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► **To cite this version:**

Indranil Biswas, Sorin Dumitrescu. NONABELIAN HODGE THEORY FOR FUJIKI CLASS C MANIFOLDS. 2020. hal-02869851

HAL Id: hal-02869851

<https://hal.science/hal-02869851>

Preprint submitted on 16 Jun 2020

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NONABELIAN HODGE THEORY FOR FUJIKI CLASS \mathcal{C} MANIFOLDS

INDRANIL BISWAS AND SORIN DUMITRESCU

ABSTRACT. The nonabelian Hodge correspondence (Corlette-Simpson correspondence), between the polystable Higgs bundles with vanishing Chern classes on a compact Kähler manifold X and the completely reducible flat connections on X , is extended to the Fujiki class \mathcal{C} manifolds.

1. INTRODUCTION

Given a compact Kähler manifold X , foundational works of Simpson and Corlette, [Si1], [Co] establish a natural equivalence between the category of local systems over X and the category of certain analytical objects called *Higgs bundles* that consist of a holomorphic vector bundle V over X together with a holomorphic section $\theta \in H^0(X, \text{End}(V) \otimes \Omega_X^1)$ such that the section $\theta \wedge \theta \in H^0(X, \text{End}(V) \otimes \Omega_X^2)$ vanishes identically (see also [Si2]). The section θ is called a *Higgs field* on V .

While local systems are topological objects on X , which correspond to flat vector bundles (or, equivalently, to equivalence classes of representations of the fundamental group of X), the Higgs bundles on X are holomorphic objects. There are notions of stability and polystability for Higgs bundles which are analogous to the corresponding notions for holomorphic vector bundles, while restricting the class of subsheaves to only those that are invariant by the Higgs field (see Section 2.1). This stability condition generalizes the stability of holomorphic vector bundles introduced by Mumford in the context of geometric invariant theory which he developed.

The above mentioned equivalence of categories exhibits a natural correspondence between the completely reducible local systems on X and the polystable Higgs bundles on X with vanishing rational Chern classes. It may be mentioned that for polystable Higgs bundles, the vanishing of the first two Chern classes implies the vanishing of all Chern classes. This correspondence is constructed via a Hermitian metric on V that satisfies the Yang–Mills–Higgs equation for a polystable (V, θ) , [Si1], and a harmonic metric on a vector bundle on X equipped with a completely reducible flat connection [Co]. The construction of these canonical metrics can be seen as a vast generalization of Hodge Theorem on existence of harmonic forms, and for this reason the above correspondence is also called a “nonabelian Hodge theorem”.

The aim here is to extend this Corlette-Simpson (nonabelian Hodge) correspondence to the more general context of compact Fujiki class \mathcal{C} manifolds; see Theorem 5.2.

2010 *Mathematics Subject Classification.* 32G13, 53C07, 58D27, 14E05.

Key words and phrases. Nonabelian Hodge theory, flat connection, Higgs bundle, Fujiki class \mathcal{C} manifold.

Recall that a manifold M is in Fujiki class \mathcal{C} if it is the image of a Kähler manifold through a holomorphic map [Fu2], or, equivalently, M is bimeromorphic to a compact Kähler manifold [Va] (see Section 2.2). The proof of Theorem 5.2 uses a well-known functoriality property of Corlette-Simpson correspondence (see Theorem 2.3) and a descent result (see Proposition 3.1) which is inspired by Theorem 1.2 in [GKPT].

2. REPRESENTATIONS, HIGGS BUNDLES AND FUJIKI CLASS \mathcal{C} MANIFOLDS

2.1. Nonabelian Hodge theory. Let X be a compact connected complex manifold. Fix a base point $x_0 \in X$ to define the fundamental group $\pi_1(X, x_0)$ of X . Take a positive integer r , and consider any homomorphism

$$\rho : \pi_1(X, x_0) \longrightarrow \mathrm{GL}(r, \mathbb{C}).$$

The homomorphism ρ is called *irreducible* if the standard action of $\rho(\pi_1(X, x_0))$ on \mathbb{C}^r does not preserve any nonzero proper subspace of \mathbb{C}^r . The homomorphism ρ is called *completely reducible* if it is a direct sum of irreducible representations.

Two homomorphisms $\rho_1, \rho_2 : \pi_1(X, x_0) \longrightarrow \mathrm{GL}(r, \mathbb{C})$ are called *equivalent* if there is an element $g \in \mathrm{GL}(r, \mathbb{C})$ such that

$$\rho_1(\gamma) = g^{-1}\rho_2(\gamma)g$$

for all $\gamma \in \pi_1(X, x_0)$. Clearly, this equivalence relation preserves irreducibility and complete reducibility. The space of equivalence classes of completely reducible homomorphisms from $\pi_1(X, x_0)$ to $\mathrm{GL}(r, \mathbb{C})$ has the structure of an affine scheme defined over \mathbb{C} , which can be seen as follows. Since X is compact, $\pi_1(X, x_0)$ is a finitely presented group; $\mathrm{GL}(r, \mathbb{C})$ being an affine algebraic group, the space of all homomorphisms $\mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}(r, \mathbb{C}))$ is a complex affine scheme. The adjoint action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}(r, \mathbb{C}))$ produces an action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}(r, \mathbb{C}))$. The geometric invariant theoretic quotient $\mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}(r, \mathbb{C}))//\mathrm{GL}(r, \mathbb{C})$ is the moduli space of equivalence classes of completely reducible homomorphisms from $\pi_1(X, x_0)$ to $\mathrm{GL}(r, \mathbb{C})$; see [Si3], [Si4]. Let $\mathcal{R}(X, r)$ denote this moduli space of equivalence classes of completely reducible homomorphisms from $\pi_1(X, x_0)$ to $\mathrm{GL}(r, \mathbb{C})$. It is known as the Betti moduli space.

A homomorphism $\rho : \pi_1(X, x_0) \longrightarrow \mathrm{GL}(r, \mathbb{C})$ produces a holomorphic vector bundle E on X of rank r equipped with a flat holomorphic connection, together with a trivialization of the fiber E_{x_0} . Equivalence classes of such homomorphisms correspond to holomorphic vector bundles of rank r equipped with a flat holomorphic connection; this is an example of Riemann–Hilbert correspondence. A connection ∇ on a vector bundle E is called *irreducible* if there is no subbundle $0 \neq F \subsetneq E$ such that ∇ preserves F . A connection ∇ on a vector bundle E is called *completely reducible* if

$$(E, \nabla) = \bigoplus_{i=1}^N (E_i, \nabla^i),$$

where each ∇^i is an irreducible connection on E_i . We note that irreducible (respectively, completely reducible) flat connections of rank r on X correspond to irreducible (respectively, completely reducible) equivalence classes of homomorphisms from $\pi_1(X, x_0)$ to $\mathrm{GL}(r, \mathbb{C})$.

A *Higgs field* on a holomorphic vector bundle V on X is a holomorphic section

$$\theta \in H^0(X, \text{End}(V) \otimes \Omega_X^1)$$

such that the section $\theta \wedge \theta \in H^0(X, \text{End}(V) \otimes \Omega_X^2)$ vanishes identically [Si1], [Si2]. If (z_1, \dots, z_d) are local holomorphic coordinates on X with respect to which the local expression of the section θ is $\sum_i \theta_i \otimes dz_i$, with θ_i being locally defined holomorphic endomorphisms of V , the above integrability condition $\theta \wedge \theta = 0$ is equivalent to the condition that $[\theta_i, \theta_j] = 0$ for all i, j .

A *Higgs bundle* on X is a holomorphic vector bundle on X together with a Higgs field on it. A homomorphism of Higgs bundles $(V_1, \theta_1) \rightarrow (V_2, \theta_2)$ is a holomorphic homomorphism

$$\Psi : V_1 \rightarrow V_2$$

such that $\theta_2 \circ \Psi = (\Psi \otimes \text{Id}_{\Omega_X^1}) \circ \theta_1$ as homomorphisms from V_1 to $V_2 \otimes \Omega_X^1$.

Assume now that X is Kähler, and fix a Kähler form ω on X . The *degree* of a torsionfree coherent analytic sheaf F on X is defined to be

$$\text{degree}(F) := \int_X c_1(\det F) \wedge \omega^{d-1} \in \mathbb{R},$$

where $d = \dim_{\mathbb{C}} X$; see [Ko, Ch. V, § 6] (also Definition 1.34 in [Br]) for determinant line bundle $\det F$. The number

$$\mu(F) := \frac{\text{degree}(F)}{\text{rank}(F)} \in \mathbb{R}$$

is called the *slope* of F .

A Higgs bundle (V, θ) on X is called *stable* (respectively, *semistable*) if for every coherent analytic subsheaf $F \subset V$ with $0 < \text{rank}(F) < \text{rank}(V)$ and $\theta(F) \subset F \otimes \Omega_X^1$, the inequality

$$\mu(F) < \mu(V) \quad (\text{respectively, } \mu(F) \leq \mu(V))$$

holds. A Higgs bundle (V, θ) is called *polystable* if it is semistable and a direct sum of stable Higgs bundles. To verify the stability (or semistability) condition it suffices to consider coherent analytic subsheaves $F \subsetneq V$ such that the quotient V/F is torsionfree [Ko, Ch. V, Proposition 7.6]. These subsheaves are reflexive (see [Ko, Ch. V, Proposition 5.22]).

Theorem 2.1 ([Si1], [Co], [Si2]). *There is a natural equivalence of categories between the following two:*

- (1) *The objects are completely reducible flat complex connections on X , and morphisms are connection preserving homomorphisms.*
- (2) *Objects are polystable Higgs bundles (V, θ) on X such that $c_1(V) = 0 = c_2(V)$, where c_i denotes the i -th Chern class with coefficient in \mathbb{Q} ; the morphisms are homomorphisms of Higgs bundles.*

In [Si2], the conditions on the Chern classes of the polystable Higgs bundle (V, θ) are $\text{degree}(V) = 0 = (ch_2(V) \cup [\omega^{d-2}]) \cap [X]$, instead of the above conditions $c_1(V) = 0 = c_2(V)$. However, since the existence of flat connection on a complex vector bundle implies that all its rational Chern classes vanish, these two sets of conditions are equivalent in the given context.

Remark 2.2. Recall that the notion of (poly)stability depends on the choice of the Kähler class of ω . In the case of vanishing first two Chern classes, these notions are actually independent of the class of ω . In fact, given an equivalence class of completely reducible homomorphism $\rho : \pi_1(X, x_0) \rightarrow \mathrm{GL}(r, \mathbb{C})$, the Higgs bundle (V, θ) of rank r associated to ρ by the equivalence of categories in Theorem 2.1 is in fact independent of the choice of ω . Indeed, the local system ρ is obtained from (V, θ) by constructing a Hermitian metric h on V that satisfies the Yang–Mills–Higgs equation $K(\nabla^h) + [\theta, \theta_h^*] = 0$, where $K(\nabla^h)$ is the curvature of the Chern connection ∇^h on V corresponding to h , and θ_h^* is the adjoint of θ with respect to h . Since this Yang–Mills–Higgs equation does not depend on the Kähler form ω , the flat connection corresponding to (V, θ) is independent of ω .

We recall a basic property of the equivalence of categories in Theorem 2.1.

Theorem 2.3 ([Si2]). *Let X and X_1 be compact connected Kähler manifolds, and let*

$$\beta : X_1 \rightarrow X$$

*be any holomorphic map. Let (E, ∇) be a completely reducible flat connection on X , and let (V, θ) be the corresponding polystable Higgs bundle on X with $c_1(V) = 0 = c_2(V)$. Then the pulled back Higgs bundle $(\beta^*V, \beta^*\theta)$ is polystable, and, moreover, the flat connection corresponding to it coincides with $(\beta^*E, \beta^*\nabla)$.*

To explain Theorem 2.3, take any completely reducible homomorphism $\rho : \pi_1(X, x_0) \rightarrow \mathrm{GL}(r, \mathbb{C})$. Let $\alpha : \tilde{X} \rightarrow \mathrm{GL}(r, \mathbb{C})/\mathrm{U}(r)$ be the ρ -equivariant harmonic map defined on the universal cover of the Kähler manifold (X, x_0) . Then $\alpha \circ \tilde{\beta}$ is $\beta^*\rho$ -equivariant harmonic, where $\tilde{\beta}$ is the lift of the map β in Theorem 2.3 to a universal covering of X_1 . Also, as noted in Remark 2.2, the Yang–Mills–Higgs equation for (V, θ) does not depend on the Kähler form ω . Theorem 2.3 follows from these facts.

Note that Remark 2.2 can be seen as a particular case of Theorem 2.3 by setting β to be the identity map of X equipped with two different Kähler structures.

A useful particular case of Theorem 2.3 is the following:

Corollary 2.4. *Let the map β in Theorem 2.3 be such that the corresponding homomorphism of fundamental groups*

$$\beta_* : \pi_1(X_1, x_1) \rightarrow \pi_1(X, \beta(x_1))$$

is trivial. For any polystable Higgs bundle (V, θ) on X of rank r with $c_1(V) = 0 = c_2(V)$,

- (1) $\beta^*V = \mathcal{O}_{X_1}^{\oplus r}$, and
- (2) $\beta^*\theta = 0$.

The equivalence of categories in Theorem 2.1, between the equivalence classes of completely reducible flat connections on X and the polystable Higgs bundles (V, θ) on X such that $c_1(V) = 0 = c_2(V)$, extend to the context of principal G -bundles, where G is any complex affine algebraic group [BG].

Let $\mathcal{H}(X, r)$ denote the moduli space of polystable Higgs bundles (V, θ) of rank r on X such that $c_1(V) = 0 = c_2(V)$, where c_i denotes the rational i -th Chern class. It

is canonically homeomorphic to the earlier defined moduli space $\mathcal{R}(X, r)$ (of equivalence classes of completely reducible homomorphisms from $\pi_1(X, x_0)$ to $\mathrm{GL}(r, \mathbb{C})$). However the complex structure of these two moduli spaces are different in general.

Now assume that X is a smooth projective variety defined over \mathbb{C} . Simpson proved the following two results in [Si2].

Theorem 2.5 ([Si2]). *There is an equivalence of categories between the following two:*

- (1) *The objects are flat complex connections on X , and morphisms are connection preserving homomorphisms.*
- (2) *Objects are semistable Higgs bundles (V, θ) on X such that $c_1(V) = 0 = c_2(V)$, where c_i denotes the i -th Chern class with coefficient in \mathbb{Q} ; the morphisms are homomorphisms of Higgs bundles.*

The equivalence of categories in Theorem 2.5 extends the one in Theorem 2.1 but by imposing the condition that X is complex projective.

Proposition 2.6 ([Si2]). *Take X and X_1 in Theorem 2.3 and Corollary 2.4 to be smooth complex projective varieties. Then Theorem 2.3 and Corollary 2.4 remain valid if polystability is replaced by semistability.*

2.2. Fujiki class \mathcal{C} manifolds. A compact complex manifold is said to be in *the Fujiki class \mathcal{C}* if it is the image of a compact Kähler space under a holomorphic map [Fu2]. A result of Varouchas, [Va, Section IV.3], asserts that a compact complex manifold M belongs to Fujiki class \mathcal{C} if and only if there is a holomorphic map

$$\phi : X \longrightarrow M$$

such that

- X is a compact Kähler manifold, and
- ϕ is bimeromorphic.

In other words, M lies in class \mathcal{C} if and only if it admits a compact Kähler modification.

3. A DESCENT RESULT FOR VECTOR BUNDLES

A holomorphic line bundle L on a compact complex Hermitian manifold (Y, ω_Y) is called numerically effective if for every $\epsilon > 0$, there is a Hermitian structure h_ϵ on L such that $\mathrm{Curv}(L, h_\epsilon) \geq -\epsilon\omega_Y$, where $\mathrm{Curv}(L, h_\epsilon)$ is the curvature of the Chern connection on L for h_ϵ ; since Y is compact, this condition does not depend on the choice of the Hermitian metric ω_Y [DPS, Definition 1.2]. A holomorphic vector bundle \mathcal{E} on Y is called numerically effective if the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$ is numerically effective [DPS, p. 305, Definition 1.9]. A holomorphic vector bundle \mathcal{E} on Y is called numerically flat if both \mathcal{E} and \mathcal{E}^* are numerically effective [DPS, p. 311, Definition 1.17].

Proposition 3.1. *Let $\phi : X \longrightarrow M$ be a proper bimeromorphic morphism between complex manifolds, and let $V \longrightarrow X$ be a holomorphic vector bundle such that for every $x \in M$,*

the restriction $V|_{\phi^{-1}(x)}$ is a numerically flat vector bundle. Then there exists a holomorphic vector bundle W on M such that $V \simeq \phi^*W$.

Proof. Set $r := \text{rank}(V)$. We shall prove the proposition in three steps.

Step 1. Assume that ϕ is the blow-up of M along a smooth center Z . This is well-known; for the convenience of the reader we give an argument in the spirit of the method of [GKPT] (Sections 4 and 5). Let $E \subset X$ be the exceptional divisor of the blow-up. The restriction $V|_E$ is a vector bundle such that the restriction to every fiber of

$$\phi|_E : E \longrightarrow Z$$

is numerically flat. As the fibers are projective spaces, it can be shown that the vector bundle $V|_E$ is trivial on the fibers of $\phi|_E$. Indeed, a numerically flat bundle admits a filtrations of holomorphic subbundles such that each successive quotient admits a unitary flat connection [DPS, p. 311, Theorem 1.18]. Since a projective space is simply connected, each successive quotient is actually trivial. On the other hand, an extension of a trivial bundle bundle on $\mathbb{C}\mathbb{P}^k$ by a trivial bundle is also trivial, because $H^1(\mathbb{C}\mathbb{P}^k, \mathcal{O}_{\mathbb{C}\mathbb{P}^k}) = 0$.

Since $\phi|_E$ is locally trivial, we see that there exists a vector bundle W_Z on Z such that $V|_E \simeq (\phi|_E)^*W_Z$. Consider now the projectivised vector bundle $\pi : \mathbb{P}(V) \longrightarrow X$. Then

$$\pi^*E \simeq \mathbb{P}(V|_E) \simeq \mathbb{P}((\phi|_E)^*W_Z) \simeq (\phi|_E)^*\mathbb{P}(W_Z)$$

is a divisor that admits a fibration onto $\mathbb{P}(W_Z)$. In fact, for any point $z \in Z$, we have

$$\mathbb{P}(V|_{\phi^{-1}(z)}) \simeq \phi^{-1}(z) \times \mathbb{P}(W_{Z,z})$$

and the fibration is given by projection onto the second factor. Since the restriction of the divisor E to $\phi^{-1}(z)$ is anti-ample, this also holds for the restriction of π^*E to the fibers of $\pi^*E \longrightarrow \mathbb{P}(W_Z)$. Now we can apply a theorem of Fujiki, [Fu1, p. 495, Theorem 2], to see that there exist a variety T and a bimeromorphic morphism $\tilde{\phi} : \mathbb{P}(V) \longrightarrow T$ such that $\tilde{\phi}|_{\pi^*E}$ is the fibration $\pi^*E \longrightarrow \mathbb{P}(W_Z)$ and the restriction of it to $\mathbb{P}(V) \setminus \pi^*E$ is an isomorphism. By construction the variety T admits a morphism onto M such that all the fibers are isomorphic to \mathbb{P}^{r-1} ; in particular, T is smooth and $\mathbb{P}(V)$ is the blowup of T along $\mathbb{P}(W_Z)$. The push-forward of $c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ onto T defines a Cartier divisor on T such that the restriction to the fibers of $T \longrightarrow M$ is the hyperplane class. Thus the corresponding direct image sheaf defines a vector bundle $W \longrightarrow M$ satisfying the condition that $V \simeq \phi^*W$.

Step 2. Assume that ϕ is the composition of smooth blowups. Set $X_0 := X$ and $X_k := M$, and for $i \in \{1, \dots, k\}$, let $\nu_i : X_{i-1} \longrightarrow X_i$ be blowups such that

$$\phi = \nu_k \circ \dots \circ \nu_1.$$

Since every ν_1 -fiber is contained in a ϕ -fiber, it is evident that the restriction of V to every ν_1 -fiber is trivial. Thus, by Step 1, there exists a vector bundle V_1 on X_1 such that $V \simeq \nu_1^*V_1$.

We shall now proceed by induction on $i \in \{1, \dots, k\}$ and assume that we have found a vector bundle $V_i \longrightarrow X_i$ such that its pull-back to X is isomorphic to V . We have to check that V_i satisfies the triviality condition with respect to the morphism ν_{i+1} : let $G \subset X_i$ be any ν_{i+1} -fiber and let $Z \subset (\nu_i \circ \dots \circ \nu_1)^{-1}(G)$ be an irreducible component that surjects

onto G . Since G is contracted by ν_{i+1} , the variety Z is contained in a ϕ -fiber. Consequently, $V|_Z$ is trivial. Since

$$V|_Z \simeq ((\nu_i \circ \dots \circ \nu_1)|_Z)^*(V_i|_G),$$

this shows that $V_i|_G$ is numerically flat. Thus we can apply Step 1 to ν_{i+1} .

Step 3. The general case. If $\nu : X' \rightarrow X$ is another bimeromorphic morphism, then the pull-back ν^*V is a vector bundle on X' that satisfies the assumption with respect to the morphism $\phi \circ \nu$. Thus it suffices to prove the statement for $\phi \circ \nu$. Since any bimeromorphic morphism between manifolds is dominated by a sequence of blowups with smooth centers, we can assume without loss generality that ϕ is a composition of blowups with smooth centers. Therefore, the proof is completed using Step 2. \square

4. NONABELIAN HODGE THEORY FOR MOISHEZON MANIFOLDS

Let M be a compact Moishezon manifold. Recall that Moishezon manifolds, defined as manifolds of maximal algebraic dimension, were introduced and studied by Moishezon in [Mo], where he proved that they are birational to smooth complex projective manifolds [Mo] (see also [Ue, p. 26, Theorem 3.6]).

Definition 4.1. Take a Higgs bundle (V, θ) on M such that $c_1(V) = 0 = c_2(V)$. The Higgs bundle (V, θ) is called *semistable* (respectively, *polystable*) if for every pair (C, τ) , where C is a compact connected Riemann surface and $\tau : C \rightarrow M$ is a holomorphic map, the pulled back Higgs bundle $(\tau^*V, \tau^*\theta)$ is semistable (respectively, polystable). Clearly, it is enough to consider only nonconstant maps τ .

When M is a smooth complex projective variety, then semistability and polystability according to Definition 4.1 coincide with those described in Section 2.1. Indeed, from Proposition 2.6 (respectively, Theorem 2.3) we know that a semistable (respectively, polystability) Higgs bundle (V, θ) on M with $c_1(V) = 0 = c_2(V)$ is semistable (respectively, polystability) in the sense of Definition 4.1. Conversely, if (V, θ) is a Higgs bundle on M with $c_1(V) = 0 = c_2(V)$ such that it is semistable (respectively, polystability) in the sense of Definition 4.1, then it is straightforward to deduce that (V, θ) is semistable (respectively, polystability).

Theorem 4.2. *Let M be a compact Moishezon manifold. There is an equivalence of categories between the following two:*

- (1) *The objects are flat complex connections on M , and morphisms are connection preserving homomorphisms.*
- (2) *Objects are Higgs bundles (V, θ) on M satisfying the following conditions: $c_1(V) = 0 = c_2(V)$, and (V, θ) is semistable. The morphisms are homomorphisms of Higgs bundles.*

Proof. Fix a holomorphic map

$$\phi : X \rightarrow M$$

from a smooth complex projective variety X such that ϕ is bimeromorphic.

Let (E, ∇) be a flat complex connection on M . Consider the flat complex connection $(\phi^*E, \phi^*\nabla)$ on X . Let (V, θ_V) be the semistable Higgs bundle on X that corresponds to it by Theorem 2.5.

We shall show that there is a holomorphic vector bundle W on M such that $\phi^*W = V$.

Let $\beta : F' \rightarrow X$ be the desingularization of a subvariety $F \subset X$ that is contained in a ϕ -fiber. Using the fact that F is contracted by the map ϕ we conclude that the homomorphism

$$\beta_* : \pi_1(F') \rightarrow \pi_1(X)$$

induced by β is trivial. Since Corollary 2.4 remains valid when polystability is replaced by semistability (see Proposition 2.6), from Corollary 2.4(1) we know that β^*V is a trivial holomorphic vector bundle. In particular, the restriction $V|_F$ is numerically flat. Therefore, by Proposition 3.1, there is a holomorphic vector bundle W on M such that $\phi^*W = V$.

Let $U \subset M$ be the open subset over which ϕ is an isomorphism. The Higgs field θ_V produces a Higgs field on $W|_U$. Now by Hartogs' extension theorem, this Higgs field on $W|_U$ extends to a Higgs field on W over M ; this extended Higgs field on W will be denoted by θ_W .

We have $c_1(W) = 0 = c_2(W)$, because $c_1(V) = 0 = c_2(V)$. We shall show that the Higgs bundle (W, θ_W) on M is semistable.

Take any pair (C, τ) , where C is a compact connected Riemann surface and $\tau : C \rightarrow M$ is a nonconstant holomorphic map. Then there is a triple $(\tilde{C}, \psi, \tilde{\tau})$ such that

- \tilde{C} is a compact connected Riemann surface,
- $\psi : \tilde{C} \rightarrow C$ is a surjective holomorphic map,
- $\tilde{\tau} : \tilde{C} \rightarrow X$ is a holomorphic map, and
- $\phi \circ \tilde{\tau} = \tau \circ \psi$.

From Theorem 2.3 and Proposition 2.6 we know that the Higgs bundle $(\tilde{\tau}^*V, \tilde{\tau}^*\theta_V)$ is semistable. Combining this with the two facts that $\phi \circ \tilde{\tau} = \tau \circ \psi$ and $(V, \theta_V) = (\phi^*W, \phi^*\theta_W)$, we conclude that the Higgs bundle $(\psi^*\tau^*W, \psi^*\tau^*\theta_W)$ is semistable. But this implies that $(\tau^*W, \tau^*\theta_W)$ is semistable. Indeed, if a subbundle $W' \subset \tau^*W$ contradicts the semistability condition for $(\tau^*W, \tau^*\theta_W)$, then $\psi^*W' \subset \psi^*\tau^*W$ contradicts the semistability condition for $(\psi^*\tau^*W, \psi^*\tau^*\theta_W)$. Therefore, the Higgs bundle (W, θ_W) is semistable.

To prove the converse, let (V, θ) be a Higgs bundle on M satisfying the following conditions: $c_1(V) = 0 = c_2(V)$, and (V, θ) is semistable. Consider the Higgs bundle $(\phi^*V, \phi^*\theta)$ on X . We evidently have $c_1(\phi^*V) = 0 = c_2(\phi^*V)$. We shall prove that $(\phi^*V, \phi^*\theta)$ is semistable.

Take any pair pair (C, τ_1) , where C is a compact connected Riemann surface and $\tau_1 : C \rightarrow X$ is a holomorphic map. Set

$$\tau = \phi \circ \tau_1.$$

Therefore, the given condition that $(\tau^*V, \tau^*\theta)$ is semistable implies that $(\tau_1^*\phi^*V, \tau_1^*\phi^*\theta)$ is semistable. But this implies that $(\phi^*V, \phi^*\theta)$ is semistable with respect to any polarization on X . Let (E', ∇') be the flat complex connection on X that corresponds to $(\phi^*V, \phi^*\theta)$ by

Theorem 2.5. Since the map ϕ is bimeromorphic, the homomorphism $\phi_* : \pi_1(X) \rightarrow \pi_1(M)$ induced by it is an isomorphism. Consequently, (E', ∇') produces a flat complex connection on M .

It is straightforward to check that the above two constructions, namely from flat connection on M to Higgs bundles on M and vice versa, are inverses of each other. \square

Proposition 4.3. *Let M be a compact Moishezon manifold. There is an equivalence of categories between the following two:*

- (1) *The objects are completely reducible flat complex connections on M , and morphisms are connection preserving homomorphisms.*
- (2) *Objects are Higgs bundles (V, θ) on M such that $c_1(V) = 0 = c_2(V)$, and $(\phi^*V, \phi^*\theta)$ is polystable; the morphisms are homomorphisms of Higgs bundles.*

Proof. The proof is very similar to the proof of Theorem 4.2.

Let (V, θ) be a Higgs bundle on M such that $c_1(V) = 0 = c_2(V)$ and $(\phi^*V, \phi^*\theta)$ is polystable. Take any pair (C, τ_1) , where C is a compact connected Riemann surface and $\tau_1 : C \rightarrow X$ is a holomorphic map. Setting $\tau = \phi \circ \tau_1$ we conclude that $(\tau^*V, \tau^*\theta) = (\tau_1^*\phi^*V, \tau_1^*\phi^*\theta)$ is polystable. This implies that $(\phi^*V, \phi^*\theta)$ is semistable. Let (E, ∇) be the complex flat connection on X that corresponds to $(\phi^*V, \phi^*\theta)$ by Theorem 2.5. If $\tau_1(C)$ is an intersection of very ample hypersurfaces on X , the homomorphism of fundamental groups induced by τ_1

$$\tau_{1*} : \pi_1(C, x_0) \rightarrow \pi_1(X, x_0)$$

is surjective. Since $(\tau_1^*\phi^*V, \tau_1^*\phi^*\theta)$ is polystable, the restriction of (E, ∇) to $\tau_1(C)$ is completely reducible. Now from the surjectivity of τ_{1*} it follows immediately that (E, ∇) is completely reducible on X . Since the homomorphism $\phi_* : \pi_1(X) \rightarrow \pi_1(M)$ induced by ϕ is an isomorphism, the completely reducible complex flat connection (E, ∇) on X produces a completely reducible complex flat connection on M .

To prove the converse, let (E, ∇) be a completely reducible complex flat connection on M . Since the homomorphism $\phi_* : \pi_1(X) \rightarrow \pi_1(M)$ induced by ϕ is an isomorphism, the pulled back flat connection $(\phi^*E, \phi^*\nabla)$ is completely reducible. Let (V, θ_V) be the polystable Higgs bundle on X corresponding to $(\phi^*E, \phi^*\nabla)$. As in the proof of Theorem 4.2, there is a Higgs bundle (W, θ_W) on M such that

$$(\phi^*W, \phi^*\theta_W) = (V, \theta_V)$$

and $c_1(W) = 0 = c_2(W)$.

In the proof of Theorem 4.2 it was shown that (W, θ_W) is semistable. To complete the proof we need to show that (W, θ_W) is polystable.

Take any pair (C, τ) , where C is a compact connected Riemann surface and $\tau : C \rightarrow M$ is a nonconstant holomorphic map. There is a triple $(\tilde{C}, \psi, \tilde{\tau})$ such that

- \tilde{C} is a compact connected Riemann surface,
- $\psi : \tilde{C} \rightarrow C$ is a surjective holomorphic map,
- $\tilde{\tau} : \tilde{C} \rightarrow X$ is a holomorphic map, and

- $\phi \circ \tilde{\tau} = \tau \circ \psi$.

We know that $(\tilde{\tau}^*\phi^*W, \tilde{\tau}^*\phi^*\theta_W)$ is polystable and the corresponding flat connection, namely $(\tilde{\tau}^*\phi^*E, \tilde{\tau}^*\phi^*\nabla)$, is completely reducible. For the homomorphism of fundamental groups induced by ψ

$$\psi_* : \pi_1(\tilde{C}) \longrightarrow \pi_1(C)$$

the image is a finite index subgroup. This implies that the flat connection $(\tau^*E, \tau^*\nabla)$ is completely reducible. Therefore, the Higgs bundle $(\tau^*W, \tau^*\theta_W)$ is polystable, so (W, θ_W) is polystable. This completes the proof. \square

5. NONABELIAN HODGE THEORY FOR FUJIKI CLASS \mathcal{C} MANIFOLDS

Let M be a compact connected complex manifold lying in Fujiki class \mathcal{C} . Fix a bimeromorphic map

$$\phi : X \longrightarrow M \tag{5.1}$$

such that X is compact Kähler. Let (V, θ) be a Higgs bundle on M such that $c_1(V) = 0 = c_2(V)$. Further assume that the pulled back Higgs bundle $(\phi^*V, \phi^*\theta)$ is polystable. As noted in Remark 2.2, this condition is independent of the choice of the Kähler form on X .

Lemma 5.1. *Let $f : Y \longrightarrow M$ be a holomorphic map from a compact Kähler manifold Y such that f is bimeromorphic. Then the pulled back Higgs bundle $(f^*V, f^*\theta)$ is also polystable.*

Proof. Consider the irreducible component of the fiber product $Y \times_M X$ that dominates M . Let Z be a desingularization of it. Let p_Y and p_X be the natural projections of Z to Y and X respectively.

Since $(\phi^*V, \phi^*\theta)$ is polystable with $c_1(\phi^*V) = 0 = c_2(\phi^*V)$, from Theorem 2.3 we know that $(p_X^*\phi^*V, p_X^*\phi^*\theta)$ is polystable; as before, this condition is independent of the choice of the Kähler form on Z . The Higgs bundle $(p_Y^*f^*V, p_Y^*f^*\theta)$ is polystable, because

$$(p_X^*\phi^*V, p_X^*\phi^*\theta) = (p_Y^*f^*V, p_Y^*f^*\theta).$$

Let (W, ∇) be the completely reducible flat complex connection on Z .

The homomorphism $p_{Y*} : \pi_1(Z, z_0) \longrightarrow \pi_1(Y, p_Y(z_0))$ induced by p_Y is an isomorphism, because the map p_Y is bimeromorphic. So (W, ∇) gives a completely reducible flat complex connection (W', ∇') on Y . The Higgs bundle on Y corresponding to (W', ∇') is isomorphic to $(f^*V, f^*\theta)$. In particular, $(f^*V, f^*\theta)$ is polystable. \square

From Lemma 5.1 it follows that the second category in the following theorem is independent of the choice of the pair (X, ϕ) .

Theorem 5.2. *Let M be a compact connected complex manifold lying in Fujiki class \mathcal{C} . There is an equivalence of categories between the following two:*

- (1) *The objects are completely reducible flat complex connections on M , and morphisms are connection preserving homomorphisms.*

- (2) *Objects are Higgs bundles (V, θ) on M such that $c_1(V) = 0 = c_2(V)$, and $(\phi^*V, \phi^*\theta)$ is polystable; the morphisms are homomorphisms of Higgs bundles.*

Proof. The homomorphism of fundamental groups induced by ϕ

$$\phi_* : \pi_1(X, x_0) \longrightarrow \pi_1(M, \phi(x_0))$$

is an isomorphism, because ϕ is bimeromorphic. Consequently, the operation of pullback, to X , of flat vector bundles on M identifies the flat bundles on M with those on X . Also, connection preserving homomorphisms between two flat bundles on M coincide with connection preserving homomorphisms between their pullback to X .

Let (V_1, θ_1) be a polystable Higgs bundle on X with $c_1(V_1) = 0 = c_2(V_1)$. Then, as shown in the proof of Theorem 4.2, using Proposition 3.1 the vector bundle V_1 descends to a holomorphic vector bundle on M , meaning there exists a bundle W_1 on M such that $V_1 = \phi^*W_1$. Since $c_1(V_1) = 0 = c_2(V_1)$, this implies that $c_1(W_1) = 0 = c_2(W_1)$.

Let $U \subset X$ be the open subset over which ϕ is an isomorphism.

The Higgs field θ_1 defines a Higgs field on $W_1|_{\phi(U)}$. Again using Hartogs' extension theorem this Higgs field on $W_1|_{\phi(U)}$ extends to a Higgs field on W_1 ; this extended Higgs field on W_1 will be denoted by θ' . It is evident that $(\phi^*W_1, \phi^*\theta') = (V_1, \theta_1)$.

If (W_2, θ'') is a polystable Higgs bundle on M with $c_1(W_2) = 0 = c_2(W_2)$, then it can be shown that

$$H^0(M, \text{Hom}((W_1, \theta'), (W_2, \theta''))) = H^0(X, \text{Hom}((V_1, \theta_1), (\phi^*W_2, \phi^*\theta''))). \quad (5.2)$$

To prove this, let (E_1, ∇_1) (respectively, (E_2, ∇_2)) be the completely reducible flat complex connection on M corresponding to (W_1, θ') (respectively, (W_2, θ'')). Then

$$H^0(X, \text{Hom}((V_1, \theta_1), (\phi^*W_2, \phi^*\theta''))) = H^0(X, \text{Hom}((\phi^*E_1, \phi^*\nabla_1), (\phi^*E_2, \phi^*\nabla_2))).$$

But

$$H^0(X, \text{Hom}((\phi^*E_1, \phi^*\nabla_1), (\phi^*E_2, \phi^*\nabla_2))) = H^0(M, \text{Hom}((E_1, \nabla_1), (E_2, \nabla_2))).$$

Hence from the isomorphism

$$H^0(M, \text{Hom}((E_1, \nabla_1), (E_2, \nabla_2))) = H^0(M, \text{Hom}((W_1, \theta'), (W_2, \theta'')))$$

we conclude that (5.2) holds. This completes the proof. \square

It is rather routine to check that the results on [BG] extend to the context of Fujiki class \mathcal{C} manifolds.

ACKNOWLEDGEMENTS

We are grateful to Andreas H\"oring who helped us providing the proof of Proposition 3.1. We also thank Yohan Brunebarbe and Carlos Simpson for useful conversations on the subject.

This work has been supported by the French government through the UCAJEDI Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR2152IDEX201. The first-named author is partially supported by a J.

C. Bose Fellowship, and school of mathematics, TIFR, is supported by 12-R&D-TFR-5.01-0500.

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