



**HAL**  
open science

# THE DUAL OF INFINITESIMAL UNITARY HOPF ALGEBRAS AND PLANAR ROOTED FORESTS

Xiaomeng Wang, Loïc Foissy, Xing Gao

► **To cite this version:**

Xiaomeng Wang, Loïc Foissy, Xing Gao. THE DUAL OF INFINITESIMAL UNITARY HOPF ALGEBRAS AND PLANAR ROOTED FORESTS. International Electronic Journal of Algebra, 2023, 10.24330/ieja.1220707 . hal-02874051

**HAL Id: hal-02874051**

**<https://hal.science/hal-02874051>**

Submitted on 18 Jun 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# THE DUAL OF INFINITESIMAL UNITARY HOPF ALGEBRAS AND PLANAR ROOTED FORESTS

XIAOMENG WANG, LOÏC FOISSY, AND XING GAO

ABSTRACT. We study the infinitesimal (in the sense of Joni and Rota) bialgebra  $H_{RT}$  of planar rooted trees introduced by Gao and Wang, whose coproduct is given by deletion of a vertex. We prove that its dual  $H_{RT}^*$  is isomorphic to a free non unitary algebra, and give two free generating sets. Giving  $H_{RT}$  a second product, we make it an infinitesimal bialgebra in the sense of Loday and Ronco, which allows to explicitly construct a projector on its space primitive elements, which freely generates  $H_{RT}$ .

## CONTENTS

1.	Introduction	1
2.	The infinitesimal unitary Hopf algebras of planar rooted forests	3
2.1.	Planar rooted forests	3
2.2.	Infinitesimal unitary Hopf algebras of planar rooted forests	3
3.	The dual of infinitesimal unitary Hopf algebra on planar rooted forests	5
4.	Primitive elements of $H_{RT}$	13
4.1.	A second product on $H_{RT}$	13
4.2.	A projection on primitive elements	14
4.3.	Dual coproduct of $\star$	20
	References	21

## 1. INTRODUCTION

Rooted trees and planar rooted trees have a very rich algebraic structure: Grossman and Larson [10] first gave rooted trees a structure of noncommutative and cocommutative algebraic structure, closely related to the Butcher group of Runge-Kutta methods [2]; then, in order to algebraically treat the operation of Renormalization in Quantum Field Theory, Connes and Kreimer introduced a commutative, noncocommutative Hopf algebra of rooted trees [3], and it was proved that the Connes-Kreimer and the Grossman-Larson Hopf algebra are in duality [12, 16]. A self-dual noncommutative version of the Connes-Kreimer Hopf algebra was simultaneously introduced by Foissy and Holtkamp [5, 13], and this object was deformed as an infinitesimal bialgebra in the sense of Loday and Ronco [15] in [7].

Recently, Gao and Wang introduced another infinitesimal coproduct  $\Delta_{RT}$  on planar rooted trees, where the usual 1-cocycle compatibility between the operator  $B^+$  (see paragraph 2.1 below) and the coproduct is modified: in the Foissy-Holtkamp case, this is the 1-cocycle condition

$$\Delta_{\mathcal{T}} \circ B^+(x) = B^+(x) \otimes \mathbb{1} + (\text{id} \otimes B^+) \circ \Delta_{\mathcal{T}}(x),$$

---

*Date:* June 18, 2020.

*2010 Mathematics Subject Classification.* 16T05, 16T10, 16T30, 05C05.



a coassociative coalgebra (possibly without counit), unless otherwise stated. Linear maps and tensor products are taken over  $\mathbf{k}$ . For any algebra  $A$ , we view  $A \otimes A$  as an  $A$ -bimodule via

$$(1) \quad a \cdot (b \otimes c) := ab \otimes c \text{ and } (b \otimes c) \cdot a := b \otimes ca.$$

## 2. THE INFINITESIMAL UNITARY HOPF ALGEBRAS OF PLANAR ROOTED FORESTS

In this section, we first recall some basic notations used throughout the paper.

**2.1. Planar rooted forests.** We expose some concepts and notations on planar rooted forests from [11, 17]. Let  $\mathcal{T}$  denote the set of planar rooted trees and  $M(\mathcal{T})$  the free monoid generated by  $\mathcal{T}$  in which the multiplication is the concatenation, denoted by  $m_{RT}$  and usually suppressed. Thus an element  $F$  in  $M(\mathcal{T})$ , called a **planar rooted forest**, is a noncommutative product of planar rooted trees in  $\mathcal{T}$ . The empty tree  $\mathbb{1}$  is the unity of  $M(\mathcal{T})$ .

Here are some examples of elements of  $\mathcal{T}$  where the root is on the bottom:

$$., \quad \downarrow, \quad \vee, \quad \downarrow, \quad \Psi, \quad \downarrow \vee, \quad \vee \downarrow, \quad \Upsilon.$$

Here are some examples of elements of  $M(\mathcal{T})$ :

$$\mathbb{1}, \quad .., \quad \downarrow \dots, \quad .\downarrow, \quad \vee \downarrow, \quad \dots.$$

Let  $H_{RT} := \mathbf{k}M(\mathcal{T})$  be the free  $\mathbf{k}$ -module spanned by  $M(\mathcal{T})$ . Denote by

$$B^+ : H_{RT} \rightarrow H_{RT}$$

the grafting map sending  $\mathbb{1}$  to  $.$  and sending a planar rooted forest in  $H_{RT}$  to its grafting on a new root, and by  $m_{RT}$  the concatenation on  $H_{RT}$ . Then  $H_{RT}$  is closed under the concatenation  $m_{RT}$  [17]. Here are some examples about  $B^+$  on  $H_{RT}$ :

$$B^+(\mathbb{1}) = ., \quad B^+(\downarrow) = \downarrow, \quad B^+(\downarrow \vee) = \downarrow \vee.$$

For  $F = T_1 \cdots T_m \in M(\mathcal{T})$  with  $T_1, \dots, T_m \in \mathcal{T}$ , we define  $\text{bre}(F) := m$  to be the **breadth** of  $F$ . Here we use the convention that  $\text{bre}(\mathbb{1}) = 0$  when  $m = 0$ . The **depth**  $\text{dep}(T)$  of a rooted tree is the maximal length of linear chains from the root to the leaves of the tree. For  $F = T_1 \cdots T_m \in M(\mathcal{T})$  with  $m \geq 0$ , we define

$$\text{dep}(F) := \max\{\text{dep}(T_i) \mid i = 1, \dots, m\}.$$

**2.2. Infinitesimal unitary Hopf algebras of planar rooted forests.** In order to provide an algebraic framework for the calculus of divided differences, Joni and Rota [14] introduced the concept of an infinitesimal bialgebra.

**Definition 2.1.** [14] An **infinitesimal bialgebra** is a triple  $(A, m, \Delta)$  where  $(A, m)$  is an associative algebra,  $(A, \Delta)$  is a coassociative coalgebra and for each  $a, b \in A$ ,

$$(2) \quad \Delta(ab) = a \cdot \Delta(b) + \Delta(a) \cdot b = \sum_{(b)} ab_{(1)} \otimes b_{(2)} + \sum_{(a)} a_{(1)} \otimes a_{(2)}b.$$

Note that we do not require that  $(A, m)$  is unitary, nor that  $(A, \Delta)$  is counitary. The concept of an infinitesimal Hopf algebra was introduced by Aguiar in order to develop and study infinitesimal bialgebras [1]. If  $A$  is an infinitesimal bialgebra, then the space  $\text{Hom}_{\mathbf{k}}(A, A)$  is still an algebra under convolution:

$$f * g := m(f \otimes g) \Delta,$$

but possibly without unity with respect to the convolution  $*$  [1]. Therefore, it is impossible to consider antipode. To solve this difficulty, Aguiar equipped the space  $\text{Hom}_{\mathbf{k}}(A, A)$  with circular convolution  $\otimes$  given by

$$f \otimes g := f * g + f + g, \text{ that is, } (f \otimes g)(a) := \sum_{(a)} f(a_{(1)})g(a_{(2)}) + f(a) + g(a) \text{ for } a \in A.$$

Note that  $f \otimes 0 = f = 0 \otimes f$  and so  $0 \in \text{Hom}_{\mathbf{k}}(A, A)$  is the unity with respect to the circular convolution  $\otimes$ .

With the help of the circular convolution, one can describe infinitesimal Hopf algebras.

**Definition 2.2.** [1] An infinitesimal bialgebra  $(A, m, \Delta)$  is called an **infinitesimal Hopf algebra** if the identity map  $\text{id} \in \text{Hom}_{\mathbf{k}}(A, A)$  is invertible with respect to the circular convolution. In this case, its inverse  $S \in \text{Hom}_{\mathbf{k}}(A, A)$  is called the **antipode** of  $A$ . It is characterized by the equations

$$(3) \quad \sum_{(a)} S(a_{(1)})a_{(2)} + S(a) + a = 0 = \sum_{(a)} a_{(1)}S(a_{(2)}) + S(a) + a \text{ for } a \in A,$$

where  $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ .

Now we recall the infinitesimal Hopf algebraic structure on top of planar rooted forests defined in [9]. The coproduct  $\Delta_{RT}$  on  $H_{RT}$  is defined recursively on depth. Let  $F$  be a forest in  $H_{RT}$ . For the initial step of  $\text{dep}(F) = 0$ , we define

$$(4) \quad \Delta_{RT}(F) := \Delta_{RT}(\mathbb{1}) = 0.$$

For the induction step of  $\text{dep}(F) \geq 1$ , we reduce to the induction on  $\text{bre}(F) \geq 1$ . If  $\text{bre}(F) = 1$ , then  $F = B^+(\bar{F})$  for some  $\bar{F} \in M(\mathcal{T})$  and define

$$(5) \quad \Delta_{RT}(F) := \Delta_{RT}B^+(\bar{F}) := \bar{F} \otimes \mathbb{1} + (\text{id} \otimes B^+)\Delta_{RT}(\bar{F}),$$

that is,  $\Delta_{RT}B^+ = \text{id} \otimes \mathbb{1} + (\text{id} \otimes B^+)\Delta_{RT}$ . Here the coproduct  $\Delta_{RT}(\bar{F})$  is defined by the induction hypothesis on depth. If  $\text{bre}(F) \geq 2$ , then  $F = T_1T_2 \cdots T_m$  with  $\text{bre}(F) = m \geq 2$  and define

$$(6) \quad \Delta_{RT}(F) := T_1 \cdot \Delta_{RT}(T_2 \cdots T_m) + \Delta_{RT}(T_1) \cdot (T_2 \cdots T_m).$$

**Remark 2.3.** Foissy [7] also studied another kind of infinitesimal Hopf algebras on planar rooted forests, using a different coproduct  $\Delta_{\mathcal{T}}$  given by

$$\Delta_{\mathcal{T}}(F) := \begin{cases} \mathbb{1} \otimes \mathbb{1}, & \text{if } F = \mathbb{1}, \\ F \otimes \mathbb{1} + (\text{id} \otimes B^+)\Delta_{\mathcal{T}}(\bar{F}), & \text{if } F = B^+(\bar{F}), \\ F_1 \cdot \Delta_{\mathcal{T}}(F_2) + \Delta_{\mathcal{T}}(F_1) \cdot F_2 - F_1 \otimes F_2, & \text{if } F = F_1F_2. \end{cases}$$

We give some examples to expose the differences between these two coproducts  $\Delta_{RT}$  and  $\Delta_{\mathcal{T}}$ . On the one hand,

$$\begin{aligned} \Delta_{RT}(\cdot) &= \mathbb{1} \otimes \mathbb{1}; \\ \Delta_{RT}(\uparrow) &= \cdot \otimes \mathbb{1} + \mathbb{1} \otimes \cdot; \\ \Delta_{RT}(\vee) &= \cdot \otimes \mathbb{1} + \cdot \otimes \cdot + \mathbb{1} \otimes \uparrow; \\ \Delta_{RT}\left(\begin{array}{c} \uparrow \\ \vee \end{array}\right) &= \uparrow \cdot \otimes \mathbb{1} + \uparrow \otimes \cdot + \cdot \otimes \uparrow + \mathbb{1} \otimes \vee. \end{aligned}$$

On the other hand,

$$\Delta_{\mathcal{T}}(\cdot) = \cdot \otimes \mathbb{1} + \mathbb{1} \otimes \cdot;$$

$$\begin{aligned}\Delta_{\mathcal{T}}(\mathfrak{!}) &= \mathfrak{!} \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{!} + \cdot \otimes \cdot; \\ \Delta_{\mathcal{T}}(\mathfrak{V}) &= \mathfrak{V} \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{V} + \cdot \otimes \cdot + \cdot \otimes \mathfrak{!}; \\ \Delta_{\mathcal{T}}\left(\begin{array}{c} \mathfrak{!} \\ \mathfrak{V} \end{array}\right) &= \begin{array}{c} \mathfrak{!} \\ \mathfrak{V} \end{array} \otimes \mathbb{1} + \mathbb{1} \otimes \begin{array}{c} \mathfrak{!} \\ \mathfrak{V} \end{array} + \mathfrak{!} \otimes \cdot + \cdot \otimes \mathfrak{V} + \mathfrak{!} \otimes \mathfrak{!}.\end{aligned}$$

Let us recall the combinatorial description of  $\Delta_{RT}$  given in [9], in terms of an order on the set  $V(F)$  of vertices of a forest  $F$  [5, 7].

**Definition 2.4.** Let  $F = T_1 \cdots T_m \in M(\mathcal{T})$  with  $T_1, \dots, T_m \in \mathcal{T}$  and  $m \geq 1$ , and let  $u, v \in V(F)$  be two vertices. Then

- (a)  $u \leq_h v$  (**being higher**) if there exists a (directed) path from  $u$  to  $v$  in  $F$ , the edges of  $F$  being oriented from roots to leaves;
- (b)  $u \leq_l v$  (**being more on the left**) if  $u$  and  $v$  are not comparable for  $\leq_h$  and one of the following assertions is satisfied:
  - (i)  $u$  is a vertex of  $T_i$  and  $v$  is a vertex of  $T_j$  with  $1 \leq j < i \leq m$ .
  - (ii)  $u$  and  $v$  are vertices of the same  $T_i$ , and  $u \leq_l v$  in the forest obtained from  $T_i$  by deleting its root;
- (c)  $u \leq_{h,l} v$  (**being higher or more on the left**) if  $u \leq_h v$  or  $u \leq_l v$ .

As usual, we denote  $u <_{h,l} v$  (resp.  $u <_l v$ ,  $u <_h v$ ) if  $u \leq_{h,l} v$  (resp.  $u \leq_l v$ ,  $u \leq_h v$ ) but  $u \neq v$ . The **induced subgraph** in  $G$  by  $V$  is the graph whose vertex set is  $V$  and whose edge set consists of all of the edges in  $G$  that have both endpoints in  $V$  [4].

Let  $F \in M(\mathcal{T})$  be a planar rooted forest. For each vertex  $v \in V(F)$ , denote by  $B_v$  the induced subgraph in  $F$  by the set  $\{u \in V(F) \mid v <_{h,l} u\}$ , and by  $R_v$  the induced subgraph in  $F$  by the set  $V(F) \setminus (V(B_v) \cup \{v\})$ . Equivalently,  $R_v$  is the induced subgraph in  $F$  by the set  $\{u \in V(F) \mid u <_{h,l} v\}$ . Note that both  $B_v$  and  $R_v$  are planar rooted forests in  $M(\mathcal{T})$ , not containing the vertex  $v$ . Then [9]

$$(7) \quad \Delta_{RT}(F) = \sum_{v \in V(F)} B_v \otimes R_v \text{ for } F \in M(\mathcal{T}).$$

**Lemma 2.5.** [9]

- (a) The quadruple  $(H_{RT}, m_{RT}, \mathbb{1}, \Delta_{RT})$  is an infinitesimal unitary bialgebra.
- (b) The quadruple  $(H_{RT}, m_{RT}, \mathbb{1}, \Delta_{RT})$  is an infinitesimal unitary Hopf algebra.

### 3. THE DUAL OF INFINITESIMAL UNITARY HOPF ALGEBRA ON PLANAR ROOTED FORESTS

In this section, we show that the dual  $H_{RT}^* = (H_{RT}^*, \Delta_{RT}^*, m_{RT}^*, \mathbb{1}^*)$  of  $H_{RT}$  is a free algebra. Let us first recall some fundamental facts.

**Lemma 3.1.** [8] Let  $V = \bigoplus_{n=1}^{\infty} V^{(n)}$  be a graded vector space and the homogeneous components of  $V$  be finite dimensional. Then

- (a) The graded dual  $V^* := \bigoplus_{n=1}^{\infty} (V^{(n)})^*$  is also a graded space, and  $V^{**} \simeq V$ .
- (b)  $V \otimes V$  is also a graded vector space with  $(V \otimes V)^{(n)} = \sum_{i=0}^n V^{(i)} \otimes V^{(n-i)}$  for all  $n \in \mathbb{N}$ . Moreover,  $(V \otimes V)^* \simeq V^* \otimes V^*$ .

The  $H_{RT}$  can be graded by number of vertices. Denote by

$$H_{RT}(n) := \mathbf{k} \left\{ F \in M(\mathcal{T}) \mid |F| = n - 1 \right\} \quad \text{for } n \geq 1,$$

where  $|F|$  is the number of vertices of  $F$ . Then

$$H_{RT} = \bigoplus_{n=1}^{\infty} H_{RT}(n) \quad \text{and} \quad H_{RT}^* = \bigoplus_{n=1}^{\infty} (H_{RT}(n))^*.$$

We now give a combinatorial description of the dual of the coproduct  $\Delta_{RT}$ . Let us propose the following concepts as a preparation.

**Definition 3.2.** Let  $T$  be a planar rooted tree. The **left path**  $\text{LP}(T)$  of  $T$  is defined to be the path from the root to the left most leaf of  $T$ .

**Example 3.3.** The following paths in green are left paths of planar rooted trees, respectively.

$$\mathbb{V}, \quad \mathbb{I}, \quad \mathbb{V}, \quad \mathbb{V}, \quad \mathbb{Y}.$$

**Definition 3.4.** Let  $T$  and  $T'$  be two planar rooted trees. A **left grafting** of  $T'$  over  $T$  is a planar rooted tree obtained by grafting  $T'$  in a vertex  $v$  of the left path  $\text{LP}(T)$  by connecting  $v$  and the root of  $T'$ , such that  $T'$  is on the left of  $T$ . Denote by  $\mathcal{L}(T', T)$  the set of all left graftings of  $T'$  over  $T$ .

**Example 3.5.** Consider  $T' = \cdot$  and  $T = \mathbb{I}$ . Then  $\mathbb{V}$  and  $\mathbb{I}$  are the two left graftings of  $T'$  over  $T$ .

In general, we propose

**Definition 3.6.** Let  $F = T_1 \cdots T_m$  be a planar rooted forest with  $T_1, \dots, T_m \in \mathcal{T}$  and  $T$  a planar rooted tree. A **left grafting** of  $F$  over  $T$  is a planar rooted tree obtained by left grafting each  $T_i$  in a vertex  $v_i$  of  $\text{LP}(T)$  such that  $v_i \leq_{\text{h,l}} v_j$  when  $i < j$ . Denoted by  $\mathcal{L}(F, T)$  the set of all left grafting of  $F$  over  $T$ .

Notice that the  $T_i$  and  $T_j$  may be grafted in the same vertex.

**Example 3.7.** Let  $F = \dots$  and  $T = \mathbb{I}$ . Then

$$\mathcal{L}(F, T) = \left\{ \mathbb{V}, \mathbb{V}, \mathbb{Y} \right\}.$$

**Definition 3.8.** Let  $F$  be a planar rooted forest and  $T$  a planar rooted tree.

Step 1: Decompose  $F = F_1 F_2$  and  $F_1 B^+(F_2) = F'_1 F'_2$ , where  $F_1, F_2, F'_1, F'_2 \in M(\mathcal{T})$ .

Step 2: Left graft  $F'_2$  over  $T$  to obtain an  $\tilde{F} \in \mathcal{L}(F'_2, T)$ , and concatenate  $F'_1$  and  $\tilde{F}$  to get  $F'_1 \tilde{F}$ , and call the concatenation  $F'_1 \tilde{F}$  a **grafting** of  $F$  over  $T$ .

Denote by  $\mathcal{G}(F, T)$  the set of all graftings of  $F$  over  $T$ .

Let us compute explicitly an example for better understanding of Definition 3.8.

**Example 3.9.** Let  $F = \dots$  and  $T = \mathbb{I}$ . Then the decomposition  $F = F_1 F_2$  as concatenation product can be

$$F = \mathbb{1}(\dots) = (\cdot)(\cdot) = (\dots)\mathbb{1}.$$

**Case 1.**  $F_1 = \mathbb{1}$  and  $F_2 = \dots$ . Then  $B^+(F_2) = \mathbb{V}$  and  $F_1 B^+(F_2) = \mathbb{V}$ . The decomposition  $F_1 B^+(F_2) = F'_1 F'_2$  can be

$$F_1 B^+(F_2) = (\mathbb{V})\mathbb{1} = \mathbb{1}(\mathbb{V}).$$

We have two subcases.

**Subcase 1.1.**  $F'_1 = \mathbb{V}$  and  $F'_2 = \mathbb{1}$ . Then  $\mathbb{V} \downarrow$  is the only one grafting of  $F$  over  $T$  in this subcase.

**Subcase 1.2.**  $F'_1 = \mathbb{1}$  and  $F'_2 = \mathbb{V}$ . Then

$$\mathcal{L}(F'_2, T) = \mathcal{L}(\mathbb{V}, \downarrow) = \left\{ \begin{array}{c} \mathbb{V} \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \mathbb{V} \\ \downarrow \\ \downarrow \end{array} \right\}$$

and  $\begin{array}{c} \mathbb{V} \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \mathbb{V} \\ \downarrow \\ \downarrow \end{array}$  are two graftings of  $F$  over  $T$  in this subcase.

**Case 2.**  $F_1 = \cdot$  and  $F_2 = \cdot\cdot$ . Then  $B^+(F_2) = \downarrow$  and  $F_1 B^+(F_2) = \cdot\downarrow$ . The decomposition  $F_1 B^+(F_2) = F'_1 F'_2$  can be

$$F_1 B^+(F_2) = (\cdot\downarrow)\mathbb{1} = (\cdot)(\downarrow) = \mathbb{1}(\cdot\downarrow).$$

We have the following three subcases.

**Subcase 2.1.**  $F'_1 = \cdot\downarrow$  and  $F'_2 = \mathbb{1}$ . Then  $\cdot\downarrow \downarrow$  is the only one grafting of  $F$  over  $T$  in this subcase.

**Subcase 2.2.**  $F'_1 = \cdot$  and  $F'_2 = \downarrow$ . Then

$$\mathcal{L}(F'_2, T) = \mathcal{L}(\downarrow, \downarrow) = \left\{ \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right\}$$

and  $\begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$  are two graftings of  $F$  over  $T$  in this subcase.

**Subcase 2.3.**  $F'_1 = \mathbb{1}$  and  $F'_2 = \cdot\downarrow$ . Then

$$\mathcal{L}(F'_2, T) = \mathcal{L}(\cdot\downarrow, \downarrow) = \left\{ \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right\}$$

and  $\begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$  are three graftings of  $F$  over  $T$  in this subcase.

**Case 3.**  $F_1 = \cdot\cdot$  and  $F_2 = \mathbb{1}$ . Then  $B^+(F_2) = \cdot$  and  $F_1 B^+(F_2) = \cdot\cdot\cdot$ . The decomposition  $F_1 B^+(F_2) = F'_1 F'_2$  can be

$$F_1 B^+(F_2) = (\cdot\cdot\cdot)\mathbb{1} = (\cdot\cdot)(\cdot) = (\cdot)(\cdot\cdot) = \mathbb{1}(\cdot\cdot\cdot).$$

There are four subcases.

**Subcase 3.1.**  $F'_1 = \cdot\cdot\cdot$  and  $F'_2 = \mathbb{1}$ . Thus  $\cdot\cdot\cdot \downarrow$  is the only one grafting of  $F$  over  $T$  in this subcase.

**Subcase 3.2.**  $F'_1 = \cdot\cdot$  and  $F'_2 = \cdot$ . Then

$$\mathcal{L}(F'_2, T) = \mathcal{L}(\cdot, \downarrow) = \left\{ \mathbb{V}, \downarrow \right\}$$

and  $\mathbb{V}, \downarrow$  are two graftings of  $F$  over  $T$  in this subcase.

**Subcase 3.3.**  $F'_1 = \cdot$  and  $F'_2 = \cdot\cdot\cdot$ . We have

$$\mathcal{L}(F'_2, T) = \mathcal{L}(\cdot\cdot\cdot, \downarrow) = \left\{ \begin{array}{c} \mathbb{V} \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right\}.$$

Then  $\begin{array}{c} \mathbb{V} \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array}$  are three graftings of  $F$  over  $T$  in this subcase.

**Subcase 3.4.**  $F'_1 = \mathbb{1}$  and  $F'_2 = \cdot\cdot\cdot$ . Then

$$\mathcal{L}(F'_2, T) = \mathcal{L}(\cdot\cdot\cdot, \downarrow) = \left\{ \begin{array}{c} \mathbb{V} \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \mathbb{V} \end{array}, \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right\}$$

and  are four graftings of  $F$  over  $T$  in this subcase.

In summary, there are nineteen graftings of  $F$  over  $T$ :

$$\mathcal{G}(\dots, \uparrow) = \left\{ \begin{array}{l} \text{[Diagram 1]}, \text{ [Diagram 2]}, \text{ [Diagram 3]}, \text{ [Diagram 4]}, \text{ [Diagram 5]}, \text{ [Diagram 6]}, \text{ [Diagram 7]}, \text{ [Diagram 8]}, \text{ [Diagram 9]}, \text{ [Diagram 10]}, \text{ [Diagram 11]}, \text{ [Diagram 12]}, \text{ [Diagram 13]}, \text{ [Diagram 14]}, \text{ [Diagram 15]}, \text{ [Diagram 16]}, \text{ [Diagram 17]}, \text{ [Diagram 18]}, \text{ [Diagram 19]} \end{array} \right\}.$$

**Proposition 3.10.** *Let  $F$  be a planar rooted forest and  $T$  be a planar rooted tree. We put  $\text{bre}(F) := k$  and we denote by  $l$  the cardinality of  $LP(T)$ . Then:*

$$a_{k,l} := |\mathcal{G}(F, T)| = \binom{k+l+2}{l+1} - 1.$$

Moreover, in  $\mathbb{Q}[[X, Y]]$ :

$$\sum_{k,l=0}^{\infty} a_{k,l} X^k Y^l = \frac{1}{(1-X)(1-Y)(1-X-Y)}.$$

*Proof.* As  $\text{bre}(F) = k$ , if we put  $F = T_1 \dots T_k$ , with  $T_1, \dots, T_k$  planar rooted trees, there are exactly  $k+1$  possibilities for writing  $F = F_1 F_2$ , which are  $(F_1^{(i)}, F_2^{(i)}) = (T_1 \dots T_{i-1}, T_i \dots T_k)$ , with  $1 \leq i \leq k+1$ . Note that  $G^{(i)} = F_1^{(i)} B^+(F_2^{(i)})$  is a forest of breadth  $i$ . Therefore, there are exactly  $\binom{i+l}{i}$  ways to write it as  $G^{(i)} = G_0 \dots G_l$ , so there are exactly  $\binom{i+l}{l}$  ways to proceed in the second step of Definition 3.8. Hence:

$$|\mathcal{G}(F, T)| = \sum_{i=1}^{k+1} \binom{i+l}{l} = \sum_{i=0}^{k+1} \binom{i+l}{l} - 1 = \sum_{j=l}^{k+l+1} \binom{j}{l} - 1 = \binom{k+l+2}{l+1} - 1.$$

Moreover:

$$\begin{aligned} \sum_{k,l=0}^{\infty} a_{k,l} X^k Y^l &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{k+l+2}{l+1} X^k Y^l - \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} X^k Y^l \\ &= \sum_{k=0}^{\infty} \frac{X^k}{Y} \sum_{l=1}^{\infty} \binom{k+l+1}{l} Y^l - \frac{1}{(1-X)(1-Y)} \\ &= \sum_{k=0}^{\infty} \frac{X^k}{Y} \left( \frac{1}{(1-Y)^{k+2}} - 1 \right) - \frac{1}{(1-X)(1-Y)} \\ &= \frac{1}{Y(1-Y)^2} \sum_{k=0}^{\infty} \left( \frac{X}{1-Y} \right)^k - \frac{1}{Y} \sum_{k=0}^{\infty} X^k - \frac{1}{(1-X)(1-Y)} \\ &= \frac{1}{Y(1-Y)^2} \frac{1}{1 - \frac{X}{1-Y}} - \frac{1}{Y(1-X)} - \frac{1}{(1-X)(1-Y)} \\ &= \frac{1}{(1-X)(1-Y)(1-X-Y)}. \end{aligned}$$

□

Note that for any  $k, l \geq 1$ ,  $a_{k,l} = a_{l,k}$ .

In general, we propose

**Definition 3.11.** Let  $F$  and  $F' = TF_1$  be two planar rooted forests with  $T \in \mathcal{T}$ . We call  $\tilde{F}F_1$  a **grafting** of  $F$  over  $F'$ , where  $\tilde{F}$  is a grafting of  $F$  over  $T$  given in Definition 3.8. Let  $\mathcal{G}(F, F')$  be the set of all graftings of  $F$  over  $F'$ .

For  $F, F', F'' \in M(\mathcal{T})$ , denote by  $n'(F, F'; F'')$  the number of ways of grafting of  $F$  over  $F'$  to obtain  $F''$ . For example,

$$n'(\dots, \uparrow; \dots) = 0, \quad n'(\dots, \uparrow; \dots) = 0, \quad n'(\dots, \uparrow; \vee \uparrow) = 1, \quad \text{and} \quad n'(\dots, \uparrow; \curvearrowright) = 1.$$

For each  $F \in M(\mathcal{T})$ , we define

$$(8) \quad Z_F : \begin{cases} H_{RT} & \longrightarrow \mathbf{k}, \\ F' & \longmapsto \delta_{F,F'} \text{ for } F' \in M(\mathcal{T}), \end{cases}$$

where  $\delta_{F,F'}$  is the Kronecker function. Then  $\{Z_F \mid F \in M(\mathcal{T})\}$  is a basis of  $H_{RT}^*$ . We denote by  $n(F, F'; F'')$  the coefficient of  $F \otimes F'$  in  $\Delta_{RT}(F'')$ , where  $F, F', F'' \in M(\mathcal{T})$ . The following result gives the relation between  $n(F, F'; F'')$  and  $n'(F, F'; F'')$ .

**Lemma 3.12.** Let  $F, F', F'' \in M(\mathcal{T})$ . Then  $n'(F, F'; F'') = n(F, F'; F'')$ . Moreover, this coefficient is 0 or 1.

*Proof.* We first show that  $n'(F, F'; F'') \leq n(F, F'; F'')$ . Let  $F'' \in \mathcal{G}(F, F')$  be a grafting of  $F$  over  $F'$  determined by the decompositions  $F = F_1F_2$  and  $F_1B^+(F_2) = F'_1F'_2$ , in which the new vertex added by  $B^+$  is denoted by  $v$ . Graphically,

$$F_1B^+(F_2) = F_1 \begin{array}{c} \textcircled{F_2} \\ | \\ v \bullet \end{array}.$$

Then  $v \in V(F'')$  and

$$V(F_1) = \{u \in V(F'') \mid v <_1 u\} \text{ and } V(F_2) = \{u \in V(F'') \mid v <_h u\}.$$

By the combinatorial description of the coproduct  $\Delta_{RT}(F'')$ , we have

$$B_v = \{u \in V(F'') \mid v <_{h,l} u\} = F_1F_2 = F.$$

Since  $V(F'') = V(F) \sqcup V(F') \sqcup \{v\}$ , we get

$$R_v = \text{the induced subgraph of } F'' \text{ by } V(F'') \setminus (\{v\} \sqcup V(B_v)) = F'.$$

Therefore a grafting  $F''$  of  $F$  over  $F'$  induces a term  $B_v \otimes R_v = F \otimes F'$  in  $\Delta_{RT}(F'')$  and so  $n'(F, F'; F'') \leq n(F, F'; F'')$ .

Next we show that  $n'(F, F'; F'') \geq n(F, F'; F'')$ . By Eq. (7), we may let  $B_v \otimes R_v$  be an item in  $\Delta_{RT}(F'')$  for some  $v \in V(F'')$ . Write  $F'' = T_1 \cdots T_m$  with  $T_1, \dots, T_m \in \mathcal{T}$ , and assume  $v \in V(T_i)$  for some  $1 \leq i \leq m$ . Let  $F := B_v$ ,  $F' := R_v$ ,

$$F_1 := \text{the induced subgraph of } F'' \text{ by } \{u \in V(F'') \mid v <_1 u\}$$

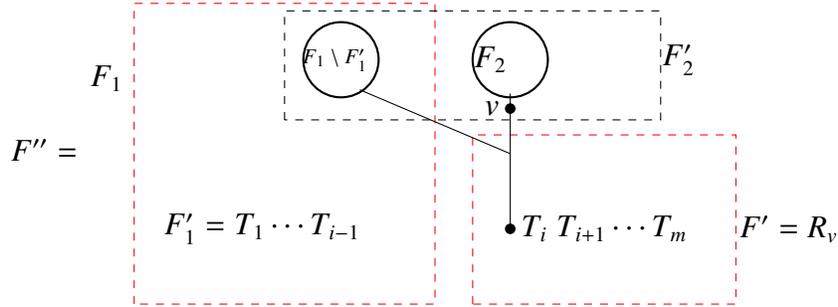
and

$$F_2 := B_v \setminus F_1 = \text{the induced subgraph of } F'' \text{ by } \{u \in V(F'') \mid v <_h u\}.$$

Since  $F = B_v$  is the induced subgraph of  $F''$  by  $\{u \in V(F'') \mid v <_{h,l} u\}$ , it follows from Definition 2.4 that  $F = B_v = F_1 F_2$ . Let  $F'_1 := T_1 \cdots T_{i-1}$  and

$$F'_2 := \text{the induced subgraph of } T_i \text{ by } \{u \in V(T_i) \mid v <_{h,l} u\} \sqcup \{v\}.$$

Then  $F''$  is a grafting of  $F$  over  $F'$  determinate by the decomposition  $F_1 B^+(F_2) = F'_1 F'_2$  (see Fig.1).



**Fig. 1** The illustration of the grafting  $F''$  of  $F$  over  $F'$ .

Hence, if  $F \otimes F'$  is a term of  $\Delta_{RT}(F'')$ , then we obtain a grafting  $F''$  of  $F$  over  $F'$  and so  $n'(F, F'; F'') \leq n(F, F'; F'')$ . Therefore  $n'(F, F'; F'') = n(F, F'; F'')$ .

If  $n(F, F'; F'') \geq 2$ , then there exist two different vertices  $u$  and  $v$  such that  $B_u \otimes R_u = F \otimes F' = B_v \otimes R_v$  by Eq. (7). Thus  $B_u = B_v$  and so  $u = v$  by the definition of  $B_u$  and  $B_v$ , a contradiction.  $\square$

The following result gives a combinatorial description of the multiplication in  $H_{RT}^*$ , dual of the coproduct  $\Delta_{RT}$ , which we denote by  $\diamond = \Delta_{RT}^*$ .

**Proposition 3.13.** *Let  $F, F' \in M(\mathcal{T})$ . The product of  $Z_F$  and  $Z_{F'}$  is*

$$(9) \quad Z_F \diamond Z_{F'} = \sum_{F'' \in M(\mathcal{T})} n'(F, F'; F'') Z_{F''} = \sum_{F'' \in \mathcal{G}(F, F')} Z_{F''}.$$

*Proof.* The second equation follows from Lemma 3.12. We are left to show the first equation. For any  $F, F' \in M(\mathcal{T})$ , we suppose

$$Z_F \diamond Z_{F'} = \sum_{F'' \in M(\mathcal{T})} a_{F, F'}^{F''} Z_{F''}.$$

For any  $F_1 \in M(\mathcal{T})$ , we have

$$\sum_{F'' \in M(\mathcal{T})} a_{F, F'}^{F''} Z_{F''}(F_1) = a_{F, F'}^{F_1} Z_{F_1}(F_1) = a_{F, F'}^{F_1},$$

and

$$\begin{aligned} Z_F \diamond Z_{F'}(F_1) &= \Delta_{RT}^*(Z_F \otimes Z_{F'})(F_1) = (Z_F \otimes Z_{F'})(\Delta_{RT}(F_1)) \\ &= (Z_F \otimes Z_{F'}) \left( \sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} \right) \\ &= \sum_{(F_1)} Z_F(F_{1(1)}) \otimes Z_{F'}(F_{1(2)}) \\ &= \sum_{(F_1)} \delta_{F, F_{1(1)}} \otimes \delta_{F', F_{1(2)}} \end{aligned}$$

$$\begin{aligned}
&= n(F, F'; F_1) \quad (\text{by Definition of } n(F, F'; F_1)) \\
&= n'(F, F'; F_1) \quad (\text{by Lemma 3.12}).
\end{aligned}$$

Thus  $a_{F, F'}^{F_1} = n'(F, F'; F_1)$ , as required.  $\square$

Let us expose an example.

**Example 3.14.** Let  $F, F'$  be the planar rooted forests in Example 3.9. Then the product of  $Z_{\dots}$  and  $Z_{\downarrow}$  is

$$\begin{aligned}
Z_{\dots} \diamond Z_{\downarrow} &= Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} + Z_{\dots \downarrow} + Z_{\dots \downarrow} + Z_{\dots \downarrow} + Z_{\dots \downarrow} \\
&+ Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow} \\
&+ Z_{\downarrow \downarrow} + Z_{\downarrow \downarrow}.
\end{aligned}$$

**Remark 3.15.** (a) We define the degree of a forest as  $\deg(F) = |F| + 1$  for  $F \in M(\mathcal{T})$ . Since a new vertex is added by the grafting operator  $B^+$  in the second step of Definition 3.8, the multiplication

$$\diamond : H_{RT}^* \otimes H_{RT}^* \rightarrow H_{RT}^*$$

is homogeneous of degree 0 by Proposition 3.13.

(b) Let  $F, F' = \overline{TF'}$  be two planar rooted forests. Then by Definition 3.11 and Proposition 3.13,

$$Z_F \diamond Z_{F'} = \sum_{F'' \in \mathcal{G}(F, F')} Z_{F''} = \sum_{F_1 F_2 = F} \sum_{F'_1 F'_2 = F_1 B^+(F_2)} \sum_{\tilde{F} \in \mathcal{L}(F'_2, T)} Z_{F'_1 \tilde{F} \overline{F'}},$$

where  $F_1, F_2, F'_1, F'_2 \in M(\mathcal{T})$ .

The following result characterizes the coproduct on  $H_{RT}^*$ .

**Lemma 3.16.** Let  $T_1 \cdots T_n \in M(\mathcal{T})$ . The coproduct of  $Z_{T_1 \cdots T_n} \in H_{RT}^*$  is

$$(10) \quad m_{RT}^*(Z_{T_1 \cdots T_n}) = \sum_{i=0}^n Z_{T_1 \cdots T_i} \otimes Z_{T_{i+1} \cdots T_n},$$

with the convention that  $Z_{T_1 T_0} = \mathbb{1}$  and  $Z_{T_{n+1} T_n} = \mathbb{1}$ .

*Proof.* Suppose

$$m_{RT}^*(Z_{T_1 \cdots T_n}) = \sum_{F', F'' \in M(\mathcal{T})} c_{F', F''} Z_{F'} \otimes Z_{F''}.$$

For any  $F_1, F_2 \in M(\mathcal{T})$ , we have

$$m_{RT}^*(Z_{T_1 \cdots T_n})(F_1 \otimes F_2) = Z_{T_1 \cdots T_n}(m_{RT}(F_1 \otimes F_2)) = Z_{T_1 \cdots T_n}(F_1 F_2) = \delta_{T_1 \cdots T_n, F_1 F_2},$$

and

$$\begin{aligned}
\left( \sum_{F', F'' \in M(\mathcal{T})} c_{F', F''} (Z_{F'} \otimes Z_{F''}) \right) (F_1 \otimes F_2) &= \sum_{F', F'' \in M(\mathcal{T})} c_{F', F''} Z_{F'}(F_1) \otimes Z_{F''}(F_2) \\
&= \sum_{F', F'' \in M(\mathcal{T})} c_{F', F''} \delta_{F', F_1} \otimes \delta_{F'', F_2} \\
&= c_{F_1, F_2}.
\end{aligned}$$

Thus  $\delta_{T_1 \dots T_n, F_1 F_2} = c_{F_1, F_2}$  and so  $c_{F_1, F_2} = 1$  if  $T_1 \dots T_n = F_1 F_2$  and  $c_{F_1, F_2} = 0$  otherwise. This completes the proof.  $\square$

For example, we have

$$m_{RT}^*(Z_{\cdot, \downarrow}) = Z_{\cdot, \downarrow} \otimes \mathbb{1} + \mathbb{1} \otimes Z_{\cdot, \downarrow} + Z_{\cdot} \otimes Z_{\downarrow}.$$

Now we are ready for our main result in this section. Denote by

$$S := \{Z_F \mid F \in M(\mathcal{T}) \text{ such that } \text{bre}(F) \text{ is even}\}.$$

**Theorem 3.17.** *The algebra  $(H_{RT}^*, \diamond)$  is the free non unitary algebra on  $S$ .*

*Proof.* We first prove that  $S$  generates  $H_{RT}^*$ . Let  $A$  be the subalgebra of  $(H_{RT}^*, \diamond)$  generated by  $S$  and let  $Z_{T_1 \dots T_m} \in H_{RT}^*$  be a basis element with  $T_1, \dots, T_m \in \mathcal{T}$ . If  $m$  is even, then  $Z_{T_1 \dots T_m} \in A$ . If  $m$  is odd, we prove that  $Z_{T_1 \dots T_m} \in A$  by induction on  $|T_1| \geq 1$ . If  $|T_1| = 1$ , then  $T_1 = \cdot$  and

$$Z_{\mathbb{1}} \diamond Z_{T_2 \dots T_m} = Z_{\cdot, T_2 \dots T_m} + \sum_{F' \in \mathcal{L}(\cdot, T_2)} Z_{F' T_3 \dots T_m} \quad (\text{by Item (b) of Remark 3.15}).$$

Since  $Z_{\mathbb{1}}, Z_{T_2 \dots T_m} \in A$  and  $Z_{F' T_3 \dots T_m} \in A$  by  $\text{bre}(F' T_3 \dots T_m) = m - 1$ , we have  $Z_{T_1 T_2 \dots T_m} = Z_{\cdot, T_2 \dots T_m} \in A$ . If  $|T_1| \geq 2$ , then  $T_1 = B^+(F)$  for some  $F \in H_{RT}$ . We have

$$\begin{aligned} Z_F \diamond Z_{T_2 \dots T_m} &= \sum_{F_1 F_2 = F} \sum_{F'_1 F'_2 = F_1 B^+(F_2)} \sum_{\tilde{F} \in \mathcal{L}(F'_2, T_2)} Z_{F'_1 \tilde{F} T_3 \dots T_m} \quad (\text{by Item (b) of Remark 3.15}) \\ &= Z_{B^+(F) T_2 \dots T_m} + \sum_{F_1 F_2 = F} \sum_{\substack{F'_1 F'_2 = F_1 B^+(F_2) \\ F'_1 = \mathbb{1}}} \sum_{\tilde{F} \in \mathcal{L}(F'_2, T_2)} Z_{\tilde{F} T_3 \dots T_m} \\ &\quad + \sum_{\substack{F_1 F_2 = F \\ F_1 \neq \mathbb{1}}} \sum_{\substack{F'_1 F'_2 = F_1 B^+(F_2) \\ F'_1 \neq \mathbb{1}}} \sum_{\tilde{F} \in \mathcal{L}(F'_2, T_2)} Z_{F'_1 \tilde{F} T_3 \dots T_m}. \end{aligned}$$

Since  $\text{bre}(\tilde{F} T_3 \dots T_m) = m - 1$ , we have  $Z_{T_2 \dots T_m}, Z_{\tilde{F} T_3 \dots T_m} \in A$  by definition of  $A$ . Moreover,  $Z_F, Z_{F'_1 \tilde{F} T_3 \dots T_m} \in A$  by the induction hypothesis. Hence  $Z_{T_1 \dots T_m} \in A$  and  $A = H_{RT}^*$ .

Next, we prove that  $(H_{RT}^*, \diamond)$  is the free algebra on  $S$ . By [6, Proposition 8], the formal series of  $H_{RT} = \bigoplus_{n=1}^{\infty} H_{RT}(n)$  is

$$\mathbf{F}(x) = \sum_{i=1}^{\infty} \dim H_{RT}^{(i)} x^i = x + x^2 + 2x^3 + 5x^4 + \dots = \frac{1 - \sqrt{1 - 4x}}{2},$$

which is also the formal series of  $H_{RT}^* = \bigoplus_{n=1}^{\infty} (H_{RT}(n))^*$ . Thus  $\mathbf{F}^2(x) = \mathbf{F}(x) - x$ . Since each planar rooted tree is a grafting operation  $B^+$  of a planar rooted forest  $F$  and vice-versa, the formal series of planar rooted trees is the same as the one of planar rooted forests, that is,

$$\mathbf{T}(x) = \sum_{i=1}^{\infty} a_i x^i = \mathbf{F}(x) = x + x^2 + 2x^3 + 5x^4 + \dots,$$

where  $a_i$  is the number of trees with  $i$  vertices. So the formal series of forests with even breadth is

$$\sum_{i=0}^{+\infty} b_i x^i = 1 + \mathbf{T}(x)\mathbf{T}(x) + \mathbf{T}(x)\mathbf{T}(x)\mathbf{T}(x)\mathbf{T}(x) + \dots = \sum_{i=0}^{+\infty} \mathbf{T}^{2i}(x) = \frac{1}{1 - \mathbf{T}^2(x)} = \frac{1}{1 - \mathbf{F}^2(x)},$$

where  $b_i$  is the number of forests of even breadth with  $i$  vertices. Let  $\mathbf{G}(x) := \sum_{i=1}^{+\infty} g_i x^i$ , where  $g_i$  is the number of forests of even breadth with degree  $i$ . Then

$$\begin{aligned} \mathbf{G}(x) &= x \sum_{i=1}^{\infty} g_i x^{i-1} = x \sum_{i=0}^{\infty} g_{i+1} x^i = x \sum_{i=0}^{\infty} b_i x^i \quad (\text{by } \deg(F) = |F| + 1) \\ &= \frac{x}{1 - \mathbf{F}^2(x)} = \frac{\mathbf{F}(x)}{1 + \mathbf{F}(x)} \quad (\text{by } x = \mathbf{F}(x) - \mathbf{F}^2(x)). \end{aligned}$$

Let  $T(S)$  be the free algebra on  $S$ . Since  $H_{RT}^*$  is generated by  $S$ , there exists a surjective algebra morphism  $\phi : T(S) \rightarrow H_{RT}^*$ . By Item (a) of Remark 3.15, the formal series of  $T(S)$  is

$$\mathbf{G}(x) + \mathbf{G}(x)\mathbf{G}(x) + \mathbf{G}(x)\mathbf{G}(x)\mathbf{G}(x) + \mathbf{G}(x)\mathbf{G}(x)\mathbf{G}(x)\mathbf{G}(x) + \cdots = \frac{\mathbf{G}(x)}{1 - \mathbf{G}(x)} = \frac{\frac{\mathbf{F}(x)}{1 + \mathbf{F}(x)}}{1 - \frac{\mathbf{F}(x)}{1 + \mathbf{F}(x)}} = \mathbf{F}(x).$$

Thus  $H_{RT}^*$  and  $T(S)$  have the same formal series and so  $\phi$  is injective. Hence  $T(S)$  is isomorphic to  $H_{RT}^*$ , as required.  $\square$

#### 4. PRIMITIVE ELEMENTS OF $H_{RT}$

In this section, we first construct a second product  $\star$  on  $H_{RT}$ , making  $H_{RT}$  a unital infinitesimal graded bialgebra in the sense of Loday and Ronco [15]. Then we give a projection of  $H_{RT}$  on its primitive elements. Finally, we characterize the dual of  $\star$ .

**4.1. A second product on  $H_{RT}$ .** The following result gives  $H_{RT}$  a second product  $\star$ .

**Proposition 4.1.** *We define a product  $\star$  on  $H_{RT}$  by*

$$\forall x, y \in H_{RT}, \quad x \star y = x \cdot \cdot \cdot y.$$

*Then  $\star$  is associative (and not unitary), and for any  $x, y \in H_{RT}$ :*

$$\Delta_{RT}(x \star y) = x \star \Delta_{RT}(y) + \Delta_{RT}(x) \star y + x \otimes y.$$

*Moreover,  $(H_{RT}, \star)$  is a graded algebra (recall that for any  $n \geq 1$ ,  $H_{RT}(n)$  is the subspace generated by forests with  $n - 1$  vertices).*

*Proof.* For any  $x, y, z \in H_{RT}$ :

$$(x \star y) \star z = x \cdot \cdot \cdot y \cdot \cdot \cdot z = x \star (y \star z),$$

so  $\star$  is associative. As  $\Delta_{RT}(\cdot) = \mathbb{1} \otimes \mathbb{1}$ , for any  $x, y \in H_{RT}$ :

$$\begin{aligned} \Delta_{RT}(x \star y) &= \Delta_{RT}(x \cdot \cdot \cdot y) \\ &= \Delta_{RT}(x \cdot \cdot \cdot) \cdot y + x \cdot \cdot \cdot \Delta_{RT}(y) \\ &= (x \cdot \Delta_{RT}(\cdot)) \cdot y + (\Delta_{RT}(x) \cdot \cdot \cdot) \cdot y + (x \cdot \cdot \cdot) \cdot \Delta_{RT}(y) \\ &= x \cdot (\mathbb{1} \otimes \mathbb{1}) \cdot y + \Delta_{RT}(x) \cdot (\cdot \cdot y) + (x \cdot \cdot \cdot) \cdot \Delta_{RT}(y) \\ &= x \otimes y + \Delta_{RT}(x) \star y + x \star \Delta_{RT}(y). \end{aligned}$$

Let  $x \in H_{RT}(k)$  and  $y \in H_{RT}(l)$ , with  $k, l \geq 1$ . Then  $x$  is a linear span of forests with  $k - 1$  vertices and  $y$  is a linear span of forests with  $l - 1$  vertices. By definition of  $\star$ ,  $x \star y$  is a linear span of forests with  $k - 1 + l - 1 + 1 = k + l - 1$  vertices, so belongs to  $H_{RT}(k + l)$ .  $\square$

**4.2. A projection on primitive elements.** The following result gives a projection of  $H_{RT}$  on its primitive elements.

**Theorem 4.2.** *We define an operator  $\theta$  on  $H_{RT}$  by:*

$$\forall x \in H_{RT}, \quad \theta(x) := \sum_{k=1}^{\infty} (-1)^{k+1} \star^{(k-1)} \circ \Delta^{(k-1)}(x),$$

where  $\Delta^{(l)} : H_{RT} \longrightarrow H_{RT}^{\otimes(l+1)}$  and  $\star^{(l)} : H_{RT}^{\otimes(l+1)} \longrightarrow H_{RT}$  are inductively defined:

$$\begin{aligned} \Delta^{(0)} &= \text{id}_{H_{RT}}, & \star^{(0)} &= \text{id}_{H_{RT}}, \\ \Delta^{(l+1)} &= (\Delta^{(l)} \otimes \text{id}_{H_{RT}}) \circ \Delta_{RT}, & \star^{(l+1)} &= \star \circ (\star^{(l)} \otimes \text{id}_{H_{RT}}). \end{aligned}$$

Then  $\theta$  is a projector on  $\text{Prim}(H_{RT}) = \text{Ker}(\Delta_{RT})$ . The kernel of  $\theta$  is

$$\text{Ker}(\theta) = H_{RT} \star H_{RT} = \mathbf{k}\{F \text{ forest with at least one tree equal to } \cdot\}.$$

*Proof.* For any forest  $F$  with  $n$  vertices,  $\Delta^{(k)}(F) = 0$  if  $k \geq n$ , so  $\theta$  is well-defined. If  $x \in \text{Prim}(H_{RT})$ ,  $\Delta^{(k)}(x) = 0$  if  $k \geq 1$ , so  $\theta(x) = x + 0 = x$ .

Let  $k \geq 2$  and  $x_1, \dots, x_k \in H_{RT}$ . By the compatibility between the product  $\star$  and the coproduct  $\Delta_{RT}$ :

$$\begin{aligned} \Delta_{RT} \circ \star^{(k-1)}(x_1 \otimes \dots \otimes x_k) &= \sum_{i=1}^k \sum_{(x_i)} x_1 \star \dots \star x_{i-1} \star x_i^{(1)} \otimes x_i^{(2)} \star x_{i+1} \star \dots \star x_k \\ &\quad + \sum_{i=1}^{k-1} x_1 \star \dots \star x_i \otimes x_{i+1} \star \dots \star x_k. \end{aligned}$$

Hence, using Sweedler's notation:

$$\begin{aligned} \Delta_{RT} \circ \theta(x) &= x^{(1)} \otimes x^{(2)} + \sum_{k=2}^{\infty} (-1)^{k+1} \sum_{(x)} \sum_{i=1}^k x^{(1)} \star \dots \star x^{(i)} \otimes x^{(i+1)} \star \dots \star x^{(k+1)} \\ &\quad + \sum_{k=2}^{\infty} (-1)^{k+1} \sum_{(x)} \sum_{i=1}^{k-1} x^{(1)} \star \dots \star x^{(i)} \otimes x^{(i+1)} \star \dots \star x^{(k)} \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{(x)} \sum_{i=1}^k x^{(1)} \star \dots \star x^{(i)} \otimes x^{(i+1)} \star \dots \star x^{(k+1)} \\ &\quad + \sum_{k=1}^{\infty} (-1)^k \sum_{(x)} \sum_{i=1}^k x^{(1)} \star \dots \star x^{(i)} \otimes x^{(i+1)} \star \dots \star x^{(k+1)} \\ &= 0, \end{aligned}$$

so  $\theta$  is a projector on  $\text{Prim}(H_{RT})$ .

By definition of  $\theta$ , for any  $x \in H_{RT}$ ,  $\theta(x) - x \in H_{RT} \star H_{RT}$ , so:

$$H_{RT} = \text{Prim}(H_{RT}) + H_{RT} \star H_{RT}.$$

Let  $x, y \in H_{RT}$ . By the compatibility between the product  $\star$  and the coproduct  $\Delta_{HR}$ , for any  $k \geq 2$ :

$$\Delta^{(k-1)}(x \star y) = \sum_{i+j=k} \sum_{(x)} \sum_{(y)} x^{(1)} \otimes \dots \otimes x^{(i)} \otimes y^{(1)} \otimes \dots \otimes y^{(j)}$$

$$+ \sum_{i+j=k+1} \sum_{(x)} \sum_{(y)} x^{(1)} \otimes \dots \otimes x^{(i-1)} \otimes x^{(i)} \star y^{(1)} \otimes y^{(2)} \otimes \dots \otimes y^{(j)}.$$

Therefore:

$$\begin{aligned} \theta(x \star y) &= x \star y + \sum_{k=2}^{\infty} (-1)^{k+1} \sum_{i+j=k} \sum_{(x)} \sum_{(y)} x^{(1)} \star \dots \star x^{(i)} \star y^{(1)} \star \dots \star y^{(j)} \\ &+ \sum_{k=2}^{\infty} (-1)^{k+1} \sum_{i+j=k+1} \sum_{(x)} \sum_{(y)} x^{(1)} \star \dots \star x^{(i)} \star y^{(1)} \star \dots \star y^{(j)} \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{(x)} \sum_{(y)} (-1)^{i+j-1} x^{(1)} \star \dots \star x^{(i)} \star y^{(1)} \star \dots \star y^{(j)} \\ &+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{(x)} \sum_{(y)} (-1)^{i+j} x^{(1)} \star \dots \star x^{(i)} \star y^{(1)} \star \dots \star y^{(j)} \\ &= 0. \end{aligned}$$

Consequently,  $\text{Prim}(H_{RT}) \cap (H_{RT} \star H_{RT}) = (0)$ , so:

$$H_{RT} = \text{Prim}(H_{RT}) \oplus (H_{RT} \star H_{RT}),$$

and the projection on  $\text{Prim}(H_{RT})$  in this direct sum is  $\theta$ .  $\square$

**Remark 4.3.**  $(H_{RT}, \star)$  is not a unitary algebra. If we consider the unitary  $\overline{H}_{RT} = \mathbf{k} \oplus H_{RT}$ , with the coproduct  $\overline{\Delta}_{RT}$  defined by

$$\forall x \in H_{RT}, \quad \overline{\Delta}_{RT}(x) = x \otimes 1 + 1 \otimes x + \Delta_{RT}(x),$$

then  $(\overline{H}_{RT}, \star, \overline{\Delta}_{RT})$  is a unitary infinitesimal bialgebra in the sense of Loday and Ronco [15], and  $\theta$  is its antipode, defined as the idempotent  $e$  in [15].

By definition of the coproduct  $\Delta_{RT}$ :

**Proposition 4.4.** *Let  $F$  be a planar rooted forest. We denote its vertices according to the order  $\leq_{h,l}$ :*

$$v_1 \leq_{h,l} \dots \leq_{h,l} v_n.$$

Then:

$$(11) \quad \theta(F) = \sum_{k=0}^n \sum_{1 \leq i_1 < \dots < i_k \leq n} (-1)^k F_{\{v_{i_k+1}, \dots, v_n\}} \bullet F_{\{v_{i_{k-1}+1}, \dots, v_{i_k-1}\}} \bullet \dots \bullet F_{\{v_{i_1+1}, \dots, v_{i_2-1}\}} \bullet F_{\{v_1, \dots, v_{i_1-1}\}}.$$

**Example 4.5.** Applying this formula for a given  $n$ , we obtain, after simplifications:

$$\begin{aligned} \text{if } n = 2, & \quad \theta(F) = F - \dots, \\ \text{if } n = 3, & \quad \theta(F) = F - F_{\{v_2, v_3\}} \bullet - \bullet F_{\{v_1, v_2\}} + \dots, \\ \text{if } n = 4, & \quad \theta(F) = F - F_{\{v_2, v_3, v_4\}} \bullet - \bullet F_{\{v_1, v_2, v_3\}} + \bullet F_{\{v_2, v_3\}} \bullet \bullet \end{aligned}$$

Consequently:

$$\begin{aligned} \theta(\mathfrak{!}) &= \mathfrak{!} - \dots, \\ \theta(\mathfrak{V}) &= \mathfrak{V} - \mathfrak{!}, & \theta(\mathfrak{!}) &= \mathfrak{!} - \mathfrak{!} \bullet - \mathfrak{!} \bullet \bullet + \dots, \\ \theta(\mathfrak{!!}) &= \mathfrak{!!} - \mathfrak{!} \bullet - \dots + \dots, & \theta(\mathfrak{V}) &= \mathfrak{V} - \bullet \mathfrak{V}, \end{aligned}$$

$$\begin{aligned}\theta(\dot{V}) &= \dot{V} - \dot{1} \cdot - \cdot V + \cdot 1 \cdot, & \theta(\dot{V}) &= \dot{V} - \cdot \dot{1}, \\ \theta(\dot{Y}) &= \dot{Y} - V \cdot - \cdot \dot{1} + \cdot 1 \cdot, & \theta(\dot{1}) &= \dot{1} - \dot{1} \cdot - \cdot \dot{1} + \cdot 1 \cdot.\end{aligned}$$

**Corollary 4.6.** Let  $V$  be the vector space  $\text{Prim}(H_{RT})$ . We put:

$$T_+(V) = \bigoplus_{n \geq 1} V^{\otimes n}.$$

We give it the concatenation product  $m_{conc}$  and the deconcatenation coproduct  $\Delta_{dec}$ :

$$\forall v_1, \dots, v_n \in V, \quad \Delta_{dec}(v_1 \dots v_n) = \sum_{i=1}^{n-1} v_1 \dots v_i \otimes v_{i+1} \dots v_n.$$

Then the following map is an algebra and a coalgebra isomorphism:

$$\Upsilon : \begin{cases} (T_+(V), m_{conc}, \Delta_{dec}) & \longrightarrow & (H_{RT}, \star, \Delta_{RT}) \\ v_1 \dots v_n & \longmapsto & v_1 \star \dots \star v_n. \end{cases}$$

*Proof.* Obviously,  $\Upsilon$  is an algebra morphism. By the compatibility between the product  $\star$  and the coproduct  $\Delta_{RT}$ , for any  $v_1, \dots, v_n \in \text{Prim}(H)$ :

$$\Delta_{RT}(v_1 \star \dots \star v_n) = \sum_{i=1}^n v_1 \star \dots \star v_i \otimes v_{i+1} \star \dots \star v_n.$$

Consequently,  $\theta$  is a coalgebra morphism.

The graduation of  $H_{RT}$  induces a graduation of  $\text{Prim}(H_{RT}) = V$ , which in turn gives a graduation of  $T_+(V)$ :

$$\forall n \geq 1, \quad T_+(V)_n = \bigoplus_{k=1}^n \bigoplus_{n_1 + \dots + n_k = n} V_{n_1} \otimes \dots \otimes V_{n_k}.$$

As the product  $\star$  is homogeneous of degree 0,  $\Upsilon$  is homogeneous of degree 0. Let us assume that  $\Upsilon$  is not injective, and let  $x \in \text{Ker}(\Upsilon)$ , nonzero, of minimal degree  $n$ . Then:

$$0 = \Delta_{RT} \circ \Upsilon(x) = (\Upsilon \otimes \Upsilon) \circ \Delta_{dec}(x).$$

Moreover,

$$\Delta_{dec}(x) \in \sum_{k=1}^{n-1} T_+(V)_k \otimes T_+(V)_l.$$

By definition of  $n$ ,  $\Upsilon|_{T_+(V)_k}$  is injective if  $k < n$ , so  $\Delta_{dec}(x) = 0$ , and  $x \in \text{Ker}(\Delta_{dec}) = V$ . Therefore,  $\Upsilon(x) = x = 0$ : this is a contradiction, and  $\Upsilon$  is injective.

In order to prove that  $\Upsilon$  is surjective, it is enough to prove that  $\text{Prim}(H_{RT})$  generates the algebra  $(H_{RT}, \star)$ . Let  $F$  be a planar rooted forest with  $n$  vertices, let us prove that it belongs to the subalgebra  $A$  generated by  $\text{Prim}(H)$  by induction on  $n$ . If  $n = 1$ , then  $F = \mathbb{1} \in V$  and this is obvious. Otherwise, let us put  $y = x - \Upsilon(x)$ . By definition of  $\Upsilon(x)$ , this is a linear span of forests of the form  $G = G_1 \cdot G_2$ , with  $n$  vertices. By the induction hypothesis,  $G_1, G_2 \in A$ , so  $G = G_1 \star G_2 \in A$  and finally  $y \in A$ . As  $\Upsilon(x) \in V \subseteq A$ ,  $x = y + \Upsilon(x) \in A$ .  $\square$

Let us simplify the writing of  $\theta(F)$ , in order to avoid the simplifications we observed in the examples.

**Definition 4.7.** Let us define a sequence of scalars  $(c(n))_{n \geq 0}$  by:

$$c(n) = \begin{cases} 1, & \text{if } n \equiv 0[3], \\ 0, & \text{if } n \equiv 1[3], \\ -1, & \text{if } n \equiv 2[3]. \end{cases}$$

Recall that a composition is a finite sequence of positive integers. If  $(n_1, \dots, n_k)$  is a composition, we can write it as:

$$(n_1, \dots, n_k) = (\underbrace{1, \dots, 1}_{\alpha_0}, \underbrace{1, \dots, 1}_{\alpha_1}, \dots, \underbrace{1, \dots, 1}_{\alpha_{p-1}}, \underbrace{1, \dots, 1}_{\alpha_p}, \underbrace{1, \dots, 1}_{\alpha_p}),$$

where  $p \geq 0$ ,  $b_1, \dots, b_p \geq 2$ ,  $\alpha_0, \dots, \alpha_p \geq 0$ . This is abbreviated as

$$(n_1, \dots, n_k) = 1^{\alpha_0} b_1 1^{\alpha_1} \dots 1^{\alpha_{p-1}} b_p 1^{\alpha_p}.$$

We then put:

$$c(n_1, \dots, n_k) = \begin{cases} c(\alpha_0), & \text{if } p = 0, \\ c(\alpha_0 + 2)c(\alpha_1 + 1) \dots c(\alpha_{p-1} + 1)c(\alpha_p + 2), & \text{if } p \geq 1. \end{cases}$$

**Corollary 4.8.** Let  $F$  be a planar rooted forest. We denote its vertices according to the order  $\leq_{h,l}$ :

$$v_1 \leq_{h,l} \dots \leq_{h,l} v_n.$$

Then:

$$\theta(F) = \sum_{n_1 + \dots + n_k = n} c(n_1, \dots, n_k) F_{\{v_{n_1 + \dots + n_{k-1} + 1}, \dots, v_{n_1 + \dots + n_k}\}} \cdots F_{\{v_1, \dots, v_{n_1}\}}.$$

*Proof.* Interpreting the trees  $\cdot$  in (11) as  $F_{\{v_i\}}$ , we can rewrite, for any forest  $F$  with  $n$  vertices:

$$(12) \quad \theta(F) = \sum_{n_1 + \dots + n_k = n} a(n_1, \dots, n_k) F_{\{v_{n_1 + \dots + n_{k-1} + 1}, \dots, v_{n_1 + \dots + n_k}\}} \cdots F_{\{v_1, \dots, v_{n_1}\}},$$

for a certain family of scalars  $a(n_1, \dots, n_k)$ , independent of  $F$ . Let us prove that  $a(n_1, \dots, n_k) = c(n_1, \dots, n_k)$  for any composition  $(n_1, \dots, n_k)$ .

*First case.* We consider the case  $p = 0$ , that is to say  $(n_1, \dots, n_k) = 1^n$ . The term  $F_{\{v_n\}} \dots F_{\{v_1\}}$  in (12) comes from the terms in (11) indexed by  $1 \leq i_1 < \dots < i_k \leq n$  with:

- $i_1 \leq 2$ .
- $i_k \geq n - 1$ .
- If  $2 \leq p \leq k$ , then  $i_p \leq i_{p-1} + 2$ .

Any such  $(i_1, \dots, i_k)$  contributes with  $(-1)^k$ . Note that, in  $\mathbb{Q}[[X]]$ :

$$\begin{aligned} \frac{1}{X} \sum_{l=1}^{\infty} (-1)^{l-1} (X + X^2)^l &= \sum_{l=1}^{\infty} (-1)^{l-1} \sum_{j_1, \dots, j_l \in \{1,2\}} X^{j_1 + \dots + j_l - 1} \\ &= \sum_{m=1}^{\infty} \left( \sum_{\substack{j_1, \dots, j_m \in \{1,2\}, \\ j_1 + \dots + j_m = m}} (-1)^{l-1} \right) X^{m-1} \\ &= \sum_{m=1}^{\infty} \left( \sum_{\substack{1 \leq i_1 < \dots < i_{l-1} \leq m-1, \\ i_1 \leq 2, i_{l-1} \geq m-2, \\ i_p \leq i_{p-1} + 2 \text{ if } 2 \leq p \leq m-1}} (-1)^{l-1} \right) X^{m-1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \left( \sum_{\substack{1 \leq i_1 < \dots < i_k \leq n, \\ i_1 \leq 2, i_k \geq n-1, \\ i_p \leq i_{p-1}+2 \text{ if } 2 \leq p \leq n}} (-1)^k \right) X^n \\
&= \sum_{n=0}^{\infty} a(1^n) X^n,
\end{aligned}$$

where in the third equality,  $i_p = j_1 + \dots + j_p$  for any  $p$  and  $(k, n) = (l-1, m-1)$  for the fourth one. Therefore:

$$\sum_{n=0}^{\infty} a(1^n) X^n = \frac{1}{X} \sum_{l=1}^{\infty} (-1)^{l-1} (X + X^2)^l = -\frac{1}{X} \sum_{l=1}^{\infty} (-X - X^2)^l = \frac{1+X}{1+X+X^2} = \frac{X^2-1}{X^3-1}.$$

Hence:

$$a(1^0) = 1, \quad a(1^1) = 0, \quad a(1^2) = -1, \quad a(1^n) = a(1^{n-3}) \text{ if } n \geq 3.$$

So  $a(1^n) = c(n)$  for any  $n \geq 0$ .

*Second case.* We now assume that  $p \geq 1$ . For any  $1 \leq i_1 < \dots < i_k \leq n$  contributing in (12) to  $a(n_1, \dots, n_k)$ , necessarily:

- If  $\alpha_0 \geq 1$ ,  $\alpha_0$  belongs to  $\{i_1, \dots, i_k\}$ .
- If  $\alpha_p \geq 1$ ,  $\alpha_0 + \dots + \alpha_{p-1} + b_1 + \dots + b_p + 1$  belongs to  $\{i_1, \dots, i_k\}$ .
- If  $1 \leq i \leq p-1$  and  $\alpha_i \geq 1$ , then  $\alpha_0 + \dots + \alpha_{i-1} + b_1 + \dots + b_{i-1} + 1$  and  $\alpha_0 + \dots + \alpha_i + b_1 + \dots + b_{i-1}$  belong to  $\{i_1, \dots, i_k\}$ .

Separating the study for each block of 1 in  $(n_1, \dots, n_k)$ , we observe that  $a(n_1, \dots, n_k)$  can be written as a product

$$a(n_1, \dots, n_k) = a^{(0)}(\alpha_0) \dots a^{(p)}(\alpha_p).$$

Mimicking the study of the first case:

$$a^{(0)}(\alpha_0) = \begin{cases} 1, & \text{if } \alpha_0 = 0, \\ -1, & \text{if } \alpha_0 = 1, \\ -a(1^{\alpha_0-1}), & \text{if } \alpha_0 \geq 2. \end{cases}$$

In all cases, we obtain that  $a^{(0)}(\alpha_0) = -a(1^{\alpha_0+2}) = -c(\alpha_0 + 2)$ . Similarly,  $a^{(p)}(\alpha_p) = -c(\alpha_p + 2)$ . If  $1 \leq i \leq p-1$ :

$$a^{(i)}(\alpha_i) = \begin{cases} 0, & \text{if } \alpha_i = 0, \\ -1, & \text{if } \alpha_i = 1, \\ a(1^{\alpha_i-2}), & \text{if } \alpha_i \geq 2. \end{cases}$$

In all cases, we obtain that  $a^{(i)}(\alpha_i) = a(1^{\alpha_0+1}) = c(\alpha_0 + 1)$ . As a consequence,  $a(n_1, \dots, n_k) = c(n_1, \dots, n_k)$ .  $\square$

For any composition  $(n_1, \dots, n_k)$ ,  $c(n_1, \dots, n_k) \in \{-1, 0, 1\}$ . Let us denote by  $t_n$  the number of compositions  $(n_1, \dots, n_k)$ , with  $n_1 + \dots + n_k = n$  and  $c(n_1, \dots, n_k) \neq 0$ .

**Proposition 4.9.** *In  $\mathbb{Q}[[X]]$ :*

$$\sum_{n=0}^{\infty} t_n X^n = \frac{1-X+2X^2}{1-X-2X^3}.$$

As a consequence, for any  $n \geq 3$ ,

$$t_n = t_{n-1} + 2t_{n-3}.$$

*Proof.* By definition,  $t_n$  is the number of compositions  $(n_1, \dots, n_k) = 1^{\alpha_0} b_1 1^{\alpha_1} \dots 1^{\alpha_{p-1}} b_p 1^{\alpha_p}$  with  $n_1 + \dots + n_k = n$ , such that:

- If  $p = 0$ , then  $\alpha_i \equiv 0[3]$  or  $\alpha_i \equiv 2[3]$ .
- if  $p \geq 1$ :
  - $\alpha_0 \equiv 0[3]$  or  $\alpha_0 \equiv 1[3]$ .
  - $\alpha_p \equiv 0[3]$  or  $\alpha_p \equiv 1[3]$ .
  - If  $1 \leq i \leq p - 1$ ,  $\alpha_i \equiv 1[3]$  or  $\alpha_i \equiv 2[3]$ .

We shall use the following formal series:

$$\begin{aligned}
 P_0(X) &= \sum_{k=0}^{\infty} X^{3k} + \sum_{k=0}^{\infty} X^{3k+2} = \frac{1 + X^2}{1 - X^3}, \\
 P_2(X) &= \sum_{k=0}^{\infty} X^{3k} + \sum_{k=0}^{\infty} X^{3k+1} = \frac{1 + X}{1 - X^3}, \\
 P_1(X) &= \sum_{k=0}^{\infty} X^{3k+1} + \sum_{k=0}^{\infty} X^{3k+2} = X P_2(X).
 \end{aligned}$$

Then:

$$\begin{aligned}
 \sum_{n=0}^{\infty} t_n X^n &= P_0(X) + \sum_{k=1}^{\infty} P_2(X) \left( \frac{X^2}{1 - X} P_1(X) \right)^{k-1} \frac{X^2}{1 - X} P_2(X) \\
 &= P_0(X) + \sum_{k=1}^{\infty} P_2(X)^{k+1} X^{k-1} \left( \frac{X^2}{1 - X} \right)^k \\
 &= P_0(X) + \frac{P_2(X)}{X} \sum_{k=1}^{\infty} \left( \frac{X^3 P_2(X)}{1 - X} \right)^k \\
 &= P_0(X) + \frac{P_2(X)}{X} \frac{\frac{X^3 P_2(X)}{1 - X}}{1 - \frac{X^3 P_2(X)}{1 - X}} \\
 &= \frac{1 - X + 2X^2}{1 - X - 2X^3}.
 \end{aligned}$$

□

Here are the first values of  $t_n$ :

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$t_n$	1	0	2	4	4	8	16	24	40	72	120	200	344	584	984	1672

**Example 4.10.** Let us consider the case  $n = 5$ . There are 8 contributing terms in (12), corresponding to the compositions:

- (5), (4, 1), (1, 4), (1, 3, 1), (2, 1, 2), (2, 1, 1, 1), (1, 1, 1, 2), (1, 1, 1, 1, 1).

Therefore, if  $F$  has 5 vertices:

$$\theta(F) = F - \bullet F_{\{|1,2,3,4\}} - F_{\{|2,3,4,5\}} \bullet + \bullet F_{\{|2,3,4\}} \bullet - F_{\{|4,5\}} \bullet F_{\{|1,2\}} + \dots F_{\{|1,2\}} + F_{\{|4,5\}} \bullet \dots - \dots$$

4.3. **Dual coproduct of  $\star$ .** The product  $\star$  on  $H_{RT}$  induces by duality a coproduct  $\blacktriangle$  on  $H_{RT}^*$ .

**Lemma 4.11.** Let  $T_1, \dots, T_k \in \mathcal{T}$ . Then, in  $H_{RT}^*$ :

$$(13) \quad \blacktriangle(Z_{T_1 \dots T_k}) = \sum_{i=1}^k \delta_{T_i, \bullet} Z_{T_1 \dots T_{i-1}} \otimes Z_{T_{i+1} \dots T_k}.$$

*Proof.* Let  $F, G \in M(\mathcal{T})$ . Then:

$$\begin{aligned} \blacktriangle(Z_{T_1 \dots T_k})(F \otimes G) &= Z_{T_1 \dots T_k}(F \star G) \\ &= Z_{T_1 \dots T_k}(F \cdot G) \\ &= \delta_{T_1 \dots T_k, F \cdot G} \\ &= \sum_{i=1}^k \delta_{T_1 \dots T_{i-1}, F} \delta_{T_i, \bullet} \delta_{T_{i+1} \dots T_k, G} \\ &= \left( \sum_{i=1}^k \delta_{T_i, \bullet} Z_{T_1 \dots T_{i-1}} \otimes Z_{T_{i+1} \dots T_k} \right) (F \otimes G), \end{aligned}$$

which implies (13). □

Dualizing Proposition 4.1:

**Proposition 4.12.** For any  $f, g \in H_{RT}^*$ :

$$\blacktriangle(f \diamond g) = f \diamond \blacktriangle(g) + \blacktriangle(f) \diamond g + f \otimes g.$$

*Proof.* Let  $f, g \in H_{RT}^*$ . For any  $x, y \in H_{RT}$ :

$$\begin{aligned} \blacktriangle(f \diamond g)(x \otimes y) &= (f \diamond g)(x \star y) \\ &= (f \otimes g) \Delta_{RT}(x \star y) \\ &= (f \otimes g)(x \star \Delta_{RT}(y) + \Delta_{RT}(x) \star y + x \otimes y) \\ &= (\blacktriangle(f) \diamond g + f \diamond \blacktriangle(g) + f \otimes g)(x \otimes y), \end{aligned}$$

which implies the result. □

Let us then dualize Theorem 4.2:

**Theorem 4.13.** The transpose of  $\theta$  is the map  $\theta^*$  given by:

$$\forall f \in H_{RT}^*, \quad \theta^*(f) := \sum_{k=1}^{\infty} (-1)^{k+1} \diamond^{(k-1)} \circ \blacktriangle^{(k-1)}(f),$$

where  $\blacktriangle^{(l)} : H_{RT}^* \longrightarrow (H_{RT}^*)^{\otimes(l+1)}$  and  $\diamond^{(l)} : (H_{RT}^*)^{\otimes(l+1)} \longrightarrow H_{RT}^*$  are inductively defined:

$$\begin{aligned} \blacktriangle^{(0)} &= \text{id}_{H_{RT}^*}, & \diamond^{(0)} &= \text{id}_{H_{RT}^*}, \\ \blacktriangle^{(l+1)} &= (\blacktriangle^{(l)} \otimes \text{id}_{H_{RT}^*}) \circ \blacktriangle_{RT}, & \diamond^{(l+1)} &= \diamond \circ (\diamond^{(l)} \otimes \text{id}_{H_{RT}^*}). \end{aligned}$$

Then  $\theta^*$  is a projector on  $\text{Ker}(\blacktriangle) = \mathbf{k}\{Z_F, \text{no tree of } F \text{ is equal to } \bullet\}$ . The kernel of  $\theta^*$  is

$$\text{Ker}(\theta^*) = H_{RT}^* \diamond H_{RT}^*.$$

*Proof.* The description of  $\theta^*$  comes from  $\diamond = \Delta_{RT}^*$  and  $\blacktriangle = \star^*$ . As  $\theta$  is the projection on  $\text{Ker}(\Delta_{RT})$  which vanishes on  $\text{Im}(\star)$ ,  $\theta^*$  is the projection on  $\text{Im}(\star)^\perp$  which vanishes on  $\text{Ker}(\Delta_{RT})^\perp$ , and:

$$\begin{aligned} \text{Im}(\star)^\perp &= \text{Ker}(\star^*) = \text{Ker}(\blacktriangle), \\ \text{Ker}(\Delta_{RT})^\perp &= \text{Im}(\Delta_{RT}^*) = \text{Im}(\diamond) = H_{RT}^* \diamond H_{RT}^*. \end{aligned}$$

The description of  $\text{Ker}(\blacktriangle)$  is immediate. □

Consequently,

$$H_{RT}^* = \text{Ker}(\blacktriangle) \oplus H_{RT}^* \diamond H_{RT}^*.$$

As  $(H_{RT}^*, \diamond)$  is a free non unitary algebra, any complement of  $H_{RT}^* \diamond H_{RT}^*$  freely generates it as an algebra. Hence:

**Corollary 4.14.** The algebra  $(H_{RT}^*, \diamond)$  is freely generated by the elements  $Z_F$ , where  $F$  is a forest with no tree equal to  $\cdot$ .

**Acknowledgments:** This work was supported by the National Natural Science Foundation of China (Grant No. 11861051).

#### REFERENCES

- [1] M. Aguiar, Infinitesimal Hopf algebras, *Contemporary Mathematics* **267** (2000), 1-30.
- [2] Ch. Brouder, Trees, renormalization and differential equations, *BIT. Numerical Mathematics* **44(3)** (2004), 425-438.
- [3] A. Connes and D. Kreimer, Hopf algebras, renormalization and non-commutative geometry, *Comm. Math. Phys.* **199(1)** (1998), 203-242.
- [4] R. Diestel, Graph Theory, Springer-Verlag, 2006.
- [5] L. Foissy, Les algèbres de Hopf des arbres enracinés décorés I, *Bull. Sci. Math.* **126** (2002), 193-239.
- [6] L. Foissy, Bidendriform bialgebras, trees, and free quasi-symmetric functions, *J. Pure Appl. Algebra* **209** (2007), 439-459.
- [7] L. Foissy, The infinitesimal Hopf algebra and the poset of planar forests, *J. Algebra. Comb.* **30** (2009), 277-309.
- [8] L. Foissy, Introduction to Hopf Algebra of Rooted Trees, available at <http://loic.foissy.free.fr/pageperso/preprint3.pdf>.
- [9] X. Gao and X. Wang, Infinitesimal unitary Hopf algebras and planar rooted forests, *Journal of algebra combinatorics* **49** (2019) 437-460.
- [10] R. Grossman and R.G. Larson, Hopf-algebraic structure of families of trees, *J. Algebra* **126(1)** (1989), 184-210.
- [11] L. Guo, An Introduction to Rota-Baxter Algebra, International Press, 2012.
- [12] M. Hoffman, Combinatorics of rooted trees and Hopf algebras, *Trans. Amer. Math. Soc.* **355(9)** (2003), 3795-3811.
- [13] R. Holtkamp, Comparison of Hopf algebras on trees. *Arch. Math. (Basel)* **80** (2003), 368-383.
- [14] S. Joni and G.-C. Rota, Coalgebras and bialgebras in combinatorics, *Stud. Appl. Math.* **61** (1979), 93-139.
- [15] J.-L. Loday and M.O. Ronco, On the structure of cofree Hopf algebras, *J. Reine Angew. Math.* **592** (2006), 123-155.
- [16] F. Panaite, Relating the Connes-Kreimer and Grossman-Larson Hopf algebras built on rooted trees, *Lett. Math. Phys.* **51(3)** (2000), 211-219.
- [17] T. J. Zhang, X. Gao and L. Guo, Hopf algebras of rooted forests, cocycles and free Rota-Baxter algebras, *Journal of Mathematical Physics* **57** (2016), 101701.

SCHOOL OF MATHEMATICS AND STATISTICS, LANZHOU UNIVERSITY, LANZHOU, GANSU 730000, P. R. CHINA  
*Email address:* wangxm2015@lzu.edu.cn

UNIV. LITTORAL CÔTE D'OPALE, UR 2597, LMPA, LABORATOIRE DE MATHÉMATIQUES PURES ET APPLIQUÉES JOSEPH  
LIOUVILLE F-62100 CALAIS, FRANCE  
*Email address:* foissy@univ-littoral.fr

SCHOOL OF MATHEMATICS AND STATISTICS, KEY LABORATORY OF APPLIED MATHEMATICS AND COMPLEX SYSTEMS, LANZHOU  
UNIVERSITY, LANZHOU, GANSU 730000, P. R. CHINA  
*Email address:* gaoxing@lzu.edu.cn