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On Furstenberg systems of aperiodic multiplicative functions of Matomäki, Radziwiłł and Tao

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Abstract

It is shown that in a class of counterexamples to Elliott's conjecture by Matomäki, Radziwiłł and Tao [23], the Chowla conjecture holds along a subsequence.

1 Introduction

The celebrated Chowla conjecture [5] from 1965 predicts that for the arithmetic Liouville function λ , we have

$$(1) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \lambda(n + a_1) \cdot \dots \cdot \lambda(n + a_k) = 0$$

for any choice of $0 \leq a_1 < \dots < a_k$, $k \geq 1$. As noticed by Sarnak [25] this is equivalent to saying that the Liouville subshift $X_\lambda \subset \{-1, 1\}^{\mathbb{N}}$ is actually the full shift and λ is a generic point for the Bernoulli measure $(1/2, 1/2)^{\otimes \mathbb{N}}$, which is the Haar measure on $\{-1, 1\}^{\mathbb{N}}$. When we consider a more general multiplicative function $\mathbf{u} : \mathbb{N} \rightarrow \mathbb{S}^1$ which is **aperiodic** (i.e. its mean along any arithmetic progression exists and equals zero), and if all powers \mathbf{u}^k ($k \geq 1$) are still aperiodic, then the analog of (1) for \mathbf{u} becomes

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \mathbf{u}^{r_1}(n + a_1) \cdot \dots \cdot \mathbf{u}^{r_k}(n + a_k) \overline{\mathbf{u}^{s_1}(n + b_1) \cdot \dots \cdot \mathbf{u}^{s_\ell}(n + b_\ell)} = 0$$

for all powers $r_u, s_t \in \mathbb{N}$ and $\{a_1, \dots, a_k\} \cap \{b_1, \dots, b_\ell\} = \emptyset$, which in turn means that \mathbf{u} is a generic point for $(\text{Leb}_{\mathbb{S}^1})^{\otimes \mathbb{N}}$, the Haar measure on $(\mathbb{S}^1)^{\mathbb{N}}$.

This more general form of Chowla conjecture is still a particular case of Elliott's conjecture [7], [8], [9] which deals with several (possibly different) multiplicative functions (one of which is aperiodic). Similar conjectures can be formulated for multiplicative functions taking values in the unit disk \mathbb{D} , but in this case we have to consider properties of \mathbf{u} relative to $|\mathbf{u}|$, where the latter point is always generic for an ergodic measure (with respect to the left shift), often called the Mirsky measure, on $[0, 1]^{\mathbb{N}}$ (see [25] for the case of the Möbius function μ or e.g. [4] for a general case).

In [23], Matomäki, Radziwiłł and Tao gave a class of examples of multiplicative and aperiodic¹ $\mathbf{u} : \mathbb{N} \rightarrow \mathbb{S}^1 \cup \{0\}$ for which the sequence

$$\left(\frac{1}{N} \sum_{1 \leq n \leq N} \mathbf{u}(n) \overline{\mathbf{u}(n+1)} \right)_{N \geq 1}$$

does not converge to zero. This disproved the general form of Chowla conjecture for aperiodic \mathbf{u} , and in particular, they disproved Elliott's conjecture. In their class $|\mathbf{u}| = \boldsymbol{\mu}^2$ but they also explained that their example could be modified to get a completely multiplicative \mathbb{S}^1 -valued aperiodic function. Finally, this led them to reformulate Elliott's conjecture in the sense that it is expected to be valid for a subclass of aperiodic functions, the so-called *strongly aperiodic functions*, and till today this stronger form of Elliott's conjecture is open and under intensive study [12], [13], [21], [26]-[29].

In this note, we will have a closer look at the counterexample given by Matomäki, Radziwiłł and Tao. To avoid some technical, rather notational, issues, we will deal with the completely multiplicative, \mathbb{S}^1 -valued version of their construction, which we call here *the MRT class* (see the complete description of such multiplicative functions in Section 3). If $\mathbf{u} \in \text{MRT}$ then it cannot be generic for the Haar measure on $(\mathbb{S}^1)^\mathbb{N}$, but still we can ask for which measures on $(\mathbb{S}^1)^\mathbb{N}$ it is quasi-generic. Each such measure yields a so-called measure-theoretic *Furstenberg system* (see Section 2). In particular, the arguments given in [23] prove that there exists an increasing sequence of integers giving rise to a Furstenberg system which is measure-theoretically isomorphic to the action of the identity map on \mathbb{S}^1 equipped with the Lebesgue measure. What kind of other dynamical systems can be obtained as Furstenberg systems for \mathbf{u} in the MRT class is a natural question. Furthermore, in the topological setting, \mathbf{u} determines a subshift $X_{\mathbf{u}} \subset (\mathbb{S}^1)^\mathbb{N}$ and we can ask for its topological entropy. Our aim is to prove the following result.

Main Theorem. *Let \mathbf{u} be in the MRT class. Then, for each $d \geq 0$, there is a Furstenberg system $(X_{\mathbf{u}}, \nu_d, S)$ of \mathbf{u} which is measure-theoretically isomorphic to the unipotent system*

$$(x_d, x_{d-1}, \dots, x_0) \mapsto (x_d, x_{d-1} + x_d, \dots, x_0 + x_1)$$

on \mathbb{T}^{d+1} equipped with the $(d+1)$ -dimensional Lebesgue measure.

Furthermore, the Bernoulli shift $((\mathbb{S}^1)^\mathbb{N}, (\text{Leb}_{\mathbb{S}^1})^{\otimes \mathbb{N}}, S)$ is also a Furstenberg system of \mathbf{u} , i.e.

(2) *the analog of the Chowla conjecture holds for \mathbf{u} along a subsequence.*

In particular,

$$(3) \quad X_{\mathbf{u}} = (\mathbb{S}^1)^\mathbb{N}, \text{ } ^2$$

and

$$(4) \quad h_{\text{top}}(X_{\mathbf{u}}, S) = \infty.$$

¹Their functions are totally aperiodic, that is, all powers \mathbf{u}^k , $k \geq 1$ are also aperiodic.

²Cf. this equality with a remark after Theorem 10 in [25] as a topological instance of validity of Chowla conjecture.

The proof of the above theorem is given in Section 4. It seems also that this result makes it legitimate to ask whether (some of) properties (2)-(4) are valid for all totally aperiodic multiplicative functions \mathbf{u} , $|\mathbf{u}| = 1$. Note also that the theorem above sheds light on Frantzikinakis' question: see Problem 3.1 on the list of problems in [3] of whether the system $(x, y) \mapsto (x, x + y)$ on \mathbb{T}^2 can be a Furstenberg system of the Liouville function. The expected answer is of course negative (as the positive answer is in conflict with the Chowla conjecture) but, as our result shows, this unipotent system is a Furstenberg system for a class of multiplicative aperiodic functions. Moreover, our Main Theorem yields positive answers to questions raised in Problem 7.3 [3] in the class of MRT functions. More than that, since the ergodic components of ν_d , $d \geq 2$, are affine extensions of irrational rotations, the Main Theorem disproves Conjecture 2 from [15].

As each $\mathbf{u} \in \text{MRT}$ is given "locally by Archimedean characters", in Section 5, we will deal with Furstenberg systems of Archimedean characters themselves and will describe their Furstenberg systems. Moreover, we will show that no $\mathbf{u} \in \text{MRT}$ satisfies the analog of Sarnak's conjecture. Finally, in Section 6 we show that the analog of logarithmic Chowla conjecture for \mathbf{u} (cf. (2)) holds along a subsequence and discuss further properties of Furstenberg systems of MRT arithmetic functions: strong stationarity and the absence of zero mean on typical short interval.

In Appendix we discuss Furstenberg systems given by ν_d from the pure ergodic theory point of view. We explain their connections with the classical theory of transformations with quasi-discrete spectrum and give a new characterization of quasi-eigenfunctions which seems to be of independent interest.

2 Furstenberg systems of a bounded arithmetic function

Assume that $u : \mathbb{N} \rightarrow \mathbb{C}$ is an arithmetic function, $|u| \leq 1$. Denote by \mathbb{D} the unit disk. On the space $M(\mathbb{D}^{\mathbb{N}})$ of Borel probability measures on $\mathbb{D}^{\mathbb{N}}$, we consider the weak*-topology, in which $\nu_m \rightarrow \nu$ if and only if $\int_{\mathbb{D}^{\mathbb{N}}} f d\nu_m \rightarrow \int_{\mathbb{D}^{\mathbb{N}}} f d\nu$ for each $f \in C(\mathbb{D}^{\mathbb{N}})$. This topology turns $M(\mathbb{D}^{\mathbb{N}})$ into a compact metrizable space. Let S denote the shift map on $\mathbb{D}^{\mathbb{N}}$, and consider in $M(\mathbb{D}^{\mathbb{N}})$ the sequence

$$(E_N(u))_{N \geq 1} := \left(\frac{1}{N} \sum_{0 \leq n < N} \delta_{S^n u} \right)_{N \geq 1}$$

of empiric probability measures. By compactness, we can choose a converging subsequence

$$E_{N_m}(u) \rightarrow \nu,$$

and we say that u is *quasi-generic for ν along the sequence (N_m)* . It is not hard to see (cf. the Krylov-Bogolyubov theorem) that such a limit point ν is S -invariant. Moreover, such a ν is always supported on the subshift X_u generated by u , that is

$$X_u := \overline{\{S^n u : n \geq 0\}}.$$

The measure-theoretic dynamical system (X_u, ν, S) is called a *Furstenberg system* of u . Denote by $V(u)$ the set of all probability measures on X_u for which u is quasi-generic:

$$V(u) := \left\{ \nu \in M(X_u, S) : \nu = \lim_{m \rightarrow \infty} E_{N_m}(u) \text{ for some } N_m \rightarrow \infty \right\}.$$

Classically, we have the following:

Proposition 2.1. ([6]) (i) $V(u) \subset M(X_u, S)$ is closed (in the weak*-topology). (ii) $V(u)$ is connected, whence either $|V(u)| = 1$ or $V(u)$ is uncountable.

Let $Z_0 : \mathbb{D}^{\mathbb{N}} \rightarrow \mathbb{D}$ be the 1-coordinate map: $Z_0(y) = y(1)$ for $y = (y(n))_{n \in \mathbb{N}} \in \mathbb{D}^{\mathbb{N}}$. In general, we define $Z_n := Z_0 \circ S^n$ as the map $y \mapsto y(n+1)$. Then, given $\nu \in V(u)$, we obtain a stationary process (Z_0, Z_1, \dots) with values in $\mathbb{D}^{\mathbb{N}}$ whose distribution is ν . Let $\kappa = (Z_0)_*(\nu)$ be the distribution of the random variable Z_0 under ν . Then by the S -invariance of ν , κ is also the distribution of each coordinate Z_n , $n \geq 0$. An example of particular interest corresponds to the i.i.d. case, arising when ν is a product measure, i.e., when ν is of the form $\kappa^{\otimes \mathbb{N}}$. We have the following observation.

Proposition 2.2. Let κ be a probability measure on \mathbb{D} . Assume that for each $d \geq 0$ there exists $\nu_d \in V(u)$ under which the distribution of (Z_0, \dots, Z_d) is $\kappa^{\otimes d+1}$. Then $\kappa^{\otimes \mathbb{N}} \in V(u)$.

Proof. By compactness, we can assume that $\nu_d \rightarrow \rho$, and by Proposition 2.1 (i), $\rho \in V(u)$. All we need to show is that

$$\mathbb{E}_\rho(Z_0^{q_1} \circ S^{j_1} \dots \circ Z_0^{q_k} \circ S^{j_k}) = \prod_{i=1}^k \mathbb{E}_\kappa Z_0^{q_i}$$

for each $k \geq 1$, $q_i \in \mathbb{Z}$ and $0 \leq j_1 < \dots < j_k$. But the equality above is true if ρ is replaced by ν_d for $d \geq j_k$, and since $Z_0^{q_1} \circ S^{j_1} \dots \circ Z_0^{q_k} \circ S^{j_k} \in C(X_u)$, the result follows. \square

Remark 2.3. If $\kappa^{\otimes \mathbb{N}} \in V(u)$ then by the variational principle (see e.g. [30, Section 8.2]) $h_{\text{top}}(X_u, S) \geq h(X_u, \kappa^{\otimes \mathbb{N}}, S) = H(\kappa)$. If the distribution of κ is continuous then immediately $H(\kappa) = +\infty$, whence $h_{\text{top}}(X_u, S) = +\infty$ in this case.

Proposition 2.2 can be useful if we want to show that the product measure yields a Furstenberg system of Bernoulli type (“Chowla holds along a subsequence”). Indeed, we only need to show the existence of Furstenberg systems which yield some finite degree of independence of the process $(Z_n)_{n \geq 0}$ and such can be firstly of zero entropy and even very non-ergodic (i.e. belonging to Erg^\perp), cf. also [10]. Together with Remark 2.3, it also gives a nice criterion to show that the topological entropy of u is infinite.

3 MRT multiplicative functions

In this section we describe more precisely the MRT class of completely multiplicative functions to which our Main Theorem applies. We start by giving a

formal definition of this class, then we resume the construction given in [23] by Matomäki, Radziwiłł and Tao of a completely multiplicative function meeting the required property. Finally, we give the key property of MRT arithmetic functions that will be used in the proof of our Main Theorem.

Definition 3.1. A completely multiplicative function $\mathbf{u} : \mathbb{N} \rightarrow \mathbb{S}^1$ belongs to the MRT class if there exist two increasing sequences of integers (t_m) and (s_m) such that, for each $m \geq 1$, we have the following properties:

- $t_m < s_{m+1} < s_{m+1}^2 \leq t_{m+1}$,
- for each prime $p \in (t_m, t_{m+1}]$, $\mathbf{u}(p) = p^{is_{m+1}}$,
- for each prime $p \leq t_m$, $|\mathbf{u}(p) - p^{is_{m+1}}| < \frac{1}{t_m^2}$.

Here is the method to get such a function. We just have to define $\mathbf{u}(p)$ for each prime p and to construct the sequences (t_m) and (s_m) , which is done inductively as follows. Start by choosing an integer $t_1 \in \mathbb{N}$ and set, for each prime $p \leq t_1$, $\mathbf{u}(p) := 1$. Now, assume that for some $m \geq 1$ we have already defined t_m and $\mathbf{u}(p)$ for each $p \leq t_m$. In the Cartesian product $\prod_{p \leq t_m} \mathbb{S}^1$, we consider the sequence of points

$$\left((p^{is})_{p \leq t_m} \right)_{s \in \mathbb{N}}.$$

Since the numbers $\log p$, $p \leq t_m$, are linearly independent over the integers, this sequence is dense in $\prod_{p \leq t_m} \mathbb{S}^1$. Thus, we can choose $s_{m+1} > t_m$ so that (6) is satisfied. We then choose $t_{m+1} \geq s_{m+1}^2$, and for $t_m < p \leq t_{m+1}$ we set $\mathbf{u}(p) := p^{is_{m+1}}$. Doing this inductively for each $m \geq 1$ gives a completely multiplicative function $\mathbf{u} \in \text{MRT}$.

Remark 3.1. It is also interesting to note that the growth of s_{m+1}/t_m is necessarily superpolynomial: in fact, it follows from Propositions 4.3 and 4.7 that, for each $\beta > 0$, $t_m < s_{m+1}^\beta$ for m large enough.

It is also shown in [23] that once $s_{m+1} > e^{t_m}$ for $m \geq 1$, the resulting \mathbf{u} is aperiodic.

We will use the following easy lemma.

Lemma 3.2. Let $\mathbf{u} \in \text{MRT}$ and (t_m) , (s_m) be as in Definition 3.1. Let $m \geq 1$ and $n \leq t_{m+1}$. If the number of prime factors of n less than or equal to t_m (counting multiplicity) is bounded by t_m , then

$$|\mathbf{u}(n) - n^{is_{m+1}}| \leq \frac{1}{t_m}.$$

Proof. We write n as a product of primes

$$n = \prod_{p \leq t_m} p^{\alpha_p(n)} \prod_{t_m < p \leq t_{m+1}} p^{\alpha_p(n)}.$$

By the complete multiplicativity of \mathbf{u} and by (5), we then have

$$\mathbf{u}(n) = \prod_{p \leq t_m} \mathbf{u}(p)^{\alpha_p(n)} \left(\prod_{t_m < p \leq t_{m+1}} p^{\alpha_p(n)} \right)^{is_{m+1}}.$$

Using (6) in the first product above, we get

$$|\mathbf{u}(n) - n^{is_{m+1}}| \leq \frac{\sum_{p \leq t_m} \alpha_p(n)}{t_m^2}.$$

In particular, if $\sum_{p \leq t_m} \alpha_p(n) \leq t_m$, then

$$|\mathbf{u}(n) - n^{is_{m+1}}| \leq \frac{1}{t_m}.$$

□

The next lemma is useful to estimate the density of the integers n for which (7) is not valid.

Lemma 3.3. *For $t \in \mathbb{N}$, denote*

$$B_t := \{n \geq 1 : \sum_{p \leq t} \alpha_p(n) \geq t\},$$

i.e. B_t is the set of integers $n \geq 1$ having at least t prime factors less than or equal to t (counting multiplicity). Then

$$(8) \quad \varepsilon_t := \sup_{N \geq 1} \frac{1}{N} \sum_{1 \leq n \leq N} 1_{B_t}(n) \xrightarrow{t \rightarrow \infty} 0.$$

Proof. For each $t \in \mathbb{N}$, set $k_t := \left\lfloor \frac{t}{\pi(t)} \right\rfloor$ (where $\pi(t)$ denotes, as usual, the number of primes up to t). Let $n \in \mathbb{N}$; if, for each prime $p \leq t$, p^{k_t} does not divide n , then the number of prime factors of n less than or equal to t (counting multiplicity) is at most $\pi(t)(k_t - 1) < t$, hence $n \notin B_t$. By contraposition, if $n \in B_t$, there exists a prime number $p \leq t$ such that $p^{k_t} | n$. Therefore, we have for each $N \geq 1$

$$\begin{aligned} \frac{1}{N} \sum_{1 \leq n \leq N} 1_{B_t}(n) &\leq \frac{1}{N} \sum_{1 \leq n \leq N} \sum_{p \leq t} 1_{p^{k_t} | n} \\ &= \sum_{p \leq t} \frac{1}{N} \sum_{1 \leq n \leq N} 1_{p^{k_t} | n} \\ &\leq \sum_{p \leq t} \frac{1}{p^{k_t}} \\ &< \sum_p \frac{1}{p^{k_t}} \xrightarrow{t \rightarrow \infty} 0 \quad (\text{since } k_t \rightarrow \infty \text{ as } t \rightarrow \infty). \end{aligned}$$

□

Using Lemma 3.2 and Lemma 3.3, we obtain the following result.

Proposition 3.4. *Let $\mathbf{u} \in \text{MRT}$ and $(t_m), (s_m)$ be as in Definition 3.1. Let (N_m) be an increasing sequence of integers with $N_m \leq t_{m+1}$ for each $m \in \mathbb{N}$. Then*

$$\frac{1}{N_m} \# \left\{ n \in \{1, \dots, N_m\} : |\mathbf{u}(n) - n^{is_{m+1}}| > \frac{1}{t_m} \right\} \leq \varepsilon_{t_m} \xrightarrow{m \rightarrow \infty} 0.$$

4 Proof of the Main Theorem

Here is an outline of the proof. First, we present in Section 4.1 a family of stationary processes, taking values in the unit circle, parametrized by an integer $d \geq 0$. Such a process generates a measure-theoretic dynamical system isomorphic to the unipotent system appearing in the statement of the theorem. It is easily characterized by two conditions: each coordinate of the process is uniformly distributed on the circle, and some deterministic function ϕ_{d+1} of the coordinates of the process is constantly equal to 1. We completely describe in Proposition 4.1 and Remark 4.2 the distribution ν_d of such a process, in particular we show that $d + 1$ consecutive coordinates are independent. Then Proposition 4.3 provides a criterion for an arithmetic function u to be quasi-generic for this probability measure ν_d . This criterion involves the functions ϕ_{d+1} , and we study in Section 4.2 two sequences of polynomials related with this family of functions. We show in Section 4.3 that for $\mathbf{u} \in \text{MRT}$, the criterion is fulfilled for each $d \geq 0$. For this we use Proposition 3.4 which allows us to replace, in the appropriate range, $\mathbf{u}(n)$ by $n^{i_{s_m}+1}$. The criterion then becomes an evaluation of some exponential sums, that we can handle with the help of Kusmin-Landau Theorem (Theorem 4.8). Finally, using Proposition 2.2, we can conclude that \mathbf{u} is also quasi-generic for the product measure $(\text{Leb}_{\mathbb{S}^1})^{\otimes \mathbb{N}}$.

4.1 Processes in the unit circle

We recall the notation $Z_n = Z_0 \circ S^n$ ($n \geq 0$) from Section 2, but we restrict ourselves to the set of sequences taking values in \mathbb{S}^1 . We define inductively a new sequence of maps $(Y^{(d)})_{d \geq 0}$ defined on $(\mathbb{S}^1)^{\mathbb{N}}$ taking also their values in \mathbb{S}^1 , by $Y^{(0)} := Z_0$, $Y^{(1)} := Z_1/Z_0$, and in general for each $d \geq 0$,

$$Y^{(d+1)} := \frac{Y^{(d)} \circ S}{Y^{(d)}}.$$

We also define the auxiliary sequence $(X_d)_{d \geq 0}$ taking values in the one-dimensional torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, by

$$e^{i2\pi X_d} := Y^{(d)}.$$

As $e^{i2\pi X_{d+1}} = Y^{(d+1)} = e^{i2\pi(X_d \circ S - X_d)}$, we get

$$(9) \quad \forall d \geq 0, \quad X_d \circ S = X_{d+1} + X_d.$$

Moreover, by an easy induction on n using the above formula, we can recover the process each Z_n from (X_d) by the relation

$$(10) \quad \forall n \geq 0, \quad Z_n = e^{i2\pi \sum_{d=0}^n \binom{n}{d} X_d}.$$

Proposition 4.1. *Let ν be a shift-invariant probability measure on $(\mathbb{S}^1)^{\mathbb{N}}$. Assume that there exists $d \geq 0$ such that, under ν ,*

- $Y^{(d)}$ is S -invariant (i.e. $Y^{(d+1)} = 1$ ν -a.s.),
- the distribution of $Y^{(d)}$ is uniform on \mathbb{S}^1 .

Then, for each n , Z_n is uniformly distributed on \mathbb{S}^1 , and Z_0, Z_1, \dots, Z_d are independent.

Proof. As $Y^{(d)}$ is S -invariant, so is X_d . Hence, in every ergodic component of ν , X_d is a.s. constant. Moreover, since the distribution of X_d under ν is the Lebesgue measure $\text{Leb}_{\mathbb{T}}$ on \mathbb{T} , in almost every ergodic component the value of X_d is irrational. Let us fix such an ergodic component, say $\tilde{\nu}$, and let α be the value taken by X_d under $\tilde{\nu}$. From (9), we get

$$\begin{aligned} (X_{d-1}, \dots, X_0) \circ S &= (X_{d-1} + X_d, X_{d-2} + X_{d-1}, \dots, X_0 + X_1) \\ &= (X_{d-1} + \alpha, X_{d-2} + X_{d-1}, \dots, X_0 + X_1). \end{aligned}$$

But when α is irrational, the map

$$(x_{d-1}, \dots, x_0) \in \mathbb{T}^d \mapsto (x_{d-1} + \alpha, x_{d-2} + x_{d-1}, \dots, x_0 + x_1)$$

is uniquely ergodic [16], with the d -fold product $(\text{Leb}_{\mathbb{T}})^{\otimes d}$ as the only ergodic invariant measure. We deduce that, under $\tilde{\nu}$, the distribution of (X_{d-1}, \dots, X_0) must be $(\text{Leb}_{\mathbb{T}})^{\otimes d}$. Integrating over the set of ergodic components, we get that under ν , the distribution of $(X_d, X_{d-1}, \dots, X_0)$ is $(\text{Leb}_{\mathbb{T}})^{\otimes (d+1)}$. Then, from Formula (10), we can write each $0 \leq n \leq d$,

$$Z_n = \tilde{Z}_{n-1} e^{i2\pi X_n},$$

where \tilde{Z}_{n-1} is measurable with respect to X_0, \dots, X_{d-1} ($\tilde{Z}_{n-1} = 1$ for $n = 0$). From this it follows that Z_n is uniformly distributed on \mathbb{S}^1 conditionally to X_0, \dots, X_{n-1} . This concludes the proof. \square

Remark 4.2. The proof shows in fact that, for each $d \geq 0$, there is a unique shift-invariant measure ν_d on $(\mathbb{S}^1)^{\mathbb{N}}$ such that whenever the assumptions of Proposition 4.1 are satisfied, then the distribution of the stationary process $(Z_n)_{n \in \mathbb{N}}$ is ν_d . Under ν_d , any $d + 1$ consecutive coordinates of the process are independent and uniformly distributed on the circle, and for each $n \geq 0$ the coordinate Z_{n+d+1} is a deterministic function of Z_n, \dots, Z_{n+d} given by the condition $Y^{(d+1)} = 1$. Moreover, looking at the process $((X_0, X_1, \dots, X_d) \circ S^n)_{n \geq 0}$, we see that the dynamical system $((\mathbb{S}^1)^{\mathbb{N}}, \nu_d, S)$ is measure-theoretically isomorphic to the unipotent system described in the statement of the Main Theorem.

It will be useful to introduce, for each $d \geq 0$, the function $\phi_d : (\mathbb{S}^1)^{d+1} \rightarrow \mathbb{S}^1$ such that $Y^{(d)} = \phi_d(Z_0, \dots, Z_d)$. For $d = 0, \dots, 4$, these functions are given by

$$(11) \quad \begin{aligned} \phi_0(z_0) &= \frac{z_0}{1} \quad ; \quad \phi_1(z_0, z_1) = \frac{z_1}{z_0} \quad ; \quad \phi_2(z_0, z_1, z_2) = \frac{z_0 z_2}{z_1^2} \quad ; \\ \phi_3(z_0, z_1, z_2, z_3) &= \frac{z_1^3 z_3}{z_0 z_2^3} \quad ; \quad \phi_4(z_0, z_1, z_2, z_3, z_4) = \frac{z_0 z_2^6 z_4}{z_1^4 z_3^4}. \end{aligned}$$

In general, we can always write the function ϕ_d as a quotient

$$(12) \quad \phi_d(z_0, \dots, z_d) = \frac{\pi_d(z_0, \dots, z_d)}{\tilde{\pi}_d(z_0, \dots, z_d)},$$

where $\pi_d(z_0, \dots, z_d)$ and $\tilde{\pi}_d(z_0, \dots, z_d)$ are products of some z_j 's (with possible multiplicity). These sequences of products are completely defined by the following recurrence property: $\pi_0(z_0) := z_0$, $\tilde{\pi}_0(z_0) := 1$, and for each $d \geq 1$,

$$(13) \quad \pi_{d+1}(z_0, \dots, z_{d+1}) := \pi_d(z_1, \dots, z_{d+1}) \tilde{\pi}_d(z_0, \dots, z_d),$$

and

$$(14) \quad \tilde{\pi}_{d+1}(z_0, \dots, z_{d+1}) := \tilde{\pi}_d(z_1, \dots, z_{d+1}) \pi_d(z_0, \dots, z_d).$$

Note that, as π_d and $\tilde{\pi}_d$ only involve a product of variables, their definition can be extended to \mathbb{C}^{d+1} .

The next proposition is a key ingredient for the identification of some Furstenberg systems of multiplicative functions.

Proposition 4.3. *Let $u = (u(n))_{n \in \mathbb{N}} \in (\mathbb{S}^1)^{\mathbb{N}}$. Assume that, for some $d \geq 1$ and some increasing sequence $(N_m) \subset \mathbb{N}$, we have*

$$(15) \quad \frac{1}{N_m} \sum_{n \leq N_m} \phi_{d+1}(u(n), u(n+1), \dots, u(n+d+1)) \xrightarrow{m \rightarrow \infty} 1,$$

and that

$$(16) \quad \forall \ell \geq 1, \quad \frac{1}{N_m} \sum_{n \leq N_m} \phi_d^\ell(u(n), u(n+1), \dots, u(n+d)) \xrightarrow{m \rightarrow \infty} 0.$$

Then, along the subsequence (N_m) , u is quasi-generic for the measure ν_d described in Remark 4.2.

Proof. Let μ be a shift-invariant measure on $(\mathbb{S}^1)^{\mathbb{N}}$ such that, along some subsequence of (N_m) , u is quasi-generic for μ . From (15), we get that

$$1 = \mathbb{E}_\mu [\phi_{d+1}(Z_0, \dots, Z_{d+1})] = \mathbb{E}_\mu [Y^{(d+1)}],$$

hence $Y^{(d+1)} = 1$ μ -a.s. And from (16), we get that for each $\ell \geq 1$,

$$0 = \mathbb{E}_\mu [\phi_d^\ell(Z_0, \dots, Z_d)] = \mathbb{E}_\mu [(Y^{(d)})^\ell],$$

which shows that, under μ , $Y^{(d)}$ is uniformly distributed on \mathbb{S}^1 . Then, with Proposition 4.1 and Remark 4.2, we conclude that $\mu = \nu_d$. \square

4.2 Special sequences of polynomials

We now introduce two sequences $(P_d)_{d \geq 0}$ and $(Q_d)_{d \geq 0}$ of polynomials in the variable n , linked to the analysis of the preceding section by the following definition: for each $d \geq 0$,

$$(17) \quad P_d(n) := \pi_d(n, n+1, \dots, n+d), \quad \text{and} \quad Q_d(n) := \tilde{\pi}_d(n, n+1, \dots, n+d).$$

The first polynomials of this family are given below (compare with the numerators and denominators in (11)).

$$\begin{aligned} P_0(n) &= n ; & Q_0(n) &= 1 ; \\ P_1(n) &= n+1 ; & Q_1(n) &= n ; \\ P_2(n) &= n(n+2) ; & Q_2(n) &= (n+1)^2 ; \\ P_3(n) &= (n+1)^3(n+3) ; & Q_3(n) &= n(n+2)^3 ; \\ P_4(n) &= n(n+2)^6(n+4) ; & Q_4(n) &= (n+1)^4(n+3)^4 . \end{aligned}$$

Note that, according to (13) and (14), these sequences of polynomials satisfy the following recurrence property:

$$(18) \quad \begin{aligned} P_{d+1}(n) &= P_d(n+1)Q_d(n), \\ Q_{d+1}(n) &= Q_d(n+1)P_d(n). \end{aligned}$$

Proposition 4.4. *For each $d \geq 1$, P_d and Q_d are both of degree 2^{d-1} , and the degree of the difference $R_d := Q_d - P_d$ is $2^{d-1} - d$.*

The computation of the degree of P_d and Q_d is a straightforward induction using (18). For the degree of R_d , we will use the following lemma.

Lemma 4.5. *Let P and R be two polynomials, of degree p and r , respectively and with $p \neq r$. Let \tilde{R} be the polynomial defined by*

$$\tilde{R}(n) := R(n+1)P(n) - R(n)P(n+1).$$

Then the degree of \tilde{R} is $r + p - 1$.

Proof. Let us write the two terms of highest degree in P and R :

$$P(n) = a_P n^p + b_P n^{p-1} + \dots$$

and

$$R(n) = a_R n^r + b_R n^{r-1} + \dots$$

where $a_P a_R \neq 0$. Then a direct computation shows that the two terms of highest degree in $R(n+1)P(n)$ are

$$R(n+1)P(n) = a_R a_P n^{r+p} + (a_R b_P + a_P b_R + r a_P a_R) n^{r+p-1} + \dots$$

Likewise, the two terms of highest degree in $R(n)P(n+1)$ are

$$R(n)P(n+1) = a_R a_P n^{r+p} + (a_R b_P + a_P b_R + p a_P a_R) n^{r+p-1} + \dots$$

It follows that the term of highest degree in $\tilde{R}(n)$ is $(r-p)a_P a_R n^{r+p-1}$. \square

End of the proof of Proposition 4.4. We show by induction on $d \geq 1$ that the degree of R_d is $2^{d-1} - d$. This is already true for $d = 1$ since $R_1 = -1$ is of degree 0. Then, assume that the result holds for some $d \geq 1$, and let us consider the polynomial R_{d+1} . We have

$$\begin{aligned} R_{d+1}(n) &= Q_{d+1}(n) - P_{d+1}(n) \\ &= Q_d(n+1)P_d(n) - P_d(n+1)Q_d(n) \\ &= (P_d(n+1) + R_d(n+1))P_d(n) - P_d(n+1)(P_d(n) + R_d(n)) \\ &= R_d(n+1)P_d(n) - R_d(n)P_d(n+1). \end{aligned}$$

But we know that the degree of P_d is 2^{d-1} , and by the induction assumption the degree of R_d is $2^{d-1} - d$. These degrees are different, therefore Lemma 4.5 applies, and shows that the degree of R_{d+1} is $2^d - (d+1)$.

\square

4.3 Furstenberg systems of MRT multiplicative functions

Now, we consider a completely multiplicative arithmetic function $\mathbf{u} = (\mathbf{u}(n))_{n \geq 1} \in \text{MRT}$. Let (t_m) and (s_m) be the associated sequences, as in Definition 3.1. Let (N_m) be an increasing sequence of integers with $N_m \leq t_{m+1}$, and let $d \geq 0$ and $\ell \geq 1$ be fixed integers. Our purpose is to apply Proposition 4.3 to \mathbf{u} , therefore we have to evaluate the expressions of the form

$$(19) \quad \frac{1}{N_m} \sum_{1 \leq n \leq N_m} \phi_d^\ell(\mathbf{u}(n), \mathbf{u}(n+1), \dots, \mathbf{u}(n+d)).$$

By Proposition 3.4, we have

$$(20) \quad \begin{aligned} & \frac{1}{N_m} \sum_{1 \leq n \leq N_m} \phi_d^\ell(\mathbf{u}(n), \dots, \mathbf{u}(n+d)) \\ &= \frac{1}{N_m} \sum_{1 \leq n \leq N_m} \phi_d^\ell(n^{is_{m+1}}, \dots, (n+d)^{is_{m+1}}) + o(1) \\ &= \frac{1}{N_m} \sum_{1 \leq n \leq N_m} e^{i\ell s_{m+1} \log \frac{P_d(n)}{Q_d(n)}} + o(1). \end{aligned}$$

Therefore, in view of applying Proposition 4.1, we can replace an expression of the form (19) by

$$\frac{1}{N_m} \sum_{1 \leq n \leq N_m} e^{i\ell s_{m+1} f_d(n)},$$

where f_d is defined by

$$(21) \quad f_d(x) := \log \frac{P_d(x)}{Q_d(x)}.$$

Note that, by Proposition 4.4, for $d \geq 1$, the d terms of highest degrees in P_d and Q_d coincide. In particular, we have

$$\lim_{x \rightarrow \infty} \frac{P_d(x)}{Q_d(x)} = 1,$$

so $f_d(x)$ is well defined for $x \in \mathbb{R}$ large enough. We will use the following results on the asymptotic behaviour of f_d .

Lemma 4.6. *For each $d \geq 1$, there exists $K_d \neq 0$ such that*

$$(22) \quad f_d(x) \sim \frac{K_d}{x^d} \quad \text{as } x \rightarrow \infty.$$

For each $d \geq 0$, there exists $L_d \neq 0$ such that

$$(23) \quad f'_d(x) \sim \frac{L_d}{x^{d+1}} \quad \text{as } x \rightarrow \infty.$$

Moreover, there exists $H_d > 0$ such that f'_d is monotone on $[H_d, +\infty)$.

Proof. For $d = 0$, note that $\overline{f_0(x)} = \log x$, so that $f'_0(x) = \frac{1}{x}$ and the result concerning f'_0 is obvious. We consider now the case $d \geq 1$. From Proposition 4.4, we can write

$$f_d(n) = \log \frac{P_d(n)}{P_d(n) + R_d(n)} = -\log \left(1 + \frac{R_d(n)}{P_d(n)} \right),$$

where $\deg P_d = 2^{d-1}$ and $\deg R_d = 2^{d-1} - d$, which yields (22).

Deriving f_d gives

$$\begin{aligned} f'_d &= \frac{Q_d}{P_d} \cdot \frac{P'_d(P_d + R_d) - P_d(P'_d + R'_d)}{Q_d^2} \\ &= \frac{P'_d R_d - P_d R'_d}{P_d Q_d}. \end{aligned}$$

Since the degrees of P_d and R_d are different, the degree of the numerator is equal to $\deg P_d + \deg R_d - 1 = 2^d - (d + 1)$. But the degree of the denominator is $\deg P_d + \deg Q_d = 2^d$, and this gives (23). Finally, as a nonzero rational fraction, the second derivative f''_d has finitely many zeros, from which we get the last claim of the lemma. \square

We can now state the precise asymptotics which, together with Proposition 4.3, will allow us to identify some Furstenberg systems of \mathbf{u} .

Proposition 4.7. *Let $d \geq 0$ be a fixed integer, and choose a real number β such that*

$$\frac{1}{d+1} < \beta < \frac{1}{d} \text{ if } d \geq 1, \quad 1 < \beta < 2 \text{ if } d = 0.$$

Set, for each $m \geq 1$, $N_m := \lfloor (s_{m+1})^\beta \rfloor$. Then

$$(24) \quad \forall \ell \geq 1, \quad \frac{1}{N_m} \sum_{1 \leq n \leq N_m} e^{i\ell s_{m+1} f_d(n)} \xrightarrow{m \rightarrow \infty} 0,$$

and

$$(25) \quad \frac{1}{N_m} \sum_{1 \leq n \leq N_m} e^{i s_{m+1} f_{d+1}(n)} \xrightarrow{m \rightarrow \infty} 1.$$

An important tool in the proof of the above proposition is the following theorem of Kusmin-Landau, which we state as in [24]. Here, if x is a real number, $\|x\|$ stands for the distance of x to the nearest integer.

Theorem 4.8 (Kusmin-Landau Theorem). *If $f : [a, b] \rightarrow \mathbb{R}$ is C^1 , f' is monotone and $\|f'\| \geq \lambda_1 > 0$ then*

$$\left| \sum_{n \in [a, b]} e^{i2\pi f(n)} \right| \leq \frac{2}{\pi \lambda_1}.$$

Proof of Proposition 4.7. Choose α such that $\frac{1}{d+1} < \alpha < \beta$. We can replace the average in (24) and in (25) by the average over $s_{m+1}^\alpha \leq n \leq s_{m+1}^\beta$. For such an n , using (22), we get

$$|s_{m+1} f_{d+1}(n)| = O \left(\frac{1}{s_{m+1}^{\alpha(d+1)-1}} \right) \xrightarrow{m \rightarrow \infty} 0,$$

and this immediately gives (25).

By Lemma 4.6, for m large enough f'_d is monotone on $[s_{m+1}^\alpha, s_{m+1}^\beta]$, and there exists $K > 0$ (depending only on ℓ and d) such that, for n in this interval,

$$\|\ell s_{m+1} f'_d(n)\| \geq K \frac{s_{m+1}}{n^{d+1}} \geq K \frac{1}{s_{m+1}^{\beta(d+1)-1}}.$$

By Kusmin-Landau Theorem, we get

$$\left| \sum_{s_{m+1}^\alpha \leq n \leq s_{m+1}^\beta} e^{i\ell s_{m+1} f_d(n)} \right| = O\left(s_{m+1}^{\beta(d+1)-1}\right),$$

and finally

$$\left| \frac{1}{N_m} \sum_{1 \leq n \leq N_m} e^{i\ell s_{m+1} f_d(n)} \right| = O\left(s_{m+1}^{\beta d-1}\right) \xrightarrow{m \rightarrow \infty} 0.$$

□

Putting together (20), Proposition 4.7 and Proposition 4.3, we get the following result.

Theorem 4.9. *For any $d \geq 0$, the measure-preserving system $((\mathbb{S}^1)^\mathbb{N}, \nu_d, S)$ appears as a Furstenberg system of \mathbf{u} .*

Using Proposition 2.2 and the fact that, under ν_d , $d+1$ consecutive coordinates of the process (Z_n) are independent and uniformly distributed on \mathbb{S}^1 , we obtain the following.

Corollary 4.10. The system $((\mathbb{S}^1)^\mathbb{N}, (\text{Leb}_{\mathbb{S}^1})^{\otimes \mathbb{N}}, S)$ is a Furstenberg system of \mathbf{u} .

This concludes the proof of the Main Theorem.

5 Archimedean characters and slowly varying arithmetic functions

We recall that an *Archimedean character* is a completely multiplicative function of the form $n \mapsto n^{it}$ with some fixed $t \in \mathbb{R}$. As

$$|(n+1)^{it} - n^{it}| = \left| e^{it \log(1+1/n)} - 1 \right| \xrightarrow{n \rightarrow \infty} 0,$$

Archimedean characters fall into the category of *slowly varying* arithmetic functions, that are bounded arithmetic functions u satisfying

$$u(n+1) - u(n) \xrightarrow{n \rightarrow \infty} 0.$$

A useful weakening of this property is the following: we say that the bounded arithmetic function u is *mean slowly varying* if

$$(26) \quad \frac{1}{N} \sum_{n \leq N} |u(n+1) - u(n)| \xrightarrow{N \rightarrow \infty} 0.$$

Note that this is equivalent to the fact that $u(n+1) - u(n) \rightarrow 0$ on a subset of n of density 1.

It is worth mentioning a result by Klurman [20, Theorem 1.8], who proved a conjecture by Kátai concerning mean slowly varying multiplicative functions (non necessarily bounded). From his theorem, it easily follows that the only mean slowly varying multiplicative functions taking values in the unit circle are the Archimedean characters.

5.1 Furstenberg systems of (mean) slowly varying functions

Proposition 5.1. *The arithmetic function $u : \mathbb{N} \rightarrow \mathbb{D}$ is mean slowly varying if and only if all Furstenberg systems of u are measure-theoretically isomorphic to the action of the identity on some probability space.*

Proof. Assume first that u is mean slowly varying, and that u is quasi-generic for some S -invariant measure ν on $\mathbb{D}^{\mathbb{N}}$ along a subsequence (N_m) . Then, by (26), we have

$$\mathbb{E}_\nu[|Z_1 - Z_0|] = \lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{n \leq N_m} |u(n+1) - u(n)| = 0.$$

It follows that $Z_1 = Z_0$ ν -a.e., and more generally by S -invariance, for each $k \in \mathbb{N}$, we also have $Z_{k+1} = Z_k$ ν -a.e. Hence, ν is concentrated on the subset of sequences with identical coordinates, and $S = \text{Id}$ ν -a.e.

Conversely, assume that (26) fails. Then there exists a subsequence (N_m) such that

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{n \leq N_m} |u(n+1) - u(n)| > 0,$$

and by compactness of $M(\mathbb{D}^{\mathbb{N}})$ we can assume that u is quasi-generic for some S -invariant measure ν along (N_m) . But then we have

$$\mathbb{E}_\nu[|Z_1 - Z_0|] = \mathbb{E}_\nu[|Z_0 \circ S - Z_0|] > 0,$$

and the Furstenberg system of u defined by ν is not measure-theoretically isomorphic to the action of the identity. \square

Proposition 5.2. *Let $u : \mathbb{N} \rightarrow \mathbb{D}$ be a slowly varying arithmetic function, and let $X_u \subset \mathbb{D}^{\mathbb{N}}$ be the subshift generated by u . Then the topological entropy of (X_u, S) is zero.*

Proof. We start by observing that, as u is slowly varying, for each $\varepsilon > 0$ there exists L_ε such that whenever $\ell \geq L_\varepsilon$,

$$(27) \quad \forall n \in \mathbb{N}, \quad \frac{1}{\ell} \sum_{n \leq j < n+\ell} |u(j+1) - u(j)| < \varepsilon.$$

Let w be a sequence in X_u . As each subblock of w is a limit of subblocks of u , it follows that (27) is still valid if we replace u by w . But this in turn implies that any $w \in X_u$ is mean slowly varying.

Now, let ν be an ergodic shift-invariant measure on X_u , and let $w \in X_u$ be ν -generic (ν -almost all sequences in X_u are ν -generic). Then w is mean slowly varying, and by Proposition 5.1, the measure-theoretic system (X_u, ν, S) is isomorphic to (X_u, ν, Id) . In particular, its Kolmogorov entropy is zero.

By the variational principle (see e.g. [30, Section 8.2], the topological entropy of (X_u, S) must be also zero. \square

Remark 5.3. As a matter of fact, the above proof shows that whenever u is a slowly varying function, any $w \in X_u$ is also a slowly varying function. More than that, we can make the following observation: by the definition of X_u , each $w \in X_u$ can be written as

$$w = \lim_{k \rightarrow \infty} S^{n_k} u,$$

where (n_k) is a non-decreasing sequence of non-negative integers. Either (n_k) is bounded, and then $w = S^n u$ for some $n \geq 0$, or $n_k \rightarrow \infty$, and then as u is slowly varying, w must be of the form (w_1, w_1, w_1, \dots) for some $w_1 \in \mathbb{D}$.

In the following remark we provide some more observations on mean slowly varying functions.

Remark 5.4. (a) Note that the subshift generated by a *mean* slowly varying function $u : \mathbb{N} \rightarrow \mathbb{D}$ can be of infinite entropy: we can modify a slowly varying function on a subset of density zero to get a mean slowly varying function generating the full subshift $(\mathbb{D}^{\mathbb{N}}, S)$.

(b) If u is mean slowly varying and if, for each $\ell \geq 1$, the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq n \leq N} u^\ell(n)$$

exists then there is only one Furstenberg system of u : there exists a shift-invariant measure ν such that $E_N(u) \xrightarrow{N \rightarrow \infty} \nu$. Indeed, note that for each $k \geq 1$, $\ell_r \geq 1, j_r \geq 0$ for $r = 1, \dots, k$, we have

$$\left| \frac{1}{N} \sum_{1 \leq n \leq N} u^{\ell_1}(n + j_1) \cdot \dots \cdot u^{\ell_k}(n + j_k) - \frac{1}{N} \sum_{1 \leq n \leq N} u^{\ell_1 + \dots + \ell_k}(n) \right| \xrightarrow{N \rightarrow \infty} 0.$$

(c) We can repeat word for word all arguments for the logarithmic averages. In particular, if u is mean slowly varying and, for each $\ell \geq 1$, the limit

$$(28) \quad \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{1 \leq n \leq N} \frac{1}{n} u^\ell(n)$$

exists, then there is only one logarithmic Furstenberg system (see Section 6.1): there exists a shift-invariant measure ν such that

$$\frac{1}{\log N} \sum_{1 \leq n \leq N} \frac{1}{n} \delta_{S^n u} \xrightarrow{N \rightarrow \infty} \nu.$$

(d) For $u(n) = n^{it}$, we have the logarithmic assumption (28) satisfied (in fact, each such limit is zero whenever $\ell \neq 0$), so there is only one logarithmic Furstenberg system, as already noticed in [15] but we have uncountably many Furstenberg systems (cf. Corollary 5.5).

As it follows from Proposition 2.1, a slowly varying function $u : \mathbb{N} \rightarrow \mathbb{D}$ either has one Furstenberg system, or it has uncountably many different Furstenberg systems. By different, we mean that we obtain different measures. The measure-theoretic dynamical systems given by these measures can however be all isomorphic which is the case in the example below.

We consider now a mean slowly varying function u such that $|u(n)| = 1$ for all $n \geq 1$. Assume, moreover, that

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{0 \leq n < N_m} \delta_{S^{n_u}} = \lim_{m \rightarrow \infty} E_{N_m}(u) = \nu$$

for an increasing sequence (N_m) . It follows from Proposition 5.1 that ν is supported on the set $\{(z, z, \dots) : z \in \mathbb{S}^1\}$. Moreover, $Z_0^k(z, z, \dots) = z^k$. Hence, we can identify ν with a measure κ defined on \mathbb{S}^1 such that

$$\lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{1 \leq n \leq N_m} u^k(n) = \int Z_0^k d\nu = \int_{\mathbb{S}^1} z^k d\kappa.$$

It follows that, varying $k \in \mathbb{Z}$, the LHS of the equation above provides the Fourier transform of the measure κ which we are seeking. In the particular case where $u(n) = n^i$, we have

$$\frac{1}{N_m} \sum_{1 \leq n \leq N_m} n^{ik} = \frac{N_m^{ik}}{1 + ik} + o(1),$$

thus the Fourier coefficients of κ are given by the limits of $\frac{N_m^{ik}}{1 + ik}$ as $m \rightarrow \infty$. The case $k = 1$ yields that $N_m^i \rightarrow c$, where $|c| = 1$, and we obtain that

$$\int_{\mathbb{S}^1} z^k d\kappa = \frac{c^k}{1 + ik}, \quad k \in \mathbb{Z}.$$

Consider, $\kappa' := \kappa * \delta_{\bar{c}}$. Then

$$\int_{\mathbb{S}^1} z^k d\kappa' = \frac{1}{1 + ik}, \quad k \in \mathbb{Z}.$$

Since $(\frac{1}{1+ik})$ is an ℓ^2 -sequence, the measure κ' is absolutely continuous with respect to the (normalized) Lebesgue measure $\text{Leb}_{\mathbb{S}^1}$, with density g equal to $g(z) = \sum_{k=-\infty}^{\infty} \frac{1}{1+ik} z^k$. In fact, noting that for all $k \in \mathbb{Z}$,

$$\int_0^1 e^{2\pi x} e^{i2\pi kx} dx = \frac{e^{2\pi} - 1}{2\pi(1 + ik)},$$

we see that, for $x \in [0, 1)$,

$$g(e^{i2\pi x}) = \frac{2\pi e^{2\pi x}}{e^{2\pi} - 1}.$$

Finally, note that the sequence (n^i) is dense in \mathbb{S}^1 as the sequence $(\log n)$ is dense in $[0, 2\pi) \bmod 2\pi$.

Corollary 5.5. The family of Furstenberg systems of $u(n) = n^i$ consists of uncountably many different systems given by all rotations of $g(z)dz$. All of them are isomorphic to the identity on the circle with Lebesgue measure (and ergodic components are Dirac measures on the circle). Moreover, $X_u = \{(z, z, \dots) : z \in \mathbb{S}^1\} \cup \{S^n u : n \in \mathbb{N}\}$.

5.2 MRT arithmetic functions do not satisfy Sarnak's conjecture

As MRT arithmetic functions mostly behave like Archimedean characters on very large intervals, we will use some ideas presented in the preceding section to prove that MRT arithmetic functions do not satisfy Sarnak's conjecture: for any MRT function \mathbf{u} there exists a zero entropy topological system which outputs a sequence having some correlation with \mathbf{u} .

Let \mathbf{u} be an MRT arithmetic function, and let (t_m) and (s_m) be as in Definition 3.1. We set, for each $m \in \mathbb{N}$, $r_m := \lfloor s_m^{3/2} \rfloor$. Then we define a new arithmetic function $v = (v(n))_{n \in \mathbb{N}} \in (\mathbb{S}^1)^{\mathbb{N}}$ by setting for each m :

$$v(n) := \begin{cases} 1 & \text{for } t_m < n \leq r_{m+1}, \\ n^{is_{m+1}} & \text{for } r_{m+1} < n \leq t_{m+1}. \end{cases}$$

Then, for each m , we have

$$|v(n+1) - v(n)| = \begin{cases} 0 & \text{if } t_m < n < r_{m+1}, \\ O\left(s_{m+1}^{-1/2}\right) & \text{if } r_{m+1} < n < t_{m+1}. \end{cases}$$

Even though v may not be slowly varying because of the possible jumps in t_m and in r_{m+1} , the property described in Equation (27) is still valid, as these jumps are bounded by 2 and are separated by gaps whose lengths tend to ∞ . Therefore, the proof of Proposition 5.2 also applies to v , and we have $h_{\text{top}}(X_v, S) = 0$.

But, since $\frac{r_{m+1}}{t_{m+1}} \xrightarrow{m \rightarrow \infty} 0$, in view of Proposition 3.4, we have

$$\frac{1}{t_{m+1}} \sum_{1 \leq n \leq t_{m+1}} \mathbf{u}(n) \overline{v(n)} \xrightarrow{m \rightarrow \infty} 1.$$

We thus have found a topological dynamical system (X_v, S) of zero topological entropy, a point $v \in X_v$ and a continuous map $f : X_v \rightarrow \mathbb{C}$ (the conjugate of the zero-coordinate map), such that the sequence $(f(S^n v))$ is not orthogonal to \mathbf{u} , in the sense that

$$\frac{1}{N} \sum_{1 \leq n \leq N} \mathbf{u}(n) f(S^n v) \not\rightarrow 0 \text{ as } N \rightarrow \infty.$$

6 Further properties of MRT functions

6.1 MRT functions satisfy logarithmic Chowla conjecture along a subsequence

The purpose of this section is to study what the Main Theorem becomes if we consider logarithmic averages instead of usual averages. A *logarithmic Furstenberg system* of an arithmetic function $u : \mathbb{N} \rightarrow \mathbb{D}$ is defined as in Section 2 as a measure-theoretic dynamical system (X_u, ν, S) , where ν is now a weak* limit of a subsequence of the *logarithmic empirical measures*

$$E_N^{\log}(u) := \frac{1}{L_N} \sum_{1 \leq n \leq L_N} \frac{1}{n} \delta_{S^{n-1}u},$$

with $L_N := 1 + \frac{1}{2} + \dots + \frac{1}{N}$ ($N \geq 1$).

We recall the classical relation between the logarithmic and usual averages, obtained by summation by parts:

$$(29) \quad E_N^{\log}(u) = \frac{1}{L_N} \sum_{1 \leq n \leq N-1} \frac{1}{n+1} E_n(u) + \frac{1}{L_N} E_N(u).$$

We recall that the weak*-topology turns $M((\mathbb{S}^1)^{\mathbb{N}})$ into a compact metrizable space, and that a possible metric is given by

$$\Delta(\nu, \kappa) = \sum_{j \geq 1} \frac{1}{2^j} \left| \int f_j d\nu - \int f_j d\kappa \right|,$$

where $(f_j)_{j \geq 1}$ is a countable family generating a dense subspace of $C((\mathbb{S}^1)^{\mathbb{N}})$. We can for example take for $(f_j)_{j \geq 1}$ the family of $\{Z_{i_1}^{r_1} \dots Z_{i_t}^{r_t} : t \geq 1, i_k \geq 0, r_k \in \mathbb{Z}\}$ taken in any order. Then, the metric is globally bounded by 2. Note also that for convex combinations, we have

$$(30) \quad \Delta \left(\int \nu_\gamma dP(\gamma), \int \kappa_\gamma dP(\gamma) \right) \leq \int \Delta(\nu_\gamma, \kappa_\gamma) dP(\gamma).$$

We need the following simple observation.

Lemma 6.1. *Assume that $\mu_k \rightarrow \mu$ and let, for $d \geq 1$, $\kappa_d := \sum_{k=d}^{d+a_d} \alpha_k^{(d)} \mu_k + \beta^{(d)} \rho_d$, where $\alpha_k^{(d)}, \beta^{(d)} \geq 0$, $\sum_{k=d}^{d+a_d} \alpha_k^{(d)} = 1 - \beta^{(d)}$. If $\beta^{(d)} \rightarrow 0$ then $\kappa_d \rightarrow \mu$.*

Proof. This follows immediately from (30). \square

Now, we consider an MRT arithmetic function \mathbf{u} , and we take (t_m) and (s_m) as in Definition 3.1. The proof of Main Theorem (see Section 4.3) yields the following:

Lemma 6.2. *Fix $d \geq 0$ and choose*

$$\begin{cases} \frac{1}{d+1} < \beta_d < \beta'_d < \frac{1}{d} & \text{if } d \geq 1, \\ 1 < \beta_0 < \beta'_0 < 2 & \text{if } d = 0. \end{cases}$$

Let $\varepsilon > 0$. Then, for each m large enough,

$$\Delta(E_N(\mathbf{u}), \nu_d) < \varepsilon$$

uniformly in $s_{m+1}^{\beta_d} \leq N \leq s_{m+1}^{\beta'_d}$.

Proof. Assume the result does not hold. Then for infinitely many integers m we can find $s_{m+1}^{\beta_d} \leq N_m \leq s_{m+1}^{\beta'_d}$ such that

$$\Delta(E_{N_m}(\mathbf{u}), \nu_d) \geq \varepsilon.$$

On the other hand, the proof of Proposition 4.7 shows that, along such a sequence (N_m) we must have

$$E_{N_m}(\mathbf{u}) \rightarrow \nu_d,$$

which yields a contradiction. \square

Now, let us fix integers $1 \leq D_1 < D_2$. We consider the convergence of logarithmic empirical measures of \mathbf{u} along the increasing sequence of integers $(N_m) := \left(\left\lfloor s_{m+1}^{1/D_1} \right\rfloor \right)$. Given a small real number $\varepsilon > 0$, set for each $D_1 \leq d \leq D_2$

$$\beta_d := \frac{1}{d+1} + \frac{\varepsilon}{2d(d+1)} \quad \text{and} \quad \beta'_d := \frac{1}{d} - \frac{\varepsilon}{2d(d+1)}.$$

Using (29), we write $E_{N_m}^{\log}(\mathbf{u})$ as a convex combination of the empirical measures $E_n(\mathbf{u})$, $1 \leq n \leq N$. We partition $\{1, \dots, N_m\}$ as $\bigsqcup_{D_1 \leq d \leq D_2} I_d^m \sqcup J^m$, where

$$I_d^m := \left\{ n \in \mathbb{N} : s_{m+1}^{\beta_d} \leq n \leq s_{m+1}^{\beta'_d} \right\},$$

and

$$J^m := \{1, \dots, N_m\} \setminus \bigsqcup_{D_1 \leq d \leq D_2} I_d^m.$$

By Lemma 6.2, for m large, $E_n(\mathbf{u})$ is close to ν_d for $n \in I_d^m$. The weight of I_d^m in the convex combination is

$$\frac{1}{L_{N_m}} \sum_{n \in I_d^m} \frac{1}{n+1} \underset{m \rightarrow \infty}{\sim} D_1(\beta'_d - \beta_d) = D_1 \left(\frac{1}{d} - \frac{1}{d+1} \right) (1 - \varepsilon).$$

It follows that the total weight of $\bigsqcup_{D_1 \leq d \leq D_2} I_d^m$ is asymptotic, as $m \rightarrow \infty$, to

$$D_1 \left(\frac{1}{D_1} - \frac{1}{D_2} \right) (1 - \varepsilon) > 1 - \varepsilon - \frac{D_1}{D_2},$$

and then for m large enough the weight of J^m is bounded by $\varepsilon + \frac{D_1}{D_2}$. In view of Lemma 6.2 and (30), any weak* limit of $E_{N_m}^{\log}(\mathbf{u})$ can be written as

$$D_1(1 - \varepsilon) \sum_{D_1 \leq d \leq D_2} \left(\frac{1}{d} - \frac{1}{d+1} \right) \nu_d + \alpha \rho,$$

where ρ is some shift-invariant probability measure on $X_{\mathbf{u}}$ and $0 \leq \alpha \leq \varepsilon + \frac{D_1}{D_2}$. Letting $\varepsilon \rightarrow 0$ and $D_2 \rightarrow \infty$, we see (cf. Lemma 6.1) that for any $D_1 \geq 1$ there exists a logarithmic Furstenberg system of \mathbf{u} whose invariant measure is

$$D_1 \sum_{d \geq D_1} \left(\frac{1}{d} - \frac{1}{d+1} \right) \nu_d.$$

Note that, under this measure, the distribution of (Z_0, \dots, Z_{D_1}) is $(\text{Leb}_{\mathbb{S}^1})^{\otimes(D_1+1)}$. By Proposition 2.2 (which is also valid in the logarithmic case), we can also find $(\text{Leb}_{\mathbb{S}^1})^{\otimes \mathbb{N}}$ as an invariant measure of a logarithmic Furstenberg system of \mathbf{u} .

Thus, we have proved the following result:

Corollary 6.3. Each $\mathbf{u} \in \text{MRT}$ satisfies the logarithmic Chowla conjecture along a subsequence.

Getting ν_0

Under the assumptions given in Definition 3.1, we can easily modify the above argument to incorporate ν_0 in the weak* limit of $E_{N_m}^{\log}(\mathbf{u})$: we just have to take $D_1 = 0$ and choose $\beta_0 := 1 + \varepsilon$, $\beta'_0 := 2 - \varepsilon$. The weak* limit we get has then the form

$$\frac{1}{2} \left(\nu_0 + \sum_{d \geq 1} \left(\frac{1}{d} - \frac{1}{d+1} \right) \nu_d \right).$$

If in the construction of the MRT sequence we add the extra assumption that

$$s_{m+1}^{a_m} \leq t_{m+1}, \quad \text{with } a_m \rightarrow \infty,$$

which is compatible with the rest, then we can now take β'_0 as large as we want, so that the weight of ν_0 in the weak* limit is as close to 1 as we want. Finally, we get a logarithmic Furstenberg system of \mathbf{u} with ν_0 as an invariant measure.

It is not clear however if we can get ν_d ($d \geq 1$) for some logarithmic Furstenberg system of an MRT function.

6.2 Absence of zero mean on typical short interval

Motivated by Matomäki-Radziwiłł's theorem [22] concerning strongly aperiodic multiplicative functions, we say that $\mathbf{u} : \mathbb{N} \rightarrow \mathbb{D}$ has *zero mean on typical short interval* if

$$\lim_{M, H \rightarrow \infty, H=o(M)} \frac{1}{M} \sum_{1 \leq m \leq M} \left| \frac{1}{H} \sum_{0 \leq h < H} u(m+h) \right| = 0.$$

Proposition 6.4. *Assume that $\liminf_{\ell \rightarrow \infty} \frac{1}{\ell} \sum_{1 \leq j \leq \ell} |u(j)| =: \alpha > 0$. If u has identity as a Furstenberg system then u has no zero mean on short intervals.*

Proof. Suppose that u has zero mean on short intervals. Assume that

$$\lim_{k \rightarrow \infty} E_{M_k}(u) = \kappa.$$

Now, the system (S, X_u, κ) is the identity if and only if $Z_0 = Z_0 \circ S$ κ -a.e.

Let $\varepsilon > 0$ and choose H_0 so that for $H > H_0$, we have

$$\limsup_{k \rightarrow \infty} \frac{1}{M_k} \sum_{1 \leq m \leq M_k} \left| \frac{1}{H} \sum_{0 \leq h < H} u(m+h) \right| < \varepsilon.$$

Take $H > H_0$. Then

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \frac{1}{M_k} \sum_{1 \leq m \leq M_k} \left| \frac{1}{H} \sum_{0 \leq h < H} u(m+h) \right| = \\ & \limsup_{k \rightarrow \infty} \frac{1}{M_k} \sum_{1 \leq m \leq M_k} \left| \frac{1}{H} \sum_{0 \leq h < H} Z_0(S^{h+m}u) \right| = \\ & \limsup_{k \rightarrow \infty} \int_{X_u} \left| \frac{1}{H} \sum_{0 \leq h < H} Z_0 \circ S^h \right| dE_{M_k}(u) = \end{aligned}$$

$$\int_{X_u} \left| \frac{1}{H} \sum_{0 \leq h < H} Z_0 \circ S^h \right| d\kappa = \int_{X_u} |Z_0| d\kappa \geq \alpha > 0,$$

a contradiction. \square

Corollary 6.5. If $u \in \text{MRT}$ then u has no zero mean on short intervals.

6.3 Strong stationarity

We now show that, for each $d \geq 0$, the stationary process (Z_n) with distribution ν_d is *strongly stationary* (see [19]), i.e. for each $s_1 < \dots < s_k$ and each $r \geq 1$, the distributions of the vectors $(Z_{s_1}, \dots, Z_{s_k})$ and $(Z_{rs_1}, \dots, Z_{rs_k})$ are the same.

Fix $d \geq 0$ and we recall that the process (Z_n) with the distribution ν_d is the same as of the process $(f \circ T_d^n)$, see (34) in Appendix for the definition of T_d and $f(x_1, \dots, x_d) = e^{i2\pi x_d}$. Then

$$f \circ T_d^s(x_1, \dots, x_d) = e^{2\pi i \left(\binom{s}{d-1} x_1 + \binom{s}{d-2} x_2 + \dots + \binom{s}{0} x_d \right)}.$$

It follows that for each $s_1 < \dots < s_k$ and each choice of integers j_1, \dots, j_k , we have

$$\mathbb{E} Z_{s_1}^{j_1} \dots Z_{s_k}^{j_k} \neq 0 \Leftrightarrow \sum_{i=1}^k j_i \binom{s_i}{\ell} = 0 \text{ for } \ell = 0, \dots, d-1 \Leftrightarrow \sum_{i=1}^k s_i^\ell = 0 \text{ for } \ell = 0, \dots, d-1.$$

Therefore,

$$\mathbb{E} Z_{s_1}^{j_1} \dots Z_{s_k}^{j_k} = 0 \Leftrightarrow \mathbb{E} Z_{rs_1}^{j_1} \dots Z_{rs_k}^{j_k} = 0 \text{ for each } r \geq 1.$$

Moreover, $\mathbb{E} Z_{s_1}^{j_1} \dots Z_{s_k}^{j_k} \neq 0$ is equivalent to $\mathbb{E} Z_{s_1}^{j_1} \dots Z_{s_k}^{j_k} = 1$. It follows that the process (Z_n) with the distribution ν_d is strongly stationary.

It follows that all Furstenberg systems which have been found in the paper are given by stationary processes which are strongly stationary. Note also that since every convex combination of strongly stationary processes remains strongly stationary, also the logarithmic Furstenberg systems determined in Section 6.1 are strongly stationary which fits perfectly to a general result of Frantzikinakis and Host [14], [15] about logarithmic Furstenberg systems of strongly aperiodic multiplicative functions.

We end by asking the following question, which has been suggested to us by both Nikos Frantzikinakis and Florian Richter: can we find a Furstenberg system of some $u \in \text{MRT}$ which is the direct product of a Bernoulli shift and a unipotent system?

7 Appendix

We aim at showing that from the dynamical point of view there is a close relation between the processes $(Y^{(d)})_{d \geq 0}$ which appeared in Section 4 and the concept of quasi-eigenfunction in ergodic theory.

7.1 Algebraic constraints and quasi-eigenfunctions

Given a standard Borel probability space (X, \mathcal{B}, μ) , let $\mathcal{M}(X)$ denote the set of all measurable functions of modulus 1 defined on X . Endowed with the pointwise multiplication and the L^2 -topology, it becomes a Polish group. Given an automorphism T on (X, \mathcal{B}, μ) , set

$$E_0(T) := \{g \in \mathcal{M}(X) : g \circ T = g\}$$

and then inductively, define

$$E_{d+1}(T) := \{f \in \mathcal{M}(X) : f \circ T/f \in E_d(T)\}, \quad d \geq 0.$$

Note that each $E_d(T)$ is a group which is called the group of d -quasi-eigenfunctions. For $w \in \mathcal{M}(X)$ and $m \geq 1$, we set

$$w^{(m)} = w \circ T^{m-1} \cdot \dots \cdot w$$

with $w^{(0)} = 1$. Assume that $g \in E_d(T)$ and let $g \circ T = hg$ with $h = g \circ T/g \in E_{d-1}(T)$. Then, for each ℓ_1, \dots, ℓ_k , we have

$$\begin{aligned} \prod_{i=1}^k g \circ T^{\ell_i} &= \prod_{i=1}^k h^{(\ell_i)} g = g^k \prod_{i=1}^k h^{(\ell_i)} = \\ g^k \prod_{i=1}^k \left(\prod_{j=0}^{\ell_i-1} h \circ T^j \right) &= g^k \prod_{i=1}^k \prod_{j=0}^{\ell_i-1} \left(\frac{h \circ T}{h} \right)^{(j)} h = \\ g^k \prod_{i=1}^k h^{\ell_i} \prod_{j=0}^{\ell_i-1} \left(\frac{h \circ T}{h} \right)^{(j)} &= \\ g^k h^{\sum_{i=1}^k \ell_i} \prod_{i=1}^k \prod_{j=0}^{\ell_i-1} \left(\frac{h \circ T}{h} \right) \circ T^{j-1} \dots \left(\frac{h \circ T}{h} \right) &= \\ g^k h^{\sum_{i=1}^k \ell_i} \prod_{i=1}^k \prod_{j=0}^{\ell_i-1} \prod_{p=0}^{j-1} \left(\frac{(h \circ T/h) \circ T}{h \circ T/h} \right)^{(p)} \frac{h \circ T}{h} &= \\ g^k h^{\sum_{i=1}^k \ell_i} \prod_{i=1}^k \prod_{j=0}^{\ell_i-1} \left(\frac{h \circ T}{h} \right)^j \prod_{p=0}^{j-1} \left(\frac{(h \circ T/h) \circ T}{h \circ T/h} \right)^{(p)} &= \\ g^k h^{\sum_{i=1}^k \ell_i} \left(\frac{h \circ T}{h} \right)^{\sum_i \frac{\ell_i(\ell_i-1)}{2}} \prod_{i=1}^k \prod_{j=0}^{\ell_i-1} \prod_{p=0}^{j-1} \left(\frac{(h \circ T/h) \circ T}{h \circ T/h} \right)^{(p)} &= \dots \end{aligned}$$

It follows that for the stationary process $(g \circ T^n)_{n \in \mathbb{N}}$ the following holds: whenever ℓ_i, ℓ'_i satisfy $\sum_{i=1}^k \ell_i^j = \sum_{i=1}^k \ell'_i{}^j$ for $j = 0, 1, \dots, d$, we have

$$(31) \quad \prod_{i=1}^k g \circ T^{\ell_i} = \prod_{i=1}^k g \circ T^{\ell'_i}$$

provided that $g \in E_d(T)$.

In fact, the processes given by quasi-eigenfunctions are the only satisfying the algebraic relation (31).

Proposition 7.1. *Assume that (Z_n) is a stationary \mathbb{S}^1 -valued process. Then for each $d \geq 0$, $Z_0 \in E_d(T)$ if and only if*

$$(32) \quad Z_{\ell_1} \cdots Z_{\ell_k} = Z_{\ell'_1} \cdots Z_{\ell'_k}$$

for each ℓ_i, ℓ'_i for which $\sum_{i=1}^k \ell_i^j = \sum_{i=1}^k \ell'_i{}^j$ for $j = 0, 1, \dots, d$ assuming that

$$(33) \quad \{\ell_1, \dots, \ell_k\} \cap \{\ell'_1, \dots, \ell'_k\} = \emptyset.$$

Proof. We can assume that $Z_n = Z_0 \circ T^n$ for an automorphism T of (X, \mathcal{B}, μ) . Now, we proceed by induction. Since the first condition ($d = 0$) is just $Z_0 \circ T^{\ell_1} \cdots \cdots \circ Z_0 \circ T^{\ell_k} = Z_0 \circ T^{\ell'_1} \cdots \cdots \circ Z_0 \circ T^{\ell'_k}$ whichever numbers we take, we have (by taking $k = 1$, $\ell_1 = 1$ and $\ell'_1 = 0$) $Z_0 \circ T = Z_0$, so $Z_0 \in E_0(T)$.

Notice that condition (33) is superfluous since if some of the numbers ℓ_i are equal to ℓ'_j , then we can just cancel out the relevant factors. So, assume now that Z_0 satisfies (32) and let $f = Z_0 \circ T/Z_0$. We need to prove that f satisfies the relation (32) up to d (if so, then by the induction assumption $f \in E_d(T)$, whence $Z_0 \in E_{d+1}(T)$). That is, we want to prove that

$$f \circ T^{\ell_1} \cdots \cdots f \circ T^{\ell_k} = f \circ T^{\ell'_1} \cdots \cdots f \circ T^{\ell'_k}$$

provided that $\sum_{i=1}^k \ell_i^j = \sum_{i=1}^k \ell'_i{}^j$ for $j = 0, 1, \dots, d$. Equivalently, we want to show that

$$\begin{aligned} & Z_0 \circ T^{\ell_1+1} \cdots \cdots Z_0 \circ T^{\ell_k+1} \cdot Z_0 \circ T^{\ell'_1} \cdots \cdots Z_0 \circ T^{\ell'_k} = \\ & Z_0 \circ T^{\ell'_1+1} \cdots \cdots Z_0 \circ T^{\ell'_k+1} \cdot Z_0 \circ T^{\ell_1} \cdots \cdots Z_0 \circ T^{\ell_k}. \end{aligned}$$

But we clearly have

$$\begin{aligned} & (\ell_1 + 1)^{d+1} + \dots + (\ell_k + 1)^{d+1} + \ell_1^{d+1} + \dots + \ell_k^{d+1} = \\ & (\ell'_1 + 1)^{d+1} + \dots + (\ell'_k + 1)^{d+1} + \ell_1^{d+1} + \dots + \ell_k^{d+1} \end{aligned}$$

(let alone if we replace $d + 1$ by a smaller j), hence the result. \square

Remark 7.2. Assume additionally that T is totally ergodic (*i.e.* all non-zero powers are ergodic). Then, following [2], T has *quasi-discrete spectrum* if $\overline{\text{span}}(\bigcup_{d \geq 0} E_d(T)) = L^2(X, \mathcal{B}, \mu)$ (sometimes, T is called an Abramov automorphism). Quasi-discrete spectrum automorphisms are basically affine automorphisms of compact, Abelian, metric groups, see [18] for more details.

Remark 7.3. Knowing that $Z_0 \in E_d(T)$ does not determine the whole process $(Z_m)_{m \in \mathbb{N}}$. Indeed, for example, if we take the ergodic decomposition of T , then the process (Z_m) (considered with respect to an ergodic component) will still satisfy the same algebraic relation even though the distribution of the process may have changed.

7.2 Algebraic constraints and n -independence

We will be interested in the (very) non-ergodic case. Given $d \geq 1$, consider the (unipotent) automorphism $T_d : \mathbb{T}^d \rightarrow \mathbb{T}^d$, defined by:

$$(34) \quad T_d(x_1, \dots, x_d) := (x_1, x_1 + x_2, \dots, x_{d-1} + x_d)$$

(hence T_1 is just the identity on \mathbb{T}). Set $f(x_1, \dots, x_d) = e^{i2\pi x_d}$. Then, $(f \circ T/f)(x_1, \dots, x_d) = e^{i2\pi x_{d-1}}$, so by induction, it is easy to show that $f \in E_d(T_d)$.

Denote by Erg^\perp the class of automorphisms disjoint from all ergodic automorphisms.

Proposition 7.4. *For each $d \geq 1$, $T_d \in \text{Erg}^\perp$. Moreover, the stationary process $(f \circ T_d^m)_{m \in \mathbb{N}}$ is d -independent, i.e. for all $m_1 < \dots < m_d$ the variables $f \circ T_d^{m_1}, \dots, f \circ T_d^{m_d}$ are independent.*

Proof. To prove that $T_d \in \text{Erg}^\perp$, we first notice that its ergodic decomposition is given by the tori (on which we consider the relevant Lebesgue measure) $\mathbb{T}_x^{d-1} = \mathbb{T}^{d-1}$ with $x \in \mathbb{T}$ irrational (and we consider Lebesgue measure³ on \mathbb{T}) on which the action $T_{d,x}$ is given by

$$(x_2, x_3, \dots, x_d) \mapsto (x_2 + x, x_2 + x_3, \dots, x_{d-1} + x_d).$$

If an ergodic automorphism S acting on (Y, \mathcal{C}, ν) is non-disjoint with $T_{d,x}$ then since $T_{d,x}$ is an ergodic compact group extension of an irrational rotation, the associated Koopman operator U_S must share a common eigenvalue with the Koopman operator $U_{T_{d,x}}$ given by $T_{d,x}$ (cf. [17], Chapter 6). As U_S can have only countably many eigenvalues and the measure on the space of ergodic components is continuous, we can assume that S is disjoint with all ergodic components. Take any joining ρ of T_d and S . Let

$$\rho = \int \rho_\gamma dP(\gamma)$$

be its ergodic decomposition. Then, for P -a.e. γ , the projection $\rho_\gamma|_Y$ of ρ_γ on Y equals ν since ν is ergodic. Moreover,

$$(35) \quad \text{Leb}_{\mathbb{T}^{\otimes d}} = \rho|_{\mathbb{T}^d} = \int \rho_\gamma|_{\mathbb{T}^d} dP(\gamma)$$

is a decomposition of $\text{Leb}_{\mathbb{T}^{\otimes d}}$ into ergodic measures. By the uniqueness of ergodic decomposition, it is (35) which is the ergodic decomposition, and (by disjointness) we obtain that

$$\rho = \int_{\mathbb{T}} (\delta_x \otimes (\text{Leb}_{\mathbb{T}})^{\otimes (d-1)}) \otimes \nu d\text{Leb}_{\mathbb{T}}(x)$$

and the first claim easily follows.

It is not hard to check that by elementary properties of Pascal triangle:

$$(36) \quad T_d^r(x_1, \dots, x_d) = \left(x_1, \dots, \binom{r}{d-1}x_1 + \binom{r}{d-2}x_2 + \dots + \binom{r}{0}x_d\right).$$

Now, choose any integers $r_1 < r_2 < \dots < r_d$ and $q_j \in \mathbb{Z}$ for $j = 1, \dots, d$. We want to show that the distribution of the vector $(f^{q_1} \circ T_d^{r_1}, \dots, f^{q_d} \circ T_d^{r_d})$ is $(\text{Leb}_{\mathbb{T}})^{\otimes d}$ and for that we need to check that

$$(37) \quad \mathbb{E}(f^{q_1} \circ T_d^{r_1} \cdot \dots \cdot f^{q_d} \circ T_d^{r_d}) = 0$$

³That is, we identify the space of ergodic components with $(\mathbb{T}, \text{Leb}_{\mathbb{T}})$.

unless $q_1 = \dots = q_d = 0$. Now, in view of (36), the negation of (37) is equivalent to

$$\sum_{j=1}^d q_j \binom{r_j}{k} = 0 \text{ for } k = 0, 1, \dots, d-1.$$

It is not hard to see that this system of linear equations is equivalent to

$$\sum_{j=1}^d q_j r_j^k = 0 \text{ for } k = 0, 1, \dots, d-1.$$

However, the determinant here is the Vandermonde determinant and since the numbers r_j are pairwise different, our claim follows. \square

Remark 7.5. By the proof of Proposition 4.1 (see also Remark 4.2), under the assumptions of Proposition 4.1, we obtain that the dynamical system corresponding to the stationary process (Z_n) is (up to isomorphism) just T_d .

Remark 7.6. If T acting on (X, \mathcal{B}, μ) is additionally ergodic then $E_0(T)$ consists of the constants while $E_1(T)$ consists of the eigenfunctions of the Koopman operator U_T acting on $L^2(X, \mathcal{B}, \mu)$. Hence, if $(Z_n)_{n \in \mathbb{N}}$ is a stationary process satisfying

$$(38) \quad Z_0 \cdot Z_2 = Z_1^2$$

then, using also Proposition 7.1, the ergodic components of the corresponding dynamical system must have discrete spectra.⁴ A prominent example of such a situation is the automorphisms $T_2(x_1, x_2) = (x_1, x_1 + x_2)$ on \mathbb{T}^2 whose ergodic components are (all) irrational rotations (and Z_0 is given by $f_2 : (x_1, x_2) \mapsto e^{i2\pi x_2}$).

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⁴Note that by Proposition 7.1 it follows that $Z_1 = cZ_0$, whence (34) is satisfied whenever $\sum_{i=1}^k \ell_i = \sum_{i=1}^k \ell'_i$.

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