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# RIGHT-ORDERABILITY VERSUS LEFT-ORDERABILITY FOR MONOIDS

FRIEDRICH WEHRUNG

ABSTRACT. We investigate the relationship between (total) left- and right-orderability for monoids. In particular, we construct a (positively) right-orderable submonoid of the free product of the cyclic group of order 7 by the free group on four generators, which is not left-orderable. Any group extension of that monoid has elements of order 7.

## 1. INTRODUCTION

The purpose of the present note is to remove an itch.

A monoid  $M$  is *right-orderable* if it has a right order, that is, a *total* order  $\leq$  such that  $x \leq y$  implies  $xz \leq yz$  whenever  $x, y, z \in M$ . *Left-orderability* is right-orderability applied to the opposite monoid of  $M$  (with multiplication defined by  $x \cdot' y \stackrel{\text{def}}{=} yx$ ). It is well known that right-orderability and left-orderability are equivalent for groups (*Proof*: let  $x \leq' y$  hold if  $y^{-1} \leq x^{-1}$ ).

Are those equivalent for arbitrary monoids? A moment's thought, backed up by a few computations, show that even for *finite* monoids this is not the case. In Example 2.9 we show a five-element counterexample.

However, this is not the main point of the above-mentioned itch. Owing to a large existing corpus of works involving *submonoids of groups*, see in particular [4, 6], we are asking the question for those particular monoids. Although this kind of question is bound to be attached a firmly set, if not outright claimed, popular belief, inducing an obvious feeling of type “this should have been done long ago”, the author of the present note was unable to find it in print anywhere. Our counterexample  $M$ , constructed in Theorem 3.2, is in fact *positively* ordered (i.e., the unit of  $M$  is its bottom element), and any group containing  $M$  has elements of order 7. Although we first define  $M$  *via* generators and relations, we soon find it convenient to describe it as the universal monoid of a finite category (with 30 non-identity arrows). The universal group of  $M$  is the free product of  $\mathbb{Z}/7\mathbb{Z}$  by the free group on four generators. The embeddability of  $M$  into a group (thus into that group) does not follow *a priori* from usual sufficient conditions such as Adjan's condition (cf. [1] or [11, Theorem 4.6]) or Dehornoy's 3-Ore condition [5]. Instead, we are applying our criterion, stated in [15, Theorem 10.1], of embeddability of the universal monoid of a category into its universal group. Further, the right-orderability of  $M$  follows from a general result (Lemma 3.1) making it possible, under certain conditions,

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to extend a right order, from the canonical image of a category  $S$  in its universal monoid  $U_{\text{mon}}(S)$ , to a right order on the full  $U_{\text{mon}}(S)$ .

Let us briefly mention some related works, whose results we will not use here but that might help putting things into context. Johnson constructs in [8] cancellative bi-orderable (i.e., both right- and left-orderable) monoids that cannot be embedded into groups. Darnel, Glass, and Rhemtulla prove in [3] that if every right order on an orderable group  $G$  is also a left order, then  $G$  is Abelian. For further insights on ordered semigroups, see Tringali [13]. For the commutative case (which will be briefly of concern in Section 2), see Evans *et al.* [7], Vetterlein [14], Whipple [16].

We will denote by  $\mathbb{N}$ ,  $\mathbb{Z}$ , and  $\mathbb{Q}$  the additive monoids of all nonnegative integers, integers, and rational numbers, respectively. We will also denote by  $F_{\text{gp}}(n)$  the free group on  $n$  generators, and by  $G * H$  the free product of any two groups  $G$  and  $H$ .

A subset  $X$  in a partially ordered set  $P$  is a *lower subset* of  $P$  if for all  $x \in X$  and all  $p \in P$ ,  $p \leq x$  implies that  $p \in X$ .

## 2. RIGHT-ORDERABLE MONOIDS

In this section we shall get started by recalling the definition of a right order (resp., a positive right order) on a monoid (Definition 2.1) and collect a few observations thereof, consisting of both positive results (Propositions 2.2 and 2.8) and counterexamples (Examples 2.4, 2.5, 2.6, and 2.9). As the existence of a translation-invariant partial order, on any given monoid, is trivial (just take the diagonal), we shall follow tradition and omit the qualifier “total” (or “linear”), saying, for example, “order”, “positive order”, “partial order” instead of “total order”, “total positive order”, and “order”, respectively.

**Definition 2.1.** A partial order  $\leq$  on a monoid  $M$  is

- *positive* if  $1 \leq x$  for every  $x \in M$ ;
- a *partial right order* if  $x \leq y$  implies that  $xz \leq yz$ , for all  $x, y, z \in M$  (we say that  $\leq$  is *right translation-invariant*);
- a *right order* if it is both a *total* order and a partial right order.

We say that the monoid  $M$  is *right-orderable* (resp., *positively right-orderable*) if it has a right order (resp., a positive right order).

Those concepts, applied to the opposite monoid of  $M$ , yield (partial, positive) *left orders* and (positively) *left-orderable monoids*, respectively. Moreover, by *orderable* we will mean the conjunction of right- and left-orderable. In particular, we will omit the prefix “right-” in the commutative case.

*Note.* In many works, right-orderability is set to define *strict* orders on a monoid (i.e.,  $x < y$  implies  $xz < yz$ ). Our current definition is thus weaker: as we will see shortly, it indeed produces finite examples. However, for right cancellative monoids the two definitions are of course equivalent.

**Proposition 2.2.** *Let  $a$  and  $b$  be idempotents in a monoid  $M$ . If  $M$  is orderable, then  $ab \in \{a, b\}$ .*

*Proof.* Let  $\leq$  be a translation-invariant order on  $M$ . We may replace  $\leq$  by its dual order and thus assume that  $1 \leq ab$ . If  $a \leq b$ , then  $b = 1b \leq ab^2 = ab \leq b^2 = b$ , thus  $b = ab$ . If  $b \leq a$ , then  $a = a1 \leq a^2b = ab \leq a^2 = a$ , thus  $a = ab$ .  $\square$

**Corollary 2.3.** *The semilattice  $\{0, 1\}^2$  is not orderable.*

This shows an important difference between the general case and the cancellative case: in the latter case, orderability is preserved under products (use a lexicographical order), which fails in the non-cancellative case by Corollary 2.3.

Any positively right-orderable monoid is *conical*, meaning that it satisfies the implication  $xy = 1 \Rightarrow y = 1$ ; in fact, it obviously satisfies the more general implication  $xyz = z \Rightarrow yz = z$ . In particular, the orderable monoid (group)  $\mathbb{Z}$  is not positively orderable. The following example shows that even for conical monoids, positive right-orderability is stronger than right-orderability. (It is obtained by “trying to turn  $\{-1, 0, 1\}$  into an additive monoid”.)

**Example 2.4.** *There exists a three-element, conical, right-orderable, and positively left-orderable, idempotent monoid without any positive partial right order.*

*Proof.* Our monoid  $M$  has underlying set  $\{1, a, b\}$  and multiplication given by

$$xy \stackrel{\text{def}}{=} \begin{cases} y, & \text{if } x = 1, \\ x, & \text{if } x \neq 1, \end{cases} \quad \text{for all } x, y \in M.$$

The inequalities  $a < 1 < b$  (resp.,  $1 < a < b$ ) define a right (resp., positive left) order on  $M$ . Any positive partial right order  $\trianglelefteq$  on  $M$  would satisfy  $a \trianglelefteq ba = b$  and  $b \trianglelefteq ab = a$ , thus  $a = b$ , a contradiction.  $\square$

Our next example is a commutative analogue of Example 2.4. Since it is commutative, we shall write it additively.

**Example 2.5.** *There exists an infinite, conical, orderable, commutative monoid without any positive partial order.*

*Proof.* We define a monoid congruence  $\equiv$  on  $\mathbb{N} \times \mathbb{N}$  by the rule

$$(x_1, y_1) \equiv (x_2, y_2) \text{ if } (x_1, y_1) = (x_2, y_2) \text{ or } (0 \notin \{x_1, y_1, x_2, y_2\} \text{ and } y_1 - x_1 = y_2 - x_2).$$

and we set  $M \stackrel{\text{def}}{=} (\mathbb{N} \times \mathbb{N}) / \equiv$ . We can identify  $M$  with the set of all bottom elements (with respect to the componentwise order of  $\mathbb{N} \times \mathbb{N}$ ) of  $\equiv$ -equivalence classes, so  $M = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid \{x, y\} \cap \{0, 1\} \neq \emptyset\}$ . Whenever  $z_1, z_2 \in M$ ,  $z_1 + z_2$  is then the bottom element of the  $\equiv$ -equivalence class of the componentwise sum of  $z_1$  and  $z_2$ . The binary relation  $\leq$  then defined on  $M$  by the rule

$$(x_1, y_1) \leq (x_2, y_2) \text{ if either } y_1 - x_1 < y_2 - x_2 \text{ or } y_1 - x_1 = y_2 - x_2 \text{ and } x_1 \leq x_2$$

is a translation-invariant order on  $M$ . (Notice that since  $M$  is not cancellative, the associated strict order is not translation-invariant: for example,  $(0, 1) < (1, 2)$  whereas  $(0, 1) + (1, 0) = (1, 2) + (1, 0) = (1, 1)$ .) Setting  $a \stackrel{\text{def}}{=} (1, 0)$ ,  $b \stackrel{\text{def}}{=} (0, 1)$ , and  $e \stackrel{\text{def}}{=} a + b = (1, 1)$ , we can illustrate the order of  $M$  by writing it as

$$\dots < 2a < 2a + e < a < a + e < 0 < e < b < b + e < 2b < 2b + e < \dots$$

Suppose that  $M$  has a positive partial order  $\trianglelefteq$ . From  $0 \trianglelefteq a \trianglelefteq a + b = e$  it follows that  $e \trianglelefteq a + e \trianglelefteq 2e = e$ , thus  $a + e = e$ , that is,  $(2, 1) = (1, 1)$ , a contradiction.  $\square$

The existence of a finite analogue of Example 2.4 was stated as an open problem in the preliminary version of the present note. A decisive clue was then quickly provided by George Bergman, who suggested the author to try constructing an orderable commutative monoid satisfying a more general form of the relations (2.1), between the parameters  $a, b, p, q$ , appearing in the proof of the example below.

The rest of the work was performed by McCune’s wonderful `Mace4` counterexample builder [10], which, in a few minutes, returned the following example.

**Example 2.6.** *There exists a nine-element orderable commutative monoid without any positive (total) order.*

*Proof.* We are looking for a finite orderable commutative monoid  $M$ , with elements  $a, b, p, q$  such that

$$2a + p = a + b \neq 2a \quad \text{and} \quad 2b + q = a + b \neq 2b. \quad (2.1)$$

Suppose that such a monoid has a translation-invariant positive order, with associated strict order  $\triangleleft$ . From  $2a + p = a + b \neq 2a$  it follows that  $0 \triangleleft p$  and  $2a \triangleleft 2a + p = a + b$ , whence (since  $a$  and  $b$  are comparable with respect to  $\triangleleft$ ) we get  $a \triangleleft b$ . Similarly, from  $2b + q = a + b \neq 2b$  we get  $b \triangleleft a$ ; a contradiction.

A `Mace4` search, for ordered commutative monoids with elements satisfying (2.1), yields the commutative monoid with addition table represented in Table 2.1, ordered via  $\bar{1} < 0 < 1 < 2 < 3 < 4 < 5 < 6 < 7$ . (It turns out that this order and its dual are the only orders of  $M$ .)

+	$\bar{1}$	0	1	2	3	4	5	6	7
$\bar{1}$	$\bar{1}$	$\bar{1}$	1	2	2	4	5	5	7
0	$\bar{1}$	0	1	2	3	4	5	6	7
1	1	1	4	5	5	5	7	7	7
2	2	2	5	5	5	7	7	7	7
3	2	3	5	5	6	7	7	7	7
4	4	4	5	7	7	7	7	7	7
5	5	5	7	7	7	7	7	7	7
6	5	6	7	7	7	7	7	7	7
7	7	7	7	7	7	7	7	7	7

TABLE 2.1. A 9-element orderable commutative monoid

Set  $a = 1$ ,  $b = 3$ ,  $p = 1$ , and  $q = \bar{1}$ . □

*Remark 2.7.* A fundamental difference between Examples 2.4 and 2.5 on the one hand, and Example 2.6 on the other hand, is that the former examples both state the non-existence of a positive *partial* order. A moment’s thought shows that Example 2.6 cannot be improved in that direction (stating the non-existence of any positive *partial* order). Indeed, it is not hard to verify that every finite orderable commutative monoid  $M$  satisfies the implication  $x + y + z = z \Rightarrow y + z = z$  (*Hint:* verify that for every  $x \in M$  there exists  $n \in \mathbb{N}$  such that  $nx = (n + 1)x$ . Denote that value by  $\infty x$ , verify that it is idempotent, and observe that  $x + y = y$  iff  $\infty x + y = y$ ). Therefore, the binary relation  $\leq^+$  on  $M$ , defined by letting  $x \leq^+ y$  hold if there exists  $z$  such that  $x + z = y$ , is a positive partial order of  $M$ .

It turns out that the situation is different in the cancellative case. Say that a monoid  $M$  has *unique roots* if it satisfies the implication  $x^n = y^n \Rightarrow x = y$ , for every positive integer  $n$ . The following result, extending Levi’s classical result [9] that every torsion-free Abelian group is orderable, shows that there is no cancellative analogue of Example 2.5.

**Proposition 2.8.** *The following are equivalent, for any cancellative commutative monoid  $M$ :*

- (i)  $M$  is positively orderable;
- (ii)  $M$  is both conical and orderable;
- (iii)  $M$  is conical and has unique roots.

*Proof.* The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are straightforward. It remains to establish the implication (iii) $\Rightarrow$ (i). Let thus  $M$  be a cancellative, conical, commutative monoid with unique roots. In order to prove the positive orderability of  $M$ , it suffices, by a standard compactness argument, to consider the case where  $M$  is finitely generated. In such a case, since the universal group  $G$  of  $M$  is torsion-free (for  $M$  has unique roots),  $G \cong \mathbb{Z}^n$  for some positive integer  $n$ . Let  $K$  be the convex hull, within  $\mathbb{Q}^n$ , of some finite generating subset of  $M \setminus \{0\}$ . From the conicality of  $M$  it follows that  $0 \notin K$ . By the Farkas-Minkowski-Weyl Theorem (see, e.g., [12, Corollary 7.1a]), there is a linear functional  $f: \mathbb{Q}^n \rightarrow \mathbb{Q}$  such that  $f(x) > 0$  for all  $x \in K$ ; whence  $f(x) > 0$  for all  $x \in M \setminus \{0\}$ . We may assume that  $f$  is the projection on the first coordinate in  $\mathbb{Q}^n$ . The order of  $M$ , defined as the restriction of the lexicographical order on  $\mathbb{Q}^n$ , is as required.  $\square$

Our next example shows that left- and right-orderability are not equivalent.

**Example 2.9.** *There exists a five-element positively right-orderable idempotent monoid that is not left-orderable.*

*Proof.* The multiplication table of our monoid  $M$ , again obtained with the help of the Mace4 program, is given by Table 2.2. The positive right order on  $M$  is given by  $1 < a < b < c < d$  (all columns of Table 2.2 are indeed ascending).

$\cdot$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$
$a$	$a$	$a$	$b$	$c$	$d$
$b$	$b$	$a$	$b$	$c$	$d$
$c$	$c$	$d$	$d$	$c$	$d$
$d$	$d$	$d$	$d$	$c$	$d$

TABLE 2.2. A 5-element right-orderable monoid

Suppose that there is a left order  $\trianglelefteq$  on  $M$ . By replacing  $\trianglelefteq$  by its dual order, we may assume that either  $1 \trianglelefteq a \trianglelefteq b$ ,  $a \trianglelefteq b \trianglelefteq 1$ , or  $b \trianglelefteq 1 \trianglelefteq a$ . If  $1 \trianglelefteq a \trianglelefteq b$ , then  $b \trianglelefteq ba \trianglelefteq b^2 = b$  thus  $b = ba = a$ , a contradiction. If  $a \trianglelefteq b \trianglelefteq 1$  then  $a = a^2 \trianglelefteq ab \trianglelefteq a$  thus  $b = ab = a$ , a contradiction. If  $b \trianglelefteq 1 \trianglelefteq a$ , then  $d = cb \trianglelefteq c \trianglelefteq ca = d$ , thus  $c = d$ , a contradiction.  $\square$

### 3. A COUNTEREXAMPLE THAT EMBEDS INTO A GROUP

The monoid underlying Example 2.9 is finite, thus it cannot yield a cancellative example (for every nontrivial cancellative right-orderable monoid is infinite). Attempting to construct a cancellative precursor of that monoid, with a similar property, we stumble on the equation  $cb = ca$ , which rules out cancellativity.

Hence, finding a cancellative, right-orderable, but not left-orderable monoid will require more work. Our construction, detailed in the proof of Theorem 3.2, will in fact yield a submonoid of a group.

Let us briefly recall some background, on *categories*, required for our construction. We will mostly follow the notation and terminology of [15]. All our categories will be small categories, viewed as certain partial semigroups with identities, for example in the sense of [4, § II.1.1] or [15, Definition 3.1]. Our categories will be understood in the source / target sense, as opposed to domain / range: hence every element  $x$  in a category  $S$  has a source  $\partial_0 x$  and a target  $\partial_1 x$ , which are the only identities in  $S$  satisfying  $x = \partial_0 x \cdot x = x \cdot \partial_1 x$ . A product  $xy$  is defined iff  $\partial_1 x = \partial_0 y$ . A monoid is a category with a unique identity, and a functor between categories is a homomorphism with respect to the partial product, source, and target operations. For identities  $a$  and  $b$  in  $S$ , the hom-set  $S(a, b)$  is  $\{x \in S \mid \partial_0 x = a \text{ and } \partial_1 x = b\}$ . We will denote by  $\varepsilon_S: S \rightarrow \mathbf{U}_{\text{mon}}(S)$  the natural functor from  $S$  to its universal monoid, and we will set  $\overline{S} \stackrel{\text{def}}{=} \varepsilon_S[S]$ . The map  $\varepsilon_S$  identifies two elements  $x$  and  $y$  of  $S$  iff either  $x = y$  or  $x$  and  $y$  are both identities [15, Lemma 3.10]. Denoting by  $\text{Idt } S$  the set of all identities of  $S$ , it follows that  $\overline{S}$  can be identified with  $(S \setminus \text{Idt } S) \cup \{1\}$ .

By [15, Lemma 3.4], any element of  $\mathbf{U}_{\text{mon}}(S)$  can be uniquely written as a product  $x_1 \cdots x_n$  (which will then be called a *reduced word*), where  $n$  is a nonnegative integer, the  $x_i$  are non-identities in  $S$ , and each product  $x_i x_{i+1}$  is undefined (i.e.,  $\partial_1 x_i \neq \partial_0 x_{i+1}$ ). We set  $n = \text{lh}(x)$ , the *length* of  $x$ .

As in [15], a category  $S$  is

- *conical* if any composition of two non-identities of  $S$  is a non-identity;
- *right cancellative* if for all  $x, y, z \in S$ , if  $xz$  and  $yz$  are both defined and  $xz = yz$ , then  $x = y$ . By [15, Corollary 4.3], this is equivalent to saying that the universal monoid  $\mathbf{U}_{\text{mon}}(S)$  of  $S$  is right cancellative.

Our next result states a criterion for the existence of certain right orders on universal monoids of categories.

**Lemma 3.1.** *The following are equivalent, for any conical, right cancellative category  $S$ :*

- (i) *The monoid  $\mathbf{U}_{\text{mon}}(S)$  has a positive right order, with respect to which  $\overline{S}$  is a lower subset of  $\mathbf{U}_{\text{mon}}(S)$ .*
- (ii) *There exists a total order  $\leq$  on  $\overline{S}$ , with least element 1, such that for all  $x, y, z \in \overline{S}$ ,  $x \leq y$  and  $yz \in \overline{S}$  together imply that  $xz \in \overline{S}$  and  $xz \leq yz$ .*

*Proof.* For any right order  $\leq$  on  $\mathbf{U}_{\text{mon}}(S)$  satisfying (i), the restriction of  $\leq$  to  $\overline{S}$  obviously satisfies (ii).

Let, conversely,  $\leq$  be a total order on  $\overline{S}$  satisfying (ii). For any elements  $x$  and  $y$  of  $\mathbf{U}_{\text{mon}}(S)$ , respectively written in reduced form as  $x_m \cdots x_1$  and  $y_n \cdots y_1$  with all  $x_i, y_j \in S \setminus \text{Idt } S$ , we define  $\nu(x, y)$  as the least  $k \in \{1, \dots, \min\{m, n\}\}$  if it exists such that  $x_k \neq y_k$ . Let  $x < y$  hold if either  $m < n$ , or  $m = n$  and for  $k \stackrel{\text{def}}{=} \nu(x, y)$  it holds that  $x_k < y_k$ . Further, let  $x \leq y$  hold if either  $x < y$  or  $x = y$ . In particular, that relation extends the original  $\leq$  on  $\overline{S}$ , whence it is consistent to denote it by the same symbol. Since  $\leq$  is just the shortlex order (with words being read from right to left) on finite sequences from  $S \setminus \text{Idt } S$ , it is a total order on  $\mathbf{U}_{\text{mon}}(S)$ .

In order to prove that  $\leq$  is right translation-invariant, it suffices to prove that  $x < y$  implies that  $xz < yz$ , for all  $x, y \in \mathbf{U}_{\text{mon}}(S)$  and all  $z \in S \setminus \text{Idt } S$ . Write again  $x$  and  $y$  as reduced words as above. In particular,  $m \leq n$ . Moreover, from  $x < y$  it follows that  $n > 0$ . If  $m = n$ , then  $k \stackrel{\text{def}}{=} \nu(x, y)$  exists and  $x_k < y_k$ .

Suppose first that  $y_1z \notin \overline{S}$ . Then  $y_n \cdots y_2 \cdot y_1 \cdot z$  is the reduced word representation of  $yz$ , thus  $\text{lh}(xz) \leq m + 1 \leq n + 1 = \text{lh}(yz)$ . It follows that if either  $m < n$  or  $x_1z \in \overline{S}$ , then  $\text{lh}(xz) < \text{lh}(yz)$ , thus  $xz < yz$ . If, on the other hand,  $m = n$  and  $x_1z \notin \overline{S}$ , then  $x_n \cdots x_2 \cdot x_1 \cdot z$  is the reduced word representation of  $xz$ , so  $\nu(xz, yz) = k + 1$  and  $(xz)_{k+1} = x_k < y_k = (yz)_{k+1}$ , whence  $xz < yz$ .

We may thus assume from now on that  $y_1z \in \overline{S}$ . It follows from the conicality of  $S$  that  $y_n \cdots y_2 \cdot y_1z$  is the reduced word representing  $yz$ , so  $\text{lh}(yz) = n$ . From  $1 \leq y_1$  (within  $\overline{S}$ ) it follows that  $z = 1z \leq y_1z$ . If  $z = y_1z$ , that is,  $\partial_0z \cdot z = y_1z$  (within  $S$ ), then, since  $S$  is right cancellative,  $y_1 = \partial_0z$  is an identity, a contradiction. Hence,

$$z < y_1z. \quad (3.1)$$

In particular, if  $x = 1$  (i.e.,  $m = 0$ ), then  $xz = z < yz$ . We may thus assume from now on that  $m > 0$ . In particular, if  $m + 2 \leq n$ , then  $\text{lh}(xz) \leq m + 1 < n = \text{lh}(yz)$ , so  $xz < yz$ .

It thus remains to check all cases where  $y_1z \in \overline{S}$  and  $0 < m \leq n \leq m + 1$ .

Suppose first that  $y_1z \in \overline{S}$  and  $m = n$ . From  $x \leq y$  it follows that  $x_1 \leq y_1$ , thus, since  $y_1z \in \overline{S}$  and by our assumption on the order  $\leq$ , we get  $x_1z \in \overline{S}$ ; so  $x_n \cdots x_2 \cdot x_1z$  is the reduced word representing  $xz$ , and so  $\text{lh}(xz) = n$ . If  $k \geq 2$ , then  $x_1 = y_1$ , so  $\nu(xz, yz) = k$  and  $(xz)_k = x_k < y_k = (yz)_k$ , and so  $xz < yz$ . If  $k = 1$ , then  $x_1 < y_1$ , thus, since  $\leq$  is right translation-invariant and  $S$  is both conical and right cancellative,  $x_1z < y_1z$ , so  $\nu(xz, yz) = 1$  and  $xz < yz$ .

It remains to deal with the case where  $y_1z \in \overline{S}$  and  $m + 1 = n$ . If  $x_1z \in \overline{S}$ , then  $x_m \cdots x_2 \cdot x_1z$  is the reduced word representing  $xz$ , so  $\text{lh}(xz) = m < n = \text{lh}(yz)$  and so  $xz < yz$ . If  $x_1z \notin \overline{S}$ , then  $x_m \cdots x_1 \cdot z$  is the reduced word representing  $xz$  and  $\text{lh}(xz) = m + 1 = n = \text{lh}(yz)$ . Using (3.1), we get  $(xz)_1 = z < y_1z = (yz)_1$ , so  $xz < yz$  again. We have thus completed the proof that  $\leq$  is a right order of  $U_{\text{mon}}(\overline{S})$ .

Since  $\overline{S}$  is the set of all elements of  $U_{\text{mon}}(S)$  with length at most 1, and since  $x \leq y$  implies  $\text{lh}(x) \leq \text{lh}(y)$ ,  $\overline{S}$  is a lower subset of  $U_{\text{mon}}(S)$ .  $\square$

**Theorem 3.2.** *There exists a positively right-orderable submonoid  $M$ , of the group  $(\mathbb{Z}/7\mathbb{Z}) * F_{\text{gp}}(4)$ , containing a triple of elements with no smallest element with respect to any partial left order of  $M$ . In particular,  $M$  is not left-orderable.*

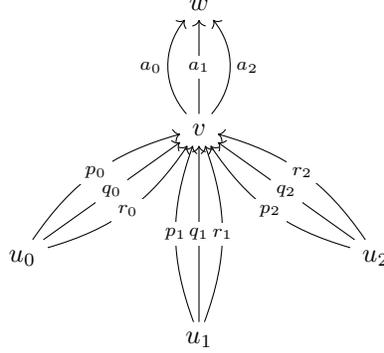
*Proof.* Our monoid  $M$  is defined by the generators  $p_i, q_i, r_i, a_i$ , for  $i \in \{0, 1, 2\}$ , subjected to the relations

$$p_i a_i = r_i a_{i+2}; \quad p_i a_{i+1} = q_i a_i; \quad q_i a_{i+1} = r_i a_i, \quad \text{for } i \in \{0, 1, 2\} \quad (3.2)$$

(where indices are taken modulo 3). It is convenient to represent  $M$  as the universal monoid of the finite category  $S$  consisting of the generating set

$$\Sigma \stackrel{\text{def}}{=} \{p_0, q_0, r_0, p_1, q_1, r_1, p_2, q_2, r_2, a_0, a_1, a_2\}$$

together with the products  $p_i a_j, q_i a_j, r_i a_j$  for  $i, j \in \{0, 1, 2\}$ , and the five identities  $u_i \stackrel{\text{def}}{=} \partial_0 p_i = \partial_0 q_i = \partial_0 r_i$ ,  $v \stackrel{\text{def}}{=} \partial_1 p_i = \partial_1 q_i = \partial_1 r_i = \partial_0 a_j$ , and  $w = \partial_1 a_j$ , for  $i, j \in \{0, 1, 2\}$ . In particular,  $\overline{S} = \{1\} \cup \Sigma \cup \{p_i a_j, q_i a_j, r_i a_j \mid i, j \in \{0, 1, 2\}\}$ . By virtue of the equations (3.2),  $\overline{S}$  has 31 elements and  $S$  has 35 elements. The category  $S$  is obviously both conical and right cancellative. It can be illustrated in Figure 3.1 (which displays neither the equations (3.2) nor the elements of  $S(u_i, w)$ ). We define  $G$  as the universal group of the category  $S$ .

FIGURE 3.1. Illustrating the category  $S$ 

**Claim 1.** *The group  $G$  is isomorphic to  $(\mathbb{Z}/7\mathbb{Z}) * F_{\text{gp}}(4)$ .*

*Proof of Claim.* If working in groups, we can eliminate  $q_i$  and  $r_i$  from the equations (3.2) and thus obtain the equivalent group presentation of  $G$  given by

$$q_i = p_i a_{i+1} a_i^{-1}; \quad r_i = p_i a_i a_{i+2}^{-1}; \quad (3.3)$$

$$a_{i+1} a_i^{-1} a_{i+1} = a_i a_{i+2}^{-1} a_i, \quad (3.4)$$

where  $i$  ranges over  $\{0, 1, 2\}$ . Now (3.4), for  $i = 0$ , yields  $a_2 = a_0 a_1^{-1} a_0 a_1^{-1} a_0$ , which, combined with (3.4) for  $i = 1$ , yields

$$a_0 a_1^{-1} a_0 a_1^{-1} a_0 a_1^{-1} a_0 a_1^{-1} a_0 a_1^{-1} a_0 = a_1 a_0^{-1} a_1,$$

which is equivalent to  $(a_0^{-1} a_1)^7 = 1$ . Setting  $c \stackrel{\text{def}}{=} a_0^{-1} a_1$ , it follows that  $c^7 = 1$  and  $a_1 = a_0 c$ . Furthermore,  $a_2 = a_0 a_1^{-1} a_0 a_1^{-1} a_0 = a_0 c^5$ . Conversely, any triple  $(a_0, a_1, a_2, c)$  with  $c^7 = 1$ ,  $a_1 = a_0 c$ , and  $a_2 = a_0 c^5$  satisfies the equations (3.4). Taking into account the equations (3.3), it follows that  $G$  is the free product of the cyclic group generated by  $c$  (which is isomorphic to  $\mathbb{Z}/7\mathbb{Z}$ ) and the free group on  $\{p_0, p_1, p_2, a_0\}$ .  $\square$  Claim 1.

Owing to (the proof of) Claim 1, we shall from now on identify  $G$  with the free product of  $\mathbb{Z}/7\mathbb{Z}$  and the free group on generators  $\{p_0, p_1, p_2, a_0\}$ , with  $c$  the canonical generator of  $\mathbb{Z}/7\mathbb{Z}$ ,  $a_1 = a_0 c$ ,  $a_2 = a_0 c^5$ , and the terms  $q_i$  and  $r_i$  given by (3.3). Easy calculations then yield

$$q_0 = p_0 a_1 a_0^{-1} = p_0 a_0 c a_0^{-1}; \quad r_0 = p_0 a_0 a_2^{-1} = p_0 a_0 c^2 a_0^{-1}; \quad (3.5)$$

$$q_1 = p_1 a_2 a_1^{-1} = p_1 a_0 c^4 a_0^{-1}; \quad r_1 = p_1 a_1 a_0^{-1} = p_1 a_0 c a_0^{-1}; \quad (3.6)$$

$$q_2 = p_2 a_0 a_2^{-1} = p_2 a_0 c^2 a_0^{-1}; \quad r_2 = p_2 a_2 a_1^{-1} = p_2 a_0 c^4 a_0^{-1}. \quad (3.7)$$

**Claim 2.** *The natural monoid homomorphism  $\varphi_S: M \rightarrow G$  is one-to-one.*

*Proof of Claim.* Composing the natural functor  $\varepsilon_S: S \rightarrow M$  with  $\varphi_S$  yields a functor  $\eta_S: S \rightarrow G$ . By [15, Theorem 10.1], it suffices to prove that the restriction of  $\eta_S$  to each hom-set of  $S$  is one-to-one. For  $i \in \{0, 1, 2\}$ , the elements of  $S(u_i, v)$  are  $p_i, q_i, r_i$ , which are mapped under  $\eta_S$  to distinct elements of  $G$  since the expressions given on the right hand sides of the equations in (3.5)–(3.7) are normal

forms within  $(\mathbb{Z}/7\mathbb{Z}) * F_{\text{gp}}(4)$ . Furthermore, the elements of  $S(v, w)$  are  $a_0, a_1, a_2$ , which are mapped under  $\eta_S$  to the distinct elements  $a_0, a_0c, a_0c^5$ , respectively.

Finally, each  $S(u_i, w)$  has exactly six elements, that can be listed as

$$p_i a_{i+2}; q_i a_{i+2}; p_i a_i = r_i a_{i+2}; p_i a_{i+1} = q_i a_i; q_i a_{i+1} = r_i a_i; r_i a_{i+1}, \quad (3.8)$$

which, owing to the relations (3.3) and (3.4), are respectively mapped by  $\eta_S$  to

$$p_i a_{i+2}; p_i a_{i+1} a_i^{-1} a_{i+2}; p_i a_i; p_i a_{i+1}; p_i a_{i+1} a_i^{-1} a_{i+1}; p_i a_i a_{i+2}^{-1} a_{i+1}.$$

Substituting in those equations the expressions  $a_1 = a_0c, a_2 = a_0c^5$ , we obtain that those elements are respectively equal to

$$\begin{aligned} p_0 a_0 c^5; p_0 a_0 c^6; p_0 a_0; p_0 a_0 c; p_0 a_0 c^2; p_0 a_0 c^3 & \quad (\text{for } i = 0), \\ p_1 a_0; p_1 a_0 c^4; p_1 a_0 c; p_1 a_0 c^5; p_1 a_0 c^2; p_1 a_0 c^6 & \quad (\text{for } i = 1), \\ p_2 a_0 c; p_2 a_0 c^3; p_2 a_0 c^5; p_2 a_0; p_2 a_0 c^2; p_2 a_0 c^4 & \quad (\text{for } i = 2). \end{aligned}$$

For each value of  $i \in \{0, 1, 2\}$ , those elements are pairwise distinct, so the restriction of  $\eta_S$  to  $S(u_i, w)$  is one-to-one.  $\square$  Claim 2.

**Claim 3.** *The monoid  $M$  has a positive right order, with respect to which  $\overline{S}$  is a lower subset.*

*Proof of Claim.* We apply Lemma 3.1. We will construct the required order on  $\overline{S}$  by initiating the order on  $\Sigma \cup \{1\}$ , by setting

$$1 < p_0 < q_0 < r_0 < p_1 < q_1 < r_1 < p_2 < q_2 < r_2 < a_0 < a_1 < a_2, \quad (3.9)$$

then extending  $\leq$  to the whole  $\overline{S}$  by using the right invariance of  $\leq$ . On elements of the form  $p_i a_j, q_i a_j, r_i a_j$  for fixed  $i$  (listed in (3.8)), there is no choice, we need to define

$$p_i a_{i+2} < q_i a_{i+2} < p_i a_i = r_i a_{i+2} < p_i a_{i+1} = q_i a_i < q_i a_{i+1} = r_i a_i < r_i a_{i+1} \quad (3.10)$$

(that part of the calculations does not involve the inequalities  $a_0 < a_1 < a_2$ ). We complete the construction by linking the chains (3.10), in the order 012, atop the chain (3.9), thus yielding the inequalities

$$\underbrace{1 < \dots < a_2}_{\text{chain (3.9)}} < \underbrace{p_0 a_2 < \dots < r_0 a_1}_{\text{chain (3.10) for } i = 0} < \underbrace{p_1 a_0 < \dots < r_1 a_2}_{\text{chain (3.10) for } i = 1} < \underbrace{p_2 a_1 < \dots < r_2 a_0}_{\text{chain (3.10) for } i = 2}. \quad (3.11)$$

We need to verify that for all  $x, y, z \in \overline{S}$ ,  $x \leq y$  and  $yz \in \overline{S}$  implies that  $xz \in \overline{S}$  and  $xz \leq yz$ . Since this is vacuous if either  $z = 1$  or  $z \in \{p_i, q_i, r_i\}$  for some  $i$  (only trivial products intervene there), the only interesting case is the one where  $z = a_j$  for some  $j$  and  $y$  belongs to the set  $\Sigma' \stackrel{\text{def}}{=} \{1, p_0, q_0, r_0, p_1, q_1, r_1, p_2, q_2, r_2\}$ . By (3.9),  $\Sigma'$  is a lower subset of  $\overline{S}$ , thus  $x$  also belongs to  $\Sigma'$ , and thus  $xz = xa_j$  belongs to  $\overline{S}$ . By the construction of the chain (3.11), we get  $xz \leq yz$ .  $\square$  Claim 3.

**Claim 4.** *There is no partial left order of  $M$  for which  $\{a_0, a_1, a_2\}$  has a smallest element.*

*Proof of Claim.* Since the defining equations (3.2) of  $M$  are invariant under translation (modulo 3) of the index  $i$ , it suffices to prove that there is no partial left order  $\triangleleft$  of  $M$  for which  $a_0 \triangleleft a_1$  and  $a_0 \triangleleft a_2$ . Using left invariance, we get

$$p_0 a_0 \triangleleft p_0 a_1 = q_0 a_0 \triangleleft q_0 a_1 = r_0 a_0 \triangleleft r_0 a_2 = p_0 a_0,$$

thus  $p_0a_0 = p_0a_1$ , a contradiction.  $\square$  Claim 4.

The conclusion of Theorem 3.2 follows from the Claims above.  $\square$

*Remark 3.3.* Recall from Bergman [2, Example 23] that there are monoids, embeddable into free groups, of which the universal group is not free—it may even have torsion; by the results of [15], this is more the rule than the exception. Now the proof of Theorem 3.2 above shows that each  $(a_i a_j^{-1})^7 = 1$  within the universal group of  $M$ —equivalently, within any group extension of  $M$ . Since  $a_0, a_1, a_2$  are pairwise distinct, it follows that *every group extension of  $M$  has torsion elements*.

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