Horn problem for quasi-hermitian Lie groups

Paul-Emile Paradan

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June 13, 2020

Abstract

In this paper, we prove some convexity results associated to orbit projection of non-compact real reductive Lie groups.

Contents

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1 Introduction

This paper is concerned with convexity properties associated to orbit projection.

Let us consider two Lie groups $G \subset \tilde{G}$ with Lie algebras $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ and corresponding projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}}^* \to \mathfrak{g}^*$. A longstanding problem has been to understand how a coadjoint orbit $\tilde{\mathcal{O}} \subset \tilde{\mathfrak{g}}^*$ decomposes under the projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}$. For this purpose, we may define

$$\Delta_G(\tilde{\mathcal{O}}) = \{\mathcal{O} \in \mathfrak{g}^*/G\,;\ \mathcal{O} \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{\mathcal{O}})\}.$$

When the Lie group G is compact and connected, the set $\mathfrak{g}^*/G$ admits a natural identification with a Weyl chamber $\mathfrak{t}^*_{\geq 0}$. In this context, we have the well-known convexity theorem [12, 1, 10, 16, 13, 34, 22].

Theorem 1.1 *Suppose that G is compact connected and that the projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}$ is proper when restricted to $\tilde{\mathcal{O}}$. Then $\Delta_G(\tilde{\mathcal{O}}) = \{\xi \in \mathfrak{t}^*_{\geq 0}\,;\ G\xi \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{\mathcal{O}})\}$ is a closed convex locally polyhedral subset of $\mathfrak{t}^*$.*

When the Lie group $\tilde{G}$ is also compact and connected, we may consider

$$\Pi(\tilde{G}, G) := \left\{(\tilde{\xi}, \xi) \in \tilde{\mathfrak{t}}^*_{\geq 0} \times \mathfrak{t}^*_{\geq 0};\ G\xi\ \subset\ \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}\big(\tilde{G}\tilde{\xi}\,\big)\right\}. \tag{1}$$

Here is another convexity theorem [14, 17, 4, 2, 3, 25, 20, 35].

Theorem 1.2 *Suppose that $G \subset \tilde{G}$ are compact connected Lie groups. Then $\Pi(\tilde{G}, G)$ is a closed convex polyhedral cone and we can parametrize its facets by cohomological means (i.e. Schubert calculus).*

In this article we obtain an extension of Theorems 1.1 and 1.2 in the case where the two groups G and $\tilde{G}$ are not compact.

Let us explain in which framework we work. We suppose that $\tilde{G}/\tilde{K}$ is a Hermitian symmetric space of a non-compact type. Among the elliptic coadjoint orbits of $\tilde{G}$, some of them are naturally Kähler $\tilde{K}$-manifolds. These orbits are called the holomorphic coadjoint orbits of $\tilde{G}$. They are the strongly elliptic coadjoint orbits closely related to the holomorphic discrete series of Harish-Chandra. These orbits intersect the Weyl chamber $\tilde{\mathfrak{t}}^*_{\geq 0}$ of $\tilde{K}$

into a sub-chamber $\tilde{\mathcal{C}}_{\rm hol}$ called the holomorphic chamber. The basic fact here is that the union

$$\mathcal{C}^0_{\tilde{G}/\tilde{K}} := \bigcup_{\tilde{a}\in\tilde{\mathcal{C}}_{\rm hol}} \tilde{G}\tilde{a}$$

is an open invariant convex cone of $\tilde{\mathfrak{g}}^*$. See §2.1 for more details.

Let $G/K \subset \tilde{G}/\tilde{K}$ be a sub-Hermitian symmetric space of a non-compact type. As the projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}}^* \to \mathfrak{g}^*$ sends the convex cone $\mathcal{C}^0_{\tilde{G}/\tilde{K}}$ inside the convex cone $\mathcal{C}^0_{G/K}$, it is natural to study the following object reminiscent of (1) :

$$\Pi_{\rm hol}(\tilde{G}, G) := \Big\{(\tilde{\xi}, \xi) \in \tilde{\mathcal{C}}_{\rm hol} \times \mathcal{C}_{\rm hol};\ G\xi \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}\big(\tilde{G}\tilde{\xi}\,\big)\Big\}. \tag{2}$$

Let $\tilde{\mu} \in \tilde{\mathcal{C}}_{\rm hol}$. We will also give a particular attention to the intersection of $\Pi_{\rm hol}(\tilde{G}, G)$ with the hyperplane $\tilde{\xi} = \tilde{\mu}$, that is to say

$$\Delta_G(\tilde{G}\tilde{\mu}) := \Big\{\xi \in \mathcal{C}_{\rm hol};\ G\xi \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}\big(\tilde{G}\tilde{\mu}\,\big)\Big\}. \tag{3}$$

Consider the case where G is embedded diagonally in $\tilde{G} := G^s$ for $s \geq 2$. The corresponding set $\Pi_{\rm hol}(G^s, G)$ is called the holomorphic Horn cone, and it is defined as follows

$$\mathrm{Horn}^s_{\rm hol}(G) := \Big\{(\xi_1, \cdots, \xi_{s+1}) \in \mathcal{C}^{s+1}_{\rm hol};\ G\xi_{s+1} \subset \sum_{j=1}^{s} G\xi_j\Big\}.$$

The first result of this article is the following Theorem.

Theorem A.

- $\Pi_{\rm hol}(\tilde{G}, G)$ is a closed convex cone of $\tilde{\mathcal{C}}_{\rm hol} \times \mathcal{C}_{\rm hol}$.
- $\mathrm{Horn}^s_{\rm hol}(G)$ is a closed convex cone of $\mathcal{C}^{s+1}_{\rm hol}$ for any $s \geq 2$.

We obtain the following corollary which corresponds to a result of Weinstein [37].

Corollary. *For any $\tilde{\mu} \in \tilde{\mathcal{C}}_{\rm hol}$, $\Delta_G(\tilde{G}\tilde{\mu})$ is a closed and convex subset of $\mathcal{C}_{\rm hol}$.*

A first description of the closed convex cone $\Pi_{\rm hol}(\tilde{G}, G)$ goes as follows. The quotient $\mathfrak{q}$ of the tangent spaces $\mathbf{T}_e G/K$ and $\mathbf{T}_e\tilde{G}/\tilde{K}$ has a natural structure of a Hermitian K-vector space. The symmetric algebra $\mathrm{Sym}(\mathfrak{q})$ of $\mathfrak{q}$ defines an admissible K-module. The irreducible representations of K (resp. $\tilde{K}$) are parametrized by a semi-group $\wedge^*_+$: (resp. $\tilde{\wedge}^*_+$). For any $\lambda \in \wedge^*_+$ (resp. $\tilde{\lambda} \in \tilde{\wedge}^*_+$), we denote by V^K_λ (resp. $V^{\tilde{K}}_{\tilde{\lambda}}$) the irreducible representation of K (resp. $\tilde{K}$) with highest weight λ (resp. $\tilde{\lambda}$). If E is a representation of K, we denote by $\big[V^K_\lambda : E\big]$ the multiplicity of V^K_λ in E.

Definition 1.3 *1. $\Pi^{\mathbb{Z}}(\tilde{G}, G)$ is the semigroup of $\tilde{\wedge}^*_+ \times \wedge^*_+$ defined by the conditions:*

$$(\tilde{\lambda}, \lambda) \in \Pi^{\mathbb{Z}}(\tilde{G}, G) \Longleftrightarrow \Big[V^K_\lambda\ :\ V^{\tilde{K}}_{\tilde{\lambda}} \otimes \mathrm{Sym}(\mathfrak{q})\Big] \neq 0.$$

2. $\Pi(\tilde{G},G)$ *is the convex cone defined as the closure of* $\mathbb{Q}^{>0}\cdot\Pi^{\mathbb{Z}}(\tilde{G},G)$.

The second result of this article is the following Theorem.

Theorem B. *We have the equality*

$$\Pi_{\rm hol}(\tilde{G},G)=\Pi(\tilde{G},G)\bigcap\tilde{\mathcal{C}}_{\rm hol}\times\mathcal{C}_{\rm hol}. \tag{4}$$

A natural question is the description of the facets of the convex cone $\Pi_{\rm hol}(\tilde{G},G)$. In order to do that, we consider the group $\tilde{K}$ endowed with the following $\tilde{K}\times K$-action : $(\tilde{k},k)\cdot\tilde{a}=\tilde{k}\tilde{a}k^{-1}$. The cotangent space $\mathbf{T}^*\tilde{K}$ is then a symplectic manifold equipped with a Hamiltonian action of $\tilde{K}\times K$. We consider now the Hamiltonian $\tilde{K}\times K$-manifold $\mathbf{T}^*\tilde{K}\times\mathfrak{q}$, and we denote by $\Delta(\mathbf{T}^*\tilde{K}\times\mathfrak{q})$ the corresponding Kirwan polyhedron.

Let $W=N(T)/T$ be the Weyl group of (K,T) and let w_0 be the longest Weyl group element. Define an involution $*:\mathfrak{t}^*\to\mathfrak{t}^*$ by $\xi^*=-w_0\xi$. A standard result permits to affirm that $(\tilde{\xi},\xi)\in\Pi(\tilde{G},G)$ is an only if $(\tilde{\xi},\xi^*)\in\Delta(\mathbf{T}^*\tilde{K}\times\mathfrak{q})$ (see §4.2).

We obtain then another version of Theorem B.

Theorem B, second version. *An element* $(\tilde{\xi},\xi)$ *belongs to* $\Pi_{\rm hol}(\tilde{G},G)$ *if and only if*

$$(\tilde{\xi},\xi)\in\tilde{\mathcal{C}}_{\rm hol}\times\mathcal{C}_{\rm hol}\quad\text{and}\quad(\tilde{\xi},\xi^*)\in\Delta(\mathbf{T}^*\tilde{K}\times\mathfrak{q}).$$

Thanks to the second version of Theorem B, a natural way to describe the facets of the cone $\Pi_{\rm hol}(\tilde{G},G)$ is to exhibit those of the Kirwan polyhedron $\Delta(\mathbf{T}^*\tilde{K}\times\mathfrak{q})$. In this later case, it can be done using Ressayre's data (see §5).

The second version of Theorem B permits also the following description of the convex subsets $\Delta_G(\tilde{G}\tilde{\mu})$, $\tilde{\mu}\in\tilde{\mathcal{C}}_{\rm hol}$. Let $\Delta_K(\tilde{K}\tilde{\mu}\times\overline{\mathfrak{q}})$ be the Kirwan polyhedron associated to the Hamiltonian action of K on $\tilde{K}\tilde{\mu}\times\overline{\mathfrak{q}}$. Here $\overline{\mathfrak{q}}$ denotes the K-module $\mathfrak{q}$ with opposite complex structure.

Theorem C. *For any* $\tilde{\mu}\in\tilde{\mathcal{C}}_{\rm hol}$, *we have* $\Delta_G(\tilde{G}\tilde{\mu})=\Delta_K(\tilde{K}\tilde{\mu}\times\overline{\mathfrak{q}})$.

Let us detail Theorem C in the case where G is embedded in $\tilde{G}=G\times G$ diagonally. We denote by $\mathfrak{p}$ the K-Hermitian space $\mathbf{T}_eG/K$. Let κ be the Killing form of the Lie algebra $\mathfrak{g}$. The vector space $\overline{\mathfrak{p}}$ is equipped with the symplectic 2-form $\Omega_{\overline{\mathfrak{p}}}(X,Y)=-\kappa(z,[X,Y])$ and the compatible complex structure $-\mathrm{ad}(z)$.

Let us denote by $\Delta_K(K\mu_1\times K\mu_2\times\overline{\mathfrak{p}})$ and by $\Delta_K(\overline{\mathfrak{p}})$ the Kirwan polyhedrons relative to the Hamiltonian actions of K on $K\mu_1\times K\mu_2\times\overline{\mathfrak{p}}$ and on $\overline{\mathfrak{p}}$. Theorem C says that for any $\mu_1,\mu_2\in\mathcal{C}_{\rm hol}$, the convex set $\Delta_G(G\mu_1\times G\mu_2)$ is equal to the Kirwan polyhedron $\Delta_K(K\mu_1\times K\mu_2\times\overline{\mathfrak{p}})$.

To any non-empty subset C of a real vector space E, we may associate its asymptotic cone $\mathrm{As}(C) \subset E$ which is the set formed by the limits $y = \lim_{k\to\infty} t_k y_k$, where (t_k) is a sequence of non-negative reals converging to 0 and $y_k \in C$.

We finally get the following description of the asymptotic cone of $\Delta_G(G\mu_1 \times G\mu_2)$.

Corollary D. *For any $\mu_1, \mu_2 \in \mathcal{C}_{\mathrm{hol}}$, the asymptotic cone of $\Delta_G(G\mu_1 \times G\mu_2)$ is equal to $\Delta_K(\overline{\mathfrak{p}})$.*

In [28] §5, we explained how to describe the cone $\Delta_K(\overline{\mathfrak{p}})$ in terms of strongly orthogonal roots.

Let us finish this introduction with few remarks on related works:

- When G is compact, equal to the maximal compact subgroup $\tilde{K}$ of $\tilde{G}$, the results of Theorems B and C were already obtained by G. Deltour in his thesis [6, 7].
- In [9], A. Eshmatov and P. Foth proposed a description of the set $\Delta_G(G\mu_1 \times G\mu_2)$. **But their computations give not the same result as ours**. From their main result (Theorem 3.2) it follows that the asymptotic cone of $\Delta_G(G\mu_1 \times G\mu_2)$ is equal to the intersection of the Kirwan polyhedron $\Delta_T(\overline{\mathfrak{p}})$ with the Weyl chamber $\mathfrak{t}^*_{\geq 0}$. But since $\Delta_K(\overline{\mathfrak{p}}) \neq \Delta_T(\overline{\mathfrak{p}}) \cap \mathfrak{t}^*_{\geq 0}$ in general, it is in contradiction with Corollary D.

Notations

In this paper, G denotes a connected real reductive Lie group : we take here the convention of Knapp [18]. We have a Cartan involution Θ on G, such that the fixed point set $K := G^\Theta$ is a connected maximal compact subgroup. We have Cartan decompositions at the level of Lie algebras $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and at the level of the group $G \simeq K \times \exp(\mathfrak{p})$. We denote by b a G-invariant non-degenerate bilinear form on $\mathfrak{g}$ that is equal to the Killing form on $[\mathfrak{g}, \mathfrak{g}]$, and that defines a K-invariant scalar product $(X, Y) := -b(X, \Theta(Y))$. We will use the K-equivariant identification $\xi \mapsto \tilde{\xi}$, $\mathfrak{g}^* \simeq \mathfrak{g}$ defined by $(\tilde{\xi}, X) := \langle \xi, X \rangle$ for $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.

When a Lie group H acts on a manifold N, the stabilizer subgroup of $n \in N$ is denoted by $H_n = \{g \in G, gn = n\}$, and its Lie algebra by $\mathfrak{h}_n$. Let us define

$$\dim_H(\mathcal{X}) = \min_{n\in\mathcal{X}} \dim(\mathfrak{h}_n) \tag{5}$$

for any subset $\mathcal{X} \subset N$.

2 The cone $\Pi_{\mathrm{hol}}(\tilde{G}, G)$: first properties

We assume here that G/K is a Hermitian symmetric space, that is to say, there exists a G-invariant complex structure on the manifold G/K, or equivalently there exists a K-invariant element $z \in \mathfrak{k}$ such that $\mathrm{ad}(z)|_{\mathfrak{p}}$ defines a complex structure on $\mathfrak{p}$: $(\mathrm{ad}(z)|_{\mathfrak{p}})^2 = -\mathrm{Id}_{\mathfrak{p}}$. This condition imposes that the ranks of G and K are equal.

We are interested in the following closed invariant convex cone of $\mathfrak{g}^*$:

$$\mathcal{C}_{G/K} = \{\xi \in \mathfrak{g}^*, \langle \xi, gz\rangle \geq 0,\ \forall g \in G\}\,.$$

2.1 The holomorphic chamber

Let T be a maximal torus of K, with Lie algebra $\mathfrak{t}$. Its dual $\mathfrak{t}^*$ can be seen as the subspace of $\mathfrak{g}^*$ fixed by T. Let us denote by $\mathfrak{g}^*_e$ the set formed by the elliptic elements: in other words $\mathfrak{g}^*_e := \mathrm{Ad}^*(G)\cdot \mathfrak{t}^*$.

Following [37], we consider the invariant open subset $\mathfrak{g}^*_{se} = \{\xi \in \mathfrak{g}^* \,|\, G_\xi \text{ is compact}\}$ of *strongly elliptic* elements. It is non-empty since the groups G and K have the same rank.

We start with the following basic facts.

Lemma 2.1
- *$\mathfrak{g}^*_{se}$ is contained in $\mathfrak{g}^*_e$.*
- *The interior $\mathcal{C}^0_{G/K}$ of the cone $\mathcal{C}_{G/K}$ is contained in $\mathfrak{g}^*_{se}$.*

Proof : The first point is due to the fact that every compact subgroup of G is conjugate to a subgroup of K.

Let $\xi \in \mathcal{C}^0_{G/K}$. There exists $\epsilon > 0$ so that

$$\langle \xi + \eta, gz\rangle \geq 0, \quad \forall g \in G, \quad \forall \|\eta\| \leq \epsilon.$$

It implies that $|\langle \eta, gz\rangle| \leq \langle \xi, z\rangle$, $\forall g \in G_\xi$ and $\forall \|\eta\| \leq \epsilon$. In other words, the adjoint orbit $G_\xi \cdot z \subset \mathfrak{g}$ is bounded. For any $g = e^X k$, with $(X, k) \in \mathfrak{p} \times K$, a direct computation shows that $\|gz\| = \|e^X z\| \geq \|[z, X]\| = \|X\|$. Then, there exists $\rho > 0$ such that $\|X\| \leq \rho$ if $e^X k \in G_\xi$ for some $k \in K$. It follows that the stabilizer subgroup G_ξ is compact. $\Box$

Let $\wedge^* \subset \mathfrak{t}^*$ be the weight lattice : $\alpha \in \wedge^*$ if $i\alpha$ is the differential of a character of T. Let $\mathfrak{R} \subset \wedge^*$ be the set of roots for the action of T on $\mathfrak{g} \otimes \mathbb{C}$. We have $\mathfrak{R} = \mathfrak{R}_c \cup \mathfrak{R}_n$ where $\mathfrak{R}_c$ and $\mathfrak{R}_n$ are respectively the set of roots for the action of T on $\mathfrak{k} \otimes \mathbb{C}$ and $\mathfrak{p} \otimes \mathbb{C}$. We fix a system of positive roots $\mathfrak{R}^+_c$ in $\mathfrak{R}_c$, and we denote by $\mathfrak{t}^*_{\geq 0}$ the corresponding Weyl chamber.

We have $\mathfrak{p} \otimes \mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where the K-module $\mathfrak{p}^\pm$ is equal to $\ker(\mathrm{ad}(z) \mp i)$. Let $\mathfrak{R}^{\pm,z}_n$ be the set of roots for the action of T on $\mathfrak{p}^\pm$. The union

$$\mathfrak{R}^+ = \mathfrak{R}^+_c \cup \mathfrak{R}^{+,z}_n \tag{6}$$

defines then a system of positive roots in $\mathfrak{R}$. We notice that $\mathfrak{R}^{+,z}_n$ is the set of roots $\beta \in \mathfrak{R}$ satisfying $\langle \beta, z\rangle = 1$. Hence $\mathfrak{R}^{+,z}_n$ is invariant relatively to the action of the Weyl group $W = N(T)/T$.

Let us recall the following classical fact concerning the parametrization of the G-orbits in $\mathcal{C}^0_{G/K}$ via the holomorphic chamber

$$\mathcal{C}_{\mathrm{hol}} := \{\xi \in \mathfrak{t}^*_{\geq 0}, (\xi, \beta) > 0,\ \forall \beta \in \mathfrak{R}^{+,z}_n\}.$$

Proposition 2.2 • *The set* $\mathcal{C}^0_{G/K} \cap \mathfrak{t}^*_{\geq 0}$ *is contained in* $\mathcal{C}_{\rm hol}$.

- *For any compact subset* $\mathcal{K}$ *of* $\mathcal{C}_{\rm hol}$, *there exists* $c_{\mathcal{K}} > 0$ *such that* $\langle \xi, gz\rangle \geq c_{\mathcal{K}}\|gz\|$, $\forall g \in G$, $\forall \xi \in \mathcal{K}$.

- *The map* $\mathcal{O} \mapsto \mathcal{O} \cap \mathfrak{t}^*_{\geq 0}$ *defines a bijective map between the set of* G*-orbits in* $\mathcal{C}^0_{G/K}$ *and the holomorphic chamber* $\mathcal{C}_{\rm hol}$.

Proof : Let $\xi \in \mathfrak{t}^*_{\geq 0} \cap \mathcal{C}^0_{G/K}$. The first point is proved if we check that $(\xi, \beta) > 0$ for any $\beta \in \mathfrak{R}^{+,z}_n$. Let $X_\beta, Y_\beta \in \mathfrak{p}$ such that $X_\beta + iY_\beta \in (\mathfrak{p} \otimes \mathbb{C})_\beta$. We choose the following normalization : the vector $h_\beta := [X_\beta, Y_\beta]$ satisfies $\langle \beta, h_\beta\rangle = 1$. We see then that $(\xi, \beta) = \frac{1}{\|h_\beta\|^2}\langle \xi, h_\beta\rangle$ for any $\xi \in \mathfrak{g}^*$. Standard computation [27] gives

$$e^{t\,{\rm ad}(X_\beta)}z = z + (\cosh(t) - 1)h_\beta + \sinh(t)Y_\beta, \quad \forall t \in \mathbb{R}.$$

By definition, we must have $\langle \xi + \eta, e^{t\,{\rm ad}(X_\beta)}z\rangle \geq 0, \forall t \in \mathbb{R}$, for any $\eta \in \mathfrak{t}^*$ small enough. It imposes that $\langle \xi, h_\beta\rangle > 0$. The first point is settled.

Now choose some maximal strongly orthogonal system $\Sigma \subset \mathfrak{R}^{+,z}_n$. The real span $\mathfrak{a}$ of the $X_\beta, \beta \in \Sigma$ is a maximal abelian subspace of $\mathfrak{p}$. Hence any element $g \in G$ can be written $g = ke^{X(t)}k'$ with $X(t) = \sum_{\beta\in\Sigma} t_\beta X_\beta$ and $k, k' \in K$. We get

$$gz = k\left(z + \sum_{\beta\in\Sigma}(\cosh(t_\beta) - 1)h_\beta + \sum_{\beta\in\Sigma}\sinh(t_\beta)Y_\beta\right) \tag{7}$$

and

$$\langle \xi, gz\rangle = \langle k^{-1}\xi, z\rangle + \sum_{\beta\in\Sigma}(\cosh(t_\beta) - 1)\langle k^{-1}\xi, h_\beta\rangle.$$

For any $\xi \in \mathcal{C}_{\rm hol}$, we define $c_\xi := \min_{\beta\in\mathfrak{R}^{+,z}_n}\langle \xi, h_\beta\rangle > 0$. Let $\pi : \mathfrak{k}^* \to \mathfrak{t}^*$ be the projection. We have $\langle k^{-1}\xi, z\rangle = \langle \pi(k^{-1}\xi), z\rangle$ and $\langle k^{-1}\xi, h_\beta\rangle = \langle \pi(k^{-1}\xi), h_\beta\rangle$. The convexity theorem of Kostant tell us that $\pi(k^{-1}\xi)$ belongs to the convex hull of $\{w\xi, w \in W\}$. It follows that $\langle k^{-1}\xi, z\rangle \geq \langle \xi, z\rangle > 0$ and $\langle k^{-1}\xi, h_\beta\rangle \geq c_\xi > 0$. We obtain then that $\langle \xi, gz\rangle \geq \frac{1}{2}\min(\langle \xi, z\rangle, c_\xi)e^{\|X(t)\|}$ for any $\xi \in \mathcal{C}_{\rm hol}$, where $\|X(t)\| = \sup|t_\beta|$. From (7), it is not difficult to see that there exists $C > 0$ such that $\|gz\| \leq Ce^{\|X(t)\|}$ for any $g = ke^{X(t)}k'$.

Let $\mathcal{K}$ be a compact subset of $\mathcal{C}_{\rm hol}$. Take $c_{\mathcal{K}} = \frac{1}{2C}\min(\min_{\xi\in\mathcal{K}}\langle \xi, z\rangle, \min_{\xi\in\mathcal{K}} c_\xi) > 0$. The previous computations show that $\langle \xi, gz\rangle \geq c_{\mathcal{K}}\|gz\|$, $\forall g \in G$, $\forall \xi \in \mathcal{K}$. The second point is proved.

The first two points show that $\mathfrak{t}^*_{\geq 0} \cap \mathcal{C}^0_{G/K} = \mathcal{C}_{\rm hol}$. The third point follows then from the fact that $\mathcal{C}^0_{G/K}$ is contained in $\mathfrak{g}^*_e$ (see Lemma 2.1). □

We have a canonical map ${\rm p} : \mathcal{C}^0_{G/K} \to \mathcal{C}_{\rm hol}$ defined by the relations $G\xi \cap \mathfrak{t}^*_{\geq 0} = \{{\rm p}(\xi)\}$, $\forall \xi \in \mathcal{C}^0_{G/K}$. The following result is needed in §4.1.

Lemma 2.3 *The map* p *is continuous.*

Proof : Let (ξ_n) be a sequence of $\mathcal{C}^0_{G/K}$ converging to $\xi_\infty \in \mathcal{C}^0_{G/K}$. Let $\xi'_n = \mathrm{p}(\xi_n)$ and $\xi'_\infty = \mathrm{p}(\xi_\infty)$: we have to prove that the sequence (ξ'_n) converges to ξ'_∞. We choose elements $g_n, g_\infty \in G$ such that $\xi_n = g_n\xi'_n, \forall n$, and $\xi_\infty = g_\infty\xi'_\infty$.

First we notice that $-b(\xi_n,\xi_n) = \|\xi'_n\|^2$, hence the sequence (ξ'_n) is bounded. We will now prove that the sequence (g_n) is bounded.

Let $\epsilon > 0$ such that $\langle \xi_\infty + \eta, gz\rangle \geq 0$, $\forall g \in G$, $\forall \|\eta\| \leq \epsilon$. If $\|\xi - \xi_\infty\| \leq \epsilon/2$, we write $\xi = \frac{1}{2}(\xi_\infty + 2(\xi - \xi_\infty)) + \frac{1}{2}\xi_\infty$, and then

$$\langle \xi, gz\rangle = \frac{1}{2}\langle \xi_\infty + 2(\xi-\xi_\infty), gz\rangle + \frac{1}{2}\langle \xi_\infty, gz\rangle \geq \frac{1}{2}\langle \xi_\infty, gz\rangle, \quad \forall g \in G.$$

Now we have $\langle \xi'_n, z\rangle = \langle \xi_n, g_n z\rangle \geq \frac{1}{2}\langle \xi_\infty, g_n z\rangle$ if n is large enough. This shows that the sequence $\langle \xi_\infty, g_n z\rangle$ is bounded. If we use the second point of Proposition 2.2, we can conclude that the sequence (g_n) is bounded.

Let $(\xi'_{\phi(n)})$ be a subsequence converging to $\ell \in \mathfrak{t}^*_{\geq 0}$. Since $(g_{\phi(n)})$ is bounded, there exists a subsequence $(g_{\phi\circ\psi(n)})$ converging to $h \in G$. From the relations $\xi_{\phi\circ\psi(n)} = g_{\phi\circ\psi(n)}\xi'_{\phi\circ\psi(n)}, \forall n \in \mathbb{N}$, we obtain $\xi_\infty = h\ell$. Then $\ell = \mathrm{p}(\xi_\infty) = \xi'_\infty$. Since every subsequence of (ξ'_n) has a subsequential limit to ξ'_∞, then the sequence (ξ'_n) converges to ξ'_∞. $\square$

2.2 The cone $\Pi_{\mathrm{hol}}(\tilde{G}, G)$ is closed

We suppose that G/K is a complex submanifold of a Hermitian symmetric space $\tilde{G}/\tilde{K}$. In other words, $\tilde{G}$ is a reductive real Lie group such that $G \subset \tilde{G}$ is a closed subgroup preserved by the Cartan involution, and $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$ containing K. We denote by $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{k}}$ the Lie algebras of $\tilde{G}$ and $\tilde{K}$ respectively. We suppose that there exists a $\tilde{K}$-invariant element $z \in \mathfrak{k}$ such that $\mathrm{ad}(z)|_{\tilde{\mathfrak{p}}}$ defines a complex structure on $\tilde{\mathfrak{p}}$: $(\mathrm{ad}(z)|_{\tilde{\mathfrak{p}}})^2 = -Id_{\tilde{\mathfrak{p}}}$.

Let $\mathcal{C}_{\tilde{G}/\tilde{K}} \subset \tilde{\mathfrak{g}}^*$ be the closed invariant cone associated to the Hermitian symmetric space $\tilde{G}/\tilde{K}$. We start with the following key fact.

Lemma 2.4 *The projection* $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}}^* \to \mathfrak{g}^*$ *sends* $\mathcal{C}^0_{\tilde{G}/\tilde{K}}$ *into* $\mathcal{C}^0_{G/K}$.

Proof : Let $\tilde{\xi} \in \mathcal{C}^0_{\tilde{G}/\tilde{K}}$ and $\xi = \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{\xi})$. Then $\langle \tilde{\xi} + \tilde{\eta}, \tilde{g}z\rangle \geq 0$, $\forall \tilde{g} \in \tilde{G}$ if $\tilde{\eta} \in \tilde{\mathfrak{g}}^*$ is small enough. It follows that $\langle \xi + \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{\eta}), gz\rangle = \langle \tilde{\xi} + \tilde{\eta}, gz\rangle \geq 0$, $\forall g \in G$ if $\tilde{\eta}$ is small enough. Since $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}$ is an open map, we can conclude that $\xi \in \mathcal{C}^0_{G/K}$. $\square$

Let $\tilde{T}$ be a maximal torus of $\tilde{K}$, with Lie algebra $\tilde{\mathfrak{t}}$. The $\tilde{G}$-orbits in the interior of $\mathcal{C}_{\tilde{G}/\tilde{K}}$ are parametrized by the holomorphic chamber $\tilde{\mathcal{C}}_{\mathrm{hol}} \subset \tilde{\mathfrak{t}}^*$. The previous Lemma says that the projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{\mathcal{O}})$ of any $\tilde{G}$-orbit $\tilde{\mathcal{O}} \subset \mathcal{C}^0_{\tilde{G}/\tilde{K}}$ is the union of G-orbits $\mathcal{O} \subset \mathcal{C}^0_{G/K}$.

So it is natural to study the following object

$$\Pi_{\rm hol}(\tilde{G},G) := \left\{(\tilde{\xi},\xi)\in\tilde{\mathcal{C}}_{\rm hol}\times\mathcal{C}_{\rm hol};\ G\xi\ \subset\ \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}\big(\tilde{G}\tilde{\xi}\,\big)\right\}. \tag{8}$$

Here is a first result.

Proposition 2.5 $\Pi_{\rm hol}(\tilde{G},G)$ *is a closed cone of* $\tilde{\mathcal{C}}_{\rm hol}\times\mathcal{C}_{\rm hol}$.

Proof : Suppose that a sequence $(\tilde{\xi}_n,\xi_n)\in\Pi_{\rm hol}(\tilde{G},G)$ converge to $(\tilde{\xi}_\infty,\xi_\infty)\in\tilde{\mathcal{C}}_{\rm hol}\times\mathcal{C}_{\rm hol}$. By definition, there exists a sequence $(\tilde{g}_n,g_n)\in\tilde{G}\times G$ such that $g_n\xi_n=\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{g}_n\tilde{\xi}_n)$. Let $\tilde{h}_n := g_n^{-1}\tilde{g}_n$, so that $\xi_n=\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{h}_n\tilde{\xi}_n)$ and $\langle\tilde{h}_n\tilde{\xi}_n,z\rangle=\langle\xi_n,z\rangle$. We use now that the sequence $\langle\xi_n,z\rangle$ is bounded, and that the sequence $\tilde{\xi}_n$ belongs to a compact subset of $\tilde{\mathcal{C}}_{\rm hol}$. Thanks to the second point of Proposition 2.2, these facts imply that $\|\tilde{h}_n^{-1}z\|$ is a bounded sequence. Hence $\tilde{h}_n$ admits a subsequence converging to $\tilde{h}_\infty$. So we get $\xi_\infty=\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{h}_\infty\tilde{\xi}_\infty)$, and that proves that $(\tilde{\xi}_\infty,\xi_\infty)\in\Pi_{\rm hol}(\tilde{G},G)$. $\Box$

2.3 Rational and weakly regular points

Let (M,Ω) be a symplectic manifold. We suppose that there exists a line bundle $\mathcal{L}$ with connection ∇ that prequantizes the 2-form Ω: in other words, $\nabla^2=-i\,\Omega$. Let K be a compact connected Lie group acting on $\mathcal{L}\to M$, and leaving the connection invariant. Let $\Phi_K : M\to\mathfrak{k}^*$ be the moment map defined by Kostant's relations

$$L_X-\nabla_X=i\langle\Phi_K,X\rangle,\quad \forall X\in\mathfrak{k}. \tag{9}$$

Here L_X is the Lie derivative acting on the sections of $\mathcal{L}\to M$.

Remark that relations (9) imply, via the equivariant Bianchi formula, the relations

$$\iota(X_M)\Omega=-d\langle\Phi_K,X\rangle,\quad \forall X\in\mathfrak{k}, \tag{10}$$

where $X_M(m):=\frac{d}{dt}|_{t=0}e^{-tX}m$ is the vector field on M generated by $X\in\mathfrak{k}$.

Definition 2.6 *Let* $\dim_K(M):=\min_{m\in M}\dim\mathfrak{k}_m$. *An element* $\xi\in\mathfrak{k}^*$ *is a weakly-regular value of* Φ_K *if for all* $m\in\Phi_K^{-1}(\xi)$ *we have* $\dim\mathfrak{k}_m=\dim_K(M)$.

When $\xi\in\mathfrak{k}^*$ is a weakly-regular value of Φ_K, the constant rank theorem tells us that $\Phi_K^{-1}(\xi)$ is a submanifold of M stable under the action of the stabilizer subgroup K_ξ. We see then that the reduced space

$$M_\xi:=\Phi_K^{-1}(\xi)/K_\xi \tag{11}$$

is a symplectic orbifold.

Let $T\subset K$ be a maximal torus with Lie algebra $\mathfrak{t}$. We consider the lattice $\wedge := \frac{1}{2\pi}\ker(\exp:\mathfrak{t}\to T)$ and the dual lattice $\wedge^*\subset\mathfrak{t}^*$ defined by $\wedge^*=\hom(\wedge,\mathbb{Z})$. We remark that $i\eta$ is a differential of a character of T if and only if $\eta \in \wedge^*$. The $\mathbb{Q}$-vector space generated by the lattice $\wedge^*$ is denoted by $\mathfrak{t}^*_{\mathbb{Q}}$: the vectors belonging to $\mathfrak{t}^*_{\mathbb{Q}}$ are designed as rational. An affine subspace $V \subset \mathfrak{t}^*$ is called rational if it is affinely generated by its rational points.

We also fix a closed positive Weyl chamber $\mathfrak{t}^*_{\geq 0}$. For each relatively open face $\sigma \subset \mathfrak{t}^*_{\geq 0}$, the stabilizer K_ξ of points $\xi \in \sigma$ under the coadjoint action, does not depend on ξ, and will be denoted K_σ. The Lie algebra $\mathfrak{g}_\sigma$ decomposes into its semi-simple and central parts $\mathfrak{k}_\sigma = [\mathfrak{k}_\sigma, \mathfrak{k}_\sigma] \oplus \mathfrak{z}_\sigma$. The subspace $\mathfrak{z}^*_\sigma$ is defined to be the annihilator of $[\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]$, or equivalently the fixed point set of the coadjoint K_σ action. Notice that $\mathfrak{z}^*_\sigma$ is a rational subspace of $\mathfrak{t}^*$, and that the face σ is an open cone of $\mathfrak{z}^*_\sigma$,

We suppose that the moment map Φ_K is *proper*. The Convexity Theorem [1, 10, 16, 34, 22] tells us that $\Delta_K(M) := \mathrm{Image}(\Phi_K) \bigcap \mathfrak{t}^*_{\geq 0}$ is a closed, convex, locally polyhedral set.

Definition 2.7 *We denote by $\Delta_K(M)^0$ the subset of $\Delta_K(M)$ formed by the* weakly-regular values *of the moment map Φ_K contained in $\Delta_K(M)$.*

We will use the following remark in the next sections.

Lemma 2.8 *The subset $\Delta_K(M)^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_K(M)$.*

Proof : Let us first explain why $\Delta_K(M)^0$ is a dense open subset of $\Delta_K(M)$. There exists a unique open face τ of the Weyl chamber $\mathfrak{t}^*_{\geq 0}$ such as $\Delta_K(M) \cap \tau$ is dense in $\Delta_K(M)$: τ is called the *principal* face in [22]. The Principal-cross-section Theorem [22] tells us that $Y_\tau := \Phi^{-1}(\tau)$ is a symplectic K_τ-manifold, with a trivial action of $[K_\tau, K_\tau]$. The line bundle $\mathcal{L}_\tau := \mathcal{L}|_{Y_\tau}$ prequantizes the symplectic structure on Y_τ, and relations (10) show that $[K_\tau, K_\tau]$ acts trivially on $\mathcal{L}_\tau$. Moreover the restriction of Φ_K on Y_τ is the moment map $\Phi_\tau : Y_\tau \to \mathfrak{z}^*_\tau$ associated to the action of the torus $Z_\tau = \exp(\mathfrak{z}_\tau)$ on $\mathcal{L}_\tau$.

Let $I \subset \mathfrak{z}^*_\tau$ be the smallest affine subspace containing $\Delta_K(M)$. Let $\mathfrak{z}_I \subset \mathfrak{z}_\tau$ be the annihilator of the subspace parallel to I : relations (10) show that $\mathfrak{z}_I$ is the generic infinitesimal stabilizer of the $\mathfrak{z}_\tau$-action on Y_τ. Hence $\mathfrak{z}_I$ is the Lie algebra of the torus $Z_I := \exp(\mathfrak{z}_I)$.

We see then that any regular value of $\Phi_\tau : Y_\tau \to I$, viewed as a map with codomain I, is a weakly-regular value of the moment map Φ_K. It explains why $\Delta_K(M)^0$ is a dense open subset of $\Delta_K(M)$.

The convex set $\Delta_K(M) \cap \tau$ is equal to $\Delta_{Z_\tau}(Y_\tau) := \mathrm{Image}(\Phi_\tau)$, it is sufficient to check that $\Delta_{Z_\tau}(Y_\tau)^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_{Z_\tau}(Y_\tau)$. The subtorus $Z_I \subset Z_\tau$ acts trivially on Y_τ, and it acts on the line bundle $\mathcal{L}_\tau$ through a character χ. Let $\eta \in \wedge^* \cap \mathfrak{t}^*_\tau$ such that $d\chi = i\eta|_{\mathfrak{z}_I}$. The affine subspace I which is equal to $\eta + (\mathfrak{z}_I)^\perp$ is rational. Since the open subset $\Delta_{Z_\tau}(Y_\tau)^0$ generates the rational affine subspace I, we can conclude that $\Delta_{Z_\tau}(Y_\tau)^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_{Z_\tau}(Y_\tau)$. $\square$

2.4 Weinstein's theorem

Let $\tilde{a} \in \tilde{\mathcal{C}}_{\rm hol}$. Consider the Hamiltonian action of the group G on the coadjoint orbit $\tilde{G}\tilde{a}$. The moment map $\Phi^{\tilde{a}}_G : \tilde{G}\tilde{a} \to \mathfrak{g}^*$ corresponds to the restriction of the projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}$ to $\tilde{G}\tilde{a}$. In this setting, the following conditions holds :

1. The action of G on $\tilde{G}\tilde{a}$ is proper.
2. The moment map $\Phi^{\tilde{a}}_G$ is a proper map since the map $\langle \Phi^{\tilde{a}}_G, z\rangle$ is proper (see the second point of Proposition 2.2).

Conditions 1. and 2. imposes that the image of $\Phi^{\tilde{a}}_G$ is contained in the open subset $\mathfrak{g}^*_{se}$ of strongly elliptic elements [30]. Thus, the G-orbits contained in the image of $\Phi^{\tilde{a}}_G$ are parametrized by the following subset of the holomorphic chamber $\mathcal{C}_{\rm hol}$:

$$\Delta_G(\tilde{G}\tilde{a}) := \mathrm{Image}(\Phi^{\tilde{a}}_G) \bigcap \mathfrak{t}^*_{\geq 0}.$$

We notice that $\Pi_{\rm hol}(\tilde{G}, G) = \bigcup_{\tilde{a}\in\tilde{\mathcal{C}}_{\rm hol}} \{\tilde{a}\} \times \Delta_G(\tilde{G}\tilde{a})$.

Like in Definition 2.6, an element $\xi \in \mathfrak{g}^*$ is a *weakly-regular* value of $\Phi^{\tilde{a}}_G$ if for all $m \in (\Phi^{\tilde{a}}_G)^{-1}(\xi)$ we have $\dim \mathfrak{g}_m = \min_{x\in\tilde{G}\tilde{a}} \dim(\mathfrak{g}_x)$. We denote by $\Delta_G(\tilde{G}\tilde{a})^0$ the set of elements $\xi \in \Delta_G(\tilde{G}\tilde{a})$ that are weakly-regular for $\Phi^{\tilde{a}}_G$.

Theorem 2.9 (Weinstein) *For any $\tilde{a} \in \tilde{\mathcal{C}}_{\rm hol}$, $\Delta_G(\tilde{G}\tilde{a})$ is a closed convex subset contained in $\mathcal{C}_{\rm hol}$.*

Proof : We recall briefly the arguments of the proof (see [37] or [30][§2]). Under Conditions 1. and 2., one checks easily that $Y_{\tilde{a}} := (\Phi^{\tilde{a}}_G)^{-1}(\mathfrak{k}^*)$ is a smooth K-invariant symplectic submanifold of $\tilde{G}\tilde{a}$ such that

$$\tilde{G}\tilde{a} \simeq G \times_K Y_{\tilde{a}}. \tag{12}$$

The moment map $\Phi^{\tilde{a}}_K : Y_{\tilde{a}} \to \mathfrak{k}^*$, which corresponds to the restriction of the map $\Phi^{\tilde{a}}_G$ to $Y_{\tilde{a}}$, is a proper map. Hence the Convexity Theorem tells us that $\Delta_K(Y_{\tilde{a}}) := \mathrm{Image}(\Phi^{\tilde{a}}_K) \bigcap \mathfrak{t}^*_{\geq 0}$ is a closed, convex, locally polyhedral set. Thanks to the isomorphism (12) we see that $\Delta_G(\tilde{G}\tilde{a})$ coincides with the closed convex subset $\Delta_K(Y_{\tilde{a}})$. The proof is completed. $\Box$

The next Lemma is used in §4.1.

Lemma 2.10 *Let $\tilde{a} \in \tilde{\mathcal{C}}_{\rm hol}$ be a rational element. Then $\Delta_G(\tilde{G}\tilde{a})^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_G(\tilde{G}\tilde{a})$.*

Proof : Thanks to (12), we know that $\Delta_G(\tilde{G}\tilde{a}) = \Delta_K(Y_{\tilde{a}})$. Relation (12) shows also that $\Delta_G(\tilde{G}\tilde{a})^0 = \Delta_K(Y_{\tilde{a}})^0$. Let $N \geq 1$ such that $\tilde{\mu} = N\tilde{a} \in \wedge^* \cap \mathcal{C}_{\rm hol}$. The stabilizer subgroup $\tilde{G}_{\tilde{\mu}}$ is compact, equal to $\tilde{K}_{\tilde{\mu}}$. Let us denote by $\mathbb{C}_{\tilde{\mu}}$ the one-dimensional representation of $\tilde{K}_{\tilde{\mu}}$ associated to $\tilde{\mu}$. The convex set $\Delta_G(\tilde{G}\tilde{a})$ is equal to $\frac{1}{N}\Delta_G(\tilde{G}\tilde{\mu})$, so it is sufficient to check that $\Delta_G(\tilde{G}\tilde{\mu})^0 \cap \mathfrak{t}^*_{\mathbb{Q}} = \Delta_K(Y_{\tilde{\mu}})^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_G(\tilde{G}\tilde{\mu}) = \Delta_K(Y_{\tilde{\mu}})$. The coadjoint orbit $\tilde{G}\tilde{\mu}$ is prequantized by the line bundle $\tilde{G} \times_{K_{\tilde{\mu}}} \mathbb{C}_{\tilde{\mu}}$ and the symplectic slice $Y_{\tilde{\mu}}$ is prequantized by the line bundle $\mathcal{L}_{\tilde{\mu}} := \tilde{G} \times_{K_{\tilde{\mu}}} \mathbb{C}_{\tilde{\mu}}|_{Y_{\tilde{\mu}}}$. Thanks to Lemma 2.8, we know that $\Delta_K(Y_{\tilde{\mu}})^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_K(Y_{\tilde{\mu}})$: the proof is complete. $\Box$

3 Holomorphic discrete series

3.1 Definition

We return to the framework of §2.1. We recall the notion of holomorphic discrete series representations associated to a Hermitian symmetric spaces G/K. Let us introduce

$$\mathcal{C}_{\rm hol}^{\rho} := \left\{\xi \in \mathfrak{t}^*_{\geq 0} |\ (\xi, \beta) \geq (2\rho_n, \beta),\ \forall \beta \in \mathfrak{R}_n^{+,z}\right\}$$

where $2\rho_n = \sum_{\beta \in \mathfrak{R}_n^{+,z}} \beta$ is W-invariant.

Lemma 3.1 *1. We have $\mathcal{C}_{\rm hol}^{\rho} \subset \mathcal{C}_{\rm hol}$.*

2. For any $\xi \in \mathcal{C}_{\rm hol}$ there exists $N \geq 1$ such that $N\xi \in \mathcal{C}_{\rm hol}^{\rho}$.

Proof : The first point is due to the fact that $(\beta_0, \beta_1) \geq 0$ for any $\beta_0, \beta_1 \in \mathfrak{R}_n^{+,z}$. The second point is obvious. □

We will be interested in the following subset of dominants weights :

$$\widehat{G}_{\rm hol} := \wedge^*_+ \bigcap \mathcal{C}_{\rm hol}^{\rho}.$$

Let $\mathrm{Sym}(\mathfrak{p})$ be the symmetric algebra of the complex K-module $(\mathfrak{p}, \mathrm{ad}(z))$.

Theorem 3.2 (Harish-Chandra) *For any $\lambda \in \widehat{G}_{\rm hol}$, there exists an irreducible unitary representation of G, denoted V_λ^G, such that the vector space of K-finite vectors is $V_\lambda^G|_K := V_\lambda^K \otimes \mathrm{Sym}(\mathfrak{p})$.*

The set $V_\lambda^G, \lambda \in \widehat{G}_{\rm hol}$ corresponds to the holomorphic discrete series representations associated to the complex structure $\mathrm{ad}(z)$.

3.2 Restriction

We come back to the framework of §2.2. We consider two compatible Hermitian symmetric spaces $G/K \subset \tilde{G}/\tilde{K}$, and we look after the restriction of holomorphic discrete series representations of $\tilde{G}$ to the subgroup G.

Let $\tilde{\lambda} \in \widehat{\tilde{G}}_{\rm hol}$. Since the representation $V_{\tilde{\lambda}}^{\tilde{G}}$ is discretely admissible relatively the circle group $\exp(\mathbb{R}z)$, it is also discretely admissible relatively to G. We can be more precise [15, 24, 21] :

Proposition 3.3 *We have an Hilbertian direct sum*

$$V_{\tilde{\lambda}}^{\tilde{G}}|_G = \bigoplus_{\lambda \in \widehat{G}_{\rm hol}} m_{\tilde{\lambda}}^{\lambda}\ V_\lambda^G,$$

where the multiplicity $m_{\tilde{\lambda}}^{\lambda} := [V_\lambda^G : V_{\tilde{\lambda}}^{\tilde{G}}]$ is finite for any λ.

The Hermitian $\tilde{K}$-vector space $\tilde{\mathfrak{p}}$, when restricted to the K-action, admits an orthogonal decomposition $\tilde{\mathfrak{p}} = \mathfrak{p} \oplus \mathfrak{q}$. Notice that the symmetric algebra $\mathrm{Sym}(\mathfrak{q})$ is an admissible K-module.

In [15], H.P. Jakobsen and M. Vergne obtained the following nice characterization of the multiplicities $[V_\lambda^G : V_{\tilde{\lambda}}^{\tilde{G}}]$. Another proof is given in [30], §4.4.

Theorem 3.4 (Jakobsen-Vergne) *Let $(\tilde{\lambda}, \lambda) \in \widehat{\tilde{G}}_{\mathrm{hol}} \times \widehat{G}_{\mathrm{hol}}$. The multiplicity $[V_\lambda^G : V_{\tilde{\lambda}}^{\tilde{G}}]$ is equal to the multiplicity of the representation V_λ^K in $\mathrm{Sym}(\mathfrak{q}) \otimes V_{\tilde{\lambda}}^{\tilde{K}}|_K$.*

3.3 Discrete analogues of $\Pi_{\mathrm{hol}}(\tilde{G}, G)$

We define the following discrete analogues of the cone $\Pi_{\mathrm{hol}}(\tilde{G}, G)$:

$$\Pi_{\mathrm{hol}}^{\mathbb{Z}}(\tilde{G}, G) := \left\{(\tilde{\lambda}, \lambda) \in \widehat{\tilde{G}}_{\mathrm{hol}} \times \widehat{G}_{\mathrm{hol}}\ ;\ [V_\lambda^G : V_{\tilde{\lambda}}^{\tilde{G}}] \neq 0\right\}. \tag{13}$$

and

$$\Pi_{\mathrm{hol}}^{\mathbb{Q}}(\tilde{G}, G) := \left\{(\tilde{\xi}, \xi) \in \tilde{\mathcal{C}}_{\mathrm{hol}} \times \mathcal{C}_{\mathrm{hol}}\ ;\ \exists N \geq 1,\ (N\xi, N\tilde{\xi}) \in \Pi_{\mathrm{hol}}^{\mathbb{Z}}(\tilde{G}, G)\right\}. \tag{14}$$

We have the following key fact.

Proposition 3.5

- *$\Pi_{\mathrm{hol}}^{\mathbb{Z}}(\tilde{G}, G)$ is a subset of $\tilde{\wedge}^* \times \wedge^*$ stable under the addition.*
- *$\Pi_{\mathrm{hol}}^{\mathbb{Q}}(\tilde{G}, G)$ is a $\mathbb{Q}$-convex cone of the $\mathbb{Q}$-vector space $\tilde{\mathfrak{t}}_{\mathbb{Q}}^* \times \mathfrak{t}_{\mathbb{Q}}^*$.*

Proof : Suppose that $a_1 := (\tilde{\lambda}_1, \lambda_1)$ and $a_2 := (\tilde{\lambda}_2, \lambda_2)$ belongs to $\Pi_{\mathrm{hol}}^{\mathbb{Z}}(\tilde{G}, G)$. Thanks to Theorem 3.4, we know that the K-modules $\mathrm{Sym}(\mathfrak{q}) \otimes (V_{\lambda_j}^K)^* \otimes V_{\tilde{\lambda}_j}^{\tilde{K}}|_K$ possess a non-zero invariant vector ϕ_j, for $j = 1, 2$.

Let $\mathbb{X} := \overline{K/T} \times \tilde{K}/\tilde{T}$ be the product of flag manifolds. The complex structure is normalized so that $\mathbf{T}_{([e],[\tilde{e}])}\mathbb{X} \simeq \mathfrak{n}_- \oplus \tilde{\mathfrak{n}}_+$, where $\mathfrak{n}_- = \sum_{\alpha<0} (\mathfrak{k}_{\mathbb{C}})_\alpha$ and $\tilde{\mathfrak{n}}_+ = \sum_{\tilde{\alpha}>0} (\tilde{\mathfrak{k}}_{\mathbb{C}})_{\tilde{\alpha}}$. We associate to each data a_j, the holomorphic line bundle $\mathcal{L}_j := K \times_T \mathbb{C}_{-\lambda_j} \boxtimes \tilde{K} \times_{\tilde{T}} \mathbb{C}_{-\tilde{\lambda}_j}$ on $\mathbb{X}$. Let $H^0(\mathbb{X}, \mathcal{L}_j)$ be the space of holomorphic sections of the line bundle $\mathcal{L}_j$. The Borel-Weil Theorem tell us that $H^0(\mathbb{X}, \mathcal{L}_j) \simeq (V_{\lambda_j}^K)^* \otimes V_{\tilde{\lambda}_j}^{\tilde{K}}|_K$, $\forall j \in \{1, 2\}$.

We have $\phi_j \in \left[\mathrm{Sym}(\mathfrak{q}) \otimes H^0(\mathbb{X}, \mathcal{L}_j)\right]^K$, $\forall j$, and then $\phi_1\phi_2 \in \mathrm{Sym}(\mathfrak{q}) \otimes H^0(\mathbb{X}, \mathcal{L}_1 \otimes \mathcal{L}_2)$ is a non-zero invariant vector. Hence $[\mathrm{Sym}(\mathfrak{q}) \otimes (V_{\lambda_1+\lambda_2}^K)^* \otimes V_{\tilde{\lambda}_1+\tilde{\lambda}_2}^{\tilde{K}}|_K]^K \neq 0$. Thanks to Theorem 3.4, we can conclude that $a_1 + a_2 = (\tilde{\lambda}_1 + \tilde{\lambda}_2, \lambda_1 + \lambda_2)$ belongs to $\Pi_{\mathrm{hol}}^{\mathbb{Z}}(\tilde{G}, G)$. The first point is proved. From the first point, one checks easily that

- $\Pi_{\mathrm{hol}}^{\mathbb{Q}}(\tilde{G}, G)$ is stable under addition,
- $\Pi_{\mathrm{hol}}^{\mathbb{Q}}(\tilde{G}, G)$ is stable by expansion by a non-negative rational number.

The second point is settled. □

3.4 Riemann-Roch numbers

We come back to the framework of §2.3.

We associate to a dominant weight $\mu \in \wedge_+^*$ the (possibly singular) symplectic reduced space $M_\mu := \Phi_K^{-1}(\mu)/K_\mu$, and the (possibly singular) line bundle over M_μ :

$$\mathcal{L}_\mu := \left(\mathcal{L}|_{\Phi_K^{-1}(\mu)} \otimes \mathbb{C}_{-\mu}\right)/K_\mu.$$

Suppose first that μ is a weakly-regular value of Φ_K. Then M_μ is an orbifold equipped with a symplectic structure Ω_μ, and $\mathcal{L}_\mu$ is a line orbi-bundle over M_μ that prequantizes the symplectic structure. By choosing an almost complex structure on M_μ compatible with Ω_μ, we get a decomposition $\wedge \mathbf{T}^* M_\mu \otimes \mathbb{C} = \oplus_{i,j} \wedge^{i,j} \mathbf{T}^* M_\mu$ of the bundle of differential forms. Using Hermitian structure in the tangent bundle $\mathbf{T}M_\mu$ of M_μ, and in the fibers of $\mathcal{L}_\mu$, we define a Dolbeault-Dirac operator

$$D_\mu^+ : \mathcal{A}^{0,+}(M_\mu, \mathcal{L}_\mu) \longrightarrow \mathcal{A}^{0,-}(M_\mu, \mathcal{L}_\mu)$$

where $\mathcal{A}^{i,j}(M_\mu, \mathcal{L}_\mu) = \Gamma(M_\mu, \wedge^{i,j}\mathbf{T}^* M_\mu \otimes \mathcal{L}_\mu)$.

Definition 3.6 *Let $\mu \in \wedge_+^*$ be a weakly-regular value of the moment map Φ_K. The Riemann-Roch number $RR(M_\mu, \mathcal{L}_\mu) \in \mathbb{Z}$ is defined as the index of the elliptic operator D_μ^+: $RR(M_\mu, \mathcal{L}_\mu) = \dim(ker(D_\mu^+)) - \dim(coker(D_\mu^+))$.*

Suppose that $\mu \notin \Delta_K(M)$. Then $M_\mu = \emptyset$ and we take $RR(M_\mu, \mathcal{L}_\mu) = 0$.

Suppose now that $\mu \in \Delta_K(M)$ is not (necessarily) a weakly-regular value of Φ_K. Take a small element $\epsilon \in \mathfrak{t}^*$ such that $\mu + \epsilon$ is a weakly-regular value of Φ_K belonging to $\Delta_K(M)$. We consider the symplectic orbifold $M_{\mu+\epsilon}$: if ϵ is small enough,

$$\mathcal{L}_{\mu,\epsilon} := \left(\mathcal{L}|_{\Phi_K^{-1}(\mu+\epsilon)} \otimes \mathbb{C}_{-\mu}\right)/K_{\mu+\epsilon}.$$

is a line orbi-bundle over $M_{\mu+\epsilon}$.

We have the following important result (see §3.4.3 in [33]).

Proposition 3.7 *Let $\mu \in \Delta_K(M) \cap \wedge^*$. The Riemann-Roch number $RR(M_{\mu+\epsilon}, \mathcal{L}_{\mu,\epsilon})$ do not depend on the choice of ϵ small enough so that $\mu + \epsilon \in \Delta_K(M)$ is a weakly-regular value of Φ_K.*

We can now introduce the following definition.

Definition 3.8 *Let $\mu \in \wedge_+^*$. We define*

$$\mathcal{Q}(M_\mu, \Omega_\mu) = \begin{cases} 0 & \text{if} \quad \mu \notin \Delta_K(M), \\ RR(M_{\mu+\epsilon}, \mathcal{L}_{\mu,\epsilon}) & \text{if} \quad \mu \in \Delta_K(M). \end{cases}$$

Above, ϵ is chosen small enough so that $\mu + \epsilon \in \Delta_K(M)$ is a weakly-regular value of Φ_K.

Let $n \geq 1$. The manifold M, equipped with the symplectic structure $n\Omega$, is prequantized by the line bundle $\mathcal{L}^{\otimes n}$: the corresponding moment map is $n\Phi_K$. For any dominant weight $\mu \in \wedge^*_+$, the symplectic reduction of $(M, n\Omega)$ relatively to the weight $n\mu$ is $(M_\mu, n\Omega_\mu)$. Like in definition 3.8, we consider the following Riemann-Roch numbers

$$\mathcal{Q}(M_\mu, n\Omega_\mu) = \begin{cases} 0 & \text{if} \quad \mu \notin \Delta_K(M), \\ RR(M_{\mu+\epsilon}, (\mathcal{L}_{\mu,\epsilon})^{\otimes n}) & \text{if} \quad \mu \in \Delta_K(M) \text{ and } \|\epsilon\| << 1. \end{cases}$$

Kawasaki-Riemann-Roch formula shows that $n \geq 1 \mapsto \mathcal{Q}(M_\mu, n\Omega_\mu)$ is a quasi-polynomial map [36, 23]. When μ is a weakly-regular value of Φ_K, we denote by $\mathrm{vol}(M_\mu) := \frac{1}{d_\mu}\int_{M_\mu}(\frac{\Omega_\mu}{2\pi})^{\frac{\dim M_\mu}{2}}$ the symplectic volume of the symplectic orbifold (M_μ, Ω_μ). Here d_μ is the generic value of the map $m \in \Phi_K^{-1}(\mu) \mapsto \mathrm{cardinal}(K_m/K_m^0)$.

The following proposition is a direct consequence of Kawasaki-Riemann-Roch formula (see [23] or §1.3.4 in [29]).

Proposition 3.9 *Let $\mu \in \Delta_K(M) \cap \wedge^*_+$ be a weakly-regular value of Φ_K. Then we have $\mathcal{Q}(M_\mu, n\Omega_\mu) \sim \mathrm{vol}(M_\mu)\, n^{\frac{\dim M_\mu}{2}}$, when $n \to \infty$. In particular the map $n \geq 1 \mapsto \mathcal{Q}(M_\mu, n\Omega_\mu)$ is non-zero.*

3.5 Quantization commutes with reduction

Let us explain the "*Quantization commutes with Reduction*" Theorem proved in [30].

We fix $\tilde{\lambda} \in \widehat{\tilde{G}}_{\mathrm{hol}}$. The coadjoint orbit $\tilde{G}\tilde{\lambda}$ is prequantized by the line bundle $\tilde{G} \times_{K_{\tilde{\lambda}}} \mathbb{C}_{\tilde{\lambda}}$, and the moment map $\Phi_G^{\tilde{\lambda}} : \tilde{G}\tilde{\lambda} \to \mathfrak{g}^*$ corresponding to the G-action on $\tilde{G} \times_{K_{\tilde{\lambda}}} \mathbb{C}_{\tilde{\lambda}}$ is equal to the restriction of the map $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}$ to $\tilde{G}\tilde{\lambda}$.

The symplectic slice $Y_{\tilde{\lambda}} = (\Phi_G^{\tilde{\lambda}})^{-1}(\mathfrak{k}^*)$ is prequantized by the line bundle $\mathcal{L}_{\tilde{\lambda}} := \tilde{G} \times_{K_{\tilde{\lambda}}} \mathbb{C}_{\tilde{\lambda}}|_{Y_{\tilde{\lambda}}}$. The moment map $\Phi_K^{\tilde{\lambda}} : Y_{\tilde{\lambda}} \to \mathfrak{k}^*$ corresponding to the K-action is equal to the restriction of $\Phi_G^{\tilde{\lambda}}$ to $Y_{\tilde{\lambda}}$.

For any $\lambda \in \widehat{G}_{\mathrm{hol}}$, we consider the (possibly singular) symplectic reduced space

$$\mathbb{X}_{\tilde{\lambda},\lambda} := (\Phi_K^{\tilde{\lambda}})^{-1}(\lambda)/K_\lambda,$$

equipped with the reduced symplectic form $\Omega_{\tilde{\lambda},\lambda}$, and the (possibly singular) line bundle

$$\mathbb{L}_{\tilde{\lambda},\lambda} := \left(\mathcal{L}_{\tilde{\lambda}}|_{(\Phi_K^{\tilde{\lambda}})^{-1}(\lambda)} \otimes \mathbb{C}_{-\lambda}\right)/K_\lambda.$$

Thanks to Definition 3.8, the geometric quantization $\mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda}, \Omega_{\tilde{\lambda},\lambda}) \in \mathbb{Z}$ of those compact symplectic spaces $(\mathbb{X}_{\tilde{\lambda},\lambda}, \Omega_{\tilde{\lambda},\lambda})$ are well-defined even if they are singular. In particular $\mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda}, \Omega_{\tilde{\lambda},\lambda}) = 0$ when $\mathbb{X}_{\tilde{\lambda},\lambda} := \emptyset$.

The following Theorem is proved in [30].

Theorem 3.10 *Let $\tilde{\lambda} \in \widehat{\tilde{G}}_{\rm hol}$. We have an Hilbertian direct sum*

$$V_{\tilde{\lambda}}^{\tilde{G}}|_G = \bigoplus_{\lambda \in \widehat{G}_{\rm hol}} \mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda}, \Omega_{\tilde{\lambda},\lambda})\ V_\lambda^G,$$

It means that for any $\lambda \in \widehat{G}_{\rm hol}$, the multiplicity of the representation V_λ^G in the restriction $V_{\tilde{\lambda}}^{\tilde{G}}|_G$ is equal to the geometric quantization $\mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda}, \Omega_{\tilde{\lambda},\lambda})$ of the (compact) reduced space $\mathbb{X}_{\tilde{\lambda},\lambda}$.

Remark 3.11 *Let $(\tilde{\lambda}, \lambda) \in \widehat{\tilde{G}}_{\rm hol} \times \widehat{G}_{\rm hol}$. Theorem 3.10. shows that*

$$\left[V_{n\lambda}^G : V_{n\tilde{\lambda}}^{\tilde{G}}\right] = \mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda}, n\Omega_{\tilde{\lambda},\lambda})$$

for any $n \geq 1$.

4 Proofs of the main results

We come back to the setting of §2.2: G/K is a complex submanifold of a Hermitian symmetric space $\tilde{G}/\tilde{K}$. It means that there exits a $\tilde{K}$-invariant element $z \in \mathfrak{k}$ such that $\mathrm{ad}(z)$ defines complex structures on $\tilde{\mathfrak{p}}$ and $\mathfrak{p}$. We consider the orthogonal decomposition $\tilde{\mathfrak{p}} = \mathfrak{p} \oplus \mathfrak{q}$, and we denote by $\mathrm{Sym}(\mathfrak{q})$ the symmetric algebra of the complex K-module $(\mathfrak{q}, \mathrm{ad}(z))$.

4.1 Proof of Theorem A

The set $\Pi_{\rm hol}(\tilde{G}, G)$ is equal to $\bigcup_{\tilde{a} \in \tilde{\mathcal{C}}_{\rm hol}} \{\tilde{a}\} \times \Delta_G(\tilde{G}\tilde{a})$. We define

$$\Pi_{\rm hol}(\tilde{G}, G)^0 := \bigcup_{\tilde{a} \in \tilde{\mathcal{C}}_{\rm hol}} \{\tilde{a}\} \times \Delta_G(\tilde{G}\tilde{a})^0.$$

We start with the following result.

Lemma 4.1 *The set $\Pi_{\rm hol}(\tilde{G}, G)^0 \bigcap \tilde{\mathfrak{t}}^*_{\mathbb{Q}} \times \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Pi_{\rm hol}(\tilde{G}, G)$.*

Proof : Let $(\tilde{\xi}, \xi) \in \Pi_{\rm hol}(\tilde{G}, G)$: take $\tilde{g} \in \tilde{G}$ such that $\xi = \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{g}\tilde{\xi})$. We consider a sequence $\tilde{\xi}_n \in \tilde{\mathcal{C}}_{\rm hol} \cap \tilde{\mathfrak{t}}^*_{\mathbb{Q}}$ converging to $\tilde{\xi}$. Then $\xi_n := \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{g}\tilde{\xi}_n)$ is a sequence of $\mathcal{C}^0_{G/K}$ converging to $\xi \in \mathcal{C}_{\rm hol}$. Since the map $\mathrm{p} : \mathcal{C}^0_{G/K} \to \mathcal{C}_{\rm hol}$ is continuous (se Lemma 2.3), the sequence $\eta_n := \mathrm{p}(\xi_n)$ converges to $\mathrm{p}(\xi) = \xi$. By definition, we have $\eta_n \in \Delta_G(\tilde{G}\,\tilde{\xi}_n)$ for any $n \in \mathbb{N}$. Since $\tilde{\xi}_n$ are rational, each subset $\Delta_G(\tilde{G}\tilde{\xi}_n)^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_G(\tilde{G}\tilde{\xi}_n)$ (see Lemma 2.10). Hence, $\forall n \in \mathbb{N}$, there exists $\zeta_n \in \Delta_G(\tilde{G}\,\tilde{\xi}_n)^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ such that $\|\zeta_n - \eta_n\| \leq 2^{-n}$. Finally, we see that $(\tilde{\xi}_n, \zeta_n)$ is a sequence of rational elements of $\Pi_{\rm hol}(\tilde{G}, G)^0$ converging to $(\xi, \tilde{\xi})$. □

The main purpose of this section is the proof of the following theorem.

Theorem 4.2 *The following inclusions hold*

$$\Pi_{\rm hol}(\tilde{G},G)^0 \bigcap \tilde{\mathfrak{t}}^*_{\mathbb{Q}}\times\mathfrak{t}^*_{\mathbb{Q}} \underset{(1)}{\subset} \Pi^{\mathbb{Q}}_{\rm hol}(\tilde{G},G) \underset{(2)}{\subset} \Pi_{\rm hol}(\tilde{G},G).$$

Lemma 4.1 and Theorem 4.2 gives the important Corollary.

Corollary 4.3 $\Pi^{\mathbb{Q}}_{\rm hol}(\tilde{G},G)$ *is dense in* $\Pi_{\rm hol}(\tilde{G},G)$.

Proof of Theorem 4.2 : Let $(\tilde{\mu},\mu)\in\Pi^{\mathbb{Q}}_{\rm hol}(\tilde{G},G)$: there exists $N\geq 1$ such that $(N\tilde{\mu},N\mu)\in\Pi^{\mathbb{Z}}_{\rm hol}(\tilde{G},G)$. The multiplicity $[V^G_{N\mu}:V^{\tilde{G}}_{N\tilde{\mu}}]$ is non-zero, and thanks to Theorem 3.10, it implies that the reduced space $\mathbb{X}_{N\tilde{\mu},N\mu}$ is non-empty. In other words $(N\tilde{\mu},N\mu)\in\Pi_{\rm hol}(\tilde{G},G)$. The inclusion (2) is proven.

Let $(\tilde{\mu},\mu)\in\Pi_{\rm hol}(\tilde{G},G)^0\bigcap\mathfrak{t}^*_{\mathbb{Q}}\times\tilde{\mathfrak{t}}^*_{\mathbb{Q}}$. There exists $N_o\geq 1$ such that $\lambda:=N_o\mu\in\widehat{G}_{\rm hol}$, $\tilde{\lambda}:=N_o\tilde{\mu}\in\widehat{\tilde{G}}_{\rm hol}$ and $\lambda\in\Delta_G(\tilde{G}\tilde{\lambda})^0$: the element λ is a weakly-regular value of the moment map $\tilde{G}\tilde{\lambda}\to\mathfrak{g}^*$. Theorem 3.10 tell us that, for any $n\geq 1$, the multiplicity $[V^G_{n\lambda}:V^{\tilde{G}}_{n\tilde{\lambda}}]$ is equal to Riemann-Roch number $\mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda},n\Omega_{\tilde{\lambda},\lambda})$. Since the map $n\mapsto\mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda},n\Omega_{\tilde{\lambda},\lambda})$ is non-zero (see Proposition 3.9), we can conclude that there exists $n_o\geq 1$ such that $[V^G_{n_o\lambda}:V^{\tilde{G}}_{n_o\tilde{\lambda}}]\neq 0$. In other words, we obtain $n_oN_o(\tilde{\mu},\mu)\in\Pi^{\mathbb{Z}}_{\rm hol}(\tilde{G},G)$ and so $(\tilde{\mu},\mu)\in\Pi^{\mathbb{Q}}_{\rm hol}(\tilde{G},G)$. The inclusion (1) is settled. $\Box$

Now we can terminate the proof of Theorem A.

Thanks to Proposition 3.5, we know that $\Pi^{\mathbb{Q}}_{\rm hol}(\tilde{G},G)$ is a $\mathbb{Q}$-convex cone. Since $\Pi_{\rm hol}(\tilde{G},G)$ is a closed subset of $\tilde{\mathcal{C}}_{\rm hol}\times\mathcal{C}_{\rm hol}$ (see Proposition 2.5), we can conclude, by a density argument, that $\Pi_{\rm hol}(\tilde{G},G)$ is a closed convex cone of $\tilde{\mathcal{C}}_{\rm hol}\times\mathcal{C}_{\rm hol}$.

4.2 The affine variety $\tilde{K}_{\mathbb{C}}\times\mathfrak{q}$

Let $\tilde{\kappa}$ be the Killing form on the Lie algebra $\tilde{\mathfrak{g}}$. We consider the $\tilde{K}$-invariant symplectic structures $\Omega_{\tilde{\mathfrak{p}}}$ on $\tilde{\mathfrak{p}}$, defined by the relation

$$\Omega_{\tilde{\mathfrak{p}}}(\tilde{Y},\tilde{Y}')=\tilde{\kappa}(z,[\tilde{Y},\tilde{Y}']),\quad\forall\tilde{Y},\tilde{Y}'\in\tilde{\mathfrak{p}}.$$

We notice that the complex structure $\mathrm{ad}(z)$ is adapted to $\Omega_{\tilde{\mathfrak{p}}}$: $\Omega_{\tilde{\mathfrak{p}}}(\tilde{Y},\mathrm{ad}(z)\tilde{Y})>0$ if $\tilde{Y}\neq 0$.

We denote by $\Omega_{\mathfrak{q}}$ the restriction of $\Omega_{\tilde{\mathfrak{p}}}$ on the symplectic subspace $\mathfrak{q}$. The moment map $\Phi_{\mathfrak{q}}$ associated to the K-action on $(\mathfrak{q},\Omega_{\mathfrak{q}})$ is defined by the relations $\langle\Phi_{\mathfrak{q}}(Y),X\rangle=\frac{-1}{2}\tilde{\kappa}([X,Y],[z,Y])$, $\forall(X,Y)\in\mathfrak{p}\times\mathfrak{q}$. In particular, $\langle\Phi_{\mathfrak{q}}(Y),z\rangle=\frac{-1}{2}\|Y\|^2$, $\forall Y\in\mathfrak{q}$, so the map $\langle\Phi_{\mathfrak{q}},z\rangle$ is proper.

The complex reductive group $\tilde{K}_{\mathbb{C}}$ is equipped with the following action of $\tilde{K}\times K$: $(\tilde{k},k)\cdot a=\tilde{k}ak^{-1}$. It has a canonical structure of a smooth affine variety. There is a diffeomorphism of the cotangent bundle $\mathbf{T}^*\tilde{K}$ with $\tilde{K}_{\mathbb{C}}$ defined as follows. We identify

$\mathbf{T}^*\tilde{K}$ with $\tilde{K}\times\tilde{\mathfrak{k}}^*$ by means of left-translation and then with $\tilde{K}\times\tilde{\mathfrak{k}}$ by means of an invariant inner product on $\tilde{\mathfrak{k}}$. The map $\varphi:\tilde{K}\times\tilde{\mathfrak{k}}\to\tilde{K}_{\mathbb{C}}$ given by $\varphi(a,X)=ae^{iX}$ is a diffeomorphism. If we use φ to transport the complex structure of $\tilde{K}_{\mathbb{C}}$ to $\mathbf{T}^*\tilde{K}$, then the resulting complex structure on $\mathbf{T}^*\tilde{K}$ is compatible with the symplectic structure on $\mathbf{T}^*\tilde{K}$, so that $\mathbf{T}^*\tilde{K}$ becomes a Kähler Hamiltonian $\tilde{K}\times K$-manifold (see [11], §3). The moment map relative to the $\tilde{K}\times K$-action is the proper map $\Phi_{\tilde{K}}\oplus\Phi_K:\mathbf{T}^*\tilde{K}\to\tilde{\mathfrak{k}}^*\oplus\mathfrak{k}^*$ defined by $\Phi_{\tilde{K}}(\tilde{a},\tilde{\eta})=-\tilde{a}\tilde{\eta}$ and $\Phi_K(\tilde{a},\tilde{\eta})=\pi_{\mathfrak{k},\tilde{\mathfrak{k}}}(\tilde{\eta})$. Here $\pi_{\mathfrak{k},\tilde{\mathfrak{k}}}:\tilde{\mathfrak{k}}^*\to\mathfrak{k}^*$ is the projection dual to the inclusion $\mathfrak{k}\hookrightarrow\tilde{\mathfrak{k}}$ of Lie algebras.

Finally we consider the Kähler Hamiltonian $\tilde{K}\times K$-manifold $\mathbf{T}^*\tilde{K}\times\mathfrak{q}$, where $\mathfrak{q}$ is equipped with the symplectic structure $\Omega_{\mathfrak{q}}$. Let us denote by $\Phi:\mathbf{T}^*\tilde{K}\times\mathfrak{q}\to\tilde{\mathfrak{k}}^*\oplus\mathfrak{k}^*$ the moment map relative to the $\tilde{K}\times K$-action :

$$\Phi(\tilde{a},\tilde{\eta},Y)=\Big(-\tilde{a}\tilde{\eta},\pi_{\mathfrak{k},\tilde{\mathfrak{k}}}(\tilde{\eta})+\Phi_{\mathfrak{q}}(Y)\Big). \tag{15}$$

Since Φ is proper map, the Convexity Theorem tell us that

$$\Delta(\mathbf{T}^*\tilde{K}\times\mathfrak{q}):=\mathrm{Image}(\Phi)\bigcap\tilde{\mathfrak{t}}^*_{\geq 0}\times\mathfrak{t}^*_{\geq 0}$$

is a closed convex locally polyhedral set.

We consider now the action of $\tilde{K}\times K$ on the affine variety $\tilde{K}_{\mathbb{C}}\times\mathfrak{q}$. The set of highest weights of $\tilde{K}_{\mathbb{C}}\times\overline{\mathfrak{q}}$ is the semigroup

$$\Delta^{\mathbb{Z}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})\subset\tilde{\wedge}^*_+\times\wedge^*_+$$

consisting of all dominant weights $(\tilde{\lambda},\lambda)$ such that the irreducible $\tilde{K}\times K$-representation $V^{\tilde{K}}_{\tilde{\lambda}}\otimes V^K_\lambda$ occurs in the coordinate ring $\mathbb{C}[\tilde{K}_{\mathbb{C}}\times\mathfrak{q}]$. A direct application of the Peter-Weyl Theorem gives the following characterization :

$$(\tilde{\lambda},\lambda)\in\Delta^{\mathbb{Z}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})\Longleftrightarrow\Big[V^{\tilde{K}}_{\tilde{\lambda}}|_K\otimes V^K_\lambda\otimes\mathrm{Sym}(\mathfrak{q})\Big]^K\neq 0. \tag{16}$$

We denote by $\Delta^{\mathbb{Q}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})$ the $\mathbb{Q}$-convex cone generated by the semigroup $\Delta^{\mathbb{Z}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})$: $(\tilde{\xi},\xi)\in\Delta^{\mathbb{Q}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})$ if and only if $\exists N\geq 1$, $N(\tilde{\xi},\xi)\in\Delta^{\mathbb{Z}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})$.

The following important fact is classical (see Theorem 4.9 in [34]).

Proposition 4.4 *The Kirwan polyhedron $\Delta(\mathbf{T}^*\tilde{K}\times\mathfrak{q})$ is equal to the closure of the $\mathbb{Q}$-convex cone $\Delta^{\mathbb{Q}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})$.*

4.3 Proof of Theorem B

Consider the semigroup $\Pi^{\mathbb{Z}}(\tilde{G},G)$ of $\tilde{\wedge}^*_+\times\wedge^*_+$ (see Definition 1.3) and the $\mathbb{Q}$-convex cone $\Pi^{\mathbb{Q}}(\tilde{G},G):=\{(\tilde{\xi},\xi)\in\tilde{\mathfrak{t}}^*_{\geq 0}\times\mathfrak{t}^*_{\geq 0}\ ;\ \exists N\geq 1,N(\tilde{\xi},\xi)\in\Pi^{\mathbb{Z}}(\tilde{G},G)\}$.

The Jakobsen-Vergne theorem says that $\Pi^{\mathbb{Z}}_{\rm hol}(\tilde{G},G) = \Pi^{\mathbb{Z}}(\tilde{G},G) \bigcap \widehat{\tilde{G}}_{\rm hol} \times \widehat{G}_{\rm hol}$. Hence, the convex cone $\Pi^{\mathbb{Q}}_{\rm hol}(\tilde{G},G)$ is equal to $\Pi^{\mathbb{Q}}(\tilde{G},G) \cap \tilde{\mathcal{C}}_{\rm hol} \times \mathcal{C}_{\rm hol}$. Thanks to (16), we know that $(\tilde{\xi},\xi) \in \Pi^{\mathbb{Q}}(\tilde{G},G)$ if and only if $(\tilde{\xi},\xi^*) \in \Delta^{\mathbb{Q}}(\tilde{K}_{\mathbb{C}} \times \mathfrak{q})$. The density results obtained in Proposition 4.4 and Corollary 4.3 gives finally Theorem B.

Theorem 4.5 *1. We have* $\Pi_{\rm hol}(\tilde{G},G) = \Pi(\tilde{G},G) \bigcap \tilde{\mathcal{C}}_{\rm hol} \times \mathcal{C}_{\rm hol}$.

2. An element $(\tilde{\xi},\xi)$ *belongs to* $\Pi_{\rm hol}(\tilde{G},G)$ *if and only if*

$$(\tilde{\xi},\xi) \in \tilde{\mathcal{C}}_{\rm hol} \times \mathcal{C}_{\rm hol} \quad \text{and} \quad (\tilde{\xi},\xi^*) \in \Delta(\mathbf{T}^*\tilde{K} \times \mathfrak{q}).$$

4.4 Proof of Theorem C

We denote by $\bar{\mathfrak{q}}$ the K-vector space $\mathfrak{q}$ equipped with the opposite symplectic form $-\Omega_{\mathfrak{q}}$ and opposite complex structure $-\mathrm{ad}(z)$. The moment map relative to the K-action on $\bar{\mathfrak{q}}$ is denoted by $\Phi_{\bar{\mathfrak{q}}} = -\Phi_{\mathfrak{q}}$.

Lemma 4.6 *Any element* $(\tilde{\xi},\xi) \in \tilde{\mathfrak{t}}^*_{\geq 0} \times \mathfrak{t}^*_{\geq 0}$ *satisfies the equivalence :*

$$(\tilde{\xi},\xi^*) \in \Delta(\mathbf{T}^*\tilde{K} \times \mathfrak{q}) \Longleftrightarrow \xi \in \Delta_K(\tilde{K}\tilde{\xi} \times \overline{\mathfrak{q}}).$$

Proof : Thanks to (15), we see immediatly that $\exists(\tilde{a},\tilde{\eta},Y) \in \mathbf{T}^*\tilde{K} \times \mathfrak{q}$ such that $(\tilde{\xi},\xi^*) = \Phi(\tilde{a},\tilde{\eta},Y)$ if and only if $\exists(\tilde{b},Z) \in \tilde{K} \times \mathfrak{q}$ such that $\xi = \pi_{\mathfrak{k},\tilde{\mathfrak{k}}}(\tilde{b}\tilde{\xi}) + \Phi_{\bar{\mathfrak{q}}}(Z)$. □

At this stage we know that $\Delta_G(\tilde{G}\tilde{\mu}) = \Delta_K(\tilde{K}\tilde{\mu} \times \overline{\mathfrak{q}}) \cap \mathcal{C}_{\rm hol}$. Hence, Theorem C will follows from the next result.

Proposition 4.7 *For any* $\tilde{\mu} \in \tilde{\mathcal{C}}_{\rm hol}$*, the Kirwan polyhedron* $\Delta_K(\tilde{K}\tilde{\mu} \times \overline{\mathfrak{q}})$ *is contained in* $\mathcal{C}_{\rm hol}$.

Proof : By definition $\mathcal{C}_{\rm hol} = \mathcal{C}^0_{G/K} \cap \mathfrak{t}^*_{\geq 0}$, so we have to prove that $\pi_{\mathfrak{k},\tilde{\mathfrak{k}}}(\tilde{K}\tilde{\mu}) + \mathrm{Image}(\Phi_{\bar{\mathfrak{q}}})$ is contained in $\mathcal{C}^0_{G/K}$. By definition $\tilde{K}\tilde{\mu} \subset \mathcal{C}^0_{\tilde{G}/\tilde{K}}$, and then $\pi_{\mathfrak{k},\tilde{\mathfrak{k}}}(\tilde{K}\tilde{\mu}) \subset \mathcal{C}^0_{G/K}$. Since $\mathcal{C}^0_{G/K} + \mathcal{C}_{G/K} \subset \mathcal{C}^0_{G/K}$, it is sufficient to check that $\mathrm{Image}(\Phi_{\bar{\mathfrak{q}}}) \subset \mathcal{C}_{G/K}$. Let $\Phi_{\tilde{\mathfrak{p}}}$ be the moment map relative to the action of $\tilde{K}$ on $(\tilde{\mathfrak{p}},\Omega_{\tilde{\mathfrak{p}}})$. As $\mathrm{Image}(\Phi_{\bar{\mathfrak{q}}}) \subset \pi_{\mathfrak{k},\tilde{\mathfrak{k}}}\big(\mathrm{Image}(-\Phi_{\tilde{\mathfrak{p}}})\big)$, the following lemma will terminate the proof of Proposition 4.7. □

Lemma 4.8 *The image of the moment map* $-\Phi_{\tilde{\mathfrak{p}}}$ *is contained in* $\mathcal{C}_{\tilde{G}/\tilde{K}}$.

Proof : Let $z^* \in \tilde{\mathfrak{t}}^*$ such that $\langle z^*, \tilde{X}\rangle = -\tilde{\kappa}(z,\tilde{X})$, $\forall \tilde{X} \in \tilde{\mathfrak{g}}$. Consider the coadjoint orbit $\tilde{\mathcal{O}} = \tilde{G}z^*$ equipped with its canonical symplectic structure $\Omega_{\tilde{\mathcal{O}}}$: the symplectic vector space $\mathbf{T}_{z^*}\tilde{\mathcal{O}}$ is canonically isomorphic to $(\tilde{\mathfrak{p}}, -\Omega_{\tilde{\mathfrak{p}}})$. In [26], McDuff proved that $(\tilde{\mathcal{O}}, \Omega_{\tilde{\mathcal{O}}})$ is diffeomorphic, as a $\tilde{K}$-symplectic manifold, to the symplectic vector space $(\tilde{\mathfrak{p}}, -\Omega_{\tilde{\mathfrak{p}}})$

(see [6, 8] for a generalization of this fact). McDuff's results shows in particular that $\mathrm{Image}(-\Phi_{\tilde{\mathfrak{p}}}) = \pi_{\tilde{\mathfrak{g}},\tilde{\mathfrak{k}}}(\tilde{\mathcal{O}})$. Our proof is completed if we check that $\pi_{\tilde{\mathfrak{g}},\tilde{\mathfrak{k}}}(\tilde{\mathcal{O}}) \subset \mathcal{C}_{\tilde{G}/\tilde{K}}$: in other words, if $\langle \pi_{\tilde{\mathfrak{g}},\tilde{\mathfrak{k}}}(\tilde{g}_0\, z^*), \tilde{g}_1 z\rangle \geq 0$, $\forall \tilde{g}_0, \tilde{g}_1 \in \tilde{G}$. But

$$\begin{aligned} 2\langle \pi_{\tilde{\mathfrak{g}},\tilde{\mathfrak{k}}}(\tilde{g}_0\, z^*), \tilde{g}_1\, z\rangle &= \langle \tilde{g}_0\, z^*, \tilde{g}_1 z + \Theta(\tilde{g}_1) z\rangle \\ &= -\tilde{\kappa}(z, \tilde{g}_0^{-1}\tilde{g}_1\, z) - \tilde{\kappa}(z, \tilde{g}_0^{-1}\Theta(\tilde{g}_1) z). \end{aligned}$$

With (7) in hand, it is not difficult to see that $-\tilde{\kappa}(z, \tilde{g}\, z) \geq 0$ for every $\tilde{g} \in \tilde{G}$. We thus verified that $\pi_{\tilde{\mathfrak{g}},\tilde{\mathfrak{k}}}(\tilde{\mathcal{O}}) \subset \mathcal{C}_{\tilde{G}/\tilde{K}}$. $\Box$

5 Facets of the cone $\Pi_{\mathrm{hol}}(\tilde{G}, G)$

We come back to the framework of §4.2. We consider the Kähler Hamiltonian $\tilde{K} \times K$-manifold $\mathbf{T}^*\tilde{K} \times \mathfrak{q}$. The moment map, $\Phi : \mathbf{T}^*\tilde{K} \times \mathfrak{q} \to \tilde{\mathfrak{k}}^* \oplus \mathfrak{k}^*$, relative to the $\tilde{K} \times K$-action is defined by (15).

In this section, we adapt to our case the result of §6 of [31] concerning the parametrization of the facets of Kirwan polyhedrons in terms of Ressayre's data.

5.1 Admissible elements

We choose maximal torus $\tilde{T} \subset \tilde{K}$ and $T \subset K$ such that $T \subset \tilde{T}$. Let $\mathfrak{R}_o$ and $\mathfrak{R}$ be respectively the set of roots for the action of T on $(\tilde{\mathfrak{g}}/\mathfrak{g}) \otimes \mathbb{C}$ and $\mathfrak{g} \otimes \mathbb{C}$. Let $\tilde{\mathfrak{R}}$ be the set of roots for the action of $\tilde{T}$ on $\tilde{\mathfrak{g}} \otimes \mathbb{C}$. Let $\mathfrak{R}^+ \subset \mathfrak{R}$ and $\tilde{\mathfrak{R}}^+ \subset \tilde{\mathfrak{R}}$ be the systems of positive roots defined in (6). Let $W, \tilde{W}$ be the Weyl groups of (T, K) and $(\tilde{T}, \tilde{K})$. Let $w_o \in W$ be the longest element.

We start by introducing the notion of admissible elements. The group $\hom(U(1), T)$ admits a natural identification with the lattice $\wedge := \frac{1}{2\pi}\ker(\exp : \mathfrak{t} \to T)$. A vector $\gamma \in \mathfrak{t}$ is called rational if it belongs to the $\mathbb{Q}$-vector space $\mathfrak{t}_{\mathbb{Q}}$ generated by $\wedge$.

We consider the $\tilde{K} \times K$-action on $N := \mathbf{T}^*\tilde{K} \times \mathfrak{q}$. We associate to any subset $\mathcal{X} \subset N$, the integer $\dim_{\tilde{K}\times K}(\mathcal{X})$ (see (5)).

Definition 5.1 *A non-zero element $(\tilde{\gamma}, \gamma) \in \tilde{\mathfrak{t}} \times \mathfrak{t}$ is called* admissible *if the elements $\tilde{\gamma}$ and γ are rational and if $\dim_{\tilde{K}\times K}(N^{(\tilde{\gamma},\gamma)}) - \dim_{\tilde{K}\times K}(N) \in \{0, 1\}$.*

If $\gamma \in \mathfrak{t}$, we denote by $\mathfrak{R}_o \cap \gamma^\perp$ the subsets of weight vanishing against γ. We start with the following lemma whose proof is left to the reader (see §6.1.1 of [31]).

Lemma 5.2 *1. $N^{(\tilde{\gamma},\gamma)} \neq \emptyset$ if and only if $\tilde{\gamma} \in \tilde{W}\gamma$.*

2. $\dim_{\tilde{K}\times K}(N) = \dim_T(\tilde{\mathfrak{g}}/\mathfrak{g}) = \dim(\mathfrak{t}) - \dim(\mathrm{Vect}(\mathfrak{R}_o))$.

3. For any $\tilde{w} \in \tilde{W}$, $\dim_{\tilde{K}\times K}(N^{(\tilde{w}\gamma,\gamma)}) = \dim_T(\tilde{\mathfrak{g}}^\gamma/\mathfrak{g}^\gamma) = \dim(\mathfrak{t}) - \dim(\mathrm{Vect}(\mathfrak{R}_o \cap \gamma^\perp))$.

The next result is a direct consequence of the previous lemma.

Lemma 5.3 *The admissible elements relative to the $\tilde{K}\times K$-action on $\mathbf{T}^*\tilde{K}\times\mathfrak{q}$ are of the form $(\tilde{w}\gamma,\gamma)$ where $\tilde{w}\in\tilde{W}$ and γ is a non-zero rational element satisfying* $\mathrm{Vect}(\mathfrak{R}_o)\cap\gamma^\perp = \mathrm{Vect}(\mathfrak{R}_o\cap\gamma^\perp)$.

5.2 Ressayre's data

Definition 5.4 *1. Consider the linear action $\rho: G\to \mathrm{GL}_{\mathbb{C}}(V)$ of a compact Lie group on a complex vector space V. For any $(\eta,a)\in\mathfrak{g}\times\mathbb{R}$, we define the vector subspace $V^{\eta=a}=\{v\in V, d\rho(\eta)v=iav\}$. Thus, for any $\eta\in\mathfrak{g}$, we have the decomposition $V=V^{\eta>0}\oplus V^{\eta=0}\oplus V^{\eta<0}$ where $V^{\eta>0}=\sum_{a>0}V^{\eta=a}$, and $V^{\eta<0}=\sum_{a<0}V^{\eta=a}$.*

2. The real number $\mathrm{Tr}_\eta(V^{\eta>0})$ is defined as the sum $\sum_{a>0}a\,\dim(V^{\eta=a})$.

We consider an admissible element $(\tilde{w}\gamma,\gamma)$. The submanifold of $N\simeq\tilde{K}_{\mathbb{C}}\times\mathfrak{q}$ fixed of by $(\tilde{w}\gamma,\gamma)$ is $N^{(\tilde{w}\gamma,\gamma)}=\tilde{w}\tilde{K}^\gamma_{\mathbb{C}}\times\mathfrak{q}^\gamma$. There is a canonical isomorphism between the manifold $N^{(\tilde{w}\gamma,\gamma)}$ equipped with the action of $\tilde{w}\tilde{K}^\gamma\tilde{w}^{-1}\times K^\gamma$ with the manifold $\tilde{K}^\gamma_{\mathbb{C}}\times\mathfrak{q}^\gamma$ equipped with the action of $\tilde{K}^\gamma\times K^\gamma$. The tangent bundle $(\mathbf{T}N|_{N^{(\tilde{w}\gamma,\gamma)}})^{(\tilde{w}\gamma,\gamma)>0}$ is isomorphic to $N^{\gamma_w}\times\tilde{\mathfrak{k}}^{\gamma>0}_{\mathbb{C}}\times\mathfrak{q}^{\gamma>0}$.

The choice of positive roots $\mathfrak{R}^+$ (resp. $\tilde{\mathfrak{R}}^+$) induces a decomposition $\mathfrak{k}_{\mathbb{C}}=\mathfrak{n}\oplus\mathfrak{t}_{\mathbb{C}}\oplus\overline{\mathfrak{n}}$ (resp. $\tilde{\mathfrak{k}}_{\mathbb{C}}=\tilde{\mathfrak{n}}\oplus\tilde{\mathfrak{t}}_{\mathbb{C}}\oplus\overline{\tilde{\mathfrak{n}}}$), where $\mathfrak{n}=\sum_{\alpha\in\mathfrak{R}^+}(\mathfrak{k}\otimes\mathbb{C})_\alpha$ (resp. $\tilde{\mathfrak{n}}=\sum_{\tilde{\alpha}\in\tilde{\mathfrak{R}}^+}(\tilde{\mathfrak{k}}\otimes\mathbb{C})_{\tilde{\alpha}}$). We consider the map

$$\rho^{\tilde{w},\gamma}:\tilde{K}^\gamma_{\mathbb{C}}\times\mathfrak{q}^\gamma\longrightarrow \hom\left(\tilde{\mathfrak{n}}^{\tilde{w}\gamma>0}\times\mathfrak{n}^{\gamma>0}\,,\,\tilde{\mathfrak{k}}^{\gamma>0}_{\mathbb{C}}\times\mathfrak{q}^{\gamma>0}\right)$$

defined by the relation

$$\rho^{\tilde{w},\gamma}(\tilde{x},v):(\tilde{X},X)\longmapsto((\tilde{w}\tilde{x})^{-1}\tilde{X}-X\,;\,X\cdot v),$$

for any $(\tilde{x},v)\in\tilde{K}^\gamma_{\mathbb{C}}\times\mathfrak{q}^\gamma$.

Definition 5.5 *$(\gamma,\tilde{w})\in\mathfrak{t}\times\tilde{W}$ is a Ressayre's data if*

1. *$(\tilde{w}\gamma,\gamma)$ is admissible,*
2. *$\exists(\tilde{x},v)$ such that $\rho^{\tilde{w},\gamma}(\tilde{x},v)$ is bijective.*

Remark 5.6 *In [31], the Ressayre's data were called* regular infinitesimal B-Ressayre's pairs.

Since the linear map $\rho^{\tilde{w},\gamma}(\tilde{x},v)$ commutes with the γ-actions, we obtain the following necessary conditions.

Lemma 5.7 *If $(\gamma,\tilde{w})\in\mathfrak{t}\times\tilde{W}$ is a Ressayre's data, then*

- *Relation (A) :* $\dim(\tilde{\mathfrak{n}}^{\tilde{w}\gamma>0}) + \dim(\mathfrak{n}^{\gamma>0}) = \dim(\tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0}) + \dim(\mathfrak{q}^{\gamma>0})$.
- *Relation (B) :* $\mathrm{Tr}_{\tilde{w}\gamma}(\tilde{\mathfrak{n}}^{\tilde{w}\gamma>0}) + \mathrm{Tr}_{\gamma}(\mathfrak{n}^{\gamma>0}) = \mathrm{Tr}_{\gamma}(\tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0}) + \mathrm{Tr}_{\gamma}(\mathfrak{q}^{\gamma>0})$.

Lemma 5.8 *Relation (B) is equivalent to*

$$\sum_{\substack{\alpha\in\mathfrak{R}^+\\ \langle\alpha,\gamma\rangle>0}} \langle\alpha,\gamma\rangle = \sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}^+\\ \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle>0}} \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle. \tag{17}$$

Proof : First one sees that $\mathrm{Tr}_{\gamma}(\mathfrak{q}^{\gamma>0}) = \mathrm{Tr}_{\gamma}(\tilde{\mathfrak{p}}^{\gamma>0}) - \mathrm{Tr}_{\gamma}(\mathfrak{p}^{\gamma>0}) = \sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}_n^+\\ \langle\tilde{\alpha},\gamma\rangle>0}} \langle\tilde{\alpha},\gamma\rangle - \sum_{\substack{\alpha\in\mathfrak{R}_n^+\\ \langle\alpha,\gamma\rangle>0}} \langle\alpha,\gamma\rangle$, and $\mathrm{Tr}_{\gamma}(\tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0}) = \mathrm{Tr}_{\tilde{w}\gamma}(\tilde{\mathfrak{k}}_{\mathbb{C}}^{\tilde{w}\gamma>0}) = \mathrm{Tr}_{\tilde{w}\gamma}(\tilde{\mathfrak{n}}^{\tilde{w}\gamma>0}) + \sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}_c^+\\ \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle>0}} \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle$. Relation (B) is equivalent to

$$\mathrm{Tr}_{\gamma}(\mathfrak{n}^{\gamma>0}) + \sum_{\substack{\alpha\in\mathfrak{R}_n^+\\ \langle\alpha,\gamma\rangle>0}} \langle\alpha,\gamma\rangle = \sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}_n^+\\ \langle\tilde{\alpha},\gamma\rangle>0}} \langle\tilde{\alpha},\gamma\rangle + \sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}_c^+\\ \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle>0}} \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle. \tag{18}$$

Since $\tilde{\mathfrak{R}}_n^+$ is invariant under the action of the Weyl group $\tilde{W}$, the right hand side of (18) is equal to $\sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}^+\\ \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle>0}} \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle$. Since the left hand side of (18) is equal to $\sum_{\substack{\alpha\in\mathfrak{R}^+\\ \langle\alpha,\gamma\rangle>0}} \langle\alpha,\gamma\rangle$, the proof of the Lemma is complete. $\Box$

5.3 Cohomological characterization of Ressayre's data

Let $\gamma \in \mathfrak{t}$ be a non-zero rational element. We denote by $B \subset K_{\mathbb{C}}$ and by $\tilde{B} \subset \tilde{K}_{\mathbb{C}}$ the Borel subgroups with Lie algebra $\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}$ and $\tilde{\mathfrak{b}} = \tilde{\mathfrak{t}}_{\mathbb{C}} \oplus \tilde{\mathfrak{n}}$. Consider the parabolic subgroup $P_\gamma \subset K_{\mathbb{C}}$ defined by

$$P_\gamma = \{g \in K_{\mathbb{C}}, \lim_{t\to\infty} \exp(-it\gamma) g \exp(it\gamma) \text{ exists}\}. \tag{19}$$

Similarly, one defines a parabolic subgroup $\tilde{P}_\gamma \subset \tilde{K}_{\mathbb{C}}$.

We work with the projective varieties $\mathcal{F}_\gamma := K_{\mathbb{C}}/P_\gamma$, $\tilde{\mathcal{F}}_\gamma := \tilde{K}_{\mathbb{C}}/\tilde{P}_\gamma$, and the canonical embedding $\iota : \mathcal{F}_\gamma \to \tilde{\mathcal{F}}_\gamma$. We associate to any $\tilde{w} \in \tilde{W}$, the Schubert cell

$$\tilde{\mathfrak{X}}^o_{\tilde{w},\gamma} := \tilde{B}[\tilde{w}] \subset \tilde{\mathcal{F}}_\gamma.$$

and the Schubert variety $\tilde{\mathfrak{X}}_{\tilde{w},\gamma} := \overline{\tilde{\mathfrak{X}}^o_{\tilde{w},\gamma}}$. If $\tilde{W}^\gamma$ denotes the subgroup of $\tilde{W}$ that fixes γ, we see that the Schubert cell $\tilde{\mathfrak{X}}^o_{\tilde{w},\gamma}$ and the Schubert variety $\tilde{\mathfrak{X}}_{\tilde{w},\gamma}$ depends only of the class of $\tilde{w}$ in $\tilde{W}/\tilde{W}^\gamma$.

On the variety $\mathcal{F}_\gamma$, we consider the Schubert cell $\mathfrak{X}^o_\gamma := B[e]$ and the Schubert variety $\mathfrak{X}_\gamma := \overline{\mathfrak{X}^o_\gamma}$.

We consider the cohomology[1] ring $H^*(\tilde{\mathcal{F}}_\gamma,\mathbb{Z})$ of $\tilde{\mathcal{F}}_\gamma$. If Y is an irreducible closed subvariety of $\tilde{\mathcal{F}}_\gamma$, we denote by $[Y]\in H^{2n_Y}(\tilde{\mathcal{F}}_\gamma,\mathbb{Z})$ its cycle class in cohomology : here $n_Y=\mathrm{codim}_{\mathbb{C}}(Y)$. Let $\iota^*: H^*(\tilde{\mathcal{F}}_\gamma,\mathbb{Z})\to H^*(\mathcal{F}_\gamma,\mathbb{Z})$ be the pull-back map in cohomology. Recall that the cohomology class $[pt]$ associated to a singleton $Y=\{pt\}\subset\mathcal{F}_\gamma$ is a basis of $H^{\max}(\mathcal{F}_\gamma,\mathbb{Z})$.

To an oriented real vector bundle $\mathcal{E}\to N$ of rank r, we can associate its Euler class $\mathrm{Eul}(\mathcal{E})\in H^{2r}(N,\mathbb{Z})$. When $\mathcal{V}\to N$ is a complex vector bundle, then $\mathrm{Eul}(\mathcal{V}_{\mathbb{R}})$ corresponds to the top Chern class $c_p(\mathcal{V})$, where p is the complex rank of $\mathcal{V}$, and $\mathcal{V}_{\mathbb{R}}$ means $\mathcal{V}$ viewed as a real vector bundle oriented by its complex structure (see [5], §21).

The isomorphism $\mathfrak{q}^{\gamma>0}\simeq\mathfrak{q}/\mathfrak{q}^{\gamma\leq 0}$ shows that $\mathfrak{q}^{\gamma>0}$ can be viewed as a P_γ-module. Let $[\mathfrak{q}^{\gamma>0}]=K_{\mathbb{C}}\times_{P_\gamma}\mathfrak{q}^{\gamma>0}$ be the corresponding complex vector bundle on $\mathcal{F}_\gamma$. We denote simply by $\mathrm{Eul}(\mathfrak{q}^{\gamma>0})$ the Euler class $\mathrm{Eul}([\mathfrak{q}^{\gamma>0}]_{\mathbb{R}})\in H^*(\mathcal{F}_\gamma,\mathbb{Z})$.

The following characterization of Ressayre's data was obtained in [31], §6. Recall that $\mathfrak{R}_o$ denotes the set of weights relative to the T-action on $(\tilde{\mathfrak{g}}/\mathfrak{g})\otimes\mathbb{C}$.

Proposition 5.9 *An element $(\gamma,\tilde{w})\in\mathfrak{t}\times\tilde{W}$ is a Ressayre's data if and only if the following conditions hold :*

- *γ is non-zero and rational.*
- $\mathrm{Vect}(\mathfrak{R}_o\cap\gamma^\perp)=\mathrm{Vect}(\mathfrak{R}_o)\cap\gamma^\perp$.
- $[\mathfrak{X}_\gamma]\cdot\iota^*([\tilde{\mathfrak{X}}_{\tilde{w},\gamma}])\cdot\mathrm{Eul}(\mathfrak{q}^{\gamma>0})=k[pt],\ k\geq 1$ *in* $H^*(\mathcal{F}_\gamma,\mathbb{Z})$.
- $\sum_{\substack{\alpha\in\mathfrak{R}^+\\ \langle\alpha,\gamma\rangle>0}}\langle\alpha,\gamma\rangle=\sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}^+\\ \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle>0}}\langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle$.

5.4 Parametrization of the facets

We can finally describe the Kirwan polyhedron $\Delta(\mathbf{T}^*\tilde{K}\times\mathfrak{q})$ (see [31], §6).

Theorem 5.10 *An element $(\tilde{\xi},\xi)\in\tilde{\mathfrak{t}}^*_{\geq 0}\times\mathfrak{t}^*_{\geq 0}$ belongs to $\Delta(\mathbf{T}^*\tilde{K}\times\mathfrak{q})$ if and only if*

$$\langle\tilde{\xi},\tilde{w}\gamma\rangle+\langle\xi,\gamma\rangle\geq 0$$

for any Ressayre's data $(\gamma,\tilde{w})\in\mathfrak{t}\times\tilde{W}$.

Theorem 5.10 and Theorem 4.5 permit us to give the following description of the convex cone $\Pi_{\mathrm{hol}}(\tilde{G},G)$.

Theorem 5.11 *An element $(\tilde{\xi},\xi)$ belongs to $\Pi_{\mathrm{hol}}(\tilde{G},G)$ if and only if $(\tilde{\xi},\xi)\in\tilde{\mathcal{C}}_{\mathrm{hol}}\times\mathcal{C}_{\mathrm{hol}}$ and*

$$\langle\tilde{\xi},\tilde{w}\gamma\rangle\geq\langle\xi,w_0\gamma\rangle$$

for any $(\gamma,\tilde{w})\in\mathfrak{t}\times\tilde{W}$ satisfying the following conditions:

[1] Here, we use singular cohomology with integer coefficients.

- *γ is non-zero and rational.*
- $\mathrm{Vect}(\mathfrak{R}_o \cap \gamma^\perp) = \mathrm{Vect}(\mathfrak{R}_o) \cap \gamma^\perp$.
- $[\mathfrak{X}_\gamma] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w},\gamma}]) \cdot \mathrm{Eul}(\mathfrak{q}^{\gamma>0}) = k[pt],\ k \geq 1$ *in* $H^*(\mathcal{F}_\gamma, \mathbb{Z})$.
- $\sum_{\substack{\alpha \in \mathfrak{R}+ \\ \langle \alpha, \gamma \rangle > 0}} \langle \alpha, \gamma \rangle = \sum_{\substack{\tilde{\alpha} \in \tilde{\mathfrak{R}}+ \\ \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w}\gamma \rangle > 0}} \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w}\gamma \rangle$.

6 Example: the holomorphic Horn cone $\mathrm{Horn}_{\mathrm{hol}}(p, q)$

Let $p \geq q \geq 1$. We consider the pseudo-unitary group $G = U(p,q) \subset GL_{p+q}(\mathbb{C})$ defined by the relation : $g \in U(p,q)$ if and only if $g\mathrm{Id}_{p,q} g^* = \mathrm{Id}_{p,q}$ where $\mathrm{Id}_{p,q}$ is the diagonal matrice $\mathrm{Diag}(\mathrm{Id}_p, -\mathrm{Id}_q)$.

We work with the maximal compact subgroup $K = U(p) \times U(q) \subset G$. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is identified with the vector space $M_{p,q}$ of $p \times q$ matrices through the map

$$X \in M_{p,q} \longmapsto \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}.$$

We work with the element $z_{p,q} = \frac{i}{2}\mathrm{Id}_{p,q}$ which belongs to the center of $\mathfrak{k}$. The adjoint action of $z_{p,q}$ on $\mathfrak{p}$ corresponds to the standard complex structure on $M_{p,q}$.

The trace on $\mathfrak{gl}_{p+q}(\mathbb{C})$ defines an identification $\mathfrak{g} \simeq \mathfrak{g}^* = \hom(\mathfrak{g}, \mathbb{R})$: to $X \in \mathfrak{g}$ we associate $\xi_X \in \mathfrak{g}^*$ defined by $\langle \xi_X, Y \rangle = -\mathrm{Tr}(XY)$. Thus, the G-invariant cone $\mathcal{C}_{G/K}$ defined by $z_{p,q}$ can be viewed as the following cone of $\mathfrak{g}$:

$$\mathcal{C}(p,q) = \left\{ X \in \mathfrak{g},\ \mathrm{Im}\left(\mathrm{Tr}(gXg^{-1}\mathrm{Id}_{p,q})\right) \geq 0,\ \forall g \in U(p,q) \right\}.$$

Let $T \subset U(p) \times U(q)$ be the maximal torus formed by the diagonal matrices. The Lie algebra $\mathfrak{t}$ is identified with $\mathbb{R}^p \times \mathbb{R}^q$ through the map $\mathbf{d} : \mathbb{R}^p \times \mathbb{R}^q \to \mathfrak{u}(p) \times \mathfrak{u}(q)$: $\mathbf{d}_x = \mathrm{Diag}(ix_1, \cdots, ix_p, ix_{p+1}, \cdots, ix_{p+q})$. The Weyl chamber is

$$\mathfrak{t}_{\geq 0} = \{ x \in \mathbb{R}^p \times \mathbb{R}^q,\ x_1 \geq \cdots \geq x_p \text{ and } x_{p+1} \geq \cdots \geq x_{p+q} \}.$$

Proposition 2.2 tells us that the $U(p,q)$ adjoint orbits in the interior of $\mathcal{C}(p,q)$ are parametrized by the holomorphic chamber

$$\mathcal{C}_{p,q} = \{ x \in \mathbb{R}^p \times \mathbb{R}^q, x_1 \geq \cdots \geq x_p > x_{p+1} \geq \cdots \geq x_{p+q} \} \subset \mathfrak{t}_{\geq 0}.$$

Definition 6.1 *The holomorphic Horn cone* $\mathrm{Horn}_{\mathrm{hol}}(p,q) := \mathrm{Horn}^2_{\mathrm{hol}}(U(p,q))$ *is defined by the relations*

$$\mathrm{Horn}_{\mathrm{hol}}(p,q) = \left\{ (A,B,C) \in (\mathcal{C}_{p,q})^3,\ U(p,q)\mathbf{d}_C \subset U(p,q)\mathbf{d}_A + U(p,q)\mathbf{d}_B \right\},$$

Let us detail the description given of $\mathrm{Horn}_{\mathrm{hol}}(p,q)$ by Theorem B. For any $n \geq 1$, we consider the semigroup $\wedge_n^+ = \{(\lambda_1 \geq \cdots \geq \lambda_n)\} \subset \mathbb{Z}^n$. If $\lambda = (\lambda', \lambda'') \in \wedge_p^+ \times \wedge_q^+$, then $V_\lambda := V_{\lambda'}^{U(p)} \otimes V_{\lambda''}^{U(q)}$ denotes the irreducible representation of $U(p) \times U(q)$ with highest weight λ. We denote by $\mathrm{Sym}(M_{p,q})$ the symmetric algebra of $M_{p,q}$.

Definition 6.2 *1.* $\mathrm{Horn}^{\mathbb{Z}}(p,q)$ *is the semigroup of* $(\wedge_p^+ \times \wedge_q^+)^3$ *defined by the conditions:*

$$(\lambda, \mu, \nu) \in \mathrm{Horn}^{\mathbb{Z}}(p,q) \Longleftrightarrow [V_\nu \,:\, V_\lambda \otimes V_\mu \otimes \mathrm{Sym}(M_{p,q})] \neq 0.$$

2. $\mathrm{Horn}(p,q)$ *is the convex cone of* $(\mathfrak{t}_{\geq 0})^3$ *defined as the closure of* $\mathbb{Q}^{>0} \cdot \mathrm{Horn}^{\mathbb{Z}}(p,q)$.

Theorem B asserts that

$$\mathrm{Horn}_{\mathrm{hol}}(p,q) = \mathrm{Horn}(p,q) \bigcap (\mathcal{C}_{p,q})^3. \tag{20}$$

In another article [32], we obtained a recursive description of the cones $\mathrm{Horn}(p,q)$. This allows us to give the following description of the holomorphic Horn cone $\mathrm{Horn}_{\mathrm{hol}}(2,2)$.

Example 6.3 *An element* $(A,B,C) \in (\mathbb{R}^4)^3$ *belongs to* $\mathrm{Horn}_{\mathrm{hol}}(2,2)$ *if and only if the following conditions holds:*

$$\boxed{\begin{array}{ccc} a_1 \geq a_2 & > & a_3 \geq a_4 \\ b_1 \geq b_2 & > & b_3 \geq b_4 \\ c_1 \geq c_2 & > & c_3 \geq c_4 \end{array}}$$

$$\boxed{a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 + b_4 = c_1 + c_2 + c_3 + c_4}$$

$$\boxed{a_1 + a_2 + b_1 + b_2 \leq c_1 + c_2}$$

$$\boxed{\begin{array}{ccc} a_2 + b_2 & \leq & c_2 \\ a_2 + b_1 & \leq & c_1 \\ a_1 + b_2 & \leq & c_1 \end{array}}$$

$$\boxed{\begin{array}{ccc} a_3 + b_3 & \geq & c_3 \\ a_3 + b_4 & \geq & c_4 \\ a_4 + b_3 & \geq & c_4 \end{array}}$$

$$\boxed{\begin{array}{ccc} a_2 + a_4 + b_2 + b_4 & \leq & c_1 + c_4 \\ a_2 + a_4 + b_2 + b_4 & \leq & c_2 + c_3 \\ a_2 + a_4 + b_1 + b_4 & \leq & c_1 + c_3 \\ a_1 + a_4 + b_2 + b_4 & \leq & c_1 + c_3 \\ a_2 + a_4 + b_2 + b_3 & \leq & c_1 + c_3 \\ a_2 + a_3 + b_2 + b_4 & \leq & c_1 + c_3 \end{array}}$$

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