Central limit theorem for a critical multi-type branching process in random environment

Emmanuelle Le Page, M. Peigné, C. Pham

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Abstract

Let $(Z_n)_{n\geq 0}$ with $Z_n = (Z_n(i,j))_{1\leq i,j\leq p}$ be a p multi-type critical branching process in random environment, and let M_n be the expectation of Z_n given a fixed environment. We prove theorems on convergence in distribution of sequences of branching processes $\left\{\frac{Z_n}{|M_n|}/|Z_n|>0\right\}$ and $\left\{\frac{\ln Z_n}{\sqrt{n}}/|Z_n|>0\right\}$. These theorems extend similar results for single-type critical branching process in random environment.

Keywords: multi-type branching process, random environments, central limit theorem, critical case.

1 Introduction

Single-type and multi-type branching processes in random environments (BPREs) are a central topic of research; they were introduced in the 1960s in order to describe the development of populations whose evolution may be randomly affected by environmental factors.

In the single-type case, the behaviour of these processes is mainly determined by the 1-dimensional random walk generated by the logarithms of the expected population sizes m_k, for $k \geq 0$, of the respective generations; they are classified in three classes - supercritical, critical and subcritical - of single-type BPREs, according to the fact that the associated random walk tends to $+\infty$, oscillates or tends to $-\infty$. Their study is closely related to the theory of fluctuations of random walks on $\mathbb{R}$ with i.i.d. increments; when $\mathbb{E}\left[|\log m_k|\right] < +\infty$, the BPRE is supercritical (resp., critical and subcritical) when $\mathbb{E}\big[|\log m_k|\big] > 0$ (resp., $\mathbb{E}\big[|\log m_k|\big] = 0$ or $\mathbb{E}\big[|\log m_k|\big] < 0$).

In this context, a huge body of papers is devoted to study of the asymptotic behaviour of the probability of non-extinction up to time n and the distribution of the population size conditioned to survival up to time n. In the critical case, the branching process is degenerate with probability one and the probability of non-extinction up to time n is equivalent to $c/\sqrt{n}$ as $n \to +\infty$, for some explicit positive constant c [15], [10]. The convergence in distribution of the process conditioned to non-extinction comprises first

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the Yaglom classical theorem; the convergence of finite-dimensional distributions of the processes was established by Lamperti and Ney [16] who showed that the limiting process is a diffusion process and described its transition function. V.I. Afanasev described the limiting process in terms of Brownian excursions [1]. These statements more or less claim that the conditional logarithmic behaviour of the BPRE given its non-extinction at the terminal time n is the same as the one of the associated random walk of conditional mean values conditioned to staying positive. These results are extended in [3] under more general assumptions, known as Spitzer's condition in fluctuation theory of random walks, and some additional moment conditions.

It is of interest to prove analogues of the above statements for multi-type BPREs $(Z_n)_{n\geq 0}$. As in the single-type case, the set of multi-type BPREs may be divided into three classes: they are supercritical (resp. critical or subcritical) when the upper Lyapunov exponent of the product of random mean matrices M_k is positive (resp. null or negative) [13]. Let us emphasize that the role of the random walk associated to the BPRE in the single-type case is played in the multi-type case by the logarithm of the norm of some $\mathbb{R}^p$-valued Markov chain whose increments are governed by i.i.d. random $p \times p$-matrices M_k, for $k \geq 0$, whose coefficients are non-negative and correspond to the expected population sizes of the respective generations, according to the types of the particles and their direct parents. Product of random matrices is the object of several investigations and many limit theorems do exist in this context (see [4] and references therein). The theory of their fluctuations is recently studied during the last decade using the promising approach initiated by V. Denisov V. Wachtel [5].

The question of the asymptotic behaviour of the probability of non-extinction up to time n is solved recently, under quite general moment assumptions and irreducibility condition on the projective action of the product of matrices M_k; as in the single-type case, it is proved that the probability of non-extinction up to time n is equivalent to $c/\sqrt{n}$ as $n \to +\infty$, for some explicit positive constant c [17], [8]. The asymptotic distribution of the size of the population conditioned to non-extinction remains open; in this paper, we prove a central limit theorem for the logarithm of the size of the population at time n, conditioned to non-extinction.

2 Preliminaries, hypotheses and statements

We fix an integer $p \geq 2$ and denote by $\mathbb{R}^p$ (resp. $\mathbb{N}^p$) the set of p-dimensional column vectors with real (resp. non-negative integer) coordinates; for any column vector x of $\mathbb{R}^p$ defined by $x = (x_i)_{1\leq i\leq p}$, we denote by $\tilde{x}$ the row vector $\tilde{x} := (x_1, \ldots, x_p)$. Let $\mathbf{1}$ (resp. $\mathbf{0}$) be the column vector of $\mathbb{R}^p$ whose all coordinates equal 1 (resp. 0). We denote by $\{e_i, 1 \leq i \leq p\}$ the canonical basis, by $\langle \cdot, \cdot \rangle$ the usual scalar product and by $|.|$ the corresponding $\mathbb{L}^1$ norm. We also consider the general linear semi-group $\mathcal{S}^+$ of $p \times p$ matrices with non-negative coefficients. We endow $\mathcal{S}^+$ with the $\mathbb{L}^1$-norm denoted also by $|.|$.

The multi-type Galton-Watson process we study here is the Markov chain $(Z_n)_{n\geq 0}$ whose states are $p \times p$ matrices with integer entries. We always assume that Z_0 is non-random. For any $1 \leq i, j \leq p$, the $(i,j)^{th}$ component $Z_n(i,j)$ of Z_n may be interpreted as the number of particles of type j in the n^{th} generation providing that the ancestor at time 0 is of type i. In particular, $\sum_{i=1}^p Z_n(i,j) = |Z_n(.,j)|$ equals the number of particles of type j in the n^{th} generation when there is 1 ancestor of each type at generation 0; this quantity equals $Z_n(\tilde{\mathbf{1}}, j)$, with the notations introduced below. Similarly, $\sum_{j=1}^p Z_n(i,j) = |Z_n(i,\cdot)|$ equals the size of the population at time n when there is one ancestor of type i at time 0.

Let us introduce Galton-Watson process in varying environment and assume that the offspring distributions of the process $(Z_n)_{n\geq 0}$ are given by a sequence of $\mathbb{N}^p$-valued random variables $(\xi_n)_{n\geq 0}$. More precisely, the distribution of the number of typed j children born to a single-typed i parent at time n is the same as the one of $\xi_n(i,j)$. Let $\xi_{n,k}, n\geq 0, k\geq 1$, be independent random variables defined on $(\Omega,\mathcal{F},\mathbb{P})$, where the $\xi_{n,k}, k\geq 1$ have the same distribution as ξ_n, for any $n\geq 0$.

The process $(Z_n)_{n\geq 0}$ is thus defined by recurrence as follows: for any $1\leq i,j\leq p$ and $n\geq 0$,

$$Z_{n+1}(i,j) := \begin{cases} \displaystyle\sum_{\ell=1}^{p} \sum_{k=1+Z_n(i,1)+\ldots+Z_n(i,\ell-1)}^{Z_n(i,1)+\ldots+Z_n(i,\ell)} \xi_{n,k}(\ell,j) & \text{when} \quad \displaystyle\sum_{\ell=1}^{p} Z_n(i,\ell)>0; \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{G}$ the set of multivariate probability generating functions $g=(g^{(i)})_{1\leq i\leq p}$ defined by:

$$g^{(i)}(s)=\sum_{\alpha\in\mathbb{N}^p} p^{(i)}(\alpha)s^{\alpha},$$

for any $s=(s_i)_{1\leq i\leq p}\in[0,1]^p$, where

1. $\alpha=(\alpha_i)_i\in\mathbb{N}^p$ and $s^{\alpha}=s_1^{\alpha_1}\ldots s_p^{\alpha_p}$;
2. $p^{(i)}(\alpha)=p^{(i)}(\alpha_1,\ldots,\alpha_p)$ is the probability that a parent of type i has α_1 children of type $1,\ldots,\alpha_p$ children of type p.

For each $1\leq i\leq p$, the distribution of the i^{th} row vector $(\xi_n(i,j))_{1\leq j\leq p}$ of the ξ_n is characterized by its generating function denoted by $g_n^{(i)}$, and set $g_n=(g_n^{(i)})_{1\leq i\leq p}$ and $\mathbf{g}=(g_n)_{n\geq 0}$. For any $s=(s_i)_{1\leq i\leq p}\in[0,1]^p$,

$$g_n^{(i)}(s)=\mathbb{E}\left[s_1^{\xi_n(i,1)}s_2^{\xi_n(i,2)}\ldots s_p^{\xi_n(i,p)}\right].$$

For a given sequence $\mathbf{g}$ in $\mathcal{G}$, we denote by $Z^{\mathbf{g}}$ the Galton-Watson process corresponding to $\mathbf{g}$ and omit the exponent $\mathbf{g}$ when there is no confusion; furthermore, we set for any $n\geq 1$,

$$g_{0,n}=g_0\circ g_1\ldots\circ g_{n-1}=(g_{0,n}^{(i)})_{1\leq i\leq p},$$

where $g_{0,n}^{(i)}=g_0^{(i)}\circ g_1\ldots\circ g_{n-1}$. For any $1\leq i\leq p$ and $s\in[0,1]^p$,

$$\mathbb{E}\left[s^{Z_n^{\mathbf{g}}(i,\cdot)}\Big/ Z_0^{\mathbf{g}}(i,\cdot)\ldots,Z_{n-1}^{\mathbf{g}}(i,\cdot)\right]=g_{n-1}(s)^{Z_{n-1}^{\mathbf{g}}(i,\cdot)},$$

which yields that

$$\mathbb{E}\left[s^{Z_n^{\mathbf{g}}(i,\cdot)}\right]=\mathbb{E}\left[s_1^{Z_n^{\mathbf{g}}(i,1)}s_2^{Z_n^{\mathbf{g}}(i,2)}\ldots s_p^{Z_n^{\mathbf{g}}(i,p)}\right]=g_{0,n}^{(i)}(s).$$

More generally, for any $z=(z_1,\ldots,z_p)\in\mathbb{N}^p$, we denote by $Z_n^{\mathbf{g}}(\tilde{z},j)$ the number of particles of type j in the n^{th} generation providing that there are z_i ancestors of type i at time 0, for any $1\leq i\leq p$. Therefore,

$$\mathbb{E}\left[s^{Z_n^{\mathbf{g}}(\tilde{z},\cdot)}\Big/ Z_0^{\mathbf{g}}(\tilde{z},\cdot),\ldots,Z_{n-1}^{\mathbf{g}}(\tilde{z},\cdot)\right]=g_{0,n}(s)^{\tilde{z}}.$$

Before going further, we introduce some necessary notations. For $n\geq 0$,

1. we denote by M_{g_n} the mean matrix $\mathbb{E}\xi_n$:

$$M_{g_n} = (M_{g_n}(i,j))_{1\leq i,j\leq p} \quad \text{with} \quad M_{g_n}(i,j) = \mathbb{E}[\xi_n(i,j)];$$

in other words,

$$M_{g_n} = \begin{pmatrix} \dfrac{\partial g_n^{(1)}(\tilde{\mathbf{1}})}{\partial s_1} & \cdots & \dfrac{\partial g_n^{(1)}(\tilde{\mathbf{1}})}{\partial s_p} \\ \vdots & & \\ \dfrac{\partial g_n^{(p)}(\tilde{\mathbf{1}})}{\partial s_1} & \cdots & \dfrac{\partial g_n^{(p)}(\tilde{\mathbf{1}})}{\partial s_p} \end{pmatrix};$$

2. for $1 \leq i \leq p$, let $B_{g_n}^{(i)}$ be the Hessian matrices

$$\begin{aligned} B_{g_n}^{(i)} &= \left(B_{g_n}^{(i)}(k,\ell)\right)_{1\leq k,\ell\leq p} := \left(\frac{\partial^2 g_n^{(i)}}{\partial s_k \partial s_\ell}(\tilde{\mathbf{1}})\right)_{1\leq k,\ell\leq p} \\ &= \left(\mathbb{E}\big[\xi_n(i,k)(\xi_n(i,\ell) - \delta_{k,\ell})\big]\right)_{1\leq k,\ell\leq p}; \end{aligned}$$

in particular, $\sigma_{g_n}^2(i,j) := \mathrm{Var}(\xi_n(i,j)) = B_{g_n}^{(i)}(j,j) + M_{g_n}(i,j) - M_{g_n}(i,j)^2$.

3. and we set $\mu_{g_n} := \sum_{i=1}^{p} |B_{g_n}^{(i)}|$ and $\eta_{g_n} := \dfrac{\mu_{g_n}}{|M_{g_n}|^2}$.

The product of matrices $M_{0,n}^{\mathbf{g}} := M_{g_0} \dots M_{g_{n-1}}$ controls the mean value of Z_n, according to the value of Z_0; indeed $\mathbb{E}[Z_n] = Z_0 M_{g_0} \dots M_{g_{n-1}}$ for any $n \geq 0$. The matrices M_{g_n}, for $n \geq 0$, have non-negative entries which plays an important role on the asymptotic behaviour of the products $M_{0,n}^{\mathbf{g}}$, for $n \geq 0$.

We consider the cone of p-dimensional row vectors

$$\mathbb{R}_+^p := \left\{\tilde{x} = (x_1, \dots, x_p) \in \mathbb{R}^p \;\middle/\; \forall i = 1, \dots, p,\ x_i \geq 0\right\},$$

and the corresponding simplex $\mathbb{X}$ defined by:

$$\mathbb{X} := \{\tilde{x} \in \mathbb{R}_+^p \;/\; |\tilde{x}| = 1\}.$$

We introduce the actions of the semi-group $\mathcal{S}^+$ on $\mathbb{R}_+^p$ and $\mathbb{X}$ as follows:

- the right and the left linear actions of $\mathcal{S}^+$ on $\mathbb{R}_+^p$ defined by:

$$(\tilde{x}, M) \mapsto \tilde{x}M \quad \text{and} \quad (\tilde{x}, M) \mapsto Mx$$

for any $\tilde{x} \in \mathbb{R}_+^p$ and $M \in \mathcal{S}^+$,

- the right and the left projective actions of $\mathcal{S}^+$ on $\mathbb{X}$ defined by:

$$(\tilde{x}, M) \mapsto \tilde{x} \cdot M := \frac{\tilde{x}M}{|\tilde{x}M|} \quad \text{and} \quad (\tilde{x}, M) \mapsto M \cdot x := \frac{Mx}{|Mx|}$$

for any $\tilde{x} \in \mathbb{X}$ and $M \in \mathcal{S}^+$.

For any $M \in \mathcal{S}^+$, let $v(M) := \min_{1\le j\le p}\Big(\sum_{i=1}^{d} M(i,j)\Big)$. Then for any $\tilde{x} \in \mathbb{R}_+^p$,

$$0 < v(M)\ |x| \le |Mx| \le |M|\ |x|.$$

We set $\mathfrak{n}(M) := \max\Big(\frac{1}{v(M)}, |M|\Big)$.

We also introduce some proper subset of $\mathcal{S}^+$ which is of interest in the sequel: for any constant $B \ge 1$, let $\mathcal{S}^+(B)$ denote the set of $p \times p$ matrices M with positive coefficients such that for any $1 \le i, j, k, l \le p$,

$$\frac{1}{B} \le \frac{M(i,j)}{M(k,l)} \le B.$$

Following [6], we introduce some proper subset of generating functions of offspring distributions; let ξ be a $\mathbb{N}^p$-valued random variable defined by $\xi = (\xi(i,j))_{1\le i,j\le p}$, with generating function g (as described above).

Notation 2.1 *Let $\varepsilon \in]0,1[$ and $K > 0$. We denote by $\mathcal{G}_{\varepsilon,K}$ the set of generating functions of multivariate offspring distributions satisfying the following non-degeneracy assumptions: for any $n \ge 0$ and $1 \le i, j \le p$,*

(1) $\mathbb{P}\big(\xi(i,j) \ge 2\big) \ge \varepsilon$,

(2) $\mathbb{P}\big(\xi(i,\cdot) = \mathbf{0}\big) \ge \varepsilon$,

(3) $\mathbb{E}\big[|\xi(i,\cdot)|^2\big] \le K < +\infty$.

D. Dolgopyat and co-authors in [6] proposed a deep and useful description of the behaviour of the process $(Z_n^{\mathbf{g}})_{n\ge 0}$ when all the generating functions g_n of the varying environment $\mathbf{g}$ belong to $\mathcal{G}_{\varepsilon,K}$. We present and extend their results in section 3.3, they play a key role in controlling the fluctuations of the Galton-Watson process in random environment, conditioned to non-extinction (see Corollary 4.3).

In random environment, we consider a sequence $\mathbf{f} = (f_n)_{n\ge 0}$ of i.i.d. random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and we set $f_{0,n} = f_0 \circ f_1 \ldots \circ f_{n-1}$ for any $n \ge 1$; as above, for any $\tilde{z} = (z_1, \ldots, z_p) \in \mathbb{N}^p$ and $s \in [0,1]^p$,

$$\mathbb{E}\Big[s^{Z_n(\tilde{z},\cdot)}\Big/ Z_0(\tilde{z},\cdot)\ldots, Z_{n-1}(\tilde{z},\cdot); f_0, \ldots, f_{n-1}\Big] = f_{0,n}(s)^{Z_{n-1}(\tilde{z},\cdot)}.$$

For $n \ge 0$, the random matrices M_n and $B_n^{(i)}$ are i.i.d. and of non-negative entries. The common law of the M_n is denoted by μ. In order to simplify the notations, we set $M_n := M_{f_n}, B_n^{(i)} = B_{f_n}^{(i)}$ and $\eta_n := \eta_{f_n}$, with the notice that η_n are non-negative real valued i.i.d. random variables. Moreover, let $M_{0,n}$ and $M_{n,0}$ denote the right and the left product of random matrices M_k for $k \ge 0$, respectively $M_{0,n} = M_{0,n}^{\mathbf{f}} = M_0 M_1 \ldots M_{n-1}$ and $M_{n,0} = M_{n-1}\ldots M_1 M_0$, with the convention that $M_{0,0} = \mathrm{I}$. Therefore,

$$\mathbb{E}\Big[Z_n\Big/ f_0, f_1, \ldots, f_{n-1}\Big] = Z_0 M_0 \ldots M_{n-1} = Z_0 M_{0,n} \qquad \mathbb{P}\text{-a.s.}.$$

For any $1 \le i \le p$, the probability of non-extinction at generation n given the environment $f_0, f_1, \ldots f_{n-1}$ is

$$\begin{aligned} q_{n,i}^{\mathbf{f}} &:= \mathbb{P}\Big(Z_n(i,\cdot) \ne \tilde{\mathbf{0}}\Big/ f_0, f_1, \ldots, f_{n-1}\Big) \\ &= 1 - f_0^{(i)}(f_1(\ldots f_{n-1}(\tilde{\mathbf{0}})\ldots)) = \tilde{e}_i(\mathbf{1} - f_0(f_1(\ldots f_{n-1}(\tilde{\mathbf{0}})\ldots))), \end{aligned}$$

where the letter i presents the unique typed i ancestor, so that

$$\mathbb{E}[q_{n,i}^{\mathbf{f}}] = \mathbb{E}\Big[\mathbb{P}(Z_n(i,\cdot) \neq \tilde{\mathbf{0}} \Big/ f_0, f_1, \ldots, f_{n-1})\Big] = \mathbb{P}(Z_n(i,\cdot) \neq \tilde{\mathbf{0}}).$$

As in the classical single-type case, the asymptotic behaviour of the quantity $\mathbb{E}[q_{n,i}^{\mathbf{f}}]$ above is controlled by the mean of the offspring distributions. By [9], if $\mathbb{E}\big[\ln^+ |M_0|\big] < +\infty$, then the sequence $\left(\frac{1}{n}\ln|M_{0,n}|\right)_{n\geq 1}$ converges $\mathbb{P}$-almost surely to some constant limit $\pi_\mu := \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}\big[\ln|M_{0,n}|\big]$. On the product space $\mathbb{X}\times\mathcal{S}^+$, we define the function ρ by setting $\rho(\tilde{x}, M) := \ln|\tilde{x}M|$ for $(\tilde{x}, M) \in \mathbb{X}\times\mathcal{S}^+$. This function satisfies the cocycle property, namely for any $M, N \in \mathcal{S}^+$ and $\tilde{x} \in \mathbb{X}$,

$$\rho(\tilde{x}, MN) = \rho(\widetilde{x}\ \cdot\ M, N) + \rho(\tilde{x}, M). \tag{2.1}$$

Under hypothesis H3(δ) introduced below, there exists a unique μ-invariant measure ν on $\mathbb{X}$ such that for any continuous function φ on $\mathbb{X}$,

$$(\mu * \nu)(\varphi) = \int_{\mathcal{S}^+}\int_{\mathbb{X}} \varphi(\tilde{x}\cdot M)\nu(d\tilde{x})\mu(\mathrm{d}M) = \int_{\mathbb{X}} \varphi(\tilde{x})\nu(d\tilde{x}) = \nu(\varphi).$$

Moreover, the upper Lyapunov exponent π_μ defined above coincides with the quantity $\int_{\mathbb{X}\times\mathcal{S}^+} \rho(\tilde{x}, M)\mu(\mathrm{d}M)\nu(d\tilde{x})$ and is finite [4].

For any $0 < \delta < 1$, we consider the following hypotheses concerning the distribution μ of the mean matrices M_n and the distributions of the random variables ξ_n at each step.

Hypotheses

H1(δ). $\mathbb{E}[|\ln \mathfrak{n}(M_1)|^{2+\delta}] < +\infty$.

H2. *(Strong irreducibility) There exists no affine subspaces $\mathcal{A}$ of $\mathbb{R}^d$ such that $\mathcal{A}\cap\mathbb{R}^p_+$ is non-empty, bounded and invariant under the action of all elements of the support of μ.*

H3(δ). *The support of μ is included in $\mathcal{S}^+(B)$ with $B = \frac{1}{\delta}$.*

H4. *The upper Lyapunov exponent π_μ equals 0.*

H5(δ). *$\mu(E_\delta) > 0$, where $E_\delta := \{M \in \mathcal{S}^+ \ / \forall \tilde{x} \in \mathbb{X},\ \ \ln|\tilde{x}M| \geq \delta\}$.*

H6. $\mathbb{E}\Big[\frac{\mu_1}{|M_1|^2}(1 + \ln^+|M_1|)\Big] < +\infty$.

Notice that the moment hypotheses H1(δ), H3(δ) and H6 are satisfied when the offspring generating functions f_n, for $n \geq 0$, belong to some $\mathcal{G}_{\varepsilon,K}$; indeed, in this case, for any $1 \leq i, j \leq p$,

$$2p\varepsilon \leq |M_1| \leq \sum_{i,j=1}^{p} \mathbb{E}[\xi_1^2(i,j)] \leq p^2K \quad \text{and} \quad \mu_1 \leq p^3K \qquad \mathbb{P}-\text{a.s.}$$

A lot of researchers investigated the behaviour of the survival probability of $(Z_n)_{n\geq 0}$ in random environment, under various sets of rather restrictive assumptions. In [17] and [8], we can find the most general statement in this context of multi-type Galton-Watson process: when hypotheses H1–H6 hold for some $\delta > 0$, there exist positive numbers $\beta_1, \ldots, \beta_p$ such that for $1 \leq i \leq p$,

$$\lim_{n\to+\infty} \sqrt{n}\mathbb{P}(Z_n(i,\cdot) \neq \tilde{\mathbf{0}}) = \beta_i. \tag{2.2}$$

Let β be the column vector in $\mathbb{R}_+^p$ with entries $\beta_1, \ldots, \beta_p$. Notice that hypothesis H6 above is weaker than the one in [8]; indeed, the key argument in [17] and [8] is based on our Lemma 4.2, whose proof is detailed in this article, which remains valid under assumptions H1–H6 (for some $\delta > 0$).

The convergence (2.2) relies on a deep understanding, developed in [19], of the behavior of the semi-markovian random walk $(S_n(\tilde{x}, a))_{n\geq 0}$ defined by $S_n(\tilde{x}, a) = a + \ln|\tilde{x}M_{0,n}|$, for any $\tilde{x} \in \mathbb{X}, a \geq 0$ and $n \geq 0$. It is well known that this Markov walk satisfies a strong law of large number and a central limit theorem; denote by $\sigma^2 := \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}[S_n^2(\tilde{x}, a)]$ its variance and recall that, under Hypotheses H1 to H5, the quantity σ^2 is positive.

Here comes the main result of the present paper; it concerns the asymptotic distribution of the random variables $\ln|Z_n(\tilde{z}, \cdot)|$ conditioned to non-extinction.

Theorem 2.2 *Assume that*

(1) there exist $\varepsilon \in]0, 1[$ and $K > 0$ such that $\mathbb{P}$-a.s, the offspring distributions f_n, for $n \geq 0$, belong to $\mathcal{G}_{\varepsilon,K}$;

(2) there exists $\delta > 0$ such that hypotheses H2, H4 *and* H5*(δ) hold.*

Then for any $\tilde{z} \in \mathbb{N}^p \setminus \{\tilde{\mathbf{0}}\}$ and $t \geq 0$,

$$\lim_{n\to+\infty} \mathbb{P}\left(\frac{\ln|Z_n(\tilde{z}, \cdot)|}{\sqrt{n}} \leq t \;\Big/\; |Z_n(\tilde{z}, \cdot)| > 0\right) = \frac{2}{\sigma\sqrt{2\pi}}\Phi^+\left(\frac{t}{\sigma}\right),$$

where Φ^+ denotes the cumulative function of the Rayleigh distribution:

$$\Phi^+\left(\frac{t}{\sigma}\right) := \int_0^t s\exp\left(-\frac{s^2}{2}\right)\mathrm{d}s.$$

The first step to prove this main theorem is to provide a limit theorem for the processes $(Z_n(\tilde{z}, j))_{n\geq 0}$, where $\tilde{z} \in \mathbb{N}^p \setminus \{\tilde{\mathbf{0}}\}$ and $1 \leq j \leq p$, conditioned to non-extinction and randomly rescaled; this statement is of intrinsic interest and holds under weaker the assumptions H1–H6 (for some $\delta > 0$).

Theorem 2.3 *Assume that hypotheses* H1–H6 *hold for some $\delta > 0$. Then, for any $\tilde{z} \in \mathbb{N}^p \setminus \{\tilde{\mathbf{0}}\}$ and $1 \leq j \leq p$, there exists a probability measure $\nu_{\tilde{z},j}$ on $\mathbb{R}^+$ such that for any non-negative continuity point t of the distribution function $s \mapsto \nu_{\tilde{z},j}([0, s])$,*

$$\lim_{n\to+\infty} \mathbb{P}\left(\frac{Z_n(\tilde{z}, j)}{|M_{0,n}e_j|} \leq t \Big/ |Z_n(\tilde{z}, \cdot)| > 0\right) = \nu_{\tilde{z},j}([0, t]).$$

Furthermore, if there exist $\varepsilon \in]0, 1[$ and $K > 0$ such that $f_n \in \mathcal{G}_{\varepsilon,K}$ for any $n \geq 0$, the probability measures $\nu_{\tilde{z},j}$ are supported on $]0, +\infty[$.

Theorem 2.2 is not a direct consequence of Theorem 2.3; we need an intermediate stage which concerns the behaviour of the processes $(\tilde{x}M_0 \ldots M_{n-1})_{n\geq 0}$, for $\tilde{x} \in \mathbb{X}$, conditioned to the event $(|Z_n(\tilde{z}, \cdot)| > 0)$. A close conditioned limit theorem involving the Rayleigh distribution also holds (see Corollary 3.6 below) but its condition is not the one required here. The following proposition fills this gap and is essential to connect the two statements above.

Proposition 2.4 *Assume that hypotheses* H1–H6 *hold for some $\delta > 0$. Then for any $\tilde{z} \in \mathbb{N}^p \setminus \{\tilde{\mathbf{0}}\}$, $\tilde{x} \in \mathbb{X}, a \geq 0$ and $t \geq 0$,*

$$\lim_{n\to+\infty} \mathbb{P}\left(\frac{S_n(\tilde{x}, a)}{\sqrt{n}} \leq t \;\Big/\; |Z_n(\tilde{z}, \cdot)| > 0\right) = \frac{2}{\sigma\sqrt{2\pi}}\Phi^+\left(\frac{t}{\sigma}\right).$$

The article is structured as follows. In section 3, we present some useful auxiliary results on product of random matrices and properties on varying environment. Section 4 is devoted to the random environment while the proofs of Theorem 2.3, Proposition 2.4 and Theorem 2.2 are detailed in sections 5, 6 and 7 respectively.

Notations. *Let $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ be two sequences of positive reals; we write*

- $u_n \overset{c}{\preceq} v_n$ *if* $u_n \leq cv_n$ *for any* $n \geq 0$,

(and simply $u_n \preceq v_n$ when $u_n \overset{c}{\preceq} v_n$ for some constant $c > 0$);

- $u_n \overset{c}{\asymp} v_n$ *when* $\frac{1}{c}u_n \leq v_n \leq cu_n$ *for any* $n \geq 0$,

(and simply $u_n \asymp v_n$ when $u_n \overset{c}{\asymp} v_n$ for some constant $c > 0$);

- $u_n \sim v_n$ *if* $\lim_{n\to+\infty} \dfrac{u_n}{v_n} = 1$.

3 Auxiliary results

In this section, we state some well known and useful results about fluctuations of products of random matrices with non-negative entries and some convergence theorems for multi-type Galton-Watson processes in varying environment.

3.1 On positive matrices and their products

Following [11], we endow $\mathbb{X}$ with a bounded distance $\mathfrak{d}$ such that any $A \in \mathcal{S}$ acts on $\mathbb{X}$ as a contraction with respect to $\mathfrak{d}$. In the following lemma, we just recall some fundamental properties of this distance.

Lemma 3.1 *There exists a distance $\mathfrak{d}$ on $\mathbb{X}$ which is compatible with the standard topology of $\mathbb{X}$ and satisfies the following properties:*

1. $\sup\{\mathfrak{d}(x,y) \ / \ \tilde{x},\tilde{y} \in \mathbb{X}\} = 1$.
2. $|x-y| \leq 2\mathfrak{d}(x,y)$ *for any* $\tilde{x},\tilde{y} \in \mathbb{X}$.
3. *For any $M \in \mathcal{S}^+$, set $[M] := \sup\{\mathfrak{d}(M\cdot x, M\cdot y) \ / \ \tilde{x},\tilde{y} \in \mathbb{X}\}$. Then,*
   (a) $\mathfrak{d}(M\cdot x, M\cdot y) \leq [M]\mathfrak{d}(x,y)$ *for any* $\tilde{x},\tilde{y} \in \mathbb{X}$;
   (b) $[MN] \leq [M][N]$ *for any* $M, N \in \mathcal{S}^+$.
4. *For any $B \geq 1$, there exists $\rho_B \in]0,1[$ such that $[M] \leq \rho_\delta$ for any $M \in \mathcal{S}^+(B)$.*

Similar statements also hold for the right action of $\mathcal{S}^+$ and $\mathcal{S}^+(B)$ on $\mathbb{X}$.

The following Property is a direct consequence of Lemma 3.1: up to some normalization, products of matrices in $S^+(B)$ converge to some rank-one matrix.

For any $M = (M(i,j))_{1\leq i,j\leq p}$ in S^+, we denote by $\overline{M}$ the matrix with entries

$$\overline{M}(i,j) = \frac{M(i,j)}{|M(\cdot,j)|} = \frac{M(i,j)}{\sum_{\ell=1}^{k} M(\ell,j)}.$$

Property 3.2 *Let $\mathbf{M} = (M_n)_{n\geq 0}$ be a sequence of matrices in $\mathcal{S}^+(B)$. Then the sequence $(\overline{M}_{0,n})_{n\geq 0}$ converges as $n \to +\infty$ towards a rank-one matrix whose columns are all equal to $\mathbf{M}_\infty = (\mathbf{M}_\infty(i))_{1\leq i\leq p}$, where, for any $1 \leq j \leq p$,*

$$\mathbf{M}_\infty(i) := \lim_{n\to+\infty} \overline{M}_{0,n}(i,j).$$

Let us also recall some important properties of matrices in $\mathcal{S}^+(B)$.

Lemma 3.3 *[9] Let $T(B)$ be the closed semi-group generated by $\mathcal{S}^+(B)$. For any $M, N \in T(B)$ and $1 \leq i, j, k, l \leq p$,*

$$M(i,j) \overset{B^2}{\asymp} M(k,l).$$

In particular, there exists $c > 1$ such that for any $M, N \in T(B)$ and for any $\tilde{x}, \tilde{y} \in \mathbb{X}$,

1. $|Mx| \overset{c}{\asymp} |M|$ *and* $|\tilde{y}M| \overset{c}{\asymp} |M|$,
2. $|\widetilde{y}Mx| \overset{c}{\asymp} |M|$,
3. $|MN| \overset{c}{\asymp} |M||N|$.

3.2 Limit theorem for products of random positive matrices

Throughout this subsection, the matrices $M_n, n \geq 0$, are i.i.d. and their law μ satisfies hypotheses H1–H5 for some $\delta > 0$. We introduce the homogenous Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{X}$ defined by the initial value $X_0 = \tilde{x} \in \mathbb{X}$ and for $n \geq 1$,

$$X_n = \tilde{x} \cdot M_{0,n}.$$

Its transition probability P is given by: for any $\tilde{x} \in \mathbb{X}$ and any bounded Borel function $\varphi : \mathbb{X} \to \mathbb{R}$,

$$P\varphi(\tilde{x}) := \int_{\mathcal{S}^+} \varphi(\tilde{x} \cdot M)\mu(\mathrm{d}M).$$

In the sequel, we are interested in the left linear action $\tilde{x} \mapsto \widetilde{x}M_{0,n}$ of the right products $M_{0,n}$, for any $\tilde{x} \in \mathbb{X}$. By simple transformation, we see that

$$\widetilde{x}M_{0,n} = e^{\ln|\widetilde{x}M_{0,n}|}\ \widetilde{x} \cdot M_{0,n},$$

which turns it natural to consider the random process $(S_n)_{n\geq 0}$ defined by: for any $\tilde{x} \in \mathbb{X}, a \in \mathbb{R}$ and $n \geq 1$,

$$S_0 = S_0(\tilde{x}, a) := a \quad \text{and} \quad S_n = S_n(\tilde{x}, a) := a + \ln|\widetilde{x}M_{0,n}|.$$

In order to simplify the notations, let $S_n(\tilde{x}) := S_n(\tilde{x}, 0)$ for any $\tilde{x} \in \mathbb{X}$ and any $n \geq 0$. By iterating the cocycle property (2.1), the basic decomposition of $S_n(\tilde{x}, a)$ arrives:

$$S_n(\tilde{x}, a) = a + \ln|\tilde{x}M_{0,n}| = a + \sum_{k=0}^{n-1} \rho(X_k, M_k).$$

It is noticeable that for any $a \in \mathbb{R}$, the sequence $(X_n, S_n)_{n\geq 0}$ is a Markov chain on $\mathbb{X} \times \mathbb{R}$ whose transition probability $\widetilde{P}$ is defined by: for any $(\tilde{x}, a) \in \mathbb{X} \times \mathbb{R}$ and any bounded Borel function $\psi : \mathbb{X} \times \mathbb{R} \to \mathbb{C}$,

$$\widetilde{P}\psi(\tilde{x}, a) = \int_{\mathcal{S}^+} \psi(\tilde{x} \cdot M, a + \rho(\tilde{x}, M))\mu(\mathrm{d}M).$$

For any $\tilde{x} \in \mathbb{X}$ and $a \geq 0$, we denote by $\mathbb{P}_{\tilde{x},a}$ the probability measure on $(\Omega, \mathcal{F}, \mathbb{P})$ conditioned to the event $(X_0 = \tilde{x}, S_0 = a)$ and $\mathbb{E}_{\tilde{x},a}$ the corresponding expectation; the index a is omitted when $a = 0$ and $\mathbb{P}_{\tilde{x}}$ denotes the corresponding probability.

We set $\mathbb{R}_*^+ := \mathbb{R}^+\backslash\{0\}$ and define $\widetilde{P}_+$ the restriction of $\widetilde{P}$ to $\mathbb{X}\times\mathbb{R}^+$: for $a > A$ and any $\tilde{x}\in\mathbb{X}$,

$$\widetilde{P}_+((\tilde{x},a),\cdot) = \mathbf{1}_{\mathbb{X}\times\mathbb{R}^+}(\cdot)\widetilde{P}((\tilde{x},a),\cdot).$$

Furthermore, we introduce

• the first (random) time at which the random process $(S_n(\tilde{x},a))_{n\geq 0}$ becomes non-positive:

$$\tau = \tau_{\tilde{x},a} := \min\{n\geq 1 \ / \ S_n(\tilde{x},a)\leq 0\}.$$

• the minimum m_n for $n\geq 1$, defined by

$$m_n = m_n(\tilde{x},a) := \min\{S_1(\tilde{x},a),\ldots,S_n(\tilde{x},a)\},$$

and we set

$$\mathbf{m}_n(\tilde{x},a) := \mathbb{P}_{\tilde{x},a}(m_n>0) = \mathbb{P}_{\tilde{x},a}(\tau>n) = \mathbb{P}(\tau_{\tilde{x},a}>n).$$

In order to simplify the notations, when $a=0$ let $\tau_{\tilde{x}} := \tau_{\tilde{x},0}$, $m_n(\tilde{x}) := m_n(\tilde{x},0)$ and $\mathbf{m}_n(\tilde{x}) := \mathbf{m}_n(\tilde{x},0)$ for any $\tilde{x}\in\mathbb{X}$ and any $n\geq 0$.

We recall some important results about the behaviour of the random products of variables $M_{0,n}$ and the distribution of τ, under $\mathbb{P}_{\tilde{x},a}$.

Proposition 3.4 *[19] Assume hypotheses* H1 – H5 *for some $\delta>0$. Then for any $\tilde{x}\in\mathbb{X}$ and $a\geq 0$, the sequence $\left(\mathbb{E}_{\tilde{x},a}[S_n;\tau>n]\right)_{n\geq 0}$ converges to some quantity $V(\tilde{x},a)$. The function V is $\widetilde{P}_+$-harmonic on $\mathbb{X}\times\mathbb{R}^+$ and satisfies the following properties:*

1. *for any $\tilde{x}\in\mathbb{X}$, the function $V(\tilde{x},\cdot)$ is increasing on $\mathbb{R}^+$;*
2. *there exist $c>0$ and $A>0$ such that for any $\tilde{x}\in\mathbb{X}$ and $a\geq 0$,*

$$\frac{1}{c}\vee(a-A)\leq V(\tilde{x},a)\leq c(1+a);$$

3. *for any $\tilde{x}\in\mathbb{X}$, the function $V(\tilde{x},.)$ satisfies $\lim_{a\to+\infty}\dfrac{V(\tilde{x},a)}{a}=1$.*

The next statement allows to control the tail of the distribution of the random variable τ.

Theorem 3.5 *[19] Assume hypotheses* H1–H5 *for some $\delta>0$. Then for any $\tilde{x}\in\mathbb{X}$ and $a\geq 0$,*

$$\mathbb{P}_{\tilde{x},a}(\tau>n)\sim\frac{2}{\sigma\sqrt{2\pi n}}V(\tilde{x},a) \text{ as } n\to+\infty,$$

where $\sigma^2>0$ is the variance of the Markov walk $(X_n,S_n)_{n\geq 0}$. Moreover, there exists a constant c such that for any $\tilde{x}\in\mathbb{X}$, $a\geq 0$ and $n\geq 1$,

$$0\leq\sqrt{n}\mathbb{P}_{\tilde{x},a}(\tau>n)\leq cV(\tilde{x},a).$$

Our hypotheses H1–H5 correspond to hypotheses P1–P5 in [19], except that hypothesis H1 is weaker than P1. Indeed, the existence of moments of order $2+\delta$ suffices. This ensures that the map $t\mapsto P_t$ in Proposition 2.3 of [19] is C^2, which suffices for this Proposition to hold. Moreover, the martingale $(M_n)_{n\geq 0}$ which approximates the process $(S_n(x))_{n\geq 0}$ belongs to $\mathbb{L}^p$ for $p=2+\delta$ (and not for any $p>2$ as stated in [19] Proposition 2.6). This last property was useful in [19] to achieve the proof of Lemma 4.5, by choosing p great

enough in such a way that $(p-1)\delta - \frac{1}{2} > 2\varepsilon$ for some fixed constant $\varepsilon > 0$. Recently, by following the same strategy as C. Pham, M. Peigné and W. Woess improved this part of the proof, by allowing various parameters (see [18], Proof of Theorem 1.6 (d)).

As a direct consequence, up to some normalisation and conditioned to the event $[m_n > 0]$, the sequence $(S_n)_{n\geq 1}$ converges weakly to the Rayleigh distribution.

Corollary 3.6 *[19] Assume that hypotheses* H1–H5 *hold for some $\delta > 0$. Then for any $\tilde{x} \in \mathbb{X}$, $a \geq 0$ and $t > 0$,*

$$\lim_{n\to+\infty} \mathbb{P}_{\tilde{x},a}\left(\frac{S_n}{\sqrt{n}} \leq t \Big/ \tau > n\right) = \Phi^+\left(\frac{t}{\sigma}\right).$$

Theorem 3.5 leads to some upper bound in the local limit theorem.

Corollary 3.7 *Assume hypotheses* H1–H5 *for some $\delta > 0$. Then there exists a positive constant c such that for any $\tilde{x} \in \mathbb{X}, a, b \geq 0$ and $n \geq 1$,*

$$0 \leq \mathbb{P}_{\tilde{x},a}\big(S_n \in [b, b+1[, \tau > n\big) \leq c\frac{(1+a)(1+b)}{n^{3/2}}.$$

Proof. We follow and adapt the strategy of Proposition 2.3 in [2]. For fixed $\tilde{x} \in \mathbb{X}$ and $a, b > 0$, we write

$$\begin{aligned}
E_n = E_n(x,a,b) &:= \Big(S_n(\tilde{x},a) \in [b, b+1[,\ \tau_{\tilde{x},a} > n\Big) \\
&= \Big(e^a|\tilde{x}M_{0,n}| \in [e^b, e^{b+1}[,\ e^a|\tilde{x}M_0| > 1, \dots,\ e^a|\tilde{x}M_{0,n}| > 1\Big) \\
&\subset \Big(\tau_{\tilde{x},a} > n/3\Big) \\
&\qquad \cap \Big(|\tilde{x}M_{0,n}| \in [e^{b-a}, e^{b-a+1}[,\ e^a|\tilde{x}M_{0,[2n/3]+1}| > 1, \dots,\ e^a|\tilde{x}M_{0,n}| > 1\Big).
\end{aligned}$$

Let us decompose $M_{0,n}$ into three parts, using the notation $M_{k,n} = M_k \dots M_{n-1}$ for any $0 \leq k < n$ and $n \geq 1$. It holds that $M_{0,n} = M'_n M''_n M'''_n$ with

$$M'_n = M_{0,[n/3]} = M_0 \dots M_{[n/3]-1}, \quad M''_n = M_{[n/3],[2n/3]} = M_{[n/3]} \dots M_{[2n/3]-1}$$

and

$$M'''_n = M_{[2n/3],n} = M_{[2n/3]} \dots M_{n-1}.$$

By Lemma 3.3, we may write, on the one hand

$$\begin{aligned}
\Big(|\tilde{x}M_{0,n}| \in [e^{b-a}, e^{b-a+1}[\Big) &= \Big(|\tilde{x}M'_nM''_nM'''_n| \in [e^{b-a}, e^{b-a+1}[\Big) \\
&\overset{\mathbb{P}\text{-a.s.}}{\subset} \Big(|M'_nM''_nM'''_n| \in [e^{b-a}, ce^{b-a+1}[\Big) \\
&\overset{\mathbb{P}\text{-a.s.}}{\subset} \Big(|M''_n| \in \Big[\frac{e^{b-a}}{|M'_n|\ |M'''_n|}, c^3\frac{e^{b-a+1}}{|M'_n|\ |M'''_n|}\Big[\Big) \\
&\overset{\mathbb{P}\text{-a.s.}}{\subset} \Big(|\tilde{x}M''_n| \in \Big[\frac{e^{b-a}}{c|M'_n|\ |M'''_n|}, c^3\frac{e^{b-a+1}}{|M'_n|\ |M'''_n|}\Big[\Big)
\end{aligned}$$

and on the other hand, for any $[2n/3] < k \leq n$,

$$\frac{1}{c}|\tilde{x}M_{0,k}|\ |M_{k+1,n}| \leq |\tilde{x}M_{0,n}|\ \mathbb{P}\text{-a.s.}$$

This yields that

$$\begin{aligned}\Big(|\tilde{x}M_{0,n}| \in [e^{b-a}, e^{b-a+1}[,\ e^a|\tilde{x}M_{0,[2n/3]+1}| > 1, \dots, e^a|\tilde{x}M_{0,n}| > 1\Big)& \\ \overset{\mathbb{P}\text{-a.s.}}{\subset} \quad \Big(|M_{[2n/3]+1,n}| < ce^{b+1}, \dots, |M_{n-1,n}| < ce^{b+1}\Big)& \\ \overset{\mathbb{P}\text{-a.s.}}{\subset} \quad \Big(|\tilde{x}M_{[2n/3]+1,n}| < ce^{b+1}, \dots, |\tilde{x}M_{n-1,n}| < ce^{b+1}\Big)&.\end{aligned}$$

Finally,

$$E_n \overset{\mathbb{P}\text{-a.s.}}{\subset} E'_n \cap E''_n \cap E'''_n$$

with

$$E'_n = E'_n(x,a,b) = \Big(\tau_{\tilde{x},a} > n/3\Big), \quad E''_n = E''_n(x,a,b) = \Big(|\tilde{x}M''_n| \in \Big[\frac{e^{b-a}}{c|M'_n|\ |M'''_n|}, c^3\frac{e^{b-a+1}}{|M'_n|\ |M'''_n|}\Big[\Big)$$

and

$$E'''_n = E'''_n(x,a,b) = \Big(|\tilde{x}M_{[2n/3]+1,n}| < ce^{b+1}, \dots, |\tilde{x}M_{n-1,n}| < ce^{b+1}\Big).$$

The events E'_n and E'''_n are measurable with respect to the σ-field $\mathcal{T}_n$ generated by $M_0, \dots, M_{[n/3]-1}$ and $M_{[2n/3]}, \dots, M_{n-1}$; consequently,

$$\begin{aligned}\mathbb{P}(E_n) &\le \mathbb{P}\Big(\mathbb{P}\big(E'_n \cap E''_n \cap E'''_n \mid \mathcal{T}_n\big)\Big) \\ &= \mathbb{E}\big[\mathbf{1}_{E'_n \cap E'''_n}\mathbb{P}(E''_n \mid \mathcal{T}_n)\big].\end{aligned}$$

The random variable $\ln|\tilde{x}M''_n|$ are independent on $\mathcal{T}_n$ and their distribution coincides with the one of $S_{[n/3]}(\tilde{x}, 0)$. Therefore, by the classical local limit theorem for product of random matrices with non-negative entries, for any $\tilde{x} \in \mathbb{X}$ and $a, b \ge 0$,

$$\mathbb{P}\big(E''_n \mid \mathcal{T}_n\big) \overset{\mathbb{P}\text{-a.s.}}{\le} \sup_{A\in\mathbb{R}} \mathbb{P}\Big(|\tilde{x}M''_n| \in \Big[A, c^4eA\Big[\Big) \preceq \frac{1}{\sqrt{n}}.$$

Since the events E'_n and E'''_n are independent, it follows that

$$\mathbb{P}(E_n) \preceq \frac{\mathbb{P}(E'_n \cap E''_n)}{\sqrt{n}} = \frac{\mathbb{P}(E'_n)\mathbb{P}(E''_n)}{\sqrt{n}}. \tag{3.1}$$

The probability of $E'_n(x,a,b)$ is controlled by Theorem 3.5: uniformly in $\tilde{x} \in \mathbb{X}$ and $a, b \ge 0$,

$$\mathbb{P}(E'_n(x,a,b)) \preceq \frac{1+a}{\sqrt{n}}. \tag{3.2}$$

To control the probability of the event $E'''_n(x,a,b)$, we introduce the stopping time

$$\tau^+ := \min\{n \ge 1 \mid S_n \ge 0\}$$

and notice that its distribution satisfies the same tail condition as τ. Since

$$(M_{[2n/3]}, M_{[2n/3]+1}, \dots, M_{n-1}) \overset{\text{dist}}{=} (M_0, M_1, \dots, M_{n-[2n/3]-1}),$$

it holds uniformly in $\tilde{x}, a, b$ that,

$$\begin{aligned}\mathbb{P}(E'''_n(x,a,b)) &\le \mathbb{P}\Big(|\tilde{x}M_0| < ce^{b+1}, \dots, |\tilde{x}M_{0,n-[2n/3]}| < ce^{b+1}\Big) \\ &= \mathbb{P}_{\tilde{x},-\ln c-b-1}(\tau^+ > n - [2n/3]) \\ &\preceq \frac{1+b}{\sqrt{n}}.\end{aligned} \tag{3.3}$$

The proof is done, by combining (3.1), (3.2) and (3.3).

□

3.3 On the probability of extinction in varying environment

In this section we state some useful results concerning multi-type Galton-Watson processes in varying environment $\mathbf{g} = (g_n)_{n\geq 0}$.

For any $1 \leq j \leq p$, the quantity $\left|M_{0,n}^{\mathbf{g}}(\cdot, j)\right|$ equals the mean number $\mathbb{E}[Z_n(\tilde{\mathbf{1}}, j)]$ of particles of type j at generation n, given that there is one ancestor of each type at time 0. By Lemma 3.3, if all the M_{g_n} belong to $\mathcal{S}^+(B)$, then $\left|M_{0,n}^{\mathbf{g}}(\cdot, j)\right| \asymp \left|M_{0,n}^{\mathbf{g}}(i, j)\right| \asymp \left|M_{0,n}^{\mathbf{g}}\right|$ for any $1 \leq i, j \leq p$ and $n \geq 0$. Furthermore, by Property 3.2, the sequence of normalized matrices $(\overline{M}_{0,n}^{\mathbf{g}})_{n\geq 0}$ converges as $n \to +\infty$ towards a rank-one matrix with common column vectors $\mathbf{M}_\infty^{\mathbf{g}} = (\mathbf{M}_\infty^{\mathbf{g}}(i))_{1\leq i\leq p}$.

The following statement brings together several results obtained by D. Dolgopyat and al. [6], O. D. Jones [12] and G. Kersting [14] in the varying environment framework. The last point of this statement is a new key to the main theorem of this paper: the conditioned central limit theorem for multi-type Galton-Watson processes in random environment.

Proposition 3.8 *Let $Z^{\mathbf{g}} = (Z_n^{\mathbf{g}})_{n\geq 0}$ be a p multi-type Galton-Watson process in varying environment $\mathbf{g} = (g_n)_{n\geq 0}$. Assume that $\mathbf{g} \in \mathcal{E}$ and that there exists $B > 1$ such that for any $n \geq 0$, the mean matrices M_{g_n} belong to $\mathcal{S}^+(B)$. Then*

1. *the extinction of the process $Z^{\mathbf{g}}$ occurs with probability $q_i^{\mathbf{g}} < 1$ for some (hence every) $i \in \{1, \ldots, p\}$ if and only if for some (hence every) $i, j \in \{1, \ldots, p\}$,*

$$\sum_{n\geq 0} \frac{1}{M_{0,n}^{\mathbf{g}}(i, j)} < +\infty. \tag{3.4}$$

2. *If for some (hence every) $i, j \in \{1, \cdots, p\}$,*

$$\sum_{n\geq 0} \frac{1}{|M_{0,n}^{\mathbf{g}}|} \frac{\sigma_{g_n}^2(i, j)}{|M_{g_{n+1}}|^2} < +\infty, \tag{3.5}$$

then there exists a non-negative random column vector $\mathcal{W}^{\mathbf{g}} = (\mathcal{W}^{\mathbf{g}}(i))_{1\leq i\leq p}$ such that $\mathbb{E}[\mathcal{W}^{\mathbf{g}}(i)] = \mathbf{M}_\infty^{\mathbf{g}}(i)$ and as $n \to +\infty$, for every $i, j \in \{1, \ldots, p\}$,

$$\mathcal{W}_n^{\mathbf{g}}(i, j) := \frac{Z_n^{\mathbf{g}}(i, j)}{\left|M_{0,n}^{\mathbf{g}}(\cdot, j)\right|} = \frac{Z_n^{\mathbf{g}}(i, j)}{|\mathbb{E}[Z_n^{\mathbf{g}}(\tilde{\mathbf{1}}, j)]|} \quad \xrightarrow{\mathbb{L}^2(\mathbb{P}_{\mathbf{g}})} \quad \mathcal{W}^{\mathbf{g}}(i). \tag{3.6}$$

3. *Under conditions (3.4) and (3.5), for any $1 \leq i \leq p$, it holds*

$$\mathbb{P}\big(\mathcal{W}^{\mathbf{g}}(i) = 0\big) = \mathbb{P}\big(\liminf_n |Z_n^{\mathbf{g}}(i, \cdot)| = 0\big) = q_i^{\mathbf{g}}. \tag{3.7}$$

Let us shortly comment on this statement.

• The first point means that condition (3.4) is equivalent to the fact that $Z^{\mathbf{g}}$ is supercritical.

• For a single-type supercritical Galton-Watson process $(Z_n)_{n\geq 0}$ in constant environment, it is well known that the sequence $(\mathcal{W}_n)_{n\geq 0}$, where $\mathcal{W}_n := Z_n/\mathbb{E}[Z_n]$, is a non-negative martingale, hence it converges $\mathbb{P}$-a.s. towards some non-negative limit $\mathcal{W}$ (and in $\mathbb{L}^1$ under some other moment conditions). The second assertion corresponds to a weak version of this property for multi-type Galton-Watson processes in varying environment, without the martingale's argument which fails here.

• The third assertion corresponds to the famous "Kesten-Stigum's theorem"; this is new result in the context of multi-type Galton-Watson processes in varying environment, the proof is detailed in subsection 3.3. We refer to [14] and references therein.

Proof. Throughout this proof, in order to simplify the notations, we omit the exponent $\mathbf{g}$, except at the end when we need to specify the environment.

Assertion 1 follows by combining Proposition 2.1 (e) and Theorem 2.2 in [6].

Assertion 2 corresponds to Theorem 1 in [12].

In [12], one may also find some conditions on $\mathbf{g}$ which ensure that equality (3.7) holds; nevertheless, as claimed by the author, "it can be difficult to check them", except in some restrictive cases. Therefore, as far as we know, Assertion 3 is a new statement; we detail its proof here, by following the strategy developed by G. Kersting (Theorem 2 (ii) in [14]) and by using some estimations obtained in [6].

By construction of the $\mathcal{W}^{\mathbf{g}}(i), 1 \leq i \leq p$, the inclusion

$$(\liminf_n |Z_n^{\mathbf{g}}(i,\cdot)| = 0) \subset (\mathcal{W}^{\mathbf{g}}(i) = 0)$$

is obvious. Hence, it suffices to show that $\mathbb{P}(\mathcal{W}^{\mathbf{g}}(i) = 0, \liminf_n |Z_n^{\mathbf{g}}(i,\cdot)| \geq 1) = 0$. We decompose the argument into two steps.

Step 1. *Comparison between* $\mathbb{P}(\liminf_n |Z_n^{\mathbf{g}}(\ell,\cdot)| = 0)$ *and* $\mathbb{P}(\mathcal{W}^{\mathbf{g}}(i) = 0)$.

By formula (7) in [6], since the sequence $\mathbf{g}_1 = (g_k)_{k\geq 1}$ belongs to $\mathcal{E}$, the functions $g_{1,n} = g_1 \circ \ldots \circ g_n$ satisfy the following property: for $s \in [0,1[^p$,

$$\sum_{k=1}^{n} \frac{1}{|M_{1,k}^{\mathbf{g}}|} \preceq \frac{1}{|\mathbf{1} - g_{1,n}(s)|} \preceq \frac{1}{|M_{1,n}^{\mathbf{g}}|} \frac{1}{|\mathbf{1} - s|} + \sum_{k=1}^{n} \frac{1}{|M_{1,k}^{\mathbf{g}}|}. \tag{3.8}$$

By convexity of g_0, there exists $c_0 \geq 1$ such that for any $1 \leq i \leq p$ and $t \in [0,1[^p$,

$$\frac{1}{c_0} \leq \frac{1 - g_0^{(i)}(t)}{|1-t|} \leq c_0. \tag{3.9}$$

(We detail the argument at the end of the present proof). Thus, by combining (3.8), (3.9) and Lemma 3.3, for any $1 \leq i \leq p$, it holds that

$$\sum_{k=1}^{n} \frac{1}{|M_{0,k}^{\mathbf{g}}|} \preceq \frac{1}{1 - g_{0,n}^{(i)}(s)} \preceq \frac{1}{|M_{0,n}^{\mathbf{g}}|} \frac{1}{|\mathbf{1} - s|} + \sum_{k=1}^{n} \frac{1}{|M_{0,k}^{\mathbf{g}}|}.$$

This yields

- on the one hand, by choosing $s = \mathbf{0}$,

$$\frac{1}{\mathbb{P}(|Z_n^{\mathbf{g}}(i,\cdot)| \geq 1)} \succeq \sum_{k=1}^{n} \frac{1}{|M_{0,k}^{\mathbf{g}}|};$$

- on the other hand, by setting $s_\lambda = (e^{-\lambda/|M_{0,n}^{\mathbf{g}}(\cdot,1)|}, 1, \ldots, 1)$ with $\lambda > 0$,

$$\frac{1}{1 - \mathbb{E}\left[e^{-\lambda \frac{Z_n^{\mathbf{g}}(i,1)}{|M_{0,n}^{\mathbf{g}}(\cdot,1)|}}\right]} = \frac{1}{1 - g_{0,n}^{(i)}(s_\lambda)} \leq \frac{1}{|M_{0,n}^{\mathbf{g}}|} \frac{1}{1 - e^{-\frac{\lambda}{|M_{0,n}^{\mathbf{g}}(\cdot,1)|}}} + \sum_{k=1}^{n} \frac{1}{|M_{0,k}^{\mathbf{g}}|}.$$

This readily implies that for any $1 \leq i, \ell \leq p$,

$$\frac{1}{1 - \mathbb{E}\left[e^{-\lambda \frac{Z_n^{\mathbf{g}}(i,1)}{|M_{0,n}^{\mathbf{g}}(\cdot,1)|}}\right]} \preceq \frac{1}{|M_{0,n}^{\mathbf{g}}|} \frac{1}{1 - e^{-\frac{\lambda}{|M_{0,n}^{\mathbf{g}}(\cdot,1)|}}} + \frac{1}{\mathbb{P}(|Z_n^{\mathbf{g}}(\ell,\cdot)| \geq 1)}.$$

By Lemma 3.3, it holds that $|M_{0,n}(\cdot,1)| \asymp |M^{\mathbf{g}}_{0,n}|$; furthermore, these quantities tend to $+\infty$ as $n \to +\infty$, by (3.4). Hence

$$\frac{1}{1-\mathbb{E}\left[e^{-\lambda \mathcal{W}^{\mathbf{g}}(i)}\right]} \preceq \frac{1}{\lambda} + \frac{1}{\mathbb{P}(\liminf_n |Z^{\mathbf{g}}_n(\ell,\cdot)| \geq 1)}.$$

Letting $\lambda \to +\infty$ yields that $\quad \dfrac{1}{1-\mathbb{P}(\mathcal{W}^{\mathbf{g}}(i)=0)} \preceq \dfrac{1}{\mathbb{P}(\liminf_n |Z^{\mathbf{g}}_n(\ell,\cdot)| \geq 1)}$.

In other words, there exists a constant $\kappa \geq 1$ such that for any $i, \ell \in \{1,\dots,p\}$,

$$\mathbb{P}(\liminf_n |Z^{\mathbf{g}}_n(\ell,\cdot)| \geq 1) \ \leq \ \kappa \mathbb{P}(\mathcal{W}^{\mathbf{g}}(i) > 0),$$

which implies that when $\mathbb{P}(\mathcal{W}^{\mathbf{g}}(i) > 0) \leq \frac{1}{2\kappa}$,

$$\mathbb{P}(\liminf_n |Z^{\mathbf{g}}_n(\ell,\cdot)| \ = 0) \ \geq \ \Big(\mathbb{P}(\mathcal{W}^{\mathbf{g}}(i) = 0\Big)^{2\kappa}. \tag{3.10}$$

To get this last inequality, we use the following elementary lemma.

Lemma 3.9 *[14] Let $\kappa \geq 1$ and A, B be two events such that $\mathbb{P}(A) \leq \kappa \mathbb{P}(B)$ and $\mathbb{P}(B) \leq \frac{1}{2\kappa}$. Then*

$$\mathbb{P}(\overline{A}) \geq \mathbb{P}(\overline{B})^{2\kappa}.$$

Step 2. *A martingale argument.*

As G. Kersting in [14], we introduce a martingale defined by: for $k \geq 0$ and any $i \in \{1,\dots,p\}$,

$$\mathcal{M}_k := \mathbb{P}(\mathcal{W}^{\mathbf{g}}(i) = 0 \Big/ Z^{\mathbf{g}}_0, \dots, Z^{\mathbf{g}}_k).$$

It is known that $\mathcal{M}_k \to \mathbf{1}_{(\mathcal{W}^{\mathbf{g}}(i)=0)}$ $\mathbb{P}$-a.s. as $k \to +\infty$ by standard martingale theory; in particular it converges $\mathbb{P}$-a.s. towards 1 on the event $(\mathcal{W}^{\mathbf{g}}(i) = 0)$.

The branching property of the process $(Z^{\mathbf{g}}_n)_{n\geq 0}$ is used to express $\mathcal{M}_k$ in another form. It is noticeable that $\mathcal{W}^{\mathbf{g}}(i)$ depends on the whole sequence $\mathbf{g}$; let us set $\mathbf{g}_k := (g_l)_{l\geq k}$ and denote $\mathcal{W}^{\mathbf{g}_k}(i)$ the random variable defined as in (3.6) but with respect to the Galton-Watson process $Z^{\mathbf{g}_k}$ corresponding to the environment $\mathbf{g}_k$. By the branching property,

$$\mathcal{M}_k = \mathbb{P}(\mathcal{W}^{\mathbf{g}_k}(1) = 0)^{Z^{\mathbf{g}}_k(i,1)} \times \ldots \times \mathbb{P}(\mathcal{W}^{\mathbf{g}_k}(p) = 0)^{Z^{\mathbf{g}}_k(i,p)},$$

so that for $1 \leq \ell \leq p$, as $k \to +\infty$,

$$\mathbb{P}(\mathcal{W}^{\mathbf{g}_k}(\ell) = 0)^{Z^{\mathbf{g}}_k(i,\ell)} \longrightarrow 1 \quad \mathbb{P}\text{-a.s. on the event} \quad (\mathcal{W}^{\mathbf{g}}(i) = 0). \tag{3.11}$$

The same property holds, replacing the event $(\mathcal{W}^{\mathbf{g}_k}(\ell) = 0)$ by $(\liminf_n |Z^{\mathbf{g}_k}_n(\ell,\cdot)| \ = 0)$, namely: for any $\ell \in \{1,\dots,p\}$, as $k \to +\infty$,

$$\mathbb{P}(\liminf_n |Z^{\mathbf{g}_k}_n(\ell,\cdot)| \ = 0)^{Z^{\mathbf{g}}_k(i,\ell)} \longrightarrow 1 \quad \mathbb{P}\text{-a.s. on the event} \quad (\mathcal{W}^{\mathbf{g}}(i) = 0). \tag{3.12}$$

Indeed, every subsequence $\Big(\mathbb{P}(\liminf_n |Z^{\mathbf{g}_{k_r}}_n(\ell,\cdot)| \ = 0)^{Z^{\mathbf{g}}_{k_r}(i,\ell)}\Big)_{r\geq 0}$ has a further subsequence which converges to 1. In order to apply inequality (3.10), we distinguish two cases.

(i) Either $\mathbb{P}(\mathcal{W}^{\mathbf{g}_{k_r}}(\ell) > 0) \leq \frac{1}{2\kappa}$ for k large enough; we may apply (3.10) and (3.11) to obtain, $\mathbb{P}$-a.s. on $(\mathcal{W}^{\mathbf{g}}(i) = 0)$,

$$\liminf_{r\to+\infty} \mathbb{P}(\liminf_n |Z^{\mathbf{g}_{k_r}}_n(\ell,\cdot)| \ = 0)^{Z^{\mathbf{g}}_{k_r}(i,\ell)} \geq \liminf_{r\to+\infty} \mathbb{P}(\mathcal{W}^{\mathbf{g}_{k_r}}(\ell) = 0)^{2\kappa Z^{\mathbf{g}}_{k_r}(i,\ell)} = 1.$$

(ii) Or there exists a further subsequence $(k'_r)_{k\geq 0}$ such that $\mathbb{P}(\mathcal{W}^{\mathbf{g}_{k'_r}}(\ell) > 0) > \frac{1}{2\kappa}$. Hence, (3.11) implies that $Z^{\mathbf{g}}_{k'_r}(i,\ell) \to 0$ $\mathbb{P}$-a.s. on $(\mathcal{W}^{\mathbf{g}}(i) = 0)$, as $r \to +\infty$; in other words $Z^{\mathbf{g}}_{k'_r}(i,\ell) = 0$ for r large enough, thus $\mathbb{P}(\liminf_n |Z_n^{\mathbf{g}_{k'_r}}(\ell,\cdot)| \ = 0)^{Z^{\mathbf{g}}_{k'_r}(i,\ell)} = 1$.

Finally, in both cases, convergence (3.12) holds. By Egorov's theorem, for any $\varepsilon > 0$ and k sufficiently large,

$$\begin{aligned}
&\mathbb{P}(\mathcal{W}^{\mathbf{g}}(i) = 0, \liminf_n |Z^{\mathbf{g}}_n(i,\cdot)| \ \geq 1)\\
&\leq \varepsilon + \mathbb{P}\left(\prod_{\ell=1}^{p} \mathbb{P}(\liminf_n |Z_n^{\mathbf{g}_k}(\ell,\cdot)| \ = 0)^{Z^{\mathbf{g}}_k(i,\ell)} \geq 1-\varepsilon; |Z^{\mathbf{g}}_k(i,\cdot)| \ \geq 1\right)\\
&\leq \varepsilon + \frac{1}{1-\varepsilon}\mathbb{E}\left[\prod_{\ell=1}^{p} \mathbb{P}(\liminf_n |Z_n^{\mathbf{g}_k}(\ell,\cdot)| \ = 0)^{Z^{\mathbf{g}}_k(i,\ell)}; |Z^{\mathbf{g}}_k(i,\cdot)| \ \geq 1\right]\\
&= \varepsilon + \frac{1}{1-\varepsilon}\mathbb{E}\left[\mathbb{P}\left(\liminf_n |Z^{\mathbf{g}}_n(i,\cdot)| \ = 0 \Big/ Z^{\mathbf{g}}_0, \ldots, Z^{\mathbf{g}}_n\right); |Z^{\mathbf{g}}_k(i,\cdot)| \ \geq 1\right]\\
&= \varepsilon + \frac{1}{1-\varepsilon}\mathbb{P}\left(\liminf_n |Z^{\mathbf{g}}_n(i,\cdot)| \ = 0; |Z^{\mathbf{g}}_k(i,\cdot)| \ \geq 1\right).
\end{aligned}$$

Letting $k \to +\infty$, we obtain that $\mathbb{P}(\mathcal{W}^{\mathbf{g}}(i) = 0, \liminf_n |Z^{\mathbf{g}}_n(i,\cdot)| \ \geq 1) \leq \varepsilon$ and the claim follows with $\varepsilon \to 0$.

□

Proof of (3.9) We denote $|\cdot|_2$ the Euclidean norm on $\mathbb{R}^p$. The second inequality is classical:

$$|1 - g_0^{(i)}(t)| \leq \left\langle \left(\nabla g_0^{(i)}\right)(\mathbf{1}), \mathbf{1} - t \right\rangle = \langle M_{g_0}(i,\cdot), \mathbf{1} - t\rangle \leq |\mathbf{1} - t|_2 \preceq |M_{g_0}||\mathbf{1} - t|.$$

To prove the first inequality, let $[t', \mathbf{1}]$ be the intersection of the cube $[0,1[^p$ with the line passing through t and $\mathbf{1}$. By convexity of $g_0^{(i)}$ on the segment $[t', \mathbf{1}]$, it holds that

$$\frac{1 - g_0^{(i)}(t)}{|\mathbf{1} - t|} \succeq \frac{1 - g_0^{(i)}(t)}{|\mathbf{1} - t|_2} \geq \frac{1 - g_0^{(i)}(t')}{|\mathbf{1} - t'|_2} \geq \frac{1 - g_0^{(i)}(t')}{\sqrt{p}}.$$

The point $t' = (t'_1, \ldots, t'_p)$ belongs to the boundary of the cube $[0,1[^p$ and at least one of its entries, say t'_j, equals 0; hence, recalling that $\xi_0(i,\cdot)$ is a $\mathbb{N}^p$-valued random variable with generating function $g_0^{(i)}$, then

$$1 - g_0^{(i)}(t') \geq 1 - g_0^{(i)}(\mathbf{1} - e_j) = 1 - \mathbb{P}(\xi_0(i,j) = 0) = \mathbb{P}(\xi_0(i,j) \geq 1) \geq \varepsilon.$$

This achieves the proof.

□

4 On the random environment

In this section, we present the random environment that we use and introduce some considerable classical change of measure and its main properties.

Why this change of measure? The bright idea introduced to study critical branching processes in random environment is to assume first that the random walk S_n is greater than some constant $-a$, then let $a \to +\infty$ (see for instance [7] and references therein). On the intermediate probability space, for almost all environment with respect to the new

probability measure, the Galton-Watson processe we consider is in varying environment and becomes a super critical process; we may thus apply Proposition 3.8 to each one of these environment (quenched version).

Recall that $\mathbf{f} = (f_n)_{n\geq 0}$ is a sequence of i.i.d. random variables with values in $\mathcal{G}$.

4.1 Construction of a new probability measure $\widehat{\mathbb{P}}_{\tilde{x},a}$

The $\widetilde{P}_+$-harmonic function V on $\mathbb{X}\times\mathbb{R}^+$ gives rise to a Markov kernel $\widetilde{P}_+^V$ on $\mathbb{X}\times\mathbb{R}^+$ defined formally by:

$$\widetilde{P}_+^V\phi = \frac{1}{V}\widetilde{P}_+(V\phi)$$

for any bounded measurable function ϕ on $\mathbb{X}\times\mathbb{R}^+$. By Proposition 3.4, there exists $A>0$ such that $V(\tilde{x},a)>0$ whenever $a>A$; thus, for any $\tilde{x}\in\mathbb{X}$, $a>A$ and $n\geq 1$,

$$(\widetilde{P}_+^V)^n\phi(\tilde{x},a) \;=\; \frac{1}{V(\tilde{x},a)}\mathbb{E}_{\tilde{x},a}\left[(V\phi)(X_n,S_n);m_n>0\right].$$

We introduce a change of probability measure on the canonical path space $((\mathbb{X}\times\mathbb{R})^{\otimes\mathbb{N}},\sigma(X_n,S_n:n\geq 0),\theta)$ [(3)] of the Markov chain $(X_n,S_n)_{n\geq 0}$ from $\mathbb{P}$ to the measure $\widehat{\mathbb{P}}_{\tilde{x},a}$ characterized by the property that

$$\widehat{\mathbb{E}}_{\tilde{x},a}[\phi(X_0,S_0,\ldots,X_k,S_k)] = \frac{1}{V(\tilde{x},a)}\mathbb{E}_{\tilde{x},a}[\phi(X_0,\ldots,S_k)V(X_k,S_k);\ m_k>0] \qquad (4.1)$$

for any positive Borel function ϕ on $(\mathbb{X}\times\mathbb{R})^{k+1}$. By Proposition 3.4 and Theorem 3.5

$$\begin{aligned}\lim_{n\to+\infty}\mathbb{E}_{\tilde{x},a}[\phi(X_0,\ldots,S_k)|m_n>0] &= \frac{1}{V(\tilde{x},a)}\mathbb{E}_{\tilde{x},a}[V(X_k,S_k)\phi(X_0,\ldots,S_k);m_k>0]\\ &= \widehat{\mathbb{E}}_{\tilde{x},a}[\phi(X_0,\ldots,S_k)], \qquad (4.2)\end{aligned}$$

which clarifies the interpretation of $\widehat{\mathbb{P}}_{\tilde{x},a}$ (see [17] section 3.2 for the details).
This probability may be extended to the whole σ-algebra $\sigma(f_n,Z_n:n\geq 0)$ as follows; the extension is done in three steps:
Step 1. the marginal distribution of $\widehat{\mathbb{P}}_{\tilde{x},a}$ on $\sigma(X_n,S_n:n\geq 0)$ is $\widehat{\mathbb{P}}_{\tilde{x},a}$ characterized by the property (4.1);
Step 2. for any $n\geq 0$, the conditional distribution of $(f_0,\cdots,f_n)$ under $\widehat{\mathbb{P}}_{\tilde{x},a}$ given $X_0=\tilde{x}_0=\tilde{x},\ldots,X_n=\tilde{x}_n,S_0=s_0=a,\ldots,S_n=s_n$ equals the one of $(f_0,\cdots,f_n)$ under $\widehat{\mathbb{P}}$; namely, for any measurable sets $G_0,\ldots,G_n$ in $\mathcal{G}$ and all $(\tilde{x}_i)_{0\leq i\leq n}$ and $(s_i)_{0\leq i\leq n}$

$$\begin{aligned}&\widehat{\mathbb{P}}_{\tilde{x},a}(f_k\in G_k,0\leq k\leq n\Big/X_i=\tilde{x}_i,S_i=s_i,0\leq i\leq n)\\ &\qquad=\mathbb{P}(f_k\in G_k,0\leq k\leq n\Big/X_i=\tilde{x}_i,S_i(\tilde{x},0)=s_i,0\leq i\leq n).\end{aligned}$$

Step 3. the conditional distribution of $(Z_n)_{n\geq 0}$ under $\widehat{\mathbb{P}}_{\tilde{x},a}$ given $\mathbf{f}=(f_0,f_1,\ldots)$ is the same as under $\mathbb{P}$,; namely, for any $n\geq 0$ and $1\leq i\leq p$,

$$\begin{aligned}\widehat{\mathbb{E}}_{\tilde{x},a}\left[s^{Z_n(i,\cdot)}\Big/Z_0,\ldots,Z_{n-1},f_0^{(i)},f_1,\ldots,f_{n-1}\right] &= \mathbb{E}\left[s^{Z_n(i,\cdot)}\Big/Z_0,\ldots,Z_{n-1},f_0^{(i)},f_1,\ldots,f_{n-1}\right]\\ &= f_{n-1}(s)^{Z_{n-1}(i,\cdot)}.\end{aligned}$$

[3] θ denotes the shift operator on $(\mathbb{X}\times\mathbb{R})^{\otimes\mathbb{N}}$ defined by $\theta\Big((x_k,s_k)_{k\geq 0}\Big)=(x_{k+1},s_{k+1})_{k\geq 0}$ for any $(x_k,s_k)_{k\geq 0}$ in $(\mathbb{X}\times\mathbb{R})^{\otimes\mathbb{N}}$

4.2 Some properties of the probability measures $\widehat{\mathbb{P}}_{\tilde{x},a}, \tilde{x} \in \mathbb{X}, a \geq 0$

The following lemma extends property (4.2) to the σ-algebra $\mathcal{F}_\infty = \sigma(\vee_{k\geq 0}\mathcal{F}_k)$ where $\mathcal{F}_k := \sigma\{f_\ell, Z_\ell \mid 0 \leq \ell \leq k\}$ for any $k \geq 0$.

Lemma 4.1 *Assume that hypotheses* H1–H5 *hold for some $\delta > 0$. Let $(Y_k)_{k\geq 0}$ be a sequence of bounded real-valued random variables adapted to the filtration $(\mathcal{F}_k)_{k\geq 0}$.*

1. *[8] For any $\tilde{x} \in \mathbb{X}$ and $a > A$,*

$$\lim_{n\to+\infty} \mathbb{E}_{\tilde{x},a}\big[Y_k \mid \tau > n\big] = \widehat{\mathbb{E}}_{\tilde{x},a}[Y_k]. \tag{4.3}$$

2. *Moreover, if $(Y_k)_{k\geq 0}$ converges in $\mathbb{L}^1(\widehat{\mathbb{P}}_{\tilde{x},a})$ to some random variable Y_∞,*

$$\lim_{n\to+\infty} \mathbb{E}_{\tilde{x},a}\big[Y_n \mid \tau > n\big] = \widehat{\mathbb{E}}_{\tilde{x},a}[Y_\infty].$$

Proof. Property (4.3) is proved in [8]. The second assertion has an analogue version in [8], where the almost-sure convergence is required; in fact, the convergence in L^1 and the boundedness of the Y_k suffice.

For any $k \in \mathbb{N}$,

$$\sqrt{n}\mathbb{E}_{\tilde{x},a}[Y_n, \tau > n] = \sqrt{n}\mathbb{E}_{\tilde{x},a}[Y_k, \tau > n] + \sqrt{n}\mathbb{E}_{\tilde{x},a}[Y_n - Y_k, \tau > n],$$

with

$$\begin{aligned}\lim_{n\to+\infty} \sqrt{n}\mathbb{E}_{\tilde{x},a}[Y_k, \tau > n] &= \lim_{n\to+\infty} \sqrt{n}\mathbb{E}_{\tilde{x},a}\big[Y_k \mid \tau > n\big]\mathbb{P}_{\tilde{x},a}(\tau > n) \\ &= \frac{2}{\sigma\sqrt{2\pi}} V(\tilde{x}, a)\widehat{\mathbb{E}}_{\tilde{x},a}[Y_k],\end{aligned}$$

by (4.3) and Theorem 3.5. Since $(\widehat{\mathbb{E}}_{\tilde{x},a}[Y_k])_{k\geq 0}$ converges to $\widehat{\mathbb{E}}_{\tilde{x},a}[Y_\infty]$ as $k \to +\infty$, it remains to prove that

$$\lim_{k\to+\infty} \lim_{n\to+\infty} \sqrt{n}\mathbb{E}_{\tilde{x},a}\big[|Y_n - Y_k|; \tau > n\big] = 0. \tag{4.4}$$

We fix $\rho > 1$ and decompose $\mathbb{E}_{\tilde{x},a}\big[|Y_n - Y_k|, \tau > n\big]$ as

$$\mathbb{E}_{\tilde{x},a}\big[|Y_n - Y_k|, \tau > n\big] = \mathbb{E}_{\tilde{x},a}\big[|Y_n - Y_k|, n < \tau < \rho n\big] + \mathbb{E}_{\tilde{x},a}\big[|Y_n - Y_k|, \tau > \rho n\big]. \tag{4.5}$$

For the first term in (4.5), since the random variables Y_n are bounded, it is clear that

$$\begin{aligned}\mathbb{E}_{\tilde{x},a}\big[|Y_n - Y_k|, n < \tau < \rho n\big] &\preceq \mathbb{P}_{\tilde{x},a}(n < \tau < \rho n) \\ &= \mathbb{P}_{\tilde{x},a}(\tau > n) - \mathbb{P}_{\tilde{x},a}(\tau > \rho n).\end{aligned}$$

Therefore, by Theorem 3.5, for any k and $\rho > 1$,

$$\begin{aligned}\limsup_{n\to+\infty} \sqrt{n}\mathbb{E}_{\tilde{x},a}\big[|Y_n - Y_k|, n < \tau < \rho n\big] &\preceq \lim_{n\to+\infty} \sqrt{n}\mathbb{P}_{\tilde{x},a}(\tau > n) - \lim_{n\to+\infty} \sqrt{n}\mathbb{P}_{\tilde{x},a}(\tau > \rho n) \\ &= \frac{2}{\sigma\sqrt{2\pi}} V(\tilde{x}, a)\left(1 - \frac{1}{\sqrt{\rho}}\right) \longrightarrow 0 \text{ as } \rho \to 1.\end{aligned}$$

For the second term in (4.5), we write

$$\begin{aligned}\mathbb{E}_{\tilde{x},a}\big[|Y_n - Y_k|, \tau > \rho n\big] &= \mathbb{E}_{\tilde{x},a}\Big[\mathbb{E}\big[|Y_n - Y_k|, \tau > \rho n \,\Big/\, \mathcal{F}_n\big]\Big] \\ &= \mathbb{E}_{\tilde{x},a}\big[|Y_n - Y_k|\mathbf{m}_{\rho n - n}(X_n, S_n), \tau > n\big] \\ &\preceq \frac{1}{\sqrt{n(\rho-1)}}\mathbb{E}_{\tilde{x},a}\big[|Y_n - Y_k|V(X_n, S_n); \tau > n\big] \\ &= \frac{1}{\sqrt{n(\rho-1)}}V(\tilde{x}, a)\widehat{\mathbb{E}}_{\tilde{x},a}\big[|Y_n - Y_k|\big].\end{aligned}$$

Hence, since $Y_n \to Y_\infty$ in $\mathbb{L}^1(\widehat{\mathbb{P}}_{\tilde{x},a})$,

$$\begin{aligned}&\limsup_{k\to+\infty}\limsup_{n\to+\infty}\sqrt{n}\mathbb{E}_{\tilde{x},a}\big[|Y_n - Y_k|, \tau > \rho n\big] \\ &\qquad\leq \frac{c}{\sqrt{\rho-1}}V(\tilde{x}, a)\limsup_{k\to+\infty}\limsup_{n\to+\infty}\widehat{\mathbb{E}}_{\tilde{x},a}\big[|Y_n - Y_k|\big] \\ &\qquad= \frac{cV(\tilde{x}, a)}{\sqrt{\rho-1}}\limsup_{k\to+\infty}\widehat{\mathbb{E}}_{\tilde{x},a}\big[|Y_\infty - Y_k|\big] = 0.\end{aligned}$$

□

The following statement plays a crucial role in the sequel. It was first proved in the multi-type context in [17] (Lemma 3.1), when the generating functions are linear-fractional; then the general case was considered in [8] (Lemma 7). We generalize these statements under weaker moment conditions.

Lemma 4.2 *Assume hypotheses* H1–H6 *hold for some* $\delta > 0$. *Then, for any* $\tilde{x} \in \mathbb{X}$ *and* $a > A$,

$$\sum_{n=0}^{+\infty}\widehat{\mathbb{E}}_{\tilde{x},a}\left[e^{-S_n}\right] < +\infty \quad \textit{and} \quad \sum_{n=0}^{+\infty}\widehat{\mathbb{E}}_{\tilde{x},a}\left[\eta_n e^{-S_n}\right] < +\infty.$$

Proof. In order to ease the arguments for proving the first part of the statement, we begin by studying the second one. We fix $\tilde{x} \in \mathbb{X}, a > A$ and $n \geq 0$ and use Corollary 3.6 to control each term $\widehat{\mathbb{E}}_{\tilde{x},a}\left[\eta_n e^{-S_n}\right]$. By the definition of the probability measure $\widehat{\mathbb{P}}_{\tilde{x},a}$,

$$\begin{aligned}\widehat{\mathbb{E}}_{\tilde{x},a}\left[\eta_n e^{-S_n}\right] &= \widehat{\mathbb{E}}_{\tilde{x},a}\left[\frac{\mu_n}{|M_n|^2}e^{-S_n}\right] \\ &\preceq \widehat{\mathbb{E}}_{\tilde{x},a}\left[\frac{\mu_n}{|M_n X_n|^2}e^{-S_n}\right] \quad \text{(by Lemma 3.3)} \\ &\preceq \int \widehat{\mathbb{E}}_{\tilde{x},a}\left[\mu_n \Big/ X_n = \tilde{y}, S_n = s, X_{n+1} = \tilde{z}, S_{n+1} = t\right] e^{s-2t} \\ &\qquad\qquad \widehat{\mathbb{P}}_{\tilde{x},a}(X_n \in \mathrm{d}\tilde{y},\ S_n \in \mathrm{d}s, X_{n+1} \in \mathrm{d}\tilde{z},\ S_{n+1} \in \mathrm{d}t) \\ &= \int \mathbb{E}\left[\mu_n \Big/ X_n = \tilde{y}, S_n = s, X_{n+1} = \tilde{z}, S_{n+1} = t\right] e^{s-2t} \\ &\qquad\qquad \widehat{\mathbb{P}}_{\tilde{x},a}(X_n \in \mathrm{d}\tilde{y},\ S_n \in \mathrm{d}s, X_{n+1} \in \mathrm{d}\tilde{z},\ S_{n+1} \in \mathrm{d}t).\end{aligned}$$

Hence, by Proposition 3.4

$$\begin{aligned}\widehat{\mathbb{E}}_{\tilde{x},a}\left(\eta_n e^{-S_n}\right) &\preceq \mathbb{E}_{\tilde{x},a}\Big(\mathbb{E}\left(\mu_n \Big/ X_n, S_n, X_{n+1}, S_{n+1}\right) e^{S_n-2S_{n+1}} V(X_{n+1}, S_{n+1}); m_{n+1}>0\Big)\\ &\preceq \mathbb{E}_{\tilde{x},a}\Big(\mathbb{E}\left(\mu_n \Big/ X_n, S_n, X_{n+1}, S_{n+1}\right) e^{S_n-2S_{n+1}} |S_{n+1}|; m_{n+1}>0\Big)\\ &\preceq \mathbb{E}_{\tilde{x},a}\Big(\mathbb{E}\left(\mu_n \Big/ X_n, S_n, X_{n+1}, S_{n+1}\right) e^{S_n-2S_{n+1}} \Big(|S_n| + \ln^+|M_n|\Big); m_{n+1}>0\Big)\end{aligned} \tag{4.6}$$

On the one hand

$$\begin{aligned}&\mathbb{E}_{\tilde{x},a}\Big(\mathbb{E}\left(\mu_n \Big/ X_n, S_n, X_{n+1}, S_{n+1}\right) e^{S_n-2S_{n+1}} |S_n|; m_{n+1}>0\Big)\\ &\quad\preceq \mathbb{E}_{\tilde{x},a}\Big(\mathbb{E}\left(\mu_n \Big/ X_n, S_n, X_{n+1}, S_{n+1}\right) \frac{e^{-S_n}|S_n|}{|M_n|^2}; m_n>0\Big)\\ &\quad= \mathbb{E}\Big(\frac{\mu_n}{|M_n|^2} e^{-S_n}|S_n|; m_n>0\Big)\\ &\quad\le \mathbb{E}_{\tilde{x},a}\left(\frac{\mu_n}{|M_n|^2}\right) \times \mathbb{E}(e^{-S_n}|S_n|; m_n>0).\end{aligned} \tag{4.7}$$

On the other hand

$$\begin{aligned}&\mathbb{E}_{\tilde{x},a}\Big(\mathbb{E}\left(\mu_n \Big/ X_n, S_n, X_{n+1}, S_{n+1}\right) e^{S_n-2S_{n+1}} \ln^+|M_n|; m_{n+1}>0\Big)\\ &\preceq \mathbb{E}_{\tilde{x},a}\Big(\mathbb{E}\left(\mu_n \Big/ X_n, S_n, X_{n+1}, S_{n+1}\right) e^{-S_n} \frac{\ln^+|M_n|}{|M_n|^2}; m_n>0\Big)\\ &\le \mathbb{E}\left(\frac{\mu_n}{|M_n|^2}\ln^+|M_n|\right) \times \mathbb{E}_{\tilde{x},a}(e^{-S_n}; m_n>0).\end{aligned} \tag{4.8}$$

By hypothesis H6, quantities $\mathbb{E}\left(\frac{\mu_n}{|M_n|^2}\right)$ and $\mathbb{E}\left(\frac{\mu_n}{|M_n|^2}\ln^+|M_n|\right)$ are both finite; furthermore, Corollary 3.7 yields

$$n^{3/2}\mathbb{E}_{\tilde{x},a}(e^{-S_n}|S_n|; m_n>0) \preceq (1+a)\sum_{b\ge 0}(1+b)^2 e^{-b} < +\infty.$$

Finally, combining (4.6), (4.7) and (4.8), we obtain that

$$\sup_{n\ge 1} n^{3/2}\widehat{\mathbb{E}}_{\tilde{x},a}\left(\eta_n e^{-S_n}\right) < +\infty$$

and the lemma follows.

□

As a direct consequence, $\widehat{\mathbb{P}}_{\tilde{x},a}$-almost surely, the environment $\mathbf{f}$ do satisfy the conclusions of Proposition 3.8.

Corollary 4.3 *Assume that hypotheses* H1–H6 *hold for some* $\delta > 0$. *Then, for* $\widehat{\mathbb{P}}_{\tilde{x},a}$-*almost all environment* $\mathbf{f} = (f_n)_{n\ge 0}$,

1. *the process* $Z^{\mathbf{f}}$ *becomes extinct with probability* $q^{\mathbf{f}}(i) < 1$ *for some (hence every)* $i \in \{1,\dots,p\}$;
2. *there exists a non-negative random column vector* $\mathcal{W}^{\mathbf{f}} = (\mathcal{W}^{\mathbf{f}}(i))_{1\le i\le p}$ *such that for every* $i, j \in \{1,\dots,p\}$, *as* $n \to +\infty$,

$$\mathcal{W}_n^{\mathbf{f}}(i,j) := \frac{Z_n^{\mathbf{f}}(i,j)}{|M_{0,n}^{\mathbf{f}} e_j|} \quad \xrightarrow{\mathbb{L}^2(\mathbb{P}_{\mathbf{f}})} \quad \mathcal{W}^{\mathbf{f}}(i). \tag{4.9}$$

3. For any $1 \leq i \leq p$, it holds

$$(\mathcal{W}^{\mathbf{f}}(i) > 0) = \big(\bigcap_{n\geq 0} |Z_n^{\mathbf{f}}(i,\cdot)| = 0)\big) \quad \mathbb{P}_{\mathbf{f}}\text{- a.s.}$$

Proof. By Lemma 4.2, for any $\tilde{x} \in \mathbb{X}$ and $a > A$,

$$\widehat{\mathbb{E}}_{\tilde{x},a}\left[\mathbb{E}\left[\sum_{n=0}^{+\infty} e^{-S_n}\Big/ f_0,\dots,f_{n-1}\right]\right] < +\infty \text{ and } \widehat{\mathbb{E}}_{\tilde{x},a}\left[\mathbb{E}\left[\sum_{n=0}^{+\infty} \eta_n e^{-S_n}\Big/ f_0,\dots,f_{n-1}\right]\right] < +\infty,$$

which yields, for $\widehat{\mathbb{P}}_{\tilde{x},a}$-almost all $\mathbf{f}$,

$$\mathbb{E}\left[\sum_{n=0}^{+\infty} e^{-S_n(\tilde{x},a)}\Big/ \mathbf{f}\right] < +\infty \quad \text{and} \quad \mathbb{E}\left[\sum_{n=0}^{+\infty} \eta_n e^{-S_n(\tilde{x},a)}\Big/ \mathbf{f}\right] < +\infty.$$

Hence, by Lemmas 4.2 and 3.3, for $\widehat{\mathbb{P}}_{\tilde{x},a}$-almost all $\mathbf{f}$ and any $1 \leq i,j \leq p$, on the one hand,

$$\sum_{n=0}^{+\infty} \frac{1}{|M_{0,n}^{\mathbf{f}}(i,j)|} < +\infty,$$

and on the other hand,

$$\sum_{n=0}^{+\infty} \frac{1}{|M_{0,n}^{\mathbf{f}}|} \frac{|B_{f_n}^{(i)}|}{|M_{f_n}|} \leq \sum_{n=0}^{+\infty} \frac{1}{|M_{0,n}^{\mathbf{f}}|}\, \eta_{f_n} < +\infty.$$

Hence, $\widehat{\mathbb{P}}_{\tilde{x},a}$-almost all environment $\mathbf{f}$ satisfy the hypotheses of Proposition 3.8, Corollary 4.3 follows immediately.

□

4.3 On the extinction of $(Z_n(\tilde{z},\cdot))_{n\geq 0}$ in random environment

The following result extends property (2.2) to Galton-Watson processes $(Z_n(\tilde{z},\cdot))_{n\geq 0}$ with any initial population $\tilde{z} \in \mathbb{N}^p$.

Recall that for any $n \geq 0$ and $i \in \{1,\dots,p\}$, the probability of extinction at time n of $(Z_n(i,\cdot))_{n\geq 0}$, given the environment $\mathbf{f}$ (or equivalently given $f_0,\dots,f_{n-1}$) equals

$$q_{n,i}^{\mathbf{f}} = \mathbb{P}(|Z_n(\tilde{e}_i,\cdot)| > 0 \;/\; f_0,\dots,f_{n-1}) = 1 - f_0^{(i)} f_1 \dots f_{n-1}(\tilde{0}).$$

For any environment $\mathbf{f}$, the sequence $(q_{n,i}^{\mathbf{f}})_{n\geq 0}$ converges to some limit, denoted $q_i^{\mathbf{f}}$.

By the branching property, for any $\tilde{z} = (z_1,\dots,z_p) \in \mathbb{N}^p$,

$$\begin{aligned} q_{n,\tilde{z}}^{\mathbf{f}} &:= \mathbb{P}(|Z_n(\tilde{z},\cdot)| > 0 \;/ f_0,\dots,f_{n-1}) \\ &= 1 - \prod_{i=1}^{p} [f_0^{(i)} f_1 \dots f_{n-1}(\tilde{0})]^{z_i} = 1 - \prod_{i=1}^{p} [1 - q_{n,i}^{\mathbf{f}}(\tilde{0})]^{z_i}. \end{aligned}$$

Let us denote $q_{\tilde{z}}^{\mathbf{f}}$ the limit of the sequence $(q_{n,\tilde{z}}^{\mathbf{f}})_{n\geq 0}$.

For any $\tilde{x} \in \mathbb{X}$ and $a > A$, it holds that

$$\mathbb{P}(|Z_n(\tilde{z},\cdot)| > 0) = \mathbb{E}[q_{n,\tilde{z}}^{\mathbf{f}}] \quad \text{and} \quad \widehat{\mathbb{P}}_{\tilde{x},a}(|Z_n(\tilde{z},\cdot)| > 0) = \widehat{\mathbb{E}}_{\tilde{x},a}[q_{n,\tilde{z}}^{\mathbf{f}}].$$

By the dominated convergence theorem,

$$\lim_{n\to+\infty} \mathbb{P}(|Z_n(\tilde{z},\cdot)|>0)) = \mathbb{P}(\cap_{n\geq 0}(|Z_n(\tilde{z},\cdot)|>0))) = \mathbb{E}[q_{\tilde{z}}^{\mathbf{f}}]$$

(resp. $\lim_{n\to+\infty} \widehat{\mathbb{P}}_{\tilde{x},a}(|Z_n(\tilde{z},\cdot)|>0) = \widehat{\mathbb{P}}(\cap_{n\geq 0}(|Z_n(\tilde{z},\cdot)|>0))) = \widehat{\mathbb{E}}[q_{\tilde{z}}^{\mathbf{f}}]$).

These two limits are related to each other in the following way.

Property 4.4 *Assume that hypotheses* H1–H6 *hold for some $\delta>0$. Then for any $\tilde{x}\in\mathbb{X}$ and $\tilde{z}\in\mathbb{R}_+^p$,*

$$\lim_{n\to+\infty} \sqrt{n}\mathbb{P}(|Z_n(\tilde{z},\cdot)|>0) = \lim_{a\to+\infty} V(\tilde{x},a)\widehat{\mathbb{E}}_{\tilde{x},a}(q_{\tilde{z}}^{\mathbf{f}}) = \langle z,\beta\rangle,$$

where $\beta=(\beta_1,\ldots,\beta_p)$ with $\beta_i>0$ given in (2.2).

Proof. Let $\tilde{z}=(z_1,\ldots,z_p)\neq\tilde{\mathbf{0}}$. By using the branching property of the Galton-Watson process,

$$|Z_n(\tilde{z},\cdot)| = \sum_{i=1}^p |Z_n(z_i\tilde{e}_i,\cdot)| = \sum_{i=1}^p\sum_{k=1}^{z_i} |Z_{n,k}(i,\cdot)|,$$

where the random variables $Z_{n,k}=(Z_{n,k}(i,j))_{i,j}$, for $n,k\geq 0$, are independent and $Z_{n,k}\overset{\text{dist}}{=}Z_n$ for any $n,k\geq 0$. Hence, by combining the independence of these random variables and property (2.2), we obtain

$$\begin{aligned}\mathbb{P}(|Z_n(\tilde{z},\cdot)|=0) &= \prod_{i=1}^p\prod_{k=1}^{z_i}\mathbb{P}(|Z_{n,k}(i,\cdot)|=0)\\ &= \prod_{i=1}^p\left(1-\frac{\beta_i}{\sqrt{n}}(1+o(n))\right)^{z_i} = 1-\frac{\langle\beta,z\rangle}{\sqrt{n}}(1+o(n)).\end{aligned}$$

Moreover, by [17], it holds that $\beta_i=\lim_{a\to+\infty} V(\tilde{x},a)\widehat{\mathbb{E}}_{\tilde{x},a}[q_i^{\mathbf{f}}]$ for any $1\leq i\leq p$; thus $\lim_{a\to+\infty} V(\tilde{x},a)\widehat{\mathbb{E}}_{\tilde{x},a}[q_{\tilde{z}}^{\mathbf{f}}]=\langle z,\beta\rangle$.

Similarly, we need to extend property (4.9) to Galton-Watson processes $(Z_n(\tilde{z},\cdot))_{n\geq 0}$ with any initial population $\tilde{z}\in\mathbb{R}_+^p$. The following statement is a direct consequence of a combination of Corollary 4.3 and the branching property.

Property 4.5 *Assume that hypotheses* H1–H6 *hold for some $\delta>0$. Then for all $\tilde{x}\in\mathbb{X}, a>A$ and $\widehat{\mathbb{P}}_{\tilde{x},a}$-almost all environment $\mathbf{f}=(f_n)_{n\geq 0}$, any $\tilde{z}\in\mathbb{R}_+^p\setminus\{\mathbf{0}\}$, and any $j\in\{1,\ldots,p\}$,*

$$\mathcal{W}_n^{\mathbf{f}}(\tilde{z},j) := \frac{Z_n^{\mathbf{f}}(\tilde{z},j)}{|M_{0,n}^{\mathbf{f}}e_j|} \quad \xrightarrow{\mathbb{L}^2(\mathbb{P}_{\mathbf{f}})} \quad \mathcal{W}^{\mathbf{f}}(\tilde{z}) := \sum_{i=1}^p\sum_{k=1}^{z_i}\mathcal{W}_k^{\mathbf{f}}(i),$$

where the random variables $\mathcal{W}_k^{\mathbf{f}}$ for $k\geq 0$, are independent copies of $\mathcal{W}^{\mathbf{f}}$.

In particular, for any $j\in\{1,\ldots,p\}$,

$$\lim_{n\to+\infty}\frac{\mathbb{E}^{\mathbf{f}}[Z_n^{\mathbf{f}}(\tilde{z},j)]}{|M_{0,n}^{\mathbf{f}}e_j|} = \lim_{n\to+\infty}\frac{\sum_{i=1}^p z_i\mathbb{E}^{\mathbf{f}}[Z_n^{\mathbf{f}}(i,j)]}{|M_{0,n}^{\mathbf{f}}e_j|} = \langle z,\mathbb{E}[\mathcal{W}^{\mathbf{f}}]\rangle.$$

Furthermore, if there exist $\varepsilon\in]0,1[$ and $K>0$ such that $f_n\in G_{\varepsilon,K}$ for any $n\geq 0$, then

$$(\mathcal{W}^{\mathbf{f}}(\tilde{z})>0) = \bigcap_{n\geq 0}(|Z_n^{\mathbf{f}}(\tilde{z},\cdot)|>0) \quad \mathbb{P}_{\mathbf{f}}\text{- a.s.} \tag{4.10}$$

5 Proof of Theorem 2.3

By a standard argument in probability theory, since the random variable $Z_n(\tilde{z},j)/|M_{0,n}e_j|$, for $n \geq 0, \tilde{z} \in \mathbb{N}^p \setminus \{\tilde{\mathbf{0}}\}$ and $1 \leq j \leq p$, are non-negative, it suffices to prove that the sequence of Laplace transform

$$\lambda \mapsto \mathbb{E}\left[\exp\left(-\lambda \frac{Z_n(\tilde{z},j)}{|M_0 \dots M_{n-1}e_j|}\right) \Big/ |Z_n(\tilde{z},\cdot)| > 0\right]$$

converges on $[0,+\infty[$ to some function which is continuous at 0.

We fix $z \in \mathbb{N}^p \setminus \{\tilde{\mathbf{0}}\}$ and $1 \leq j \leq p$. For any $\lambda \geq 0$,

$$\mathbb{E}\left[\exp - \left(\lambda \frac{-Z_n(\tilde{z},j)}{|M_0 \dots M_{n-1}e_j|}\right) \Big/ |Z_n(\tilde{z},\cdot)| > 0\right] = \frac{\sqrt{n}\mathbb{E}\left[\exp\left(-\lambda \frac{Z_n(\tilde{z},j)}{|M_0 \dots M_{n-1}e_j|}\right), |Z_n(\tilde{z},\cdot)| > 0\right]}{\sqrt{n}\mathbb{P}(|Z_n(\tilde{z},\cdot)| > 0)}.$$

By Property 4.4, it suffices to prove that the sequence $(\phi_{n,\tilde{z},j})_{n\geq 0}$ defined by

$$\forall \lambda \geq 0, \quad \phi_{n,\tilde{z},j}(\lambda) := \sqrt{n}\mathbb{E}\left[\exp\left(-\lambda \frac{Z_n(\tilde{z},j)}{|M_0 \dots M_{n-1}e_j|}\right), |Z_n(\tilde{z},\cdot)| > 0\right]$$

converges to some function $\phi_{\tilde{z},j} : \mathbb{R}^+ \to [0,1]$ such that

$$\lim_{\lambda \to 0^+} \phi_{\tilde{z},j}(\lambda) = \phi_{\tilde{z},j}(0) = \langle z, \beta \rangle.$$

A candidate for this limit is

$$\phi_{\tilde{z},j}(\lambda) = \frac{2}{\sigma\sqrt{2\pi}} \sum_{k=0}^{+\infty} \mathbb{E}_{\tilde{x},a}\left[V(X_k,0)\mathbf{1}_{(T_k=k)}\Psi(\lambda, X_k, 0, Z_k(\tilde{z},\cdot), \mathbf{f}\circ\theta^k)\right],$$

where

$$\Psi(\lambda,\tilde{x}',a',\tilde{z}',\mathbf{g}) := \widehat{\mathbb{E}}_{\tilde{x}',a'}\left[\exp\Big(-\lambda \frac{\mathcal{W}^{\mathbf{g}}(\tilde{z}')}{\langle \alpha, \mathbb{E}(\mathcal{W}^{\mathbf{g}})\rangle}\Big)\mathbf{1}_{\cap_{n>1}(|Z_n^{\mathbf{g}}(\tilde{z}',\cdot)|>0)}\right] \tag{5.1}$$

for any $\lambda \geq 0, \tilde{x}' \in \mathbb{X}, a' > 0, z' \in \mathbb{N}^p \setminus \{0\}$ and $\mathbf{g} \in \mathcal{G}^{\mathbb{N}}$. For any $n \geq 1$, we set $T_n = \max\{k \ / \ 0 \leq k \leq n \text{ such that } S_k = m_n\}$; the random variable T_n satisfy the following simple properties:

- $T_n \leq n$ for any $n \geq 1$;
- T_n does not depend on the value of S_0;
- let $m_{k,n} := \min\{S_{k+1} - S_k, \dots, S_n - S_k\}$, then for any $0 \leq k \leq n$,

$$(T_n = k) = (T_k = k) \cap (m_{k,n} > 0).$$

These random variable yields to the following decomposition

$$\begin{aligned}
\phi_{n,\tilde{z},j}(\lambda) &= \sqrt{n}\mathbb{E}\left[\exp\left(-\lambda\frac{Z_n(\tilde{z},j)}{|M_0\dots M_{n-1}e_j|}\right);|Z_n(\tilde{z},\cdot)|>0\right] \\
&= \sqrt{n}\mathbb{E}_{\tilde{x},a}\left[\exp\left(-\lambda\frac{Z_n(\tilde{z},j)}{|M_0\dots M_{n-1}e_j|}\right);|Z_n(\tilde{z},\cdot)|>0,T_n=n\right] \\
&\qquad + \sqrt{n}\sum_{k=0}^{n-1}\mathbb{E}_{\tilde{x},a}\left[\exp\left(-\lambda\frac{Z_n(\tilde{z},j)}{|M_0\dots M_{n-1}e_j|}\right);|Z_n(\tilde{z},\cdot)|>0,T_n=k\right] \\
&= \underbrace{\sqrt{n}\mathbb{E}_{\tilde{x},a}\left[\exp\left(-\lambda\frac{Z_n(\tilde{z},j)}{|M_0\dots M_{n-1}e_j|}\right);|Z_n(\tilde{z},\cdot)|>0,T_n=n\right]}_{\Sigma_1(n,\lambda)} \\
&\qquad + \underbrace{\sqrt{n}\sum_{k=0}^{n-1}\mathbb{E}_{\tilde{x},a}\left[\exp\left(-\lambda\frac{Z_n(\tilde{z},j)}{|M_0\dots M_{n-1}e_j|}\right);|Z_n(\tilde{z},\cdot)|>0,T_k=k,m_{k,n}>0\right]}_{\Sigma_2(n,\lambda)}.
\end{aligned}$$

The following lemma shows that

$$\lim_{n\to+\infty}\Sigma_1(n,\lambda)=0,\quad \text{uniformly in}\quad \lambda\geq 0. \tag{5.2}$$

Lemma 5.1 *There exists a positive constant c such that for any $n\geq 1,\tilde{x}\in\mathbb{X},a>0$ and $z\in\mathbb{N}^p\setminus\{\mathbf{0}\}$,*

$$\mathbb{P}_{\tilde{x},a}(|Z_n(\tilde{z},\cdot)|>0,T_n=n)\leq c\frac{|z|}{n^{3/2}}.$$

The term $\Sigma_2(n,\lambda)$ may be decomposed as follows: for $1\leq\ell\leq n-1$ fixed,

$$\begin{aligned}
\Sigma_2(n,\lambda) &= \underbrace{\sqrt{n}\sum_{k=0}^{\ell}\mathbb{E}_{\tilde{x},a}\left[\exp\left(-\lambda\frac{Z_n(\tilde{z},j)}{|M_0\dots M_{n-1}e_j|}\right);|Z_n(\tilde{z},\cdot)|>0,T_k=k,m_{k,n}>0\right]}_{\Sigma_{2,1}(n,\ell,\lambda)} \\
&\qquad + \underbrace{\sqrt{n}\sum_{k=\ell+1}^{n-1}\mathbb{E}_{\tilde{x},a}\left[\exp\left(-\lambda\frac{Z_n(\tilde{z},j)}{|M_0\dots M_{n-1}e_j|}\right);|Z_n(\tilde{z},\cdot)|>0,T_k=k,m_{k,n}>0\right]}_{\Sigma_{2,2}(n,\ell,\lambda)}
\end{aligned}$$

and we study separately the two terms $\Sigma_{2,1}(n,\ell)$ and $\Sigma_{2,2}(n,\ell)$. Firstly,

$$\begin{aligned}
&\Sigma_{2,1}(n,\ell,\lambda) \\
&\quad= \sum_{k=0}^{\ell}\frac{\sqrt{n}}{\sqrt{n-k}}\mathbb{E}_{\tilde{x},a}\left[\sqrt{n-k}\exp\left(-\lambda\frac{Z_n(\tilde{z},j)}{|M_0\dots M_{n-1}e_j|}\right);|Z_n(\tilde{z},\cdot)|>0,T_k=k,m_{k,n}>0\right] \\
&\quad= \sum_{k=0}^{\ell}\frac{\sqrt{n}}{\sqrt{n-k}}\int df_0\dots df_{k-1}\ \delta_{M_{0,k}}(dM)\ \int\mathbb{P}\big(Z_k(\tilde{z},\cdot)\in dZ\mid f_0,\dots,f_{k-1}\big)\mathbf{1}_{(T_k=k)}\mathbf{1}_{(|Z|>0)} \\
&\qquad\times\sqrt{n-k}\ \mathbb{E}_{\tilde{x}\cdot M,0}\left[\exp\left(-\lambda\frac{Z_{n-k}^{\mathbf{f}\circ\theta^k}(Z,j)}{|MM_{0,n-k}\circ\theta^k e_j|}\right);|Z_{n-k}^{\mathbf{f}\circ\theta^k}(Z,\cdot)|>0,\tau\circ\theta^k>n-k\right].
\end{aligned}$$

Let us fix $0 \le k \le \ell$ and set $Y_{n,k}(\lambda, Z, j) := \exp\left(-\lambda \frac{Z^{\mathbf{f}\circ\theta^k}_{n-k}(Z,j)}{|MM_{0,n-k}\circ\theta^k e_j|}\right)\mathbf{1}_{(|Z^{\mathbf{f}\circ\theta^k}_{n-k}(Z,\cdot)|>0)}$ for $n > k$. By Property 4.5, for $\widehat{\mathbb{P}}_{\tilde{x}\cdot M,0}$-almost all environment $\mathbf{g}$,

- the sequence $\left(\frac{Z^{\mathbf{g}}_{n-k}(Z,j)}{|M^{\mathbf{g}}_{0,n-k}e_j|}\right)_{n>k}$ converges in $\mathbb{L}^2(\mathbb{P}_{\mathbf{g}})$ to $\mathcal{W}^{\mathbf{g}}(Z) = \sum_{i=1}^{p}\sum_{l=1}^{Z_i}\mathcal{W}^{\mathbf{g}}_l(i)$

where the $W^{\mathbf{g}}_l$, for $l \ge 1$ are independent copies of $\mathcal{W}^{\mathbf{g}}$;

- let $\alpha_i := |M(\cdot, i)|$ for $1 \le i \le p$ and $\alpha = (\alpha_i)_{1\le i\le p}$, it holds that

$$\begin{aligned}\frac{|MM^{\mathbf{g}}_{0,n-k}e_j|}{|M^{\mathbf{g}}_{0,n-k}e_j|} &= \frac{\sum_{i=1}^p \alpha_i M^{\mathbf{g}}_{0,n-k}(i,j)}{\sum_{i=1}^p M^{\mathbf{g}}_{0,n-k}(i,j)}\\ &= \frac{\sum_{i=1}^p \alpha_i \mathbb{E}^{\mathbf{g}}[Z_{n-k}(i,j)]}{\mathbb{E}^{\mathbf{g}}|Z_{n-k}(\cdot,j)|} \quad \longrightarrow \langle \alpha, \mathbb{E}[\mathcal{W}^{\mathbf{g}}]\rangle \quad \text{as} \quad n \to +\infty;\end{aligned}$$

- $\lim_{n\to+\infty} \mathbf{1}_{(|Z^{\mathbf{g}}_{n-k}(Z,\cdot)|>0)} = \mathbf{1}_{\cap_{n>k}(|Z^{\mathbf{g}}_{n-k}(Z,\cdot)|>0))}$.

Hence, for any $0 \le k \le \ell$, the sequences $\left(Y_{n,k}(\lambda, Z, j)\right)_{n>k}$ converge in $\mathbb{L}^1(\widehat{\mathbb{P}}_{\tilde{x}\cdot M,0})$ to the random variable

$$Y_{\infty,k}(\lambda, Z) := \exp\left(-\lambda \frac{\mathcal{W}^{\mathbf{f}\circ\theta^k}(Z)}{\langle \alpha, \mathbb{E}(\mathcal{W}^{\mathbf{f}\circ\theta^k})\rangle}\right)\mathbf{1}_{\bigcap_{n>k}(|Z^{\mathbf{f}\circ\theta^k}_{n-k}(Z,\cdot)|>0))}.$$

Lemma 4.1 yields that

$$\lim_{n\to+\infty} \sqrt{n-k}\ \mathbb{E}_{\tilde{x}\cdot M,0}\left[Y_{n,k}(\lambda)\Big/\tau\circ\theta^k > n-k\right] = \widehat{\mathbb{E}}_{\tilde{x}\cdot M,0}[Y_{\infty,k}(\lambda, Z)].$$

Consequently

$$\begin{aligned}&\lim_{n\to+\infty} \Sigma_{2,1}(n,\ell,\lambda)\\ &= \sum_{k=0}^{\ell}\int df_0\dots df_{k-1}\ \delta_{M_{0,k}}(dM)\ \int_{(|Z|>0)} \mathbb{P}(Z_k(\tilde{z},\cdot)\in dZ|f_0,\dots,f_{k-1})\mathbf{1}_{(T_k=k)}\\ &\qquad\times \frac{2}{\sigma\sqrt{2\pi}} V(\tilde{x}\cdot M,0)\ \widehat{\mathbb{E}}_{\tilde{x}\cdot M,0}\left[\exp\left(-\lambda\frac{\mathcal{W}^{\mathbf{f}\circ\theta^k}(Z)}{\langle\alpha,\mathbb{E}(\mathcal{W}^{\mathbf{f}\circ\theta^k})\rangle}\right)\mathbf{1}_{\cap_{n>k}(|Z^{\mathbf{f}\circ\theta^k}_{n-k}(Z,\cdot)|>0)}\right]\\ &= \frac{2}{\sigma\sqrt{2\pi}}\sum_{k=0}^{\ell}\mathbb{E}_{\tilde{x},a}\left[V(X_k,0)\mathbf{1}_{(T_k=k)}\Psi(\lambda,X_k,0,Z_k(\tilde{z},\cdot),\mathbf{f}\circ\theta^k)\right]\end{aligned}$$

where Ψ is defined in (5.1). Notice that, by Lemma 5.1, for any $k \ge 1$,

$$\begin{aligned}0 \le \mathbb{E}_{\tilde{x},a}&\left[V(X_k,0)\Psi(\lambda,X_k,0,Z_k(\tilde{z},\cdot),\mathbf{f}\circ\theta^k)\right]\\ &\preceq \mathbb{P}_{\tilde{x},a}(T_k=k,|Z_k(\tilde{z},\cdot)|>0) \preceq \frac{|z|}{k^{3/2}}\end{aligned} \tag{5.3}$$

so that uniformly in $\lambda \ge 0$,

$$\lim_{\ell\to+\infty}\lim_{n\to+\infty}\Sigma_{2,1}(n,\ell,\lambda) = \frac{2}{\sigma\sqrt{2\pi}}\sum_{k=0}^{+\infty}\mathbb{E}_{\tilde{x},a}\left[V(X_k,0)\Psi(\lambda,X_k,0,Z_k(\tilde{z},\cdot),\mathbf{f}\circ\theta^k)\right] \tag{5.4}$$

exists and is finite.

Let us control the term $\Sigma_{2,2}(n,\lambda)$.

$$
\begin{aligned}
&\Sigma_{2,2}(n,\lambda)\\
&= \sqrt{n}\sum_{k=l+1}^{n-1}\int df_0\dots df_{k-1}\ \delta_{M_{0,k}}(dM)\int \mathbb{P}(Z_k(\tilde{z},\cdot)\in dZ|f_0,\dots,f_{k-1})\mathbf{1}_{(T_k=k)}\mathbf{1}_{(|Z|>0)}\\
&\qquad\qquad \times \mathbb{E}_{\tilde{x}\cdot M,0}\left[\exp\left(-\lambda\frac{Z_{n-k}(Z,j)}{|MM_{0,n-k}e_j|}\right);|Z_{n-k}(Z,\cdot)|>0,\tau>n-k\right]\\
&\le \sqrt{n}\sum_{k=l+1}^{n-1}\int df_0\dots df_{k-1}\delta_{M_{0,k}}(dM)\\
&\qquad\qquad \int \mathbb{P}(Z_k(\tilde{z},\cdot)\in dZ|f_0,\dots,f_{k-1})\mathbf{1}_{(T_k=k)}\mathbf{1}_{(|Z|>0)}\mathbb{P}_{\tilde{x}\cdot M,0}(\tau\circ\theta^k>n-k)\\
&= \sqrt{n}\sum_{k=l+1}^{n-1}\mathbb{E}_{\tilde{x},a}\left[\mathbb{P}_{X_k,0}(\tau\circ\theta^k>n-k);T_k=k;|Z_k(\tilde{z},\cdot)|>0\right].
\end{aligned}
$$

By Theorem 3.5 and Proposition 3.4,

$$
\begin{aligned}
\Sigma_{2,2}(n,\lambda) &\preceq \sum_{k=l+1}^{n-1}\frac{\sqrt{n}}{\sqrt{n-k}}\mathbb{P}_{\tilde{x},a}(T_k=k;|Z_k(\tilde{z},\cdot)|>0)\\
&\le |z|\sum_{k=l+1}^{n-1}\frac{\sqrt{n}}{\sqrt{n-k}\ \ k^{3/2}}\\
&\preceq |z|\left(\frac{1}{\sqrt{l}}+\frac{1}{\sqrt{n}}\right)
\end{aligned}
$$

which readily implies, uniformly in $\lambda\ge 0$,

$$
\limsup_{l\to+\infty}\limsup_{n\to+\infty}\Sigma_{2,2}(n,\lambda)=0 \tag{5.5}
$$

We conclude by combining (5.2), (5.4) and (5.5). In particular, since the above convergences are uniform in $\lambda\ge 0$, it holds that $\lim_{\lambda\to 0^+}\phi_{\tilde{z},j}(\lambda)=\phi_{\tilde{z},j}(0)$.

Finally, let us prove that $\nu_{\tilde{z},j}(\{0\})=0$ when the offspring generating functions belong to $\mathcal{G}_{\varepsilon,K}$. It suffices to prove that the Laplace transform of $\nu_{\tilde{z},j}$ (or equivalently the function $\phi_{\tilde{z},j}$) tends to 0 as $\lambda\to+\infty$. Indeed, by (4.10), $\widehat{\mathbb{P}}_{\tilde{x},a}$-almost surely,

$$
\begin{aligned}
\Psi(\lambda,X_k,0,Z_k(\tilde{z},\cdot),\mathbf{f}\circ\theta^k) &= \widehat{\mathbb{E}}_{\tilde{X}_k,0}\left[\exp\Big(-\lambda\frac{\mathcal{W}^{\mathbf{g}}(\tilde{z})}{\langle\alpha,\mathbb{E}(\mathcal{W}^{\mathbf{f}\circ\theta^{\mathbf{k}}})\rangle}\Big)\mathbf{1}_{(\mathcal{W}^{\mathbf{f}}(\tilde{z})>0)}\right]\Big/_{\tilde{z}=Z_k(\tilde{x},\cdot)}\\
&\longrightarrow 0 \quad\text{as}\quad \lambda\to+\infty.
\end{aligned}
$$

Hence, by combining the Lebesgue dominated convergence theorem and (5.3),

$$
\lim_{\lambda\to+\infty}\phi_{z,j}(\lambda)=0.
$$

This achieves the proof.

□

It remains to prove Lemma 5.1.

Proof of Lemma 5.1. By the branching property, for any $\tilde{z} \in \mathbb{N}^p \setminus \{\tilde{\mathbf{0}}\}$ and $\mathbb{P}$-almost all environment $\mathbf{f}$,

$$Z_n^{\mathbf{f}}(\tilde{z}, \cdot) = \sum_{i=1}^{p} \sum_{k=z_1+\ldots+z_{i-1}}^{z_1+\ldots+z_i} Z_{n,k}^{\mathbf{f}}(i, \cdot),$$

where the $Z_{n,k}^{\mathbf{f}}$, for $k \geq 1$, are i.i.d. copies of $Z_n^{\mathbf{f}}$; in particular, if $|Z_n^{\mathbf{f}}(\tilde{z}, \cdot)| > 0$, then there exist i and k such that $1 \leq i \leq p$ and $z_1 + \ldots + z_{i-1} \leq k \leq z_1 + \ldots + z_i$ and $|Z_{n,k}^{\mathbf{f}}(i, \cdot)| > 0$. Hence, noticing that T_n does not depend on the value of (X_0, S_0) and using Lemma 3.3, for any $\tilde{x} \in \mathbb{X}$ and $a > A$,

$$\begin{aligned}
\mathbb{P}_{\tilde{x},a}(|Z_n(\tilde{z},\cdot)| > 0, T_n = n) &= \mathbb{P}_{\tilde{x},0}(|Z_n(\tilde{z},\cdot)| > 0, T_n = n) \\
&\leq \sum_{i=1}^{p} z_i \mathbb{P}_{\tilde{x},0}(|Z_n(i,\cdot)| > 0, T_n = n) \\
&\leq \sum_{i=1}^{p} z_i \mathbb{E}_{\tilde{x},0}\big[|Z_n(i,\cdot)|; T_n = n\big] \\
&= \sum_{i=1}^{p} z_i \mathbb{E}_{\tilde{x},0}\big[\mathbb{E}[|Z_n(i,\cdot)|/f_0, \ldots, f_{n-1}]; T_n = n\big] \\
&= \sum_{i=1}^{p} z_i \mathbb{E}_{\tilde{x},0}\big[|M_{0,n}|; T_n = n\big] \\
&= |z| \mathbb{E}_{\tilde{x},0}\big[|M_{0,n}|; T_n = n\big] \\
&= |z| \ \mathbb{E}_{\tilde{x},0}\big[|M_{0,n}|; S_n \leq S_0, S_n \leq S_1, \ldots, S_n \leq S_{n-1}\big] \\
&\leq |z| \ \mathbb{E}\big[|M_{0,n}|; |M_{0,n}| \leq c, |M_{0,n}| \leq c|M_{0,1}| \ldots, |M_{0,n}| \leq c|M_{0,n-1}|\big] \\
&= |z| \ \mathbb{E}\big[|M_{n,0}|; |M_{n,0}| \leq c, |M_{n,0}| \leq c|M_{n,n-1}|, \ldots, |M_{n,0}| \leq c|M_{n,1}|\big] \\
&\qquad \text{since } (M_0, \ldots, M_{n-1}) \overset{\text{dist.}}{=} (M_{n-1}, \ldots, M_0)
\end{aligned}$$

$$\begin{aligned}
&\leq |z| \ \mathbb{E}\big[|M_{n,0}|; |M_{n,0}| \leq c, |M_{n-1,0}| \leq c^2, \ldots, |M_{1,0}| \leq c^2\big] \\
&\leq c \ |z| \ \mathbb{E}\big[|M_{n,0}x|; |M_{n,0}x| \leq c^2, |M_{n-1,0}x| \leq c^2, \ldots, |M_{1,0}x| \leq c^2\big] \\
&= c^3 \ |z| \ \mathbb{E}\left[\frac{1}{c^2}|M_{n,0}x|; \frac{1}{c^2}|M_{n,0}x| \leq 1, \frac{1}{c^2}|M_{n-1,0}x| \leq 1, \ldots, \frac{1}{c^2}|M_{1,0}x| \leq 1\right] \\
&= c^3 \ |z| \ \mathbb{E}_{x,-\ln c^2}\big[\exp(S'_n); S'_n \leq 0; S'_{n-1} \leq 0, \ldots, S'_1 \leq 0\big] \\
&= c^3 \ |z| \ \mathbb{E}_{x,-\ln c^2}\big[\exp(S'_n); \tau' > n\big]
\end{aligned}$$

with $S'_n = S'_n(x,a) = a + \ln|M_{0,n}x|$ for any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$ and $\tau' = \tau'_{x,a} = \min\{n \geq 1 : S'_n(x,a) > 0\}$.

Similar statements as Theorem 3.5, Proposition 3.4 and Corollary 3.7 also exist for the sequence $(S'_n(x,a))_{n\geq 0}$ and the stopping time τ'; in particular, there exists a positive constant c' such that for any $\tilde{x} \in \mathbb{X}, a, b \in \mathbb{R}$ and $n \geq 1$,

$$0 \leq \mathbb{P}_{x,a}(S'_n \in]b-1, b], \tau' > n) \leq c' \frac{(1+|a|)(1+|b|)}{n^{3/2}}.$$

Therefore,

$$\begin{aligned}
&\mathbb{P}_{\tilde{x},a}(|Z_n(\tilde{z},\cdot)|>0, T_n=n)\\
&\qquad\leq c^3\ |z|\ \mathbb{E}_{x,-\ln c^2}\left[\exp\left(S'_n\right);\tau'>n\right]\\
&\qquad= c^3\ |z|\ \sum_{b\leq 0} e^b\ \mathbb{P}_{x,-\ln c^2}\left(S'_n\in]b-1,b], \tau'>n\right)\\
&\qquad\leq c^3\ (1+|\ln c^2|)\ \ c'\ |z|\ \left(\sum_{b\leq 0} e^b(1+|b|)\right)\frac{1}{n^{3/2}}.
\end{aligned}$$

□

6 Proof of Proposition 2.4

We fix $t\in\mathbb{R}, \tilde{x}\in X, a>A$ and $\tilde{z}\in\mathbb{N}^p\setminus\{\tilde{\mathbf{0}}\}$. By Property 4.4, we have to prove that the sequence

$$\left(\sqrt{n}\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}}\leq t\ , |Z_n(\tilde{z},\cdot)|>0\right)\right)_{n\geq 0}$$

converges as $n\to+\infty$ and identify its limit.

For any $b\geq 0, \rho\in]0,1[$ and $m\in\{1,\dots,[\rho n]\}$, we may decompose the quantity

$$\sqrt{n}\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}}\leq t\ , |Z_n(\tilde{z},\cdot)|>0\right)$$

as

$$\begin{aligned}
&\underbrace{\sqrt{n}\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}}\leq t, |Z_n(\tilde{z},\cdot)|>0, \tau_{\tilde{x},b}\leq n\right)}_{A_n(b)} + \sqrt{n}\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}}\leq t, |Z_n(\tilde{z},\cdot)|>0, \tau_{\tilde{x},b}>n\right)\\
&\quad= A_n(b) + \sqrt{n}\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}}\leq t, |Z_{[\rho n]}(\tilde{z},\cdot)|>0, \tau_{\tilde{x},b}>n\right)\\
&\qquad\quad - \underbrace{\sqrt{n}\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}}\leq t, |Z_{[\rho n]}(\tilde{z},\cdot)|>0, |Z_n(\tilde{z},\cdot)|=0, \tau_{\tilde{x},b}>n\right)}_{B_n(b,\rho)}\\
&\quad= A_n(b) - B_n(b,\rho) + \underbrace{\sqrt{n}\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}}\leq t, |Z_m(\tilde{z},\cdot)|>0, \tau_{\tilde{x},b}>n\right)}_{C_n(b,\rho,m)}\\
&\qquad\quad - \underbrace{\sqrt{n}\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}}\leq t, |Z_m(\tilde{z},\cdot)|>0, |Z_{[\rho n]}(\tilde{z},\cdot)|=0, \tau_{\tilde{x},b}>n\right)}_{D_n(b,\rho,m)}\\
&\quad= A_n(b) - B_n(b,\rho) + C_n(b,\rho,m) - D_n(b,\rho,m).
\end{aligned}$$

We control these terms one by one.

Step 1. *The sequence $(A_n(b))_{n\geq 0}$ converges to $A(b)\geq 0$ and $\lim_{b\to+\infty} A(b)=0$.*

This is a direct consequence of the following inequality: for any $n\geq 1$ and $b\geq 0$,

$$\sqrt{n}\mathbb{P}\left(|Z_n(\tilde{z},\cdot)|>0, \tau_{\tilde{x},b}\leq n\right)\leq c\ |z|\ (1+b)e^{-b},$$

for some positive constant c. Indeed, by (3.8) and (Lemma 3.3, for all $\tilde{x} \in \mathbb{X}$ and $1 \leq k \leq n$, it holds $\mathbb{P}$-a.s. that

$$\mathbb{P}(|Z_n(i,\cdot)| > 0 \mid f_0, \cdots, f_{n-1}) = 1 - f_{0,n}^{(i)}(\tilde{0}) \leq \left(\sum_{l=1}^{n} \frac{1}{|M_{0,l}|} \right)^{-1} \leq c|\tilde{x} M_{0,k}|$$

so that

$$\mathbb{P}(|Z_n(i,\cdot)| > 0 \mid f_0, \cdots, f_{n-1}) \leq c e^{m_n(\tilde{x},0)}.$$

This yields

$$\begin{aligned}
&\sqrt{n}\mathbb{P}\left(|Z_n(\tilde{z},\cdot)| > 0, \tau_{\tilde{x},b} \leq n\right) \\
&\quad \leq \sqrt{n} \sum_{i=1}^{p} z_i \mathbb{E}\left[\mathbb{P}\left(|Z_n(i,\cdot)| > 0 \Big/ f_0, \cdots, f_{n-1}\right), \tau_{\tilde{x},b} \leq n\right] \\
&\quad \leq c\ \sqrt{n}\ |z|\ \mathbb{E}\left[e^{m_n(\tilde{x},0)}; \tau_{\tilde{x},b} \leq n\right] \\
&\quad \leq c\ \sqrt{n}\ |z|\ \mathbb{E}\left[e^{m_n(\tilde{x},0)}; m_n(\tilde{x},0) < -b\right] \\
&\quad = c\sqrt{n}\ |z|\ \sum_{k=0}^{+\infty} e^{-k-b} \mathbb{P}\left(-k-1-b \leq m_n(\tilde{x},0) < -k-b\right) \\
&\quad \leq c\sqrt{n}\ |z|\ \sum_{k=0}^{+\infty} e^{-k-b} \mathbb{P}\left(m_n(\tilde{x},0) \geq -k-1-b\right) \\
&\quad = c\sqrt{n}\ |z|\ \sum_{k=0}^{+\infty} e^{-k-b} \mathbb{P}_{\tilde{x},k+1+b}(\tau > n) \\
&\quad \preceq |z|\ e^{-b} \sum_{k=0}^{+\infty} (b+k+2) e^{-k} \preceq |z|\ (1+b) e^{-b} \quad \text{by Proposition 3.4 and Theorem 3.5.}
\end{aligned}$$

Step 2. *For any $b \geq 0, \rho \in]0,1[$ and $0 \leq m \leq [\rho n]$, the sequence $(D_n(b,\rho,m))_{n\geq 0}$ converges to 0.*

It suffices to prove that

$$\lim_{n \to +\infty} \sqrt{n}\mathbb{P}\left(|Z_m(\tilde{z},\cdot)| > 0, |Z_{[\rho n]}(\tilde{z},\cdot)| = 0, \tau_{\tilde{x},b} > n\right) = 0. \tag{6.1}$$

For $1 \leq m \leq [\rho n]$,

$$\begin{aligned}
&\mathbb{P}_{\tilde{x},b}\left(|Z_m(\tilde{z},\cdot)| > 0, |Z_{[\rho n]}(\tilde{z},\cdot)| = 0, \tau > n\right) \\
&\quad = \mathbb{P}_{\tilde{x},b}\left(|Z_m(\tilde{z},\cdot)| > 0, \tau > n\right) - \mathbb{P}_{\tilde{x},b}\left(|Z_{[\rho n]}(\tilde{z},\cdot)| > 0, \tau > n\right) \\
&\quad = \mathbb{E}_{\tilde{x},b}\Big[\mathbb{P}(|Z_m(\tilde{z},\cdot)| > 0 \mid f_0, \ldots, f_{n-1}) - \mathbb{P}(|Z_{[\rho n]}(\tilde{z},\cdot)| > 0 \mid f_0, \ldots, f_{[\rho n]-1});\ \tau > n)\Big] \\
&\quad = \mathbb{E}_{\tilde{x},b}\left[q^{\mathbf{f}}_{m,\tilde{z}} - q^{\mathbf{f}}_{[\rho n],\tilde{z}};\ \tau > n\right].
\end{aligned}$$

Hence, by Theorem 3.5,

$$\begin{aligned}\mathbb{P}_{\tilde{x},b}(|Z_m(\tilde{z},\cdot)|>0,&|Z_{[\rho n]}(\tilde{z},\cdot)|=0,\tau>n)\\ &\le c\mathbb{E}_{\tilde{x},b}\left[(q^{\mathbf{f}}_{m,\tilde{z}}-q^{\mathbf{f}}_{[\rho n],\tilde{z}})\frac{V(X_{[\rho n]},S_{[\rho n]})}{\sqrt{n-[\rho n]}};\tau>[\rho n]\right]\\ &\le \frac{c}{\sqrt{n(1-\rho)}}\mathbb{E}\left[(q^{\mathbf{f}}_{m,\tilde{z}}-q^{\mathbf{f}}_{[\rho n],\tilde{z}})V(X_{[\rho n]},S_{[\rho n]});\tau>[\rho n]\right]\\ &= \frac{c}{\sqrt{n(1-\rho)}}V(\tilde{x},a)\widehat{\mathbb{E}}_{\tilde{x},b}\left[q^{\mathbf{f}}_{m,\tilde{z}}-q^{\mathbf{f}}_{[\rho n],\tilde{z}}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{m\to+\infty}\lim_{n\to+\infty}\sqrt{n}\mathbb{P}_{\tilde{x},b}(|Z_m(\tilde{z},\cdot)|>0,|Z_{[\rho n]}(\tilde{z},\cdot)|=0,\tau>n)&\\ \le \frac{c}{\sqrt{1-\rho}}V(\tilde{x},a)\lim_{m\to+\infty}\underbrace{\lim_{n\to+\infty}\widehat{\mathbb{E}}_{\tilde{x},b}[q^{\mathbf{f}}_{m,\tilde{z}}-q^{\mathbf{f}}_{[\rho n],\tilde{z}}]}_{\widehat{\mathbb{E}}_{\tilde{x},b}[q^{\mathbf{f}}_{m,\tilde{z}}-q^{\mathbf{f}}_{\tilde{z}}]}}&=0.\end{aligned}$$

where the last equality is a direct consequence of the preamble of subsection 4.3.

Step 3. *For any $b\ge 0$ and $\rho\in]0,1[$, the sequence $(B_n(b,\rho))_{n\ge0}$ converges to 0.*

We write

$$\begin{aligned}0\le B_n(b,\rho)&=\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}}\le t,|Z_{[\rho n]}(\tilde{z},\cdot)|>0,|Z_n(\tilde{z},\cdot)|=0,\tau_{\tilde{x},b}>n\right)\\ &\le \mathbb{P}\left(|Z_{[\rho n]}(\tilde{z},\cdot)|>0,|Z_n(\tilde{z},\cdot)|=0,\tau_{\tilde{x},b}>n\right)\\ &=\mathbb{P}_{\tilde{x},b}\left(|Z_{[\rho n]}(\tilde{z},\cdot)|>0,\tau>n\right)-\mathbb{P}_{\tilde{x},b}\left(|Z_n(\tilde{z},\cdot)|>0,\tau>n\right).\end{aligned}$$

By Lemma 4.1 and Theorem 3.5 ,

$$\lim_{n\to+\infty}\sqrt{n}\mathbb{P}_{\tilde{x},b}\left(|Z_n(\tilde{z},\cdot)|>0,\tau>n\right)=\frac{2}{\sigma\sqrt{2\pi}}V(\tilde{x},b)\widehat{\mathbb{E}}_{\tilde{x},b}\left[q^{\mathbf{f}}_{\tilde{z}}\right]$$

and it suffices to check that the sequence $\left(\sqrt{n}\mathbb{P}_{\tilde{x},b}\left(|Z_{[\rho n]}(\tilde{z},\cdot)|>0,\tau>n\right)\right)_{n\ge0}$ converges to the same limit. Indeed, for $1\le m\le[\rho n]$,

$$\begin{aligned}&\sqrt{n}\mathbb{P}_{\tilde{x},b}\left(|Z_{[\rho n]}(\tilde{z},\cdot)|>0,\tau>n\right)\\ &\quad=\sqrt{n}\mathbb{P}_{\tilde{x},b}\left(|Z_m(\tilde{z},\cdot)|>0,\tau>n\right)-\sqrt{n}\mathbb{P}_{\tilde{x},b}\left(|Z_m(\tilde{z},\cdot)|>0,|Z_{[\rho n]}(\tilde{z},\cdot)|=0,\tau>n\right)\end{aligned}$$

with

- $\lim_{n\to+\infty}\sqrt{n}\mathbb{P}_{\tilde{x},b}\left(|Z_m(\tilde{z},\cdot)|>0,\tau>n\right)=\frac{2V(\tilde{x},b)}{\sigma\sqrt{2\pi}}\widehat{\mathbb{P}}_{\tilde{x},b}(|Z_m(\tilde{z},\cdot)|>0)$, by Lemma 4.1 and Theorem 3.5;
- $\lim_{n\to+\infty}\sqrt{n}\mathbb{P}_{\tilde{x},b}\left(|Z_m(\tilde{z},\cdot)|>0,|Z_{[\rho n]}(\tilde{z},\cdot)|=0,\tau>n\right)=0$ by (6.1) of Step 2.

Hence

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sqrt{n}\mathbb{P}_{\tilde{x},b}\left(|Z_{[\rho n]}(\tilde{z},\cdot)|>0,\tau>n\right)=\frac{2V(\tilde{x},b)}{\sigma\sqrt{2\pi}}\widehat{\mathbb{P}}_{\tilde{x},b}\Big(q^{\mathbf{f}}_{\tilde{z}}\Big)$$

and the proof is complete.

Step 4. *For any $b\ge0$ and $\rho\in]0,1[$,*

$$\lim_{m\to+\infty}\lim_{n\to+\infty}C_n(b,\rho,m)=\frac{2}{\sigma\sqrt{2\pi}}V(\tilde{x},b)\ \widehat{\mathbb{E}}_{x,b}[q^{\mathbf{f}}_{\tilde{z}}]\ \Phi^+\left(\frac{t}{\sigma}\right).$$

Assume that $n \geq 2m$. On the one hand, when $t < 0$, the quantity $t + \frac{b-a}{\sqrt{n}}$ becomes negative when n is great enough, in which case $\left(\frac{S_n}{\sqrt{n}} \leq t + \frac{b-a}{\sqrt{n}}\right) \cap (\tau > n) = \emptyset$; therefore, the above limit holds in this case. On the other hand, when $t \geq 0$,

$$\begin{aligned}
&\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}} \leq t, |Z_m(\tilde{z},\cdot)| > 0, \tau_{\tilde{x},b} > n\right) \\
&\quad = \mathbb{P}_{\tilde{x},b}\left(\frac{S_n}{\sqrt{n}} \leq t + \frac{b-a}{\sqrt{n}}, |Z_m(\tilde{z},\cdot)| > 0, \tau > n\right) \\
&\quad = \mathbb{E}_{\tilde{x},b}\left[\mathbb{P}_{\tilde{x},b}\left(\frac{S_n}{\sqrt{n}} \leq t + \frac{b-a}{\sqrt{n}}, |Z_m(\tilde{z},\cdot)| > 0, \tau > n \;\Big/\; f_0, \ldots, f_{m-1}, Z_0, \ldots, Z_m\right)\right] \\
&\quad = \mathbb{E}_{\tilde{x},b}\left[|Z_m(\tilde{z},\cdot)| > 0, \tau > m,\right. \\
&\qquad \left.\mathbb{P}_{\tilde{x},b}\left(\frac{S_m + S_{n-m} \circ \theta^m(X_m, S_m)}{\sqrt{n-m}} \leq t_{n,m}, \tau_{(X_m,S_m)} > n - m \;\Big/\; f_0, \ldots, f_{m-1}, Z_0, \ldots, Z_m\right)\right],
\end{aligned}$$

where $t_{n,m} = \left(t + \frac{b-a}{\sqrt{n}}\right)\frac{\sqrt{n}}{\sqrt{n-m}}$. By Corollary 3.6, as $n \to +\infty$,

$$\sqrt{n}\mathbb{P}_{\tilde{x},b}\left(\frac{S_n}{\sqrt{n}} \leq s, \tau > n\right) \to \frac{2V(\tilde{x},a)}{\sigma\sqrt{2\pi}}\Phi^+\left(\frac{s}{\sigma}\right) = \frac{2V(\tilde{x},a)}{\sigma\sqrt{2\pi}}\left(1 - \exp\left(-\frac{s^2}{2\sigma^2}\right)\right),$$

then

$$\begin{aligned}
&\sqrt{n-m}\ \mathbb{P}\Big(\frac{S_m(\tilde{x},a) + S_{n-m} \circ \theta^m(X_m,S_m)}{\sqrt{n-m}} \leq t_{n,m}, \tau_{(X_m,S_m)} > n - m \Big/ f_0, \ldots, f_{m-1}, Z_0, \ldots, Z_m\Big) \\
&\qquad = \frac{2}{\sigma\sqrt{2\pi}} V(X_m, S_m)\left(1 - \exp\left(-\frac{t_{n,m}^2}{2\sigma^2}\right)\right)(1 + o(n-m)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sqrt{n}\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}} \leq t, |Z_m(\tilde{z},\cdot)| > 0, \tau_{\tilde{x},b} > n\right) \\
&= \frac{2}{\sigma\sqrt{2\pi}}\frac{\sqrt{n}}{\sqrt{n-m}}\mathbb{E}_{\tilde{x},b}\left[V(X_m,S_m)(1 + o(n-m))\left(1 - \exp\left(-\frac{t_{n,m}^2}{2\sigma^2}\right)\right), |Z_m(\tilde{z},\cdot)| > 0, \tau > m\right] \\
&= \frac{2}{\sigma\sqrt{2\pi}}\frac{\sqrt{n}}{\sqrt{n-m}} V(\tilde{x},b)\ \widehat{\mathbb{E}}_{\tilde{x},b}\left[\left(1 - \exp\left(-\frac{t_{n,m}^2}{2\sigma^2}\right)\right)(1 + o(n-m)); |Z_m(\tilde{z},\cdot)| > 0\right] \\
&\qquad \longrightarrow \frac{2}{\sigma\sqrt{2\pi}}\left(1 - \exp\left(-\frac{t^2}{2\sigma^2}\right)\right) V(\tilde{x},b)\ \widehat{\mathbb{P}}_{\tilde{x},b}(|Z_m(\tilde{z},\cdot)| > 0) \quad \text{as} \quad n \to +\infty.
\end{aligned}$$

Finally

$$\begin{aligned}
&\lim_{m\to+\infty}\lim_{n\to+\infty}\sqrt{n}\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}} \leq t, |Z_m(\tilde{z},\cdot)| > 0, \tau_{\tilde{x},b} > n\right) \\
&\qquad = \frac{2}{\sigma\sqrt{2\pi}}\left(1 - \exp\left(-\frac{t^2}{2\sigma^2}\right)\right) V(\tilde{x},b)\ \widehat{\mathbb{P}}_{\tilde{x},b}\left(\bigcap_{m\geq 0}[|Z_m(\tilde{z},\cdot)| > 0]\right) \\
&\qquad = \frac{2}{\sigma\sqrt{2\pi}}\Phi^+\left(\frac{t}{\sigma}\right) V(\tilde{x},b)\ \widehat{\mathbb{P}}_{\tilde{x},b}\left(q_{\tilde{z}}^{\mathbf{f}}\right).
\end{aligned}$$

Step 5. *Conclusion*

By the four previous steps and Property 4.4, letting $n \to +\infty$, then $m \to +\infty$ and at last $b \to +\infty$, we obtain that

$$\lim_{n\to+\infty} \sqrt{n}\mathbb{P}\left(\frac{S_n(\tilde{x},a)}{\sqrt{n}} \leq t \ , |Z_n(\tilde{z},\cdot)| > 0\right) = \frac{2}{\sigma\sqrt{2\pi}}\Phi^+\left(\frac{t}{\sigma}\right).$$

□

7 Proof of Theorem 2.2

Let $\varepsilon > 0$. Then

$$\begin{aligned}
&\mathbb{P}\Big(\Big|\frac{\ln|Z_n(\tilde{z},\cdot)|}{\sqrt{n}} - \frac{S_n(\tilde{x},a)}{\sqrt{n}}\Big| \geq \varepsilon \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\Big) \\
&\qquad = \mathbb{P}\left(|\ln|Z_n(\tilde{z},\cdot)| - S_n(\tilde{x},a)| \geq \varepsilon\sqrt{n} \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\right) \\
&\qquad = \mathbb{P}\left(\frac{|Z_n(\tilde{z},\cdot)|}{e^{S_n(\tilde{x},a)}} \geq e^{\varepsilon\sqrt{n}} \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\right) \\
&\qquad\qquad + \mathbb{P}\left(\frac{|Z_n(\tilde{z},\cdot)|}{e^{S_n(\tilde{x},a)}} \leq e^{-\varepsilon\sqrt{n}} \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\right) \\
&\qquad \leq \sum_{j=1}^{p} \mathbb{P}\left(\frac{Z_n(\tilde{z},j)}{e^{S_n(\tilde{x},a)}} \geq e^{\varepsilon\sqrt{n}}/p \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\right) \\
&\qquad\qquad + \mathbb{P}\left(\frac{Z_n(\tilde{z},1)}{e^{S_n(\tilde{x},a)}} \leq e^{-\varepsilon\sqrt{n}} \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\right) \\
&\qquad \leq \sum_{j=1}^{p} \mathbb{P}\left(\frac{Z_n(\tilde{z},j)}{|M_{0,n}e_j|} \geq c\ e^{a+\varepsilon\sqrt{n}}/p \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\right) \\
&\qquad\qquad + \mathbb{P}\left(\frac{Z_n(\tilde{z},1)}{|M_{0,n}e_1|} \leq \frac{1}{c}\ e^{a-\varepsilon\sqrt{n}} \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\right) \\
&\qquad\qquad\qquad \text{(where } c \text{ is the constant which appears in Lemma 3.3)} \\
&\qquad = \sum_{j=1}^{p} \left(1 - \mathbb{P}\left(\frac{|Z_n(\tilde{z},j)|}{|M_{0,n}e_j|} < c\ e^{a+\varepsilon\sqrt{n}}/p \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\right)\right) \\
&\qquad\qquad + \mathbb{P}\left(\frac{Z_n(\tilde{z},1)}{|M_{0,n}e_1|} \leq \frac{1}{c}\ e^{a-\varepsilon\sqrt{n}} \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\right).
\end{aligned}$$

Fix $A > 1$ then there exists a number n_A great enough such that $c\ e^{a+\varepsilon\sqrt{n}_A}/p > A$ and $\frac{1}{c}\ e^{a-\varepsilon\sqrt{n}_A} < 1/A$; without loss of generality, we assume $\nu_{\tilde{z},j}(\{\frac{1}{A}, A\}) = 0$. Hence, for any $n \geq n_A$,

$$\begin{aligned}
&\mathbb{P}\Big(\Big|\frac{\ln|Z_n(\tilde{z},\cdot)|}{\sqrt{n}} - \frac{S_n(\tilde{x},a)}{\sqrt{n}}\Big| \geq \varepsilon \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\Big) \\
&\quad \leq \sum_{j=1}^{p} \left(1 - \mathbb{P}\left(\frac{|Z_n(\tilde{z},j)|}{|M_{0,n}e_j|} < A \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\right)\right) + \mathbb{P}\left(\frac{Z_n(\tilde{z},1)}{|M_{0,n}e_1|} \leq \frac{1}{A} \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\right);
\end{aligned}$$

hence, by Theorem 2.3,

$$\limsup_{n\to+\infty} \mathbb{P}\Big(\Big|\frac{\ln|Z_n(\tilde{z},\cdot)|}{\sqrt{n}} - \frac{S_n(\tilde{x},a)}{\sqrt{n}}\Big| \geq \varepsilon \ \Big/ \ |Z_n(\tilde{z},\cdot)| > 0\Big) \leq \sum_{j=1}^{p} (1 - \nu_{\tilde{z},j}([0,A])) + \nu_{\tilde{z},1}([0,1/A]).$$

Since the $\nu_{\tilde{z},j}$ are probability measures on $]0,+\infty[$, it holds $\nu_{\tilde{z},j}([0,A]) \to 1$ and $\nu_{\tilde{z},j}([0,1/A]) \to 0$ for any $1 \le j \le p$, as $A \to +\infty$. This yields

$$\lim_{n\to+\infty} \mathbb{P}\Big(\Big|\frac{\ln|Z_n(\tilde{z},\cdot)|}{\sqrt{n}} - \frac{S_n(\tilde{x},a)}{\sqrt{n}}\Big/ \ge \varepsilon \;\Big|\; |Z_n(\tilde{z},\cdot)| > 0\Big) = 0.$$

We complete the proof by combining Proposition 2.4 and Slutsky's lemma.

□

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