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## Horn problem for quasi-hermitian Lie groups

Paul-Emile Paradan

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# Horn problem for quasi-hermitian Lie groups

Paul-Emile Paradan\*

June 13, 2020

## Abstract

In this paper, we prove some convexity results associated to orbit projection of non-compact real reductive Lie groups.

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## 1 Introduction

This paper is concerned with convexity properties associated to orbit projection.

Let us consider two Lie groups  $G \subset \tilde{G}$  with Lie algebras  $\mathfrak{g} \subset \tilde{\mathfrak{g}}$  and corresponding projection  $\pi_{\mathfrak{g}, \tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}}^* \rightarrow \mathfrak{g}^*$ . A longstanding problem has been to understand how a coadjoint orbit  $\tilde{\mathcal{O}} \subset \tilde{\mathfrak{g}}^*$  decomposes under the projection  $\pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}$ . For this purpose, we may define

$$\Delta_G(\tilde{\mathcal{O}}) = \{\mathcal{O} \in \mathfrak{g}^*/G; \mathcal{O} \subset \pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}(\tilde{\mathcal{O}})\}.$$

When the Lie group  $G$  is compact and connected, the set  $\mathfrak{g}^*/G$  admits a natural identification with a Weyl chamber  $\mathfrak{t}_{\geq 0}^*$ . In this context, we have the well-known convexity theorem [12, 1, 10, 16, 13, 34, 22].

**Theorem 1.1** *Suppose that  $G$  is compact connected and that the projection  $\pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}$  is proper when restricted to  $\tilde{\mathcal{O}}$ . Then  $\Delta_G(\tilde{\mathcal{O}}) = \{\xi \in \mathfrak{t}_{\geq 0}^*; G\xi \subset \pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}(\tilde{\mathcal{O}})\}$  is a closed convex locally polyhedral subset of  $\mathfrak{t}^*$ .*

When the Lie group  $\tilde{G}$  is also compact and connected, we may consider

$$(1) \quad \Pi(\tilde{G}, G) := \left\{ (\tilde{\xi}, \xi) \in \tilde{\mathfrak{t}}_{\geq 0}^* \times \mathfrak{t}_{\geq 0}^*; G\xi \subset \pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}(\tilde{G}\tilde{\xi}) \right\}.$$

Here is another convexity theorem [14, 17, 4, 2, 3, 25, 20, 35].

**Theorem 1.2** *Suppose that  $G \subset \tilde{G}$  are compact connected Lie groups. Then  $\Pi(\tilde{G}, G)$  is a closed convex polyhedral cone and we can parametrize its facets by cohomological means (i.e. Schubert calculus).*

In this article we obtain an extension of Theorems 1.1 and 1.2 in the case where the two groups  $G$  and  $\tilde{G}$  are not compact.

Let us explain in which framework we work. We suppose that  $\tilde{G}/\tilde{K}$  is a Hermitian symmetric space of a non-compact type. Among the elliptic coadjoint orbits of  $\tilde{G}$ , some of them are naturally Kähler  $\tilde{K}$ -manifolds. These orbits are called the holomorphic coadjoint orbits of  $\tilde{G}$ . They are the strongly elliptic coadjoint orbits closely related to the holomorphic discrete series of Harish-Chandra. These orbits intersect the Weyl chamber  $\tilde{\mathfrak{t}}_{\geq 0}^*$  of  $\tilde{K}$

into a sub-chamber  $\tilde{\mathcal{C}}_{\text{hol}}$  called the holomorphic chamber. The basic fact here is that the union

$$\mathcal{C}_{\tilde{G}/\tilde{K}}^0 := \bigcup_{\tilde{a} \in \tilde{\mathcal{C}}_{\text{hol}}} \tilde{G}\tilde{a}$$

is an open invariant convex cone of  $\tilde{\mathfrak{g}}^*$ . See §2.1 for more details.

Let  $G/K \subset \tilde{G}/\tilde{K}$  be a sub-Hermitian symmetric space of a non-compact type. As the projection  $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}}^* \rightarrow \mathfrak{g}^*$  sends the convex cone  $\mathcal{C}_{\tilde{G}/\tilde{K}}^0$  inside the convex cone  $\mathcal{C}_{G/K}^0$ , it is natural to study the following object reminiscent of (1) :

$$(2) \quad \Pi_{\text{hol}}(\tilde{G}, G) := \left\{ (\tilde{\xi}, \xi) \in \tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}}; G\xi \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{G}\tilde{\xi}) \right\}.$$

Let  $\tilde{\mu} \in \tilde{\mathcal{C}}_{\text{hol}}$ . We will also give a particular attention to the intersection of  $\Pi_{\text{hol}}(\tilde{G}, G)$  with the hyperplane  $\tilde{\xi} = \tilde{\mu}$ , that is to say

$$(3) \quad \Delta_G(\tilde{G}\tilde{\mu}) := \left\{ \xi \in \mathcal{C}_{\text{hol}}; G\xi \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{G}\tilde{\mu}) \right\}.$$

Consider the case where  $G$  is embedded diagonally in  $\tilde{G} := G^s$  for  $s \geq 2$ . The corresponding set  $\Pi_{\text{hol}}(G^s, G)$  is called the holomorphic Horn cone, and it is defined as follows

$$\text{Horn}_{\text{hol}}^s(G) := \left\{ (\xi_1, \dots, \xi_{s+1}) \in \mathcal{C}_{\text{hol}}^{s+1}; G\xi_{s+1} \subset \sum_{j=1}^s G\xi_j \right\}.$$

The first result of this article is the following Theorem.

**Theorem A.**

- $\Pi_{\text{hol}}(\tilde{G}, G)$  is a closed convex cone of  $\tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}}$ .
- $\text{Horn}_{\text{hol}}^s(G)$  is a closed convex cone of  $\mathcal{C}_{\text{hol}}^{s+1}$  for any  $s \geq 2$ .

We obtain the following corollary which corresponds to a result of Weinstein [37].

**Corollary.** *For any  $\tilde{\mu} \in \tilde{\mathcal{C}}_{\text{hol}}$ ,  $\Delta_G(\tilde{G}\tilde{\mu})$  is a closed and convex subset of  $\mathcal{C}_{\text{hol}}$ .*

A first description of the closed convex cone  $\Pi_{\text{hol}}(\tilde{G}, G)$  goes as follows. The quotient  $\mathfrak{q}$  of the tangent spaces  $\mathbf{T}_e G/K$  and  $\mathbf{T}_e \tilde{G}/\tilde{K}$  has a natural structure of a Hermitian  $K$ -vector space. The symmetric algebra  $\text{Sym}(\mathfrak{q})$  of  $\mathfrak{q}$  defines an admissible  $K$ -module. The irreducible representations of  $K$  (resp.  $\tilde{K}$ ) are parametrized by a semi-group  $\Lambda_+^*$  : (resp.  $\tilde{\Lambda}_+^*$ ). For any  $\lambda \in \Lambda_+^*$  (resp.  $\tilde{\lambda} \in \tilde{\Lambda}_+^*$ ), we denote by  $V_\lambda^K$  (resp.  $V_{\tilde{\lambda}}^{\tilde{K}}$ ) the irreducible representation of  $K$  (resp.  $\tilde{K}$ ) with highest weight  $\lambda$  (resp.  $\tilde{\lambda}$ ). If  $E$  is a representation of  $K$ , we denote by  $[V_\lambda^K : E]$  the multiplicity of  $V_\lambda^K$  in  $E$ .

**Definition 1.3** 1.  $\Pi^{\mathbb{Z}}(\tilde{G}, G)$  is the semigroup of  $\tilde{\Lambda}_+^* \times \Lambda_+^*$  defined by the conditions:

$$(\tilde{\lambda}, \lambda) \in \Pi^{\mathbb{Z}}(\tilde{G}, G) \iff [V_\lambda^K : V_{\tilde{\lambda}}^{\tilde{K}} \otimes \text{Sym}(\mathfrak{q})] \neq 0.$$

2.  $\Pi(\tilde{G}, G)$  is the convex cone defined as the closure of  $\mathbb{Q}^{>0} \cdot \Pi^{\mathbb{Z}}(\tilde{G}, G)$ .

The second result of this article is the following Theorem.

**Theorem B.** *We have the equality*

$$(4) \quad \Pi_{\text{hol}}(\tilde{G}, G) = \Pi(\tilde{G}, G) \cap \tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}}.$$

A natural question is the description of the facets of the convex cone  $\Pi_{\text{hol}}(\tilde{G}, G)$ . In order to do that, we consider the group  $\tilde{K}$  endowed with the following  $\tilde{K} \times K$ -action :  $(\tilde{k}, k) \cdot \tilde{a} = \tilde{k}\tilde{a}k^{-1}$ . The cotangent space  $\mathbf{T}^*\tilde{K}$  is then a symplectic manifold equipped with a Hamiltonian action of  $\tilde{K} \times K$ . We consider now the Hamiltonian  $\tilde{K} \times K$ -manifold  $\mathbf{T}^*\tilde{K} \times \mathfrak{q}$ , and we denote by  $\Delta(\mathbf{T}^*\tilde{K} \times \mathfrak{q})$  the corresponding Kirwan polyhedron.

Let  $W = N(T)/T$  be the Weyl group of  $(K, T)$  and let  $w_0$  be the longest Weyl group element. Define an involution  $*$  :  $\mathfrak{t}^* \rightarrow \mathfrak{t}^*$  by  $\xi^* = -w_0\xi$ . A standard result permits to affirm that  $(\tilde{\xi}, \xi) \in \Pi(\tilde{G}, G)$  is an only if  $(\tilde{\xi}, \xi^*) \in \Delta(\mathbf{T}^*\tilde{K} \times \mathfrak{q})$  (see §4.2).

We obtain then another version of Theorem B.

**Theorem B, second version.** *An element  $(\tilde{\xi}, \xi)$  belongs to  $\Pi_{\text{hol}}(\tilde{G}, G)$  if and only if*

$$(\tilde{\xi}, \xi) \in \tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}} \quad \text{and} \quad (\tilde{\xi}, \xi^*) \in \Delta(\mathbf{T}^*\tilde{K} \times \mathfrak{q}).$$

Thanks to the second version of Theorem B, a natural way to describe the facets of the cone  $\Pi_{\text{hol}}(\tilde{G}, G)$  is to exhibit those of the Kirwan polyhedron  $\Delta(\mathbf{T}^*\tilde{K} \times \mathfrak{q})$ . In this later case, it can be done using Ressayre's data (see §5).

The second version of Theorem B permits also the following description of the convex subsets  $\Delta_G(\tilde{G}\tilde{\mu})$ ,  $\tilde{\mu} \in \tilde{\mathcal{C}}_{\text{hol}}$ . Let  $\Delta_K(\tilde{K}\tilde{\mu} \times \bar{\mathfrak{q}})$  be the Kirwan polyhedron associated to the Hamiltonian action of  $K$  on  $\tilde{K}\tilde{\mu} \times \bar{\mathfrak{q}}$ . Here  $\bar{\mathfrak{q}}$  denotes the  $K$ -module  $\mathfrak{q}$  with opposite complex structure.

**Theorem C.** *For any  $\tilde{\mu} \in \tilde{\mathcal{C}}_{\text{hol}}$ , we have  $\Delta_G(\tilde{G}\tilde{\mu}) = \Delta_K(\tilde{K}\tilde{\mu} \times \bar{\mathfrak{q}})$ .*

Let us detail Theorem C in the case where  $G$  is embedded in  $\tilde{G} = G \times G$  diagonally. We denote by  $\mathfrak{p}$  the  $K$ -Hermitian space  $\mathbf{T}_e G/K$ . Let  $\kappa$  be the Killing form of the Lie algebra  $\mathfrak{g}$ . The vector space  $\bar{\mathfrak{p}}$  is equipped with the symplectic 2-form  $\Omega_{\bar{\mathfrak{p}}}(X, Y) = -\kappa(z, [X, Y])$  and the compatible complex structure  $-\text{ad}(z)$ .

Let us denote by  $\Delta_K(K\mu_1 \times K\mu_2 \times \bar{\mathfrak{p}})$  and by  $\Delta_K(\bar{\mathfrak{p}})$  the Kirwan polyhedrons relative to the Hamiltonian actions of  $K$  on  $K\mu_1 \times K\mu_2 \times \bar{\mathfrak{p}}$  and on  $\bar{\mathfrak{p}}$ . Theorem C says that for any  $\mu_1, \mu_2 \in \mathcal{C}_{\text{hol}}$ , the convex set  $\Delta_G(G\mu_1 \times G\mu_2)$  is equal to the Kirwan polyhedron  $\Delta_K(K\mu_1 \times K\mu_2 \times \bar{\mathfrak{p}})$ .

To any non-empty subset  $C$  of a real vector space  $E$ , we may associate its asymptotic cone  $\text{As}(C) \subset E$  which is the set formed by the limits  $y = \lim_{k \rightarrow \infty} t_k y_k$ , where  $(t_k)$  is a sequence of non-negative reals converging to 0 and  $y_k \in C$ .

We finally get the following description of the asymptotic cone of  $\Delta_G(G\mu_1 \times G\mu_2)$ .

**Corollary D.** *For any  $\mu_1, \mu_2 \in \mathcal{C}_{\text{hol}}$ , the asymptotic cone of  $\Delta_G(G\mu_1 \times G\mu_2)$  is equal to  $\Delta_K(\bar{\mathfrak{p}})$ .*

In [28] §5, we explained how to describe the cone  $\Delta_K(\bar{\mathfrak{p}})$  in terms of strongly orthogonal roots.

Let us finish this introduction with few remarks on related works:

- When  $G$  is compact, equal to the maximal compact subgroup  $\tilde{K}$  of  $\tilde{G}$ , the results of Theorems B and C were already obtained by G. Deltour in his thesis [6, 7].
- In [9], A. Eshmatov and P. Foth proposed a description of the set  $\Delta_G(G\mu_1 \times G\mu_2)$ . **But their computations give not the same result as ours.** From their main result (Theorem 3.2) it follows that the asymptotic cone of  $\Delta_G(G\mu_1 \times G\mu_2)$  is equal to the intersection of the Kirwan polyhedron  $\Delta_T(\bar{\mathfrak{p}})$  with the Weyl chamber  $\mathfrak{t}_{\geq 0}^*$ . But since  $\Delta_K(\bar{\mathfrak{p}}) \neq \Delta_T(\bar{\mathfrak{p}}) \cap \mathfrak{t}_{\geq 0}^*$  in general, it is in contradiction with Corollary D.

## Notations

In this paper,  $G$  denotes a connected real reductive Lie group : we take here the convention of Knapp [18]. We have a Cartan involution  $\Theta$  on  $G$ , such that the fixed point set  $K := G^\Theta$  is a connected maximal compact subgroup. We have Cartan decompositions at the level of Lie algebras  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  and at the level of the group  $G \simeq K \times \exp(\mathfrak{p})$ . We denote by  $b$  a  $G$ -invariant non-degenerate bilinear form on  $\mathfrak{g}$  that is equal to the Killing form on  $[\mathfrak{g}, \mathfrak{g}]$ , and that defines a  $K$ -invariant scalar product  $(X, Y) := -b(X, \Theta(Y))$ . We will use the  $K$ -equivariant identification  $\xi \mapsto \tilde{\xi}$ ,  $\mathfrak{g}^* \simeq \mathfrak{g}$  defined by  $(\tilde{\xi}, X) := \langle \xi, X \rangle$  for  $\xi \in \mathfrak{g}^*$  and  $X \in \mathfrak{g}$ .

When a Lie group  $H$  acts on a manifold  $N$ , the stabilizer subgroup of  $n \in N$  is denoted by  $H_n = \{g \in G, gn = n\}$ , and its Lie algebra by  $\mathfrak{h}_n$ . Let us define

$$(5) \quad \dim_H(\mathcal{X}) = \min_{n \in \mathcal{X}} \dim(\mathfrak{h}_n)$$

for any subset  $\mathcal{X} \subset N$ .

## 2 The cone $\Pi_{\text{hol}}(\tilde{G}, G)$ : first properties

We assume here that  $G/K$  is a Hermitian symmetric space, that is to say, there exists a  $G$ -invariant complex structure on the manifold  $G/K$ , or equivalently there exists a  $K$ -invariant element  $z \in \mathfrak{k}$  such that  $\text{ad}(z)|_{\mathfrak{p}}$  defines a complex structure on  $\mathfrak{p}$  :  $(\text{ad}(z)|_{\mathfrak{p}})^2 = -\text{Id}_{\mathfrak{p}}$ . This condition imposes that the ranks of  $G$  and  $K$  are equal.

We are interested in the following closed invariant convex cone of  $\mathfrak{g}^*$ :

$$\mathcal{C}_{G/K} = \{\xi \in \mathfrak{g}^*, \langle \xi, gz \rangle \geq 0, \forall g \in G\}.$$

## 2.1 The holomorphic chamber

Let  $T$  be a maximal torus of  $K$ , with Lie algebra  $\mathfrak{t}$ . Its dual  $\mathfrak{t}^*$  can be seen as the subspace of  $\mathfrak{g}^*$  fixed by  $T$ . Let us denote by  $\mathfrak{g}_e^*$  the set formed by the elliptic elements: in other words  $\mathfrak{g}_e^* := \text{Ad}^*(G) \cdot \mathfrak{t}^*$ .

Following [37], we consider the invariant open subset  $\mathfrak{g}_{se}^* = \{\xi \in \mathfrak{g}^* \mid G_\xi \text{ is compact}\}$  of *strongly elliptic* elements. It is non-empty since the groups  $G$  and  $K$  have the same rank.

We start with the following basic facts.

**Lemma 2.1** •  $\mathfrak{g}_{se}^*$  is contained in  $\mathfrak{g}_e^*$ .

- The interior  $\mathcal{C}_{G/K}^0$  of the cone  $\mathcal{C}_{G/K}$  is contained in  $\mathfrak{g}_{se}^*$ .

*Proof* : The first point is due to the fact that every compact subgroup of  $G$  is conjugate to a subgroup of  $K$ .

Let  $\xi \in \mathcal{C}_{G/K}^0$ . There exists  $\epsilon > 0$  so that

$$\langle \xi + \eta, gz \rangle \geq 0, \quad \forall g \in G, \quad \forall \|\eta\| \leq \epsilon.$$

It implies that  $|\langle \eta, gz \rangle| \leq \langle \xi, z \rangle$ ,  $\forall g \in G_\xi$  and  $\forall \|\eta\| \leq \epsilon$ . In other words, the adjoint orbit  $G_\xi \cdot z \subset \mathfrak{g}$  is bounded. For any  $g = e^X k$ , with  $(X, k) \in \mathfrak{p} \times K$ , a direct computation shows that  $\|gz\| = \|e^X z\| \geq \|[z, X]\| = \|X\|$ . Then, there exists  $\rho > 0$  such that  $\|X\| \leq \rho$  if  $e^X k \in G_\xi$  for some  $k \in K$ . It follows that the stabilizer subgroup  $G_\xi$  is compact.  $\square$

Let  $\Lambda^* \subset \mathfrak{t}^*$  be the weight lattice :  $\alpha \in \Lambda^*$  if  $i\alpha$  is the differential of a character of  $T$ . Let  $\mathfrak{R} \subset \Lambda^*$  be the set of roots for the action of  $T$  on  $\mathfrak{g} \otimes \mathbb{C}$ . We have  $\mathfrak{R} = \mathfrak{R}_c \cup \mathfrak{R}_n$  where  $\mathfrak{R}_c$  and  $\mathfrak{R}_n$  are respectively the set of roots for the action of  $T$  on  $\mathfrak{k} \otimes \mathbb{C}$  and  $\mathfrak{p} \otimes \mathbb{C}$ . We fix a system of positive roots  $\mathfrak{R}_c^+$  in  $\mathfrak{R}_c$ , and we denote by  $\mathfrak{t}_{\geq 0}^*$  the corresponding Weyl chamber.

We have  $\mathfrak{p} \otimes \mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$  where the  $K$ -module  $\mathfrak{p}^\pm$  is equal to  $\ker(\text{ad}(z) \mp i)$ . Let  $\mathfrak{R}_n^{\pm, z}$  be the set of roots for the action of  $T$  on  $\mathfrak{p}^\pm$ . The union

$$(6) \quad \mathfrak{R}^+ = \mathfrak{R}_c^+ \cup \mathfrak{R}_n^{+, z}$$

defines then a system of positive roots in  $\mathfrak{R}$ . We notice that  $\mathfrak{R}_n^{+, z}$  is the set of roots  $\beta \in \mathfrak{R}$  satisfying  $\langle \beta, z \rangle = 1$ . Hence  $\mathfrak{R}_n^{+, z}$  is invariant relatively to the action of the Weyl group  $W = N(T)/T$ .

Let us recall the following classical fact concerning the parametrization of the  $G$ -orbits in  $\mathcal{C}_{G/K}^0$  via the holomorphic chamber

$$\mathcal{C}_{\text{hol}} := \{\xi \in \mathfrak{t}_{\geq 0}^*, \langle \xi, \beta \rangle > 0, \forall \beta \in \mathfrak{R}_n^{+, z}\}.$$

**Proposition 2.2** • The set  $\mathcal{C}_{G/K}^0 \cap \mathfrak{t}_{\geq 0}^*$  is contained in  $\mathcal{C}_{\text{hol}}$ .

- For any compact subset  $\mathcal{K}$  of  $\mathcal{C}_{\text{hol}}$ , there exists  $c_{\mathcal{K}} > 0$  such that  $\langle \xi, gz \rangle \geq c_{\mathcal{K}} \|gz\|$ ,  $\forall g \in G, \forall \xi \in \mathcal{K}$ .
- The map  $\mathcal{O} \mapsto \mathcal{O} \cap \mathfrak{t}_{\geq 0}^*$  defines a bijective map between the set of  $G$ -orbits in  $\mathcal{C}_{G/K}^0$  and the holomorphic chamber  $\mathcal{C}_{\text{hol}}$ .

*Proof* : Let  $\xi \in \mathfrak{t}_{\geq 0}^* \cap \mathcal{C}_{G/K}^0$ . The first point is proved if we check that  $\langle \xi, \beta \rangle > 0$  for any  $\beta \in \mathfrak{R}_n^{+,z}$ . Let  $X_\beta, Y_\beta \in \mathfrak{p}$  such that  $X_\beta + iY_\beta \in (\mathfrak{p} \otimes \mathbb{C})_\beta$ . We choose the following normalization : the vector  $h_\beta := [X_\beta, Y_\beta]$  satisfies  $\langle \beta, h_\beta \rangle = 1$ . We see then that  $\langle \xi, \beta \rangle = \frac{1}{\|h_\beta\|^2} \langle \xi, h_\beta \rangle$  for any  $\xi \in \mathfrak{g}^*$ . Standard computation [27] gives

$$e^{t \text{ad}(X_\beta)} z = z + (\cosh(t) - 1)h_\beta + \sinh(t)Y_\beta, \quad \forall t \in \mathbb{R}.$$

By definition, we must have  $\langle \xi + \eta, e^{t \text{ad}(X_\beta)} z \rangle \geq 0, \forall t \in \mathbb{R}$ , for any  $\eta \in \mathfrak{t}^*$  small enough. It imposes that  $\langle \xi, h_\beta \rangle > 0$ . The first point is settled.

Now choose some maximal strongly orthogonal system  $\Sigma \subset \mathfrak{R}_n^{+,z}$ . The real span  $\mathfrak{a}$  of the  $X_\beta, \beta \in \Sigma$  is a maximal abelian subspace of  $\mathfrak{p}$ . Hence any element  $g \in G$  can be written  $g = ke^{X(t)}k'$  with  $X(t) = \sum_{\beta \in \Sigma} t_\beta X_\beta$  and  $k, k' \in K$ . We get

$$(7) \quad gz = k \left( z + \sum_{\beta \in \Sigma} (\cosh(t_\beta) - 1)h_\beta + \sum_{\beta \in \Sigma} \sinh(t_\beta)Y_\beta \right)$$

and

$$\langle \xi, gz \rangle = \langle k^{-1}\xi, z \rangle + \sum_{\beta \in \Sigma} (\cosh(t_\beta) - 1) \langle k^{-1}\xi, h_\beta \rangle.$$

For any  $\xi \in \mathcal{C}_{\text{hol}}$ , we define  $c_\xi := \min_{\beta \in \mathfrak{R}_n^{+,z}} \langle \xi, h_\beta \rangle > 0$ . Let  $\pi : \mathfrak{k}^* \rightarrow \mathfrak{t}^*$  be the projection. We have  $\langle k^{-1}\xi, z \rangle = \langle \pi(k^{-1}\xi), z \rangle$  and  $\langle k^{-1}\xi, h_\beta \rangle = \langle \pi(k^{-1}\xi), h_\beta \rangle$ . The convexity theorem of Kostant tell us that  $\pi(k^{-1}\xi)$  belongs to the convex hull of  $\{w\xi, w \in W\}$ . It follows that  $\langle k^{-1}\xi, z \rangle \geq \langle \xi, z \rangle > 0$  and  $\langle k^{-1}\xi, h_\beta \rangle \geq c_\xi > 0$ . We obtain then that  $\langle \xi, gz \rangle \geq \frac{1}{2} \min(\langle \xi, z \rangle, c_\xi) e^{\|X(t)\|}$  for any  $\xi \in \mathcal{C}_{\text{hol}}$ , where  $\|X(t)\| = \sup |t_\beta|$ . From (7), it is not difficult to see that there exists  $C > 0$  such that  $\|gz\| \leq Ce^{\|X(t)\|}$  for any  $g = ke^{X(t)}k'$ .

Let  $\mathcal{K}$  be a compact subset of  $\mathcal{C}_{\text{hol}}$ . Take  $c_{\mathcal{K}} = \frac{1}{2C} \min(\min_{\xi \in \mathcal{K}} \langle \xi, z \rangle, \min_{\xi \in \mathcal{K}} c_\xi) > 0$ . The previous computations show that  $\langle \xi, gz \rangle \geq c_{\mathcal{K}} \|gz\|$ ,  $\forall g \in G, \forall \xi \in \mathcal{K}$ . The second point is proved.

The first two points show that  $\mathfrak{t}_{\geq 0}^* \cap \mathcal{C}_{G/K}^0 = \mathcal{C}_{\text{hol}}$ . The third point follows then from the fact that  $\mathcal{C}_{G/K}^0$  is contained in  $\mathfrak{g}_e^*$  (see Lemma 2.1).  $\square$

We have a canonical map  $\mathfrak{p} : \mathcal{C}_{G/K}^0 \rightarrow \mathcal{C}_{\text{hol}}$  defined by the relations  $G\xi \cap \mathfrak{t}_{\geq 0}^* = \{\mathfrak{p}(\xi)\}$ ,  $\forall \xi \in \mathcal{C}_{G/K}^0$ . The following result is needed in §4.1.



**Lemma 2.3** *The map  $p$  is continuous.*

*Proof :* Let  $(\xi_n)$  be a sequence of  $\mathcal{C}_{G/K}^0$  converging to  $\xi_\infty \in \mathcal{C}_{G/K}^0$ . Let  $\xi'_n = p(\xi_n)$  and  $\xi'_\infty = p(\xi_\infty)$ : we have to prove that the sequence  $(\xi'_n)$  converges to  $\xi'_\infty$ . We choose elements  $g_n, g_\infty \in G$  such that  $\xi_n = g_n \xi'_n, \forall n$ , and  $\xi_\infty = g_\infty \xi'_\infty$ .

First we notice that  $-b(\xi_n, \xi_n) = \|\xi'_n\|^2$ , hence the sequence  $(\xi'_n)$  is bounded. We will now prove that the sequence  $(g_n)$  is bounded.

Let  $\epsilon > 0$  such that  $\langle \xi_\infty + \eta, gz \rangle \geq 0, \forall g \in G, \forall \|\eta\| \leq \epsilon$ . If  $\|\xi - \xi_\infty\| \leq \epsilon/2$ , we write  $\xi = \frac{1}{2}(\xi_\infty + 2(\xi - \xi_\infty)) + \frac{1}{2}\xi_\infty$ , and then

$$\langle \xi, gz \rangle = \frac{1}{2} \langle \xi_\infty + 2(\xi - \xi_\infty), gz \rangle + \frac{1}{2} \langle \xi_\infty, gz \rangle \geq \frac{1}{2} \langle \xi_\infty, gz \rangle, \quad \forall g \in G.$$

Now we have  $\langle \xi'_n, z \rangle = \langle \xi_n, g_n z \rangle \geq \frac{1}{2} \langle \xi_\infty, g_n z \rangle$  if  $n$  is large enough. This shows that the sequence  $\langle \xi_\infty, g_n z \rangle$  is bounded. If we use the second point of Proposition 2.2, we can conclude that the sequence  $(g_n)$  is bounded.

Let  $(\xi'_{\phi(n)})$  be a subsequence converging to  $\ell \in \mathfrak{t}_{\geq 0}^*$ . Since  $(g_{\phi(n)})$  is bounded, there exists a subsequence  $(g_{\phi \circ \psi(n)})$  converging to  $h \in G$ . From the relations  $\xi_{\phi \circ \psi(n)} = g_{\phi \circ \psi(n)} \xi'_{\phi \circ \psi(n)}, \forall n \in \mathbb{N}$ , we obtain  $\xi_\infty = h\ell$ . Then  $\ell = p(\xi_\infty) = \xi'_\infty$ . Since every subsequence of  $(\xi'_n)$  has a subsequential limit to  $\xi'_\infty$ , then the sequence  $(\xi'_n)$  converges to  $\xi'_\infty$ .  $\square$

## 2.2 The cone $\Pi_{\text{hol}}(\tilde{G}, G)$ is closed

We suppose that  $G/K$  is a complex submanifold of a Hermitian symmetric space  $\tilde{G}/\tilde{K}$ . In other words,  $\tilde{G}$  is a reductive real Lie group such that  $G \subset \tilde{G}$  is a closed subgroup preserved by the Cartan involution, and  $\tilde{K}$  is a maximal compact subgroup of  $\tilde{G}$  containing  $K$ . We denote by  $\tilde{\mathfrak{g}}$  and  $\tilde{\mathfrak{k}}$  the Lie algebras of  $\tilde{G}$  and  $\tilde{K}$  respectively. We suppose that there exists a  $\tilde{K}$ -invariant element  $z \in \tilde{\mathfrak{k}}$  such that  $\text{ad}(z)|_{\tilde{\mathfrak{p}}}$  defines a complex structure on  $\tilde{\mathfrak{p}} : (\text{ad}(z)|_{\tilde{\mathfrak{p}}})^2 = -Id_{\tilde{\mathfrak{p}}}$ .

Let  $\mathcal{C}_{\tilde{G}/\tilde{K}} \subset \tilde{\mathfrak{g}}^*$  be the closed invariant cone associated to the Hermitian symmetric space  $\tilde{G}/\tilde{K}$ . We start with the following key fact.

**Lemma 2.4** *The projection  $\pi_{\mathfrak{g}, \tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}}^* \rightarrow \mathfrak{g}^*$  sends  $\mathcal{C}_{\tilde{G}/\tilde{K}}^0$  into  $\mathcal{C}_{G/K}^0$ .*

*Proof :* Let  $\tilde{\xi} \in \mathcal{C}_{\tilde{G}/\tilde{K}}^0$  and  $\xi = \pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}(\tilde{\xi})$ . Then  $\langle \tilde{\xi} + \tilde{\eta}, \tilde{g}z \rangle \geq 0, \forall \tilde{g} \in \tilde{G}$  if  $\tilde{\eta} \in \tilde{\mathfrak{g}}^*$  is small enough. It follows that  $\langle \xi + \pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}(\tilde{\eta}), gz \rangle = \langle \tilde{\xi} + \tilde{\eta}, \tilde{g}z \rangle \geq 0, \forall g \in G$  if  $\tilde{\eta}$  is small enough. Since  $\pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}$  is an open map, we can conclude that  $\xi \in \mathcal{C}_{G/K}^0$ .  $\square$

Let  $\tilde{T}$  be a maximal torus of  $\tilde{K}$ , with Lie algebra  $\tilde{\mathfrak{t}}$ . The  $\tilde{G}$ -orbits in the interior of  $\mathcal{C}_{\tilde{G}/\tilde{K}}$  are parametrized by the holomorphic chamber  $\tilde{\mathcal{C}}_{\text{hol}} \subset \tilde{\mathfrak{t}}^*$ . The previous Lemma says that the projection  $\pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}(\tilde{\mathcal{O}})$  of any  $\tilde{G}$ -orbit  $\tilde{\mathcal{O}} \subset \mathcal{C}_{\tilde{G}/\tilde{K}}^0$  is the union of  $G$ -orbits  $\mathcal{O} \subset \mathcal{C}_{G/K}^0$ .

So it is natural to study the following object

$$(8) \quad \Pi_{\text{hol}}(\tilde{G}, G) := \left\{ (\tilde{\xi}, \xi) \in \tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}}; G\xi \subset \pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}(\tilde{G}\tilde{\xi}) \right\}.$$

Here is a first result.

**Proposition 2.5**  $\Pi_{\text{hol}}(\tilde{G}, G)$  is a closed cone of  $\tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}}$ .

*Proof:* Suppose that a sequence  $(\tilde{\xi}_n, \xi_n) \in \Pi_{\text{hol}}(\tilde{G}, G)$  converge to  $(\tilde{\xi}_\infty, \xi_\infty) \in \tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}}$ . By definition, there exists a sequence  $(\tilde{g}_n, g_n) \in \tilde{G} \times G$  such that  $g_n \xi_n = \pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}(\tilde{g}_n \tilde{\xi}_n)$ . Let  $\tilde{h}_n := g_n^{-1} \tilde{g}_n$ , so that  $\xi_n = \pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}(\tilde{h}_n \tilde{\xi}_n)$  and  $\langle \tilde{h}_n \tilde{\xi}_n, z \rangle = \langle \xi_n, z \rangle$ . We use now that the sequence  $\langle \xi_n, z \rangle$  is bounded, and that the sequence  $\tilde{\xi}_n$  belongs to a compact subset of  $\tilde{\mathcal{C}}_{\text{hol}}$ . Thanks to the second point of Proposition 2.2, these facts imply that  $\|\tilde{h}_n^{-1} z\|$  is a bounded sequence. Hence  $\tilde{h}_n$  admits a subsequence converging to  $\tilde{h}_\infty$ . So we get  $\xi_\infty = \pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}(\tilde{h}_\infty \tilde{\xi}_\infty)$ , and that proves that  $(\tilde{\xi}_\infty, \xi_\infty) \in \Pi_{\text{hol}}(\tilde{G}, G)$ .  $\square$

### 2.3 Rational and weakly regular points

Let  $(M, \Omega)$  be a symplectic manifold. We suppose that there exists a line bundle  $\mathcal{L}$  with connection  $\nabla$  that prequantizes the 2-form  $\Omega$ : in other words,  $\nabla^2 = -i\Omega$ . Let  $K$  be a compact connected Lie group acting on  $\mathcal{L} \rightarrow M$ , and leaving the connection invariant. Let  $\Phi_K : M \rightarrow \mathfrak{k}^*$  be the moment map defined by Kostant's relations

$$(9) \quad L_X - \nabla_X = i\langle \Phi_K, X \rangle, \quad \forall X \in \mathfrak{k}.$$

Here  $L_X$  is the Lie derivative acting on the sections of  $\mathcal{L} \rightarrow M$ .

Remark that relations (9) imply, via the equivariant Bianchi formula, the relations

$$(10) \quad \iota(X_M)\Omega = -d\langle \Phi_K, X \rangle, \quad \forall X \in \mathfrak{k},$$

where  $X_M(m) := \frac{d}{dt}|_{t=0} e^{-tX} m$  is the vector field on  $M$  generated by  $X \in \mathfrak{k}$ .

**Definition 2.6** Let  $\dim_K(M) := \min_{m \in M} \dim \mathfrak{k}_m$ . An element  $\xi \in \mathfrak{k}^*$  is a weakly-regular value of  $\Phi_K$  if for all  $m \in \Phi_K^{-1}(\xi)$  we have  $\dim \mathfrak{k}_m = \dim_K(M)$ .

When  $\xi \in \mathfrak{k}^*$  is a weakly-regular value of  $\Phi_K$ , the constant rank theorem tells us that  $\Phi_K^{-1}(\xi)$  is a submanifold of  $M$  stable under the action of the stabilizer subgroup  $K_\xi$ . We see then that the reduced space

$$(11) \quad M_\xi := \Phi_K^{-1}(\xi)/K_\xi$$

is a symplectic orbifold.

Let  $T \subset K$  be a maximal torus with Lie algebra  $\mathfrak{t}$ . We consider the lattice  $\wedge := \frac{1}{2\pi} \ker(\exp : \mathfrak{t} \rightarrow T)$  and the dual lattice  $\wedge^* \subset \mathfrak{t}^*$  defined by  $\wedge^* = \text{hom}(\wedge, \mathbb{Z})$ . We remark

that  $i\eta$  is a differential of a character of  $T$  if and only if  $\eta \in \Lambda^*$ . The  $\mathbb{Q}$ -vector space generated by the lattice  $\Lambda^*$  is denoted by  $\mathfrak{t}_{\mathbb{Q}}^*$ : the vectors belonging to  $\mathfrak{t}_{\mathbb{Q}}^*$  are designed as rational. An affine subspace  $V \subset \mathfrak{t}^*$  is called rational if it is affinely generated by its rational points.

We also fix a closed positive Weyl chamber  $\mathfrak{t}_{\geq 0}^*$ . For each relatively open face  $\sigma \subset \mathfrak{t}_{\geq 0}^*$ , the stabilizer  $K_{\xi}$  of points  $\xi \in \sigma$  under the coadjoint action, does not depend on  $\xi$ , and will be denoted  $K_{\sigma}$ . The Lie algebra  $\mathfrak{g}_{\sigma}$  decomposes into its semi-simple and central parts  $\mathfrak{k}_{\sigma} = [\mathfrak{k}_{\sigma}, \mathfrak{k}_{\sigma}] \oplus \mathfrak{z}_{\sigma}$ . The subspace  $\mathfrak{z}_{\sigma}^*$  is defined to be the annihilator of  $[\mathfrak{k}_{\sigma}, \mathfrak{k}_{\sigma}]$ , or equivalently the fixed point set of the coadjoint  $K_{\sigma}$  action. Notice that  $\mathfrak{z}_{\sigma}^*$  is a rational subspace of  $\mathfrak{t}^*$ , and that the face  $\sigma$  is an open cone of  $\mathfrak{z}_{\sigma}^*$ ,

We suppose that the moment map  $\Phi_K$  is *proper*. The Convexity Theorem [1, 10, 16, 34, 22] tells us that  $\Delta_K(M) := \text{Image}(\Phi_K) \cap \mathfrak{t}_{\geq 0}^*$  is a closed, convex, locally polyhedral set.

**Definition 2.7** We denote by  $\Delta_K(M)^0$  the subset of  $\Delta_K(M)$  formed by the weakly-regular values of the moment map  $\Phi_K$  contained in  $\Delta_K(M)$ .

We will use the following remark in the next sections.

**Lemma 2.8** The subset  $\Delta_K(M)^0 \cap \mathfrak{t}_{\mathbb{Q}}^*$  is dense in  $\Delta_K(M)$ .

*Proof* : Let us first explain why  $\Delta_K(M)^0$  is a dense open subset of  $\Delta_K(M)$ . There exists a unique open face  $\tau$  of the Weyl chamber  $\mathfrak{t}_{\geq 0}^*$  such as  $\Delta_K(M) \cap \tau$  is dense in  $\Delta_K(M)$  :  $\tau$  is called the *principal* face in [22]. The Principal-cross-section Theorem [22] tells us that  $Y_{\tau} := \Phi^{-1}(\tau)$  is a symplectic  $K_{\tau}$ -manifold, with a trivial action of  $[K_{\tau}, K_{\tau}]$ . The line bundle  $\mathcal{L}_{\tau} := \mathcal{L}|_{Y_{\tau}}$  prequantizes the symplectic structure on  $Y_{\tau}$ , and relations (10) show that  $[K_{\tau}, K_{\tau}]$  acts trivially on  $\mathcal{L}_{\tau}$ . Moreover the restriction of  $\Phi_K$  on  $Y_{\tau}$  is the moment map  $\Phi_{\tau} : Y_{\tau} \rightarrow \mathfrak{z}_{\tau}^*$  associated to the action of the torus  $Z_{\tau} = \exp(\mathfrak{z}_{\tau})$  on  $\mathcal{L}_{\tau}$ .

Let  $I \subset \mathfrak{z}_{\tau}^*$  be the smallest affine subspace containing  $\Delta_K(M)$ . Let  $\mathfrak{z}_I \subset \mathfrak{z}_{\tau}$  be the annihilator of the subspace parallel to  $I$  : relations (10) show that  $\mathfrak{z}_I$  is the generic infinitesimal stabilizer of the  $\mathfrak{z}_{\tau}$ -action on  $Y_{\tau}$ . Hence  $\mathfrak{z}_I$  is the Lie algebra of the torus  $Z_I := \exp(\mathfrak{z}_I)$ .

We see then that any regular value of  $\Phi_{\tau} : Y_{\tau} \rightarrow I$ , viewed as a map with codomain  $I$ , is a weakly-regular value of the moment map  $\Phi_K$ . It explains why  $\Delta_K(M)^0$  is a dense open subset of  $\Delta_K(M)$ .

The convex set  $\Delta_K(M) \cap \tau$  is equal to  $\Delta_{Z_{\tau}}(Y_{\tau}) := \text{Image}(\Phi_{\tau})$ , it is sufficient to check that  $\Delta_{Z_{\tau}}(Y_{\tau})^0 \cap \mathfrak{t}_{\mathbb{Q}}^*$  is dense in  $\Delta_{Z_{\tau}}(Y_{\tau})$ . The subtorus  $Z_I \subset Z_{\tau}$  acts trivially on  $Y_{\tau}$ , and it acts on the line bundle  $\mathcal{L}_{\tau}$  through a character  $\chi$ . Let  $\eta \in \Lambda^* \cap \mathfrak{t}_{\tau}^*$  such that  $d\chi = i\eta|_{\mathfrak{z}_I}$ . The affine subspace  $I$  which is equal to  $\eta + (\mathfrak{z}_I)^{\perp}$  is rational. Since the open subset  $\Delta_{Z_{\tau}}(Y_{\tau})^0$  generates the rational affine subspace  $I$ , we can conclude that  $\Delta_{Z_{\tau}}(Y_{\tau})^0 \cap \mathfrak{t}_{\mathbb{Q}}^*$  is dense in  $\Delta_{Z_{\tau}}(Y_{\tau})$ .  $\square$

## 2.4 Weinstein's theorem

Let  $\tilde{a} \in \tilde{\mathcal{C}}_{\text{hol}}$ . Consider the Hamiltonian action of the group  $G$  on the coadjoint orbit  $\tilde{G}\tilde{a}$ . The moment map  $\Phi_G^{\tilde{a}} : \tilde{G}\tilde{a} \rightarrow \mathfrak{g}^*$  corresponds to the restriction of the projection  $\pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}$  to  $\tilde{G}\tilde{a}$ . In this setting, the following conditions holds :

1. The action of  $G$  on  $\tilde{G}\tilde{a}$  is proper.
2. The moment map  $\Phi_G^{\tilde{a}}$  is a proper map since the map  $\langle \Phi_G^{\tilde{a}}, z \rangle$  is proper (see the second point of Proposition 2.2).

Conditions 1. and 2. imposes that the image of  $\Phi_G^{\tilde{a}}$  is contained in the open subset  $\mathfrak{g}_{se}^*$  of strongly elliptic elements [30]. Thus, the  $G$ -orbits contained in the image of  $\Phi_G^{\tilde{a}}$  are parametrized by the following subset of the holomorphic chamber  $\mathcal{C}_{\text{hol}}$  :

$$\Delta_G(\tilde{G}\tilde{a}) := \text{Image}(\Phi_G^{\tilde{a}}) \cap \mathfrak{t}_{\geq 0}^*.$$

We notice that  $\Pi_{\text{hol}}(\tilde{G}, G) = \bigcup_{\tilde{a} \in \tilde{\mathcal{C}}_{\text{hol}}} \{\tilde{a}\} \times \Delta_G(\tilde{G}\tilde{a})$ .

Like in Definition 2.6, an element  $\xi \in \mathfrak{g}^*$  is a *weakly-regular* value of  $\Phi_G^{\tilde{a}}$  if for all  $m \in (\Phi_G^{\tilde{a}})^{-1}(\xi)$  we have  $\dim \mathfrak{g}_m = \min_{x \in \tilde{G}\tilde{a}} \dim(\mathfrak{g}_x)$ . We denote by  $\Delta_G(\tilde{G}\tilde{a})^0$  the set of elements  $\xi \in \Delta_G(\tilde{G}\tilde{a})$  that are weakly-regular for  $\Phi_G^{\tilde{a}}$ .

**Theorem 2.9 (Weinstein)** *For any  $\tilde{a} \in \tilde{\mathcal{C}}_{\text{hol}}$ ,  $\Delta_G(\tilde{G}\tilde{a})$  is a closed convex subset contained in  $\mathcal{C}_{\text{hol}}$ .*

*Proof :* We recall briefly the arguments of the proof (see [37] or [30][§2]). Under Conditions 1. and 2., one checks easily that  $Y_{\tilde{a}} := (\Phi_G^{\tilde{a}})^{-1}(\mathfrak{k}^*)$  is a smooth  $K$ -invariant symplectic submanifold of  $\tilde{G}\tilde{a}$  such that

$$(12) \quad \tilde{G}\tilde{a} \simeq G \times_K Y_{\tilde{a}}.$$

The moment map  $\Phi_K^{\tilde{a}} : Y_{\tilde{a}} \rightarrow \mathfrak{k}^*$ , which corresponds to the restriction of the map  $\Phi_G^{\tilde{a}}$  to  $Y_{\tilde{a}}$ , is a proper map. Hence the Convexity Theorem tells us that  $\Delta_K(Y_{\tilde{a}}) := \text{Image}(\Phi_K^{\tilde{a}}) \cap \mathfrak{t}_{\geq 0}^*$  is a closed, convex, locally polyhedral set. Thanks to the isomorphism (12) we see that  $\Delta_G(\tilde{G}\tilde{a})$  coincides with the closed convex subset  $\Delta_K(Y_{\tilde{a}})$ . The proof is completed.  $\square$

The next Lemma is used in §4.1.

**Lemma 2.10** *Let  $\tilde{a} \in \tilde{\mathcal{C}}_{\text{hol}}$  be a rational element. Then  $\Delta_G(\tilde{G}\tilde{a})^0 \cap \mathfrak{t}_{\mathbb{Q}}^*$  is dense in  $\Delta_G(\tilde{G}\tilde{a})$ .*

*Proof :* Thanks to (12), we know that  $\Delta_G(\tilde{G}\tilde{a}) = \Delta_K(Y_{\tilde{a}})$ . Relation (12) shows also that  $\Delta_G(\tilde{G}\tilde{a})^0 = \Delta_K(Y_{\tilde{a}})^0$ . Let  $N \geq 1$  such that  $\tilde{\mu} = N\tilde{a} \in \wedge^* \cap \mathcal{C}_{\text{hol}}$ . The stabilizer subgroup  $\tilde{G}_{\tilde{\mu}}$  is compact, equal to  $\tilde{K}_{\tilde{\mu}}$ . Let us denote by  $\mathbb{C}_{\tilde{\mu}}$  the one-dimensional representation of  $\tilde{K}_{\tilde{\mu}}$  associated to  $\tilde{\mu}$ . The convex set  $\Delta_G(\tilde{G}\tilde{a})$  is equal to  $\frac{1}{N}\Delta_G(\tilde{G}\tilde{\mu})$ , so it is sufficient to check that  $\Delta_G(\tilde{G}\tilde{\mu})^0 \cap \mathfrak{t}_{\mathbb{Q}}^* = \Delta_K(Y_{\tilde{\mu}})^0 \cap \mathfrak{t}_{\mathbb{Q}}^*$  is dense in  $\Delta_G(\tilde{G}\tilde{\mu}) = \Delta_K(Y_{\tilde{\mu}})$ . The coadjoint orbit  $\tilde{G}\tilde{\mu}$  is prequantized by the line bundle  $\tilde{G} \times_{K_{\tilde{\mu}}} \mathbb{C}_{\tilde{\mu}}$  and the symplectic slice  $Y_{\tilde{\mu}}$  is prequantized by the line bundle  $\mathcal{L}_{\tilde{\mu}} := \tilde{G} \times_{K_{\tilde{\mu}}} \mathbb{C}_{\tilde{\mu}}|_{Y_{\tilde{\mu}}}$ . Thanks to Lemma 2.8, we know that  $\Delta_K(Y_{\tilde{\mu}})^0 \cap \mathfrak{t}_{\mathbb{Q}}^*$  is dense in  $\Delta_K(Y_{\tilde{\mu}})$  : the proof is complete.  $\square$

### 3 Holomorphic discrete series

#### 3.1 Definition

We return to the framework of §2.1. We recall the notion of holomorphic discrete series representations associated to a Hermitian symmetric spaces  $G/K$ . Let us introduce

$$\mathcal{C}_{\text{hol}}^\rho := \{ \xi \in \mathfrak{t}_{\geq 0}^* \mid (\xi, \beta) \geq (2\rho_n, \beta), \forall \beta \in \mathfrak{A}_n^{+,z} \}$$

where  $2\rho_n = \sum_{\beta \in \mathfrak{A}_n^{+,z}} \beta$  is  $W$ -invariant.

**Lemma 3.1** 1. We have  $\mathcal{C}_{\text{hol}}^\rho \subset \mathcal{C}_{\text{hol}}$ .

2. For any  $\xi \in \mathcal{C}_{\text{hol}}$  there exists  $N \geq 1$  such that  $N\xi \in \mathcal{C}_{\text{hol}}^\rho$ .

*Proof* : The first point is due to the fact that  $(\beta_0, \beta_1) \geq 0$  for any  $\beta_0, \beta_1 \in \mathfrak{A}_n^{+,z}$ . The second point is obvious.  $\square$

We will be interested in the following subset of dominants weights :

$$\widehat{G}_{\text{hol}} := \wedge_+^* \bigcap \mathcal{C}_{\text{hol}}^\rho.$$

Let  $\text{Sym}(\mathfrak{p})$  be the symmetric algebra of the complex  $K$ -module  $(\mathfrak{p}, \text{ad}(z))$ .

**Theorem 3.2 (Harish-Chandra)** For any  $\lambda \in \widehat{G}_{\text{hol}}$ , there exists an irreducible unitary representation of  $G$ , denoted  $V_\lambda^G$ , such that the vector space of  $K$ -finite vectors is  $V_\lambda^G|_K := V_\lambda^K \otimes \text{Sym}(\mathfrak{p})$ .

The set  $V_\lambda^G, \lambda \in \widehat{G}_{\text{hol}}$  corresponds to the holomorphic discrete series representations associated to the complex structure  $\text{ad}(z)$ .

#### 3.2 Restriction

We come back to the framework of §2.2. We consider two compatible Hermitian symmetric spaces  $G/K \subset \widetilde{G}/\widetilde{K}$ , and we look after the restriction of holomorphic discrete series representations of  $\widetilde{G}$  to the subgroup  $G$ .

Let  $\tilde{\lambda} \in \widehat{\widetilde{G}}_{\text{hol}}$ . Since the representation  $V_{\tilde{\lambda}}^{\widetilde{G}}$  is discretely admissible relatively the circle group  $\exp(\mathbb{R}z)$ , it is also discretely admissible relatively to  $G$ . We can be more precise [15, 24, 21] :

**Proposition 3.3** We have an Hilbertian direct sum

$$V_{\tilde{\lambda}}^{\widetilde{G}}|_G = \bigoplus_{\lambda \in \widehat{G}_{\text{hol}}} m_\lambda^{\tilde{\lambda}} V_\lambda^G,$$

where the multiplicity  $m_\lambda^{\tilde{\lambda}} := [V_\lambda^G : V_{\tilde{\lambda}}^{\widetilde{G}}]$  is finite for any  $\lambda$ .

The Hermitian  $\tilde{K}$ -vector space  $\tilde{\mathfrak{p}}$ , when restricted to the  $K$ -action, admits an orthogonal decomposition  $\tilde{\mathfrak{p}} = \mathfrak{p} \oplus \mathfrak{q}$ . Notice that the symmetric algebra  $\text{Sym}(\mathfrak{q})$  is an admissible  $K$ -module.

In [15], H.P. Jakobsen and M. Vergne obtained the following nice characterization of the multiplicities  $[V_\lambda^G : V_{\tilde{\lambda}}^{\tilde{G}}]$ . Another proof is given in [30], §4.4.

**Theorem 3.4 (Jakobsen-Vergne)** *Let  $(\tilde{\lambda}, \lambda) \in \widehat{G}_{\text{hol}} \times \widehat{G}_{\text{hol}}$ . The multiplicity  $[V_\lambda^G : V_{\tilde{\lambda}}^{\tilde{G}}]$  is equal to the multiplicity of the representation  $V_\lambda^K$  in  $\text{Sym}(\mathfrak{q}) \otimes V_{\tilde{\lambda}}^{\tilde{K}}|_K$ .*

### 3.3 Discrete analogues of $\Pi_{\text{hol}}(\tilde{G}, G)$

We define the following discrete analogues of the cone  $\Pi_{\text{hol}}(\tilde{G}, G)$ :

$$(13) \quad \Pi_{\text{hol}}^{\mathbb{Z}}(\tilde{G}, G) := \left\{ (\tilde{\lambda}, \lambda) \in \widehat{G}_{\text{hol}} \times \widehat{G}_{\text{hol}} ; [V_\lambda^G : V_{\tilde{\lambda}}^{\tilde{G}}] \neq 0 \right\}.$$

and

$$(14) \quad \Pi_{\text{hol}}^{\mathbb{Q}}(\tilde{G}, G) := \left\{ (\tilde{\xi}, \xi) \in \tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}} ; \exists N \geq 1, (N\xi, N\tilde{\xi}) \in \Pi_{\text{hol}}^{\mathbb{Z}}(\tilde{G}, G) \right\}.$$

We have the following key fact.

#### Proposition 3.5

- $\Pi_{\text{hol}}^{\mathbb{Z}}(\tilde{G}, G)$  is a subset of  $\tilde{\Lambda}^* \times \Lambda^*$  stable under the addition.
- $\Pi_{\text{hol}}^{\mathbb{Q}}(\tilde{G}, G)$  is a  $\mathbb{Q}$ -convex cone of the  $\mathbb{Q}$ -vector space  $\tilde{\mathfrak{k}}_{\mathbb{Q}}^* \times \mathfrak{k}_{\mathbb{Q}}^*$ .

*Proof :* Suppose that  $a_1 := (\tilde{\lambda}_1, \lambda_1)$  and  $a_2 := (\tilde{\lambda}_2, \lambda_2)$  belongs to  $\Pi_{\text{hol}}^{\mathbb{Z}}(\tilde{G}, G)$ . Thanks to Theorem 3.4, we know that the  $K$ -modules  $\text{Sym}(\mathfrak{q}) \otimes (V_{\lambda_j}^K)^* \otimes V_{\tilde{\lambda}_j}^{\tilde{K}}|_K$  possess a non-zero invariant vector  $\phi_j$ , for  $j = 1, 2$ .

Let  $\mathbb{X} := \overline{K/T} \times \tilde{K}/\tilde{T}$  be the product of flag manifolds. The complex structure is normalized so that  $\mathbf{T}_{([e], [\tilde{e}])} \mathbb{X} \simeq \mathfrak{n}_- \oplus \tilde{\mathfrak{n}}_+$ , where  $\mathfrak{n}_- = \sum_{\alpha < 0} (\mathfrak{k}_{\mathbb{C}})_{\alpha}$  and  $\tilde{\mathfrak{n}}_+ = \sum_{\tilde{\alpha} > 0} (\tilde{\mathfrak{k}}_{\mathbb{C}})_{\tilde{\alpha}}$ . We associate to each data  $a_j$ , the holomorphic line bundle  $\mathcal{L}_j := K \times_T \mathbb{C}_{-\lambda_j} \boxtimes \tilde{K} \times_{\tilde{T}} \mathbb{C}_{-\tilde{\lambda}_j}$  on  $\mathbb{X}$ . Let  $H^0(\mathbb{X}, \mathcal{L}_j)$  be the space of holomorphic sections of the line bundle  $\mathcal{L}_j$ . The Borel-Weil Theorem tell us that  $H^0(\mathbb{X}, \mathcal{L}_j) \simeq (V_{\lambda_j}^K)^* \otimes V_{\tilde{\lambda}_j}^{\tilde{K}}|_K, \forall j \in \{1, 2\}$ .

We have  $\phi_j \in [\text{Sym}(\mathfrak{q}) \otimes H^0(\mathbb{X}, \mathcal{L}_j)]^K, \forall j$ , and then  $\phi_1 \phi_2 \in \text{Sym}(\mathfrak{q}) \otimes H^0(\mathbb{X}, \mathcal{L}_1 \otimes \mathcal{L}_2)$  is a non-zero invariant vector. Hence  $[\text{Sym}(\mathfrak{q}) \otimes (V_{\lambda_1 + \lambda_2}^K)^* \otimes V_{\tilde{\lambda}_1 + \tilde{\lambda}_2}^{\tilde{K}}|_K]^K \neq 0$ . Thanks to Theorem 3.4, we can conclude that  $a_1 + a_2 = (\tilde{\lambda}_1 + \tilde{\lambda}_2, \lambda_1 + \lambda_2)$  belongs to  $\Pi_{\text{hol}}^{\mathbb{Z}}(\tilde{G}, G)$ . The first point is proved. From the first point, one checks easily that

- $\Pi_{\text{hol}}^{\mathbb{Q}}(\tilde{G}, G)$  is stable under addition,
- $\Pi_{\text{hol}}^{\mathbb{Q}}(\tilde{G}, G)$  is stable by expansion by a non-negative rational number.

The second point is settled.  $\square$

### 3.4 Riemann-Roch numbers

We come back to the framework of §2.3.

We associate to a dominant weight  $\mu \in \Lambda_+^*$  the (possibly singular) symplectic reduced space  $M_\mu := \Phi_K^{-1}(\mu)/K_\mu$ , and the (possibly singular) line bundle over  $M_\mu$  :

$$\mathcal{L}_\mu := \left( \mathcal{L}|_{\Phi_K^{-1}(\mu)} \otimes \mathbb{C}_{-\mu} \right) / K_\mu.$$

Suppose first that  $\mu$  is a weakly-regular value of  $\Phi_K$ . Then  $M_\mu$  is an orbifold equipped with a symplectic structure  $\Omega_\mu$ , and  $\mathcal{L}_\mu$  is a line orbi-bundle over  $M_\mu$  that prequantizes the symplectic structure. By choosing an almost complex structure on  $M_\mu$  compatible with  $\Omega_\mu$ , we get a decomposition  $\wedge \mathbf{T}^* M_\mu \otimes \mathbb{C} = \oplus_{i,j} \wedge^{i,j} \mathbf{T}^* M_\mu$  of the bundle of differential forms. Using Hermitian structure in the tangent bundle  $\mathbf{T}M_\mu$  of  $M_\mu$ , and in the fibers of  $\mathcal{L}_\mu$ , we define a Dolbeaut-Dirac operator

$$D_\mu^+ : \mathcal{A}^{0,+}(M_\mu, \mathcal{L}_\mu) \longrightarrow \mathcal{A}^{0,-}(M_\mu, \mathcal{L}_\mu)$$

where  $\mathcal{A}^{i,j}(M_\mu, \mathcal{L}_\mu) = \Gamma(M_\mu, \wedge^{i,j} \mathbf{T}^* M_\mu \otimes \mathcal{L}_\mu)$ .

**Definition 3.6** *Let  $\mu \in \Lambda_+^*$  be a weakly-regular value of the moment map  $\Phi_K$ . The Riemann-Roch number  $RR(M_\mu, \mathcal{L}_\mu) \in \mathbb{Z}$  is defined as the index of the elliptic operator  $D_\mu^+$ :  $RR(M_\mu, \mathcal{L}_\mu) = \dim(\ker(D_\mu^+)) - \dim(\operatorname{coker}(D_\mu^+))$ .*

Suppose that  $\mu \notin \Delta_K(M)$ . Then  $M_\mu = \emptyset$  and we take  $RR(M_\mu, \mathcal{L}_\mu) = 0$ .

Suppose now that  $\mu \in \Delta_K(M)$  is not (necessarily) a weakly-regular value of  $\Phi_K$ . Take a small element  $\epsilon \in \mathfrak{t}^*$  such that  $\mu + \epsilon$  is a weakly-regular value of  $\Phi_K$  belonging to  $\Delta_K(M)$ . We consider the symplectic orbifold  $M_{\mu+\epsilon}$ : if  $\epsilon$  is small enough,

$$\mathcal{L}_{\mu,\epsilon} := \left( \mathcal{L}|_{\Phi_K^{-1}(\mu+\epsilon)} \otimes \mathbb{C}_{-\mu} \right) / K_{\mu+\epsilon}.$$

is a line orbi-bundle over  $M_{\mu+\epsilon}$ .

We have the following important result (see §3.4.3 in [33]).

**Proposition 3.7** *Let  $\mu \in \Delta_K(M) \cap \Lambda^*$ . The Riemann-Roch number  $RR(M_{\mu+\epsilon}, \mathcal{L}_{\mu,\epsilon})$  do not depend on the choice of  $\epsilon$  small enough so that  $\mu + \epsilon \in \Delta_K(M)$  is a weakly-regular value of  $\Phi_K$ .*

We can now introduce the following definition.

**Definition 3.8** *Let  $\mu \in \Lambda_+^*$ . We define*

$$\mathcal{Q}(M_\mu, \Omega_\mu) = \begin{cases} 0 & \text{if } \mu \notin \Delta_K(M), \\ RR(M_{\mu+\epsilon}, \mathcal{L}_{\mu,\epsilon}) & \text{if } \mu \in \Delta_K(M). \end{cases}$$

*Above,  $\epsilon$  is chosen small enough so that  $\mu + \epsilon \in \Delta_K(M)$  is a weakly-regular value of  $\Phi_K$ .*

Let  $n \geq 1$ . The manifold  $M$ , equipped with the symplectic structure  $n\Omega$ , is prequantized by the line bundle  $\mathcal{L}^{\otimes n}$ : the corresponding moment map is  $n\Phi_K$ . For any dominant weight  $\mu \in \Lambda_+^*$ , the symplectic reduction of  $(M, n\Omega)$  relatively to the weight  $n\mu$  is  $(M_\mu, n\Omega_\mu)$ . Like in definition 3.8, we consider the following Riemann-Roch numbers

$$\mathcal{Q}(M_\mu, n\Omega_\mu) = \begin{cases} 0 & \text{if } \mu \notin \Delta_K(M), \\ RR(M_{\mu+\epsilon}, (\mathcal{L}_{\mu,\epsilon})^{\otimes n}) & \text{if } \mu \in \Delta_K(M) \text{ and } \|\epsilon\| \ll 1. \end{cases}$$

Kawasaki-Riemann-Roch formula shows that  $n \geq 1 \mapsto \mathcal{Q}(M_\mu, n\Omega_\mu)$  is a quasi-polynomial map [36, 23]. When  $\mu$  is a weakly-regular value of  $\Phi_K$ , we denote by  $\text{vol}(M_\mu) := \frac{1}{d_\mu} \int_{M_\mu} \left(\frac{\Omega_\mu}{2\pi}\right)^{\frac{\dim M_\mu}{2}}$  the symplectic volume of the symplectic orbifold  $(M_\mu, \Omega_\mu)$ . Here  $d_\mu$  is the generic value of the map  $m \in \Phi_K^{-1}(\mu) \mapsto \text{cardinal}(K_m/K_m^0)$ .

The following proposition is a direct consequence of Kawasaki-Riemann-Roch formula (see [23] or §1.3.4 in [29]).

**Proposition 3.9** *Let  $\mu \in \Delta_K(M) \cap \Lambda_+^*$  be a weakly-regular value of  $\Phi_K$ . Then we have  $\mathcal{Q}(M_\mu, n\Omega_\mu) \sim \text{vol}(M_\mu) n^{\frac{\dim M_\mu}{2}}$ , when  $n \rightarrow \infty$ . In particular the map  $n \geq 1 \mapsto \mathcal{Q}(M_\mu, n\Omega_\mu)$  is non-zero.*

### 3.5 Quantization commutes with reduction

Let us explain the “*Quantization commutes with Reduction*” Theorem proved in [30].

We fix  $\tilde{\lambda} \in \widehat{G}_{\text{hol}}$ . The coadjoint orbit  $\tilde{G}\tilde{\lambda}$  is prequantized by the line bundle  $\tilde{G} \times_{K_{\tilde{\lambda}}} \mathbb{C}_{\tilde{\lambda}}$ , and the moment map  $\Phi_{\tilde{G}}^{\tilde{\lambda}} : \tilde{G}\tilde{\lambda} \rightarrow \mathfrak{g}^*$  corresponding to the  $G$ -action on  $\tilde{G} \times_{K_{\tilde{\lambda}}} \mathbb{C}_{\tilde{\lambda}}$  is equal to the restriction of the map  $\pi_{\mathfrak{g}, \tilde{\mathfrak{g}}}$  to  $\tilde{G}\tilde{\lambda}$ .

The symplectic slice  $Y_{\tilde{\lambda}} = (\Phi_{\tilde{G}}^{\tilde{\lambda}})^{-1}(\mathfrak{k}^*)$  is prequantized by the line bundle  $\mathcal{L}_{\tilde{\lambda}} := \tilde{G} \times_{K_{\tilde{\lambda}}} \mathbb{C}_{\tilde{\lambda}}|_{Y_{\tilde{\lambda}}}$ . The moment map  $\Phi_K^{\tilde{\lambda}} : Y_{\tilde{\lambda}} \rightarrow \mathfrak{k}^*$  corresponding to the  $K$ -action is equal to the restriction of  $\Phi_{\tilde{G}}^{\tilde{\lambda}}$  to  $Y_{\tilde{\lambda}}$ .

For any  $\lambda \in \widehat{G}_{\text{hol}}$ , we consider the (possibly singular) symplectic reduced space

$$\mathbb{X}_{\tilde{\lambda}, \lambda} := (\Phi_K^{\tilde{\lambda}})^{-1}(\lambda)/K_\lambda,$$

equipped with the reduced symplectic form  $\Omega_{\tilde{\lambda}, \lambda}$ , and the (possibly singular) line bundle

$$\mathbb{L}_{\tilde{\lambda}, \lambda} := \left( \mathcal{L}_{\tilde{\lambda}}|_{(\Phi_K^{\tilde{\lambda}})^{-1}(\lambda)} \otimes \mathbb{C}_{-\lambda} \right) / K_\lambda.$$

Thanks to Definition 3.8, the geometric quantization  $\mathcal{Q}(\mathbb{X}_{\tilde{\lambda}, \lambda}, \Omega_{\tilde{\lambda}, \lambda}) \in \mathbb{Z}$  of those compact symplectic spaces  $(\mathbb{X}_{\tilde{\lambda}, \lambda}, \Omega_{\tilde{\lambda}, \lambda})$  are well-defined even if they are singular. In particular  $\mathcal{Q}(\mathbb{X}_{\tilde{\lambda}, \lambda}, \Omega_{\tilde{\lambda}, \lambda}) = 0$  when  $\mathbb{X}_{\tilde{\lambda}, \lambda} := \emptyset$ .

The following Theorem is proved in [30].



**Theorem 3.10** *Let  $\tilde{\lambda} \in \widehat{G}_{\text{hol}}$ . We have an Hilbertian direct sum*

$$V_{\tilde{\lambda}}^{\tilde{G}}|_G = \bigoplus_{\lambda \in \widehat{G}_{\text{hol}}} \mathcal{Q}(\mathbb{X}_{\tilde{\lambda}, \lambda}, \Omega_{\tilde{\lambda}, \lambda}) V_{\lambda}^G,$$

*It means that for any  $\lambda \in \widehat{G}_{\text{hol}}$ , the multiplicity of the representation  $V_{\lambda}^G$  in the restriction  $V_{\tilde{\lambda}}^{\tilde{G}}|_G$  is equal to the geometric quantization  $\mathcal{Q}(\mathbb{X}_{\tilde{\lambda}, \lambda}, \Omega_{\tilde{\lambda}, \lambda})$  of the (compact) reduced space  $\mathbb{X}_{\tilde{\lambda}, \lambda}$ .*

**Remark 3.11** *Let  $(\tilde{\lambda}, \lambda) \in \widehat{G}_{\text{hol}} \times \widehat{G}_{\text{hol}}$ . Theorem 3.10. shows that*

$$\left[ V_{n\tilde{\lambda}}^G : V_{n\lambda}^{\tilde{G}} \right] = \mathcal{Q}(\mathbb{X}_{\tilde{\lambda}, \lambda}, n\Omega_{\tilde{\lambda}, \lambda})$$

*for any  $n \geq 1$ .*

## 4 Proofs of the main results

We come back to the setting of §2.2:  $G/K$  is a complex submanifold of a Hermitian symmetric space  $\tilde{G}/\tilde{K}$ . It means that there exists a  $\tilde{K}$ -invariant element  $z \in \mathfrak{k}$  such that  $\text{ad}(z)$  defines complex structures on  $\tilde{\mathfrak{p}}$  and  $\mathfrak{p}$ . We consider the orthogonal decomposition  $\tilde{\mathfrak{p}} = \mathfrak{p} \oplus \mathfrak{q}$ , and we denote by  $\text{Sym}(\mathfrak{q})$  the symmetric algebra of the complex  $K$ -module  $(\mathfrak{q}, \text{ad}(z))$ .

### 4.1 Proof of Theorem A

The set  $\Pi_{\text{hol}}(\tilde{G}, G)$  is equal to  $\bigcup_{\tilde{a} \in \tilde{\mathcal{C}}_{\text{hol}}} \{\tilde{a}\} \times \Delta_G(\tilde{G}\tilde{a})$ . We define

$$\Pi_{\text{hol}}(\tilde{G}, G)^0 := \bigcup_{\tilde{a} \in \tilde{\mathcal{C}}_{\text{hol}}} \{\tilde{a}\} \times \Delta_G(\tilde{G}\tilde{a})^0.$$

We start with the following result.

**Lemma 4.1** *The set  $\Pi_{\text{hol}}(\tilde{G}, G)^0 \cap \tilde{\mathfrak{t}}_{\mathbb{Q}}^* \times \mathfrak{t}_{\mathbb{Q}}^*$  is dense in  $\Pi_{\text{hol}}(\tilde{G}, G)$ .*

*Proof :* Let  $(\tilde{\xi}, \xi) \in \Pi_{\text{hol}}(\tilde{G}, G)$ : take  $\tilde{g} \in \tilde{G}$  such that  $\xi = \pi_{\mathfrak{g}, \tilde{g}}(\tilde{g}\tilde{\xi})$ . We consider a sequence  $\tilde{\xi}_n \in \tilde{\mathcal{C}}_{\text{hol}} \cap \tilde{\mathfrak{t}}_{\mathbb{Q}}^*$  converging to  $\tilde{\xi}$ . Then  $\xi_n := \pi_{\mathfrak{g}, \tilde{g}}(\tilde{g}\tilde{\xi}_n)$  is a sequence of  $\mathcal{C}_{G/K}^0$  converging to  $\xi \in \mathcal{C}_{\text{hol}}$ . Since the map  $p : \mathcal{C}_{G/K}^0 \rightarrow \mathcal{C}_{\text{hol}}$  is continuous (see Lemma 2.3), the sequence  $\eta_n := p(\xi_n)$  converges to  $p(\xi) = \xi$ . By definition, we have  $\eta_n \in \Delta_G(\tilde{G}\tilde{\xi}_n)$  for any  $n \in \mathbb{N}$ . Since  $\tilde{\xi}_n$  are rational, each subset  $\Delta_G(\tilde{G}\tilde{\xi}_n)^0 \cap \tilde{\mathfrak{t}}_{\mathbb{Q}}^*$  is dense in  $\Delta_G(\tilde{G}\tilde{\xi}_n)$  (see Lemma 2.10). Hence,  $\forall n \in \mathbb{N}$ , there exists  $\zeta_n \in \Delta_G(\tilde{G}\tilde{\xi}_n)^0 \cap \tilde{\mathfrak{t}}_{\mathbb{Q}}^*$  such that  $\|\zeta_n - \eta_n\| \leq 2^{-n}$ . Finally, we see that  $(\tilde{\xi}_n, \zeta_n)$  is a sequence of rational elements of  $\Pi_{\text{hol}}(\tilde{G}, G)^0$  converging to  $(\xi, \tilde{\xi})$ .  $\square$

The main purpose of this section is the proof of the following theorem.

**Theorem 4.2** *The following inclusions hold*

$$\Pi_{\text{hol}}(\tilde{G}, G)^0 \cap \tilde{\mathfrak{t}}_{\mathbb{Q}}^* \times \mathfrak{t}_{\mathbb{Q}}^* \underset{(1)}{\subset} \Pi_{\text{hol}}^{\mathbb{Q}}(\tilde{G}, G) \underset{(2)}{\subset} \Pi_{\text{hol}}(\tilde{G}, G).$$

Lemma 4.1 and Theorem 4.2 gives the important Corollary.

**Corollary 4.3**  $\Pi_{\text{hol}}^{\mathbb{Q}}(\tilde{G}, G)$  *is dense in*  $\Pi_{\text{hol}}(\tilde{G}, G)$ .

*Proof of Theorem 4.2 :* Let  $(\tilde{\mu}, \mu) \in \Pi_{\text{hol}}^{\mathbb{Q}}(\tilde{G}, G)$ : there exists  $N \geq 1$  such that  $(N\tilde{\mu}, N\mu) \in \Pi_{\text{hol}}^{\mathbb{Z}}(\tilde{G}, G)$ . The multiplicity  $[V_{N\tilde{\mu}}^G : V_{N\mu}^{\tilde{G}}]$  is non-zero, and thanks to Theorem 3.10, it implies that the reduced space  $\mathbb{X}_{N\tilde{\mu}, N\mu}$  is non-empty. In other words  $(N\tilde{\mu}, N\mu) \in \Pi_{\text{hol}}(\tilde{G}, G)$ . The inclusion (2) is proven.

Let  $(\tilde{\mu}, \mu) \in \Pi_{\text{hol}}(\tilde{G}, G)^0 \cap \tilde{\mathfrak{t}}_{\mathbb{Q}}^* \times \mathfrak{t}_{\mathbb{Q}}^*$ . There exists  $N_o \geq 1$  such that  $\lambda := N_o\mu \in \widehat{G}_{\text{hol}}$ ,  $\tilde{\lambda} := N_o\tilde{\mu} \in \widehat{\tilde{G}}_{\text{hol}}$  and  $\lambda \in \Delta_G(\tilde{G}\tilde{\lambda})^0$ : the element  $\lambda$  is a weakly-regular value of the moment map  $\tilde{G}\tilde{\lambda} \rightarrow \mathfrak{g}^*$ . Theorem 3.10 tell us that, for any  $n \geq 1$ , the multiplicity  $[V_{n\lambda}^G : V_{n\tilde{\lambda}}^{\tilde{G}}]$  is equal to Riemann-Roch number  $\mathcal{Q}(\mathbb{X}_{\tilde{\lambda}, \lambda}, n\Omega_{\tilde{\lambda}, \lambda})$ . Since the map  $n \mapsto \mathcal{Q}(\mathbb{X}_{\tilde{\lambda}, \lambda}, n\Omega_{\tilde{\lambda}, \lambda})$  is non-zero (see Proposition 3.9), we can conclude that there exists  $n_o \geq 1$  such that  $[V_{n_o\lambda}^G : V_{n_o\tilde{\lambda}}^{\tilde{G}}] \neq 0$ . In other words, we obtain  $n_o N_o(\tilde{\mu}, \mu) \in \Pi_{\text{hol}}^{\mathbb{Z}}(\tilde{G}, G)$  and so  $(\tilde{\mu}, \mu) \in \Pi_{\text{hol}}^{\mathbb{Q}}(\tilde{G}, G)$ . The inclusion (1) is settled.  $\square$

Now we can terminate the proof of Theorem A.

Thanks to Proposition 3.5, we know that  $\Pi_{\text{hol}}^{\mathbb{Q}}(\tilde{G}, G)$  is a  $\mathbb{Q}$ -convex cone. Since  $\Pi_{\text{hol}}(\tilde{G}, G)$  is a closed subset of  $\tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}}$  (see Proposition 2.5), we can conclude, by a density argument, that  $\Pi_{\text{hol}}(\tilde{G}, G)$  is a closed convex cone of  $\tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}}$ .

## 4.2 The affine variety $\tilde{K}_{\mathbb{C}} \times \mathfrak{q}$

Let  $\tilde{\kappa}$  be the Killing form on the Lie algebra  $\tilde{\mathfrak{g}}$ . We consider the  $\tilde{K}$ -invariant symplectic structures  $\Omega_{\tilde{\mathfrak{p}}}$  on  $\tilde{\mathfrak{p}}$ , defined by the relation

$$\Omega_{\tilde{\mathfrak{p}}}(\tilde{Y}, \tilde{Y}') = \tilde{\kappa}(z, [\tilde{Y}, \tilde{Y}']), \quad \forall \tilde{Y}, \tilde{Y}' \in \tilde{\mathfrak{p}}.$$

We notice that the complex structure  $\text{ad}(z)$  is adapted to  $\Omega_{\tilde{\mathfrak{p}}}$ :  $\Omega_{\tilde{\mathfrak{p}}}(\tilde{Y}, \text{ad}(z)\tilde{Y}) > 0$  if  $\tilde{Y} \neq 0$ .

We denote by  $\Omega_{\mathfrak{q}}$  the restriction of  $\Omega_{\tilde{\mathfrak{p}}}$  on the symplectic subspace  $\mathfrak{q}$ . The moment map  $\Phi_{\mathfrak{q}}$  associated to the  $K$ -action on  $(\mathfrak{q}, \Omega_{\mathfrak{q}})$  is defined by the relations  $\langle \Phi_{\mathfrak{q}}(Y), X \rangle = \frac{-1}{2}\tilde{\kappa}([X, Y], [z, Y]), \forall (X, Y) \in \mathfrak{p} \times \mathfrak{q}$ . In particular,  $\langle \Phi_{\mathfrak{q}}(Y), z \rangle = \frac{-1}{2}\|Y\|^2, \forall Y \in \mathfrak{q}$ , so the map  $\langle \Phi_{\mathfrak{q}}, z \rangle$  is proper.

The complex reductive group  $\tilde{K}_{\mathbb{C}}$  is equipped with the following action of  $\tilde{K} \times K$ :  $(\tilde{k}, k) \cdot a = \tilde{k}ak^{-1}$ . It has a canonical structure of a smooth affine variety. There is a diffeomorphism of the cotangent bundle  $\mathbf{T}^*\tilde{K}$  with  $\tilde{K}_{\mathbb{C}}$  defined as follows. We identify

$\mathbf{T}^* \tilde{K}$  with  $\tilde{K} \times \tilde{\mathfrak{k}}^*$  by means of left-translation and then with  $\tilde{K} \times \tilde{\mathfrak{k}}$  by means of an invariant inner product on  $\tilde{\mathfrak{k}}$ . The map  $\varphi : \tilde{K} \times \tilde{\mathfrak{k}} \rightarrow \tilde{K}_{\mathbb{C}}$  given by  $\varphi(a, X) = ae^{iX}$  is a diffeomorphism. If we use  $\varphi$  to transport the complex structure of  $\tilde{K}_{\mathbb{C}}$  to  $\mathbf{T}^* \tilde{K}$ , then the resulting complex structure on  $\mathbf{T}^* \tilde{K}$  is compatible with the symplectic structure on  $\mathbf{T}^* \tilde{K}$ , so that  $\mathbf{T}^* \tilde{K}$  becomes a Kähler Hamiltonian  $\tilde{K} \times K$ -manifold (see [11], §3). The moment map relative to the  $\tilde{K} \times K$ -action is the proper map  $\Phi_{\tilde{K}} \oplus \Phi_K : \mathbf{T}^* \tilde{K} \rightarrow \tilde{\mathfrak{k}}^* \oplus \mathfrak{k}^*$  defined by  $\Phi_{\tilde{K}}(\tilde{a}, \tilde{\eta}) = -\tilde{a}\tilde{\eta}$  and  $\Phi_K(\tilde{a}, \tilde{\eta}) = \pi_{\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}}(\tilde{\eta})$ . Here  $\pi_{\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}} : \tilde{\mathfrak{k}}^* \rightarrow \mathfrak{k}^*$  is the projection dual to the inclusion  $\mathfrak{k} \hookrightarrow \tilde{\mathfrak{k}}$  of Lie algebras.

Finally we consider the Kähler Hamiltonian  $\tilde{K} \times K$ -manifold  $\mathbf{T}^* \tilde{K} \times \mathfrak{q}$ , where  $\mathfrak{q}$  is equipped with the symplectic structure  $\Omega_{\mathfrak{q}}$ . Let us denote by  $\Phi : \mathbf{T}^* \tilde{K} \times \mathfrak{q} \rightarrow \tilde{\mathfrak{k}}^* \oplus \mathfrak{k}^*$  the moment map relative to the  $\tilde{K} \times K$ -action :

$$(15) \quad \Phi(\tilde{a}, \tilde{\eta}, Y) = \left( -\tilde{a}\tilde{\eta}, \pi_{\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}}(\tilde{\eta}) + \Phi_{\mathfrak{q}}(Y) \right).$$

Since  $\Phi$  is proper map, the Convexity Theorem tell us that

$$\Delta(\mathbf{T}^* \tilde{K} \times \mathfrak{q}) := \text{Image}(\Phi) \cap \tilde{\mathfrak{k}}_{\geq 0}^* \times \mathfrak{k}_{\geq 0}^*$$

is a closed convex locally polyhedral set.

We consider now the action of  $\tilde{K} \times K$  on the affine variety  $\tilde{K}_{\mathbb{C}} \times \mathfrak{q}$ . The set of highest weights of  $\tilde{K}_{\mathbb{C}} \times \mathfrak{q}$  is the semigroup

$$\Delta^{\mathbb{Z}}(\tilde{K}_{\mathbb{C}} \times \mathfrak{q}) \subset \tilde{\Lambda}_+^* \times \Lambda_+^*$$

consisting of all dominant weights  $(\tilde{\lambda}, \lambda)$  such that the irreducible  $\tilde{K} \times K$ -representation  $V_{\tilde{\lambda}}^{\tilde{K}} \otimes V_{\lambda}^K$  occurs in the coordinate ring  $\mathbb{C}[\tilde{K}_{\mathbb{C}} \times \mathfrak{q}]$ . A direct application of the Peter-Weyl Theorem gives the following characterization :

$$(16) \quad (\tilde{\lambda}, \lambda) \in \Delta^{\mathbb{Z}}(\tilde{K}_{\mathbb{C}} \times \mathfrak{q}) \iff \left[ V_{\tilde{\lambda}}^{\tilde{K}}|_K \otimes V_{\lambda}^K \otimes \text{Sym}(\mathfrak{q}) \right]^K \neq 0.$$

We denote by  $\Delta^{\mathbb{Q}}(\tilde{K}_{\mathbb{C}} \times \mathfrak{q})$  the  $\mathbb{Q}$ -convex cone generated by the semigroup  $\Delta^{\mathbb{Z}}(\tilde{K}_{\mathbb{C}} \times \mathfrak{q})$ :  $(\tilde{\xi}, \xi) \in \Delta^{\mathbb{Q}}(\tilde{K}_{\mathbb{C}} \times \mathfrak{q})$  if and only if  $\exists N \geq 1, N(\tilde{\xi}, \xi) \in \Delta^{\mathbb{Z}}(\tilde{K}_{\mathbb{C}} \times \mathfrak{q})$ .

The following important fact is classical (see Theorem 4.9 in [34]).

**Proposition 4.4** *The Kirwan polyhedron  $\Delta(\mathbf{T}^* \tilde{K} \times \mathfrak{q})$  is equal to the closure of the  $\mathbb{Q}$ -convex cone  $\Delta^{\mathbb{Q}}(\tilde{K}_{\mathbb{C}} \times \mathfrak{q})$ .*

### 4.3 Proof of Theorem B

Consider the semigroup  $\Pi^{\mathbb{Z}}(\tilde{G}, G)$  of  $\tilde{\Lambda}_+^* \times \Lambda_+^*$  (see Definition 1.3) and the  $\mathbb{Q}$ -convex cone  $\Pi^{\mathbb{Q}}(\tilde{G}, G) := \{(\tilde{\xi}, \xi) \in \tilde{\mathfrak{k}}_{\geq 0}^* \times \mathfrak{k}_{\geq 0}^* ; \exists N \geq 1, N(\tilde{\xi}, \xi) \in \Pi^{\mathbb{Z}}(\tilde{G}, G)\}$ .

The Jakobsen-Vergne theorem says that  $\Pi_{\text{hol}}^{\mathbb{Z}}(\tilde{G}, G) = \Pi^{\mathbb{Z}}(\tilde{G}, G) \cap \widehat{G}_{\text{hol}} \times \widehat{G}_{\text{hol}}$ . Hence, the convex cone  $\Pi_{\text{hol}}^{\mathbb{Q}}(\tilde{G}, G)$  is equal to  $\Pi^{\mathbb{Q}}(\tilde{G}, G) \cap \tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}}$ . Thanks to (16), we know that  $(\tilde{\xi}, \xi) \in \Pi^{\mathbb{Q}}(\tilde{G}, G)$  if and only if  $(\tilde{\xi}, \xi^*) \in \Delta^{\mathbb{Q}}(\tilde{K}_{\mathbb{C}} \times \mathfrak{q})$ . The density results obtained in Proposition 4.4 and Corollary 4.3 gives finally Theorem B.

**Theorem 4.5** 1. We have  $\Pi_{\text{hol}}(\tilde{G}, G) = \Pi(\tilde{G}, G) \cap \tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}}$ .

2. An element  $(\tilde{\xi}, \xi)$  belongs to  $\Pi_{\text{hol}}(\tilde{G}, G)$  if and only if

$$(\tilde{\xi}, \xi) \in \tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}} \quad \text{and} \quad (\tilde{\xi}, \xi^*) \in \Delta(\mathbf{T}^* \tilde{K} \times \mathfrak{q}).$$

#### 4.4 Proof of Theorem C

We denote by  $\bar{\mathfrak{q}}$  the  $K$ -vector space  $\mathfrak{q}$  equipped with the opposite symplectic form  $-\Omega_{\bar{\mathfrak{q}}}$  and opposite complex structure  $-\text{ad}(z)$ . The moment map relative to the  $K$ -action on  $\bar{\mathfrak{q}}$  is denoted by  $\Phi_{\bar{\mathfrak{q}}} = -\Phi_{\mathfrak{q}}$ .

**Lemma 4.6** Any element  $(\tilde{\xi}, \xi) \in \tilde{\mathfrak{t}}_{\geq 0}^* \times \mathfrak{t}_{\geq 0}^*$  satisfies the equivalence :

$$(\tilde{\xi}, \xi^*) \in \Delta(\mathbf{T}^* \tilde{K} \times \mathfrak{q}) \iff \xi \in \Delta_K(\tilde{K} \tilde{\xi} \times \bar{\mathfrak{q}}).$$

*Proof* : Thanks to (15), we see immediatly that  $\exists(\tilde{a}, \tilde{\eta}, Y) \in \mathbf{T}^* \tilde{K} \times \mathfrak{q}$  such that  $(\tilde{\xi}, \xi^*) = \Phi(\tilde{a}, \tilde{\eta}, Y)$  if and only if  $\exists(\tilde{b}, Z) \in \tilde{K} \times \mathfrak{q}$  such that  $\xi = \pi_{\tilde{\mathfrak{t}}, \tilde{\mathfrak{k}}}(\tilde{b} \tilde{\xi}) + \Phi_{\bar{\mathfrak{q}}}(Z)$ .  $\square$

At this stage we know that  $\Delta_G(\tilde{G} \tilde{\mu}) = \Delta_K(\tilde{K} \tilde{\mu} \times \bar{\mathfrak{q}}) \cap \mathcal{C}_{\text{hol}}$ . Hence, Theorem C will follow from the next result.

**Proposition 4.7** For any  $\tilde{\mu} \in \tilde{\mathcal{C}}_{\text{hol}}$ , the Kirwan polyhedron  $\Delta_K(\tilde{K} \tilde{\mu} \times \bar{\mathfrak{q}})$  is contained in  $\mathcal{C}_{\text{hol}}$ .

*Proof* : By definition  $\mathcal{C}_{\text{hol}} = \mathcal{C}_{G/K}^0 \cap \mathfrak{t}_{\geq 0}^*$ , so we have to prove that  $\pi_{\tilde{\mathfrak{t}}, \tilde{\mathfrak{k}}}(\tilde{K} \tilde{\mu}) + \text{Image}(\Phi_{\bar{\mathfrak{q}}})$  is contained in  $\mathcal{C}_{G/K}^0$ . By definition  $\tilde{K} \tilde{\mu} \subset \mathcal{C}_{\tilde{G}/\tilde{K}}^0$ , and then  $\pi_{\tilde{\mathfrak{t}}, \tilde{\mathfrak{k}}}(\tilde{K} \tilde{\mu}) \subset \mathcal{C}_{G/K}^0$ . Since  $\mathcal{C}_{G/K}^0 + \mathcal{C}_{G/K} \subset \mathcal{C}_{G/K}^0$ , it is sufficient to check that  $\text{Image}(\Phi_{\bar{\mathfrak{q}}}) \subset \mathcal{C}_{G/K}$ . Let  $\Phi_{\tilde{\mathfrak{p}}}$  be the moment map relative to the action of  $\tilde{K}$  on  $(\tilde{\mathfrak{p}}, \Omega_{\tilde{\mathfrak{p}}})$ . As  $\text{Image}(\Phi_{\bar{\mathfrak{q}}}) \subset \pi_{\tilde{\mathfrak{t}}, \tilde{\mathfrak{k}}}(\text{Image}(-\Phi_{\tilde{\mathfrak{p}}}))$ , the following lemma will terminate the proof of Proposition 4.7.  $\square$

**Lemma 4.8** The image of the moment map  $-\Phi_{\tilde{\mathfrak{p}}}$  is contained in  $\mathcal{C}_{\tilde{G}/\tilde{K}}$ .

*Proof* : Let  $z^* \in \tilde{\mathfrak{t}}^*$  such that  $\langle z^*, \tilde{X} \rangle = -\tilde{\kappa}(z, \tilde{X})$ ,  $\forall \tilde{X} \in \tilde{\mathfrak{g}}$ . Consider the coadjoint orbit  $\tilde{\mathcal{O}} = \tilde{G}z^*$  equipped with its canonical symplectic structure  $\Omega_{\tilde{\mathcal{O}}}$ : the symplectic vector space  $\mathbf{T}_{z^*} \tilde{\mathcal{O}}$  is canonically isomorphic to  $(\tilde{\mathfrak{p}}, -\Omega_{\tilde{\mathfrak{p}}})$ . In [26], McDuff proved that  $(\tilde{\mathcal{O}}, \Omega_{\tilde{\mathcal{O}}})$  is diffeomorphic, as a  $\tilde{K}$ -symplectic manifold, to the symplectic vector space  $(\tilde{\mathfrak{p}}, -\Omega_{\tilde{\mathfrak{p}}})$

(see [6, 8] for a generalization of this fact). McDuff's results shows in particular that  $\text{Image}(-\Phi_{\tilde{\mathfrak{p}}}) = \pi_{\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}}}(\tilde{\mathcal{O}})$ . Our proof is completed if we check that  $\pi_{\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}}}(\tilde{\mathcal{O}}) \subset \mathcal{C}_{\tilde{G}/\tilde{K}}$ : in other words, if  $\langle \pi_{\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}}}(\tilde{g}_0 z^*), \tilde{g}_1 z \rangle \geq 0$ ,  $\forall \tilde{g}_0, \tilde{g}_1 \in \tilde{G}$ . But

$$\begin{aligned} 2\langle \pi_{\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}}}(\tilde{g}_0 z^*), \tilde{g}_1 z \rangle &= \langle \tilde{g}_0 z^*, \tilde{g}_1 z + \Theta(\tilde{g}_1)z \rangle \\ &= -\tilde{\kappa}(z, \tilde{g}_0^{-1} \tilde{g}_1 z) - \tilde{\kappa}(z, \tilde{g}_0^{-1} \Theta(\tilde{g}_1)z). \end{aligned}$$

With (7) in hand, it is not difficult to see that  $-\tilde{\kappa}(z, \tilde{g} z) \geq 0$  for every  $\tilde{g} \in \tilde{G}$ . We thus verified that  $\pi_{\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}}}(\tilde{\mathcal{O}}) \subset \mathcal{C}_{\tilde{G}/\tilde{K}}$ .  $\square$

## 5 Facets of the cone $\Pi_{\text{hol}}(\tilde{G}, G)$

We come back to the framework of §4.2. We consider the Kähler Hamiltonian  $\tilde{K} \times K$ -manifold  $\mathbf{T}^* \tilde{K} \times \mathfrak{q}$ . The moment map,  $\Phi : \mathbf{T}^* \tilde{K} \times \mathfrak{q} \rightarrow \tilde{\mathfrak{k}}^* \oplus \mathfrak{k}^*$ , relative to the  $\tilde{K} \times K$ -action is defined by (15).

In this section, we adapt to our case the result of §6 of [31] concerning the parametrization of the facets of Kirwan polyhedrons in terms of Ressayre's data.

### 5.1 Admissible elements

We choose maximal torus  $\tilde{T} \subset \tilde{K}$  and  $T \subset K$  such that  $T \subset \tilde{T}$ . Let  $\mathfrak{R}_o$  and  $\mathfrak{R}$  be respectively the set of roots for the action of  $T$  on  $(\tilde{\mathfrak{g}}/\mathfrak{g}) \otimes \mathbb{C}$  and  $\mathfrak{g} \otimes \mathbb{C}$ . Let  $\tilde{\mathfrak{R}}$  be the set of roots for the action of  $\tilde{T}$  on  $\tilde{\mathfrak{g}} \otimes \mathbb{C}$ . Let  $\mathfrak{R}^+ \subset \mathfrak{R}$  and  $\tilde{\mathfrak{R}}^+ \subset \tilde{\mathfrak{R}}$  be the systems of positive roots defined in (6). Let  $W, \tilde{W}$  be the Weyl groups of  $(T, K)$  and  $(\tilde{T}, \tilde{K})$ . Let  $w_o \in W$  be the longest element.

We start by introducing the notion of admissible elements. The group  $\text{hom}(U(1), T)$  admits a natural identification with the lattice  $\wedge := \frac{1}{2\pi} \ker(\exp : \mathfrak{t} \rightarrow T)$ . A vector  $\gamma \in \mathfrak{t}$  is called rational if it belongs to the  $\mathbb{Q}$ -vector space  $\mathfrak{t}_{\mathbb{Q}}$  generated by  $\wedge$ .

We consider the  $\tilde{K} \times K$ -action on  $N := \mathbf{T}^* \tilde{K} \times \mathfrak{q}$ . We associate to any subset  $\mathcal{X} \subset N$ , the integer  $\dim_{\tilde{K} \times K}(\mathcal{X})$  (see (5)).

**Definition 5.1** *A non-zero element  $(\tilde{\gamma}, \gamma) \in \tilde{\mathfrak{t}} \times \mathfrak{t}$  is called admissible if the elements  $\tilde{\gamma}$  and  $\gamma$  are rational and if  $\dim_{\tilde{K} \times K}(N^{(\tilde{\gamma}, \gamma)}) - \dim_{\tilde{K} \times K}(N) \in \{0, 1\}$ .*

If  $\gamma \in \mathfrak{t}$ , we denote by  $\mathfrak{R}_o \cap \gamma^\perp$  the subsets of weight vanishing against  $\gamma$ . We start with the following lemma whose proof is left to the reader (see §6.1.1 of [31]).

**Lemma 5.2** 1.  $N^{(\tilde{\gamma}, \gamma)} \neq \emptyset$  if and only if  $\tilde{\gamma} \in \tilde{W}\gamma$ .

2.  $\dim_{\tilde{K} \times K}(N) = \dim_T(\tilde{\mathfrak{g}}/\mathfrak{g}) = \dim(\mathfrak{t}) - \dim(\text{Vect}(\mathfrak{R}_o))$ .

3. For any  $\tilde{w} \in \tilde{W}$ ,  $\dim_{\tilde{K} \times K}(N^{(\tilde{w}\tilde{\gamma}, \gamma)}) = \dim_T(\tilde{\mathfrak{g}}^\gamma/\mathfrak{g}^\gamma) = \dim(\mathfrak{t}) - \dim(\text{Vect}(\mathfrak{R}_o \cap \gamma^\perp))$ .

The next result is a direct consequence of the previous lemma.

**Lemma 5.3** *The admissible elements relative to the  $\tilde{K} \times K$ -action on  $\mathbf{T}^* \tilde{K} \times \mathfrak{q}$  are of the form  $(\tilde{w}\gamma, \gamma)$  where  $\tilde{w} \in \tilde{W}$  and  $\gamma$  is a non-zero rational element satisfying  $\text{Vect}(\mathfrak{R}_o) \cap \gamma^\perp = \text{Vect}(\mathfrak{R}_o \cap \gamma^\perp)$ .*

## 5.2 Ressayre's data

**Definition 5.4** 1. Consider the linear action  $\rho : G \rightarrow \text{GL}_{\mathbb{C}}(V)$  of a compact Lie group on a complex vector space  $V$ . For any  $(\eta, a) \in \mathfrak{g} \times \mathbb{R}$ , we define the vector subspace  $V^{\eta=a} = \{v \in V, d\rho(\eta)v = iav\}$ . Thus, for any  $\eta \in \mathfrak{g}$ , we have the decomposition  $V = V^{\eta>0} \oplus V^{\eta=0} \oplus V^{\eta<0}$  where  $V^{\eta>0} = \sum_{a>0} V^{\eta=a}$ , and  $V^{\eta<0} = \sum_{a<0} V^{\eta=a}$ .

2. The real number  $\text{Tr}_\eta(V^{\eta>0})$  is defined as the sum  $\sum_{a>0} a \dim(V^{\eta=a})$ .

We consider an admissible element  $(\tilde{w}\gamma, \gamma)$ . The submanifold of  $N \simeq \tilde{K}_{\mathbb{C}} \times \mathfrak{q}$  fixed of by  $(\tilde{w}\gamma, \gamma)$  is  $N^{(\tilde{w}\gamma, \gamma)} = \tilde{w}\tilde{K}_{\mathbb{C}}^\gamma \times \mathfrak{q}^\gamma$ . There is a canonical isomorphism between the manifold  $N^{(\tilde{w}\gamma, \gamma)}$  equipped with the action of  $\tilde{w}\tilde{K}^\gamma \tilde{w}^{-1} \times K^\gamma$  with the manifold  $\tilde{K}_{\mathbb{C}}^\gamma \times \mathfrak{q}^\gamma$  equipped with the action of  $\tilde{K}^\gamma \times K^\gamma$ . The tangent bundle  $(\mathbf{TN}|_{N^{(\tilde{w}\gamma, \gamma)}})^{(\tilde{w}\gamma, \gamma)>0}$  is isomorphic to  $N^{\gamma w} \times \tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0} \times \mathfrak{q}^{\gamma>0}$ .

The choice of positive roots  $\mathfrak{R}^+$  (resp.  $\tilde{\mathfrak{R}}^+$ ) induces a decomposition  $\mathfrak{k}_{\mathbb{C}} = \mathfrak{n} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \bar{\mathfrak{n}}$  (resp.  $\tilde{\mathfrak{k}}_{\mathbb{C}} = \tilde{\mathfrak{n}} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \bar{\tilde{\mathfrak{n}}}$ ), where  $\mathfrak{n} = \sum_{\alpha \in \mathfrak{R}^+} (\mathfrak{k} \otimes \mathbb{C})_\alpha$  (resp.  $\tilde{\mathfrak{n}} = \sum_{\tilde{\alpha} \in \tilde{\mathfrak{R}}^+} (\tilde{\mathfrak{k}} \otimes \mathbb{C})_{\tilde{\alpha}}$ ). We consider the map

$$\rho^{\tilde{w}, \gamma} : \tilde{K}_{\mathbb{C}}^\gamma \times \mathfrak{q}^\gamma \longrightarrow \text{hom} \left( \tilde{\mathfrak{n}}^{\tilde{w}\gamma>0} \times \mathfrak{n}^{\gamma>0}, \tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0} \times \mathfrak{q}^{\gamma>0} \right)$$

defined by the relation

$$\rho^{\tilde{w}, \gamma}(\tilde{x}, v) : (\tilde{X}, X) \longmapsto ((\tilde{w}\tilde{x})^{-1}\tilde{X} - X; X \cdot v),$$

for any  $(\tilde{x}, v) \in \tilde{K}_{\mathbb{C}}^\gamma \times \mathfrak{q}^\gamma$ .

**Definition 5.5**  $(\gamma, \tilde{w}) \in \mathfrak{t} \times \tilde{W}$  is a Ressayre's data if

1.  $(\tilde{w}\gamma, \gamma)$  is admissible,
2.  $\exists(\tilde{x}, v)$  such that  $\rho^{\tilde{w}, \gamma}(\tilde{x}, v)$  is bijective.

**Remark 5.6** In [31], the Ressayre's data were called regular infinitesimal B-Ressayre's pairs.

Since the linear map  $\rho^{\tilde{w}, \gamma}(\tilde{x}, v)$  commutes with the  $\gamma$ -actions, we obtain the following necessary conditions.

**Lemma 5.7** If  $(\gamma, \tilde{w}) \in \mathfrak{t} \times \tilde{W}$  is a Ressayre's data, then

- Relation (A) :  $\dim(\tilde{\mathfrak{n}}^{\tilde{w}\gamma>0}) + \dim(\mathfrak{n}^{\gamma>0}) = \dim(\tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0}) + \dim(\mathfrak{q}^{\gamma>0})$ .
- Relation (B) :  $\mathrm{Tr}_{\tilde{w}\gamma}(\tilde{\mathfrak{n}}^{\tilde{w}\gamma>0}) + \mathrm{Tr}_{\gamma}(\mathfrak{n}^{\gamma>0}) = \mathrm{Tr}_{\gamma}(\tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0}) + \mathrm{Tr}_{\gamma}(\mathfrak{q}^{\gamma>0})$ .

**Lemma 5.8** Relation (B) is equivalent to

$$(17) \quad \sum_{\substack{\alpha \in \mathfrak{A}_n^+ \\ \langle \alpha, \gamma \rangle > 0}} \langle \alpha, \gamma \rangle = \sum_{\substack{\tilde{\alpha} \in \tilde{\mathfrak{A}}_n^+ \\ \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w} \gamma \rangle > 0}} \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w} \gamma \rangle.$$

*Proof* : First one sees that  $\mathrm{Tr}_{\gamma}(\mathfrak{q}^{\gamma>0}) = \mathrm{Tr}_{\gamma}(\tilde{\mathfrak{p}}^{\gamma>0}) - \mathrm{Tr}_{\gamma}(\mathfrak{p}^{\gamma>0}) = \sum_{\substack{\tilde{\alpha} \in \tilde{\mathfrak{A}}_n^+ \\ \langle \tilde{\alpha}, \gamma \rangle > 0}} \langle \tilde{\alpha}, \gamma \rangle - \sum_{\substack{\alpha \in \mathfrak{A}_n^+ \\ \langle \alpha, \gamma \rangle > 0}} \langle \alpha, \gamma \rangle$ , and  $\mathrm{Tr}_{\gamma}(\tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0}) = \mathrm{Tr}_{\tilde{w}\gamma}(\tilde{\mathfrak{k}}_{\mathbb{C}}^{\tilde{w}\gamma>0}) = \mathrm{Tr}_{\tilde{w}\gamma}(\tilde{\mathfrak{n}}^{\tilde{w}\gamma>0}) + \sum_{\substack{\tilde{\alpha} \in \tilde{\mathfrak{A}}_c^+ \\ \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w} \gamma \rangle > 0}} \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w} \gamma \rangle$ . Relation (B) is equivalent to

$$(18) \quad \mathrm{Tr}_{\gamma}(\mathfrak{n}^{\gamma>0}) + \sum_{\substack{\alpha \in \mathfrak{A}_n^+ \\ \langle \alpha, \gamma \rangle > 0}} \langle \alpha, \gamma \rangle = \sum_{\substack{\tilde{\alpha} \in \tilde{\mathfrak{A}}_n^+ \\ \langle \tilde{\alpha}, \gamma \rangle > 0}} \langle \tilde{\alpha}, \gamma \rangle + \sum_{\substack{\tilde{\alpha} \in \tilde{\mathfrak{A}}_c^+ \\ \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w} \gamma \rangle > 0}} \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w} \gamma \rangle.$$

Since  $\tilde{\mathfrak{A}}_n^+$  is invariant under the action of the Weyl group  $\tilde{W}$ , the right hand side of (18) is equal to  $\sum_{\substack{\tilde{\alpha} \in \tilde{\mathfrak{A}}_n^+ \\ \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w} \gamma \rangle > 0}} \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w} \gamma \rangle$ . Since the left hand side of (18) is equal to  $\sum_{\substack{\alpha \in \mathfrak{A}_n^+ \\ \langle \alpha, \gamma \rangle > 0}} \langle \alpha, \gamma \rangle$ , the proof of the Lemma is complete.  $\square$

### 5.3 Cohomological characterization of Ressayre's data

Let  $\gamma \in \mathfrak{t}$  be a non-zero rational element. We denote by  $B \subset K_{\mathbb{C}}$  and by  $\tilde{B} \subset \tilde{K}_{\mathbb{C}}$  the Borel subgroups with Lie algebra  $\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}$  and  $\tilde{\mathfrak{b}} = \mathfrak{t}_{\mathbb{C}} \oplus \tilde{\mathfrak{n}}$ . Consider the parabolic subgroup  $P_{\gamma} \subset K_{\mathbb{C}}$  defined by

$$(19) \quad P_{\gamma} = \{g \in K_{\mathbb{C}}, \lim_{t \rightarrow \infty} \exp(-it\gamma)g \exp(it\gamma) \text{ exists}\}.$$

Similarly, one defines a parabolic subgroup  $\tilde{P}_{\gamma} \subset \tilde{K}_{\mathbb{C}}$ .

We work with the projective varieties  $\mathcal{F}_{\gamma} := K_{\mathbb{C}}/P_{\gamma}$ ,  $\tilde{\mathcal{F}}_{\gamma} := \tilde{K}_{\mathbb{C}}/\tilde{P}_{\gamma}$ , and the canonical embedding  $\iota : \mathcal{F}_{\gamma} \rightarrow \tilde{\mathcal{F}}_{\gamma}$ . We associate to any  $\tilde{w} \in \tilde{W}$ , the Schubert cell

$$\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}^o := \tilde{B}[\tilde{w}] \subset \tilde{\mathcal{F}}_{\gamma}.$$

and the Schubert variety  $\tilde{\mathfrak{X}}_{\tilde{w}, \gamma} := \overline{\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}^o}$ . If  $\tilde{W}^{\gamma}$  denotes the subgroup of  $\tilde{W}$  that fixes  $\gamma$ , we see that the Schubert cell  $\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}^o$  and the Schubert variety  $\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}$  depends only of the class of  $\tilde{w}$  in  $\tilde{W}/\tilde{W}^{\gamma}$ .

On the variety  $\mathcal{F}_{\gamma}$ , we consider the Schubert cell  $\mathfrak{X}_{\gamma}^o := B[e]$  and the Schubert variety  $\mathfrak{X}_{\gamma} := \overline{\mathfrak{X}_{\gamma}^o}$ .

We consider the cohomology<sup>1</sup> ring  $H^*(\tilde{\mathcal{F}}_\gamma, \mathbb{Z})$  of  $\tilde{\mathcal{F}}_\gamma$ . If  $Y$  is an irreducible closed subvariety of  $\tilde{\mathcal{F}}_\gamma$ , we denote by  $[Y] \in H^{2n_Y}(\tilde{\mathcal{F}}_\gamma, \mathbb{Z})$  its cycle class in cohomology : here  $n_Y = \text{codim}_{\mathbb{C}}(Y)$ . Let  $\iota^* : H^*(\tilde{\mathcal{F}}_\gamma, \mathbb{Z}) \rightarrow H^*(\mathcal{F}_\gamma, \mathbb{Z})$  be the pull-back map in cohomology. Recall that the cohomology class  $[pt]$  associated to a singleton  $Y = \{pt\} \subset \mathcal{F}_\gamma$  is a basis of  $H^{\max}(\mathcal{F}_\gamma, \mathbb{Z})$ .

To an oriented real vector bundle  $\mathcal{E} \rightarrow N$  of rank  $r$ , we can associate its Euler class  $\text{Eul}(\mathcal{E}) \in H^{2r}(N, \mathbb{Z})$ . When  $\mathcal{V} \rightarrow N$  is a complex vector bundle, then  $\text{Eul}(\mathcal{V}_{\mathbb{R}})$  corresponds to the top Chern class  $c_p(\mathcal{V})$ , where  $p$  is the complex rank of  $\mathcal{V}$ , and  $\mathcal{V}_{\mathbb{R}}$  means  $\mathcal{V}$  viewed as a real vector bundle oriented by its complex structure (see [5], §21).

The isomorphism  $\mathfrak{q}^{\gamma>0} \simeq \mathfrak{q}/\mathfrak{q}^{\gamma \leq 0}$  shows that  $\mathfrak{q}^{\gamma>0}$  can be viewed as a  $P_\gamma$ -module. Let  $[\mathfrak{q}^{\gamma>0}] = K_{\mathbb{C}} \times_{P_\gamma} \mathfrak{q}^{\gamma>0}$  be the corresponding complex vector bundle on  $\mathcal{F}_\gamma$ . We denote simply by  $\text{Eul}(\mathfrak{q}^{\gamma>0})$  the Euler class  $\text{Eul}([\mathfrak{q}^{\gamma>0}]_{\mathbb{R}}) \in H^*(\mathcal{F}_\gamma, \mathbb{Z})$ .

The following characterization of Ressayre's data was obtained in [31], §6. Recall that  $\mathfrak{R}_o$  denotes the set of weights relative to the  $T$ -action on  $(\tilde{\mathfrak{g}}/\mathfrak{g}) \otimes \mathbb{C}$ .

**Proposition 5.9** *An element  $(\gamma, \tilde{w}) \in \mathfrak{t} \times \tilde{W}$  is a Ressayre's data if and only if the following conditions hold :*

- $\gamma$  is non-zero and rational.
- $\text{Vect}(\mathfrak{R}_o \cap \gamma^\perp) = \text{Vect}(\mathfrak{R}_o) \cap \gamma^\perp$ .
- $[\mathfrak{X}_\gamma] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}]) \cdot \text{Eul}(\mathfrak{q}^{\gamma>0}) = k[pt]$ ,  $k \geq 1$  in  $H^*(\mathcal{F}_\gamma, \mathbb{Z})$ .
- $\sum_{\substack{\alpha \in \mathfrak{R}^+ \\ \langle \alpha, \gamma \rangle > 0}} \langle \alpha, \gamma \rangle = \sum_{\substack{\tilde{\alpha} \in \mathfrak{R}^+ \\ \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w} \gamma \rangle > 0}} \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w} \gamma \rangle$ .

#### 5.4 Parametrization of the facets

We can finally describe the Kirwan polyhedron  $\Delta(\mathbf{T}^* \tilde{K} \times \mathfrak{q})$  (see [31], §6).

**Theorem 5.10** *An element  $(\tilde{\xi}, \xi) \in \tilde{\mathfrak{t}}_{\geq 0}^* \times \mathfrak{t}_{\geq 0}^*$  belongs to  $\Delta(\mathbf{T}^* \tilde{K} \times \mathfrak{q})$  if and only if*

$$\langle \tilde{\xi}, \tilde{w} \gamma \rangle + \langle \xi, \gamma \rangle \geq 0$$

for any Ressayre's data  $(\gamma, \tilde{w}) \in \mathfrak{t} \times \tilde{W}$ .

Theorem 5.10 and Theorem 4.5 permit us to give the following description of the convex cone  $\Pi_{\text{hol}}(\tilde{G}, G)$ .

**Theorem 5.11** *An element  $(\tilde{\xi}, \xi)$  belongs to  $\Pi_{\text{hol}}(\tilde{G}, G)$  if and only if  $(\tilde{\xi}, \xi) \in \tilde{\mathcal{C}}_{\text{hol}} \times \mathcal{C}_{\text{hol}}$  and*

$$\langle \tilde{\xi}, \tilde{w} \gamma \rangle \geq \langle \xi, w_0 \gamma \rangle$$

for any  $(\gamma, \tilde{w}) \in \mathfrak{t} \times \tilde{W}$  satisfying the following conditions:

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<sup>1</sup>Here, we use singular cohomology with integer coefficients.



- $\gamma$  is non-zero and rational.
- $\text{Vect}(\mathfrak{A}_o \cap \gamma^\perp) = \text{Vect}(\mathfrak{A}_o) \cap \gamma^\perp$ .
- $[\mathfrak{X}_\gamma] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}]) \cdot \text{Eul}(\mathfrak{q}^{\gamma > 0}) = k[pt]$ ,  $k \geq 1$  in  $H^*(\mathcal{F}_\gamma, \mathbb{Z})$ .
- $\sum_{\substack{\alpha \in \mathfrak{A}^+ \\ \langle \alpha, \gamma \rangle > 0}} \langle \alpha, \gamma \rangle = \sum_{\substack{\tilde{\alpha} \in \tilde{\mathfrak{A}}^+ \\ \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w} \gamma \rangle > 0}} \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w} \gamma \rangle$ .

## 6 Example: the holomorphic Horn cone $\text{Horn}_{\text{hol}}(p, q)$

Let  $p \geq q \geq 1$ . We consider the pseudo-unitary group  $G = U(p, q) \subset GL_{p+q}(\mathbb{C})$  defined by the relation :  $g \in U(p, q)$  if and only if  $g \text{Id}_{p,q} g^* = \text{Id}_{p,q}$  where  $\text{Id}_{p,q}$  is the diagonal matrix  $\text{Diag}(\text{Id}_p, -\text{Id}_q)$ .

We work with the maximal compact subgroup  $K = U(p) \times U(q) \subset G$ . We have the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , where  $\mathfrak{p}$  is identified with the vector space  $M_{p,q}$  of  $p \times q$  matrices through the map

$$X \in M_{p,q} \longmapsto \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}.$$

We work with the element  $z_{p,q} = \frac{i}{2} \text{Id}_{p,q}$  which belongs to the center of  $\mathfrak{k}$ . The adjoint action of  $z_{p,q}$  on  $\mathfrak{p}$  corresponds to the standard complex structure on  $M_{p,q}$ .

The trace on  $\mathfrak{gl}_{p+q}(\mathbb{C})$  defines an identification  $\mathfrak{g} \simeq \mathfrak{g}^* = \text{hom}(\mathfrak{g}, \mathbb{R})$ : to  $X \in \mathfrak{g}$  we associate  $\xi_X \in \mathfrak{g}^*$  defined by  $\langle \xi_X, Y \rangle = -\text{Tr}(XY)$ . Thus, the  $G$ -invariant cone  $\mathcal{C}_{G/K}$  defined by  $z_{p,q}$  can be viewed as the following cone of  $\mathfrak{g}$ :

$$\mathcal{C}(p, q) = \{X \in \mathfrak{g}, \text{Im}(\text{Tr}(gXg^{-1}\text{Id}_{p,q})) \geq 0, \forall g \in U(p, q)\}.$$

Let  $T \subset U(p) \times U(q)$  be the maximal torus formed by the diagonal matrices. The Lie algebra  $\mathfrak{t}$  is identified with  $\mathbb{R}^p \times \mathbb{R}^q$  through the map  $\mathbf{d} : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathfrak{u}(p) \times \mathfrak{u}(q) : \mathbf{d}_x = \text{Diag}(ix_1, \dots, ix_p, ix_{p+1}, \dots, ix_{p+q})$ . The Weyl chamber is

$$\mathfrak{t}_{\geq 0} = \{x \in \mathbb{R}^p \times \mathbb{R}^q, x_1 \geq \dots \geq x_p \text{ and } x_{p+1} \geq \dots \geq x_{p+q}\}.$$

Proposition 2.2 tells us that the  $U(p, q)$  adjoint orbits in the interior of  $\mathcal{C}(p, q)$  are parametrized by the holomorphic chamber

$$\mathcal{C}_{p,q} = \{x \in \mathbb{R}^p \times \mathbb{R}^q, x_1 \geq \dots \geq x_p > x_{p+1} \geq \dots \geq x_{p+q}\} \subset \mathfrak{t}_{\geq 0}.$$

**Definition 6.1** *The holomorphic Horn cone  $\text{Horn}_{\text{hol}}(p, q) := \text{Horn}_{\text{hol}}^2(U(p, q))$  is defined by the relations*

$$\text{Horn}_{\text{hol}}(p, q) = \{(A, B, C) \in (\mathcal{C}_{p,q})^3, U(p, q)\mathbf{d}_C \subset U(p, q)\mathbf{d}_A + U(p, q)\mathbf{d}_B\},$$

Let us detail the description given of  $\text{Horn}_{\text{hol}}(p, q)$  by Theorem B. For any  $n \geq 1$ , we consider the semigroup  $\wedge_n^+ = \{(\lambda_1 \geq \dots \geq \lambda_n)\} \subset \mathbb{Z}^n$ . If  $\lambda = (\lambda', \lambda'') \in \wedge_p^+ \times \wedge_q^+$ , then  $V_\lambda := V_{\lambda'}^{U(p)} \otimes V_{\lambda''}^{U(q)}$  denotes the irreducible representation of  $U(p) \times U(q)$  with highest weight  $\lambda$ . We denote by  $\text{Sym}(M_{p,q})$  the symmetric algebra of  $M_{p,q}$ .

**Definition 6.2** 1.  $\text{Horn}^{\mathbb{Z}}(p, q)$  is the semigroup of  $(\wedge_p^+ \times \wedge_q^+)^3$  defined by the conditions:

$$(\lambda, \mu, \nu) \in \text{Horn}^{\mathbb{Z}}(p, q) \iff [V_\nu : V_\lambda \otimes V_\mu \otimes \text{Sym}(M_{p,q})] \neq 0.$$

2.  $\text{Horn}(p, q)$  is the convex cone of  $(\mathfrak{t}_{\geq 0})^3$  defined as the closure of  $\mathbb{Q}^{>0} \cdot \text{Horn}^{\mathbb{Z}}(p, q)$ .

Theorem B asserts that

$$(20) \quad \text{Horn}_{\text{hol}}(p, q) = \text{Horn}(p, q) \cap (\mathcal{C}_{p,q})^3.$$

In another article [32], we obtained a recursive description of the cones  $\text{Horn}(p, q)$ . This allows us to give the following description of the holomorphic Horn cone  $\text{Horn}_{\text{hol}}(2, 2)$ .

**Example 6.3** An element  $(A, B, C) \in (\mathbb{R}^4)^3$  belongs to  $\text{Horn}_{\text{hol}}(2, 2)$  if and only if the following conditions holds:

$\begin{array}{l} a_1 \geq a_2 > a_3 \geq a_4 \\ b_1 \geq b_2 > b_3 \geq b_4 \\ c_1 \geq c_2 > c_3 \geq c_4 \end{array}$
--

$a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 + b_4 = c_1 + c_2 + c_3 + c_4$
---

$a_1 + a_2 + b_1 + b_2 \leq c_1 + c_2$
--

$\begin{array}{l} a_2 + b_2 \leq c_2 \\ a_2 + b_1 \leq c_1 \\ a_1 + b_2 \leq c_1 \end{array}$
---

$\begin{array}{l} a_3 + b_3 \geq c_3 \\ a_3 + b_4 \geq c_4 \\ a_4 + b_3 \geq c_4 \end{array}$
---

$\begin{array}{l} a_2 + a_4 + b_2 + b_4 \leq c_1 + c_4 \\ a_2 + a_4 + b_2 + b_4 \leq c_2 + c_3 \\ a_2 + a_4 + b_1 + b_4 \leq c_1 + c_3 \\ a_1 + a_4 + b_2 + b_4 \leq c_1 + c_3 \\ a_2 + a_4 + b_2 + b_3 \leq c_1 + c_3 \\ a_2 + a_3 + b_2 + b_4 \leq c_1 + c_3 \end{array}$
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