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ON SOME SUBSPACES OF THE EXTERIOR ALGEBRA OF A SIMPLE LIE ALGEBRA

JEAN-YVES CHARBONNEL

Abstract. In this article, we are interested in some subspaces of the exterior algebra of a simple Lie algebra $\mathfrak{g}$. In particular, we prove that some graded subspaces of degree d generate the $\mathfrak{g}$-module $\bigwedge^d(\mathfrak{g})$ for some integers d.

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1. Introduction

In this note, the base field $\mathbb{k}$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a simple Lie algebra of finite dimension, ℓ is its rank, and G is its adjoint group.

1.1. **Main results.** Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$ and $\mathfrak{u}$ the nilpotent radical of $\mathfrak{b}$. Set $\mathrm{b}_{\mathfrak{g}} := \dim \mathfrak{b}$ and $n := \mathrm{b}_{\mathfrak{g}} - \ell$. For k a nonnegative integer, let $\bigwedge^k(\mathfrak{g})$ be the component of degree k of the exterior algebra $\bigwedge(\mathfrak{g})$ of $\mathfrak{g}$. The adjoint action of G in $\mathfrak{g}$ induces an action of G in $\bigwedge(\mathfrak{g})$. For all k, $\bigwedge^k(\mathfrak{g})$ is invariant under this action.

For $\mathfrak{p}$ parabolic subalgebra of $\mathfrak{g}$, containing $\mathfrak{b}$, denote by $\mathfrak{p}_{\mathrm{u}}$ the nilpotent radical of $\mathfrak{p}$, $\mathfrak{l}$ the reductive factor of $\mathfrak{p}$, containing $\mathfrak{h}$, $\mathfrak{z}$ the center of $\mathfrak{l}$ and $\mathfrak{p}_{-,\mathrm{u}}$ the complement to $\mathfrak{p}$ in $\mathfrak{g}$, invariant under the adjoint action of $\mathfrak{h}$. Let $\mathfrak{d}$ be the derived algebra of $\mathfrak{l}$ and $\mathfrak{d}_1, \dots, \mathfrak{d}_{\mathrm{n}}$ its simple factors. Set:

$$E_{\mathfrak{p}} := \mathfrak{p}_{-,\mathrm{u}} \oplus \mathfrak{z} \oplus \mathfrak{p}_{\mathrm{u}}, \quad n_i := \dim \mathfrak{d}_i \cap \mathfrak{u},$$

$$\mathbb{I}_k := \{(j_1, \dots, j_{\mathrm{n}}) \in \mathbb{N}^{\mathrm{n}} \mid j_1 \leq n_1, \dots, j_{\mathrm{n}} \leq n_{\mathrm{n}}, j_1 + \cdots + j_{\mathrm{n}} = k\}$$

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for k positive integer. Denote by $V'_{k,\mathfrak{p}}$ and $V_{k,\mathfrak{p}}$ the subspaces of $\bigwedge^k(\mathfrak{g})$,

$$V'_{k,\mathfrak{p}} := \bigoplus_{(j_1,\dots,j_n)\in\mathbb{I}_k} \bigwedge^{j_1}(\mathfrak{d}_1)\wedge\cdots\wedge\bigwedge^{j_n}(\mathfrak{d}_n) \quad \text{and} \quad V_{k,\mathfrak{p}} := \bigoplus_{i=0}^{k} \bigwedge^{i}(E_{\mathfrak{p}})\wedge V'_{k-i,\mathfrak{p}}.$$

The goal of this note is the following theorem:

Theorem 1.1. *Let $k = 1,\dots,n$. Then $\bigwedge^k(\mathfrak{g})$ is the G-submodule of $\bigwedge^k(\mathfrak{g})$ generated by $V_{k,\mathfrak{p}}$.*

This result arises from the study of the commuting variety of $\mathfrak{g}$ (see [Ch20]). One of the main step of the proof is to consider the orthogonal complements to some subspaces of $\bigwedge^k(\mathfrak{g})$ in $\bigwedge^k(\mathfrak{g})$ with respect to the canonical extension of the Killing form of $\mathfrak{g}$ to $\bigwedge^k(\mathfrak{g})$.

1.2. **Notations.** • Let $\Bbbk^* := \Bbbk\setminus\{0\}$. For E a finite set, its cardinality is denoted by $|E|$. For k,m positive integers, set:

$$\mathbb{N}^m_k := \{(j_1,\dots,j_m)\in\mathbb{N}^m \mid j_1+\cdots+j_m = k\}.$$

As usual, for $i = (i_1,\dots,i_m)$ in $\mathbb{N}^m$,

$$|i| := i_1+\cdots+i_m.$$

• For V vector space, denote by $\bigwedge(V)$ the exterior algebra of V. This algebra has a natural gradation. For i integer, denote by $\bigwedge^i(V)$ the space of degree i of $\bigwedge(V)$. In particular, for i negative, $\bigwedge^i(V)$ is equal to $\{0\}$. As $\mathfrak{g}$ is a G-module for the adjoint action, so is $\bigwedge^i(\mathfrak{g})$ for all i.

Lemma 1.2. *Let A be a subgroup of G, k a positive integer, i a positive integer smaller than k, V a subspace of $\bigwedge^i(\mathfrak{g})$ and W the A-submodule of $\bigwedge^i(\mathfrak{g})$ generated by V. Then, for all A-submodule W' of $\bigwedge^{k-i}(\mathfrak{g})$, $W\wedge W'$ is the A-submodule of $\bigwedge^k(\mathfrak{g})$ generated by $V\wedge W'$.*

Proof. Let W'' be the A-submodule of $\bigwedge^k(\mathfrak{g})$ generated by $V\wedge W'$. Let ω and ω' be in W and W' respectively. For some $\omega_1,\dots,\omega_m$ in V and $g_1,\dots,g_m$ in A,

$$\omega = g_1.\omega_1+\cdots+g_m.\omega_m,$$

whence

$$\omega\wedge\omega' = g_1.(\omega_1\wedge g_1^{-1}.\omega')+\cdots+g_m.(\omega_m\wedge g_m^{-1}.\omega')$$

and $W'' = W\wedge W'$. □

• The Killing form of $\mathfrak{g}$ is denoted by $\langle .,.\rangle$. For k positive integer, the Killing form of $\mathfrak{g}$ has a natural extension to $\bigwedge^k(\mathfrak{g})$ and this extension is not degenerate.

• For $\mathfrak{a}$ a semisimple Lie algebra, denote by $\mathrm{b}_{\mathfrak{a}}$ the dimension of its Borel subalgebras and $\ell_{\mathfrak{a}}$ its rank.

• Let $\mathcal{R}$ be the root system of $\mathfrak{h}$ in $\mathfrak{g}$, $\mathcal{R}_+$ the positive root system of $\mathcal{R}$ defined by $\mathfrak{b}$ and Π the basis of $\mathcal{R}_+$. For α in $\mathcal{R}$, H_α is the coroot of α, the corresponding root subspace is denoted by $\mathfrak{g}^\alpha$ and a generator x_α of $\mathfrak{g}^\alpha$ is chosen so that $\langle x_\alpha, x_{-\alpha}\rangle = 1$.

• We consider on Π its structure of Dynkin diagram. As $\mathfrak{g}$ is simple, Π is connected and has three extremities when Π has type D_ℓ, E_6, E_7, E_8, one extremity when Π has type A_1 and 2 otherwise. The elements $\beta_1,\dots,\beta_\ell$ of Π are ordered as in [Bou02, Ch. VI].

• Let X be a subset of Π. We denote by $\mathcal{R}_X$ the root subsystem of $\mathcal{R}$ generated by X and we set

$$< X >:= \mathcal{R}_+ \cap \mathcal{R}_X \quad \text{so that} \quad \mathcal{R}_X =< X > \cup - < X > .$$

Let $\mathfrak{p}_X$ be the parabolic subalgebra of $\mathfrak{g}$,

$$\mathfrak{p}_X := \mathfrak{b} \oplus \bigoplus_{\alpha \in <X>} \mathfrak{g}^{-\alpha},$$

$\mathfrak{p}_{X,\mathfrak{u}}$ its nilpotent radical, $\mathfrak{l}_X$ the reductive factor of $\mathfrak{p}_X$ containing $\mathfrak{h}$, $\mathfrak{z}_X$ the center of $\mathfrak{l}_X$, $\mathfrak{d}_X$ the derived algebra of $\mathfrak{l}_X$, $\mathfrak{p}_{X,-,\mathfrak{u}}$ the complement to $\mathfrak{p}_X$ in $\mathfrak{g}$, invariant under $\mathrm{ad}\,\mathfrak{h}$ and E_X the sum of $\mathfrak{z}_X$, $\mathfrak{p}_{X,\mathfrak{u}}$, $\mathfrak{p}_{X,-,\mathfrak{u}}$. When X is empty, $\mathfrak{p}_X$ is the Borel subalgebra $\mathfrak{b}$.

• Let X be a nonempty subset of Π and $X_1, \dots, X_{\mathrm{n}_X}$ its connected components. For $i = 1, \dots, \mathrm{n}_X$, denote by n_i the cardinality of $< X_i >$ and $\mathfrak{d}_i$ the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^{\pm\beta}$, $\beta \in X_i$. Then $\mathfrak{d}_1, \dots, \mathfrak{d}_{\mathrm{n}_X}$ are the simple factors of $\mathfrak{d}_X$. For k positive integer, set:

$$V_{k,\mathfrak{p}_X} := \bigoplus_{j_1=0}^{n_1} \cdots \bigoplus_{j_{\mathrm{n}_X}=0}^{n_{\mathrm{n}_X}} \bigwedge^{j_1}(\mathfrak{d}_1) \wedge \cdots \wedge \bigwedge^{j_{\mathrm{n}_X}}(\mathfrak{d}_{\mathrm{n}_X}) \wedge \bigwedge^{k-j_1-\cdots-j_{\mathrm{n}_X}}(E_X)$$

and denote by $V_{k,X}$ the G-submodule of $\bigwedge^k(\mathfrak{g})$ generated by $V_{k,\mathfrak{p}_X}$.

2. Orthogonal complement

Let $\ell \geq 2$ and X a nonempty subset of Π. Set:

$$\mathfrak{p} := \mathfrak{p}_X, \quad \mathfrak{p}_\mathfrak{u} := \mathfrak{p}_{X,\mathfrak{u}}, \quad \mathfrak{l} := \mathfrak{l}_X, \quad \mathfrak{p}_{-,\mathfrak{u}} := \mathfrak{p}_{X,-,\mathfrak{u}},$$

$$\mathfrak{p}_{\pm,\mathfrak{u}} := \mathfrak{p}_\mathfrak{u} \oplus \mathfrak{p}_{-,\mathfrak{u}}, \quad \mathfrak{p}_- := \mathfrak{l} \oplus \mathfrak{p}_{-,\mathfrak{u}}, \quad d := \dim \mathfrak{p}_\mathfrak{u}.$$

2.1. **General fact.** Let A be a subgroup of G. For k positive integer and W subspace of $\bigwedge^k(\mathfrak{g})$, denote by $W^\perp$ the orthogonal complement to W in $\bigwedge^k(\mathfrak{g})$. As the bilinear form on $\bigwedge^k(\mathfrak{g})$, defined by the Killing form, is not degenerate,

$$\dim W + \dim W^\perp = \dim \bigwedge^k(\mathfrak{g}).$$

Lemma 2.1. *Let k be a positive integer smaller than* $\dim \mathfrak{g}$. *Let V be a subspace of $\bigwedge^k(\mathfrak{g})$. Denote by W the A-submodule of $\bigwedge^k(\mathfrak{g})$ generated by V. Then $W^\perp$ is the biggest A-submodule of $\bigwedge^k(\mathfrak{g})$, contained in $V^\perp$.*

Proof. Denote by $W^\#$ the biggest A-submodule contained in $V^\perp$. As W is a A-module, so is $W^\perp$. Then $W^\perp$ is contained in $W^\#$. Moreover, V is contained in the orthogonal complement to $W^\#$ in $\bigwedge^k(\mathfrak{g})$. Hence W is orthogonal to $W^\#$ since the orthogonal complement to $W^\#$ is a A-module. As a result, $W^\# = W^\perp$. □

2.2. **Orthogonality.** Let V be a finite dimensional vector space with a non degenerate symmetric bilinear form on V. For k positive integer, it induces a non degenerate symmetric bilinear form on $\bigwedge^k(V)$. Let $V_1, \dots, V_m$ be pairwise orhogonal subspaces of V such that V is the direct sum of these subspaces. For $i = (i_1, \dots, i_m)$ in $\mathbb{N}_k^m$, set:

$$C_{i,V} := \bigwedge^{i_1}(V_1) \wedge \cdots \wedge \bigwedge^{i_m}(V_m).$$

If V_m is the direct sum of two isotropic subspaces $V_{m,+}$ and $V_{m,-}$, for $i = (i_1, \dots, i_{m+1})$ in $\mathbb{N}_k^{m+1}$, set:

$$i^* := (i_1, \dots, i_{m-1}, i_{m+1}, i_m) \quad \text{and}$$
$$C'_{i,V} := \bigwedge^{i_1}(V_1) \wedge \cdots \wedge \bigwedge^{i_{m-1}}(V_{m-1}) \wedge \bigwedge^{i_m}(V_{m,+}) \wedge \bigwedge^{i_{m+1}}(V_{m,-}).$$

Lemma 2.2. *Let k be a positive integer.*

(i) *For i, i' in $\mathbb{N}_k^m$, if $i \neq i'$ then $C_{i,V}$ is orthogonal to $C_{i',V}$.*

(ii) *Suppose that V_m is the direct sum of two isotropic subspaces $V_{m,+}$ and $V_{m,-}$. For i, i' in $\mathbb{N}_k^{m+1}$, if $i' \neq i^*$ then $C'_{i,V}$ is orthogonal to $C'_{i',V}$.*

Proof. Denote by $\langle ., . \rangle$ the symmetric bilinear form on V and $\bigwedge^k(V)$. As $V_1, \dots, V_m$ are pairwise orthogonal and V is the direct sum of these subspaces, for $i = 1, \dots, m$, the restriction to $V_i \times V_i$ of $\langle ., . \rangle$ is non degenerate. For $j = 1, \dots, m$, let n_j be the dimension of the sum

$$V_1 \oplus \cdots \oplus V_j$$

and $v_1, \dots, v_{n_m}$ an orthonormal basis of V such that $\{v_1, \dots, v_{n_j}\}$ is contained in the union of $V_1, \dots, V_j$ for $j = 1, \dots, m$.

(i) Let i and i' be in $\mathbb{N}_k^m$ such that $i \neq i'$. If $k > n_m$, there is nothing to prove. Suppose $k \leq n_m$. For $j = j_1, \dots, j_k$ in $\{1, \dots, n_m\}$ such that $1 \leq j_1 < \cdots < j_k \leq n_m$, set:

$$w_j := v_{j_1} \wedge \cdots \wedge v_{j_k}.$$

Setting $n_0 := 0$, w_j is in $C_{i,V}$ if and only if

$$|\{l \in \{1, \dots, k\} \mid n_{s-1} + 1 \leq j_l \leq n_s\}| = i_s$$

for $s = 1, \dots, m$. Denote by I_i the set of j satisfying this condition so that $w_j,\ j \in I_i$ is a basis of $C_{i,V}$.

Let (j, j') be in $I_i \times I_{i'}$. By definition,

$$\langle w_j, w_{j'} \rangle = \det(\langle v_{j_l}, v_{j'_{l'}} \rangle,\ 1 \leq l, l' \leq k).$$

As the basis $v_1, \dots, v_{n_{m'}}$ is orthonormal,

$$\langle v_{j_l}, v_{j'_{l'}} \rangle = \delta_{j_l, j'_{l'}}$$

with $\delta_{s,s'}$ the Kronecker symbol. As a result, if all the lines of the above matix are all different from 0 then

$$|\{l \in \{1, \dots, k\} \mid n_{s-1} + 1 \leq j_l \leq n_s\}| = |\{l \in \{1, \dots, k\} \mid n_{s-1} + 1 \leq j'_l \leq n_s\}|$$

for $s = 1, \dots, m$ since $V_1, \dots, V_m$ are pairwise orthogonal. Then $\langle w_j, w_{j'} \rangle = 0$ since $i \neq i'$, whence the assertion.

(ii) Let i and i' be in $\mathbb{N}_k^{m+1}$ such that $i' \neq i^*$. By (i), we can suppose that $i_s = i'_s$ for $s = 1, \dots, m-1$. Since $V_{m,+}$ and $V_{m,-}$ are isotropic, they have the same dimension m_0 and V_m has a basis $u_1, \dots, u_{2m_0}$ such that

$$\{u_1, \dots, u_{m_0}\} \subset V_{m,+}, \quad \{u_{m_0+1}, \dots, u_{2m_0}\} \subset V_{m,-}, \quad \langle u_s, u_{s'+m_0}\rangle = \delta_{s,s'}$$

for $1 \leq s, s' \leq m_0$. Let $v'_1, \dots, v'_{n_m}$ be the basis of V such that $v'_l = v_l$ for $l = 1, \dots, n_{m-1}$ and $v'_{l'} = u_{l'-n_{m-1}}$ for $l' = n_{m-1}+1, \dots, n_m$. For $j = j_1, \dots, j_k$ in $\{1, \dots, n_m\}$ such that $1 \leq j_1 < \cdots < j_k \leq n_m$, set:

$$w'_j = v'_{j_1} \wedge \cdots \wedge v'_{j_k}.$$

Then w'_j is in $C'_{i,V}$ if and only if

$$|\{l \in \{1, \dots, k\} \mid n_{m-1}+1 \leq j_l \leq n_{m-1}+m_0\}| = i_m, \quad |\{l \in \{1, \dots, k\} \mid n_{m-1}+m_0+1 \leq j_l \leq n_m\}| = i_{m+1},$$

$$|\{l \in \{1, \dots, k\} \mid n_{s-1}+1 \leq j_l \leq n_s\}| = i_s$$

for $s = 1, \dots, m-1$. Denote by I_i the set of j satisfying this condition so that w'_j, $j \in I_i$ is a basis of $C'_{i,V}$.

Let (j, j') be in $I_i \times I_{i'}$. By definition,

$$\langle w'_j, w'_{j'}\rangle = \det(\langle v'_{j_l}, v'_{j'_{l'}}\rangle,\ 1 \leq l, l' \leq k).$$

Then

$$j_l \leq n_{m-1} \Longrightarrow \langle v'_{j_l}, v'_{j'_{l'}}\rangle = \delta_{j_j, j'_{l'}},$$

$$j_l > n_{m-1} \quad \text{and} \quad j'_{l'} > n_{m-1} \Longrightarrow \langle v'_{j_l}, v'_{j'_{l'}}\rangle = \delta_{|j_j - j'_{l'}|, m_0}.$$

As a result, if all the lines of the above matix are all different from 0 then

$$|\{l \in \{1, \dots, k\} \mid n_{m-1}+1 \leq j_l \leq n_{m-1}+m_0\}| = |\{l \in \{1, \dots, k\} \mid n_{m-1}+m_0+1 \leq j'_l \leq n_m\}|$$

since $i_s = i'_s$ for $s \leq m-1$ and $V_1, \dots, V_m$ are pairwise orthogonal. Then $\langle w_j, w_{j'}\rangle = 0$ since $i' \neq i^*$, whence the assertion. □

For $i = (i_1, i_2, i_3)$ in $\mathbb{N}^3$, set:

$$C_i := \bigwedge^{i_1}(\mathfrak{l}) \wedge \bigwedge^{i_2}(\mathfrak{p}_{\mathrm{u}}) \wedge \bigwedge^{i_3}(\mathfrak{p}_{-,\mathrm{u}})$$

and denote by i^* the element (i_1, i_3, i_2) of $\mathbb{N}^3$.

Corollary 2.3. *Let k be a positive integer.*

(i) *For i, i' in $\mathbb{N}_k^3$, C_i is orthogonal to $C_{i'}$ if $i^* \neq i'$.*

(ii) *For i in $\mathbb{N}_k^3$, the orthogonal complement to C_i in $\bigwedge^k(\mathfrak{g})$ is equal to*

$$\bigoplus_{i' \in \mathbb{N}_k^3 \setminus \{i^*\}} C_{i'}.$$

Proof. (i) Let i and i' be in $\mathbb{N}_k^3$ such that $i^* \neq i'$. By Lemma 2.1(ii), C_i is orthogonal to $C_{i'}$ since $\mathfrak{l}$ and $\mathfrak{p}_{\pm,\mathrm{u}}$ are orthogonal and $\mathfrak{p}_{\mathrm{u}}$ and $\mathfrak{p}_{-,\mathrm{u}}$ are isotropic.

(ii) Since $\bigwedge^k(\mathfrak{g})$ is the direct sum of $C_{i'}$, $i' \in \mathbb{N}_k^3$, the orthogonal complement to C_i in $\bigwedge^k(\mathfrak{g})$ is the direct sum of $C_{i'}$, $i' \in \mathbb{N}_k^{n+3} \setminus \{i^*\}$ by (i). □

Corollary 2.4. *Let $k = 1, \dots, d$.*
(i) *The orthogonal complement to $\bigwedge^k(\mathfrak{p}_{-,\mathfrak{u}})$ in $\bigwedge^k(\mathfrak{g})$ is equal to $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$.*
(ii) *The orthogonal complement to $\bigwedge^k(\mathfrak{p}_{\pm,\mathfrak{u}})$ in $\bigwedge^k(\mathfrak{g})$ is equal to $\mathfrak{l} \wedge \bigwedge^{k-1}(\mathfrak{g})$.*

Proof. (i) Let I_1 be the subset of $\mathbb{N}_k^3$,

$$I_1 := \{(i_1, i_2, i_3) \in \mathbb{N}_k^3 \mid i_1 = i_2 = 0\}.$$

The complement to I_1^* in $\mathbb{N}_k^3$ is equal to

$$\{(i_1, i_2, i_3) \in \mathbb{N}_k^3 \mid i_1 > 0 \quad \text{or} \quad i_3 > 0\},$$

whence the assertion by Corollary 2.3 since

$$\bigwedge^k(\mathfrak{p}_{-,\mathfrak{u}}) = \bigoplus_{i \in I_1} C_i.$$

(ii) Let I_2 be the subset of $\mathbb{N}_k^3$,

$$I_2 := \{(i_1, i_2, i_3) \in \mathbb{N}_k^3 \mid i_1 = 0\}.$$

The complement to I_2^* in $\mathbb{N}_k^3$ is equal to

$$\{(i_1, i_2, i_3) \in \mathbb{N}_k^3 \mid i_1 > 0\},$$

whence the assertion by Corollary 2.3 since $\bigwedge^k(\mathfrak{p}_{\pm,\mathfrak{u}})$ is the sum of C_i, $i \in I_2$. □

3. Action of the unipotent radical of a parabolic subgroup

Let $\ell \geq 2$ and X a subset of Π. Set:

$$\mathfrak{p} := \mathfrak{p}_X, \quad \mathfrak{p}_\mathfrak{u} := \mathfrak{p}_{X,\mathfrak{u}}, \quad \mathfrak{l} := \mathfrak{l}_X, \quad \mathfrak{p}_{-,\mathfrak{u}} := \mathfrak{p}_{X,-,\mathfrak{u}},$$

$$\mathfrak{p}_{\pm,\mathfrak{u}} := \mathfrak{p}_\mathfrak{u} \oplus \mathfrak{p}_{-,\mathfrak{u}}, \quad \mathfrak{p}_- := \mathfrak{l} \oplus \mathfrak{p}_{-,\mathfrak{u}}, \quad d := \dim \mathfrak{p}_\mathfrak{u}.$$

Denote by L and H the connected closed subgroups of G whose Lie algebras are $\mathfrak{l}$ and $\mathfrak{h}$ respectively. Let P and P_- be the normalizers of $\mathfrak{p}$ and $\mathfrak{p}_-$ in G and $P_\mathfrak{u}$ and $P_{-,\mathfrak{u}}$ their unipotent radicals.

3.1. Invariant subspaces.

Let $k = 1, \dots, d$, W_k the biggest $P_\mathfrak{u}$-submodule of $\bigwedge^k(\mathfrak{g})$ contained in $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$ and $V_{k,\mathfrak{u}}$ the $P_\mathfrak{u}$-submodule of $\bigwedge^k(\mathfrak{g})$ generated by $\bigwedge^k(\mathfrak{p}_{-,\mathfrak{u}})$.

Lemma 3.1. *Let $W_{k,0}$ be the subspace of elements of $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$ invariant under $\mathfrak{u}$.*
(i) *The subspace W_k of $\bigwedge^k(\mathfrak{g})$ is invariant under $\mathfrak{u}$.*
(ii) *The subspace $W_{k,0}$ of $\bigwedge^k(\mathfrak{g})$ is contained in W_k and generated by highest weight vectors.*
(iii) *The subspace $\mathrm{U}(\mathfrak{u}_-).W_{k,0}$ of $\bigwedge^k(\mathfrak{g})$ is the biggest G-submodule of $\bigwedge^k(\mathfrak{g})$ contained in $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$.*

Proof. (i) Denote by W'_k the L-submodule of $\bigwedge^k(\mathfrak{g})$ generated by W_k. As $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$ is invariant under L, W'_k is contained in $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$. For x in $\mathfrak{p}_\mathfrak{u}$ and g in L,

$$x.g.W_k = g.\mathrm{Ad}\, g^{-1}(x).W_k \subset W'_k,$$

since $\mathfrak{p}_\mathfrak{u}$ is invaraint under the adjoint action of L in $\mathfrak{g}$. Then W'_k is invariant under $P_\mathfrak{u}$, whence $W_k = W'_k$. As a result, W_k is a $\mathfrak{u}$-submodule of $\bigwedge^k(\mathfrak{g})$ since $\mathfrak{u}$ is contained in $\mathfrak{p}$.

(ii) For ω in $W_{k,0}$, the subspace of $\bigwedge^k(\mathfrak{g})$ generated by ω is a $P_\mathfrak{u}$-submodule contained in $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$. Hence $W_{k,0}$ is contained in W_k. Moreover, for x in $\mathfrak{u}$ and g in H,

$$x.g.\omega = g.\mathrm{Ad}\, g^{-1}(x).\omega = 0.$$

Hence $W_{k,0}$ is invariant under H. As a result, $W_{k,0}$ is generated by highest weight vectors.

(iii) By (ii), $\mathrm{U}(\mathfrak{u}_-).W_{k,0}$ is the G-submodule of $\bigwedge^k(\mathfrak{g})$ generated by $W_{k,0}$. As $\mathfrak{u}_-$ is contained in $\mathfrak{p}_-$, $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$ is a $\mathrm{U}(\mathfrak{u}_-)$-submodule of $\bigwedge^k(\mathfrak{g})$ so that $\mathrm{U}(\mathfrak{u}_-).W_{k,0}$ is contained in $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$. Since a G-submodule of $\bigwedge^k(\mathfrak{g})$ is generated by highest weight vectors, $\mathrm{U}(\mathfrak{u}_-).W_{k,0}$ is the biggest G-submodule of $\bigwedge^k(\mathfrak{g})$ contained in $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$. □

Corollary 3.2. (i) *The subspace W_k of $\bigwedge^k(\mathfrak{g})$ is the biggest G-submodule contained in $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$.*

(ii) *The subspace $V_{k,\mathfrak{u}}$ of $\bigwedge^k(\mathfrak{g})$ is a G-submodule of $\bigwedge^k(\mathfrak{g})$.*

Proof. (i) Denote by $\widetilde{W}_k$ the biggest G-submodule of $\bigwedge^k(\mathfrak{g})$ contained in $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$. Then $\widetilde{W}_k$ is contained in W_k. Let $W_{k,1}$ be a complement to $\widetilde{W}_k$ in $\bigwedge^k(\mathfrak{g})$, invariant under G. Then W_k is the direct sum of $\widetilde{W}_k$ and $W_k \cap W_{k,1}$. By Lemma 3.1(i), $W_k \cap W_{k,1}$ is invariant under $\mathfrak{u}$. Then, by Lie's Theorem, $W_{k,0} \cap W_{k,1} \neq \{0\}$ if $W_k \cap W_{k,1} \neq \{0\}$. Hence $W_k = \widetilde{W}_k$ since $W_{k,0}$ is contained in $\widetilde{W}_k$ by Lemma 3.1(iii).

(ii) By Corollary 2.4(i) and Lemma 2.1, W_k is the orthogonal complement to $V_{k,\mathfrak{u}}$ in $\bigwedge^k(\mathfrak{g})$. Hence $V_{k,\mathfrak{u}}$ is a G-module by (i). □

3.2. **A particular case.** In this subsection, for some β in Π, $X := \Pi \setminus \{\beta\}$. Denote by $\mathfrak{h}_\beta$ the orthogonal complement to H_β in $\mathfrak{h}$. Let Z be the subset of elements α of $< X >$ such that $\beta + \alpha$ is a root. Set:

$$Y := \mathcal{R}_+ \setminus (< X > \cup \{\beta\}), \quad Z' :=< X > \setminus Z, \quad E := \bigoplus_{\alpha \in Y} \mathfrak{g}^\alpha, \quad E_- := \bigoplus_{\alpha \in Y} \mathfrak{g}^{-\alpha},$$

$$\mathfrak{u}_0 := \bigoplus_{\alpha \in Z} \mathfrak{g}^\alpha, \quad \mathfrak{u}_{0,+} := \bigoplus_{\alpha \in Z'} \mathfrak{g}^\alpha, \quad \mathfrak{u}_{0,0} := \bigoplus_{\alpha \in Z} \mathfrak{g}^{-\alpha}, \quad \mathfrak{u}_{0,-} := \bigoplus_{\alpha \in Z'} \mathfrak{g}^{-\alpha}.$$

Then

$$\mathfrak{g} := E_- \oplus \mathfrak{g}^{-\beta} \oplus \mathfrak{u}_{0,-} \oplus \mathfrak{u}_{0,0} \oplus \mathbb{k}H_\beta \oplus \mathfrak{h}_\beta \oplus \mathfrak{u}_0 \oplus \mathfrak{u}_{0,+} \oplus \mathfrak{g}^\beta \oplus E.$$

For $i = (i_1, \ldots, i_{10})$, set:

$$C_i := \bigwedge^{i_1}(E_-) \wedge \bigwedge^{i_2}(\mathfrak{g}^{-\beta}) \wedge \bigwedge^{i_3}(\mathfrak{u}_{0,-}) \wedge \bigwedge^{i_4}(\mathfrak{u}_{0,0}) \wedge \bigwedge^{i_5}(\mathbb{k}H_\beta) \wedge$$
$$\bigwedge^{i_6}(\mathfrak{h}_\beta) \wedge \bigwedge^{i_7}(\mathfrak{u}_0) \wedge \bigwedge^{i_8}(\mathfrak{u}_{0,+}) \wedge \bigwedge^{i_9}(\mathfrak{g}^\beta) \wedge \bigwedge^{i_{10}}(E).$$

For k positive integer, $\bigwedge^k(\mathfrak{g})$ is the direct sum of C_i, $i \in \mathbb{N}_k^{10}$.

For α in Z, denote by ω'_α and ω_α the elements of $\bigwedge^2(\mathfrak{g})$,

$$\omega'_\alpha := H_\beta \wedge [x_\beta, x_\alpha] + 2x_\beta \wedge x_\alpha, \quad \omega_\alpha := H_\beta \wedge [x_{-\beta}, x_{-\alpha}] + c_\alpha x_{-\beta} \wedge x_{-\alpha}, \quad \text{with}$$

$$c_\alpha := -\frac{1}{2}\langle H_\beta, H_\beta\rangle\langle [x_\beta, x_\alpha], [x_{-\beta}, x_{-\alpha}]\rangle$$

so that ω_α is orthogonal to ω'_α.

Lemma 3.3. *Let $k = 1, \dots, d$. Denote by I the subset of elements i of $\mathbb{N}_k^{10}$ such that $i_1 + \cdots + i_8 \geq 2$, M' the subspace of elements μ_α, $\alpha \in Z$ of $\bigwedge^{k-1}(E)^Z$ such that*

$$\sum_{\alpha\in Z}[x_\beta, x_\alpha] \wedge \mu_\alpha = 0$$

and M the image of M' by the map

$$\bigwedge^{k-1}(E)^Z \longrightarrow \bigwedge^k(\mathfrak{g}), \qquad (\mu_\alpha,\ \alpha \in Z) \longmapsto \sum_{\alpha\in Z} x_\alpha \wedge \mu_\alpha.$$

The space W_k is contained in the subspace of $\bigwedge^k(\mathfrak{g})$ generated by M, $\omega'_\alpha \wedge \bigwedge^{k-2}(E)$, $\alpha \in Z$, C_i, $i \in I$.

Proof. By Corollary 2.4(i) and Corollary 3.2, W_k is the biggest G-module contained in $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$. Denoting by I' the subset of elements i of $\mathbb{N}_k^{10}$ such that

$$i_1 + \cdots + i_8 > 0,$$

$\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$ is the sum of C_i, $i \in I'$. Then for i in I and x in $\mathfrak{g}$, $x.C_i$ is contained in $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$. The complement to I in I' is equal to the subset of elements i of I' such that $i_9 + i_{10} = k - 1$. For i in $I' \setminus I$ such that $i_5 = i_7 = 0$, $x_\beta.C_i$ is contained in $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$ since

$$[x_\beta, E_-] \subset \mathfrak{p}_-, \quad [x_\beta, \mathfrak{g}^{-\beta}] \subset \mathfrak{p}_-, \quad [x_\beta, \mathfrak{u}_{0,-} + \mathfrak{u}_{0,0} + \mathfrak{h}_\beta + \mathfrak{u}_{0,+}] = \{0\}.$$

For i in $I' \setminus I$,

$$i_7 = 1 \quad \text{and} \quad i_9 = 0 \Longrightarrow x_\beta.C_i \subset C_i \oplus \bigwedge^k(E),$$
$$i_7 = 1 \quad \text{and} \quad i_9 = 1 \Longrightarrow x_\beta.C_i \subset C_i \oplus \mathfrak{g}^\beta \wedge \bigwedge^{k-1}(E),$$
$$i_5 = 1 \quad \text{and} \quad i_9 = 0 \Longrightarrow x_\beta.C_i \subset C_i \oplus \mathfrak{g}^\beta \wedge \bigwedge^{k-1}(E),$$
$$i_5 = 1 \quad \text{and} \quad i_9 = 1 \Longrightarrow x_\beta.C_i \subset C_i.$$

As a result, for ω and μ_α, $\alpha \in Z$ in $\bigwedge^{k-1}(E)$ and μ'_α, $\alpha \in Z$ in $\bigwedge^{k-2}(E)$ such that

$$\omega' + H_\beta \wedge \omega + \sum_{\alpha\in Z} x_\alpha \wedge \mu_\alpha + \sum_{\alpha\in Z} x_\beta \wedge x_\alpha \wedge \mu'_\alpha \in W_k$$

for some ω' in the sum of C_i, $i \in I$,

$$-2\omega + \sum_{\alpha\in Z}[x_\beta, x_\alpha] \wedge \mu'_\alpha = 0 \quad \text{and} \quad \sum_{\alpha\in Z}[x_\beta, x_\alpha] \wedge \mu_\alpha = 0$$

since $x_\beta.W_k$ is contained in $\mathfrak{p}_- \wedge \bigwedge^{k-1}(\mathfrak{g})$, whence the lemma. □

For $i = (i_1, \dots, i_{10})$ in $\mathbb{N}^{10}$, denote by i^* the element of $\mathbb{N}^{10}$,

$$i^* := (i_{10}, i_9, i_8, i_7, i_5, i_6, i_4, i_3, i_2, i_1).$$

By Lemma 2.2(ii), for k positive integer and i, i' in $\mathbb{N}_k^{10}$, C_i is orthogonal to $C_{i'}$ if and only if $i' \neq i^*$.

Corollary 3.4. *Let $k = 1, \dots, d$ and α in Z.*

(i) *The space $\omega_\alpha \wedge \bigwedge^{k-2}(E_-)$ is contained in $V_{k,\mathfrak{u}}$.*

(ii) *The space $\mathfrak{g}^\alpha \wedge \bigwedge^{k-1}(E_-)$ is contained in $V_{k,\mathfrak{u}}$.*

(iii) *The spaces $H_\alpha \wedge \bigwedge^{k-2}(E_-)$ and $\mathfrak{g}^{-\alpha} \wedge \bigwedge^{k-1}(E_-)$ are contained in $V_{k,\mathfrak{u}}$.*

(iv) *The space $\mathfrak{h}_\beta \wedge \bigwedge^{k-1}(E_-)$ is contained in $V_{k,\mathfrak{u}}$.*

Proof. (i) Let I_1 be the subset of elements i of $\mathbb{N}_k^{10}$ such that

$$(i_1 = k-1 \quad \text{and} \quad i_5 = 1) \quad \text{or} \quad (i_1 = k-2,\ i_2 = 1,\ i_4 = 1).$$

Then $\omega_\alpha \wedge \bigwedge^{k-2}(E_-)$ is contained in the sum of C_i, $i \in I_1$. Hence $\omega_\alpha \wedge \bigwedge^{k-2}(E_-)$ is orthogonal to C_i for all i in I and for i such that $i_7 = 1$ and $i_{10} = k-1$. By Corollary 2.3(i), for all γ in Z, $\omega_\alpha \wedge \bigwedge^{k-2}(E_-)$ is orthogonal to $\omega'_\gamma \wedge \bigwedge^{k-2}(E)$ since ω_α and ω'_γ are orthogonal, whence the assertion by Lemma 3.3 since $V_{k,\mathfrak{u}}$ is the orthogonal complement to W_k in $\bigwedge^k(\mathfrak{g})$ by Lemma 2.1.

(ii) The space $\mathfrak{g}^\alpha \wedge \bigwedge^{k-1}(E_-)$ is contained in C_i with i in $\mathbb{N}_k^{10}$ such that $i_7 = 1$ and $i_1 = k-1$. Hence $\mathfrak{g}^\alpha \wedge \bigwedge^{k-1}(E_-)$ is orthogonal to C_j for all j in I. Moreover, it is orthogonal to C_j for j in $\mathbb{N}_k^{10}$ such that $j_7 = 1$ and $j_{10} = k-1$ and $C_{j'}$ for j' in I_1. As a result, $\mathfrak{g}^\alpha \wedge \bigwedge^{k-1}(E_-)$ is orthogonal to W_k by Lemma 3.3, whence the assertion since $V_{k,\mathfrak{u}}$ is the orthogonal complement to W_k in $\bigwedge^k(\mathfrak{g})$ by Lemma 2.1.

(iii) By (ii) and Corollary 3.2(ii), for ω in $\bigwedge^{k-1}(E_-)$,

$$V_{k,\mathfrak{u}} \ni x_{-\alpha}.(x_\alpha \wedge \omega) = -H_\alpha \wedge \omega + x_\alpha \wedge x_{-\alpha}.\omega \quad \text{and}$$

$$x_{-\alpha}.(H_\alpha \wedge \omega) = 2x_{-\alpha} \wedge \omega + H_\alpha \wedge x_{-\alpha}.\omega.$$

As E_- is invariant under the adjoint action of $x_{-\alpha}$, $x_\alpha \wedge x_{-\alpha}.\omega$ is in $V_{k,\mathfrak{u}}$ by (ii), whence the assertion.

(iv) The space $\mathfrak{h}_\beta \wedge \bigwedge^{k-1}(E_-)$ is equal to C_i for i such that $i_1 = k-1$ and $i_6 = 1$. Then C_i is orthogonal to C_j for j in I. Moreover, it is orthogonal to $\mathfrak{g}^\alpha \wedge \bigwedge^{k-1}(E)$ for all α in Z and $H_\beta \wedge \bigwedge^{k-1}(\mathfrak{p}_\mathfrak{u})$ since $\mathfrak{h}_\beta$ is orthogonal to H_β and $\mathfrak{u}$. As a result, $\mathfrak{h}_\beta \wedge \bigwedge^{k-1}(E_-)$ is orthogonal to W_k by Lemma 3.3, whence the assertion since $V_{k,\mathfrak{u}}$ is the orthogonal complement to W_k in $\bigwedge^k(\mathfrak{g})$ by Lemma 2.1. □

Denote by $\mathfrak{d}$ the derived algebra of $\mathfrak{l}$.

Proposition 3.5. *Let $k = 1, \dots, d$, $i = 0, \dots, k-1$.*

(i) *Let M be a $P_\mathfrak{u}$-submodule of $\bigwedge^i(\mathfrak{g})$. Then the $P_\mathfrak{u}$-submodule of $\bigwedge^k(\mathfrak{g})$ generated by $\bigwedge^{k-i}(\mathfrak{p}_{-,\mathfrak{u}}) \wedge M$ contains $\bigwedge^{k-i-1}(\mathfrak{p}_{-,\mathfrak{u}}) \wedge \mathfrak{d} \wedge M$.*

(ii) *Let N be a $P_{-,\mathfrak{u}}$-submodule of $\bigwedge^i(\mathfrak{g})$. Then the $P_{-,\mathfrak{u}}$-submodule of $\bigwedge^k(\mathfrak{g})$ generated by $\bigwedge^{k-i}(\mathfrak{p}_\mathfrak{u}) \wedge N$ contains $\bigwedge^{k-i-1}(\mathfrak{p}_\mathfrak{u}) \wedge \mathfrak{d} \wedge N$.*

Proof. (i) By Lemma 1.2, it is sufficient to prove that $V_{k-i,\mathfrak{u}}$ contains $\bigwedge^{k-i-1}(\mathfrak{p}_{-,\mathfrak{u}}) \wedge \mathfrak{d}$ since M is a $P_{\mathfrak{u}}$-module. For α in Z,

$$H_\alpha \in \frac{\beta(H_\alpha)}{2} H_\beta + \mathfrak{h}_\beta.$$

So, by Corollary 3.4,(iii) and (iv), $H_\beta \wedge \bigwedge^{k-i-1}(E_-)$ is contained in $V_{k-i,\mathfrak{u}}$. Then, by Corollary 3.4(i),

$$V_{k-i,\mathfrak{u}} \supset \mathfrak{g}^{-\beta} \wedge \mathfrak{g}^{-\alpha} \wedge \bigwedge^{k-i-2}(E_-).$$

As a result, by Corollary 3.4(iii), for all α in Z, $\mathfrak{g}^{-\alpha} \wedge \bigwedge^{k-i-1}(\mathfrak{p}_{-,\mathfrak{u}})$ is contained in $V_{k-i,\mathfrak{u}}$. As $\mathfrak{g}$ is simple, the $\mathfrak{l}$-submodule of $\mathfrak{g}$, generated by $\mathfrak{g}^\alpha$, $\alpha \in Z$ is equal to $\mathfrak{d}$. By Corollary 3.2(ii), $V_{k-i,\mathfrak{u}}$ is a $\mathfrak{l}$-module. Then, by Lemma 1.2, $V_{k-i,\mathfrak{u}}$ contains $\bigwedge^{k-i-1}(\mathfrak{p}_{-,\mathfrak{u}}) \wedge \mathfrak{d}$ since $\bigwedge^{k-i-1}(\mathfrak{p}_{-,\mathfrak{u}})$ is a $\mathfrak{l}$-module, whence the assertion.

(ii) For some automorphism g of $\mathfrak{g}$, $g(\mathfrak{p}_{\mathfrak{u}}) = \mathfrak{p}_{-,\mathfrak{u}}$, $g(\mathfrak{p}_{-,\mathfrak{u}}) = \mathfrak{p}_{\mathfrak{u}}$, $g(\mathfrak{h}) = \mathfrak{h}$. Then $\mathfrak{l}$ and $\mathfrak{d}$ are invariant under g. A a result, by (i), the $P_{-,\mathfrak{u}}$-submodule of $\bigwedge^k(\mathfrak{g})$ generated by $\bigwedge^{k-i}(\mathfrak{p}_{\mathfrak{u}}) \wedge N$ contains $\bigwedge^{k-i-1}(\mathfrak{p}_{\mathfrak{u}}) \wedge \mathfrak{d} \wedge N$ since $g(N)$ is a $P_{\mathfrak{u}}$-submodule of $\bigwedge^i(\mathfrak{g})$. □

4. Proof of Theorem 1.1

Let $\ell \geq 2$ and X a nonempty subset of Π, different from Π. Set:

$$\mathfrak{p} := \mathfrak{p}_X, \quad \mathfrak{p}_{\mathfrak{u}} := \mathfrak{p}_{X,\mathfrak{u}}, \quad \mathfrak{l} := \mathfrak{l}_X, \quad \mathfrak{z} := \mathfrak{z}_X, \quad \mathfrak{d} := \mathfrak{d}_X, \quad \mathfrak{n} := \mathfrak{n}_X,$$

$$\mathfrak{p}_{-,\mathfrak{u}} := \mathfrak{p}_{X,-,\mathfrak{u}}, \quad \mathfrak{p}_{\pm,\mathfrak{u}} := \mathfrak{p}_{\mathfrak{u}} \oplus \mathfrak{p}_{-,\mathfrak{u}}, \quad E := E_X, \quad \mathfrak{p}_- := \mathfrak{l} \oplus \mathfrak{p}_{-,\mathfrak{u}}, \quad d := \dim \mathfrak{p}_{\mathfrak{u}}.$$

Recall that $\mathfrak{d}_1, \ldots, \mathfrak{d}_{\mathfrak{n}}$ are the simple factors of $\mathfrak{d}$ and for $i = 1, \ldots, \mathfrak{n}$, n_i is the number of positive roots α such that $\mathfrak{g}^\alpha$ is contained in $\mathfrak{d}_i$. Let $P_{\mathfrak{u}}$ and $P_{-,\mathfrak{u}}$ be as in Section 3. For $k = 1, \ldots, n$, set $V_{k,\mathfrak{p}} := V_{k,\mathfrak{p}_X}$ and $V_{k,X} := V_k$.

4.1. **A partial result.**

Let n' be the sum $n_1 + \cdots + n_{\mathfrak{n}}$. For $k = 1, \ldots, n$, denote by V'_k the subspace of $\bigwedge^k(\mathfrak{g})$,

$$V'_k := \bigoplus_{j=0}^{n'} \bigwedge^j(\mathfrak{d}) \wedge \bigwedge^{k-j}(E).$$

Proposition 4.1. *Suppose $|X| = \ell - 1$. Let $k = 1, \ldots, n$. Then $\bigwedge^k(\mathfrak{g})$ is the G-submodule of $\bigwedge^k(\mathfrak{g})$ generated by V'_k.*

Proof. For $k = 1, \ldots, d$, denote by E_k the G-submodule of $\bigwedge^k(\mathfrak{g})$ generated by $\bigwedge^k(E)$. For $k = 1, \ldots, n'$, $V'_k = \bigwedge^k(\mathfrak{g})$ and for $k > n'$,

$$V'_k = \bigwedge^{n'}(\mathfrak{g}) \wedge \bigwedge^{k-n'}(E).$$

So, by Lemma 1.2, it is sufficient to prove that $E_k = \bigwedge^k(\mathfrak{g})$ for $k = 1, \ldots, d$ since $n = n' + d$.

Prove the assertion by induction on k. For $k = 1$ the assertion is true since $\mathfrak{g}$ is simple. Suppose $k > 1$ and the assertion true for $k - 1$. As $|X| = \ell - 1$, $\mathfrak{z}$ has dimension 1 and

$$\bigwedge^k(E) = \bigwedge^k(\mathfrak{p}_{\pm,\mathfrak{u}}) \oplus \mathfrak{z} \wedge \bigwedge^{k-1}(\mathfrak{p}_{\pm,\mathfrak{u}}) =$$

$$\bigoplus_{j=0}^{k} \bigwedge^{k-j}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \bigwedge^{j}(\mathfrak{p}_{\mathrm{u}}) \oplus \bigoplus_{j=0}^{k-1} \bigwedge^{k-1-j}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{z} \wedge \bigwedge^{j}(\mathfrak{p}_{\mathrm{u}}).$$

For $j = 0, \dots, d$, $\bigwedge^{j}(\mathfrak{p}_{\mathrm{u}})$ and $\bigwedge^{j+1}(\mathfrak{p}_{\mathrm{u}}) + \mathfrak{z} \wedge \bigwedge^{j}(\mathfrak{p}_{\mathrm{u}})$ are P_{u}-submodules of $\bigwedge^{j}(\mathfrak{g})$ and $\bigwedge^{j+1}(\mathfrak{g})$ respectively. Then, by Proposition 3.5(i), E_k contains

$$\bigwedge^{k-j-1}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{d} \wedge \bigwedge^{j}(\mathfrak{p}_{\mathrm{u}}) \quad \text{and} \quad \bigwedge^{k-j'-2}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{z} \wedge \mathfrak{d} \wedge \bigwedge^{j'}(\mathfrak{p}_{\mathrm{u}})$$

for $j = 0, \dots, k-1$ and $j' = 0, \dots, k-2$. Hence E_k contains

$$\bigwedge^{k-j-1}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{g} \wedge \bigwedge^{j}(\mathfrak{p}_{\mathrm{u}}) \quad \text{and} \quad \bigwedge^{k-j'-2}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{z} \wedge \mathfrak{g} \wedge \bigwedge^{j'}(\mathfrak{p}_{\mathrm{u}})$$

for $j = 0, \dots, k-1$ and $j' = 0, \dots, k-2$ since $\bigwedge^{k}(E)$ contains

$$\bigwedge^{k-j}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \bigwedge^{j}(\mathfrak{p}_{\mathrm{u}}), \quad \bigwedge^{k-j-1}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \bigwedge^{j+1}(\mathfrak{p}_{\mathrm{u}}),$$

$$\bigwedge^{k-j-1}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{z} \wedge \bigwedge^{j}(\mathfrak{p}_{\mathrm{u}}), \quad \bigwedge^{k-j'-2}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{z} \wedge \bigwedge^{j'+1}(\mathfrak{p}_{\mathrm{u}})$$

for $j = 0, \dots, k-1$ and $j' = 0, \dots, k-2$. As a result, E_k contains $\mathfrak{g} \wedge \bigwedge^{k-1}(E)$. Then, by Lemma 1.2 and the induction hypothesis, $E_k = \bigwedge^{k}(\mathfrak{g})$, whence the proposition. □

Remark 4.2. When X is connected, $V_{k,\mathfrak{p}} = V'_{k,\mathfrak{p}}$. Then, in this case under the assumption $|X| = \ell - 1$, $V_k = \bigwedge^{k}(\mathfrak{g})$ by Proposition 4.1.

4.2. **A first particular case.** In this subsection, $|X| = \ell - 1$ so that $\mathrm{n} \in \{1, 2, 3\}$. As a matter of fact, $\mathrm{n} = 3$ only for type D and E. As in Subsection 4.1, $n' = n_1 + \cdots + n_{\mathrm{n}}$. For $i = (i_1, \dots, i_{\mathrm{n}})$ and $k = 0, \dots, n$, set:

$$\mathfrak{D}_i := \bigwedge^{i_1}(\mathfrak{d}_1) \wedge \cdots \wedge \bigwedge^{i_{\mathrm{n}}}(\mathfrak{d}_{\mathrm{n}}) \quad \text{and} \quad \mathbb{I}_k := \{(i_1, \dots, i_{\mathrm{n}}) \in \mathbb{N}_k^{\mathrm{n}} \mid 0 \le i_1 \le n_1, \dots, 0 \le i_{\mathrm{n}} \le n_{\mathrm{n}}\}.$$

For l, l' nonnegative integers and i in $\mathbb{N}^{\mathrm{n}}$, set:

$$V_{l,l',i} := \bigwedge^{l}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{D}_i \wedge \bigwedge^{l'}(\mathfrak{p}_{\mathrm{u}}) \quad \text{and}$$

$$V'_{l,l',i} := \bigwedge^{l}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{z} \wedge \mathfrak{D}_i \wedge \bigwedge^{l'}(\mathfrak{p}_{\mathrm{u}}).$$

For j in $\mathbb{N}^{\mathrm{n}}$ and $t = 0, \dots, |j|$, denote by Δ_j and $\Delta_{j,t}$ the subsets of $\mathbb{N}^{\mathrm{n}}$,

$$\Delta_j := \{j' \in \mathbb{N}^{\mathrm{n}} \mid j'_1 \le j_1, \dots, j'_{\mathrm{n}} \le j_{\mathrm{n}}\} \quad \text{and} \quad \Delta_{j,t} := \Delta_j \cap \mathbb{N}_t^{\mathrm{n}}.$$

Lemma 4.3. *Let $k = 1, \dots, n$, (l, l') in $\mathbb{N}^2$ such that $l + l' \le 2d$, i in $\mathbb{I}_{k-l-l'}$ and i' in $\mathbb{I}_{k-l-l'-1}$.*

(i) *Suppose that $V_{l,l'+s,j}$ is contained in V_k for all nonnegative integer s such that $l + l' + s \le 2d$ and all j in $\Delta_{i,|i|-s}$. Then $\mathfrak{d} \wedge V_{l-1,l',i}$ is contained in V_k.*

(ii) *Suppose that $V_{l+s,l',j}$ is contained in V_k for all nonnegative integer s such that $l + l' + s \le 2d$ and all j in $\Delta_{i,|i|-s}$. Then $\mathfrak{d} \wedge V_{l,l'-1,i}$ is contained in V_k.*

(iii) *Suppose that $V'_{l,l'+s,j}$ is contained in V_k for all nonnegative integer s such that $l + l' + s \le 2d$ and all j in $\Delta_{i',|i'|-s}$. Then $\mathfrak{d} \wedge V'_{l-1,l',i'}$ is contained in V_k.*

(iv) *Suppose that $V'_{l+s,l',j}$ is contained in V_k for all nonnegative integer s such that $l + l' + s \le 2d$ and all j in $\Delta_{i',|i'|-s}$. Then $\mathfrak{d} \wedge V'_{l,l'-1,i'}$ is contained in V_k.*

Proof. For m in $\mathbb{N}$ and j in $\mathbb{N}^{\mathrm{n}}$, set:

$$M_{m,j} := \bigoplus_{t=0}^{|j|} \bigoplus_{\iota\in\Delta_{j,t}} \mathfrak{D}_\iota \wedge \bigwedge^{m+|j|-t}(\mathfrak{p}_{\mathrm{u}}), \quad M_{m,j,-} := \bigoplus_{t=0}^{|j|} \bigoplus_{\iota\in\Delta_{j,t}} \mathfrak{D}_\iota \wedge \bigwedge^{m+|j|-t}(\mathfrak{p}_{-,\mathrm{u}}),$$

$$M'_{m,j} := \bigoplus_{t=0}^{|j|} \bigoplus_{\iota\in\Delta_{j,t}} \mathfrak{z} \wedge \mathfrak{D}_\iota \wedge \bigwedge^{m+|j|-t}(\mathfrak{p}_{\mathrm{u}}), \quad M'_{m,j,-} := \bigoplus_{t=0}^{|j|} \bigoplus_{\iota\in\Delta_{j,t}} \mathfrak{z} \wedge \mathfrak{D}_\iota \wedge \bigwedge^{m+|j|-t}(\mathfrak{p}_{-,\mathrm{u}}).$$

Then $M_{m,j}$ is a P_{u}-submodule of $\bigwedge^{m+|j|}(\mathfrak{g})$, $M_{m,j,-}$ is a $P_{-,\mathrm{u}}$-submodule of $\bigwedge^{m+|j|}(\mathfrak{g})$, $M'_{m,j}$ is a P_{u}-submodule of $\bigwedge^{m+|j|+1}(\mathfrak{g})$, $M'_{m,j,-}$ is a $P_{-,\mathrm{u}}$-submodule of $\bigwedge^{m+|j|+1}(\mathfrak{g})$.

(i) By hypothesis,

$$V_k \supset \bigwedge^{l}(\mathfrak{p}_{-,\mathrm{u}}) \wedge M_{l',i} \supset V_{l,l',i}.$$

Then by Proposition 3.5(i), $\mathfrak{d} \wedge V_{l-1,l',i}$ is contained in V_k.

(ii) By hypothesis,

$$V_k \supset \bigwedge^{l'}(\mathfrak{p}_{\mathrm{u}}) \wedge M_{l,i,-} \supset V_{l,l',i}.$$

Then by Proposition 3.5(ii), $\mathfrak{d} \wedge V_{l,l'-1,i}$ is contained in V_k.

(iii) By hypothesis,

$$V_k \supset \bigwedge^{l}(\mathfrak{p}_{-,\mathrm{u}}) \wedge M'_{l',i'} \supset V'_{l,l',i'}.$$

Then by Proposition 3.5(i), $\mathfrak{d} \wedge V'_{l-1,l',i'}$ is contained in V_k.

(iv) By hypothesis,

$$V_k \supset \bigwedge^{l'}(\mathfrak{p}_{\mathrm{u}}) \wedge M'_{l',i',-} \supset V'_{l,l',i'}.$$

Then by Proposition 3.5(ii), $\mathfrak{d} \wedge V'_{l,l'-1,i'}$ is contained in V_k. □

Corollary 4.4. *Let $k = 1, \dots, n$, (l, l') in $\mathbb{N}^2$ such that $l + l' < 2d$, i in $\mathbb{I}_{k-l-l'-1}$ and i' in $\mathbb{I}_{k-l-l'-2}$. Then $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are contained in V_k.*

Proof. Since $V_{l,l',i}$ is contained in $V_{k-1,\mathfrak{p}}$, $\mathfrak{p}_{\mathrm{u}} \wedge V_{l,l',i}$ and $\mathfrak{p}_{-,\mathrm{u}} \wedge V_{l,l',i}$ are contained in $V_{k,\mathfrak{p}}$. Moreover, for all nonnegative integer s such that $l+l'+s+1 \leq 2d$ and all j in $\Delta_{i,|i|-s}$, $\mathfrak{p}_{\mathrm{u}} \wedge V_{l,l'+s,j}$, $\mathfrak{p}_{-,\mathrm{u}} \wedge V_{l,l'+s,j}$, $\mathfrak{p}_{\mathrm{u}} \wedge V_{l+s,l',j}$, $\mathfrak{p}_{-,\mathrm{u}} \wedge V_{l+s,l',j}$ are contained in $V_{k,\mathfrak{p}}$. Then, by Lemma 4.3,(i) and (ii), $\mathfrak{d} \wedge V_{l,l',i}$ is contained in V_k.

Since $V'_{l,l',i'}$ is contained in $V_{k-1,\mathfrak{p}}$, $\mathfrak{p}_{\mathrm{u}} \wedge V'_{l,l',i'}$ and $\mathfrak{p}_{-,\mathrm{u}} \wedge V'_{l,l',i'}$ are contained in $V_{k,\mathfrak{p}}$. Moreover, for all nonnegative integer s such that $l+l'+s+1 \leq 2d$ and all j in $\Delta_{i',|i'|-s}$, $\mathfrak{p}_{\mathrm{u}} \wedge V'_{l,l'+s,j}$, $\mathfrak{p}_{-,\mathrm{u}} \wedge V'_{l,l'+s,j}$, $\mathfrak{p}_{\mathrm{u}} \wedge V'_{l+s,l',j}$, $\mathfrak{p}_{-,\mathrm{u}} \wedge V'_{l+s,l',j}$ are contained in $V_{k,\mathfrak{p}}$. Then, by Lemma 4.3,(iii) and (iv), $\mathfrak{d} \wedge V'_{l,l',i'}$ is contained in V_k. By definition,

$$V_{k,\mathfrak{p}} \supset \mathfrak{p}_{-,\mathrm{u}} \wedge V_{k-1,\mathfrak{p}} + \mathfrak{z} \wedge V_{k-1,\mathfrak{p}} + \mathfrak{p}_{\mathrm{u}} \wedge V_{k-1,\mathfrak{p}}.$$

Hence $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are containe in V_k. □

Proposition 4.5. *Let $k = 1, \dots, n$. Suppose that one of the following condition is satisfied:*

(1) Π *is exceptional,*
(2) Π *has type* D_ℓ *and* $\mathrm{n} = 3$,
(3) Π *has classical type,* $\mathrm{n} = 2$, $2d + n_1$ *and* $2d + n_2$ *are bigger than* n.

Then $V_k = \bigwedge^k(\mathfrak{g})$.

Proof. Prove the proposition by induction on k. For $k = 1$, it is true since $\mathfrak{g}$ is simple. Suppose $k > 1$ and the proposition true for $k - 1$. By Lemma 1.2 and the induction hypothesis, it is sufficient to prove that $\mathfrak{g} \wedge V_{k-1,\mathfrak{p}}$ is contained in V_k. As a matter of fact, we have to prove that $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are contained in V_k for (l, l') in $\mathbb{N}^2$ such that $l + l' \leq 2d$, i in $\mathbb{I}_{k-l-l'-1}$ and i' in $\mathbb{I}_{k-l-l'-2}$ since

$$V_{k-1,\mathfrak{p}} = \bigoplus_{t=0}^{2d} \bigoplus_{(l,l')\in\mathbb{N}_t^2} \bigoplus_{i\in\mathbb{I}_{k-t-1}} V_{l,l',i} \oplus \bigoplus_{t=0}^{2d} \bigoplus_{(l,l')\in\mathbb{N}_t^2} \bigoplus_{i'\in\mathbb{I}_{k-t-2}} V'_{l,l',i'}.$$

Let (l, l') be in $\mathbb{N}^2$ such that $l + l' \leq 2d$, i in $\mathbb{I}_{k-l-l'-1}$ and i' in $\mathbb{I}_{k-l-l'-2}$. If Condition (1) or Condition (2) is satisfied, then $l + l' < 2d$ by Proposition A.1. As a result, by Corollary 4.4, $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are contained in V_k. If Condition (3) is satisfied, $l + l' < 2d$ or $l + l' = 2d$, $i_1 < n_1$, $i_2 < n_2$. In the first case, by Corollary 4.4, $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are contained in V_k. In the second case, $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are contained in $V_{k,\mathfrak{p}}$, whence the proposition. □

Remark 4.6. By the proof of Proposition 4.5, when $\mathrm{n} = 2$, for $k = 1, \ldots, \inf\{2d+n_1-1, 2d+n_2-1\}$, $V_k = \bigwedge^k(\mathfrak{g})$.

4.3. **A second particular case.** In this subsection, $|X| = \ell - 1$, Π has classical type, $\mathrm{n} = 2$ and $2d + n_1 \leq n$. By Proposition A.1, $2d + n_2 > n$, $\ell \geq 6$ for Π of type A_ℓ, $\ell \geq 7$ for Π of type B_ℓ or C_ℓ, $\ell \geq 8$ for Π of type D_ℓ.

For $i = (i_0, i_1, i_2, i_3, i_4)$ in $\mathbb{N}^5$, set:

$$C_i := \bigwedge^{i_0}(\mathfrak{z}) \wedge \bigwedge^{i_1}(\mathfrak{d}_1) \wedge \bigwedge^{i_2}(\mathfrak{d}_2) \wedge \bigwedge^{i_3}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \bigwedge^{i_4}(\mathfrak{p}_{\mathrm{u}}).$$

Let $k = 2d + n_1, \ldots, n$ and $j = k - 2d - n_1$. Set:

$$\iota := (0, n_1, j, d, d), \quad \iota' := (1, n_1, j-1, d, d),$$

$$\iota_+ := (0, n_1+1, j-1, d, d), \quad \iota'_+ := (1, n_1+1, j-2, d, d), \quad \kappa := (0, n_1+1, j, d-1, d),$$

$$\kappa' := (1, n_1+1, j-1, d-1, d), \quad \kappa_- := (0, n_1+1, j, d, d-1), \quad \kappa'_- := (1, n_1+1, j-1, d, d-1).$$

Lemma 4.7. *Denote by M_ι and $M_{\iota'}$ the G-submodules of $\bigwedge^k(\mathfrak{g})$ generated by C_ι and $C_{\iota'}$ respectively.*

(i) *The subspace M_ι of $\bigwedge^k(\mathfrak{g})$ contains C_κ and C_{κ_-}, and $M_{\iota'}$ contains $C_{\kappa'}$ and $C'_{\kappa'_-}$.*

(ii) *The spaces C_{ι_+} and $C_{\iota'_+}$ are contained in M_ι and $M_{\iota'}$ respectively.*

Proof. (i) The subspaces of $\bigwedge^{k-d}(\mathfrak{g})$,

$$\bigwedge^{n_1}(\mathfrak{d}_1) \wedge \bigwedge^{j}(\mathfrak{d}_2) \wedge \bigwedge^{d}(\mathfrak{p}_{\mathrm{u}}) \quad \text{and} \quad \bigwedge^{n_1}(\mathfrak{d}_1) \wedge \bigwedge^{j}(\mathfrak{d}_2) \wedge \mathfrak{z} \wedge \bigwedge^{d}(\mathfrak{p}_{\mathrm{u}}),$$

are invariant under P_{u}. So, by Proposition 3.5(i), M_ι and $M_{\iota'}$ contain C_κ and $C_{\kappa'}$ respectively. The subspaces of $\bigwedge^{k-d}(\mathfrak{g})$,

$$\bigwedge^{d}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \bigwedge^{n_1}(\mathfrak{d}_1) \wedge \bigwedge^{j}(\mathfrak{d}_2) \quad \text{and} \quad \bigwedge^{d}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \bigwedge^{n_1}(\mathfrak{d}_1) \wedge \bigwedge^{j}(\mathfrak{d}_2) \wedge \mathfrak{z},$$

are invariant under $P_{-\mathrm{u}}$. So, by Proposition 3.5(ii), M_ι and $M_{\iota'}$ contain C_{κ_-} and $C_{\kappa'_-}$ respectively.

(ii) For $i = (i_0, i_1, i_2, i_3, i_4)$ in $\mathbb{N}^5$, set: $i^* := (i_0, i_1, i_2, i_4, i_3,)$. By corollary 2.3(i), for i, j in $\mathbb{N}^5$, C_i is orthogonal to C_j if and only if $j \neq i^*$.

Denote by $M_\iota^\perp$ and $C_\iota^\perp$ the orthogonal complements to M_ι and C_ι in $\bigwedge^k(\mathfrak{g})$ respectively. By Lemma 2.1, $M_\iota^\perp$ is the biggest G-module contained in $C_\iota^\perp$. Suppose that C_{ι_+} is not contained in M_ι. A contradiction is expected. As $\bigwedge^k(\mathfrak{g})$ is the direct sum of C_i, $i \in \mathbb{N}_k^5$, $C_\iota^\perp$ is the direct sum of C_i, $i \in \mathbb{N}_k^5 \setminus \{\iota\}$ since $\iota^* = \iota$. By (i), $M_\iota^\perp$ is contained in the sum of C_i, $i \in \mathbb{N}_k^5 \setminus \{\iota, \kappa, \kappa_-\}$. Since $\iota_+^* = \iota_+$, the orthogonal complement to C_{ι_+} is the sum of C_i, $i \in \mathbb{N}_k^5 \setminus \{\iota_+\}$. Then $M_\iota^\perp$ is not contained in the direct sum of C_i, $i \in \mathbb{N}_k^5 \setminus \{\iota, \kappa, \kappa_-, \iota_+\}$ since C_{ι_+} is not contained in M_ι. Hence for some subspace M of $\bigwedge^{n_1+1}(\mathfrak{d}_1) \wedge \bigwedge^{j-1}(\mathfrak{d}_2)$,

$$M \neq \{0\} \quad \text{and} \quad M_\iota^\perp \supset \bigwedge^d(\mathfrak{p}_{-,\mathrm{u}}) \wedge M \wedge \bigwedge^d(\mathfrak{p}_\mathrm{u})$$

since $\bigwedge^d(\mathfrak{p}_{-,\mathrm{u}}) \wedge \bigwedge^d(\mathfrak{p}_\mathrm{u})$ has dimension 1. As a result, by Proposition 3.5, (i) and (ii),

$$M_\iota^\perp \supset \bigwedge^{d-1}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{d}_2 \wedge M \wedge \bigwedge^d(\mathfrak{p}_\mathrm{u})$$
$$M_\iota^\perp \supset \bigwedge^d(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{d}_2 \wedge M \wedge \bigwedge^{d-1}(\mathfrak{p}_\mathrm{u})$$

since $M \wedge \bigwedge^d(\mathfrak{p}_\mathrm{u})$ is a P_u-submodule of $\bigwedge^{d+j+n_1}(\mathfrak{g})$ and $M \wedge \bigwedge^d(\mathfrak{p}_{-,\mathrm{u}})$ is a $P_{-,\mathrm{u}}$-submodule of $\bigwedge^{d+j+n_1}(\mathfrak{g})$. As j is smaller than $\dim \mathfrak{d}_2$ and M is different from zero, $\mathfrak{d}_2 \wedge M \neq \{0\}$. Then $C_\kappa + C_{\kappa,-}$ is not contained in M_ι since $C_\kappa + C_{\kappa_-}$ is orthogonal to C_i for all i in $\mathbb{N}_k^5 \setminus \{\kappa, \kappa_-\}$, whence the contradiction.

Denote by $M_{\iota'}^\perp$ and $C_{\iota'}^\perp$ the orthogonal complements to $M_{\iota'}$ and $C_{\iota'}$ in $\bigwedge^k(\mathfrak{g})$ respectively. By Lemma 2.1, $M_{\iota'}^\perp$ is the biggest G-module contained in $C_{\iota'}^\perp$. Suppose that $C_{\iota'_+}$ is not contained in $M_{\iota'}$. A contradiction is expected. As $\bigwedge^k(\mathfrak{g})$ is the direct sum of C_i, $i \in \mathbb{N}_k^5$, $C_{\iota'}^\perp$ is the direct sum of C_i, $i \in \mathbb{N}_k^5 \setminus \{\iota'\}$ since $\iota'^* = \iota'$. By (i), $M_{\iota'}^\perp$ is contained in the sum of C_i, $i \in \mathbb{N}_k^5 \setminus \{\iota', \kappa', \kappa'_-\}$. Since ${\iota'_+}^* = \iota'_+$, the orthogonal complement to $C_{\iota'_+}$ is the sum of C_i, $i \in \mathbb{N}_k^5 \setminus \{\iota'_+\}$. Then $M_{\iota'}^\perp$ is not contained in the direct sum of C_i, $i \in \mathbb{N}_k^5 \setminus \{\iota', \kappa', \kappa'_-, \iota'_+\}$ since $C_{\iota'_+}$ is not contained in $M_{\iota'}$. Hence for some subspace M' of $\bigwedge^{n_1+1}(\mathfrak{d}_1) \wedge \bigwedge^{j-2}(\mathfrak{d}_2)$,

$$M' \neq \{0\} \quad \text{and} \quad M_{\iota'}^\perp \supset \bigwedge^d(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{z} \wedge M' \wedge \bigwedge^d(\mathfrak{p}_\mathrm{u})$$

since $\mathfrak{z} \wedge \bigwedge^d(\mathfrak{p}_{-,\mathrm{u}}) \wedge \bigwedge^d(\mathfrak{p}_\mathrm{u})$ has dimension 1. As a result, by Proposition 3.5, (i) and (ii),

$$M_{\iota'}^\perp \supset \bigwedge^{d-1}(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{z} \wedge \mathfrak{d}_2 \wedge M' \wedge \bigwedge^d(\mathfrak{p}_\mathrm{u})$$
$$M_{\iota'}^\perp \supset \bigwedge^d(\mathfrak{p}_{-,\mathrm{u}}) \wedge \mathfrak{z} \wedge \mathfrak{d}_2 \wedge M' \wedge \bigwedge^{d-1}(\mathfrak{p}_\mathrm{u})$$

since $\mathfrak{z} \wedge M' \wedge \bigwedge^d(\mathfrak{p}_\mathrm{u})$ is a P_u-submodule of $\bigwedge^{d+j+n_1}(\mathfrak{g})$ and $\mathfrak{z} \wedge M' \wedge \bigwedge^d(\mathfrak{p}_{-,\mathrm{u}})$ is a $P_{-,\mathrm{u}}$-submodule of $\bigwedge^{d+i+n_1}(\mathfrak{g})$. As j is smaller than $\dim \mathfrak{d}_2$ and M' is different from zero, $\mathfrak{d}_2 \wedge M' \neq \{0\}$. Then $C_{\kappa'} + C_{\kappa',-}$ is not contained in $M_{\iota'}$ since $C_{\kappa'} + C_{\kappa'_-}$ is orthogonal to C_i for all i in $\mathbb{N}_k^5 \setminus \{\kappa', \kappa'_-\}$, whence the contradiction. □

Proposition 4.8. *For $k = 2d + n_1, \ldots, n$, V_k is equal to $\bigwedge^k(\mathfrak{g})$.*

Proof. Prove the proposition by induction on k. Let (l, l') be in $\mathbb{N}^2$ such that $l+l' \leq 2d$, $i \in \mathbb{I}_{k-l-l'-1}$, $i' \in \mathbb{I}_{k-l-l'-2}$. If $l + l' < 2d$ then $\mathfrak{g} \wedge V_{l,l',i}$ and $\mathfrak{g} \wedge V'_{l,l',i'}$ are contained in V_k by Corollary 4.4. If $l = l' = d$ and $i_1 < n_1$ then $\mathfrak{g} \wedge V_{l,l',i}$ is contained in $V_{k,\mathfrak{p}}$ since $2d + n_2 > n$ by Proposition A.1. If

$l = l' = d$ and $i'_1 < n_1$ then $\mathfrak{g} \wedge V'_{l,l',i'}$ is contained in $V_{k,\mathfrak{p}}$ since $2d + n_2 > n$ by Proposition A.1. As a result, for $k = 2d + n_1$, by Lemma 1.2 and Remark 4.6, $V_k = \bigwedge^k(\mathfrak{g})$.

Suppose $k > 2d + n_1$, $V_{k-1} = \bigwedge^{k-1}(\mathfrak{g})$, $i = (n_1, k - 2d - n_1 - 1)$ and $i' = (n_1, k - 2d - n_1 - 2)$. By definition, $V_{k,\mathfrak{p}}$ contains the subspaces of $\bigwedge^k(\mathfrak{g})$,

$$\mathfrak{p}_{-,\mathrm{u}} \wedge V_{d,d,i}, \quad \mathfrak{p}_{\mathrm{u}} \wedge V_{d,d,i}, \quad \mathfrak{d}_2 \wedge V_{d,d,i}, \quad \mathfrak{p}_{-,\mathrm{u}} \wedge V'_{d,d,i'}, \quad \mathfrak{p}_{\mathrm{u}} \wedge V'_{d,d,i'}, \quad \mathfrak{d}_2 \wedge V'_{d,d,i'}$$

since i_2 and i'_2 are smaller than n_2. By Lemma 4.7(ii), V_k contains $\mathfrak{d}_1 \wedge V_{d,d,i}$ and $\mathfrak{d}_1 \wedge V'_{d,d',i'}$. Then V_k contains $\mathfrak{g} \wedge V_{d,d,i}$ and $\mathfrak{g} \wedge V'_{d,d,i'}$. As a result, by our previous remark, V_k contains $\mathfrak{g} \wedge V_{k-1,\mathfrak{p}}$, whence the proposition by Lemma 1.2 and the induction hypothesis. □

4.4. **The general case.** First, we consider the case when X contains the extremities of Π.

Lemma 4.9. *Suppose $\ell \geq 2$, $\mathrm{n} \geq 2$ and the extremities of Π contained in X. If $|X|$ is smaller than $\ell - 1$ then for some β in $\Pi \setminus X$, $Y := \Pi \setminus \{\beta\}$ has two connected components, X is contained in Y and a connected component of Y is a connected component of X.*

Proof. Suppose $|X| < \ell - 1$. We consider the following cases:

(1) Π has not type D, E,
(2) Π has type D_ℓ,
(3) Π has type E_6,
(4) Π has type E_7,
(5) Π has type E_8.

(1) Let X_1 be the connected component of X containing β_1. There is only one element β of $\Pi \setminus X$ not orthogonal to X_1. Then $Y := \Pi \setminus \{\beta\}$ has two connected components and X_1 is a connected component of Y.

(2) Let X_1 be the connected component of X containing β_1. As β_ℓ and $\beta_{\ell-1}$ are in X, for some i smaller than $\ell - 2$, β_i is not in X since $|X| < \ell - 1$. Then there is only one element β in $\Pi \setminus X$ not orthogonal to X_1 so that $Y := \Pi \setminus \{\beta\}$ has two connected components and X_1 is a connected component of Y.

(3) As $\beta_1, \beta_2, \beta_6$ are in X, β_3 or β_5 is not in X since $|X| < \ell - 1$. Setting $Y_i := \Pi \setminus \{\beta_i\}$ for $i = 3, 5$, Y_i has two connected components and for some i, X is contained in Y_i and a connected component of X is a connected component of Y_i.

(4) As $\beta_1, \beta_2, \beta_7$ are in X, β_3 or β_5 or β_6 is not in X since $|X| < \ell - 1$. Setting $Y_i := \Pi \setminus \{\beta_i\}$ for $i = 3, 5, 6$, Y_i has two connected components and for some i, X is contained in Y_i and a connected component of X is a connected component of Y_i.

(5) As $\beta_1, \beta_2, \beta_8$ are in X, β_3 or β_5 or β_6 or β_7 is not in X since $|X| < \ell - 1$. Setting $Y_i := \Pi \setminus \{\beta_i\}$ for $i = 3, 5, 6, 7$, Y_i has two connected components and for some i, X is contained in Y_i and a connected component of X is a connected component of Y_i. □

Proposition 4.10. *Let $k = 1, \ldots, n$. Suppose that Theorem 1.1 is true for the simple algebras of rank smaller than ℓ and X contains the extremities of Π. Then $V_k = \bigwedge^k(\mathfrak{g})$.*

Proof. As X contains the extremities of Π and is different from Π, $\ell \geq 3$ and $\mathrm{n} \geq 2$. By Proposition 4.5, Remark 4.6 and Proposition 4.8, $V_k = \bigwedge^k(\mathfrak{g})$ when $|X| = \ell - 1$. In particular, $V_k = \bigwedge^k(\mathfrak{g})$ when $\ell = 3$. Suppose $\ell > 3$ and $|X| < \ell - 1$.

Let Y be as in Lemma 4.9. Then $\mathfrak{d}_Y$ has two simple factors $\mathfrak{d}_1$ and $\mathfrak{a}$ and $\mathfrak{d}_1$ is a simple factor of $\mathfrak{d}$. Denote by $V_{k,Y}$ the G-submodule of $\bigwedge^k(\mathfrak{g})$ generated by $V_{k,\mathfrak{p}_Y}$. Then, by Proposition 4.5, Remark 4.6 and Proposition 4.8, $V_{k,Y} = \bigwedge^k(\mathfrak{g})$. The intersection $\mathfrak{q} := \mathfrak{a} \cap \mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{a}$. Let E' be the intersection of E and $\mathfrak{a}$. Then E is the direct sum of E' and E_Y. As a result, setting $n_* := \mathfrak{b}_\mathfrak{a} - \ell_\mathfrak{a}$,

$$V_{k,\mathfrak{p}} = \bigoplus_{i=0}^{n_1} \bigoplus_{j=0}^{n_*} \bigwedge^i(\mathfrak{d}_1) \wedge V_{j,\mathfrak{q}} \wedge \bigwedge^{k-i-j}(E_Y).$$

Let A be the connected closed subgroup of G whose Lie algebra is $\mathfrak{a}$. By the hypothesis, for $j = 1, \dots, n_*$, the A-submodule of $\bigwedge^j(\mathfrak{a})$, generated by $V_{j,\mathfrak{q}}$, is equal to $\bigwedge^j(\mathfrak{a})$. Hence, by Lemma 1.2, $V_{k,\mathfrak{p}_Y}$ is contained in V_k since $\mathfrak{d}_1$ and E_Y are invariant under A, whence $V_k = \bigwedge^k(\mathfrak{g})$. □

To finish the proof of Theorem 1.1, we have to consider the case when X does not contain all the extremities of Π.

Lemma 4.11. *Suppose that X does not contain all the extremities of Π.*

(i) *There exists a sequence*

$$X_0 \subset \cdots \subset X_m = \Pi$$

of connected subsets of Π satisfying the following conditions:

(1) *for $i = 1, \dots, m$, $|X_i \setminus X_{i-1}| = 1$,*

(2) *X contains the extremities of X_0.*

(ii) *For $i = 0, \dots, m$, let $\mathfrak{a}_i$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^{\pm\beta}$, $\beta \in X_i$. Then $\mathfrak{a}_i$ is a simple algebra and $\mathfrak{p}_i := \mathfrak{p} \cap \mathfrak{a}_i$ is a parabolic subalgebra of $\mathfrak{a}_i$.*

(iii) *For $i = 0, \dots, m$, E is the direct sum of $E_i := E \cap \mathfrak{a}_i$ and E_{X_i}.*

Proof. (i) Define X_i by induction on i. Let X_0 be a connected subset of Π, containing X of minimal cardinality. By minimality of $|X_0|$, X contains the extremities of X_0. Suppose $i > 0$ and X_{i-1} defined. If $X_{i-1} = \Pi$ there is nothing to do. Suppose $X_{i-1} \neq \Pi$. As Π is connected, there is some β in $\Pi \setminus X_{i-1}$, not orthogonal to an extremity of X_{i-1}. Then $X_i := X_{i-1} \cup \{\beta\}$ is a connected subset of Π since so is X_{i-1}, whence the assertion.

(ii) As X_i is connected, $\mathfrak{a}_i$ is a simple algebra. For α in $< X_i >$, $\mathfrak{g}^\alpha$ is contained in $\mathfrak{a}_i \cap \mathfrak{p}$. Hence $\mathfrak{p}_i$ contains the Borel subalgebra of $\mathfrak{a}_i$ generated by $\mathfrak{h} \cap \mathfrak{a}_i$ and $\mathfrak{g}^\beta$, $\beta \in X_i$, whence the assertion.

(iii) Let α be a positive root such that $\mathfrak{g}^\alpha$ is contained in E. If α is in $< X_i >$ then $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ are contained in E_i. Otherwise, $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ are contained in E_{X_i} by definition.

Let z be in $E \cap \mathfrak{h}$. By definition, $\mathfrak{h} \cap E_{X_i}$ is the orthogonal complement to $\mathfrak{a}_i$ in $\mathfrak{h}$. Then $z = z_1 + z_2$ with z_1 in $\mathfrak{a}_i \cap \mathfrak{h}$ and z_2 in E_{X_i}. Hence z_1 is orthogonal to $\mathfrak{a}_i \cap \mathfrak{p}$. As a result, z_1 is in E_i and z is in $E_i + E_{X_i}$, whence the assertion. □

We can now give the proof of Theorem 1.1.

Proof. Prove the theorem by induction on ℓ. First of all, for X empty subset of Π, $V_{k,\mathfrak{p}_X} = \bigwedge^k(\mathfrak{g})$. For $\ell = 1$, $n = 1$. Hence the theorem is true in this case and we can suppose X nonempty and $\ell \geq 2$. By Proposition 4.1 and Remark 4.2, $V_k = \bigwedge^k(\mathfrak{g})$ when X is connected. In particular, the

theorem is true in rank 2. Then, by Proposition 4.10, the theorem is true for $\ell = 3$ since in this case X contains all the extremities of Π when it is not connected.

Suppose $\ell > 3$ and the theorem true for the simple algebras of rank smaller than ℓ. By induction hypothesis and Proposition 4.10, $V_k = \bigwedge^k(\mathfrak{g})$ when X contains all the extremities of Π. So, we can suppose that X does not contain all the extremities of Π. Let $X_0, \dots, X_m$ be as in Lemma 4.11. For $i = 0, \dots, m$, set $e_i := |< X_i >|$ and prove by induction on i the inclusion

$$\bigoplus_{j=0}^{e_i} \bigwedge^j(\mathfrak{a}_i) \wedge \bigwedge^{k-j}(E_{X_i}) \subset V_k.$$

For $i = 0, \dots, n$, denote by A_i the connected closed subgroup of G whose Lie algebra is $\mathfrak{a}_i$. By Lemma 4.11,(ii) and (iii),

$$V_{k,\mathfrak{p}} = \bigoplus_{j=0}^{e_i} V_{j,\mathfrak{p}_i} \wedge \bigwedge^{k-j}(E_{X_i}),$$

for $i = 0, \dots, m$. Then, by Proposition 4.10, the induction hypothesis and Lemma 1.2, the inclusion is true for $i = 0$ since E_{X_0} is invariant under A_0. Suppose $i > 0$ and the inclusion true for $i - 1$. Let E'_{X_i} be the intersection of $E_{X_{i-1}}$ and $\mathfrak{a}_i$. Denote by $\mathfrak{q}_i$ the parabolic subalgebra of $\mathfrak{a}_i$ containing $\mathfrak{b} \cap \mathfrak{a}_i$ and such that $\mathfrak{a}_{i-1}$ is the derived algebra of the reductive factor of $\mathfrak{q}_i$ containing $\mathfrak{h} \cap \mathfrak{a}_i$. Then

$$E_{X_{i-1}} = E'_{X_i} \oplus E_{X_i}, \quad \mathfrak{a}_i = \mathfrak{a}_{i-1} \oplus E'_{X_i},$$

$$\bigwedge^j(\mathfrak{a}_{i-1}) \wedge \bigwedge^{k-j}(E_{X_{i-1}}) = \bigoplus_{l=0}^{k-j} \bigwedge^j(\mathfrak{a}_{i-1}) \wedge \bigwedge^l(E'_{X_i}) \wedge \bigwedge^{k-j-l}(E_{X_{i-1}})$$

for $j = 0, \dots, e_{i-1}$. As a result,

$$\bigoplus_{j=0}^{e_{i-1}} \bigwedge^j(\mathfrak{a}_{i-1}) \wedge \bigwedge^{k-j}(E_{X_{i-1}}) = \bigoplus_{j=0}^{e_i} V_{j,\mathfrak{q}_i} \wedge \bigwedge^{k-j}(E_{X_i}).$$

By Proposition 4.1 and Remark 4.2, for $j = 0, \dots, e_i$, the A_i-submodule of $\bigwedge^j(\mathfrak{a}_i)$ generated by $V_{j,\mathfrak{q}_i}$ is equal to $\bigwedge^j(\mathfrak{a}_i)$ since $\mathfrak{a}_{i-1}$ is simple and $\ell_{\mathfrak{a}_i} - \ell_{\mathfrak{a}_{i-1}} = 1$. Then, by Lemma 1.2, the A_i-submodule of $\bigwedge^k(\mathfrak{g})$ generated by

$$\bigoplus_{j=0}^{e_{i-1}} \bigwedge^j(\mathfrak{a}_{i-1}) \wedge \bigwedge^{k-j}(E_{X_{i-1}})$$

is equal to

$$\bigoplus_{j=0}^{e_i} \bigwedge^j(\mathfrak{a}_i) \wedge \bigwedge^{k-j}(E_{X_i})$$

since E_{X_i} is invariant under A_i, whence the assertion and the theorem since for $i = m$ the sum is equal to $\bigwedge^k(\mathfrak{g})$. □

Appendix A. Some remarks on root systems

Let β be in Π and $X := \Pi \setminus \{\beta\}$. Set $\mathfrak{p}_{\mathrm{u}} := \mathfrak{p}_{\mathrm{u},X}$ and $d := \dim \mathfrak{p}_{\mathrm{u},X}$. The goal of the section is the following proposition:

Proposition A.1. (i) *Suppose Π of type A_ℓ and X not connected. Then $\beta = \beta_{s+1}$ for some s in $\{1, \dots, \ell-2\}$,*

$$\mathrm{n} = 2, \quad n_1 = \frac{s(s+1)}{2}, \quad n_2 = \frac{(\ell-s-1)(\ell-s)}{2}.$$

Moreover, if $2d + n_1 \leq n$ then

$$\ell \geq 6, \quad s \leq \frac{1}{6}(2\ell - 3 - \sqrt{4\ell^2 + 12\ell + 9}), \quad 2d + n_2 > n.$$

(ii) *Suppose Π of type B_ℓ or C_ℓ and X not connected. Then $\beta = \beta_{s+1}$ for some s in $\{1, \dots, \ell-2\}$,*

$$\mathrm{n} = 2, \quad n_1 = \frac{s(s+1)}{2}, \quad n_2 = (\ell-s-1)^2.$$

Moreover, if $2d + n_1 \leq n$ then

$$\ell \geq 7, \quad s \leq \frac{1}{10}(8\ell - 9 - \sqrt{24\ell^2 + 16\ell + 1}), \quad 2d + n_2 > n.$$

(iii) *Suppose Π of type D_ℓ. If $\beta = \beta_{\ell-2}$ then $2d > n$. If β is different from $\beta_{\ell-2}$ and X is not connected, then*

$$\mathrm{n} = 2, \quad n_1 = \frac{s(s+1)}{2}, \quad n_2 = (\ell-s-1)^2.$$

Moreover, if $2d + n_1 \leq n$ then

$$\ell \geq 8, \quad s \leq \frac{1}{10}(8\ell - 13 - \sqrt{24\ell^2 - 8\ell + 9}), \quad 2d + n_2 > n.$$

(iv) *Suppose that Π is exceptional. If $2d \leq n$ then X is connected.*

We prove the proposition case by case. So, in the classical case, we suppose $\ell \geq 3$ and X not connected.

A.1. **Type A_ℓ.** As X is not connected, $\mathrm{n} = 2$ and $\beta = \beta_{s+1}$ for some s in $\{1, \dots, \ell-2\}$. Then

$$n_1 = \frac{s(s+1)}{2}, \quad n_2 = \frac{(\ell-s-1)(\ell-s)}{2}, \quad d = n - n_1 - n_2,$$

$$n - 2d - n_1 = \frac{1}{2}(3s^2 + (-4\ell + 3)s + \ell^2 - 3\ell).$$

If $n - 2d - n_1 \geq 0$ then

$$s \leq \frac{1}{6}(4\ell - 3 - \sqrt{4\ell^2 + 12\ell + 9}) \quad \text{or} \quad s \geq \frac{1}{6}(4\ell - 3 + \sqrt{4\ell^2 + 12\ell + 9}).$$

As $s \geq 1$, the first inequality is possible only if $\ell \geq 6$. The second inequality is impossible since its right hand side is bigger than $\ell - 2$ and s is at most $\ell - 2$.

By the above equalities,

$$n - 2d - n_2 = 3s^2 + (-2\ell + 3)s - 2\ell.$$

If the left hand side is nonnegative then

$$s \leq \frac{1}{6}(2\ell - 3 - \sqrt{4\ell^2 + 12\ell + 9}) \quad \text{or} \quad s \geq \frac{1}{6}(2\ell - 3 + \sqrt{4\ell^2 + 12\ell + 9}).$$

The first inequality is impossible since its right hand side is negative. The second inequality is possible only if $\ell \geq 7$ since $s \leq \ell - 2$. Moreover, it is not possible to have $n \geq 2d + n_1$ and $n \geq 2d + n_2$ since

$$\frac{1}{6}(2\ell - 3 + \sqrt{4\ell^2 + 12\ell + 9}) > \frac{1}{6}(4\ell - 3 - \sqrt{4\ell^2 + 12\ell + 9}),$$

whence Assertion (i) of Proposition A.1.

A.2. **Type** B_ℓ **or** C_ℓ**.** As X is not connected, $\mathrm{n} = 2$ and $\beta = \beta_{s+1}$ for some s in $\{1, \ldots, \ell - 2\}$. Then

$$n_1 = \frac{s(s+1)}{2}, \quad n_2 = (\ell - s - 1)^2, \quad d = n - n_1 - n_2,$$

$$n - 2d - n_1 = \frac{1}{2}(5s^2 + (-8\ell + 9)s + 2\ell^2 - 8\ell + 4).$$

If $n - 2d - n_1 \geq 0$ then

$$s \leq \frac{1}{10}(8\ell - 9 - \sqrt{24\ell^2 + 16\ell + 1}) \quad \text{or} \quad s \geq \frac{1}{10}(8\ell - 9 + \sqrt{24\ell^2 + 16\ell + 1}).$$

As $s \geq 1$, the first inequality is possible only if $\ell \geq 7$. The second inequality is impossible since its right hand side is bigger than $\ell - 2$ and s is at most $\ell - 2$.

By the above equalities,

$$n - 2d - n_2 = 2s^2 + (-2\ell + 5)s - 4\ell + 4.$$

If the left hand side is nonnegative then

$$s \leq \frac{1}{4}(2\ell - 5 - \sqrt{4\ell^2 + 12\ell - 7}) \quad \text{or} \quad s \geq \frac{1}{4}(2\ell - 5 + \sqrt{4\ell^2 + 12\ell - 9}).$$

The first inequality is impossible since its right hand side is negative. The second inequality is impossible since

$$s \leq \ell - 2 \quad \text{and} \quad \frac{1}{4}(2\ell - 5 + \sqrt{4\ell^2 + 12\ell - 9}) > \ell - 2,$$

whence Assertion (ii) of Proposition A.1.

A.3. **Type** D_ℓ**.** As X is not connected, β is different from $\beta_1, \beta_{\ell-1}, \beta_\ell$. If $\beta = \beta_{\ell-2}$ then X has three connected components and

$$d = \ell(\ell - 1) - 2 - \frac{1}{2}(\ell - 3)(\ell - 2).$$

In this case $n < 2d$. Suppose $\ell \geq 5$ and $\beta = \beta_{s+1}$ for some s in $\{1, \ldots, \ell - 4\}$. Then

$$n_1 = \frac{s(s+1)}{2}, \quad n_2 = (\ell - s - 1)(\ell - s - 2), \quad d = n - n_1 - n_2,$$

$$n - 2d - n_1 = 5s^2 - s(4\ell - 7) + \ell^2 - 5\ell + 4.$$

If $n - 2d - n_1 \geq 0$ then

$$s \leq \frac{1}{10}(8\ell - 13 - \sqrt{24\ell^2 - 8\ell + 9}) \quad \text{or} \quad s \geq \frac{1}{10}(8\ell - 13 + \sqrt{24\ell^2 - 8\ell + 9}).$$

As $s \geq 1$, the first inequality is possible only if $\ell \geq 8$. The second inequality is impossible since its right hand side is bigger than $\ell - 4$ and s is at most $\ell - 4$.

By the above equalities,

$$n - 2d - n_2 = 2s^2 + (-2\ell + 4)s - 2\ell + 2.$$

If the left hand side is nonnegative then

$$s \leq -1 \quad \text{or} \quad s \geq \ell - 1.$$

These inequalities are impossible since s is positive and smaller than $\ell - 3$, whence Assertion (iii) of Proposition A.1.

A.4. **The exceptional case.** Set $\mathfrak{l} := \mathfrak{l}_X$, $\mathfrak{d} := \mathfrak{d}_X$. Then $2d = \dim \mathfrak{g} - \dim \mathfrak{l}$. For each case, we give all the possible dimensions of $\mathfrak{l}$ when $|X| = \ell - 1$.

(a) The algebra $\mathfrak{g}$ has type G_2. Then X is connected, whence Assertion (iv) of Proposition A.1 for this case.

(b) The algebra $\mathfrak{g}$ has type F_4. In this case $n = 24$ and

$$\dim \mathfrak{l} \in \{12, 22\} \quad \text{whence} \quad 2d \in \{40, 30\}$$

and Assertion (iv) of Proposition A.1 for this case.

(c) The algebra $\mathfrak{g}$ has type E_6. In this case $n = 36$ and

$$\dim \mathfrak{l} \in \{20, 28, 36, 46\} \quad \text{whence} \quad 2d \in \{58, 50, 42, 32\}$$

and Assertion (iv) of Proposition A.1 for this case since $\mathfrak{d}$ is simple of type D_5 when $2d = 32$.

(d) The algebra $\mathfrak{g}$ has type E_7. In this case $n = 63$ and

$$\dim \mathfrak{l} \in \{27, 33, 39, 49, 67, 79\} \quad \text{whence} \quad 2d \in \{106, 100, 94, 84, 66, 54\}$$

and Assertion (iv) of Proposition A.1 for this case since $\mathfrak{d}$ is simple of type E_6 when $2d = 54$.

(e) The algebra $\mathfrak{g}$ has type E_8. In this case $n = 120$ and

$$\dim \mathfrak{l} \in \{36, 40, 52, 54, 64, 82, 92, 134\} \quad \text{whence} \quad 2d \in \{212, 208, 196, 194, 184, 166, 156, 114\}$$

and Assertion (iv) of Proposition A.1 for this case since $\mathfrak{d}$ is simple of type E_7 when $2d = 114$.

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