$\mathrm{Horn}(p, q)$

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Abstract

In this article, we obtain a recursive description of the Horn cone $\mathrm{Horn}(p, q)$ with respect to the integers p and q, as in the classical Horn's conjecture.

Contents

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1 Introduction

When G is Lie group, a natural problem is to understand how the sum of two adjoint orbits decomposes into a union of adjoint orbits. Let $\mathfrak{g}$ be the the Lie algebra of G and let $\mathfrak{g}/G$ be the set of adjoint orbits. The *Horn cone* is defined as follows

$$\mathrm{Horn}(G) = \{(\mathcal{O}, \mathcal{O}', \mathcal{O}'') \in (\mathfrak{g}/G)^3,\ \mathcal{O}'' \subset \mathcal{O} + \mathcal{O}'\}.$$

Consider the case where G is a compact connected Lie group. Let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t}$. The set $\mathfrak{g}/G$ admits a canonical identification with a Weyl chamber $\mathfrak{t}_{\geqslant 0} \subset \mathfrak{t}$. In this setting, the Horn cone $\mathrm{Horn}(G) \subset (\mathfrak{t}_{\geqslant 0})^3$ has been at the center of numerous studies [13, 15, 16, 4, 2, 3, 14, 27] that we summarize by the following theorem. We refer the reader to the survey articles [5, 18] for details.

Theorem 1.1 *If G is a compact connected Lie group,* $\mathrm{Horn}(G)$ *is a polyhedral convex cone and one can parametrize the equation of its facets by cohomological means.*

1.1 Horn's conjecture

When G is the unitary group $U(n)$, the convex polyhedral cone[1] $\mathrm{Horn}(n)$ has a nice feature which was predicted by A. Horn in the 60s : it admits a recursive description relative to the integer $n \geqslant 1$ [13].

Denote the set of cardinality r subsets $I = \{i_1 < i_2 < \cdots < i_r\}$ of $[n] = \{1, \ldots, n\}$ by $\mathcal{P}_r^n$. To each $I \in \mathcal{P}_r^n$ we associate a weakly decreasing sequence of non-negative integers

$$\lambda(I) = (\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_r) \;\in\; \mathbb{Z}_{\geqslant 0}^r \tag{1}$$

where $\lambda_a = n - r + a - i_a$ for $a \in [r]$.

Let $\mathbf{d} : \mathbb{R}^n \to \mathfrak{u}(n)$ be the map that sends $X = (x_1, \ldots, x_n)$ to the diagonal matrix $\mathbf{d}_X = \mathrm{Diag}(ix_1, \ldots, ix_n)$. The map $\mathbf{d}$ induces a one to one correspondence between $\mathcal{C}_n = \{(x_1 \geqslant \cdots \geqslant x_n)\} \subset \mathbb{R}^n$ and the set of $U(n)$-adjoint orbits. If $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $I \subset [n]$, we define $|\, X\,|_I = \sum_{i \in I} x_i$ and $|\, X\,| = \sum_{i=1}^n x_i$.

Definition 1.2 *Let $n \geqslant 1$.*

$$\mathrm{Horn}(n) = \{(A, B, C) \in (\mathcal{C}_n)^3,\ U(n)\mathbf{d}_C \subset U(n)\mathbf{d}_A + U(n)\mathbf{d}_B\}.$$

The following Horn's conjecture [13] was settled in the affirmative by combining the work of A. Klyachko [15] with the work of A. Knutson and T. Tao [16] on the "saturation" problem. We refer the reader to Fulton's survey article [10] for details.

Theorem 1.3 (Horn's conjecture) *An element $(A, B, C) \in (\mathcal{C}_n)^3$ belongs to* $\mathrm{Horn}(n)$ *if and only if the following conditions holds*

- $|\, A\,| + |\, B\,| = |\, C\,|$,
- $\forall r \in [n-1]$, $\forall I, J, K \in \mathcal{P}_r^n$, *we have*

$$|\, A\,|_I + |\, B\,|_J \leqslant |\, C\,|_K \quad \text{if} \quad (\lambda(I), \lambda(J), \lambda(K)) \in \mathrm{Horn}(r).$$

[1] We note $\mathrm{Horn}(U(n))$ simply by $\mathrm{Horn}(n)$.

1.2 Holomorphic Horn cone $\mathrm{Horn}_{\mathrm{hol}}(p,q)$

Let $p \geqslant q \geqslant 1$. We begin by recalling the definition of the holomorphic Horn cone $\mathrm{Horn}_{\mathrm{hol}}(p,q)$ associated with the pseudo-unitary group $U(p,q)$.

The Lie group $U(p,q) \subset GL_{p+q}(\mathbb{C})$ is defined by the relations $g\mathrm{Id}_{p,q}g^* = \mathrm{Id}_{p,q}$, where $\mathrm{Id}_{p,q}$ is the diagonal matrice $\mathrm{Diag}(\mathrm{Id}_p, -\mathrm{Id}_q)$. The Lie algebra $\mathfrak{u}(p,q)$ of $U(p,q)$ admits the following invariant convex cone

$$\mathcal{C}(p,q) = \left\{X \in \mathfrak{u}(p,q),\ \mathrm{Im}(\mathrm{Tr}(gXg^{-1}\mathrm{Id}_{p,q})) \geqslant 0,\ \forall g \in U(p,q)\right\}.$$

Let us consider

$$\mathcal{C}_{p,q} = \{x \in \mathbb{R}^p \times \mathbb{R}^q, x_1 \geqslant \cdots \geqslant x_p > x_{p+1} \geqslant \cdots \geqslant x_{p+q}\} \subset \mathcal{C}_p \times \mathcal{C}_q$$

and the map $\mathbf{d} : \mathbb{R}^p \times \mathbb{R}^q \to \mathfrak{u}(p,q)$. A well-know result says that for any $U(p,q)$-orbit $\mathcal{O}$ contained in the interior of $\mathcal{C}(p,q)$, there exists a unique $X \in \mathcal{C}_{p,q}$ such that $\mathcal{O} = U(p,q)\mathbf{d}_X$ (see [28, 21]). In other words, the map $\mathbf{d}$ realizes a one to one map between $\mathcal{C}_{p,q}$ and the set of $U(p,q)$-orbits in the interior of the invariant convex cone $\mathcal{C}(p,q)$. The holomorphic Horn cone is then defined as follows :

$$\mathrm{Horn}_{\mathrm{hol}}(p,q) = \left\{(A,B,C) \in (\mathcal{C}_{p,q})^3,\ U(p,q)\mathbf{d}_C \subset U(p,q)\mathbf{d}_A + U(p,q)\mathbf{d}_B\right\}.$$

In a companion paper [23], we have proved that $\mathrm{Horn}_{\mathrm{hol}}(p,q)$ is a closed convex cone of $(\mathcal{C}_{p,q})^3$, and we have explained a way to compute it. In order to detail this result, we need some additional notations. For any $n \geqslant 1$, we consider the semigroup $\wedge_n^+ = \{(\lambda_1 \geqslant \cdots \geqslant \lambda_n)\} \subset \mathbb{Z}^n$. If $\lambda = (\lambda', \lambda'') \in \wedge_p^+ \times \wedge_q^+$, then $V_\lambda := V_{\lambda'}^{U(p)} \otimes V_{\lambda''}^{U(q)}$ denotes the irreducible representation of $U(p) \times U(q)$ with highest weight λ. We denote by $M_{p,q}$ the vector space of $p \times q$ complex matrices, and by $\mathrm{Sym}(M_{p,q})$ the symmetric algebra of $M_{p,q}$.

If H is a representation of $U(p) \times U(q)$, we denote by $[V_\nu : H]$ the multiplicity of V_ν in H.

Definition 1.4 *1.* $\mathrm{Horn}^{\mathbb{Z}}(p,q)$ *is the semigroup of* $(\wedge_p^+ \times \wedge_q^+)^3$ *defined by the conditions:*

$$(\lambda,\mu,\nu) \in \mathrm{Horn}^{\mathbb{Z}}(p,q) \Longleftrightarrow [V_\nu : V_\lambda \otimes V_\mu \otimes \mathrm{Sym}(M_{p,q})] \neq 0.$$

2. $\mathrm{Horn}(p,q)$ *is the convex cone of* $(\mathcal{C}_p \times \mathcal{C}_q)^3$ *defined as the closure of* $\mathbb{Q}^{>0} \cdot \mathrm{Horn}^{\mathbb{Z}}(p,q)$.

The following result is proved in [23].

Theorem 1.5 *We have*

$$\mathrm{Horn}_{\mathrm{hol}}(p,q) = \mathrm{Horn}(p,q) \bigcap (\mathcal{C}_{p,q})^3.$$

1.3 Statement of the main result

We now explain the main purpose of this paper that concerns a recursive description of the convex polyhedral cones $\mathrm{Horn}(p, q)$ as in Horn's conjecture. We need another notations.

1. If $A = (A', A'') \in \mathbb{R}^p \times \mathbb{R}^q$ and $I = I' \times I'' \subset [p] \times [q]$, we define $|\,A\,|_I = |\,A'\,|_{I'} + |\,A''\,|_{I''}$ and $|\,A\,| = |\,A'\,| + |\,A''\,|$.
2. If $I = I' \times I'' \subset [n] \times [m]$ then $\lambda(I) = (\lambda(I'), \lambda(I'')) \in \wedge_n^+ \times \wedge_m^+$.
3. Let $\mathbf{1}_n = (1, \ldots, 1) \in \mathbb{Z}^n$.

The main result of this paper is the following theorem.

Theorem 1.6 *Let $p \geqslant q \geqslant 1$. An element $(A, B, C) \in (\mathcal{C}_p \times \mathcal{C}_q)^3$ belongs to* $\mathrm{Horn}(p, q)$ *if and only if the following conditions holds:*

- $\boxed{|\,A\,| + |\,B\,| = |\,C\,|}$.
- $\boxed{|\,A'\,| + |\,B'\,| \leqslant |\,C'\,|}$.
- *For any $r \in [p-1]$, for any $I', J', K' \in \mathcal{P}_r^p$, we have :*

$$\boxed{|\,A'\,|_{I'} + |\,B'\,|_{J'} \leqslant |\,C'\,|_{K'}} \quad \text{if } (\lambda(I'), \lambda(J'), \lambda(K')) \in \mathrm{Horn}(r).$$
$$\boxed{|\,A'\,|_{I'} + |\,B'\,|_{J'} \geqslant |\,C'\,|_{K'}} \quad \text{if } (\lambda(I'), \lambda(J'), \lambda(K') + (q+p-r)\mathbf{1}_r) \in \mathrm{Horn}(r).$$

- *For any $s \in [q-1]$, for any $I'', J'', K'' \in \mathcal{P}_s^q$, we have :*

$$\boxed{|\,A''\,|_{I''} + |\,B''\,|_{J''} \geqslant |\,C''\,|_{K''}} \quad \text{if } (\lambda(I''), \lambda(J''), \lambda(K'') + (q-s)\mathbf{1}_s) \in \mathrm{Horn}(s).$$
$$\boxed{|\,A''\,|_{I''} + |\,B''\,|_{J''} \leqslant |\,C''\,|_{K''}} \quad \text{if } (\lambda(I''), \lambda(J''), \lambda(K'') - p\mathbf{1}_s) \in \mathrm{Horn}(s).$$

- *For any $(r, s) \in [p-1] \times [q-1]$ with $r \geqslant s$, for any $I, J, K \in \mathcal{P}_r^p \times \mathcal{P}_s^q$, we have*

$$\boxed{|\,A\,|_I + |\,B\,|_J \leqslant |\,C\,|_K} \quad \text{if } \big(\lambda(I), \lambda(J), \lambda(K) + (0, (r-p)\mathbf{1}_s)\big) \in \mathrm{Horn}(r, s).$$

1.4 Examples

The convex cones Horn(1, 1), Horn(2, 1) and Horn(2, 2) admit the following descriptions.

Proposition 1.7 *An element* $(A, B, C) \in (\mathbb{R} \times \mathbb{R})^3$ *belongs to* Horn(1, 1) *if and only if the following conditions holds:*

$$\boxed{a_1 + a_2 + b_1 + b_2 = c_1 + c_2}$$

$$\boxed{a_1 + b_1 \leqslant c_1}$$

Proposition 1.8 *An element* $(A, B, C) \in (\mathcal{C}_2 \times \mathbb{R})^3$ *belongs to* Horn(2, 1) *if and only if the following conditions holds:*

$$\boxed{a_1 + a_2 + a_3 + b_1 + b_2 + b_3 = c_1 + c_2 + c_3}$$

$$\boxed{a_1 + a_2 + b_1 + b_2 \leqslant c_1 + c_2}$$

$$\boxed{\begin{array}{rcl} a_2 + b_2 & \leqslant & c_2 \\ a_2 + b_1 & \leqslant & c_1 \\ a_1 + b_2 & \leqslant & c_1 \\ a_1 + b_1 & \geqslant & c_2 \end{array}}$$

Proposition 1.9 *An element* $(A, B, C) \in (\mathcal{C}_2 \times \mathcal{C}_2)^3$ *belongs to* Horn(2, 2) *if and only if the following conditions holds:*

$$\boxed{a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 + b_4 = c_1 + c_2 + c_3 + c_4}$$

$$\boxed{a_1 + a_2 + b_1 + b_2 \leqslant c_1 + c_2}$$

$$\boxed{\begin{array}{rcl} a_2 + b_2 & \leqslant & c_2 \\ a_2 + b_1 & \leqslant & c_1 \\ a_1 + b_2 & \leqslant & c_1 \end{array}}$$

$$\boxed{\begin{array}{rcl} a_3 + b_3 & \geqslant & c_3 \\ a_3 + b_4 & \geqslant & c_4 \\ a_4 + b_3 & \geqslant & c_4 \end{array}}$$

$$\boxed{\begin{array}{rcl} a_2+a_4+b_2+b_4 & \leqslant & c_1+c_4 \\ a_2+a_4+b_2+b_4 & \leqslant & c_2+c_3 \\ a_2+a_4+b_1+b_4 & \leqslant & c_1+c_3 \\ a_1+a_4+b_2+b_4 & \leqslant & c_1+c_3 \\ a_2+a_4+b_2+b_3 & \leqslant & c_1+c_3 \\ a_2+a_3+b_2+b_4 & \leqslant & c_1+c_3 \end{array}}$$

1.5 Outline of the article

The recursive description of $\mathrm{Horn}(p,q)$ is obtained by studying the Hamiltonian action of $(U(p)\times U(q))^3$ on the manifold[2] $(GL_p\times GL_q)^2\times\mathbb{C}^p\otimes\mathbb{C}^q$. Let $\mathrm{S}(p,q)\subset(\mathcal{C}_p\times\mathcal{C}_q)^3$ be the corresponding Kirwan polyhedron.

In §2, we study the general framework of a Hamiltonian action of a compact Lie group K^3 on $(K_{\mathbb{C}}\times K_{\mathbb{C}})^2\times E$: here E is a K-module such that the coordinate ring $\mathbb{C}[E]$ does not admit non-constant invariant vectors. We explain how to parameterize the facets of the Kirwan polyhedron $\Delta((K_{\mathbb{C}}\times K_{\mathbb{C}})^2\times E)$ in terms of Ressayre's data [22]. This parametrization requires two steps : determination of the admissible elements which are the potential vectors orthogonal to the facets, and computation of cohomological conditions on flag varieties.

In §3, we check that the semigroup $\mathrm{Horn}^{\mathbb{Z}}(p,q)$ is saturated. It is a direct consequence of the Darksen-Weyman saturation theorem [7].

In §4, we determine the admissible elements relative to the action of $(U(p)\times U(q))^3$ on $(GL_p\times GL_q)^2\times\mathbb{C}^p\otimes\mathbb{C}^q$, and we detailed the cohomological conditions in this particular case. The formulas need the computation of certain Euler classes which we carry over to §5.

In §6, we calculate (recursively) the facets of the Kirwan polyhedron $\mathrm{S}(p,q)$. In the last subsection, we complete the proof of our main result.

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2 The K^3-manifold $K_{\mathbb{C}}\times K_{\mathbb{C}}\times E$

In this section, we briefly recall the result of §6 of [22] concerning the parametrization of the facets of Kirwan polyhedrons in terms of Ressayre's

[2] We use the notation GL_n for the Lie group $GL(\mathbb{C}^n)$.

data.

Let K be a compact connected Lie group with complexification $K_{\mathbb{C}}$. Let $T \subset K$ be a maximal torus with Lie algebra $\mathfrak{t}$. We consider the lattice $\wedge := \frac{1}{2\pi}\ker(\exp : \mathfrak{t} \to T)$ and the dual lattice $\wedge^* \subset \mathfrak{t}^*$ defined by $\wedge^* = \hom(\wedge, \mathbb{Z})$. We remark that $i\eta$ is a differential of a character of T if and only if $\eta \in \wedge^*$. The $\mathbb{Q}$-vector space generated by the lattice $\wedge^*$ is denoted by $\mathfrak{t}^*_{\mathbb{Q}}$: the vectors belonging to $\mathfrak{t}^*_{\mathbb{Q}}$ are designed as rational. Let $\mathfrak{t}^*_{\geqslant 0}$ be a Weyl chamber. The set $\wedge^*_+ := \wedge^* \cap \mathfrak{t}^*_{\geqslant 0}$ parametrizes the irreducible representations of K: for any $\mu \in \wedge^*_+$, we denote by V_μ the irreducible representation of K with highest weight μ.

When K acts linearly on a vector space H, we denote by H^K the subspace of invariant vectors under the K-action.

Let E be a K-module such that $\mathbb{C}[E]^K = \mathbb{C}$: hence the coordinate ring $\mathbb{C}[E]$ has finite K-multiplicities. We consider the following $K \times K \times K$ action on the affine variety $K_{\mathbb{C}} \times K_{\mathbb{C}} \times E$:

$$(k_1, k_2, k_3) \cdot (x, y, v) = (k_1 x k_3^{-1}, k_2 y k_3^{-1}, k_3 v).$$

The coordinate ring $\mathbb{C}[K_{\mathbb{C}} \times K_{\mathbb{C}} \times E]$, viewed as a K^3-module, admits the following decomposition

$$\mathbb{C}[K_{\mathbb{C}} \times K_{\mathbb{C}} \times E] = \sum_{\lambda,\mu,\nu \in \wedge^*_+} m_E(\lambda, \mu, \nu)\ V^1_\lambda \otimes V^2_\mu \otimes V^3_\nu,$$

where $m_E(\lambda, \mu, \nu) = \dim[V_\lambda \otimes V_\mu \otimes V_\nu \otimes \mathrm{Sym}(E)]^K$.

Definition 2.1 *We define the following sets :*

- *The semigroup $\Delta^{\mathbb{Z}}(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E) \subset (\wedge^*_+)^3$ is defined as follows: $(\lambda, \mu, \nu) \in \Delta^{\mathbb{Z}}(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E) \Longleftrightarrow m_E(\lambda, \mu, \nu) \neq 0$.*
- *The convex cone $\Delta(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E) \subset (\mathfrak{t}^*_{\geqslant 0})^3$ is the closure of $\mathbb{Q}^{>0} \cdot \Delta^{\mathbb{Z}}(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E)$.*

Let us explain why the complex K^3-manifold $N = K_{\mathbb{C}} \times K_{\mathbb{C}} \times E$ admits a symplectic structure Ω_N compatible with the complex structure, and a moment map $\Phi : N \to (\mathfrak{k}^*)^3$ associated to the action of K^3 on (N, Ω_N).

Let h_E be a K-invariant hermitian structure on E. We equip E with the 2-form $\Omega_E = -\mathrm{Im}(h_E)$. The moment map Φ_E relative to the action of K on the symplectic vector space (E, Ω_E) is defined by

$$\langle \Phi_E(v), X \rangle = \tfrac{1}{2}\Omega_E(Xv, v), \quad \forall v \in V,\ \forall X \in \mathfrak{k}.$$

The hypothesis $\mathbb{C}[E]^K = \mathbb{C}$ implies that Φ_E is a proper map.

There is a diffeomorphism of the cotangent bundle $\mathbf{T}^*K$ with $K_\mathbb{C}$ defined as follows. We identify $\mathbf{T}^*K$ with $K \times \mathfrak{k}^*$ by means of left-translation and then with $K \times \mathfrak{k}$ by means of an invariant inner product on $\mathfrak{k}$. The map $\varphi : K \times \mathfrak{k} \to K_\mathbb{C}$ given by $\varphi(k, X) = ke^{iX}$ is a diffeomorphism. If we use φ to transport the canonical symplectic 2-form of $\mathbf{T}^*K$ to $K_\mathbb{C}$, then the resulting 2-form $\Omega_{K_\mathbb{C}}$ on $K_\mathbb{C}$ is compatible with the complex structure (see [12], §3).

Finally, the K^3-manifold $K_\mathbb{C} \times K_\mathbb{C} \times E \simeq \mathbf{T}^*K \times \mathbf{T}^*K \times E$ carries the symplectic 2-form $\Omega_N := \Omega_{K_\mathbb{C}} \times \Omega_{K_\mathbb{C}} \times \Omega_E$ which is compatible with the complex structure. The moment map relative to the K^3-action on (N, Ω_N) is the proper map $\Phi = \Phi_1 \oplus \Phi_2 \oplus \Phi_3 : \mathbf{T}^*K \times \mathbf{T}^*K \times E \to \mathfrak{k}^* \oplus \mathfrak{k}^* \oplus \mathfrak{k}^*$ defined by

$$\Phi(g_1, \xi_1, g_2, \xi_2, v) = (-g_1\xi_1, -g_2\xi_2, \xi_1 + \xi_2 + \Phi_E(v)). \tag{2}$$

By definition, the Kirwan polyhedron $\Delta(\mathbf{T}^*K \times \mathbf{T}^*K \times E)$ is the intersection of the image of Φ with $(\mathfrak{t}^*_{\geqslant 0})^3$. The following result is classical (see Theorem 4.9 in [25]).

Proposition 2.2 *The Kirwan polyhedron $\Delta(\mathbf{T}^*K \times \mathbf{T}^*K \times E)$ is equal to $\Delta(K_\mathbb{C} \times K_\mathbb{C} \times E)$.*

2.1 Admissible elements

Definition 2.3 *When a Lie group G acts on a manifold N, the stabilizer subgroup of $n \in N$ is denoted by $G_n = \{g \in G, gn = n\}$, and its Lie algebra by $\mathfrak{g}_n$. Let us define $\dim_G(\mathcal{X}) = \min_{n\in\mathcal{X}} \dim(\mathfrak{g}_n)$ for any subset $\mathcal{X} \subset N$.*

We start by introducing the notion of admissible elements. The group $\hom(U(1), T)$ admits a natural identification with the lattice $\wedge := \frac{1}{2\pi}\ker(\exp : \mathfrak{t} \to T)$. A vector $\gamma \in \mathfrak{t}$ is called rational if it belongs to the $\mathbb{Q}$-vector space $\mathfrak{t}_\mathbb{Q}$ generated by $\wedge$.

We consider the K^3-action on $N := \mathbf{T}^*K \times \mathbf{T}^*K \times E$.

Definition 2.4 *A non-zero element $(\gamma_1, \gamma_2, \gamma_3) \in \mathfrak{t}^3$ is called* admissible *if the elements γ_i are rational and if $\dim_{K^3}(N^{(\gamma_1,\gamma_2,\gamma_3)}) - \dim_{K^3}(N) \in \{0, 1\}$.*

Let $\mathfrak{R}$ be the set of roots for (K, T), and let $W = N(T)/T$ be the Weyl group. The set of weights for the T-action on E is denoted $\mathfrak{R}_E$. If $\gamma \in \mathfrak{t}$, we denote by $(\mathfrak{R} \cup \mathfrak{R}_E) \cap \gamma^\perp$ the subsets of weight vanishing against γ.

If $w = (w_1, w_2, w_3) \in W^3$ and $\gamma \in \mathfrak{t}$, we write $\gamma_w = (w_1\gamma, w_2\gamma, w_3\gamma)$. We start with the following lemma whose proof is left to the reader.

Lemma 2.5 *1.* $N^{(\gamma_1,\gamma_2,\gamma_3)} \neq \emptyset$ *if and only if* $\gamma_1, \gamma_2 \in W\gamma_3$.

2. $\dim_{K^3}(N) = \dim_T(\mathfrak{k} \times E) = \dim(\mathfrak{t}) - \dim(\mathrm{Vect}(\mathfrak{R} \cup \mathfrak{R}_E))$.

3. $\dim_{K^3}(N^{\gamma_w}) = \dim_T(\mathfrak{k}^\gamma \times E^\gamma) = \dim(\mathfrak{t}) - \dim(\mathrm{Vect}((\mathfrak{R} \cup \mathfrak{R}_E) \cap \gamma^\perp))$.

The following result is a direct consequence of the previous lemma.

Lemma 2.6 *The admissible elements relative to the* K^3*-action on* $\mathbf{T}^*K \times \mathbf{T}^*K \times E$ *are of the form* γ_w *where* $w \in W^3$ *and* γ *is a non-zero rational element satisfying* $\mathrm{Vect}(\mathfrak{R} \cup \mathfrak{R}_E) \cap \gamma^\perp = \mathrm{Vect}((\mathfrak{R} \cup \mathfrak{R}_E) \cap \gamma^\perp)$.

2.2 Ressayre's data

Definition 2.7 *1. Consider the linear action* $\rho : G \to \mathrm{GL}_\mathbb{C}(V)$ *of a compact Lie group on a complex vector space* V*. For any* $(\eta, a) \in \mathfrak{g} \times \mathbb{R}$*, we define the vector subspace* $V^{\eta=a} = \{v \in V, d\rho(\eta)v = iav\}$*. Thus, for any* $\eta \in \mathfrak{g}$*, we have the decomposition* $V = V^{\eta>0} \oplus V^{\eta=0} \oplus V^{\eta<0}$ *where* $V^{\eta>0} = \sum_{a>0} V^{\eta=a}$*, and* $V^{\eta<0} = \sum_{a<0} V^{\eta=a}$*.*

2. The real number $\mathrm{Tr}_\eta(V^{\eta>0})$ *is defined as the sum* $\sum_{a>0} a \dim(V^{\eta=a})$*.*

We consider an admissible element $\gamma_w = (w_1\gamma, w_2\gamma, w_3\gamma)$. The submanifold fixed by γ_w is $N^{\gamma_w} = w_1 K_\mathbb{C}^\gamma w_3^{-1} \times w_2 K_\mathbb{C}^\gamma w_3^{-1} \times E^{w_3\gamma}$. There is a canonical isomorphism of the manifold N^{γ_w} equipped with the action of $w_1 K^\gamma w_1^{-1} \times w_2 K^\gamma w_2^{-1} \times w_3 K^\gamma w_3^{-1}$ with the manifold $K_\mathbb{C}^\gamma \times K_\mathbb{C}^\gamma \times E^\gamma$ equipped with the action of $K^\gamma \times K^\gamma \times K^\gamma$. The tangent bundle $(\mathbf{T}N|_{N^{\gamma_w}})^{\gamma_w>0}$ is isomorphic to $N^{\gamma_w} \times \mathfrak{k}_\mathbb{C}^{\gamma>0} \times \mathfrak{k}_\mathbb{C}^{\gamma>0} \times E^{\gamma>0}$.

The choice of positive roots $\mathfrak{R}^+$ induces a decomposition $\mathfrak{k}_\mathbb{C} = \mathfrak{n} \oplus \mathfrak{t}_\mathbb{C} \oplus \overline{\mathfrak{n}}$, where $\mathfrak{n} = \sum_{\alpha \in \mathfrak{R}^+} (\mathfrak{k} \otimes \mathbb{C})_\alpha$. We consider the map

$$\rho^{\gamma,w} : K_\mathbb{C}^\gamma \times K_\mathbb{C}^\gamma \times E^\gamma \longrightarrow \hom\left(\mathfrak{n}^{w_1\gamma>0} \times \mathfrak{n}^{w_2\gamma>0} \times \mathfrak{n}^{w_3\gamma>0}, \mathfrak{k}_\mathbb{C}^{\gamma>0} \times \mathfrak{k}_\mathbb{C}^{\gamma>0} \times E^{\gamma>0}\right)$$

defined by the relation

$$\rho^{\gamma,w}(x,y,v) : (X,Y,Z) \mapsto ((w_1x)^{-1}X - w_3^{-1}Z\,;\, (w_2y)^{-1}Y - w_3^{-1}Z\,;\, (w_3^{-1}Z)\cdot v),$$

for any $(x,y,v) \in K_\mathbb{C}^\gamma \times K_\mathbb{C}^\gamma \times E^\gamma$.

Definition 2.8 $(\gamma, w) \in \mathfrak{t} \times W^3$ *is a Ressayre's data if*

1. γ_w *is admissible,*

2. $\exists(x,y,v)$ such that $\rho^{\gamma,w}(x,y,v)$ is bijective.

Remark 2.9 *In [22], the Ressayre's data were called* regular infinitesimal B-Ressayre's pairs.

Since the linear map $\rho^{\gamma,w}(x,y,v)$ commutes with the γ-actions, we obtain the following necessary conditions.

Lemma 2.10 *If $(\gamma,w)\in\mathfrak{t}\times W^3$ is a Ressayre's data, then*

- *Relation (A) : $\sum_{i=1}^3 \dim(\mathfrak{n}^{w_i\gamma>0}) = 2\dim(\mathfrak{k}_{\mathbb{C}}^{\gamma>0}) + \dim(E^{\gamma>0})$.*
- *Relation (B) : $\sum_{i=1}^3 \mathrm{Tr}_{w_i\gamma}(\mathfrak{n}^{w_i\gamma>0}) = 2\,\mathrm{Tr}_\gamma(\mathfrak{k}_{\mathbb{C}}^{\gamma>0}) + \mathrm{Tr}_\gamma(E^{\gamma>0})$.*

2.3 Cohomological characterization of Ressayre's data

Let $\gamma\in\mathfrak{t}$ be a rational element. We denote by $B\subset K_{\mathbb{C}}$ the Borel subgroup with Lie algebra $\mathfrak{b}=\mathfrak{t}_{\mathbb{C}}\oplus\mathfrak{n}$. Consider the parabolic subgroup $P_\gamma\subset K_{\mathbb{C}}$ defined by

$$P_\gamma=\{g\in K_{\mathbb{C}}, \lim_{t\to\infty}\exp(-it\gamma)g\exp(it\gamma) \text{ exists}\}. \tag{3}$$

We work with the projective variety $\mathcal{F}_\gamma := K_{\mathbb{C}}/P_\gamma$. We associate to any $w\in W$, the Schubert cell

$$\mathfrak{X}^o_{w,\gamma} := B[w]\subset\mathcal{F}_\gamma,$$

and the Schubert variety $\mathfrak{X}_{w,\gamma} := \overline{\mathfrak{X}^o_{w,\gamma}}$. If W^γ denotes the subgroup of W that fixes γ, we see that the Schubert cell $\mathfrak{X}^o_{w,\gamma}$ and the Schubert variety $\mathfrak{X}_{w,\gamma}$ depends only of the class of w in W/W^γ.

We consider the cohomology[3] ring $H^*(\mathcal{F}_\gamma,\mathbb{Z})$ of $\mathcal{F}_\gamma$. If Y is an irreducible closed subvariety of $\mathcal{F}_\gamma$, we denote by $[Y]\in H^{2n_Y}(\mathcal{F}_\gamma,\mathbb{Z})$ its cycle class in cohomology : here $n_Y=\mathrm{codim}(Y)$. Recall that the cohomology class $[pt]$ associated to a singleton $\{pt\}\subset\mathcal{F}_\gamma$ is a basis of $H^{\max}(\mathcal{F}_\gamma,\mathbb{Z})$.

To an oriented real vector bundle $\mathcal{E}\to N$ of rank r, we can associate its Euler class $\mathrm{Eul}(\mathcal{E})\in H^r(N,\mathbb{Z})$. When $\mathcal{E}\to N$ is a complex vector bundle, then $\mathrm{Eul}(\mathcal{E}_{\mathbb{R}})$ corresponds to the top Chern class $c_p(\mathcal{E})$. Here p is the complex rank of $\mathcal{E}$, and $\mathcal{E}_{\mathbb{R}}$ means $\mathcal{E}$ viewed as a real vector bundle oriented by its complex structure (see [6], §21).

The isomorphism $E^{\gamma>0}\simeq E/E^{\gamma\leq 0}$ shows that $E^{\gamma>0}$ can be viewed as a P_γ-module. Let $\mathcal{E}^{\gamma>0}=K_{\mathbb{C}}\times_{P_\gamma}E^{\gamma>0}$ be the corresponding complex vector

[3] Here, we use singular cohomology with integer coefficients.

bundle on $\mathcal{F}_\gamma$. In the following proposition, we denote simply by $\mathrm{Eul}(E^{\gamma>0})$ the Euler class $\mathrm{Eul}(\mathcal{E}_{\mathbb{R}}^{\gamma>0}) \in H^*(\mathcal{F}_\gamma, \mathbb{Z})$.

The following characterization of Ressayre's data was obtained in [22], §6.

Proposition 2.11 *$(\gamma, w) \in \mathfrak{t} \times W^3$ is a Ressayre's data if and only if*

1. *γ_w is admissible,*
2. *Relation (B) holds,*
3. *The following relation holds in $H^*(\mathcal{F}_\gamma, \mathbb{Z})$:*

$$(4) \qquad [\mathfrak{X}_{w_1,\gamma}] \cdot [\mathfrak{X}_{w_2,\gamma}] \cdot [\mathfrak{X}_{w_3,\gamma}] \cdot \mathrm{Eul}(E^{\gamma>0}) = k[pt], \quad k \geqslant 1.$$

Remark 2.12 *Notice that relation (A) is equivalent to $\sum_{i=1}^3 \mathrm{codim}(\mathfrak{X}_{w_i,\gamma}) + \dim(E^{\gamma>0}) = \dim(\mathcal{F}_\gamma)$, hence relation (A) follows from (4).*

2.4 Convex cone $\Delta(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E)$: equations of the facets

The following result is proved in [22], §6.

Theorem 2.13 *Let E be a K-module such that $\mathbb{C}[E]^K = \mathbb{C}$. An element $(\xi_1, \xi_2, \xi_3) \in (\mathfrak{t}^*_{\geqslant 0})^3$ belongs to $\Delta(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E)$ if and only*

$$(5) \qquad \langle \xi_1, w_1\gamma \rangle + \langle \xi_2, w_2\gamma \rangle + \langle \xi_3, w_3\gamma \rangle \geqslant 0$$

for any $(\gamma, w) \in \mathfrak{t} \times W^3$ that is a Ressayre's data, that is to say satisfying the following properties:

a) γ is a non-zero rational element.

b) $\mathrm{Vect}(\mathfrak{R} \cup \mathfrak{R}_E) \cap \gamma^\perp = \mathrm{Vect}((\mathfrak{R} \cup \mathfrak{R}_E) \cap \gamma^\perp)$.

c) $[\mathfrak{X}_{w_1,\gamma}] \cdot [\mathfrak{X}_{w_2,\gamma}] \cdot [\mathfrak{X}_{w_3,\gamma}] \cdot \mathrm{Eul}(E^{\gamma>0}) = k[pt]$, $k \geqslant 1$ in $H^*(\mathcal{F}_\gamma, \mathbb{Z})$.

d) Relation (B) holds : $\sum_{i=1}^3 \mathrm{Tr}_{w_i\gamma}(\mathfrak{n}^{w_i\gamma>0}) = 2\,\mathrm{Tr}_\gamma(\mathfrak{k}_{\mathbb{C}}^{\gamma>0}) + \mathrm{Tr}_\gamma(E^{\gamma>0})$.

2.5 Remark on the saturation property

The semigroup $\Delta^{\mathbb{Z}}(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E) \subset (\wedge^*_+)^3$ is called *saturated* if for any $\theta \in (\wedge^*_+)^3$ and any $N \geqslant 1$ we have $N\theta \in \Delta^{\mathbb{Z}}(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E)$ only if $\theta \in \Delta^{\mathbb{Z}}(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E)$.

Proposition 2.14 *1. We have*

$$\mathbb{Q}^{>0} \cdot \Delta^{\mathbb{Z}}(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E) = \Delta(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E) \cap (\mathfrak{t}^*_{\mathbb{Q}})^3.$$

2. The semigroup $\Delta^{\mathbb{Z}}(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E)$ is saturated if and only if

$$\Delta^{\mathbb{Z}}(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E) = \Delta(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E) \cap (\wedge^*_+)^3.$$

Proof. Let us prove the first point. The inclusion $\mathbb{Q}^{>0} \cdot \Delta^{\mathbb{Z}}(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E) \subset \Delta(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E) \cap (\mathfrak{t}^*_{\mathbb{Q}})^3$ follows from the definitions. Let us explain why the opposite inclusion is a consequence of the $[Q, R] = 0$ theorem.

We consider the proper moment map $\Phi : K_{\mathbb{C}} \times K_{\mathbb{C}} \times E \to (\mathfrak{k}^*)^3$. For any $\mu = (\mu_1, \mu_2, \mu_3) \in (\wedge^*_+)^3$, we denote by $m_E(\mu)$ the multiplicity of $V_\mu = V^{K,1}_{\mu_1} \otimes V^{K,2}_{\mu_2} \otimes V^{K,3}_{\mu_3}$ in the coordinate ring $\mathbb{C}[K_{\mathbb{C}} \times K_{\mathbb{C}} \times E]$, and we consider the reduced space

$$\mathcal{M}_\mu := \Phi^{-1}(K\mu_1 \times K\mu_2 \times K\mu_2)/K \times K \times K.$$

that is equipped with the line bundle

$$\mathcal{L}_\mu = \Phi^{-1}(K\mu_1 \times K\mu_2 \times K\mu_2) \times_{K_{\mu_1} \times K_{\mu_2} \times K_{\mu_2}} (\mathbb{C}_{-\mu_1} \otimes \mathbb{C}_{-\mu_2} \otimes \mathbb{C}_{-\mu_3}) .$$

Suppose that $\mathcal{M}_\mu$ is non-empty. Then $\mathcal{M}_\mu$ is a complex-projective variety, a projective embedding being given by the Kodaira map $\mathcal{M}_\mu \to \mathbb{P}(H^0(\mathcal{M}_\mu, \mathcal{L}^{\otimes k}_\mu))$ for all sufficiently large k (see Theorem 2.17 in [24]). Moreover, the $[Q, R]$ theorem says that for all $k \geqslant 1$, we have $m_E(k\mu) = \dim H^0(\mathcal{M}_\mu, \mathcal{L}^{\otimes k}_\mu)$: hence $m_E(k\mu) \neq 0$ for k sufficiently large.

Let $\xi \in \Delta(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E) \cap (\mathfrak{t}^*_{\mathbb{Q}})^3$: let $N \geqslant 1$ such that $\mu_o := N\xi \in (\wedge^*_+)^3$. By definition, the reduced space $\mathcal{M}_{\mu_o}$ is non-empty. So there exists, $k_o \geqslant 1$ such that $m_E(k_o\mu_o) \neq 0$, i.e. $k_o\mu_o \in \Delta^{\mathbb{Z}}(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E)$. We have proved that $\xi = \frac{1}{k_o N} k_o\mu \in \mathbb{Q}^{>0} \cdot \Delta^{\mathbb{Z}}(K_{\mathbb{C}} \times K_{\mathbb{C}} \times E)$.

The first point is settled and the second one is an immediate consequence of the first one. □

3 Saturated semigroups

For any $n \geqslant 1$, we consider the semigroup $\wedge_n^+ = \{(\lambda_1 \geqslant \cdots \geqslant \lambda_n)\} \subset \mathbb{Z}^n$ that parametrizes the irreducible representations of the unitary group $U(n)$. When $\lambda \in \wedge_n^+$, the notation $\lambda \geqslant 0$ (resp. $\lambda \leqslant 0$) means that $\lambda_n \geqslant 0$ (resp. $\lambda_1 \leqslant 0$), and length(λ) is the number of non-zero coordinates λ_i. To $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n) \in \wedge_n^+$, we associate $\lambda^* = (-\lambda_n \geqslant \cdots \geqslant -\lambda_1) \in \wedge_n^+$: the representation $V_{\lambda^*}^{U(n)}$ is then the dual of $V_\lambda^{U(n)}$.

Let start by recalling the properties of the semigroup $\mathrm{Horn}^{\mathbb{Z}}(n)$ of $(\wedge_n^+)^3$ defined by the relations

$$(\lambda, \mu, \nu) \in \mathrm{Horn}^{\mathbb{Z}}(n) \Longleftrightarrow \left[V_\nu^{U(n)} : V_\lambda^{U(n)} \otimes V_\mu^{U(n)}\right] \neq 0.$$

First, the convex cone of $(\mathcal{C}_n)^3$ defined as the closure of $\mathbb{Q}^{>0} \cdot \mathrm{Horn}^{\mathbb{Z}}(n)$ corresponds to $\mathrm{Horn}(n)$ (see Definition 1.2). Moreover, thanks to the saturation Theorem of A. Knutson and T. Tao [16], we know that an element $(\lambda, \mu, \nu) \in (\wedge_n^+)^3$ belongs to the semigroup $\mathrm{Horn}^{\mathbb{Z}}(n)$ if and only if $(\lambda, \mu, \nu) \in \mathrm{Horn}(n)$ (see Proposition 2.14).

In the rest of this section we work with the compact Lie group $K = U(p) \times U(q)$, so that $K_{\mathbb{C}} = GL_p \times GL_q$. If $\lambda = (\lambda', \lambda'') \in \wedge_p^+ \times \wedge_q^+$, then $V_\lambda := V_{\lambda'}^{U(p)} \otimes V_{\lambda''}^{U(q)}$ denotes the irreducible representation of $U(p) \times U(q)$ with highest weight λ.

Recall that for any $p, q \geqslant 1$, $M_{p,q}$ denotes the vector space of $p \times q$ complex matrices.

The purpose of this section is the study of the following semigroups of $(\wedge_p^+ \times \wedge_q^+)^3$.

Definition 3.1 *Let $(\lambda, \mu, \nu) \in (\wedge_p^+ \times \wedge_q^+)^3$.*

- *The semigroup $\mathrm{Horn}^{\mathbb{Z}}(p, q)$ is defined by the conditions:*

$$(\lambda, \mu, \nu) \in \mathrm{Horn}^{\mathbb{Z}}(p, q) \Longleftrightarrow [V_\nu : V_\lambda \otimes V_\mu \otimes \mathrm{Sym}(M_{p,q})] \neq 0.$$

- *The semigroup $\mathrm{Q}^{\mathbb{Z}}(p, q)$ is defined by the conditions:*

$$(\lambda, \mu, \nu) \in \mathrm{Q}^{\mathbb{Z}}(p, q) \Longleftrightarrow \begin{cases} \lambda \leqslant 0, \\ \mu \leqslant 0, \\ [V_\lambda \otimes V_\mu \otimes V_\nu \otimes \mathrm{Sym}(M_{p,q})]^{U(p) \times U(q)} \neq 0. \end{cases}$$

- *The semigroup $\mathrm{S}^{\mathbb{Z}}(p, q)$ is defined by the conditions:*

$$(\lambda, \mu, \nu) \in \mathrm{S}^{\mathbb{Z}}(p, q) \Longleftrightarrow [V_\lambda \otimes V_\mu \otimes V_\nu \otimes \mathrm{Sym}(\mathbb{C}^p \otimes \mathbb{C}^q)]^{U(p) \times U(q)} \neq 0.$$

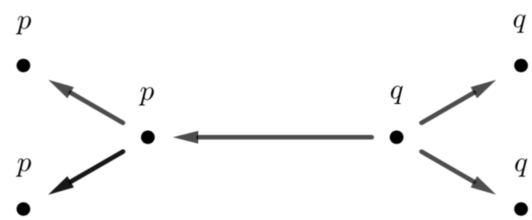

Figure 1: Quiver Q with dimension vector $\mathbf{v}_{p,q}$

3.1 The semigroup $\mathrm{Q}^{\mathbb{Z}}(p,q)$

Let $p, q \geqslant 1$. We consider the quiver Q of Figure 1, with dimension vector $\mathbf{v}_{p,q} = (p,p,p,q,q,q)$. The vector space

$$\mathrm{Rep}(Q, \mathbf{v}_{p,q}) = (M_{p,p} \times M_{q,q})^2 \times M_{p,q}$$

admits a natural action of the algebraic group $\mathrm{GL}(Q, \mathbf{v}_{p,q}) = (GL_p \times GL_q)^3$ that we recall. Take $g = (g_1, g_2, g_3) \in \mathrm{GL}(Q, \mathbf{v}_{p,q})$ with $g_i = (g_i', g_i'') \in GL_p \times GL_q$ and $(X_1, X_2, Y) \in \mathrm{Rep}(Q, \mathbf{v})$ where $X_i = (X_i', X_i'') \in M_{p,p} \times M_{q,q}$ and $Y \in M_{p,q}$. Then $g \cdot X = (x_1, y_2, y)$ where $x_i = (g_i' X_i' (g_3')^{-1}, g_i'' X_i'' (g_3'')^{-1})$ and $y = g_3' Y (g_3'')^{-1}$.

We consider the multipicity map $m : (\wedge_p^+ \times \wedge_q^+)^3 \to \mathbb{N}$ defined by

$$m(\lambda, \mu, \nu) = \dim \left[V_\lambda \otimes V_\mu \otimes V_\nu \otimes \mathrm{Sym}(M_{p,q})\right]^{GL_p \times GL_q}.$$

Lemma 3.2 *The coordinate ring $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{v}_{p,q})]$, viewed as $\mathrm{GL}(Q, \mathbf{v}_{p,q})$-module, admits the following decomposition*

$$\mathbb{C}[\mathrm{Rep}(Q, \mathbf{v}_{p,q})] = \sum_{\lambda \leqslant 0, \mu \leqslant 0, \nu} m(\lambda, \mu, \nu)\, V_\lambda^1 \otimes V_\mu^2 \otimes V_\nu^3.$$

Proof. It is due to the fact that the $\mathrm{GL}(Q, \mathbf{v}_{p,q})$-module $\mathbb{C}[(M_{p,p} \times M_{q,q})^2]$ admits the decomposition $\mathbb{C}[(M_{p,p} \times M_{q,q})^2] = \sum_{\lambda \leqslant 0, \mu \leqslant 0} V_\lambda^1 \otimes V_\mu^2 \otimes V_{\lambda *}^3 \otimes V_{\mu *}^3$.
□

The previous lemma shows that $\mathrm{Q}^{\mathbb{Z}}(p,q)$ corresponds to the semigroup of highest weights associated to the action of the group $\mathrm{GL}(Q, \mathbf{v}_{p,q})$ on the coordinate ring $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{v}_{p,q})]$.

Proposition 3.3 *The semigroup $\mathrm{Q}^{\mathbb{Z}}(p,q)$ is saturated.*

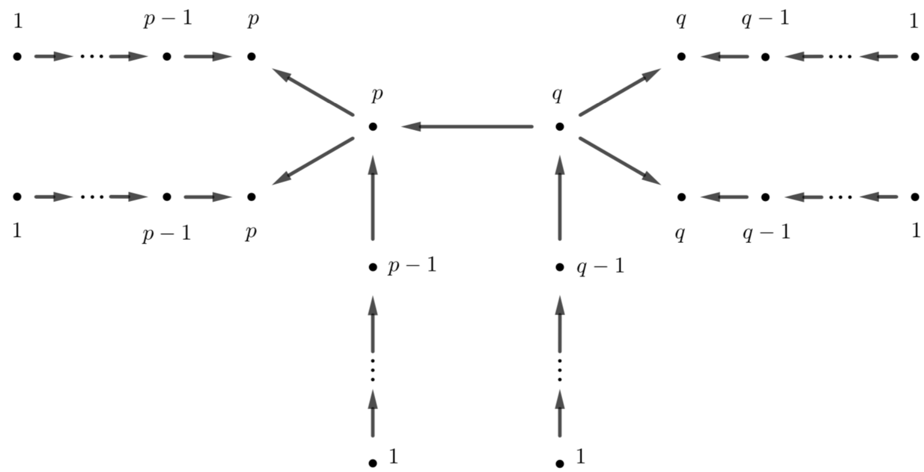

Figure 2: Quiver $\widetilde{Q}$ with dimension vector $\widetilde{\mathbf{v}}_{p,q}$

Proof. This is a direct consequence of the Derksen-Weyman saturation theorem [7], which asserts that, for a quiver without cycles, the semigroup of weights of semi-invariants is saturated. Indeed, augment the quiver $(Q, \mathbf{v}_{p,q})$ to the quiver $(\widetilde{Q}, \widetilde{\mathbf{v}}_{p,q})$ (see Figure 2). Then, using the Cauchy formula for the decomposition of $\otimes_{k=1}^{n-1}\mathbb{C}[(\mathbb{C}^k)^* \otimes \mathbb{C}^{k+1}]$ under the action of $\prod_{k=1}^n GL_k$, one sees that there is a bijective morphism between the semigroup of weights of semi-invariants of the coordinate ring $\mathbb{C}[\mathrm{Rep}(\widetilde{Q}, \widetilde{\mathbf{v}}_{p,q})]$ under the action of $(\prod_{k=1}^p GL_k)^3 \times (\prod_{\ell=1}^q GL_\ell)^3$ and the semigroup $\mathrm{Q}^{\mathbb{Z}}(p,q)$. □

3.2 The semigroups $\mathrm{Horn}^{\mathbb{Z}}(p,q)$ and $\mathrm{S}^{\mathbb{Z}}(p,q)$

We use the involution $\Theta : (\wedge_p^+ \times \wedge_q^+)^3 \to (\wedge_p^+ \times \wedge_q^+)^3$ that sends (λ, μ, ν) to

$$\left(\begin{pmatrix} \lambda' \\ (\lambda'')^* \end{pmatrix}, \begin{pmatrix} \mu' \\ (\mu'')^* \end{pmatrix}, \begin{pmatrix} (\nu')^* \\ \nu'' \end{pmatrix}\right).$$

Let us denote by $\mathbf{1} \in \wedge_p^+ \times \wedge_q^+$ the vector $(1, \ldots, 1, 1, \ldots, 1)$. The main purpose of this section is the following result.

Proposition 3.4 *1. For any $(\lambda, \mu, \nu) \in \mathrm{Horn}^{\mathbb{Z}}(p,q)$, there exists $k_o \geqslant 0$ such that $(\lambda - k\mathbf{1}, \mu - k\mathbf{1}, \nu^* + 2k\mathbf{1}) \in \mathrm{Q}^{\mathbb{Z}}(p,q)$, $\forall k \geqslant k_o$.*

2. $(\lambda, \mu, \nu) \in \mathrm{Horn}^{\mathbb{Z}}(p,q)$ if and only if $\Theta(\lambda, \mu, \nu) \in \mathrm{S}^{\mathbb{Z}}(p,q)$.

3. The semigroups $\mathrm{Horn}^{\mathbb{Z}}(p,q)$ *and* $\mathrm{S}^{\mathbb{Z}}(p,q)$ *are saturated.*

Proof. Let $(\lambda,\mu,\nu) \in \mathrm{Horn}^{\mathbb{Z}}(p,q)$: it means that $[V_\lambda \otimes V_\mu \otimes V_{\nu^*} \otimes \mathrm{Sym}(M_{p,q})]^{GL_p\times GL_q} \neq 0$ and so $(\lambda,\mu,\nu^*) \in \mathrm{Q}^{\mathbb{Z}}(p,q)$ if $\lambda \leqslant 0$ and $\mu \leqslant 0$. We notice that $(\lambda + k\mathbf{1}, \mu + k\mathbf{1}, \nu + 2k\mathbf{1}) \in \mathrm{Horn}^{\mathbb{Z}}(p,q)$, $\forall k \in \mathbb{Z}$. Let $k_o = \sup(|\lambda_1|,|\mu_1|)$: we see that $\lambda - k\mathbf{1} \leqslant 0$ and $\mu - k\mathbf{1} \leqslant 0$ if $k \geqslant k_o$, and consequently $(\lambda - k\mathbf{1}, \mu - k\mathbf{1}, \nu^* + 2k\mathbf{1}) \in \mathrm{Q}^{\mathbb{Z}}(p,q)$ if $k \geqslant k_o$. The first point is settled.

Cauchy formulas give the decompositions

$$\mathrm{Sym}(M_{p,q}) = \mathrm{Sym}(\mathbb{C}^p \otimes (\mathbb{C}^q)^*) = \sum_{\substack{\mathrm{length}(a)\leqslant \inf(p,q)\\ a\geqslant 0}} V_a^{GL_p} \otimes V_{a^*}^{GL_q},$$

$$\mathrm{Sym}(\mathbb{C}^p \otimes \mathbb{C}^q) = \sum_{\substack{\mathrm{length}(a)\leqslant \inf(p,q)\\ a\geqslant 0}} V_a^{GL_p} \otimes V_a^{GL_q}.$$

For the second point, we have to compare the following cases :

- $(\lambda,\mu,\nu) \in \mathrm{Horn}^{\mathbb{Z}}(p,q)$ if there exists $a \geqslant 0$ with $\mathrm{length}(a) \leqslant \inf(p,q)$ such that both conditions hold

$$\big[V_{\lambda'} \otimes V_{\mu'} \otimes V_{(\nu')^*} \otimes V_a\big]^{GL_p} \neq 0$$
$$\big[V_{\lambda''} \otimes V_{\mu''} \otimes V_{(\nu'')^*} \otimes V_{a^*}\big]^{GL_q} \neq 0.$$

- $(\lambda,\mu,\nu) \in \mathrm{S}^{\mathbb{Z}}(p,q)$ if there exists $a \geqslant 0$ with $\mathrm{length}(a) \leqslant \inf(p,q)$ such that both conditions hold

$$\big[V_{\lambda'} \otimes V_{\mu'} \otimes V_{\nu'} \otimes V_a\big]^{GL_p} \neq 0$$
$$\big[V_{\lambda''} \otimes V_{\mu''} \otimes V_{\nu''} \otimes V_a\big]^{GL_q} \neq 0.$$

It is then immediate to conclude that $(\lambda,\mu,\nu) \in \mathrm{Horn}^{\mathbb{Z}}(p,q)$ if and only if $\Theta(\lambda,\mu,\nu) \in \mathrm{S}^{\mathbb{Z}}(p,q)$.

Let us check that $\mathrm{Horn}^{\mathbb{Z}}(p,q)$ is saturated. Let (λ,μ,ν) such that $N(\lambda,\mu,\nu) \in \mathrm{Horn}^{\mathbb{Z}}(p,q)$ for some $N \geqslant 1$. If we use the first point, we know that there exists $k_o \geqslant 0$, such that

$$N(\lambda,\mu,\nu^*) - k(\mathbf{1},\mathbf{1},-2\cdot\mathbf{1}) \in \mathrm{Q}^{\mathbb{Z}}(p,q), \quad \forall k \geqslant k_o.$$

Take $k = Nk_o \geqslant k_o$: we obtain that $N(\lambda - k_o\mathbf{1}, \mu - k_o\mathbf{1}, \nu^* + 2k_o\mathbf{1}) \in \mathrm{Q}^{\mathbb{Z}}(p,q)$. It follows that $(\lambda - k_o\mathbf{1}, \mu - k_o\mathbf{1}, \nu^* + 2k_o\mathbf{1}) \in \mathrm{Q}^{\mathbb{Z}}(p,q)$ because the semigroup $\mathrm{Q}^{\mathbb{Z}}(p,q)$ is saturated. Finally, the last relation implies that $(\lambda,\mu,\nu) \in \mathrm{Horn}^{\mathbb{Z}}(p,q)$.

We have verified that the semigroup $\mathrm{Horn}^{\mathbb{Z}}(p,q)$ is saturated, and the second point also allows us to conclude that the semigroup $\mathrm{S}^{\mathbb{Z}}(p,q)$ is saturated. $\square$

3.3 Final remarks

We have seen that the semigroups $\mathrm{Horn}^{\mathbb{Z}}(p,q)$, $\mathrm{S}^{\mathbb{Z}}(p,q)$ and $\mathrm{Q}^{\mathbb{Z}}(p,q)$ are all related. Thus, the associated convex cones $\mathrm{Horn}(p,q) = \overline{\mathbb{Q}^{>0} \cdot \mathrm{Horn}^{\mathbb{Z}}(p,q)}$, $\mathrm{S}(p,q) = \overline{\mathbb{Q}^{>0} \cdot \mathrm{S}^{\mathbb{Z}}(p,q)}$ and $\mathrm{Q}(p,q) = \overline{\mathbb{Q}^{>0} \cdot \mathrm{Q}^{\mathbb{Z}}(p,q)}$ are also interdependent. The calculation of one entails those of the others. In this paper, we obtain a recursive description of the Horn cone $\mathrm{Horn}(p,q)$ through the calculation of $\mathrm{S}(p,q)$.

Let Q_o be a quiver without cycle which is equipped with a dimension vector $\mathbf{v_o}$. In [1], V. Baldoni, M. Vergne and M. Walter have proposed a recursive description of the cone generated by the highest weights associated to the action of the group $\mathrm{GL}(Q_o,\mathbf{v_o})$ on the coordinate ring $\mathbb{C}[\mathrm{Rep}(Q_o,\mathbf{v_o})]$. By applying their result to the quiver $(Q,\mathbf{v}_{p,q})$ in Figure 1, this should also allow a recursive description of the Horn cone $\mathrm{Horn}(p,q)$.

4 Convex cone $\mathrm{S}(p,q)$

In this section, we apply the results of §2 to the case where $K_{\mathbb{C}} = GL_p \times GL_q$ and the $K_{\mathbb{C}}$-module is $E = \mathbb{C}^p \otimes \mathbb{C}^q$. The coordinate ring $\mathbb{C}[(GL_p \times GL_q)^2 \times \mathbb{C}^p \otimes \mathbb{C}^q]$, viewed as a $(GL_p \times GL_q)^3$-module, admits the following decomposition

$$\mathbb{C}[(GL_p \times GL_q)^2 \times \mathbb{C}^p \otimes \mathbb{C}^q] = \sum_{\lambda,\mu,\nu \in \wedge_p^+ \times \wedge_q^+} m(\lambda,\mu,\nu)\ V_\lambda^1 \otimes V_\mu^2 \otimes V_\nu^3,$$

where $m(\lambda,\mu,\nu) = \dim[V_\lambda \otimes V_\mu \otimes V_\nu \otimes \mathrm{Sym}(\mathbb{C}^p \otimes \mathbb{C}^q)]^{GL_p \times GL_q}$.

We see that the semigroup $\Delta^{\mathbb{Z}}((GL_p \times GL_q)^2 \times \mathbb{C}^p \otimes \mathbb{C}^q)$ corresponds to $\mathrm{S}^{\mathbb{Z}}(p,q)$. Hence we will denote by $\mathrm{S}(p,q)$ the convex cone $\Delta((GL_p \times GL_q)^2 \times \mathbb{C}^p \otimes \mathbb{C}^q)$: it corresponds to the Kirwan polyhedron relative to the Hamiltonian action of $U(p) \times U(q)$ on $(GL_p \times GL_q)^2 \times \mathbb{C}^p \otimes \mathbb{C}^q$.

4.1 Admissible elements for $(GL_p \times GL_q)^2 \times \mathbb{C}^p \otimes \mathbb{C}^q$

Let $T \simeq U(1)^p \times U(1)^q$ be the maximal torus of $K = U(p) \times U(q)$ formed by the diagonal matrices. The Lie algebra $\mathfrak{t}$ admits a canonical identification with $\mathbb{R}^p \times \mathbb{R}^q$ through the map $\mathbf{d}$ and the Weyl group of (K, T) is isomorphic to $W = \mathfrak{S}_p \times \mathfrak{S}_q$.

The set of roots for (K, T) is $\mathfrak{R} = \{\epsilon_i - \epsilon_j, 1 \leqslant i \neq j \leqslant p\} \cup \{\epsilon'_k - \epsilon'_l, 1 \leqslant k \neq l \leqslant q\}$. The set of weights for the T-action on $E = \mathbb{C}^p \otimes \mathbb{C}^q$ is $\mathfrak{R}_E = \{\epsilon_i + \epsilon'_k, 1 \leqslant i \leqslant p, 1 \leqslant k \leqslant q\}$. Let us denote $\mathfrak{R}_o = \mathfrak{R} \cup \mathfrak{R}_E$. We first notice that $\mathfrak{R}_o^\perp = \mathbb{R}\gamma_0^0$, with $\gamma_0^0 = \mathbf{1}_p \oplus -\mathbf{1}_q \in \mathfrak{t}$.

Definition 4.1 *For any $(r, s) \in \{0, \ldots, p\} \times \{0, \ldots, q\}$, we define $\gamma_s^r = \gamma^r \oplus \gamma_s \in \mathfrak{t}$ where $\gamma^r = (\underbrace{0, \ldots, 0}_{r \text{ times}}, 1, \ldots, 1)$ and $\gamma_s = (-1, \ldots, -1, \underbrace{0, \ldots, 0}_{s \text{ times}})$.*

Lemma 4.2 *Let $\gamma \in \mathfrak{t}$ be a non-zero rational element such that* $\mathrm{Vect}(\mathfrak{R}_o) \cap \gamma^\perp = \mathrm{Vect}(\mathfrak{R}_o \cap \gamma^\perp)$. *There exists $(r, s) \notin \{(0, 0), (p, q)\}$, $w \in W$ and $(a, b) \in \mathbb{Q}^{\geqslant 0} \times \mathbb{Q}$ such that $\gamma = a(w\gamma_s^r) + b\gamma_0^0$.*

Proof. For any $t \in \mathbb{R}$, we define

$$\gamma(t) = \sum_{\substack{1 \leqslant i \leqslant p \\ \gamma_i = t}} e_i - \sum_{\substack{1 \leqslant k \leqslant q \\ \gamma_k = -t}} e'_k.$$

We notice that $\gamma(t)$ is orthogonal to $\mathfrak{R}_o \cap \gamma^\perp$. If $\mathrm{Vect}(\mathfrak{R}_o) \cap \gamma^\perp = \mathrm{Vect}(\mathfrak{R}_o \cap \gamma^\perp)$ holds we get that $\gamma(t) \in (\mathrm{Vect}(\mathfrak{R}_o) \cap \gamma^\perp)^\perp = \mathbb{R}\gamma_0^0 + \mathbb{R}\gamma$ for all $t \in \mathbb{R}$. Take t_o such that $\gamma(t_o) \neq 0$. Two situations holds.

1. $\gamma(t_o) \in \mathbb{R}\gamma_0^0$. This case only occurs if $\gamma \in \mathbb{Q}\gamma_0^0$.
2. $\gamma(t_o) \notin \mathbb{R}\gamma_0^0$. Then there exists $(r, s) \notin \{(0, 0), (p, q)\}$, $w \in W$ and $(x, y) \in \mathbb{Q} - \{0\} \times \mathbb{Q}$ such that $\gamma(t_o) = w\gamma_s^r$ and $\gamma(t_o) = x\gamma + y\gamma_0^0$: hence $\gamma = \frac{1}{x}(w\gamma_s^r) - \frac{y}{x}\gamma_0^0$. If $\frac{1}{x} > 0$, the proof is completed. If $\frac{1}{x} < 0$, we use that $-\gamma_s^r = w_o\gamma_{q-s}^{p-r} - \gamma_0^0$ for some $w_o \in W$ in order to come back to the previous case. □

In order to describe the facets of the convex cone $\mathrm{S}(p, q)$, we must consider the following admissible elements:

- $\pm(\gamma_0^0, \gamma_0^0, \gamma_0^0)$,
- $(w_1\gamma_s^r, w_2\gamma_s^r, w_3\gamma_s^r)$ where $w_1, w_2, w_3 \in W$ and $(r, s) \notin \{(0, 0), (p, q)\}$.

4.2 Admissible elements $\pm(\gamma_0^0, \gamma_0^0, \gamma_0^0)$

The admissible elements $\pm(\gamma_0^0, \gamma_0^0, \gamma_0^0)$ act on $(GL_p \times GL_q)^2 \times \mathbb{C}^p \otimes \mathbb{C}^q$ trivially. Hence $\pm(\gamma_0^0, \gamma_0^0, \gamma_0^0)$ are Ressayre's data, and inequalities (5) are $\pm\left(\langle A, \gamma_0^0\rangle + \langle B, \gamma_0^0\rangle + \langle C, \gamma_0^0\rangle\right) \geqslant 0$. In other words,

$$\boxed{|A'| + |B'| + |C'| = |A''| + |B''| + |C''|}. \tag{6}$$

4.3 Admissible element $(w_1\gamma_s^r, w_2\gamma_s^r, w_3\gamma_s^r)$

Recall the relations that a Ressayre's data (γ, w_1, w_2, w_3) must satisfy (see Lemma 2.10):

$$\text{Relation } (A): \quad \sum_{i=1}^{3} \dim(\mathfrak{n}^{w_i\gamma>0}) = 2\dim(\mathfrak{k}_{\mathbb{C}}^{\gamma>0}) + \dim((\mathbb{C}^p \otimes \mathbb{C}^q)^{\gamma>0}),$$

$$\text{Relation } (B): \quad \sum_{i=1}^{3} \mathrm{Tr}_{w_i\gamma}(\mathfrak{n}^{w_i\gamma>0}) = 2\,\mathrm{Tr}_{\gamma}(\mathfrak{k}_{\mathbb{C}}^{\gamma>0}) + \mathrm{Tr}_{\gamma}((\mathbb{C}^p \otimes \mathbb{C}^q)^{\gamma>0}).$$

It is immediate to see that for $(\gamma_s^r, w_1, w_2, w_3)$, Relations (A) and (B) are equivalent.

We associate to $(w_1, w_2, w_3) \in (\mathfrak{S}_p \times \mathfrak{S}_q)^3$ the following subsets :

- Those of cardinal r : $I' = w_1'(\{1, \dots, r\})$, $J' = w_2'(\{1, \dots, r\})$, and $K' = w_3'(\{1, \dots, r\})$.
- Those of cardinal s : $I'' = w_1''(\{q-s+1, \dots, q\})$, $J'' = w_2''(\{q-s+1, \dots, q\})$, and $K'' = w_3''(\{q-s+1, \dots, q\})$.

Inequality (5) becomes $|A'|_{(I')^c} + |B'|_{(J')^c} + |C'|_{(K')^c} \geqslant |A''|_{(I'')^c} + |B''|_{(J'')^c} + |C''|_{(K'')^c}$ which is equivalent to

$$(7) \qquad \boxed{|\,A'\,|_{I'} + |\,B'\,|_{J'} + |\,C'\,|_{K'} \leqslant |\,A''\,|_{I''} + |\,B''\,|_{J''} + |\,C''\,|_{K''}},$$

thanks to (6).

4.4 Schubert classes

For any $m, n \geqslant 0$, let $\mathbb{G}(m,n)$ denote the Grassmannian of complex m-dimensional linear subspaces of $\mathbb{C}^{m+n}$. The singular cohomology of $\mathbb{G}(m,n)$ with integers coefficients is denoted $H^*(\mathbb{G}(m,n), \mathbb{Z})$.

Let $m, n \geqslant 1$. When a partition λ is included in a $m \times n$ rectangle, we write $\lambda \subset m \times n : n \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_m \geqslant 0$.

Denote the set of cardinality m subsets $I = \{i_1 < i_2 < \cdots < i_m\}$ of $[m+n] = \{1, \dots, m+n\}$ by $\mathcal{P}_m^{m+n}$. To each $I \in \mathcal{P}_m^{m+n}$ we associate $\lambda(I) = (\lambda_1 \geqslant \lambda_2 \cdots \geqslant \lambda_m) \subset m \times n$ where $\lambda_a = n + a - i_a$ for $a \in [m]$. The map $I \mapsto \lambda(I)$ is one to one map between $\mathcal{P}_m^{m+n}$ and the set of partitions of size $m \times n$. The inverse map is denoted by $\lambda \subset m \times n \mapsto I(\lambda) \in \mathcal{P}_m^{m+n}$.

We work with the flag $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^{n+m-1} \subset \mathbb{C}^{n+m}$. For any partition $\lambda \subset m \times n$, we define the Schubert cell

$$\mathfrak{X}_\lambda^o = \{F \in \mathbb{G}_{m,n},\ \mathbb{C}^{k-1} \cap F \neq \mathbb{C}^k \cap F \ \text{ if and only if } \ k \in I(\lambda)\}.$$

and the Schubert variety $\mathfrak{X}_\lambda = \overline{\mathfrak{X}^o_\lambda}$. When the partition is $(k, 0, \ldots, 0)$ with $1 \leqslant k \leqslant n$, the corresponding Schubert variety is denoted by $\mathfrak{X}_k$.

Let $B_{n+m} \subset GL_{n+m}$ be the Borel subgroup formed by the upper-triangular matrices. The following facts are well-known (see e.g. [11, 8, 20]) :

1. $\mathfrak{X}^o_\lambda = B_{m+n} \cdot \mathrm{Vect}(e_i, i \in I(\lambda))$,
2. $\mathbb{G}_{m,n} = \bigcup_{\lambda \subset m \times n} \mathfrak{X}^o_\lambda$,
3. $\mathrm{codim}(\mathfrak{X}^o_\lambda) = |\lambda| = \sum_k \lambda_k$.

Since the Schubert cells define a complex cellular decomposition of the Grassmannian, an immediate consequence is that the fundamental class of the Schubert varieties, the *Schubert classes* $\sigma_\lambda = [\mathfrak{X}_\lambda] \in H^{2|\lambda|}(\mathbb{G}_{m,n}, \mathbb{Z})$, where $\lambda \subset m \times n$, form a basis of the cohomology with integers coefficients :

$$H^*(\mathbb{G}(m,n), \mathbb{Z}) = \bigoplus_{\lambda \subset m \times n} \mathbb{Z}\sigma_\lambda.$$

When $\lambda = (k, 0, \ldots, 0)$, we denote by $\sigma_k \in H^{2k}(\mathbb{G}_{m,n}, \mathbb{Z})$ the corresponding Schubert class.

We finish this section by recalling that the Grassmannian $\mathbb{G}_{m,n}$ admits the following complex vector bundles :

1. A canonical vector bundle of rank m : $\mathbb{E}_{m,n}$.
2. A vector bundle of rank n, denoted $\mathbb{E}^\perp_{m,n}$, such that $\mathbb{E}_{m,n} \oplus \mathbb{E}^\perp_{m,n}$ is a trivial bundle of rank $m + n$.

4.5 Cohomological conditions

Theorem 2.13 tells us that an element $(A, B, C) \in (\mathcal{C}_p \times \mathcal{C}_q)^3$ belongs to $\mathrm{S}(p, q) = \Delta((GL_p \times GL_q)^2 \times \mathbb{C}^p \otimes \mathbb{C}^q)$ if and only (6) holds and (7) holds for any $(w_1, w_2, w_3) \in (\mathfrak{S}_p \times \mathfrak{S}_q)^3$ and any couple $(r, s) \in \{0, \ldots, p\} \times \{0, \ldots, q\} - \{(p, q), (0, 0)\}$, satisfying the relation

$$[\mathfrak{X}_{w_1,\gamma^r_s}] \cdot [\mathfrak{X}_{w_2,\gamma^r_s}] \cdot [\mathfrak{X}_{w_3,\gamma^r_s}] \cdot \mathrm{Eul}(\mathcal{V}^r_s) = k[pt], \;\; k \geqslant 1 \tag{8}$$

in $H^*(\mathcal{F}_{\gamma^r_s}, \mathbb{Z})$.

Let us detailed (8). We fix $(r, s) \notin \{(0, 0)\,, (p, q)\}$. The parabolic subgroup $P_{\gamma^r_s} \subset GL_p \times GL_q$ associated to γ^r_s by (3) is equal to $P_{\gamma^r} \times P_{\gamma_s}$ where

$$P_{\gamma^r} = \left(\begin{array}{c|c} GL_r & * \\ \hline 0 & GL_{p-r} \end{array}\right) \subset GL_p \quad \text{and} \quad P_{\gamma_s} = \left(\begin{array}{c|c} GL_{q-s} & * \\ \hline 0 & GL_s \end{array}\right) \subset GL_q.$$

Let $\mathbb{C}^r \subset \mathbb{C}^p$ and $\mathbb{C}^{q-s} \subset \mathbb{C}^q$ denote respectively the subspaces $\mathrm{Vect}(e_i, 1 \leqslant i \leqslant r)$ and $\mathrm{Vect}(e_j, 1 \leqslant j \leqslant q-s)$. We use on $\mathbb{C}^p$ and $\mathbb{C}^q$ the canonical bilinear forms $(x,y) \mapsto \sum_k x_k y_k$. Then $(\mathbb{C}^r)^\perp \subset \mathbb{C}^p$ and $(\mathbb{C}^{q-s})^\perp \subset \mathbb{C}^q$ are respectively the subspaces $\mathrm{Vect}(e_i, r+1 \leqslant i \leqslant p)$ and $\mathrm{Vect}(e_j, q-s+1 \leqslant j \leqslant q)$.

The flag variety $\mathcal{F}_{\gamma_s^r} = GL_p/P_{\gamma^r} \times GL_q/P_{\gamma_s}$ admits a canonical identification with $\mathbb{G}(r, p-r) \times \mathbb{G}(q-s, s)$ through the map

$$([g],[h]) \in GL_p/P_{\gamma^r} \times GL_q/P_{\gamma_s} \longmapsto \big(g(\mathbb{C}^r), h(\mathbb{C}^{q-s})\big) \in \mathbb{G}(r,p-r) \times \mathbb{G}(q-s,s).$$

Let $B_p \subset GL_p$ and $B_q \subset GL_q$ be the Borel subgroups formed by the upper-triangular matrices. For any $w = (w', w'') \in \mathfrak{S}_p \times \mathfrak{S}_q$, we consider the Schubert cell $\mathfrak{X}^o_{w,\gamma_s^r} = B_p[w'] \times B_q[w''] \subset \mathbb{G}(p-r, r) \times \mathbb{G}(s, q-s)$ and the Schubert variety

$$\mathfrak{X}_{w,\gamma_s^r} = \overline{B_p[w']} \times \overline{B_q[w'']} = \mathfrak{X}_{\mu'} \times \mathfrak{X}_{\mu''}$$

where $\mu' = \lambda(w'\{1,\dots,r\}) \subset r \times p-r$ and $\mu'' = \lambda(w''\{1,\dots,q-s\}) \subset q-s \times s$.

If we associate to $(w_1, w_2, w_3) \in (\mathfrak{S}_p \times \mathfrak{S}_q)^3$, the subsets

- $I' = w_1'(\{1,\dots,r\})$, $J' = w_2'(\{1,\dots,r\})$, and $K' = w_3'(\{1,\dots,r\})$,
- $I'' = w_1''(\{q-s+1,\dots,q\})$, $J'' = w_2''(\{q-s+1,\dots,q\})$, and $K'' = w_3''(\{q-s+1,\dots,q\})$,

the term $[\mathfrak{X}_{w_1,\gamma_s^r}] \cdot [\mathfrak{X}_{w_2,\gamma_s^r}] \cdot [\mathfrak{X}_{w_3,\gamma_s^r}] \in H^*(\mathbb{G}(r,p-r) \times \mathbb{G}(q-s,s), \mathbb{Z})$ is then equal to the tensor product of the cohomology classes[4]

$$\sigma_{\lambda(I')} \cdot \sigma_{\lambda(J')} \cdot \sigma_{\lambda(K')} \in H^*(\mathbb{G}(r,p-r),\mathbb{Z}),$$
$$\sigma_{\lambda((I'')^c)} \cdot \sigma_{\lambda((J'')^c)} \cdot \sigma_{\lambda((K'')^c)} \in H^*(\mathbb{G}(q-s,s),\mathbb{Z}).$$

The subspace $(\mathbb{C}^p \otimes \mathbb{C}^q)^{\gamma_s^r > 0}$ is equal to $(\mathbb{C}^r)^\perp \otimes (\mathbb{C}^{q-s})^\perp$. Hence the vector bundle $\mathcal{V}_s^r$ is equal to the tensor product $\mathbb{E}^\perp_{r,p-r} \boxtimes \mathbb{E}^\perp_{q-s,s}$. Let $\mathrm{Eul}(\mathcal{V}_s^r) \in H^{2(p-r)s}(\mathbb{G}(r,p-r) \times \mathbb{G}(q-s,s),\mathbb{Z})$ be its Euler class.

Finally, the cohomological condition (8) says that the product

$$\big(\sigma_{\lambda(I')} \otimes \sigma_{\lambda((I'')^c)}\big) \cdot \big(\sigma_{\lambda(J')} \otimes \sigma_{\lambda((J'')^c)}\big) \cdot \big(\sigma_{\lambda(K')} \otimes \sigma_{\lambda((K'')^c)}\big) \cdot \mathrm{Eul}(\mathcal{V}_s^r)$$

is a non zero multiple of $[pt] \in H^{\max}(\mathbb{G}(r,p-r) \times \mathbb{G}(q-s,s),\mathbb{Z})$.

[4] X^c denotes the complement of X.

5 Computation of the Euler class $\mathrm{Eul}(\mathcal{V}_s^r)$

Before giving a formula for the Euler class $\mathrm{Eul}(\mathcal{V}_s^r)$, we need to recall some well-known facts.

5.1 Polynomial representations

We are concerned here with the polynomial representations of GL_m. When a representation $\pi_V : GL_m \to GL(V)$ is polynomial, its character is an invariant polynomial $\chi_V \in \mathbb{C}[M_{m,m}]$. We denote then by s_V the restriction of χ_V to the diagonal matrices.

Let $R_+(GL_m)$ denotes the polynomial representation ring of GL_m, and let $\bigwedge_m = \mathbb{Z}[x_1, \ldots, x_m]^{\mathfrak{S}_m}$ be the ring of symmetric polynomials, with integral coefficients, in m variables. The map $V \in R_+(GL_m) \mapsto s_V \in \bigwedge_m$ is a ring isomorphism.

For any partition λ of length m, we associate the irreducible polynomial representation[5] $V_\lambda^{GL_m}$ of the group GL_m and the Schur polynomial $\mathbf{s}_\lambda := s_{V_\lambda} \in \bigwedge_m$. Recall that the Schur polynomials $\mathbf{s}_\lambda$ determine a $\mathbb{Z}$-basis of $\bigwedge_m$.

We recall the following classical fact (for a proof see §3.2.2 in [20]).

Theorem 5.1 *The map $\phi_{m,n} : \bigwedge_m \longrightarrow H^*(\mathbb{G}_{m,n}, \mathbb{Z})$ defined by the relations*

$$\phi_{m,n}(\mathbf{s}_\lambda) = \begin{cases} \sigma_\lambda & \text{if} \quad \lambda_1 \leqslant n, \\ 0 & \text{if} \quad \lambda_1 > n. \end{cases}$$

is a ring morphism.

Remark 5.2 *Since $V \in R_+(GL_m) \mapsto s_V \in \bigwedge_m$ is a ring isomorphism, we will also denote by $\phi_{m,n} : R_+(GL_m) \longrightarrow H^*(\mathbb{G}_{m,n}, \mathbb{Z})$ the ring morphism $V \mapsto \phi_{m,n}(s_V)$.*

If $k \geqslant 1$, we denote by 1^k the partition $(1, \ldots, 1, 0, \ldots, 0)$ where there are k-times 1.

Example 5.3

$$\phi_{m,n}(\mathrm{Sym}^k(\mathbb{C}^m)) = \begin{cases} \sigma_k & \text{if} \quad 1 \leqslant k \leqslant n, \\ 0 & \text{if} \quad k > n. \end{cases}$$

$$\phi_{m,n}\left(\bigwedge^k \mathbb{C}^m\right) = \begin{cases} \sigma_{1^k} & \text{if} \quad 1 \leqslant k \leqslant m, \\ 0 & \text{if} \quad k > m. \end{cases}$$

[5] When the group is understood, we use the notation V_λ.

5.2 Duality I

We associate to a partition $\lambda \subset m \times n$ it's complementary partition $\widehat{\lambda} \subset m \times n : \widehat{\lambda}_k = n - \lambda_{m+1-k},\ 1 \leqslant k \leqslant m$. Recall that

- If $\lambda = \lambda(I)$ then $\widehat{\lambda} = \lambda(\widetilde{I})$ where $\widetilde{I} = \{n + m + 1 - i\,;\, i \in I\}$.
- If V_λ is the irreducible polynomial representation of GL_m associated to $\lambda \subset m \times n$, then $(V_\lambda)^* = V_{\widehat{\lambda}} \otimes \det^{-n}$.

The cohomology class $[pt] \in H^{2nm}(\mathbb{G}_{m,n}, \mathbb{Z})$ of top degree associated to a singleton $\{pt\}$ is a basis of $H^{2nm}(\mathbb{G}_{m,n}, \mathbb{Z})$.

We recall the following classical fact (for a proof see §3.2.2 in [20]).

Proposition 5.4 *Let $\lambda', \lambda \subset m \times n$ be two partitions such that $|\lambda| + |\lambda'| = nm$. Then, the following relations hold in $H^*(\mathbb{G}_{m,n}, \mathbb{Z})$:*

$$\sigma_\lambda \cdot \sigma_{\lambda'} = \begin{cases} [pt] & \text{if} \quad \lambda' = \widehat{\lambda}, \\ 0 & \text{if} \quad \lambda' \neq \widehat{\lambda}. \end{cases}$$

The next corollary follows from Theorem 5.1 and Proposition 5.4.

Corollary 5.5 *Let $\lambda_1, \lambda_2, \lambda_3 \subset m \times n$. The following assertions are equivalent :*

- $\sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdot \sigma_{\lambda_3} = k[pt],\ k \geqslant 1$ *in* $H^*(\mathbb{G}_{m,n}, \mathbb{Z})$.
- $\left[V_{\widehat{\lambda}_3} : V_{\lambda_1} \otimes V_{\lambda_2}\right] \neq 0$.
- $\left[V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \otimes \det^{-n}\right]^{GL_m} \neq 0$.

5.3 Duality II

Taking the transpose of the Young diagram defines a bijective map $\lambda \subset m \times n \longmapsto \lambda^\vee \subset n \times m$. The following lemma is useful in our computations.

Lemma 5.6 *If the partition $\lambda \subset m \times n$ is equal to $\lambda(I)$ then $\lambda^\vee = \lambda(I^\vee)$ where $I^\vee = \widetilde{(I^c)}$.*

The canonical bilinear form on $\mathbb{C}^{n+m}$ permits to define the map $\delta : \mathbb{G}_{n,m} \longrightarrow \mathbb{G}_{m,n}$ that sends F to $F^\perp$. Let $\delta^* : H^*(\mathbb{G}_{m,n}) \to H^*(\mathbb{G}_{n,m})$ denotes the pullback map in cohomology.

Lemma 5.7 *For any partition $\lambda \subset m \times n$, we have $\delta^*(\sigma_\lambda) = \sigma_{\lambda^\vee}$.*

5.4 Morphism $\phi_{m,n}$: geometric definition

To a polynomial representation $V \in R_+(GL_m)$, we associated the polynomial map $\pi_V : M_{m,m} \longrightarrow \mathrm{End}(V)$ and the invariant polynomial $\chi_V(X) := \mathrm{Tr}_V(\pi_V(X))$.

Let $\mathbb{E}_{m,n} \to \mathbb{G}_{m,n}$ be the canonical vector bundle of rank m. Let $\Omega_{m,n} \in \mathcal{A}^2(\mathbb{G}_{m,n}, \mathrm{End}(\mathbb{E}_{m,n}))$ be its curvature. The Chern-Weil homomorphism associates to the invariant polynomial χ_V the closed form $\chi_V\left(\frac{i}{2\pi}\Omega_{m,n}\right)$ of even degree on $\mathbb{G}_{m,n}$. We denote by $H^*(\mathbb{G}_{m,n})$ the de Rham cohomology of $\mathbb{G}_{m,n}$. We have a natural (injective) morphism $H^*(\mathbb{G}_{m,n}, \mathbb{Z}) \to H^*(\mathbb{G}_{m,n})$.

Here is a geometric definition of the map $\phi_{m,n}$ [26].

Theorem 5.8 *For any $V \in R_+(GL_m)$, $\phi_{m,n}(V) \in H^*(\mathbb{G}_{m,n})$ is the class defined by the closed form $\chi_V\left(\frac{i}{2\pi}\Omega_{m,n}\right)$.*

If $\mathcal{V} \to N$ is a complex vector bundle, we denote by $c_k(\mathcal{V})$ its k-Chern class. In the next lemma, we recall the computation of the Chern classes of the vector bundles $\mathbb{E}_{m,n}$ and $\mathbb{E}^{\perp}_{m,n}$.

Lemma 5.9 *The following relations holds in $H^*(\mathbb{G}_{m,n})$.*

$$c_k(\mathbb{E}_{m,n}) = \begin{cases} \sigma_{1^k} & \text{if} \quad 1 \leqslant k \leqslant m, \\ 0 & \text{if} \quad k > m. \end{cases}$$

$$c_k(\mathbb{E}^{\perp}_{m,n}) = \begin{cases} \sigma_{k} & \text{if} \quad 1 \leqslant k \leqslant n, \\ 0 & \text{if} \quad k > n. \end{cases}$$

Proof : If $k > m = \mathrm{rank}(\mathbb{E}_{m,n})$, then $c_k(\mathbb{E}_{m,n}) = 0$. If $1 \leqslant k \leqslant m$, then $c_k(\mathbb{E}_{m,n}) = \phi_{m,n}(\bigwedge^k \mathbb{C}^m) = \sigma_{1^k}$. For the second point, let us use the isomorphism $\delta : \mathbb{G}_{m,n} \to \mathbb{G}_{n,m}$. We see that the vector bundle $\mathbb{E}^{\perp}_{m,n}$ is isomorphic to $\delta^{-1}(\mathbb{E}_{n,m})$. Then $c_k(\mathbb{E}^{\perp}_{m,n}) = \delta^*(c_k(\mathbb{E}_{n,m})) = \delta^*(\sigma_{1^k}) = \sigma_k$ for any $1 \leqslant k \leqslant n$. $\square$

5.5 Cauchy formula

We fix some integers $m, n, m', n' \geqslant 1$.

We consider the vector bundles $\mathbb{E}_{m,n} \to \mathbb{G}_{m,n}$ and $\mathbb{E}_{m',n'} \to \mathbb{G}_{m',n'}$. We can form the bundles $\mathbb{E}_{m,n} \boxtimes \mathbb{E}_{m',n'}$ and $\mathbb{E}^{\perp}_{m,n} \boxtimes \mathbb{E}^{\perp}_{m',n'}$ on $\mathbb{G}_{m,n} \times \mathbb{G}_{m',n'}$. The purpose of this section is the computation of their Euler classes.

For any partition $\lambda \subset m' \times m$, we define $\tilde{\lambda} = \widehat{\lambda}^{\vee} \subset m \times m'$.

Proposition 5.10 *The following relation holds in $H^{2mm'}(\mathbb{G}_{m,n} \times \mathbb{G}_{m',n'})$:*

$$\mathrm{Eul}(\mathbb{E}_{m,n} \boxtimes \mathbb{E}_{m',n'}) = \sum_{\lambda \subset m' \times m} \sigma_{\tilde{\lambda}} \otimes \sigma_\lambda,$$

where $\sigma_\lambda \in H^{2|\lambda|}(\mathbb{G}_{m',n'})$ and $\sigma_{\tilde{\lambda}} \in H^{2(mm'-|\lambda|)}(\mathbb{G}_{m,n})$. For a partition $\lambda \subset m' \times m$, the product $\sigma_{\tilde{\lambda}} \otimes \sigma_\lambda$ does not vanish if only if the following conditions hold :

- $n' \geqslant \lambda_1$,
- $\sharp\{1 \leqslant k \leqslant m', \lambda_k = m\} \geqslant m' - n$.

Proof : The Euler class $\mathrm{Eul}(\mathbb{E}_{m,n} \boxtimes \mathbb{E}_{m',n'})$ is equal to the top Chern class $c_{mm'}(\mathbb{E}_{m,n} \boxtimes \mathbb{E}_{m',n'}) \in H^{2mm'}(\mathbb{G}_{m,n} \times \mathbb{G}_{m',n'})$. The curvature of the vector bundle $\mathbb{E}_{m,n} \boxtimes \mathbb{E}_{m',n'}$ is equal to $\Omega_{m,n} \otimes \mathrm{Id}' + \mathrm{Id} \otimes \Omega_{m',n'}$ where $\mathrm{Id} \in \mathrm{End}(\mathbb{E}_{m,n})$ and $\mathrm{Id}' \in \mathrm{End}(\mathbb{E}_{m',n'})$ are the identity maps. In order to compute

$$c_{mm'}(\mathbb{E}_{m,n} \boxtimes \mathbb{E}_{m',n'}) = \det\left(\tfrac{i}{2\pi}\Omega_{m,n} \otimes \mathrm{Id}' + \mathrm{Id} \otimes \tfrac{i}{2\pi}\Omega_{m',n'}\right)$$

we use the following Cauchy formula (see [19])

$$\prod_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant m'}} (x_i + x'_j) = \sum_{\lambda \subset m' \times m} s_{\tilde{\lambda}}(x) s_\lambda(x').$$

The previous relation implies that

$$\det\left(X \otimes \mathrm{Id}' + \mathrm{Id} \otimes X'\right) = \sum_{\lambda \subset m' \times m} \chi_{V_{\tilde{\lambda}}}(X) \chi_{V_\lambda}(X'),$$

for all $(X, X') \in M_{m,m} \times M_{m',m'}$. Finally, we obtain thank to Theorem 5.8, the following relation

$$\begin{aligned} c_{mm'}(\mathbb{E}_{m,n} \boxtimes \mathbb{E}_{m',n'}) &= \sum_{\lambda \subset m' \times m} \chi_{V_{\tilde{\lambda}}}(\tfrac{i}{2\pi}\Omega_{m,n}) \chi_{V_\lambda}(\tfrac{i}{2\pi}\Omega_{m',n'}) \\ &= \sum_{\lambda \subset m' \times m} \sigma_{\tilde{\lambda}} \otimes \sigma_\lambda. \end{aligned}$$

Let us analyse when $\sigma_{\tilde{\lambda}} \otimes \sigma_\lambda \neq 0$. From the definition, we see that for any partition $\lambda \subset m' \times m$, we have $\tilde{\lambda}_j = \sharp\{1 \leqslant k \leqslant m', \lambda_k \leqslant m - j\}$, $\forall j \in \{1, \ldots, m\}$. In particular we get $\tilde{\lambda}_1 = m' - \sharp\{1 \leqslant k \leqslant m', \lambda_k = m\}$ since $\lambda_k \leqslant m$, $\forall k$.

The element $\sigma_\lambda \in H^{2|\lambda|}(\mathbb{G}_{m',n'})$ does not vanish if and only if $\lambda_1 \leqslant n'$, and $\sigma_{\tilde{\lambda}} \in H^{2(mm'-|\lambda|)}(\mathbb{G}_{m,n})$ does not vanish if and only if $\tilde{\lambda}_1 \leqslant n$. Our proof is completed since the relation $\tilde{\lambda}_1 \leqslant n$ is equivalent to $\sharp\{1 \leqslant k \leqslant m', \lambda_k = m\} \geqslant m' - n$. $\square$

We apply our formula to the case where $p \geqslant q \geqslant 1$, $1 < r < p$ and $1 < s < q$. The following relation holds in $H^{2(p-r)s}(\mathbb{G}_{p-r,r} \times \mathbb{G}_{s,q-s})$:

$$\mathrm{Eul}(\mathbb{E}_{p-r,r} \boxtimes \mathbb{E}_{s,q-s}) = \sum_{\lambda \subset s \times p-r} \sigma_{\tilde{\lambda}} \otimes \sigma_\lambda, \tag{9}$$

where $\sigma_\lambda \in H^{2|\lambda|}(\mathbb{G}_{p-r,r})$ and $\sigma_{\tilde{\lambda}} \in H^{2((p-r)s-|\lambda|)}(\mathbb{G}_{s,q-s})$. For a partition $\lambda \subset s \times p-r$, the product $\sigma_{\tilde{\lambda}} \otimes \sigma_\lambda$ does not vanish if only if $q - s \geqslant \lambda_1$, and $\sharp\{1 \leqslant k \leqslant s, \lambda_k = p - r\} \geqslant s - r$.

Lemma 5.11 *Suppose that* $p \geqslant q$. *Then* $\mathrm{Eul}(\mathbb{E}_{p-r,r} \boxtimes \mathbb{E}_{s,q-s}) = 0$ *if* $s > r$.

Proof : Suppose that $s - r > 0$. If $\mathrm{Eul}(\mathbb{E}_{p-r,r} \boxtimes \mathbb{E}_{s,q-s}) \neq 0$, there exists a partition $\lambda \subset s \times p - r$ such that $\sigma_{\tilde{\lambda}} \otimes \sigma_\lambda \neq 0$: hence $q - s \geqslant \lambda_1$ and $\sharp\{1 \leqslant k \leqslant s, \lambda_k = p - r\} \geqslant s - r > 0$. We obtain $q - s \geqslant \lambda_1 = p - r$, so $q \geqslant p + s - r > p$, which is in contradiction with our hypothesis. $\square$

We consider now the vector bundle $\mathcal{V}_s^r = \mathbb{E}_{r,p-r}^\perp \boxtimes \mathbb{E}_{q-s,s}^\perp$ on $\mathbb{G}_{r,p-r} \times \mathbb{G}_{q-s,s}$.

Proposition 5.12 *Let* $p \geqslant q \geqslant 1$, $1 < r < p$ *and* $1 < s < q$.

- *If* $s > r$, *then* $\mathrm{Eul}(\mathcal{V}_s^r) = 0$
- *If* $s \leqslant r$, *the following relation holds in* $H^{2(p-r)s}(\mathbb{G}_{r,p-r} \times \mathbb{G}_{q-s,s})$:

  $$\mathrm{Eul}(\mathcal{V}_s^r) = \sum_{\lambda \subset s \times p-r} \sigma_{\hat{\lambda}} \otimes \sigma_{\lambda^\vee},$$

  where $\sigma_{\lambda^\vee} \in H^{2|\lambda|}(\mathbb{G}_{q-s,s})$ *and* $\sigma_{\hat{\lambda}} \in H^{2((p-r)s-|\lambda|)}(\mathbb{G}_{r,p-r})$. *For a partition* $\lambda \subset s \times p - r$, *the product* $\sigma_{\hat{\lambda}} \otimes \sigma_{\lambda^\vee}$ *does not vanish if and only if* $q - s \geqslant \lambda_1$.

Proof : Let $\delta \times \delta : \mathbb{G}_{r,p-r} \times \mathbb{G}_{q-s,s} \to \mathbb{G}_{p-r,r} \times \mathbb{G}_{s,q-s}$ be the product of "duality" maps (see §5.3). Since $\mathbb{E}_{r,p-r}^\perp \boxtimes \mathbb{E}_{q-s,s}^\perp \simeq (\delta \times \delta)^{-1}(\mathbb{E}_{p-r,r} \boxtimes \mathbb{E}_{q-s,s})$, we have $\mathrm{Eul}(\mathcal{V}_s^r) = \delta^* \times \delta^*(\mathrm{Eul}(\mathbb{E}_{p-r,r} \boxtimes \mathbb{E}_{s,q-s}))$. We can use Lemmas 5.11 and 5.7, and (9) to complete the proof. $\square$

6 Convex cone $\mathrm{S}(p,q)$: equations of the facets

Theorem 2.13 tells us that an element $(A,B,C) \in (\mathcal{C}_p \times \mathcal{C}_q)^3$ belongs to $\mathrm{S}(p,q) = \Delta((GL_p \times GL_q)^2 \times \mathbb{C}^p \otimes \mathbb{C}^q)$ if and only

$$\boxed{|A'| + |B'| + |C'| = |A''| + |B''| + |C''|}$$

and

$$\boxed{|\,A'\,|_{I'} + |\,B'\,|_{J'} + |\,C'\,|_{K'} \leqslant |\,A''\,|_{I''} + |\,B''\,|_{J''} + |\,C''\,|_{K''}}, \tag{10}$$

for any couple $(r,s) \in \{0,\ldots,p\} \times \{0,\ldots,q\} - \{(p,q),(0,0)\}$, for any $I', J', K' \in \mathcal{P}^p_r$ and any $I'', J'', K'' \in \mathcal{P}^p_s$, such that the product

$$\left(\sigma_{\lambda(I')} \otimes \sigma_{\lambda((I'')^c)}\right) \cdot \left(\sigma_{\lambda(J')} \otimes \sigma_{\lambda((J'')^c)}\right) \cdot \left(\sigma_{\lambda(K')} \otimes \sigma_{\lambda((K'')^c)}\right) \cdot \mathrm{Eul}(\mathcal{V}^r_s) \tag{11}$$

is a non zero multiple of $[pt] \in H^{\max}(\mathbb{G}(r,p-r) \times \mathbb{G}(q-s,s),\mathbb{Z})$.

In the following sections, we study each case according to the parameter (r,s).

6.1 $r = 0$ and $s = q$

Here, $\mathbb{G}(r,p-r) \times \mathbb{G}(q-s,s) = \{pt\}$ and $\mathcal{V}^0_q \neq 0$, hence the cohomological condition (11) does not hold.

6.2 $r = p$ and $s = 0$

Here, $\mathbb{G}(r,p-r) \times \mathbb{G}(q-s,s) = \{pt\}$ and $\mathcal{V}^p_0 = 0$, hence the cohomological condition (11) holds. Relation (10) becomes $\boxed{|\,A'\,| + |\,B'\,| + |\,C'\,| \leqslant 0}$.

6.3 $0 < r < p$ and $s = 0$

Here $\mathcal{V}^r_0 = 0$ and the cohomological condition (11) becomes $\sigma_{\lambda(I')} \cdot \sigma_{\lambda(J')} \cdot \sigma_{\lambda(K')} = k[pt]$, $k \geqslant 1$, in $H^*(\mathbb{G}(r,p-r))$ which is equivalent to asking that

$$\left[V_{\lambda(I')} \otimes V_{\lambda(J')} \otimes V_{\lambda(K')} \otimes \det{}^{-(p-r)}\right]^{GL_r} \neq 0.$$

Here inequality (10) becomes $\boxed{|\,A'\,|_{I'} + |\,B'\,|_{J'} + |\,C'\,|_{K'} \leqslant 0}$.

6.4 $0 < r < p$ and $s = q$

Here $\mathcal{V}_q^r$ is the vector bundle $\mathbb{E}_{r,p-r}^{\perp} \otimes \mathbb{C}^q$ on $\mathbb{G}(r, p-r)$. Since

$$\mathrm{Eul}(\mathcal{V}_q^r) = \mathrm{Eul}(\mathbb{E}_{r,p-r}^{\perp})^q = \left(c_{p-r}(\mathbb{E}_{r,p-r}^{\perp})\right)^q = (\sigma_{p-r})^q \quad \in H^*(\mathbb{G}(r, p-r)),$$

condition (11) is $\sigma_{\lambda(I')} \cdot \sigma_{\lambda(J')} \cdot \sigma_{\lambda(K')} \cdot (\sigma_{p-r})^q = k[pt],\ k \geqslant 1$. If we take the image of the previous relation through $\delta : H^*(\mathbb{G}(r, p-r)) \to H^*(\mathbb{G}(p-r, r))$, we obtain

$$\sigma_{\lambda(\widetilde{(I')^c})} \cdot \sigma_{\lambda(\widetilde{(J')^c})} \cdot \sigma_{\lambda(\widetilde{(K')^c})} \cdot (\sigma_{1^{p-r}})^q = k[pt],\ k \geqslant 1, \quad \text{in} \quad H^*(\mathbb{G}(p-r, r))$$

that is equivalent to

$$\left[V_{\lambda(\widetilde{(I')^c})} \otimes V_{\lambda(\widetilde{(J')^c})} \otimes V_{\lambda(\widetilde{(K')^c})} \otimes \det{}^{q-r}\right]^{GL_{p-r}} \neq 0.$$

Here inequality (10) is equivalent to $\boxed{\mid A' \mid_{(I')^c} + \mid B' \mid_{(J')^c} + \mid C' \mid_{(K')^c} \geqslant 0}$.

6.5 $r = 0$ and $0 < s < q$

Here $\mathcal{V}_s^0$ is the vector bundle $\mathbb{C}^p \otimes \mathbb{E}_{q-s,s}^{\perp}$ on $\mathbb{G}(p-s, s)$. Since $\mathrm{Eul}(\mathcal{V}_s^0) = (\sigma_s)^p$, condition (11) is $\sigma_{\lambda((I'')^c)} \cdot \sigma_{\lambda((J'')^c)} \cdot \sigma_{\lambda((K'')^c)} \cdot (\sigma_s)^p = k[pt],\ k \geqslant 1$. If we take the image of the previous relation through $\delta : H^*(\mathbb{G}(q-s, s)) \to H^*(\mathbb{G}(s, q-s))$, we obtain

$$\sigma_{\lambda(\widetilde{I''})} \cdot \sigma_{\lambda(\widetilde{J''})} \cdot \sigma_{\lambda(\widetilde{K''})} \cdot (\sigma_{1^s})^p = k[pt],\ k \geqslant 1, \quad \text{in} \quad H^*(\mathbb{G}(s, q-s))$$

that is equivalent to

$$\left[V_{\lambda(\widetilde{I''})} \otimes V_{\lambda(\widetilde{J''})} \otimes V_{\lambda(\widetilde{K''})} \otimes \det{}^{p-(q-s)}\right]^{GL_s} \neq 0.$$

Here inequality (10) becomes $\boxed{\mid A'' \mid_{I''} + \mid B' \mid_{J''} + \mid C' \mid_{K''} \geqslant 0}$.

6.6 $r = p$ and $0 < s < q$

Here $\mathcal{V}_s^p = 0$ and the cohomological condition (11) becomes $\sigma_{\lambda((I'')^c)} \cdot \sigma_{\lambda((J'')^c)} \cdot \sigma_{\lambda((K'')^c)} = k[pt],\ k \geqslant 1$, in $H^*(\mathbb{G}(q-s, s))$ that is equivalent to

$$\left[V_{\lambda((I'')^c)} \otimes V_{\lambda((J'')^c)} \otimes V_{\lambda((K'')^c)} \otimes \det{}^{-s}\right]^{GL_{q-s}} \neq 0.$$

Here inequality (10) is equivalent to $\boxed{\mid A'' \mid_{(I'')^c} + \mid B'' \mid_{(J'')^c} + \mid C'' \mid_{(K'')^c} \leqslant 0}$.

6.7 $0 < r < p$ and $0 < s < q$

We know that $\mathrm{Eul}(\mathcal{V}_s^r) = 0$ if $s > r$, hence condition (11) does not hold if $s > r$. Suppose now that $s \leqslant r$. Thanks to Proposition 5.12, we see that condition (11) is equivalent to asking that the sum

$$\sum_{\lambda \subset s \times p-r} \left(\sigma_{\lambda((I')^c)} \cdot \sigma_{\lambda((J')^c)} \cdot \sigma_{\lambda((K')^c)} \cdot \sigma_{\widehat{\lambda}}\right) \otimes \left(\sigma_{\lambda(I'')} \cdot \sigma_{\lambda(J'')} \cdot \sigma_{\lambda(K'')} \cdot \sigma_{\lambda^\vee}\right)$$

is equal to $k[pt], k \geqslant 1$ in $H^*(\mathbb{G}(r, p-r)) \otimes H^*(\mathbb{G}(q-s, s))$. Hence (11) holds if and only if there exists a partition $\lambda \subset s \times p - r$ such that

$$\sigma_{\lambda(I')} \cdot \sigma_{\lambda(J')} \cdot \sigma_{\lambda(K')} \cdot \sigma_{\widehat{\lambda}} = \ell'[pt] \quad \text{in } H^*(\mathbb{G}(r, p-r), \mathbb{Z}), \tag{12}$$

$$\sigma_{\lambda((I'')^c)} \cdot \sigma_{\lambda((J'')^c)} \cdot \sigma_{\lambda((K'')^c)} \cdot \sigma_{\lambda^\vee} = \ell''[pt] \quad \text{in } H^*(\mathbb{G}(q-s, s), \mathbb{Z}), \tag{13}$$

for some $\ell', \ell'' \geqslant 1$.

Lemma 6.1 *If $s \leqslant r$, then (12) and (13) hold if and only if there exists a partition $\mu \subset s \times p - r$, such that both conditions hold*

$$\left[V_{\lambda(I')} \otimes V_{\lambda(J')} \otimes V_{\lambda(K')} \otimes V_\mu \otimes \det{}^{-(p-r)}\right]^{GL_r} \neq 0, \tag{14}$$

$$\left[V_{\lambda(I'')} \otimes V_{\lambda(J'')} \otimes V_{\lambda(K'')} \otimes V_\mu \otimes \det{}^{-(p-r)-2(q-s)}\right]^{GL_s} \neq 0. \tag{15}$$

Proof : Let $\lambda \subset s \times p - r$ satisfying (12) and (13). Let $\mu = \widehat{\lambda} \subset s \times p - r$. Condition (12) is then equivalent[6] to (14).

Take now the image of (13) through the map $\delta : H^*(\mathbb{G}(q-s, s), \mathbb{Z}) \to H^*(\mathbb{G}(s, q-s), \mathbb{Z})$: we obtain $\sigma_{\lambda(\widetilde{I''})} \cdot \sigma_{\lambda(\widetilde{J''})} \cdot \sigma_{\lambda(\widetilde{K''})} \cdot \sigma_\lambda = \ell''[pt]$, $\ell'' \geqslant 1$ in $H^*(\mathbb{G}(s, q-s), \mathbb{Z})$, that is equivalent to

$$[V_{\lambda(\widetilde{I''})} \otimes V_{\lambda(\widetilde{J''})} \otimes V_{\lambda(\widetilde{K''})} \otimes V_\lambda \otimes \det{}^{-(q-s)}]^{GL_s} \neq 0. \tag{16}$$

Since we have the following relations between representations of GL_s :

1. $V_{\lambda(\widetilde{I''})} = (V_{\lambda(I'')})^* \otimes \det^{q-s}$ (with the same relations for J'' and K''),

2. $V_\lambda = (V_\mu)^* \otimes \det^{p-r}$,

condition (16) is equivalent to (15). $\square$

[6] Since the length of μ is less than $r \geqslant s$, V_μ is well defined as an irreducible representation of GL_r.

6.8 Summary

Let us summarize the computations done in the previous sections.

An element $(A,B,C) \in (\mathcal{C}_p \times \mathcal{C}_q)^3$ belongs to $\mathrm{S}(p,q)$ if and only the following conditions hold

- $\boxed{|A'|+|B'|+|C'| = |A''|+|B''|+|C''|}$.
- $\boxed{|\,A'\,|+|\,B'\,|+|\,C'\,| \leqslant 0}$.
- For any $0 < r < p$, for any $I', J', K' \in \mathcal{P}_r^p$, we have :

$$\boxed{|\,A'\,|_{I'}+|\,B'\,|_{J'}+|\,C'\,|_{K'} \leqslant 0} \quad \text{if } \left[V_{\lambda(I')} \otimes V_{\lambda(J')} \otimes V_{\lambda(K')} \otimes \det{}^{-p+r}\right]^{GL_r} \neq 0,$$

$$\boxed{|\,A'\,|_{I'}+|\,B'\,|_{J'}+|\,C'\,|_{K'} \geqslant 0} \quad \text{if } \left[V_{\lambda(\tilde{I}')} \otimes V_{\lambda(\tilde{J}')} \otimes V_{\lambda(\widetilde{K}')} \otimes \det{}^{q-p+r}\right]^{GL_r} \neq 0.$$

- For any $0 < s < q$, for any $I'', J'', K'' \in \mathcal{P}_s^q$, we have :

$$\boxed{|\,A''\,|_{I''}+|\,B''\,|_{J''}+|\,C''\,|_{K''} \leqslant 0} \quad \text{if } \left[V_{\lambda(I'')} \otimes V_{\lambda(J'')} \otimes V_{\lambda(K'')} \otimes \det{}^{-q+s}\right]^{GL_s} \neq 0.$$

$$\boxed{|\,A''\,|_{I''}+|\,B''\,|_{J''}+|\,C''\,|_{K''} \geqslant 0} \quad \text{if } \left[V_{\lambda(\widetilde{I''})} \otimes V_{\lambda(\widetilde{J''})} \otimes V_{\lambda(\widetilde{K''})} \otimes \det{}^{p-q+s}\right]^{GL_s} \neq 0.$$

- For any $(r,s) \in [p-1] \times [q-1]$ with $r \geqslant s$, for any $I, J, K \in \mathcal{P}_r^p \times \mathcal{P}_s^q$, we have

$$\boxed{|\,A'\,|_{I'}+|\,B'\,|_{J'}+|\,C'\,|_{K'} \leqslant |\,A''\,|_{I''}+|\,B''\,|_{J''}+|\,C''\,|_{K''}}$$

  if there exists a partition $\mu \subset s \times p-r$, such that both conditions hold

$$\left[V_{\lambda(I')} \otimes V_{\lambda(J')} \otimes V_{\lambda(K')} \otimes V_\mu \otimes \det{}^{-(p-r)}\right]^{GL_r} \neq 0,$$

$$\left[V_{\lambda(I'')} \otimes V_{\lambda(J'')} \otimes V_{\lambda(K'')} \otimes V_\mu \otimes \det{}^{-(p-r)-2(q-s)}\right]^{GL_s} \neq 0.$$

6.9 Proof of the main result

In the previous section, we described the facets of the convex cone $\mathrm{S}(p,q)$. We will now exploit the fact that $(A,B,C) \in \mathrm{Horn}(p,q)$ if and only if

$$\Theta(A,B,C) = \left(\begin{pmatrix} A' \\ (A')^* \end{pmatrix}, \begin{pmatrix} B' \\ (B'')^* \end{pmatrix}, \begin{pmatrix} (C')^* \\ C'' \end{pmatrix} \right) \in \mathrm{S}(p,q).$$

In what follows, we crucially use that the semigroups $\mathrm{Horn}^{\mathbb{Z}}(n)$ and $\mathrm{Horn}^{\mathbb{Z}}(p,q)$ are saturated. Hence $\mathrm{Horn}^{\mathbb{Z}}(n) = \mathrm{Horn}(n) \cap (\wedge_n^+)^3$ and $\mathrm{Horn}^{\mathbb{Z}}(p,q) = \mathrm{Horn}(p,q) \cap (\wedge_p^+ \times \wedge_q^+)^3$. Recall also the identity $|A^*|_I = -|A|_{\tilde{I}}$ which will be used several times.

Let us compute the image of the facets of $\mathrm{S}(p,q)$ through the linear map Θ :

- The image of $|A'| + |B'| + |C'| = |A''| + |B''| + |C''|$ through Θ is $\boxed{|\,A\,| + |\,B\,| = |\,C\,|}$.
- The image of the half space $|\,A'\,| + |\,B'\,| + |\,C'\,| \leqslant 0$ through the map Θ is $\boxed{|\,A'\,| + |\,B'\,| \leqslant |\,C'\,|}$.
- The image of the half space $|\,A'\,|_{I'} + |\,B'\,|_{J'} + |\,C'\,|_{K'} \leqslant 0$ through the map Θ is $\boxed{|\,A'\,|_{I'} + |\,B'\,|_{J'} \leqslant |\,C'\,|_{\widetilde{K'}}}$, and condition
$$\left[V_{\lambda(I')} \otimes V_{\lambda(J')} \otimes V_{\lambda(K')} \otimes \det{}^{-p+r}\right]^{GL_r} \neq 0$$
is equivalent to $(\lambda(I'), \lambda(J'), \lambda(\widetilde{K'})) \in \mathrm{Horn}(r)$.
- The image of of the half space $|\,A'\,|_{I'} + |\,B'\,|_{J'} + |\,C'\,|_{K'} \geqslant 0$ through the map Θ is $\boxed{|\,A'\,|_{I'} + |\,B'\,|_{J'} \geqslant |\,C'\,|_{\widetilde{K'}}}$, and condition
$$\left[V_{\lambda(\tilde{I}')} \otimes V_{\lambda(\tilde{J}')} \otimes V_{\lambda(\widetilde{K'})} \otimes \det{}^{q-p+r}\right]^{GL_r} \neq 0$$
is equivalent to $(\lambda(I'), \lambda(J'), \lambda(\widetilde{K'}) + (q+p-r)\mathbf{1}_r) \in \mathrm{Horn}(r)$.
- The image of of the half space $|\,A''\,|_{I''} + |\,B''\,|_{J''} + |\,C''\,|_{K''} \leqslant 0$ through the map Θ is $\boxed{|\,A''\,|_{\widetilde{I''}} + |\,B''\,|_{\widetilde{J''}} \geqslant |\,C''\,|_{K''}}$ and condition
$$\left[V_{\lambda(I'')} \otimes V_{\lambda(J'')} \otimes V_{\lambda(K'')} \otimes \det{}^{-q+s}\right]^{GL_s} \neq 0$$
is equivalent to $(\lambda(\widetilde{I''}), \lambda(\widetilde{J''}), \lambda(K'') + (q-s)\mathbf{1}_s) \in \mathrm{Horn}(s)$.
- The image of of the half space $|\,A''\,|_{I''} + |\,B''\,|_{J''} + |\,C''\,|_{K''} \geqslant 0$ through the map Θ is $\boxed{|\,A''\,|_{\widetilde{I''}} + |\,B''\,|_{\widetilde{J''}} \leqslant |\,C''\,|_{K''}}$ and condition
$$\left[V_{\lambda(\widetilde{I''})} \otimes V_{\lambda(\widetilde{J''})} \otimes V_{\lambda(\widetilde{K''})} \otimes \det{}^{p-q+s}\right]^{GL_s} \neq 0$$
is equivalent to $(\lambda(\widetilde{I''}), \lambda(\widetilde{J''}), \lambda(K'') - p\mathbf{1}_s) \in \mathrm{Horn}(s)$.

- The image of of the half space

$$|A'|_{I'}+|B'|_{J'}+|C'|_{K'}\leqslant|A''|_{I''}+|B''|_{J''}+|C''|_{K''}$$

through the map Θ is

$$\boxed{|A'|_{I'}+|A''|_{\widetilde{I''}}+|B'|_{J'}+|B''|_{\widetilde{J''}}\leqslant|C'|_{\widetilde{K'}}+|C''|_{K''}}$$

and conditions

$$\left[V_{\lambda(I')}\otimes V_{\lambda(J')}\otimes V_{\lambda(K')}\otimes V_\mu\otimes\det{}^{-(p-r)}\right]^{GL_r}\neq 0,$$
$$\left[V_{\lambda(I'')}\otimes V_{\lambda(J'')}\otimes V_{\lambda(K'')}\otimes V_\mu\otimes\det{}^{-(p-r)-2(q-s)}\right]^{GL_s}\neq 0.$$

are equivalent to

$$\left[V_{\lambda(I')}\otimes V_{\lambda(J')}\otimes V^*_{\lambda(\widetilde{K'})}\otimes V_\mu\right]^{GL_r}\neq 0,$$
$$\left[V_{\lambda(\widetilde{I''})}\otimes V_{\lambda(\widetilde{J''})}\otimes V^*_{\lambda(K'')}\otimes\det{}^{(p-r)}\otimes V^*_\mu\right]^{GL_s}\neq 0.$$

The existence of a partition $\mu\in s\times p-r$ satisfying the previous relations is equivalent to asking that

$$\left(\begin{pmatrix}\lambda(I')\\ \lambda(\widetilde{I''})\end{pmatrix},\begin{pmatrix}\lambda(J')\\ \lambda(\widetilde{J''})\end{pmatrix},\begin{pmatrix}\lambda(\widetilde{K'})\\ \lambda(K'')-(p-r)\mathbf{1}_s\end{pmatrix}\right)\in(\wedge_r^+\times\wedge_s^+)^3$$

belongs to $\mathrm{Horn}(r,s)$.

The proof of Theorem 1.6 is complete.

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