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RIGHT-ORDERABILITY VERSUS LEFT-ORDERABILITY FOR MONOIDS

FRIEDRICH WEHRUNG

ABSTRACT. We investigate the relationship between (total) left- and right-orderability for monoids, in particular illustrating the finite case by various structural observations and counterexamples, also highlighting the particular role played by *positive* orderability. Moreover, we construct a non-left-orderable, positively right-orderable submonoid of the free product of the cyclic group of order 7 with the free group on four generators. Any group extension of that monoid has elements of order 7.

1. INTRODUCTION

The purpose of the present note is to remove an itch.

A monoid M is *right-orderable* if it has a right order, that is, a *total* order \leq such that $x \leq y$ implies $xz \leq yz$ whenever $x, y, z \in M$. *Left-orderability* is right-orderability applied to the opposite monoid of M (with multiplication defined by $x \cdot' y \stackrel{\text{def}}{=} yx$). It is well known that right-orderability and left-orderability are equivalent for groups (*Proof*: let $x \leq' y$ hold if $y^{-1} \leq x^{-1}$).

Are they equivalent for arbitrary monoids? A moment's thought, backed up by a few computations, shows that even for *finite* monoids this is not the case. In Example 2.10(1) we describe a four-element counterexample.

However, this is not the main point of the above-mentioned itch. Owing to a large existing corpus of works involving *submonoids of groups*, see in particular [6, 8], we are asking the question for those particular monoids. One would expect that question to have been answered long ago, however the author of the present note has not been able to find such a result in the literature. Our counterexample M , constructed in Theorem 4.2, is in fact *positively* ordered (i.e., the unit of M is its bottom element), and any group containing M has elements of order 7. Although we first define M *via* generators and relations, we soon find it convenient to describe it as the universal monoid of a finite category (with 30 non-identity arrows). The universal group of M is the free product of $\mathbb{Z}/7\mathbb{Z}$ with the free group on four generators. The embeddability of M into a group (thus into that group) does not follow *a priori* from usual sufficient conditions such as Adjan's condition (cf. [1] or [13, Theorem 4.6]) or Dehornoy's 3-Ore condition [7]. Instead, we are applying our criterion, stated in [19, Theorem 10.1], of embeddability of the universal monoid of a category into its universal group. Further, the right-orderability of M follows from a general result (Lemma 4.1) making it possible, under certain conditions,

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to extend a right order, from the canonical image of a category S in its universal monoid $U_{\text{mon}}(S)$, to a right order on the full $U_{\text{mon}}(S)$.

Let us briefly mention some related works, whose results we will not use here but that might help putting things into context. Johnson constructs in [10] cancellative bi-orderable (i.e., having a two-sided order) monoids that cannot be embedded into groups. Darnel, Glass, and Rhemtulla prove in [5] that if every right order on a right-orderable group G is also a left order, then G is Abelian. For further insights on ordered semigroups, see Tringali [17]. For the commutative case (which will be of concern in Sections 2 and 3), see Evans *et al.* [9], Vetterlein [18], Whipple [20].

The finite model property for the variety of all distributive ℓ -monoids, established in Section 4.2 of Almudena Colacito's PhD thesis [3], highlights an interesting connection between orderability in finite monoids and in infinite (e.g., free, lattice-ordered. . .) groups. Actually, the main (infinite, group-embeddable) example underlying Section 4 was first obtained by testing, on finite right-orderable monoids, variants of the defining equations (4.2) for that example. This helps motivating our Sections 2 and 3, mostly focused on various aspects of finite orderable monoids. Nonetheless, aside from the main definition (Definition 2.1), those sections can be read independently from Section 4 (our group-embeddable counterexample).

We claim no particular depth for the results of this note, rather hoping that they could, aside from removing a few itches, lead to further-reaching investigations, as well on the finite case as on the group-embeddable case, and their connections.

We will denote by \mathbb{N} , \mathbb{Z} , and \mathbb{Q} the additive monoids of all nonnegative integers, integers, and rational numbers, respectively. We will also denote by $F_{\text{gp}}(n)$ the free group on n generators, and by $G * H$ the free product of any two groups G and H .

A subset X in a partially ordered set P is a *lower subset* of P if for all $x \in X$ and all $p \in P$, $p \leq x$ implies that $p \in X$.

2. RIGHT-ORDERABLE MONOIDS

In this section we shall get started by formally stating the definition of a right order (resp., a positive right order) on a monoid (Definition 2.1) and collect a few observations about those, consisting of both positive results and counterexamples, focusing on the finite case. As the existence of a translation-invariant partial order, on any given monoid, is trivial (just take the diagonal), we shall follow tradition and omit the qualifier “total” (or “linear”), for example saying “order”, “positive order”, “partial order” instead of “total order”, “total positive order”, and “order”, respectively.

Definition 2.1. A partial order \leq on a monoid M is

- *positive* if $1 \leq x$ for every $x \in M$;
- a *partial right order* if $x \leq y$ implies that $xz \leq yz$, for all $x, y, z \in M$ (we say that \leq is *right translation-invariant*);
- a *right order* if it is both a *total* order and a *partial right order*.

We say that the monoid M is *right-orderable* (resp., *positively right-orderable*) if it has a right order (resp., a positive right order).

Those concepts, applied to the opposite monoid of M , yield (partial, positive) *left orders* and (positively) *left-orderable monoids*, respectively. An order is a *bi-order* if it is both a right- and left-order, and we define *bi-orderability* as the existence of a bi-order.

We will omit either prefix “right-”, “left-”, or “bi-” in the commutative case.

In many works, right-orderability is defined to mean the existence of a right translation-invariant *strict* order on a monoid (i.e., $x < y$ implies $xz < yz$). Our current definition is thus weaker: as we will see shortly, it indeed produces finite examples. For right cancellative monoids the two definitions are of course equivalent.

Note. Bi-orderability, as defined above, is (strictly) stronger than the conjunction of right- and left-orderability. A well known example illustrating this is the braid group B_3 , which is right-orderable (thus, since it is a group, left-orderable as well) but not bi-orderable (cf. Propositions II.1.2 and II.1.9 in [8]). Finite both right- and left-orderable, but not bi-orderable, monoids are given in Examples 3.4 (which is idempotent, that is, all of its elements are idempotent) and 3.5 (with is positively left- and right-orderable).

The following observation shows that existence of translation-invariant orders reflects on the idempotents. The specialization of Proposition 2.2(2) to idempotent monoids follows from Couceiro *et al.* [4, Lemma 2.5].

Proposition 2.2. *The following statements hold, for any monoid M :*

- (1) *If M is idempotent and has both a positive partial right order \leq_r and a positive partial left order \leq_l , then M is commutative. In particular, M is bi-orderable.*
- (2) *If M is bi-orderable, then $ab \in \{a, b\}$ for all idempotent $a, b \in M$.*
- (3) *If M is positively bi-orderable, then $ab = ba \in \{a, b\}$ for all idempotent $a, b \in M$ (we say that the idempotents of M form a chain).*

Proof. Ad (1). Let $x, y \in M$. Then $yx \leq_r xyx \leq_r yxyx = yx$, whence $xyx = yx$. Likewise, $xy \leq_l xyx \leq_l xyxy = xy$, whence $xyx = xy$. Therefore, $xy = yx$.

Ad (2). Let \leq be a bi-order of M . We may replace \leq by its dual order and thus assume that $1 \leq ab$. If $a \leq b$, then $b = 1b \leq ab^2 = ab \leq b^2 = b$, thus $b = ab$. If $b \leq a$, then $a = a1 \leq a^2b = ab \leq a^2 = a$, thus $a = ab$.

Ad (3). By (2), the set E of all idempotent elements in M is a submonoid of M , and $ab \in \{a, b\}$. By (1), E is commutative. \square

Remark 2.3. Binary operations \cdot satisfying $x \cdot y \in \{x, y\}$, for all x, y in their domain, are often called *quasitrivial*. Proposition 2.2(2) says that if M is bi-orderable, then the restriction of the multiplication of M to its idempotents is quasitrivial. A quasitrivial monoid arises in this way iff it is bi-orderable. A 4-element, quasitrivial, non left-orderable monoid is given in [4, Fig.5]. A deep result of Saitô [14, Theorem 4.11] characterizes the bi-orderability of an idempotent semigroup in terms of a finite list of forbidden finite subsemigroups.

Proposition 2.2(2) implies that the Boolean semilattice $\{0, 1\}^2$ is not orderable. This is superseded by the following observation by George Bergman, which shows an important difference between the general case and the cancellative case.

Proposition 2.4. *Let M and M' be monoids. Then $M \times M'$ is right-orderable iff M and M' are both right-orderable and at least one of them is right cancellative.*

Proof. If M and M' are both right ordered and M is right cancellative, then the lexicographical product on $M \times M'$ is a right order. Conversely, let $M \times M'$ be right ordered and suppose, by way of contradiction, that there are $x, y, z \in M$ such

that $x \neq y$ and $xz = yz$ (denote that element by u), together with $x', y', z' \in M'$ such that $x' \neq y'$ and $x'z' = y'z'$ (denote that element by u'). We may assume that $(x, u') < (y, u')$. Since $(x, u') = (x, x')(1, z') = (x, y')(1, z')$ and $(y, u') = (y, x')(1, z') = (y, y')(1, z')$, it follows that $(x, x') < (y, y')$ and $(x, y') < (y, x')$. If $(u, x') < (u, y')$, then, since $(u, x') = (y, x')(z, 1)$ and $(u, y') = (x, y')(z, 1)$, we get $(y, x') < (x, y')$, a contradiction. If $(u, y') < (u, x')$, then, since $(u, y') = (y, y')(z, 1)$ and $(u, x') = (x, x')(z, 1)$, we get $(y, y') < (x, x')$, a contradiction again. \square

Say that a monoid M is *conical* if it satisfies the implication¹ $xy = 1 \Rightarrow y = 1$.

Proposition 2.5. *The following statements hold, for any finite right-orderable monoid M :*

- (1) *For every $x \in M$ there exists $m \in \mathbb{N}$ such that $x^m = x^{m+1}$. Denoting² that element of M by x^ω , it is idempotent and it satisfies the equations $xx^\omega = x^\omega x = x^\omega$. Further, for any $y \in M$, $xy = y$ iff $x^\omega y = y$, and $yx = y$ iff $yx^\omega = y$.*
- (2) *M is conical.*
- (3) *If, in addition, M is commutative, then it satisfies the implication $xyz = z \Rightarrow yz = z$.*

Proof. Ad (1). If $1 \leq x$, then $x^n \leq x^{n+1}$ for all $n \in \mathbb{N}$. If $x \leq 1$, then $x^{n+1} \leq x^n$ for all $n \in \mathbb{N}$. In both cases, since M is finite, there exists m such that $x^m = x^{m+1}$. Denoting this element by x^ω , it satisfies, by definition, $xx^\omega = x^\omega x = x^\omega = x^\omega x^\omega$. If $xy = y$, then $x^n y = y$ for all n , whence $x^\omega y = y$. Conversely, if $x^\omega y = y$, then $xy = xx^\omega y = x^\omega y = y$. Similarly, $yx = y$ iff $yx^\omega = y$.

Ad (2). If $xy = 1$, then $x^n y^n = 1$ for all $n \in \mathbb{N}$; so $x^\omega y^\omega = 1$, and so $y = 1y = x^\omega y^\omega y = x^\omega y^\omega = 1$.

Ad (3). Let $xyz = z$ within M . Then $(xy)^\omega z = z$ by (1). Since M is commutative, $(xy)^\omega = y^\omega x^\omega$, so $yz = y(xy)^\omega z = yy^\omega x^\omega z = y^\omega x^\omega z = (xy)^\omega z = z$. \square

Remark 2.6. By using a more elaborate argument, Proposition 2.5(3) can be extended to the case where *any two idempotents of M commute*. (By Example 2.10(2), the commuting idempotents assumption cannot be dispensed with.) That extension follows in turn from Proposition 2.2(2) together with a more general result, due to Shevrin [16, Corollary 3.38], valid in the class of all *epigroups* (aka *completely π -regular semigroups*).

Proposition 2.5(3) says that in the commutative case, the binary relation \leq^+ , defined by letting $x \leq^+ y$ hold if there exists z such that $x + z = y$, is a positive partial order of the monoid. In contrast to Remark 2.6, the following example, obtained with the help of McCune's wonderful Mace4 counterexample builder [12], shows that this observation fails to extend to the non-commutative case, even if we assume that the idempotents form a chain.

Example 2.7. *A 10-element bi-orderable monoid, in which the idempotents form a chain, without any positive partial bi-order.*

¹Here and elsewhere we use math sans serif fonts x, y, z to mark syntactic variables, as distinct from monoid elements x, y, z .

²The notation x^ω is customary in semigroup theory, see for example Shevrin [16].

Proof. The monoid M , with multiplication table represented in Table 2.1, can be bi-ordered *via* the inequalities $i < 1 < a < b < c < d < e < f < g < \infty$. Its idempotents, namely $1, i, a, \infty$, form a chain (e.g., $ia = ai = a$).

\cdot	i	1	a	b	c	d	e	f	g	∞
i	i	i	a	b	c	d	e	f	g	∞
1	i	1	a	b	c	d	e	f	g	∞
a	a	a	a	b	d	d	e	f	g	∞
b	b	b	e	f	f	g	∞	∞	∞	∞
c	b	c	e	f	f	g	∞	∞	∞	∞
d	b	d	e	f	f	g	∞	∞	∞	∞
e	e	e	e	f	g	g	∞	∞	∞	∞
f	f	f	∞	∞	∞	∞	∞	∞	∞	∞
g	f	g	∞	∞	∞	∞	∞	∞	∞	∞
∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

TABLE 2.1. A bi-orderable monoid, in which the idempotents form a chain, with no positive partial bi-order

Suppose that M has a positive partial bi-order \trianglelefteq . Then $c^2 \trianglelefteq cac \trianglelefteq caci = c^2$, so $f = c^2 = cac = g$, a contradiction. \square

Let us now introduce a source of future examples.

Notation 2.8. For any set X , denote by $X^{(1)}$ the monoid obtained by adjoining a unit to the right zero semigroup with universe X . Hence $X^{(1)} = X \cup \{1\}$ and

$$xy = \begin{cases} x, & \text{if } y = 1, \\ y, & \text{if } y \neq 1 \end{cases} \quad \text{for all } x, y \in X \cup \{1\}.$$

The monoid $X^{(1)}$ is both conical and idempotent. More can be said.

Proposition 2.9. *The following statements hold for any set X :*

- (1) $X^{(1)}$ is positively right-orderable.
- (2) $X^{(1)}$ is orderable iff it is left-orderable iff $\text{card } X \leq 2$.
- (3) $X^{(1)}$ is positively orderable iff it has a positive partial left order iff $\text{card } X \leq 1$.

Proof. Ad (1). Any total order of the set $X \cup \{1\}$ is a right order of $X^{(1)}$.

Ad (2), (3). If $X = \emptyset$ or X is a singleton, then $X^{(1)}$ is the one-element semilattice or the two-element semilattice, respectively; thus it is positively orderable. If $X = \{a, b\}$ with $a \neq b$, then the inequalities $a < 1 < b$ define a bi-order of $X^{(1)}$.

Let \leq be a left order of $X^{(1)}$, and suppose that X has at least three distinct elements. We may replace \leq by its dual order and thus assume that there are $a, b \in X$ such that $1 < a < b$. From $1 < a$ it follows that $b \leq ba$, that is, $b \leq a$, a contradiction.

Let \leq be a positive partial left order of $X^{(1)}$. We prove that any two elements a, b of X are equal. From $1 \leq a$ it follows that $b = b1 \leq ba = a$. Similarly, $a \leq b$; whence $a = b$. \square

A direct application of Proposition 2.9 thus yields the following examples.

Examples 2.10. Let a, b, c be distinct objects. Then

- (1) $\{a, b, c\}^{(1)}$ is a positively right-orderable, but not left-orderable, idempotent monoid.
- (2) $\{a, b\}^{(1)}$ is a bi-orderable, positively right-orderable, idempotent monoid without any positive partial left order.

Any positively partially right-orderable monoid is conical; in fact, it obviously satisfies the more general implication $xyz = z \Rightarrow yz = z$ (*Proof*: observe the inequalities $z \leq yz \leq xyz$). In particular, the orderable monoid (group) \mathbb{Z} is not positively orderable. Our next example is commutative and we will thus write it additively.

Example 2.11. *An infinite, conical, orderable, commutative monoid without any positive partial order.*

Proof. We define a monoid congruence \equiv on $\mathbb{N} \times \mathbb{N}$ by the rule

$$(x_1, y_1) \equiv (x_2, y_2) \text{ if } (x_1, y_1) = (x_2, y_2) \text{ or } (0 \notin \{x_1, y_1, x_2, y_2\} \text{ and } y_1 - x_1 = y_2 - x_2).$$

and we set $M \stackrel{\text{def}}{=} (\mathbb{N} \times \mathbb{N}) / \equiv$. We can identify M with the set of all bottom elements (with respect to the componentwise order of $\mathbb{N} \times \mathbb{N}$) of \equiv -equivalence classes, so $M = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid \{x, y\} \cap \{0, 1\} \neq \emptyset\}$. Whenever $z_1, z_2 \in M$, $z_1 + z_2$ is then the bottom element of the \equiv -equivalence class of the componentwise sum of z_1 and z_2 . The binary relation \leq , defined on M by the rule

$$(x_1, y_1) \leq (x_2, y_2) \text{ if either } y_1 - x_1 < y_2 - x_2 \text{ or } y_1 - x_1 = y_2 - x_2 \text{ and } x_1 \leq x_2,$$

is a translation-invariant order of M . (*Notice that since M is not cancellative, the associated strict order is not translation-invariant: for example, $(0, 1) < (1, 2)$ whereas $(0, 1) + (1, 0) = (1, 2) + (1, 0) = (1, 1)$.) Setting $a \stackrel{\text{def}}{=} (1, 0)$, $b \stackrel{\text{def}}{=} (0, 1)$, and $e \stackrel{\text{def}}{=} a + b = (1, 1)$, we can illustrate the order of M by writing it as*

$$\dots < 2a < 2a + e < a < a + e < 0 < e < b < b + e < 2b < 2b + e < \dots$$

Suppose that M has a positive partial order \trianglelefteq . From $0 \trianglelefteq a \trianglelefteq a + b = e$ it follows that $e \trianglelefteq a + e \trianglelefteq 2e = e$, thus $a + e = e$, that is, $(2, 1) = (1, 1)$, a contradiction. \square

Example 2.11, being infinite, naturally induces further itches. Our next section will be aimed at removing some of those.

3. MORE ON POSITIVE ORDERABILITY VERSUS ORDERABILITY

The existence of a finite analogue of Example 2.11 was stated as an open problem in the preliminary version of the present note. A decisive clue was then quickly provided by George Bergman, who suggested the author to try constructing an orderable commutative monoid satisfying a more general form of the relations (3.1), between the parameters a, b, p, q , appearing in the proof of the example below. The rest of the work was performed by the Mace4 counterexample builder [12], which returned the following example.

Example 3.1. *A nine-element orderable commutative monoid without any positive (total) order.*

+	$\bar{1}$	0	1	2	3	4	5	6	∞
$\bar{1}$	$\bar{1}$	$\bar{1}$	1	2	2	4	5	5	∞
0	$\bar{1}$	0	1	2	3	4	5	6	∞
1	1	1	4	5	5	5	∞	∞	∞
2	2	2	5	5	5	∞	∞	∞	∞
3	2	3	5	5	6	∞	∞	∞	∞
4	4	4	5	∞	∞	∞	∞	∞	∞
5	5	5	∞	∞	∞	∞	∞	∞	∞
6	5	6	∞	∞	∞	∞	∞	∞	∞
∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

TABLE 3.1. An orderable, but not positively orderable, commutative monoid

Proof. The monoid M , with multiplication table represented in Table 3.1, can be ordered *via* the inequalities $\bar{1} < 0 < 1 < 2 < 3 < 4 < 5 < 6 < \infty$.

The elements $a = 1$, $b = 3$, $p = 1$, and $q = \bar{1}$ satisfy the relations

$$2a + p = a + b \neq 2a \quad \text{and} \quad 2b + q = a + b \neq 2b. \tag{3.1}$$

Suppose that M has a positive order, with associated strict order \triangleleft . From $2a + p = a + b \neq 2a$ it follows that $0 \triangleleft p$ and $2a \triangleleft 2a + p = a + b$, whence (since a and b are comparable with respect to \triangleleft) we get $a \triangleleft b$. Similarly, from $2b + q = a + b \neq 2b$ we get $b \triangleleft a$; a contradiction. \square

The idempotents in the example M above, namely $0, \bar{1}, \infty$, form a chain, which should be no surprise by Proposition 2.2(2) together with the commutativity of M .

Remark 3.2. A fundamental difference between Examples 2.7, 2.10(2), and 2.11 on the one hand, and Example 3.1 on the other hand, is that the former examples all state the non-existence of a positive *partial* order. Example 3.1 cannot be improved in that direction (stating the non-existence of any positive *partial* order): indeed, by Proposition 2.5(3), every finite orderable commutative monoid M satisfies the implication $x + y + z = z \Rightarrow y + z = z$, so the binary relation \leq^+ on M , defined by letting $x \leq^+ y$ hold if there exists z such that $x + z = y$, is a positive partial order of M .

It turns out that the situation is different in the cancellative case. Say that a monoid M *has unique roots* if it satisfies the implication $x^n = y^n \Rightarrow x = y$, for every positive integer n . The following result, slightly extending Levi’s classical result [11] that every torsion-free Abelian group is orderable, shows that there is no cancellative analogue of Example 2.11.

Proposition 3.3. *The following are equivalent, for any cancellative commutative monoid M :*

- (i) M is positively orderable;
- (ii) M is conical and orderable;
- (iii) M is conical and has unique roots.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are straightforward. It remains to establish the implication (iii) \Rightarrow (i). Let thus M be a cancellative, conical, commutative monoid with unique roots. In order to prove the positive orderability of M , it suffices,

by a standard compactness argument, to consider the case where M is finitely generated. In such a case, since the universal group G of M is torsion-free (because M has unique roots), $G \cong \mathbb{Z}^n$ for some positive integer n . Let K be the convex hull, within \mathbb{Q}^n , of some finite generating subset of $M \setminus \{0\}$. From the conicality of M it follows that $0 \notin K$. By the Farkas-Minkowski-Weyl Theorem (see, e.g., [15, Corollary 7.1a]), there is a linear functional $f: \mathbb{Q}^n \rightarrow \mathbb{Q}$ such that $f(x) > 0$ for all $x \in K$; whence $f(x) > 0$ for all $x \in M \setminus \{0\}$. We may assume that f is the projection on the first coordinate in \mathbb{Q}^n . The order of M , defined as the restriction of the lexicographical order of \mathbb{Q}^n , is as required. \square

The following Examples 3.4 and 3.5 were both obtained with the help of Mace4.

Example 3.4. *A four-element, left-orderable, positively right-orderable, non bi-orderable idempotent monoid.*

Proof. Consider the monoid M with multiplication table represented in Table 3.2. The inequalities $1 <_r a <_r b <_r c$ define a positive right order of M , while the

\cdot	1	a	b	c
1	1	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	b	b	c

TABLE 3.2. A left-orderable, positively right-orderable, non bi-orderable idempotent monoid

inequalities $c <_l 1 <_l a <_l b$ define a left order of M . Every element of M is idempotent, and $ca = b \notin \{a, c\}$. By Proposition 2.2(2), M is not bi-orderable. \square

Example 3.5. *A five-element, positively left- and right-orderable, non bi-orderable monoid, in which the idempotents form a chain.*

Proof. The monoid M , with multiplication table represented in Table 3.3, is positively right-orderable *via* the inequalities $1 <_r a <_r b <_r c <_r \infty$, and positively left-orderable *via* the inequalities $1 <_l a <_l c <_l b <_l \infty$. The idempotents of M , namely $1, a, \infty$, form a chain. Suppose that M has a bi-order \leq , say with

\cdot	1	a	b	c	∞
1	1	a	b	c	∞
a	a	a	∞	c	∞
b	b	b	∞	∞	∞
c	c	∞	∞	∞	∞
∞	∞	∞	∞	∞	∞

TABLE 3.3. A positively left- and right-orderable, non bi-orderable monoid

$b < c$. Then $ba \leq bc$, so $b < \infty$, and $ab \leq ac$, so $\infty < c$. Now if $1 < \infty$, then $c = c1 \leq c\infty = \infty$, a contradiction. If $\infty < 1$, then $\infty = b\infty \leq b1 = b$, a contradiction again. \square

By Proposition 2.2(1), there is no “best of two worlds” example combining the idempotency of the monoid in Example 3.4 with the positive left- and right-orderability of the monoid in Example 3.5.

4. A COUNTEREXAMPLE THAT EMBEDS INTO A GROUP

The monoid underlying Example 2.10(1) (right-orderable but not left-orderable) is finite, thus (since every nontrivial cancellative right-orderable monoid is infinite) it cannot yield a cancellative example. Finding a cancellative, right-orderable, but not left-orderable monoid turns out to require more work. Our construction, detailed in the proof of Theorem 4.2, will in fact yield a submonoid of a group.

Let us briefly recall some background, on *categories*, required for our construction. All our categories will be small categories, viewed as certain partial semigroups with identity elements (in short *identities*), for example in the sense of [6, § II.1.1] or [19, Definition 3.1]. Following tradition from existing works such as [6], we will understand categories in the source / target sense, as opposed to domain / range: objects are identified with their identity morphisms, and morphisms are composed like functions written to the right of their arguments. Hence every element x in a category S has a source $\partial_0 x$ and a target $\partial_1 x$, which are the only identities in S satisfying $x = \partial_0 x \cdot x = x \cdot \partial_1 x$. A product $x \cdot y$ (in short xy) is defined iff $\partial_1 x = \partial_0 y$. A product $(xy)z$ is defined iff $x(yz)$ is defined, iff both xy and yz are defined, and then $(xy)z = x(yz)$. A monoid is a category with a unique identity, and a functor between categories is a homomorphism with respect to the partial product, source, and target operations. For identities a and b in S , the hom-set $S(a, b)$ is $\{x \in S \mid \partial_0 x = a \text{ and } \partial_1 x = b\}$. We will denote by $\varepsilon_S: S \rightarrow U_{\text{mon}}(S)$ the natural functor from S to its universal monoid, and we will set $\overline{S} \stackrel{\text{def}}{=} \varepsilon_S[S]$. The map ε_S identifies two elements x and y of S iff either $x = y$ or x and y are both identities [19, Lemma 3.10]. Denoting by $\text{Idt } S$ the set of all identities of S , it follows that \overline{S} can be identified with $(S \setminus \text{Idt } S) \cup \{1\}$.

By [19, Lemma 3.4], any element of $U_{\text{mon}}(S)$ can be uniquely written as a product $x_1 \cdots x_n$ (which will then be called a *reduced word*), where n is a nonnegative integer, the x_i are non-identities in S , and each product $x_i x_{i+1}$ is undefined in S (i.e., $\partial_1 x_i \neq \partial_0 x_{i+1}$). We set $n = \text{lh}(x)$, the *length* of x .

As in [19], a category S is

- *conical* if any composition of two non-identities of S is a non-identity;
- *right cancellative* if for all $x, y, z \in S$, if xz and yz are both defined and $xz = yz$, then $x = y$. By [19, Corollary 4.3], this is equivalent to saying that the universal monoid $U_{\text{mon}}(S)$ of S is right cancellative.

Our next result states an extension result of a right order from a category to its universal monoid.

Lemma 4.1. *Let S be a conical, right cancellative category, and let \leq be a partial order on \overline{S} , with least element 1, such that for all $x, y, z \in \overline{S}$, $x \leq y$ and $yz \in \overline{S}$ together imply that $xz \in \overline{S}$ and $xz \leq yz$. Then \leq is the restriction to \overline{S} of a positive partial right order \preceq on the monoid $U_{\text{mon}}(S)$, with respect to which \overline{S} is a lower subset of $U_{\text{mon}}(S)$, such that if \leq is a total order on \overline{S} , then \preceq is a total order on $U_{\text{mon}}(S)$.*

Note (Following a comment by George Bergman). The existence of an order \leq on \overline{S} as above puts quite a constraint on the category S . Let an identity a of S be a *midway object* if there are non-identities x and y such that $a = \partial_0 x = \partial_1 y$. Then S has at most one midway object. Indeed, let a and b be midway objects. There are non-identities x and y with targets a and b , respectively. We may assume that

$x \leq y$. Then for any non-identity z with source b , $yz \in \overline{S}$, thus, by assumption, $xz \in \overline{S}$, whence $a = \partial_1 x = \partial_0 z = \partial_1 y = b$.

For example, the category S constructed in the proof of Theorem 4.2 has exactly one midway object, namely v (cf. Figure 4.1).

Proof. For any elements x and y of $U_{\text{mon}}(S)$, respectively written in reduced form as $x_m \cdots x_1$ and $y_n \cdots y_1$ with all $x_i, y_j \in S \setminus \text{Idt } S$, we define $\nu(x, y)$ as the least $k \in \{1, \dots, \min\{m, n\}\}$ if it exists such that $x_k \neq y_k$. Let $x \triangleleft y$ hold if either $m < n$, or $m = n$ and for $k \stackrel{\text{def}}{=} \nu(x, y)$ it holds that $x_k < y_k$. Further, let $x \trianglelefteq y$ hold if either $x \triangleleft y$ or $x = y$. Hence \trianglelefteq is just the shortlex order, with words being read from right to left, on finite sequences from $S \setminus \text{Idt } S$. It is a partial order on $U_{\text{mon}}(S)$ whose restriction to \overline{S} is \leq . Since $x \trianglelefteq y$ implies $\text{lh}(x) \leq \text{lh}(y)$, $\overline{S} = \{x \in U_{\text{mon}}(S) \mid \text{lh}(x) \leq 1\}$ is a lower subset of $U_{\text{mon}}(S)$. If \leq is total, then so is \trianglelefteq .

In order to prove that \trianglelefteq is right translation-invariant, it suffices to prove that $x \triangleleft y$ implies that $xz \triangleleft yz$, for all $x, y \in U_{\text{mon}}(S)$ and all $z \in S \setminus \text{Idt } S$. Write again x and y as reduced words as above. In particular, $m \leq n$. Moreover, from $x \triangleleft y$ it follows that $n > 0$. If $m = n$, then $k \stackrel{\text{def}}{=} \nu(x, y)$ exists and $x_k < y_k$.

Suppose first that $y_1 z \notin \overline{S}$. Then $y_n \cdots y_2 \cdot y_1 \cdot z$ is the reduced word representation of yz , thus $\text{lh}(xz) \leq m + 1 \leq n + 1 = \text{lh}(yz)$. It follows that if either $m < n$ or $x_1 z \in \overline{S}$, then $\text{lh}(xz) < \text{lh}(yz)$, thus $xz \triangleleft yz$. If, on the other hand, $m = n$ and $x_1 z \notin \overline{S}$, then $x_n \cdots x_2 \cdot x_1 \cdot z$ is the reduced word representation of xz , so $\nu(xz, yz) = k + 1$ and $(xz)_{k+1} = x_k < y_k = (yz)_{k+1}$, whence $xz \triangleleft yz$.

We may thus assume from now on that $y_1 z \in \overline{S}$. It follows from the conicality of S that $y_n \cdots y_2 \cdot y_1 z$ is the reduced word representing yz , so $\text{lh}(yz) = n$. From $1 \leq y_1$ (within \overline{S}) it follows that $z = 1z \leq y_1 z$. If $z = y_1 z$, that is, $\partial_0 z \cdot z = y_1 z$ (within S), then, since S is right cancellative, $y_1 = \partial_0 z$ is an identity, a contradiction. Hence,

$$z < y_1 z. \quad (4.1)$$

In particular, if $x = 1$ (i.e., $m = 0$), then $xz = z \triangleleft yz$. We may thus assume from now on that $m > 0$. In particular, if $m + 2 \leq n$, then $\text{lh}(xz) \leq m + 1 < n = \text{lh}(yz)$, so $xz \triangleleft yz$.

It thus remains to check all cases where $y_1 z \in \overline{S}$ and $0 < m \leq n \leq m + 1$.

Suppose first that $y_1 z \in \overline{S}$ and $m = n$. From $x \trianglelefteq y$ it follows that $x_1 \leq y_1$, thus, since $y_1 z \in \overline{S}$ and by our assumption on the order \leq , we get $x_1 z \in \overline{S}$; so $x_n \cdots x_2 \cdot x_1 z$ is the reduced word representing xz , and so $\text{lh}(xz) = n$. If $k \geq 2$, then $x_1 = y_1$, so $\nu(xz, yz) = k$ and $(xz)_k = x_k < y_k = (yz)_k$, and so $xz \triangleleft yz$. If $k = 1$, then $x_1 < y_1$, thus, since \leq is right translation-invariant and S is both conical and right cancellative, $x_1 z < y_1 z$, so $\nu(xz, yz) = 1$ and $xz \triangleleft yz$.

It remains to deal with the case where $y_1 z \in \overline{S}$ and $m + 1 = n$. If $x_1 z \in \overline{S}$, then $x_m \cdots x_2 \cdot x_1 z$ is the reduced word representing xz , so $\text{lh}(xz) = m < n = \text{lh}(yz)$ and so $xz \triangleleft yz$. If $x_1 z \notin \overline{S}$, then $x_m \cdots x_1 \cdot z$ is the reduced word representing xz and $\text{lh}(xz) = m + 1 = n = \text{lh}(yz)$. Using (4.1), we get $(xz)_1 = z < y_1 z = (yz)_1$, so $xz \triangleleft yz$ again. We have thus completed the proof that \trianglelefteq is a right order of $U_{\text{mon}}(S)$. \square

Theorem 4.2. *There exists a positively right-orderable submonoid M , of the group $(\mathbb{Z}/7\mathbb{Z}) * F_{\text{gp}}(4)$, containing a triple of elements with no smallest element with respect to any partial left order of M . In particular, M is not left-orderable.*

Proof. Our monoid M is defined by the generators p_i, q_i, r_i, a_i , for $i \in \{0, 1, 2\}$, subjected to the relations

$$p_i a_i = r_i a_{i+2}; p_i a_{i+1} = q_i a_i; q_i a_{i+1} = r_i a_i, \quad \text{for } i \in \{0, 1, 2\} \quad (4.2)$$

(where indices are taken modulo 3). It is convenient to represent M as the universal monoid of the finite category S consisting of the generating set

$$\Sigma \stackrel{\text{def}}{=} \{p_0, q_0, r_0, p_1, q_1, r_1, p_2, q_2, r_2, a_0, a_1, a_2\}$$

together with the products $p_i a_j, q_i a_j, r_i a_j$ for $i, j \in \{0, 1, 2\}$, and the five identities $u_i \stackrel{\text{def}}{=} \partial_0 p_i = \partial_0 q_i = \partial_0 r_i, v \stackrel{\text{def}}{=} \partial_1 p_i = \partial_1 q_i = \partial_1 r_i = \partial_0 a_j$, and $w = \partial_1 a_j$, for $i, j \in \{0, 1, 2\}$. In particular, $\overline{S} = \{1\} \cup \Sigma \cup \{p_i a_j, q_i a_j, r_i a_j \mid i, j \in \{0, 1, 2\}\}$. By virtue of the equations (4.2), \overline{S} has 31 elements and S has 35 elements. The category S is obviously both conical and right cancellative. It can be partly illustrated in Figure 4.1 (which displays neither the equations (4.2) nor the elements of $S(u_i, w)$). We define G as the universal group of the category S .

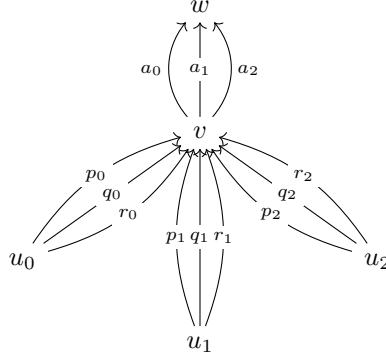


FIGURE 4.1. Illustrating the category S

Claim 1. *The group G is isomorphic to $(\mathbb{Z}/7\mathbb{Z}) * F_{\text{gp}}(4)$.*

Proof of Claim. If working in groups, we can eliminate q_i and r_i from the equations (4.2) and thus obtain the equivalent group presentation of G given by

$$q_i = p_i a_{i+1} a_i^{-1}; r_i = p_i a_i a_{i+2}^{-1}; \quad (4.3)$$

$$a_{i+1} a_i^{-1} a_{i+1} = a_i a_{i+2}^{-1} a_i, \quad (4.4)$$

where i ranges over $\{0, 1, 2\}$. Now (4.4), for $i = 0$, yields $a_2 = a_0 a_1^{-1} a_0 a_1^{-1} a_0$, which, combined with (4.4) for $i = 1$, yields

$$a_0 a_1^{-1} a_0 a_1^{-1} a_0 a_1^{-1} a_0 a_1^{-1} a_0 a_1^{-1} a_0 = a_1 a_0^{-1} a_1,$$

which is equivalent to $(a_0^{-1} a_1)^7 = 1$. Setting $c \stackrel{\text{def}}{=} a_0^{-1} a_1$, it follows that $c^7 = 1$ and $a_1 = a_0 c$. Furthermore, $a_2 = a_0 a_1^{-1} a_0 a_1^{-1} a_0 = a_0 c^5$. Conversely, any triple (a_0, a_1, a_2, c) with $c^7 = 1$, $a_1 = a_0 c$, and $a_2 = a_0 c^5$ satisfies the equations (4.4). Taking into account the equations (4.3), it follows that G is the free product of the cyclic group generated by c (which is isomorphic to $\mathbb{Z}/7\mathbb{Z}$) with the free group on $\{p_0, p_1, p_2, a_0\}$. \square Claim 1.

Owing to (the proof of) Claim 1, we shall from now on identify G with the free product of $\mathbb{Z}/7\mathbb{Z}$ with the free group on generators $\{p_0, p_1, p_2, a_0\}$, with c the canonical generator of $\mathbb{Z}/7\mathbb{Z}$, $a_1 = a_0c$, $a_2 = a_0c^5$, and the terms q_i and r_i given by (4.3). Easy calculations then yield

$$q_0 = p_0a_1a_0^{-1} = p_0a_0ca_0^{-1}; \quad r_0 = p_0a_0a_2^{-1} = p_0a_0c^2a_0^{-1}; \quad (4.5)$$

$$q_1 = p_1a_2a_1^{-1} = p_1a_0c^4a_0^{-1}; \quad r_1 = p_1a_1a_0^{-1} = p_1a_0ca_0^{-1}; \quad (4.6)$$

$$q_2 = p_2a_0a_2^{-1} = p_2a_0c^2a_0^{-1}; \quad r_2 = p_2a_2a_1^{-1} = p_2a_0c^4a_0^{-1}. \quad (4.7)$$

Claim 2. *The natural monoid homomorphism $\varphi_S: M \rightarrow G$ is one-to-one.*

Proof of Claim. Composing the natural functor $\varepsilon_S: S \rightarrow M$ with φ_S yields the natural functor $\eta_S: S \rightarrow G$. By [19, Theorem 10.1], it suffices to prove that the restriction of η_S to each hom-set of S is one-to-one. For $i \in \{0, 1, 2\}$, the elements of $S(u_i, v)$ are p_i, q_i, r_i , which are mapped under η_S to distinct elements of G since the expressions given on the right hand sides of the equations in (4.5)–(4.7) are normal forms within $(\mathbb{Z}/7\mathbb{Z}) * \mathbb{F}_{\text{gp}}(4)$. Furthermore, the elements of $S(v, w)$ are a_0, a_1, a_2 , which are mapped under η_S to the distinct elements a_0, a_0c, a_0c^5 , respectively.

Finally, each $S(u_i, w)$ has exactly six elements, that can be listed as

$$p_ia_{i+2}; q_ia_{i+2}; p_ia_i = r_ia_{i+2}; p_ia_{i+1} = q_ia_i; q_ia_{i+1} = r_ia_i; r_ia_{i+1}, \quad (4.8)$$

which, owing to the relations (4.3) and (4.4), are respectively mapped by η_S to

$$p_ia_{i+2}; p_ia_{i+1}a_i^{-1}a_{i+2}; p_ia_i; p_ia_{i+1}; p_ia_{i+1}a_i^{-1}a_{i+1}; p_ia_ia_{i+2}^{-1}a_{i+1}.$$

Substituting in those equations the expressions $a_1 = a_0c$ and $a_2 = a_0c^5$, we obtain that those elements are respectively equal to

$$\begin{aligned} & p_0a_0c^5; p_0a_0c^6; p_0a_0; p_0a_0c; p_0a_0c^2; p_0a_0c^3 \quad (\text{for } i = 0), \\ & p_1a_0; p_1a_0c^4; p_1a_0c; p_1a_0c^5; p_1a_0c^2; p_1a_0c^6 \quad (\text{for } i = 1), \\ & p_2a_0c; p_2a_0c^3; p_2a_0c^5; p_2a_0; p_2a_0c^2; p_2a_0c^4 \quad (\text{for } i = 2). \end{aligned}$$

For each value of $i \in \{0, 1, 2\}$, those elements are pairwise distinct, so the restriction of η_S to $S(u_i, w)$ is one-to-one. \square Claim 2.

Claim 3. *The monoid M has a positive right order, with respect to which \overline{S} is a lower subset.*

Proof of Claim. We apply Lemma 4.1. We will construct the required order of \overline{S} by initializing the order on $\Sigma \cup \{1\}$, by setting

$$1 < p_0 < q_0 < r_0 < p_1 < q_1 < r_1 < p_2 < q_2 < r_2 < a_0 < a_1 < a_2, \quad (4.9)$$

then extending \leq to the whole \overline{S} by using the right invariance of \leq . On elements of the form p_ia_j, q_ia_j, r_ia_j for fixed i (listed in (4.8)), there is no choice, we need to define

$$p_ia_{i+2} < q_ia_{i+2} < p_ia_i = r_ia_{i+2} < p_ia_{i+1} = q_ia_i < q_ia_{i+1} = r_ia_i < r_ia_{i+1} \quad (4.10)$$

(that part of the calculations does not involve the inequalities $a_0 < a_1 < a_2$). We complete the construction by linking the chains (4.10), in the order 012, atop the chain (4.9), thus yielding the inequalities

$$\underbrace{1 < \cdots < a_2}_{\text{chain (4.9)}} < \underbrace{p_0a_2 < \cdots < r_0a_1}_{\text{chain (4.10) for } i = 0} < \underbrace{p_1a_0 < \cdots < r_1a_2}_{\text{chain (4.10) for } i = 1} < \underbrace{p_2a_1 < \cdots < r_2a_0}_{\text{chain (4.10) for } i = 2}. \quad (4.11)$$

We need to verify that for all $x, y, z \in \overline{S}$, $x \leq y$ and $yz \in \overline{S}$ implies that $xz \in \overline{S}$ and $xz \leq yz$. Since this is vacuous if either $z = 1$ or $z \in \{p_i, q_i, r_i\}$ for some i (only trivial products intervene there), the only interesting case is the one where $z = a_j$ for some j and y belongs to the set $\Sigma' \stackrel{\text{def}}{=} \{1, p_0, q_0, r_0, p_1, q_1, r_1, p_2, q_2, r_2\}$. By (4.9), Σ' is a lower subset of \overline{S} , thus x also belongs to Σ' , and thus $xz = xa_j$ belongs to \overline{S} . By the construction of the chain (4.11), we get $xz \leq yz$. \square Claim 3.

Claim 4. *There is no partial left order of M for which $\{a_0, a_1, a_2\}$ has a smallest element.*

Proof of Claim. Since the defining equations (4.2) of M are invariant under translation (modulo 3) of the index i , it suffices to prove that there is no partial left order \trianglelefteq of M for which $a_0 \triangleleft a_1$ and $a_0 \triangleleft a_2$. Using left invariance, we get

$$p_0a_0 \trianglelefteq p_0a_1 = q_0a_0 \trianglelefteq q_0a_1 = r_0a_0 \trianglelefteq r_0a_2 = p_0a_0,$$

thus $p_0a_0 = p_0a_1$, a contradiction. \square Claim 4.

The conclusion of Theorem 4.2 follows from the Claims above. \square

Remark 4.3. Recall from Bergman [2, Example 23] that there are monoids, embeddable into free groups, of which the universal group is not free—it may even have torsion; by the results of [19], this is more the rule than the exception. Now the proof of Theorem 4.2 above shows that each $(a_i a_j^{-1})^7 = 1$ within the universal group of M —equivalently, within any group extension of M . Since a_0, a_1, a_2 are pairwise distinct, it follows that *every group extension of M has torsion elements.*

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