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A CURIOSITY ABOUT $(-1)^{[e]} + (-1)^{[2e]} + \dots + (-1)^{[Ne]}$.

FRANCESCO AMOROSO AND MOUBINOOL OMARJEE

ABSTRACT. Let α be an irrational real number; the behavior of the sum $S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \dots + (-1)^{[N\alpha]}$ depends on the continued fraction expansion of $\alpha/2$. Since the continued fraction expansion of $\sqrt{2}/2$ has bounded partial quotients, $S_N(\sqrt{2}) = O(\log(N))$ and this bound is best possible. The partial quotients of the continued fraction expansion of e grow slowly and thus $S_N(2e) = O(\frac{\log(N)^2}{\log \log(N)^2})$, again best possible. The partial quotients of the continued fraction expansion of $e/2$ behave similarly as those of e . Surprisingly enough $S_N(e) = O(\frac{\log(N)}{\log \log(N)})$.

1. INTRODUCTION

Let α be an irrational real number; we are interested in the asymptotic behavior of the sum

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \dots + (-1)^{[N\alpha]}.$$

The origin of this question seems to go back to [12], where it is remarked that $S_N(\sqrt{2}) = O(\log N)$. More accurate estimates for $S_N(\sqrt{2})$ are available in [5] and were already implicit in [9], where the authors gave an unexpected explicit formula¹ for $S_N(\sqrt{2})$ in terms of the continued fraction expansion² $\sqrt{2} = [1; \bar{2}]$.

The behavior of $S_N(\alpha)$ is closely related to the uniform distribution mod 1 of the sequence $(n\alpha/2)_{n \in \mathbb{N}}$. Indeed, $[n\alpha]$ is even if and only if the fractional part $\{n\alpha/2\}$ is in $[0, 1/2)$. Thus,

$$\begin{aligned} (1.1) \quad S_N(\alpha) &= |\{n = 1, \dots, N \mid [n\alpha] \text{ even}\}| - |\{n = 1, \dots, N \mid [n\alpha] \text{ odd}\}| \\ &= 2|\{n = 1, \dots, N \mid \{n\alpha/2\} \in [0, 1/2)\}| - N \\ &= 2D_N(\alpha/2, 1/2). \end{aligned}$$

Here D_N is the local discrepancy:

$$D_N(\alpha, x) = |\{n = 1, \dots, N \mid \{n\alpha\} \in [0, x)\}| - Nx$$

for $\alpha \in \mathbb{R}$ and $x \in [0, 1]$. A lazy way to bound $D_N(\alpha)$ is to put in the picture the global discrepancy³

$$D_N(\alpha) := \sup_{0 \leq x < y \leq 1} \left| |\{n = 1, \dots, N \mid \{n\alpha\} \in [x, y)\}| - N(y - x) \right|.$$

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¹which can be viewed as an equality between non absolutely convergent Fourier series.

²Here and below, $\overline{a_1, \dots, a_k}$ means $a_1, \dots, a_k, a_1, \dots, a_k, \dots$.

³Note that some authors, as [3], divide by N in the definition of D_N .

Thus $|D_N(\alpha, 1/2)| \leq D_N(\alpha)$. For an irrational α , the sequence $(n\alpha)$ is uniform distribution mod 1 by a well known theorem attributed ([3], p.21) independently to Bohl, Sierpiński and Weyl. This means that $D_N(\alpha) = o(N)$. More precise estimates for D_N depend on the diophantine approximation properties of α . We recall that the irrationality exponent $\mu(\alpha)$ of an irrational $\alpha \in \mathbb{R}$ is the infimum (possibly $+\infty$) of the set of positive real numbers μ such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $p, q \in \mathbb{Z}$ with $q > 0$ we have

$$\left| \alpha - \frac{p}{q} \right| > \frac{C_\varepsilon}{q^{\mu+\varepsilon}}.$$

It is well known that $\mu(\alpha) \geq 2$ with equality for almost all α . It is also well known that μ is invariant by integral Möbius transformations $\alpha \mapsto \frac{a\alpha+b}{c\alpha+d}$ ($a, b, c, d \in \mathbb{Z}$, $ad - bc \neq 0$).

From⁴ [3, Theorem 3.2, p.123] $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 1 - \frac{1}{\mu(\alpha)-1}$. In particular, if $\mu(\alpha) = 2$ we have $D_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 0$. A more precise result holds for irrational numbers α whose continued fraction expansion has bounded partial quotients (and hence irrationality measure 2). In this case we have (see [3, Theorem 3.4, p.125]) $D_N(\alpha) = O(\log N)$.

This last estimate cannot be improved. Indeed the global discrepancy D_N of every infinite sequence $(u_n)_n$ is⁵ $\Omega(\log N)$ (see [13, Theorem 1, p.45]).

Nevertheless, we can construct irrational α such that $|D_N(\alpha, 1/2)|$ is as small as we wish. Our first result is:

Theorem 1.1. *Let $\delta: \mathbb{N} \rightarrow \mathbb{R}^+$ be a function which tends to infinity. Then there exists an irrational number α such that*

$$D_N(\alpha, 1/2) = O(\delta(N)).$$

Equivalently, we can find an irrational α such that⁶ $S_n(\alpha) = O(\delta(N))$.

By [2, Theorem 8, p.237], for any irrational α there exists a positive constant $A = A(\alpha)$ such that $|\sum_{n=1}^N f(n\alpha)| \geq AN$, where $f(t) = \{t\} - 1/2$. By Theorem (1.1) we cannot replace in this statement $\{t\} - 1/2$ with $(-1)^{\lfloor t \rfloor}$, even taking instead of N any function $\delta(N)$ which tends to infinity. See also [10] for a related question.

We then show that for some *classical number* the local discrepancy $D_N(\alpha, 1/2)$ can be substantially smaller than $D_N(\alpha)$ and even $o(\log N)$.

Theorem 1.2.

$$(1.2) \quad \overline{\lim}_{N \rightarrow +\infty} D_N(e/2) \left(\frac{\log \log N}{\log N} \right)^2 = \frac{1}{8}$$

and

$$(1.3) \quad \overline{\lim}_{N \rightarrow +\infty} |D_N(e/2, 1/2)| \frac{\log \log N}{\log N} = \frac{3}{2}.$$

⁴The authors state this result in terms of the *type* of α which is equal to $\mu(\alpha) - 1$.

⁵Here Ω is the Landau symbol: if f, g are two functions with $g > 0$ then $f = \Omega(g)$ means $f \neq o(g)$

⁶Note that for any irrational α we have $\overline{\lim} |S_n(\alpha)| = +\infty$ ([7, Theorem 1]).

Let's come back to the sum in the title. The question of providing good bound for $S_N(e)$ goes back to H. Pépin [7], who, in the nice self-contained treatment of this matter [8], already get $S_N(e) = O((\log N)^2)$. Equations (1.1) and (1.2) show that $S_N(e)$ is smaller than what one would expect:

$$(1.4) \quad S_N(e) = (-1)^{[e]} + (-1)^{[2e]} + \dots + (-1)^{[Ne]} = O(\log(N)/\log \log(N)).$$

Note that

$$S_N(2e) = (-1)^{[2e]} + (-1)^{[4e]} + \dots + (-1)^{[2Ne]} = O((\log(N)/\log \log(N))^2)$$

is best possible, by (1.1) and by (1.6) of Theorem 1.3 below.

Bounds for $S_N(\alpha)$ are useful to study the convergence of sums of the shape $\sum_n (-1)^{[n\alpha]} u_n$. Let $\Delta u_n = u_{n+1} - u_n$. By partial sommation (as in the solution to [6] proposed by R. Tauraso [14]) we see that such a sum converges if $S_N(\alpha)u_N \rightarrow 0$ and $\sum_N S_N(\alpha)\Delta u_N$ converges. By (1.4) both conditions are satisfied when $\alpha = e$ and $u_n = \frac{\log \log(n+1)}{\log(n+1)^2}$. To get more precise and general results, it might be suitable to make a second partial summation, since the arithmetic mean of $S_N(\alpha)$ behave more regularly.

The gain of the factor $\frac{\log \log N}{\log N}$ in (1.3) heavily depends on the particular structure of the continued expansion of $e/2$. Let us give a short explication. Both estimates (1.2) and (1.3) for the global and local discrepancy of $(ne/2)$ depend on the partial quotients $\{a_n\}_{n \geq 1}$ of the continued fraction expansion of $e/2$. This sequence is unbounded. But in the estimate (1.3) only the a_n with $n \not\equiv 2 \pmod 3$ come in. The corresponding sequence is now bounded. This phenomenon does not occur if we replace $e/2$ by e , as the following theorem shows.

Theorem 1.3.

$$(1.5) \quad \overline{\lim}_{N \rightarrow +\infty} D_N(e) \left(\frac{\log \log N}{\log N} \right)^2 = \frac{1}{4}$$

and

$$(1.6) \quad \overline{\lim}_{N \rightarrow +\infty} |D_N(e, 1/2)| \left(\frac{\log \log N}{\log N} \right)^2 = \frac{1}{4}.$$

Relations (1.3) and (1.6) show that the order of growth of $\alpha \mapsto D_N(\alpha, 1/2)$ is *not* invariant with respect to Möbius transformations, contrary to what happen for the global discrepancy.

Although our theorems are straightforward applications of known results ([1] and [11]), it seems that they deserve to be remarked.

2. COMPUTATIONS

Proof of of Theorem 1.1. The proof is an easy application of [11, Example, p.1497]. Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a function taking odd values and which increases to infinity sufficiently fast. We choose

$$\alpha = \alpha_f = [0; 1, 1, f(1), 1, 1, f(2), \dots].$$

Let a_j and q_m be the partial quotients and the denominators of the convergents of α . For $N \in \mathbb{N}$ we define $m(N) \in \mathbb{N}$ by the property

$$q_{m(N)} \leq N < q_{m(N)} + 1.$$

Put

$$a_j^+ = \begin{cases} a_j, & \text{if } q_{j-1} \text{ is even and } j \text{ is odd;} \\ 0, & \text{otherwise} \end{cases}$$

and

$$a_j^- = \begin{cases} a_j, & \text{if } q_{j-1} \text{ and } j \text{ are even;} \\ 0, & \text{otherwise.} \end{cases}$$

Define the following sums:

$$S_m^+ = \frac{1}{4} \sum_{\substack{2|j \leq m \\ 2|q_j}} a_{j+1} = \frac{1}{4} \sum_{j=0}^m a_{j+1}^+, \quad S_m^- = \frac{1}{4} \sum_{\substack{2|j \leq m \\ 2|q_j}} a_{j+1} = \frac{1}{4} \sum_{j=0}^m a_{j+1}^-.$$

Then from [11, Example, p.1497] we have (as in the deduction of Corollary 1.2 from Theorem 1.1 in [1]):

$$(2.1) \quad \overline{\lim}_{N \rightarrow +\infty} D_N(e/2, 1/2)/S_{m(N)}^+ = - \underline{\lim}_{N \rightarrow +\infty} D_N(e/2, 1/2)/S_{m(N)}^- = 1.$$

From the usual recursive definition of q_m we easily see that⁷ q_{j-1} is even iff $j \equiv 0 \pmod{3}$. Thus

$$\{a_j^+\}_{j \geq 1} = \{\overline{1, 0, 0}\}, \quad \{a_j^-\}_{j \geq 1} = \{\overline{0, 1, 0}\}$$

and

$$(2.2) \quad S_m^+ \sim S_m^- \sim \frac{1}{4} \sum_{k=1}^{\lfloor m/3 \rfloor} 1 \sim \frac{m}{12}.$$

Moreover, from the recursive definition of q_m we have

$$q_m \geq \prod_{j=1}^{\lfloor m/3 \rfloor} f(j).$$

Thus, if f grows sufficiently fast, for $N \in \mathbb{N}$ we have $q_{12\lceil \delta(N) \rceil} \geq N$ and, by definition, $m(N) \leq \lceil 12\delta(N) \rceil$. By (2.1) and (2.2) we have $D_N(\alpha, 1/2) = O(\delta(N))$ as desired.

□

⁷ To check this property we can of course reduce modulo 2 all the partial coefficients, thus reduce ourselves to compute the well-known convergents of the golden ratio.

Proof of of Theorem 1.2. To prove (1.2) we follow the proof of [1, Theorem 3.2(2), p.286] taking now (cf (2.3)) a_1, a_2, \dots be the partial quotients of the continued fraction expansion⁸ of $e/2$

$$(2.3) \quad e/2 = [1; 2, \mathbf{1, 3, 1, 1, 1, 3, 3, 3, 1, 3, 1, 3, 5, 3, 1, 5, 1, 3, 7, 3, 1, 7, 1, 3, \dots}].$$

We easily find

$$\sum_{i=0}^m a_{i+1} \sim 2 \sum_{k=1}^{[m/6]} (2k-1) \sim \frac{1}{18} m^2$$

and

$$\sum_{i=0}^m \log a_{i+1} \sim 2 \sum_{k=1}^{[m/6]} \log(2k-1) \sim \frac{1}{3} m \log m.$$

Thus

$$\overline{\lim}_{N \rightarrow +\infty} D_N(e/2) \left(\frac{\log \log N}{\log N} \right)^2 = \frac{\frac{1}{18}}{4(\frac{1}{3})^2} = \frac{1}{8}.$$

To prove (1.3) we apply again the formula in [11, Example, p.1497]. Let a_j and q_m be the partial quotients and the denominators of the convergents of (2.3). Let $m(N)$, a_j^\pm and S_m^\pm be as in the the proof of Theorem 1.1. Then

$$(2.4) \quad \overline{\lim}_{N \rightarrow +\infty} D_N(e/2, 1/2) / S_{m(N)}^+ = - \underline{\lim}_{N \rightarrow +\infty} D_N(e/2, 1/2) / S_{m(N)}^- = 1.$$

From (2.3) and from the usual recursive definition of q_m we see (cf note⁷) that q_{j-1} is even iff $j \equiv 2 \pmod{3}$. Thus

$$\{a_j^+\}_{j \geq 1} = \{2, \overline{0, 3, 0, 0, 0, 3}\}, \quad \{a_j^-\}_{j \geq 1} = \{2, \overline{0, 0, 1, 0, 1, 0}\}$$

and

$$S_m^+ \sim \frac{1}{4} \sum_{k=1}^{[m/6]} (3+3) \sim \frac{1}{4} m, \quad S_m^- \sim \frac{1}{4} \sum_{k=1}^{[m/6]} (1+1) \sim \frac{1}{12} m.$$

Moreover (cf (2.3))

$$\log q_m \sim \sum_{i=1}^m \log a_i \sim 2 \sum_{k=1}^{[m/6]} \log(2k-1) \sim 2 \frac{m}{6} \log m = \frac{1}{3} m \log m$$

which, by definition of $m(N)$, easily implies $m(N) \sim 3 \frac{\log N}{\log \log N}$. Replacing these estimates in (2.6) we get

$$\overline{\lim}_{N \rightarrow +\infty} D_N(e/2, 1/2) / \left(\frac{3}{4} \frac{\log N}{\log \log N} \right) = - \underline{\lim}_{N \rightarrow +\infty} D_N(e/2, 1/2) / \left(\frac{3}{12} \frac{\log N}{\log \log N} \right) = 1.$$

Equation (1.3) follows. □

⁸which can be easily computed from the well-known Euler continued fraction of e , e.g. by known algorithms [4].

Proof of of Theorem 1.3. Equation (1.5) is a special case of [1, Theorem 3.2(2), p.286]. The deduction of (1.6) follows the same lines as that of (1.3). Let a_j and q_m be the partial quotients and the denominators of the convergents of the continued fraction expansion of e

$$(2.5) \quad e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$$

Let $m(N)$, a_j^\pm and S_m^\pm as in the the proof of Theorem 1.1. From [11, Example, p.1497]:

$$(2.6) \quad \overline{\lim}_{N \rightarrow +\infty} D_N(e, 1/2)/S_{m(N)}^+ = - \underline{\lim}_{N \rightarrow +\infty} D_N(e, 1/2)/S_{m(N)}^- = 1.$$

From (2.5) and from the usual recursive definition of q_m we easily see that q_{j-1} is even iff $j \equiv 0, 4 \pmod{6}$. Thus (cf (2.5))

$$\begin{aligned} \{a_j^+\}_{j \geq 1} &= \{1, 0, 1, 0, 4, 0, 1, 0, 1, 0, 8, 0, 1, 0, 1, 0, 12, 0, \dots\}; \\ \{a_j^-\}_{j \geq 1} &= \{0, 2, 0, 0, 0, 0, 0, 6, 0, 0, 0, 0, 0, 10, 0, 0, 0, 0, \dots\} \end{aligned}$$

and

$$S_m^+ \sim \frac{1}{4} \sum_{k=1}^{\lfloor m/6 \rfloor} (1 + 1 + 4k) \sim \frac{1}{72} m^2, \quad S_m^- \sim \frac{1}{4} \sum_{k=1}^{\lfloor m/6 \rfloor} (4k - 2) \sim \frac{1}{72} m^2.$$

Moreover (cf again (2.5))

$$\log q_m \sim \sum_{i=1}^m \log a_i \sim \sum_{k=1}^{\lfloor m/3 \rfloor} \log(2k) \sim \frac{1}{3} m \log m$$

which implies $m(N) \sim 3 \frac{\log N}{\log \log N}$. Replacing these estimates in (2.6) we get

$$\overline{\lim}_{N \rightarrow +\infty} D_N(e, 1/2) / \left(\frac{1}{8} \left(\frac{\log N}{\log \log N}\right)^2\right) = - \underline{\lim}_{N \rightarrow +\infty} D_N(e, 1/2) / \left(\frac{1}{8} \left(\frac{\log N}{\log \log N}\right)^2\right) = 1.$$

Equation (1.6) follows. □

REFERENCES

- [1] C. Baxa and J. Schoissengeier, “Minimum and maximum order of magnitude of the discrepancy of $(n\alpha)$ ”, *Acta Arithmetica* **67**, 281–290 (1994)
- [2] G. H. Hardy and J. E Littlewood, “The lattice-points of a right-angled triangle. (Second memoir.)” *Abh. Math. Sem. Univ. Hamburg* **1** 211–248 (1922).
- [3] L. Kuipers and H. Niederreiter “Uniform Distribution of Sequences”. Dover Books on Mathematics. 2006.
- [4] P. Liardet and P. Stambul, “Algebraic computations with continued fractions”. *J. Number Theory* **73** 92–121 (1998).
- [5] K. O’Byrant, B. Reznick and M. Serbinowska “Almost Alternating Sums”, *The Amer. Math. Monthly*, **113**, 673–688 (2006)
- [6] M. Omarjee, “Problem 11384.” *Amer. Math. Monthly* **115**, 757 (2008)
- [7] H. Pépin, “Question et Réponses. Q86”, *Revue de mathématiques spéciales*, Octobre 1987, page 116.
- [8] H. Pépin, “Question et Réponses. R86”, *Revue de mathématiques spéciales*, Janvier 1989, pages 229–232.
- [9] A. Perelli e U. Zannier, “On the parity of $[n\sqrt{2}]$ ”. *Boll. Un. Mat. Ital. A (6)*, **2** 77–83 (1983).

- [10] A. Perelli and U. Zannier, “An Ω result in uniform distribution theory”. *Quart. J. Math. Oxford Ser. (2)* **38** 95–102(1987).
- [11] L. Roçadas and J. Schoissengeier, “On the local discrepancy of $(n\alpha)$ -sequences”, *J. Number Theory* **131**, 1492–1497 (2011)
- [12] H. D. Ruderman and D. Borwein, “Problems and Solutions: Solutions of Advanced Problems: 6105”, *Amer. Math. Monthly* **85**, 207–208 (1978).
- [13] W. M. Schmidt, “Irregularities of distribution.VII” , *Acta Arith.* **21** 45–50 (1972).
- [14] R. Tauraso, “Solution to Problem 11384 (American Mathematical Monthly, Vol.115, October 2008)”. <https://www.mat.uniroma2.it/~tauraso/AMM/AMM11384.pdf>

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