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# Perfectly Matched Layers on Cubic Domains for Pauli's Equations

Laurence Halpern <sup>\*</sup>      Jeffrey Rauch <sup>†</sup>

## Abstract

This article proves the well posedness of the boundary value problem that arises when PML algorithms are applied to Pauli's equations with a three dimensional rectangle as computational domain. The absorptions are positive near the boundary and zero far from the boundary so are always  $x$ -dependent. At the flat parts of the boundary of the rectangle, the natural absorbing boundary conditions are imposed. The difficulty addressed is the analysis of the resulting variable coefficient problem on the rectangular solid with its edges and corners. The Laplace transform is analysed. It turns on the analysis of a boundary value problem formally obtained by complex stretching. Existence is proved by deriving a boundary value problems for a complex stretched Helmholtz equation on smoothed domains. This is the first stability proof with  $x$ -dependent absorptions on a bounded domain whose boundary is not smooth.

**Keywords.** Hyperbolic boundary value problem, PML, trihedral corner, dissipative boundary conditions, PML, Bérenger, Pauli system.

**AMS Subject Classification.** 35F46, 35J25, 35L20, 35L53, 35Q40, 65N12.

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## 1 Introduction

This paper analyses initial boundary value problems that arise when one uses perfectly matched absorbing layers in the time domain. The most common configuration is a three dimensional rectangular solid surrounded by a larger rectangular solid computational domain. The inner solid contains the sources and is the region where the computed values are required. In the region between the rectangles, perfectly matched layers are interposed. Boundary conditions at the exterior boundary are imposed that are designed to be weakly reflecting. In addition to perfect matching, an advantage of the PML strategy is its ease of implementation including at the corners. To our knowledge, the present work is the first to prove well posedness for such a PML with non constant absorptions  $\sigma_j$  in the presence of trihedral corners. That problem poses two fundamental challenges.

Even for a system with a very simple energy estimate like Pauli's equations, the split equations of Berenger and also the stretched system that is

at the heart of its analysis do not have simple estimates. Such estimates are crucial for constructing solutions and express stability. In practice the split system needs to be discretised and the stability of the discretisation analysed. This article does not study that problem. A recent survey for the constant coefficient half space case is [12].

The Pauli system shares the Lorentz invariance, symmetry, and three dimensionality of Maxwell's equations. It has two advantages. It is a  $2 \times 2$  system as opposed to a  $6 \times 6$  system. More importantly, the generator is elliptic. The analysis extends with almost no modifications to the Dirac system. The Pauli operator is

$$L := \partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_2 + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \partial_3 := \partial_t + \sum_{j=1}^3 A_j \partial_j. \quad (1.1)$$

Introduce the notations with  $\xi \in \mathbb{C}^3$ ,

$$L(\partial_t, \partial_x) := \partial_t + A(\partial_x), \quad A(\xi) := A_1 \xi_1 + A_2 \xi_2 + A_3 \xi_3. \quad (1.2)$$

**Definition 1.1** For  $A \in \text{Hom}(\mathbb{C}^k)$ , with spectrum disjoint from  $i\mathbb{R}$ ,  $\mathcal{E}^+(A)$  (resp.  $\mathcal{E}^-(A)$ ) denotes the spectral subspace corresponding to eigenvalues with strictly positive (resp. strictly negative) real part. Denote by  $\pi^\pm(A)$  the corresponding spectral projections onto those spaces. As our interest is the Pauli system,  $\mathcal{E}^\pm(\xi)$  and  $\pi^\pm(\xi)$  are shorthands for  $\mathcal{E}^\pm(A(\xi))$  and  $\pi^\pm(A(\xi))$  for  $\xi \in \mathbb{C}^3$  so that  $A(\xi)$  has no purely imaginary eigenvalues.

**Definition 1.2** Denote by  $\mathcal{Q} = \mathcal{Q}(L_1, L_2, L_3)$  the rectangle

$$\mathcal{Q} := \{x \in \mathbb{R}^3 : |x_j| < L_j/2, \quad j = 1, 2, 3\}.$$

$\mathcal{Q}$  has six open faces  $G_k$  with  $1 \leq k \leq 6$ . For  $j = 1, 2, 3$ ,

$$\begin{aligned} G_j &:= \{x_j = -L_j/2, \text{ and, } |x_i| < L_i/2 \text{ for } j \neq i\}, \\ G_{j+3} &:= \{x_j = L_j/2, \text{ and, } |x_i| < L_i/2 \text{ for } j \neq i\}. \end{aligned}$$

For a point  $x \in G_k$ ,  $\nu(x)$  denotes the outward unit normal to  $\mathcal{Q}$ . The split equations involve non negative **absorption coefficients**  $\sigma_j \in C_0^\infty(\mathbb{R})$  for  $j = 1, 2, 3$ .

**Example 1.1** For the usual implementations of the PML method, there is an  $\ell < 1$  so that the absorptions vanish in  $\ell\mathcal{Q}$ , the sources are supported in  $\ell\mathcal{Q}$  and the values of the solution on  $\ell\mathcal{Q}$  are those of interest.

The Pauli system is

$$\left(\partial_t + A_1\partial_1 + A_2\partial_2 + A_3\partial_3\right)u = f \quad \text{on } \mathcal{Q}. \quad (1.3)$$

**Definition 1.3** *Bérenger's method has unknown that is a triple  $(U^1, U^2, U^3)$  with  $U^j$  taking values in  $\mathbb{C}^2$ . On  $\mathbb{R} \times \mathcal{Q}$ ,  $(U^1, U^2, U^3)$  satisfy the **split equations**,*

$$\begin{aligned} (\partial_t + \sigma_1(x_1))U^1 + A_1\partial_1(U^1 + U^2 + U^3) &= f_1, \\ (\partial_t + \sigma_2(x_2))U^2 + A_2\partial_2(U^1 + U^2 + U^3) &= f_2, \\ (\partial_t + \sigma_3(x_3))U^3 + A_3\partial_3(U^1 + U^2 + U^3) &= f_3. \end{aligned} \quad (1.4)$$

The  $j^{\text{th}}$  equation has the  $\partial_j$  derivative. The  $f_j$  are constrained to satisfy  $f = \sum_j f_j$  and to vanish on a neighborhood of  $\partial\mathcal{Q}$ . A choice respecting the symmetry of the problem is  $f_j = f/3$  for  $j = 1, 2, 3$ .

The boundary of  $\mathcal{Q}$  is not perfectly transparent. In favorable cases like the Pauli system, waves are expected to decay in the layers so little signal reaches  $\partial\mathcal{Q}$  and the reflections cause small errors. In practice rather thin layers suffice. With  $x$ -dependent absorptions and computations in the time domain, proving exponential decay in the layers is an outstanding open problem. See Remark 3.3 for the easier time harmonic case.

The split equations are not symmetric and they have a lower order term that depends on  $x$  through the absorption coefficients  $\sigma_j$ . They do not have simple *a priori* estimates showing that they yield a well posed pure initial value problem. Petit-Bergez [22, 15] proved that since the Pauli system generates a  $C_0$ -semigroup on  $L^2(\mathbb{R}^3)$  and has elliptic generator, it follows that the split equations on  $\mathbb{R}^3$  also generate a  $C_0$ -semigroup. This contrasts to the loss of one derivative for the split Maxwell equations proved by Arbarbanel and Gottlieb [1].

The Pauli system is symmetric hyperbolic. The most strongly dissipative boundary condition for the Pauli system is  $u \in \mathcal{E}^+(\nu)$ . Thanks to the symmetry and ellipticity there is an  $M_0$  so that for  $M > M_0$  and  $f \in e^{Mt}L^2(\mathbb{R} \times \mathcal{Q})$  the boundary value problem  $Lu = f$  with boundary condition  $u \in \mathcal{E}^+(\nu)$  on the  $\mathbb{R} \times G_k$  has a unique solution  $u \in e^{Mt}L^2(\mathbb{R} \times \mathcal{Q})$  with  $u \in e^{Mt}L^2(\mathbb{R} \times \partial\mathcal{Q})$  (see Part I of [16]). The substantial difficulty here is the presence of the edges and corner of  $\mathcal{Q}$ .

For the Pauli system, if one imposed a conservative rather than dissipative boundary condition, then waves that arrive at the external boundary are

totally reflected back to the interior. This behavior would be approximately inherited by the stretched system and is to be avoided. Less obvious is that our proof breaks down for non dissipative conditions. Even for the unstretched problem uniqueness of solutions for a conservative condition on domains with trihedral corners is not known. This underscores the difficulties of domains with trihedral corners.

The Laplace transform of solutions of (1.4) are

$$\begin{aligned}(\tau + \sigma_1(x_1))\widehat{U}^1 + A_1\partial_1(\widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3) &= \widehat{f}_1, \\(\tau + \sigma_2(x_2))\widehat{U}^2 + A_2\partial_2(\widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3) &= \widehat{f}_2, \\(\tau + \sigma_3(x_3))\widehat{U}^3 + A_3\partial_3(\widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3) &= \widehat{f}_3.\end{aligned}\tag{1.5}$$

Define for  $j = 1, 2, 3$ ,

$$\widetilde{\partial}_j := \frac{\tau}{\tau + \sigma_j(x_j)} \frac{\partial}{\partial x_j}, \quad \text{and} \quad u := \widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3.\tag{1.6}$$

These yield the **stretched equation**

$$\left(\tau + A_1\widetilde{\partial}_1 + A_2\widetilde{\partial}_2 + A_3\widetilde{\partial}_3\right)u = F := \sum_{j=1}^3 \frac{\tau}{\tau + \sigma_j(x_j)} \widehat{f}_j.\tag{1.7}$$

**Definition 1.4** *The operator*

$$L(\tau, \widetilde{\partial}_x) := \tau + \sum_{j=1}^3 A_j \widetilde{\partial}_j = \tau + \sum_{j=1}^3 A_j \frac{\tau}{\tau + \sigma_j(x_j)} \frac{\partial}{\partial x_j}$$

*is called the stretched Pauli operator.*

**Definition 1.5 i.** *If  $\Omega \subset \mathbb{R}^3$  is open and  $K \subset \Omega$  is compact,*

$$C_K^\infty(\Omega) := \left\{ f \in C^\infty(\Omega) ; \text{supp } f \subset K \right\}.$$

**ii.** *Similarly,*

$$L_K^2(\Omega) := \left\{ f \in L^2(\Omega) ; \text{supp } f \subset K \right\}.$$

The next result solving the stretched equation is the main result of the paper. It allows one to prove the stability of Bérenger's split method. In our experience other versions of PML lead also to this same stretched system.

**Theorem 1.6** For each  $\ell \in ]0, 1[$  there exist  $C, M_1$  so that for all  $M \geq M_1$ , and, holomorphic  $F : \{\operatorname{Re} \tau > M\} \rightarrow L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q})$ , there is a unique holomorphic function  $u : \{\operatorname{Re} \tau > M\} \rightarrow H^1(\mathcal{Q})$  satisfying the stretched boundary value problem on  $\mathcal{Q}$ ,

$$L(\tau, \tilde{\partial}_x)u = F \quad \text{on } \mathcal{Q}, \quad u \in \mathcal{E}^+(\nu) \quad \text{on } G_k, \quad 1 \leq k \leq 6.$$

It satisfies for all  $\operatorname{Re} \tau > M$ ,

$$(\operatorname{Re} \tau) \|u\|_{L^2(\mathcal{Q})} + (\operatorname{Re} \tau)^{1/2} \|u\|_{L^2(\partial\mathcal{Q})} + \frac{\operatorname{Re} \tau}{|\tau|} \|\nabla_x u\| \leq C \|F\|_{L^2(\mathcal{Q})}.$$

**Remark 1.1** The gradient estimate degenerates as  $\operatorname{Im} \tau \rightarrow \infty$  with  $\operatorname{Re} \tau$  fixed. A second hyperbolic aspect is that the boundary values are estimated in  $L^2$  and not in  $H^{1/2}$ .

Theorem 1.6 allows us to analyse the split equations with the absorbing boundary condition  $U^1 + U^2 + U^3 \in \mathcal{E}^+(\nu)$  on the  $G_k$ . It is the first existence theorem for the split equations with non constant  $\sigma_j$  in domains whose boundary is not smooth. Since standard practice uses cubes with non constant  $\sigma_j$  is it the first justification, beyond extensive practical experience, that the Bérenger algorithm is stable.

**Theorem 1.7** There are strictly positive constants  $C, M$  so that if  $\lambda > M$ ,  $f \in e^{\lambda t} L^2(\mathbb{R} \times \mathcal{Q})$  with support in  $[0, \infty[ \times \ell\overline{\mathcal{Q}}$ , then there is one and only one  $(U^1, U^2, U^3) \in e^{\lambda t} L^2(\mathbb{R} \times \mathcal{Q})$  supported in  $t \geq 0$  that satisfies (1.4), and the boundary condition  $U^1 + U^2 + U^3 \in \mathcal{E}^+(\nu)$  on each  $G_k$ . The function  $U^1 + U^2 + U^3$  satisfies,

$$\begin{aligned} & \lambda \|e^{-\lambda t} (U^1 + U^2 + U^3)\|_{L^2(\mathbb{R} \times \mathcal{Q})} + \\ & \lambda^{1/2} \|e^{-\lambda t} (U^1 + U^2 + U^3)\|_{L^2(\mathbb{R} \times \partial\mathcal{Q})} \leq C \|e^{-\lambda t} f\|_{L^2(\mathbb{R} \times \mathcal{Q})}. \end{aligned} \quad (1.8)$$

The split unknowns satisfy the weaker estimate

$$\|e^{-\lambda t} \{\lambda U^j, \partial_t U^j\}\|_{L^2(\mathbb{R}; H^{-1}(\mathcal{Q}))} \leq C \|e^{-\lambda t} f\|_{L^2(\mathbb{R} \times \mathcal{Q})}. \quad (1.9)$$

**Remark 1.2 i.** It is wise to think of Bérenger's algorithm as a method that inputs  $f$  and outputs  $U^1 + U^2 + U^3$ . Estimate 1.8 shows that the output satisfies bounds as strong as strictly dissipative boundary value problems for symmetric hyperbolic systems. This behavior is known for the pure initial value problem (see Theorem 1.3 of Bécache and Joly [7] for the split Maxwell

equations with constant  $\sigma_j$ , and for Bérenger transmission problem even with variable  $\sigma_j$  in [15]).

**ii.** *The estimates of Theorem 1.7 permit exponential growth in time. Even for sources compactly supported in time. Practical experience with Bérenger’s method for equations closely tied to the wave equation (e.g. Maxwell and Pauli) show no growth in time even with variable  $\sigma_j$ . Interesting bounds uniform in time are proved for the case of constant  $\sigma_j$  for sufficiently regular solutions by Bécache-Joly, Diaz-Joly, and Baffet-Grote-Imperiale-Kachanovska [7, 13, 5]. It is not known whether the uniform bounds of these authors imply that the  $L^2(\mathcal{Q})$  norm  $U^1 + U^2 + U^3$  or any of its derivatives is uniformly bounded. Uniform bounds in time is an important and wide open problem. The present paper proves estimates resembling those at the expense of exponential growth and very substantial difficulty. Appelo-Hagstrom-Kreiss [4] analyse the problem of exponential growth with constant parameters by explicit formulas in Fourier. They propose stabilization methods. Variable coefficients and corner domains are beyond that strategy.*

The paper is organized as follows. Section 2 presents the Pauli system and most importantly the stretched Pauli system that is satisfied by the Laplace transform of  $\widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3$ . Theorems 1.6 and 2.5 assert existence and uniqueness for the boundary value problems for the stretched system on  $\mathcal{Q}$  as well as smoothed versions  $\mathcal{Q}_\delta$  converging to  $\mathcal{Q}$  as  $\delta \rightarrow 0$ . With important  $\delta$ -independent estimates. An important step is showing that the solutions on  $\mathcal{Q}_\delta$  satisfy an additional boundary condition stated in Corollary 2.12. It is routine to show that solutions of the stretched system are solutions of a Helmholtz type equation. The stability theorem for the stretched boundary value problem implies stability for other implementations of perfect matching. Respecting the history, we present the details for Bérenger’s splitting. The second boundary condition yields a coercive elliptic boundary value problem That problem is solved in Theorem 2.5. The estimates are derived by the energy method tied to a family of complex quadratic forms. The real and imaginary parts play key roles. The family is singular in the limit  $\delta \rightarrow 0$ . In spite of this apparent failure of uniform coercivity, estimates uniform in  $\delta$  and  $\tau$  are proved. On solutions, the form is much smaller than its upper bound.

The backward implication, that the solution of the Helmholtz system solves the stretched system from which it is derived is subtle. It is proved only in the smoothed domain  $\mathcal{Q}_\delta$  and where the smoothness of solutions is important for the argument. We have considered the option of skipping the



smoothing and using layer potential methods developed for the study of Lipschitz domains. Since the hard harmonic analysis would need to be adapted to the new problems, the smoothing is both more elementary and shorter.

Section 4 derives the main theorems from the Helmholtz existence results. Section 4.1 proves unique solvability of the stretched Pauli system on  $\mathcal{Q}_\delta$  stated in Theorem 2.5. Holomorphy in  $\tau$  is crucial for uniqueness. Section 4.2 proves Theorem 1.6 by passing to the limit  $\delta \rightarrow 0$  to solve the stretched system on  $\mathcal{Q}$ . Section 4.3 derives Theorem 1.7.

The proof is long and technical. The hypothesis  $\sigma_j \in C^\infty$  avoids some inessential difficulties. The proof uses  $H^2$  regularity for the Helmholtz problem on  $\mathcal{Q}_\delta$ . Absorptions  $\sigma_j \in L^\infty$  suffice for  $H^1$  regularity and  $\sigma_j$  Lipschitzian is sufficient for  $H^2(\mathcal{Q}_\delta)$ . Standard practice involves such Lipschitzian absorptions. This strengthening of the results is left to the interested reader.

## 2 The Pauli system and smoothed domains $\mathcal{Q}_\delta$

### 2.1 Pauli system and its symbol

The coefficients of the Pauli system (1.1) satisfy,

$$A_j^2 = I, \quad A_i A_j + A_j A_i = 0 \quad \text{for } i \neq j. \quad (2.1)$$

These identities imply the connections to the Laplacian,

$$\left( \sum_j A_j \partial_j \right)^2 = \Delta, \quad \left( \sum_j A_j \partial_j - \tau \right) \left( \sum_j A_j \partial_j + \tau \right) = \Delta - \tau^2. \quad (2.2)$$

**Proposition 2.1 i.** For all  $(\tau, \xi) \in \mathbb{C}^{1+3}$ ,

$$\det L(\tau, \xi) = \tau^2 - \sum_{j=1}^3 \xi_j^2. \quad (2.3)$$

**ii.** For  $\xi \in \mathbb{R}^3 \setminus 0$ , the  $2 \times 2$  hermitian symmetric matrix  $A(\xi)$  has eigenvalues  $\pm|\xi|$  with one dimensional eigenspaces

$$\mathcal{E}^-(A(\xi)) = \mathbb{C}(\xi_1 - |\xi|, \xi_2 - i\xi_3), \quad \text{and,} \quad \mathcal{E}^+(A(\xi)) = \mathbb{C}(\xi_1 + |\xi|, \xi_2 - i\xi_3).$$

**iii.** For all  $\xi, \eta \in \mathbb{C}^3$ ,

$$A(\xi)A(\eta) + A(\eta)A(\xi) = 2 \left( \sum_i \xi_i \eta_i \right) I. \quad (2.4)$$

**Proof.** Write from (1.2),

$$L(\tau, \xi) = \tau + A(\xi) = \begin{pmatrix} \tau + \xi_1 & \xi_2 + i\xi_3 \\ \xi_2 - i\xi_3 & \tau - \xi_1 \end{pmatrix}$$

This implies directly. **i.**

**ii.** For  $\xi \in \mathbb{R}^3 \setminus 0$ , (2.3) shows that the eigenvalues of  $A(\xi)$  are  $\pm|\xi|$ . The first column yields the formula for  $\mathcal{E}^-(\xi)$  in **ii.** The other choice of sign yields  $\mathcal{E}^+(\xi)$ .

**iii.** Expand

$$A(\xi)A(\eta) = \left( \sum_i A_i \xi_i \right) \left( \sum_j A_j \eta_j \right) = \sum_{i,j} A_i A_j \xi_i \eta_j.$$

Symmetrizing yields,

$$A(\xi)A(\eta) + A(\eta)A(\xi) = \sum_{i,j} A_i A_j \xi_i \eta_j + \sum_{i,j} A_i A_j \eta_i \xi_j.$$

In the last sum interchange the roll of  $i, j$  to find

$$A(\xi)A(\eta) + A(\eta)A(\xi) = \sum_{i,j} A_i A_j \xi_i \eta_j + \sum_{i,j} A_j A_i \eta_j \xi_i.$$

Separate out the terms with  $i = j$  to find

$$A(\xi)A(\eta) + A(\eta)A(\xi) = 2 \sum_i A_i^2 \xi_i \eta_i + \sum_{i \neq j} (A_i A_j + A_j A_i) \eta_i \xi_j.$$

Equation (2.1) yields (2.4). □

**Example 2.1** Define  $\mathcal{Z} := \{\xi \in \mathbb{C}^3 : \sum_j \xi_j^2 = 0\}$ . For  $\xi \in \mathbb{C}^3 \setminus \mathcal{Z}$ , the spectrum of  $A(\xi)$  consists of two simple eigenvalues differing by a factor  $-1$ .

The eigenvalues  $\pm|\xi|$  for  $\xi \in \mathbb{R}^3 \setminus 0$  extend to holomorphic eigenvalues  $\lambda^\pm(\xi) = \pm(\sum \xi_j^2)^{1/2}$  the domain

$$\left\{ \xi \in \mathbb{C}^3 \setminus 0 : |\operatorname{Im} \xi| < |\operatorname{Re} \xi| \right\}. \quad (2.5)$$

In this case  $\sum \xi_j^2 \in \mathbb{C} \setminus [-\infty, 0]$ .

**Proposition 2.2 i.** *The eigenprojections  $\pi^\pm(\xi)$  for  $\xi \in \mathbb{R}^3 \setminus 0$  extend to holomorphic functions on the domain (2.5), satisfying with notation from Example 2.1,*

$$\pi^\pm(\xi) A(\xi) = A(\xi) \pi^\pm(\xi) = \lambda^\pm(\xi) \pi^\pm(\xi). \quad (2.6)$$

They are given by

$$\pi^\pm(\xi) = \frac{1}{2} \left( \sum \xi_j^2 \right)^{-1/2} \left( A(\xi) \pm \left( \sum \xi_j^2 \right)^{1/2} I \right).$$

ii. For  $\xi, \eta$  belonging to (2.5),

$$\pi^\pm(\eta) A(\xi) \pi^\pm(\eta) = \left( \sum \eta_j^2 \right)^{-1/2} \left( \sum \xi_i \eta_i \right) \pi^\pm(\eta). \quad (2.7)$$

**Proof. i.** The formulas

$$A(\xi) = \lambda^+ (\pi^+(A(\xi)) - \pi^-(A(\xi))), \quad \text{and,} \quad I = \pi^+(A(\xi)) + \pi^-(A(\xi)),$$

together with  $(\pi^\pm(\eta))^2 = \pi^\pm(\eta)$ , imply the formulas for  $\pi^\pm(A(\xi))$  in i.

ii. Multiply (2.4) on the left and right by  $\pi^\pm(\eta)$  to find

$$2 \pi^\pm(\eta) \left( \sum \xi_i \eta_i \right) \pi^\pm(\eta) = \pi^\pm(\eta) A(\xi) A(\eta) \pi^\pm(\eta) + \pi^\pm(\eta) A(\eta) A(\xi) \pi^\pm(\eta).$$

Use (2.6) twice and  $(\pi^\pm(\eta))^2 = \pi^\pm(\eta)$  to find,

$$\begin{aligned} 2 \left( \sum \xi_i \eta_i \right) \pi^\pm(\eta) &= \pi^\pm(\eta) A(\xi) \lambda^\pm(\eta) \pi^\pm(\eta) + \lambda^\pm(\eta) \pi^\pm(\eta) A(\xi) \pi^\pm(\eta) \\ &= 2 \lambda^\pm(\eta) \pi^\pm(\eta) A(\xi) \pi^\pm(\eta). \end{aligned}$$

This completes the proof.  $\square$

## 2.2 The stretched system on $\mathcal{Q}_\delta$ , Theorem 2.5 statement

The stretched equation (1.7) resembles the Laplace transform of the original system. For  $\tau$  real and positive it comes from the original transformed system by a change of variable, called *coordinate stretching* (see Section 2.3.2, and Chew-Weedon [10]). The stretched equations are sometimes expressed using auxiliary variables  $\psi_j$  defined as the solutions of

$$(\partial_t + \sigma_j(x_j)) \psi_j = \partial_t u, \quad \psi_j = 0 \quad \text{for} \quad t < 0.$$

Then  $\tilde{\partial}_j \hat{u} = \partial_j \hat{\psi}_j$ .

Theorem 1.6 is proved by solving the stretched equation on smoothed truncated domains and passing to the limit.

**Definition 2.3** *The singular set of the boundary of  $\mathcal{Q}$  is*

$$\mathcal{S} := \left\{ x \in \partial\mathcal{Q} : \exists i \neq j, x \in \overline{G}_i \cap \overline{G}_j \right\}.$$

*Introduce for  $0 < \delta < 1$  bounded smooth approximations  $\mathcal{Q}_\delta$  of  $\mathcal{Q}$ . Smooth the edges and corners of  $\mathcal{Q}$  on a  $\delta/2$ -neighborhood of  $\mathcal{S}$  to yield bounded smooth convex sets  $\mathcal{Q}_\delta$ . Do this so that for  $\delta_1 < \delta_2$ ,  $\mathcal{Q}_{\delta_1} \supset \mathcal{Q}_{\delta_2}$ .*

*The symbol  $\Omega$  is often used to denote elements of the family of sets  $\{\mathcal{Q}_\delta : \delta \in ]0, 1[ \}$ .*

**Definition 2.4** *For  $\tau$  with  $\operatorname{Re} \tau > 0$  and  $\nu \in \mathbb{R}^3$  define*

$$\tilde{\nu}(\tau, x) := \left( \frac{\nu_1 \tau}{\tau + \sigma_1(x)}, \frac{\nu_2 \tau}{\tau + \sigma_2(x)}, \frac{\nu_3 \tau}{\tau + \sigma_3(x)} \right).$$

In the next discussion this is used with  $\nu$  equal to the outward unit normal to  $\partial\mathcal{Q}_\delta$ . Next choose a boundary condition for the stretched equations on  $\mathcal{Q}_\delta$ . On the flat parts of  $\partial\mathcal{Q}_\delta$  one has  $u \in \mathcal{E}^+(\nu)$ . On the curved parts of the boundary and for  $\tau > 0$  and real, the stretched problem is symmetric hyperbolic and the normal matrix is  $A(\tilde{\nu}(\tau, x))$ . The maximally dissipative condition is  $u \in \mathcal{E}^+(\tilde{\nu})$ . If  $u(\tau)$  is holomorphic and satisfies this condition for  $\tau > 0$  then by analytic continuation it holds for general  $\tau$ . Therefore,  $u \in \mathcal{E}^+(\tilde{\nu})$  is a natural maximally dissipative condition for  $\tau$  complex.

The main result for the stretched system on  $\mathcal{Q}_\delta$  is the following.

**Theorem 2.5** *For  $0 < \ell < 1$  there exist  $C, M_1$  so that for all  $\delta \in (0, 1)$ ,  $M \geq M_1$ , and holomorphic  $F : \{\operatorname{Re} \tau > M\} \rightarrow C_{\ell\overline{\mathcal{Q}}}^\infty(\mathcal{Q}_\delta)$ , there is a unique holomorphic  $u^\delta : \{\operatorname{Re} \tau > M\} \rightarrow H^2(\mathcal{Q}_\delta)$  satisfying*

$$L(\tau, \tilde{\partial}_x)u^\delta = F, \quad \text{on } \mathcal{Q}_\delta, \quad u^\delta|_{\partial\mathcal{Q}_\delta} \in \mathcal{E}^+(A(\tilde{\nu}(\tau, x))). \quad (2.8)$$

*In addition,*

$$\begin{aligned} (\operatorname{Re} \tau) \|u^\delta\|_{L^2(\mathcal{Q}_\delta)} + (\operatorname{Re} \tau)^{1/2} \|u^\delta\|_{L^2(\partial\mathcal{Q}_\delta)} \\ + \frac{\operatorname{Re} \tau}{|\tau|} \|\nabla_x u^\delta\|_{L^2(\mathcal{Q}_\delta)} \leq C \|F(\tau)\|_{L_{\ell\overline{\mathcal{Q}}}^2(\mathcal{Q}_\delta)}. \end{aligned} \quad (2.9)$$

**Strategy of proof.** Theorem 2.5 is proved by solving carefully constructed Helmholtz equations and boundary conditions on  $\mathcal{Q}_\delta$ . The boundary conditions, automatically satisfied by solutions of the stretched problems, are identified in the next section. On  $\mathcal{Q}_\delta$  the solutions are smooth. The smoothness is used to prove that the solution of the Helmholtz problem on  $\mathcal{Q}_\delta$  solves the stretched equations when proving Theorem 2.5. Taking the limit  $\delta \rightarrow 0$  yields Theorem 1.6.

## 2.3 Second boundary condition for the Helmholtz BVP

This section concerns solutions of the stretched Pauli boundary value problem. Theorem 2.5 is proved by solving a Helmholtz boundary value problem. The hypotheses of Theorem 2.5 yield the boundary condition  $u \in \mathcal{E}^+(\tilde{\nu})$ . Corollary 2.12 of this section yields a crucial second boundary condition. Example 3.1 shows that it is a natural boundary condition for a weak formulation. Section 4.1 includes a proof of the converse implication that the Helmholtz equation plus the two boundary conditions imply the stretched Pauli equations.

### 2.3.1 Neumann identity for the unstretched Pauli system

**Definition 2.6** For  $\underline{x} \in \partial\Omega$  the **Weingarten map** (see for example [17]) is the real self adjoint map of the tangent space  $T_{\underline{x}}(\partial\Omega)$  to itself that is the differential of the unit exterior normal  $\nu$ . It maps  $T_{\underline{x}}(\partial\Omega) \ni \mathbf{v} \rightarrow \mathbf{v} \cdot \nabla \nu$ . Its eigenvalues are the **principal curvatures** of  $\partial\Omega$  at  $\underline{x}$ . The **mean curvature**, denoted  $H_{\Omega}(\underline{x})$ , is the average of the two principal curvatures.

Extend  $\nu$  to a smooth unit vector field defined on a neighborhood of  $\partial\Omega$  so as to be constant on normal lines to the boundary. Then  $\pi^{\pm}(\nu(x))$  is well defined and smooth for  $x$  in a neighborhood of  $\partial\Omega$ .

The term  $H_{\mathcal{Q}_{\delta}}(\underline{x})$  is equal to zero except for a  $\delta$  neighborhood of  $\mathcal{S}$  where it attains values  $\sim 1/\delta$ . The identity of the next proposition is simple in the case of flat boundaries.

**Proposition 2.7** If  $u \in H^2(\Omega)$  satisfies the boundary condition  $\pi^{-}(\nu)u = 0$  on  $\partial\Omega$ , then,

$$\pi^{+}(\nu) \sum_{j=1}^3 A_j \partial_j u = \pi^{+}(\nu)(\nu \cdot \partial_x + 2H_{\Omega})u, \quad \text{on } \partial\Omega. \quad (2.10)$$

**Proof of Proposition 2.7.** An invariance argument shows that it is sufficient to treat the case where  $\underline{x} = 0$ ,  $\nu(\underline{x}) = (-1, 0, 0)$  and the  $x_j$ -axes for  $j \geq 2$  are principal curvature directions of  $\partial\Omega$ .

Denote by  $\mathbf{e}_j$ ,  $j = 1, 2, 3$  the standard basis for  $\mathbb{R}^3$ . The principal curvatures corresponding to the tangent directions  $\mathbf{e}_2$  and  $\mathbf{e}_3$  are denoted  $\kappa_2$  and  $\kappa_3$ . The mean curvature is  $H := (\kappa_2 + \kappa_3)/2$ . At  $\underline{x}$  the outward unit normal is  $-\mathbf{e}_1$ . At  $\underline{x}$  the principal curvature formulas are  $\partial_2 \nu = -\kappa_2 \mathbf{e}_1$  and  $\partial_3 \nu = -\kappa_3 \mathbf{e}_1$ .

**First simplifications of the left hand side of (2.10).** The operator on the left is  $\pi^+(\nu)(A_1\partial_1 + A_2\partial_2 + A_3\partial_3)$ . On the  $x_1$  axis,  $\nu = (-1, 0, 0)$ , so  $\pi^+(\nu(x))A_1 = -\pi^+(\nu(x))$ . On that axis the operator is

$$-\pi^+(\nu)\partial_1 + \pi^+(\nu)(A_2\partial_2 + A_3\partial_3) = \pi^+(\nu(x))\nu(x) \cdot \partial_x + \pi^+(\nu)(A_2\partial_2 + A_3\partial_3). \quad (2.11)$$

**Second simplifications.** Consider the two summands  $\pi^+(\nu)A_j\partial_j u$  with  $j \geq 2$ . On the  $x_1$ -axis, part **ii** of Proposition 2.2 implies that

$$\pi^+(\nu)A_2\pi^+(\nu) = \pi^+(\nu)A_3\pi^+(\nu) = 0. \quad (2.12)$$

Using the boundary condition yields

$$\partial_j u = \partial_j(\pi^+(\nu)u + \pi^-(\nu)u) = \partial_j(\pi^+(\nu)u) \text{ at } \underline{x}. \quad (2.13)$$

For  $j \in \{2, 3\}$  if  $Z$  is a vector field on a neighborhood of  $\underline{x}$  that is tangent to the boundary and satisfies  $Z(\underline{x}) = \partial_j$  then

$$\partial_j u(\underline{x}) = Z(u|_{\partial\Omega})(\underline{x}).$$

Since  $\pi^+(\nu)u = u$  on the boundary it follows that

$$\partial_j u(\underline{x}) = Z(\pi^+(\nu)u|_{\partial\Omega})(\underline{x}) = \left(\partial_j(\pi^+(\nu)u)\right)(\underline{x}).$$

Using (2.12) in the last of the following equalities yields

$$\begin{aligned} \pi^+(\nu)A_j\partial_j u(\underline{x}) &= \pi^+(\nu)A_j\left(\partial_j[\pi^+(\nu)u]\right)(\underline{x}) \\ &= \pi^+(\nu)A_j\left(\partial_j\pi^+(\nu)u(\underline{x}) + \pi^+(\nu)\partial_j u(\underline{x})\right) \\ &= \pi^+(\nu)A_j(\partial_j\pi^+(\nu))u(\underline{x}). \end{aligned} \quad (2.14)$$

**The perturbation theory step.** Use perturbation theory to compute the term  $\partial_j\pi^+$  in the last expression. Denote by  $Q(\xi)$  the partial inverse of  $A(\xi) - |\xi|I$  associated to the eigenvalue  $+|\xi|$ . It is defined by

$$Q(\xi)(A(\xi) - |\xi|I) = I - \pi^+(\xi), \quad Q(\xi)\pi^+(\xi) = 0.$$

Writing

$$A(\xi) - |\xi|I = (|\xi|\pi^+ - |\xi|\pi^-) - (|\xi|\pi^+ + |\xi|\pi^-) = -2|\xi|\pi^-$$

shows that  $Q = (-2|\xi|)^{-1}\pi^-(\xi)$ .

First order perturbation theory ( Theorem 3.I.2 in [23], or formulas (II.2.13), (II.2.33) in [19]) implies that

$$\frac{\partial}{\partial x_j} \left( \pi^+(A(\nu)) \right) = -\pi^+(\nu) \left( \frac{\partial A(\nu)}{\partial x_j} \right) Q(\nu) - Q(\nu) \left( \frac{\partial A(\nu)}{\partial x_j} \right) \pi^+(\nu). \quad (2.15)$$

**Endgame.** When (2.15) is injected in (2.14) the contribution of the first term vanishes thanks to (2.12). Turn next to

$$\frac{\partial}{\partial x_j} A(\nu(x)) = A \left( \frac{\partial \nu}{\partial x_j} \right).$$

The principal curvature formulas imply that at  $\underline{x}$ ,

$$\frac{\partial \nu}{\partial x_j} = \kappa_j(\underline{x}) \mathbf{e}_j, \quad \text{for } j = 2, 3, \quad \text{so,} \quad A \left( \frac{\partial \nu}{\partial x_j} \right) = \kappa_j(\underline{x}) A_j.$$

Therefore (2.14) yields

$$\pi^+(\nu) A_j \partial_j u(\underline{x}) = \kappa_j(\underline{x}) \pi^+(\nu) A_j \pi^-(\nu) A_j \pi^+(\nu).$$

Compute using (2.12) and omitting the argument  $\nu(\underline{x})$  for ease of reading yields

$$\pi^+ A_j \pi^- A_j \pi^+ = \pi^+ A_j (\pi^- + \pi^+) A_j \pi^+ = \pi^+ A_j A_j \pi^+ = \pi^+ \pi^+ = \pi^+.$$

Therefore

$$\pi^+(\nu(\underline{x})) A_j \partial_j u(\underline{x}) = \kappa_j(\underline{x}) \pi^+(\nu(\underline{x})) u, \quad \text{for } j = 2, 3. \quad (2.16)$$

The sum of the terms (2.16) is equal to  $(\kappa_2 + \kappa_3) \pi^+ u = 2H_\Omega \pi^+ u$ . This yields

$$\pi^+(\nu(\underline{x})) \left( \nu \cdot \nabla_x + 2H_\Omega(\underline{x}) \right) u.$$

This completes the proof of (2.10).  $\square$

### 2.3.2 Transverse identity for stretched Pauli for $\tau \in ]m, \infty[$

**Definition 2.8 i.** For  $\tau \in \mathbb{C} \setminus \{0\}$  the coordinate stretchings  $X_j(\tau, x_j)$  are defined as the solutions of the ordinary differential equation in  $x_j$ ,

$$\frac{\partial X_j}{\partial x_j} = \frac{\tau + \sigma_j(x_j)}{\tau}, \quad X_j(0) = 0. \quad (2.17)$$

ii. For real  $\tau > 0$ ,  $\partial_j X_j > 0$  and  $x \mapsto X(\tau, x)$  is a diffeomorphism from  $\mathbb{R}^3$  onto itself. Denote by  $\underline{\Omega} \subset \mathbb{R}_X^d$  the image of  $\Omega \subset \mathbb{R}_x^d$ .

**Example 2.2** In the standard implementation of Example 1.1, the  $\sigma_j$  vanish on  $\ell\mathcal{Q}$ . Therefore  $X$  is equal to the identity on that set.

Compute for real  $\tau > 0$ ,

$$\frac{\partial}{\partial x_j} = \sum_k \frac{\partial X_k}{\partial x_j} \frac{\partial}{\partial X_k} = \frac{\tau + \sigma_j(x_j)}{\tau} \frac{\partial}{\partial X_j}, \quad \frac{\tau}{\tau + \sigma_j(x_j)} \frac{\partial}{\partial x_j} = \frac{\partial}{\partial X_j}. \quad (2.18)$$

Equation (2.18) gives a geometric interpretation of the stretched operator  $L(\tau, \tilde{\partial})$  for  $\tau \in \mathbb{R}_+$ . It shows that  $\tilde{\partial}_j$  in the  $x$  coordinates is equal to  $\partial/\partial X_j$  in the  $X$  coordinates. Therefore if  $u(x)$  and  $v(X)$  are related by  $v(X(\tau, x)) = u(x)$  then  $L(\tau, \tilde{\partial})u(x) = (L(\tau, \partial_X)v)(X(\tau, x))$ .

To find the conormals to  $\underline{\Omega}$ , compute

$$\sum_j \nu_j dx_j = \sum_j \nu_j \sum_k \frac{\partial x_j}{\partial X_k} dX_k = \sum_j \nu_j \frac{\partial x_j}{\partial X_j} dX_j = \sum_j \frac{\nu_j \tau}{\tau + \sigma_j} dX_j.$$

$\sum_j \nu_j dx_j$  annihilates the tangent space to  $\partial\Omega$  at  $x$ . The map  $x \rightarrow X$  takes the tangent space to  $\Omega$  to the tangent space to  $\underline{\Omega}$ . Therefore,  $\sum_j \nu_j \tau / (\tau + \sigma_j) dX_j$  annihilates the tangent space to  $\underline{\Omega}$  at  $X(x)$ . It is therefore a conormal to  $\underline{\Omega}$ . The unit conormal  $\nu_{\underline{\Omega}}(X)$  is

$$\nu_{\underline{\Omega}}(X) = \left( \sum_j \frac{\nu_j^2(x(X)) \tau^2}{(\tau + \sigma_j(x(X)))^2} \right)^{-1/2} \left( \sum_{j=1}^3 \frac{\nu_j(x(X)) \tau}{\tau + \sigma_j(x(X))} dX_j \right).$$

**Definition 2.9** For  $\text{Re } \tau > 0$  and  $x$  on a neighborhood of  $\partial\Omega$ , define the first order differential operator  $V$  by

$$V(\tau, x, \partial) := \left( \sum_j \frac{\nu_j^2 \tau^2}{(\tau + \sigma_j)^2} \right)^{-1/2} \sum_j \frac{\nu_j \tau^2}{(\tau + \sigma_j)^2} \frac{\partial}{\partial x_j}. \quad (2.19)$$

**Remark 2.1 i.** For  $\tau \in ]0, \infty[$ ,  $V$  is a unit vector field transverse to  $\partial\Omega$  since its scalar product with the unit outward normal  $\nu \cdot \partial$  is strictly positive.

**ii.** For  $\tau$  not real, the coefficients of  $V$  are not real, so  $V$  is not a vector field.

**iii.** There is an  $R > 0$  independent of  $\delta$  so that for  $|\tau| > R$ ,  $\partial\Omega$  is non characteristic for  $V$ . Indeed,  $V - \nu \cdot \partial$  has coefficients  $O(1/\tau)$  and the boundary is noncharacteristic for  $\nu \cdot \partial$ .



**Corollary 2.10** *There is an  $m > 0$  so that if  $\tau \in ]m, \infty[$  and  $u \in H^2(\Omega)$  satisfies the boundary condition*

$$u \in \mathcal{E}^+(\tilde{\nu}(\tau, x)) \quad \text{on} \quad \partial\Omega, \quad (2.20)$$

*then with  $V(\tau, x, \partial)$  from (2.19),*

$$\pi^+(\tilde{\nu}) \sum_{j=1}^3 A_j \tilde{\partial}_j u = \pi^+(\tilde{\nu}) \left( V(\tau, \underline{x}, \partial) + 2H_{\underline{\Omega}}(X(\tau, \underline{x})) \right) u \quad \text{on} \quad \partial\Omega. \quad (2.21)$$

**Remark 2.2** *The normal matrix of the stretched system is equal to  $A(\tilde{\nu})$ . For positive  $\tau$ , the boundary condition in (2.20) is the natural maximally absorbing one.*

**Proof of Corollary 2.10.** Define  $v : \underline{\Omega} \rightarrow \mathbb{C}^2$  by  $v(X) := u(x(X))$ . Since  $u$  satisfies the stretched Pauli system on a neighborhood of  $\partial\Omega$ , (2.18) implies that  $v$  satisfies the unstretched Pauli system on a neighborhood of  $\partial\underline{\Omega}$ .

The unstretched differential equation satisfied by  $v$  has principle symbol  $\sum_j A_j \partial/\partial X_j$ . The symbol at any outward conormal vector to  $\underline{\Omega}$  is equal to a positive multiple of  $\sum_j A_j \nu_j \tau / (\tau + \sigma_j)$ . This sum is equal to the symbol of the stretched operator on  $\Omega$  at the conormal  $\nu$  to  $\Omega$ . Thus the positive eigenspace of the unstretched symbol at  $\nu_{\underline{\Omega}}(X)$  is equal to the positive eigenspace of the stretched operator at  $\nu_{\Omega}(x)$ .

The boundary condition satisfied by  $u$  asserts that

$$u \in \mathcal{E}^+(A(\tilde{\nu})) = \mathcal{E}^+(A(\nu_{\underline{\Omega}})).$$

Therefore  $v$  satisfies the boundary condition  $v|_{\partial\underline{\Omega}} \in \mathcal{E}^+(A(\nu_{\underline{\Omega}}))$ . The function  $v$  on  $\underline{\Omega}$  therefore satisfies the hypotheses of Proposition 2.7 on  $\underline{\Omega}$ . That Proposition implies that for  $X \in \partial\underline{\Omega}$ ,

$$\pi^+(\tilde{\nu}(x(X))) \sum_{j=1}^3 A_j \tilde{\partial}_j u = \pi^+(\tilde{\nu}(x(X))) \left( \nu_{\underline{\Omega}} \cdot \partial_X + 2H_{\underline{\Omega}}(X) \right) v.$$

Equation (2.18) shows that

$$\nu_{\underline{\Omega}} \cdot \partial_X = \left( \sum_j \frac{\nu_j^2}{(\tau + \sigma_j)^2} \right)^{-1/2} \sum_j \frac{\nu_j}{\tau + \sigma_j} \frac{\tau}{\tau + \sigma_j} \frac{\partial}{\partial x_j} = V.$$

Inserting in the preceding equation yields (2.21).  $\square$

### 2.3.3 Transverse identity for stretched Pauli for $\tau \notin \mathbb{R}$

Part **iv** of the next proposition is the key identity for complex  $\tau$ .

**Proposition 2.11 i.** *There is an  $R_1 > 1$  so that for  $|\tau| > R_1$  the spectrum of  $A(\tilde{\nu}(\tau, x))$  consists of one simple eigenvalue in  $|z - 1| < 1$  and a second in  $|z - (-1)| < 1$ . Then the map  $\tau \mapsto \pi^\pm(A(\tilde{\nu}(\tau, x)))$  is analytic in  $|\tau| > R_1$ .*

**ii.** *There is an  $R_2 \geq R_1$  so that the function  $\tau \mapsto \nu_\Omega(X(\tau, x))$  from  $]m, \infty[$  to  $C^\infty(\partial\Omega)$  has a holomorphic extension to  $\{|\tau| > R_2\}$ .*

**iii.** *There is an  $R_3 \geq R_2$  so that the function  $\tau \mapsto H_\Omega(X(\tau, x))$  from  $]m, \infty[$  to  $C^\infty(\partial\Omega)$  has a holomorphic extension to  $\{|\tau| > R_3\}$ .*

**iv.** *If  $|\tau| > R_3$  and  $\tau \mapsto u(\tau) \in H^2(\Omega)$  satisfies  $u \in \mathcal{E}^+(\tilde{\nu})$  on  $\partial\Omega$  and is holomorphic on a connected open subset of  $|\tau| > R_3$  that meets the real axis in an open set, then (2.21) holds on that open set.*

**Proof. i.** For  $|\tau|$  large one has uniformly for  $x \in \partial\Omega$ ,

$$\tilde{\nu} = \left( \frac{\nu_1 \tau}{\tau + \sigma_1}, \frac{\nu_2 \tau}{\tau + \sigma_2}, \frac{\nu_3 \tau}{\tau + \sigma_3} \right) = \nu + O(|\tau|^{-1}),$$

The assertions in **i** follows from Part **ii** of Proposition 2.1.

**ii.** It suffices to construct the analytic continuation for points in a neighborhood of each  $\underline{X} \in \partial\Omega$ . Suppose that  $\underline{X} = X(\tau, \underline{x})$  with  $\tau > 0$  and  $\underline{x} \in \partial\Omega$  and  $X$  the stretching transformation defined by (2.17). The map  $\tau \mapsto X(\tau, \cdot)$  is holomorphic on  $\tau \neq 0$  with values in  $C^\infty(\partial\Omega; \mathbb{C})$ . In addition,  $\partial X / \partial x = I + O(1/\tau)$ , so  $\partial X / \partial x$  is invertible for  $|\tau| > R$ .

Suppose that  $x(\alpha_1, \alpha_2)$  is a parametrization of a neighborhood of  $\underline{x}$  in  $\partial\Omega$ . Then for  $\tau > 0$ ,  $X(\tau, x(\alpha_1, \alpha_2))$  is a parametrization of a neighborhood of  $\underline{X}$  in  $\partial\Omega$ . For those  $\tau$  the tangent space to  $\partial\Omega$  is spanned by the independent vectors  $\partial X(\tau, x(\alpha))/\partial\alpha_i$ ,  $1 = 1, 2$ . Thanks to the invertibility of  $\partial X / \partial x$ , the formula

$$\text{Span} \left\{ \frac{\partial X(\tau, x(\alpha))}{\partial\alpha_1}, \frac{\partial X(\tau, x(\alpha))}{\partial\alpha_2} \right\} = \text{Span} \left\{ \frac{\partial X}{\partial x} \frac{\partial x}{\partial\alpha_1}, \frac{\partial X}{\partial x} \frac{\partial x}{\partial\alpha_2} \right\} \quad (2.22)$$

shows that the tangent space has a holomorphic continuation to  $|\tau| > R$ .

For real  $\tau$  a normal vector to  $\Omega$  at  $X(\tau, x(\alpha))$  is given by

$$\frac{\partial X}{\partial x} \frac{\partial x}{\partial\alpha_1} \wedge \frac{\partial X}{\partial x} \frac{\partial x}{\partial\alpha_2}.$$

It is nonvanishing because  $\partial X/\partial x$  is invertible and the vectors  $\partial x/\partial \alpha_j$  are independent. The unit normal vector is given by

$$\nu(X(\tau, x(\alpha))) = \frac{\frac{\partial X}{\partial x} \frac{\partial x}{\partial \alpha_1} \wedge \frac{\partial X}{\partial x} \frac{\partial x}{\partial \alpha_2}}{[\sum_i ((\frac{\partial X}{\partial x} \frac{\partial x}{\partial \alpha_1} \wedge \frac{\partial X}{\partial x} \frac{\partial x}{\partial \alpha_2})_i)^2]^{1/2}}$$

Since  $\partial X/\partial x = I + O(1/\tau)$  it follows that one can choose  $R > 0$  so that

$$\sum_i \left( \left( \frac{\partial X}{\partial x} \frac{\partial x}{\partial \alpha_1} \wedge \frac{\partial X}{\partial x} \frac{\partial x}{\partial \alpha_2} \right)_i \right)^2$$

has strictly positive real part for  $|\tau| > R$ . With that choice the expression for  $\nu(X(\tau, x(\alpha)))$  yields an analytic continuation of the unit normal vector to  $|\tau| > R$ . For nonreal values of  $\tau$ ,  $\nu(X(\tau, x(\alpha)))$  need not be real and need not be of unit length.

**iii.** For  $\tau$  real the Weingarten map is the map from  $T_{\underline{X}}(\partial\Omega)$  to itself that maps the two basis vectors as follows,

$$\frac{\partial X(\tau, x(\alpha))}{\partial \alpha_j} \rightarrow \frac{\partial \nu_{\underline{\Omega}}(X(\tau, x(\alpha)))}{\partial \alpha_j}, \quad j = 1, 2. \quad (2.23)$$

The holomorphic extension of  $\nu$  implies that the Weingarten map extends holomorphically to a family of linear map of the holomorphic family of two dimensional spaces (2.22) to itself.

For  $\tau$  real the mean curvature  $H_{\underline{\Omega}}$  is equal to one half of the trace of the Weingarten map. The preceding paragraph shows that this trace has a holomorphic continuation proving **iii**.

**iv.** The difference of the two sides of (2.21) is holomorphic on a connected set. Corollary 2.10 implies that it vanishes for  $\tau$  on the open intersection with the real axis. By analytic continuation it vanishes identically.  $\square$

**Corollary 2.12** *If  $u \in H^2(\Omega)$  satisfies  $L(\tau, \tilde{\partial})u = 0$  on  $\partial\Omega$  and  $u \in \mathcal{E}^+(\tilde{\nu})$  on  $\partial\Omega$ , then*

$$\pi^+(\tilde{\nu}) \left( V(\tau, x, \partial) + \tau + 2H_{\underline{\Omega}}(X(\tau, \underline{x})) \right) u = 0 \quad \text{on} \quad \partial\Omega. \quad (2.24)$$

**Proof.** Equation (2.21) implies that

$$\pi^+(\tilde{\nu}) L(\tau, \tilde{\partial})u = \pi^+(\tilde{\nu}) \left( V(\tau, x, \partial) + \tau + 2H_{\underline{\Omega}}(X(\tau, \underline{x})) \right) u \quad \text{on} \quad \partial\Omega.$$

Since  $L(\tau, \tilde{\partial})u = 0$  on  $\partial\Omega$ ,  $u$  satisfies (2.24).  $\square$

### 3 The Pauli-Helmoltz system

#### 3.1 The Helmholtz operator

Derive a Helmholtz equation that is satisfied by all solutions of the stretched equations. For  $i \neq j$  the anticommutation formulas (2.1) imply that

$$A_i \tilde{\partial}_i A_j \tilde{\partial}_j + A_j \tilde{\partial}_j A_i \tilde{\partial}_i = 0, \quad \text{for } i \neq j.$$

Indeed, when the derivatives fall on variable coefficients they yield zero. Define

$$\tilde{\partial}_j^2 := \left( \frac{\tau}{\tau + \sigma_j(x_j)} \partial_j \right) \left( \frac{\tau}{\tau + \sigma_j(x_j)} \partial_j \right)$$

where the order of the operators inside the parentheses is important. The following stretched versions of (2.2) hold,

$$\begin{aligned} \left( \sum_j A_j \tilde{\partial}_j \right)^2 &= - \sum_j \tilde{\partial}_j^2, \\ \left( \sum_j A_j \tilde{\partial}_j - \tau \right) \left( \sum_j A_j \tilde{\partial}_j + \tau \right) &= \sum_j \tilde{\partial}_j^2 - \tau^2. \end{aligned} \quad (3.1)$$

The second equation in (3.1) shows that where a function  $u$  satisfies  $L(\tau, \tilde{\partial})u = 0$ , it satisfies the elliptic equation  $(\sum_j \tilde{\partial}_j^2 - \tau^2)u = 0$ . The next lemma gives a divergence form equation.

**Definition 3.1** *Define*

$$p(\tau, x, \partial)u := \sum_{j=1}^3 \partial_j \frac{(\tau + \sigma_{j+1}(x_{j+1}))(\tau + \sigma_{j+2}(x_{j+2}))}{\tau(\tau + \sigma_j(x_j))} \partial_j u, \quad (3.2)$$

and

$$\Pi(\tau, x) := \prod_{i=1}^3 \frac{\tau + \sigma_i(x_i)}{\tau}. \quad (3.3)$$

**Lemma 3.2** *As operators on  $H_{loc}^2(\mathcal{Q})$ ,*

$$\Pi(\tau, x) \left( \sum_j A_j \tilde{\partial}_j - \tau \right) \left( \sum_j A_j \tilde{\partial}_j + \tau \right) = p(\tau, x, \partial) - \tau^2 \Pi(\tau, x). \quad (3.4)$$

**Proof.** Expanding the product on the left using the anticommutation relations (2.1) yields

$$\sum_j \left( \prod_{i=1}^3 \frac{\tau + \sigma_i(x_i)}{\tau} \right) \left( \frac{\tau}{\tau + \sigma_j(x_j)} \partial_j \right) \left( \frac{\tau}{\tau + \sigma_j(x_j)} \partial_j \right) - \tau^2 \prod_{i=1}^3 \frac{\tau + \sigma_i(x_i)}{\tau}.$$

The factor before the first derivative on the left is equal to

$$\left( \prod_{i=1}^3 \frac{\tau + \sigma_i(x_i)}{\tau} \right) \frac{\tau}{\tau + \sigma_j(x_j)} = \frac{(\tau + \sigma_{j+1}(x_{j+1}))(\tau + \sigma_{j+2}(x_{j+2}))}{\tau^2}.$$

This function does not depend on  $x_j$  so commutes with  $\partial_j$ .

$$\begin{aligned} & \left( \prod_{i=1}^3 \frac{\tau + \sigma_i(x_i)}{\tau} \right) \left( \frac{\tau}{\tau + \sigma_j(x_j)} \partial_j \right) \left( \frac{\tau}{\tau + \sigma_j(x_j)} \partial_j \right) \\ &= \partial_j \left( \frac{(\tau + \sigma_{j+1}(x_{j+1}))(\tau + \sigma_{j+2}(x_{j+2}))}{\tau^2} \frac{\tau}{\tau + \sigma_j(x_j)} \right) \partial_j \\ &= \partial_j \left( \frac{(\tau + \sigma_{j+1}(x_{j+1}))(\tau + \sigma_{j+2}(x_{j+2}))}{\tau(\tau + \sigma_j(x_j))} \right) \partial_j. \end{aligned}$$

This completes the proof.  $\square$

**Remark 3.1 i.** The factors in the product on the left of (3.4) are

$$\sum_j A_j \tilde{\partial}_j + \tau = L(\tau, \tilde{\partial}), \quad \text{and,} \quad \sum_j A_j \tilde{\partial}_j - \tau = L(-\tau, \tilde{\partial}). \quad (3.5)$$

ii. Since

$$\left| \Pi(\tau, x) - 1 \right| = \left| \prod_{i=1}^3 \frac{\tau + \sigma_i(x_i)}{\tau} - 1 \right| \lesssim \frac{1}{|\tau|} \quad (3.6)$$

the coefficients of the operator on the right of (3.4) differ from those of the classical Helmholtz operator  $\Delta - \tau^2$  by  $O(|\tau|^{-1})$ .

**Definition 3.3** • For vectors  $\alpha, \beta$  in  $\mathbb{C}^k$  define  $\alpha \cdot \beta := \sum_j \alpha_j \beta_j$ .

• Define the continuous bilinear form  $a : H^1(\mathcal{Q}; \mathbb{C}^2) \times H^1(\mathcal{Q}; \mathbb{C}^2) \rightarrow \mathbb{C}$  associated to  $-p$  by

$$a(u, v) = \int_{\mathcal{Q}} \sum_{j=1}^3 \frac{(\tau + \sigma_{j+1}(x_{j+1}))(\tau + \sigma_{j+2}(x_{j+2}))}{\tau(\tau + \sigma_j(x_j))} \partial_j u \cdot \partial_j v \, dx. \quad (3.7)$$

• If  $\Omega \subset \mathcal{Q}$  is open the formula with integration over  $\Omega$  defines a continuous form from  $H_{loc}^1(\Omega) \times H_{compact}^1(\Omega) \rightarrow \mathbb{C}$ .

**Remark 3.2 i.** If  $u \in H_{loc}^1(\Omega)$  and  $f \in H_{loc}^{-1}(\Omega)$  then  $u$  satisfies  $pu = f$  on  $\Omega$  if and only if

$$\forall \phi \in C_0^\infty(\Omega), \quad a(u, \phi) = - \int_{\Omega} f \cdot \phi \, dx .$$

**ii.** Multiplying numerator and denominator of the coefficient of  $\partial_j$  in (3.7) by  $\tau + \sigma_j$  shows that

$$a(u, v) = \int_{\Omega} \Pi(\tau, x) \sum_{j=1}^3 \frac{\tau^2}{(\tau + \sigma_j)^2} \partial_j u \cdot \partial_j v \, dx . \quad (3.8)$$

**iii.** If  $u \in H^2(\Omega)$ , an integration by parts yields

$$a(u, v) = - \int_{\Omega} pu \cdot v \, dx + \int_{\partial\Omega} \Pi(\tau, x) \sum_{j=1}^3 \frac{\nu_j \tau^2}{(\tau + \sigma_j)^2} \partial_j u \cdot v \, d\Sigma . \quad (3.9)$$

To solve the stretched equation, start by using (3.4) to show that any solution must satisfy the Helmholtz equation

$$\left( p(\tau, x, \partial) - \tau^2 \Pi(\tau, x) \right) u = \Pi(\tau, x) \left( \sum A_j \tilde{\partial}_j - \tau \right) F . \quad (3.10)$$

**Remark 3.3** There is an extensive literature on using the PML technology for the solution of time harmonic scattering problems for the wave equation beginning with Collino-Monk and Lassas-Somersalo [11, 20, 21, 8, 9]. All depend on choosing  $\sigma_j$  constant outside a compact set and then relying on an explicit Green's function for the Helmholtz operator  $\tau^2 - p$  with  $\tau = i\omega$  and  $x$  outside that compact set. Rellich's Uniqueness Theorem and the exponential decay of the Green's function drives the analysis. The operator  $p$  and the form  $a(\cdot, \cdot)$  appear in those articles. Variable  $\sigma_j$ , corners, and absorbing boundary conditions at trihedral corners have no analogue in their work. This time harmonic work is related to the method of complex scaling in Scattering Theory introduced by Balslev-Coombes [6] and raised to high art by Sjöstrand and a brilliant school (see [14]).

## 3.2 The Helmholtz boundary value problem, Theorem 3.7

Equation (3.10) is supplemented by boundary conditions. The goal is to prove Theorem 2.5 so have  $u \in \mathcal{E}^+(\tilde{\nu})$ , equivalently  $\pi^-(\tilde{\nu})u = 0$ . Corollary

2.12 provides the second boundary condition. The present section is devoted to studying the resulting Helmholtz boundary value problem,

$$\begin{aligned} \left( \tau^2 \Pi(\tau, x) - p(\tau, x, \partial) \right) u &= f \quad \text{on } \mathcal{Q}_\delta, \\ \pi^-(\tilde{v}(\tau, x)) u &= g_1 \quad \text{on } \partial \mathcal{Q}_\delta, \\ \pi^+(\tilde{v}(\tau, x)) \left( V(\tau, x, \partial) + \tau + 2H_{\underline{\Omega}}(X(\tau, \underline{x})) \right) u &= g_2 \quad \text{on } \partial \mathcal{Q}_\delta. \end{aligned} \quad (3.11)$$

Here  $g_1$  and  $g_2$  are functions on  $\partial\Omega$  that take values in  $\mathcal{E}^-(A(\tilde{v}(\tau, x)))$  and  $\mathcal{E}^+(A(\tilde{v}(\tau, x)))$  respectively.

**Definition 3.4** For  $S \in \text{Hom } \mathbb{C}^k$  denote by  $S^\dagger$  the transposed matrix so that  $Su \cdot v = u \cdot S^\dagger v$  for all vectors  $u, v \in \mathbb{C}^k$ .

For  $|\text{Im } \xi| < |\text{Re } \xi|$ ,  $A(\xi)$  has two eigenvalues  $\lambda^\pm(\xi)$  and spectral representation

$$A(\xi) = \lambda^+ \pi^+(\xi) + \lambda^- \pi^-(\xi), \quad \text{so,} \quad A(\xi)^\dagger = \lambda^+ \pi^+(\xi)^\dagger + \lambda^- \pi^-(\xi)^\dagger.$$

Therefore  $\lambda^\pm$  are eigenvalues of  $A(\xi)^\dagger$  and  $\pi^\pm(\xi)^\dagger$  are the corresponding spectral projections.

**Definition 3.5** Define the transposed boundary value problem as,

$$\begin{aligned} \left( \tau^2 \Pi(\tau, x) - p(\tau, x, \partial) \right) u &= f \quad \text{on } \mathcal{Q}_\delta, \\ \pi^-(\tilde{v}(\tau, x))^\dagger u &= g_1 \quad \text{on } \partial \mathcal{Q}_\delta, \\ \pi^+(\tilde{v}(\tau, x))^\dagger \left( V(\tau, x, \partial) + \tau + 2H_{\underline{\Omega}}(X(\tau, \underline{x})) \right) u &= g_2 \quad \text{on } \partial \mathcal{Q}_\delta. \end{aligned} \quad (3.12)$$

The  $g_1$  and  $g_2$  are functions on  $\partial\Omega$  taking values in  $\mathcal{E}^-(A(\tilde{v})^\dagger)$  and  $\mathcal{E}^+(A(\tilde{v})^\dagger)$  respectively.

The annihilator of the range of the direct problem is equal to the nullspace of the transposed problem (see Section 3.3.3).

**Lemma 3.6** There is an  $R > 0$  independent of  $\delta$  to that for  $|\tau| > R$  the boundary value problems (3.11) and (3.12) satisfy Lopatinski's condition for all  $x \in \partial \mathcal{Q}_\delta$ .

**Proof.** Treat only (3.11). The proof for the other is nearly identical.

The Lopatinski condition concerns only the leading order parts of the operators. In addition as  $|\tau| \rightarrow \infty$ ,  $p(\tau, x, \partial) \rightarrow \Delta$  and  $V(\tau, x, \partial) \rightarrow \nu_{\mathcal{Q}_\delta} \cdot \partial$ . Thus it suffices to prove Lopatinski's condition for the constant coefficient half space problems,

$$\begin{aligned} \Delta u &= f \quad \text{on } \nu \cdot x < 0, \\ \pi^-(\nu)u &= g_1 \quad \text{on } \nu \cdot x = 0, \\ \pi^+(\nu)(\nu \cdot \partial_x u) &= g_2 \quad \text{on } \nu \cdot x = 0. \end{aligned} \tag{3.13}$$

Though (3.11) is not rotation invariant, (3.13) is rotation invariant. It therefore suffices to consider (3.13) with  $\nu = (-1, 0, 0)$ . That yields the boundary value problem

$$\begin{aligned} \Delta u &= f \quad \text{on } x_1 > 0, \\ u_1 &= g_1 \quad \text{on } x_1 = 0, \\ \partial_{x_1} u_2 &= g_2 \quad \text{on } x_1 = 0. \end{aligned} \tag{3.14}$$

This is the Dirichlet problem for  $u_1$  and the Neumann problem for  $u_2$ . Lopatinski's condition is known for each of them.  $\square$

**Theorem 3.7** *There is an  $M > 0$  so that if  $\operatorname{Re} \tau > M$  and  $0 < \delta < 1$ , then the continuous linear map*

$$H^2(\mathcal{Q}_\delta) \ni u \mapsto (f, g_1, g_2) \in L^2(\mathcal{Q}_\delta) \times H^{3/2}(\partial\mathcal{Q}_\delta; \mathcal{E}^-(\tilde{\nu})) \times H^{1/2}(\partial\mathcal{Q}_\delta; \mathcal{E}^+(\tilde{\nu}))$$

*defined by (3.11) is one to one and onto.*

**Strategy of the proof of Theorem 3.7.** The theory of elliptic boundary value problems satisfying Lopatinski's condition implies the following facts, see [2, 3].

- The kernel of the map is a finite dimensional subset of  $C^\infty(\overline{\mathcal{Q}_\delta})$ .
- The range is closed with finite codimension.
- The annihilator of the range is a subspace of  $C^\infty(\overline{\mathcal{Q}_\delta}) \times C^\infty(\partial\mathcal{Q}_\delta) \times C^\infty(\partial\mathcal{Q}_\delta)$ .

*To prove the theorem it suffices to prove that the kernel and the annihilator of the range are both trivial.*

### 3.3 Main a priori estimate, Theorem 3.8

**Theorem 3.8** *There are constants  $C, M$  independent of  $\delta \in ]0, 1[$  and  $\tau \in \{\operatorname{Re} \tau > M\}$  so that if  $u \in H^2(\mathcal{Q}_\delta)$  satisfies the direct problem (3.11) (resp.*



the transposed problem (3.12) with  $g_1 = 0$  and  $g_2 = 0$  then

$$|\tau| (\operatorname{Re} \tau) \|u\|_{L^2(\mathcal{Q}_\delta)}^2 + \frac{\operatorname{Re} \tau}{|\tau|} \left( \|\beta|^{1/2} u\|_{L^2(\partial\mathcal{Q}_\delta)}^2 + \|\nabla_x u\|_{L^2(\mathcal{Q}_\delta)}^2 \right) \leq C \left| \int_{\mathcal{Q}_\delta} f \bar{u} \, dx \right|. \quad (3.15)$$

**Definition 3.9** • Using the analytic continuation  $H_{\underline{\Omega}}(X(\tau, x))$  from Part iii of Proposition 2.11, define  $\Phi, \beta \in C^\infty(\{\operatorname{Re} \tau > M\} \times ]0, 1[ \times \partial\Omega)$  by

$$\begin{aligned} \Phi(\tau, x) &:= \Pi(\tau, x) \left( \sum_j \frac{\nu_j^2 \tau^2}{(\tau + \sigma_j)^2} \right)^{1/2}, \\ \beta(\tau, \delta, x) &:= \tau + 2H_{\underline{\Omega}}(X(\tau, x)). \end{aligned} \quad (3.16)$$

• With  $\operatorname{Re} \tau > M$  and  $a(u, v)$  from (3.7), define continuous bilinear forms  $\mathcal{A}(\tau, \cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$  by

$$\mathcal{A}(\tau, u, v) := a(u, v) + \int_{\Omega} \tau^2 \Pi(\tau, x) u \cdot v \, dx + \int_{\partial\Omega} \Phi \beta u \cdot v \, d\Sigma. \quad (3.17)$$

The proof of Theorem 3.8 relies on two estimates for  $\mathcal{A}$ . The first is a lower bound for  $\mathcal{A}(u, \bar{u})$  that holds for all  $u \in H^1(\Omega)$ . The second is an upper bound that relies on the boundary conditions. The dependence of  $\mathcal{A}$  on  $\Omega$  and therefore  $\delta$  is suppressed. Similarly, the dependence of  $\mathcal{A}$  on  $\tau$  is usually not indicated.

**Lemma 3.10** If  $u \in H^2(\Omega)$  and  $v \in H^1(\Omega)$ , define  $f := (\tau^2 \Pi - p)u$ . Then,

$$\mathcal{A}(\tau, u, v) - \int_{\Omega} f \cdot v \, dx = \int_{\partial\Omega} \Phi(\tau, x) (V + \beta(\tau, \delta, x)) u \cdot v \, d\Sigma. \quad (3.18)$$

**Proof.** The differential operator appearing in the boundary term of Green's formula (3.9) is related to the operator  $V(\tau, x, \partial)$  associated to the natural boundary condition for the stretched Pauli system by

$$\begin{aligned} \Pi(\tau, x) \sum_{j=1}^3 \frac{\nu_j \tau^2}{(\tau + \sigma_j)^2} \partial_j &= \Pi(\tau, x) \left( \sum_{j=1}^3 \left( \frac{\nu_j \tau^2}{(\tau + \sigma_j)^2} \right)^2 \right)^{1/2} V(\tau, x, \partial) \\ &= \Phi(\tau, x) V(\tau, x, \partial). \end{aligned}$$

Equation (3.9) shows that

$$\begin{aligned} a(u, v) + \int_{\Omega} \tau^2 \Pi(\tau, x) u \cdot v \, dx - \int_{\Omega} f \cdot v \, dx = \\ \int_{\partial\Omega} \Pi(\tau, x) \sum_{j=1}^3 \frac{\nu_j \tau^2 \partial_j u}{(\tau + \sigma_j)^2} \cdot v \, d\Sigma = \int_{\partial\Omega} \Phi(\tau, x) Vu \cdot v \, d\Sigma. \end{aligned} \quad (3.19)$$

Adding  $\int_{\partial\Omega} \Phi \beta u \cdot v \, d\Sigma$  to both sides proves (3.18).  $\square$

**Example 3.1** *If on  $\partial\Omega$ ,  $u$  satisfies*

$$\pi^-(\tilde{v})u = 0, \quad \text{and} \quad \left( V + \tau + 2H_{\underline{\Omega}}(X(\tau, \underline{x})) \right) u = 0,$$

*and  $v$  satisfies  $\pi^+(\tilde{v})^\dagger v = 0$ , then the boundary term in Lemma 3.10 vanishes. This yields a weak formulation, and a mixed finite element approach to the boundary value problem for  $u$ .*

### 3.3.1 Lower bound for $|\mathcal{A}(u, \bar{u})|$

**Proposition 3.11** *There are constants  $C, M > 0$  independent of  $\delta \in ]0, 1[$  so that for any  $\tau \in \{ \operatorname{Re} \tau \geq M \}$ , and  $u \in H^1(\mathcal{Q}_\delta)$ ,*

$$\begin{aligned} |\tau| (\operatorname{Re} \tau) \|u\|_{L^2(\mathcal{Q}_\delta)}^2 + \\ \frac{\operatorname{Re} \tau}{|\tau|} \left( \|\beta^{1/2} u\|_{L^2(\partial\mathcal{Q}_\delta)}^2 + \|\nabla_x u\|_{L^2(\mathcal{Q}_\delta)}^2 \right) \lesssim |\mathcal{A}(u, \bar{u})|. \end{aligned} \quad (3.20)$$

**Remark 3.4** *In (3.35), we show that  $H_{\underline{\Omega}} = H_{\Omega} + O(1/\tau)$ . Since  $\beta = \tau + 2H_{\underline{\Omega}}(\tau, x)$  it follows that there is an  $M$  independent of  $\delta$  so that for  $\operatorname{Re} \tau > M$*

$$|\tau| + H_{\Omega}(x) \leq |\beta(\tau, \delta, x)| \leq |\tau| + 3H_{\Omega}(x).$$

**Proof. Step 1.  $\mathcal{A}_0$  and its real and imaginary parts.** Denote by  $\mathcal{A}_0$  the form that one would have if  $\sigma_j = 0$  for all  $j$ ,

$$\begin{aligned} \mathcal{A}_0(\tau, u, v) &:= \int_{\mathcal{Q}_\delta} \tau^2 u \cdot v \, dx + \int_{\partial\mathcal{Q}_\delta} \beta u \cdot v \, d\Sigma + \int_{\mathcal{Q}_\delta} \nabla_x u \cdot \nabla_x v \, dx, \\ \mathcal{A}_0(\tau, u, \bar{u}) &:= \int_{\mathcal{Q}_\delta} \tau^2 |u|^2 \, dx + \int_{\partial\mathcal{Q}_\delta} \beta |u|^2 \, d\Sigma + \int_{\mathcal{Q}_\delta} |\nabla_x u|^2 \, dx. \end{aligned}$$

The real part of  $\mathcal{A}_0$  is

$$\begin{aligned} \operatorname{Re} \mathcal{A}_0(u, \bar{u}) &= \left( (\operatorname{Re} \tau)^2 - (\operatorname{Im} \tau)^2 \right) \|u\|_{L^2(\mathcal{Q}_\delta)}^2 \\ &\quad + \|(\operatorname{Re} \beta)^{1/2} u\|_{L^2(\partial \mathcal{Q}_\delta)}^2 + \|\nabla_x u\|_{L^2(\mathcal{Q}_\delta)}^2. \end{aligned} \quad (3.21)$$

Use  $\operatorname{Im} \tau^2 = 2 (\operatorname{Im} \tau)(\operatorname{Re} \tau)$  to find,

$$\begin{aligned} \operatorname{Im} \int_{\mathcal{Q}_\delta} \tau^2 |u|^2 dx &= (\operatorname{Im} \tau) \int_{\mathcal{Q}_\delta} 2 \operatorname{Re} \tau |u|^2 dx, \\ \operatorname{Im} \int_{\partial \mathcal{Q}_\delta} \beta |u|^2 d\Sigma &= (\operatorname{Im} \tau) \int_{\partial \mathcal{Q}_\delta} |u|^2 d\sigma. \end{aligned}$$

Combining shows that for  $0 \neq \operatorname{Im} \tau$ ,

$$\frac{\operatorname{Im} \mathcal{A}_0(u, \bar{u})}{\operatorname{Im} \tau} = 2 (\operatorname{Re} \tau) \|u\|_{L^2(\mathcal{Q}_\delta)}^2 + \|u\|_{L^2(\partial \mathcal{Q}_\delta)}^2. \quad (3.22)$$

**Step 2. Proof for  $\mathcal{A}_0$ .** • The bound (3.20) is proved by combining (3.21) and (3.22). Care is needed where the terms on the right of (3.21) do not have the same sign. Where  $|\operatorname{Im} \tau| < \operatorname{Re} \tau/2$ , (3.21) implies (3.20).

• It suffices to consider the complementary set  $\{|\operatorname{Im} \tau| \geq \operatorname{Re} \tau/2\}$ . In that parameter range (3.22) implies

$$(\operatorname{Re} \tau) \|u\|_{L^2(\mathcal{Q}_\delta)}^2 + \|u\|_{L^2(\partial \mathcal{Q}_\delta)}^2 \lesssim \frac{|\operatorname{Im} \mathcal{A}_0(u, \bar{u})|}{|\tau|}. \quad (3.23)$$

Multiplying by  $|\tau|^2 / \operatorname{Re} \tau$  yields

$$|\tau|^2 \|u\|_{L^2(\mathcal{Q}_\delta)}^2 + \frac{|\tau|^2}{\operatorname{Re} \tau} \|u\|_{L^2(\partial \mathcal{Q}_\delta)}^2 \lesssim \frac{|\tau|}{\operatorname{Re} \tau} |\operatorname{Im} \mathcal{A}_0(u, \bar{u})|. \quad (3.24)$$

Therefore,

$$\left| (\operatorname{Re} \tau)^2 - (\operatorname{Im} \tau)^2 \right| \|u\|_{L^2(\mathcal{Q}_\delta)}^2 \leq |\tau|^2 \|u\|_{L^2(\mathcal{Q}_\delta)}^2 \lesssim \frac{|\tau|}{\operatorname{Re} \tau} |\mathcal{A}_0(u, \bar{u})|.$$

Using this in (3.21) yields for  $|\tau| > M_1$ ,

$$\|\nabla_x u\|_{L^2(\mathcal{Q}_\delta)}^2 + \|(\operatorname{Re} \beta)^{1/2} u\|_{L^2(\partial \mathcal{Q}_\delta)}^2 \lesssim \frac{|\tau|}{\operatorname{Re} \tau} |\mathcal{A}_0(u, \bar{u})|. \quad (3.25)$$

Adding (3.24) and (3.25) yields

$$\begin{aligned} |\tau|^2 \|u\|_{L^2(\mathcal{Q}_\delta)}^2 + \|\nabla_x u\|_{L^2(\mathcal{Q}_\delta)}^2 + \|(\operatorname{Re} \beta)^{1/2} u\|_{L^2(\partial \mathcal{Q}_\delta)}^2 + \\ + \frac{|\tau|^2}{\operatorname{Re} \tau} \|u\|_{L^2(\partial \mathcal{Q}_\delta)}^2 \lesssim \frac{|\tau|}{\operatorname{Re} \tau} |\mathcal{A}_0(u, \bar{u})|. \end{aligned}$$

Multiply by  $(\operatorname{Re} \tau)/|\tau|$  and use  $|\beta| \leq (\operatorname{Re} \beta) + |\tau|$  to find

$$\begin{aligned} |\tau|(\operatorname{Re} \tau)\|u\|_{L^2(\mathcal{Q}_\delta)}^2 + \frac{\operatorname{Re} \tau}{|\tau|} \left( \|\beta^{1/2} u\|_{L^2(\partial\mathcal{Q}_\delta)}^2 + \|\nabla_x u\|_{L^2(\mathcal{Q}_\delta)}^2 \right) \\ \lesssim |\mathcal{A}_0(u, \bar{u})|. \end{aligned} \quad (3.26)$$

**Step 3. Perturbation argument.** For  $\tau \neq 0$ ,  $\tau + \sigma_j(x_j) = \tau(1 + \sigma_j/\tau)$ . Write

$$\begin{aligned} a(u, \bar{u}) - a_0(u, \bar{u}) &= \int_{\Omega} \left( \frac{(\tau + \sigma_{j+1})(\tau + \sigma_{j+2})}{\tau(\tau + \sigma_j)} - 1 \right) |\partial_j u|^2 dx \\ &\quad + \tau^2 \int_{\Omega} (\Pi(\tau, x) - 1) |u|^2 dx + \int_{\partial\Omega} (\Phi(\tau, x) - 1) |\beta| |u|^2 d\Sigma. \end{aligned}$$

Since  $|\Pi - I| + |\Phi - I| = O(1/\tau)$ , this yields

$$\begin{aligned} |\mathcal{A}(u, \bar{u}) - \mathcal{A}_0(u, \bar{u})| &\lesssim |\tau| \|u\|_{L^2(\mathcal{Q}_\delta)}^2 + \\ &\quad + \frac{1}{|\tau|} \|\beta^{1/2} u\|_{L^2(\partial\mathcal{Q}_\delta)}^2 + \frac{1}{|\tau|} \|\nabla_x u\|_{L^2(\mathcal{Q}_\delta)}^2 \lesssim \frac{1}{\operatorname{Re} \tau} |\mathcal{A}_0(u, \bar{u})|, \end{aligned} \quad (3.27)$$

where inequality (3.26) for  $\mathcal{A}_0$  is used in the last inequality. The triangle inequality and estimate (3.27) imply

$$|\mathcal{A}(u, \bar{u})| \geq \mathcal{A}_0(u, \bar{u}) - |\mathcal{A}(u, \bar{u}) - \mathcal{A}_0(u, \bar{u})| \geq \left(1 - \frac{c}{\operatorname{Re} \tau}\right) |\mathcal{A}_0(u, \bar{u})|.$$

For  $\operatorname{Re} \tau > 2c$  this yields (3.20) completing the proof of Proposition 3.11.  $\square$

### 3.3.2 Upper bound for $|\mathcal{A}(u, \bar{u})|$ , proof of Theorem 3.8

**Proposition 3.12** *If  $u \in H^2(\Omega)$  is a solution of the Helmholtz boundary value problem (3.11) (resp. the transposed problem (3.12)) with  $g_1 = 0$  and  $g_2 = 0$ , then with constant independent of  $\delta \in ]0, 1[$  and  $|\tau| > 1$ ,*

$$|\mathcal{A}(u, \bar{u})| \lesssim \left| \int_{\Omega} f \bar{u} dx \right| + \frac{1}{|\tau|} \left( \|\beta^{1/2} u\|_{L^2(\partial\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 \right).$$

**Proof of Proposition 3.12.** For (3.11) write

$$(V + \beta)u = (\pi^+(\tilde{\nu}) + \pi^-(\tilde{\nu}))(V + \beta)u = \pi^-(\tilde{\nu})(V + \beta)u.$$

For the transposed boundary value problem (3.12) write

$$(V + \beta)u = (\pi^+(\tilde{\nu})^\dagger + \pi^-(\tilde{\nu})^\dagger)(V + \beta)u = \pi^-(\tilde{\nu})^\dagger(V + \beta)u.$$

Continuing the computation for (3.11), Lemma 3.10 yields for  $u, v \in H^1(\Omega)$ ,

$$\mathcal{A}(u, v) = \int_{\Omega} f \cdot v \, dx - \int_{\partial\Omega} \Phi \pi^-(\tilde{\nu}) (V(\tau, x, \partial) + \beta) u \cdot v \, d\Sigma.$$

With  $v = \bar{u}$  this is

$$\mathcal{A}(u, \bar{u}) = \int_{\Omega} f \cdot \bar{u} \, dx - \int_{\partial\Omega} \Phi \pi^-(\tilde{\nu}) (V(\tau, x, \partial) + \beta) u \cdot \bar{u} \, d\Sigma. \quad (3.28)$$

The difficult step is to derive an upper bound for

$$\int_{\partial\Omega} \Phi \pi^-(\tilde{\nu}) (V(\tau, x, \partial) + \beta) u \cdot \bar{u} \, d\Sigma.$$

The boundary condition  $\pi^-(\tilde{\nu})u = 0$  implies  $\pi^+(\tilde{\nu})u = u$  so,

$$\int_{\partial\Omega} \Phi \pi^-(\tilde{\nu}) (V(\tau, x, \partial) + \beta) u \cdot \bar{u} \, d\Sigma = \int_{\partial\Omega} \Phi \pi^-(\tilde{\nu}) (V(\tau, x, \partial) + \beta) u \cdot \overline{\pi^+(\tilde{\nu})u} \, d\Sigma.$$

Write

$$\overline{\pi^+(\tilde{\nu})u} = \overline{\pi^+(\tilde{\nu})\bar{u}} = \pi^+(\tilde{\nu})^\dagger \bar{u} + \left( \overline{\pi^+(\tilde{\nu})} - \pi^+(\tilde{\nu})^\dagger \right) \bar{u}.$$

When this is inserted the  $(\pi^+(\tilde{\nu}))^\dagger \bar{u}$  term yields zero. Therefore

$$\begin{aligned} & \int_{\partial\Omega} \Phi \pi^-(\tilde{\nu}) (V(\tau, x, \partial) + \beta) u \cdot \bar{u} \, d\Sigma \\ &= \int_{\partial\Omega} \Phi \pi^-(\tilde{\nu}) (V(\tau, x, \partial) + \beta) u \cdot \left( \overline{\pi^+(\tilde{\nu})} - \pi^+(\tilde{\nu})^\dagger \right) \bar{u} \, d\Sigma \\ &= \int_{\partial\Omega} \Phi (V(\tau, x, \partial) + \beta) u \cdot \pi^-(\tilde{\nu})^\dagger \left( \overline{\pi^+(\tilde{\nu})} - \pi^+(\tilde{\nu})^\dagger \right) \bar{u} \, d\Sigma \\ &= \int_{\partial\Omega} \Phi (V(\tau, x, \partial) + \beta) u \cdot w \, d\Sigma \end{aligned} \quad (3.29)$$

with

$$w := \pi^-(\tilde{\nu})^\dagger \left( \overline{\pi^+(\tilde{\nu})} - \pi^+(\tilde{\nu})^\dagger \right) \bar{u}.$$

For the transposed problem the difficult boundary term is

$$\int_{\partial\Omega} \Phi \pi^-(\tilde{\nu})^\dagger (V(\tau, x, \partial) + \beta) u \cdot \bar{u} \, d\Sigma = \int_{\partial\Omega} \Phi (V(\tau, x, \partial) + \beta) u \cdot w \, d\Sigma$$

with

$$w := \pi^-(\tilde{\nu}) \left( \overline{\pi^+(\tilde{\nu})^\dagger} - \pi^+(\tilde{\nu}) \right) \bar{u}.$$

The estimates in the two cases are virtually identical. The details are presented only for the direct problem. For the direct problem define  $\mathbf{m} \in C^\infty(\{\operatorname{Re} \tau > M\} \times \partial\Omega)$  by

$$\mathbf{m}(\tau, x) := \tau \pi^-(\tilde{\nu})^\dagger \left( \overline{\pi^+(\tilde{\nu})} - \pi^+(\tilde{\nu})^\dagger \right), \quad \text{so,} \quad w = \frac{1}{\tau} \mathbf{m} \bar{u}. \quad (3.30)$$

Need an upper bound for (3.29). Equation (3.30) shows that this is equal to

$$\frac{1}{\tau} \left( \int_{\partial\Omega} \Phi V u \cdot \mathbf{m} \bar{u} \, dx + \Phi \beta u \cdot \mathbf{m} \bar{u} \, d\Sigma \right). \quad (3.31)$$

The next lemma gathers estimates for  $V$  and  $\mathbf{m}$ .

**Lemma 3.13** *There are constants  $C, M$  so that for all  $\operatorname{Re} \tau > M$ , and,  $0 < \delta < 1$ , the following hold.*

- i.  $\operatorname{supp} \mathbf{m} \subset \{x \in \partial\Omega : \operatorname{dist}(x, \mathcal{S}) < \delta\}$ .
- ii.  $\|\mathbf{m}(\tau, x)\|_{L^\infty(\partial\Omega)} \leq C$ .
- iii.  $\|\nabla_x \mathbf{m}(\tau, x)\|_{L^\infty(\partial\Omega)} \leq C|\beta|$ .
- iv. For all  $u \in H^{1/2}(\partial\Omega)$ ,

$$\|\mathbf{m} u\|_{H^{1/2}(\partial\Omega)} \lesssim \|\beta(\tau, x)|^{1/2} u\|_{L^2(\partial\Omega)} + \|u\|_{H^{1/2}(\partial\Omega)}.$$

- v. For all  $u \in H^1(\Omega)$ ,  $\|Vu\|_{H^{-1/2}(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}$ .

**Proof of Lemma. i.** For most points  $x \in \partial\Omega$ , one has  $x \in G_j$  for some  $j$ ,  $\nu = \pm \mathbf{e}_j$ , and  $A(\nu) = \pm A_j$  The spectral representation is

$$A(\nu) = \pi^+(\nu) - \pi^-(\nu), \quad \pi^\pm(\nu) = \pi^\pm(\nu)^* = \pi^\pm(\nu)^2, \quad \pi^\pm(\nu)\pi^\mp(\nu) = 0.$$

These imply the spectral representations

$$A(\nu)^\dagger = \pi^+(\nu)^\dagger - \pi^-(\nu)^\dagger, \quad \text{and,} \quad \overline{A(\nu)} = \overline{\pi^+(\nu)} - \overline{\pi^-(\nu)}.$$

For  $j \in \{1, 2, 4, 5\}$ ,  $A(\nu)$  is real and hermitian symmetric,  $A(\nu) = A(\nu)^\dagger = \overline{A(\nu)}$ . Comparing the spectral representations yields  $\pi^\pm(\nu) = \pi^\pm(\nu)^\dagger = \overline{\pi^\pm(\nu)}$ . Since  $\tilde{\nu}$  is a scalar multiple of  $\nu$  this yields

$$\pi^\pm(\nu) = \pi^\pm(\nu)^\dagger = \overline{\pi^\pm(\nu)} = \pi^\pm(\tilde{\nu}) = \pi^\pm(\tilde{\nu})^\dagger = \overline{\pi^\pm(\tilde{\nu})}. \quad (3.32)$$

It follows that  $\mathbf{m} = 0$  at such points.

For  $j \in \{3, 6\}$  and  $x \in G_j$ ,  $A(\nu)^\dagger = \overline{A(\nu)} = -A(\nu)$ . Comparing the spectral representations as above implies that

$$\pi^\pm(\nu) = \pi^\mp(\nu)^\dagger = \overline{\pi^\mp(\nu)} = \pi^\pm(\tilde{\nu}) = \pi^\mp(\tilde{\nu})^\dagger = \overline{\pi^\mp(\tilde{\nu})}. \quad (3.33)$$

Therefore  $\mathfrak{m} = 0$  at these flat parts of the boundary too.

These results for all  $G_j$  show that  $\mathfrak{m}$  is supported on the rounded edges of  $\partial\Omega$  proving **i**.

**ii.** Compute

$$\frac{\tau}{\tau + \sigma_j} = \frac{1}{1 + \sigma_j/\tau} = 1 - \frac{\sigma_j}{\tau} + \left(\frac{\sigma_j}{\tau}\right)^2 - \dots.$$

It follows that as  $|\tau| \rightarrow \infty$ ,

$$\tilde{\nu} - \nu = O(1/|\tau|), \quad \text{so,} \quad \pi^+(\tilde{\nu}) - \pi^+(\nu) = O(1/|\tau|).$$

To estimate the size of  $\mathfrak{m}$  write

$$\overline{\pi^+(\tilde{\nu})} - \pi^+(\tilde{\nu})^\dagger = \left(\overline{\pi^+(\tilde{\nu})} - \overline{\pi^+(\nu)}\right) + \left(\overline{\pi^+(\nu)} - \pi^+(\tilde{\nu})^\dagger\right).$$

The first summand is  $O(1/\tau)$ . Equations (3.32) and (3.33) imply that the second is also  $O(1/|\tau|)$ . It follows that  $\mathfrak{m}$  is bounded uniformly in  $\tau, \delta$ , proving **ii**.

**iii.** Use the notations from Proposition 2.11. Then  $\tau \mapsto \nu(\tau, \cdot)$  is analytic in  $|\tau| > R$  with values in  $C^\infty(\partial\Omega)$ .

Expand the stretchings in  $z = 1/\tau$  about  $z = 0$ . The transformation satisfies

$$\frac{\partial X_j(\tau, x_j)}{\partial x_j} = \frac{\tau + \sigma_j(x_j)}{\tau} = 1 + z\sigma_j(x_j), \quad X_j(\tau, 0) = 0. \quad (3.34)$$

Thus  $X$  is analytic on a neighborhood of  $z = 0$  with  $X(0, x) = x$ . The derivative with respect to  $x$  satisfies  $D_x X = I + O(z)$ . It follows that

$$\nu(\tau, x) = \nu(\infty, x) + O(1/\tau), \quad \text{and,} \quad \nabla_x \nu(\tau, x) = \nabla_x \nu(\infty, x) + O(z).$$

At  $\tau = \infty$  the  $\nabla_x \nu$  restricted to the tangent space is the Weingarten map of  $\partial\Omega$  from Definition 2.6. At  $\tau = \infty$ , the eigenvalues are nonnegative. Therefore

$$\begin{aligned} H_{\underline{\Omega}}(\tau, x) &= H_{\Omega}(x) + O(1/\tau), \\ |\nabla_x \nu(\infty, x)| &\lesssim \max\{\kappa_1, \kappa_2\} \leq 2H_{\Omega}(x), \\ |\nabla_x \nu(\tau, x)| &\lesssim |H_{\underline{\Omega}}(\tau, x)| + |\tau|^{-1} \lesssim |\beta(\tau, \delta, x)|. \end{aligned} \quad (3.35)$$

Since  $|\nabla_x \mathbf{m}| \lesssim |\nabla_x \nu|$  this proves **iii**.

**iv**. Estimates **ii**, **iii** imply that with constants independent of  $\tau, \delta$  and all  $u$ ,

$$\begin{aligned} \|\mathbf{m} u\|_{L^2(\partial\Omega)} &\lesssim \|u\|_{L^2(\partial\Omega)}, \\ \|\mathbf{m} u\|_{H^1(\partial\Omega)} &\lesssim \|\beta u\|_{L^2(\partial\Omega)} + \|u\|_{H^1(\partial\Omega)}. \end{aligned} \quad (3.36)$$

To prove the second, apply the product rule with vector fields  $\partial$  that are tangent to the boundary to find  $\partial(\mathbf{m}u) = \mathbf{m}\partial u + (\partial\mathbf{m})u$ . Therefore

$$\|\partial(\mathbf{m}u)\|_{L^2(\partial\Omega)} \leq \|\mathbf{m}\|_{L^\infty(\partial\Omega)} \|\partial u\|_{L^2(\partial\Omega)} + \|(\partial\mathbf{m})u\|_{L^2(\partial\Omega)}.$$

Using **iii** in the second summand proves (3.36)

Denote by  $\Delta_{\partial\Omega}$  the Laplace-Betrami operator of  $\partial\Omega$ . The estimates (3.36) are the cases  $\theta = 0, 1$  of

$$\|\mathbf{m} u\|_{H^\theta(\partial\Omega)} \lesssim \left( |\beta(\tau, x)| + |\Delta_\Omega|^{1/2} \right)^\theta \|u\|_{L^2(\partial\Omega)}.$$

Interpolation implies the estimate for  $0 \leq \theta \leq 1$ . Use the case  $\theta = 1/2$ . For self adjoint  $B_j \geq 0$  with  $B_1$  bounded and  $u \in \mathcal{D}(B_2)$ ,

$$\begin{aligned} \|\sqrt{B_1 + B_2} u\|^2 &= (\sqrt{B_1 + B_2} u, \sqrt{B_1 + B_2} u) = ((B_1 + B_2)u, u) \\ &= (B_1 u, u) + (B_2 u, u) = \|\sqrt{B_1} u\|^2 + \|\sqrt{B_2} u\|^2. \end{aligned}$$

With  $B_1 = |\beta(\tau, x)|$  and  $B_2 = |\Delta_\Omega|^{1/2}$  this yields

$$\left\| \left( |\beta(\tau, x)| + |\Delta_\Omega|^{1/2} \right)^{1/2} u \right\|_{L^2(\partial\Omega)}^2 = \|\beta(\tau, x) u\|_{L^2(\partial\Omega)}^2 + \|\Delta_\Omega^{1/4} u\|_{L^2(\partial\Omega)}^2.$$

Using this in the  $\theta = 1/2$  estimate proves **iv**.

**v**. With constants independent of  $\delta, \tau$  with  $|\tau| > R$ , one has for all  $u \in H^1(\Omega)$ ,

$$\int_\Omega |\nabla_x u|^2 dx \leq C \left( -\operatorname{Re} \int_\Omega p(\tau, x, \partial) u \cdot \bar{u} dx \right).$$

It follows that for  $|\tau| > R$  and  $0 < \delta < 1$ , the operator  $1 - p(\tau, x, \partial)$  is an isomorphism of  $H^1(\Omega)$  to  $H_0^{-1}(\Omega)$ , and with constants independent of  $\tau, \delta$ ,

$$\|u\|_{H^1(\Omega)} \lesssim \|(1 - p)u\|_{H_0^{-1}(\Omega)} \lesssim \|u\|_{H^1(\Omega)}.$$

Therefore,

$$\begin{aligned} \|pu\|_{H_0^{-1}(\Omega)} &\leq \|(1 - p)u\|_{H_0^{-1}(\Omega)} + \|u\|_{H_0^{-1}(\Omega)} \\ &\lesssim \|u\|_{H^1(\Omega)} + \|u\|_{H_0^{-1}(\Omega)} \lesssim \|u\|_{H^1(\Omega)}. \end{aligned} \quad (3.37)$$



Using (2.19), (3.9), and (3.16) shows that for all  $u, v \in H^1(\Omega)$

$$a(u, v) - \int_{\Omega} (p(\tau, x, \partial) u) \cdot v \, dx = \int_{\partial\Omega} \Phi(\tau, x) V u \cdot v \, d\Sigma.$$

For  $\phi \in H^{1/2}(\partial\Omega)$  choose  $v \in H^1(\Omega)$  with  $\|v\|_{H^1(\Omega)} \lesssim \|\phi\|_{H^{1/2}(\partial\Omega)}$  to find,

$$\begin{aligned} \left| \int_{\partial\Omega} \Phi(\tau, x) V u \cdot \phi \, d\Sigma \right| &= \left| a(u, v) - \int_{\Omega} (p u) \cdot v \, dx \right| \\ &\lesssim \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|p u\|_{H_0^{-1}(\Omega)} \|v\|_{H^1(\Omega)} \\ &\lesssim (\|\nabla u\|_{L^2(\Omega)} + \|p u\|_{H_0^{-1}(\Omega)}) \|\phi\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

Using this in the upper bound for  $|\int \Phi V u \cdot \phi \, d\Sigma|$ , shows that

$$\left| \int_{\partial\Omega} \Phi(\tau, x) V u \cdot \phi \, d\Sigma \right| \lesssim \|u\|_{H^1(\Omega)} \|\phi\|_{H^{1/2}(\partial\Omega)}.$$

Since  $\Phi$  and  $1/\Phi$  as well as their derivatives are uniformly bounded, this proves **v**.  $\square$

**End of proof of Proposition 3.12.** The second term on the right in (3.31) is estimated as

$$\left| \int_{\partial\Omega} \Phi \beta u \cdot \mathbf{m} \bar{u} \, d\Sigma \right| \lesssim \int_{\partial\Omega} |\beta| |u|^2 \, d\Sigma = \| |\beta|^{1/2} u \|_{L^2(\partial\Omega)}^2. \quad (3.38)$$

The first summand is estimated as

$$\left| \int_{\partial\Omega} \Phi V u \cdot \mathbf{m} \bar{u} \, d\Sigma \right| \lesssim \|V u\|_{H^{-1/2}(\partial\Omega)} \|\mathbf{m} \bar{u}\|_{H^{1/2}(\partial\Omega)}. \quad (3.39)$$

For  $\|\mathbf{m} u\|_{H^{1/2}(\partial\Omega)}$  use Part **iv** of the lemma in (3.39) to find,

$$\begin{aligned} \left| \int_{\partial\Omega} \Phi V u \cdot \mathbf{m} \bar{u} \, d\Sigma \right| &\lesssim \\ &(\|p u\|_{H_0^{-1}(\Omega)} + \|\nabla u\|_{L^2(\Omega)}) (\| |\beta|^{1/2} u \|_{L^2(\partial\Omega)} + \|u\|_{H^{1/2}(\partial\Omega)}). \end{aligned} \quad (3.40)$$

Use this, (3.37), and,  $\|u\|_{H^{1/2}(\partial\Omega)} \lesssim \|u\|_{H^1(\Omega)}$  in (3.40) to find,

$$\left| \int_{\partial\Omega} \Phi V u \cdot \mathbf{m} \bar{u} \, d\Sigma \right| \lesssim \| |\beta|^{1/2} u \|_{L^2(\partial\Omega)}^2 + \|u\|_{H^1(\Omega)}^2. \quad (3.41)$$

Adding the estimates (3.38), and (3.41) for the terms on the right of (3.28) proves Proposition 3.12.  $\square$

**Proof of Theorem 3.8.** Combine the lower and upper bounds for  $|\mathcal{A}(u, \bar{u})|$  from Propositions 3.11 and 3.12 to find,

$$\begin{aligned} |\tau| (\operatorname{Re} \tau) \|u\|_{L^2(\mathcal{Q}_\delta)}^2 + \frac{\operatorname{Re} \tau}{|\tau|} \left( \|\beta^{1/2} u\|_{L^2(\partial \mathcal{Q}_\delta)}^2 + \|\nabla_x u\|_{L^2(\mathcal{Q}_\delta)}^2 \right) \\ \leq C \left| \int_{\mathcal{Q}_\delta} f \bar{u} \, dx \right| + \frac{C}{|\tau|} \left( \|\beta^{1/2} u\|_{L^2(\partial \mathcal{Q}_\delta)}^2 + \|u\|_{L^2(\mathcal{Q}_\delta)}^2 \right). \end{aligned}$$

Choose  $M = 2C$ . Then for  $\operatorname{Re} \tau > M$ , the second summand on the right can be absorbed in the left hand side yielding (3.15). This completes the proof of Theorem 3.8.  $\square$

### 3.3.3 Proof of Theorem 3.7

Follow the strategy described after the statement of Theorem 3.7.

**Proof that the map  $u \mapsto (f, g_1, g_2)$  has trivial kernel.** If  $u \in C^\infty(\overline{\mathcal{Q}_\delta})$  is in the kernel, it follows that  $u \in H^2(\mathcal{Q}_\delta)$  and satisfies the homogeneous boundary value problem with sources  $f, g_1, g_2$  equal to zero. Theorem 3.8 implies that  $u = 0$ .

**Proof that the annihilator of the range, is  $\{0\}$ .** • Need the following Green's identity for  $u, v \in H^2(\Omega)$ ,

$$\begin{aligned} \int_{\Omega} (\tau^2 \Pi(\tau, x) - p(\tau, x, \partial)) u \cdot v \, dx - u \cdot (\tau^2 \Pi(\tau, x) - p(\tau, x, \partial)) v \, dx \\ = - \int_{\partial \Omega} \Phi(\tau, x) \left( (V + \beta(\tau, x)) u \cdot v - u \cdot (V + \beta(\tau, x)) v \right) d\Sigma. \end{aligned} \quad (3.42)$$

To prove (3.42), subtract (3.18) from the same identity with  $u$  and  $v$  interchanged.

- Equations for the annihilators. The function

$$(\underline{u}, \underline{g}_1, \underline{g}_2) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial \Omega; \mathcal{E}^-) \times C^\infty(\partial \Omega; \mathcal{E}^+)$$

annihilates the range if and only if  $\forall u \in H^2(\Omega)$ ,

$$\begin{aligned} \int_{\Omega} (\tau^2 \Pi(\tau, x) - p(\tau, x, \partial)) u \cdot \underline{u} \, dx + \int_{\partial \Omega} \pi^-(\tilde{v}) u \cdot \underline{g}_1 \, d\Sigma \\ + \int_{\partial \Omega} \pi^+(\tilde{v}) \left( V + \tau + 2H_{\underline{\Omega}} \right) u \cdot \underline{g}_2 \, d\Sigma = 0. \end{aligned} \quad (3.43)$$

The operator  $\tau^2\Pi(\tau, x) - p$  is equal to its own transpose. Therefore, taking  $u$  that vanish on a neighborhood of  $\partial\Omega$  implies that

$$(\tau^2\Pi(\tau, x) - p(\tau, x, \partial))\underline{u} = 0 \quad \text{on } \Omega. \quad (3.44)$$

This together with (3.42) shows that (3.43) holds if and only if

$$\begin{aligned} 0 = & \int_{\partial\Omega} \pi^+(\tilde{\nu}) \left( (V + \tau + 2H_{\underline{\Omega}})u \cdot \underline{g}_2 + \pi^-(\tilde{\nu})u \cdot \underline{g}_1 \right. \\ & \left. - \Phi(\tau, x) \left( (V + \tau + 2H_{\underline{\Omega}})u \cdot \underline{u} - u \cdot (V + \tau + 2H_{\underline{\Omega}})\underline{u} \right) d\Sigma. \end{aligned} \quad (3.45)$$

Equation (3.45) is used first on test functions  $u$  that satisfy  $(V + \tau + 2H_{\underline{\Omega}})u = 0$  on  $\partial\Omega$ . That constraint leaves  $u|_{\partial\Omega}$  arbitrary. Of those test functions first consider those that satisfy  $\pi^-(\tilde{\nu})u|_{\partial\Omega} = 0$ . For those one finds

$$\int_{\partial\Omega} \Phi(\tau, x) u \cdot (V + \tau + 2H_{\underline{\Omega}})\underline{u} d\Sigma = 0.$$

Since the  $\Phi$  factor is scalar and nowhere vanishing it follows that for arbitrary  $\phi \in C^\infty(\partial\Omega)$ ,

$$\int_{\partial\Omega} \pi^+(\tilde{\nu})\phi \cdot (V + \tau + 2H_{\underline{\Omega}})\underline{u} d\Sigma = 0.$$

This shows that  $\underline{u}$  satisfies the transposed boundary condition

$$\pi^+(\tilde{\nu})^\dagger (V + \tau + 2H_{\underline{\Omega}})\underline{u} = 0, \quad \text{on } \partial\Omega. \quad (3.46)$$

Next take  $u$  satisfying  $\pi^+(\tilde{\nu})u|_{\partial\Omega} = 0$ . Then  $u|_{\partial\Omega} = \pi^-(\tilde{\nu})u$ . This yields

$$\int_{\partial\Omega} \Phi(\tau, x) \left( \pi^-(\tilde{\nu})u \cdot (V + \tau + 2H_{\underline{\Omega}})\underline{u} \right) + \pi^-(\tilde{\nu})u \cdot \underline{g}_1 d\Sigma.$$

The set of functions  $\pi^-(\tilde{\nu})u|_{\partial\Omega}$  includes the set of  $\pi^-(\tilde{\nu})\psi$  for an arbitrary  $\psi \in C^\infty(\partial\Omega; \mathbb{C}^2)$ . It follows that on  $\partial\Omega$ ,

$$\pi^-(\tilde{\nu})^\dagger \left( \Phi(\tau, x) \left( (V + \tau + 2H_{\underline{\Omega}})\underline{u} + \underline{g}_1 \right) \right) = 0 \quad \text{on } \partial\Omega. \quad (3.47)$$

Next extract the information from test functions that satisfy  $u|_{\partial\Omega} = 0$ . For such test functions,  $[V + \tau + 2H]_{\partial\Omega}$  can be chosen as an arbitrary element  $\psi \in C^\infty(\partial\Omega; \mathbb{C}^2)$ . This yields

$$- \int_{\partial\Omega} \Phi(\tau, x) \psi \cdot \underline{u} d\Sigma + \int_{\partial\Omega} \pi^+(\tilde{\nu})\psi \cdot \underline{g}_2 d\Sigma = 0.$$

First take those  $\psi$  that satisfy  $\pi^+(\tilde{\nu})\psi = 0$ . That is equivalent to  $\psi = \pi^-(\tilde{\nu})\phi$  for arbitrary  $\phi$ . That yields

$$\int_{\partial\Omega} \Phi(\tau, x) \pi^-(\tilde{\nu})\phi \cdot \underline{u} \, d\Sigma = 0.$$

This is equivalent to the Dirichlet boundary condition for  $\underline{u}$ ,

$$\pi^-(\tilde{\nu})^\dagger \underline{u} = 0, \quad \text{on } \partial\Omega. \quad (3.48)$$

Finally, consider  $\psi$  with  $\pi^-(\tilde{\nu})\psi = 0$ . Equivalently  $\psi = \pi^+(\tilde{\nu})\phi$  for arbitrary  $\phi$ . This yields

$$\int_{\partial\Omega} \pi^+(\tilde{\nu})\phi \cdot \left( -\Phi(\tau, x)\underline{u} + \underline{g}_2 \right) d\Sigma = 0.$$

Since  $\phi$  is arbitrary this is equivalent to

$$\pi^+(\tilde{\nu})^\dagger \left( -\Phi(\tau, x)\underline{u} + \underline{g}_2 \right) = 0 \quad \text{on } \partial\Omega. \quad (3.49)$$

• Proof that  $\underline{u} = 0$ ,  $\underline{g}_1 = 0$ , and  $\underline{g}_2 = 0$ . The three equations (3.44), (3.46), and (3.48) assert that  $\underline{u}$  is a smooth solution of the transposed boundary value problem with zero sources. Theorem 3.8 implies that  $\underline{u} = 0$ .

From the fact that  $\underline{u} = 0$ , (3.47) implies that  $(\pi^-(\tilde{\nu}))^\dagger \underline{g}_1 = 0$ . In addition  $\underline{g}_1$  takes values in  $\mathcal{E}^-(\tilde{\nu})$ . There is an  $R_2$  so that for  $|\tau| > R_1$ ,  $\pi^-(\tilde{\nu})^\dagger$  is injective on  $\mathcal{E}^-(\tilde{\nu}(\tau, x))$  for all  $x \in \partial\Omega$ . For those  $\tau$ , conclude that  $\underline{g}_1 = 0$ .

An entirely analogous argument using (3.49) shows that  $\underline{g}_2 = 0$ . This completes the proof that the annihilator of the range is equal to  $\{0\}$ .  $\square$

### 3.4 Analyticity in $\tau$ of the Helmholtz solution

Use the shorthand  $\mathcal{E}^\pm(\tau, x)$  for  $\mathcal{E}^\pm(\tilde{\nu}(\tau, x))$ . The vector spaces  $\mathcal{E}^\pm(\tau, x)$  depends analytically on  $\tau$ . The next example shows that defining what it means to depend analytically on  $\tau$  has pitfalls.

**Example 3.2 i.** *The subspace  $\mathbb{U}(\tau) \subset \mathbb{C}^2$  spanned by  $(1, \tau^2)$  depends analytically on  $\tau$  for any reasonable definition including the one below.*

**ii.** *The unit vectors spanning  $\mathbb{U}(\tau)$  are*

$$e^{i\theta(\tau)} \frac{(1, \tau^2)}{(1 + |\tau|^4)^{1/2}}, \quad \theta \in \mathbb{R}.$$

No choice of  $\theta$  makes this holomorphic.

iii. Orthogonal projection onto  $\mathbb{U}(\tau)$  has matrix equal to

$$\frac{1}{1 + |\tau|^4} \begin{pmatrix} 1 & \bar{\tau}^2 \\ \tau^2 & |\tau|^4 \end{pmatrix}.$$

It is not a holomorphic function of  $\tau$ .

The analytic dependence of  $\mathcal{E}^\pm(\tau, x)$  is expressed as follows. For each  $(\tau, x)$ ,  $\mathbb{C}^2 = \mathcal{E}^+(\tau, x) \oplus \mathcal{E}^-(\tau, x)$ . For  $\tau$  near a fixed  $\underline{\tau}$  and all  $x \in \partial\Omega$ ,  $\pi^+(\tilde{\nu})$  is an isomorphism from  $\mathcal{E}^+(\tau, x) \rightarrow \mathcal{E}^+(\underline{\tau}, x)$ . Define the linear transformation  $R^+(\tau, x) \in \text{Hom}(\mathbb{C}^2)$  to be the inverse of this isomorphism for  $v \in \mathcal{E}^+(\underline{\tau}, x)$  and equal to zero on  $\mathcal{E}^-(\underline{\tau}, x)$ . An analogous definition yields  $R^-(\tau, x)$ . Then  $R^\pm(\tau, x) \in \text{Hom}(\mathcal{E}^\pm(\underline{\tau}, x) : \mathbb{C}^2)$  depend analytically on  $\tau$ . For  $\tau$  near  $\underline{\tau}$  and all  $x \in \partial\Omega$ ,

$$\mathcal{E}^+(\tau, x) = R^+(\tau, x) \mathcal{E}^+(\underline{\tau}, x).$$

This is a local trivialization of  $\mathcal{E}^+(\tau, x)$  that depends analytically on  $\tau$ . Considering different  $\underline{\tau}$  the change of trivialization formulas are analytic in  $\tau$ . This is the definition of analytic dependence.

The boundary value problem (3.11) has source terms  $g_j$  that takes values in  $\mathcal{E}^\pm(\tau, x)$ . The local representation allows one to suppress the  $\tau$  dependence as follows. For  $\tau$  near  $\underline{\tau}$ , a section  $g$  of  $\mathcal{E}^+(\tau, x)$  is uniquely represented as  $R^+(\tau, x)\underline{g}$  where  $\underline{g}$  takes values in the  $\tau$  dependent space  $\mathcal{E}^+(\underline{\tau}, x)$ . The boundary value problem takes the form

$$\begin{aligned} (\tau^2 \Pi(\tau, x) - p(\tau, x, \partial))u &= f \quad \text{on } \Omega, \\ \pi^+(\tilde{\nu}(\tau, x))u &= R^-(\tau, x)\underline{g}_1 \quad \text{on } \partial\Omega, \\ \pi^+(\tilde{\nu}(\tau, x))(V + \tau + 2H_{\underline{\Omega}}(X(\tau, \underline{x})))u &= R^+(\tau, x)\underline{g}_2 \quad \text{on } \partial\Omega. \end{aligned} \quad (3.50)$$

Here  $\underline{g}_1$  takes values in  $\mathcal{E}^-(\underline{\tau}, x)$  and  $\underline{g}_2$  takes values in  $\mathcal{E}^+(\underline{\tau}, x)$ . In this form, the source terms  $\underline{g}_j$  belong to a  $\tau$ -independent space and the coefficients of the operators depend differentiably on  $\tau, x$  and analytically on  $\tau$ .

**Definition 3.14** A  $\tau$ -dependent section  $g_1(\tau) \in H^{3/2}(\mathcal{E}^-(\tau, x))$  depends analytically on  $\tau$  when the corresponding functions  $\underline{g}_1(\tau) \in H^{3/2}(\mathcal{E}^-(\underline{\tau}, x))$  depend analytically on  $\tau$ . A similar definition applies for  $g_2(\tau) \in H^{1/2}(\mathcal{E}^-(\tau, x))$ .

**Theorem 3.15** If the source terms

$$(f, g_1, g_2) \in L^2(\Omega) \times H^{3/2}(\mathcal{E}^-(\tau, x)) \times H^{1/2}(\mathcal{E}^+(\tau, x))$$

depend analytically on  $\tau$  on  $\operatorname{Re} \tau > M$ , then the corresponding solution  $u(\tau, \cdot)$  of (3.11) is an analytic function of  $\tau$  with values in  $H^2(\Omega)$ .

**Proof.** Standard elliptic theory shows that writing  $\tau = a + ib$  the map  $a, b \mapsto u$  is infinitely differentiable with values in  $H^2(\Omega)$ . The derivatives satisfy the system obtained by differentiating, with respect to  $a, b$ , the system and boundary conditions satisfied by  $u$ .

To prove analyticity it suffices to show that  $w := \partial u / \partial \bar{\tau} = 0$ . Since all the coefficients and the  $f, g_1, g_2$  are analytic, differentiating the boundary value problem with respect to  $\bar{\tau}$  shows that  $w$  satisfies

$$\begin{aligned} \left( \tau^2 \Pi(\tau, x) - p(\tau, x, \partial) \right) w &= 0 \quad \text{on } \mathcal{Q}_\delta, \\ \pi^-(\tilde{\nu}(\tau, x)) w &= 0 \quad \text{on } \partial \mathcal{Q}_\delta, \\ \pi^+(\tilde{\nu}(\tau, x)) \left( V(\tau, x, \partial) + \tau + 2H_{\underline{\Omega}}(X(\tau, \underline{x})) \right) w &= 0 \quad \text{on } \partial \mathcal{Q}_\delta. \end{aligned}$$

Theorem 3.8, implies that  $w = 0$ . □

## 4 Proofs of the Main Theorems

### 4.1 The stretched equation on $\mathcal{Q}_\delta$ , Theorem 2.5

**Proof of Theorem 2.5. Uniqueness.** Multiply the differential equation  $L(\tau, \tilde{\partial})u^\delta = F$  from (2.8) by  $\Pi(\tau, x)(\tau - \sum A_j \tilde{\partial}_j)$  and use (3.4) to find the first line in the Helmholtz boundary value problem

$$\begin{aligned} \left( \tau^2 \Pi(\tau, x) - p(\tau, x, \partial) \right) u^\delta &= \Pi(\tau, x) \left( \tau - \sum A_j \tilde{\partial}_j \right) F, \\ \pi^-(\tilde{\nu}) u^\delta &= 0, \quad \text{on } \partial \mathcal{Q}_\delta, \\ \pi^+(\tilde{\nu}) \left( V(\tau, x, \partial) + \tau + 2H_{\underline{\mathcal{Q}}_\delta} \right) u^\delta &= 0, \quad \text{on } \partial \mathcal{Q}_\delta. \end{aligned} \tag{4.1}$$

The second line is part of (3.4). The last line follows from part **iv** of Proposition 2.11 since  $F = 0$  on a neighborhood of  $\partial \mathcal{Q}_\delta$  and  $u^\delta \in H^2(\mathcal{Q}_\delta)$ .

The hypotheses of Theorem 3.8 are satisfied. Apply the estimate of that Theorem with  $f = 0$  to conclude that  $u = 0$ .

**Existence.** For  $\operatorname{Re} \tau > M$ , Theorem 3.7 implies that the boundary value problem (4.1) has a unique solution  $u^\delta \in H^2(\mathcal{Q}_\delta)$ . Theorem 3.15 implies that  $u$  is holomorphic with values in  $H^2(\mathcal{Q}_\delta)$ .

Since  $F \in L^2_{\ell\overline{\mathcal{Q}}}(\mathbb{R}^3)$  it follows that the source term  $f := \Pi(\tau, x)(\tau - \sum A_j \tilde{\partial}_j)F$  belongs to  $L^2(\mathbb{R}^3)$  with  $\text{supp } F \subset \ell\overline{\mathcal{Q}}$ . Estimate

$$\begin{aligned} \left| \int_{\mathcal{Q}_\delta} f \bar{u} \, dx \right| &= \left| \int_{\mathcal{Q}_\delta} F (\tau - \sum A_j \tilde{\partial}_j)^*(\Pi \bar{u}) \, dx \right| \\ &\lesssim \|F\|_{L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q}_\delta)} \left( \|\tau u\|_{L^2(\mathcal{Q}_\delta)} + \|\nabla_x u\|_{L^2(\mathcal{Q}_\delta)} \right). \end{aligned}$$

Estimate the two terms on the right as follows. Write

$$\begin{aligned} C \|\mu^{-1} F\|_{L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q}_\delta)} \|\mu \tau u\|_{L^2(\mathcal{Q}_\delta)} &\leq \frac{C^2 \mu^{-2}}{2} \|F\|_{L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q}_\delta)}^2 + \frac{\mu^2 |\tau|^2}{2} \|u\|_{L^2(\mathcal{Q}_\delta)}^2. \\ C \|\epsilon^{-1} F\|_{L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q}_\delta)} \|\epsilon \nabla_x u\|_{L^2(\mathcal{Q}_\delta)} &\leq \frac{C^2 \epsilon^{-2}}{2} \|F\|_{L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q}_\delta)}^2 + \frac{\epsilon^2}{2} \|\nabla_x u\|_{L^2(\mathcal{Q}_\delta)}^2. \end{aligned}$$

Choose  $\mu, \epsilon$  so that  $\mu^2 |\tau|^2 = |\tau|(\text{Re } \tau)$  and  $\epsilon^2 = (\text{Re } \tau)/|\tau|$ . Then,

$$\begin{aligned} C \|F\|_{L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q}_\delta)} \|\tau u\|_{L^2(\mathcal{Q}_\delta)} &\leq \frac{C^2 |\tau|}{2 \text{Re } \tau} \|F\|_{L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q}_\delta)}^2 + \frac{|\tau|(\text{Re } \tau)}{2} \|u\|_{L^2(\mathcal{Q}_\delta)}^2. \\ C \|F\|_{L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q}_\delta)} \|\nabla_x u\|_{L^2(\mathcal{Q}_\delta)} &\leq \frac{C^2 |\tau|}{2 \text{Re } \tau} \|F\|_{L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q}_\delta)}^2 + \frac{\text{Re } \tau}{2 |\tau|} \|\nabla_x u\|_{L^2(\mathcal{Q}_\delta)}^2. \end{aligned}$$

Absorbing the two right hand terms, Theorem 3.8 shows that with constant independent of  $\delta$ ,

$$\begin{aligned} |\tau|(\text{Re } \tau) \|u^\delta\|_{L^2(\mathcal{Q}_\delta)}^2 + |\tau| \|u^\delta\|_{L^2(\partial\mathcal{Q}_\delta)}^2 \\ + \frac{\text{Re } \tau}{|\tau|} \|\nabla u^\delta\|_{L^2(\mathcal{Q}_\delta)}^2 \lesssim \frac{|\tau|}{\text{Re } \tau} \|F\|_{L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q}_\delta)}^2. \end{aligned} \quad (4.2)$$

Multiplying by  $(\text{Re } \tau)/|\tau|$  yields (2.9).

To complete the proof it suffices to show that  $u^\delta$  satisfies the stretched boundary value problem (1.7) on  $\mathcal{Q}_\delta$ . Need to reverse the steps that lead from the stretched equations to the Helmholtz boundary value problem. The proof uses the  $H^2(\mathcal{Q}_\delta)$  regularity that requires the smoothness of  $\mathcal{Q}_\delta$ . Define

$$w := (A(\tilde{\partial}) + \tau)u \in H^1(\mathcal{Q}_\delta).$$

Need to show that the stretched equation,  $w = F$ , is satisfied.

The Helmholtz equation implies that  $w - F \in H^1(\mathcal{Q}_\delta)$  satisfies

$$(A(\tilde{\partial}) - \tau)(w - F) = 0, \quad \text{on } \mathcal{Q}_\delta. \quad (4.3)$$

Part **iv** of Proposition 2.11 shows that the derivative boundary condition satisfied by  $u^\delta$  is equivalent to

$$\pi^+(A(\tilde{\nu})) A(\tilde{\nu})^{-1} (w - F) = 0 \quad \text{on } \partial\mathcal{Q}_\delta.$$

Since  $\pi^+(A(\tilde{\nu}))$  and  $A(\tilde{\nu})$  commute, this is equivalent to

$$\pi^+(A(\tilde{\nu})) (w - F) = 0 \quad \text{on } \partial\mathcal{Q}_\delta. \quad (4.4)$$

When  $\tau$  is real and large, the pair of equations (4.3), (4.4) is a strictly dissipative boundary value problem with vanishing sources on the smooth domain  $\mathcal{Q}_\delta$  with noncharacteristic boundary. The solution is in  $H^1(\mathcal{Q}_\delta)$ . That the solution vanishes follows by a direct integration by parts showing that

$$\|w - F\|_{L^2(\mathcal{Q}_\delta)}^2 \lesssim \operatorname{Re} \int_{\mathcal{Q}_\delta} \left( (\tau - A(\partial))(w - F), (w - F) \right)_{\mathbb{C}^6} dx = 0.$$

The map  $\tau \mapsto (w - F)(\tau)$  is holomorphic for  $\operatorname{Re} \tau$  large. It has just been proved that it vanishes on  $]m, \infty[$  for  $m$  large. By analytic continuation, it follows that  $w - F = 0$  for all  $\operatorname{Re} \tau > M$ .

Thus the stretched equation is satisfied on  $\mathcal{Q}_\delta$  for  $\operatorname{Re} \tau > M$ . This completes the proof of existence.  $\square$

## 4.2 The stretched equation on $\mathcal{Q}$ , Theorem 1.6

### Proof of Theorem 1.6. Uniqueness.

The solution with vanishing data is holomorphic in  $\operatorname{Re} \tau$  large. To prove that it vanishes it is sufficient to prove that it vanishes for  $\tau \in ]m, \infty[$  for  $m$  large.

For  $\tau$  real and large, the stretched equation,  $L(\tau, \tilde{\partial})u(\tau) = 0$  is symmetric positive in the sense of Friedrichs, that is

$$L(\tau, \tilde{\partial}) + L(\tau, \tilde{\partial})^* \geq C_1(\tau - C_2)I, \quad C_1 > 0.$$

In addition,  $u(\tau) \in H^1(\mathcal{Q}_\delta)$  satisfies strictly dissipative boundary conditions on each smooth faces  $G_j$ . Therefore a straightforward integration by parts shows that

$$\|u(\tau)\|_{L^2(\mathcal{Q})}^2 \leq C_1(\tau - C_2) \int_{\mathcal{Q}} (L(\tau, \tilde{\partial})u, u) dx = 0.$$



**Existence.** Use Theorem 2.5. Solve on  $\mathcal{Q}_\delta$  and pass to the limit  $\delta \rightarrow 0$ . At the same time one must smooth the source term  $f$  in order to apply Theorem 2.5.

Choose  $0 < \underline{\epsilon} < \text{dist}(\ell\overline{\mathcal{Q}}, \partial\mathcal{Q})/2$ . Define  $K'$  to be the set of points at distance  $\underline{\epsilon}$  from  $\ell\mathcal{Q}$ . Then  $K' \subset \mathcal{Q}$  is compact. For  $\epsilon < \underline{\epsilon}$ , define  $F_\epsilon := j_\epsilon * F$  where  $j_\epsilon$  is a smooth mollification kernel on  $\mathbb{R}^3$  with support in the ball of radius  $\epsilon$  at the origin. The source term  $F_\epsilon \in C_{K'}^\infty(\mathcal{Q})$ . For  $\delta$  sufficiently small  $K' \subset \mathcal{Q}_\delta$  and Theorem 2.5 applies.

Define  $\delta(n) = 2^{-n}$ , and  $u^{\delta(n)} \in H^2(\mathcal{Q}_{\delta(n)})$  to be the solution from Theorem 2.5 with source term equal to  $F_{\delta(n)}$ . Then with  $C$  independent of  $n$ ,

$$\begin{aligned} (\text{Re } \tau)^2 \|u^{\delta(n)}\|_{L^2(\mathcal{Q}_{\delta(n)})}^2 + (\text{Re } \tau) \|u^{\delta(n)}\|_{L^2(\partial\mathcal{Q}_{\delta(n)})}^2 \\ + \frac{(\text{Re } \tau)^2}{|\tau|^2} \|\nabla_x u^{\delta(n)}\|_{L^2(\mathcal{Q}_{\delta(n)})}^2 \leq C \|F_{\delta(n)}\|_{L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q}_{\delta(n)})}^2. \end{aligned} \quad (4.5)$$

Extract a subsequence that converges weakly in  $H^1(\mathcal{Q}_{\delta(1)})$  to a limit  $v_1$ . Extract a further subsequence that converges weakly in  $H^1(\mathcal{Q}_{\delta(2)})$  to a limit  $v_2$ . And so forth. For each  $n > 1$ , one has  $v_n = v_{n-1}$  on  $\mathcal{Q}_{\delta(n-1)}$ . Define  $v \in H^1(\mathcal{Q})$  by  $v = v_n$  on  $\mathcal{Q}_{\delta(n)}$ . Using that  $\mathcal{Q}_{\delta(n)} \nearrow \mathcal{Q}$  and  $\partial\mathcal{Q}_{\delta(n)} \cap G_j \nearrow G_j$  conclude that for each  $n$ ,  $u_k$  converges weakly to  $v$  in  $H^1(\mathcal{Q}_{\delta(n)})$  with

$$\begin{aligned} (\text{Re } \tau)^2 \|v\|_{L^2(\mathcal{Q})}^2 + (\text{Re } \tau) \|v\|_{L^2(\partial\mathcal{Q})}^2 \\ + \frac{(\text{Re } \tau)^2}{|\tau|^2} \|\nabla_x v\|_{L^2(\mathcal{Q})}^2 \leq C \|F\|_{L^2_{\ell\overline{\mathcal{Q}}}(\mathcal{Q})}^2. \end{aligned} \quad (4.6)$$

The differential equation  $L(\tau, \tilde{\partial})v = F$  on  $\mathcal{Q}$  follows from the equations  $L(\tau, \tilde{\partial})u_k = F_k$  on  $\mathcal{Q}_{\delta(n(k))}$  on passing to the limit  $k \rightarrow \infty$ . Similarly, the boundary condition

$$\pi^+(\nu)v = 0, \quad \text{on } G_k$$

follows on passing to the limit in

$$\pi^+(\nu)u_{\delta(n)}|_{G_k \cap \partial\mathcal{Q}_{\delta(n)}} = 0.$$

For any  $\underline{\delta} > 0$  the holomorphy of  $\tau \mapsto v(\tau)$  from  $\text{Re } \tau > M$  to  $L^2(\mathcal{Q}_{\underline{\delta}})$  follows from the fact that it is the weak limit of bounded family of holomorphic functions. Therefore, for any  $\delta$ ,  $v : \{\text{Re } \tau > M\} \rightarrow L^2(\mathcal{Q}_\delta)$  is holomorphic.

To show that  $v$  is holomorphic with values in  $L^2(\mathcal{Q})$  it is sufficient to show that  $\tau \mapsto \ell(v(\tau))$  is holomorphic for each  $\ell$  in the dual of  $L^2(\mathcal{Q})$ .

Since  $v \in L^\infty(\{\operatorname{Re} \tau > M\}; L^2(\mathcal{Q}))$ , it suffices to show that  $\ell(v(\tau))$  is holomorphic for  $\ell$  in a dense subset. Indeed if  $\ell$  is the limit of  $\ell_j$  for which the result is true, estimate

$$|\ell(v(\tau)) - \ell_j(v(\tau))| \leq \|\ell - \ell_j\| \sup_{\operatorname{Re} \tau > M} \|v(\tau)\|_{H^1(\mathcal{Q})}, \quad \text{on } \operatorname{Re} \tau > M.$$

This proves that  $\ell(v(\tau))$  is the uniform limit of the holomorphic functions  $\ell_j(v(\tau))$ .

Take the dense set to be the linear functionals  $v \mapsto \int v \cdot \phi \, dx$  with  $\phi \in C_0^\infty(\mathcal{Q})$ . For each such  $\phi$ ,  $\phi \in C_0^\infty(\mathcal{Q}_\delta)$  for  $\delta$  small. That  $\ell(v(\tau))$  is holomorphic then follows from the fact that  $v$  is holomorphic with values in  $L^2(\mathcal{Q}_\delta)$ . This completes the proof of the Theorem.  $\square$

### 4.3 Bérenger's equation on $\mathbb{R}_t \times \mathcal{Q}$ , Theorem 1.7

Need the Paley-Wiener Theorem for functions with values in a Hilbert space  $H$  (see [18]).

**Theorem 4.1** *The Laplace transforms of functions  $F \in e^{Mt} L^2(\mathbb{R}; H)$  with  $\operatorname{supp} F \subset \{t \geq 0\}$  are exactly the functions  $G(\tau)$  holomorphic in  $\operatorname{Re} \tau > M$  with values in  $H$  and so that*

$$\sup_{\lambda > M} \int_{\operatorname{Re} \tau = \lambda} \|\widehat{F}(\tau)\|_H^2 |d\tau| < \infty.$$

In this case the function  $\widehat{F}(\tau)$  has trace at  $\operatorname{Re} \tau = M$  that satisfies

$$\int e^{-2Mt} \|F(t)\|_H^2 dt = \sup_{\lambda > M} \int_{\operatorname{Re} \tau = \lambda} \|\widehat{F}(\tau)\|_H^2 |d\tau| = \int_{\operatorname{Re} \tau = M} \|\widehat{F}(\tau)\|_H^2 |d\tau|.$$

**Proof of Theorem 1.7. Uniqueness.** Need to show that if  $U^1, U^2, U^3$  is a solution with source  $f = 0$ , then  $U^j = 0$ . Denote by  $\widehat{U}^j$  the Laplace transform that is holomorphic in  $\{\operatorname{Re} \tau > M\}$  with values in  $L^2(\mathcal{Q})$ .

The function  $v(\tau) := \sum \widehat{U}^j$  is holomorphic with values in  $L^2(\mathcal{Q})$  and satisfies the stretched equation

$$\tau v + \sum A_j \widetilde{\partial}_j v = 0.$$

In addition,  $v|_{G_k}$  is holomorphic with values in  $L^2(G_k)$ . The boundary condition satisfied by  $\sum U^j$  implies that  $v$  satisfies the boundary condition

$$v|_{G_k} \in \mathcal{E}^+(\nu), \quad 1 \leq k \leq 6.$$

The stretched operator is elliptic. When  $\tau \in ]m, \infty[$  the stretched operator is symmetric and positive in the sense of Friedrichs. The uniqueness theorem for such strictly dissipative symmetric and elliptic problems with trihedral corners from Part I of [16] implies that  $\widehat{u}(\tau) = 0$  for  $\tau \in ]m, \infty[$ . By analytic continuation,  $v(\tau) = 0$  on  $\{\operatorname{Re} \tau > M\}$ .

The Laplace transform of the split equation yields

$$(\tau + \sigma_1(x_1))\widehat{U}^j = -A_1\partial_1 v = 0.$$

This implies that  $\widehat{U}^j$  vanishes and therefore that  $U^j = 0$ . This completes the proof of uniqueness.

**Existence.** The solution  $u(t, x)$  is constructed by finding its Laplace transform. Denote by  $U^1(t, x)$ ,  $U^2(t, x)$ , and,  $U^3(t, x)$  the unknowns to be found. Denote by  $v(\tau, x)$  the function of  $\tau$  that will be the Laplace transform of  $U^1(t, x) + U^2(t, x) + U^3(t, x)$ . Define  $v(\tau, x)$  to be the solution of the stretched equation

$$\tau v + \sum_{j=1}^3 A_j \widetilde{\partial}_j v = F(\tau) := \sum_{j=1}^3 \frac{\tau \widehat{f}_j(\tau)}{\tau + \sigma_j(x)}. \quad (4.7)$$

constructed in Theorem 1.6. Then  $v$  holomorphic in  $\operatorname{Re} \tau > M$  with values in  $H^1(\mathcal{Q})$  and  $v|_{G_k}$  is holomorphic with values in  $L^2(G_k)$ . In addition,

$$\begin{aligned} (\operatorname{Re} \tau) \|v(\tau)\|_{L^2(\mathcal{Q})} + (\operatorname{Re} \tau)^{1/2} \|v(\tau)\|_{L^2(\partial\mathcal{Q})} + \frac{\operatorname{Re} \tau}{|\tau|} \|\nabla_x v(\tau)\|_{L^2(\mathcal{Q})} \\ \leq C \|F(\tau)\|_{L_K^2(\mathcal{Q})} \leq C \|\widehat{f}(\tau)\|_{L_K^2(\mathcal{Q})}. \end{aligned} \quad (4.8)$$

Define  $V^j$  destined to be the Laplace transforms of the  $U^j$  by the analogue of (1.5),

$$(\tau + \sigma_j(x_j))V^j + A_j \partial_j v = \widehat{f}_j, \quad j = 1, 2, 3. \quad (4.9)$$

Multiplying by  $\tau/(\tau + \sigma_j(x_j))$  yields

$$\tau V^j + A_j \widetilde{\partial}_j v = \frac{\tau \widehat{f}_j}{\tau + \sigma_j}, \quad j = 1, 2, 3.$$

Summing yields

$$\tau(V^1 + V^2 + V^3) + \sum A_j \widetilde{\partial}_j v = \sum_{j=1}^3 \frac{\tau \widehat{f}_j}{\tau + \sigma_j(x_j)} = F.$$

Subtracting from (4.7) yields

$$\tau(V^1 + V^2 + V^3 - v) = 0 \quad \text{so,} \quad v = V^1 + V^2 + V^3.$$

The Paley-Wiener theorem implies that

$$\sup_{\lambda > M} \int \|\widehat{f}(\tau)\|^2 |d\tau| \leq \int e^{2Mt} \|f(t)\|_{L^2_{i\mathbb{Q}}(\mathcal{Q})}^2 dt.$$

Equation (4.8) together with the Paley-Wiener Theorem implies that  $v$  is the Laplace transform of a function  $u \in e^{Mt}L^2(\mathbb{R}; L^2(\mathcal{Q}))$  supported in  $t \geq 0$ . Moreover,

$$\begin{aligned} \int_0^\infty e^{2Mt} \left( M \|u(t)\|_{L^2(\mathcal{Q})}^2 + M^{1/2} \|u(t)|_{\partial\mathcal{O}}\|_{L^2(\partial\mathcal{Q})}^2 \right) dt \\ \lesssim \int_0^\infty e^{2Mt} \|f(t)\|_{L^2_{i\mathbb{Q}}(\mathcal{Q})}^2 dt. \end{aligned}$$

Similarly the Paley-Wiener Theorem implies that  $V^j(\tau)$  is the Laplace transform of a function  $U^j(t) \in e^{Mt}L^2(\mathbb{R}; H^{-1}(\mathcal{Q}))$  supported in  $t \geq 0$  and satisfying

$$\int_0^\infty e^{2Mt} \|MU^j(t), \partial_t U^j(t)\|_{H^{-1}(\mathcal{Q})}^2 dt \lesssim \int_0^\infty e^{2Mt} \|f(t)\|_{L^2_{i\mathbb{Q}}(\mathcal{Q})}^2 dt.$$

The fact that  $v = \sum V^j$  implies that  $u = \sum U^j$ . Equation (4.9) implies that  $(U^1, U^2, U^3)$  satisfies the Bérenger split equations. The last two estimates are exactly those required in Theorem 1.7.

Denoting by  $\mathbb{L}$  the Laplace transform, one has

$$\mathbb{L}(\pi^-(\nu)u|_{G_j}) = \pi^-(\nu)(\mathbb{L}(u|_{G_j})) = \pi^-(\nu)v|_{G_j} = 0.$$

This proves the boundary condition  $\pi^-(\nu)u|_{G_j} = 0$ . This completes the proof that the  $U^j$  satisfy the boundary value problem and estimates of Theorem 1.7.  $\square$

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