



HAL
open science

Diffusion limit for a stochastic kinetic problem with unbounded driving process

Shmuel Rakotonirina-Ricquebourg

► **To cite this version:**

Shmuel Rakotonirina-Ricquebourg. Diffusion limit for a stochastic kinetic problem with unbounded driving process. 2020. hal-02866901

HAL Id: hal-02866901

<https://hal.science/hal-02866901>

Preprint submitted on 12 Jun 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Diffusion limit for a stochastic kinetic problem with unbounded driving process

Shmuel Rakotonirina--Ricquebourg*

June 12, 2020

Abstract This paper studies the limit of a kinetic evolution equation involving a small parameter and driven by a random process which also scales with the small parameter. In order to prove the convergence in distribution to the solution of a stochastic diffusion equation while removing a boundedness assumption on the driving random process, we adapt the method of perturbed test functions to work with stopped martingales problems.

Contents

1	Introduction	2
2	Assumptions and main result	4
2.1	Driving random term	5
2.1.1	Assumption on moments	6
2.1.2	Mixing property	7
2.2	Main result	7
2.3	Strategy of the proof of Theorem 2.1	8
3	Preliminary results	9
3.1	Resolvent operator	9
3.1.1	Additional assumptions	9
3.1.2	Results on the resolvent operator	10
3.2	Properties of the covariance operator	13
3.3	Behavior of the stopping time for the driving process	14
3.4	Pathwise solutions	16
3.5	Estimate in $L^2(\mathcal{M}^{-1})$	17

*Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, F-69622 Villeurbanne, France

4	Martingale problems and perturbed test functions	20
4.1	Generator and martingales	20
4.2	The perturbed test functions method	28
4.2.1	Framework for the perturbed test functions method	29
4.2.2	Consistency result	29
4.2.3	Construction of the first corrector function φ_1	30
4.2.4	Construction of the second corrector function φ_2	32
4.2.5	Controls on the second corrector function	33
4.2.6	Good test function property	35
5	Dynamics associated with the limiting equation	36
6	Tightness of the coupled stopped process	38
6.1	Proof of the first claim (i)	39
6.2	Proof of the second claim (ii)	42
7	Identification of the limit points	45
7.1	Convergence of the auxiliary process	46
7.2	Convergence of the stopped martingale problems	47
7.3	Identification of the limit point	49
7.4	Convergence of the unstopped process	52
8	Strong convergence	53

1 Introduction

Our aim in this work is to study the scaling limit of a stochastic kinetic equation in the diffusion approximation regime, both in Partial Differential Equation (PDE) and probabilistic senses. For deterministic problems, this is a thoroughly studied field in the literature, starting historically with [26, 1]. Kinetic models with small parameters appear in various situations, for example when studying semi-conductors [18] and discrete velocity models [27] or as a limit of a particle system, either with a single particle [19] or multiple ones [32]. It is important to understand the limiting equations, which are in general much easier to simulate numerically. For instance, in the asymptotic regime we study, the velocity variable disappears at the limit.

When a random term with the correct scaling (here t/ε^2) is added to a differential equation, it is classical that, when $\varepsilon \rightarrow 0$, the solution may converge in distribution to a diffusion process, which solves a Stochastic Differential Equation (SDE) driven by a white noise in time. This is a diffusive limit in the probabilistic sense. Such convergence has been proved initially by Has'minskii [20, 21] and then, using the martingale approach and perturbed test functions, in the classical article [29] (see also [25, 14, 17, 31, 6]). The use of perturbed test functions in the context of PDEs with diffusive limits also concerns various situations, for instance in the context of viscosity solutions [15], nonlinear Schrödinger equations [28, 7, 9, 6], a parabolic PDE [30] or, as in this paper, kinetic SPDEs [10, 11, 8].

In this article, we consider the following equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} L f^\varepsilon + \frac{1}{\varepsilon} f^\varepsilon \overline{m}^\varepsilon, \quad (1)$$

$$f^\varepsilon(0) = f_0^\varepsilon. \quad (2)$$

where f^ε is defined on $\mathbb{R}^+ \times \mathbb{T}^d \times V$, L is a linear operator (see (3) below) and the source term \overline{m}^ε is a random process defined on $\mathbb{R}^+ \times \mathbb{T}^d$ (satisfying assumptions given in Section 2.1). The goal of this article is to study the limit $\varepsilon \rightarrow 0$ of its solution f^ε , and to generalize previous results of [10].

The solution $f^\varepsilon(t, x, v)$ is interpreted as a probability distribution function of particles, having position x and velocity $a(v)$ at time t . The variable v belongs to a measure space (V, μ) , where μ is a probability measure. The function a models the velocity.

The Bhatnagar-Gross-Krook operator L expresses the particle interactions, defined on $L^1(V, \mu)$ by

$$L f = \rho \mathcal{M} - f, \quad (3)$$

where $\rho \doteq \int_V f d\mu$ and $\mathcal{M} \in L^1(V)$.

The source term \overline{m}^ε is defined as

$$\overline{m}^\varepsilon(t, x) = \overline{m}(t/\varepsilon^2, x), \quad (4)$$

where \overline{m} is a random process, not depending on ε .

In the deterministic case $\overline{m}^\varepsilon = 0$, such a problem occurs in various physical situations [12]. The density $\rho^\varepsilon \doteq \int_V f^\varepsilon d\mu$ converges to the solution of the linear parabolic equation

$$\partial_t \rho - \operatorname{div}(K \nabla \rho) = 0, \quad (5)$$

on $\mathbb{R}^+ \times \mathbb{T}^d$. This is a diffusive limit in the PDE sense, since the limit equation is a diffusion equation.

In this article, the diffusion limit of (1) is considered simultaneously in the PDE and in the probabilistic sense. The main result, Theorem 2.1, establishes that, under appropriate assumptions, the density $\rho^\varepsilon = \int_V f^\varepsilon d\mu$ converges in distribution in $C([0, T], H^{-\sigma}(\mathbb{T}^d))$ for any $\sigma > 0$ and in $L^2([0, T], L^2(\mathbb{T}^d))$ to the solution of the stochastic linear diffusion equation

$$d\rho = \operatorname{div}(K \nabla \rho) dt + \rho \circ Q^{1/2} dW(t),$$

with K as in (5). The equation is written in Stratonovitch form and is driven by a cylindrical Wiener process W , the covariance operator Q being trace-class. As usual in the context of diffusion limit, the stochastic equation involves a Stratonovitch product. The diffusive limit in the stochastic case has been first proved in [10], under a restrictive condition on the driving random term: \overline{m} is bounded almost surely. The boundedness of \overline{m} is a strong assumption, which is not satisfied by an Ornstein-Uhlenbeck process for instance. The contribution of this article is to relax this assumption: we prove the convergence under a moment bound assumption for the driving process.

The main tools of [10] are the perturbed test function method and the concept of solution in the martingale sense. Our general strategy for the proof is similar, therefore those tools are also used here. The main novelty is the introduction of stopping times to obtain the estimates required to establish tightness and convergence. Indeed, relaxing the conditions on \bar{m} implies that moments of the solutions are not controlled (exponential moments for \bar{m} would be necessary). The strategy from [10] needs to be substantially modified: the martingale problem approach is combined with the use of stopping times. At the limit, the stopping times persist, thus the limit processes solves the limit martingale problem only up to a stopping time. We manage to identify them nonetheless as a stopped version of the global solution.

This article is organized as follows: in Section 2, we set some notation, the assumptions on the driving random term and the main result, Theorem 2.1. Section 3 states some auxiliary results that are used in the later sections. In Section 4, we introduce the notion of martingale problem and the perturbed test function method that are used to prove the convergence. In Section 6, we prove the tightness of the family of processes $((\rho^\varepsilon, \tau_\Lambda^\varepsilon, \zeta^\varepsilon, \tau_\Lambda^\varepsilon))_\varepsilon$ stopped at the random time τ_Λ^ε . Section 7 takes the limit when $\varepsilon \rightarrow 0$ in the martingale problems and establishes the convergence of ρ^ε in $C([0, T], H^{-\sigma}(\mathbb{T}^d))$. In Section 8, we prove the convergence in a stronger sense, namely in $L^2([0, T], L^2(\mathbb{T}^d))$, using an additional assumption and an averaging lemma.

2 Assumptions and main result

Assumption 1. The operator L is defined on $L^1(V, \mu)$ by (3), with $\mathcal{M} \in L^1(V, \mu)$ such that $\inf_V \mathcal{M} > 0$ and $\int_V \mathcal{M} d\mu = 1$.

Let us define the spaces $L^2(\mathcal{M}^{-1})$ and L_x^2 and the associated inner products:

$$L^2(\mathcal{M}^{-1}) \doteq L^2(\mathbb{T}^d \times V, dx \mathcal{M}^{-1}(v) d\mu(v)), (f, g)_{L^2(\mathcal{M}^{-1})} \doteq \int_{\mathbb{T}^d} \int_V \frac{f(x, v) g(x, v)}{\mathcal{M}(v)} d\mu(v) dx,$$

$$L_x^2 \doteq L^2(\mathbb{T}^d, dx), (f, g)_{L_x^2} \doteq \int_{\mathbb{T}^d} f(x) g(x) dx.$$

We also define the norms $\|\cdot\|_{L^2(\mathcal{M}^{-1})}$ and $\|\cdot\|_{L_x^2}$ associated with these inner products.

Note that L is an orthogonal projection in $L^2(\mathcal{M}^{-1})$, hence

$$\forall f \in L^2(\mathcal{M}^{-1}), \|Lf\|_{L^2(\mathcal{M}^{-1})} \leq \|f\|_{L^2(\mathcal{M}^{-1})}.$$

Assumption 2. The function a is bounded ($a \in L^\infty(V, \mu; \mathbb{R}^d)$), centered for $\mathcal{M} d\mu$, namely

$$\int_V a(v) \mathcal{M}(v) d\mu(v) = 0, \tag{6}$$

and the following matrix is symmetric and positive definite

$$K \doteq \int_V a(v) \otimes a(v) \mathcal{M}(v) d\mu(v) > 0. \tag{7}$$

The following assumption is not required to get the convergence in $C([0, T], H_x^{-\sigma})$ but is used in Section 8 to retrieve a stronger convergence (in $L^2([0, T], L_x^2)$). It is exactly the assumption of Theorem 2.3 in [3].

Assumption 3. We have $(V, d\mu) = (\mathbb{R}^n, \psi(v)dv)$ for some function $\psi \in H^1(\mathbb{R}^n)$, $a \in \text{Lip}_{loc}(\mathbb{R}^n; \mathbb{R}^d)$ and there exists $C \geq 0$ and $\sigma^* \in (0, 1]$ such that

$$\forall u \in S^{d-1}, \forall \lambda \in \mathbb{R}, \forall \delta > 0, \int_{\lambda < a(v) \cdot u < \lambda + \delta} (|\psi(v)|^2 + |\nabla \psi(v)|^2) dv \leq C \delta^{\sigma^*}.$$

If ψ is not compactly supported, assume moreover that ∇a is globally bounded.

Assumption 4. We have

$$\sup_{\varepsilon \in (0, 1)} \mathbb{E} \left[\|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^{24} \right] < \infty. \quad (8)$$

and ρ_0^ε converges in distribution in L_x^2 to ρ_0 .

Remark. The moments of order 24 in Assumption 4 are useful in Sections 4.1 and 4.2.

2.1 Driving random term

Consider the normed space

$$E \doteq C^{2[d/2]+4}(\mathbb{T}^d),$$

where the norm is given by

$$\|\cdot\|_E = \sum_{|\beta| \leq 2[d/2]+4} \sup_{x \in \mathbb{T}^d} \left| \frac{\partial^{|\beta|} \cdot}{\partial x^\beta} \right|,$$

where $\beta \in \mathbb{N}^d$, $|\beta| = \sum_{i=1}^d \beta_i$ and

$$\frac{\partial^{|\beta|}}{\partial x^\beta} = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_d^{\beta_d}}.$$

Assumption 5. The family of process $(m(\cdot, n))_{n \in E}$ is a E -valued, càdlàg, stochastically continuous and homogeneous Markov process with initial condition $m(0, n) = n$. It admits a unique centered stationary distribution ν

$$\int_E \|n\|_E d\nu(n) < \infty \text{ and } \int_E n d\nu(n) = 0.$$

The driving process \bar{m} is the stationary Markov process associated with $(m(\cdot, n))_{n \in E}$, meaning that for all $t \in \mathbb{R}^+$, the distribution of $\bar{m}(t)$ is ν . It is adapted to a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ satisfying the usual conditions (complete and right-continuous).

For $\theta \in L^1(E) \doteq L^1(E, \nu)$, set

$$\langle \theta \rangle \doteq \int_E \theta d\nu.$$

Note that most of the arguments below only require $\bar{m}(t) \in C^1(\mathbb{T}^d)$. However, in Section 6, we use the compact embedding $H^{|d/2|+2}(\mathbb{T}^d) \subset C^1(\mathbb{T}^d)$ and in Section 3.2, we need $\bar{m}(t) \in C^{2s}(\mathbb{T}^d)$ with $H^s(\mathbb{T}^d) \subset C^1(\mathbb{T}^d)$, hence $s = [d/2] + 2$.

Definition 1. For $\varepsilon > 0$, the random process \bar{m}^ε is defined by (4) where \bar{m} is defined by Assumption 5. Let $\mathcal{F}_t^\varepsilon = \mathcal{F}_{t/\varepsilon^2}$ so that \bar{m}^ε is adapted to the filtration $(\mathcal{F}_t^\varepsilon)_{t \in \mathbb{R}^+}$.

2.1.1 Assumption on moments

From now on, we depart from the setting of [10]. In the previous works [10, 11], it is assumed that there exists $C_* \in \mathbb{R}^+$ such that, almost surely,

$$\forall t \in \mathbb{R}^+, \|\bar{m}(t)\|_E \leq C_*.$$

The main novelty of this article is that we relax this assumption into Assumptions 6 and 7 concerning moments.

Assumption 6. There exists $\gamma \in (4, \infty)$ such that

$$\mathbb{E} \left[\sup_{t \in [0,1]} \|\bar{m}(t)\|_E^\gamma \right] < \infty.$$

The condition $\gamma > 4$ is required below in Assumption 7, where we also assume that the moments on $m(t, n)$ depend polynomially on n .

Assumption 7. There exists $b \in [0, \frac{\gamma}{2} - 2)$ such that

$$\sup_{n \in E} \sup_{t \in \mathbb{R}^+} \frac{\mathbb{E} \left[\|m(t, n)\|_E^2 \right]^{\frac{1}{2}}}{1 + \|n\|_E^b} < \infty,$$

and such that ν has a finite $8(b+2)$ -order moment, namely

$$\int_E \|n\|_E^{8(b+2)} d\nu(n) < \infty.$$

For instance, if m is an Ornstein-Uhlenbeck process

$$dm(t) = -\theta m(t)dt + \sigma dW(t),$$

with W a E -valued Wiener process, then m satisfies Assumptions 6 and 7.

Moreover, any process satisfying the boundedness assumption in [10] also satisfies Assumptions 6 and 7.

2.1.2 Mixing property

Assumption 8 (Mixing property). There exists a nonnegative integrable function $\gamma_{mix} \in L^1(\mathbb{R}^+)$ such that, for all $n_1, n_2 \in E$, there exists a coupling $(m^*(\cdot, n_1), m^*(\cdot, n_2))$ of $(m(\cdot, n_1), m(\cdot, n_2))$ such that

$$\forall t \in \mathbb{R}^+, \mathbb{E} \left[\|m^*(t, n_1) - m^*(t, n_2)\|_E^2 \right]^{1/2} \leq \gamma_{mix}(t) \|n_1 - n_2\|_E.$$

Typically, γ_{mix} is expected to be of the form $\gamma_{mix}(t) = C_{mix} e^{-\beta_{mix} t}$ for some $\beta_{mix} > 0$. In the example where m is an Ornstein-Uhlenbeck process, consider $m^*(\cdot, n_1)$ and $m^*(\cdot, n_2)$ driven by the same Wiener process W . Owing to Gronwall's Lemma, it is straightforward to prove that this coupling satisfies Assumption 8 and that γ_{mix} decays exponentially fast.

We also need Assumptions 9 and 10 concerning the transition semi-group associated to the homogeneous Markov process $(m(\cdot, n))_{n \in E}$. Since those assumptions are quite technical, we postpone their statement in Section 3.1.1.

2.2 Main result

For $x, y \in \mathbb{T}^d$, define the kernel

$$k(x, y) = \mathbb{E} \left[\int_{\mathbb{R}} \bar{m}(0)(x) \bar{m}(t)(y) dt \right], \quad (9)$$

and for $f \in L_x^2$ and $x \in \mathbb{T}^d$, let us recall from [10]

$$Qf(x) = \int_{\mathbb{T}^d} k(x, y) f(y) dy. \quad (10)$$

Theorem 2.1. *Let Assumptions 1, 2 and 4 to 10 be satisfied. Let W be a cylindrical Wiener process on L_x^2 , ρ_0 be a random variable in L_x^2 and ρ be the weak solution of the linear stochastic diffusion equation*

$$d\rho = \operatorname{div}(K\nabla\rho)dt + \rho Q^{1/2} \circ dW(t), \quad (11)$$

with initial condition $\rho(0) = \rho_0$, in the sense of Definition 5. Also assume that $\rho^\varepsilon(0)$ converges in distribution to ρ_0 in L_x^2 . Then, for all $\sigma > 0$ and $T > 0$, the density ρ^ε converges in distribution in $C([0, T], H_x^{-\sigma})$ to ρ .

Let Assumptions 1 to 10 be satisfied. Then ρ^ε also converges in distribution in $L^2([0, T], L_x^2)$ to ρ .

The noise in (11) involves a Stratonovitch product, which is usual in the context of diffusion limit. Written with a Itô product, the limit becomes

$$d\rho = \operatorname{div}(K\nabla\rho)dt + \frac{1}{2}F\rho dt + \rho Q^{1/2}dW(t), \quad (12)$$

where F is the trace of Q , namely

$$F(x) = k(x, x). \quad (13)$$

This equation is well-posed, as discussed after Definition 5.

This is the same limit than in [10]. Compared with [10], we obtain a stronger convergence result, namely a convergence in $L^2([0, T], L_x^2)$ under additional assumptions.

2.3 Strategy of the proof of Theorem 2.1

A standard strategy to prove the convergence of ρ^ε when $\varepsilon \rightarrow 0$ (see [10, 11, 8]) is first to establish the tightness of the family $(\rho^\varepsilon)_{\varepsilon>0}$, and then the uniqueness of the limit point of this family and solves (18). The tightness usually comes from estimates on moments of trajectories. It is the case in [10], where the boundedness of \bar{m} is used to get an estimate on $\mathbb{E} \left[\sup_{t \in [0, T]} \|f^\varepsilon(t)\|_{L^2}^p \right]$ for all $T > 0$ and $p \geq 1$. However, without an almost sure bound on \bar{m} , we do not manage to get this estimate. Instead, we introduce a stopping time τ_Λ^ε depending on a parameter Λ such that the estimate holds for $f^{\varepsilon, \tau_\Lambda^\varepsilon} \doteq f^\varepsilon(t \wedge \tau_\Lambda^\varepsilon)$.

More precisely, define a first stopping time

$$\tau^\varepsilon \doteq \inf \{t \in \mathbb{R}^+ \mid \|\bar{m}^\varepsilon(t)\|_E > \varepsilon^{-\alpha}\}, \quad (14)$$

for some parameter α . Let $C_x^1 \doteq C^1(\mathbb{T}^d)$ and define the hitting time of a threshold Λ by $z \in C([0, T], C_x^1)$

$$\tau_\Lambda(z) \doteq \inf \left\{ t \in \mathbb{R}^+ \mid \|z(t)\|_{C_x^1} \geq \Lambda \right\}. \quad (15)$$

Then, define the auxiliary process

$$\zeta^\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t \bar{m}^\varepsilon(s) ds = \varepsilon \int_0^{t/\varepsilon^2} \bar{m}(s) ds \in E \subset C_x^1. \quad (16)$$

Observe that $\frac{1}{\varepsilon} \bar{m}^\varepsilon = \partial_t \zeta^\varepsilon$.

We can now define

$$\tau_\Lambda^\varepsilon \doteq \tau^\varepsilon \wedge \tau_\Lambda(\zeta^\varepsilon). \quad (17)$$

The times τ^ε and $\tau_\Lambda(\zeta^\varepsilon)$ have different asymptotic behaviors. On the one hand, Lemma 3.4 states that $\tau^\varepsilon \rightarrow \infty$ in probability. On the other hand, Section 7.1 establishes that ζ^ε converges in distribution, when $\varepsilon \rightarrow 0$, to a Wiener process ζ . Thus, we prove that, for all Λ outside of a countable set, $\tau_\Lambda(\zeta^\varepsilon)$ converges in distribution to $\tau_\Lambda(\zeta)$. Hence, τ_Λ^ε converges in distribution to $\tau_\Lambda(\zeta)$.

In Section 3.5, we prove an estimate on $f^{\varepsilon, \tau_\Lambda^\varepsilon}$ depending only on T , Λ and f_0^ε . This estimate leads to prove the tightness of the family of stopped processes $(\rho^{\varepsilon, \tau_\Lambda^\varepsilon})_{\varepsilon>0}$. Then, we identify the limit points of this family using the notions of martingale problems and perturbed test functions, and we deduce the convergence of the stopped process to a limit ρ_Λ .

Since $\tau^\varepsilon \rightarrow \tau_\Lambda(\zeta)$ and we expect $\rho^\varepsilon \rightarrow \rho$, it is convenient to study the process $(\rho^\varepsilon, \zeta^\varepsilon)$ to be able to write the limit of $\rho^{\varepsilon, \tau_\Lambda^\varepsilon}$ as $\rho^{\tau_\Lambda(\zeta)}$. Moreover, to prove that ρ^ε indeed

converges to ρ and that ρ satisfies (11), we need $\tau_\Lambda(\zeta)$ to be a stopping time for the limit process. Thus, we need to consider the convergence in distribution of the couple $(\rho^\varepsilon, \zeta^\varepsilon)$ in $C([0, T], H_x^{-\sigma}) \times C([0, T], C_x^1)$ to the solution (ρ, ζ) of

$$\begin{cases} d\rho = \operatorname{div}(K\nabla\rho)dt + \frac{1}{2}F\rho dt + \rho Q^{1/2}dW(t) \\ d\zeta = Q^{1/2}dW(t), \end{cases} \quad (18)$$

with initial condition $\rho(0) = \rho_0$ and $\zeta(0) = 0$. In this framework, $\tau_\Lambda(\zeta^\varepsilon)$ is a stopping time for $(\rho^\varepsilon, \zeta^\varepsilon)$ and $\tau_\Lambda(\zeta)$ is a stopping time for the limit (ρ, ζ) .

We first state in Section 3 some consequences of our assumptions in Section 2.1 and introduce the stopping times. In Section 4, we define the martingale problem solved by the process $(\rho^\varepsilon, \zeta^\varepsilon)$ and set up the perturbed test functions strategy.

In Section 6, we prove the tightness of the stopped process in $C([0, T], H_x^{-\sigma}) \times C([0, T], C_x^1)$, using the perturbed test functions of Section 4. Then, in Section 7, we establish the convergence of the martingale problems when $\varepsilon \rightarrow 0$ to identify the limit as a solution of a stopped martingale problem, and deduce the convergence of the original process $(\rho^\varepsilon, \zeta^\varepsilon)$ in $C([0, T], H_x^{-\sigma}) \times C([0, T], C_x^1)$.

In Section 8, we prove the tightness of the stopped process in $L^2([0, T], L_x^2)$ under the assumptions of Theorem 2.1, using an averaging lemma. Combined with the previous results, we deduce the convergence of the original process ρ^ε in $L^2([0, T], L_x^2)$.

3 Preliminary results

3.1 Resolvent operator

3.1.1 Additional assumptions

Denote by $(P_t)_{t \in \mathbb{R}^+}$ the transition semi-group on E associated to the homogeneous Markov process \bar{m} and let B denote its infinitesimal generator

$$\forall n \in E, B\theta(n) = \lim_{t \rightarrow 0} \frac{P_t\theta(n) - \theta(n)}{t}.$$

The usual framework for Markov processes and their transition semi-groups is to consider continuous bounded test functions $\theta \in C_b(E)$, so that $P_t\theta$ is a contraction semi-group (see [14]). Here, we need unbounded test functions (see Section 4.2), thus consider the action of the semi-group on $C(E) \cap L^1(E)$. We also consider the domain of B

$$D(B) \doteq \{\theta \in C(E) \cap L^1(E) \mid \forall n \in E, B\theta(n) \text{ exists and } B\theta \in C(E) \cap L^1(E)\}.$$

We need a continuity property for the semi-group $(P_t)_{t \in \mathbb{R}^+}$. Define first the resolvent operator.

Definition 2. For $\lambda \in [0, \infty)$ and $\theta \in C(E) \cap L^1(E)$ such that $\int_0^\infty |P_t\theta(n)| dt < \infty$ for all $n \in E$, define the resolvent: for all $n \in E$

$$R_\lambda\theta(n) \doteq \int_0^\infty e^{-\lambda t} P_t\theta(n) dt.$$

Assumption 9. The family $(P_t)_{t \in \mathbb{R}^+}$ is a semi-group on $C(E) \cap L^1(E)$. Moreover, for all $(\lambda_i)_{1 \leq i \leq 4} \in [0, \infty)^4$ and $(\theta_i)_{1 \leq i \leq 4} \in C(E) \cap L^1(E)^4$ such that $R_{\lambda_i} \theta_i$ are well-defined by Definition 2, we have

$$\forall j \in \llbracket 1, 4 \rrbracket, \Pi_{i=1}^j R_{\lambda_i} \theta_i \in D(B).$$

In addition, we assume that for $\lambda \in [0, \infty)$ and θ such that $R_\lambda \theta \in D(B)$, the commutation formula holds

$$B \int_0^\infty e^{-\lambda t} P_t \theta(\cdot) dt = \int_0^\infty e^{-\lambda t} B P_t \theta(\cdot) dt.$$

The second part of Assumption 9 is satisfied under a continuity property for the semi-group $(P_t)_{t \in \mathbb{R}^+}$. Indeed, consider the following computations

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{P_s - \text{id}}{s} \int_0^\infty e^{-\lambda t} P_t \theta(\cdot) dt &= \lim_{s \rightarrow 0} \int_0^\infty e^{-\lambda t} \frac{P_{s+t} - P_t}{s} \theta(\cdot) dt \\ &= \int_0^\infty e^{-\lambda t} \lim_{s \rightarrow 0} \frac{P_{s+t} - P_t}{s} \theta(\cdot) dt \end{aligned}$$

To justify the first equality, it is sufficient to assume point-wise continuity of P_t for all t on the space $C(E) \cap L^1(E)$. The second equality is a consequence of the bounded convergence theorem.

Note that by means of Assumption 9, $-R_0$ is the inverse of B . Indeed, for θ such that $R_0 \theta \in D(B)$, we have

$$B \int_0^\infty P_t \theta(\cdot) dt = \int_0^\infty B P_t \theta(\cdot) dt = \int_0^\infty \partial_t P_t \theta(\cdot) dt = -\theta.$$

We sometimes use functions having at most polynomial growth. Our last assumption is that B preserves this property.

Assumption 10. If $\theta \in D(B)$ has at most polynomial growth, then $B\theta$ has at most polynomial growth with the same degree. Namely, there exists $C_B \in (0, \infty)$ such that, for any $\theta \in D(B)$ and $k \in \mathbb{N}$,

$$\sup_{n \in E} \frac{|B\theta(n)|}{1 + \|n\|_E^k} \leq C_B \sup_{n \in E} \frac{|\theta(n)|}{1 + \|n\|_E^k}$$

3.1.2 Results on the resolvent operator

We introduce a class of pseudo-linear (respectively pseudo-quadratic) functions, which behave like linear (respectively quadratic) functions for our purposes.

Definition 3. A function $\theta \in \text{Lip}(E)$ such that $\langle \theta \rangle = 0$, is called pseudo-linear. Denote by $[\theta]_{\text{Lip}}$ its Lipschitz constant.

A function $\theta : E \rightarrow \mathbb{R}$ is called pseudo-quadratic if there exists a function $b_\theta : E^2 \rightarrow \mathbb{R}$ satisfying

- for all $n \in E$, $\theta(n) = b_\theta(n, n)$,

- for all $n \in E$, $b_\theta(n, \cdot)$ and $b_\theta(\cdot, n)$ are Lipschitz continuous,
- the mappings $n \mapsto [b_\theta(n, \cdot)]_{\text{Lip}}$ and $n \mapsto [b_\theta(\cdot, n)]_{\text{Lip}}$ have at most linear growth.

If θ is a pseudo-quadratic function, then let

$$[\theta]_{\text{quad}} \doteq \sup_{n_1 \neq n_2 \in E} \frac{|\theta(n_2) - \theta(n_1)|}{(1 + \|n_1\|_E + \|n_2\|_E) \|n_2 - n_1\|_E} < \infty.$$

Let E^* denote the dual space of E . Any element $\theta \in E^*$ is pseudo-linear.

A consequence of the mixing property (Assumption 8) is that the pseudo-linear and the pseudo-quadratic functions introduced in Definition 3 satisfy the conditions of Definition 2.

Lemma 3.1. *Let θ be a pseudo-linear function. Then, for all $\lambda \geq 0$, $R_\lambda \theta$ is well-defined and is pseudo-linear. Moreover, let*

$$C_\lambda = \int_0^\infty e^{-\lambda t} \gamma_{\text{mix}}(t) dt \text{ and } C'_\lambda = \left(1 \vee \int \|n_2\|_E d\nu(n_2)\right) C_\lambda.$$

Then, we have

$$[R_\lambda \theta]_{\text{Lip}} \leq [\theta]_{\text{Lip}} C_\lambda,$$

and for $n \in E$,

$$|R_\lambda \theta(n)| \leq C'_\lambda [\theta]_{\text{Lip}} (1 + \|n\|_E). \quad (19)$$

Let θ be a pseudo-quadratic function. Then, for $\lambda \geq 0$, $R_\lambda [\theta - \langle \theta \rangle]$ is well-defined. Moreover, there exists $C''_\lambda \in (0, \infty)$ depending only on C_λ and b such that, for $n \in E$,

$$|R_\lambda [\theta - \langle \theta \rangle](n)| \leq C''_\lambda [\theta]_{\text{quad}} (1 + \|n\|_E^{b+1})$$

where b is defined in Assumption 7.

Proof. Let $n_1, n_2 \in E$ and denote by $(m^*(\cdot, n_1), m^*(\cdot, n_2))$ the coupling introduced in Assumption 8. If θ is Lipschitz continuous, then for all $t \in \mathbb{R}^+$, Assumption 8 leads to

$$|P_t \theta(n_1) - P_t \theta(n_2)| \leq [\theta]_{\text{Lip}} \gamma_{\text{mix}}(t) \|n_1 - n_2\|_E. \quad (20)$$

Recall that P_t is ν -invariant, i.e. $\nu P_t = \nu$, hence we have

$$\begin{aligned} |P_t \theta(n_1) - \langle \theta \rangle| &= \left| \int_E (P_t \theta(n_1) - P_t \theta(n_2)) d\nu(n_2) \right| \\ &\leq \gamma_{\text{mix}}(t) [\theta]_{\text{Lip}} \int_E \|n_1 - n_2\|_E d\nu(n_2) \\ &\leq \left(1 \vee \int \|n_2\|_E d\nu(n_2)\right) \gamma_{\text{mix}}(t) [\theta]_{\text{Lip}} (1 + \|n_1\|_E). \end{aligned} \quad (21)$$

Assume that θ is pseudo-linear. Since $\langle \theta \rangle = 0$, (21) implies that $R_\lambda \theta$ is well-defined for all $\lambda \in [0, \infty)$, and (20) implies that $R_\lambda \theta$ is $([\theta]_{\text{Lip}} C_\lambda)$ -Lipschitz continuous. Moreover, by means of Fubini's Theorem,

$$\begin{aligned} \int_E R_\lambda \theta(n) d\nu(n) &= \int_E \int_0^\infty e^{-\lambda t} P_t \theta(n) dt d\nu(n) \\ &= \int_0^\infty e^{-\lambda t} \int_E P_t \theta(n) d\nu(n) dt \\ &= \int_0^\infty e^{-\lambda t} \int_E \theta(n) d\nu(n) dt \\ &= 0. \end{aligned}$$

This concludes the proof that $R_\lambda \theta$ is pseudo-linear. Finally, (19) is a straightforward consequence of (21).

Now assume that θ is a pseudo-quadratic function: using Assumptions 7 and 8 and Cauchy-Schwarz inequality, we have for all $n_1, n_2 \in E$

$$|P_t \theta(n_1) - P_t \theta(n_2)| \leq C [\theta]_{\text{quad}} \left(1 + \|n_1\|_E^b + \|n_2\|_E^b\right) \gamma_{\text{mix}}(t) \|n_1 - n_2\|_E,$$

for some constant C depending on b . Since P_t is ν -invariant and ν has a finite moment of order $b + 1$ by Assumption 7, we get

$$|P_t \theta(n) - \langle \theta \rangle| \leq C' [\theta]_{\text{quad}} \gamma_{\text{mix}}(t) (1 + \|n\|_E^{b+1}),$$

for some constant C' depending on b . Integrating with respect to t gives the announced result. \square

Let us recall notation from [10]. For $\rho, \rho' \in L_x^2$, denote by $\psi_{\rho, \rho'} \in E^*$ the continuous linear form

$$\forall n \in E, \psi_{\rho, \rho'}(n) \doteq (\rho n, \rho')_{L_x^2}$$

The linear form $\psi_{\rho, \rho'}$ is pseudo-linear and

$$[\psi_{\rho, \rho'}]_{\text{Lip}} = \|\psi_{\rho, \rho'}\|_{E^*} \leq \|\rho\|_{L_x^2} \|\rho'\|_{L_x^2}.$$

Hence, by Lemma 3.1, we have for $\rho, \rho' \in L_x^2$, $\lambda \geq 0$ and $n \in E$

$$|R_\lambda \psi_{\rho, \rho'}(n)| \leq C'_\lambda \|\rho\|_{L_x^2} \|\rho'\|_{L_x^2} (1 + \|n\|_E).$$

Thus, for all $n \in E$, $(\rho, \rho') \mapsto R_\lambda \psi_{\rho, \rho'}(n)$ is a continuous bilinear form on L_x^2 . By means of Riesz Representation Theorem, there exists a continuous linear map $\tilde{R}_\lambda(n) : L_x^2 \rightarrow L_x^2$ such that

$$\forall \rho, \rho' \in L_x^2, \forall n \in E, R_\lambda \psi_{\rho, \rho'}(n) = (\tilde{R}_\lambda(n) \rho, \rho')_{L_x^2}.$$

By a slight abuse of notation, denote $R_\lambda(n) = \tilde{R}_\lambda(n)$. For $\varphi \in C^1(L_x^2)$, the linear mapping $D\varphi(\rho)$ can also be identified as an element of L_x^2 :

$$\forall \rho, h \in L_x^2, D\varphi(\rho)(h) = (h, D\varphi(\rho))_{L_x^2},$$

so that we can define $D\varphi(\rho)(R_\lambda(n)h)$ for $\rho, h \in L_x^2$.

Now consider ρ and ρ' in dual Sobolev spaces H_x^k and H_x^{-k} (for $k \in \mathbb{N}$ such that $E \subset C_x^k$, namely $k \leq 2[d/2] + 2$). We also may define $R_\lambda(n) : H_x^k \rightarrow H_x^k$ in a compatible way.

3.2 Properties of the covariance operator

Recall that k , F and Q are defined by equations (9), (10) and (13).

Lemma 3.2. *The kernel k is symmetric and in $L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$. Moreover, Q is a bounded, self-adjoint, compact and non-negative operator on L_x^2 .*

Proof. Since \bar{m} is stationary, we have the identity

$$k(x, y) = \int_E \psi_x(n) R_0 \psi_y(n) d\nu(n) + \int_E R_0 \psi_x(n) \psi_y(n) d\nu(n), \quad (22)$$

with $\psi_x(n) = n(x)$ for all $n \in E$ and $x \in \mathbb{T}^d$. The functions ψ_x and ψ_y are continuous linear forms, thus we have

$$\sup_{x, y \in \mathbb{T}^d, n \in E} \frac{|\psi_x R_0 \psi_y(n)|}{1 + \|n\|_E^2} < \infty.$$

Owing to Assumption 6, $\int_E \|n\|_E^2 d\nu(n) < \infty$, thus k is well-defined, bounded and symmetric. It implies that Q is a bounded operator on L_x^2 and is self-adjoint and compact (see for instance [13] XI.6).

The proof of non-negativity of Q is given in [10]. \square

By means of Lemma 3.2, the operator $Q^{1/2}$ can be defined ($L_x^2 \rightarrow L_x^2$). Note that Q is trace-class, that $Q^{1/2}$ is Hilbert-Schmidt on L_x^2 and that

$$\|Q^{1/2}\|_{\mathcal{L}_2}^2 = \text{Tr } Q = \int_{\mathbb{T}^d} F(x) dx.$$

Let $(F_i)_i$ be a orthonormal and complete system of eigenvectors of $Q^{1/2}$, and $(\sqrt{q_i})_i$ their associated eigenvalues.

Lemma 3.3. *For all i , $F_i \in C_x^1$ and*

$$\sum_i q_i \|F_i\|_{C_x^1}^2 < \infty.$$

Proof. Let $s = \lfloor \frac{d}{2} \rfloor + 2$, so that we have the continuous embeddings $H_x^s \subset C_x^1$. It is thus sufficient to prove that $\sum_i q_i \|F_i\|_{H_x^s}^2 < \infty$.

Since $\bar{m}(t) \in E = C_x^{2s}$ and is mixing, is it straightforward to prove that $k \in H_x^{2s}$ using a differentiation under the integral sign in (9).

For $|\beta| \leq s$, we multiply the identity $q_i F_i = Q F_i$ by $\frac{\partial^{2|\beta|} F_i}{\partial x^{2\beta}}$ and integrate by parts both sides of the equality to get

$$(-1)^{|\beta|} q_i \left\| \frac{\partial^{|\beta|} F_i}{\partial x^\beta} \right\|_{L_x^2}^2 = \int \int \frac{\partial^{2|\beta|} k(x, y)}{\partial x^{2\beta}} F_i(x) F_i(y) dx dy.$$

Using (22), we have

$$\begin{aligned} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\partial^{2|\beta|} k(x, y)}{\partial x^{2\beta}} F_i(x) F_i(y) dx dy = \\ \int_E \int_{\mathbb{T}^d} \frac{\partial^{2|\beta|} n}{\partial x^{2\beta}}(x) F_i(x) dx \int_{\mathbb{T}^d} R_0 \psi_y(n) F_i(y) dy d\nu(n) \\ + \int_E \int_{\mathbb{T}^d} \frac{\partial^{2|\beta|} (R_0 \psi_x(n))}{\partial x^{2\beta}} F_i(x) dx \int_{\mathbb{T}^d} n(y) F_i(y) dy d\nu(n). \end{aligned}$$

We sum with respect to i , use the Parseval identity and the Cauchy-Schwarz inequality to get

$$\begin{aligned} \sum_i q_i \left\| \frac{\partial^{|\beta|} F_i}{\partial x^\beta} \right\|_{L_x^2}^2 &\leq 2 \int_E \|R_0 \psi_x(n)\|_{H_x^{2|\beta|}} \|n\|_{H_x^{2|\beta|}} d\nu(n) \\ &\leq C \int_E (1 + \|n\|_E^2) d\nu(n), \end{aligned}$$

for some constant C , using Lemma 3.1. This upper bound is finite by Assumption 7. Summing with respect to β concludes the proof. \square

3.3 Behavior of the stopping time for the driving process

Recall that τ^ε is defined by (14): $\tau^\varepsilon \doteq \inf \{t \in \mathbb{R}^+ \mid \|\bar{m}^\varepsilon(t)\|_E > \varepsilon^{-\alpha}\}$.

In this section, we establish Lemma 3.4 below. Its objectives are twofold. On the one hand, it shows that \bar{m}^ε is almost surely bounded on any interval $[0, T]$, which is useful to justify the well-posedness of (1). On the other hand, it gives us an estimate for $\varepsilon^\alpha \|\bar{m}^{\varepsilon, \tau^\varepsilon}(t)\|_E$ uniform in t and ε for some small α . This estimate will prove useful for Sections 4.1, 6 and 7. Therefore, it is a key result of this paper.

Lemma 3.4. *Let Assumptions 5 and 6 be satisfied and let $T > 0$. Then almost surely*

$$\sup_{t \in [0, T]} \|\bar{m}^\varepsilon(t)\|_E < \infty.$$

Moreover, let $\alpha > \frac{2}{\gamma}$ and define the $(\mathcal{F}_t^\varepsilon)_t$ -stopping time τ^ε by (14). Then, we have

$$\forall T > 0, \mathbb{P}(\tau^\varepsilon < T) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Remark. Equation (14) implies that for $t \in \mathbb{R}^+$,

$$\|\bar{m}^{\varepsilon, \tau^\varepsilon}(t)\|_E = \|\bar{m}^\varepsilon(t \wedge \tau^\varepsilon)\|_E \leq \varepsilon^{-\alpha} \vee \|\bar{m}(0)\|_E. \quad (23)$$

In particular, on the event $\{\|\bar{m}(0)\|_E > \varepsilon^{-\alpha}\}$, we have $\tau^\varepsilon = 0$ and $\bar{m}^{\varepsilon, \tau^\varepsilon}(t) = \bar{m}(0)$. Thus, one does not necessarily have the estimate $\|\bar{m}^{\varepsilon, \tau^\varepsilon}(t)\|_E \leq \varepsilon^{-\alpha}$.

Proof. Let $(S_k)_k$ be the identically distributed random variables defined by

$$\forall k \in \mathbb{N}_0, S_k \doteq \sup_{t \in [k, k+1]} \|\bar{m}(t)\|_E.$$

By means of Assumptions 5 to 7, for all $k \in \mathbb{N}$, $\mathbb{E}[S_k^\gamma] = \mathbb{E}[S_0^\gamma] < \infty$. Thus, almost surely, $S_k < \infty$ for all $k \in \mathbb{N}_0$. This yields the first result:

$$\mathbb{P}\text{-a.s.}, \forall T > 0, \sup_{t \in [0, T]} \|\bar{m}^\varepsilon(t)\|_E \leq \sup_{k \leq T\varepsilon^{-2}+1} S_k < \infty.$$

Since $\alpha > \frac{2}{\gamma}$, there exists δ such that $\frac{\alpha}{2} > \delta > \frac{1}{\gamma}$. Then, the Markov inequality yields

$$\sum_{k \in \mathbb{N}} \mathbb{P}(S_k \geq k^\delta) \leq \sum_{k \in \mathbb{N}} \frac{\mathbb{E}[S_k^\gamma]}{k^{\delta\gamma}} = \mathbb{E}[S_0^\gamma] \sum_{k \in \mathbb{N}} \frac{1}{k^{\delta\gamma}} < \infty.$$

By means of the Borel-Cantelli Lemma, almost surely, there exists a random integer $k_0 \in \mathbb{N}$ such that

$$\mathbb{P}\text{-a.s.}, \forall k > k_0, S_k < k^\delta.$$

Define the random variable $Z \doteq \sup_{k \leq k_0} S_k$. Then Z is almost surely finite and

$$\mathbb{P}\text{-a.s.}, \forall t \in \mathbb{R}^+, \|\bar{m}(t)\|_E \leq S_{[t]} \leq Z + [t]^\delta \leq Z + t^\delta.$$

Finally, for $T > 0$, using that $\alpha > 2\delta$, we get

$$\mathbb{P}(\tau^\varepsilon < T) = \mathbb{P}\left(\sup_{t \in [0, T]} \|\bar{m}^\varepsilon(t)\|_E > \varepsilon^{-\alpha}\right) \leq \mathbb{P}\left(Z + (T\varepsilon^{-2})^\delta > \varepsilon^{-\alpha}\right) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

□

In the sequel, α will be required to satisfy the constraint

$$\alpha < \frac{1}{b+2}. \quad (24)$$

Combined with the condition $\alpha > \frac{2}{\gamma}$ appearing in Lemma 3.4, this motivates the condition $\gamma > 2(b+2)$ in Assumption 7.

3.4 Pathwise solutions

By means of Lemma 3.4, we are in position to prove the existence and uniqueness of pathwise solutions of (1) and (2) (namely solutions when ω is fixed).

Proposition 3.5. *Let Assumptions 5 and 6 be satisfied. Let $T > 0$ and $\varepsilon > 0$. Then for any $f_0^\varepsilon \in L^2(\mathcal{M}^{-1})$, there exists, almost surely, a unique solution f^ε of (1) in $C([0, T]; L^2(\mathcal{M}^{-1}))$, in the sense that*

$$\mathbb{P}\text{-a.s.}, \forall t \in [0, T], f^\varepsilon(t) = e^{-\frac{t}{\varepsilon}A} f_0^\varepsilon + \int_0^t e^{-\frac{t-s}{\varepsilon}A} \left(\frac{1}{\varepsilon^2} Lf^\varepsilon(s) + \frac{1}{\varepsilon} f^\varepsilon(s) \overline{m}^\varepsilon(s) \right) ds$$

where A is the operator defined by

$$\begin{aligned} D(A) &\doteq \{f \in L^2(\mathcal{M}^{-1}) \mid (x, v) \mapsto a(v) \cdot \nabla_x f(x, v) \in L^2(\mathcal{M}^{-1})\} \\ Af(x, v) &\doteq a(v) \cdot \nabla_x f(x, v). \end{aligned}$$

Note that here ε is fixed. Thus, the proof is standard, based on a fixed-point theorem.

Proof. Let $\omega \in \Omega$ and $\varepsilon > 0$. For $f \in C([0, T], L^2(\mathcal{M}^{-1}))$, let

$$\Phi(f) = e^{-\frac{t}{\varepsilon}A} f_0^\varepsilon + \int_0^t e^{-\frac{t-s}{\varepsilon}A} \left(\frac{1}{\varepsilon^2} Lf(s) + \frac{1}{\varepsilon} f(s) \overline{m}^\varepsilon(s) \right) ds.$$

Owing to the Banach fixed-point theorem, it is sufficient to prove that Φ is a contraction for some Banach norm on $C([0, T], L^2(\mathcal{M}^{-1}))$. For $r \in [0, \infty)$, we consider the following Banach norm

$$\forall f \in C([0, T], L^2(\mathcal{M}^{-1})), \|f\|_r = \sup_{t \in [0, T]} e^{-rt} \|f(t)\|_{L^2(\mathcal{M}^{-1})}.$$

Since the semi-group associated to A is given by

$$\forall f \in L^2(\mathcal{M}^{-1}), \forall x \in \mathbb{T}^d, \forall v \in V, e^{tA} f(x, v) = f(x + ta(v), v),$$

for $f \in L^2(\mathcal{M}^{-1})$, we have for all $t \in \mathbb{R}$ and $f \in L^2(\mathcal{M}^{-1})$,

$$\|e^{tA} f\|_{L^2(\mathcal{M}^{-1})} = \|f\|_{L^2(\mathcal{M}^{-1})}.$$

Thus, for $t \in [0, T]$, and $f, g \in C([0, T], L^2(\mathcal{M}^{-1}))$, we get

$$\begin{aligned} \|\Phi(f)(t) - \Phi(g)(t)\|_{L^2(\mathcal{M}^{-1})} &\leq \frac{1}{\varepsilon^2} \int_0^t \|L(f - g)(s)\|_{L^2(\mathcal{M}^{-1})} ds \\ &\quad + \frac{1}{\varepsilon} \int_0^t \|(f - g)(s) \overline{m}^\varepsilon(s)\|_{L^2(\mathcal{M}^{-1})} ds. \end{aligned}$$

By Lemma 3.4, since ω is fixed, \overline{m}^ε is bounded in E on $[0, T]$. Since $\|Lh\|_{L^2(\mathcal{M}^{-1})} \leq \|h\|_{L^2(\mathcal{M}^{-1})}$ for $h \in L^2(\mathcal{M}^{-1})$, we get

$$\|\Phi(f)(t) - \Phi(g)(t)\|_{L^2(\mathcal{M}^{-1})} \leq \left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \sup_{t \in [0, T]} \|\overline{m}^\varepsilon(t)\|_E \right) \int_0^t e^{rs} \|f - g\|_r ds.$$

Hence, we have

$$e^{-rt} \|\Phi(f)(t) - \Phi(g)(t)\|_{L^2(\mathcal{M}^{-1})} \leq \left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \sup_{t \in [0, T]} \|\bar{m}^\varepsilon(t)\|_E \right) \frac{1 - e^{-rt}}{r} \|f - g\|_r,$$

and

$$\|\Phi(f) - \Phi(g)\|_r \leq \frac{1}{r} \left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \sup_{t \in [0, T]} \|\bar{m}^\varepsilon(t)\|_E \right) \|f - g\|_r.$$

By taking r large enough, we get that Φ is contracting, which concludes the proof. \square

3.5 Estimate in $L^2(\mathcal{M}^{-1})$

In this section, we obtain an upper bound on $\|f^{\varepsilon, \tau_\Lambda^\varepsilon}\|_{L^2(\mathcal{M}^{-1})}$. Note that in the case where the driving process \bar{m} is bounded, [10] establishes a similar upper bound without introducing a stopping time. Here, the unboundedness of the stopped process $\bar{m}^{\varepsilon, \tau^\varepsilon}$ requires more intricate arguments and an additional stopping time $\tau_\Lambda(\zeta^\varepsilon)$. One of these arguments is the introduction of a weight \mathcal{M}^ε that depends on ε .

Proposition 3.6. *Assume that $f_0^\varepsilon \in L^2(\mathcal{M}^{-1})$. For $\Lambda > 0$ and $\varepsilon > 0$, define ζ^ε by (16) and $\tau_\Lambda(\zeta^\varepsilon)$ by (15).*

Then almost surely, for all $t \in [0, T]$ and $\varepsilon \in (0, (4\|a\|_{L^\infty} \Lambda)^{-1}]$,

$$\|f^{\varepsilon, \tau_\Lambda(\zeta^\varepsilon)}(t)\|_{L^2(\mathcal{M}^{-1})}^2 + \frac{1}{\varepsilon^2} \int_0^{t \wedge \tau_\Lambda(\zeta^\varepsilon)} \|L f^{\varepsilon, \tau_\Lambda(\zeta^\varepsilon)}(s)\|_{L^2(\mathcal{M}^{-1})}^2 ds \leq C_\Lambda(T) \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^2, \quad (25)$$

for some $C_\Lambda(T) > 0$ depending only on Λ , $\|a\|_{L^\infty}$ and T .

Note that $\tau_\Lambda(\zeta^\varepsilon) > 0$ almost surely since $\zeta^\varepsilon(0) = 0$.

Remark. The condition $\varepsilon \in (0, (4\|a\|_{L^\infty} \Lambda)^{-1}]$ only reads: we fix Λ , then take ε small enough ($\varepsilon \rightarrow 0$) depending on the fixed Λ . From now on, we always assume $\varepsilon \leq (4\|a\|_{L^\infty} \Lambda)^{-1} < 1$. In particular, we denote by \sup_ε the supremum with respect to $\varepsilon \in (0, (4\|a\|_{L^\infty} \Lambda)^{-1}) \subset (0, 1)$.

In most of the paper, we neglect the integral term of the left-hand side of (25) and we only use

$$\|f^{\varepsilon, \tau_\Lambda(\zeta^\varepsilon)}(t)\|_{L^2(\mathcal{M}^{-1})}^2 \leq C_\Lambda(T) \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^2.$$

Equation (25) will prove useful in Section 8.

Let us introduce some notation. For any variable u , $x \lesssim_u y$ means that there exists C such that $x \leq Cy$ where C depends only on u , a , \mathcal{M} , B , ν , γ , α , γ_{mix} , b and $\mathbb{P}_{f_0^\varepsilon}$ the distribution of f_0^ε . With this notation, (25) yields

$$\|f^{\varepsilon, \tau_\Lambda(\zeta^\varepsilon)}(t)\|_{L^2(\mathcal{M}^{-1})}^2 \lesssim_\Lambda \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^2.$$

and

$$\frac{1}{\varepsilon^2} \int_0^{t \wedge \tau_\Lambda(\zeta^\varepsilon)} \|L f^{\varepsilon, \tau_\Lambda(\zeta^\varepsilon)}(s)\|_{L^2(\mathcal{M}^{-1})}^2 ds \lesssim_\Lambda \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^2.$$

Proof. Define the time-dependent weight

$$\mathcal{M}^\varepsilon(t, x, v) = e^{2\zeta^\varepsilon(t, x)} \mathcal{M}(v),$$

and the associate weighted norm

$$\|f\|_{L^2(\mathcal{M}^\varepsilon(t)^{-1})} \doteq \left(\int \int \frac{|f^\varepsilon(x, v)|^2}{\mathcal{M}^\varepsilon(t, x, v)} dx d\mu(v) \right)^{\frac{1}{2}}.$$

We have, for $t \in \mathbb{R}^+$,

$$\begin{aligned} \frac{1}{2} \partial_t \|f^\varepsilon(t)\|_{L^2(\mathcal{M}^\varepsilon(t)^{-1})}^2 &= \int \int \left(\frac{f^\varepsilon(t, x, v)}{\mathcal{M}^\varepsilon(t, x, v)} \left(-\frac{1}{\varepsilon} A f^\varepsilon + \frac{1}{\varepsilon^2} L f^\varepsilon + \frac{1}{\varepsilon} f^\varepsilon \bar{m}^\varepsilon \right) (t, x, v) \right. \\ &\quad \left. - \frac{|f^\varepsilon(t, x, v)|^2}{2 |\mathcal{M}^\varepsilon(t, x, v)|^2} \partial_t \mathcal{M}^\varepsilon(t, x, v) \right) dx d\mu(v) \\ &= \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon \end{aligned}$$

with

$$\begin{aligned} \mathcal{A}_\varepsilon &= \frac{1}{\varepsilon^2} \int \int \frac{f^\varepsilon(t, x, v)}{\mathcal{M}^\varepsilon(t, x, v)} \left(L f^\varepsilon + \varepsilon f^\varepsilon \bar{m}^\varepsilon - \frac{\varepsilon^2}{2} \frac{f^\varepsilon}{\mathcal{M}^\varepsilon} \partial_t \mathcal{M}^\varepsilon \right) (t, x, v) dx d\mu(v) \\ \mathcal{B}_\varepsilon &= -\frac{1}{\varepsilon} \int \int \frac{f^\varepsilon(t, x, v)}{\mathcal{M}^\varepsilon(t, x, v)} A f^\varepsilon(t, x, v) dx d\mu(v). \end{aligned}$$

On the one hand, the weight \mathcal{M}^ε has been chosen in order to satisfy $\varepsilon \bar{m}^\varepsilon - \frac{\varepsilon^2}{2} \frac{\partial_t \mathcal{M}^\varepsilon}{\mathcal{M}^\varepsilon} = 0$. Moreover, since $f^\varepsilon = \rho^\varepsilon \mathcal{M} - L f^\varepsilon$ and $\int_V L f^\varepsilon d\mu = 0$, we get

$$\begin{aligned} \mathcal{A}_\varepsilon &= \frac{1}{\varepsilon^2} \int \int \frac{f^\varepsilon(t, x, v)}{\mathcal{M}^\varepsilon(t, x, v)} L f^\varepsilon(t, x, v) dx d\mu(v) \\ &= \frac{1}{\varepsilon^2} \int_{\mathbb{T}^d} e^{-2\zeta^\varepsilon(t, x)} \rho^\varepsilon(t, x) \int_V L f^\varepsilon(t, x, v) d\mu(v) dx \\ &\quad - \frac{1}{\varepsilon^2} \int \int \frac{|L f^\varepsilon(t, x, v)|^2}{\mathcal{M}^\varepsilon(t, x, v)} d\mu(v) dx \\ &= -\frac{1}{\varepsilon^2} \int \int \frac{|L f^\varepsilon(t, x, v)|^2}{\mathcal{M}^\varepsilon(t, x, v)} d\mu(v) dx = -\frac{1}{\varepsilon^2} \|L f^\varepsilon(t)\|_{L^2(\mathcal{M}^\varepsilon(t)^{-1})}^2. \end{aligned}$$

On the other hand, by means of an integration by parts (we take a primitive of $f^\varepsilon \partial_{x_i} f^\varepsilon$ and a derivative of $\frac{1}{\mathcal{M}^\varepsilon}$), we write

$$\begin{aligned} \mathcal{B}_\varepsilon &= -\frac{1}{\varepsilon} \int \int a(v) \cdot \frac{f^\varepsilon(t, x, v) \nabla_x f^\varepsilon(t, x, v)}{\mathcal{M}^\varepsilon(t, x, v)} dx d\mu(v) \\ &= -\frac{1}{\varepsilon} \int \int a(v) \cdot \frac{\frac{1}{2} |f^\varepsilon(t, x, v)|^2 \nabla_x \mathcal{M}^\varepsilon(t, x, v)}{|\mathcal{M}^\varepsilon(t, x, v)|^2} dx d\mu(v) \\ &= -\frac{1}{2\varepsilon} \int \int \left| \frac{f^\varepsilon(t, x, v)}{\mathcal{M}^\varepsilon(t, x, v)} \right|^2 A \mathcal{M}^\varepsilon(t, x, v) dx d\mu(v). \end{aligned}$$

Then, by definition of \mathcal{M}^ε and A , we have

$$\mathcal{B}_\varepsilon = -\frac{1}{\varepsilon} \int_{\mathbb{T}^d} \nabla_x \zeta^\varepsilon(t, x) \cdot \int_V \frac{|f^\varepsilon(t, x, v)|^2}{\mathcal{M}^\varepsilon(t, x, v)} a(v) d\mu(v) dx.$$

Using once again the identity $f^\varepsilon = \rho^\varepsilon \mathcal{M} - Lf^\varepsilon$, we get

$$\begin{aligned} \mathcal{B}_\varepsilon &= -\frac{1}{\varepsilon} \int_{\mathbb{T}^d} e^{-2\zeta^\varepsilon(t, x)} |\rho^\varepsilon(t, x)|^2 \nabla_x \zeta^\varepsilon(t, x) \cdot \int_V a(v) \mathcal{M}(v) d\mu(v) dx \\ &\quad - \frac{1}{\varepsilon} \int_{\mathbb{T}^d} \nabla_x \zeta^\varepsilon(t, x) \cdot \int_V \frac{|Lf^\varepsilon(t, x, v)|^2}{\mathcal{M}^\varepsilon(t, x, v)} a(v) d\mu(v) dx \\ &\quad + \frac{2}{\varepsilon} \int \int e^{-2\zeta^\varepsilon(t, x)} \nabla_x \zeta^\varepsilon(t, x) \cdot a(v) \rho^\varepsilon(t, x) Lf^\varepsilon(t, x, v) d\mu(v) dx \\ &= \mathcal{B}_\varepsilon^1 + \mathcal{B}_\varepsilon^2 + \mathcal{B}_\varepsilon^3. \end{aligned}$$

- Since a is centered for $\mathcal{M}\mu$, $\mathcal{B}_\varepsilon^1 = 0$.
- For $t \leq \tau_\Lambda(\zeta^\varepsilon)$, we have $\|\zeta^\varepsilon(t)\|_{C_x^1} \leq \Lambda$ and we assumed $\Lambda \leq (4\|a\|_{L^\infty} \varepsilon)^{-1}$. Thus, we get

$$\forall t \leq \tau_\Lambda(\zeta^\varepsilon), |\mathcal{B}_\varepsilon^2| \leq \frac{1}{4\varepsilon^2} \|Lf^\varepsilon(t)\|_{L^2(\mathcal{M}^\varepsilon(t)^{-1})}^2.$$

- Using the Young inequality, we have

$$\begin{aligned} |\mathcal{B}_\varepsilon^3| &\leq \frac{1}{4\varepsilon^2} \|Lf^\varepsilon(t)\|_{L^2(\mathcal{M}^\varepsilon(t)^{-1})}^2 \\ &\quad + 4\|a\|_{L^2(\mathcal{M})}^2 \|\nabla_x \zeta^\varepsilon(t)\|_{C(\mathbb{T}^d)}^2 \int_{\mathbb{T}^d} e^{-2\zeta^\varepsilon(t, x)} |\rho^\varepsilon(t, x)|^2 dx, \end{aligned}$$

with $\|a\|_{L^2(\mathcal{M})}^2 \doteq \int_V |a(v)|^2 \mathcal{M}(v) d\mu(v)$. Now using the Cauchy-Schwarz inequality and the identity $\int_V \mathcal{M}(v) d\mu(v) = 1$, we have

$$|\rho^\varepsilon(t, x)|^2 \leq \int_V \frac{|f^\varepsilon(t, x, v)|^2}{\mathcal{M}^\varepsilon(t, x, v)} d\mu(v) \int_V \mathcal{M}^\varepsilon(t, x, v) d\mu(v) = \|f^\varepsilon(t)\|_{L^2(\mathcal{M}^\varepsilon(t)^{-1})}^2 e^{2\zeta^\varepsilon(t, x)},$$

hence

$$|\mathcal{B}_\varepsilon^3| \leq \frac{1}{4\varepsilon^2} \|Lf^\varepsilon(t)\|_{L^2(\mathcal{M}^\varepsilon(t)^{-1})}^2 + 4\|a\|_{L^2(\mathcal{M})}^2 \|\nabla_x \zeta^\varepsilon(t)\|_{C(\mathbb{T}^d)}^2 \|f^\varepsilon(t)\|_{L^2(\mathcal{M}^\varepsilon(t)^{-1})}^2.$$

We finally get, for $t \leq \tau_\Lambda(\zeta^\varepsilon)$,

$$\begin{aligned} \partial_t \|f^\varepsilon(t)\|_{L^2(\mathcal{M}^\varepsilon(t)^{-1})}^2 &\leq -\frac{1}{2\varepsilon^2} \|Lf^\varepsilon(t)\|_{L^2(\mathcal{M}^\varepsilon(t)^{-1})}^2 \\ &\quad + 4\|a\|_{L^2(\mathcal{M})}^2 \|\nabla_x \zeta^\varepsilon(t)\|_{C(\mathbb{T}^d)}^2 \|f^\varepsilon(t)\|_{L^2(\mathcal{M}^\varepsilon(t)^{-1})}^2. \end{aligned}$$

For $t \leq \tau_\Lambda(\zeta^\varepsilon)$, Gronwall's Lemma implies

$$\begin{aligned} \|f^\varepsilon(t)\|_{L^2(\mathcal{M}^{\varepsilon(t)-1})}^2 + \int_0^t \frac{1}{2\varepsilon^2} \|Lf^\varepsilon(t)\|_{L^2(\mathcal{M}^{\varepsilon(t)-1})}^2 dt \\ \leq \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^2 e^{4\|a\|_{L^2(\mathcal{M})}^2 \int_0^t \|\nabla_x \zeta^\varepsilon(s)\|_{C(\mathbb{T}^d)}^2 ds}. \end{aligned}$$

Since, for $t \in \mathbb{R}^+$, we have

$$\|\cdot\|_{L^2(\mathcal{M}^{\varepsilon(t)-1})}^2 \geq \|\cdot\|_{L^2(\mathcal{M}^{-1})}^2 e^{-2\|\zeta^\varepsilon(t)\|_{C(\mathbb{T}^d)}},$$

we get, for $t \leq \tau_\Lambda(\zeta^\varepsilon)$,

$$\begin{aligned} \|f^\varepsilon(t)\|_{L^2(\mathcal{M}^{-1})}^2 + \int_0^t \frac{1}{2\varepsilon^2} \|Lf^\varepsilon(t)\|_{L^2(\mathcal{M}^{-1})}^2 dt \\ \leq \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^2 \exp\left(2 \sup_{s \in [0,t]} \|\zeta^\varepsilon(s)\|_{C(\mathbb{T}^d)} + 4\|a\|_{L^2(\mathcal{M})}^2 \int_0^t \|\nabla_x \zeta^\varepsilon(s)\|_{C(\mathbb{T}^d)}^2 ds\right). \end{aligned}$$

To conclude, it is sufficient to recall that for $t \leq \tau_\Lambda(\zeta^\varepsilon)$, we have $\|\zeta^\varepsilon(t)\|_{C_x^1} \leq \Lambda$. \square

4 Martingale problems and perturbed test functions

The proof of Theorem 2.1 heavily relies on the notion of martingale problems as introduced in [34]. To identify a limit point of $(\mathbb{P}_{\rho^\varepsilon})_{\varepsilon>0}$, we characterize it by a family of martingales and take the limit when $\varepsilon \rightarrow 0$ in their martingale properties.

The characterization of the distribution of a solution of a SPDE in terms of martingales is based on the Markov property satisfied by this solution. However, we expect a limit point ρ_Λ of the stopped process $\rho^{\varepsilon, \tau_\Lambda}$ to be stopped at some $\tau_\Lambda(\zeta)$, as mentioned in Section 3.5. Since $\tau_\Lambda(\zeta)$ is not a stopping time for the filtration generated by ρ_Λ , this latter process should not be Markov. Thus, we need to consider the convergence of the couple $(\rho^\varepsilon, \zeta^\varepsilon)$ instead of just ρ^ε . We will see more precisely in Section 7 at which point this matter occurs.

4.1 Generator and martingales

Also note that $(f^\varepsilon, \zeta^\varepsilon)$ is not a Markov process. As in [10], we consider the coupled process with \bar{m}^ε and thus consider the $L^2(\mathcal{M}^{-1}) \times C_x^1 \times E$ -valued Markov process $X^\varepsilon \doteq (f^\varepsilon, \zeta^\varepsilon, \bar{m}^\varepsilon)$.

Denote by \mathcal{L}^ε the infinitesimal generator of X^ε . Since f^ε is solution of (1) and since $\partial_t \zeta^\varepsilon = \frac{1}{\varepsilon} \bar{m}^\varepsilon$, the infinitesimal generator has an expression of the type

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon} \mathcal{L}_1 + \frac{1}{\varepsilon^2} \mathcal{L}_2 \tag{26}$$

with

$$\begin{aligned}\mathcal{L}_1\varphi(f, z, n) &= D_f\varphi(f, z, n)(-Af + nf) + D_z\varphi(f, z, n)(n) \\ \mathcal{L}_2\varphi(f, z, n) &= D_f\varphi(f, z, n)(Lf) + B\varphi(f, z, n),\end{aligned}$$

where B is the infinitesimal generator of \overline{m} . The domain of this generator contains the class of good test functions defined below. The terminology of "good test function" is inherited from [10], although our definition is a little more restrictive.

Definition 4. A function $\varphi : L^2(\mathcal{M}^{-1}) \times C_x^1 \times E \rightarrow \mathbb{R}$ is called a good test function if

- It is continuously differentiable on $L^2(\mathcal{M}^{-1}) \times C_x^1 \times E$ with respect to the first and second variables.
- For $\ell \in \{1, 2\}$, $B(\varphi(f, z, \cdot)^\ell)$ is defined for all $(f, z) \in L^2(\mathcal{M}^{-1}) \times C_x^1$, and

$$B(\varphi^\ell) : L^2(\mathcal{M}^{-1}) \times C_x^1 \times E \rightarrow \mathbb{R}$$

is continuous.

- If we identify the differential D_f with the gradient, then for $f \in L^2(\mathcal{M}^{-1})$, $z \in C_x^1$ and $n \in E$, we have

$$D_f\varphi(f, z, n) \in H^1(\mathbb{T}^d \times V, dx\mathcal{M}^{-1}(v)d\mu(v)). \quad (27)$$

- The functions φ , $D_z\varphi$, $D_f\varphi$ and $AD_f\varphi$ have at most polynomial growth in the following sense: there exists $C_\varphi > 0$ such that for $f, h \in L^2(\mathcal{M}^{-1})$, $z \in C_x^1$ and $n_1, n_2 \in E$, we have

$$\begin{aligned}|\varphi(f, z, n_1)| &\leq C_\varphi (1 + S_1^3) \left(1 + S_2^{b+2}\right) \\ |D_f\varphi(f, z, n_1)(Ah)| &\leq C_\varphi (1 + S_1^3) \left(1 + S_2^{b+2}\right) \\ |D_f\varphi(f, z, n_1)(n_2h)| &\leq C_\varphi (1 + S_1^3) \left(1 + S_2^{b+2}\right) \\ |D_f\varphi(f, z, n_1)(Lh)| &\leq C_\varphi (1 + S_1^3) \left(1 + S_2^{b+2}\right) \\ |D_z\varphi(f, z, n_1)(n_2)| &\leq C_\varphi (1 + S_1^3) \left(1 + S_2^{b+2}\right),\end{aligned} \quad (28)$$

where $S_1 = \|f\|_{L^2(\mathcal{M}^{-1})} \vee \|h\|_{L^2(\mathcal{M}^{-1})}$ and $S_2 = \|n_1\|_E \vee \|n_2\|_E$.

See Section 4.2 for a justification of the need to consider growth as appearing in (28).

A consequence of (27) is that $AD_f\varphi$ is well-defined. Thus, for $f, h \in L^2(\mathcal{M}^{-1})$, $z \in C_x^1$ and $n \in E$, we can define

$$D_f\varphi(f, z, n)(Ah) \doteq -(AD_f\varphi(f, z, n), h)_{L^2(\mathcal{M}^{-1})},$$

even though Ah is not necessarily defined in $L^2(\mathcal{M}^{-1})$.

The class of test-function introduced in Definition 4 is chosen such that the Proposition 4.1 holds.

Proposition 4.1. *Let φ be a good test function in the sense of Definition 4. Define for all $t \geq 0$*

$$M_\varphi^\varepsilon(t) \doteq \varphi(X^\varepsilon(t)) - \varphi(X^\varepsilon(0)) - \int_0^t \mathcal{L}^\varepsilon \varphi(X^\varepsilon(s)) ds, \quad (29)$$

and consider the stopping time τ_Λ^ε defined by (17).

Then $M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}$ is a càdlàg $(\mathcal{F}_t^\varepsilon)_t$ -martingale and

$$\begin{aligned} \forall t \in \mathbb{R}^+, \mathbb{E} \left[\left| M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}(t) \right|^2 \right] &= \mathbb{E} \left[\int_0^{t \wedge \tau_\Lambda^\varepsilon} (\mathcal{L}^\varepsilon(\varphi^2) - 2\varphi \mathcal{L}^\varepsilon \varphi)(X^\varepsilon(s)) ds \right] \\ &= \frac{1}{\varepsilon^2} \mathbb{E} \left[\int_0^{t \wedge \tau_\Lambda^\varepsilon} (B(\varphi^2) - 2\varphi B\varphi)(X^\varepsilon(s)) ds \right]. \end{aligned}$$

This result is expected to hold as in the standard framework [11]. However, due to the presence of stopping times, the proof is very technical.

Proof. Note that in this section, ε is fixed, it is therefore not required to prove bounds which are uniform with respect to ε .

Let φ be a good test function. Observe that φ and φ^2 are in the domain of \mathcal{L}^ε , by means of Definition 4.

Let $s, t \in \mathbb{R}^+$, $\delta > 0$ and let $s = t_1 < \dots < t_n = t$ be a subdivision of $[s, t]$ such that $\max_i |t_{i+1} - t_i| = \delta$. Let g be a $\mathcal{F}_s^\varepsilon$ -measurable and bounded function. To simplify notation, let

$$f_i \doteq f^{\varepsilon, \tau_\Lambda^\varepsilon}(t_i), \zeta_i \doteq \zeta^{\varepsilon, \tau_\Lambda^\varepsilon}(t_i), m_i \doteq \overline{m}^{\varepsilon, \tau_\Lambda^\varepsilon}(t_i).$$

Then, we have

$$\begin{aligned} &\mathbb{E} \left[\left(M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}(t) - M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}(s) \right) g \right] \\ &= \mathbb{E} \left[\left(\varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t)) - \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(s)) - \int_{s \wedge \tau_\Lambda^\varepsilon}^{t \wedge \tau_\Lambda^\varepsilon} \mathcal{L}^\varepsilon \varphi(X^\varepsilon(u)) du \right) g \right] \\ &= r_f + r_z + r_n, \end{aligned}$$

where

$$\begin{aligned} r_f &= \sum_{i=1}^{n-1} \mathbb{E} \left[\left(\varphi(f_{i+1}, \zeta_{i+1}, m_{i+1}) - \varphi(f_i, \zeta_{i+1}, m_{i+1}) \right. \right. \\ &\quad \left. \left. - \int_{t_i \wedge \tau_\Lambda^\varepsilon}^{t_{i+1} \wedge \tau_\Lambda^\varepsilon} D_f \varphi(X^\varepsilon(u)) \left(-\frac{1}{\varepsilon} A f^\varepsilon(u) + \frac{1}{\varepsilon^2} L f^\varepsilon(u) + \frac{1}{\varepsilon} f^\varepsilon(u) \overline{m}^\varepsilon(u) \right) du \right) g \right], \\ r_z &= \sum_{i=1}^{n-1} \mathbb{E} \left[\left(\varphi(f_i, \zeta_{i+1}, m_{i+1}) - \varphi(f_i, \zeta_i, m_{i+1}) \right. \right. \\ &\quad \left. \left. - \int_{t_i \wedge \tau_\Lambda^\varepsilon}^{t_{i+1} \wedge \tau_\Lambda^\varepsilon} D_z \varphi(X^\varepsilon(u)) \left(\frac{1}{\varepsilon} \overline{m}^\varepsilon(u) \right) du \right) g \right], \end{aligned}$$

and

$$r_n = \sum_{i=1}^{n-1} \mathbb{E} \left[\left(\varphi(f_i, \zeta_i, m_{i+1}) - \varphi(f_i, \zeta_i, m_i) - \int_{t_i \wedge \tau_\Lambda^\varepsilon}^{t_{i+1} \wedge \tau_\Lambda^\varepsilon} \frac{1}{\varepsilon^2} B\varphi(X^\varepsilon(u)) du \right) g \right].$$

Straightforward computations lead to

$$r_f = \mathbb{E} \left[\left(\int_s^t r'_f(u) du \right) g \right], r_z = \mathbb{E} \left[\left(\int_s^t r'_z(u) du \right) g \right]$$

with

$$r'_f(u) = \sum_{i=1}^{n-1} \mathbb{1}_{[t_i \wedge \tau_\Lambda^\varepsilon, t_{i+1} \wedge \tau_\Lambda^\varepsilon]}(u) [D_f \varphi(f^\varepsilon, \tau_\Lambda^\varepsilon(u), \zeta_{i+1}, m_{i+1}) - D_f \varphi(X^\varepsilon, \tau_\Lambda^\varepsilon(u))] (\partial_t f^\varepsilon, \tau_\Lambda^\varepsilon(u)),$$

$$r'_z(u) = \sum_{i=1}^{n-1} \mathbb{1}_{[t_i \wedge \tau_\Lambda^\varepsilon, t_{i+1} \wedge \tau_\Lambda^\varepsilon]}(u) [D_z \varphi(f_i, \zeta^\varepsilon, \tau_\Lambda^\varepsilon(u), m_{i+1}) - D_z \varphi(X^\varepsilon, \tau_\Lambda^\varepsilon(u))] (\partial_t \zeta^\varepsilon, \tau_\Lambda^\varepsilon(u)).$$

Let us now check that $r_n = \mathbb{E} \left[\left(\int_s^t r'_n(u) du \right) g \right]$ with

$$r'_n(u) = \frac{1}{\varepsilon^2} \sum_{i=1}^{n-1} \mathbb{1}_{\{t_i \wedge \tau_\Lambda^\varepsilon, t_{i+1} \wedge \tau_\Lambda^\varepsilon\}}(u) [B\varphi(f_i, \zeta_i, \bar{m}^\varepsilon, \tau_\Lambda^\varepsilon(u)) - B\varphi(X^\varepsilon, \tau_\Lambda^\varepsilon(u))].$$

For $\theta \in C(E) \cap L^1(E)$, the Markov property for \bar{m} yields

$$\mathbb{E} [\theta(\bar{m}(t)) \mid \mathcal{F}_s] = P_{t-s} \theta(\bar{m}(s)).$$

Usually, this property is written for θ deterministic, continuous and bounded, but it is straightforward to check that it is still satisfied when $\theta \in C(E) \cap L^1(E)$ \mathcal{F}_s -measurable. The standard proof to show that \bar{m} solves the martingale problem associated to B (see for example [11], Theorem B.3) can be applied, and we get that, for $\theta \in D(B)$,

$$t \mapsto \theta(\bar{m}(t)) - \theta(\bar{m}(0)) - \int_0^t B\theta(\bar{m}(u)) du$$

is an integrable \mathcal{F}_t -martingale. By rescaling the time to retrieve \bar{m}^ε , stopping the martingale at τ_Λ^ε and using a conditioning argument (g , f_i and ζ_i are \mathcal{F}_{t_i} -measurable), we get

$$\mathbb{E} [(\varphi(f_i, \zeta_i, m_{i+1}) - \varphi(f_i, \zeta_i, m_i)) g] = \mathbb{E} \left[g \int_{t_i \wedge \tau_\Lambda^\varepsilon}^{t_{i+1} \wedge \tau_\Lambda^\varepsilon} \frac{1}{\varepsilon^2} B\varphi(X^\varepsilon(u)) du \right].$$

Hence, we can write $r_n = \mathbb{E} \left[\left(\int_s^t r'_n(u) du \right) g \right]$ as claimed above.

Since the estimates given by (23) and Proposition 3.6 are uniform for $t \in [0, T]$, we can use (28) with $S_1 \lesssim \Lambda \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}$ and $S_2 \leq \varepsilon^{-\alpha} \vee \|\bar{m}(0)\|_E$. This leads to

$$\sup_{u \in [0, T]} |r'_*(u)|^2 \lesssim_{\varphi, \Lambda, \varepsilon} \left(1 + \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^6 \right) \left(1 + \|\bar{m}(0)\|_E^{2(b+2)} \right), \quad (30)$$

where $r'_* \in \{r'_f, r'_z, r'_n\}$. Hence, the Cauchy-Schwarz inequality, Assumptions 4 and 7 yield

$$\begin{aligned} \mathbb{E} \left[\sup_{u \in [0, T]} |r'_*(u)|^2 \right] &\lesssim_{\varphi, \Lambda, \varepsilon} \mathbb{E} \left[\left(1 + \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^{12} \right) \right]^{\frac{1}{2}} \mathbb{E} \left[\left(1 + \|\bar{m}(0)\|_E^{4(b+2)} \right) \right]^{\frac{1}{2}} \\ &< \infty. \end{aligned} \quad (31)$$

Thus, the terms r'_* are uniformly integrable with respect to (u, ω) . Recall that $f^{\varepsilon, \tau_\Lambda^\varepsilon}$ and $\zeta^{\varepsilon, \tau_\Lambda^\varepsilon}$ are almost surely continuous and that $\bar{m}^{\varepsilon, \tau_\Lambda^\varepsilon}$ is stochastically continuous. Then, the terms r'_* converge to 0 in probability when $\delta \rightarrow 0$. By uniform integrability, the terms r_* converge to 0, which proves that $M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}$ is a $(\mathcal{F}_t^\varepsilon)_t$ -martingale. Note that we only used moments of order 12 and $4(b+2)$, instead of 24 and $8(b+2)$ as assumed in Assumptions 4 and 7. Hence, this proof can be adapted to establish that $M_{\varphi^2}^{\varepsilon, \tau_\Lambda^\varepsilon}$ is also a $(\mathcal{F}_t^\varepsilon)_t$ -martingale.

It remains to prove the formulas for the variance. This is done in several steps, following Appendix B of [11]. Since φ and φ^2 belong to the domain of \mathcal{L}^ε , the process

$$N^\varepsilon(t) = \int_0^t (\mathcal{L}^\varepsilon(\varphi^2) - 2\varphi\mathcal{L}^\varepsilon\varphi)(X^\varepsilon(s))ds = \frac{1}{\varepsilon^2} \int_0^t (B(\varphi^2) - 2\varphi B\varphi)(X^\varepsilon(s))ds,$$

is well-defined.

The proof of the second equality is straightforward: since $D = \mathcal{L}^\varepsilon - \frac{1}{\varepsilon^2}B$ is a first order differential operator, we have $D(\varphi^2) - 2\varphi D\varphi = 0$.

Let $0 = t_0 < t_1 < \dots < t_n = T$ be a subdivision of $[0, T]$ of step $\max |t_{i+1} - t_i| = \delta$.

Step 1: We claim that the following convergence is satisfied in $\mathbb{L}^2 \doteq \mathbb{L}^2(\Omega)$

$$N^{\varepsilon, \tau_\Lambda^\varepsilon}(t) = \lim_{\delta \rightarrow 0} \sum_i \mathbb{E} [N^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_{i+1}) - N^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_i) \mid \mathcal{F}_{t_i}^\varepsilon]. \quad (32)$$

Let $\Delta_i \doteq N^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_{i+1}) - N^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_i) - \mathbb{E} [N^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_{i+1}) - N^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_i) \mid \mathcal{F}_{t_i}^\varepsilon]$ so that (32) is equivalent to $\sum_i \Delta_i \xrightarrow{\delta \rightarrow 0} 0$ in \mathbb{L}^2 . Note that $\mathbb{E} [\Delta_i \Delta_j] = 0$ for $i \neq j$. Hence, we have $\mathbb{E} [|\sum_i \Delta_i|^2] = \mathbb{E} [\sum_i |\Delta_i|^2]$. Using that a conditional expectation is an orthogonal projection in \mathbb{L}^2 , we get

$$\begin{aligned} \mathbb{E} [|\Delta_i|^2] &\leq \mathbb{E} \left[|N^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_{i+1}) - N^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_i)|^2 \right] \\ &\lesssim_\varepsilon \mathbb{E} \left[\left| \int_{t \wedge t_i \wedge \tau_\Lambda^\varepsilon}^{t \wedge t_{i+1} \wedge \tau_\Lambda^\varepsilon} (B(\varphi^2) - 2\varphi B\varphi)(X^\varepsilon(s))ds \right|^2 \right], \end{aligned}$$

By means of Definition 4 and Assumption 10,

$$\left| (B(\varphi^2) - 2\varphi B\varphi)(X^\varepsilon(s)) \right|^2 \lesssim_\varphi \left(1 + \|f^\varepsilon(s)\|_{L^2(\mathcal{M}^{-1})}^{12} \right) \left(1 + \|\bar{m}^\varepsilon(s)\|_E^{4(b+2)} \right).$$

As in (30) and (31) (using moments of order 24 and $8(b+2)$ instead of 12 and $4(b+2)$), Proposition 3.6 and (23) lead to

$$\mathbb{E} \left[\sup_{s \in [0, T]} |(B(\varphi^2) - 2\varphi B\varphi)(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(s))|^2 \right] \lesssim_{\varphi, \Lambda, \varepsilon} 1.$$

Since $t \wedge t_i \wedge \tau_{\Lambda}^{\varepsilon} - t \wedge t_{i+1} \wedge \tau_{\Lambda}^{\varepsilon} \leq t_{i+1} - t_i$, we get

$$\mathbb{E} \left[|\Delta_i|^2 \right] \lesssim_{\varphi, \Lambda, \varepsilon} (t_{i+1} - t_i)^2,$$

which then yields $\mathbb{E} \left[\left| \sum_i \Delta_i \right|^2 \right] \lesssim_{\varphi, \Lambda, \varepsilon} T\delta \xrightarrow{\delta \rightarrow 0} 0$, which proves (32).

Step 2: We claim that

$$\mathbb{E} \left[\sum_i |R_{t_i, t_{i+1}}| \right] \lesssim_{\varphi, \Lambda, \varepsilon} \delta^{1/2}, \quad (33)$$

where, for $0 \leq t < t' \leq T$,

$$\begin{aligned} R_{t, t'} &= \left| M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t') - M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t) \right|^2 - \left| \varphi(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t')) - \varphi(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)) \right|^2 \\ &= \left| \int_{t \wedge \tau_{\Lambda}^{\varepsilon}}^{t' \wedge \tau_{\Lambda}^{\varepsilon}} \mathcal{L}^{\varepsilon} \varphi(X^{\varepsilon}(s)) ds \right|^2 - 2 \left(\varphi(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t')) - \varphi(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)) \right) \int_{t \wedge \tau_{\Lambda}^{\varepsilon}}^{t' \wedge \tau_{\Lambda}^{\varepsilon}} \mathcal{L}^{\varepsilon} \varphi(X^{\varepsilon}(s)) ds. \end{aligned} \quad (34)$$

We can write

$$\begin{aligned} \left| \varphi(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t')) - \varphi(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)) \right|^2 &= M_{\varphi^2}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t') - M_{\varphi^2}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t) \\ &\quad - 2\varphi(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)) \left(M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t') - M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t) \right) \\ &\quad + \int_{t \wedge \tau_{\Lambda}^{\varepsilon}}^{t' \wedge \tau_{\Lambda}^{\varepsilon}} \mathcal{L}^{\varepsilon}(\varphi^2)(X^{\varepsilon}(s)) ds \\ &\quad - 2\varphi(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)) \int_{t \wedge \tau_{\Lambda}^{\varepsilon}}^{t' \wedge \tau_{\Lambda}^{\varepsilon}} \mathcal{L}^{\varepsilon} \varphi(X^{\varepsilon}(s)) ds. \end{aligned} \quad (35)$$

As established in the first part of the proof, $M_{\varphi^2}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}$ and $M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}$ are $(\mathcal{F}_s^{\varepsilon})_s$ -martingales. Moreover, $\varphi(X_t^{\varepsilon, \tau_{\Lambda}^{\varepsilon}})$ is $\mathcal{F}_t^{\varepsilon}$ -measurable. Thus, taking the expectation in (35) yields

$$\begin{aligned} \mathbb{E} \left[\left| \varphi(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t')) - \varphi(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)) \right|^2 \right] &= \\ \mathbb{E} \left[\int_{t \wedge \tau_{\Lambda}^{\varepsilon}}^{t' \wedge \tau_{\Lambda}^{\varepsilon}} \left(\mathcal{L}^{\varepsilon}(\varphi^2)(X^{\varepsilon}(s)) ds - 2\varphi(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)) \mathcal{L}^{\varepsilon} \varphi(X^{\varepsilon}(s)) \right) ds \right]. \end{aligned}$$

As in Step 1, by Definition 4, Assumption 10, (23), and Proposition 3.6, the integrand is bounded by $(1 + \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^6) (1 + \|\bar{m}(0)\|_E^{2(b+2)})$ (up to a constant depending on φ , Λ and ε) and we get

$$\mathbb{E} \left[\left| \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t')) - \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t)) \right|^2 \right] \lesssim_{\varphi, \Lambda, \varepsilon} t' - t, \quad (36)$$

owing to the Cauchy-Schwarz inequality, Assumptions 4 and 7. Young's inequality with a parameter $\eta > 0$ yields

$$\begin{aligned} \mathbb{E} [|R_{t,t'}|] &\lesssim_{\varphi, \Lambda, \varepsilon} \left(1 + \frac{1}{\eta}\right) \mathbb{E} \left[\left| \int_{t \wedge \tau_\Lambda^\varepsilon}^{t' \wedge \tau_\Lambda^\varepsilon} \mathcal{L}^\varepsilon \varphi(X^\varepsilon(s)) ds \right|^2 \right] \\ &\quad + \eta \mathbb{E} \left[\left| \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t')) - \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t)) \right|^2 \right]. \end{aligned}$$

Similarly, we get

$$\mathbb{E} \left[\left| \int_{t \wedge \tau_\Lambda^\varepsilon}^{t' \wedge \tau_\Lambda^\varepsilon} \mathcal{L}^\varepsilon \varphi(X^\varepsilon(s)) ds \right|^2 \right] \lesssim_{\varphi, \Lambda, \varepsilon} (t' - t)^2.$$

Choosing $\eta = (t' - t)^{1/2}$ yields $\mathbb{E} [|R_{t,t'}|] \lesssim_{\varphi, \varepsilon} (t' - t)^{3/2}$, which gives (33).

Step 3: We claim that $\mathbb{E} \left[\left| M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}(t) \right|^2 \right] = \mathbb{E} [N^{\varepsilon, \tau_\Lambda^\varepsilon}(t)]$.

Taking conditional expectation in (34) leads to

$$\begin{aligned} \sum_i \mathbb{E} \left[\left| M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_{i+1}) - M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_i) \right|^2 \mid \mathcal{F}_{t_i}^\varepsilon \right] &= \\ &= \sum_i \mathbb{E} [R_{t \wedge t_i, t \wedge t_{i+1}} \mid \mathcal{F}_{t_i}^\varepsilon] \\ &\quad + \sum_i \mathbb{E} \left[\left| \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_{i+1})) - \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_i)) \right|^2 \mid \mathcal{F}_{t_i}^\varepsilon \right]. \end{aligned}$$

Using (35) and the martingale property on $M_{\varphi^2}^{\varepsilon, \tau_\Lambda^\varepsilon}$ and $M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}$, the last term can be rewritten as

$$\sum_i \int_{t \wedge t_i \wedge \tau_\Lambda^\varepsilon}^{t \wedge t_{i+1} \wedge \tau_\Lambda^\varepsilon} \mathcal{L}^\varepsilon(\varphi^2)(X^\varepsilon(s)) ds - 2 \sum_i \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_i)) \int_{t \wedge t_i \wedge \tau_\Lambda^\varepsilon}^{t \wedge t_{i+1} \wedge \tau_\Lambda^\varepsilon} \mathcal{L}^\varepsilon \varphi(X^\varepsilon(s)) ds$$

Then, (32) yields

$$\sum_i \mathbb{E} \left[\left| M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_{i+1}) - M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_i) \right|^2 \mid \mathcal{F}_{t_i}^\varepsilon \right] = N^{\varepsilon, \tau_\Lambda^\varepsilon}(t) + r_1 + r_2$$

where

$$r_1 = \sum_i \mathbb{E} [R_{t \wedge t_i, t \wedge t_{i+1}} | \mathcal{F}_{t_i}^\varepsilon]$$

$$r_2 = 2 \sum_i \mathbb{E} \left[\int_{t \wedge t_i \wedge \tau_\Lambda^\varepsilon}^{t \wedge t_{i+1} \wedge \tau_\Lambda^\varepsilon} (\varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(s)) - \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_i))) \mathcal{L}^\varepsilon \varphi(X^\varepsilon(s)) ds | \mathcal{F}_{t_i}^\varepsilon \right].$$

By means of (33), $r_1 \rightarrow 0$ in \mathbb{L}^1 . For r_2 , we have

$$\begin{aligned} \mathbb{E} [|r_2|] &\leq 2 \mathbb{E} \left[\sum_i \int_{t \wedge t_i \wedge \tau_\Lambda^\varepsilon}^{t \wedge t_{i+1} \wedge \tau_\Lambda^\varepsilon} |\varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(s)) - \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t_i))| |\mathcal{L}^\varepsilon \varphi(X^\varepsilon(s))| ds \right] \\ &\leq 2 \mathbb{E} \left[\sum_i \int_{t_i}^{t_{i+1}} |\varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(s)) - \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t_i))| |\mathcal{L}^\varepsilon \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(s))| ds \right] \\ &\leq 2 \sum_i \int_{t_i}^{t_{i+1}} \mathbb{E} [|\varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(s)) - \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t_i))| |\mathcal{L}^\varepsilon \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(s))|] ds \\ &\leq 2 \sum_i \int_{t_i}^{t_{i+1}} \mathbb{E} [|\varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(s)) - \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t_i))|^2]^{1/2} \mathbb{E} [|\mathcal{L}^\varepsilon \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(s))|^2]^{1/2} ds. \end{aligned}$$

As above, one can show $\mathbb{E} [\sup_s |\mathcal{L}^\varepsilon \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(s))|^2] \lesssim_{\varphi, \Lambda, \varepsilon} 1$. Thus, (36) yields

$$\begin{aligned} \mathbb{E} [|r_2|] &\lesssim_{\varphi, \Lambda, \varepsilon} \sum_i \int_{t_i}^{t_{i+1}} \mathbb{E} [|\varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(s)) - \varphi(X^{\varepsilon, \tau_\Lambda^\varepsilon}(t_i))|^2]^{1/2} ds \\ &\lesssim_{\varphi, \Lambda, \varepsilon} \sum_i \int_{t_i}^{t_{i+1}} (s - t_i)^{1/2} ds \\ &\lesssim_{\varphi, \Lambda, \varepsilon} \sum_i (t_{i+1} - t_i)^{3/2} ds \\ &\lesssim_{\varphi, \Lambda, \varepsilon} T \delta^{1/2} \xrightarrow{\delta \rightarrow 0} 0. \end{aligned}$$

Thus, in \mathbb{L}^1 , we have

$$\lim_{\delta \rightarrow 0} \sum_i \mathbb{E} \left[\left| M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_{i+1}) - M_\varphi^{\varepsilon, \tau_\Lambda^\varepsilon}(t \wedge t_i) \right|^2 | \mathcal{F}_{t_i}^\varepsilon \right] = N^{\varepsilon, \tau_\Lambda^\varepsilon}(t). \quad (37)$$

In particular, the expectation converges. Then, the martingale property and the tower

property $\mathbb{E}[\mathbb{E}[\cdot | \mathcal{F}_s]] = \mathbb{E}[\cdot]$ yield

$$\begin{aligned}
\mathbb{E}[N^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)] &= \lim_{\delta \rightarrow 0} \mathbb{E} \left[\sum_i \mathbb{E} \left[\left| M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t \wedge t_{i+1}) - M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t \wedge t_i) \right|^2 \mid \mathcal{F}_{t_i}^{\varepsilon} \right] \right] \\
&= \lim_{\delta \rightarrow 0} \mathbb{E} \left[\sum_i \mathbb{E} \left[\left| M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t \wedge t_{i+1}) \right|^2 - \left| M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t \wedge t_i) \right|^2 \mid \mathcal{F}_{t_i}^{\varepsilon} \right] \right] \\
&= \lim_{\delta \rightarrow 0} \sum_i \mathbb{E} \left[\left| M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t \wedge t_{i+1}) \right|^2 - \left| M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t \wedge t_i) \right|^2 \right] \\
&= \mathbb{E} \left[\left| M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t) \right|^2 \right].
\end{aligned}$$

This concludes the proof that

$$\forall t \in \mathbb{R}^+, \mathbb{E} \left[\left| M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t) \right|^2 \right] = \mathbb{E} \left[\int_0^{t \wedge \tau_{\Lambda}^{\varepsilon}} (\mathcal{L}^{\varepsilon}(\varphi^2) - 2\varphi \mathcal{L}^{\varepsilon} \varphi)(X^{\varepsilon}(s)) ds \right].$$

and the proof of Proposition 4.1. \square

Remark. Note that if \bar{m} had continuous paths, then $M_{\varphi}^{\varepsilon}$ would be a continuous martingale and (37) would mean that $N^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}$ is the quadratic variation of $M_{\varphi}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}$.

A similar proof leads to the following Proposition, where we take weaker stopping times but add some conditions on φ . The proof is omitted.

Proposition 4.2. *Let φ be a good test function. The conclusion of Proposition 4.1 holds in the following cases.*

- The function φ does not depend on f and $\tau_{\Lambda}^{\varepsilon}$ is replaced by τ^{ε} .
- The function φ is bounded uniformly in n and does not depend on z and $\tau_{\Lambda}^{\varepsilon}$ is replaced by $\tau_{\Lambda}(\zeta^{\varepsilon})$.
- The function φ is bounded and depends only on n and $\tau_{\Lambda}^{\varepsilon}$ is replaced by $+\infty$.

4.2 The perturbed test functions method

We use the perturbed test functions method as in [29] to exhibit a generator \mathcal{L} such that a possible limit point $(\rho_{\Lambda}, \zeta_{\Lambda})$ of $((\rho^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}, \zeta^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}))_{\varepsilon}$ solves the martingale problem associated to \mathcal{L} until some limit stopping time depending on Λ . Given a test function φ , two corrector functions φ_1 and φ_2 are constructed, so that

$$\forall (f, z, n) \in L^2(\mathcal{M}^{-1}) \times C_x^1 \times E, \varphi^{\varepsilon}(f, z, n) = \varphi(\rho, z) + \varepsilon \varphi_1(f, z, n) + \varepsilon^2 \varphi_2(f, z, n), \quad (38)$$

satisfies

$$\mathcal{L}^{\varepsilon} \varphi^{\varepsilon} = \mathcal{L} \varphi + o(1), \quad (39)$$

when $\varepsilon \rightarrow 0$. Then, we prove that φ^ε is a good test function and that we can take the limit when $\varepsilon \rightarrow 0$ in the martingale problem associated to \mathcal{L}^ε (Proposition 4.1) to obtain a stopped martingale problem solved by a limit point.

Based on the decomposition (26), a sufficient condition to prove (39) for φ^ε of the form (38) is to solve the following equations (40) to (42) and to check that (43) holds when $\varepsilon \rightarrow 0$.

$$\mathcal{L}_2\varphi = 0 \tag{40}$$

$$\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0 \tag{41}$$

$$\mathcal{L}_1\varphi_1 + \mathcal{L}_2\varphi_2 = \mathcal{L}\varphi \tag{42}$$

$$\mathcal{L}_1\varphi_2 = O(1). \tag{43}$$

The properties of the resolvent operators R_λ are employed to invert \mathcal{L}_2 .

4.2.1 Framework for the perturbed test functions method

For a martingale problem to be relevant, it is sufficient that the class of test functions satisfying the martingale problem is separating, namely that if some random variables X and X' satisfy $\mathbb{E}[\varphi(X)] = \mathbb{E}[\varphi(X')]$ for all $\varphi \in \Phi$, then we have $X \stackrel{d}{=} X'$. In this work, we use the following class

$$\Theta = \{(\rho, z) \mapsto \psi((\rho, \xi)_{L_x^2}) \chi(z) \mid \psi \in C^3(\mathbb{R}), \psi'' \in C_b^1(\mathbb{R}), \xi \in H_x^3, \chi \in C_b^3(C_x^1)\},$$

where $\rho = \int_V f d\mu$. The class Θ is indeed separating because it separates points (see [14], Theorem 4.5).

Note that the test functions depend only on ρ and z , because we expect the limit equation to be satisfied by ρ and z . It is confirmed by Section 4.2.2. To simplify the notation, for $\varphi \in \Theta$, we sometimes write $\varphi(f, z, n) \doteq \varphi(\rho, z)$ and $\varphi(\rho, z) = \Psi(\rho)\chi(z)$, where $\Psi(\rho) = \psi((\rho, \xi)_{L_x^2})$.

Proposition 4.3. *There exists an operator \mathcal{L} whose domain contains Θ and, for all $\varphi \in \Theta$, there exist two good test functions φ_1 and φ_2 such that, for all $(f, z, n) \in L^2(\mathcal{M}^{-1}) \times C_x^1 \times E$, we have*

$$|\varphi_1(f, z, n)| \lesssim_\varphi (1 + \|f\|_{L^2(\mathcal{M}^{-1})}^2)(1 + \|n\|_E) \tag{44}$$

$$|\varphi_2(f, z, n)| \lesssim_\varphi (1 + \|f\|_{L^2(\mathcal{M}^{-1})}^2)(1 + \|n\|_E^{b+1}) \tag{45}$$

$$|\mathcal{L}^\varepsilon\varphi^\varepsilon - \mathcal{L}\varphi|(f, z, n) \lesssim \varepsilon(1 + \|f\|_{L^2(\mathcal{M}^{-1})}^3)(1 + \|n\|_E^{b+2}). \tag{46}$$

Moreover, $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$ is a good test function.

Moreover, if φ depends only on z , then φ_1 , φ_2 and φ^ε depend only on z and n .

4.2.2 Consistency result

Since we already expect the limit equation to be satisfied by ρ , equation (40) will not give us extra information. Hence, this section only present a consistency result, namely that (40) forces φ to depend on f through ρ .

In fact, let φ depend on f and z but not on n . Since φ does not depend on n , $B\varphi = 0$. Hence, (40) can be written, for all $f \in L^2(\mathcal{M}^{-1})$ and $z \in C_x^1$,

$$D_f\varphi(f, z, n)(Lf) = 0. \quad (47)$$

For $t \in \mathbb{R}^+$ and $f \in L^2(\mathcal{M}^{-1})$, define

$$g(t, f) = \rho\mathcal{M} + e^{-t}(f - \rho\mathcal{M}), \quad (48)$$

and observe that $\partial_t g(t, f) = Lg(t, f)$ with $g(0, f) = f$. Owing to (47), the mapping $t \mapsto \varphi(g(t, f), z)$ is constant. Since $g(t, f) \xrightarrow[t \rightarrow \infty]{} \rho\mathcal{M}$, by continuity of φ , we get $\varphi(f, z, n) = \varphi(\rho\mathcal{M}, z, n)$, which depends on f only through ρ .

4.2.3 Construction of the first corrector function φ_1

The first corrector function φ_1 is defined as the solution of (41): the formal solution to Poisson equation will provide an expression for φ_1 , then we will check that this expression indeed solves (41).

Let $g(t, f)$ be defined by (48) and $m(t, n)$ be defined in Section 2.1 (Markov process of infinitesimal generator B starting from n). The process $((g(t, f), z, m(t, n)))_{t \in \mathbb{R}^+}$ is a $L^2(\mathcal{M}^{-1}) \times C_x^1 \times E$ -valued Markov process of generator \mathcal{L}_2 starting from (f, z, n) . Denote by $(Q_t)_{t \in \mathbb{R}^+}$ its transition semi-group. Note that this semi-group does not have a unique invariant distribution, since for any ρ fixed, $\delta_{\rho\mathcal{M}} \otimes \delta_z \otimes \nu$ is an invariant distribution. However on every space $\{(f', z', n) \in L^2(\mathcal{M}^{-1}) \times C_x^1 \times E \mid \int_V f' d\mu = \rho, z' = z\}$, this measure is the unique invariant distribution. Indeed, $\delta_{\rho\mathcal{M}}$, δ_z and ν are respectively the unique invariant distributions of each marginal process (on the corresponding subspaces), and $\delta_{\rho\mathcal{M}} \otimes \delta_z \otimes \nu$ is the only coupling of these three marginal distributions.

For $\Phi : L^2(\mathcal{M}^{-1}) \times C_x^1 \times E \rightarrow \mathbb{R}$, denote by

$$\langle \Phi \rangle_{\rho, z} \doteq \int_E \Phi(\rho\mathcal{M}, z, n) d\nu(n) = \int_{L^2(\mathcal{M}^{-1}) \times C_x^1 \times E} \Phi d(\delta_{\rho\mathcal{M}} \otimes \delta_z \otimes \nu)$$

the integral against this invariant distribution. For $\varphi \in \Theta$, $\varphi(\rho, z) = \Psi(\rho)\chi(z)$, let us compute $\mathcal{L}_1\varphi$.

We have

$$\begin{aligned} \mathcal{L}_1\varphi(f, z, n) &= D_f\varphi(f, z, n)(-Af + nf) + D_z\varphi(f, z, n)(n) \\ &= D\Psi(\rho)(-\overline{Af} + n\rho)\chi(z) + \Psi(\rho)D\chi(z)(n), \end{aligned}$$

where $\overline{h} = \int_{\mathbb{T}^d} h(x) dx$.

Owing to (6), one can write, for all $\rho \in L_x^2$, $\overline{A(\rho\mathcal{M})} = 0$. Moreover, since ν is centered by Assumption 5, any term linear in n vanishes when integrating with respect to ν . Hence, we have checked that

$$\forall \rho \in L_x^2, \forall z \in C_x^1, \langle \mathcal{L}_1\varphi \rangle_{\rho, z} = 0.$$

Using the expansion of $\mathcal{L}_1\varphi$, for all f, z, n , we have

$$\begin{aligned} \int_0^\infty Q_t \mathcal{L}_1 \varphi(f, z, n) dt &= \int_0^\infty \mathbb{E} [\mathcal{L}_1 \varphi(g(t, f), z, m(t, n))] dt \\ &= \int_0^\infty \left(-D\Psi(\rho)(\overline{Ag(t, f)})\chi(z) \right. \\ &\quad \left. + \mathbb{E} [D\Psi(\rho)(\rho m(t, n))] \chi(z) \right. \\ &\quad \left. + \Psi(\rho) D\chi(z)(m(t, n)) \right) dt, \end{aligned}$$

owing to the identity $\overline{g(t, f)} = \rho$. Equation (48) yields

$$\overline{Ag(t, f)} = e^{-t} \overline{Af},$$

since $\overline{A\rho\mathcal{M}} = 0$. Thus, owing to Definition 2, we define

$$\begin{aligned} \varphi_1(f, z, n) &= \int_0^\infty Q_t \mathcal{L}_1 \varphi(f, z, n) dt \\ &= D\Psi(\rho)(-\overline{Af} + R_0(n)\rho)\chi(z) + \Psi(\rho)R_0 [D\chi(z)](n). \end{aligned} \quad (49)$$

It is straightforward to check that φ_1 defined by (49) solves (41). Moreover, it satisfies the condition (44). It remains to prove that φ_1 is a good test function. Owing to Assumption 9 and (49), $\varphi_1 \in D(B)$ and $\varphi_1^2 \in D(B)$. For $h \in L^2(\mathcal{M}^{-1})$, we have

$$\begin{aligned} D_f \varphi_1(f, z, n)(h) &= D^2\Psi(\rho)(-\overline{Af} + R_0(n)\rho, \bar{h})\chi(z) + D\Psi(\rho)(-\overline{Ah} + R_0(n)\bar{h})\chi(z) \\ &\quad + D\Psi(\rho)(\bar{h})R_0 [D\chi(z)](n), \end{aligned}$$

hence $D_f \varphi_1(f, z, n)(Ah)$ is well-defined (as in Definition 4) and $\varphi_1, D_f \varphi_1(f, z, n)(h)$ and $D_f \varphi_1(f, z, n)(Ah)$ have at most polynomial growth in the sense of (28). For $n_2 \in E$, we have

$$\begin{aligned} D_z \varphi_1(f, z, n)(n_2) &= D\Psi(\rho)(-\overline{Af} + R_0(n)\rho)D\chi(z)(n_2) \\ &\quad + \Psi(\rho)D [z' \mapsto R_0 [D\chi(z')] (n)](z)(n_2). \end{aligned}$$

Using Lemma 3.1 and the assumption $\chi \in C_b^3(C_x^1)$, we write

$$\begin{aligned} D [R_0 [D\chi(\cdot)] (n)](z)(n_2) &= D \left[z' \mapsto \int_0^\infty P_t D\chi(z')(n) dt \right] (z)(n_2) \\ &= \int_0^\infty P_t [D^2\chi(z)(\cdot, n_2)](n) \\ &= R_0 [D^2\chi(z)(\cdot, n_2)](n). \end{aligned}$$

This leads to

$$D_z \varphi_1(f, z, n)(n_2) = D\Psi(\rho)(-\overline{Af} + R_0(n)\rho)D\chi(z)(n_2) + \Psi(\rho)R_0 [D^2\chi(z)(\cdot, n_2)](n).$$

Once again using Lemma 3.1 and that $\chi \in C_b^3(C_x^1)$, one checks that $D_z \varphi_1$ has at most polynomial growth in the sense of (28). Thus φ_1 satisfies (28).

4.2.4 Construction of the second corrector function φ_2

The second corrector φ_2 is defined as a solution of (42). To solve (42), we need the centering condition $\langle \mathcal{L}\varphi - \mathcal{L}_1\varphi_1 \rangle_{\rho,z} = 0$. This identity will be the definition of $\mathcal{L}\varphi$.

First, let us compute $\mathcal{L}_1\varphi_1$. Using the derivative calculated in (49), $\mathcal{L}_1\varphi_1$ can be written as

$$\mathcal{L}_1\varphi_1(f, z, n) = c(f, z) + \ell(f, z, n) + q(f, z, n)$$

where c , ℓ and q are defined by

$$c(f, z) = D^2\Psi(\rho)(\overline{Af}, \overline{Af})\chi(z) + D\Psi(\rho)(\overline{A^2f})\chi(z) \quad (50)$$

$$\begin{aligned} \ell(f, z, n) = & -D^2\Psi(\rho)(\overline{Af}, R_0(n)\rho + n\rho)\chi(z) \\ & - D\Psi(\rho)(\overline{A(nf)} + R_0(n)(\overline{Af}))\chi(z) \\ & - D\Psi(\rho)(\overline{Af})(R_0[D\chi(z)](n) + D\chi(z)(n)) \end{aligned} \quad (51)$$

$$\begin{aligned} q(f, z, n) = & D^2\Psi(\rho)(n\rho, R_0(n)\rho)\chi(z) + D\Psi(\rho)(R_0(n)(n\rho))\chi(z) \\ & + D\Psi(\rho)(n\rho)R_0[D\chi(z)](n) + D\Psi(\rho)(R_0(n)\rho)D\chi(z)(n) \\ & + \Psi(\rho)R_0[D^2\chi(z)(\cdot, n)](n). \end{aligned} \quad (52)$$

Note that, for fixed f and z , c does not depend on n , ℓ is pseudo-linear in n and q is pseudo-quadratic in n as introduced in Definition 3.

The function $\ell(f, z, \cdot)$ is indeed pseudo-linear as a sum of continuous linear and pseudo-linear forms, yielding $\langle \ell \rangle_{\rho,z} = 0$ for all ρ and z . Using also that $\overline{A\rho\mathcal{M}} = 0$, we get an explicit definition of \mathcal{L} :

$$\begin{aligned} \mathcal{L}\varphi(\rho, z) \doteq \langle \mathcal{L}_1\varphi_1 \rangle_{\rho,z} = & D\Psi(\rho)(\overline{A^2\rho\mathcal{M}})\chi(z) \\ & + \int D^2\Psi(\rho)(n\rho, R_0(n)\rho)d\nu(n)\chi(z) \\ & + \int D\Psi(\rho)(R_0(n)(n\rho))d\nu(n)\chi(z) \\ & + \int D\Psi(\rho)(n\rho)R_0[D\chi(z)](n)d\nu(n) \\ & + \int D\Psi(\rho)(R_0(n)\rho)D\chi(z)(n)d\nu(n) \\ & + \Psi(\rho) \int R_0[D^2\chi(z)(\cdot, n)](n)d\nu(n). \end{aligned} \quad (53)$$

Note that by taking $\chi = 1$, we obtain the same expression of \mathcal{L} as in [10].

Since the centering condition for the Poisson equation (42) is satisfied by construction

of \mathcal{L} , the second corrector function φ_2 can be defined as follows: for all f, z, n ,

$$\begin{aligned}\varphi_2(f, z, n) &\doteq \int_0^\infty Q_t \left(\mathcal{L}_1 \varphi_1 - \langle \mathcal{L}_1 \varphi_1 \rangle_{\rho, z} \right) (f, z, n) dt \\ &= \int_0^\infty Q_t \left(c - \langle c \rangle_{\rho, z} \right) (f, z, n) dt \\ &\quad + \int_0^\infty Q_t \ell(f, z, n) dt \\ &\quad + \int_0^\infty Q_t \left(q - \langle q \rangle_{\rho, z} \right) (f, z, n) dt \\ &\doteq \varphi_2^c(f, z, n) + \varphi_2^\ell(f, z, n) + \varphi_2^q(f, z, n).\end{aligned}$$

Once again, one can check that φ_2 satisfies (42). It only remains to prove (45), (46) and that φ^ε is a good test function. Since

$$\mathcal{L}^\varepsilon \varphi^\varepsilon = \mathcal{L} \varphi + \varepsilon \mathcal{L}_1 \varphi_2, \quad (54)$$

equation (46) comes from an estimate on $\mathcal{L}_1 \varphi_2(f, n)$ in terms of f, n , and φ .

4.2.5 Controls on the second corrector function

The aim of this section is to prove some estimates for $\varphi_2(f, z, n)$ and its derivatives to establish that (28), (45) and (46) are satisfied. Let $f, h \in L^2(\mathcal{M}^{-1})$, $z \in C_x^1$ and $n, n_2 \in E$ and let $S_1 = \|f\|_{L^2(\mathcal{M}^{-1})} \vee \|h\|_{L^2(\mathcal{M}^{-1})}$ and $S_2 = \|n\|_E \vee \|n_2\|_E$.

Estimates on φ_2^c We have, using $\langle c \rangle_{\rho, z} = c(\rho\mathcal{M}, z)$,

$$c(f, z) - c(\rho\mathcal{M}, z) = D^2\Psi(\rho)(\overline{Af}, \overline{Af})\chi(z) + D\Psi(\rho)(\overline{A^2(f - \rho\mathcal{M})})\chi(z).$$

Recall that $\overline{Ag(t, f)} = e^{-t}\overline{Af}$. Hence, using (48), we get

$$\begin{aligned}Q_t \left(c - \langle c \rangle_{\rho, z} \right) (f, z, n) &= \mathbb{E}[c(g(t, f), z) - c(\rho\mathcal{M}, z)] \\ &= e^{-2t} D^2\Psi(\rho)(\overline{Af}, \overline{Af})\chi(z) \\ &\quad + e^{-t} D\Psi(\rho)(\overline{A^2(f - \rho\mathcal{M})})\chi(z).\end{aligned}$$

By integration, we get

$$\varphi_2^c(f, z, n) = \frac{1}{2} D^2\Psi(\rho)(\overline{Af}, \overline{Af})\chi(z) + D\Psi(\rho)(\overline{A^2(f - \rho\mathcal{M})})\chi(z). \quad (55)$$

Moreover, we obtain

$$\begin{aligned}
D_f \varphi_2^c(f, z, n)(h) &= \frac{1}{2} D^3 \Psi(\rho)(\overline{Af}, \overline{Af}, \overline{h}) \chi(z) \\
&\quad + D^2 \Psi(\rho)(\overline{Af}, \overline{Ah}) \chi(z) \\
&\quad + D^2 \Psi(\rho)(\overline{A^2(f - \rho \mathcal{M})}, \overline{h}) \chi(z) \\
&\quad + D \Psi(\rho)(\overline{A^2(h - \overline{h} \mathcal{M})}) \chi(z), \\
D_z \varphi_2^c(f, z, n)(n_2) &= \frac{1}{2} D \Psi(\rho)(\overline{Af}, \overline{Af}) D \chi(z)(n) \\
&\quad + D \Psi(\rho)(\overline{A^2(f - \rho \mathcal{M})}) D \chi(z)(n).
\end{aligned}$$

Recall that $\|f - \rho \mathcal{M}\|_{L^2(\mathcal{M}^{-1})}^2 + \|\rho\|_{L_x^2}^2 = \|f\|_{L^2(\mathcal{M}^{-1})}^2$, hence $\|f - \rho \mathcal{M}\|_{L^2(\mathcal{M}^{-1})} \leq \|f\|_{L^2(\mathcal{M}^{-1})}$. Then, since $\Psi(\rho) = \psi((\rho, \xi)_{L_x^2})$ and $\psi'' \in C_b^1(\mathbb{R})$, we get that φ_2^c satisfies (28). More precisely, the following estimates hold:

$$\begin{aligned}
|\varphi_2^c(f, z, n)| &\lesssim_\varphi 1 + \|f\|_{L^2(\mathcal{M}^{-1})}^2 \\
|\mathcal{L}_1 \varphi_2^c(f, z, n)| &\lesssim_\varphi (1 + \|f\|_{L^2(\mathcal{M}^{-1})}^3)(1 + \|n\|_E).
\end{aligned}$$

Estimates on φ_2^ℓ Using (48), (51) and that $\overline{A\rho\mathcal{M}} = \overline{A(n\rho)\mathcal{M}} = 0$, we get

$$\forall (f, z, n), \ell(g(t, f), z, n) = e^{-t} \ell(f, z, n).$$

Thus, we have

$$Q_t \ell(f, z, n) = \mathbb{E}[\ell(g(t, f), z, m(t, n))] = e^{-t} \mathbb{E}[\ell(f, z, m(t, n))] = e^{-t} P_t \ell(f, z, n),$$

and by integrating with respect to t , we get

$$\varphi_2^\ell(f, z, n) = R_1 \ell(f, z, n).$$

Moreover, from Lemma 3.1 and (51), it is straightforward to check that

$$[\ell(f, \cdot)]_{\text{Lip}} \lesssim_\varphi (1 + \|f\|_{L^2(\mathcal{M}^{-1})}^2).$$

Hence, Lemma 3.1 yields

$$\left| \varphi_2^\ell(f, z, n) \right| \lesssim_\varphi (1 + \|f\|_{L^2(\mathcal{M}^{-1})}^2)(1 + \|n\|_E).$$

Since the operator R_1 acts only on the variable n , it commutes with the derivatives D_f and D_z in the following sense:

$$\begin{aligned}
D_f [R_1 \ell](f, z, n)(h) &= R_1 [D_f \ell(f, z, \cdot)](h)(n) \\
D_z [R_1 \ell](f, z, n)(n_2) &= R_1 [D_z \ell(f, z, \cdot)](n_2)(n).
\end{aligned}$$

Thus, after calculating the derivatives of ℓ , we get estimates on the derivatives of φ_2^ℓ the same way we got estimates on φ_2^ℓ . This leads to

$$\begin{aligned} \left| D_f \varphi_2^\ell(f, z, n)(Ah) \right| &\lesssim_\varphi (1 + S_1^3)(1 + S_2) \\ \left| D_f \varphi_2^\ell(f, z, n)(n_2 f) \right| &\lesssim_\varphi (1 + S_1^3)(1 + S_2^2) \\ \left| D_z \varphi_2^\ell(f, z, n)(n_2) \right| &\lesssim_\varphi (1 + S_1^2)(1 + S_2^2), \end{aligned}$$

hence φ_2^ℓ satisfies (28). Finally, the following estimates hold

$$\begin{aligned} \left| \varphi_2^\ell(f, z, n) \right| &\lesssim_\varphi (1 + \|f\|_{L^2(\mathcal{M}^{-1})}^2)(1 + \|n\|_E) \\ \left| \mathcal{L}_1 \varphi_2^\ell(f, z, n) \right| &\lesssim_\varphi (1 + \|f\|_{L^2(\mathcal{M}^{-1})}^3)(1 + \|n\|_E^2). \end{aligned}$$

Estimates on φ_2^q The function q depends of f only through ρ . Since $\overline{g(t, f)} = \rho$ does not depend on t , we get $Q_t q = P_t q$ and

$$\varphi_2^q(f, z, n) = R_0 \left[q - \langle q \rangle_{\rho, z} \right] (f, z, n)$$

It is straightforward to compute the derivatives of q with respect to f and z from (52). One can deduce estimates for $[q(f, z, \cdot)]_{\text{quad}}$ and for the first order derivatives $[D_f q(f, z, \cdot)(n_2 f)]_{\text{quad}}$, $[D_f q(f, z, \cdot)(Af)]_{\text{quad}}$ and $[D_z q(f, z, \cdot)(n_2)]_{\text{quad}}$. Reasoning as for R_1 , the resolvent R_0 acts only on n , and thus commutes with D_f and D_z . Thus, Lemma 3.1 with $\lambda = 0$ proves that φ_2^q satisfies (28). Finally, the following estimates hold

$$\begin{aligned} |\varphi_2^q(f, z, n)| &\lesssim_\varphi (1 + \|f\|_{L^2(\mathcal{M}^{-1})}^2)(1 + \|n\|_E^{b+1}) \\ |\mathcal{L}_1 \varphi_2^q(f, z, n)| &\lesssim_\varphi (1 + \|f\|_{L^2(\mathcal{M}^{-1})}^3)(1 + \|n\|_E^{b+2}). \end{aligned}$$

This concludes the proof that φ_2 satisfies (28) and the proof of the estimates of Proposition 4.3 on φ_2 and $\mathcal{L}_1 \varphi_2$.

4.2.6 Good test function property

It only remains to prove that φ^ε is a good test function. The estimates (28) are satisfied by $\varepsilon \varphi_1$ and $\varepsilon^2 \varphi_2$, hence by their sum φ^ε . Moreover, using the notation introduced in Section 3.1.1, φ^ε can be written as

$$\begin{aligned} \varphi^\varepsilon(f, z, n) &= \varphi(\rho, z) - \varepsilon D\Psi(\rho)(\overline{Af})\chi(z) + \varepsilon R_0 [D\Psi(\rho)(\cdot\rho)](n)\chi(z) \\ &\quad + \varepsilon \Psi(\rho) R_0 [D\chi(z)](n) + \varepsilon^2 \varphi_2^c(f, z) + \varepsilon^2 R_1 \ell(f, z, n) \\ &\quad + \varepsilon^2 R_0 \left[q - \langle q \rangle_{\rho, z} \right] (f, z, n). \end{aligned}$$

Observe that each term either does not depend on n or can be written $R_\lambda \theta$ with θ as in Definition 2. As a consequence, owing to Assumption 9, any product of at most two of these terms belongs to $D(B)$. Thus, $\varphi^\varepsilon \in D(B)$ and $(\varphi^\varepsilon)^2 \in D(B)$. This concludes the proof that φ^ε is a good test function, and the proof of Proposition 4.3.

5 Dynamics associated with the limiting equation

In this section, we show that the operator \mathcal{L} is the generator of the limit equation (18) and that the martingale problem associated to \mathcal{L} characterizes the solution of (18).

Definition 5. Let $\rho_0 \in L_x^2$ and let $\sigma > 0$. A process (ρ, ζ) is said to be a weak solution to (18) in L_x^2 if the following assertions are satisfied

- (i) $\rho(0) = \rho_0$,
- (ii) $\rho \in L^\infty([0, T], L_x^2) \cap C([0, T], H_x^{-\sigma})$ a.s. and $\zeta \in C([0, T], C_x^1)$ a.s.,
- (iii) there exists $(B_i)_i$ a sequence of independent standard Brownian motions such that (ρ, ζ) is adapted to the filtration generated by $(B_i)_i$ and such that, for all $\xi \in L_x^2$ and $t \in [0, T]$, we have a.s.

$$\begin{aligned} (\rho(t), \xi)_{L_x^2} &= (\rho_0, \xi)_{L_x^2} + \int_0^t (\rho(s), \operatorname{div}(K\nabla\xi))_{L_x^2} ds + \int_0^t \left(\frac{1}{2}F\rho(s), \xi\right)_{L_x^2} ds \\ &\quad + \sum_i \sqrt{q_i} \int_0^t (F_i\rho(s), \xi)_{L_x^2} dB_i(s) \end{aligned} \quad (56)$$

$$\zeta(t) = \sum_i \sqrt{q_i} F_i B_i(t). \quad (57)$$

Note that the sum in (57) does converge in $C([0, T], C_x^1)$ owing to Lemma 3.3.

The solution to this equation exists and is unique in distribution. The existence can be proved using energy estimates, Itô formula and regularization argument. The uniqueness comes from pathwise uniqueness which derives from the same arguments. We do not give details concerning existence and uniqueness, however, in the proof of the following Proposition, we established the aforementioned energy estimate 59.

Proposition 5.1. *Let $\sigma > 0$ and let $(\rho, \zeta) \in C([0, T], H_x^{-\sigma}) \times C([0, T], C_x^1)$.*

If (ρ, ζ) is the weak solution to (18) in $H_x^{-\sigma}$, then, for any test function $\varphi \in \Theta$, the process

$$M_\varphi(t) = \varphi(\rho(t), \zeta(t)) - \varphi(\rho_0, 0) - \int_0^t \mathcal{L}\varphi(\rho(s), \zeta(s)) ds$$

is a martingale for the filtration generated by (ρ, ζ) .

Conversely, if for all $\varphi \in \Theta$, M_φ and M_{φ^2} are martingales, then (ρ, ζ) is the weak solution of (18) in $H_x^{-\sigma}$.

Proof. Let us first prove that \mathcal{L} is the generator associated to (18). The expression of $\mathcal{L}\varphi$ is given by (53). First note that $\overline{A^2\rho\mathcal{M}} = \operatorname{div}(K\nabla\rho)$, which is the first term of (18).

The third term of (53) is associated to the second term of (18):

$$\begin{aligned}\int D\Psi(\rho)(R_0(n)(n\rho))d\nu(n) &= \mathbb{E} \left[\int_0^\infty D\Psi(\rho)(\rho\bar{m}(0)\bar{m}(t))dt \right] \\ &= \frac{1}{2} \mathbb{E} \left[\int_{\mathbb{R}} D\Psi(\rho)(\rho\bar{m}(0)\bar{m}(t))dt \right] \\ &= \frac{1}{2} D\Psi(\rho)(\rho F).\end{aligned}$$

To rewrite the second term of (53), assume first that the bilinear form $D^2\Psi(\rho)$ on L_x^2 admits a kernel k_ρ . Then, we have

$$\begin{aligned}\int D^2\Psi(\rho)(n\rho, R_0(n)\rho)d\nu(n) &= \frac{1}{2} \mathbb{E} \left[\int_0^\infty D^2\Psi(\rho)(\rho\bar{m}(0), \rho\bar{m}(t))dt \right] \\ &= \frac{1}{2} \mathbb{E} \left[\int_0^\infty \int \int k_\rho(x, y) \rho(x) \bar{m}(0)(x) \rho(y) \bar{m}(t)(y) dx dy dt \right] \\ &= \frac{1}{2} \int \int k_\rho(x, y) k(x, y) \rho(x) \rho(y) dx dy.\end{aligned}$$

Owing to Mercer's Theorem (see [16]), the kernel k can be expressed in terms of the eigenvectors and eigenvalues of Q :

$$\forall x, y, k(x, y) = \sum_i q_i F_i(x) F_i(y).$$

It is straightforward to check that $k^{(1/2)}(x, y) = \sum_i q_i^{1/2} F_i(x) F_i(y)$, $x, y \in \mathbb{T}^d$, defines a kernel for $Q^{1/2}$ and satisfies $k(x, y) = \int k^{(1/2)}(x, z) k^{(1/2)}(y, z) dz$. Thus, we have

$$\begin{aligned}\int D^2\Psi(\rho)(n\rho, R_0(n)\rho)d\nu(n) &= \frac{1}{2} \int \int \int k_\rho(x, y) k^{(1/2)}(x, z) \rho(x) k^{(1/2)}(y, z) \rho(y) dx dy dz \\ &= \frac{1}{2} \text{Tr} \left[(\rho Q^{1/2}) D^2\Psi(\rho) (\rho Q^{1/2})^* \right].\end{aligned}\tag{58}$$

By density of the functions whose second derivative admits a kernel k_ρ in C^2 , this formula holds for all test functions $\varphi \in \Theta$. Using similar reasoning for the three remaining terms, we get

$$\begin{aligned}\mathcal{L}\varphi(\rho, \zeta) &= D\Psi(\rho)(\text{div}(K\nabla\rho) + \frac{1}{2}F\rho)\chi(\zeta) \\ &\quad + \frac{1}{2} \text{Tr} \left[(\rho Q^{1/2}, Q^{1/2}) \begin{pmatrix} D^2\Psi(\rho)\chi(\zeta) & D\Psi(\rho) \otimes D\chi(\zeta) \\ D\Psi(\rho) \otimes D\chi(\zeta) & \Psi(\rho) D^2\chi(\zeta) \end{pmatrix} (\rho Q^{1/2}, Q^{1/2})^* \right],\end{aligned}$$

which is the generator of (18). Once moment estimates for ρ have been obtained in L_x^2 , integrability of M_φ is ensured. In addition, estimates on $\varphi(\rho(t), \zeta(t))$ and $\mathcal{L}\varphi(\rho(t), \zeta(t))$ (uniformly in $t \in [0, T]$) are also obtained, since φ and $\mathcal{L}\varphi$ have at most quadratic growth. Then, the proof that M_φ is a martingale follows the same strategy as for the proof of

Proposition 4.1. This proof is omitted. It thus remains to prove the moment estimates for ρ .

We apply Itô's formula, equation (56) and we take the expectation (so that the martingale part vanishes), to get

$$\begin{aligned} \frac{1}{2} \mathbb{E} \left[(\rho(t), \xi)_{L_x^2}^2 \right] &= \frac{1}{2} \mathbb{E} \left[(\rho_0, \xi)_{L_x^2}^2 \right] + \mathbb{E} \int_0^t (\rho(s), \operatorname{div}(K \nabla \xi))_{L_x^2} (\rho(s), \xi)_{L_x^2} ds \\ &\quad + \mathbb{E} \int_0^t \left(\frac{1}{2} F \rho(s), \xi \right)_{L_x^2} (\rho(s), \xi)_{L_x^2} ds + \frac{1}{2} \sum_i q_i \mathbb{E} \int_0^t (F_i \rho(s), \xi)_{L_x^2}^2 ds. \end{aligned}$$

Then, we evaluate at $\xi = e_\ell$ with $\ell \in \mathbb{Z}^d$ and e_ℓ the Fourier basis $e_\ell(x) = \exp(2i\pi \ell \cdot x)$. Let $\lambda_\ell = 4\pi^2 \ell \cdot K \ell$ so that $\operatorname{div}(K \nabla e_\ell) = -\lambda_\ell e_\ell$. We sum this formula for $|\ell| \leq L$. Let P_L be the orthogonal projector on the space generated by $\{e_\ell \mid |\ell| \leq L\}$. Since $\lambda_\ell \geq 0$, we get

$$\begin{aligned} \frac{1}{2} \mathbb{E} \left[\|P_L \rho(t)\|_{L_x^2}^2 \right] &\leq \frac{1}{2} \mathbb{E} \left[\|P_L \rho_0\|_{L_x^2}^2 \right] + \mathbb{E} \int_0^t \frac{1}{2} \|P_L(F \rho(s))\|_{L_x^2} \|P_L \rho(s)\|_{L_x^2} ds \\ &\quad + \frac{1}{2} \sum_i \mathbb{E} \int_0^t q_i \|P_L(F_i \rho(s))\|_{L_x^2}^2 ds \\ &\leq \frac{1}{2} \mathbb{E} \left[\|P_L \rho_0\|_{L_x^2}^2 \right] + \frac{1}{2} \left(\|F\|_{L^\infty} + \sum_i q_i \|F_i\|_{L^\infty}^2 \right) \mathbb{E} \int_0^t \|\rho(s)\|_{L_x^2}^2 ds. \end{aligned}$$

Taking $L \rightarrow \infty$, using Lemma 3.3 and Gronwall's Lemma, we get

$$\mathbb{E} \left[\|\rho(t)\|_{L_x^2}^2 \right] \lesssim \mathbb{E} \left[\|\rho_0\|_{L_x^2}^2 \right]. \quad (59)$$

This concludes the proof of the moment estimates for ρ , hence the proof that M_φ is a martingale.

Conversely, assume that for all $\varphi \in \Theta$, M_φ and M_{φ^2} are martingales. It holds in particular for regular and bounded test functions φ . It is then standard that a solution to this martingale problem is the Markov process of generator \mathcal{L} (see for example chapter 4 of [14]), based on Lévy's martingale representation theorem in Hilbert spaces (see [5], Theorem 8.2). This concludes the proof since we already proved that \mathcal{L} is the generator associated to (18). \square

6 Tightness of the coupled stopped process

In this section, we prove the following Proposition.

Proposition 6.1. *Let $\Lambda \in (0, \infty)$. The family of processes $((\rho^\varepsilon, \tau_\Lambda(\zeta^\varepsilon)), \zeta^\varepsilon, \tau_\Lambda(\zeta^\varepsilon))_\varepsilon$ is tight in the space $C([0, T], H_x^{-\sigma}) \times C([0, T], C_x^1)$ for any $\sigma > 0$. Moreover, the family $(\zeta^\varepsilon)_\varepsilon$ is tight in $C([0, T], C_x^1)$.*

To simplify the notation, we write $C_T H_x^{-\sigma} \times C_T C_x^1$ for $C([0, T], H_x^{-\sigma}) \times C([0, T], C_x^1)$.

Owing to Slutsky's Lemma (see [2], Theorem 4.1) and to Lemma 3.4, Proposition 6.1 is equivalent to the tightness of $((\rho^{\varepsilon, \tau_\Lambda^\varepsilon}, \zeta^{\varepsilon, \tau_\Lambda^\varepsilon}))_\varepsilon$ and $(\zeta^{\varepsilon, \tau^\varepsilon})_\varepsilon$.

Since these processes are pathwise continuous, we have the following inequality between the modulus of continuity w for continuous functions and the modulus of continuity w' for càdlàg functions (see [2], equation 14.11):

$$w_X(\delta) \leq 2w'_X(\delta),$$

with, for a càdlàg function X ,

$$w_X(\delta) \doteq \sup_{0 \leq t \leq s \leq t + \delta \leq T} \|X(s) - X(t)\|$$

$$w'_X(\delta) \doteq \sup_{(t_i)_i} \max_i \sup_{t_i \leq t \leq s < t_{i+1}} \|X(s) - X(t)\|,$$

where $(t_i)_i$ is a subdivision of $[0, T]$. Therefore, the tightness in the Skorokhod space $D_T H_x^{-\sigma} \times D_T C_x^1$ (respectively $D_T C_x^1$) implies the tightness in $C_T H_x^{-\sigma} \times C_T C_x^1$ (respectively in $C_T C_x^1$).

Owing to Theorem 3.1. of [24], tightness in the Skorokhod space follows from the following claims, which are proved in Sections 6.1 and 6.2 respectively.

- (i) For all $\eta > 0$, there exists some compact sets $K_\eta \subset H_x^{-\sigma}$ and $K'_\eta \subset C_x^1$ such that for all $\varepsilon > 0$,

$$\mathbb{P}(\forall t \in [0, T], \rho^{\varepsilon, \tau_\Lambda^\varepsilon}(t) \in K_\eta) > 1 - \eta \quad (60)$$

$$\mathbb{P}(\forall t \in [0, T], \zeta^{\varepsilon, \tau_\Lambda^\varepsilon}(t) \in K'_\eta) > 1 - \eta \quad (61)$$

$$\mathbb{P}(\forall t \in [0, T], \zeta^{\varepsilon, \tau^\varepsilon}(t) \in K'_\eta) > 1 - \eta. \quad (62)$$

- (ii) If φ is a sum of a finite number of bounded functions $\varphi_i \in \Theta$, then $(\varphi(\rho^{\varepsilon, \tau_\Lambda^\varepsilon}, \zeta^{\varepsilon, \tau_\Lambda^\varepsilon}))_\varepsilon$ is tight in $D([0, T], \mathbb{R})$.

For any $\tilde{\varphi} \in \Theta$ with $\psi = 1$, $(\tilde{\varphi}(\zeta^{\varepsilon, \tau^\varepsilon}))_\varepsilon$ is tight in $D([0, T], \mathbb{R})$.

We ask of φ to be a finite sum of test functions because Theorem 3.1. of [24] requires the class of test functions to separate points and to be closed under addition, but Θ does not satisfies the latter condition.

6.1 Proof of the first claim (i)

Using Proposition 3.6 and the Markov inequality, we have for $K > 0$,

$$\mathbb{P}(\exists t \in [0, T], \|\rho^{\varepsilon, \tau_\Lambda^\varepsilon}(t)\|_{L_x^2} > K) \leq \frac{\mathbb{E} \left[\sup_{t \in [0, T]} \|\rho^{\varepsilon, \tau_\Lambda^\varepsilon}(t)\|_{L_x^2} \right]}{K} \lesssim_\Lambda \frac{\mathbb{E} \left[\|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})} \right]}{K}.$$

Note that stopping the processes at $\tau_\Lambda(\zeta^\varepsilon)$ is necessary at this point. Owing to the compact embedding $L_x^2 \subset H_x^{-\sigma}$ for $\sigma > 0$, we get (60).

Since (61) is a consequence of (62), it remains to prove (62). Owing to Ascoli's Theorem, we have a compact embedding of the Hölder space $C_x^{1,\delta} \subset C_x^1$ for any $\delta > 0$. Moreover, with $s = \lfloor d/2 \rfloor + 2$, we have a continuous embedding $H_x^s \subset C_x^{1,\delta}$ for any $\delta \in (0, s - \frac{d}{2} - 1]$. Then (62) is a consequence of Proposition 6.2 below and of the Markov inequality.

Proposition 6.2. *Recall that τ^ε is defined by (14). Then, for all $T > 0$, we have*

$$\sup_\varepsilon \mathbb{E} \left[\sup_{t \in [0, T]} \|\zeta^{\varepsilon, \tau^\varepsilon}(t)\|_{H_x^{\lfloor d/2 \rfloor + 2}}^2 \right] < \infty.$$

Proof. The idea of this proof is to express ζ^ε (and its derivatives) as a sum of a small term and a martingale, and then to estimate the martingale using Doob's Maximal Inequality. This argument is used two times in a row, and the estimates heavily rely on Assumptions 9 and 10.

Since $\zeta^\varepsilon(t) \in E = C_x^{2\lfloor d/2 \rfloor + 4} \subset H_x^{\lfloor d/2 \rfloor + 2}$, it is sufficient to prove that for all multi-indices β of length $|\beta| \leq \lfloor d/2 \rfloor + 2$, we have

$$\sup_\varepsilon \mathbb{E} \left[\sup_{t \in [0, T]} \left\| \frac{\partial^{|\beta|} \zeta^{\varepsilon, \tau^\varepsilon}(t)}{\partial x^\beta} \right\|_{L_x^2}^2 \right] < \infty.$$

Fix such a β and let $\varepsilon > 0$. First note that

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left\| \frac{\partial^{|\beta|} \zeta^{\varepsilon, \tau^\varepsilon}(t)}{\partial x^\beta} \right\|_{L_x^2}^2 \right] \leq \int \mathbb{E} \left[\sup_{t \in [0, T]} \left| \frac{\partial^{|\beta|} \zeta^{\varepsilon, \tau^\varepsilon}(t, x)}{\partial x^\beta} \right|^2 \right] dx. \quad (63)$$

For $x \in \mathbb{T}^d$, define $\theta_{x, \beta} \in E^*$ by

$$\forall n \in E, \theta_{x, \beta}(n) = \frac{\partial^{|\beta|} n}{\partial x^\beta}(x).$$

Since \bar{m}^ε is almost surely an E -valued càdlàg function, the derivative and the integral commute in the following computation:

$$\frac{\partial^{|\beta|} \zeta^\varepsilon(t, x)}{\partial x^\beta} = \frac{1}{\varepsilon} \int_0^t \frac{\partial^{|\beta|} \bar{m}^\varepsilon(s, x)}{\partial x^\beta} ds = \frac{1}{\varepsilon} \int_0^t \theta_{x, \beta}(\bar{m}^\varepsilon(s)) ds.$$

Owing to the identity $\langle \theta_{x, \beta} \rangle = 0$, Lemma 3.1 and Assumption 9, the function $\psi_x \doteq -R_0 \theta_{x, \beta}$ is well-defined, is Lipschitz continuous with $[\psi_x]_{\text{Lip}} \lesssim [\theta_{x, \beta}]_{\text{Lip}} = 1$ and $\psi_x, \psi_x^2 \in D(B)$. Therefore Proposition 4.2 states that

$$\begin{aligned} M_{\varepsilon \psi_x}^\varepsilon(t) &= \varepsilon \psi_x(\bar{m}^\varepsilon(t)) - \varepsilon \psi_x(\bar{m}(0)) - \frac{1}{\varepsilon^2} \int_0^t \varepsilon B \psi_x(\bar{m}^\varepsilon(s)) ds \\ &= \varepsilon \psi_x(\bar{m}^\varepsilon(t)) - \varepsilon \psi_x(\bar{m}(0)) - \frac{\partial^{|\beta|} \zeta^\varepsilon(t, x)}{\partial x^\beta} \end{aligned}$$

defines a square-integrable martingale such that

$$\mathbb{E} \left[\left| M_{\varepsilon\psi_x}^{\varepsilon, \tau^\varepsilon}(t) \right|^2 \right] = \mathbb{E} \left[\int_0^{t \wedge \tau^\varepsilon} (B(\psi_x^2) - 2\psi_x B\psi_x)(\bar{m}^\varepsilon(s)) ds \right]. \quad (64)$$

Since $[\psi_x]_{\text{Lip}} \lesssim [\theta_{x,\beta}]_{\text{Lip}} = 1$ and $\alpha < 1$ in (24), we have

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\varepsilon\psi_x(\bar{m}^{\varepsilon, \tau^\varepsilon}(t))|^2 \right] \lesssim \varepsilon^2 \mathbb{E} \left[\varepsilon^{-2\alpha} \vee \|\bar{m}(0)\|_E^2 \right] \lesssim 1,$$

and by Doob's Maximal Inequality, we get

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} \left| \frac{\partial^{|\beta|} \zeta^{\varepsilon, \tau^\varepsilon}(t, x)}{\partial x^\beta} \right|^2 \right]^{\frac{1}{2}} &\lesssim 1 + \mathbb{E} \left[\sup_{t \in [0, T]} \left| M_{\varepsilon\psi_x}^{\varepsilon, \tau^\varepsilon}(t) \right|^2 \right]^{\frac{1}{2}} \\ &\lesssim 1 + \mathbb{E} \left[\left| M_{\varepsilon\psi_x}^{\varepsilon, \tau^\varepsilon}(T) \right|^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (65)$$

Owing to Proposition 4.2, we have

$$\mathbb{E} \left[\left| M_{\varepsilon\psi_x}^{\varepsilon, \tau^\varepsilon}(T) \right|^2 \right] = \mathbb{E} \left[\int_0^{T \wedge \tau^\varepsilon} (B(\psi_x^2) - 2\psi_x B\psi_x)(\bar{m}^\varepsilon(s)) ds \right],$$

For now, we only know that the right-hand side is of order $\varepsilon^{-2\alpha}$, by (14) and (64). To retrieve an estimate uniform in ε , we use the same martingale argument as before. Let

$$\tilde{\theta}_{x,\beta} \doteq B(\psi_x^2) - 2\psi_x B\psi_x = B((R_0\theta_{x,\beta})^2) + 2\theta_{x,\beta} R_0\theta_{x,\beta},$$

so that

$$\mathbb{E} \left[\left| M_{\varepsilon\psi_x}^{\varepsilon, \tau^\varepsilon}(T) \right|^2 \right] = \mathbb{E} \left[\int_0^{T \wedge \tau^\varepsilon} \tilde{\theta}_{x,\beta}(\bar{m}^\varepsilon(s)) ds \right]. \quad (66)$$

Since $\theta_{x,\beta}$ and $R_0\theta_{x,\beta}$ are pseudo-linear functions, the function $\theta_{x,\beta} R_0\theta_{x,\beta}$ is pseudo-quadratic. Thus, by Lemma 3.1 and Assumption 9, the function

$$\tilde{\psi}_x = (R_0\theta_{x,\beta})^2 - 2R_0[\theta_{x,\beta} R_0\theta_{x,\beta} - \langle \theta_{x,\beta} R_0\theta_{x,\beta} \rangle],$$

is well-defined and satisfies $\tilde{\psi}_x, \tilde{\psi}_x^2 \in D(B)$ and $B\tilde{\psi}_x = \tilde{\theta}_{x,\beta} - 2\langle \theta_{x,\beta} R_0\theta_{x,\beta} \rangle$. As before, introduce the martingale process

$$\begin{aligned} M_{\varepsilon^2\tilde{\psi}_x}^\varepsilon(t) &= \varepsilon^2\tilde{\psi}_x(\bar{m}^\varepsilon(t)) - \varepsilon^2\tilde{\psi}_x(\bar{m}(0)) - \frac{1}{\varepsilon^2} \int_0^t \varepsilon^2 B\tilde{\psi}_x(\bar{m}^\varepsilon(s)) ds \\ &= \varepsilon^2\tilde{\psi}_x(\bar{m}^\varepsilon(t)) - \varepsilon^2\tilde{\psi}_x(\bar{m}(0)) - \int_0^t \tilde{\theta}_{x,\beta}(\bar{m}^\varepsilon(s)) ds + 2t \langle \theta_{x,\beta} R_0\theta_{x,\beta} \rangle. \end{aligned}$$

Owing to Lemma 3.1, we have

$$\begin{aligned}\forall n \in E, \left| \tilde{\psi}_x(n) \right| &\lesssim (1 + \|n\|_E^{b+1} + \|n\|_E^2) \\ \langle \theta_{x,\beta} R_0 \theta_{x,\beta} \rangle &\lesssim 1.\end{aligned}$$

Using the conditions $\alpha(b+1) < 2$ and $\alpha < 1$ in (24), and using the finiteness of moments of order $2(b+1)$ and 4 of $\bar{m}(0)$ in Assumption 7, we get

$$\mathbb{E} \left[\int_0^{T \wedge \tau^\varepsilon} \tilde{\theta}_{x,\beta}(\bar{m}^\varepsilon(s)) ds \right] \lesssim_T 1 + \mathbb{E} \left[\left| M_{\varepsilon^2 \tilde{\psi}_x}^{\varepsilon, \tau^\varepsilon}(T) \right| \right], \quad (67)$$

where, owing to Proposition 4.2,

$$\mathbb{E} \left[\left| M_{\varepsilon^2 \tilde{\psi}_x}^{\varepsilon, \tau^\varepsilon}(T) \right|^2 \right] = \varepsilon^2 \mathbb{E} \left[\int_0^{T \wedge \tau^\varepsilon} \left(B(\tilde{\psi}_x^2) - 2\tilde{\psi}_x B\tilde{\psi}_x \right) (\bar{m}^\varepsilon(s)) ds \right].$$

Owing to Assumption 10, we have

$$\forall n \in E, \left| \left(B(\tilde{\psi}_x^2) - 2\tilde{\psi}_x B\tilde{\psi}_x \right) (n) \right| \lesssim (1 + \|n\|_E^{2(b+1)} + \|n\|_E^4).$$

Since $\alpha(b+1) < 1$ and $2\alpha < 1$ in (24) and since $\bar{m}(0)$ has finite moments of order $2(b+1)$ and 4 in Assumption 7, we get

$$\mathbb{E} \left[\left| M_{\varepsilon^2 \tilde{\psi}_x}^{\varepsilon, \tau^\varepsilon}(T) \right|^2 \right] \lesssim_T 1. \quad (68)$$

Gathering the estimates (65), (66), (67) and (68), we obtain the required result

$$\sup_\varepsilon \sup_{x \in \mathbb{T}^d} \mathbb{E} \left[\sup_{t \in [0, T]} \left| \frac{\partial^{|\beta|} \zeta^{\varepsilon, \tau^\varepsilon}(t, x)}{\partial x^\beta} \right|^2 \right] \lesssim_T 1.$$

This concludes the proof by (63). \square

Proposition 6.2, together with the compact embedding $H_x^s \subset C_x^1$ and the Markov inequality, proves that (62) holds, hence (61). This concludes the proof of (i).

6.2 Proof of the second claim (ii)

As in [11], we prove (ii) using the Aldous criterion ([23], Theorem 4.5 p356).

Let $\varphi = \sum_i \varphi_i$ be the sum of a finite number of bounded functions $\varphi_i \in \Theta$. We set $X^\varepsilon = (f^\varepsilon, \zeta^\varepsilon, \bar{m}^\varepsilon)$ and $\bar{X}^\varepsilon = (\rho^\varepsilon, \zeta^\varepsilon)$. Recall that if $\tilde{\varphi} \in \Theta$ depends only on z , then the perturbed test function $\tilde{\varphi}^\varepsilon$ defined by Proposition 4.3 depends only on n and z . Using Proposition 4.2, this allows us to stop the processes only at τ^ε instead of τ_Λ^ε while keeping the same estimates. Therefore, the proof of the tightness of $(\tilde{\varphi}(\zeta^{\varepsilon, \tau^\varepsilon}))_\varepsilon$ is the same as of $(\varphi(\bar{X}^{\varepsilon, \tau_\Lambda^\varepsilon}))_\varepsilon$, and is thus omitted. It only remains to prove $(\varphi(\bar{X}^{\varepsilon, \tau_\Lambda^\varepsilon}))_\varepsilon$ is tight.

The Aldous criterion gives a sufficient condition for the tightness of the family $(\varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}))_{\varepsilon}$ in $D([0, T], \mathbb{R})$: since φ is bounded, is it sufficient to prove that

$$\forall \eta > 0, \lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \sup_{\substack{\tau_1, \tau_2 \leq T \\ \tau_1 \leq \tau_2 \leq \tau_1 + \delta}} \mathbb{P} \left(\left| \varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_2)) - \varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_1)) \right| > \eta \right) = 0, \quad (69)$$

where τ_1, τ_2 are any $(\mathcal{F}_t^{\varepsilon})_{t \in \mathbb{R}^+}$ -stopping times.

Define the perturbed test function $\varphi^{\varepsilon} = \sum_i \varphi_i^{\varepsilon}$. This sum satisfies the estimates (44) to (46). Then, define

$$\theta^{\varepsilon}(t) = \varphi(\overline{X}^{\varepsilon}(0)) + \varphi^{\varepsilon}(X^{\varepsilon}(t)) - \varphi^{\varepsilon}(X^{\varepsilon}(0)) \quad (70)$$

$$= \varphi(\overline{X}^{\varepsilon}(0)) + \int_0^t \mathcal{L}^{\varepsilon} \varphi^{\varepsilon}(X^{\varepsilon}(s)) ds + M_{\varphi^{\varepsilon}}^{\varepsilon}(t), \quad (71)$$

where $M_{\varphi^{\varepsilon}}^{\varepsilon}$ is defined by Proposition 4.1, so that

$$\begin{aligned} \varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_2)) - \varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_1)) &= (\theta^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_2) - \theta^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_1)) - \left(\varphi^{\varepsilon}(X^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_2)) - \varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_2)) \right) \\ &\quad + \left(\varphi^{\varepsilon}(X^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_1)) - \varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_1)) \right). \end{aligned}$$

Using (23), Propositions 3.6 and 4.3, we get

$$\begin{aligned} &\left| \varphi^{\varepsilon}(X^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(t)) - \varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(t)) \right| \\ &\lesssim_{\varphi, \Lambda} (1 + \|f_0^{\varepsilon}\|_{L^2(\mathcal{M}^{-1})}^2) (\varepsilon(1 + \varepsilon^{-\alpha} \vee \|\overline{m}(0)\|) + \varepsilon^2(1 + \varepsilon^{-\alpha(b+1)} \vee \|\overline{m}(0)\|^{b+1})). \end{aligned}$$

Since $\alpha < 1$ and $\alpha(b+1) < 2$ in (24), we get

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left| \varphi^{\varepsilon}(X^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(t)) - \varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(t)) \right| \right] \xrightarrow{\varepsilon \rightarrow 0} 0,$$

hence, when $\varepsilon \rightarrow 0$,

$$\sup_{\tau_1, \tau_2} \mathbb{E} \left[\left| \varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_2)) - \varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_1)) \right| \right] \leq \sup_{\tau_1, \tau_2} \mathbb{E} [|\theta^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_2) - \theta^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_1)|] + o(1).$$

Using the Markov inequality, we get

$$\sup_{\tau_1, \tau_2} \mathbb{P} \left(\left| \varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_2)) - \varphi(\overline{X}^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_1)) \right| > \eta \right) \leq \sup_{\tau_1, \tau_2} \frac{\mathbb{E} [|\theta^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_2) - \theta^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_1)|]}{\eta} + o(1).$$

Therefore, it is sufficient to prove that

$$\sup_{\tau_1, \tau_2, \varepsilon} \mathbb{E} [|\theta^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_2) - \theta^{\varepsilon, \tau_{\bar{\Lambda}}^{\varepsilon}}(\tau_1)|] \xrightarrow{\delta \rightarrow 0} 0, \quad (72)$$

to deduce (69) and then to use Aldous criterion.

Owing to (71), we have

$$|\theta^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(\tau_2) - \theta^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(\tau_1)| \leq \int_{\tau_1 \wedge \tau_{\Lambda}^{\varepsilon}}^{\tau_2 \wedge \tau_{\Lambda}^{\varepsilon}} |\mathcal{L}^{\varepsilon} \varphi^{\varepsilon}(X^{\varepsilon}(s))| ds + \left| M_{\varphi^{\varepsilon}}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(\tau_2) - M_{\varphi^{\varepsilon}}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(\tau_1) \right|. \quad (73)$$

Using once again (23), Propositions 3.6 and 4.3, we get

$$\begin{aligned} |\mathcal{L}^{\varepsilon} \varphi^{\varepsilon}(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t))| &\lesssim_{\varphi, \Lambda} \left| \mathcal{L} \varphi(\overline{X}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(s)) \right| + \varepsilon(1 + \|f_0^{\varepsilon}\|_{L^2(\mathcal{M}^{-1})}^3)(1 + \varepsilon^{-\alpha(b+2)} \vee \|\overline{m}(0)\|^{b+2}) \\ &\lesssim_{\varphi, \Lambda} 1 + \|f_0^{\varepsilon}\|_{L^2(\mathcal{M}^{-1})}^2 + \varepsilon(1 + \|f_0^{\varepsilon}\|_{L^2(\mathcal{M}^{-1})}^3)(1 + \varepsilon^{-\alpha(b+2)} \vee \|\overline{m}(0)\|^{b+2}). \end{aligned}$$

Using the Cauchy-Schwarz inequality, the condition $\alpha(b+2) < 1$ in (24), Assumptions 4 and 7, we get

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\mathcal{L}^{\varepsilon} \varphi^{\varepsilon}(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t))| \right] \lesssim_{\varphi, \Lambda} 1. \quad (74)$$

Thus, we get

$$\sup_{\varepsilon} \sup_{\tau_1, \tau_2} \mathbb{E} \left[\int_{\tau_1 \wedge \tau_{\Lambda}^{\varepsilon}}^{\tau_2 \wedge \tau_{\Lambda}^{\varepsilon}} |\mathcal{L}^{\varepsilon} \varphi^{\varepsilon}(X^{\varepsilon}(s))| ds \right] \leq \sup_{\varepsilon} \sup_{\tau_1, \tau_2} \delta \mathbb{E} \left[\sup_{t \in [0, T]} |\mathcal{L}^{\varepsilon} \varphi^{\varepsilon}(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t))| \right] \xrightarrow{\delta \rightarrow 0} 0.$$

The last term of (73) is controlled using martingale arguments. Owing to Proposition 4.1, $M_{\varphi^{\varepsilon}}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}$ is indeed a square-integrable martingale and

$$\begin{aligned} \mathbb{E} \left[\left| M_{\varphi^{\varepsilon}}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(\tau_2) - M_{\varphi^{\varepsilon}}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(\tau_1) \right|^2 \right] &= \mathbb{E} \left[\left| M_{\varphi^{\varepsilon}}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(\tau_2) \right|^2 - \left| M_{\varphi^{\varepsilon}}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(\tau_1) \right|^2 \right] \\ &= \frac{1}{\varepsilon^2} \mathbb{E} \left[\int_{\tau_1 \wedge \tau_{\Lambda}^{\varepsilon}}^{\tau_2 \wedge \tau_{\Lambda}^{\varepsilon}} (B((\varphi^{\varepsilon})^2) - 2\varphi^{\varepsilon} B\varphi^{\varepsilon})(X^{\varepsilon}(s)) ds \right] \\ &= \mathbb{E} \left[\int_{\tau_1 \wedge \tau_{\Lambda}^{\varepsilon}}^{\tau_2 \wedge \tau_{\Lambda}^{\varepsilon}} \sum_{i=-2}^2 \varepsilon^i r_i(X^{\varepsilon}(s)) ds \right] \end{aligned}$$

where the terms r_i are obtained by writing $\varphi^{\varepsilon} = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$ and expanding $B((\varphi^{\varepsilon})^2) - 2\varphi^{\varepsilon} B\varphi^{\varepsilon}$. The terms containing φ vanish, using $B\varphi = 0$, $B(\varphi^2) = 0$ and $B\varphi\varphi_j = \varphi B\varphi_j$ (since φ does not depend on n). Using Assumption 10, the remaining terms satisfy

$$\begin{aligned} r_{-2} &= r_{-1} = 0, \\ r_0(f, z, n) &= [B(\varphi_1^2) - 2\varphi_1 B\varphi_1](f, z, n) \lesssim_{\varphi} (1 + \|f\|_{L^2(\mathcal{M}^{-1})}^2)(1 + \|n\|_E^2), \\ r_1(f, z, n) &= [2B(\varphi_1\varphi_2) - \varphi_1 B\varphi_2 - \varphi_2 B\varphi_1](f, z, n) \lesssim_{\varphi} (1 + \|f\|_{L^2(\mathcal{M}^{-1})}^3)(1 + \|n\|_E^{b+2}), \\ r_2(f, z, n) &= [B(\varphi_2^2) - 2\varphi_2 B\varphi_2](f, z, n) \lesssim_{\varphi} (1 + \|f\|_{L^2(\mathcal{M}^{-1})}^4)(1 + \|n\|_E^{2(b+1)}). \end{aligned}$$

As for (74), using that $\alpha(b+2) < 1$ in (24), we have for $i \in \{1, 2\}$

$$\mathbb{E} \left[\sup_{t \in [0, T]} \varepsilon^i r_i(X^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)) \right] \lesssim_{\varphi, \Lambda} 1,$$

and

$$\sup_{\varepsilon} \sup_{\tau_1, \tau_2} \mathbb{E} \left[\int_{\tau_1 \wedge \tau_\Lambda^\varepsilon}^{\tau_2 \wedge \tau_\Lambda^\varepsilon} \varepsilon^i r_i(X^\varepsilon(s)) ds \right] \xrightarrow{\delta \rightarrow 0} 0.$$

We need to be more cautious when dealing with r_0 , since there are no ε left to compensate the $\varepsilon^{-2\alpha}$ that would appear from bounding $\overline{m}^{\varepsilon, \tau^\varepsilon}$ from above using Proposition 3.6. The idea is to use estimates for $f^{\varepsilon, \tau_\Lambda^\varepsilon}$ and \overline{m}^ε (instead of $\overline{m}^{\varepsilon, \tau_\Lambda^\varepsilon}$), using that for $s \leq \tau_\Lambda^\varepsilon$, $\overline{m}^{\varepsilon, \tau_\Lambda^\varepsilon}(s) = \overline{m}^\varepsilon(s)$. We write

$$\begin{aligned} \mathbb{E} \left[\int_{\tau_1 \wedge \tau_\Lambda^\varepsilon}^{\tau_2 \wedge \tau_\Lambda^\varepsilon} r_0(X^\varepsilon(s)) ds \right] &\lesssim_{\varphi} \mathbb{E} \left[\int_{\tau_1 \wedge \tau_\Lambda^\varepsilon}^{\tau_2 \wedge \tau_\Lambda^\varepsilon} (1 + \|f^{\varepsilon, \tau_\Lambda^\varepsilon}(s)\|_{L^2(\mathcal{M}^{-1})}^2)(1 + \|\overline{m}^\varepsilon(s)\|_E^2) ds \right] \\ &\lesssim_{\varphi, \Lambda} \mathbb{E} \left[\int_{\tau_1}^{\tau_2} (1 + \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^2)(1 + \|\overline{m}^\varepsilon(s)\|_E^2) ds \right] \\ &\lesssim_{\varphi, \Lambda} \int_0^T \mathbb{E} \left[\mathbb{1}_{[\tau_1, \tau_2]}(s) (1 + \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^2)(1 + \|\overline{m}^\varepsilon(s)\|_E^2) \right] ds. \end{aligned}$$

Then, we use the Hölder inequality to write

$$\begin{aligned} \mathbb{E} \left[\int_{\tau_1 \wedge \tau_\Lambda^\varepsilon}^{\tau_2 \wedge \tau_\Lambda^\varepsilon} r_0(X^\varepsilon(s)) ds \right] &\lesssim_{\varphi, \Lambda} \int_0^T \mathbb{E} \left[\mathbb{1}_{[\tau_1, \tau_2]}(s) \right]^{\frac{1}{3}} \mathbb{E} \left[1 + \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^6 \right]^{\frac{1}{3}} \mathbb{E} \left[1 + \|\overline{m}^\varepsilon(s)\|_E^6 \right]^{\frac{1}{3}} ds \\ &\lesssim_{\varphi, \Lambda} \int_0^T \mathbb{E} \left[\mathbb{1}_{[\tau_1, \tau_2]}(s) \right]^{\frac{1}{3}} ds \mathbb{E} \left[1 + \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^6 \right]^{\frac{1}{3}} \mathbb{E} \left[1 + \|\overline{m}^\varepsilon(0)\|_E^6 \right]^{\frac{1}{3}}, \end{aligned}$$

by stationarity of \overline{m} . Using the Cauchy-Schwarz inequality, Assumptions 4 and 7, we get

$$\begin{aligned} \mathbb{E} \left[\int_{\tau_1 \wedge \tau_\Lambda^\varepsilon}^{\tau_2 \wedge \tau_\Lambda^\varepsilon} r_0(X^\varepsilon(s)) ds \right] &\lesssim_{\varphi, \Lambda} \int_0^T \mathbb{E} \left[\mathbb{1}_{[\tau_1, \tau_2]}(s) \right]^{\frac{1}{2}} ds \\ &\lesssim_{\varphi, \Lambda, T} \left(\int_0^T \mathbb{E} \left[\mathbb{1}_{[\tau_1, \tau_2]}(s) \right] ds \right)^{\frac{1}{2}} \\ &\lesssim_{\varphi, \Lambda, T} \delta^{\frac{1}{2}} \rightarrow 0, \end{aligned}$$

uniformly in ε , τ_1 and τ_2 . This concludes the proof of (72).

We are now in position to apply Aldous' criterion, which proves that the family $((\rho^{\varepsilon, \tau_\Lambda(\zeta^\varepsilon)}, \zeta^{\varepsilon, \tau_\Lambda(\zeta^\varepsilon)})_\varepsilon)$ is tight in $C_T H_x^{-\sigma} \times C_T C_x^1$. This concludes the proof of (ii), and of Proposition 6.1.

7 Identification of the limit points

In this section, we establish the first convergence result stated in Theorem 2.1.

We start by proving the convergence of the auxiliary process ζ^ε in Section 7.1, using the convergence of a simplified martingale problem. Then, in Section 7.2, we determine the stopped martingale problem solved by a limit point of the stopped process. In Section 7.3, we use this stopped martingale to identify the limit point of the stopped process. We conclude on the convergence of the unstopped process in Section 7.4.

7.1 Convergence of the auxiliary process

Proving the convergence of ζ^ε is much simpler than for the coupled process \bar{X}^ε . Indeed, as seen in particular in Proposition 6.1, the only stopping time we need is τ^ε , and $\tau^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} +\infty$. Therefore, the convergence of martingale problems is a little more intricate than the proof used in [10], but it remains straightforward.

Proposition 7.1. *The process ζ^ε converges in distribution in $C_T C_x^1$ to a Wiener process of covariance Q when $\varepsilon \rightarrow 0$.*

Proof. Owing to the tightness established in Proposition 6.1, there exists a sequence $\varepsilon_i \xrightarrow{i \rightarrow \infty} 0$ and $\zeta \in C_T C_x^1$ such that ζ^{ε_i} converges in distribution to ζ when $i \rightarrow \infty$. We start by proving that ζ solves the martingale problem associated with the generator \mathcal{L} .

Let $\varphi \in \Theta$ with $\psi = 1$. Let $0 \leq s \leq s_1 \leq \dots \leq s_n \leq t$, let g be a continuous bounded function and for $z \in C_T C_x^1$, let $G(z) = g(z(s_1), \dots, z(s_n))$ and

$$\Phi(z) = \left(\varphi(z(t)) - \varphi(z(s)) - \int_s^t \mathcal{L}\varphi(z(u)) du \right) G(z).$$

Note that G and Φ are continuous and bounded on $C_T C_x^1$, so $\mathbb{E}[\Phi(\zeta^{\varepsilon_i})] \xrightarrow{i \rightarrow \infty} \mathbb{E}[\Phi(\zeta)]$.

Let us establish that $\mathbb{E}[\Phi(\zeta^{\varepsilon_i})]$ also converges to 0.

Let φ^{ε_i} be the perturbed test function introduced in Proposition 4.3 associated to φ . Since φ^{ε_i} is a good test function, and since $G(\zeta^{\varepsilon_i, \tau^{\varepsilon_i}})$ is $\mathcal{F}_s^{\varepsilon_i}$ -measurable, Proposition 4.1 yields

$$\mathbb{E} \left[\left(\varphi^{\varepsilon_i}(\zeta^{\varepsilon_i, \tau^{\varepsilon_i}}(t)) - \varphi^{\varepsilon_i}(\zeta^{\varepsilon_i, \tau^{\varepsilon_i}}(s)) - \int_{s \wedge \tau^{\varepsilon_i}}^{t \wedge \tau^{\varepsilon_i}} \mathcal{L}^{\varepsilon_i} \varphi^{\varepsilon_i}(\zeta^{\varepsilon_i}(u)) du \right) G(\zeta^{\varepsilon_i, \tau^{\varepsilon_i}}) \right] = 0.$$

Owing to (38), this leads to

$$|\mathbb{E}[\Phi(\zeta^{\varepsilon_i, \tau^{\varepsilon_i}})]| \lesssim_g \sum_{j=1}^4 \mathbb{E}[|r_j|],$$

with

$$\begin{aligned} r_1 &= \varepsilon_i (\varphi_1(\zeta^{\varepsilon_i, \tau^{\varepsilon_i}}(t), \bar{m}^{\varepsilon_i, \tau^{\varepsilon_i}}(t)) - \varphi_1(\zeta^{\varepsilon_i, \tau^{\varepsilon_i}}(s), \bar{m}^{\varepsilon_i, \tau^{\varepsilon_i}}(s))), \\ r_2 &= \varepsilon_i^2 (\varphi_2(\zeta^{\varepsilon_i, \tau^{\varepsilon_i}}(t), \bar{m}^{\varepsilon_i, \tau^{\varepsilon_i}}(t)) - \varphi_2(\zeta^{\varepsilon_i, \tau^{\varepsilon_i}}(s), \bar{m}^{\varepsilon_i, \tau^{\varepsilon_i}}(s))), \\ r_3 &= - \int_{s \wedge \tau^{\varepsilon_i}}^{t \wedge \tau^{\varepsilon_i}} (\mathcal{L}^{\varepsilon_i} \varphi^{\varepsilon_i}(\zeta^{\varepsilon_i}(u)) - \mathcal{L}\varphi(\zeta^{\varepsilon_i}(u))) du, \\ r_4 &= \int_{t \wedge \tau^{\varepsilon_i}}^t \mathcal{L}\varphi(\zeta^{\varepsilon_i, \tau^{\varepsilon_i}}(u)) du - \int_{s \wedge \tau^{\varepsilon_i}}^s \mathcal{L}\varphi(\zeta^{\varepsilon_i, \tau^{\varepsilon_i}}(u)) du. \end{aligned}$$

Using (24), (44), (45), (46), Assumptions 4 and 7, we have for $j \in \{1, 2, 3\}$, $\mathbb{E}[|r_j|] \xrightarrow{\varepsilon \rightarrow 0} 0$. It remains to prove that $\mathbb{E}[|r_4|] \rightarrow 0$. The term r_4 does not appear in [10], but is simple to manage since $\tau^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \infty$. The Cauchy-Schwarz inequality and Lemma 3.4 lead to

$$\begin{aligned} \mathbb{E}[|r_4|]^2 &\lesssim_\varphi \mathbb{E} \left[|t - t \wedge \tau^{\varepsilon_i}|^2 + |s - s \wedge \tau^{\varepsilon_i}|^2 \right] \\ &\lesssim_\varphi T^2 \mathbb{P}(\tau^{\varepsilon_i} < T) \xrightarrow{i \rightarrow \infty} 0 \end{aligned}$$

Thus, we get $\mathbb{E}[\Phi(\zeta^{\varepsilon_i, \tau^{\varepsilon_i}})] \xrightarrow{i \rightarrow \infty} 0$, hence $\mathbb{E}[\Phi(\zeta)] = 0$. The same proof can be adapted when replacing φ by φ^2 . Therefore, the processes M_φ and M_{φ^2} defined in Proposition 5.1 are martingales. Owing to Proposition 5.1, ζ satisfies (57) and is a Q -Wiener process.

This limit point being unique in distribution, ζ^ε converges in distribution to this Wiener process. \square

7.2 Convergence of the stopped martingale problems

In this section, we use Proposition 7.1 to establish the convergence of the stopped martingale problems satisfied by $X^{\varepsilon, \tau_\Lambda^\varepsilon}$. The proof is similar to the proof of Proposition 7.1, but this time the stopping time persists when $\varepsilon \rightarrow 0$ because of the fixed threshold Λ .

Let us introduce the path space $\underline{\Omega} = C_T H_x^{-\sigma} \times C_T C_x^1 \times C_T C_x^1$, equipped with its Borel σ -algebra. We denote by $(\underline{\rho}, \underline{\zeta}, \underline{\zeta}')$ the canonical process on $\underline{\Omega}$ and by $(\underline{\mathcal{F}}_t)_{t \in \mathbb{R}^+}$ its associated filtration.

Define $\mathbb{P}_{\varepsilon, \Lambda}$ the distribution of $(\rho^{\varepsilon, \tau_\Lambda^\varepsilon}, \zeta^{\varepsilon, \tau_\Lambda^\varepsilon}, \zeta^\varepsilon)$ and $\mathbb{E}_{\varepsilon, \Lambda}$ the expectation under this distribution (on $\underline{\Omega}$). By Proposition 6.1, the family $(\mathbb{P}_{\varepsilon, \Lambda})_\varepsilon$ is tight. Thus, in this section, we consider a sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ such that $\varepsilon_i \rightarrow 0$ and $\mathbb{P}_{\varepsilon_i, \Lambda} \rightarrow \mathbb{P}_{0, \Lambda}$ weakly when $i \rightarrow \infty$, for some limit point $\mathbb{P}_{0, \Lambda}$. Note that under $\mathbb{P}_{0, \Lambda}$, owing to Proposition 7.1, $\underline{\zeta}'$ is a Q -Wiener process whose distribution \mathbb{P}_Q does not depend on Λ .

We now state two continuity lemmas.

Lemma 7.2. *For any fixed $\Lambda \in \mathbb{R}^+$, the mapping $\tau_\Lambda(\cdot)$ defined by (15) is lower semi-continuous on $C_T C_x^1$. Moreover, it is continuous at every z such that $\tau(z)$ is continuous at Λ .*

Lemma 7.3. *The set $\{\Lambda \geq 0 \mid \mathbb{P}_Q(\tau(\underline{\zeta}') \text{ is not continuous at } \Lambda) > 0\}$ is at most countable. Let \mathfrak{L} be its complementary.*

We refer to [22] (Lemma 3.5, 3.6 and Appendix) for the proofs of Lemmas 7.2 and 7.3. These results can be applied here since $\|\underline{\zeta}'\|_{C_x^1}$ is a continuous finite dimensional process and its distribution \mathbb{P}_Q under $\mathbb{P}_{0, \Lambda}$ does not depend on Λ .

Owing to Lemma 7.3, there exist arbitrarily large numbers $\Lambda \in \mathfrak{L}$ and for all $\Lambda \in \mathfrak{L}$, $\tau(\underline{\zeta}')$ is $\mathbb{P}_{0, \Lambda}$ -almost surely continuous at Λ and by Lemma 7.2, $\tau_\Lambda(\cdot)$ is $\mathbb{P}_{0, \Lambda}$ -a.s. continuous at $\underline{\zeta}'$. From now on, it is assumed that $\Lambda \in \mathfrak{L}$.

Proposition 7.4. *Let $\Lambda \in \mathfrak{L}$. For all $\varphi \in \Theta$, the process*

$$t \mapsto \varphi(\underline{\rho}(t), \underline{\zeta}(t)) - \varphi(\underline{\rho}(0), \underline{\zeta}(0)) - \int_0^{t \wedge \tau_\Lambda(\underline{\zeta}')} \mathcal{L}\varphi(\underline{\rho}(u), \underline{\zeta}(u)) du$$

is a $(\underline{\mathcal{F}}_t)_{t \in \mathbb{R}^+}$ -martingale under $\mathbb{P}_{0, \Lambda}$.

Proof. Let $\varphi \in \Theta$. As for Proposition 7.1, let $0 \leq s \leq s_1 \leq \dots \leq s_n \leq t$, let g be a continuous bounded function, and let

$$G(\underline{\rho}, \underline{\zeta}, \underline{\zeta}') = g(\underline{\rho}(s_1), \underline{\zeta}(s_1), \underline{\zeta}'(s_1), \dots, \underline{\rho}(s_n), \underline{\zeta}(s_n), \underline{\zeta}'(s_n)),$$

and

$$\Phi(\underline{\rho}, \underline{\zeta}, \underline{\zeta}') = \left(\varphi(\underline{\rho}(t), \underline{\zeta}(t)) - \varphi(\underline{\rho}(s), \underline{\zeta}(s)) - \int_{s \wedge \tau_\Lambda(\underline{\zeta}')}^{t \wedge \tau_\Lambda(\underline{\zeta}')} \mathcal{L}\varphi(\underline{\rho}(u), \underline{\zeta}(u)) du \right) G(\underline{\rho}, \underline{\zeta}, \underline{\zeta}').$$

As for Proposition 7.1, we establish that $\mathbb{E}_{\varepsilon_i, \Lambda}[\Phi(\underline{\rho}, \underline{\zeta}, \underline{\zeta}')]]$ converges, when $i \rightarrow \infty$, to both $\mathbb{E}_{0, \Lambda}[\Phi(\underline{\rho}, \underline{\zeta}, \underline{\zeta}')]]$ and 0.

On the one hand, since Φ is continuous $\mathbb{P}_{0, \Lambda}$ -almost everywhere, $\mathbb{P}_{\varepsilon_i, \Lambda} \circ \Phi^{-1} \rightarrow \mathbb{P}_{0, \Lambda} \circ \Phi^{-1}$ weakly when $i \rightarrow \infty$ (see [4] Proposition IX.5.7). Moreover, $(\mathbb{P}_{\varepsilon_i, \Lambda} \circ \Phi^{-1})_{\varepsilon_i}$ is uniformly integrable. Indeed, using (53), we have

$$\sup_{\varepsilon} \mathbb{E}_{\varepsilon, \Lambda} \left[|\Phi(\underline{\rho}, \underline{\zeta}, \underline{\zeta}')|^2 \right] \lesssim_{T, \Lambda, \varphi, g} \sup_{\varepsilon} \mathbb{E} \left[1 + \|f_0^\varepsilon\|_{L^2(\mathcal{M}^{-1})}^4 \right] < \infty.$$

Uniform integrability and convergence in distribution yield (see [2], Theorem 5.4)

$$\mathbb{E}_{\varepsilon_i, \Lambda}[\Phi(\underline{\rho}, \underline{\zeta}, \underline{\zeta}')] \xrightarrow{i \rightarrow \infty} \mathbb{E}_{0, \Lambda}[\Phi(\underline{\rho}, \underline{\zeta}, \underline{\zeta}')] .$$

On the other hand, define the perturbed test function φ^{ε_i} as in Proposition 4.3. As for Proposition 7.1, we have

$$\mathbb{E} \left[\left(\varphi^{\varepsilon_i}(X^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}(t)) - \varphi^{\varepsilon_i}(X^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}(s)) - \int_{s \wedge \tau_\Lambda^{\varepsilon_i}}^{t \wedge \tau_\Lambda^{\varepsilon_i}} \mathcal{L}^{\varepsilon_i} \varphi^{\varepsilon_i}(X^{\varepsilon_i}(u)) du \right) G(\underline{\rho}^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}, \underline{\zeta}^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}, \underline{\zeta}^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}) \right] = 0,$$

and

$$|\mathbb{E}_{\varepsilon_i, \Lambda}[\Phi(\underline{\rho}, \underline{\zeta}, \underline{\zeta}')] | = \left| \mathbb{E} \left[\Phi(\underline{\rho}^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}, \underline{\zeta}^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}, \underline{\zeta}^{\varepsilon_i}) \right] \right| \lesssim_g \sum_{j=1}^4 \mathbb{E} [|r_j|],$$

with

$$\begin{aligned}
r_1 &= \varepsilon_i(\varphi_1(X^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}(t)) - \varphi_1(X^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}(s))) \rightarrow 0 \\
r_2 &= \varepsilon_i^2(\varphi_2(X^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}(t)) - \varphi_2(X^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}(s))) \rightarrow 0 \\
r_3 &= - \int_{s \wedge \tau_\Lambda^{\varepsilon_i}}^{t \wedge \tau_\Lambda^{\varepsilon_i}} \left(\mathcal{L}^{\varepsilon_i} \varphi^{\varepsilon_i}(X^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}(u)) - \mathcal{L} \varphi(\bar{X}^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}(u)) \right) du \rightarrow 0 \\
r_4 &= \int_{t \wedge \tau_\Lambda^{\varepsilon_i}}^{t \wedge \tau_\Lambda(\zeta^{\varepsilon_i})} \mathcal{L} \varphi(\bar{X}^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}(u)) du - \int_{s \wedge \tau_\Lambda^{\varepsilon_i}}^{s \wedge \tau_\Lambda(\zeta^{\varepsilon_i})} \mathcal{L} \varphi(\bar{X}^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}(u)) du.
\end{aligned}$$

For the last term r_4 , we have

$$\begin{aligned}
\mathbb{E}[|r_4|]^2 &\lesssim_{\varphi, \Lambda} \mathbb{E} \left[|t \wedge \tau_\Lambda(\zeta^{\varepsilon_i}) - t \wedge \tau_\Lambda^{\varepsilon_i}|^2 + |s \wedge \tau_\Lambda(\zeta^{\varepsilon_i}) - s \wedge \tau_\Lambda^{\varepsilon_i}|^2 \right] \\
&\lesssim_{\varphi, \Lambda} T^2 \mathbb{P}(\tau^{\varepsilon_i} < T \wedge \tau_\Lambda(\zeta^{\varepsilon_i})) \quad \text{using (17)} \\
&\lesssim_{\varphi, \Lambda} T^2 \mathbb{P}(\tau^{\varepsilon_i} < T) \xrightarrow{i \rightarrow \infty} 0.
\end{aligned}$$

Thus, we get $\mathbb{E}_{\varepsilon_i, \Lambda}[\Phi(\underline{\rho}, \underline{\zeta}, \underline{\zeta}')] \xrightarrow{i \rightarrow \infty} 0$, which concludes the proof of Proposition 7.4. \square

7.3 Identification of the limit point

In Section 7.1, solving the martingale problem is sufficient to characterize the distribution of the Markov process as a solution of a limit equation, under a uniqueness condition. However, the limit point $\mathbb{P}_{0, \Lambda}$ solves a martingale problem only until a stopping time $\tau_\Lambda(\zeta')$. The goal of this section is to explain how to identify $\mathbb{P}_{0, \Lambda}$ using this stopped martingale problem.

Let us come back to the space Ω to state more precise results. Recall that the distribution of $(\rho^{\varepsilon, \tau_\Lambda^{\varepsilon}}, \zeta^{\varepsilon, \tau_\Lambda^{\varepsilon}}, \zeta^{\varepsilon})$ is $\mathbb{P}_{\varepsilon, \Lambda}$, and define $(\rho_\Lambda, \zeta_\Lambda, \zeta')$ following the limit distribution $\mathbb{P}_{0, \Lambda}$ (we assume Ω is large enough to define such a process). Recall that $\bar{X}^{\varepsilon, \tau_\Lambda^{\varepsilon}} \doteq (\rho^{\varepsilon, \tau_\Lambda^{\varepsilon}}, \zeta^{\varepsilon, \tau_\Lambda^{\varepsilon}})$. Define $\bar{X}_\Lambda \doteq (\rho_\Lambda, \zeta_\Lambda)$ and \bar{X} a solution of (18).

In this section, we construct a process Y_Λ that extends \bar{X}_Λ after the stopping time $\tau_\Lambda(\zeta')$ (in distribution) and that solves the martingale problem associated to \mathcal{L} . It is similar to the proof of Theorem 6.1.2 in [34], but we adapt this proof to see precisely how $\tau_\Lambda(\zeta')$ is linked to the extended process.

Extension after a stopping time We first need a result to assert that $\tau_\Lambda(\zeta')$ is a hitting time for \bar{X}_Λ . Note that until here, we did not use ζ^{ε} when considering the coupled process $(\rho^{\varepsilon}, \zeta^{\varepsilon})$. But had we considered ρ^{ε} alone, the stopping time $\tau_\Lambda(\zeta')$ would not be a hitting time for ρ_Λ (as a matter of fact, $\tau_\Lambda(\zeta')$ is not even a stopping time for the filtration generated by ρ_Λ).

Lemma 7.5. *Let $\Lambda \in \mathfrak{L}$.*

The processes ζ_Λ and $(\zeta')^{\tau_\Lambda(\zeta')}$ are indistinguishable. In particular, $\tau_\Lambda(\zeta_\Lambda) = \tau_\Lambda(\zeta')$. Moreover, the processes ρ_Λ and $\rho_\Lambda^{\tau_\Lambda(\zeta')}$ are indistinguishable.

This result was expected, given the construction of the stopping times and the fact that ζ_Λ and ζ' are the limit of the same process, respectively with and without a stopping time. The choice $\Lambda \in \mathfrak{L}$ is here necessary to retrieve this result by taking the limit $\varepsilon \rightarrow 0$.

Proof. Since $\tau^{\varepsilon_i} \rightarrow \infty$ in probability by Lemma 3.4, Slutsky's Lemma yields the following convergence in distribution

$$(\zeta^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}, \zeta^{\varepsilon_i, \tau^{\varepsilon_i}}, \zeta^{\varepsilon_i}) \xrightarrow{i \rightarrow \infty} (\zeta_\Lambda, \zeta', \zeta')$$

Now, for $z_1, z_2, z_3 \in C_T C_x^1$, let

$$\Phi(z_1, z_2, z_3) = \left\| z_1 - z_2^{\tau_\Lambda(z_3)} \right\|_{C_T C_x^1}.$$

Owing to Lemma 7.2, the mapping Φ is almost surely continuous at $(\zeta_\Lambda, \zeta', \zeta')$. Thus $\Phi(\zeta^{\varepsilon_i, \tau_\Lambda^{\varepsilon_i}}, \zeta^{\varepsilon_i, \tau^{\varepsilon_i}}, \zeta^{\varepsilon_i}) = 0$ converges in distribution to $\Phi(\zeta_\Lambda, \zeta', \zeta')$. Hence, we have almost surely $\zeta_\Lambda = (\zeta')^{\tau_\Lambda(\zeta')}$.

The proof for ρ_Λ uses similar arguments with $\Phi(\rho, z) = \left\| \rho - \rho^{\tau_\Lambda(z)} \right\|_{C_T H_x^{-\sigma}}$. \square

From now on, for any process $Y = (\rho, \zeta)$, we write $\tau_\Lambda(Y) = \tau_\Lambda(\zeta)$ so that $\tau_\Lambda(\overline{X}_\Lambda) = \tau_\Lambda(\zeta') \in [0, \infty]$. We shorten the notation to $\tau_\Lambda \doteq \tau_\Lambda(\overline{X}_\Lambda)$. Introduce the measurable function S_Λ that stops a process at the level Λ , namely $S_\Lambda(Y) = Y^{\tau_\Lambda(Y)}$. Owing to Lemma 7.5, we have $S_\Lambda(\overline{X}_\Lambda) = \overline{X}_\Lambda$.

This section is devoted to *extend* \overline{X}_Λ after τ_Λ into a solution of the martingale problem associated to \mathcal{L} . Namely, we define a process Y_Λ such that $S_\Lambda(Y_\Lambda) \stackrel{d}{=} \overline{X}_\Lambda$ and such that Y_Λ solves the aforementioned martingale problem.

Fix $\omega' \in \Omega$. Define the process $\overline{X}_{\Lambda, \omega'}$ as follows:

- $\forall \omega \in \Omega, \forall t \leq \tau_\Lambda(\omega'), \overline{X}_{\Lambda, \omega'}(t)(\omega) = \overline{X}_\Lambda(t)(\omega')$. Note that $\tau_\Lambda(\overline{X}_{\Lambda, \omega'}) = \tau_\Lambda(\omega')$ almost surely. In particular, the distribution of $S_\Lambda(\overline{X}_{\Lambda, \omega'})$ is the Dirac distribution at $\overline{X}_\Lambda(\omega')$.
- On $[\tau_\Lambda(\omega'), T]$ (this interval can be empty), $\overline{X}_{\Lambda, \omega'}(\omega)$ is the solution of (18) starting at time $\tau_\Lambda(\omega')$ from the initial state $\overline{X}_\Lambda(\tau_\Lambda(\omega'))(\omega')$.

It is straightforward to check that

$$\omega' \mapsto \mathbb{P}(\overline{X}_{\Lambda, \omega'} \in \mathcal{C})$$

is measurable for $\mathcal{C} = \{Y \in C_T H_x^{-\sigma} \times C_T C_x^1 \mid Y(t_1) \in \Gamma_1, \dots, Y(t_n) \in \Gamma_n\}$ with $0 \leq t_1 < \dots < t_n \leq T$ and Γ_i measurable. Since those sets generate the Borel σ -algebra of $C_T H_x^{-\sigma} \times C_T C_x^1$, and since a pointwise limit of measurable functions is measurable, we can take the limit when the subdivision become thinner to get that the mapping is still measurable for any measurable \mathcal{C} . Thus, we can define a mapping $\mathcal{C} \mapsto \mathbb{E}'\mathbb{P}(\overline{X}_{\Lambda, \omega'} \in \mathcal{C})$, where \mathbb{E}' denotes the integration with respect to ω' . It is also straightforward to check that this

mapping is a probability measure, thus we can define on Ω a process Y_Λ following this distribution, namely

$$\mathbb{P}(Y_\Lambda \in \mathcal{C}) = \mathbb{E}'\mathbb{P}(\bar{X}_{\Lambda, \omega'} \in \mathcal{C}).$$

In particular, since $S_\Lambda^{-1}(\mathcal{C})$ is a measurable set, we have

$$\begin{aligned} \mathbb{P}(S_\Lambda(Y_\Lambda) \in \mathcal{C}) &= \mathbb{E}'\mathbb{P}(S_\Lambda(\bar{X}_{\Lambda, \omega'}) \in \mathcal{C}) \\ &= \mathbb{E}'\mathbb{1}_{\{\bar{X}_{\Lambda, \omega'} \in \mathcal{C}\}} \\ &= \mathbb{P}(\bar{X}_\Lambda \in \mathcal{C}), \end{aligned}$$

hence Y_Λ extends \bar{X}_Λ as announced beforehand, in the sense that $S_\Lambda(Y_\Lambda) \stackrel{d}{=} \bar{X}_\Lambda$. Moreover, for any measurable function Φ such that $\mathbb{E}'\mathbb{E}[|\Phi(\bar{X}_{\Lambda, \omega'})|] < \infty$, we have

$$\mathbb{E}[\Phi(Y_\Lambda)] = \mathbb{E}'\mathbb{E}[\Phi(\bar{X}_{\Lambda, \omega'})]. \quad (75)$$

Identification of the extended process It remains to prove that Y_Λ solves the martingale problem associated to \mathcal{L} .

For $\varphi \in \Theta$, and a process $Y \in C_T H_x^{-\sigma} \times C_T C_x^1$, define the process

$$M^Y(t) = \varphi(Y(t)) - \varphi(Y(0)) - \int_0^t \mathcal{L}\varphi(Y(u))du.$$

Let $0 \leq s_1 \leq \dots \leq s_n \leq s < t$ and g be a bounded measurable function. Let $G : Y \mapsto g(Y(s_1), \dots, Y(s_n))$.

Owing to Proposition 5.1, for almost all $\omega' \in \Omega$, the process

$$N_{\Lambda, \omega'}(t) = M^{\bar{X}_{\Lambda, \omega'}}(t) - M^{\bar{X}_{\Lambda, \omega'}}(t \wedge \tau_\Lambda(\omega'))$$

satisfies the martingale property

$$\mathbb{E}[N_{\Lambda, \omega'}(t)G(\bar{X}_{\Lambda, \omega'})] = \mathbb{E}[N_{\Lambda, \omega'}(s)G(\bar{X}_{\Lambda, \omega'})].$$

Indeed, for $t \in [0, \tau(\omega')]$, $N_{\Lambda, \omega'}(t) = 0$ and after the time $\tau(\omega')$, this process solves the martingale problem starting at time $\tau(\omega')$ by construction. Using (59) and that φ and $\mathcal{L}\varphi$ have at most quadratic growth, it is straightforward to establish

$$\mathbb{E}'\mathbb{E}[|N_{\Lambda, \omega'}(t)G(\bar{X}_{\Lambda, \omega'})|] < \infty.$$

Thus, (75) and the identity above yield

$$\mathbb{E}[(M^{Y_\Lambda}(t) - M^{Y_\Lambda}(t \wedge \tau_\Lambda(Y_\Lambda)))G(Y_\Lambda)] = \mathbb{E}[(M^{Y_\Lambda}(s) - M^{Y_\Lambda}(s \wedge \tau_\Lambda(Y_\Lambda)))G(Y_\Lambda)],$$

which can be rewritten as

$$\begin{aligned} \mathbb{E}[M^{Y_\Lambda}(t)G(Y_\Lambda)] &= \\ &\mathbb{E}[M^{Y_\Lambda}(s)\mathbb{1}_{\{\tau_\Lambda(Y_\Lambda) \leq s\}}G(Y_\Lambda)] + \mathbb{E}[M^{Y_\Lambda}(t \wedge \tau_\Lambda(Y_\Lambda))\mathbb{1}_{\{\tau_\Lambda(Y_\Lambda) > s\}}G(Y_\Lambda)]. \end{aligned} \quad (76)$$

Using that the process Y_Λ and $S_\Lambda(Y_\Lambda)$ are equal until the time $\tau_\Lambda(Y_\Lambda) = \tau_\Lambda(S_\Lambda(Y_\Lambda))$, and that $S_\Lambda(Y_\Lambda)$ and \bar{X}_Λ are equal in distribution, we get for the second term

$$\begin{aligned} \mathbb{E} \left[M^{Y_\Lambda}(t \wedge \tau_\Lambda(Y_\Lambda)) \mathbb{1}_{\{\tau_\Lambda(Y_\Lambda) > s\}} G(Y_\Lambda) \right] \\ &= \mathbb{E} \left[M^{S_\Lambda(Y_\Lambda)}(t \wedge \tau_\Lambda(S_\Lambda(Y_\Lambda))) \mathbb{1}_{\{\tau_\Lambda(S_\Lambda(Y_\Lambda)) > s\}} G(S_\Lambda(Y_\Lambda)) \right] \\ &= \mathbb{E} \left[M^{\bar{X}_\Lambda}(t \wedge \tau_\Lambda) \mathbb{1}_{\{\tau_\Lambda > s\}} G(\bar{X}_\Lambda) \right]. \end{aligned}$$

Owing to Proposition 7.4, $t \mapsto M^{\bar{X}_\Lambda}(t \wedge \tau_\Lambda)$ is a martingale for the filtration $\mathcal{F}^{\bar{X}_\Lambda}$ generated by \bar{X}_Λ . Moreover $\mathbb{1}_{\{\tau_\Lambda > s\}} G(\bar{X}_\Lambda)$ is $\mathcal{F}_s^{\bar{X}_\Lambda}$ -measurable, hence the martingale property yields

$$\mathbb{E} \left[M^{\bar{X}_\Lambda}(t \wedge \tau_\Lambda) \mathbb{1}_{\{\tau_\Lambda > s\}} G(\bar{X}_\Lambda) \right] = \mathbb{E} \left[M^{\bar{X}_\Lambda}(s) \mathbb{1}_{\{\tau_\Lambda > s\}} G(\bar{X}_\Lambda) \right].$$

Using again that $S_\Lambda(Y_\Lambda) \stackrel{d}{=} \bar{X}_\Lambda$, we get

$$\mathbb{E} \left[M^{\bar{X}_\Lambda}(s) \mathbb{1}_{\{\tau_\Lambda > s\}} G(\bar{X}_\Lambda) \right] = \mathbb{E} \left[M^{Y_\Lambda}(s) \mathbb{1}_{\{\tau_\Lambda(Y_\Lambda) > s\}} G(Y_\Lambda) \right].$$

Finally, owing to (76), we have

$$\begin{aligned} \mathbb{E} \left[M^{Y_\Lambda}(t) G(Y_\Lambda) \right] &= \mathbb{E} \left[M^{Y_\Lambda}(s) \mathbb{1}_{\{\tau_\Lambda(Y_\Lambda) \leq s\}} G(Y_\Lambda) \right] + \mathbb{E} \left[M^{Y_\Lambda}(s) \mathbb{1}_{\{\tau_\Lambda(Y_\Lambda) > s\}} G(Y_\Lambda) \right] \\ &= \mathbb{E} \left[M^{Y_\Lambda}(s) G(Y_\Lambda) \right], \end{aligned}$$

which proves that Y_Λ solves the martingale associated to \mathcal{L} . Owing to Proposition 5.1, it solves (18) and since the solution is unique $Y_\Lambda \stackrel{d}{=} \bar{X}$ the solution of (18). Therefore, the limit point is unique (and does not depend on Λ). This concludes the proof that $X^{\varepsilon, \tau^\varepsilon}$ converges in distribution to \bar{X} .

7.4 Convergence of the unstopped process

This section is devoted to the proof that the process $\bar{X}^\varepsilon \doteq (\rho^\varepsilon, \zeta^\varepsilon)$ converges in distribution to $\bar{X} \doteq (\rho, \zeta)$ solution of (18), in $C_T H_x^{-\sigma} \times C_T C_x^1$.

Let Φ be a continuous bounded mapping from $C_T H_x^{-\sigma} \times C_T C_x^1$ to \mathbb{R} . There exists a sequence ε_i such that $\varepsilon_i \rightarrow 0$ when $i \rightarrow \infty$ and

$$\limsup_{\varepsilon \rightarrow 0} |\mathbb{E} [\Phi(\bar{X}^\varepsilon)] - \mathbb{E} [\Phi(\bar{X})]| = \lim_{i \rightarrow \infty} |\mathbb{E} [\Phi(\bar{X}^{\varepsilon_i})] - \mathbb{E} [\Phi(\bar{X})]|.$$

Let $\Lambda \in \mathfrak{L}$. Owing to Proposition 6.1, up to the extraction of another subsequence, we can assume that $(\bar{X}^{\varepsilon_i, \tau^{\varepsilon_i}}, \zeta^{\varepsilon_i})$ converges in distribution to some (\bar{X}_Λ, ζ) in $(C_T H^{-\sigma} \times C_T C_x^1) \times C_T C_x^1$. Now we write

$$\begin{aligned} |\mathbb{E} [\Phi(\bar{X}^{\varepsilon_i})] - \mathbb{E} [\Phi(\bar{X})]| &\leq \left| \mathbb{E} [\Phi(\bar{X}^{\varepsilon_i})] - \mathbb{E} \left[\Phi(\bar{X}^{\varepsilon_i, \tau_\Lambda(\zeta^{\varepsilon_i})}) \right] \right| \\ &\quad + \left| \mathbb{E} \left[\Phi(\bar{X}^{\varepsilon_i, \tau_\Lambda(\zeta^{\varepsilon_i})}) \right] - \mathbb{E} [\Phi(\bar{X})] \right|. \end{aligned}$$

First, we have

$$\left| \mathbb{E} [\Phi(\bar{X}^{\varepsilon_i})] - \mathbb{E} [\Phi(\bar{X}^{\varepsilon_i, \tau_\Lambda(\zeta^{\varepsilon_i})})] \right| \lesssim_{\Phi} \mathbb{P}(\tau_\Lambda(\zeta^{\varepsilon_i}) \leq T).$$

By Lemmas 7.2 and 7.3, since $\Lambda \in \mathcal{L}$, $\tau_\Lambda(\zeta^{\varepsilon_i}) \wedge 2T$ converges in distribution to $\tau_\Lambda(\zeta') \wedge 2T$. Then, by Portmanteau's Theorem for closed sets, we have

$$\limsup_i \mathbb{P}(\tau_\Lambda(\zeta^{\varepsilon_i}) \leq T) \leq \mathbb{P}(\tau_\Lambda(\zeta') \leq T).$$

Since Φ is a continuous bounded function, we have

$$\lim_i \left| \mathbb{E} [\Phi(\bar{X}^{\varepsilon_i, \tau_\Lambda(\zeta^{\varepsilon_i})})] - \mathbb{E} [\Phi(\bar{X})] \right| = \left| \mathbb{E} [\Phi(\bar{X}_\Lambda)] - \mathbb{E} [\Phi(\bar{X})] \right|.$$

Recall that $\bar{X}_\Lambda \stackrel{d}{=} S_\Lambda(Y_\Lambda)$, and that $Y_\Lambda \stackrel{d}{=} \bar{X}$ (by Section 7.3). Thus, we get

$$\begin{aligned} \left| \mathbb{E} [\Phi(\bar{X}_\Lambda)] - \mathbb{E} [\Phi(\bar{X})] \right| &= \left| \mathbb{E} [\Phi(S_\Lambda(Y_\Lambda))] - \mathbb{E} [\Phi(Y_\Lambda)] \right| \\ &\lesssim_{\Phi} \mathbb{P}(\tau_\Lambda(Y_\Lambda) \leq T). \end{aligned}$$

Since $\tau_\Lambda(Y_\Lambda) \stackrel{d}{=} \tau_\Lambda(\bar{X}_\Lambda) = \tau_\Lambda(\zeta')$ by Lemma 7.5, we finally get for $\Lambda \in \mathcal{L}$

$$\limsup_{\varepsilon \rightarrow 0} \left| \mathbb{E} [\Phi(\bar{X}^\varepsilon)] - \mathbb{E} [\Phi(\bar{X})] \right| \lesssim_{\Phi} \mathbb{P}(\tau_\Lambda(\zeta') \leq T).$$

Since $\zeta' \in C_T C_x^1$, we have $\mathbb{P}(\tau_\Lambda(\zeta') \leq T) \xrightarrow{\Lambda \rightarrow \infty} 0$. Recall that we can take this limit since \mathcal{L} contains arbitrarily large Λ 's. Therefore, we have

$$\mathbb{E} [\Phi(\bar{X}^\varepsilon)] \xrightarrow{\varepsilon \rightarrow 0} \mathbb{E} [\Phi(\bar{X})].$$

This concludes the proof that \bar{X}^ε converges in distribution to \bar{X} , and in particular that ρ^ε converges in distribution to ρ in $C_T H_x^{-\sigma}$.

8 Strong convergence

In this section, we establish the second convergence result stated in Theorem 2.1, namely the convergence in $L_T^2 L_x^2$. Given Section 7 and Proposition 6.1, it is sufficient to prove that the sequence $(\rho^{\varepsilon, \tau_\Lambda^\varepsilon})_{\varepsilon > 0}$ is tight in $L_T^2 L_x^2$.

Recall that w_ρ denotes the modulus of continuity of a $H_x^{-\sigma}$ -valued continuous process ρ . Then, using Theorem 5 in [33], the set

$$K_R \doteq \left\{ \rho \in L_T^2 L_x^2 \mid \|\rho\|_{L_T^2 H_x^{\sigma'}} \leq R \text{ and } \forall \delta > 0, w_\rho(\delta) < \eta(\delta) \right\}$$

where $R > 0$, $\sigma' > 0$ and $\eta(\delta) \xrightarrow{\delta \rightarrow 0} 0$, is compact in $L_T^2 L_x^2$. Using Prokhorov's Theorem, the tightness of $(\rho^{\varepsilon, \tau_\Lambda^\varepsilon})_{\varepsilon > 0}$ in $L_T^2 L_x^2$ will follow if we prove that for all $\eta > 0$, there exists $R > 0$ and $\sigma' > 0$ such that

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathbb{P} \left(w_{\rho^{\varepsilon, \tau_\Lambda^\varepsilon}}(\delta) > \eta \right) = 0, \quad (77)$$

and

$$\sup_{\varepsilon} \mathbb{P} \left(\|\rho^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}\|_{L_T^2 H_x^{\sigma'}} > R \right) < \eta. \quad (78)$$

Equation (77) is a direct consequence of (69). It remains to prove (78). Owing to the Markov Inequality, it is sufficient to prove that, for some $\sigma' > 0$, we have

$$\sup_{\varepsilon} \mathbb{E} \left[\|\rho^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}\|_{L_T^2 H_x^{\sigma'}} \right] \lesssim_{\Lambda} 1. \quad (79)$$

Let $g^{\varepsilon} = \varepsilon \partial_t f^{\varepsilon} + a(v) \cdot \nabla_x f^{\varepsilon}$. Owing to Assumption 3, we can use an averaging lemma (Theorem 2.3 in [3] with $f(t) = f^{\varepsilon}(\varepsilon t)$, $g(t) = g^{\varepsilon}(\varepsilon t)$ and $h = 0$ until the time $T \wedge \tau_{\Lambda}^{\varepsilon}$) and by rescaling the time, we get

$$\begin{aligned} \|\rho^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}\|_{L_T^2 H_x^{1/4}}^2 &= \int_0^{T \wedge \tau_{\Lambda}^{\varepsilon}} \|\rho^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)\|_{H_x^{\sigma^*/4}}^2 dt \\ &\lesssim \varepsilon \|f_0^{\varepsilon}\|_{L_x^2}^2 + \int_0^{T \wedge \tau_{\Lambda}^{\varepsilon}} \|f^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)\|_{L^2(\mathcal{M}^{-1})}^2 dt + \int_0^{T \wedge \tau_{\Lambda}^{\varepsilon}} \|g^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)\|_{L^2(\mathcal{M}^{-1})}^2 dt, \end{aligned}$$

where, using the Cauchy-Schwarz inequality,

$$\begin{aligned} \|g^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)\|_{L^2(\mathcal{M}^{-1})} &= \left\| f^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t) \overline{m}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t) + \frac{1}{\varepsilon} L f^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t) \right\|_{L^2(\mathcal{M}^{-1})} \\ &\leq \|f^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)\|_{L^2(\mathcal{M}^{-1})} \|\overline{m}^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)\|_{L_x^2} + \frac{1}{\varepsilon} \|L f^{\varepsilon, \tau_{\Lambda}^{\varepsilon}}(t)\|_{L^2(\mathcal{M}^{-1})}. \end{aligned}$$

Then Assumption 4, (23), and Proposition 3.6 lead to (79) with $\sigma' = \frac{\sigma^*}{4}$. Since the sets K_R are compacts, Prokhorov's Theorem yields, using (77) and (78), that $(\rho^{\varepsilon, \tau_{\Lambda}^{\varepsilon}})_{\varepsilon > 0}$ is tight in $L_T^2 L_x^2$.

Given Section 7, this concludes the proof of the convergence in distribution of ρ^{ε} in $L_T^2 L_x^2$ to ρ the solution of (18).

References

- [1] Alain Bensoussan, Jacques-L. Lions, and George C. Papanicolaou. Boundary layers and homogenization of transport processes. *Publ. Res. Inst. Math. Sci.*, 15(1):53–157, 1979.
- [2] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
- [3] F. Bouchut and L. Desvillettes. Averaging lemmas without time Fourier transform and application to discretized kinetic equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 129(1):19–36, 1999.

- [4] Nicolas Bourbaki. *Integration. II. Chapters 7–9*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004. Translated from the 1963 and 1969 French originals by Sterling K. Berberian.
- [5] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*, volume 152 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2014.
- [6] A. de Bouard and M. Gazeau. A diffusion approximation theorem for a nonlinear PDE with application to random birefringent optical fibers. *Ann. Appl. Probab.*, 22(6):2460–2504, 2012.
- [7] Anne de Bouard and Arnaud Debussche. The nonlinear Schrödinger equation with white noise dispersion. *J. Funct. Anal.*, 259(5):1300–1321, 2010.
- [8] Arnaud Debussche, Angelo Rosello, and Julien Vovelle. Diffusion-approximation for a kinetic spray-like system with random forcing. *arXiv preprint arXiv:2005.09374*, 2020.
- [9] Arnaud Debussche and Yoshio Tsutsumi. 1D quintic nonlinear Schrödinger equation with white noise dispersion. *J. Math. Pures Appl. (9)*, 96(4):363–376, 2011.
- [10] Arnaud Debussche and Julien Vovelle. Diffusion limit for a stochastic kinetic problem. *Commun. Pure Appl. Anal.*, 11(6):2305–2326, 2012.
- [11] Arnaud Debussche and Julien Vovelle. Diffusion-approximation in stochastically forced kinetic equations. *Tunisian Journal of Mathematics*, 3(1):1–53, 2020.
- [12] P. Degond, T. Goudon, and F. Poupaud. Diffusion limit for nonhomogeneous and non-micro-reversible processes. *Indiana Univ. Math. J.*, 49(3):1175–1198, 2000.
- [13] Nelson Dunford and Jacob T Schwartz. *Linear Operators: Spectral Theory*, volume 2. Interscience Publishers, 1964.
- [14] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986. Characterization and convergence.
- [15] Lawrence C Evans. The perturbed test function method for viscosity solutions of nonlinear pde. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 111(3-4):359–375, 1989.
- [16] JC Ferreira and VA Menegatto. Eigenvalues of integral operators defined by smooth positive definite kernels. *Integral Equations and Operator Theory*, 64(1):61–81, 2009.
- [17] Jean-Pierre Fouque, Josselin Garnier, George Papanicolaou, and Knut Sørensen. *Wave propagation and time reversal in randomly layered media*, volume 56 of *Stochastic Modelling and Applied Probability*. Springer, New York, 2007.

- [18] F. Golse and F. Poupaud. Limite fluide des équations de Boltzmann des semi-conducteurs pour une statistique de Fermi-Dirac. *Asymptotic Anal.*, 6(2):135–160, 1992.
- [19] Thierry Goudon and Mathias Rousset. Stochastic acceleration in an inhomogeneous time random force field. *Applied Mathematics Research Express*, 2009(1):1–46, 2009.
- [20] R. Z. Has'minskiĭ. A limit theorem for solutions of differential equations with a random right hand part. *Teor. Veroyatnost. i Primenen.*, 11:444–462, 1966.
- [21] R. Z. Has'minskiĭ. Stochastic processes defined by differential equations with a small parameter. *Teor. Veroyatnost. i Primenen.*, 11:240–259, 1966.
- [22] Martina Hofmanová and Jan Seidler. On weak solutions of stochastic differential equations. *Stoch. Anal. Appl.*, 30(1):100–121, 2012.
- [23] Jean Jacod and Albert N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.
- [24] Adam Jakubowski. On the Skorokhod topology. *Ann. Inst. H. Poincaré Probab. Statist.*, 22(3):263–285, 1986.
- [25] Harold J. Kushner. *Approximation and weak convergence methods for random processes, with applications to stochastic systems theory*, volume 6 of *MIT Press Series in Signal Processing, Optimization, and Control*. MIT Press, Cambridge, MA, 1984.
- [26] Edward W. Larsen and Joseph B. Keller. Asymptotic solution of neutron transport problems for small mean free paths. *J. Mathematical Phys.*, 15:75–81, 1974.
- [27] Pierre Louis Lions and Giuseppe Toscani. Diffusive limit for finite velocity Boltzmann kinetic models. *Rev. Mat. Iberoamericana*, 13(3):473–513, 1997.
- [28] Renaud Marty et al. On a splitting scheme for the nonlinear schrödinger equation in a random medium. *Communications in Mathematical Sciences*, 4(4):679–705, 2006.
- [29] G. C. Papanicolaou, D. Stroock, and S. R. S. Varadhan. Martingale approach to some limit theorems. pages ii+120 pp. Duke Univ. Math. Ser., Vol. III, 1977.
- [30] E. Pardoux and A. L. Piatnitski. Homogenization of a nonlinear random parabolic partial differential equation. *Stochastic Process. Appl.*, 104(1):1–27, 2003.
- [31] Grigoris Pavliotis and Andrew Stuart. *Multiscale methods: averaging and homogenization*. Springer Science & Business Media, 2008.
- [32] Frédéric Poupaud and Alexis Vasseur. Classical and quantum transport in random media. *Journal de mathématiques pures et appliquées*, 82(6):711–748, 2003.

- [33] Jacques Simon. Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.
- [34] Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.