \partial_x(\mu \partial_x u) = 0$

energy conservation $\partial_t E + \partial_x(E u + p u) - \partial_x(\mu u \partial_x u) - \frac{\gamma}{Pr} \partial_x(\mu \partial_x e) = 0$

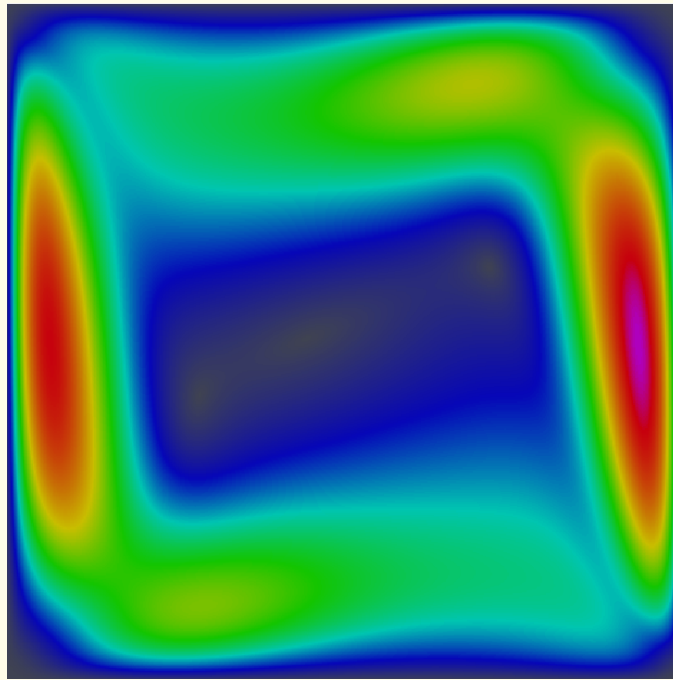
viscous work $\partial_x(\mu u \partial_x u)$

Navier-Stokes-Fourier with lattice Boltzmann schemes 68



The viscous and thermal modes merge together for a critical wave number [P. Lallemand and L.-S. Luo, Phys. Rev. E, 2003]
 Linear analysis of D2Q13 lattice Boltzmann scheme for advective acoustics and tuning the parameters of the D2Q13 [Pierre Lallemand and FD, CaMwA, 2013]

De Vahl Davis thermal test case for natural convection 69



Rayleigh number = 10^5 , Prandtl number = 0.71

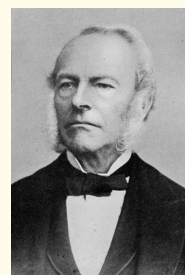
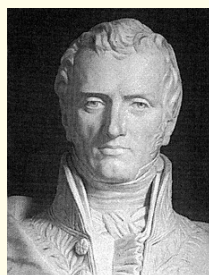
D2Q13 lattice Boltzmann scheme with a single particle distribution
Iso-velocity curves for the modulus of the fluid speed

[Pierre Lallemand and FD, CiCP, 2015]

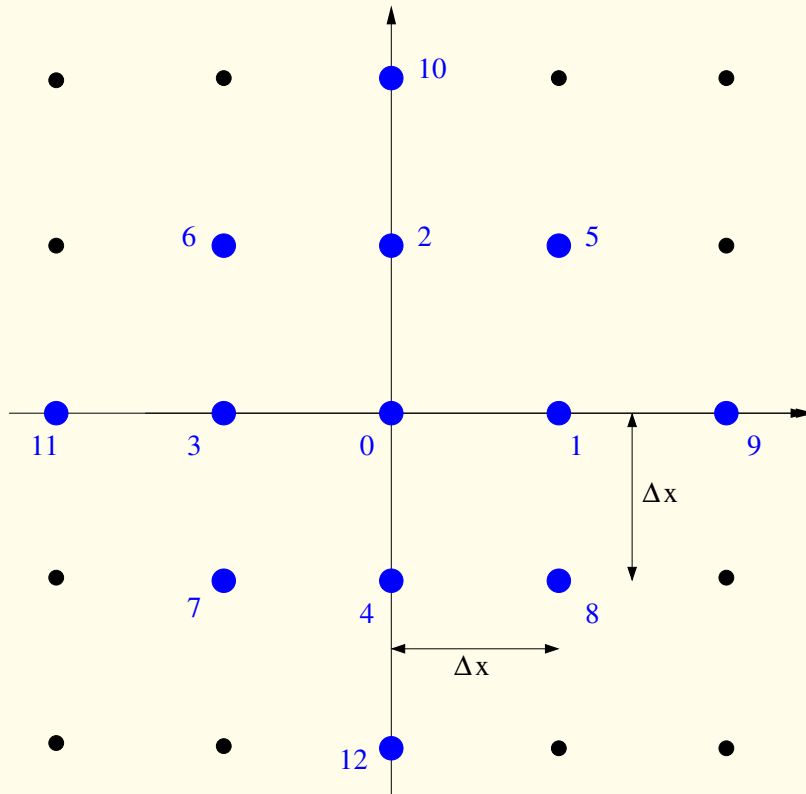
Question

70

What ability for a MRT Boltzmann scheme
to recover formally the one-dimensional or two-dimensional
Navier-Stokes-Fourier equations ?



D2Q13



| | |
|---------------|---|
| ρ | 0 |
| J_x | 1 |
| J_y | 1 |
| ε | 2 |
| xx | 2 |
| xy | 2 |
| q_x | 3 |
| q_y | 3 |
| r_x | 5 |
| r_y | 5 |
| h | 4 |
| x_{xe} | 4 |
| h_2 | 6 |

D2Q13: isothermal operator matrix Λ

$$\Lambda = \begin{bmatrix} \rho & J_x & J_y & \varepsilon & xx & xy & q_x & q_y & r_x & r_y & h & x_{xe} & h_2 \\ 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * \partial_x & 0 & 0 & * \partial_x & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * \partial_y & 0 & 0 & * \partial_y & * \partial_y & * \partial_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & 0 \\ 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & * \partial_x & * \partial_y & * \partial_x & * \partial_y & 0 & 0 & 0 \\ 0 & * \partial_y & * \partial_x & 0 & 0 & 0 & * \partial_y & * \partial_x & * \partial_y & * \partial_x & 0 & 0 & 0 \\ 0 & 0 & 0 & * \partial_x & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & * \partial_x & * \partial_x & 0 \\ 0 & 0 & 0 & * \partial_y & * \partial_y & * \partial_x & 0 & 0 & 0 & 0 & * \partial_y & * \partial_y & 0 \\ 0 & 0 & 0 & 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & * \partial_x & * \partial_x & * \partial_x \\ 0 & 0 & 0 & 0 & * \partial_y & * \partial_x & 0 & 0 & 0 & 0 & * \partial_y & * \partial_y & * \partial_y \\ 0 & 0 & 0 & 0 & 0 & 0 & * \partial_x & * \partial_y & * \partial_x & * \partial_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * \partial_x & * \partial_y & * \partial_x & * \partial_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \partial_x & * \partial_y & 0 & 0 & 0 \end{bmatrix}$$

D2Q13: momentum-velocity operator matrix Λ

$$\Lambda = \begin{bmatrix} \rho & J_x & J_y & \varepsilon & xx & xy & q_x & q_y & r_x & r_y & h & x_{xe} & h_2 \\ 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * \partial_x & 0 & 0 & * \partial_x & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * \partial_y & 0 & 0 & * \partial_y & * \partial_y & * \partial_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & 0 \\ 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & * \partial_x & * \partial_y & * \partial_x & * \partial_y & 0 & 0 & 0 \\ 0 & * \partial_y & * \partial_x & 0 & 0 & 0 & * \partial_y & * \partial_x & * \partial_y & * \partial_x & 0 & 0 & 0 \\ 0 & 0 & 0 & * \partial_x & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & * \partial_x & * \partial_x & 0 \\ 0 & 0 & 0 & * \partial_y & * \partial_y & * \partial_x & 0 & 0 & 0 & 0 & * \partial_y & * \partial_y & 0 \\ 0 & 0 & 0 & 0 & * \partial_x & * \partial_y & 0 & 0 & 0 & 0 & * \partial_x & * \partial_x & * \partial_x \\ 0 & 0 & 0 & 0 & * \partial_y & * \partial_x & 0 & 0 & 0 & 0 & * \partial_y & * \partial_y & * \partial_y \\ 0 & 0 & 0 & 0 & 0 & 0 & * \partial_x & * \partial_y & * \partial_x & * \partial_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * \partial_x & * \partial_y & * \partial_x & * \partial_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \partial_x & * \partial_y & 0 & 0 & 0 \end{bmatrix}$$

D2Q13: first order equations

$$\begin{aligned} \partial_t \rho + \partial_x J_x + \partial_y J_y &= 0 & \partial_t \rho + \partial_x J_x + \partial_y J_y &= 0 \\ \partial_t J_x + \partial_x \left(\frac{14}{13} \lambda^2 \rho + \frac{1}{26} \varepsilon + \frac{1}{2} \Phi_{xx} \right) + \partial_y \Phi_{xy} &= 0 & \partial_t J_x + \partial_x (\rho u^2 + p) + \partial_y (\rho u v) &= 0 \\ \partial_t J_y + \partial_x \Phi_{xy} + \partial_y \left(\frac{14}{13} \lambda^2 \rho + \frac{1}{26} \varepsilon - \frac{1}{2} \Phi_{xx} \right) &= 0 & \partial_t J_y + \partial_x (\rho u v) + \partial_x (\rho v^2 + p) &= 0 \\ \partial_t \varepsilon + 11 \lambda^2 (\partial_x J_x + \partial_y J_y) + 13 (\partial_x \Phi_{qx} + \partial_y \Phi_{qy}) &= 0 & \partial_t E + \partial_x (E u + p u) + \partial_y (E v + p v) &= 0 \end{aligned}$$

then $\frac{14}{13} \lambda^2 \rho + \frac{1}{26} \varepsilon + \frac{1}{2} \Phi_{xx} = \rho u^2 + p, \quad \Phi_{xy} = \rho u v$
 $\frac{14}{13} \lambda^2 \rho + \frac{1}{26} \varepsilon - \frac{1}{2} \Phi_{xx} = \rho v^2 + p \quad \text{and} \quad \Phi_{xx} = \rho (u^2 - v^2)$

$\frac{28}{13} \lambda^2 \rho + \frac{1}{13} \varepsilon = \rho (u^2 + v^2) + 2p$ is conserved
 then $p = \rho e$ and $\gamma = 2$

then $\varepsilon = 26 E - 28 \lambda^2 \rho$

D2Q13: first order equations (ii)

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$$\varepsilon = 26 E - 28 \lambda^2 \rho, \quad \rho = \frac{1}{26} \varepsilon - \frac{1}{2} \rho (u^2 + v^2) + \frac{14}{13} \lambda^2 \rho = \rho e$$

$$\partial_t \varepsilon + \partial_x (11 \lambda^2 J_x + 13 \Phi_{qx}) + \partial_y (11 \lambda^2 J_y + 13 \Phi_{qy}) = 0$$

$$\partial_t (26 E - 28 \lambda^2 \rho) + \partial_x (\varepsilon u + 26 \rho u) + \partial_y (\varepsilon v + 26 \rho v) = 0$$

then $\Phi_{qx} = \frac{1}{13} \varepsilon u + 2 \rho u - \frac{11}{13} \lambda^2 \rho u$

$$\Phi_{qy} = \frac{1}{13} \varepsilon v + 2 \rho v - \frac{11}{13} \lambda^2 \rho v$$

Φ : vector of moments at equilibrium

$$\Phi = (\Phi_{xx}, \Phi_{xy}, \Phi_{qx}, \Phi_{qy}, \Phi_{rx}, \Phi_{ry}, \Phi_h, \Phi_{xxe}, \Phi_{h2})^t$$

D2Q13: towards second order equations

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Φ : vector of moments at equilibrium

$$\Phi = (\Phi_{xx}, \Phi_{xy}, \Phi_{qx}, \Phi_{qy}, \Phi_{rx}, \Phi_{ry}, \Phi_h, \Phi_{xxe}, \Phi_{h2})^t$$

conservative first order fluxes

$$\Gamma_1 = \begin{bmatrix} \partial_x J_x + \partial_y J_y \\ \partial_x (\rho u^2 + p) + \partial_y (\rho u v) \\ \partial_x (\rho u v) + \partial_x (\rho v^2 + p) \\ \partial_x (\varepsilon u + 26 \rho u) + \partial_y (\varepsilon v + 26 \rho v) \end{bmatrix}$$

$$\Psi_1 = d\Phi(W) \cdot \Gamma_1 - (C W + D \Phi(W)) \in \mathbb{R}^9$$

viscous fluxes $-\Delta t \Gamma_2 = -\Delta t B \Sigma \Psi_1 = \text{physical fluxes ?}$

$$= \begin{bmatrix} 0 \\ \partial_j \sigma_{xj} \equiv \partial_x (2 \mu \partial_x u + (\zeta - \mu) (\partial_x u + \partial_y v)) + \partial_y (\mu (\partial_x v + \partial_y u)) \\ \partial_j \sigma_{yj} \equiv \partial_x (\mu (\partial_x v + \partial_y u)) + \partial_y ((\zeta - \mu) (\partial_x u + \partial_y v) + 2 \mu \partial_y v) \\ 26 [\partial_j (u_i \sigma_{ij}) + \frac{\gamma}{Pr} (\partial_x (\mu \partial_x e) + \partial_y (\mu \partial_y e))] \end{bmatrix}$$

D2Q13: towards second order equations (ii)

No term in $\partial_x \rho$ and $\partial_y \rho$ is the physical viscous fluxes:

nonlinear system of $3 \times 2 \times 2 = 12$ equations

possible non-isotropic solution

$$\Phi_{rx} = \frac{31}{6} \rho \lambda^4 u + \frac{7}{6} \lambda^2 (u^2 + 6v^2) \rho u - \frac{21}{2} e u$$

$$\Phi_{ry} = \frac{31}{6} \rho \lambda^4 v + \frac{7}{6} \lambda^2 (6u^2 + v^2) \rho v - \frac{21}{2} e v$$

$$\Phi_h = 140 \lambda^4 \rho - 361 \lambda^2 \rho e - \frac{361}{2} \lambda^2 \rho (u^2 + v^2) + \frac{77}{2} \rho (u^2 + v^2)^2 + 308 (u^2 + v^2 + e) e$$

$$\Phi_{xxe} = -\frac{65}{12} \lambda^2 \rho (u^2 - v^2) + \frac{17}{12} \rho (u^4 - v^4) + \frac{17}{2} \rho (u^2 + v^2)$$

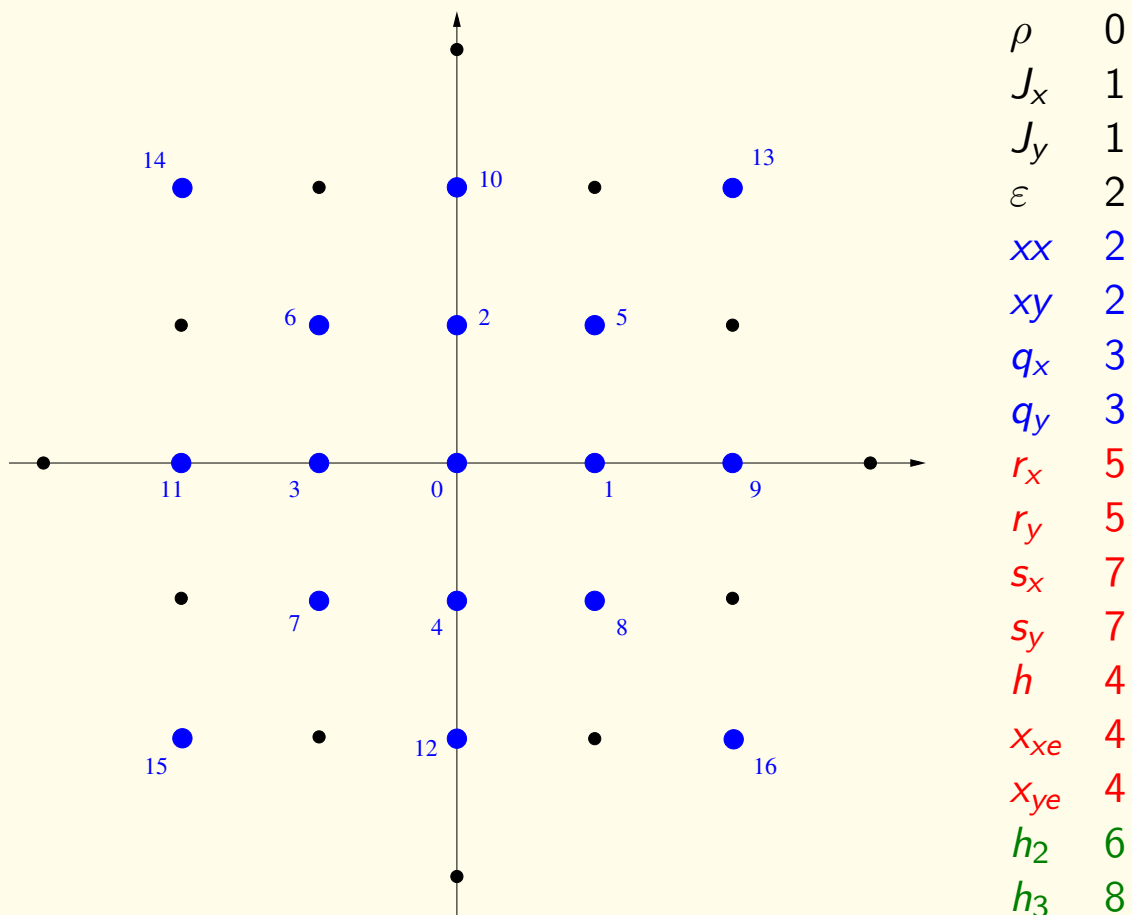
but there are **important remaining discrepancies...**

$3 \times 2 \times 2 \times 3 = 36$ equations to recover the viscous fluxes

possible simple formulas for the viscosities:

$$\mu = 2 \rho e \sigma_x \Delta t, \quad \zeta = \mu$$

D2Q17



D2Q17 : non null elements of the Λ matrix

$$\begin{bmatrix} 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

D2Q17: first order equations

$$\partial_t \rho + \partial_x J_x + \partial_y J_y = 0 \qquad \partial_t \rho + \partial_x J_x + \partial_y J_y = 0$$

$$\begin{aligned} \partial_t J_x + \partial_x \left(\frac{30}{17} \lambda^2 \rho + \frac{1}{34} \varepsilon + \frac{1}{2} \Phi_{xx} \right) + \partial_y \Phi_{xy} &= 0 \\ \partial_t J_x + \partial_x (\rho u^2 + p) + \partial_y (\rho u v) &= 0 \end{aligned}$$

$$\begin{aligned} \partial_t J_y + \partial_x \Phi_{xy} + \partial_y \left(\frac{30}{17} \lambda^2 \rho + \frac{1}{34} \varepsilon - \frac{1}{2} \Phi_{xx} \right) &= 0 \\ \partial_t J_y + \partial_x (\rho u v) + \partial_x (\rho v^2 + p) &= 0 \end{aligned}$$

$$\begin{aligned} \partial_t \varepsilon + \frac{109}{3} \lambda^2 (\partial_x J_x + \partial_y J_y) + \frac{17}{3} (\partial_x q_x + \partial_y q_y) &= 0 \\ \partial_t E + \partial_x (E u + p u) + \partial_y (E v + p v) &= 0 \end{aligned}$$

then $\frac{30}{17} \lambda^2 \rho + \frac{1}{34} \varepsilon + \frac{1}{2} \Phi_{xx} = \rho u^2 + p, \qquad \Phi_{xy} = \rho u v$
 $\frac{30}{17} \lambda^2 \rho + \frac{1}{34} \varepsilon - \frac{1}{2} \Phi_{xx} = \rho v^2 + p \quad \text{and} \quad \Phi_{xx} = \rho (u^2 - v^2)$

$\frac{60}{17} \lambda^2 \rho + \frac{1}{17} \varepsilon = \rho (u^2 + v^2) + 2p$ is conserved
 then $\rho = \rho e$ and $\gamma = 2$

then $\varepsilon = 34 E - 60 \lambda^2 \rho$

D2Q17: first order equations (ii)

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$$\varepsilon = 34 E - 60 \lambda^2 \rho, \quad p = \frac{1}{34} \varepsilon - \frac{1}{2} \rho (u^2 + v^2) + \frac{30}{17} \lambda^2 \rho \rho e$$

$$\partial_t \varepsilon + \frac{109}{3} \lambda^2 (\partial_x J_x + \partial_y J_y) + \frac{17}{3} (\partial_x q_x + \partial_y q_y) = 0$$

$$\partial_t (34 E - 60 \lambda^2 \rho) + \partial_x (\varepsilon u + 34 p u) + \partial_y (\varepsilon v + 34 p v) = 0$$

then $\Phi_{qx} = \frac{6}{17} \varepsilon u + \frac{71}{17} \lambda^2 \rho u - 3 \rho (u^2 + v^2) u$

$$\Phi_{qy} = \frac{6}{17} \varepsilon v + \frac{71}{17} \lambda^2 \rho v - 3 \rho (u^2 + v^2) v$$

Φ : vector of moments at equilibrium

$$\Phi = \begin{pmatrix} \Phi_{xx}, \Phi_{xy}, \Phi_{qx}, \Phi_{qy} \\ \Phi_{rx}, \Phi_{ry}, \Phi_{sx}, \Phi_{sy}, \Phi_h, \Phi_{xxe}, \Phi_{xye} \\ \Phi_{h2}, \Phi_{h3} \end{pmatrix}$$

D2Q17: second order equations

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Linear system of $3 \times 2 \times 8 = 48$ equations

for $4 \times 7 = 28$ partial derivatives

Possible reconstruction of the nonlinear functions :

$$\Phi_{sx} + \frac{31}{2} \lambda^2 \Phi_{rx} = \frac{221}{4} \rho \lambda^6 u - \frac{101}{4} \rho \lambda^4 u^3 + \frac{27}{2} \rho \lambda^4 u v^2 - \frac{249}{4} \rho \lambda^4 u e$$

$$\Phi_{sy} + \frac{31}{2} \lambda^2 \Phi_{ry} = \frac{221}{4} \rho \lambda^6 v - \frac{101}{4} \rho \lambda^4 v^3 + \frac{27}{2} \rho \lambda^4 u^2 v - \frac{249}{4} \rho \lambda^4 v e$$

$$\Phi_h = 620 \rho \lambda^4 + \frac{109}{2} \rho (u^2 + v^2)^2 + 436 \rho (u^2 + v^2) e \\ + 436 \rho e^2 - \frac{969}{2} \rho \lambda^2 (u^2 + v^2 + 2e)$$

$$\Phi_{xxe} = -\frac{65}{12} \rho \lambda^2 (u^2 - v^2) + \frac{17}{12} \rho (u^4 - v^4) + \frac{17}{2} \rho (u^2 - v^2) e$$

$$\Phi_{xye} = -\frac{65}{12} \rho \lambda^2 u v + \frac{17}{24} \rho (u^2 + v^2) u v$$

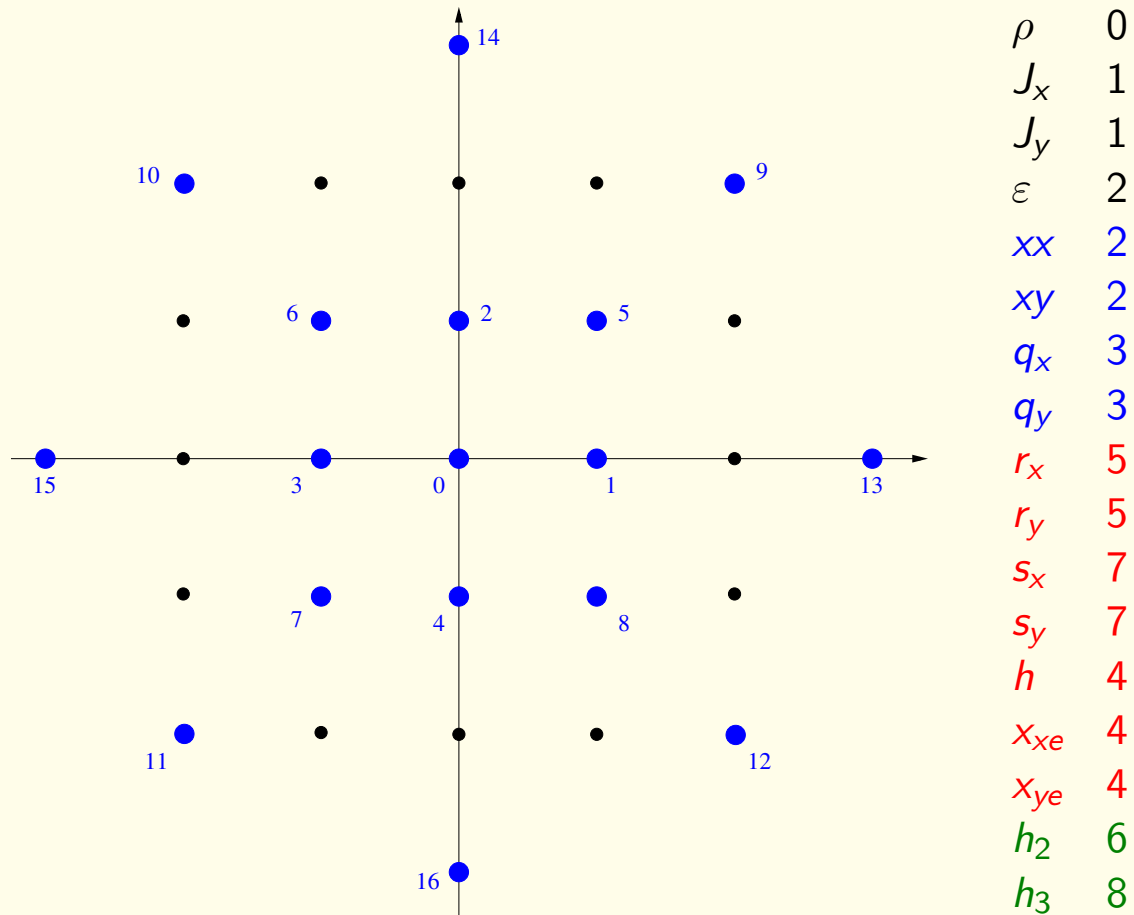
Viscosities: $\mu = \rho e \sigma_x \Delta t$ and $\zeta = 0$

Satisfy the relation $\sigma_x = \sigma_q$

Prandtl number : $Pr = 1$

and all the equations are satisfied :-)

D2V17 of Philippi and Hegele (2006)



D2V17 : non null elements of the Λ matrix

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | * | * | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| * | 0 | 0 | * | * | * | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| * | 0 | 0 | * | * | * | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | * | * | 0 | 0 | 0 | * | * | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | * | * | 0 | 0 | 0 | * | * | * | * | * | * | 0 | 0 | 0 | 0 | 0 |
| 0 | * | * | 0 | 0 | 0 | * | * | * | * | * | * | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | * | * | * | 0 | 0 | 0 | 0 | 0 | 0 | * | * | * | 0 | 0 |
| 0 | 0 | 0 | * | * | * | 0 | 0 | 0 | 0 | 0 | 0 | * | * | * | 0 | 0 |
| 0 | 0 | 0 | 0 | * | * | 0 | 0 | 0 | 0 | 0 | 0 | * | * | * | * | 0 |
| 0 | 0 | 0 | 0 | * | * | 0 | 0 | 0 | 0 | 0 | 0 | * | * | * | * | 0 |
| 0 | 0 | 0 | 0 | * | * | 0 | 0 | 0 | 0 | 0 | 0 | 0 | * | * | * | * |
| 0 | 0 | 0 | 0 | 0 | 0 | * | * | * | * | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | * | * | * | * | * | * | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | * | * | * | * | * | * | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | * | * | * | * | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | * | * | 0 | 0 | 0 | 0 | 0 |

D2V17: first order equations

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$$\begin{aligned} \partial_t \rho + \partial_x J_x + \partial_y J_y &= 0 & \partial_t \rho + \partial_x J_x + \partial_y J_y &= 0 \\ \partial_t J_x + \partial_x \left(\frac{1}{2} \Phi_{xx} + \frac{1}{34} \varepsilon + \frac{40}{17} \rho \lambda^2 \right) + \partial_y \Phi_{xy} &= 0 & \partial_t J_x + \partial_x (\rho u^2 + p) + \partial_y (\rho u v) &= 0 \\ \partial_t J_y + \partial_x \Phi_{xy} + \partial_y \left(-\frac{1}{2} \Phi_{xx} + \frac{1}{34} \varepsilon + \frac{40}{17} \rho \lambda^2 \right) - \frac{1}{2} \Phi_{xx} &= 0 & \partial_t J_y + \partial_x (\rho u v) + \partial_y (\rho v^2 + p) &= 0 \\ \partial_t \varepsilon + \frac{95}{2} \lambda^2 (\partial_x J_x + \partial_y J_y) + \frac{17}{2} (\partial_x \Phi_{qx} + \partial_y \Phi_{qy}) &= 0 & \partial_t E + \partial_x (E u + p u) + \partial_y (E v + p v) &= 0 \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{2} \Phi_{xx} + \frac{1}{34} \varepsilon + \frac{40}{17} \rho \lambda^2 &= \rho u^2 + p \\ -\frac{1}{2} \Phi_{xx} + \frac{1}{34} \varepsilon + \frac{40}{17} \rho \lambda^2 &= \rho v^2 + p \\ \Phi_{xy} = \rho u v \text{ and } \Phi_{xx} &= \rho (u^2 - v^2) \end{aligned}$$

$$\frac{1}{34} \varepsilon + \frac{40}{17} \rho \lambda^2 = \frac{1}{2} \rho (u^2 + v^2) + p$$

So $\frac{1}{2} \rho (u^2 + v^2) + p$ must be **conserved** and $\gamma = 2$

In consequence, $\frac{1}{34} \varepsilon + \frac{40}{17} \rho \lambda^2 = E \equiv \frac{1}{2} \rho (u^2 + v^2) + p e$

$$\varepsilon = 34 E - 80 \lambda^2 \rho$$

D2V17: first order equations (ii)

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$$\varepsilon = 34 E - 80 \lambda^2 \rho = 17 \rho (u^2 + v^2) + 34 \rho e - 80 \rho \lambda^2$$

$$\begin{aligned} \partial_t \varepsilon + \frac{95}{2} \lambda^2 (\partial_x J_x + \partial_y J_y) + \frac{17}{2} (\partial_x \Phi_{qx} + \partial_y \Phi_{qy}) &= 0 \\ \partial_t E + \partial_x (E u + p u) + \partial_y (E v + p v) &= 0 \\ \partial_t \varepsilon + \partial_x (\varepsilon u + 34 p u) + \partial_y (\varepsilon v + 34 p v) &= 0 \end{aligned}$$

Then

$$\frac{95}{2} \lambda^2 J_x + \frac{17}{2} \Phi_{qx} = \varepsilon u + 34 p u$$

$$\frac{95}{2} \lambda^2 J_y + \frac{17}{2} \Phi_{qy} = \varepsilon v + 34 p v$$

and

$$\Phi_{qx} = \frac{2}{17} \varepsilon u + 4 p u - \frac{95}{17} \lambda^2 J_x$$

$$\Phi_{qy} = \frac{2}{17} \varepsilon v + 4 p v - \frac{95}{17} \lambda^2 J_y$$

$$\Phi_{qx} = (2 |\mathbf{u}|^2 + 8 e - 15 \lambda^2) \rho u$$

$$\Phi_{qy} = (2 |\mathbf{u}|^2 + 8 e - 15 \lambda^2) \rho v$$

Reconstruction of the nonlinear functions :

$$\Phi_{rx} + \frac{5}{3} \lambda^2 \Phi_{sx} = \frac{1}{9} \rho \lambda^2 u^3 - \frac{8}{3} \rho \lambda^2 u v^2 - \frac{7}{3} \rho \lambda^2 u e + \frac{35}{9} \rho \lambda^4 u$$

$$\Phi_{ry} + \frac{5}{3} \lambda^2 \Phi_{sy} = -\frac{8}{3} \rho \lambda^2 u^2 v + \frac{1}{9} \rho \lambda^2 v^3 - \frac{7}{3} \rho \lambda^2 v e + \frac{35}{9} \rho \lambda^4 v$$

$$\Phi_h = \frac{19}{2} \rho (u^2 + v^2)^2 + 76 \rho (u^2 + v^2) e - \frac{185}{2} \rho \lambda^2 (u^2 + v^2) + 76 \rho e^2 - \frac{185}{2} \rho \lambda^2 e + 100 \rho \lambda^4$$

$$\Phi_{xxe} = \frac{41}{36} \rho (u^4 - v^4) + \frac{41}{6} \rho (u^2 - v^2) e - \frac{365}{36} \rho \lambda^2 (u^2 - v^2)$$

$$\Phi_{xye} = \frac{17}{24} \rho (u^2 + v^2) u v + \frac{17}{4} \rho u v e - \frac{65}{12} \rho \lambda^2 u v$$

Viscosities: $\mu = \rho e \sigma_x \Delta t$ and $\zeta = 0$

Satisfy the relation $\sigma_x = \sigma_q$

Prandtl number : $Pr = 1$

as for the D2Q17 lattice Boltzmann scheme

Isothermal Navier-Stokes

detailed analysis of the [errors of D2Q9](#)

analysis of the “third order errors” for [D3Q19](#) and [D3Q27](#)

second order [exact](#) simulations with [D2Q13](#) and [D3Q33](#)

has to be confirmed by numerical experiments !

Navier-Stokes-Fourier

lot of discrepancies for [D2Q13](#)

good properties of [D2Q17](#) and [D2V17](#) for a fixed physics

to be done: three dimensions !

For [two distributions](#) or more,

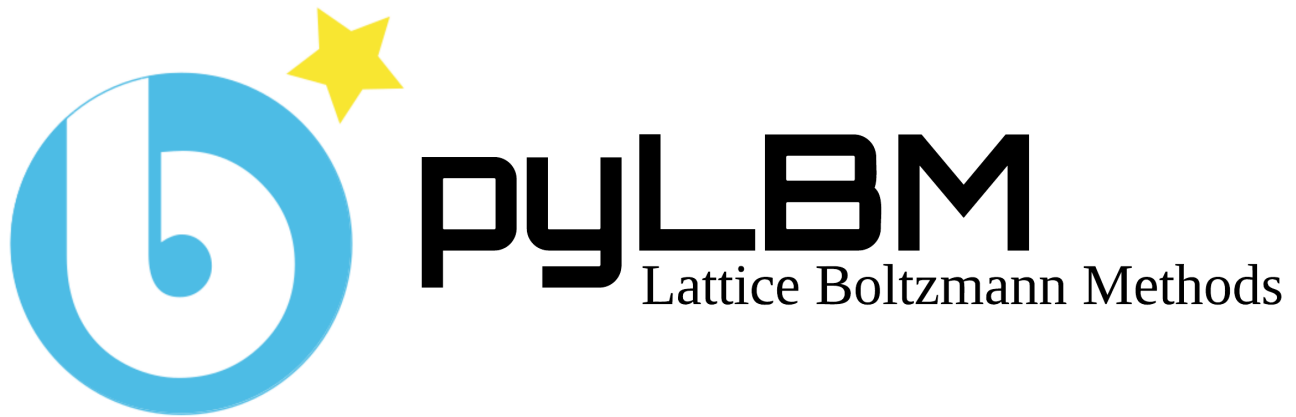
the same kind of analysis can be done

[D1Q3Q3](#), [D2Q9Q5](#), [D2Q9Q9](#), [D2Q13Q*](#), ...

Nonlinear Taylor expansion used up to [second order](#)

implementation in the new version of the software [pyLBM](#)

www.imo.universite-paris-saclay.fr/~graille/pylbn.php



pypi.org/project/pylbn

github.com/pylbn



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