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# THE DUAL OF INFINITESIMAL UNITARY HOPF ALGEBRAS AND PLANAR ROOTED FORESTS

XIAOMENG WANG, LOÏC FOISSY, AND XING GAO

**ABSTRACT.** We study the infinitesimal (in the sense of Joni and Rota) bialgebra  $H_{RT}$  of planar rooted trees introduced by Gao and Wang, whose coproduct is given by deletion of a vertex. We prove that its dual  $H_{RT}^*$  is isomorphic to a free non unitary algebra, and give two free generating sets. Giving  $H_{RT}$  a second product, we make it an infinitesimal bialgebra in the sense of Loday and Ronco, which allows to explicitly construct a projector on its space primitive elements, which freely generates  $H_{RT}$ .

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## 1. INTRODUCTION

Rooted trees and planar rooted trees have a very rich algebraic structure: Grossman and Larson [10] first gave rooted trees a structure of noncommutative and cocommutative algebraic structure, closely related to the Butcher group of Runge-Kutta methods [2]; then, in order to algebraically treat the operation of Renormalization in Quantum Field Theory, Connes and Kreimer introduced a commutative, noncocommutative Hopf algebra of rooted trees [3], and it was proved that the Connes-Kreimer and the Grossman-Larson Hopf algebra are in duality [12, 16]. A self-dual noncommutative version of the Connes-Kreimer Hopf algebra was simultaneously introduced by Foissy and Holtkamp [5, 13], and this object was deformed as an infinitesimal bialgebra in the sense of Loday and Ronco [15] in [7].

Recently, Gao and Wang introduced another infinitesimal coproduct  $\Delta_{RT}$  on planar rooted trees, where the usual 1-cocycle compatibility between the operator  $B^+$  (see paragraph 2.1 below) and the coproduct is modified: in the Foissy-Holtkamp case, this is the 1-cocycle condition

$$\Delta_{\mathcal{T}} \circ B^+(x) = B^+(x) \otimes \mathbb{1} + (\text{id} \otimes B^+) \circ \Delta_{\mathcal{T}}(x),$$

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a coassociative coalgebra (possibly without counit), unless otherwise stated. Linear maps and tensor products are taken over  $\mathbf{k}$ . For any algebra  $A$ , we view  $A \otimes A$  as an  $A$ -bimodule via

$$(1) \quad a \cdot (b \otimes c) := ab \otimes c \text{ and } (b \otimes c) \cdot a := b \otimes ca.$$

## 2. THE INFINITESIMAL UNITARY HOPF ALGEBRAS OF PLANAR ROOTED FORESTS

In this section, we first recall some basic notations used throughout the paper.

**2.1. Planar rooted forests.** We expose some concepts and notations on planar rooted forests from [11, 17]. Let  $\mathcal{T}$  denote the set of planar rooted trees and  $M(\mathcal{T})$  the free monoid generated by  $\mathcal{T}$  in which the multiplication is the concatenation, denoted by  $m_{RT}$  and usually suppressed. Thus an element  $F$  in  $M(\mathcal{T})$ , called a **planar rooted forest**, is a noncommutative product of planar rooted trees in  $\mathcal{T}$ . The empty tree  $\mathbb{1}$  is the unity of  $M(\mathcal{T})$ .

Here are some examples of elements of  $\mathcal{T}$  where the root is on the bottom:

$$., \quad \downarrow, \quad \vee, \quad \downarrow, \quad \Psi, \quad \downarrow \vee, \quad \vee \downarrow, \quad \Upsilon.$$

Here are some examples of elements of  $M(\mathcal{T})$ :

$$\mathbb{1}, \quad .., \quad \downarrow \dots, \quad \downarrow \downarrow, \quad \vee \downarrow, \quad \dots.$$

Let  $H_{RT} := \mathbf{k}M(\mathcal{T})$  be the free  $\mathbf{k}$ -module spanned by  $M(\mathcal{T})$ . Denote by

$$B^+ : H_{RT} \rightarrow H_{RT}$$

the grafting map sending  $\mathbb{1}$  to  $.$  and sending a planar rooted forest in  $H_{RT}$  to its grafting on a new root, and by  $m_{RT}$  the concatenation on  $H_{RT}$ . Then  $H_{RT}$  is closed under the concatenation  $m_{RT}$  [17]. Here are some examples about  $B^+$  on  $H_{RT}$ :

$$B^+(\mathbb{1}) = ., \quad B^+(\downarrow) = \downarrow, \quad B^+(\downarrow \vee) = \downarrow \vee.$$

For  $F = T_1 \cdots T_m \in M(\mathcal{T})$  with  $T_1, \dots, T_m \in \mathcal{T}$ , we define  $\text{bre}(F) := m$  to be the **breadth** of  $F$ . Here we use the convention that  $\text{bre}(\mathbb{1}) = 0$  when  $m = 0$ . The **depth**  $\text{dep}(T)$  of a rooted tree is the maximal length of linear chains from the root to the leaves of the tree. For  $F = T_1 \cdots T_m \in M(\mathcal{T})$  with  $m \geq 0$ , we define

$$\text{dep}(F) := \max\{\text{dep}(T_i) \mid i = 1, \dots, m\}.$$

**2.2. Infinitesimal unitary Hopf algebras of planar rooted forests.** In order to provide an algebraic framework for the calculus of divided differences, Joni and Rota [14] introduced the concept of an infinitesimal bialgebra.

**Definition 2.1.** [14] An **infinitesimal bialgebra** is a triple  $(A, m, \Delta)$  where  $(A, m)$  is an associative algebra,  $(A, \Delta)$  is a coassociative coalgebra and for each  $a, b \in A$ ,

$$(2) \quad \Delta(ab) = a \cdot \Delta(b) + \Delta(a) \cdot b = \sum_{(b)} ab_{(1)} \otimes b_{(2)} + \sum_{(a)} a_{(1)} \otimes a_{(2)}b.$$

Note that we do not require that  $(A, m)$  is unitary, nor that  $(A, \Delta)$  is counitary. The concept of an infinitesimal Hopf algebra was introduced by Aguiar in order to develop and study infinitesimal bialgebras [1]. If  $A$  is an infinitesimal bialgebra, then the space  $\text{Hom}_{\mathbf{k}}(A, A)$  is still an algebra under convolution:

$$f * g := m(f \otimes g) \Delta,$$

but possibly without unity with respect to the convolution  $*$  [1]. Therefore, it is impossible to consider antipode. To solve this difficulty, Aguiar equipped the space  $\text{Hom}_{\mathbf{k}}(A, A)$  with circular convolution  $\otimes$  given by

$$f \otimes g := f * g + f + g, \text{ that is, } (f \otimes g)(a) := \sum_{(a)} f(a_{(1)})g(a_{(2)}) + f(a) + g(a) \text{ for } a \in A.$$

Note that  $f \otimes 0 = f = 0 \otimes f$  and so  $0 \in \text{Hom}_{\mathbf{k}}(A, A)$  is the unity with respect to the circular convolution  $\otimes$ .

With the help of the circular convolution, one can describe infinitesimal Hopf algebras.

**Definition 2.2.** [1] An infinitesimal bialgebra  $(A, m, \Delta)$  is called an **infinitesimal Hopf algebra** if the identity map  $\text{id} \in \text{Hom}_{\mathbf{k}}(A, A)$  is invertible with respect to the circular convolution. In this case, its inverse  $S \in \text{Hom}_{\mathbf{k}}(A, A)$  is called the **antipode** of  $A$ . It is characterized by the equations

$$(3) \quad \sum_{(a)} S(a_{(1)})a_{(2)} + S(a) + a = 0 = \sum_{(a)} a_{(1)}S(a_{(2)}) + S(a) + a \text{ for } a \in A,$$

where  $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ .

Now we recall the infinitesimal Hopf algebraic structure on top of planar rooted forests defined in [9]. The coproduct  $\Delta_{RT}$  on  $H_{RT}$  is defined recursively on depth. Let  $F$  be a forest in  $H_{RT}$ . For the initial step of  $\text{dep}(F) = 0$ , we define

$$(4) \quad \Delta_{RT}(F) := \Delta_{RT}(\mathbb{1}) = 0.$$

For the induction step of  $\text{dep}(F) \geq 1$ , we reduce to the induction on  $\text{bre}(F) \geq 1$ . If  $\text{bre}(F) = 1$ , then  $F = B^+(\bar{F})$  for some  $\bar{F} \in M(\mathcal{T})$  and define

$$(5) \quad \Delta_{RT}(F) := \Delta_{RT}B^+(\bar{F}) := \bar{F} \otimes \mathbb{1} + (\text{id} \otimes B^+)\Delta_{RT}(\bar{F}),$$

that is,  $\Delta_{RT}B^+ = \text{id} \otimes \mathbb{1} + (\text{id} \otimes B^+)\Delta_{RT}$ . Here the coproduct  $\Delta_{RT}(\bar{F})$  is defined by the induction hypothesis on depth. If  $\text{bre}(F) \geq 2$ , then  $F = T_1T_2 \cdots T_m$  with  $\text{bre}(F) = m \geq 2$  and define

$$(6) \quad \Delta_{RT}(F) := T_1 \cdot \Delta_{RT}(T_2 \cdots T_m) + \Delta_{RT}(T_1) \cdot (T_2 \cdots T_m).$$

**Remark 2.3.** Foissy [7] also studied another kind of infinitesimal Hopf algebras on planar rooted forests, using a different coproduct  $\Delta_{\mathcal{T}}$  given by

$$\Delta_{\mathcal{T}}(F) := \begin{cases} \mathbb{1} \otimes \mathbb{1}, & \text{if } F = \mathbb{1}, \\ F \otimes \mathbb{1} + (\text{id} \otimes B^+)\Delta_{\mathcal{T}}(\bar{F}), & \text{if } F = B^+(\bar{F}), \\ F_1 \cdot \Delta_{\mathcal{T}}(F_2) + \Delta_{\mathcal{T}}(F_1) \cdot F_2 - F_1 \otimes F_2, & \text{if } F = F_1F_2. \end{cases}$$

We give some examples to expose the differences between these two coproducts  $\Delta_{RT}$  and  $\Delta_{\mathcal{T}}$ . On the one hand,

$$\begin{aligned} \Delta_{RT}(\cdot) &= \mathbb{1} \otimes \mathbb{1}; \\ \Delta_{RT}(\uparrow) &= \cdot \otimes \mathbb{1} + \mathbb{1} \otimes \cdot; \\ \Delta_{RT}(\vee) &= \cdot \otimes \mathbb{1} + \cdot \otimes \cdot + \mathbb{1} \otimes \uparrow; \\ \Delta_{RT}\left(\begin{array}{c} \uparrow \\ \vee \end{array}\right) &= \uparrow \cdot \otimes \mathbb{1} + \uparrow \otimes \cdot + \cdot \otimes \uparrow + \mathbb{1} \otimes \vee. \end{aligned}$$

On the other hand,

$$\Delta_{\mathcal{T}}(\cdot) = \cdot \otimes \mathbb{1} + \mathbb{1} \otimes \cdot;$$

$$\begin{aligned}\Delta_{\mathcal{T}}(\mathfrak{!}) &= \mathfrak{!} \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{!} + \cdot \otimes \cdot; \\ \Delta_{\mathcal{T}}(\mathfrak{V}) &= \mathfrak{V} \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{V} + \cdot \otimes \cdot + \cdot \otimes \mathfrak{!}; \\ \Delta_{\mathcal{T}}\left(\begin{array}{c} \mathfrak{!} \\ \mathfrak{V} \end{array}\right) &= \begin{array}{c} \mathfrak{!} \\ \mathfrak{V} \end{array} \otimes \mathbb{1} + \mathbb{1} \otimes \begin{array}{c} \mathfrak{!} \\ \mathfrak{V} \end{array} + \mathfrak{!} \otimes \cdot + \cdot \otimes \mathfrak{V} + \mathfrak{!} \otimes \mathfrak{!}.\end{aligned}$$

Let us recall the combinatorial description of  $\Delta_{RT}$  given in [9], in terms of an order on the set  $V(F)$  of vertices of a forest  $F$  [5, 7].

**Definition 2.4.** Let  $F = T_1 \cdots T_m \in M(\mathcal{T})$  with  $T_1, \dots, T_m \in \mathcal{T}$  and  $m \geq 1$ , and let  $u, v \in V(F)$  be two vertices. Then

- (a)  $u \leq_h v$  (**being higher**) if there exists a (directed) path from  $u$  to  $v$  in  $F$ , the edges of  $F$  being oriented from roots to leaves;
- (b)  $u \leq_l v$  (**being more on the left**) if  $u$  and  $v$  are not comparable for  $\leq_h$  and one of the following assertions is satisfied:
  - (i)  $u$  is a vertex of  $T_i$  and  $v$  is a vertex of  $T_j$  with  $1 \leq j < i \leq m$ .
  - (ii)  $u$  and  $v$  are vertices of the same  $T_i$ , and  $u \leq_l v$  in the forest obtained from  $T_i$  by deleting its root;
- (c)  $u \leq_{h,l} v$  (**being higher or more on the left**) if  $u \leq_h v$  or  $u \leq_l v$ .

As usual, we denote  $u <_{h,l} v$  (resp.  $u <_l v$ ,  $u <_h v$ ) if  $u \leq_{h,l} v$  (resp.  $u \leq_l v$ ,  $u \leq_h v$ ) but  $u \neq v$ . The **induced subgraph** in  $G$  by  $V$  is the graph whose vertex set is  $V$  and whose edge set consists of all of the edges in  $G$  that have both endpoints in  $V$  [4].

Let  $F \in M(\mathcal{T})$  be a planar rooted forest. For each vertex  $v \in V(F)$ , denote by  $B_v$  the induced subgraph in  $F$  by the set  $\{u \in V(F) \mid v <_{h,l} u\}$ , and by  $R_v$  the induced subgraph in  $F$  by the set  $V(F) \setminus (V(B_v) \cup \{v\})$ . Equivalently,  $R_v$  is the induced subgraph in  $F$  by the set  $\{u \in V(F) \mid u <_{h,l} v\}$ . Note that both  $B_v$  and  $R_v$  are planar rooted forests in  $M(\mathcal{T})$ , not containing the vertex  $v$ . Then [9]

$$(7) \quad \Delta_{RT}(F) = \sum_{v \in V(F)} B_v \otimes R_v \text{ for } F \in M(\mathcal{T}).$$

**Lemma 2.5.** [9]

- (a) The quadruple  $(H_{RT}, m_{RT}, \mathbb{1}, \Delta_{RT})$  is an infinitesimal unitary bialgebra.
- (b) The quadruple  $(H_{RT}, m_{RT}, \mathbb{1}, \Delta_{RT})$  is an infinitesimal unitary Hopf algebra.

### 3. THE DUAL OF INFINITESIMAL UNITARY HOPF ALGEBRA ON PLANAR ROOTED FORESTS

In this section, we show that the dual  $H_{RT}^* = (H_{RT}^*, \Delta_{RT}^*, m_{RT}^*, \mathbb{1}^*)$  of  $H_{RT}$  is a free algebra. Let us first recall some fundamental facts.

**Lemma 3.1.** [8] Let  $V = \bigoplus_{n=1}^{\infty} V^{(n)}$  be a graded vector space and the homogeneous components of  $V$  be finite dimensional. Then

- (a) The graded dual  $V^* := \bigoplus_{n=1}^{\infty} (V^{(n)})^*$  is also a graded space, and  $V^{**} \simeq V$ .
- (b)  $V \otimes V$  is also a graded vector space with  $(V \otimes V)^{(n)} = \sum_{i=0}^n V^{(i)} \otimes V^{(n-i)}$  for all  $n \in \mathbb{N}$ . Moreover,  $(V \otimes V)^* \simeq V^* \otimes V^*$ .

The  $H_{RT}$  can be graded by number of vertices. Denote by

$$H_{RT}(n) := \mathbf{k} \left\{ F \in M(\mathcal{T}) \mid |F| = n - 1 \right\} \quad \text{for } n \geq 1,$$

where  $|F|$  is the number of vertices of  $F$ . Then

$$H_{RT} = \bigoplus_{n=1}^{\infty} H_{RT}(n) \quad \text{and} \quad H_{RT}^* = \bigoplus_{n=1}^{\infty} (H_{RT}(n))^*.$$

We now give a combinatorial description of the dual of the coproduct  $\Delta_{RT}$ . Let us propose the following concepts as a preparation.

**Definition 3.2.** Let  $T$  be a planar rooted tree. The **left path**  $\text{LP}(T)$  of  $T$  is defined to be the path from the root to the left most leaf of  $T$ .

**Example 3.3.** The following paths in green are left paths of planar rooted trees, respectively.

$$\mathbb{V}, \quad \mathbb{I}, \quad \mathbb{V}, \quad \mathbb{V}, \quad \mathbb{Y}.$$

**Definition 3.4.** Let  $T$  and  $T'$  be two planar rooted trees. A **left grafting** of  $T'$  over  $T$  is a planar rooted tree obtained by grafting  $T'$  in a vertex  $v$  of the left path  $\text{LP}(T)$  by connecting  $v$  and the root of  $T'$ , such that  $T'$  is on the left of  $T$ . Denote by  $\mathcal{L}(T', T)$  the set of all left graftings of  $T'$  over  $T$ .

**Example 3.5.** Consider  $T' = \cdot$  and  $T = \mathbb{I}$ . Then  $\mathbb{V}$  and  $\mathbb{I}$  are the two left graftings of  $T'$  over  $T$ .

In general, we propose

**Definition 3.6.** Let  $F = T_1 \cdots T_m$  be a planar rooted forest with  $T_1, \dots, T_m \in \mathcal{T}$  and  $T$  a planar rooted tree. A **left grafting** of  $F$  over  $T$  is a planar rooted tree obtained by left grafting each  $T_i$  in a vertex  $v_i$  of  $\text{LP}(T)$  such that  $v_i \leq_{\text{h,l}} v_j$  when  $i < j$ . Denoted by  $\mathcal{L}(F, T)$  the set of all left grafting of  $F$  over  $T$ .

Notice that the  $T_i$  and  $T_j$  may be grafted in the same vertex.

**Example 3.7.** Let  $F = \dots$  and  $T = \mathbb{I}$ . Then

$$\mathcal{L}(F, T) = \left\{ \mathbb{V}, \mathbb{V}, \mathbb{Y} \right\}.$$

**Definition 3.8.** Let  $F$  be a planar rooted forest and  $T$  a planar rooted tree.

Step 1: Decompose  $F = F_1 F_2$  and  $F_1 B^+(F_2) = F'_1 F'_2$ , where  $F_1, F_2, F'_1, F'_2 \in M(\mathcal{T})$ .

Step 2: Left graft  $F'_2$  over  $T$  to obtain an  $\tilde{F} \in \mathcal{L}(F'_2, T)$ , and concatenate  $F'_1$  and  $\tilde{F}$  to get  $F'_1 \tilde{F}$ , and call the concatenation  $F'_1 \tilde{F}$  a **grafting** of  $F$  over  $T$ .

Denote by  $\mathcal{G}(F, T)$  the set of all graftings of  $F$  over  $T$ .

Let us compute explicitly an example for better understanding of Definition 3.8.

**Example 3.9.** Let  $F = \dots$  and  $T = \mathbb{I}$ . Then the decomposition  $F = F_1 F_2$  as concatenation product can be

$$F = \mathbb{1}(\dots) = (\cdot)(\cdot) = (\dots)\mathbb{1}.$$

**Case 1.**  $F_1 = \mathbb{1}$  and  $F_2 = \dots$ . Then  $B^+(F_2) = \mathbb{V}$  and  $F_1 B^+(F_2) = \mathbb{V}$ . The decomposition  $F_1 B^+(F_2) = F'_1 F'_2$  can be

$$F_1 B^+(F_2) = (\mathbb{V})\mathbb{1} = \mathbb{1}(\mathbb{V}).$$

We have two subcases.

**Subcase 1.1.**  $F'_1 = \mathbb{V}$  and  $F'_2 = \mathbb{1}$ . Then  $\mathbb{V} \downarrow$  is the only one grafting of  $F$  over  $T$  in this subcase.

**Subcase 1.2.**  $F'_1 = \mathbb{1}$  and  $F'_2 = \mathbb{V}$ . Then

$$\mathcal{L}(F'_2, T) = \mathcal{L}(\mathbb{V}, \downarrow) = \left\{ \begin{array}{c} \mathbb{V} \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \mathbb{V} \\ \downarrow \\ \downarrow \end{array} \right\}$$

and  $\begin{array}{c} \mathbb{V} \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \mathbb{V} \\ \downarrow \\ \downarrow \end{array}$  are two graftings of  $F$  over  $T$  in this subcase.

**Case 2.**  $F_1 = \cdot$  and  $F_2 = \cdot\cdot$ . Then  $B^+(F_2) = \downarrow$  and  $F_1 B^+(F_2) = \cdot\downarrow$ . The decomposition  $F_1 B^+(F_2) = F'_1 F'_2$  can be

$$F_1 B^+(F_2) = (\cdot\downarrow)\mathbb{1} = (\cdot)(\downarrow) = \mathbb{1}(\cdot\downarrow).$$

We have the following three subcases.

**Subcase 2.1.**  $F'_1 = \cdot\downarrow$  and  $F'_2 = \mathbb{1}$ . Then  $\cdot\downarrow \downarrow$  is the only one grafting of  $F$  over  $T$  in this subcase.

**Subcase 2.2.**  $F'_1 = \cdot$  and  $F'_2 = \downarrow$ . Then

$$\mathcal{L}(F'_2, T) = \mathcal{L}(\downarrow, \downarrow) = \left\{ \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right\}$$

and  $\begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$  are two graftings of  $F$  over  $T$  in this subcase.

**Subcase 2.3.**  $F'_1 = \mathbb{1}$  and  $F'_2 = \cdot\downarrow$ . Then

$$\mathcal{L}(F'_2, T) = \mathcal{L}(\cdot\downarrow, \downarrow) = \left\{ \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right\}$$

and  $\begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$  are three graftings of  $F$  over  $T$  in this subcase.

**Case 3.**  $F_1 = \cdot\cdot$  and  $F_2 = \mathbb{1}$ . Then  $B^+(F_2) = \cdot$  and  $F_1 B^+(F_2) = \cdot\cdot\cdot$ . The decomposition  $F_1 B^+(F_2) = F'_1 F'_2$  can be

$$F_1 B^+(F_2) = (\cdot\cdot\cdot)\mathbb{1} = (\cdot\cdot)(\cdot) = (\cdot)(\cdot\cdot) = \mathbb{1}(\cdot\cdot\cdot).$$

There are four subcases.

**Subcase 3.1.**  $F'_1 = \cdot\cdot\cdot$  and  $F'_2 = \mathbb{1}$ . Thus  $\cdot\cdot\cdot \downarrow$  is the only one grafting of  $F$  over  $T$  in this subcase.

**Subcase 3.2.**  $F'_1 = \cdot\cdot$  and  $F'_2 = \cdot$ . Then

$$\mathcal{L}(F'_2, T) = \mathcal{L}(\cdot, \downarrow) = \left\{ \mathbb{V}, \downarrow \right\}$$

and  $\mathbb{V}, \downarrow$  are two graftings of  $F$  over  $T$  in this subcase.

**Subcase 3.3.**  $F'_1 = \cdot$  and  $F'_2 = \cdot\cdot\cdot$ . We have

$$\mathcal{L}(F'_2, T) = \mathcal{L}(\cdot\cdot\cdot, \downarrow) = \left\{ \begin{array}{c} \mathbb{V} \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right\}.$$

Then  $\begin{array}{c} \mathbb{V} \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$  are three graftings of  $F$  over  $T$  in this subcase.

**Subcase 3.4.**  $F'_1 = \mathbb{1}$  and  $F'_2 = \cdot\cdot\cdot$ . Then

$$\mathcal{L}(F'_2, T) = \mathcal{L}(\cdot\cdot\cdot, \downarrow) = \left\{ \begin{array}{c} \mathbb{V} \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right\}$$





Note that for any  $k, l \geq 1$ ,  $a_{k,l} = a_{l,k}$ .

In general, we propose

**Definition 3.11.** Let  $F$  and  $F' = TF_1$  be two planar rooted forests with  $T \in \mathcal{T}$ . We call  $\tilde{F}F_1$  a **grafting** of  $F$  over  $F'$ , where  $\tilde{F}$  is a grafting of  $F$  over  $T$  given in Definition 3.8. Let  $\mathcal{G}(F, F')$  be the set of all graftings of  $F$  over  $F'$ .

For  $F, F', F'' \in M(\mathcal{T})$ , denote by  $n'(F, F'; F'')$  the number of ways of grafting of  $F$  over  $F'$  to obtain  $F''$ . For example,

$$n'(\dots, \uparrow; \dots) = 0, \quad n'(\dots, \uparrow; \dots) = 0, \quad n'(\dots, \uparrow; \vee \uparrow) = 1, \quad \text{and} \quad n'(\dots, \uparrow; \curvearrowright) = 1.$$

For each  $F \in M(\mathcal{T})$ , we define

$$(8) \quad Z_F : \begin{cases} H_{RT} & \longrightarrow \mathbf{k}, \\ F' & \longmapsto \delta_{F,F'} \text{ for } F' \in M(\mathcal{T}), \end{cases}$$

where  $\delta_{F,F'}$  is the Kronecker function. Then  $\{Z_F \mid F \in M(\mathcal{T})\}$  is a basis of  $H_{RT}^*$ . We denote by  $n(F, F'; F'')$  the coefficient of  $F \otimes F'$  in  $\Delta_{RT}(F'')$ , where  $F, F', F'' \in M(\mathcal{T})$ . The following result gives the relation between  $n(F, F'; F'')$  and  $n'(F, F'; F'')$ .

**Lemma 3.12.** Let  $F, F', F'' \in M(\mathcal{T})$ . Then  $n'(F, F'; F'') = n(F, F'; F'')$ . Moreover, this coefficient is 0 or 1.

*Proof.* We first show that  $n'(F, F'; F'') \leq n(F, F'; F'')$ . Let  $F'' \in \mathcal{G}(F, F')$  be a grafting of  $F$  over  $F'$  determined by the decompositions  $F = F_1F_2$  and  $F_1B^+(F_2) = F'_1F'_2$ , in which the new vertex added by  $B^+$  is denoted by  $v$ . Graphically,

$$F_1B^+(F_2) = F_1 \begin{array}{c} \textcircled{F_2} \\ | \\ v \bullet \end{array}.$$

Then  $v \in V(F'')$  and

$$V(F_1) = \{u \in V(F'') \mid v <_1 u\} \text{ and } V(F_2) = \{u \in V(F'') \mid v <_h u\}.$$

By the combinatorial description of the coproduct  $\Delta_{RT}(F'')$ , we have

$$B_v = \{u \in V(F'') \mid v <_{h,l} u\} = F_1F_2 = F.$$

Since  $V(F'') = V(F) \sqcup V(F') \sqcup \{v\}$ , we get

$$R_v = \text{the induced subgraph of } F'' \text{ by } V(F'') \setminus (\{v\} \sqcup V(B_v)) = F'.$$

Therefore a grafting  $F''$  of  $F$  over  $F'$  induces a term  $B_v \otimes R_v = F \otimes F'$  in  $\Delta_{RT}(F'')$  and so  $n'(F, F'; F'') \leq n(F, F'; F'')$ .

Next we show that  $n'(F, F'; F'') \geq n(F, F'; F'')$ . By Eq. (7), we may let  $B_v \otimes R_v$  be an item in  $\Delta_{RT}(F'')$  for some  $v \in V(F'')$ . Write  $F'' = T_1 \cdots T_m$  with  $T_1, \dots, T_m \in \mathcal{T}$ , and assume  $v \in V(T_i)$  for some  $1 \leq i \leq m$ . Let  $F := B_v$ ,  $F' := R_v$ ,

$$F_1 := \text{the induced subgraph of } F'' \text{ by } \{u \in V(F'') \mid v <_1 u\}$$

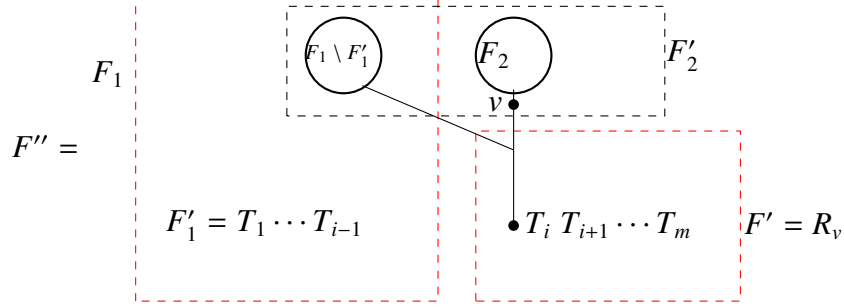
and

$$F_2 := B_v \setminus F_1 = \text{the induced subgraph of } F'' \text{ by } \{u \in V(F'') \mid v <_h u\}.$$

Since  $F = B_v$  is the induced subgraph of  $F''$  by  $\{u \in V(F'') \mid v <_{h,l} u\}$ , it follows from Definition 2.4 that  $F = B_v = F_1 F_2$ . Let  $F'_1 := T_1 \cdots T_{i-1}$  and

$$F'_2 := \text{the induced subgraph of } T_i \text{ by } \{u \in V(T_i) \mid v <_{h,l} u\} \sqcup \{v\}.$$

Then  $F''$  is a grafting of  $F$  over  $F'$  determinate by the decomposition  $F_1 B^+(F_2) = F'_1 F'_2$  (see Fig.1).



**Fig. 1** The illustration of the grafting  $F''$  of  $F$  over  $F'$ .

Hence, if  $F \otimes F'$  is a term of  $\Delta_{RT}(F'')$ , then we obtain a grafting  $F''$  of  $F$  over  $F'$  and so  $n'(F, F'; F'') \leq n(F, F'; F'')$ . Therefore  $n'(F, F'; F'') = n(F, F'; F'')$ .

If  $n(F, F'; F'') \geq 2$ , then there exist two different vertices  $u$  and  $v$  such that  $B_u \otimes R_u = F \otimes F' = B_v \otimes R_v$  by Eq. (7). Thus  $B_u = B_v$  and so  $u = v$  by the definition of  $B_u$  and  $B_v$ , a contradiction.  $\square$

The following result gives a combinatorial description of the multiplication in  $H_{RT}^*$ , dual of the coproduct  $\Delta_{RT}$ , which we denote by  $\diamond = \Delta_{RT}^*$ .

**Proposition 3.13.** *Let  $F, F' \in M(\mathcal{T})$ . The product of  $Z_F$  and  $Z_{F'}$  is*

$$(9) \quad Z_F \diamond Z_{F'} = \sum_{F'' \in M(\mathcal{T})} n'(F, F'; F'') Z_{F''} = \sum_{F'' \in \mathcal{G}(F, F')} Z_{F''}.$$

*Proof.* The second equation follows from Lemma 3.12. We are left to show the first equation. For any  $F, F' \in M(\mathcal{T})$ , we suppose

$$Z_F \diamond Z_{F'} = \sum_{F'' \in M(\mathcal{T})} a_{F, F'}^{F''} Z_{F''}.$$

For any  $F_1 \in M(\mathcal{T})$ , we have

$$\sum_{F'' \in M(\mathcal{T})} a_{F, F'}^{F''} Z_{F''}(F_1) = a_{F, F'}^{F_1} Z_{F_1}(F_1) = a_{F, F'}^{F_1},$$

and

$$\begin{aligned} Z_F \diamond Z_{F'}(F_1) &= \Delta_{RT}^*(Z_F \otimes Z_{F'})(F_1) = (Z_F \otimes Z_{F'})(\Delta_{RT}(F_1)) \\ &= (Z_F \otimes Z_{F'}) \left( \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} \right) \\ &= \sum_{(F_1)} Z_F(F_{1(1)}) \otimes Z_{F'}(F_{1(2)}) \\ &= \sum_{(F_1)} \delta_{F, F_{1(1)}} \otimes \delta_{F', F_{1(2)}} \end{aligned}$$

$$\begin{aligned}
&= n(F, F'; F_1) \quad (\text{by Definition of } n(F, F'; F_1)) \\
&= n'(F, F'; F_1) \quad (\text{by Lemma 3.12}).
\end{aligned}$$

Thus  $a_{F, F'}^{F_1} = n'(F, F'; F_1)$ , as required.  $\square$

Let us expose an example.

**Example 3.14.** Let  $F, F'$  be the planar rooted forests in Example 3.9. Then the product of  $Z_{\dots}$  and  $Z_{\downarrow}$  is

$$\begin{aligned}
Z_{\dots} \diamond Z_{\downarrow} &= Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} + Z_{\dots \downarrow} + Z_{\dots \downarrow} + Z_{\dots \downarrow} + Z_{\dots \downarrow} \\
&+ Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} \\
&+ Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow}.
\end{aligned}$$

**Remark 3.15.** (a) We define the degree of a forest as  $\deg(F) = |F| + 1$  for  $F \in M(\mathcal{T})$ . Since a new vertex is added by the grafting operator  $B^+$  in the second step of Definition 3.8, the multiplication

$$\diamond : H_{RT}^* \otimes H_{RT}^* \rightarrow H_{RT}^*$$

is homogeneous of degree 0 by Proposition 3.13.

(b) Let  $F, F' = \overline{TF'}$  be two planar rooted forests. Then by Definition 3.11 and Proposition 3.13,

$$Z_F \diamond Z_{F'} = \sum_{F'' \in \mathcal{G}(F, F')} Z_{F''} = \sum_{F_1 F_2 = F} \sum_{F'_1 F'_2 = F_1 B^+(F_2)} \sum_{\tilde{F} \in \mathcal{L}(F'_2, T)} Z_{F'_1 \tilde{F} \overline{F'}},$$

where  $F_1, F_2, F'_1, F'_2 \in M(\mathcal{T})$ .

The following result characterizes the coproduct on  $H_{RT}^*$ .

**Lemma 3.16.** Let  $T_1 \cdots T_n \in M(\mathcal{T})$ . The coproduct of  $Z_{T_1 \cdots T_n} \in H_{RT}^*$  is

$$(10) \quad m_{RT}^*(Z_{T_1 \cdots T_n}) = \sum_{i=0}^n Z_{T_1 \cdots T_i} \otimes Z_{T_{i+1} \cdots T_n},$$

with the convention that  $Z_{T_1 T_0} = \mathbb{1}$  and  $Z_{T_{n+1} T_n} = \mathbb{1}$ .

*Proof.* Suppose

$$m_{RT}^*(Z_{T_1 \cdots T_n}) = \sum_{F', F'' \in M(\mathcal{T})} c_{F', F''} Z_{F'} \otimes Z_{F''}.$$

For any  $F_1, F_2 \in M(\mathcal{T})$ , we have

$$m_{RT}^*(Z_{T_1 \cdots T_n})(F_1 \otimes F_2) = Z_{T_1 \cdots T_n}(m_{RT}(F_1 \otimes F_2)) = Z_{T_1 \cdots T_n}(F_1 F_2) = \delta_{T_1 \cdots T_n, F_1 F_2},$$

and

$$\begin{aligned}
\left( \sum_{F', F'' \in M(\mathcal{T})} c_{F', F''} (Z_{F'} \otimes Z_{F''}) \right) (F_1 \otimes F_2) &= \sum_{F', F'' \in M(\mathcal{T})} c_{F', F''} Z_{F'}(F_1) \otimes Z_{F''}(F_2) \\
&= \sum_{F', F'' \in M(\mathcal{T})} c_{F', F''} \delta_{F', F_1} \otimes \delta_{F'', F_2} \\
&= c_{F_1, F_2}.
\end{aligned}$$

Thus  $\delta_{T_1 \dots T_n, F_1 F_2} = c_{F_1, F_2}$  and so  $c_{F_1, F_2} = 1$  if  $T_1 \dots T_n = F_1 F_2$  and  $c_{F_1, F_2} = 0$  otherwise. This completes the proof.  $\square$

For example, we have

$$m_{RT}^*(Z_{\cdot, \downarrow}) = Z_{\cdot, \downarrow} \otimes \mathbb{1} + \mathbb{1} \otimes Z_{\cdot, \downarrow} + Z_{\cdot} \otimes Z_{\downarrow}.$$

Now we are ready for our main result in this section. Denote by

$$S := \{Z_F \mid F \in M(\mathcal{T}) \text{ such that } \text{bre}(F) \text{ is even}\}.$$

**Theorem 3.17.** *The algebra  $(H_{RT}^*, \diamond)$  is the free non unitary algebra on  $S$ .*

*Proof.* We first prove that  $S$  generates  $H_{RT}^*$ . Let  $A$  be the subalgebra of  $(H_{RT}^*, \diamond)$  generated by  $S$  and let  $Z_{T_1 \dots T_m} \in H_{RT}^*$  be a basis element with  $T_1, \dots, T_m \in \mathcal{T}$ . If  $m$  is even, then  $Z_{T_1 \dots T_m} \in A$ . If  $m$  is odd, we prove that  $Z_{T_1 \dots T_m} \in A$  by induction on  $|T_1| \geq 1$ . If  $|T_1| = 1$ , then  $T_1 = \cdot$  and

$$Z_{\mathbb{1}} \diamond Z_{T_2 \dots T_m} = Z_{\cdot, T_2 \dots T_m} + \sum_{F' \in \mathcal{L}(\cdot, T_2)} Z_{F' T_3 \dots T_m} \quad (\text{by Item (b) of Remark 3.15}).$$

Since  $Z_{\mathbb{1}}, Z_{T_2 \dots T_m} \in A$  and  $Z_{F' T_3 \dots T_m} \in A$  by  $\text{bre}(F' T_3 \dots T_m) = m - 1$ , we have  $Z_{T_1 T_2 \dots T_m} = Z_{\cdot, T_2 \dots T_m} \in A$ . If  $|T_1| \geq 2$ , then  $T_1 = B^+(F)$  for some  $F \in H_{RT}$ . We have

$$\begin{aligned} Z_F \diamond Z_{T_2 \dots T_m} &= \sum_{F_1 F_2 = F} \sum_{F'_1 F'_2 = F_1 B^+(F_2)} \sum_{\tilde{F} \in \mathcal{L}(F'_2, T_2)} Z_{F'_1 \tilde{F} T_3 \dots T_m} \quad (\text{by Item (b) of Remark 3.15}) \\ &= Z_{B^+(F) T_2 \dots T_m} + \sum_{F_1 F_2 = F} \sum_{\substack{F'_1 F'_2 = F_1 B^+(F_2) \\ F'_1 = \mathbb{1}}} \sum_{\tilde{F} \in \mathcal{L}(F'_2, T_2)} Z_{\tilde{F} T_3 \dots T_m} \\ &\quad + \sum_{\substack{F_1 F_2 = F \\ F_1 \neq \mathbb{1}}} \sum_{\substack{F'_1 F'_2 = F_1 B^+(F_2) \\ F'_1 \neq \mathbb{1}}} \sum_{\tilde{F} \in \mathcal{L}(F'_2, T_2)} Z_{F'_1 \tilde{F} T_3 \dots T_m}. \end{aligned}$$

Since  $\text{bre}(\tilde{F} T_3 \dots T_m) = m - 1$ , we have  $Z_{T_2 \dots T_m}, Z_{\tilde{F} T_3 \dots T_m} \in A$  by definition of  $A$ . Moreover,  $Z_F, Z_{F'_1 \tilde{F} T_3 \dots T_m} \in A$  by the induction hypothesis. Hence  $Z_{T_1 \dots T_m} \in A$  and  $A = H_{RT}^*$ .

Next, we prove that  $(H_{RT}^*, \diamond)$  is the free algebra on  $S$ . By [6, Proposition 8], the formal series of  $H_{RT} = \bigoplus_{n=1}^{\infty} H_{RT}(n)$  is

$$\mathbf{F}(x) = \sum_{i=1}^{\infty} \dim H_{RT}^{(i)} x^i = x + x^2 + 2x^3 + 5x^4 + \dots = \frac{1 - \sqrt{1 - 4x}}{2},$$

which is also the formal series of  $H_{RT}^* = \bigoplus_{n=1}^{\infty} (H_{RT}(n))^*$ . Thus  $\mathbf{F}^2(x) = \mathbf{F}(x) - x$ . Since each planar rooted tree is a grafting operation  $B^+$  of a planar rooted forest  $F$  and vice-versa, the formal series of planar rooted trees is the same as the one of planar rooted forests, that is,

$$\mathbf{T}(x) = \sum_{i=1}^{\infty} a_i x^i = \mathbf{F}(x) = x + x^2 + 2x^3 + 5x^4 + \dots,$$

where  $a_i$  is the number of trees with  $i$  vertices. So the formal series of forests with even breadth is

$$\sum_{i=0}^{+\infty} b_i x^i = 1 + \mathbf{T}(x)\mathbf{T}(x) + \mathbf{T}(x)\mathbf{T}(x)\mathbf{T}(x)\mathbf{T}(x) + \dots = \sum_{i=0}^{+\infty} \mathbf{T}^{2i}(x) = \frac{1}{1 - \mathbf{T}^2(x)} = \frac{1}{1 - \mathbf{F}^2(x)},$$

where  $b_i$  is the number of forests of even breadth with  $i$  vertices. Let  $\mathbf{G}(x) := \sum_{i=1}^{+\infty} g_i x^i$ , where  $g_i$  is the number of forests of even breadth with degree  $i$ . Then

$$\begin{aligned} \mathbf{G}(x) &= x \sum_{i=1}^{\infty} g_i x^{i-1} = x \sum_{i=0}^{\infty} g_{i+1} x^i = x \sum_{i=0}^{\infty} b_i x^i \quad (\text{by } \deg(F) = |F| + 1) \\ &= \frac{x}{1 - \mathbf{F}^2(x)} = \frac{\mathbf{F}(x)}{1 + \mathbf{F}(x)} \quad (\text{by } x = \mathbf{F}(x) - \mathbf{F}^2(x)). \end{aligned}$$

Let  $T(S)$  be the free algebra on  $S$ . Since  $H_{RT}^*$  is generated by  $S$ , there exists a surjective algebra morphism  $\phi : T(S) \rightarrow H_{RT}^*$ . By Item (a) of Remark 3.15, the formal series of  $T(S)$  is

$$\mathbf{G}(x) + \mathbf{G}(x)\mathbf{G}(x) + \mathbf{G}(x)\mathbf{G}(x)\mathbf{G}(x) + \mathbf{G}(x)\mathbf{G}(x)\mathbf{G}(x)\mathbf{G}(x) + \cdots = \frac{\mathbf{G}(x)}{1 - \mathbf{G}(x)} = \frac{\frac{\mathbf{F}(x)}{1 + \mathbf{F}(x)}}{1 - \frac{\mathbf{F}(x)}{1 + \mathbf{F}(x)}} = \mathbf{F}(x).$$

Thus  $H_{RT}^*$  and  $T(S)$  have the same formal series and so  $\phi$  is injective. Hence  $T(S)$  is isomorphic to  $H_{RT}^*$ , as required.  $\square$

#### 4. PRIMITIVE ELEMENTS OF $H_{RT}$

In this section, we first construct a second product  $\star$  on  $H_{RT}$ , making  $H_{RT}$  a unital infinitesimal graded bialgebra in the sense of Loday and Ronco [15]. Then we give a projection of  $H_{RT}$  on its primitive elements. Finally, we characterize the dual of  $\star$ .

**4.1. A second product on  $H_{RT}$ .** The following result gives  $H_{RT}$  a second product  $\star$ .

**Proposition 4.1.** *We define a product  $\star$  on  $H_{RT}$  by*

$$\forall x, y \in H_{RT}, \quad x \star y = x \cdot \cdot \cdot y.$$

*Then  $\star$  is associative (and not unitary), and for any  $x, y \in H_{RT}$ :*

$$\Delta_{RT}(x \star y) = x \star \Delta_{RT}(y) + \Delta_{RT}(x) \star y + x \otimes y.$$

*Moreover,  $(H_{RT}, \star)$  is a graded algebra (recall that for any  $n \geq 1$ ,  $H_{RT}(n)$  is the subspace generated by forests with  $n - 1$  vertices).*

*Proof.* For any  $x, y, z \in H_{RT}$ :

$$(x \star y) \star z = x \cdot \cdot \cdot y \cdot \cdot \cdot z = x \star (y \star z),$$

so  $\star$  is associative. As  $\Delta_{RT}(\cdot) = \mathbb{1} \otimes \mathbb{1}$ , for any  $x, y \in H_{RT}$ :

$$\begin{aligned} \Delta_{RT}(x \star y) &= \Delta_{RT}(x \cdot \cdot \cdot y) \\ &= \Delta_{RT}(x \cdot \cdot \cdot) \cdot y + x \cdot \cdot \cdot \Delta_{RT}(y) \\ &= (x \cdot \Delta_{RT}(\cdot \cdot \cdot)) \cdot y + (\Delta_{RT}(x) \cdot \cdot \cdot) \cdot y + (x \cdot \cdot \cdot) \cdot \Delta_{RT}(y) \\ &= x \cdot (\mathbb{1} \otimes \mathbb{1}) \cdot y + \Delta_{RT}(x) \cdot (\cdot \cdot \cdot y) + (x \cdot \cdot \cdot) \cdot \Delta_{RT}(y) \\ &= x \otimes y + \Delta_{RT}(x) \star y + x \star \Delta_{RT}(y). \end{aligned}$$

Let  $x \in H_{RT}(k)$  and  $y \in H_{RT}(l)$ , with  $k, l \geq 1$ . Then  $x$  is a linear span of forests with  $k - 1$  vertices and  $y$  is a linear span of forests with  $l - 1$  vertices. By definition of  $\star$ ,  $x \star y$  is a linear span of forests with  $k - 1 + l - 1 + 1 = k + l - 1$  vertices, so belongs to  $H_{RT}(k + l)$ .  $\square$

**4.2. A projection on primitive elements.** The following result gives a projection of  $H_{RT}$  on its primitive elements.

**Theorem 4.2.** *We define an operator  $\theta$  on  $H_{RT}$  by:*

$$\forall x \in H_{RT}, \quad \theta(x) := \sum_{k=1}^{\infty} (-1)^{k+1} \star^{(k-1)} \circ \Delta^{(k-1)}(x),$$

where  $\Delta^{(l)} : H_{RT} \longrightarrow H_{RT}^{\otimes(l+1)}$  and  $\star^{(l)} : H_{RT}^{\otimes(l+1)} \longrightarrow H_{RT}$  are inductively defined:

$$\begin{aligned} \Delta^{(0)} &= \text{id}_{H_{RT}}, & \star^{(0)} &= \text{id}_{H_{RT}}, \\ \Delta^{(l+1)} &= (\Delta^{(l)} \otimes \text{id}_{H_{RT}}) \circ \Delta_{RT}, & \star^{(l+1)} &= \star \circ (\star^{(l)} \otimes \text{id}_{H_{RT}}). \end{aligned}$$

Then  $\theta$  is a projector on  $\text{Prim}(H_{RT}) = \text{Ker}(\Delta_{RT})$ . The kernel of  $\theta$  is

$$\text{Ker}(\theta) = H_{RT} \star H_{RT} = \mathbf{k}\{F \text{ forest with at least one tree equal to } \bullet\}.$$

*Proof.* For any forest  $F$  with  $n$  vertices,  $\Delta^{(k)}(F) = 0$  if  $k \geq n$ , so  $\theta$  is well-defined. If  $x \in \text{Prim}(H_{RT})$ ,  $\Delta^{(k)}(x) = 0$  if  $k \geq 1$ , so  $\theta(x) = x + 0 = x$ .

Let  $k \geq 2$  and  $x_1, \dots, x_k \in H_{RT}$ . By the compatibility between the product  $\star$  and the coproduct  $\Delta_{RT}$ :

$$\begin{aligned} \Delta_{RT} \circ \star^{(k-1)}(x_1 \otimes \dots \otimes x_k) &= \sum_{i=1}^k \sum_{(x_i)} x_1 \star \dots \star x_{i-1} \star x_i^{(1)} \otimes x_i^{(2)} \star x_{i+1} \star \dots \star x_k \\ &\quad + \sum_{i=1}^{k-1} x_1 \star \dots \star x_i \otimes x_{i+1} \star \dots \star x_k. \end{aligned}$$

Hence, using Sweedler's notation:

$$\begin{aligned} \Delta_{RT} \circ \theta(x) &= x^{(1)} \otimes x^{(2)} + \sum_{k=2}^{\infty} (-1)^{k+1} \sum_{(x)} \sum_{i=1}^k x^{(1)} \star \dots \star x^{(i)} \otimes x^{(i+1)} \star \dots \star x^{(k+1)} \\ &\quad + \sum_{k=2}^{\infty} (-1)^{k+1} \sum_{(x)} \sum_{i=1}^{k-1} x^{(1)} \star \dots \star x^{(i)} \otimes x^{(i+1)} \star \dots \star x^{(k)} \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{(x)} \sum_{i=1}^k x^{(1)} \star \dots \star x^{(i)} \otimes x^{(i+1)} \star \dots \star x^{(k+1)} \\ &\quad + \sum_{k=1}^{\infty} (-1)^k \sum_{(x)} \sum_{i=1}^k x^{(1)} \star \dots \star x^{(i)} \otimes x^{(i+1)} \star \dots \star x^{(k+1)} \\ &= 0, \end{aligned}$$

so  $\theta$  is a projector on  $\text{Prim}(H_{RT})$ .

By definition of  $\theta$ , for any  $x \in H_{RT}$ ,  $\theta(x) - x \in H_{RT} \star H_{RT}$ , so:

$$H_{RT} = \text{Prim}(H_{RT}) + H_{RT} \star H_{RT}.$$

Let  $x, y \in H_{RT}$ . By the compatibility between the product  $\star$  and the coproduct  $\Delta_{HR}$ , for any  $k \geq 2$ :

$$\Delta^{(k-1)}(x \star y) = \sum_{i+j=k} \sum_{(x)} \sum_{(y)} x^{(1)} \otimes \dots \otimes x^{(i)} \otimes y^{(1)} \otimes \dots \otimes y^{(j)}$$

$$+ \sum_{i+j=k+1} \sum_{(x)} \sum_{(y)} x^{(1)} \otimes \dots \otimes x^{(i-1)} \otimes x^{(i)} \star y^{(1)} \otimes y^{(2)} \otimes \dots \otimes y^{(j)}.$$

Therefore:

$$\begin{aligned} \theta(x \star y) &= x \star y + \sum_{k=2}^{\infty} (-1)^{k+1} \sum_{i+j=k} \sum_{(x)} \sum_{(y)} x^{(1)} \star \dots \star x^{(i)} \star y^{(1)} \star \dots \star y^{(j)} \\ &+ \sum_{k=2}^{\infty} (-1)^{k+1} \sum_{i+j=k+1} \sum_{(x)} \sum_{(y)} x^{(1)} \star \dots \star x^{(i)} \star y^{(1)} \star \dots \star y^{(j)} \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{(x)} \sum_{(y)} (-1)^{i+j-1} x^{(1)} \star \dots \star x^{(i)} \star y^{(1)} \star \dots \star y^{(j)} \\ &+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{(x)} \sum_{(y)} (-1)^{i+j} x^{(1)} \star \dots \star x^{(i)} \star y^{(1)} \star \dots \star y^{(j)} \\ &= 0. \end{aligned}$$

Consequently,  $\text{Prim}(H_{RT}) \cap (H_{RT} \star H_{RT}) = (0)$ , so:

$$H_{RT} = \text{Prim}(H_{RT}) \oplus (H_{RT} \star H_{RT}),$$

and the projection on  $\text{Prim}(H_{RT})$  in this direct sum is  $\theta$ .  $\square$

**Remark 4.3.**  $(H_{RT}, \star)$  is not a unitary algebra. If we consider the unitary  $\overline{H}_{RT} = \mathbf{k} \oplus H_{RT}$ , with the coproduct  $\overline{\Delta}_{RT}$  defined by

$$\forall x \in H_{RT}, \quad \overline{\Delta}_{RT}(x) = x \otimes 1 + 1 \otimes x + \Delta_{RT}(x),$$

then  $(\overline{H}_{RT}, \star, \overline{\Delta}_{RT})$  is a unitary infinitesimal bialgebra in the sense of Loday and Ronco [15], and  $\theta$  is its antipode, defined as the idempotent  $e$  in [15].

By definition of the coproduct  $\Delta_{RT}$ :

**Proposition 4.4.** *Let  $F$  be a planar rooted forest. We denote its vertices according to the order  $\leq_{h,l}$ :*

$$v_1 \leq_{h,l} \dots \leq_{h,l} v_n.$$

Then:

$$(11) \quad \theta(F) = \sum_{k=0}^n \sum_{1 \leq i_1 < \dots < i_k \leq n} (-1)^k F_{\{v_{i_k+1}, \dots, v_n\}} \bullet F_{\{v_{i_{k-1}+1}, \dots, v_{i_k-1}\}} \bullet \dots \bullet F_{\{v_{i_1+1}, \dots, v_{i_2-1}\}} \bullet F_{\{v_1, \dots, v_{i_1-1}\}}.$$

**Example 4.5.** Applying this formula for a given  $n$ , we obtain, after simplifications:

$$\begin{aligned} \text{if } n = 2, & \quad \theta(F) = F - \dots, \\ \text{if } n = 3, & \quad \theta(F) = F - F_{\{v_2, v_3\}} \bullet - \bullet F_{\{v_1, v_2\}} + \dots, \\ \text{if } n = 4, & \quad \theta(F) = F - F_{\{v_2, v_3, v_4\}} \bullet - \bullet F_{\{v_1, v_2, v_3\}} + \bullet F_{\{v_2, v_3\}} \bullet \bullet \end{aligned}$$

Consequently:

$$\begin{aligned} \theta(\mathfrak{!}) &= \mathfrak{!} - \dots, \\ \theta(\mathfrak{V}) &= \mathfrak{V} - \mathfrak{!}, & \theta(\mathfrak{!}) &= \mathfrak{!} - \mathfrak{!} \bullet - \mathfrak{!} \bullet \bullet + \dots, \\ \theta(\mathfrak{!!}) &= \mathfrak{!!} - \mathfrak{!} \bullet - \dots + \dots, & \theta(\mathfrak{V}) &= \mathfrak{V} - \bullet \mathfrak{V}, \end{aligned}$$



$$\begin{aligned}\theta(\dot{V}) &= \dot{V} - \dot{1} \cdot - \cdot V + \cdot 1 \cdot, & \theta(\dot{V}) &= \dot{V} - \cdot \dot{1}, \\ \theta(\dot{Y}) &= \dot{Y} - V \cdot - \cdot \dot{1} + \cdot 1 \cdot, & \theta(\dot{1}) &= \dot{1} - \dot{1} \cdot - \cdot \dot{1} + \cdot 1 \cdot.\end{aligned}$$

**Corollary 4.6.** Let  $V$  be the vector space  $\text{Prim}(H_{RT})$ . We put:

$$T_+(V) = \bigoplus_{n \geq 1} V^{\otimes n}.$$

We give it the concatenation product  $m_{conc}$  and the deconcatenation coproduct  $\Delta_{dec}$ :

$$\forall v_1, \dots, v_n \in V, \quad \Delta_{dec}(v_1 \dots v_n) = \sum_{i=1}^{n-1} v_1 \dots v_i \otimes v_{i+1} \dots v_n.$$

Then the following map is an algebra and a coalgebra isomorphism:

$$\Upsilon : \begin{cases} (T_+(V), m_{conc}, \Delta_{dec}) & \longrightarrow & (H_{RT}, \star, \Delta_{RT}) \\ v_1 \dots v_n & \longmapsto & v_1 \star \dots \star v_n. \end{cases}$$

*Proof.* Obviously,  $\Upsilon$  is an algebra morphism. By the compatibility between the product  $\star$  and the coproduct  $\Delta_{RT}$ , for any  $v_1, \dots, v_n \in \text{Prim}(H)$ :

$$\Delta_{RT}(v_1 \star \dots \star v_n) = \sum_{i=1}^n v_1 \star \dots \star v_i \otimes v_{i+1} \star \dots \star v_n.$$

Consequently,  $\theta$  is a coalgebra morphism.

The graduation of  $H_{RT}$  induces a graduation of  $\text{Prim}(H_{RT}) = V$ , which in turn gives a graduation of  $T_+(V)$ :

$$\forall n \geq 1, \quad T_+(V)_n = \bigoplus_{k=1}^n \bigoplus_{n_1 + \dots + n_k = n} V_{n_1} \otimes \dots \otimes V_{n_k}.$$

As the product  $\star$  is homogeneous of degree 0,  $\Upsilon$  is homogeneous of degree 0. Let us assume that  $\Upsilon$  is not injective, and let  $x \in \text{Ker}(\Upsilon)$ , nonzero, of minimal degree  $n$ . Then:

$$0 = \Delta_{RT} \circ \Upsilon(x) = (\Upsilon \otimes \Upsilon) \circ \Delta_{dec}(x).$$

Moreover,

$$\Delta_{dec}(x) \in \sum_{k=1}^{n-1} T_+(V)_k \otimes T_+(V)_l.$$

By definition of  $n$ ,  $\Upsilon|_{T_+(V)_k}$  is injective if  $k < n$ , so  $\Delta_{dec}(x) = 0$ , and  $x \in \text{Ker}(\Delta_{dec}) = V$ . Therefore,  $\Upsilon(x) = x = 0$ : this is a contradiction, and  $\Upsilon$  is injective.

In order to prove that  $\Upsilon$  is surjective, it is enough to prove that  $\text{Prim}(H_{RT})$  generates the algebra  $(H_{RT}, \star)$ . Let  $F$  be a planar rooted forest with  $n$  vertices, let us prove that it belongs to the subalgebra  $A$  generated by  $\text{Prim}(H)$  by induction on  $n$ . If  $n = 1$ , then  $F = \mathbb{1} \in V$  and this is obvious. Otherwise, let us put  $y = x - \Upsilon(x)$ . By definition of  $\Upsilon(x)$ , this is a linear span of forests of the form  $G = G_1 \cdot G_2$ , with  $n$  vertices. By the induction hypothesis,  $G_1, G_2 \in A$ , so  $G = G_1 \star G_2 \in A$  and finally  $y \in A$ . As  $\Upsilon(x) \in V \subseteq A$ ,  $x = y + \Upsilon(x) \in A$ .  $\square$

Let us simplify the writing of  $\theta(F)$ , in order to avoid the simplifications we observed in the examples.

**Definition 4.7.** Let us define a sequence of scalars  $(c(n))_{n \geq 0}$  by:

$$c(n) = \begin{cases} 1, & \text{if } n \equiv 0[3], \\ 0, & \text{if } n \equiv 1[3], \\ -1, & \text{if } n \equiv 2[3]. \end{cases}$$

Recall that a composition is a finite sequence of positive integers. If  $(n_1, \dots, n_k)$  is a composition, we can write it as:

$$(n_1, \dots, n_k) = (\underbrace{1, \dots, 1}_{\alpha_0}, \underbrace{1, \dots, 1}_{\alpha_1}, \dots, \underbrace{1, \dots, 1}_{\alpha_{p-1}}, \underbrace{1, \dots, 1}_{\alpha_p}, \underbrace{1, \dots, 1}_{\alpha_p}),$$

where  $p \geq 0$ ,  $b_1, \dots, b_p \geq 2$ ,  $\alpha_0, \dots, \alpha_p \geq 0$ . This is abbreviated as

$$(n_1, \dots, n_k) = 1^{\alpha_0} b_1 1^{\alpha_1} \dots 1^{\alpha_{p-1}} b_p 1^{\alpha_p}.$$

We then put:

$$c(n_1, \dots, n_k) = \begin{cases} c(\alpha_0), & \text{if } p = 0, \\ c(\alpha_0 + 2)c(\alpha_1 + 1) \dots c(\alpha_{p-1} + 1)c(\alpha_p + 2), & \text{if } p \geq 1. \end{cases}$$

**Corollary 4.8.** Let  $F$  be a planar rooted forest. We denote its vertices according to the order  $\leq_{h,l}$ :

$$v_1 \leq_{h,l} \dots \leq_{h,l} v_n.$$

Then:

$$\theta(F) = \sum_{n_1 + \dots + n_k = n} c(n_1, \dots, n_k) F_{\{v_{n_1 + \dots + n_{k-1} + 1}, \dots, v_{n_1 + \dots + n_k}\}} \cdots F_{\{v_1, \dots, v_{n_1}\}}.$$

*Proof.* Interpreting the trees  $\cdot$  in (11) as  $F_{\{v_i\}}$ , we can rewrite, for any forest  $F$  with  $n$  vertices:

$$(12) \quad \theta(F) = \sum_{n_1 + \dots + n_k = n} a(n_1, \dots, n_k) F_{\{v_{n_1 + \dots + n_{k-1} + 1}, \dots, v_{n_1 + \dots + n_k}\}} \cdots F_{\{v_1, \dots, v_{n_1}\}},$$

for a certain family of scalars  $a(n_1, \dots, n_k)$ , independent of  $F$ . Let us prove that  $a(n_1, \dots, n_k) = c(n_1, \dots, n_k)$  for any composition  $(n_1, \dots, n_k)$ .

*First case.* We consider the case  $p = 0$ , that is to say  $(n_1, \dots, n_k) = 1^n$ . The term  $F_{\{v_n\}} \dots F_{\{v_1\}}$  in (12) comes from the terms in (11) indexed by  $1 \leq i_1 < \dots < i_k \leq n$  with:

- $i_1 \leq 2$ .
- $i_k \geq n - 1$ .
- If  $2 \leq p \leq k$ , then  $i_p \leq i_{p-1} + 2$ .

Any such  $(i_1, \dots, i_k)$  contributes with  $(-1)^k$ . Note that, in  $\mathbb{Q}[[X]]$ :

$$\begin{aligned} \frac{1}{X} \sum_{l=1}^{\infty} (-1)^{l-1} (X + X^2)^l &= \sum_{l=1}^{\infty} (-1)^{l-1} \sum_{j_1, \dots, j_l \in \{1,2\}} X^{j_1 + \dots + j_l - 1} \\ &= \sum_{m=1}^{\infty} \left( \sum_{\substack{j_1, \dots, j_m \in \{1,2\}, \\ j_1 + \dots + j_m = m}} (-1)^{l-1} \right) X^{m-1} \\ &= \sum_{m=1}^{\infty} \left( \sum_{\substack{1 \leq i_1 < \dots < i_{l-1} \leq m-1, \\ i_1 \leq 2, i_{l-1} \geq m-2, \\ i_p \leq i_{p-1} + 2 \text{ if } 2 \leq p \leq m-1}} (-1)^{l-1} \right) X^{m-1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \left( \sum_{\substack{1 \leq i_1 < \dots < i_k \leq n, \\ i_1 \leq 2, i_k \geq n-1, \\ i_p \leq i_{p-1} + 2 \text{ if } 2 \leq p \leq n}} (-1)^k \right) X^n \\
&= \sum_{n=0}^{\infty} a(1^n) X^n,
\end{aligned}$$

where in the third equality,  $i_p = j_1 + \dots + j_p$  for any  $p$  and  $(k, n) = (l-1, m-1)$  for the fourth one. Therefore:

$$\sum_{n=0}^{\infty} a(1^n) X^n = \frac{1}{X} \sum_{l=1}^{\infty} (-1)^{l-1} (X + X^2)^l = -\frac{1}{X} \sum_{l=1}^{\infty} (-X - X^2)^l = \frac{1+X}{1+X+X^2} = \frac{X^2-1}{X^3-1}.$$

Hence:

$$a(1^0) = 1, \quad a(1^1) = 0, \quad a(1^2) = -1, \quad a(1^n) = a(1^{n-3}) \text{ if } n \geq 3.$$

So  $a(1^n) = c(n)$  for any  $n \geq 0$ .

*Second case.* We now assume that  $p \geq 1$ . For any  $1 \leq i_1 < \dots < i_k \leq n$  contributing in (12) to  $a(n_1, \dots, n_k)$ , necessarily:

- If  $\alpha_0 \geq 1$ ,  $\alpha_0$  belongs to  $\{i_1, \dots, i_k\}$ .
- If  $\alpha_p \geq 1$ ,  $\alpha_0 + \dots + \alpha_{p-1} + b_1 + \dots + b_p + 1$  belongs to  $\{i_1, \dots, i_k\}$ .
- If  $1 \leq i \leq p-1$  and  $\alpha_i \geq 1$ , then  $\alpha_0 + \dots + \alpha_{i-1} + b_1 + \dots + b_{i-1} + 1$  and  $\alpha_0 + \dots + \alpha_i + b_1 + \dots + b_{i-1}$  belong to  $\{i_1, \dots, i_k\}$ .

Separating the study for each block of 1 in  $(n_1, \dots, n_k)$ , we observe that  $a(n_1, \dots, n_k)$  can be written as a product

$$a(n_1, \dots, n_k) = a^{(0)}(\alpha_0) \dots a^{(p)}(\alpha_p).$$

Mimicking the study of the first case:

$$a^{(0)}(\alpha_0) = \begin{cases} 1, & \text{if } \alpha_0 = 0, \\ -1, & \text{if } \alpha_0 = 1, \\ -a(1^{\alpha_0-1}), & \text{if } \alpha_0 \geq 2. \end{cases}$$

In all cases, we obtain that  $a^{(0)}(\alpha_0) = -a(1^{\alpha_0+2}) = -c(\alpha_0 + 2)$ . Similarly,  $a^{(p)}(\alpha_p) = -c(\alpha_p + 2)$ . If  $1 \leq i \leq p-1$ :

$$a^{(i)}(\alpha_i) = \begin{cases} 0, & \text{if } \alpha_i = 0, \\ -1, & \text{if } \alpha_i = 1, \\ a(1^{\alpha_i-2}), & \text{if } \alpha_i \geq 2. \end{cases}$$

In all cases, we obtain that  $a^{(i)}(\alpha_i) = a(1^{\alpha_0+1}) = c(\alpha_0 + 1)$ . As a consequence,  $a(n_1, \dots, n_k) = c(n_1, \dots, n_k)$ .  $\square$

For any composition  $(n_1, \dots, n_k)$ ,  $c(n_1, \dots, n_k) \in \{-1, 0, 1\}$ . Let us denote by  $t_n$  the number of compositions  $(n_1, \dots, n_k)$ , with  $n_1 + \dots + n_k = n$  and  $c(n_1, \dots, n_k) \neq 0$ .

**Proposition 4.9.** *In  $\mathbb{Q}[[X]]$ :*

$$\sum_{n=0}^{\infty} t_n X^n = \frac{1-X+2X^2}{1-X-2X^3}.$$

As a consequence, for any  $n \geq 3$ ,

$$t_n = t_{n-1} + 2t_{n-3}.$$

*Proof.* By definition,  $t_n$  is the number of compositions  $(n_1, \dots, n_k) = 1^{\alpha_0} b_1 1^{\alpha_1} \dots 1^{\alpha_{p-1}} b_p 1^{\alpha_p}$  with  $n_1 + \dots + n_k = n$ , such that:

- If  $p = 0$ , then  $\alpha_i \equiv 0[3]$  or  $\alpha_i \equiv 2[3]$ .
- if  $p \geq 1$ :
  - $\alpha_0 \equiv 0[3]$  or  $\alpha_0 \equiv 1[3]$ .
  - $\alpha_p \equiv 0[3]$  or  $\alpha_p \equiv 1[3]$ .
  - If  $1 \leq i \leq p - 1$ ,  $\alpha_i \equiv 1[3]$  or  $\alpha_i \equiv 2[3]$ .

We shall use the following formal series:

$$P_0(X) = \sum_{k=0}^{\infty} X^{3k} + \sum_{k=0}^{\infty} X^{3k+2} = \frac{1 + X^2}{1 - X^3},$$

$$P_2(X) = \sum_{k=0}^{\infty} X^{3k} + \sum_{k=0}^{\infty} X^{3k+1} = \frac{1 + X}{1 - X^3},$$

$$P_1(X) = \sum_{k=0}^{\infty} X^{3k+1} + \sum_{k=0}^{\infty} X^{3k+2} = X P_2(X).$$

Then:

$$\begin{aligned} \sum_{n=0}^{\infty} t_n X^n &= P_0(X) + \sum_{k=1}^{\infty} P_2(X) \left( \frac{X^2}{1 - X} P_1(X) \right)^{k-1} \frac{X^2}{1 - X} P_2(X) \\ &= P_0(X) + \sum_{k=1}^{\infty} P_2(X)^{k+1} X^{k-1} \left( \frac{X^2}{1 - X} \right)^k \\ &= P_0(X) + \frac{P_2(X)}{X} \sum_{k=1}^{\infty} \left( \frac{X^3 P_2(X)}{1 - X} \right)^k \\ &= P_0(X) + \frac{P_2(X)}{X} \frac{\frac{X^3 P_2(X)}{1 - X}}{1 - \frac{X^3 P_2(X)}{1 - X}} \\ &= \frac{1 - X + 2X^2}{1 - X - 2X^3}. \end{aligned}$$

□

Here are the first values of  $t_n$ :

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$t_n$	1	0	2	4	4	8	16	24	40	72	120	200	344	584	984	1672

**Example 4.10.** Let us consider the case  $n = 5$ . There are 8 contributing terms in (12), corresponding to the compositions:

$$(5), \quad (4, 1), \quad (1, 4), \quad (1, 3, 1), \quad (2, 1, 2), \quad (2, 1, 1, 1), \quad (1, 1, 1, 2), \quad (1, 1, 1, 1, 1).$$

Therefore, if  $F$  has 5 vertices:

$$\theta(F) = F - \bullet F_{\{|1,2,3,4\}} - F_{\{|2,3,4,5\}} \bullet + \bullet F_{\{|2,3,4\}} \bullet - F_{\{|4,5\}} \bullet F_{\{|1,2\}} + \dots F_{\{|1,2\}} + F_{\{|4,5\}} \bullet \dots - \dots$$

4.3. **Dual coproduct of  $\star$ .** The product  $\star$  on  $H_{RT}$  induces by duality a coproduct  $\blacktriangle$  on  $H_{RT}^*$ .

**Lemma 4.11.** Let  $T_1, \dots, T_k \in \mathcal{T}$ . Then, in  $H_{RT}^*$ :

$$(13) \quad \blacktriangle(Z_{T_1 \dots T_k}) = \sum_{i=1}^k \delta_{T_i, \bullet} Z_{T_1 \dots T_{i-1}} \otimes Z_{T_{i+1} \dots T_k}.$$

*Proof.* Let  $F, G \in M(\mathcal{T})$ . Then:

$$\begin{aligned} \blacktriangle(Z_{T_1 \dots T_k})(F \otimes G) &= Z_{T_1 \dots T_k}(F \star G) \\ &= Z_{T_1 \dots T_k}(F \cdot G) \\ &= \delta_{T_1 \dots T_k, F \cdot G} \\ &= \sum_{i=1}^k \delta_{T_1 \dots T_{i-1}, F} \delta_{T_i, \bullet} \delta_{T_{i+1} \dots T_k, G} \\ &= \left( \sum_{i=1}^k \delta_{T_i, \bullet} Z_{T_1 \dots T_{i-1}} \otimes Z_{T_{i+1} \dots T_k} \right) (F \otimes G), \end{aligned}$$

which implies (13). □

Dualizing Proposition 4.1:

**Proposition 4.12.** For any  $f, g \in H_{RT}^*$ :

$$\blacktriangle(f \diamond g) = f \diamond \blacktriangle(g) + \blacktriangle(f) \diamond g + f \otimes g.$$

*Proof.* Let  $f, g \in H_{RT}^*$ . For any  $x, y \in H_{RT}$ :

$$\begin{aligned} \blacktriangle(f \diamond g)(x \otimes y) &= (f \diamond g)(x \star y) \\ &= (f \otimes g) \Delta_{RT}(x \star y) \\ &= (f \otimes g)(x \star \Delta_{RT}(y) + \Delta_{RT}(x) \star y + x \otimes y) \\ &= (\blacktriangle(f) \diamond g + f \diamond \blacktriangle(g) + f \otimes g)(x \otimes y), \end{aligned}$$

which implies the result. □

Let us then dualize Theorem 4.2:

**Theorem 4.13.** The transpose of  $\theta$  is the map  $\theta^*$  given by:

$$\forall f \in H_{RT}^*, \quad \theta^*(f) := \sum_{k=1}^{\infty} (-1)^{k+1} \diamond^{(k-1)} \circ \blacktriangle^{(k-1)}(f),$$

where  $\blacktriangle^{(l)} : H_{RT}^* \longrightarrow (H_{RT}^*)^{\otimes(l+1)}$  and  $\diamond^{(l)} : (H_{RT}^*)^{\otimes(l+1)} \longrightarrow H_{RT}^*$  are inductively defined:

$$\begin{aligned} \blacktriangle^{(0)} &= \text{id}_{H_{RT}^*}, & \diamond^{(0)} &= \text{id}_{H_{RT}^*}, \\ \blacktriangle^{(l+1)} &= (\blacktriangle^{(l)} \otimes \text{id}_{H_{RT}^*}) \circ \blacktriangle_{RT}, & \diamond^{(l+1)} &= \diamond \circ (\diamond^{(l)} \otimes \text{id}_{H_{RT}^*}). \end{aligned}$$

Then  $\theta^*$  is a projector on  $\text{Ker}(\blacktriangle) = \mathbf{k}\{Z_F, \text{no tree of } F \text{ is equal to } \bullet\}$ . The kernel of  $\theta^*$  is

$$\text{Ker}(\theta^*) = H_{RT}^* \diamond H_{RT}^*.$$

*Proof.* The description of  $\theta^*$  comes from  $\diamond = \Delta_{RT}^*$  and  $\blacktriangle = \star^*$ . As  $\theta$  is the projection on  $Ker(\Delta_{RT})$  which vanishes on  $Im(\star)$ ,  $\theta^*$  is the projection on  $Im(\star)^\perp$  which vanishes on  $Ker(\Delta_{RT})^\perp$ , and:

$$\begin{aligned} Im(\star)^\perp &= Ker(\star^*) = Ker(\blacktriangle), \\ Ker(\Delta_{RT})^\perp &= Im(\Delta_{RT}^*) = Im(\diamond) = H_{RT}^* \diamond H_{RT}^*. \end{aligned}$$

The description of  $Ker(\blacktriangle)$  is immediate. □

Consequently,

$$H_{RT}^* = Ker(\blacktriangle) \oplus H_{RT}^* \diamond H_{RT}^*.$$

As  $(H_{RT}^*, \diamond)$  is a free non unitary algebra, any complement of  $H_{RT}^* \diamond H_{RT}^*$  freely generates it as an algebra. Hence:

**Corollary 4.14.** The algebra  $(H_{RT}^*, \diamond)$  is freely generated by the elements  $Z_F$ , where  $F$  is a forest with no tree equal to  $\cdot$ .

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