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ON SOME SUBSPACES OF THE EXTERIOR ALGEBRA OF A SIMPLE LIE ALGEBRA

JEAN-YVES CHARBONNEL

ABSTRACT. In this article, we are interested in some subspaces of the exterior algebra of a simple Lie algebra \mathfrak{g} . In particular, we prove that some graded subspaces of degree d generate the \mathfrak{g} -module $\wedge^d(\mathfrak{g})$ for some integers d .

CONTENTS

1. Introduction	1
2. Orthogonal complement	3
3. Action of the unipotent radical of a parabolic subgroup	6
4. Proof of Theorem 1.1	10
Appendix A. Some remarks on root systems	18
References	20

1. INTRODUCTION

In this note, the base field \mathbb{k} is algebraically closed of characteristic 0, \mathfrak{g} is a simple Lie algebra of finite dimension, ℓ is its rank, and G is its adjoint group.

1.1. Main results. Let \mathfrak{b} be a Borel subalgebra of \mathfrak{g} , \mathfrak{h} a Cartan subalgebra of \mathfrak{g} , contained in \mathfrak{b} and \mathfrak{u} the nilpotent radical of \mathfrak{b} . Set $b_{\mathfrak{g}} := \dim \mathfrak{b}$ and $n := b_{\mathfrak{g}} - \ell$. For k a nonnegative integer, let $\wedge^k(\mathfrak{g})$ be the component of degree k of the exterior algebra $\wedge(\mathfrak{g})$ of \mathfrak{g} . The adjoint action of G in \mathfrak{g} induces an action of G in $\wedge(\mathfrak{g})$. For all k , $\wedge^k(\mathfrak{g})$ is invariant under this action.

For \mathfrak{p} parabolic subalgebra of \mathfrak{g} , containing \mathfrak{b} , denote by $\mathfrak{p}_{\mathfrak{u}}$ the nilpotent radical of \mathfrak{p} , \mathfrak{l} the reductive factor of \mathfrak{p} , containing \mathfrak{h} , \mathfrak{z} the center of \mathfrak{l} and $\mathfrak{p}_{-\mathfrak{u}}$ the complement to \mathfrak{p} in \mathfrak{g} , invariant under the adjoint action of \mathfrak{h} . Let \mathfrak{d} be the derived algebra of \mathfrak{l} and $\mathfrak{d}_1, \dots, \mathfrak{d}_n$ its simple factors. Set:

$$E_{\mathfrak{p}} := \mathfrak{p}_{-\mathfrak{u}} \oplus \mathfrak{z} \oplus \mathfrak{p}_{\mathfrak{u}}, \quad n_i := \dim \mathfrak{d}_i \cap \mathfrak{u},$$

$$\mathbb{I}_k := \{(j_1, \dots, j_n) \in \mathbb{N}^n \mid j_1 \leq n_1, \dots, j_n \leq n_n, j_1 + \dots + j_n = k\}$$

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for k positive integer. Denote by $V'_{k,p}$ and $V_{k,p}$ the subspaces of $\wedge^k(\mathfrak{g})$,

$$V'_{k,p} := \bigoplus_{(j_1, \dots, j_n) \in \mathbb{I}_k} \wedge^{j_1}(\mathfrak{d}_1) \wedge \cdots \wedge \wedge^{j_n}(\mathfrak{d}_n) \quad \text{and} \quad V_{k,p} := \bigoplus_{i=0}^k \wedge^i(E_p) \wedge V'_{k-i,p}.$$

The goal of this note is the following theorem:

Theorem 1.1. *Let $k = 1, \dots, n$. Then $\wedge^k(\mathfrak{g})$ is the G -submodule of $\wedge^k(\mathfrak{g})$ generated by $V_{k,p}$.*

This result arises from the study of the commuting variety of \mathfrak{g} (see [Ch20]). One of the main step of the proof is to consider the orthogonal complements to some subspaces of $\wedge^k(\mathfrak{g})$ in $\wedge^k(\mathfrak{g})$ with respect to the canonical extension of the Killing form of \mathfrak{g} to $\wedge^k(\mathfrak{g})$.

1.2. Notations. • Let $\mathbb{k}^* := \mathbb{k} \setminus \{0\}$. For E a finite set, its cardinality is denoted by $|E|$. For k, m positive integers, set:

$$\mathbb{N}_k^m := \{(j_1, \dots, j_m) \in \mathbb{N}^m \mid j_1 + \cdots + j_m = k\}.$$

As usual, for $i = (i_1, \dots, i_m)$ in \mathbb{N}^m ,

$$|i| := i_1 + \cdots + i_m.$$

• For V vector space, denote by $\wedge(V)$ the exterior algebra of V . This algebra has a natural gradation. For i integer, denote by $\wedge^i(V)$ the space of degree i of $\wedge(V)$. In particular, for i negative, $\wedge^i(V)$ is equal to $\{0\}$. As \mathfrak{g} is a G -module for the adjoint action, so is $\wedge^i(\mathfrak{g})$ for all i .

Lemma 1.2. *Let A be a subgroup of G , k a positive integer, i a positive integer smaller than k , V a subspace of $\wedge^i(\mathfrak{g})$ and W the A -submodule of $\wedge^i(\mathfrak{g})$ generated by V . Then, for all A -submodule W' of $\wedge^{k-i}(\mathfrak{g})$, $W \wedge W'$ is the A -submodule of $\wedge^k(\mathfrak{g})$ generated by $V \wedge W'$.*

Proof. Let W'' be the A -submodule of $\wedge^k(\mathfrak{g})$ generated by $V \wedge W'$. Let ω and ω' be in W and W' respectively. For some $\omega_1, \dots, \omega_m$ in V and g_1, \dots, g_m in A ,

$$\omega = g_1 \cdot \omega_1 + \cdots + g_m \cdot \omega_m,$$

whence

$$\omega \wedge \omega' = g_1 \cdot (\omega_1 \wedge g_1^{-1} \cdot \omega') + \cdots + g_m \cdot (\omega_m \wedge g_m^{-1} \cdot \omega')$$

and $W'' = W \wedge W'$. □

• The Killing form of \mathfrak{g} is denoted by $\langle \cdot, \cdot \rangle$. For k positive integer, the Killing form of \mathfrak{g} has a natural extension to $\wedge^k(\mathfrak{g})$ and this extension is not degenerate.

• For \mathfrak{a} a semisimple Lie algebra, denote by $\mathfrak{b}_{\mathfrak{a}}$ the dimension of its Borel subalgebras and $\ell_{\mathfrak{a}}$ its rank.

• Let \mathcal{R} be the root system of \mathfrak{h} in \mathfrak{g} , \mathcal{R}_+ the positive root system of \mathcal{R} defined by \mathfrak{b} and Π the basis of \mathcal{R}_+ . For α in \mathcal{R} , H_{α} is the coroot of α , the corresponding root subspace is denoted by \mathfrak{g}^{α} and a generator x_{α} of \mathfrak{g}^{α} is chosen so that $\langle x_{\alpha}, x_{-\alpha} \rangle = 1$.

• We consider on Π its structure of Dynkin diagram. As \mathfrak{g} is simple, Π is connected and has three extremities when Π has type D_{ℓ} , E_6 , E_7 , E_8 , one extremity when Π has type A_1 and 2 otherwise. The elements $\beta_1, \dots, \beta_{\ell}$ of Π are ordered as in [Bou02, Ch. VI].

• Let X be a subset of Π . We denote by \mathcal{R}_X the root subsystem of \mathcal{R} generated by X and we set

$$\langle X \rangle := \mathcal{R}_+ \cap \mathcal{R}_X \quad \text{so that} \quad \mathcal{R}_X = \langle X \rangle \cup -\langle X \rangle.$$

Let \mathfrak{p}_X be the parabolic subalgebra of \mathfrak{g} ,

$$\mathfrak{p}_X := \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle X \rangle} \mathfrak{g}^{-\alpha},$$

$\mathfrak{p}_{X,u}$ its nilpotent radical, I_X the reductive factor of \mathfrak{p}_X containing \mathfrak{h} , \mathfrak{z}_X the center of I_X , \mathfrak{d}_X the derived algebra of I_X , $\mathfrak{p}_{X,-u}$ the complement to \mathfrak{p}_X in \mathfrak{g} , invariant under $\text{ad } \mathfrak{h}$ and E_X the sum of \mathfrak{z}_X , $\mathfrak{p}_{X,u}$, $\mathfrak{p}_{X,-u}$. When X is empty, \mathfrak{p}_X is the Borel subalgebra \mathfrak{b} .

• Let X be a nonempty subset of Π and X_1, \dots, X_{n_X} its connected components. For $i = 1, \dots, n_X$, denote by n_i the cardinality of $\langle X_i \rangle$ and \mathfrak{d}_i the subalgebra of \mathfrak{g} generated by $\mathfrak{g}^{\pm\beta}$, $\beta \in X_i$. Then $\mathfrak{d}_1, \dots, \mathfrak{d}_{n_X}$ are the simple factors of \mathfrak{d}_X . For k positive integer, set:

$$V_{k, \mathfrak{p}_X} := \bigoplus_{j_1=0}^{n_1} \cdots \bigoplus_{j_{n_X}=0}^{n_{n_X}} \wedge^{j_1}(\mathfrak{d}_1) \wedge \cdots \wedge \wedge^{j_{n_X}}(\mathfrak{d}_{n_X}) \wedge \wedge^{k-j_1-\cdots-j_{n_X}}(E_X)$$

and denote by $V_{k,X}$ the G -submodule of $\wedge^k(\mathfrak{g})$ generated by V_{k, \mathfrak{p}_X} .

2. ORTHOGONAL COMPLEMENT

Let $\ell \geq 2$ and X a nonempty subset of Π . Set:

$$\mathfrak{p} := \mathfrak{p}_X, \quad \mathfrak{p}_u := \mathfrak{p}_{X,u}, \quad I := I_X, \quad \mathfrak{p}_{-u} := \mathfrak{p}_{X,-u},$$

$$\mathfrak{p}_{\pm,u} := \mathfrak{p}_u \oplus \mathfrak{p}_{-u}, \quad \mathfrak{p}_- := I \oplus \mathfrak{p}_{-u}, \quad d := \dim \mathfrak{p}_u.$$

2.1. General fact. Let A be a subgroup of G . For k positive integer and W subspace of $\wedge^k(\mathfrak{g})$, denote by W^\perp the orthogonal complement to W in $\wedge^k(\mathfrak{g})$. As the bilinear form on $\wedge^k(\mathfrak{g})$, defined by the Killing form, is not degenerate,

$$\dim W + \dim W^\perp = \dim \wedge^k(\mathfrak{g}).$$

Lemma 2.1. *Let k be a positive integer smaller than $\dim \mathfrak{g}$. Let V be a subspace of $\wedge^k(\mathfrak{g})$. Denote by W the A -submodule of $\wedge^k(\mathfrak{g})$ generated by V . Then W^\perp is the biggest A -submodule of $\wedge^k(\mathfrak{g})$, contained in V^\perp .*

Proof. Denote by $W^\#$ the biggest A -submodule contained in V^\perp . As W is a A -module, so is W^\perp . Then W^\perp is contained in $W^\#$. Moreover, V is contained in the orthogonal complement to $W^\#$ in $\wedge^k(\mathfrak{g})$. Hence W is orthogonal to $W^\#$ since the orthogonal complement to $W^\#$ is a A -module. As a result, $W^\# = W^\perp$. \square

2.2. Orthogonality. Let V be a finite dimensional vector space with a non degenerate symmetric bilinear form on V . For k positive integer, it induces a non degenerate symmetric bilinear form on $\wedge^k(V)$. Let V_1, \dots, V_m be pairwise orthogonal subspaces of V such that V is the direct sum of these subspaces. For $i = (i_1, \dots, i_m)$ in \mathbb{N}_k^m , set:

$$C_{i,V} := \wedge^{i_1}(V_1) \wedge \dots \wedge \wedge^{i_m}(V_m).$$

If V_m is the direct sum of two isotropic subspaces $V_{m,+}$ and $V_{m,-}$, for $i = (i_1, \dots, i_{m+1})$ in \mathbb{N}_k^{m+1} , set:

$$i^* := (i_1, \dots, i_{m-1}, i_{m+1}, i_m) \quad \text{and} \\ C'_{i,V} := \wedge^{i_1}(V_1) \wedge \dots \wedge \wedge^{i_{m-1}}(V_{m-1}) \wedge \wedge^{i_m}(V_{m,+}) \wedge \wedge^{i_{m+1}}(V_{m,-}).$$

Lemma 2.2. *Let k be a positive integer.*

(i) *For i, i' in \mathbb{N}_k^m , if $i \neq i'$ then $C_{i,V}$ is orthogonal to $C_{i',V}$.*

(ii) *Suppose that V_m is the direct sum of two isotropic subspaces $V_{m,+}$ and $V_{m,-}$. For i, i' in \mathbb{N}_k^{m+1} , if $i' \neq i^*$ then $C'_{i,V}$ is orthogonal to $C'_{i',V}$.*

Proof. Denote by $\langle \cdot, \cdot \rangle$ the symmetric bilinear form on V and $\wedge^k(V)$. As V_1, \dots, V_m are pairwise orthogonal and V is the direct sum of these subspaces, for $i = 1, \dots, m$, the restriction to $V_i \times V_i$ of $\langle \cdot, \cdot \rangle$ is non degenerate. For $j = 1, \dots, m$, let n_j be the dimension of the sum

$$V_1 \oplus \dots \oplus V_j$$

and v_1, \dots, v_{n_m} an orthonormal basis of V such that $\{v_1, \dots, v_{n_j}\}$ is contained in the union of V_1, \dots, V_j for $j = 1, \dots, m$.

(i) Let i and i' be in \mathbb{N}_k^m such that $i \neq i'$. If $k > n_m$, there is nothing to prove. Suppose $k \leq n_m$. For $j = j_1, \dots, j_k$ in $\{1, \dots, n_m\}$ such that $1 \leq j_1 < \dots < j_k \leq n_m$, set:

$$w_j := v_{j_1} \wedge \dots \wedge v_{j_k}.$$

Setting $n_0 := 0$, w_j is in $C_{i,V}$ if and only if

$$|\{l \in \{1, \dots, k\} \mid n_{s-1} + 1 \leq j_l \leq n_s\}| = i_s$$

for $s = 1, \dots, m$. Denote by I_i the set of j satisfying this condition so that w_j , $j \in I_i$ is a basis of $C_{i,V}$.

Let (j, j') be in $I_i \times I_{i'}$. By definition,

$$\langle w_j, w_{j'} \rangle = \det(\langle v_{j_l}, v_{j'_l} \rangle, 1 \leq l, l' \leq k).$$

As the basis v_1, \dots, v_{n_m} is orthonormal,

$$\langle v_{j_l}, v_{j'_l} \rangle = \delta_{j_l, j'_l}$$

with $\delta_{s,s'}$ the Kronecker symbol. As a result, if all the lines of the above matrix are all different from 0 then

$$|\{l \in \{1, \dots, k\} \mid n_{s-1} + 1 \leq j_l \leq n_s\}| = |\{l \in \{1, \dots, k\} \mid n_{s-1} + 1 \leq j'_l \leq n_s\}|$$

for $s = 1, \dots, m$ since V_1, \dots, V_m are pairwise orthogonal. Then $\langle w_j, w_{j'} \rangle = 0$ since $i \neq i'$, whence the assertion.

(ii) Let i and i' be in \mathbb{N}_k^{m+1} such that $i' \neq i^*$. By (i), we can suppose that $i_s = i'_s$ for $s = 1, \dots, m-1$. Since $V_{m,+}$ and $V_{m,-}$ are isotropic, they have the same dimension m_0 and V_m has a basis u_1, \dots, u_{2m_0} such that

$$\{u_1, \dots, u_{m_0}\} \subset V_{m,+}, \quad \{u_{m_0+1}, \dots, u_{2m_0}\} \subset V_{m,-}, \quad \langle u_s, u_{s'+m_0} \rangle = \delta_{s,s'}$$

for $1 \leq s, s' \leq m_0$. Let v'_1, \dots, v'_{n_m} be the basis of V such that $v'_l = v_l$ for $l = 1, \dots, n_{m-1}$ and $v'_{l'} = u_{l'-n_{m-1}}$ for $l' = n_{m-1} + 1, \dots, n_m$. For $j = j_1, \dots, j_k$ in $\{1, \dots, n_m\}$ such that $1 \leq j_1 < \dots < j_k \leq n_m$, set:

$$w'_j = v'_{j_1} \wedge \dots \wedge v'_{j_k}.$$

Then w'_j is in $C'_{i,V}$ if and only if

$$|\{l \in \{1, \dots, k\} \mid n_{m-1} + 1 \leq j_l \leq n_{m-1} + m_0\}| = i_m, \quad |\{l \in \{1, \dots, k\} \mid n_{m-1} + m_0 + 1 \leq j_l \leq n_m\}| = i_{m+1},$$

$$|\{l \in \{1, \dots, k\} \mid n_{s-1} + 1 \leq j_l \leq n_s\}| = i_s$$

for $s = 1, \dots, m-1$. Denote by I_i the set of j satisfying this condition so that w'_j , $j \in I_i$ is a basis of $C'_{i,V}$.

Let (j, j') be in $I_i \times I_{i'}$. By definition,

$$\langle w'_j, w'_{j'} \rangle = \det(\langle v'_{j_l}, v'_{j'_l} \rangle, 1 \leq l, l' \leq k).$$

Then

$$j_l \leq n_{m-1} \implies \langle v'_{j_l}, v'_{j'_l} \rangle = \delta_{j_l, j'_l},$$

$$j_l > n_{m-1} \quad \text{and} \quad j'_l > n_{m-1} \implies \langle v'_{j_l}, v'_{j'_l} \rangle = \delta_{|j_l - j'_l|, m_0}.$$

As a result, if all the lines of the above matrix are all different from 0 then

$$|\{l \in \{1, \dots, k\} \mid n_{m-1} + 1 \leq j_l \leq n_{m-1} + m_0\}| = |\{l \in \{1, \dots, k\} \mid n_{m-1} + m_0 + 1 \leq j'_l \leq n_m\}|$$

since $i_s = i'_s$ for $s \leq m-1$ and V_1, \dots, V_m are pairwise orthogonal. Then $\langle w_j, w_{j'} \rangle = 0$ since $i' \neq i^*$, whence the assertion. \square

For $i = (i_1, i_2, i_3)$ in \mathbb{N}^3 , set:

$$C_i := \wedge^{i_1}(\mathbb{1}) \wedge \wedge^{i_2}(\mathfrak{p}_u) \wedge \wedge^{i_3}(\mathfrak{p}_{-,u})$$

and denote by i^* the element (i_1, i_3, i_2) of \mathbb{N}^3 .

Corollary 2.3. *Let k be a positive integer.*

(i) *For i, i' in \mathbb{N}_k^3 , C_i is orthogonal to $C_{i'}$ if $i^* \neq i'$.*

(ii) *For i in \mathbb{N}_k^3 , the orthogonal complement to C_i in $\wedge^k(\mathfrak{g})$ is equal to*

$$\bigoplus_{i' \in \mathbb{N}_k^3 \setminus \{i^*\}} C_{i'}.$$

Proof. (i) Let i and i' be in \mathbb{N}_k^3 such that $i^* \neq i'$. By Lemma 2.1(ii), C_i is orthogonal to $C_{i'}$ since $\mathbb{1}$ and $\mathfrak{p}_{\pm, u}$ are orthogonal and \mathfrak{p}_u and $\mathfrak{p}_{-, u}$ are isotropic.

(ii) Since $\wedge^k(\mathfrak{g})$ is the direct sum of $C_{i'}$, $i' \in \mathbb{N}_k^3$, the orthogonal complement to C_i in $\wedge^k(\mathfrak{g})$ is the direct sum of $C_{i'}$, $i' \in \mathbb{N}_k^{n+3} \setminus \{i^*\}$ by (i). \square

Corollary 2.4. *Let $k = 1, \dots, d$.*

- (i) *The orthogonal complement to $\wedge^k(\mathfrak{p}_{-,u})$ in $\wedge^k(\mathfrak{g})$ is equal to $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$.*
- (ii) *The orthogonal complement to $\wedge^k(\mathfrak{p}_{\pm,u})$ in $\wedge^k(\mathfrak{g})$ is equal to $\mathfrak{l} \wedge \wedge^{k-1}(\mathfrak{g})$.*

Proof. (i) Let I_1 be the subset of \mathbb{N}_k^3 ,

$$I_1 := \{(i_1, i_2, i_3) \in \mathbb{N}_k^3 \mid i_1 = i_2 = 0\}.$$

The complement to I_1^* in \mathbb{N}_k^3 is equal to

$$\{(i_1, i_2, i_3) \in \mathbb{N}_k^3 \mid i_1 > 0 \text{ or } i_3 > 0\},$$

whence the assertion by Corollary 2.3 since

$$\wedge^k(\mathfrak{p}_{-,u}) = \bigoplus_{i \in I_1} C_i.$$

(ii) Let I_2 be the subset of \mathbb{N}_k^3 ,

$$I_2 := \{(i_1, i_2, i_3) \in \mathbb{N}_k^3 \mid i_1 = 0\}.$$

The complement to I_2^* in \mathbb{N}_k^3 is equal to

$$\{(i_1, i_2, i_3) \in \mathbb{N}_k^3 \mid i_1 > 0\},$$

whence the assertion by Corollary 2.3 since $\wedge^k(\mathfrak{p}_{\pm,u})$ is the sum of C_i , $i \in I_2$. □

3. ACTION OF THE UNIPOTENT RADICAL OF A PARABOLIC SUBGROUP

Let $\ell \geq 2$ and X a subset of Π . Set:

$$\mathfrak{p} := \mathfrak{p}_X, \quad \mathfrak{p}_u := \mathfrak{p}_{X,u}, \quad \mathfrak{l} := \mathfrak{l}_X, \quad \mathfrak{p}_{-,u} := \mathfrak{p}_{X,-,u},$$

$$\mathfrak{p}_{\pm,u} := \mathfrak{p}_u \oplus \mathfrak{p}_{-,u}, \quad \mathfrak{p}_- := \mathfrak{l} \oplus \mathfrak{p}_{-,u}, \quad d := \dim \mathfrak{p}_u.$$

Denote by L and H the connected closed subgroups of G whose Lie algebras are \mathfrak{l} and \mathfrak{h} respectively. Let P and P_- be the normalizers of \mathfrak{p} and \mathfrak{p}_- in G and P_u and $P_{-,u}$ their unipotent radicals.

3.1. Invariant subspaces. Let $k = 1, \dots, d$, W_k the biggest P_u -submodule of $\wedge^k(\mathfrak{g})$ contained in $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$ and $V_{k,u}$ the P_u -submodule of $\wedge^k(\mathfrak{g})$ generated by $\wedge^k(\mathfrak{p}_{-,u})$.

Lemma 3.1. *Let $W_{k,0}$ be the subspace of elements of $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$ invariant under u .*

- (i) *The subspace W_k of $\wedge^k(\mathfrak{g})$ is invariant under u .*
- (ii) *The subspace $W_{k,0}$ of $\wedge^k(\mathfrak{g})$ is contained in W_k and generated by highest weight vectors.*
- (iii) *The subspace $U(u_-).W_{k,0}$ of $\wedge^k(\mathfrak{g})$ is the biggest G -submodule of $\wedge^k(\mathfrak{g})$ contained in $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$.*

Proof. (i) Denote by W'_k the L -submodule of $\wedge^k(\mathfrak{g})$ generated by W_k . As $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$ is invariant under L , W'_k is contained in $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$. For x in \mathfrak{p}_u and g in L ,

$$x.g.W_k = g.Adg^{-1}(x).W_k \subset W'_k,$$

since \mathfrak{p}_u is invariant under the adjoint action of L in \mathfrak{g} . Then W'_k is invariant under P_u , whence $W_k = W'_k$. As a result, W_k is a u -submodule of $\wedge^k(\mathfrak{g})$ since u is contained in \mathfrak{p} .

(ii) For ω in $W_{k,0}$, the subspace of $\wedge^k(\mathfrak{g})$ generated by ω is a P_u -submodule contained in $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$. Hence $W_{k,0}$ is contained in W_k . Moreover, for x in u and g in H ,

$$x.g.\omega = g.Adg^{-1}(x).\omega = 0.$$

Hence $W_{k,0}$ is invariant under H . As a result, $W_{k,0}$ is generated by highest weight vectors.

(iii) By (ii), $U(u_-).W_{k,0}$ is the G -submodule of $\wedge^k(\mathfrak{g})$ generated by $W_{k,0}$. As u_- is contained in \mathfrak{p}_- , $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$ is a $U(u_-)$ -submodule of $\wedge^k(\mathfrak{g})$ so that $U(u_-).W_{k,0}$ is contained in $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$. Since a G -submodule of $\wedge^k(\mathfrak{g})$ is generated by highest weight vectors, $U(u_-).W_{k,0}$ is the biggest G -submodule of $\wedge^k(\mathfrak{g})$ contained in $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$. \square

Corollary 3.2. (i) *The subspace W_k of $\wedge^k(\mathfrak{g})$ is the biggest G -submodule contained in $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$.*

(ii) *The subspace $V_{k,u}$ of $\wedge^k(\mathfrak{g})$ is a G -submodule of $\wedge^k(\mathfrak{g})$.*

Proof. (i) Denote by \widetilde{W}_k the biggest G -submodule of $\wedge^k(\mathfrak{g})$ contained in $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$. Then \widetilde{W}_k is contained in W_k . Let $W_{k,1}$ be a complement to \widetilde{W}_k in $\wedge^k(\mathfrak{g})$, invariant under G . Then W_k is the direct sum of \widetilde{W}_k and $W_k \cap W_{k,1}$. By Lemma 3.1(i), $W_k \cap W_{k,1}$ is invariant under u . Then, by Lie's Theorem, $W_{k,0} \cap W_{k,1} \neq \{0\}$ if $W_k \cap W_{k,1} \neq \{0\}$. Hence $W_k = \widetilde{W}_k$ since $W_{k,0}$ is contained in \widetilde{W}_k by Lemma 3.1(iii).

(ii) By Corollary 2.4(i) and Lemma 2.1, W_k is the orthogonal complement to $V_{k,u}$ in $\wedge^k(\mathfrak{g})$. Hence $V_{k,u}$ is a G -module by (i). \square

3.2. A particular case. In this subsection, for some β in Π , $X := \Pi \setminus \{\beta\}$. Denote by \mathfrak{h}_β the orthogonal complement to H_β in \mathfrak{h} . Let Z be the subset of elements α of $\langle X \rangle$ such that $\beta + \alpha$ is a root. Set:

$$Y := \mathcal{R}_+ \setminus (\langle X \rangle \cup \{\beta\}), \quad Z' := \langle X \rangle \setminus Z, \quad E := \bigoplus_{\alpha \in Y} \mathfrak{g}^\alpha, \quad E_- := \bigoplus_{\alpha \in Y} \mathfrak{g}^{-\alpha},$$

$$u_0 := \bigoplus_{\alpha \in Z} \mathfrak{g}^\alpha, \quad u_{0,+} := \bigoplus_{\alpha \in Z'} \mathfrak{g}^\alpha, \quad u_{0,0} := \bigoplus_{\alpha \in Z} \mathfrak{g}^{-\alpha}, \quad u_{0,-} := \bigoplus_{\alpha \in Z'} \mathfrak{g}^{-\alpha}.$$

Then

$$\mathfrak{g} := E_- \oplus \mathfrak{g}^{-\beta} \oplus u_{0,-} \oplus u_{0,0} \oplus \mathbb{k}H_\beta \oplus \mathfrak{h}_\beta \oplus u_0 \oplus u_{0,+} \oplus \mathfrak{g}^\beta \oplus E.$$

For $i = (i_1, \dots, i_{10})$, set:

$$C_i := \wedge^{i_1}(E_-) \wedge \wedge^{i_2}(\mathfrak{g}^{-\beta}) \wedge \wedge^{i_3}(u_{0,-}) \wedge \wedge^{i_4}(u_{0,0}) \wedge \wedge^{i_5}(\mathbb{k}H_\beta) \wedge$$

$$\wedge^{i_6}(\mathfrak{h}_\beta) \wedge \wedge^{i_7}(u_0) \wedge \wedge^{i_8}(u_{0,+}) \wedge \wedge^{i_9}(\mathfrak{g}^\beta) \wedge \wedge^{i_{10}}(E).$$

For k positive integer, $\wedge^k(\mathfrak{g})$ is the direct sum of C_i , $i \in \mathbb{N}_k^{10}$.

For α in Z , denote by ω'_α and ω_α the elements of $\wedge^2(\mathfrak{g})$,

$$\omega'_\alpha := H_\beta \wedge [x_\beta, x_\alpha] + 2x_\beta \wedge x_\alpha, \quad \omega_\alpha := H_\beta \wedge [x_{-\beta}, x_{-\alpha}] + c_\alpha x_{-\beta} \wedge x_{-\alpha}, \quad \text{with}$$

$$c_\alpha := -\frac{1}{2} \langle H_\beta, H_\beta \rangle \langle [x_\beta, x_\alpha], [x_{-\beta}, x_{-\alpha}] \rangle$$

so that ω_α is orthogonal to ω'_α .

Lemma 3.3. *Let $k = 1, \dots, d$. Denote by I the subset of elements i of \mathbb{N}_k^{10} such that $i_1 + \dots + i_8 \geq 2$, M' the subspace of elements μ_α , $\alpha \in Z$ of $\wedge^{k-1}(E)^Z$ such that*

$$\sum_{\alpha \in Z} [x_\beta, x_\alpha] \wedge \mu_\alpha = 0$$

and M the image of M' by the map

$$\wedge^{k-1}(E)^Z \longrightarrow \wedge^k(\mathfrak{g}), \quad (\mu_\alpha, \alpha \in Z) \longmapsto \sum_{\alpha \in Z} x_\alpha \wedge \mu_\alpha.$$

The space W_k is contained in the subspace of $\wedge^k(\mathfrak{g})$ generated by M , $\omega'_\alpha \wedge \wedge^{k-2}(E)$, $\alpha \in Z$, C_i , $i \in I$.

Proof. By Corollary 2.4(i) and Corollary 3.2, W_k is the biggest G -module contained in $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$. Denoting by I' the subset of elements i of \mathbb{N}_k^{10} such that

$$i_1 + \dots + i_8 > 0,$$

$\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$ is the sum of C_i , $i \in I'$. Then for i in I and x in \mathfrak{g} , $x.C_i$ is contained in $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$. The complement to I in I' is equal to the subset of elements i of I' such that $i_9 + i_{10} = k - 1$. For i in $I' \setminus I$ such that $i_5 = i_7 = 0$, $x_\beta.C_i$ is contained in $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$ since

$$[x_\beta, E_-] \subset \mathfrak{p}_-, \quad [x_\beta, \mathfrak{g}^{-\beta}] \subset \mathfrak{p}_-, \quad [x_\beta, \mathfrak{u}_{0,-} + \mathfrak{u}_{0,0} + \mathfrak{h}_\beta + \mathfrak{u}_{0,+}] = \{0\}.$$

For i in $I' \setminus I$,

$$\begin{aligned} i_7 = 1 \quad \text{and} \quad i_9 = 0 &\implies x_\beta.C_i \subset C_i \oplus \wedge^k(E), \\ i_7 = 1 \quad \text{and} \quad i_9 = 1 &\implies x_\beta.C_i \subset C_i \oplus \mathfrak{g}^\beta \wedge \wedge^{k-1}(E), \\ i_5 = 1 \quad \text{and} \quad i_9 = 0 &\implies x_\beta.C_i \subset C_i \oplus \mathfrak{g}^\beta \wedge \wedge^{k-1}(E), \\ i_5 = 1 \quad \text{and} \quad i_9 = 1 &\implies x_\beta.C_i \subset C_i. \end{aligned}$$

As a result, for ω and μ_α , $\alpha \in Z$ in $\wedge^{k-1}(E)$ and μ'_α , $\alpha \in Z$ in $\wedge^{k-2}(E)$ such that

$$\omega' + H_\beta \wedge \omega + \sum_{\alpha \in Z} x_\alpha \wedge \mu_\alpha + \sum_{\alpha \in Z} x_\beta \wedge x_\alpha \wedge \mu'_\alpha \in W_k$$

for some ω' in the sum of C_i , $i \in I$,

$$-2\omega + \sum_{\alpha \in Z} [x_\beta, x_\alpha] \wedge \mu'_\alpha = 0 \quad \text{and} \quad \sum_{\alpha \in Z} [x_\beta, x_\alpha] \wedge \mu_\alpha = 0$$

since $x_\beta.W_k$ is contained in $\mathfrak{p}_- \wedge \wedge^{k-1}(\mathfrak{g})$, whence the lemma. \square

For $i = (i_1, \dots, i_{10})$ in \mathbb{N}^{10} , denote by i^* the element of \mathbb{N}^{10} ,

$$i^* := (i_{10}, i_9, i_8, i_7, i_5, i_6, i_4, i_3, i_2, i_1).$$

By Lemma 2.2(ii), for k positive integer and i, i' in \mathbb{N}_k^{10} , C_i is orthogonal to $C_{i'}$ if and only if $i' \neq i^*$.

Corollary 3.4. *Let $k = 1, \dots, d$ and α in Z .*

- (i) *The space $\omega_\alpha \wedge \wedge^{k-2}(E_-)$ is contained in $V_{k,u}$.*
- (ii) *The space $g^\alpha \wedge \wedge^{k-1}(E_-)$ is contained in $V_{k,u}$.*
- (iii) *The spaces $H_\alpha \wedge \wedge^{k-2}(E_-)$ and $g^{-\alpha} \wedge \wedge^{k-1}(E_-)$ are contained in $V_{k,u}$.*
- (iv) *The space $h_\beta \wedge \wedge^{k-1}(E_-)$ is contained in $V_{k,u}$.*

Proof. (i) Let I_1 be the subset of elements i of \mathbb{N}_k^{10} such that

$$(i_1 = k - 1 \quad \text{and} \quad i_5 = 1) \quad \text{or} \quad (i_1 = k - 2, i_2 = 1, i_4 = 1).$$

Then $\omega_\alpha \wedge \wedge^{k-2}(E_-)$ is contained in the sum of C_i , $i \in I_1$. Hence $\omega_\alpha \wedge \wedge^{k-2}(E_-)$ is orthogonal to C_i for all i in I and for i such that $i_7 = 1$ and $i_{10} = k - 1$. By Corollary 2.3(i), for all γ in Z , $\omega_\alpha \wedge \wedge^{k-2}(E_-)$ is orthogonal to $\omega'_\gamma \wedge \wedge^{k-2}(E)$ since ω_α and ω'_γ are orthogonal, whence the assertion by Lemma 3.3 since $V_{k,u}$ is the orthogonal complement to W_k in $\wedge^k(\mathfrak{g})$ by Lemma 2.1.

(ii) The space $g^\alpha \wedge \wedge^{k-1}(E_-)$ is contained in C_i with i in \mathbb{N}_k^{10} such that $i_7 = 1$ and $i_1 = k - 1$. Hence $g^\alpha \wedge \wedge^{k-1}(E_-)$ is orthogonal to C_j for all j in I . Moreover, it is orthogonal to C_j for j in \mathbb{N}_k^{10} such that $j_7 = 1$ and $j_{10} = k - 1$ and $C_{j'}$ for j' in I_1 . As a result, $g^\alpha \wedge \wedge^{k-1}(E_-)$ is orthogonal to W_k by Lemma 3.3, whence the assertion since $V_{k,u}$ is the orthogonal complement to W_k in $\wedge^k(\mathfrak{g})$ by Lemma 2.1.

(iii) By (ii) and Corollary 3.2(ii), for ω in $\wedge^{k-1}(E_-)$,

$$V_{k,u} \ni x_{-\alpha} \cdot (x_\alpha \wedge \omega) = -H_\alpha \wedge \omega + x_\alpha \wedge x_{-\alpha} \cdot \omega \quad \text{and}$$

$$x_{-\alpha} \cdot (H_\alpha \wedge \omega) = 2x_{-\alpha} \wedge \omega + H_\alpha \wedge x_{-\alpha} \cdot \omega.$$

As E_- is invariant under the adjoint action of $x_{-\alpha}$, $x_\alpha \wedge x_{-\alpha} \cdot \omega$ is in $V_{k,u}$ by (ii), whence the assertion.

(iv) The space $h_\beta \wedge \wedge^{k-1}(E_-)$ is equal to C_i for i such that $i_1 = k - 1$ and $i_6 = 1$. Then C_i is orthogonal to C_j for j in I . Moreover, it is orthogonal to $g^\alpha \wedge \wedge^{k-1}(E)$ for all α in Z and $H_\beta \wedge \wedge^{k-1}(\mathfrak{p}_u)$ since h_β is orthogonal to H_β and u . As a result, $h_\beta \wedge \wedge^{k-1}(E_-)$ is orthogonal to W_k by Lemma 3.3, whence the assertion since $V_{k,u}$ is the orthogonal complement to W_k in $\wedge^k(\mathfrak{g})$ by Lemma 2.1. \square

Denote by \mathfrak{d} the derived algebra of \mathfrak{l} .

Proposition 3.5. *Let $k = 1, \dots, d$, $i = 0, \dots, k - 1$.*

(i) *Let M be a P_u -submodule of $\wedge^i(\mathfrak{g})$. Then the P_u -submodule of $\wedge^k(\mathfrak{g})$ generated by $\wedge^{k-i}(\mathfrak{p}_{-,u}) \wedge M$ contains $\wedge^{k-i-1}(\mathfrak{p}_{-,u}) \wedge \mathfrak{d} \wedge M$.*

(ii) *Let N be a $P_{-,u}$ -submodule of $\wedge^i(\mathfrak{g})$. Then the $P_{-,u}$ -submodule of $\wedge^k(\mathfrak{g})$ generated by $\wedge^{k-i}(\mathfrak{p}_u) \wedge N$ contains $\wedge^{k-i-1}(\mathfrak{p}_u) \wedge \mathfrak{d} \wedge N$.*

Proof. (i) By Lemma 1.2, it is sufficient to prove that $V_{k-i,u}$ contains $\bigwedge^{k-i-1}(\mathfrak{p}_{-,u}) \wedge \mathfrak{d}$ since M is a P_u -module. For α in Z ,

$$H_\alpha \in \frac{\beta(H_\alpha)}{2} H_\beta + \mathfrak{h}_\beta.$$

So, by Corollary 3.4,(iii) and (iv), $H_\beta \wedge \bigwedge^{k-i-1}(E_-)$ is contained in $V_{k-i,u}$. Then, by Corollary 3.4(i),

$$V_{k-i,u} \supset \mathfrak{g}^{-\beta} \wedge \mathfrak{g}^{-\alpha} \wedge \bigwedge^{k-i-2}(E_-).$$

As a result, by Corollary 3.4(iii), for all α in Z , $\mathfrak{g}^{-\alpha} \wedge \bigwedge^{k-i-1}(\mathfrak{p}_{-,u})$ is contained in $V_{k-i,u}$. As \mathfrak{g} is simple, the \mathfrak{l} -submodule of \mathfrak{g} , generated by \mathfrak{g}^α , $\alpha \in Z$ is equal to \mathfrak{d} . By Corollary 3.2(ii), $V_{k-i,u}$ is a \mathfrak{l} -module. Then, by Lemma 1.2, $V_{k-i,u}$ contains $\bigwedge^{k-i-1}(\mathfrak{p}_{-,u}) \wedge \mathfrak{d}$ since $\bigwedge^{k-i-1}(\mathfrak{p}_{-,u})$ is a \mathfrak{l} -module, whence the assertion.

(ii) For some automorphism g of \mathfrak{g} , $g(\mathfrak{p}_u) = \mathfrak{p}_{-,u}$, $g(\mathfrak{p}_{-,u}) = \mathfrak{p}_u$, $g(\mathfrak{h}) = \mathfrak{h}$. Then \mathfrak{l} and \mathfrak{d} are invariant under g . As a result, by (i), the $P_{-,u}$ -submodule of $\bigwedge^k(\mathfrak{g})$ generated by $\bigwedge^{k-i}(\mathfrak{p}_u) \wedge N$ contains $\bigwedge^{k-i-1}(\mathfrak{p}_u) \wedge \mathfrak{d} \wedge N$ since $g(N)$ is a P_u -submodule of $\bigwedge^i(\mathfrak{g})$. \square

4. PROOF OF THEOREM 1.1

Let $\ell \geq 2$ and X a nonempty subset of Π , different from Π . Set:

$$\mathfrak{p} := \mathfrak{p}_X, \quad \mathfrak{p}_u := \mathfrak{p}_{X,u}, \quad \mathfrak{l} := \mathfrak{l}_X, \quad \mathfrak{z} := \mathfrak{z}_X, \quad \mathfrak{d} := \mathfrak{d}_X, \quad \mathfrak{n} := \mathfrak{n}_X,$$

$$\mathfrak{p}_{-,u} := \mathfrak{p}_{X,-,u}, \quad \mathfrak{p}_{\pm,u} := \mathfrak{p}_u \oplus \mathfrak{p}_{-,u}, \quad E := E_X, \quad \mathfrak{p}_- := \mathfrak{l} \oplus \mathfrak{p}_{-,u}, \quad d := \dim \mathfrak{p}_u.$$

Recall that $\mathfrak{d}_1, \dots, \mathfrak{d}_n$ are the simple factors of \mathfrak{d} and for $i = 1, \dots, n$, n_i is the number of positive roots α such that \mathfrak{g}^α is contained in \mathfrak{d}_i . Let P_u and $P_{-,u}$ be as in Section 3. For $k = 1, \dots, n$, set $V_{k,p} := V_{k,p_X}$ and $V_{k,X} := V_k$.

4.1. A partial result. Let n' be the sum $n_1 + \dots + n_n$. For $k = 1, \dots, n$, denote by V'_k the subspace of $\bigwedge^k(\mathfrak{g})$,

$$V'_k := \bigoplus_{j=0}^{n'} \bigwedge^j(\mathfrak{d}) \wedge \bigwedge^{k-j}(E).$$

Proposition 4.1. *Suppose $|X| = \ell - 1$. Let $k = 1, \dots, n$. Then $\bigwedge^k(\mathfrak{g})$ is the G -submodule of $\bigwedge^k(\mathfrak{g})$ generated by V'_k .*

Proof. For $k = 1, \dots, d$, denote by E_k the G -submodule of $\bigwedge^k(\mathfrak{g})$ generated by $\bigwedge^k(E)$. For $k = 1, \dots, n'$, $V'_k = \bigwedge^k(\mathfrak{g})$ and for $k > n'$,

$$V'_k = \bigwedge^{n'}(\mathfrak{g}) \wedge \bigwedge^{k-n'}(E).$$

So, by Lemma 1.2, it is sufficient to prove that $E_k = \bigwedge^k(\mathfrak{g})$ for $k = 1, \dots, d$ since $n = n' + d$.

Prove the assertion by induction on k . For $k = 1$ the assertion is true since \mathfrak{g} is simple. Suppose $k > 1$ and the assertion true for $k - 1$. As $|X| = \ell - 1$, \mathfrak{z} has dimension 1 and

$$\bigwedge^k(E) = \bigwedge^k(\mathfrak{p}_{\pm,u}) \oplus \mathfrak{z} \wedge \bigwedge^{k-1}(\mathfrak{p}_{\pm,u}) =$$

$$\bigoplus_{j=0}^k \wedge^{k-j}(\mathfrak{p}_{-u}) \wedge \wedge^j(\mathfrak{p}_u) \oplus \bigoplus_{j=0}^{k-1} \wedge^{k-1-j}(\mathfrak{p}_{-u}) \wedge \mathfrak{z} \wedge \wedge^j(\mathfrak{p}_u).$$

For $j = 0, \dots, d$, $\wedge^j(\mathfrak{p}_u)$ and $\wedge^{j+1}(\mathfrak{p}_u) + \mathfrak{z} \wedge \wedge^j(\mathfrak{p}_u)$ are P_u -submodules of $\wedge^j(\mathfrak{g})$ and $\wedge^{j+1}(\mathfrak{g})$ respectively. Then, by Proposition 3.5(i), E_k contains

$$\wedge^{k-j-1}(\mathfrak{p}_{-u}) \wedge \mathfrak{d} \wedge \wedge^j(\mathfrak{p}_u) \quad \text{and} \quad \wedge^{k-j'-2}(\mathfrak{p}_{-u}) \wedge \mathfrak{z} \wedge \mathfrak{d} \wedge \wedge^{j'}(\mathfrak{p}_u)$$

for $j = 0, \dots, k-1$ and $j' = 0, \dots, k-2$. Hence E_k contains

$$\wedge^{k-j-1}(\mathfrak{p}_{-u}) \wedge \mathfrak{g} \wedge \wedge^j(\mathfrak{p}_u) \quad \text{and} \quad \wedge^{k-j'-2}(\mathfrak{p}_{-u}) \wedge \mathfrak{z} \wedge \mathfrak{g} \wedge \wedge^{j'}(\mathfrak{p}_u)$$

for $j = 0, \dots, k-1$ and $j' = 0, \dots, k-2$ since $\wedge^k(E)$ contains

$$\begin{aligned} & \wedge^{k-j}(\mathfrak{p}_{-u}) \wedge \wedge^j(\mathfrak{p}_u), \quad \wedge^{k-j-1}(\mathfrak{p}_{-u}) \wedge \wedge^{j+1}(\mathfrak{p}_u), \\ & \wedge^{k-j-1}(\mathfrak{p}_{-u}) \wedge \mathfrak{z} \wedge \wedge^j(\mathfrak{p}_u), \quad \wedge^{k-j'-2}(\mathfrak{p}_{-u}) \wedge \mathfrak{z} \wedge \wedge^{j'+1}(\mathfrak{p}_u) \end{aligned}$$

for $j = 0, \dots, k-1$ and $j' = 0, \dots, k-2$. As a result, E_k contains $\mathfrak{g} \wedge \wedge^{k-1}(E)$. Then, by Lemma 1.2 and the induction hypothesis, $E_k = \wedge^k(\mathfrak{g})$, whence the proposition. \square

Remark 4.2. When X is connected, $V_{k,p} = V'_{k,p}$. Then, in this case under the assumption $|X| = \ell - 1$, $V_k = \wedge^k(\mathfrak{g})$ by Proposition 4.1.

4.2. A first particular case. In this subsection, $|X| = \ell - 1$ so that $n \in \{1, 2, 3\}$. As a matter of fact, $n = 3$ only for type D and E. As in Subsection 4.1, $n' = n_1 + \dots + n_n$. For $i = (i_1, \dots, i_n)$ and $k = 0, \dots, n$, set:

$$\mathfrak{D}_i := \wedge^{i_1}(\mathfrak{d}_1) \wedge \dots \wedge \wedge^{i_n}(\mathfrak{d}_n) \quad \text{and} \quad \mathbb{I}_k := \{(i_1, \dots, i_n) \in \mathbb{N}_k^n \mid 0 \leq i_1 \leq n_1, \dots, 0 \leq i_n \leq n_n\}.$$

For l, l' nonnegative integers and i in \mathbb{N}^n , set:

$$V_{l,l',i} := \wedge^l(\mathfrak{p}_{-u}) \wedge \mathfrak{D}_i \wedge \wedge^{l'}(\mathfrak{p}_u) \quad \text{and}$$

$$V'_{l,l',i} := \wedge^l(\mathfrak{p}_{-u}) \wedge \mathfrak{z} \wedge \mathfrak{D}_i \wedge \wedge^{l'}(\mathfrak{p}_u).$$

For j in \mathbb{N}^n and $t = 0, \dots, |j|$, denote by Δ_j and $\Delta_{j,t}$ the subsets of \mathbb{N}^n ,

$$\Delta_j := \{j' \in \mathbb{N}^n \mid j'_1 \leq j_1, \dots, j'_n \leq j_n\} \quad \text{and} \quad \Delta_{j,t} := \Delta_j \cap \mathbb{N}_t^n.$$

Lemma 4.3. Let $k = 1, \dots, n$, (l, l') in \mathbb{N}^2 such that $l + l' \leq 2d$, i in $\mathbb{I}_{k-l-l'}$ and i' in $\mathbb{I}_{k-l-l'-1}$.

(i) Suppose that $V_{l,l'+s,j}$ is contained in V_k for all nonnegative integer s such that $l + l' + s \leq 2d$ and all j in $\Delta_{i,|i|-s}$. Then $\mathfrak{d} \wedge V_{l-1,l',i}$ is contained in V_k .

(ii) Suppose that $V_{l+s,l',j}$ is contained in V_k for all nonnegative integer s such that $l + l' + s \leq 2d$ and all j in $\Delta_{i,|i|-s}$. Then $\mathfrak{d} \wedge V_{l,l'-1,i}$ is contained in V_k .

(iii) Suppose that $V'_{l,l'+s,j}$ is contained in V_k for all nonnegative integer s such that $l + l' + s \leq 2d$ and all j in $\Delta_{i',|i'|-s}$. Then $\mathfrak{d} \wedge V'_{l-1,l',i'}$ is contained in V_k .

(iv) Suppose that $V'_{l+s,l',j}$ is contained in V_k for all nonnegative integer s such that $l + l' + s \leq 2d$ and all j in $\Delta_{i',|i'|-s}$. Then $\mathfrak{d} \wedge V'_{l,l'-1,i'}$ is contained in V_k .

Proof. For m in \mathbb{N} and j in \mathbb{N}^n , set:

$$\begin{aligned} M_{m,j} &:= \bigoplus_{t=0}^{|j|} \bigoplus_{i \in \Delta_{j,t}} \mathfrak{D}_i \wedge \bigwedge^{m+|j|-t}(\mathfrak{p}_u), & M_{m,j,-} &:= \bigoplus_{t=0}^{|j|} \bigoplus_{i \in \Delta_{j,t}} \mathfrak{D}_i \wedge \bigwedge^{m+|j|-t}(\mathfrak{p}_{-,u}), \\ M'_{m,j} &:= \bigoplus_{t=0}^{|j|} \bigoplus_{i \in \Delta_{j,t}} \mathfrak{z} \wedge \mathfrak{D}_i \wedge \bigwedge^{m+|j|-t}(\mathfrak{p}_u), & M'_{m,j,-} &:= \bigoplus_{t=0}^{|j|} \bigoplus_{i \in \Delta_{j,t}} \mathfrak{z} \wedge \mathfrak{D}_i \wedge \bigwedge^{m+|j|-t}(\mathfrak{p}_{-,u}). \end{aligned}$$

Then $M_{m,j}$ is a P_u -submodule of $\bigwedge^{m+|j|}(\mathfrak{g})$, $M_{m,j,-}$ is a $P_{-,u}$ -submodule of $\bigwedge^{m+|j|}(\mathfrak{g})$, $M'_{m,j}$ is a P_u -submodule of $\bigwedge^{m+|j|+1}(\mathfrak{g})$, $M'_{m,j,-}$ is a $P_{-,u}$ -submodule of $\bigwedge^{m+|j|+1}(\mathfrak{g})$.

(i) By hypothesis,

$$V_k \supset \bigwedge^l(\mathfrak{p}_{-,u}) \wedge M_{l',i} \supset V_{l,l',i}.$$

Then by Proposition 3.5(i), $\mathfrak{d} \wedge V_{l-1,l',i}$ is contained in V_k .

(ii) By hypothesis,

$$V_k \supset \bigwedge^{l'}(\mathfrak{p}_u) \wedge M_{l,i,-} \supset V_{l,l',i}.$$

Then by Proposition 3.5(ii), $\mathfrak{d} \wedge V_{l,l'-1,i}$ is contained in V_k .

(iii) By hypothesis,

$$V_k \supset \bigwedge^l(\mathfrak{p}_{-,u}) \wedge M'_{l',i'} \supset V'_{l,l',i'}.$$

Then by Proposition 3.5(i), $\mathfrak{d} \wedge V'_{l-1,l',i'}$ is contained in V_k .

(iv) By hypothesis,

$$V_k \supset \bigwedge^{l'}(\mathfrak{p}_u) \wedge M'_{l',i',-} \supset V'_{l,l',i'}.$$

Then by Proposition 3.5(ii), $\mathfrak{d} \wedge V'_{l,l'-1,i'}$ is contained in V_k . \square

Corollary 4.4. *Let $k = 1, \dots, n$, (l, l') in \mathbb{N}^2 such that $l + l' < 2d$, i in $\mathbb{I}_{k-l-l'-1}$ and i' in $\mathbb{I}_{k-l-l'-2}$. Then $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are contained in V_k .*

Proof. Since $V_{l,l',i}$ is contained in $V_{k-1,p}$, $\mathfrak{p}_u \wedge V_{l,l',i}$ and $\mathfrak{p}_{-,u} \wedge V_{l,l',i}$ are contained in $V_{k,p}$. Moreover, for all nonnegative integer s such that $l + l' + s + 1 \leq 2d$ and all j in $\Delta_{i,|i|-s}$, $\mathfrak{p}_u \wedge V_{l,l'+s,j}$, $\mathfrak{p}_{-,u} \wedge V_{l,l'+s,j}$, $\mathfrak{p}_u \wedge V_{l+s,l',j}$, $\mathfrak{p}_{-,u} \wedge V_{l+s,l',j}$ are contained in $V_{k,p}$. Then, by Lemma 4.3,(i) and (ii), $\mathfrak{d} \wedge V_{l,l',i}$ is contained in V_k .

Since $V'_{l,l',i'}$ is contained in $V_{k-1,p}$, $\mathfrak{p}_u \wedge V'_{l,l',i'}$ and $\mathfrak{p}_{-,u} \wedge V'_{l,l',i'}$ are contained in $V_{k,p}$. Moreover, for all nonnegative integer s such that $l + l' + s + 1 \leq 2d$ and all j in $\Delta_{i',|i'|-s}$, $\mathfrak{p}_u \wedge V'_{l,l'+s,j}$, $\mathfrak{p}_{-,u} \wedge V'_{l,l'+s,j}$, $\mathfrak{p}_u \wedge V'_{l+s,l',j}$, $\mathfrak{p}_{-,u} \wedge V'_{l+s,l',j}$ are contained in $V_{k,p}$. Then, by Lemma 4.3,(iii) and (iv), $\mathfrak{d} \wedge V'_{l,l',i'}$ is contained in V_k . By definition,

$$V_{k,p} \supset \mathfrak{p}_{-,u} \wedge V_{k-1,p} + \mathfrak{z} \wedge V_{k-1,p} + \mathfrak{p}_u \wedge V_{k-1,p}.$$

Hence $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are contained in V_k . \square

Proposition 4.5. *Let $k = 1, \dots, n$. Suppose that one of the following condition is satisfied:*

- (1) Π is exceptional,
- (2) Π has type D_ℓ and $n = 3$,
- (3) Π has classical type, $n = 2$, $2d + n_1$ and $2d + n_2$ are bigger than n .

Then $V_k = \bigwedge^k(\mathfrak{g})$.

Proof. Prove the proposition by induction on k . For $k = 1$, it is true since \mathfrak{g} is simple. Suppose $k > 1$ and the proposition true for $k - 1$. By Lemma 1.2 and the induction hypothesis, it is sufficient to prove that $\mathfrak{g} \wedge V_{k-1,p}$ is contained in V_k . As a matter of fact, we have to prove that $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are contained in V_k for (l, l') in \mathbb{N}^2 such that $l + l' \leq 2d$, i in $\mathbb{I}_{k-l-l'-1}$ and i' in $\mathbb{I}_{k-l-l'-2}$ since

$$V_{k-1,p} = \bigoplus_{t=0}^{2d} \bigoplus_{(l,l') \in \mathbb{N}_t^2} \bigoplus_{i \in \mathbb{I}_{k-t-1}} V_{l,l',i} \oplus \bigoplus_{t=0}^{2d} \bigoplus_{(l,l') \in \mathbb{N}_t^2} \bigoplus_{i' \in \mathbb{I}_{k-t-2}} V'_{l,l',i'}.$$

Let (l, l') be in \mathbb{N}^2 such that $l + l' \leq 2d$, i in $\mathbb{I}_{k-l-l'-1}$ and i' in $\mathbb{I}_{k-l-l'-2}$. If Condition (1) or Condition (2) is satisfied, then $l + l' < 2d$ by Proposition A.1. As a result, by Corollary 4.4, $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are contained in V_k . If Condition (3) is satisfied, $l + l' < 2d$ or $l + l' = 2d$, $i_1 < n_1$, $i_2 < n_2$. In the first case, by Corollary 4.4, $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are contained in V_k . In the second case, $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are contained in $V_{k,p}$, whence the proposition. \square

Remark 4.6. By the proof of Proposition 4.5, when $n = 2$, for $k = 1, \dots, \inf\{2d+n_1-1, 2d+n_2-1\}$, $V_k = \bigwedge^k(\mathfrak{g})$.

4.3. A second particular case. In this subsection, $|X| = \ell - 1$, Π has classical type, $n = 2$ and $2d + n_1 \leq n$. By Proposition A.1, $2d + n_2 > n$, $\ell \geq 6$ for Π of type A_ℓ , $\ell \geq 7$ for Π of type B_ℓ or C_ℓ , $\ell \geq 8$ for Π of type D_ℓ .

For $i = (i_0, i_1, i_2, i_3, i_4)$ in \mathbb{N}^5 , set:

$$C_i := \bigwedge^{i_0}(\mathfrak{z}) \wedge \bigwedge^{i_1}(\mathfrak{d}_1) \wedge \bigwedge^{i_2}(\mathfrak{d}_2) \wedge \bigwedge^{i_3}(\mathfrak{p}_{-,u}) \wedge \bigwedge^{i_4}(\mathfrak{p}_u).$$

Let $k = 2d + n_1, \dots, n$ and $j = k - 2d - n_1$. Set:

$$\iota := (0, n_1, j, d, d), \quad \iota' := (1, n_1, j-1, d, d),$$

$$\begin{aligned} \iota_+ &:= (0, n_1 + 1, j-1, d, d), & \iota'_+ &:= (1, n_1 + 1, j-2, d, d), & \kappa &:= (0, n_1 + 1, j, d-1, d), \\ \kappa' &:= (1, n_1 + 1, j-1, d-1, d), & \kappa_- &:= (0, n_1 + 1, j, d, d-1), & \kappa'_- &:= (1, n_1 + 1, j-1, d, d-1). \end{aligned}$$

Lemma 4.7. Denote by M_ι and $M_{\iota'}$ the G -submodules of $\bigwedge^k(\mathfrak{g})$ generated by C_ι and $C_{\iota'}$ respectively.

- (i) The subspace M_ι of $\bigwedge^k(\mathfrak{g})$ contains C_κ and C_{κ_-} , and $M_{\iota'}$ contains $C_{\kappa'}$ and $C_{\kappa'_-}$.
- (ii) The spaces C_{ι_+} and $C_{\iota'_+}$ are contained in M_ι and $M_{\iota'}$ respectively.

Proof. (i) The subspaces of $\bigwedge^{k-d}(\mathfrak{g})$,

$$\bigwedge^{n_1}(\mathfrak{d}_1) \wedge \bigwedge^j(\mathfrak{d}_2) \wedge \bigwedge^d(\mathfrak{p}_u) \quad \text{and} \quad \bigwedge^{n_1}(\mathfrak{d}_1) \wedge \bigwedge^j(\mathfrak{d}_2) \wedge \mathfrak{z} \wedge \bigwedge^d(\mathfrak{p}_u),$$

are invariant under P_u . So, by Proposition 3.5(i), M_ι and $M_{\iota'}$ contain C_κ and $C_{\kappa'}$ respectively. The subspaces of $\bigwedge^{k-d}(\mathfrak{g})$,

$$\bigwedge^d(\mathfrak{p}_{-,u}) \wedge \bigwedge^{n_1}(\mathfrak{d}_1) \wedge \bigwedge^j(\mathfrak{d}_2) \quad \text{and} \quad \bigwedge^d(\mathfrak{p}_{-,u}) \wedge \bigwedge^{n_1}(\mathfrak{d}_1) \wedge \bigwedge^j(\mathfrak{d}_2) \wedge \mathfrak{z},$$

are invariant under P_{-u} . So, by Proposition 3.5(ii), M_ι and $M_{\iota'}$ contain C_{κ_-} and $C_{\kappa'_-}$ respectively.

(ii) For $i = (i_0, i_1, i_2, i_3, i_4)$ in \mathbb{N}^5 , set: $i^* := (i_0, i_1, i_2, i_4, i_3)$. By corollary 2.3(i), for i, j in \mathbb{N}^5 , C_i is orthogonal to C_j if and only if $j \neq i^*$.

Denote by M_i^\perp and C_i^\perp the orthogonal complements to M_i and C_i in $\wedge^k(\mathfrak{g})$ respectively. By Lemma 2.1, M_i^\perp is the biggest G -module contained in C_i^\perp . Suppose that C_{i^*} is not contained in M_i . A contradiction is expected. As $\wedge^k(\mathfrak{g})$ is the direct sum of C_i , $i \in \mathbb{N}_k^5$, C_i^\perp is the direct sum of C_i , $i \in \mathbb{N}_k^5 \setminus \{i\}$ since $i^* = i$. By (i), M_i^\perp is contained in the sum of C_i , $i \in \mathbb{N}_k^5 \setminus \{i, \kappa, \kappa_-\}$. Since $i^* = i$, the orthogonal complement to C_{i^*} is the sum of C_i , $i \in \mathbb{N}_k^5 \setminus \{i^*\}$. Then M_i^\perp is not contained in the direct sum of C_i , $i \in \mathbb{N}_k^5 \setminus \{i, \kappa, \kappa_-, i^*\}$ since C_{i^*} is not contained in M_i . Hence for some subspace M of $\wedge^{n_1+1}(\mathfrak{d}_1) \wedge \wedge^{j-1}(\mathfrak{d}_2)$,

$$M \neq \{0\} \quad \text{and} \quad M_i^\perp \supset \wedge^d(\mathfrak{p}_{-,u}) \wedge M \wedge \wedge^d(\mathfrak{p}_u)$$

since $\wedge^d(\mathfrak{p}_{-,u}) \wedge \wedge^d(\mathfrak{p}_u)$ has dimension 1. As a result, by Proposition 3.5, (i) and (ii),

$$M_i^\perp \supset \wedge^{d-1}(\mathfrak{p}_{-,u}) \wedge \mathfrak{d}_2 \wedge M \wedge \wedge^d(\mathfrak{p}_u)$$

$$M_i^\perp \supset \wedge^d(\mathfrak{p}_{-,u}) \wedge \mathfrak{d}_2 \wedge M \wedge \wedge^{d-1}(\mathfrak{p}_u)$$

since $M \wedge \wedge^d(\mathfrak{p}_u)$ is a P_u -submodule of $\wedge^{d+j+n_1}(\mathfrak{g})$ and $M \wedge \wedge^d(\mathfrak{p}_{-,u})$ is a $P_{-,u}$ -submodule of $\wedge^{d+j+n_1}(\mathfrak{g})$. As j is smaller than $\dim \mathfrak{d}_2$ and M is different from zero, $\mathfrak{d}_2 \wedge M \neq \{0\}$. Then $C_\kappa + C_{\kappa_-}$ is not contained in M_i since $C_\kappa + C_{\kappa_-}$ is orthogonal to C_i for all i in $\mathbb{N}_k^5 \setminus \{\kappa, \kappa_-\}$, whence the contradiction.

Denote by $M_{i'}^\perp$ and $C_{i'}^\perp$ the orthogonal complements to $M_{i'}$ and $C_{i'}$ in $\wedge^k(\mathfrak{g})$ respectively. By Lemma 2.1, $M_{i'}^\perp$ is the biggest G -module contained in $C_{i'}^\perp$. Suppose that $C_{i'^*}$ is not contained in $M_{i'}$. A contradiction is expected. As $\wedge^k(\mathfrak{g})$ is the direct sum of C_i , $i \in \mathbb{N}_k^5$, $C_{i'}^\perp$ is the direct sum of C_i , $i \in \mathbb{N}_k^5 \setminus \{i'\}$ since $i'^* = i'$. By (i), $M_{i'}^\perp$ is contained in the sum of C_i , $i \in \mathbb{N}_k^5 \setminus \{i', \kappa', \kappa'_-\}$. Since $i'^* = i'$, the orthogonal complement to $C_{i'^*}$ is the sum of C_i , $i \in \mathbb{N}_k^5 \setminus \{i'^*\}$. Then $M_{i'}^\perp$ is not contained in the direct sum of C_i , $i \in \mathbb{N}_k^5 \setminus \{i', \kappa', \kappa'_-, i'^*\}$ since $C_{i'^*}$ is not contained in $M_{i'}$. Hence for some subspace M' of $\wedge^{n_1+1}(\mathfrak{d}_1) \wedge \wedge^{j-2}(\mathfrak{d}_2)$,

$$M' \neq \{0\} \quad \text{and} \quad M_{i'}^\perp \supset \wedge^d(\mathfrak{p}_{-,u}) \wedge \mathfrak{z} \wedge M' \wedge \wedge^d(\mathfrak{p}_u)$$

since $\mathfrak{z} \wedge \wedge^d(\mathfrak{p}_{-,u}) \wedge \wedge^d(\mathfrak{p}_u)$ has dimension 1. As a result, by Proposition 3.5, (i) and (ii),

$$M_{i'}^\perp \supset \wedge^{d-1}(\mathfrak{p}_{-,u}) \wedge \mathfrak{z} \wedge \mathfrak{d}_2 \wedge M' \wedge \wedge^d(\mathfrak{p}_u)$$

$$M_{i'}^\perp \supset \wedge^d(\mathfrak{p}_{-,u}) \wedge \mathfrak{z} \wedge \mathfrak{d}_2 \wedge M' \wedge \wedge^{d-1}(\mathfrak{p}_u)$$

since $\mathfrak{z} \wedge M' \wedge \wedge^d(\mathfrak{p}_u)$ is a P_u -submodule of $\wedge^{d+j+n_1}(\mathfrak{g})$ and $\mathfrak{z} \wedge M' \wedge \wedge^d(\mathfrak{p}_{-,u})$ is a $P_{-,u}$ -submodule of $\wedge^{d+j+n_1}(\mathfrak{g})$. As j is smaller than $\dim \mathfrak{d}_2$ and M' is different from zero, $\mathfrak{d}_2 \wedge M' \neq \{0\}$. Then $C_{\kappa'} + C_{\kappa'_-}$ is not contained in $M_{i'}$ since $C_{\kappa'} + C_{\kappa'_-}$ is orthogonal to C_i for all i in $\mathbb{N}_k^5 \setminus \{\kappa', \kappa'_-\}$, whence the contradiction. \square

Proposition 4.8. For $k = 2d + n_1, \dots, n$, V_k is equal to $\wedge^k(\mathfrak{g})$.

Proof. Prove the proposition by induction on k . Let (l, l') be in \mathbb{N}^2 such that $l+l' \leq 2d$, $i \in \mathbb{I}_{k-l-l'-1}$, $i' \in \mathbb{I}_{k-l-l'-2}$. If $l+l' < 2d$ then $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V_{l,l',i'}$ are contained in V_k by Corollary 4.4. If $l+l' = 2d$ and $i_1 < n_1$ then $\mathfrak{g} \wedge V_{l,l',i}$ is contained in $V_{k,p}$ since $2d + n_2 > n$ by Proposition A.1. If

$l = l' = d$ and $i'_1 < n_1$ then $g \wedge V'_{l,l',i'}$ is contained in $V_{k,p}$ since $2d + n_2 > n$ by Proposition A.1. As a result, for $k = 2d + n_1$, by Lemma 1.2 and Remark 4.6, $V_k = \wedge^k(\mathfrak{g})$.

Suppose $k > 2d + n_1$, $V_{k-1} = \wedge^{k-1}(\mathfrak{g})$, $i = (n_1, k - 2d - n_1 - 1)$ and $i' = (n_1, k - 2d - n_1 - 2)$. By definition, $V_{k,p}$ contains the subspaces of $\wedge^k(\mathfrak{g})$,

$$\mathfrak{p}_{-,u} \wedge V_{d,d,i}, \quad \mathfrak{p}_u \wedge V_{d,d,i}, \quad \mathfrak{d}_2 \wedge V_{d,d,i}, \quad \mathfrak{p}_{-,u} \wedge V'_{d,d,i'}, \quad \mathfrak{p}_u \wedge V'_{d,d,i'}, \quad \mathfrak{d}_2 \wedge V'_{d,d,i'}$$

since i_2 and i'_2 are smaller than n_2 . By Lemma 4.7(ii), V_k contains $\mathfrak{d}_1 \wedge V_{d,d,i}$ and $\mathfrak{d}_1 \wedge V'_{d,d,i'}$. Then V_k contains $g \wedge V_{d,d,i}$ and $g \wedge V'_{d,d,i'}$. As a result, by our previous remark, V_k contains $g \wedge V_{k-1,p}$, whence the proposition by Lemma 1.2 and the induction hypothesis. \square

4.4. The general case. First, we consider the case when X contains the extremities of Π .

Lemma 4.9. *Suppose $\ell \geq 2$, $n \geq 2$ and the extremities of Π contained in X . If $|X|$ is smaller than $\ell - 1$ then for some β in $\Pi \setminus X$, $Y := \Pi \setminus \{\beta\}$ has two connected components, X is contained in Y and a connected component of Y is a connected component of X .*

Proof. Suppose $|X| < \ell - 1$. We consider the following cases:

- (1) Π has not type D, E,
- (2) Π has type D_ℓ ,
- (3) Π has type E_6 ,
- (4) Π has type E_7 ,
- (5) Π has type E_8 .

(1) Let X_1 be the connected component of X containing β_1 . There is only one element β of $\Pi \setminus X$ not orthogonal to X_1 . Then $Y := \Pi \setminus \{\beta\}$ has two connected components and X_1 is a connected component of Y .

(2) Let X_1 be the connected component of X containing β_1 . As β_ℓ and $\beta_{\ell-1}$ are in X , for some i smaller than $\ell - 2$, β_i is not in X since $|X| < \ell - 1$. Then there is only one element β in $\Pi \setminus X$ not orthogonal to X_1 so that $Y := \Pi \setminus \{\beta\}$ has two connected components and X_1 is a connected component of Y .

(3) As $\beta_1, \beta_2, \beta_6$ are in X , β_3 or β_5 is not in X since $|X| < \ell - 1$. Setting $Y_i := \Pi \setminus \{\beta_i\}$ for $i = 3, 5$, Y_i has two connected components and for some i , X is contained in Y_i and a connected component of X is a connected component of Y_i .

(4) As $\beta_1, \beta_2, \beta_7$ are in X , β_3 or β_5 or β_6 is not in X since $|X| < \ell - 1$. Setting $Y_i := \Pi \setminus \{\beta_i\}$ for $i = 3, 5, 6$, Y_i has two connected components and for some i , X is contained in Y_i and a connected component of X is a connected component of Y_i .

(5) As $\beta_1, \beta_2, \beta_8$ are in X , β_3 or β_5 or β_6 or β_7 is not in X since $|X| < \ell - 1$. Setting $Y_i := \Pi \setminus \{\beta_i\}$ for $i = 3, 5, 6, 7$, Y_i has two connected components and for some i , X is contained in Y_i and a connected component of X is a connected component of Y_i . \square

Proposition 4.10. *Let $k = 1, \dots, n$. Suppose that Theorem 1.1 is true for the simple algebras of rank smaller than ℓ and X contains the extremities of Π . Then $V_k = \wedge^k(\mathfrak{g})$.*

Proof. As X contains the extremities of Π and is different from Π , $\ell \geq 3$ and $n \geq 2$. By Proposition 4.5, Remark 4.6 and Proposition 4.8, $V_k = \wedge^k(\mathfrak{g})$ when $|X| = \ell - 1$. In particular, $V_k = \wedge^k(\mathfrak{g})$ when $\ell = 3$. Suppose $\ell > 3$ and $|X| < \ell - 1$.

Let Y be as in Lemma 4.9. Then \mathfrak{d}_Y has two simple factors \mathfrak{d}_1 and \mathfrak{a} and \mathfrak{d}_1 is a simple factor of \mathfrak{d} . Denote by $V_{k,Y}$ the G -submodule of $\bigwedge^k(\mathfrak{g})$ generated by V_{k,p_Y} . Then, by Proposition 4.5, Remark 4.6 and Proposition 4.8, $V_{k,Y} = \bigwedge^k(\mathfrak{g})$. The intersection $\mathfrak{q} := \mathfrak{a} \cap \mathfrak{p}$ is a parabolic subalgebra of \mathfrak{a} . Let E' be the intersection of E and \mathfrak{a} . Then E is the direct sum of E' and E_Y . As a result, setting $n_* := b_{\mathfrak{a}} - \ell_{\mathfrak{a}}$,

$$V_{k,p} = \bigoplus_{i=0}^{n_1} \bigoplus_{j=0}^{n_*} \bigwedge^i(\mathfrak{d}_1) \wedge V_{j,q} \wedge \bigwedge^{k-i-j}(E_Y).$$

Let A be the connected closed subgroup of G whose Lie algebra is \mathfrak{a} . By the hypothesis, for $j = 1, \dots, n_*$, the A -submodule of $\bigwedge^j(\mathfrak{a})$, generated by $V_{j,q}$, is equal to $\bigwedge^j(\mathfrak{a})$. Hence, by Lemma 1.2, V_{k,p_Y} is contained in V_k since \mathfrak{d}_1 and E_Y are invariant under A , whence $V_k = \bigwedge^k(\mathfrak{g})$. \square

To finish the proof of Theorem 1.1, we have to consider the case when X does not contain all the extremities of Π .

Lemma 4.11. *Suppose that X does not contain all the extremities of Π .*

(i) *There exists a sequence*

$$X_0 \subset \dots \subset X_m = \Pi$$

of connected subsets of Π satisfying the following conditions:

- (1) *for $i = 1, \dots, m$, $|X_i \setminus X_{i-1}| = 1$,*
- (2) *X contains the extremities of X_0 .*

(ii) *For $i = 0, \dots, m$, let \mathfrak{a}_i be the subalgebra of \mathfrak{g} generated by $\mathfrak{g}^{\pm\beta}$, $\beta \in X_i$. Then \mathfrak{a}_i is a simple algebra and $\mathfrak{p}_i := \mathfrak{p} \cap \mathfrak{a}_i$ is a parabolic subalgebra of \mathfrak{a}_i .*

(iii) *For $i = 0, \dots, m$, E is the direct sum of $E_i := E \cap \mathfrak{a}_i$ and E_{X_i} .*

Proof. (i) Define X_i by induction on i . Let X_0 be a connected subset of Π , containing X of minimal cardinality. By minimality of $|X_0|$, X contains the extremities of X_0 . Suppose $i > 0$ and X_{i-1} defined. If $X_{i-1} = \Pi$ there is nothing to do. Suppose $X_{i-1} \neq \Pi$. As Π is connected, there is some β in $\Pi \setminus X_{i-1}$, not orthogonal to an extremity of X_{i-1} . Then $X_i := X_{i-1} \cup \{\beta\}$ is a connected subset of Π since so is X_{i-1} , whence the assertion.

(ii) As X_i is connected, \mathfrak{a}_i is a simple algebra. For α in $\langle X_i \rangle$, \mathfrak{g}^α is contained in $\mathfrak{a}_i \cap \mathfrak{p}$. Hence \mathfrak{p}_i contains the Borel subalgebra of \mathfrak{a}_i generated by $\mathfrak{h} \cap \mathfrak{a}_i$ and \mathfrak{g}^β , $\beta \in X_i$, whence the assertion.

(iii) Let α be a positive root such that \mathfrak{g}^α is contained in E . If α is in $\langle X_i \rangle$ then \mathfrak{g}^α and $\mathfrak{g}^{-\alpha}$ are contained in E_i . Otherwise, \mathfrak{g}^α and $\mathfrak{g}^{-\alpha}$ are contained in E_{X_i} by definition.

Let z be in $E \cap \mathfrak{h}$. By definition, $\mathfrak{h} \cap E_{X_i}$ is the orthogonal complement to \mathfrak{a}_i in \mathfrak{h} . Then $z = z_1 + z_2$ with z_1 in $\mathfrak{a}_i \cap \mathfrak{h}$ and z_2 in E_{X_i} . Hence z_1 is orthogonal to $\mathfrak{a}_i \cap \mathfrak{p}$. As a result, z_1 is in E_i and z is in $E_i + E_{X_i}$, whence the assertion. \square

We can now give the proof of Theorem 1.1.

Proof. Prove the theorem by induction on ℓ . First of all, for X empty subset of Π , $V_{k,p_X} = \bigwedge^k(\mathfrak{g})$. For $\ell = 1$, $n = 1$. Hence the theorem is true in this case and we can suppose X nonempty and $\ell \geq 2$. By Proposition 4.1 and Remark 4.2, $V_k = \bigwedge^k(\mathfrak{g})$ when X is connected. In particular, the

theorem is true in rank 2. Then, by Proposition 4.10, the theorem is true for $\ell = 3$ since in this case X contains all the extremities of Π when it is not connected.

Suppose $\ell > 3$ and the theorem true for the simple algebras of rank smaller than ℓ . By induction hypothesis and Proposition 4.10, $V_k = \bigwedge^k(\mathfrak{g})$ when X contains all the extremities of Π . So, we can suppose that X does not contain all the extremities of Π . Let X_0, \dots, X_m be as in Lemma 4.11. For $i = 0, \dots, m$, set $e_i := |\langle X_i \rangle|$ and prove by induction on i the inclusion

$$\bigoplus_{j=0}^{e_i} \bigwedge^j(\mathfrak{a}_i) \wedge \bigwedge^{k-j}(E_{X_i}) \subset V_k.$$

For $i = 0, \dots, n$, denote by A_i the connected closed subgroup of G whose Lie algebra is \mathfrak{a}_i . By Lemma 4.11,(ii) and (iii),

$$V_{k,p} = \bigoplus_{j=0}^{e_i} V_{j,p_i} \wedge \bigwedge^{k-j}(E_{X_i}),$$

for $i = 0, \dots, m$. Then, by Proposition 4.10, the induction hypothesis and Lemma 1.2, the inclusion is true for $i = 0$ since E_{X_0} is invariant under A_0 . Suppose $i > 0$ and the inclusion true for $i - 1$. Let E'_{X_i} be the intersection of $E_{X_{i-1}}$ and \mathfrak{a}_i . Denote by \mathfrak{q}_i the parabolic subalgebra of \mathfrak{a}_i containing $\mathfrak{b} \cap \mathfrak{a}_i$ and such that \mathfrak{a}_{i-1} is the derived algebra of the reductive factor of \mathfrak{q}_i containing $\mathfrak{h} \cap \mathfrak{a}_i$. Then

$$E_{X_{i-1}} = E'_{X_i} \oplus E_{X_i}, \quad \mathfrak{a}_i = \mathfrak{a}_{i-1} \oplus E'_{X_i},$$

$$\bigwedge^j(\mathfrak{a}_{i-1}) \wedge \bigwedge^{k-j}(E_{X_{i-1}}) = \bigoplus_{l=0}^{k-j} \bigwedge^j(\mathfrak{a}_{i-1}) \wedge \bigwedge^l(E'_{X_i}) \wedge \bigwedge^{k-j-l}(E_{X_{i-1}})$$

for $j = 0, \dots, e_{i-1}$. As a result,

$$\bigoplus_{j=0}^{e_{i-1}} \bigwedge^j(\mathfrak{a}_{i-1}) \wedge \bigwedge^{k-j}(E_{X_{i-1}}) = \bigoplus_{j=0}^{e_i} V_{j,\mathfrak{q}_i} \wedge \bigwedge^{k-j}(E_{X_i}).$$

By Proposition 4.1 and Remark 4.2, for $j = 0, \dots, e_i$, the A_i -submodule of $\bigwedge^j(\mathfrak{a}_i)$ generated by V_{j,\mathfrak{q}_i} is equal to $\bigwedge^j(\mathfrak{a}_i)$ since \mathfrak{a}_{i-1} is simple and $\ell_{\mathfrak{a}_i} - \ell_{\mathfrak{a}_{i-1}} = 1$. Then, by Lemma 1.2, the A_i -submodule of $\bigwedge^k(\mathfrak{g})$ generated by

$$\bigoplus_{j=0}^{e_{i-1}} \bigwedge^j(\mathfrak{a}_{i-1}) \wedge \bigwedge^{k-j}(E_{X_{i-1}})$$

is equal to

$$\bigoplus_{j=0}^{e_i} \bigwedge^j(\mathfrak{a}_i) \wedge \bigwedge^{k-j}(E_{X_i})$$

since E_{X_i} is invariant under A_i , whence the assertion and the theorem since for $i = m$ the sum is equal to $\bigwedge^k(\mathfrak{g})$. \square

APPENDIX A. SOME REMARKS ON ROOT SYSTEMS

Let β be in Π and $X := \Pi \setminus \{\beta\}$. Set $\mathfrak{p}_u := \mathfrak{p}_{u,X}$ and $d := \dim \mathfrak{p}_{u,X}$. The goal of the section is the following proposition:

Proposition A.1. (i) *Suppose Π of type A_ℓ and X not connected. Then $\beta = \beta_{s+1}$ for some s in $\{1, \dots, \ell - 2\}$,*

$$n = 2, \quad n_1 = \frac{s(s+1)}{2}, \quad n_2 = \frac{(\ell - s - 1)(\ell - s)}{2}.$$

Moreover, if $2d + n_1 \leq n$ then

$$\ell \geq 6, \quad s \leq \frac{1}{6}(2\ell - 3 - \sqrt{4\ell^2 + 12\ell + 9}), \quad 2d + n_2 > n.$$

(ii) *Suppose Π of type B_ℓ or C_ℓ and X not connected. Then $\beta = \beta_{s+1}$ for some s in $\{1, \dots, \ell - 2\}$,*

$$n = 2, \quad n_1 = \frac{s(s+1)}{2}, \quad n_2 = (\ell - s - 1)^2.$$

Moreover, if $2d + n_1 \leq n$ then

$$\ell \geq 7, \quad s \leq \frac{1}{10}(8\ell - 9 - \sqrt{24\ell^2 + 16\ell + 1}), \quad 2d + n_2 > n.$$

(iii) *Suppose Π of type D_ℓ . If $\beta = \beta_{\ell-2}$ then $2d > n$. If β is different from $\beta_{\ell-2}$ and X is not connected, then*

$$n = 2, \quad n_1 = \frac{s(s+1)}{2}, \quad n_2 = (\ell - s - 1)^2.$$

Moreover, if $2d + n_1 \leq n$ then

$$\ell \geq 8, \quad s \leq \frac{1}{10}(8\ell - 13 - \sqrt{24\ell^2 - 8\ell + 9}), \quad 2d + n_2 > n.$$

(iv) *Suppose that Π is exceptional. If $2d \leq n$ then X is connected.*

We prove the proposition case by case. So, in the classical case, we suppose $\ell \geq 3$ and X not connected.

A.1. **Type A_ℓ .** As X is not connected, $n = 2$ and $\beta = \beta_{s+1}$ for some s in $\{1, \dots, \ell - 2\}$. Then

$$n_1 = \frac{s(s+1)}{2}, \quad n_2 = \frac{(\ell - s - 1)(\ell - s)}{2}, \quad d = n - n_1 - n_2,$$

$$n - 2d - n_1 = \frac{1}{2}(3s^2 + (-4\ell + 3)s + \ell^2 - 3\ell).$$

If $n - 2d - n_1 \geq 0$ then

$$s \leq \frac{1}{6}(4\ell - 3 - \sqrt{4\ell^2 + 12\ell + 9}) \quad \text{or} \quad s \geq \frac{1}{6}(4\ell - 3 + \sqrt{4\ell^2 + 12\ell + 9}).$$

As $s \geq 1$, the first inequality is possible only if $\ell \geq 6$. The second inequality is impossible since its right hand side is bigger than $\ell - 2$ and s is at most $\ell - 2$.

By the above equalities,

$$n - 2d - n_2 = 3s^2 + (-2\ell + 3)s - 2\ell.$$

If the left hand side is nonnegative then

$$s \leq \frac{1}{6}(2\ell - 3 - \sqrt{4\ell^2 + 12\ell + 9}) \quad \text{or} \quad s \geq \frac{1}{6}(2\ell - 3 + \sqrt{4\ell^2 + 12\ell + 9}).$$

The first inequality is impossible since its right hand side is negative. The second inequality is possible only if $\ell \geq 7$ since $s \leq \ell - 2$. Moreover, it is not possible to have $n \geq 2d + n_1$ and $n \geq 2d + n_2$ since

$$\frac{1}{6}(2\ell - 3 + \sqrt{4\ell^2 + 12\ell + 9}) > \frac{1}{6}(4\ell - 3 - \sqrt{4\ell^2 + 12\ell + 9}),$$

whence Assertion (i) of Proposition A.1.

A.2. **Type B_ℓ or C_ℓ .** As X is not connected, $n = 2$ and $\beta = \beta_{s+1}$ for some s in $\{1, \dots, \ell - 2\}$. Then

$$n_1 = \frac{s(s+1)}{2}, \quad n_2 = (\ell - s - 1)^2, \quad d = n - n_1 - n_2,$$

$$n - 2d - n_1 = \frac{1}{2}(5s^2 + (-8\ell + 9)s + 2\ell^2 - 8\ell + 4).$$

If $n - 2d - n_1 \geq 0$ then

$$s \leq \frac{1}{10}(8\ell - 9 - \sqrt{24\ell^2 + 16\ell + 1}) \quad \text{or} \quad s \geq \frac{1}{10}(8\ell - 9 + \sqrt{24\ell^2 + 16\ell + 1}).$$

As $s \geq 1$, the first inequality is possible only if $\ell \geq 7$. The second inequality is impossible since its right hand side is bigger than $\ell - 2$ and s is at most $\ell - 2$.

By the above equalities,

$$n - 2d - n_2 = 2s^2 + (-2\ell + 5)s - 4\ell + 4.$$

If the left hand side is nonnegative then

$$s \leq \frac{1}{4}(2\ell - 5 - \sqrt{4\ell^2 + 12\ell - 7}) \quad \text{or} \quad s \geq \frac{1}{4}(2\ell - 5 + \sqrt{4\ell^2 + 12\ell - 9}).$$

The first inequality is impossible since its right hand side is negative. The second inequality is impossible since

$$s \leq \ell - 2 \quad \text{and} \quad \frac{1}{4}(2\ell - 5 + \sqrt{4\ell^2 + 12\ell - 9}) > \ell - 2,$$

whence Assertion (ii) of Proposition A.1.

A.3. **Type D_ℓ .** As X is not connected, β is different from $\beta_1, \beta_{\ell-1}, \beta_\ell$. If $\beta = \beta_{\ell-2}$ then X has three connected components and

$$d = \ell(\ell - 1) - 2 - \frac{1}{2}(\ell - 3)(\ell - 2).$$

In this case $n < 2d$. Suppose $\ell \geq 5$ and $\beta = \beta_{s+1}$ for some s in $\{1, \dots, \ell - 4\}$. Then

$$n_1 = \frac{s(s+1)}{2}, \quad n_2 = (\ell - s - 1)(\ell - s - 2), \quad d = n - n_1 - n_2,$$

$$n - 2d - n_1 = 5s^2 - s(4\ell - 7) + \ell^2 - 5\ell + 4.$$

If $n - 2d - n_1 \geq 0$ then

$$s \leq \frac{1}{10}(8\ell - 13 - \sqrt{24\ell^2 - 8\ell + 9}) \quad \text{or} \quad s \geq \frac{1}{10}(8\ell - 13 + \sqrt{24\ell^2 - 8\ell + 9}).$$

As $s \geq 1$, the first inequality is possible only if $\ell \geq 8$. The second inequality is impossible since its right hand side is bigger than $\ell - 4$ and s is at most $\ell - 4$.

By the above equalities,

$$n - 2d - n_2 = 2s^2 + (-2\ell + 4)s - 2\ell + 2.$$

If the left hand side is nonnegative then

$$s \leq -1 \quad \text{or} \quad s \geq \ell - 1.$$

These inequalities are impossible since s is positive and smaller than $\ell - 3$, whence Assertion (iii) of Proposition A.1.

A.4. The exceptional case. Set $\mathfrak{l} := \mathfrak{l}_X$, $\mathfrak{d} := \mathfrak{d}_X$. Then $2d = \dim \mathfrak{g} - \dim \mathfrak{l}$. For each case, we give all the possible dimensions of \mathfrak{l} when $|X| = \ell - 1$.

(a) The algebra \mathfrak{g} has type G_2 . Then X is connected, whence Assertion (iv) of Proposition A.1 for this case.

(b) The algebra \mathfrak{g} has type F_4 . In this case $n = 24$ and

$$\dim \mathfrak{l} \in \{12, 22\} \quad \text{whence} \quad 2d \in \{40, 30\}$$

and Assertion (iv) of Proposition A.1 for this case.

(c) The algebra \mathfrak{g} has type E_6 . In this case $n = 36$ and

$$\dim \mathfrak{l} \in \{20, 28, 36, 46\} \quad \text{whence} \quad 2d \in \{58, 50, 42, 32\}$$

and Assertion (iv) of Proposition A.1 for this case since \mathfrak{d} is simple of type D_5 when $2d = 32$.

(d) The algebra \mathfrak{g} has type E_7 . In this case $n = 63$ and

$$\dim \mathfrak{l} \in \{27, 33, 39, 49, 67, 79\} \quad \text{whence} \quad 2d \in \{106, 100, 94, 84, 66, 54\}$$

and Assertion (iv) of Proposition A.1 for this case since \mathfrak{d} is simple of type E_6 when $2d = 54$.

(e) The algebra \mathfrak{g} has type E_8 . In this case $n = 120$ and

$$\dim \mathfrak{l} \in \{36, 40, 52, 54, 64, 82, 92, 134\} \quad \text{whence} \quad 2d \in \{212, 208, 196, 194, 184, 166, 156, 114\}$$

and Assertion (iv) of Proposition A.1 for this case since \mathfrak{d} is simple of type E_7 when $2d = 114$.

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