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Colin Guillarmou, Antti Kupiainen, Rémi Rhodes, Vincent Vargas. CONFORMAL BOOTSTRAP IN LIOUVILLE THEORY. 2020. hal-02866510v1

HAL Id: hal-02866510

<https://hal.science/hal-02866510v1>

Preprint submitted on 12 Jun 2020 (v1), last revised 24 Nov 2020 (v2)

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CONFORMAL BOOTSTRAP IN LIOUVILLE THEORY

COLIN GUILLARMOU, ANTTI KUPIAINEN, RÉMI RHODES, AND VINCENT VARGAS

ABSTRACT. Liouville conformal field theory (denoted LCFT) is a 2-dimensional conformal field theory depending on a parameter $\gamma \in \mathbb{R}$ and studied since the eighties in theoretical physics. In the case of the theory on the 2-sphere, physicists proposed closed formulae for the n -point correlation functions using symmetries and representation theory, called the DOZZ formula (for $n = 3$) and the conformal bootstrap (for $n > 3$). In a recent work, the three last authors introduced with F. David a probabilistic construction of LCFT for $\gamma \in (0, 2]$ and proved the DOZZ formula for this construction. In this sequel work, we give the first mathematical proof that the probabilistic construction of LCFT on the 2-sphere is equivalent to the conformal bootstrap for $\gamma \in (0, \sqrt{2})$. Our proof combines the analysis of a natural semi-group, tools from scattering theory and the use of the Virasoro algebra in the context of the probabilistic approach (the so-called conformal Ward identities).

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1. INTRODUCTION

1.1. Overview. Conformal Field Theory in dimension 2. There are essentially two approaches to Quantum Field Theory (QFT) in the physics literature. In the first approach the quantum fields are (generalized) functions $\hat{\mathcal{O}}(t, x)$ on the space-time \mathbb{R}^{d+1} ($d = 3$ for the Standard Model of physics) taking values in operators acting in a Hilbert space \mathcal{H} of physical states. Matrix elements of products of field operators at different points $\langle \psi | \prod_{k=1}^N \hat{\mathcal{O}}_k(t_k, x_k) | \psi \rangle$ where $\psi \in \mathcal{H}$ (the “vacuum” state) are (generalized) functions on $\mathbb{R}^{N(d+1)}$. Physical principles (positivity of energy) imply that these matrix elements should have an analytic continuation to the *Euclidean* domain where $t_k \in i\mathbb{R}$ and be given there as correlation functions of *random fields* $\mathcal{O}(y)$ defined on $y \in \mathbb{R}^n$ where $n = d + 1$. In the second approach, based on a path integral and due to Feynman, these correlation functions are formally given as integrals over a space of (generalized) functions on \mathbb{R}^n with the formal integration measure given explicitly in terms of an action functional of the fields. The Euclidean formulation serves also as a setup of the theory of second order phase transitions in statistical mechanics systems where now $n \leq 3$. In this case one expects the correlation functions to possess an additional symmetry under the conformal transformations of \mathbb{R}^n and the QFT is now a Conformal Field Theory (CFT).

In practice most of the information on QFT obtained by the physicists has been perturbative and given in terms of a formal power series expansion in parameters perturbing a Gaussian measure (and pictorially described by Feynman diagrams). In CFT however there is another, nonperturbative, approach going under the name of *Conformal bootstrap*. In this approach one postulates a set of special *primary* fields $\mathcal{O}_\alpha(y)$ (or operators $\mathcal{O}_\alpha(t, x)$ in the Hilbert space formulation) whose correlation functions transform as tensors under the conformal group. Furthermore one postulates a rule called the *operator product expansion* allowing to expand a product of two primary fields inside a correlation function (or a product of two operators in the Hilbert space formulation) as a sum of primary fields with explicit coefficients depending on the three point correlation functions, the so called *structure constants* of the CFT.

In the case of two dimensional conformal field theories ($d = 1$ or $n = 2$ above) the conformal symmetry constrains possible CFT’s particularly strongly and Belavin, Polyakov and Zamolodchikov [BPZ84] (BPZ from now on) showed the power of the bootstrap approach by producing explicit expressions for the correlation functions of a large family of CFT’s of interest to statistical physics among them the Ising model. In a nutshell, BPZ argued that one could parametrize CFT’s by a unique parameter c called the central charge and they found the correlation functions for certain rational values of c where the number of primary fields is finite (the minimal models). During the last decade the bootstrap approach has also led to spectacular predictions of critical exponents in the physically interesting three dimensional case [PoRyVi].

Giving a rigorous mathematical meaning to these two approaches and relating them has been a huge challenge for mathematicians. On an axiomatic level the transition from the operator theory on Hilbert space to the Euclidean probabilistic theory was understood early on and for the converse the crucial concept of *reflection positivity* was isolated [OsSh73, OsSh75]. Reflection positivity is a property of the probability law underlying the random fields that allows for a construction of a canonical Hilbert space where operators representing the symmetries of the theory act. Reflection positivity is one of the crucial inputs in the present paper.

However on a more concrete level of explicit examples of QFT’s mathematical progress has been slower. The (Euclidean) path integral approach was addressed by constructive field theory in dimensions $d + 1 \leq 4$

using probabilistic methods but detailed information has been restricted to the cases that are small perturbations of a Gaussian measure. In particular the 2d CFT's have been beyond this approach so far. A different probabilistic approach to conformal invariance has been developed during the past twenty years following the introduction by Schramm [Sch00] of random curves called Schramm-Loewner evolution (SLE). This approach, centred around the geometric description of critical models of statistical physics, has led to exact statements on the interfaces of percolation or the critical Ising model; following the introduction of SLE and the work of Smirnov, probabilists also managed to justify and construct the CFT correlation functions of the scaling limit of the 2d Ising model [ChSm12, CHI15] (see also the review [Pel19] for the construction of CFT correlations via SLE observables).

Making a mathematical theory of the BPZ approach triggered in the 80's and 90's intense research in the field of Vertex Operator Algebras introduced by Borcherds [Bo86] and Frenkel-Lepowsky-Meurman [FLM89] (see also the book [Hu97] and the article [HuKo07] for more recent developments on this formalism). Even if the theory of vertex operator algebras was quite successful to rigorously formalize numerous CFTs, the approach suffers certain limitations at the moment. First, correlations are defined as formal power series (convergence issues are not tackled in the first place and are often difficult); second, many fundamental CFTs have still not been formalized within this approach, among which the CFTs with uncountable collections of primary fields and in particular Liouville conformal field theory (LCFT in short) studied in this paper. Moreover, the theory of Vertex Operator Algebras, which is based on axiomatically implementing the operator product expansion point of view of physics, does not elucidate the link to the the path integral approach or to the models of statistical physics at critical temperature (if any).

In their seminal work, BPZ were in fact motivated by the quest to compute the correlations in LCFT. LCFT had been introduced previously by Polyakov under the form of a path integral in his approach to bosonic string theory [Po81]. Since then, LCFT has appeared in the physics literature in a wide range of fields including random planar maps (discrete quantum gravity, see the review [Ko11]) and the supersymmetric Yang-Mills theory (via the AGT correspondence [AGT10]). Recently, there has been a large effort in probability theory to make sense of Polyakov's path integral formulation of LCFT within the framework of random conformal geometry and the scaling limit of random planar maps: see [LeG13, Mier13, MS20a, MS20b, MS20c, DiDuDuFa, DuFaGwPfs, MiGw] for the construction of a random metric space describing (at least at the conjectural level) the limit of random planar maps and [DMS14, NS] for exact results on their link with LCFT¹. In particular, the three last authors of the present paper in collaboration with F. David [DKRV16] have constructed the path integral formulation of LCFT on the Riemann sphere using probability theory. This was extended to higher genus surfaces in [DRV16, GRV19]. In this paper, we will be concerned with LCFT on the Riemann sphere.

LCFT depends on two parameters $\gamma \in (0, 2)$ and $\mu > 0$ ². In this paper we prove that for $\gamma \in (0, \sqrt{2})$ the probabilistic construction of LCFT matches the bootstrap construction envisioned in [BPZ84]. The proof is based on the spectral analysis of a self-adjoint unbounded operator, the *Hamiltonian* of LCFT, defined on the canonical Hilbert space of LCFT obtained from reflection positivity³. The Hamiltonian is defined for all $\gamma \in (0, 2)$. For $\gamma \in (0, \sqrt{2})$ it is of the form of a Schrödinger operator acting in the L^2 -space of an infinite dimensional measure space. It has a non trivial potential term which is a positive function for $\gamma \in (0, \sqrt{2})$. For $\gamma \in [\sqrt{2}, 2)$ the potential is no more a measurable function and this case requires essential novel ideas (note however that the Hamiltonian still is defined and self adjoint). Although the range $\gamma \in (0, \sqrt{2})$ is not totally satisfactory⁴, LCFT in this range of γ is a highly nontrivial CFT with an uncountable family of primary fields and a nontrivial OPE and we believe the proof of conformal bootstrap in this case provides a

¹It is beyond the scope of this introduction to state and comment all the exciting results that have been obtained recently in this flourishing field of probability theory.

²The non interacting case $\mu = 0$ which corresponds to Gaussian free field theory has been extensively studied by Kang-Makarov [KM13].

³The importance of understanding the spectral analysis of the Hamiltonian of LCFT was stressed by Teschner in [Te01] which was an inspiration for us.

⁴The authors of [GRSS20] informed us of an ongoing project to extend their results on the torus conformal blocks to the case of the sphere; provided this extension is performed, Theorem 1.1 below will also hold in the case $\gamma = \sqrt{2}$ by a continuity argument. In the framework of statistical physics, this corresponds to the uniform spanning tree model.

highly nontrivial test case where a mathematical justification for this beautiful idea from physics has been achieved. Also, the methodology developed in this paper will enable to prove a similar statement for LCFT on the torus [GKRV].

Probabilistic approach of Liouville CFT. LCFT, which depends on two parameters $\gamma \in (0, 2)$ and $\mu > 0$ (called cosmological constant), is the theory of a distribution valued random field ϕ defined on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The physically and mathematically interesting fields in LCFT are exponentials of the so-called Liouville field ϕ , also called “vertex operators” $V_\alpha(z) = e^{\alpha\phi(z)}$ in physics, and their n -point correlation functions $\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_{\gamma, \mu}$. Here, the points $z_i \in \mathbb{C}$ are distinct, $\alpha_i \in \mathbb{C}$ and $\langle \cdot \rangle_{\gamma, \mu}$ denotes an expectation (or average) with respect to a measure defined on an appropriate functional space. One specific feature of LCFT on the sphere is that the underlying measure of the theory has infinite mass, i.e. $\langle 1 \rangle_{\gamma, \mu} = \infty$. In the physics literature, the formal path integral definition of the correlations for distinct $z_i \in \mathbb{C}$ and $\alpha_i \in \mathbb{C}$ is given by

$$(1.1) \quad \left\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \right\rangle_{\gamma, \mu} := \int_{\phi: \mathbb{C} \rightarrow \mathbb{R}} \left(\prod_{i=1}^n e^{\alpha_i \phi(z_i)} \right) e^{-S_L(\phi)} \mathbb{D}\phi,$$

where

- $\mathbb{D}\phi$ denotes the formal “Lebesgue measure” on the space of functions $\phi: \mathbb{C} \rightarrow \mathbb{R}$ obeying the asymptotics $\phi(x) \underset{x \rightarrow \infty}{\sim} -2Q \ln|x|$ with $Q = \frac{\gamma}{2} + \frac{\mu}{2}$.
- the Liouville action is given by: $S_L(\phi) := \frac{1}{4\pi} \int_{\mathbb{C}} (|\nabla\phi(x)|^2 + 4\pi\mu e^{\gamma\phi(x)}) dx$, where dx is the standard flat Lebesgue measure and ∇ the standard (flat) gradient.

LCFT is an *interacting* field theory because of the non quadratic term $4\pi\mu e^{\gamma\phi(x)}$ in the action. This interaction term is related to the uniformization of Riemann surfaces, as the classical action $S_L(\phi)$ has critical points leading to metrics with constant Gaussian curvature. Since the Lebesgue measure on infinite dimensional space is ill defined, the measure $e^{-S_L(\phi)} \mathbb{D}\phi$ and the formula (1.1) for the correlations are formal.

The recent work [DKRV16] gives a rigorous mathematical meaning to the measure $e^{-S_L(\phi)} \mathbb{D}\phi$ via probability theory as we now describe. We first introduce the Gaussian Free Field (GFF) X on $\hat{\mathbb{C}}$; we suppose the GFF is defined on the probability space $(\Omega, \Sigma, \mathbb{P})$ (with expectation $\mathbb{E}[\cdot]$). For all $s < 0$, the GFF is a random distribution in the Sobolev space $W^s(\hat{\mathbb{C}})$ on the sphere $\hat{\mathbb{C}}$ defined by

$$W^s(\hat{\mathbb{C}}) = \left\{ f \in \mathcal{S}'(\mathbb{C}) \mid \sum_{k \geq 0} (1 + \lambda_k)^s |\langle f, u_k \rangle_{L_g^2}|^2 < \infty \right\}$$

where $(u_k)_{k \geq 0}$ denotes the orthonormal basis of eigenfunctions of the Laplacian on $\hat{\mathbb{C}}$ equipped with a Riemannian metric with volume measure $g(x)dx$, with eigenvalues λ_k , and $L_g^2 := L^2(\hat{\mathbb{C}}, g(x)dx)$ (with scalar product $\langle \cdot, \cdot \rangle_{L_g^2}$). LCFT correlation functions turn out to have an explicit dependence on g [DKRV16] and using this freedom one can use the GFF with the following covariance kernel

$$(1.2) \quad \mathbb{E}[X(x)X(y)] = \ln \frac{1}{|x-y|} + \ln|x|_+ + \ln|y|_+ := G(x, y)$$

with $|x|_+ = \max(|x|, 1)$ (in this case the background metric is $g(x) = \frac{1}{|x|_+^4}$). Note that, since the GFF is not defined pointwise, the above covariance is an abuse of notation which is standard in probability theory. With these notations, the rigorous definition of the Liouville field ϕ is $\phi(x) = c + X(x) - 2Q \ln|x|_+$ under the measure

$$(1.3) \quad e^{-2Qc} e^{-\mu e^{\gamma c} M_\gamma(\mathbb{C})} dc \otimes \mathbb{P}^5,$$

where as before $\mu > 0, \gamma \in (0, 2)$, $Q = \frac{\gamma}{2} + \frac{\mu}{2}$ and $M_\gamma(dx) := e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2]} \frac{dx}{|x|_+^4}$ is a random measure on \mathbb{C} . Hence we set for F continuous and non negative on $W^s(\hat{\mathbb{C}})$

$$(1.4) \quad \langle F(\phi) \rangle_{\gamma, \mu} = \int_{\mathbb{R}} e^{-2Qc} \mathbb{E}[F(c + X - 2Q \ln|\cdot|_+) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{C})}] dc.$$

⁵In the recent paper [KRV20], the authors considered the measure $2 e^{-2Qc} e^{-\mu e^{\gamma c} M_\gamma(\mathbb{C})} dc \otimes \mathbb{P}$ instead.

Since the GFF is not defined pointwise, the measure M_γ is defined via a renormalization procedure. Specifically, it is given by the following limit called Gaussian multiplicative chaos (GMC, originally introduced by Kahane)

$$(1.5) \quad M_\gamma(dx) := \lim_{\epsilon \rightarrow 0} e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2} \mathbb{E}[X_\epsilon(x)^2]} \frac{dx}{|x|_+^4}$$

where $X_\epsilon(x) = X * \theta_\epsilon$ is the mollification of X with an approximation $(\theta_\epsilon)_{\epsilon > 0}$ of the Dirac mass δ_0 ; indeed, one can show that the limit (1.5) exists in probability in the space of Radon measures on $\hat{\mathbb{C}}$ and that the limit does not depend on the mollifier θ_ϵ : see [RoVa10, RhVa14, Be17] for example. The condition $\gamma \in (0, 2)$ stems from the fact that the random measure M_γ is different from zero if and only if $\gamma \in (0, 2)$. For γ fixed in $(0, 2)$, LCFTs associated to different $\mu > 0$ and $\mu' > 0$ yield equivalent theories for ϕ : hence, LCFT depends essentially on the parameter γ . Finally, the variable $c \in \mathbb{R}$ is absolutely crucial here. It stems from the fact that the GFF on $\hat{\mathbb{C}}$ is defined modulo a constant but in LCFT one has to include the constant as a dynamical variable to ensure conformal invariance.

The measure (1.3) defines an infinite measure on the Sobolev space $W^s(\hat{\mathbb{C}})$ for all $s < 0$. Following [DKRV16], the n -point correlations can be defined for *real* valued α_i via the following limit

$$(1.6) \quad \left\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \right\rangle_{\gamma, \mu} := \lim_{\epsilon \rightarrow 0} \left\langle \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \right\rangle_{\gamma, \mu}$$

where $z_1, \dots, z_n \in \mathbb{C}$ are distinct,

$$(1.7) \quad V_{\alpha, \epsilon}(z) = |z|_+^{-4\Delta_\alpha} e^{\alpha c} e^{\alpha X_\epsilon(z) - \frac{\alpha^2}{2} \mathbb{E}[X_\epsilon(z)^2]}$$

and Δ_α is called the *conformal weight* of V_α

$$(1.8) \quad \Delta_\alpha = \frac{\alpha}{2} \left(Q - \frac{\alpha}{2} \right), \quad \alpha \in \mathbb{C}.$$

The limit (1.6) exists and is non trivial if and only if the following bounds hold

$$(1.9) \quad \sum_{i=1}^n \alpha_i > 2Q, \quad \alpha_i < Q, \quad \forall i = 1, \dots, n \quad (\text{Seiberg bounds}).$$

One of the main results of [DKRV16] is that the limit (1.6) admits the following representation in terms of the moments of GMC

$$(1.10) \quad \left\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \right\rangle_{\gamma, \mu} = \gamma^{-1} \left(\prod_{1 \leq j < j' \leq n} \frac{1}{|z_j - z_{j'}|^{\alpha_j \alpha_{j'}}} \right) \mu^{-s} \Gamma(s) \mathbb{E}[Z^{-s}]$$

where $s = \frac{\sum_{i=1}^n \alpha_i - 2Q}{\gamma}$, Γ is the standard Gamma function and (recall that $|x|_+ = \max(|x|, 1)$)

$$Z = \int_{\mathbb{C}} \left(\prod_{i=1}^n \frac{|x|_+^{\gamma \alpha_i}}{|x - z_i|^{\gamma \alpha_i}} \right) M_\gamma(dx).$$

We stress that the formula (1.10) is valid for correlations with $n \geq 3$, which can be seen at the level of the Seiberg bounds (1.9)⁶⁷. Also it was proved in [DKRV16, Th 3.5] that these correlation functions are *conformally covariant*. More precisely, if z_1, \dots, z_n are n distinct points in \mathbb{C} then for a Möbius map $\psi(z) = \frac{az+b}{cz+d}$ (with $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$)

$$(1.11) \quad \left\langle \prod_{i=1}^n V_{\alpha_i}(\psi(z_i)) \right\rangle_{\gamma, \mu} = \prod_{i=1}^n |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \left\langle \prod_{k=1}^n V_{\alpha_i}(z_i) \right\rangle_{\gamma, \mu}.$$

⁶In fact, one can extend the probabilistic construction (1.10) a bit beyond the Seiberg bounds but the extended bounds also imply $n \geq 3$. We will not discuss these extended bounds in this paper.

⁷For special values of the α_i , expression (1.10) is equivalent to certain moments of a natural Gibbs measure associated to X : this observation is the base of a recent work in physics by Cao-Rosso-Santachiara-Le Doussal [CRSL] which builds on [DKRV16].

Because of relation (1.11), the vertex operators are *primary fields* in the language of physics. Let us now turn to the conformal bootstrap approach to LCFT.

DOZZ formula and Conformal Bootstrap. The conformal bootstrap construction of LCFT postulates an exact formula for the 3-point correlation functions, the celebrated DOZZ formula [DoOt94, ZaZa96] discovered by Dorn, Otto, Zamolodchikov and Zamolodchikov in the 90's and defines the higher order correlation functions via a recursive procedure. In contrast to the probabilistic construction of the correlations, the bootstrap formalism involves the correlations when the vertices $V_\alpha(z)$ have their parameter α in the line $\mathcal{S} := Q + i\mathbb{R}^+$, called in physics the *spectrum* of LCFT⁸ (for a mathematician, the way physicists define the spectrum can be confusing; indeed, we will see later that one can interpret the spectrum of LCFT as a labelling of generalized eigenstates of an unbounded and self-adjoint operator whose mathematical spectrum is in fact $2(\Delta_\alpha)_{\alpha \in \mathcal{S}} = [\frac{Q^2}{2}, \infty)$ where Δ_α is defined via (1.8): see Theorem 1.3 for an exact statement). More specifically, let us denote the correlations in the bootstrap formalism by $\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_{\gamma, \mu}^{\text{Boot}}$. With this notation, the bootstrap assumption leads to the following definitions:

- **DOZZ proposal for the 3-point function:** the 3-point correlation functions can first be reduced to taking the points $(z_1, z_2, z_3) = (0, 1, \infty)$ by using conformal covariance (1.11) of the correlations functions under the action of Möbius transformations (in $\text{PSL}_2(\mathbb{C})$). Then they are defined for $\alpha_j \in \mathcal{S}$ in [DoOt94, ZaZa96] by

$$(1.12) \quad \langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty) \rangle_{\gamma, \mu}^{\text{Boot}} := \frac{1}{2} C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)^9$$

where $C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)$ is an explicit special function involving the Barnes Gamma function, which is analytic in the variables $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ (with a countable number of poles): see appendix A for the definition.

- **Bootstrap to compute the 4-point function from the 3-point function:** the 4-point correlations are defined for $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{S}$ from the 3-point function by the expression

$$(1.13) \quad \begin{aligned} & \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty) \rangle_{\gamma, \mu}^{\text{Boot}} \\ & := \frac{1}{8\pi} \int_0^\infty C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, Q - iP) C_{\gamma, \mu}^{\text{DOZZ}}(Q + iP, \alpha_3, \alpha_4) |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P(z)|^2 dP \end{aligned}$$

where \mathcal{F}_P are holomorphic functions in z called (spherical) *conformal blocks*. The conformal blocks are universal in the sense that they only depend on the conformal weights $\Delta_{\alpha_i} = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ and the central charge of LCFT $c_L = 1 + 6Q^2$ ¹⁰. We will not give an exact definition of \mathcal{F}_P in this introduction as it requires the introduction of material from representation theory of the *Virasoro algebra*: see (1.72) and (1.73) for the exact definition. More generally, the n -point function can be obtained by a similar iterative process in terms of the m -point functions for $m < n$.

While based on physics heuristics, there are strong theoretical motivations and supporting evidences that the the DOZZ proposal (1.12) and the bootstrap formula (1.13) are the right formulas LCFT should satisfy. Since the DOZZ formula is analytic in $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$, one should expect that the probabilistic and the bootstrap construction coincide on the domain of validity of the probabilistic construction, i.e. for real $\alpha_1, \alpha_2, \alpha_3$ satisfying the Seiberg bounds (1.9). This is indeed the case: in a recent work [KRV20], the last three authors proved that the probabilistically constructed 3-point correlation functions indeed satisfy the DOZZ formula.

The present work aims to complete the program of relating the probabilistic construction of LCFT to the bootstrap construction by proving the relation (1.13). In order to relate both constructions, one must also perform an analytic continuation of (1.13); more specifically, the DOZZ formulas and the conformal blocks entering the definition (1.13) can be analytically continued from $\alpha_i \in \mathcal{S}$ to real α_i satisfying the

⁸See [CuTh, BCT82, GeNe84] for arguments in physics about what the spectrum of LCFT should be. We will prove this fact in further details later in the paper: see Theorem 1.3.

⁹The $\frac{1}{2}$ has been added here because of our convention to define the Liouville field: see the footnote in (1.3).

¹⁰This dependence is not explicit in our notations but the reader should keep in mind that a fully explicit notation for $\mathcal{F}_P(z)$ would rather be $\mathcal{F}_P(c_L, \Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{\alpha_3}, \Delta_{\alpha_4}, z)$.

condition $\alpha_i < Q$ for all $i \in \llbracket 1, 4 \rrbracket$ along with $\alpha_1 + \alpha_2 > Q$ and $\alpha_3 + \alpha_4 > Q$. For these real values of α_i we still denote $C_{\gamma, \mu}^{\text{DOZZ}}$ and \mathcal{F}_P the analytically continued expressions and the analytically continued $\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty) \rangle_{\gamma, \mu}^{\text{Boot}}$ is therefore still given by expression (1.13). The main result of this paper is the following theorem proving that the conformal bootstrap formula (1.13) is the right one, in the sense that the probabilistic construction of the 4-point function produces the same exact formula:

Theorem 1.1. *Let $\gamma \in (0, \sqrt{2})$ and $\alpha_i < Q$ for all $i \in \llbracket 1, 4 \rrbracket$. Then the following identity holds for $\alpha_1 + \alpha_2 > Q$ and $\alpha_3 + \alpha_4 > Q$*

$$(1.14) \quad \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty) \rangle_{\gamma, \mu} = \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty) \rangle_{\gamma, \mu}^{\text{Boot}}.$$

The condition $\gamma \in (0, \sqrt{2})$ is a limitation of our proof but the conditions $\alpha_1 + \alpha_2 > Q$ and $\alpha_3 + \alpha_4 > Q$ are essential. Indeed, if $\alpha_1 + \alpha_2 < Q$ the analytic continuation of (1.13) from $\alpha_i \in \mathcal{S}$ requires adding an extra term (cf. the discussion on the so-called discrete terms in [ZaZa96]).

Although the conformal bootstrap Theorem 1.1 is stated for the 4-point correlation function, we emphasize that our method is constructive and allows to prove the relation between the n -point functions and the m -point functions for $m < n$. In order to keep the length of the paper reasonable, we will not state these generalizations explicitly since the bootstrap formulas for the higher order correlations are rather complicated.

Conformal blocks and relations with the AGT conjecture. Let us mention that it is not at all obvious that the bootstrap definition of the four point correlation function, i.e. the right hand side of (1.14), exists for real α_i satisfying the condition $\alpha_i < Q$ for all $i \in \llbracket 1, 4 \rrbracket$ along with $\alpha_1 + \alpha_2 > Q$ and $\alpha_3 + \alpha_4 > Q$. Indeed, first the conformal blocks are defined via a series expansion

$$(1.15) \quad \mathcal{F}_P(z) = \sum_{n=0}^{\infty} \beta_n z^n$$

where the coefficients β_n , which have a strong representation theoretic content, are non explicit and given in terms of the inverse matrix of a scalar product on a finite dimensional space (a subspace of the vector space which appears in a representation of the Virasoro algebra): see (1.73) for the exact definition of β_n . Hence, it is not at all obvious that the series (1.15) converges for $|z| < 1$. Second, it is not clear that the integral in $P \in \mathbb{R}^+$ of expression (1.13) is convergent. As a matter of fact, in the course of the proof of Theorem 1.1, we establish both that the radius of convergence¹¹ of (1.15) is 1 for almost all P and that the integral (1.13) makes sense.

To the best of our knowledge, the proof of the convergence of the conformal blocks is new and we expect that the result holds for all P , although we do not need such a strong statement for our purpose. Let us mention here the recent work of Ghosal-Remy-Sun-Sun [GRSS20] which establishes a probabilistic formula involving moments of a GMC type variable for the so-called toric conformal blocks in the case of the torus \mathbb{T}^2 , which play an analogous role for LCFT on \mathbb{T}^2 to the spherical conformal blocks considered in Theorem 1.1 (in particular, they obtain existence of the blocks for all values of the relevant parameters).

The AGT correspondence [AGT10] between $4d$ supersymmetric Yang-Mills and the bootstrap construction of LCFT (on any Riemann surface) conjectures that $\mathcal{F}_P(z)$ coincides with special cases of Nekrasov's partition function. Since Nekrasov's partition function for $\mathcal{N} = 2$ gauge theory in $4d$ is explicit, a proof of the conjecture would give an explicit formula for β_n in (1.15). However, even admitting this conjecture, it remains difficult to show that the radius of convergence of the conformal blocks \mathcal{F}_P is 1 because Nekrasov's partition function gives a representation of β_n in terms of a complicated sum over pairs of Young diagrams which is hard to study directly: see for instance the work of Felder-Müller-Lennert [FLM89] for convergence statements on the Nekrasov partition function. The AGT conjecture has nonetheless been proved in the case of the torus (for the corresponding toric conformal blocks) by Negut [Ne16] following the works of Maulik-Okounkov [MaOk12] and Schiffman-Vasserot [ScVa13]. However, these papers consider conformal

¹¹We acknowledge here an argument that was given to us by Slava Rychkov in private communications. Convergence of conformal blocks defining series is topical in physics, see [PRER12, HoRy13, KQR20].

blocks (1.15) and Nekrasov's partition function as formal power series in the z variable and do not address the issue of convergence; also, their method does not extend to the case of the Riemann sphere or to Riemann surfaces of genus greater or equal to 2 (see also Fateev-Litvinov [FL10] and Alba-Fateev-Litvinov-Tarnopolsky [AFLT11] for arguments in the physics literature which support the AGT conjecture on the torus or the Riemann sphere).

Crossing Symmetry. Let us say a few more words on the bootstrap construction and the relation with what is known as crossing symmetry. The bootstrap construction is based axiomatically on the existence of a decomposition that should be seen as an asymptotic expansion of the “product” $V_{\alpha_1}(0)V_{\alpha_2}(z)$ as $z \rightarrow 0$: in the language of physics, it is called an operator product expansion of $V_{\alpha_1}(0)V_{\alpha_2}(z)$ around $z = 0$. Our proof of identity (1.14) is in fact based on a rigorous formulation of this operator product expansion as a decomposition in a Hilbert space. More specifically, we rigorously construct the “product” as a vector in an appropriate Hilbert space (the vector is given by expression (1.22) below); the mathematical version of the operator product expansion then corresponds to a decomposition of the vector on an appropriate basis of the Hilbert space. Within the bootstrap methodology, a similar decomposition is axiomatically supposed to hold for $V_{\alpha_2}(z)V_{\alpha_3}(1)$ when $z \rightarrow 1$. Both of these expansions ($z \rightarrow 0$ or $z \rightarrow 1$) can be inserted into the 4 point correlation functions and equality of the resulting expressions is axiomatically supposed and referred to as *crossing symmetry* in the physics literature. More specifically, the bootstrap construction of LCFT assumes that the following crossing symmetry identity holds

$$(1.16) \quad \int_{\mathbb{R}^+} C_{\gamma,\mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, Q - iP) C_{\gamma,\mu}^{\text{DOZZ}}(\alpha_3, \alpha_4, Q + iP) |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P(z)|^2 dP \\ = \int_{\mathbb{R}^+} C_{\gamma,\mu}^{\text{DOZZ}}(\alpha_3, \alpha_2, Q - iP) C_{\gamma,\mu}^{\text{DOZZ}}(\alpha_1, \alpha_4, Q + iP) |1-z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_3} - \Delta_{\alpha_2})} |\tilde{\mathcal{F}}_P(1-z)|^2 dP$$

where $\tilde{\mathcal{F}}_P$ is obtained from \mathcal{F}_P by flipping the parameter α_1 with α_3 . The crossing symmetry identity (1.16), which is essential for the bootstrap construction of LCFT to be meaningful, is a very strong constraint and seems very hard to prove directly. However, it follows directly from our work since one has by conformal covariance (1.11) of the probabilistic construction of the correlations

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_3}(\infty) \rangle_{\gamma,\mu} = \langle V_{\alpha_3}(0)V_{\alpha_2}(1-z)V_{\alpha_1}(1)V_{\alpha_3}(\infty) \rangle_{\gamma,\mu}.$$

This yields the following immediate corollary:

Corollary 1.2. *The bootstrap construction of LCFT satisfies crossing symmetry for $\gamma \in (0, \sqrt{2})$.*

Let us also mention that Teschner has given strong arguments in [Te01] which suggest that the bootstrap construction of LCFT satisfies crossing symmetry.

Acknowledgements. C. Guillarmou acknowledges that this project has received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme, grant agreement No 725967. A. Kupiainen is supported by the Academy of Finland and ERC Advanced Grant 741487. R. Rhodes is partially supported by the Institut Universitaire de France (IUF). The authors wish to thank Zhen-Qing Chen, Naotaka Kajino for discussions on Dirichlet forms, Ctirad Klimcik and Yi-Zhi Huang for explaining the links with Vertex Operator Algebras, Slava Rychkov for fruitful discussions on the conformal bootstrap approach, Alex Strohmaier, Tanya Christiansen and Jan Dereziński for discussions on the scattering part and Baptiste Cerle for comments on earlier versions of this manuscript.

1.2. Outline of the proof. In the sequel, we suppose that the GFF on the Riemann sphere X is defined on a probability space $(\Omega, \Sigma, \mathbb{P})$ (with expectation $\mathbb{E}[\cdot]$) where $\Omega = \Omega_{\mathbb{T}} \times \Omega_{\mathbb{D}} \times \Omega_{\mathbb{D}^c}$, $\Sigma = \Sigma_{\mathbb{T}} \otimes \Sigma_{\mathbb{D}} \otimes \Sigma_{\mathbb{D}^c}$ and \mathbb{P} is a product measure $\mathbb{P} = \mathbb{P}_{\mathbb{T}} \otimes \mathbb{P}_{\mathbb{D}} \otimes \mathbb{P}_{\mathbb{D}^c}$. At the level of random variables, the GFF decomposes as the sum of three independent variables

$$(1.17) \quad X = P\varphi + X_{\mathbb{D}} + X_{\mathbb{D}^c}$$

where $P\varphi$ is the harmonic extension of the GFF restricted to the circle $\varphi = X|_{\mathbb{T}}$ defined on $(\Omega_{\mathbb{T}}, \Sigma_{\mathbb{T}}, \mathbb{P}_{\mathbb{T}})$ and $X_{\mathbb{D}}, X_{\mathbb{D}^c}$ are two independent GFFs on \mathbb{D} and \mathbb{D}^c with Dirichlet boundary conditions defined respectively

on the probability spaces $(\Omega_{\mathbb{D}}, \Sigma_{\mathbb{D}}, \mathbb{P}_{\mathbb{D}})$ and $(\Omega_{\mathbb{D}^c}, \Sigma_{\mathbb{D}^c}, \mathbb{P}_{\mathbb{D}^c})$ ¹², i.e. with covariance the Green kernel $G_{\mathbb{D}}(\cdot, \cdot)$ (and $G_{\mathbb{D}^c}(\cdot, \cdot)$) associated to the Laplacian with Dirichlet boundary conditions. At this stage, let us just mention the following symmetry property for the distribution of the two GFFs $X_{\mathbb{D}}, X_{\mathbb{D}^c}$

$$(1.18) \quad X_{\mathbb{D}^c}\left(\frac{1}{\bar{z}}\right) \stackrel{(Law)}{=} X_{\mathbb{D}}(z).$$

The GFF satisfies a spatial domain Markov property which implies that we may view it as a Markov process in the “time” variable $t \rightarrow X(e^{-t}\cdot)$ with state space the Sobolev space $W^s(\mathbb{T})$ with $s < 0$, i.e. generalized functions $f(\theta) = \sum_{n \in \mathbb{Z}} f_n e^{in\theta}$ such that

$$\|f\|_{W^s(\mathbb{T})}^2 := \sum_{n \in \mathbb{Z}} |f_n|^2 (|n| + 1)^{2s} < \infty.$$

The process gives rise to an Ornstein-Uhlenbeck semigroup on $L^2(\Omega_{\mathbb{T}})$ whose generator is a positive self-adjoint operator \mathbf{P} . The operator \mathbf{P} is essentially an infinite dimensional harmonic oscillator: see (1.32) below for the exact definition. This operator \mathbf{P} has only discrete spectrum consisting of eigenvalues with finite multiplicity. Similarly one may look at the Liouville field in the “time” variable $t \mapsto \phi(e^{-t}\cdot)$ under the LCFT measure (1.4): in this context, LCFT inherits the Markov property from the GFF and by a Feynman-Kac formula (see Section 2.6) it gives rise to a Markov semigroup in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ (equipped with the measure $dc \otimes \mathbb{P}_{\mathbb{T}}$) generated for $\gamma \in (0, \sqrt{2})$ by the operator

$$(1.19) \quad \mathbf{H} = -\frac{1}{2}\partial_c^2 + \frac{1}{2}Q^2 + \mathbf{P} + \mu e^{\gamma c}V$$

where V is a positive potential whose expression is

$$(1.20) \quad V = \int_0^{2\pi} e^{\gamma\varphi(\theta) - \frac{\gamma^2}{2}\mathbb{E}[\varphi(\theta)^2]} d\theta.$$

In expression (1.20), the measure $e^{\gamma\varphi(\theta) - \frac{\gamma^2}{2}\mathbb{E}[\varphi(\theta)^2]} d\theta$, which is the GMC associated to φ , is again defined via a renormalization procedure similar to (1.5). The measure $e^{\gamma\varphi(\theta) - \frac{\gamma^2}{2}\mathbb{E}[\varphi(\theta)^2]} d\theta$ is different from 0 if and only if $\gamma \in [0, \sqrt{2})$ and V belongs to $L^2(\Omega_{\mathbb{T}})$ if and only if $\gamma \in (0, 1)$. For $\gamma \in [\sqrt{2}, 2)$, LCFT also gives rise to a Markov semigroup in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ but the interaction term is no longer a function and in particular there is no simple expression for the corresponding operator. The operator (1.19) in the absence of the c -variable, i.e. the operator $\mathbf{P} + \mu V$, was first studied in [Ho71] and was shown to be essentially self-adjoint on an appropriate domain provided $\gamma \in (0, 1)$.

In the sequel, we will denote by $\langle \cdot, \cdot \rangle_2$ the standard inner product associated to $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ and $\|\cdot\|_2$ the associated norm. The connection between the correlation functions of LCFT and the semigroup $e^{-t\mathbf{H}}$ comes from *reflection positivity*, a property intimately linked to the decomposition (1.17) and the symmetry property (1.18) which allows to express correlation functions as scalar products in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ by conditioning on the variable φ . In particular for the 4-point function we can first use its conformal covariance (1.11) under Möbius transformations to reduce to the case where the points are $(0, z, z', \infty)$ with $|z| < 1$ and $|z'| > 1$. We then write for $\alpha_1 + \alpha_2 > Q$ and $\alpha_3 + \alpha_4 > Q$

$$(1.21) \quad \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(z')V_{\alpha_4}(\infty) \rangle_{\gamma, \mu} = |z'|^{-4\Delta_{\alpha_3}} \langle U_{\alpha_1, \alpha_2}(0, z) | U_{\alpha_4, \alpha_3}(0, \frac{1}{z'}) \rangle_2$$

where $U_{\alpha_1, \alpha_2}(0, z)$ and $U_{\alpha_4, \alpha_3}(0, \frac{1}{z'})$ are vectors in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ given by the following probabilistic formula for all $|z_1|, |z_2| < 1$ and all real $\alpha, \beta < Q$

$$(1.22) \quad U_{\alpha, \beta}(z_1, z_2) = e^{(\alpha+\beta-Q)c} e^{\alpha P\varphi(z_1) + \beta P\varphi(z_2) + \alpha\beta G_{\mathbb{D}}(z_1, z_2)} (1 - |z_1|^2)^{\frac{\alpha^2}{2}} (1 - |z_2|^2)^{\frac{\beta^2}{2}} \\ \times \mathbb{E}_{\varphi} \left[e^{-\mu e^{\gamma c} \int_{\mathbb{D}} e^{\gamma\alpha G_{\mathbb{D}}(x, z_1) + \gamma\beta G_{\mathbb{D}}(x, z_2)} M_{\gamma}(dx)} \right]$$

where $\mathbb{E}_{\varphi}[\cdot] := \mathbb{E}[\cdot | \Sigma_{\mathbb{T}}]$ denotes the conditional expectation with respect to the GFF on the circle (recall $P\varphi$ is the harmonic extension of φ in (1.17)). Let us emphasize here that (1.21) and (1.22) are valid in the full range $\gamma \in (0, 2)$. The $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ vector $U_{\alpha, \beta}(0, z)$ should be seen as a rigorous definition of the product of

¹²With a slight abuse of notations, we will assume that these spaces are canonically embedded in the product space (Ω, Σ) and we will identify them with the respective images of the respective embeddings.

the two vertex operators $V_\alpha(0)V_\beta(z)$ where the case $\beta = 0$ corresponds to a single vertex operator $V_\alpha(0)$; in the sequel we will denote $U_\alpha(0) := U_{\alpha,0}(0, z)$ hence we have (using $P\varphi(0) = 0$)

$$(1.23) \quad U_\alpha(0) = e^{(\alpha-Q)c} \mathbb{E}_\varphi \left[e^{-\mu e^{\gamma c} \int_{\mathbb{D}} e^{\gamma \alpha G_{\mathbb{D}}(x,0)} M_\gamma(dx)} \right].$$

Our proof of the bootstrap formula (which works in the case $\gamma \in (0, \sqrt{2})$) can then be summarized as follows:

- First, we show that $U_\alpha(0)$, defined probabilistically in (1.23) for real $\alpha < Q$, can be analytically extended to the half-plane $\text{Re}(\alpha) \leq Q$ and they are *generalized eigenfunctions* of \mathbf{H} , more precisely $(\mathbf{H} - 2\Delta_\alpha)U_\alpha(0) = 0$. They are not in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ but belong to weighted $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ -spaces with exponential weights with respect to c . The operator \mathbf{H} has only absolutely continuous spectrum, equal to $[Q^2/2, \infty)$, and we prove that \mathbf{H} can be diagonalized by using the continuous family $U_{Q+iP}(0)$ for $P \in \mathbb{R}^+$ ¹³ and the so-called *descendant fields* obtained by the action of an algebra of operators, called the Virasoro algebra and generated by the Virasoro generators $(L_n)_{n \in \mathbb{Z}}$ and $(\tilde{L}_n)_{n \in \mathbb{Z}}$, on the family $U_{Q+iP}(0)$. This action can be encoded by non-trivial pairs of Young diagrams $(\nu, \tilde{\nu})$ so that descendant fields, denoted $\Psi_{Q+iP, \nu, \tilde{\nu}}$, can be parametrized by those pairs of Young diagrams. When this pair is null ($\nu = \tilde{\nu} = \emptyset$), we have $\Psi_{Q+iP, 0, 0} = U_{Q+iP}(0)$ and these fields are called the *primary fields*. The descendant fields are not really orthonormal, which means that the decomposition of the spectral measure E_P of \mathbf{H} in terms of $U_{Q+iP}(0)$ and its descendants $(\Psi_{Q+iP, \nu, \tilde{\nu}})_{\nu, \tilde{\nu}}$ involves some matrix coefficients $F_{Q+iP}(\nu, \nu')$ (Gram Matrix) called the Shapovalov form in representation theory; this aspect needs a bit of representation theory of the Virasoro algebra. The next subsections summarize this part.
- Second, we write the $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ -product of the vectors $U_{\alpha_1, \alpha_2}(0, z)$ and $U_{\alpha_4, \alpha_3}(0, \frac{1}{\bar{z}'})$ under the form

$$\begin{aligned} & \langle U_{\alpha_1, \alpha_2}(0, z) | U_{\alpha_4, \alpha_3}(0, \frac{1}{\bar{z}'} \rangle_2 = \int_{\mathbb{R}^+} \langle dE_P U_{\alpha_1, \alpha_2}(0, z) | U_{\alpha_4, \alpha_3}(0, \frac{1}{\bar{z}'} \rangle_2 \\ & = \frac{1}{2\pi} \sum_{\nu, \tilde{\nu}, \nu', \tilde{\nu}'} \int_{\mathbb{R}^+} \langle U_{\alpha_1, \alpha_2}(0, z) | \Psi_{Q+iP, \nu', \tilde{\nu}'} \rangle_2 \langle \Psi_{Q+iP, \nu, \tilde{\nu}} | U_{\alpha_3, \alpha_4}(0, \frac{1}{\bar{z}'} \rangle_2 F_{Q+iP}^{-1}(\nu, \nu') F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}') dP \end{aligned}$$

using the spectral resolution E_P of \mathbf{H} (the spectral theorem for \mathbf{H}). Here, $\Psi_{Q+iP, \nu', \tilde{\nu}'}$ are not in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ but $U_{\alpha_1, \alpha_2}(0, z) \overline{\Psi_{Q+iP, \nu', \tilde{\nu}'}} \in L^1(\mathbb{R} \times \Omega_{\mathbb{T}})$ so that the pairing $\langle U_{\alpha_1, \alpha_2}(0, z) | \Psi_{Q+iP, \nu', \tilde{\nu}'} \rangle_2$ makes sense.

- Third, for each fixed $P \in \mathbb{R}^+$, we establish the so-called Ward identities: they encode the conformal symmetries of LCFT. Concretely they express the scalar products of descendant fields

$$(1.24) \quad \langle U_{\alpha_1, \alpha_2}(0, z) | \Psi_{Q+iP, \nu, \tilde{\nu}} \rangle_2$$

in terms of differential operators applied to the scalar product involving primary fields

$$\langle U_{\alpha_1, \alpha_2}(0, z) | U_{Q+iP}(0) \rangle_2.$$

This is an important step in our proof and we refer to Proposition 1.10 below for an exact statement. The main issue here is that we have no probabilistic representation of the descendants $\Psi_{\alpha, \nu, \tilde{\nu}}$ for α on the spectrum line $Q + i\mathbb{R}^+$. So we prove that the descendant fields can be analytically continued to real values of $\alpha < Q$ provided that α is very negative: this is what we call the *probabilistic region* as for those α 's we are able to prove a probabilistic formula involving the Stress Energy Tensor (SET) for the descendant fields $\Psi_{\alpha, \nu, \nu'}$ of $U_\alpha(0)$. One can then exploit this probabilistic formula to prove Proposition 1.10 using Gaussian integration by parts. The case $\alpha = Q + iP$ can then be deduced by an analytic continuation argument. The Ward identities of Proposition 1.10, along with a proper recombination of the terms in the spectral decomposition (see (1.59) below for the spectral decomposition) enable to recover the formula of the conformal blocks. On the other side, our proof shows convergence almost everywhere in P of the series (1.15) that defines the conformal blocks.

¹³For $P \in \mathbb{R}^+$, one can check that $\alpha = Q + iP$ produces the eigenvalue $2\Delta_\alpha = \frac{Q^2 + P^2}{2}$, hence in the spectrum $[Q^2/2, \infty)$ of \mathbf{H} .

In the next subsections of the introduction, we shall explain more in details these few aspects and state precisely the needed Theorems we are going to prove in the paper.

1.3. Spectral resolution of the Hamiltonian \mathbf{H} . One of the main mathematical inputs in our proof comes from stationary *scattering theory*. Let us start here by a toy model explaining the intuition about the spectrum of \mathbf{H} , first elaborated by Curtright and Thorn [CuTh] (see also Teschner [Te01] for a nice discussion of the scattering picture). Consider first the trivial case $\mu = 0$. The spectrum of the operator $-\frac{1}{2}\partial_c^2 + \frac{1}{2}Q^2 + \mathbf{P}$ is $[\frac{1}{2}Q^2, \infty)$ and a complete set of generalized eigenfunctions diagonalizing \mathbf{H} is given by $e^{iPc}\Phi$ with $P \in \mathbb{R}$ and Φ an eigenfunction of \mathbf{P} . Next consider the operator where V is replaced by 1. The operator $-\frac{1}{2}\partial_c^2 + \mu e^{\gamma c}$ is a Schrödinger operator with potential tending to 0 as $c \rightarrow -\infty$ and to ∞ as $c \rightarrow \infty$. It describes scattering from a wall and it has a complete set of eigenfunctions f_P , $P \in \mathbb{R}^+$

$$f_P(c) \sim \begin{cases} e^{iPc} + R(P)e^{-iPc} & c \rightarrow -\infty \\ 0 & c \rightarrow \infty \end{cases}$$

where $R(P)$ is an explicit coefficient called the reflection coefficient¹⁴. Obviously the eigenfunctions of the $V = 1$ operator are $f_P\Phi$. For $V \neq 1$ the c and the φ variables are coupled. Since V is positive and \mathbf{P} has compact resolvent we expect the above picture will stay qualitatively the same as for $V = 1$ and indeed we prove that \mathbf{H} has a spectral resolution in terms of a complete set of generalized eigenfunctions $\Psi_{Q+iP, \mathbf{m}, \mathbf{n}}$, where $P \in \mathbb{R}^+$ and the labels \mathbf{m}, \mathbf{n} are the same as for the eigenfunctions of \mathbf{P} : this means that one can write the Identity (and more generally a functional calculus, i.e. $F(\mathbf{H})$ for $F \in C_b^0(\mathbb{R})$) in terms of these eigenfunctions – see Theorem 1.3.

We will now give the precise definition of the operator (1.19) and state our result on its spectral properties. Given two independent sequences of i.i.d. standard Gaussians $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$, the GFF on the unit circle is the random Fourier series

$$(1.25) \quad \varphi(\theta) = \sum_{n \neq 0} \varphi_n e^{in\theta}$$

where for $n > 0$

$$(1.26) \quad \varphi_n := \frac{1}{2\sqrt{n}}(x_n + iy_n), \quad \varphi_{-n} := \overline{\varphi_n}.$$

One can easily check that $\mathbb{E}[\|\varphi\|_{W^s(\mathbb{T})}^2] < \infty$ for any $s < 0$ so that the series (1.25) defines a random element in $W^s(\mathbb{T})$. Moreover, by a standard computation, one can check that it is a centered Gaussian field with covariance kernel given by¹⁵

$$(1.27) \quad \mathbb{E}[\varphi(\theta)\varphi(\theta')] = \ln \frac{1}{|e^{i\theta} - e^{i\theta'}|}.$$

The underlying probability space here is $\Omega_{\mathbb{T}} = (\mathbb{R}^2)^{\mathbb{N}^*}$. It is equipped with the cylinder sigma-algebra $\Sigma_{\mathbb{T}} = \mathcal{B}^{\otimes \mathbb{N}^*}$, where \mathcal{B} stands for the Borel sigma-algebra on \mathbb{R}^2 and the product measure

$$(1.28) \quad \mathbb{P}_{\mathbb{T}} := \bigotimes_{n \geq 1} \frac{1}{2\pi} e^{-\frac{1}{2}(x_n^2 + y_n^2)} dx_n dy_n.$$

Here $\mathbb{P}_{\mathbb{T}}$ is supported on $W^s(\mathbb{T})$ for any $s < 0$.

The Hilbert space $L^2(\Omega_{\mathbb{T}}, \mathbb{P}_{\mathbb{T}})$ (denoted from now on by $L^2(\Omega_{\mathbb{T}})$) has the structure of Fock space. Let $\mathcal{P} \subset L^2(\Omega_{\mathbb{T}})$ be the linear span of the functions of the form $F(x_1, y_1, \dots, x_N, y_N)$ for some $N \geq 1$ where F is a polynomial on \mathbb{R}^{2N} . On \mathcal{P} we define the annihilation and creation operators

$$(1.29) \quad \mathbf{X}_n = \partial_{x_n}, \quad \mathbf{X}_n^* = -\partial_{x_n} + x_n,$$

$$(1.30) \quad \mathbf{Y}_n = \partial_{y_n}, \quad \mathbf{Y}_n^* = -\partial_{y_n} + y_n.$$

¹⁴This coefficient is a simplified version of the (quantum) reflection coefficient which appears in the proof of the DOZZ formula [KRV19].

¹⁵Since the field φ is not defined pointwise, recall that this is a slight abuse of notation.

where $*$ denotes the adjoint. These are closable (see e.g. [ReSi1], VIII.11) and adjoint of each other and form a representation of the algebra of canonical commutation relations on $L^2(\Omega_{\mathbb{T}})$:

$$(1.31) \quad [\mathbf{X}_n, \mathbf{X}_m^*] = \delta_{nm} = [\mathbf{Y}_n, \mathbf{Y}_m^*]$$

with other commutators vanishing. The operator \mathbf{P} is then given on \mathcal{P} as

$$(1.32) \quad \mathbf{P} = \sum_{n=1}^{\infty} n(\mathbf{X}_n^* \mathbf{X}_n + \mathbf{Y}_n^* \mathbf{Y}_n)$$

(only finite number of terms in the sum contributes when acting on \mathcal{P}) and extends to an unbounded self-adjoint positive operator on $L^2(\Omega_{\mathbb{T}})$. Let $\mathbf{k} = (k_1, k_2, \dots) \in \mathbb{N}^{\mathbb{N}^+}$ be such that $k_n = 0$ for all n large enough and \mathbf{l} similarly. The set of such sequences, i.e. sequences of integers with only a finite number of positive integers, will be denoted \mathcal{N} . Define the polynomials (here $1 \in L^2(\Omega_{\mathbb{T}})$ is the constant function)

$$(1.33) \quad \hat{\psi}_{\mathbf{k}\mathbf{l}} = \prod_n (\mathbf{X}_n^*)^{k_n} (\mathbf{Y}_n^*)^{l_n} 1.$$

Equivalently, $\hat{\psi}_{\mathbf{k}\mathbf{l}} = \prod_n \text{He}_{k_n}(x_n) \text{He}_{l_n}(y_n)$ where $(\text{He}_k)_k$ are the standard Hermite polynomials. Then, using (1.31), one checks that these are eigenstates of \mathbf{P} :

$$(1.34) \quad \mathbf{P} \hat{\psi}_{\mathbf{k}\mathbf{l}} = (|\mathbf{k}| + |\mathbf{l}|) \hat{\psi}_{\mathbf{k}\mathbf{l}} := \lambda_{\mathbf{k}\mathbf{l}} \hat{\psi}_{\mathbf{k}\mathbf{l}}.$$

where we use the notation

$$(1.35) \quad |\mathbf{k}| := \sum_{n=1}^{\infty} n k_n$$

for $\mathbf{k} \in \mathcal{N}$. It is also well known that the family $\{\psi_{\mathbf{k}\mathbf{l}} = \hat{\psi}_{\mathbf{k}\mathbf{l}} / \|\hat{\psi}_{\mathbf{k}\mathbf{l}}\|_{L^2(\Omega_{\mathbb{T}})}\}$ (where $\|\cdot\|_{L^2(\Omega_{\mathbb{T}})}$ is the standard norm in $L^2(\Omega_{\mathbb{T}})$) forms an orthonormal basis of $L^2(\Omega_{\mathbb{T}})$.

For all $\gamma \in (0, \sqrt{2})$, one can define a GMC type random measure associated to the exponential of $\gamma\varphi$. More precisely, define the regularized field

$$(1.36) \quad \varphi^{(N)}(\theta) = \sum_{|n| \leq N} \varphi_n e^{in\theta}$$

which is a.s. a smooth function. Then the GMC measure m_γ can be defined as the following weak limit of Radon measures

$$(1.37) \quad m_\gamma(d\theta) = \lim_{N \rightarrow \infty} e^{\gamma\varphi^{(N)}(\theta) - \frac{\gamma^2}{2} \mathbb{E}[\varphi^{(N)}(\theta)^2]} d\theta$$

where the above limit exists almost surely, see [Ka85, RhVa14, Be17] for instance on the topic. The multiplication operator V in (1.19) is then defined as the total mass of the measure:

$$(1.38) \quad V = m_\gamma(\mathbb{T}) = \int_0^{2\pi} m_\gamma(d\theta).$$

It is known ([RhVa14]) that for $\gamma < \sqrt{2}$, $V \in L^{\frac{2}{\gamma^2} - \varepsilon}(\Omega_{\mathbb{T}})$ for all $\varepsilon > 0$.

If $\gamma \in (0, 1)$ the domain of V contains \mathcal{P} since for $F \in \mathcal{P}$ we have $F \in L^q(\Omega_{\mathbb{T}})$ for all $q < \infty$ and therefore $VF \in L^2(\Omega_{\mathbb{T}})$. The Liouville Hamiltonian (1.19) maps $C_0^\infty(\mathbb{R}) \otimes \mathcal{P}$ to $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ and since $C_0^\infty(\mathbb{R}) \otimes \mathcal{P}$ is dense in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ it is possible to extend \mathbf{H} into a self-adjoint operator on some domain $\mathcal{D}(\mathbf{H})$ by completion of the corresponding quadratic form.

If $\gamma \in [1, \sqrt{2})$, \mathcal{P} is no longer contained in the domain $\mathcal{D}(\mathbf{H})$ but the domain of the quadratic form still contains $C_0^\infty(\mathbb{R}) \otimes \mathcal{P}$ so that \mathbf{H} can be extended into a self-adjoint operator similarly. In both cases, this extension, that we also denote \mathbf{H} , is called the Friedrichs extension: see subsection 2.4 for the exact definition. If $\gamma \in [\sqrt{2}, 2)$, the potential V makes sense only as a measure and furthermore this measure is not only absolutely continuous with respect to the Gaussian measure. This case is problematic and will not be treated in this paper.

Here is our main result on its spectrum (here $c_- = \min(c, 0)$):

Theorem 1.3. *Let $\gamma \in (0, \sqrt{2})$. The Liouville Hamiltonian (1.19) defined on $C_0^\infty(\mathbb{R}) \otimes \mathcal{P}$ has a canonical self-adjoint extension to a domain $\mathcal{D}(\mathbf{H}) \subset L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ (the Friedrichs extension); the spectrum of \mathbf{H} is absolutely continuous and given by the half-line $[\frac{Q^2}{2}, \infty)$. Each $E \in [\frac{Q^2}{2}, \infty)$ is of finite multiplicity (in the sense of absolutely continuous spectrum) and there is a complete family of generalized eigenstates $\Psi_{Q+iP, \mathbf{k}, \mathbf{l}} \in \cap_{\varepsilon > 0} e^{-\varepsilon c_-} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ labeled by $P \in \mathbb{R}^+$ and $\mathbf{k}, \mathbf{l} \in \mathcal{N}$ such that (recall the definition of $\lambda_{\mathbf{k}\mathbf{l}}$ in (1.34))*

$$\mathbf{H}\Psi_{Q+iP, \mathbf{k}, \mathbf{l}} = \left(\frac{Q^2}{2} + \frac{P^2}{2} + \lambda_{\mathbf{k}\mathbf{l}} \right) \Psi_{Q+iP, \mathbf{k}, \mathbf{l}}.$$

Moreover $\Psi_{Q+iP, \mathbf{k}, \mathbf{l}}$ is a complete family diagonalizing \mathbf{H} in the sense that for each $u_1, u_2 \in e^{\delta c_-} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for some $\delta > 0$

$$(1.39) \quad \langle u_1 | u_2 \rangle_2 = \frac{1}{2\pi} \sum_{\mathbf{k}, \mathbf{l} \in \mathcal{N}} \int_0^\infty \langle u_1 | \Psi_{Q+iP, \mathbf{k}, \mathbf{l}} \rangle_2 \langle \Psi_{Q+iP, \mathbf{k}, \mathbf{l}} | u_2 \rangle_2 dP$$

The $\Psi_{Q+iP, \mathbf{k}, \mathbf{l}}$ are *scattering states* and will be discussed more in Sect 1.6. Let us make a useful analogy with the Laplacian in \mathbb{R}^n : the $\Psi_{Q+iP, \mathbf{k}, \mathbf{l}}$ are similar to the plane waves $e^{iP\omega \cdot x}$ which diagonalize $\Delta_{\mathbb{R}^n}$, with $\omega \in \mathbb{S}^{n-1}$ and $P \in \mathbb{R}^+$ – those are not in $L^2(\mathbb{R}^n)$ but rather in $e^{\varepsilon|x|} L^2(\mathbb{R}^n)$, just as $\Psi_{Q+iP, \mathbf{k}, \mathbf{l}}$ belongs to $e^{-\varepsilon c_-} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ in our case. The Fock space $L^2(\Omega_{\mathbb{T}})$ plays in some sense the role played by $L^2(\mathbb{S}^{n-1})$ for the diagonalization of $\Delta_{\mathbb{R}^n}$, the main difference is that for \mathbb{R}^n each spectral value $E \in \mathbb{R}^+$ has infinite multiplicity (as an element of the continuous spectrum). Actually, in terms of spectrum, our operator \mathbf{H} is more similar to a Laplacian on a half-cylinder, i.e. $-\partial_c^2 + \Delta_{\mathbb{S}^{n-1}}$ on $\mathbb{R}^+ \times \mathbb{S}^{n-1}$ with Dirichlet condition at 0. Although we shall not need it, the formula (1.39) actually extends to all $u_1, u_2 \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$, but in that case, since $\Psi_{Q+iP, \mathbf{k}, \mathbf{l}}$ are not in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ the quantity $\langle u_j, \Psi_{Q+iP, \mathbf{k}, \mathbf{l}} \rangle_2$ does not make sense; however the sum and integral (1.39) make sense similarly to the way Plancherel formula holds on $L^2(\mathbb{R}^n)$ using approximation of u_1, u_2 by sequences in $e^{\delta c_-} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for some $\delta > 0$.

We notice however that proving Theorem 1.3 involves a good amount of work (the whole Section 4). The main difficulty comes from the fact that the potential V appearing in \mathbf{H} lives on an infinite dimensional space (here $\Omega_{\mathbb{T}}$) and moreover it is not bounded above and below by positive constants: one rather has $V \in L^p(\Omega_{\mathbb{T}})$ for some $p > 1$ for $\gamma < \sqrt{2}$. This weak regularity and unboundedness of the potential make the problem tricky.

1.4. Diagonalization of the free Hamiltonian using the Virasoro algebra. We start by explaining the diagonalization of the free (i.e. non interacting) Hamiltonian \mathbf{H}^0 which corresponds to the case $\mu = 0$ in (1.19). That can be done directly by using the orthonormal basis of Hermite polynomials $\psi_{\mathbf{k}\mathbf{l}}$ of $L^2(\Omega_{\mathbb{T}})$ combined with Plancherel formula for the Fourier transform on the real line: for each $u_1, u_2 \in e^{\delta c_-} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for some $\delta > 0$, one has

$$(1.40) \quad \langle u_1 | u_2 \rangle_2 = \frac{1}{2\pi} \sum_{\mathbf{k}, \mathbf{l} \in \mathcal{N}} \int_{\mathbb{R}} \langle u_1 | e^{iPc} \psi_{\mathbf{k}\mathbf{l}} \rangle_2 \langle e^{iPc} \psi_{\mathbf{k}\mathbf{l}} | u_2 \rangle_2 dP.$$

It will be useful however to use another basis for $L^2(\Omega_{\mathbb{T}})$ which respects its underlying complex analytic structure and will be useful in constructing a representation on $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ of two commuting Virasoro algebras. This will be crucial for our argument in order to identify the combinatorial structure of the Liouville Hamiltonian. To do this, we will use the complex coordinates (1.26) and denote for $n > 0$

$$\partial_n := \frac{\partial}{\partial \varphi_n} = \sqrt{n}(\partial_{x_n} - i\partial_{y_n}) \quad \text{and} \quad \partial_{-n} := \frac{\partial}{\partial \varphi_{-n}} = \sqrt{n}(\partial_{x_n} + i\partial_{y_n}).$$

We define on $C^\infty(\mathbb{R}) \otimes \mathcal{P}$ the following operators for $n > 0$:

$$\begin{aligned} \mathbf{A}_n &= \frac{i}{2} \partial_n, & \mathbf{A}_{-n} &= \frac{i}{2} (\partial_{-n} - 2n\varphi_n) \\ \tilde{\mathbf{A}}_n &= \frac{i}{2} \partial_{-n}, & \tilde{\mathbf{A}}_{-n} &= \frac{i}{2} (\partial_n - 2n\varphi_{-n}) \\ \mathbf{A}_0 &= \tilde{\mathbf{A}}_0 = \frac{i}{2} (\partial_c + Q). \end{aligned}$$

These are closable operators in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ satisfying

$$(1.41) \quad \mathbf{A}_n^* = \mathbf{A}_{-n}, \quad \tilde{\mathbf{A}}_n^* = \tilde{\mathbf{A}}_{-n}.$$

Furthermore $\mathbf{A}_n \mathbf{1} = 0$ and $\tilde{\mathbf{A}}_n \mathbf{1} = 0$ for $n > 0$ and we have the commutation relations

$$(1.42) \quad [\mathbf{A}_n, \mathbf{A}_m] = \frac{n}{2} \delta_{n,-m} = [\tilde{\mathbf{A}}_n, \tilde{\mathbf{A}}_m], \quad [\mathbf{A}_n, \tilde{\mathbf{A}}_m] = 0.$$

Thus \mathbf{A}_n and $\tilde{\mathbf{A}}_n$ ($n > 0$) are annihilation operators and $\tilde{\mathbf{A}}_{-n}$ creation operators. As before let $\mathbf{k}, \mathbf{l} \in \mathcal{N}$ and define the polynomials

$$(1.43) \quad \hat{\pi}_{\mathbf{k}\mathbf{l}} = \prod_{n>0} \mathbf{A}_{-n}^{k_n} \tilde{\mathbf{A}}_{-n}^{l_n} \mathbf{1}$$

Then $\hat{\pi}_{\mathbf{k}\mathbf{l}}$ and $\hat{\pi}_{\mathbf{k}'\mathbf{l}'}$ are orthogonal if $\mathbf{k} \neq \mathbf{k}'$ or $\mathbf{l} \neq \mathbf{l}'$ and the $\hat{\pi}_{\mathbf{k}\mathbf{l}}$'s with $|\mathbf{k}| + |\mathbf{l}| = N$ span the eigenspace of \mathbf{P} with eigenvalue N . We denote $\pi_{\mathbf{k}\mathbf{l}} := \hat{\pi}_{\mathbf{k}\mathbf{l}} / \|\hat{\pi}_{\mathbf{k}\mathbf{l}}\|_{L^2(\Omega_{\mathbb{T}})}$ the normalized eigenvectors.¹⁶

To summarize, this basis realizes a factorisation of $L^2(\Omega_{\mathbb{T}})$ as a tensor product of two Fock spaces: let \mathcal{F} be the completion of the linear span of $\prod_{n>0} \mathbf{A}_{-n}^{k_n} \mathbf{1}$ and $\tilde{\mathcal{F}}$ analogously. Then the map

$$(1.44) \quad U : \mathcal{F} \otimes \tilde{\mathcal{F}} \rightarrow L^2(\Omega_{\mathbb{T}})$$

where $U(\mathbf{A}_{-n}^{k_n} \mathbf{1} \otimes \tilde{\mathbf{A}}_{-n}^{l_n} \mathbf{1}) = \hat{\pi}_{\mathbf{k}\mathbf{l}}$ is unitary.

Define the *normal ordered product* by $\mathbf{A}_n \mathbf{A}_m := \mathbf{A}_n \mathbf{A}_m$ if $m > 0$ and $\mathbf{A}_m \mathbf{A}_n$ if $n > 0$ (i.e. annihilation operators are on the right) and then for all $n \in \mathbb{Z}$

$$(1.45) \quad \mathbf{L}_n^0 := -i(n+1)Q\mathbf{A}_n + \sum_{m \in \mathbb{Z}} : \mathbf{A}_{n-m} \mathbf{A}_m :$$

$$(1.46) \quad \tilde{\mathbf{L}}_n^0 := -i(n+1)Q\tilde{\mathbf{A}}_n + \sum_{m \in \mathbb{Z}} : \tilde{\mathbf{A}}_{n-m} \tilde{\mathbf{A}}_m : .$$

These are well defined on $C^\infty(\mathbb{R}) \otimes \mathcal{P}$ (since only a finite number of terms contribute) and they are closable operators satisfying

$$(1.47) \quad (\mathbf{L}_n^0)^* = \mathbf{L}_{-n}^0, \quad (\tilde{\mathbf{L}}_n^0)^* = \tilde{\mathbf{L}}_{-n}^0.$$

Furthermore the vector space $C^\infty(\mathbb{R}) \otimes \mathcal{P}$ is stable under \mathbf{L}_n^0 and $\tilde{\mathbf{L}}_n^0$ for all $n \in \mathbb{Z}$; on $C^\infty(\mathbb{R}) \otimes \mathcal{P}$ the \mathbf{L}_n^0 satisfy the commutation relations of the *Virasoro Algebra*:

$$(1.48) \quad [\mathbf{L}_n^0, \mathbf{L}_m^0] = (n-m)\mathbf{L}_{n+m}^0 + \frac{c_L}{12}(n^3 - n)\delta_{n,-m}$$

where the central charge is

$$c_L = 1 + 6Q^2.$$

These commutation relations can be checked by using the fact, on $C^\infty(\mathbb{R}) \otimes \mathcal{P}$, only finitely many terms contribute in (1.48) and using the commutation relation (1.42). $\tilde{\mathbf{L}}_n^0$ satisfy the same commutation relations (1.48) and commute with the \mathbf{L}_n^0 's. Note also that

$$(1.49) \quad \mathbf{L}_0^0 = \frac{1}{4}(-\partial_c^2 + Q^2) + 2 \sum_{n>0} \mathbf{A}_{-n} \mathbf{A}_n$$

$$(1.50) \quad \tilde{\mathbf{L}}_0^0 = \frac{1}{4}(-\partial_c^2 + Q^2) + 2 \sum_{n>0} \tilde{\mathbf{A}}_{-n} \tilde{\mathbf{A}}_n,$$

so that one can easily check that the $\mu = 0$ Hamiltonian $\mathbf{H}^0 := -\frac{1}{2}\partial_c^2 + \frac{1}{2}Q^2 + \mathbf{P}$ has the following decomposition

$$\mathbf{H}^0 = \mathbf{L}_0^0 + \tilde{\mathbf{L}}_0^0.$$

Remark 1.4. *In the terminology of representation theory, we have a unitary representation of two commuting Virasoro Algebras on $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ (unitary in the sense that (1.47) holds) and this representation is reducible as we will see below by constructing stable sub-representations.*

¹⁶Explicitly: $\hat{\pi}_{\mathbf{k}\mathbf{l}} = \prod_{n>0} (-in)^{k_n+l_n} \varphi_n^{k_n} \varphi_{-n}^{l_n} +$ lower order terms.

First, for $\alpha \in \mathbb{C}$, we define the function on $\mathbb{R} \times L^2(\Omega_{\mathbb{T}})$

$$(1.51) \quad \psi_{\alpha}(c, \varphi) := e^{(\alpha-Q)c}.$$

For $\alpha \in \mathbb{C}$, these are eigenstates of \mathbf{H}^0 , they never belong to $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ but rather to some weighted spaces $e^{\beta|c|}L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for $\beta > |\operatorname{Re}(\alpha) - Q|$, we then call them generalized eigenstates. We have

$$(1.52) \quad \begin{aligned} \mathbf{L}_0^0 \psi_{\alpha} &= \tilde{\mathbf{L}}_0^0 \psi_{\alpha} = \Delta_{\alpha} \psi_{\alpha} \\ \mathbf{L}_n^0 \psi_{\alpha} &= \tilde{\mathbf{L}}_n^0 \psi_{\alpha} = 0, \quad n > 0, \end{aligned}$$

where Δ_{α} is the conformal weight (1.8). In particular $\Delta_{Q+iP} = \frac{1}{4}(P^2 + Q^2)$. By definition ψ_{α} is a *highest weight state* with highest weight Δ_{α} for both algebras. Before defining the so-called descendants of ψ_{α} , we introduce the following definition:

Definition 1.5. A sequence of integers $\nu = (\nu_i)_{i \geq 0}$ is called a *Young diagram* if the mapping $i \mapsto \nu_i$ is non-increasing and if $\nu_i = 0$ for i sufficiently large. We denote by \mathcal{T} the set of all Young diagrams. We will sometimes write $\nu = (\nu_i)_{i \in [1, k]}$ where k is the last integer i such that $\nu_i > 0$ and denote by $|\nu| := \sum_{i \geq 1} \nu_i$ the length of the Young diagram¹⁷.

Given two Young diagrams $\nu = (\nu_i)_{i \in [1, k]}$ and $\tilde{\nu} = (\tilde{\nu}_i)_{i \in [1, j]}$ we denote

$$\mathbf{L}_{-\nu}^0 = \mathbf{L}_{-\nu_k}^0 \cdots \mathbf{L}_{-\nu_1}^0, \quad \tilde{\mathbf{L}}_{-\tilde{\nu}}^0 = \tilde{\mathbf{L}}_{-\tilde{\nu}_j}^0 \cdots \tilde{\mathbf{L}}_{-\tilde{\nu}_1}^0$$

and define

$$(1.53) \quad \psi_{\alpha, \nu, \tilde{\nu}} = \mathbf{L}_{-\nu}^0 \tilde{\mathbf{L}}_{-\tilde{\nu}}^0 \psi_{\alpha}.$$

The vectors $\psi_{\alpha, \nu, \tilde{\nu}}$ are called the *descendants* of ψ_{α} and also belong to some weighted spaces $e^{\beta|c|}L^2$. Then

$$(1.54) \quad \psi_{\alpha, \nu, \tilde{\nu}} = \mathcal{Q}_{\alpha, \nu, \tilde{\nu}} \psi_{\alpha}$$

where $\mathcal{Q}_{\alpha, \nu, \tilde{\nu}}$ is a polynomial: indeed, this follows from the fact that $[\mathbf{A}_n, e^{(\alpha-Q)c}] = [\tilde{\mathbf{A}}_n, e^{(\alpha-Q)c}] = 0$ for all $n \neq 0$, $[\partial_c, e^{(\alpha-Q)c}] = e^{(\alpha-Q)c}(\alpha - Q)\operatorname{Id}$ and that finitely many applications of \mathbf{A}_n and $\tilde{\mathbf{A}}_n$ to $\mathbf{1}$ is a polynomial in \mathcal{P} . Since the operators \mathbf{L}_n and $\tilde{\mathbf{L}}_n$ are built out of the \mathbf{A}_k and $\tilde{\mathbf{A}}_k$ respectively they respect the decomposition (1.44) and in Section 2.7 we show that for $\alpha = Q + iP$ the scalar products in $L^2(\Omega_{\mathbb{T}})$ factorise as follows

$$(1.55) \quad \langle \mathcal{Q}_{Q+iP, \nu, \tilde{\nu}} | \mathcal{Q}_{Q+iP, \nu', \tilde{\nu}'} \rangle_{L^2(\Omega_{\mathbb{T}})} = \delta_{|\nu|, |\nu'|} \delta_{|\tilde{\nu}|, |\tilde{\nu}'|} F_{Q+iP}(\nu, \nu') F_{Q+iP}(\tilde{\nu}, \tilde{\nu}')$$

where the matrices $F_{Q+iP}(\nu, \nu')$ for $|\nu| = |\nu'| = j$ are invertible for each $j \in \mathbb{N}$ and they depend on $P + iQ$: more precisely they are polynomials in the weight $\Delta_{Q+iP} = \frac{1}{4}(P^2 + Q^2)$ and in the central charge c_L . In this context, F_{Q+iP} is called the *Schapovalov form* associated to the conformal weight Δ_{Q+iP} . By using (1.52) and the commutation relations (1.48) with $n = 0$, we directly see that

$$\mathbf{L}_0^0 \psi_{Q+iP, \nu, \tilde{\nu}} = (\Delta_{Q+iP} + |\nu|) \psi_{Q+iP, \nu, \tilde{\nu}}, \quad \tilde{\mathbf{L}}_0^0 \psi_{Q+iP, \nu, \tilde{\nu}} = (\Delta_{Q+iP} + |\tilde{\nu}|) \psi_{Q+iP, \nu, \tilde{\nu}}$$

and thus since $\mathbf{H}^0 = \mathbf{L}_0^0 + \tilde{\mathbf{L}}_0^0$

$$\mathbf{H}^0 \psi_{Q+iP, \nu, \tilde{\nu}} = \left(\frac{1}{2}(P^2 + Q^2) + |\nu| + |\tilde{\nu}| \right) \psi_{Q+iP, \nu, \tilde{\nu}}.$$

The polynomials $\mathcal{Q}_{Q+iP, \nu, \tilde{\nu}}$ with $|\nu| + |\tilde{\nu}| = N$ span the same space as $\pi_{\mathbf{k}\mathbf{l}}$ or $\psi_{\mathbf{k}\mathbf{l}}$ above with $|\mathbf{k}| + |\mathbf{l}| = N$. Let M_{Q+iP}^N be the matrix of change of basis from the basis $\mathcal{Q}_{Q+iP, \nu, \tilde{\nu}}$ indexed by $\nu, \tilde{\nu} \in \mathcal{T}$ such that $|\nu| + |\tilde{\nu}| = N$ to the basis $\psi_{\mathbf{k}\mathbf{l}}$ (1.34) indexed by $\mathbf{k}, \mathbf{l} \in \mathcal{N}$ such that $|\mathbf{k}| + |\mathbf{l}| = N$:

$$(1.56) \quad \forall \mathbf{k}, \mathbf{l} \in \mathcal{N} \text{ with } \lambda_{\mathbf{k}\mathbf{l}} = N, \quad \psi_{\mathbf{k}\mathbf{l}} = \sum_{\nu, \tilde{\nu}, |\nu| + |\tilde{\nu}| = N} M_{Q+iP, \mathbf{k}\mathbf{l}, \nu, \tilde{\nu}}^N \mathcal{Q}_{Q+iP, \nu, \tilde{\nu}}.$$

We have, as matrices on the set $|\nu| + |\tilde{\nu}| = |\nu'| + |\tilde{\nu}'| = N$

$$(1.57) \quad ((M_{Q+iP}^N)^* M_{Q+iP}^N)_{\nu, \tilde{\nu}, \nu', \tilde{\nu}'} = \delta_{|\nu|, |\nu'|} \delta_{|\tilde{\nu}|, |\tilde{\nu}'|} F_{Q+iP}^{-1}(\nu, \nu') F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}')$$

¹⁷This length should not be confused with the length (1.35) of a sequence of integers.

where $F_{Q+iP}^{-1}(\nu, \nu') := 0$ when $|\nu| \neq |\nu'|$, while if $|\nu| = |\nu'| = j \in \mathbb{N}$ we set $F_{Q+iP}^{-1}(\nu, \nu')$ to be the matrix elements of the inverse of the matrix $(F_{Q+iP}(\nu, \nu'))_{\nu, \nu' \in \mathcal{T}_j}$ with $\mathcal{T}_j := \{\nu \in \mathcal{T} \mid |\nu| = j\}$ the set of Young diagrams of length j .

Then, using the representation (1.40) and decomposing the basis $\psi_{\mathbf{k}l}$ in terms of the new basis $\mathcal{Q}_{Q+iP, \nu, \nu'}$, we directly obtain that

$$(1.58) \quad \langle u_1 | u_2 \rangle_2 = \frac{1}{2\pi} \sum_{\nu, \tilde{\nu}, \nu', \tilde{\nu}' \in \mathcal{T}} \int_{\mathbb{R}} \langle u_1 | \psi_{Q+iP, \nu', \tilde{\nu}'} \rangle_2 \langle \psi_{Q+iP, \nu, \tilde{\nu}} | u_2 \rangle_2 F_{P+iQ}^{-1}(\nu, \nu') F_{P+iQ}^{-1}(\tilde{\nu}, \tilde{\nu}') dP.$$

1.5. Diagonalization of the Hamiltonian using the Virasoro algebra. Now we rewrite the diagonalization of the Liouville Hamiltonian for $\mu \neq 0$ in Theorem 1.3 using the the Free Field case. Let us define for $\nu, \tilde{\nu} \in \mathcal{T}$ with $|\nu| + |\tilde{\nu}| = N$

$$\Psi_{Q+iP, \nu, \tilde{\nu}} = \sum_{\mathbf{k}, \mathbf{l}, \lambda_{\mathbf{k}l} = N} (M_{Q+iP}^N)_{\mathbf{k}, \mathbf{l}, \nu, \tilde{\nu}}^{-1} \Psi_{Q+iP, \mathbf{k}, \mathbf{l}}.$$

Notice that in the above definition, we have used the same general notation Ψ for two different variables; one can distinguish them by looking at the subscripts which belong to different spaces. Then using Theorem 1.3, we get

$$\mathbf{H} \Psi_{Q+iP, \nu, \tilde{\nu}} = \left(\frac{Q^2}{2} + \frac{P^2}{2} + |\nu| + |\tilde{\nu}| \right) \Psi_{Q+iP, \nu, \tilde{\nu}}$$

and, using in addition (1.57), we see that for all $u_1, u_2 \in e^{\delta c} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ with $\delta > 0$, we have the decomposition

$$(1.59) \quad \langle u_1 | u_2 \rangle_2 = \frac{1}{2\pi} \sum_{\nu, \tilde{\nu}, \nu', \tilde{\nu}' \in \mathcal{T}} \int_0^\infty \langle u_1 | \Psi_{Q+iP, \nu', \tilde{\nu}'} \rangle_2 \langle \Psi_{Q+iP, \nu, \tilde{\nu}} | u_2 \rangle_2 F_{Q+iP}^{-1}(\nu, \nu') F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}') dP.$$

Remark 1.6. *For the reader with a background on representations of the Virasoro algebra, this result is consistent with the existence of a one-parameter (here μ) family of Virasoro representation modules $\mathcal{V}_{\Delta_{Q+iP, \mu}}$ on $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ (and a copy $\tilde{\mathcal{V}}_{\Delta_{Q+iP, \mu}}$ of $\mathcal{V}_{\Delta_{Q+iP, \mu}}$) such that*

$$(1.60) \quad L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) = \int_{[0, \infty)}^{\oplus} \mathcal{V}_{\Delta_{Q+iP, \mu}} \otimes \tilde{\mathcal{V}}_{\Delta_{Q+iP, \mu}} dP.$$

We will not construct such modules in this paper as it turns out that we can bypass their use by means of analytic continuation of the vectors $\Psi_{Q+iP, \nu, \tilde{\nu}}$ in the variable $\alpha = Q + iP$ as discussed in the following section where we explain the link between the Liouville semigroup and LCFT.

1.6. Liouville Semigroup, LCFT and Conformal Bootstrap. The basic building block of LCFT is the Gaussian Free Field (GFF) X on $\hat{\mathbb{C}}$ with covariance given by (1.2). Let us note here only that X can be viewed as a random element in a Sobolev space $W^s(\hat{\mathbb{C}})$ for $s < 0$ so that the functions $\omega \rightarrow \int X(\omega, x) f(x) dx$ with $f \in C^\infty(\hat{\mathbb{C}})$ are measurable. Recall the definition of LCFT expectations (1.4) for non negative $F : W^s(\hat{\mathbb{C}}) \rightarrow \mathbb{C}$. Let us recall here the following proposition which is the change of coordinates formula for LCFT (or KPZ formula, proved in [DKRV16, Th. 3.5]):

Proposition 1.7. *Let $\psi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a Möbius map and $\langle |F(\phi)| \rangle_{\gamma, \mu} < \infty$. Then*

$$(1.61) \quad \langle F(\phi \circ \psi + Q \ln |\psi'|) \rangle_{\gamma, \mu} = \langle F(\phi) \rangle_{\gamma, \mu}.$$

The connection between LCFT and the Hilbert space $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ discussed in the previous subsection comes from a positivity property of the linear functional $\langle \cdot \rangle_{\gamma, \mu}$ called *reflection positivity*. Briefly, for $B \subset \hat{\mathbb{C}}$ Borel set, let \mathcal{A}_B be the sigma-algebra in $W^s(\hat{\mathbb{C}})$ generated by the functions $g \mapsto \langle g, c + f \rangle$ with $f \in C_0^\infty(B)$ and $c \in \mathbb{R}$ ($\langle \cdot, \cdot \rangle$ stands for duality bracket) and let \mathcal{F}_B denote the \mathcal{A}_B -measurable complex valued functions $F : W^s(\hat{\mathbb{C}}) \rightarrow \mathbb{R}$. Let $\theta : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be the reflection at the equator:

$$(1.62) \quad \theta(z) = 1/\bar{z}$$

and extend θ to $F : W^s(\hat{\mathbb{C}}) \rightarrow \mathbb{C}$ by

$$(1.63) \quad (\Theta F)(g) = F(g \circ \theta).$$

Let $\mathbb{D} = \{|z| < 1\}$ be the unit disk. Recall that $\phi = c + X - 2Q \ln |\cdot|_+$ and let $\mathcal{F}_{\mathbb{D}}^2$ denote the subset of $\mathcal{F}_{\mathbb{D}}$ made up of those F such that $\langle |F(c+X)\Theta F(c+X)| \rangle_{\gamma, \mu} < \infty$. For $F, G \in \mathcal{F}_{\mathbb{D}}^2$ we define

$$(1.64) \quad (F, G)_{\mathbb{D}} := \langle \Theta F(c+X) \overline{G(c+X)} \rangle_{\gamma, \mu}.$$

Reflection positivity is the statement:

Proposition 1.8. *The sesquilinear form (1.64) is non-negative: for all $F \in \mathcal{F}_{\mathbb{D}}^2$*

$$(F, F)_{\mathbb{D}} \geq 0.$$

The canonical Hilbert space $\mathcal{H}_{\mathbb{D}}$ of LCFT is then defined as the completion of $\mathcal{F}_{\mathbb{D}}^2/\mathcal{N}_0$, where \mathcal{N}_0 is the null space $\mathcal{N}_0 = \{F \in \mathcal{F}_{\mathbb{D}} \mid (F, F)_{\mathbb{D}} = 0\}$, with respect to the sesquilinear form (1.64). In Section 2, we construct a map

$$(1.65) \quad U : \mathcal{F}_{\mathbb{D}} \rightarrow L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$$

which descends to a unitary map from $\mathcal{H}_{\mathbb{D}}$ onto $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$, denoted also by U .

The dilation $z \in \mathbb{C} \rightarrow s_q(z) = qz$ maps \mathbb{D} to itself for $|q| \leq 1$ and it extends to a map on distributions $X \in W^s(\hat{\mathbb{C}})$ by $X \rightarrow X \circ s_q$. We then define for $F \in \mathcal{F}_{\mathbb{D}}$

$$(1.66) \quad (S_q F)(X) = F(c + X \circ s_q + Q \log |q|).$$

In Proposition 2.5 we show that S_q extends to $\mathcal{H}_{\mathbb{D}}$ and defines a strongly continuous contraction semigroup

$$(1.67) \quad S_q S_{q'} = S_{qq'}$$

with the operator norm $\|S_q\| \leq 1$. Taking $q = e^{-t}$ with $t \geq 0$ we then prove that

$$(1.68) \quad U S_{e^{-t}} U^{-1} = e^{-t\mathbf{H}}$$

where \mathbf{H} is the self-adjoint operator defined in (1.19) generating a contraction semigroup $e^{-t\mathbf{H}}$. These observations lead to the formula (1.21) with

$$U_{\alpha, \beta}(z_1, z_2) := \lim_{\epsilon \rightarrow 0} U(V_{\alpha, \epsilon}(z_1) V_{\beta, \epsilon}(z_2)),$$

this limit being computed with the Girsanov transform applied to the field $X_{\mathbb{D}}$ to get the exact formula (1.22).

The completeness relation (1.59) then yields

$$(1.69) \quad \begin{aligned} & \langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(z') V_{\alpha_4}(\infty) \rangle_{\gamma, \mu} = |z'|^{-4\Delta_{\alpha_3}} \langle U_{\alpha_1, \alpha_2}(0, z) | U_{\alpha_4, \alpha_3}(0, \frac{1}{z'}) \rangle_2 = \\ & |z'|^{-4\Delta_{\alpha_3}} \sum_{\nu, \tilde{\nu}, \nu', \tilde{\nu}'} \int_0^\infty \langle U_{\alpha_1, \alpha_2}(0, z) | \Psi_{Q+iP, \nu, \tilde{\nu}} \rangle_2 \langle \Psi_{Q+iP, \nu, \tilde{\nu}} | U_{\alpha_4, \alpha_3}(0, \frac{1}{z'}) \rangle_2 F_{Q+iP}^{-1}(\nu, \nu') F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}') dP. \end{aligned}$$

Now, a crucial step in the proof is, for each $\nu, \tilde{\nu}$, the analytic extension of the eigenvectors $\Psi_{Q+iP, \nu, \tilde{\nu}}$ of \mathbf{H} in the parameter $\alpha := Q + iP$:

Proposition 1.9. *For each $\nu, \tilde{\nu} \in \mathcal{T}$ with $|\nu| + |\tilde{\nu}| = \lambda$, there is a connected open set $W_\lambda \subset \{\text{Re}(\alpha) \leq Q\}$ containing $(-\infty, Q - A] \cup (Q + i\mathbb{R})$ for some $A > 0$ such that*

$$\alpha \in W_\lambda \mapsto \Psi_{\alpha, \nu, \tilde{\nu}} \in \bigcup_{\beta \leq 0} e^{\beta c} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$$

is analytic except possibly at a discrete set $D \subset Q + i\mathbb{R}$. More precisely, on each compact subset $K \subset W_\lambda$, there is $\beta < 0$ such that $\alpha \mapsto \Psi_{\alpha, \nu, \tilde{\nu}} \in e^{\beta c} L^2$ is analytic in $K \setminus D$.

This Proposition is a consequence¹⁸ of Propositions 3.16 and 3.18. In fact, we show even more there, since we prove that the $\Psi_{Q+iP, \nu, \tilde{\nu}}$ admit a meromorphic extension across the imaginary line $\text{Re}(\alpha) = Q$ (where α lives in some Riemann surface). Quite remarkably, one can give probabilistic expressions of the eigenvectors $\Psi_{\alpha, \nu, \tilde{\nu}}$ for real negative $\alpha \in (-\infty, Q - A]$. Exploiting these expressions along with conformal

¹⁸We emphasize that the parameter α in Propositions 3.18 and 3.16 is not the same as the α_λ of Proposition 1.9 when $|\nu| + |\tilde{\nu}| > 0$: they are related by $(\alpha - Q)^2 = (\alpha_\lambda - Q)^2 - 2\lambda$.

invariance properties (the so-called Ward identities), we can give *exact* analytic expressions for the scalar products $\langle \Psi_{\alpha, \nu, \tilde{\nu}} | U_{\alpha_1, \alpha_2}(0, z) \rangle_2$ when $\alpha < Q - A$, and these expressions admits analytic continuation in $\alpha \in W_\lambda$ up to our line of interest $Q + i\mathbb{R}$. This is the content of our second main result. For $\nu = (\nu_i)_{i \in [1, k]}$ a Young diagram and some real $\Delta, \Delta', \Delta''$ we set

$$v(\Delta, \Delta', \Delta'', \nu) := \prod_{j=1}^k (\nu_j \Delta' - \Delta + \Delta'' + \sum_{u < j} \nu_u).$$

With this notation, we can state the following key result:

Proposition 1.10 (Ward identities). *For all P such that $\alpha = Q + iP$ belongs to W_λ , thus in particular for $P > 0$, the scalar product $\langle \Psi_{Q+iP, \nu, \tilde{\nu}} | U_{\alpha_1, \alpha_2}(0, z) \rangle_2$ is explicitly given by the following expression*

$$(1.70) \quad \langle \Psi_{Q+iP, \nu, \tilde{\nu}} | U_{\alpha_1, \alpha_2}(0, z) \rangle_2 = v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \tilde{\nu}) v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \nu) \\ \times \frac{1}{2} C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, Q + iP) \bar{z}^{|\nu|} z^{|\tilde{\nu}|} |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})}$$

where Δ_α are conformal weights (1.8) and $C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, Q + iP)$ are the DOZZ structure constants (see appendix A for the definition).

As an immediate corollary of the above Proposition and the completeness relation (1.69), we get the main result of this paper by letting z' go to 1:

$$(1.71) \quad \langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle \\ = \frac{1}{8\pi} \int_{\mathbb{R}^+} C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, Q - iP) C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_3, \alpha_4, Q + iP) |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_P(z)|^2 dP$$

where \mathcal{F}_P are the so-called (spherical) holomorphic conformal blocks given by

$$(1.72) \quad \mathcal{F}_P(z) := \sum_{n=0}^{\infty} \beta_n z^n$$

where the β_n have the following expression

$$(1.73) \quad \beta_n := \sum_{|\nu|, |\nu'|=n} v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \nu) F_{Q+iP}^{-1}(\nu, \nu') v(\Delta_{\alpha_4}, \Delta_{\alpha_3}, \Delta_{Q+iP}, \nu').$$

We prove, using the spectral decomposition and positivity arguments, that the sum (1.72) converges for almost all P (with respect to the Lebesgue measure) in $z \in \mathbb{D}$.

1.7. Organization of the paper. The paper is organized as follows. In Section 2, we will introduce the relevant material on the Gaussian Free Field and the construction of the Liouville semigroup; we will also recall the OS reconstruction theorem which relates LCFT to the Liouville semigroup. In Section 3, by general arguments on semigroups, we will prove the decomposition 1.3. In Section 4, we will prove the main result of the paper Theorem 1.1 using the material proved in Sections 3, 5 and 6. In Sections 5 and 6, we will prove the Ward identities for the correlations of LCFT. Finally, in the appendix, we will recall the DOZZ formula and gather auxiliary results (analyticity of vertex operators).

1.8. Notations and conventions: All sesquilinear forms are supposed to be linear in their first argument, antilinear in their second one. In the paper, many different scalar products will appear; here we recall the main notations:

- $\langle \cdot, \cdot \rangle_2$ will denote the scalar product in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$, the space of (complex) square integrable functions which depend on (c, φ) and $\langle \cdot, \cdot \rangle_{L^2(\Omega_{\mathbb{T}})}$ will denote the scalar product in $L^2(\Omega_{\mathbb{T}})$. The associated norms will be denoted $\|\cdot\|_2$ and $\|\cdot\|_{L^2(\Omega_{\mathbb{T}})}$ respectively. More generally, we will also consider the standard spaces $L^p(\mathbb{R} \times \Omega_{\mathbb{T}})$ and $L^p(\Omega_{\mathbb{T}})$ for $p \geq 1$ where the norms are denoted $\|\cdot\|_p$ and $\|\cdot\|_{L^p(\Omega_{\mathbb{T}})}$.
- $(\cdot, \cdot)_{\mathbb{D}}$ will denote the scalar product associated to reflexion positivity. The completion with respect to this scalar product (after taking the quotient by null sets) will be denoted $\mathcal{H}_{\mathbb{D}}$.
- For functions f, g defined on the circle, we define $\langle f, g \rangle_{\mathbb{T}} := \int_0^{2\pi} f(\theta) g(\theta) d\theta$ and for functions f, g defined on the unit disk, we define $\langle f, g \rangle_{\mathbb{D}} := \int_{\mathbb{D}} f(x) g(x) dx$.

- We will work with different Green functions: the (Dirichlet) Green function $G_{\mathbb{D}}(x, y) = \ln \frac{|1-x\bar{y}|}{|x-y|}$ for $x, y \in \mathbb{D}$ and the Green function on the unit circle $G_{\mathbb{T}}(e^{i\theta}, e^{i\theta'}) = \ln \frac{1}{|e^{i\theta} - e^{i\theta'}|}$. We will also denote $G_{\mathbb{D}}f(x) = \int_{\mathbb{D}} G_{\mathbb{D}}(x, y)f(y)dy$ for some function f on \mathbb{D} and $G_{\mathbb{T}}f(e^{i\theta}) = \int_0^{2\pi} G_{\mathbb{T}}(e^{i\theta}, e^{i\theta'})f(e^{i\theta'})d\theta'$ for some function f on \mathbb{T} .

We will also denote the expectation of LCFT by $\langle \cdot \rangle_{\gamma, \mu}$. When we consider a quantity with respect to the GFF theory (which corresponds to $\mu = 0$), we will put a subscript or superscript 0: for example the associated Hamiltonian will be denoted \mathbf{H}^0 or the unitary map U_0 .

2. RADIAL QUANTIZATION

The main purpose of this section is to explain the isometry U in (1.65) mapping the LCFT observables to states in the Hilbert space $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ and the relation (1.68) between the two semigroups. We start by a discussion of the various GFF's.

2.1. Gaussian Free Fields. Recall that the GFF on the Riemann sphere X decomposes as the sum of three independent variables

$$(2.1) \quad X = P(\varphi) + X_{\mathbb{D}} + X_{\mathbb{D}^c}$$

where P is the harmonic extension of the circle GFF φ defined on $\Omega_{\mathbb{T}}$ and $X_{\mathbb{D}}, X_{\mathbb{D}^c}$ are two independent GFFs defined on \mathbb{D} and \mathbb{D}^c with Dirichlet boundary conditions. We now recall the exact definition of the three variables.

Dirichlet GFF on the unit disk. The Dirichlet GFF $X_{\mathbb{D}}$ on the unit disk \mathbb{D} is the centered Gaussian distribution (in the sense of Schwartz) with covariance kernel $G_{\mathbb{D}}$ given by

$$(2.2) \quad G_{\mathbb{D}}(x, x') := \mathbb{E}[X_{\mathbb{D}}(x)X_{\mathbb{D}}(x')] = \ln \frac{|1 - x\bar{x}'|}{|x - x'|}.$$

Here, $G_{\mathbb{D}}$ is the Green function of the negative of the Laplacian $\Delta_{\mathbb{D}}$ with Dirichlet boundary condition on $\mathbb{T} = \partial\mathbb{D}$ and $X_{\mathbb{D}}$ can be realized as an expansion in eigenfunctions $\Delta_{\mathbb{D}}$ with Gaussian coefficients. However, it will be convenient for us to use another realization based on the following observation. Let, for $n \in \mathbb{Z}$

$$X_n(t) = \int_0^{2\pi} e^{-in\theta} X_{\mathbb{D}}(e^{-t+i\theta}) \frac{d\theta}{2\pi}.$$

Then we deduce from (2.2)

$$(2.3) \quad \mathbb{E}[X_n(t)X_m(t')] = \begin{cases} \frac{1}{2|n|} \delta_{n, -m} (e^{-|t-t'||n|} - e^{-(t+t')|n|}) & n \neq 0 \\ t \wedge t' & n = m = 0 \end{cases}$$

Thus $\{X_n\}_{n \geq 0}$ are independent Gaussian processes with $X_{-n} = \bar{X}_n$ and X_0 is Brownian motion. We can and will realize them in a probability space $\Omega_{\mathbb{D}}$ s.t. $X_n(t)$ have continuous sample paths. Then for fixed t

$$(2.4) \quad X_{\mathbb{D}}(e^{-t+i\theta}) = \sum_{n \in \mathbb{Z}} X_n(t) e^{in\theta}.$$

takes values in $W^s(\mathbb{T})$ for $s < 0$ a.s. and we can take the map $t \in \mathbb{R}^+ \rightarrow X_{\mathbb{D}}(e^{-t+i\cdot}) \in W^s(\mathbb{T})$ to be continuous a.s. in $\Omega_{\mathbb{D}}$.

The Dirichlet GFF $X_{\mathbb{D}^c}$ on the complement \mathbb{D}^c of \mathbb{D} can be constructed in the same way. Recalling the reflection (1.62) $\theta: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ with respect to the unit circle $\theta(z) = 1/\bar{z}$ we have the relation in law

$$(2.5) \quad X_{\mathbb{D}^c} \stackrel{law}{=} X_{\mathbb{D}} \circ \theta$$

or in other words $X_{\mathbb{D}^c}(e^{t+i\cdot}) \stackrel{law}{=} X_{\mathbb{D}}(e^{-t+i\cdot})$, $t \geq 0$.

Harmonic extension of the GFF on \mathbb{T} . The second ingredient we need for the decomposition of the GFF (2.1) is the harmonic extension $P\varphi$ of the circle GFF defined on $z \in \mathbb{D}$ by

$$(2.6) \quad (P\varphi)(z) = \sum_{n \geq 1} (\varphi_n z^n + \bar{\varphi}_n \bar{z}^n)$$

and on $z \in \mathbb{D}^c$ by $(P\varphi)(1/\bar{z})$ so that we have

$$P\varphi = (P\varphi) \circ \theta.$$

$P\varphi$ is a.s. a smooth field in the complement of the equator with covariance kernel given for $z, u \in \mathbb{D}$

$$\mathbb{E}[(P\varphi)(z)(P\varphi)(u)] = \frac{1}{2} \sum_{n>0} \frac{1}{n} ((z\bar{u})^n + (\bar{z}u)^n) = -\ln|1 - z\bar{u}|$$

and for $z \in \mathbb{D}, u \in \mathbb{D}^c$

$$\mathbb{E}[(P\varphi)(z)(P\varphi)(u)] = -\ln|1 - z/u|.$$

Harmonic extension of the GFF on \mathbb{T} . One can check that adding the covariances in the previous subsections we get that the field X defined by (2.1) has the covariance

$$(2.7) \quad \mathbb{E}[X(x)X(y)] = \ln \frac{1}{|x-y|} + \ln|x|_+ + \ln|y|_+$$

Furthermore X is given as a process $X_t \in W^s(\mathbb{T})$

$$X_t(\theta) = X_{\mathbb{D}}(e^{-t+i\theta})1_{t>0} + X_{\mathbb{D}^c}(e^{-t+i\theta})1_{t<0} + (P\varphi)(e^{-|t|+i\theta})$$

In the sequel, we will sometimes write $X_{\mathbb{D}}^{(1)} = X_{\mathbb{D}}$ and $X_{\mathbb{D}}^{(2)} = X_{\mathbb{D}^c} \circ \theta$ which are two independent GFFs in the unit disk.

2.2. Reflection positivity.

Reflection positivity of the GFF. Recall that $\langle \cdot | \cdot \rangle_2$ denotes the scalar product in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$, the notation $\mathbb{E}_{\varphi} = \mathbb{E}[\cdot | \Sigma_{\mathbb{T}}]$ and the definitions introduced in Subsection 1.6. Let $F, G \in \mathcal{F}_{\mathbb{D}}$ be nonnegative. The sesquilinear form (1.64) becomes at $\mu = 0$

$$(2.8) \quad (F, G)_{\mathbb{D},0} = \int_{\mathbb{R}} e^{-2Qc} \mathbb{E}[(\Theta F)(c+X) \overline{G(c+X)}] dc = \int_{\mathbb{R}} e^{-2Qc} \mathbb{E}[F(c+X^{(2)}) \overline{G(c+X^{(1)})}] dc$$

where $X^{(i)} = X_{\mathbb{D}}^{(i)} + P\varphi$. Hence by independence of $X_{\mathbb{D}}^{(i)}$

$$(2.9) \quad \begin{aligned} (F, G)_{\mathbb{D},0} &= \int_{\mathbb{R}} e^{-2Qc} \mathbb{E}[\mathbb{E}_{\varphi}[F(c+X_{\mathbb{D}}^{(2)} + P\varphi)] \overline{\mathbb{E}_{\varphi}[G(c+X_{\mathbb{D}}^{(1)} + P\varphi)}] dc \\ &= \langle U_0 F | U_0 G \rangle_2 \end{aligned}$$

where the map $U_0 : \mathcal{F}_{\mathbb{D}} \rightarrow L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ is given by

$$(2.10) \quad (U_0 F)(c, \varphi) = e^{-Qc} \mathbb{E}_{\varphi}[F(c+X_{\mathbb{D}} + P\varphi)].$$

We then have

Proposition 2.1. *The sesquilinear form (2.8) extends to*

$$\mathcal{F}_{\mathbb{D}}^{0,2} := \{F \in \mathcal{F}_{\mathbb{D}} \mid \|U_0 F\|_2 < \infty\}$$

and this extension is non negative

$$(2.11) \quad (F, F)_{\mathbb{D},0} \geq 0$$

for all $F \in \mathcal{F}_{\mathbb{D}}^{0,2}$ and the map U_0 in (2.10) descends to a unitary map $U_0 : \overline{\mathcal{F}_{\mathbb{D}}^{0,2} / \mathcal{N}_0^0} \rightarrow L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ where $\mathcal{N}_0^0 := \{F \in \mathcal{F}_{\mathbb{D}}^2 \mid (F, F)_{\mathbb{D},0} = 0\}$.

Proof. By (2.9) U_0 extends to an isometry so we need to show it is onto. Denote $\langle f, g \rangle_{\mathbb{D}} = \int_{\mathbb{D}} f(x)g(x)dx$ and $\langle f, g \rangle_{\mathbb{T}} = \int_0^{2\pi} f(\theta)g(\theta)d\theta$ and consider $F \in \mathcal{F}_{\mathbb{D}}$ of the form

$$(2.12) \quad F(c + X) = \rho(\langle c + X, g \rangle_{\mathbb{D}}) e^{\langle c + X, f \rangle_{\mathbb{D}} - \frac{1}{2} \langle f, G_{\mathbb{D}} f \rangle_{\mathbb{D}}}$$

where $\rho \in C_0^\infty(\mathbb{R})$, $g, f \in C_0^\infty(\mathbb{D})$. We take g rotation invariant i.e. $g(re^{i\theta}) = g(r)$ and f s.t. $\int_0^{2\pi} f(re^{i\theta})d\theta = 0$ for all $r \in [0, 1]$. Then $\langle c, f \rangle_{\mathbb{D}} = 0$ and $\langle P\varphi, g \rangle_{\mathbb{D}} = 0$ and we get

$$(2.13) \quad \begin{aligned} (U_0 F)(c, \varphi) &= e^{-Qc} e^{\langle P\varphi, f \rangle_{\mathbb{D}}} \mathbb{E}[\rho(c + \langle X_{\mathbb{D}}, g \rangle_{\mathbb{D}}) e^{\langle X_{\mathbb{D}}, f \rangle_{\mathbb{D}} - \frac{1}{2} \langle f, G_{\mathbb{D}} f \rangle_{\mathbb{D}}}] \\ &= e^{-Qc} e^{\langle P\varphi, f \rangle_{\mathbb{D}}} \mathbb{E}[\rho(c + \langle X_{\mathbb{D}}, g \rangle_{\mathbb{D}})] \end{aligned}$$

where we observed that $\langle X_{\mathbb{D}}, g \rangle_{\mathbb{D}}$ and $\langle X_{\mathbb{D}}, f \rangle_{\mathbb{D}}$ are independent as their covariance vanishes. Indeed, by rotation invariance of g , the function $\mathcal{O}(r, \theta) := \int_{\mathbb{D}} g(x)G_{\mathbb{D}}(x, re^{i\theta})dx$ does not depend on θ hence

$$\begin{aligned} \mathbb{E}[\langle X_{\mathbb{D}}, g \rangle_{\mathbb{D}} \langle X_{\mathbb{D}}, f \rangle_{\mathbb{D}}] &= \int_{\mathbb{D}} \int_{\mathbb{D}} G_{\mathbb{D}}(x, y) g(x) f(y) dy \\ &= \int_0^1 r \int_0^{2\pi} f(re^{i\theta}) \mathcal{O}(r, \theta) d\theta dr \\ &= \int_0^1 r \mathcal{O}(r, 0) \int_0^{2\pi} f(re^{i\theta}) d\theta dr = 0. \end{aligned}$$

Let $h \in C^\infty(\mathbb{T})$ and $f_\epsilon \in C_0^\infty(\mathbb{D})$ and g_ϵ be given by $g_\epsilon(re^{i\theta}) = \epsilon^{-1} \eta(\frac{1-r}{\epsilon})$, $f_\epsilon = hg_\epsilon$ where η is a smooth bump with support on $[1, 2]$ and total mass one. Then $\lim_{\epsilon \rightarrow 0} \langle P\varphi, f_\epsilon \rangle_{\mathbb{D}} = \langle \varphi, h \rangle_{\mathbb{T}}$ and $\lim_{\epsilon \rightarrow 0} \mathbb{E}(\langle X_{\mathbb{D}}, g_\epsilon \rangle_{\mathbb{D}}^2) = 0$ so that

$$\lim_{\epsilon \rightarrow 0} (U_0 F_\epsilon)(c, \varphi) = e^{-Qc} \rho(c) e^{\langle \varphi, h \rangle_{\mathbb{T}}}$$

where the convergence is in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. Thus the functions $e^{-Qc} \rho(c) e^{\langle \varphi, h \rangle_{\mathbb{T}}}$ are in the image of U_0 for all $\rho \in C_0^\infty(\mathbb{R})$ and $h \in C^\infty(\mathbb{T})$. Since the linear span of these is dense in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ the claim follows. \square

Remark 2.2. Note that this argument shows U_0 extends from $\mathcal{F}_{\mathbb{D}}^{0,2}$ to functionals of form $F(c + X|_{\mathbb{T}})$ and then

$$(U_0 F)(c, \varphi) = e^{-Qc} F(c + \varphi).$$

Reflection positivity of LCFT. Next we want to show reflection positivity for the LCFT expectation (1.4). The GMC measure M_γ can be defined as the martingale limit

$$(2.14) \quad M_\gamma(dx) = \lim_{N \rightarrow \infty} e^{\gamma X_N(x) - \frac{\gamma^2}{2} \mathbb{E}[X_N(x)^2]} |x|_+^{-4} dx.$$

where in X_N we cut off the series (2.4) and (1.25) defining $X_{\mathbb{D}}^{(i)}$ and φ respectively to finite number of terms $|n| \leq N$. We claim that

$$M_\gamma(\hat{\mathbb{C}}) = M_\gamma^{(1)}(\mathbb{D}) + M_\gamma^{(2)}(\mathbb{D})$$

where $M_\gamma^{(i)}$ are the GMC measures of the fields $X^{(i)} = X_{\mathbb{D}}^{(i)} + P\varphi$, $i = 1, 2$. Indeed,

$$\int_{\mathbb{D}^c} e^{\gamma X_N(x) - \frac{\gamma^2}{2} \mathbb{E}[X_N(x)^2]} |x|^{-4} dx = \int_{\mathbb{D}^c} e^{\gamma X_N^{(2)}(\frac{1}{x}) - \frac{\gamma^2}{2} \mathbb{E}[X_N^{(2)}(\frac{1}{x})^2]} |x|^{-4} dx = \int_{\mathbb{D}} e^{\gamma X_N^{(2)}(x) - \frac{\gamma^2}{2} \mathbb{E}[X_N^{(2)}(x)^2]} dx.$$

Thus, for nonnegative $F, G \in \mathcal{F}_{\mathbb{D}}$

$$(F, G)_{\mathbb{D}} = \langle \Theta F \bar{G} \rangle_{\gamma, \mu} = \langle U_0(IF) | U_0(IG) \rangle_2$$

where

$$I = e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D})}.$$

Let $\mathcal{F}_{\mathbb{D}}^2$ be the set of $F \in \mathcal{F}_{\mathbb{D}}$ such that $\|UF\|_2 < \infty$. From the above considerations, we arrive at:

Proposition 2.3. *The sesquilinear form (1.64) extends to $\mathcal{F}_{\mathbb{D}}^2$, is nonnegative and given by*

$$(2.15) \quad (F, G)_{\mathbb{D}} = \langle UF | UG \rangle_2$$

for all $F, G \in \mathcal{F}_{\mathbb{D}}^2$ where

$$(2.16) \quad (UF)(c, \varphi) = (U_0(FI))(c, \varphi) = e^{-Qc} \mathbb{E}_{\varphi} [F(c + X) e^{-\mu e^{\gamma c} M_{\gamma}(\mathbb{D})}],$$

$X = X_{\mathbb{D}} + P\varphi$ and M_{γ} is its GMC measure. Define $\mathcal{N}_0 := \{F \in \mathcal{F}_{\mathbb{D}}^2 \mid (F, F)_{\mathbb{D}} = 0\}$. Then U descends to a unitary map

$$U : \mathcal{H}_{\mathbb{D}} \rightarrow L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$$

with $\mathcal{H}_{\mathbb{D}} := \overline{\mathcal{F}_{\mathbb{D}}^2 / \mathcal{N}_0}$ (the completion with respect to $(\cdot, \cdot)_{\mathbb{D}}$).

Proof. We need to show U is onto. This follows from $U(I^{-1}F) = U_0F$ and the fact that U_0 is onto. \square

Remark 2.4. *From Remark 2.2 we conclude that U extends from $\mathcal{F}_{\mathbb{D}}$ to functionals $F(c + X|_{\mathbb{T}})$ for which*

$$(UF)(c, \varphi) = F(c + \varphi)U1$$

or, in other words, for $f \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$

$$(2.17) \quad U^{-1}f = (U1)^{-1}f.$$

2.3. Dilation Semigroup. For $q \in \mathbb{C}$ and f a function (or a distribution) on $\hat{\mathbb{C}}$ define the dilation map s_q by

$$(s_q f)(z) = f(qz).$$

Let $|q| \leq 1$. Then we can define the action of s_q on measurable functions on $W^s(\hat{\mathbb{C}})$ by

$$S_q F : u \in W^s(\hat{\mathbb{C}}) \mapsto S_q F(u) := F(s_q u + Q \ln |q|).$$

The reason for the $Q \ln |q|$ -factor is the Möbius invariance property of LCFT [DKRV16]. Recall that Möbius maps $\psi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ act on volume measures $dv_g(x) = g(x)dx$ as $\psi^* v_g = e^{\varphi} v_g$ where

$$\varphi = \ln g \circ \psi + 2 \ln |\psi'|.$$

We have then:

Proposition 2.5. *The map S_q extends to a contraction on $S_q : \mathcal{H}_{\mathbb{D}} \rightarrow \mathcal{H}_{\mathbb{D}}$ i.e for all $F \in \mathcal{H}_{\mathbb{D}}$*

$$(2.18) \quad (S_q F, S_q F)_{\mathbb{D}} \leq (F, F)_{\mathbb{D}}.$$

The adjoint of S_q is $S_q^* = S_{\bar{q}}$ i.e. for all $F, G \in \mathcal{H}_{\mathbb{D}}$

$$(2.19) \quad (S_q F, G)_{\mathbb{D}} = (F, S_{\bar{q}} G)_{\mathbb{D}}.$$

Finally the map $q \in \mathbb{D} \rightarrow S_q$ is strongly continuous and satisfies the group property

$$(2.20) \quad S_q S_{q'} = S_{qq'}$$

so that $q \in \mathbb{D} \rightarrow S_q$ is a strongly continuous contraction semigroup.

Proof. Let us start with (2.43). It suffices to consider $F, G \in \mathcal{F}_{\mathbb{D}}^2$ real. Then

$$\begin{aligned} (S_q F, G)_{\mathbb{D}} &= \langle F((c + X(\frac{1}{q\bar{x}}) + Q \ln |q|)_{|x| \leq 1}) G((c + X(x))_{|x| \leq 1}) \rangle_{\gamma, \mu} \\ &= \langle F((\phi \circ \psi_{\bar{q}})(\frac{1}{x}) + Q \ln |\psi'_{\bar{q}}(\frac{1}{x})| + 2Q \ln |\psi_{\bar{q}}(\frac{1}{x})| + 2Q \ln |q|)_{|x| \leq 1}) \\ &\quad \times G((c + (\phi \circ \psi_{\bar{q}})(\bar{q}x) + Q \ln |\psi'_{\bar{q}}(x)| + Q \ln |q|)_{|x| \leq 1}) \rangle_{\gamma, \mu} \end{aligned}$$

where $\psi_{\bar{q}}(u) = \frac{u}{\bar{q}}$ and thus $\psi'_{\bar{q}} = \frac{1}{\bar{q}}$. We apply Proposition 1.7 (change of coordinate formula) with $\psi_{\bar{q}}$:

$$\begin{aligned} (S_q F, G)_{\mathbb{D}} &= \langle F((\phi \circ \psi_{\bar{q}})\left(\frac{1}{\bar{x}}\right) + Q \ln |\psi'_{\bar{q}}\left(\frac{1}{\bar{x}}\right)| + 2Q \ln |\psi_{\bar{q}}\left(\frac{1}{\bar{x}}\right)| + 2Q \ln |q|)_{|x| \leq 1}) \\ &\quad \times G((c + (\phi \circ \psi_{\bar{q}})(\bar{q}x) + Q \ln |\psi'_{\bar{q}}(x)| + Q \ln |q|)_{|x| \leq 1}) \rangle_{\gamma, \mu} \\ &= \langle F(\phi\left(\frac{1}{\bar{x}}\right) + 2Q \ln |\psi_{\bar{q}}\left(\frac{1}{\bar{x}}\right)| + 2Q \ln |q|)_{|x| \leq 1}) G((c + \phi(\bar{q}x) + Q \ln |q|)_{|x| \leq 1}) \rangle_{\gamma, \mu} \\ &= (F, S_{\bar{q}} G)_{\mathbb{D}}. \end{aligned}$$

The group property (2.20) is obvious.

To prove the contraction, denote for $F \in \mathcal{F}_{\mathbb{D}}$, the seminorm $\|F\|_{\mathbb{D}} := (F, F)_{\mathbb{D}}^{\frac{1}{2}}$. Then we have

$$\|S_q F\|_{\mathbb{D}} = (S_q F, S_q F)_{\mathbb{D}}^{\frac{1}{2}} = (F, S_{|q|^2} F)_{\mathbb{D}}^{\frac{1}{2}} \leq \|F\|_{\mathbb{D}}^{\frac{1}{2}} \|S_{|q|^2} F\|_{\mathbb{D}}^{\frac{1}{2}}.$$

Iterating this inequality we obtain

$$\|S_q F\|_{\mathbb{D}} \leq \|F\|_{\mathbb{D}}^{1-2^{-k}} \|S_{|q|^{2k}} F\|_{\mathbb{D}}^{2^{-k}}.$$

Recall that

$$(G, G)_{\mathbb{D}} = \langle U_0(IG) | U_0(IG) \rangle_2 = \int_{\mathbb{R}} e^{-2Qc} \mathbb{E}[\mathbb{E}_{\varphi}[IG]^2] dc$$

and then by Cauchy-Schwartz applied to $\mathbb{E}_{\varphi}[\cdot]$

$$\mathbb{E}[\mathbb{E}_{\varphi}[IG]^2] = \mathbb{E}[\mathbb{E}_{\varphi}[I^{\frac{1}{2}} G I^{\frac{1}{2}}]^2] \leq \mathbb{E}[\mathbb{E}_{\varphi}[IG^2] \mathbb{E}_{\varphi}[I]]$$

so that

$$(G, G)_{\mathbb{D}} \leq \langle U_0(IG^2) | U_0 I \rangle_2 = \langle UG^2 | U1 \rangle_2 = \langle G^2 \rangle_{\gamma, \mu}.$$

Hence

$$\|S_q F\|_{\mathbb{D}} \leq \|F\|_{\mathbb{D}}^{1-2^{-k}} \langle (S_{|q|^{2k}} F)^2 \rangle_{\gamma, \mu}^{2^{-k-1}} = \|F\|_{\mathbb{D}}^{1-2^{-k}} \langle F^2 \rangle_{\gamma, \mu}^{2^{-k-1}}$$

where we used again the Möbius invariance of $\langle \cdot \rangle_{\gamma, \mu}$. Taking $k \rightarrow \infty$ we conclude $\|S_q F\|_{\mathbb{D}} \leq \|F\|_{\mathbb{D}}$ for $F \in \mathcal{F}_{\mathbb{D}}$ which satisfy $\langle F^2 \rangle_{\gamma, \mu} < \infty$. Such F form a dense set in $\mathcal{F}_{\mathbb{D}}$. Indeed, let $F \in \mathcal{F}_{\mathbb{D}}$ with $\|F\|_{\mathbb{D}} < \infty$ and let $F_R = F 1_{|F| < R}$. Then $\langle F_R^2 \rangle_{\gamma, \mu} < \infty$ and

$$\|F - F_R\|_{\mathbb{D}}^2 = \|F 1_{|F| \geq R}\|_{\mathbb{D}}^2 \leq \|F\|_{\mathbb{D}}^2 \|1_{|F| \geq R}\|_{\mathbb{D}}^2$$

and $\|1_{|F| \geq R}\|_{\mathbb{D}}^2 = \langle 1_{F \geq R} \Theta 1_{|F| \geq R} \rangle \leq \frac{1}{R^2} \langle |F| \Theta |F| \rangle \rightarrow 0$ as $R \rightarrow \infty$.

Hence (2.18) holds for all $F \in \mathcal{F}_{\mathbb{D}}$ with $\|F\|_{\mathbb{D}} < \infty$. This implies S_q maps the null space \mathcal{N}_0 to \mathcal{N}_0 and thus S_q extends to $\mathcal{H}_{\mathbb{D}}$ so that (2.18) holds.

Finally to prove strong continuity, by the semigroup property it suffices to prove it at $q = 1$ and by the contractive property we need to prove it only on a dense set. Since

$$\|S_q F - F\|_{\mathbb{D}}^2 = \|S_q F\|_{\mathbb{D}}^2 + \|F\|_{\mathbb{D}}^2 - (S_q F, F)_{\mathbb{D}} - (F, S_q F)_{\mathbb{D}} \leq 2\|F\|_{\mathbb{D}}^2 - (S_q F, F)_{\mathbb{D}} - (F, S_q F)_{\mathbb{D}}$$

it suffices to prove $(S_q F, F)_{\mathbb{D}} \rightarrow (F, F)_{\mathbb{D}}$ as $q \rightarrow 1$ on a dense set of F . Take $F = GI^{-1}$ so that $UF = U_0 G$. Then

$$(F, S_q F)_{\mathbb{D}} = \int_{\mathbb{R}} e^{-2Qc} \mathbb{E}(\Theta G G e^{-\mu e^{\gamma c} M_{\gamma}(\mathbb{D} \setminus |q|\mathbb{D})}) dc$$

which converges as $q \rightarrow 1$ to (F, F) (use $\mathbb{P}(M_{\gamma}(\mathbb{D} \setminus |q|\mathbb{D}) > \epsilon) \rightarrow 0$ as $q \rightarrow 1$). \square

In particular we can form two one-parameter (semi) groups from S_q . Taking $q = e^{-t}$ we define $T_t = S_{e^{-t}}$. Then $T_{t+s} = T_t T_s$ so T_t is a strongly continuous contraction semigroup on the Hilbert space $\mathcal{H}_{\mathbb{D}}$. Hence by the Hille-Yosida theorem

$$(2.21) \quad U S_{e^{-t}} U^{-1} = e^{-t \mathbf{H}_*}$$

where the generator \mathbf{H}_* (in the case $\mu = 0$, we will write \mathbf{H}_*^0) is a positive operator with domain $\mathcal{D}(\mathbf{H}_*)$ consisting of $\psi \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ such that $\lim_{t \rightarrow 0} \frac{1}{t}(e^{-t \mathbf{H}_*} - 1)\psi$ exists in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. \mathbf{H}_* is the *Hamiltonian*

of LCFT. Taking $q = e^{i\alpha}$ we get that $\alpha \rightarrow S_{e^{i\alpha}}$ is a strongly continuous unitary group so that by Stone's theorem

$$US_{e^{i\alpha}}U^{-1} = e^{i\alpha\Pi_*}$$

where Π_* is the self adjoint *momentum* operator of LCFT. As we will have no use for Π_* in this paper we will concentrate on \mathbf{H}_* from now on. Let us emphasize here that it is defined in the full range $\gamma \in (0, 2)$. Our next goal is to show that for $\gamma \in (0, \sqrt{2})$: $\mathbf{H}_* = \mathbf{H}$, where the Hamiltonian \mathbf{H} will be defined as the Friedrichs extension of (1.19).

2.4. Quadratic forms and the Friedrichs extension of \mathbf{H} . In this subsection, we suppose that $\gamma \in (0, \sqrt{2})$ which ensures that $V \in L^{1+\epsilon}(\Omega_{\mathbb{T}})$ for some $\epsilon > 0$. Here we consider the quadratic forms associated to (1.19) and construct the Friedrichs extension of \mathbf{H} . Recall that the underlying measure on the space $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ is $dc \times \mathbb{P}_{\mathbb{T}}$, with $\Omega_{\mathbb{T}} = (\mathbb{R}^2)^{\mathbb{N}^*}$, $\Sigma_{\mathbb{T}} = \mathcal{B}^{\otimes \mathbb{N}^*}$ (where \mathcal{B} stands for the Borel sigma-algebra on \mathbb{R}^2) and the probability measure $\mathbb{P}_{\mathbb{T}}$ defined by (1.28). Also, recall the GFF on the unit circle $\varphi : \Omega \rightarrow W^s(\mathbb{T})$ (with $s < 0$) defined by (1.25). Finally let us denote by \mathcal{S} the set of smooth functions depending on finitely many coordinates, i.e. of the form $F(x_1, y_1, \dots, x_n, y_n)$ with $n \geq 1$ and $F \in C^\infty((\mathbb{R}^2)^n)$, with at most polynomial growth at infinity for F and its derivatives. Obviously \mathcal{S} is dense in $L^2(\Omega_{\mathbb{T}}, \Sigma_{\mathbb{T}}, \mathbb{P}_{\mathbb{T}})$.

Let us introduce the bilinear form (with associated quadratic form still denoted by \mathcal{Q}) for $\mu \geq 0$

$$(2.22) \quad \mathcal{Q}(u, v) := \frac{1}{2} \mathbb{E} \int_{\mathbb{R}} \left(\partial_c u \partial_c \bar{v} + Q^2 u \bar{v} + 2(\mathbf{P}u) \bar{v} + 2\mu e^{\gamma c} V u \bar{v} \right) dc$$

with V defined by (1.38) for $\gamma^2 < 2$ and furthermore $V \geq 0$. Here u, v belong to the domain $\mathcal{D}(\mathcal{Q})$ of the quadratic form, namely the completion for the \mathcal{Q} -norm in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ of the space

$$(2.23) \quad \mathcal{C} = \text{Span}\{\psi(c)F \mid \psi \in C_c^\infty(\mathbb{R}) \text{ and } F \in \mathcal{S}\}.$$

The completion is the vector space consisting of equivalence classes of Cauchy sequences of \mathcal{C} for the norm $\|u\|_{\mathcal{Q}} := \sqrt{\mathcal{Q}(u, u)}$ under the equivalence relation $u \sim v$ iff $\|u_n - v_n\|_{\mathcal{Q}} \rightarrow 0$ as $n \rightarrow \infty$. This space is a Hilbert space, which embeds injectively in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ by the map $j : [u] \mapsto \lim_{n \rightarrow \infty} u_n$. Indeed, u_n is Cauchy for $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ since $\|u_n - u_m\|_{L^2} \leq \sqrt{2} Q^{-1} \|u_n - u_m\|_{\mathcal{Q}}$, it thus converges in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. Moreover $\|\lim_n u_n\|_{L^2} \leq \sqrt{2} Q^{-1} \lim_n \|u_n\|_{\mathcal{Q}}$ thus j is bounded. Finally if $j([u]) = 0$, then for $(u_n)_n$ a representative Cauchy sequence of $[u]$, we have $u_n \rightarrow 0$ in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ and using

$$\frac{1}{2} \|\partial_c(u_n - u_m)\|_2^2 + \|\mathbf{P}^{1/2}(u_n - u_m)\|_2^2 + \mu \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}}(u_n - u_m)\|_2^2 \leq \mathcal{Q}(u_n - u_m, u_n - u_m),$$

one has the convergence in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ of $\partial_c u_n \rightarrow v$, $\mathbf{P}^{1/2} u_n \rightarrow w$ and $(e^{\gamma c} V)^{1/2} u \rightarrow z$ for some $v, w, z \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. For each $\varphi \in \mathcal{C}$, we have as $n \rightarrow \infty$

$$\langle \partial_c u_n, \varphi \rangle_2 = \langle u_n, -\partial_c \varphi \rangle \rightarrow 0, \quad \langle \mathbf{P}^{1/2} u_n, \varphi \rangle_2 = \langle u_n, \mathbf{P}^{1/2} \varphi \rangle_2 \rightarrow 0, \quad \langle (e^{\gamma c} V)^{1/2} u_n, \varphi \rangle_2 = \langle u_n, (e^{\gamma c} V)^{1/2} \varphi \rangle_2 \rightarrow 0$$

by using that $(e^{\gamma c} V)^{1/2} \varphi \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$, thus $v = w = z = 0$ by density of \mathcal{C} in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. This implies that $\|u_n\|_{\mathcal{Q}} \rightarrow 0$ and thus j is injective. Obviously \mathcal{Q} is closed on $\mathcal{D}(\mathcal{Q})$ and lower semi-bounded $\mathcal{Q}(u) \geq Q^2 \|u\|_2^2 / 2$ so that, by [ReSi1, Theorem 8.15], there is a unique self-adjoint operator denoted \mathbf{H} called the *Friedrichs extension* and with domain $\mathcal{D}(\mathbf{H})$, such that $\mathcal{Q}(u, v) = \langle \mathbf{H}u | v \rangle_2$ for $v \in \mathcal{D}(\mathcal{Q})$ and $u \in \mathcal{D}(\mathbf{H})$. Recall that this operator is defined as follows: $\mathcal{D}(\mathbf{H})$ is made of those $u \in \mathcal{D}(\mathcal{Q})$ such that there exists some constant $C > 0$ satisfying $\forall v \in \mathcal{D}(\mathcal{Q}), \mathcal{Q}(u, v) \leq C \|v\|_2$, in which case there exists an element denoted $\mathbf{H}u \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ such that

$$\mathcal{Q}(u, v) = \langle \mathbf{H}u | v \rangle_2.$$

If we let $\mathcal{D}(\mathcal{Q})'$ be the dual to $\mathcal{D}(\mathcal{Q})$ (i.e. the space of bounded conjugate linear functionals on $\mathcal{D}(\mathcal{Q})$), the injection $L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \subset \mathcal{D}(\mathcal{Q})'$ is continuous and the operator \mathbf{H} can be extended as a bounded isomorphism

$$\mathbf{H} : \mathcal{D}(\mathcal{Q}) \rightarrow \mathcal{D}(\mathcal{Q})'.$$

Moreover $\mathcal{D}(\mathbf{H}) = \{u \in \mathcal{D}(\mathcal{Q}) \mid \mathbf{H}u \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})\}$ and $\mathbf{H}^{-1} : L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow \mathcal{D}(\mathbf{H})$ is bounded. Furthermore, by the spectral theorem, it generates a strongly continuous contraction semigroup of self-adjoint operators $(e^{-t\mathbf{H}})_{t \geq 0}$ on $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. When $\mu = 0$, we write \mathcal{Q}_0 and \mathbf{H}^0 respectively instead of \mathcal{Q} and \mathbf{H} .

We define the space

$$\mathcal{D}(e^{\gamma c}V) := \{u \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \mid e^{\gamma c}Vu \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})\}.$$

The multiplication operator $e^{\gamma c}V$ is closed on $\mathcal{D}(e^{\gamma c}V)$: indeed, if $\varphi_n \in \mathcal{D}(e^{\gamma c}V)$ converges to φ in L^2 and $e^{\gamma c}V\varphi_n \rightarrow \varphi'$ in L^2 , then $\varphi' = e^{\gamma c}V\varphi$ almost everywhere. If $\gamma < 1$ (thus $V \in L^p(\Omega_{\mathbb{T}})$ for some $p > 2$), $\mathcal{D}(e^{\gamma c}V)$ clearly contains the dense space \mathcal{C} and furthermore \mathcal{C} is a core for $e^{\gamma c}V$ defined on $\mathcal{D}(e^{\gamma c}V)$. For $\gamma \in [1, \sqrt{2})$, since V is no longer in $L^{2+\epsilon}(\Omega_{\mathbb{T}})$ for some $\epsilon > 0$, $\mathcal{D}(e^{\gamma c}V)$ does not contain \mathcal{C} anymore. Yet, it contains the set $e^{-\gamma c}V^{-1} \times \mathcal{C}$, which can be seen to be a dense subset of $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ by using that $V^{-1} \in L^p(\Omega_{\mathbb{T}})$ for all $p < \infty$. It is then straightforward to check that, for $F \in \mathcal{D}(\mathbf{H}^0) \cap \mathcal{D}(e^{\gamma c}V)$,

$$\mathbf{H}F = (\mathbf{H}^0 + \mu e^{\gamma c}V)F.$$

2.5. Dynamics of the GFF. The goal of this subsection is to prove the relation $\mathbf{H}_*^0 = \mathbf{H}^0$ in the case $\mu = 0$ i.e. we want to show

Proposition 2.6. *For all $f \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ and all $t \geq 0$*

$$(2.24) \quad U_0 S e^{-t} U_0^{-1} f = e^{-t\mathbf{H}^0} f$$

Proof. Recalling (2.4) we have the independent sum

$$(2.25) \quad X_{\mathbb{D}}(z) = B_{-\log|z|} + Y(z)$$

where B_t is Brownian motion and Y has zero average on circles. We then have, for $|q| = e^{-t}$,

$$(2.26) \quad (U_0 S_q U_0^{-1} f)(c, \varphi) = e^{-Qc} \mathbb{E}_{\varphi} [e^{Q(c+B_t-Qt)} f(c+B_t-Qt, P\varphi \circ s_q + Y \circ s_q)].$$

First we consider functions of the form (recall the notations page 18)

$$f(c, \varphi) = e^{\frac{1}{2\pi} \langle \varphi, h \rangle_{\mathbb{T}} - \frac{1}{2(2\pi)^2} \langle h, G_{\mathbb{T}} h \rangle_{\mathbb{T}}} \psi(c)$$

where $\psi \in C_0^{\infty}(\mathbb{R})$ and $h \in C^{\infty}(\mathbb{T})$ with $h(\theta) = \sum_{n \neq 0} h_n e^{in\theta}$ and $h_{-n} = \bar{h}_n$. Recall that

$$G_{\mathbb{T}} h(e^{i\theta}) = \pi \sum_{n \geq 1} \frac{1}{n} (h_n e^{in\theta} + h_{-n} e^{-in\theta})$$

and for $g(\theta) = \sum_{n \neq 0} g_n e^{in\theta}$ with $g_{-n} = \bar{g}_n$

$$\langle h, G_{\mathbb{T}} g \rangle_{\mathbb{T}} = 2\pi^2 \sum_{n \geq 1} \frac{1}{n} (h_n g_{-n} + h_{-n} g_n).$$

Then by independence

$$(U_0 S_q U_0^{-1} f)(c, \varphi) = e^{\frac{1}{2\pi} \langle P\varphi \circ s_q, h \rangle_{\mathbb{T}} - \frac{1}{2(2\pi)^2} \langle h, G_{\mathbb{T}} h \rangle_{\mathbb{T}}} \mathbb{E}[e^{Q(B_t-Qt)} \psi(c+B_t-Qt)] \mathbb{E}[e^{\frac{1}{2\pi} \langle Y \circ s_q, h \rangle_{\mathbb{T}}}]$$

where $\langle P\varphi \circ s_q, h \rangle_{\mathbb{T}} = \int_0^{2\pi} (P\varphi)(qe^{i\theta}) h(\theta) d\theta$ and $\langle Y \circ s_q, h \rangle_{\mathbb{T}}$ similarly. Let G , G_Y and $G_{P\varphi}$ be the covariances of X , Y and $P\varphi$ respectively. From the decomposition in independent fields for the GFF

$$X = P\varphi + X_{\mathbb{D}} = P\varphi + Y + B$$

we get, for $|z| = |z'| = 1$,

$$\begin{aligned} G_Y(qz, qz') &= G(qz, qz') - G_{P\varphi}(qz, qz') + \log(|qz| \vee |qz'|) \\ &= G(z, z') - G_{P\varphi}(qz, qz') \\ &= G_{\mathbb{T}}(z, z') - G_{P\varphi}(qz, qz'). \end{aligned}$$

Hence using $\mathbb{E}[e^{\frac{1}{2\pi} \langle Y \circ s_q, h \rangle_{\mathbb{T}}}] = e^{\frac{1}{2(2\pi)^2} \langle h, G_Y \circ s_q h \rangle_{\mathbb{T}}}$ we conclude

$$(U_0 S_q U_0^{-1} f)(c, \varphi) = e^{\frac{1}{2\pi} \langle \varphi, h_q \rangle_{\mathbb{T}} - \frac{1}{2(2\pi)^2} \langle h_q, G_{\mathbb{T}} h_q \rangle_{\mathbb{T}}} \mathbb{E}[e^{Q(B_t-Qt)} \psi(c+B_t-Qt)]$$

where

$$h_q = \sum_{n > 0} (\bar{q}^n h_n e^{in\theta} + q^n h_{-n} e^{-in\theta}).$$

The expectation is given by the Girsanov transform

$$\mathbb{E}[e^{Q(B_t - Qt)} \psi(c + B_t - Qt)] = e^{-\frac{Q^2}{2}t} \mathbb{E}[\psi(c + B_t)] = (e^{\frac{1}{2}(-\partial_c^2 + Q^2)t} \psi)(c).$$

Hence we have obtained

$$(2.27) \quad (U_0 S_q U_0^{-1} f)(c, \varphi) = e^{\frac{1}{2\pi} \langle \varphi, h_q \rangle_{\mathbb{T}} - \frac{1}{2(2\pi)^2} \langle h_q, G_{\mathbb{T}} h_q \rangle_{\mathbb{T}}} (e^{\frac{1}{2} \log |q| (-\partial_c^2 + Q^2)} \psi)(c).$$

Now we use this identity to compute $U_0 S_q U_0^{-1}$ over a dense set. Let

$$(2.28) \quad \Psi_h(\varphi) = e^{\frac{1}{2\pi} \langle \varphi, h \rangle_{\mathbb{T}} - \frac{1}{2(2\pi)^2} \langle h, G_{\mathbb{T}} h \rangle_{\mathbb{T}}} = e^{\sum_{n \geq 1} (\varphi_n h_{-n} + \varphi_{-n} h_n - \frac{1}{4n} (h_n h_{-n} + h_n h_{-n}))}.$$

Then

$$(2.29) \quad \langle \Psi_h | \Psi_g \rangle_{L^2(\Omega_{\mathbb{T}})} = \mathbb{E}[e^{\frac{1}{2\pi} \langle \varphi, h+g \rangle_{\mathbb{T}}} e^{-\frac{1}{2(2\pi)^2} \langle h, G_{\mathbb{T}} h \rangle_{\mathbb{T}} - \frac{1}{2(2\pi)^2} \langle g, G_{\mathbb{T}} g \rangle_{\mathbb{T}}}] = e^{\frac{1}{(2\pi)^2} \langle h, G_{\mathbb{T}} g \rangle_{\mathbb{T}}} = e^{\sum_{n \geq 1} \frac{1}{2n} (h_n g_{-n} + g_n h_{-n})}.$$

As before let $\mathbf{k} = (k_1, k_2, \dots) \in \mathbb{N}^{\mathbb{N}_+}$ be such that $k_i = 0$ for all i large enough and define polynomials

$$(2.30) \quad \xi_{\mathbf{k}, \mathbf{l}} = \prod_{n \geq 1} \frac{(\sqrt{2n} \partial_{h_{-n}})^{k_n}}{\sqrt{k_n!}} \frac{(\sqrt{2n} \partial_{h_n})^{l_n}}{\sqrt{l_n!}} \Psi_h|_{h=0}.$$

Differentiating (2.29) we get

$$\langle \xi_{\mathbf{k}, \mathbf{l}} | \xi_{\mathbf{k}', \mathbf{l}'} \rangle_{L^2(\Omega_{\mathbb{T}})} = \delta_{\mathbf{k}, \mathbf{k}'} \delta_{\mathbf{l}, \mathbf{l}'}$$

From (2.28) we see that the highest power monomial in $\xi_{\mathbf{k}, \mathbf{l}}$ is equal to $\prod_{n \geq 1} \frac{\sqrt{2n}^{k_n}}{\sqrt{k_n!}} \frac{\sqrt{2n}^{l_n}}{\sqrt{l_n!}} \prod_{n \geq 1} \varphi_n^{k_n} \bar{\varphi}_n^{l_n}$ and thus $\xi_{\mathbf{k}, \mathbf{l}} = i^{|\mathbf{k}|+|\mathbf{l}|} \pi_{\mathbf{k}, \mathbf{l}}$ is a complete set of orthonormal eigenfunctions of \mathbf{P} (recall $\pi_{\mathbf{k}, \mathbf{l}}$ was introduced as the normalization of (1.43)). In conclusion, by differentiating identity (2.27) we get for $q = e^{-t}$

$$U_0 S_q U_0^{-1} (\pi_{\mathbf{k}, \mathbf{l}} \psi) = q^{\sum n k_n} \bar{q}^{\sum n l_n} \pi_{\mathbf{k}, \mathbf{l}} e^{\frac{1}{2} \log |q| (-\partial_c^2 + Q^2)} \psi = e^{-t \mathbf{H}^0} (\pi_{\mathbf{k}, \mathbf{l}} \psi).$$

Thus we have shown that the semigroups $U_0 S_{e^{-t}} U_0^{-1}$ and $e^{-t \mathbf{H}^0}$ agree on the dense set of functions $\pi_{\mathbf{k}, \mathbf{l}} \psi$ with $\psi \in C_0^\infty(\mathbb{R})$. The claim (2.24) in the case $\mu = 0$ follows. \square

Decompose $W^s(\mathbb{T}) = \mathbb{R} \oplus W_0^s(\mathbb{T})$ where $f \in W_0^s(\mathbb{T})$ has zero average: $\int_0^{2\pi} f(\theta) d\theta = 0$. From the decomposition (2.25) we deduce that \mathbf{H}^0 has the following expression

$$(2.31) \quad (e^{-t \mathbf{H}^0} f)(c, \varphi) = e^{-\frac{Q^2 t}{2}} \mathbb{E}_\varphi[f(c + B_t, (P\varphi + Y)(e^{-t+i\cdot}))]$$

In the sequel, we will denote $\varphi_t(\theta) := (P\varphi + Y)(e^{-t+i\theta})$.

Finally, we have the simple:

Proposition 2.7. *The following properties hold:*

- (1) *The measure $dc \times \mathbb{P}_{\mathbb{T}}$ is invariant for $e^{\frac{Q^2 t}{2}} e^{-t \mathbf{H}^0}$.*
- (2) *$e^{-t \mathbf{H}^0}$ extends to a continuous semigroup on $L^p(\mathbb{R} \times \Omega_{\mathbb{T}})$ for all $p \in [1, +\infty]$ with norm $e^{-\frac{Q^2}{2} t}$ and it is strongly continuous for $p \in [1, +\infty)$.*
- (3) *$e^{-t \mathbf{H}^0}$ extends to a strongly continuous semigroup on $e^{-\alpha c} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for all $\alpha \in \mathbb{R}$ with norm $e^{(\frac{\alpha^2}{2} - \frac{Q^2}{2}) t}$.*

Proof. 1) This is a consequence of (2.31): indeed the processes B and Y are independent and describe two dynamics for which the measures dc and \mathbb{P} are respectively invariant. 2) follows from (2.31), Jensen's inequality and the fact that $dc \otimes \mathbb{P}_{\mathbb{T}}$ is invariant for \mathbf{H}_*^0 . 3) The map $K : f \rightarrow e^{-\alpha c} f$ is unitary from $L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\alpha c} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. We have $K e^{-t \mathbf{H}^0} K^{-1} = e^{t(\frac{\alpha^2}{2} - \alpha \partial_c)} e^{-t \mathbf{H}^0}$ which implies the claim. \square

Remark 2.8. *The basis (1.33) for the eigenfunctions of \mathbf{P} consists of products of eigenfunctions for each harmonic oscillator X_n and Y_n which are Hermite polynomials in the variables x_n and y_n . We do not use this representation since the $\pi_{\mathbf{k}, \mathbf{l}}$ basis matches better with the Virasoro representation theory as we'll discuss in the next subsection.*

Remark 2.9. List the eigenvalues $\lambda = |\mathbf{k}| + |\mathbf{l}|$ of \mathbf{P} in increasing order $\lambda_1 < \lambda_2 < \dots$ and let P_i be the corresponding spectral projectors. Since each λ_i is of finite multiplicity and $\lambda_i \rightarrow \infty$ as $i \rightarrow \infty$ the semigroup $e^{-t\mathbf{P}} = \sum_i e^{-t\lambda_i} P_i$ and the resolvent $(z - \mathbf{P})^{-1} = \sum_i (z - \lambda_i)^{-1} P_i$ are compact if $t > 0$ and $\Im z \neq 0$ since they are norm convergent limits of finite rank operators.

2.6. Dynamics of LCFT. In this subsection we suppose that $\gamma \in (0, 1)$. The goal of this subsection is to prove the relation $\mathbf{H}_* = \mathbf{H}$ for $\mu > 0$ where \mathbf{H} denotes the Friedrichs extension, i.e. we want to show that the LCFT semigroup (2.21) agrees with the semigroup generated by \mathbf{H} :

$$(2.32) \quad e^{-t\mathbf{H}_*} f = e^{-t\mathbf{H}} f$$

for all $f \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$.

Recall that V defined in (1.38) belongs to $V \in L^r(\Omega_{\mathbb{T}})$ for $r < \frac{2}{\gamma^2}$. Next consider the semigroup $e^{-t\mathbf{H}_*}$. We start by writing the chaos measure in terms of the process $Z_t = (c + B_t, \varphi_t)$ on $W^s(\mathbb{T})$ given by (2.31). The function $V : W^s(\mathbb{T}) \rightarrow \mathbb{R}^+$ given by

$$V(\varphi) = \lim_{N \rightarrow \infty} \int_{\mathbb{T}} e^{\gamma\varphi^{(N)}(\theta) - \frac{\gamma^2}{2} \mathbb{E}[\varphi^{(N)}(\theta)^2]} d\theta$$

is measurable since the limit is in $L^r(\Omega_{\mathbb{T}})$ for $r < \frac{2}{\gamma^2}$ and we get that conditionally on $\varphi = \varphi_0$ (by making the change of variables $dx = r dr d\theta = e^{-2s} ds d\theta$ with $r = |x| = e^{-s}$ and $-\frac{\gamma^2}{2} \mathbb{E}[B_s^2] - 2s = -\gamma Qs$)

$$e^{\gamma c} M_{\gamma}(\mathbb{D}) \stackrel{\text{law}}{=} \int_0^{\infty} e^{\gamma(c+B_s-Qs)} V(\varphi_s) ds.$$

Let now $f \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. Recall that from (2.17) we have $U^{-1}f = \Psi^{-1}f$ where

$$\Psi(c, \varphi) = (U1)(c, \varphi) = (U_0 e^{-\mu e^{\gamma c} M_{\gamma}(\mathbb{D})})(c, \varphi) = e^{-Qc} \mathbb{E}_{\varphi} [e^{-\mu \int_0^{\infty} e^{\gamma(c+B_s-Qs)} V(\varphi_s) ds}]$$

By the Markov property of $(B_t - Qt, \varphi_t)$ we have the equality

$$\Psi(c + B_t - Qt, \varphi_t) = e^{-Q(c+B_t-Qt)} \mathbb{E} [e^{-\mu \int_t^{\infty} e^{\gamma(c+B_s-Qs)} V(\varphi_s) ds} | B_t - Qt, \varphi_t]$$

By exploiting this Markov property we get the following Feynman-Kac formula for $e^{-t\mathbf{H}_*} f$

$$(2.33) \quad \begin{aligned} e^{-t\mathbf{H}_*} f &= U S_{e^{-t}} U^{-1} f \\ &= e^{-Qc} \mathbb{E}_{\varphi} [f(c + B_t - Qt, \varphi_t) \Psi^{-1}(c + B_t - Qt, \varphi_t) e^{-\mu \int_0^t e^{\gamma(c+B_s-Qs)} V(\varphi_s) ds} \\ &\quad \times \mathbb{E} [e^{-\mu \int_t^{\infty} e^{\gamma(c+B_s-Qs)} V(\varphi_s) ds} | B_t - Qt, \varphi_t]] \\ &= e^{-Qc} \mathbb{E}_{\varphi} [e^{Q(c+B_t-Qt)} f(c + B_t - Qt, \varphi_t) e^{-\mu \int_0^t e^{\gamma(c+B_s-Qs)} V(\varphi_s) ds}] \\ &= e^{-\frac{Q^2 t}{2}} \mathbb{E}_{\varphi} [f(c + B_t, \varphi_t) e^{-\mu \int_0^t e^{\gamma(c+B_s)} V(\varphi_s) ds}], \end{aligned}$$

where in the last line we used the Girsanov formula. As a consequence of this formula and similarly to \mathbf{H}^0 we have

Proposition 2.10. *The following properties hold:*

- (1) $e^{-t\mathbf{H}_*}$ extends to a continuous semigroup on $L^p(\mathbb{R} \times \Omega_{\mathbb{T}})$ for all $p \in [1, +\infty]$ with norm $e^{-\frac{Q^2}{2}t}$ and it is strongly continuous for $p \in [1, +\infty)$.
- (2) $e^{-t\mathbf{H}_*}$ extends to a strongly continuous semigroup on $e^{-\alpha c} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for all $\alpha \in \mathbb{R}$ with norm $e^{(\frac{\alpha^2}{2} - \frac{Q^2}{2})t}$.

Proof. Using in turn (2.33) and $V \geq 0$, we see that $\|e^{-t\mathbf{H}_*} f\|_p \leq \|e^{-t\mathbf{H}^0} |f|\|_p$ so that the claim 1) follows from Proposition 2.7. The same argument works for 2). \square

Recall Subsection 2.4 where we defined the domains $\mathcal{D}(\mathbf{H}^0)$ (with $\mu = 0$) and $\mathcal{D}(e^{\gamma c} V)$.

Proposition 2.11. *For $\gamma \in (0, \sqrt{2})$ and $\mu > 0$, we have the relation $\mathbf{H}_* = \mathbf{H}$. Furthermore, if $\gamma \in (0, 1)$, \mathbf{H} is essentially self-adjoint on $\mathcal{D}(\mathbf{H}^0) \cap \mathcal{D}(e^{\gamma c} V)$, \mathcal{C} is a core for $(\mathbf{H}^0, \mathcal{D}(\mathbf{H}^0))$ and for $(e^{\gamma c} V, \mathcal{D}(e^{\gamma c} V))$.*

Proof. Let us first establish the claim for $\gamma \in (0, 1)$. Let $\mathcal{A} := \{f \mid f = e^{-t\mathbf{H}_*}g, g \in L^\infty(\mathbb{R} \times \Omega_{\mathbb{T}}) \cap L^1(\mathbb{R} \times \Omega_{\mathbb{T}})\}$. By the spectral theorem, using the spectral projectors $\mathbf{1}_{[0, N]}(\mathbf{H}_*)$ for large N , we see that $(\mathbf{H}_* + i)e^{-t\mathbf{H}_*}(L^2(\mathbb{R} \times \Omega_{\mathbb{T}}))$ is dense in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ so that, by density of $L^\infty(\mathbb{R} \times \Omega_{\mathbb{T}}) \cap L^1(\mathbb{R} \times \Omega_{\mathbb{T}})$ in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ and boundedness of $(\mathbf{H}_* + i)e^{-t\mathbf{H}_*}$, \mathcal{A} is dense in L^2 . So \mathbf{H}_* is essentially self-adjoint on \mathcal{A} by [ReSi1, Coro. of Th. VIII.3].

Now we claim that $\mathcal{A} \subset \mathcal{D}(e^{\gamma c}V)$. Indeed, fix $p, q > 1$ such that $\mathbb{E}[V^{2p}] < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. We have by Hölder inequality for $f := e^{-t\mathbf{H}_*}g \in \mathcal{A}$

$$\begin{aligned} \|e^{\gamma c}Vf\|_2^2 &= \int e^{2\gamma c} \mathbb{E}[V^2 |e^{-t\mathbf{H}_*}g|^2] dc \\ &\leq \mathbb{E}[V^{2p}]^{1/p} \int e^{2\gamma c} \mathbb{E}[|e^{-t\mathbf{H}_*}g|^{2q}]^{1/q} dc. \end{aligned}$$

By using in turn Jensen and Hölder inequalities (with conjugate p', q') in (2.33) we get (with $c_+ = 0 \vee c$)

$$\begin{aligned} \mathbb{E}[|e^{-t\mathbf{H}_*}g|^{2q}]^{1/q} &\leq e^{-Q^2 t} \mathbb{E}\left[\mathbb{E}_\varphi[|g(c+B_t, \varphi_t)|^{2q}] \mathbb{E}_\varphi[\exp(-\mu \int_0^t e^{\gamma(c+B_s)} V(\varphi_s) ds)^{2q-1}]^{1/q}\right] \\ &\leq e^{-Q^2 t} \mathbb{E}\left[\mathbb{E}_\varphi[|g(c+B_t, \varphi_t)|^{2q}]^{p'}\right]^{\frac{1}{qp'}} \mathbb{E}\left[\mathbb{E}_\varphi[\exp(-\mu \int_0^t e^{\gamma(c+B_s)} V(\varphi_s) ds)^{(2q-1)q'}]\right]^{\frac{1}{qq'}} \\ &\leq e^{-Q^2 t} \|g\|_{L^{2qp'}(\Omega_{\mathbb{T}})}^2 \left(\mathbb{E}[\exp(-\mu(2q-1)q' \int_0^t e^{\gamma(c+B_s)} V(\varphi_s) ds)]\right)^{1/(qq')}. \end{aligned}$$

Now we want to evaluate the c decay of the last expectation. For arbitrary $n > 0$, using the inequality $u^n e^{-u} \leq C_n$ for $u \geq 0$ and for some constant $C_n > 0$ (in what follows, C will denote a generic constant and we will indicate as subscripts the parameters it depends on), we deduce (with $c_+ = 0 \vee c$)

$$\begin{aligned} \mathbb{E}[\exp(-\mu(2q-1)q' \int_0^t e^{\gamma(c+B_s)} V(\varphi_s) ds)] &\leq C_{\mu, n, q, q'} e^{-n\gamma c_+} \mathbb{E}[(\int_0^t e^{\gamma B_s} V(\varphi_s) ds)^{-n}] \\ &\leq C_{\mu, n, q, q', t} e^{-n\gamma c_+} \end{aligned}$$

where in the last line we used the fact that GMC measures admit moments of negative order [RhVa14]. Summarizing, for arbitrary $n > 0$,

$$\begin{aligned} \|e^{\gamma c}Vf\|_2^2 &\leq C_{\mu, n, q, q', t} \mathbb{E}[V^{2p}]^{1/p} \int \|g\|_{L^{2qp'}(\Omega_{\mathbb{T}})}^2 e^{2\gamma c - \frac{\gamma n}{qq'} c_+} dc \\ &\leq C_{\mu, n, q, q', t} \mathbb{E}[V^{2p}]^{1/p} \|g\|_{2qp'}^2 \left(\int e^{2\gamma c - \frac{\gamma n}{qq'} c_+} dc\right)^{1 - \frac{1}{qp'}} \end{aligned}$$

where the last line was obtained with Jensen and by choosing $n > 2qq'$. By noticing that $\|g\|_{2qp'}^2 < \infty$ because $g \in L^\infty(\mathbb{R} \times \Omega_{\mathbb{T}}) \cap L^1(\mathbb{R} \times \Omega_{\mathbb{T}})$, we deduce $\|e^{\gamma c}Vf\|_2^2 < \infty$. This shows that $\mathcal{A} \subset \mathcal{D}(e^{\gamma c}V)$.

Now we show that $\mathcal{A} \subset \mathcal{D}(\mathbf{H}^0)$. For this, we first observe that \mathcal{C} defined by (2.23) is a core for \mathbf{H}^0 : indeed, this results from the density of $\mathbf{H}^0(\mathcal{C})$ in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ by [ReSi2, Th. X.26]. To prove this density, expand each $F \in L^2$ along the Hermite polynomials (1.33) (which belong to \mathcal{S})

$$F = \sum_{\mathbf{kl}} F_{\mathbf{kl}}(c) \psi_{\mathbf{kl}} \quad \text{with} \quad \sum_{\mathbf{kl}} \int |F_{\mathbf{kl}}(c)|^2 dc < \infty.$$

For each N and each \mathbf{k}, \mathbf{l} find a sequence of smooth compactly supported functions $g_{\mathbf{kl}}^{(N)}$ such that

$$\begin{aligned} (-\partial_c^2 + Q^2 + 2\lambda_{\mathbf{kl}})g_{\mathbf{kl}}^{(N)} &\rightarrow 2F_{\mathbf{kl}} \quad \text{as } N \rightarrow \infty \text{ in } L^2(\mathbb{R}) \\ \int_{\mathbb{R}} |(-\partial_c^2 + Q^2 + 2\lambda_{\mathbf{kl}})g_{\mathbf{kl}}^{(N)}(c)|^2 dc &\leq 4 \int |F_{\mathbf{kl}}(c)|^2 dc \end{aligned}$$

Then set $F^{(N)} = \sum_{|\mathbf{k}|+|\mathbf{l}| \leq N} g_{\mathbf{kl}}^{(N)}(c) \psi_{\mathbf{kl}}$. Then $\mathbf{H}^0 F^{(N)} = \sum_{|\mathbf{k}|+|\mathbf{l}| \leq N} (-\partial_c^2 + Q^2 + 2\lambda_{\mathbf{kl}})g_{\mathbf{kl}}^{(N)}(c) \psi_{\mathbf{kl}}$ converges to F as $N \rightarrow \infty$ in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$.

Finally, observe that $\mathcal{C} \subset \mathcal{D}(e^{\gamma c}V)$ and that $\mathbf{H}_*g = \mathbf{H}^0g + e^{\gamma c}Vg$ for $g \in \mathcal{C}$ (this is easily seen by differentiating (2.33) with respect to t). Therefore, for each $f \in \mathcal{A}$ and $g \in \mathcal{C}$

$$\langle f | \mathbf{H}^0 g \rangle_2 = \langle f | \mathbf{H}_* g \rangle_2 - \langle f | e^{\gamma c} V g \rangle_2 = \langle \mathbf{H}_* f | g \rangle_2 - \langle e^{\gamma c} V f | g \rangle_2 \leq C \|g\|_2.$$

Since \mathcal{C} is a core for \mathbf{H}^0 the above relation extends to $g \in \mathcal{D}(\mathbf{H}^0)$ and this shows that $f \in \mathcal{D}(\mathbf{H}^0)$. As \mathbf{H} is an extension of \mathbf{H}_* restricted to $\mathcal{D}(\mathbf{H}^0) \cap \mathcal{D}(e^{\gamma c V})$, this completes the proof of our claim.

It turns out that we are not able to establish self-adjointness of \mathbf{H} for $\gamma \in [1, \sqrt{2})$, mainly because \mathcal{C} is no longer a core for $e^{\gamma c V}$. So we will just show that the semigroup associated to \mathbf{H} satisfies the Feynman-Kac formula. For this, we will first regularize the potential to make it smoother so as to establish easily the Feynman-Kac formula for the regularized potential. Then we will pass to the limit by removing the cutoff on the potential.

We will use the following convention: recall that $\varphi_t(\theta) := (P\varphi + Y)(e^{-t+i\theta})$. We will denote by $\varphi_{n,t}$ the Fourier modes of the field φ_t , namely

$$(2.34) \quad \varphi_t(\theta) := \sum_{n \in \mathbb{Z}} \varphi_{n,t} e^{in\theta}.$$

To regularize the field, we will use the frequency cutoff approximation for $N \geq 1$

$$(2.35) \quad \varphi_t^{(N)}(\theta) := \sum_{n \in \mathbb{Z}, |n| \leq N} \varphi_{n,t} e^{in\theta}.$$

We denote by $V^{(N)} := V(\varphi_0^{(N)})$ the regularized potential. We have the relation

$$V^{(N)} = \int_0^{2\pi} e^{\gamma \varphi^{(N)}(\theta) - \frac{\gamma^2}{2} \mathbb{E}[(\varphi^{(N)}(\theta))^2]} d\theta$$

with $\varphi^{(N)} = \varphi_0^{(N)}$ the field defined by (1.36). Let us denote by $\mathcal{Q}^{(N)}$ the quadratic form defined by (2.22) with V replaced by $V^{(N)}$ and $\mathbf{H}^{(N)}$ the associated operator (i.e. the Friedrichs extension of $\mathbf{H}_0 + \mu e^{\gamma c V^{(N)}}$ defined on \mathcal{C}). Let us now prove that the semigroup $e^{-t\mathbf{H}^{(N)}}$ satisfies the Feynman-Kac formula. For this, we use Kato's strong Trotter product formula (see [ReSi1, Theorem S.21 page 379]) applied to the self-adjoint operators \mathbf{H}_0 and $e^{\gamma c V^{(N)}}$: since the domain $\mathcal{D}(\mathcal{Q}^{(N)})$ of the quadratic form $\mathcal{Q}^{(N)}$ (defined in Subsection 2.4) is dense in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ and satisfies $\mathcal{D}(\mathcal{Q}^{(N)}) = \mathcal{D}(\mathcal{Q}_0) \cap \mathcal{D}(\mathcal{Q}_{V^{(N)}})$, where $\mathcal{D}(\mathcal{Q}_{V^{(N)}}) = \{f \mid \|e^{\gamma c/2} (V^{(N)})^{1/2} f\|_2 < +\infty\}$ is the domain of the quadratic form associated to the operator of multiplication by $e^{\gamma c V^{(N)}}$, we have the identity

$$\lim_{n \rightarrow \infty} (e^{-\frac{t}{n} \mathbf{H}_0} e^{-\frac{t}{n} \mu e^{\gamma c V^{(N)}}})^n = e^{-t\mathbf{H}^{(N)}}$$

where the limit is understood in the strong sense (i.e. convergence in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ when this relation is applied to $f \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$). Now we compute the limit in the left-hand side. For $f \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ we have

$$(2.36) \quad (e^{-\frac{t}{n} \mathbf{H}_0} e^{-\frac{t}{n} \mu e^{\gamma c V^{(N)}}})^n f = e^{-Q^2 t/2} \mathbb{E}_{\varphi} [f(c + B_t, \varphi_t) e^{-\mu R_t^{n,(N)}}]$$

with $R_t^{n,(N)}$ the Riemann sum

$$R_t^{n,(N)} := \frac{t}{n} \sum_{k=1}^n e^{\gamma(c+B_{kt/n})} V(\varphi_{kt/n}^{(N)}).$$

The right-hand side of (2.36) converges in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ towards the same expression with $R_t^{n,(N)}$ replaced by $\int_0^t e^{\gamma(c+B_s)} V(\varphi_s^{(N)}) ds$: indeed this can be established by using Jensen and the fact that, almost surely, the Riemann sum $R_t^{n,(N)}$ converges almost surely (for all fixed c) towards the integral $\int_0^t e^{\gamma(c+B_s)} V(\varphi_s^{(N)}) ds$ since the process $s \mapsto e^{\gamma(c+B_s)} V(\varphi_s^{(N)})$ is continuous. This provides the Feynman-Kac representation

$$(2.37) \quad e^{-t\mathbf{H}^{(N)}} f = e^{-Q^2 t/2} \mathbb{E}_{\varphi} [f(c + B_t, \varphi_t) e^{-\mu \int_0^t e^{\gamma(c+B_s)} V(\varphi_s^{(N)}) ds}].$$

The next step is to pass to the limit as $N \rightarrow \infty$. For this we need to show that $\mathbf{H}^{(N)}$ converges towards both \mathbf{H} and \mathbf{H}_* in the strong resolvent sense (see the definition in [ReSi1, page 284]). We use the criterion [ReSi1, Theorem VIII.19] which tells us that it is enough to check, for some fixed $\alpha \in \mathbb{C}$ with $\text{Im}(\alpha) \neq 0$ and $\text{Re}(\alpha) > 0$ and for all $f \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$, that the resolvent $\mathbf{R}_{\alpha}^{(N)} f$ converges towards the resolvents associated to both \mathbf{H} and \mathbf{H}_* .

Convergence in the strong resolvent sense towards \mathbf{H}_* is obvious thanks to the Feynman-Kac formula (2.37): indeed, since $\gamma < 2$, standard GMC theory (see [RhVa14] for general theory, here direct L^2 -computations also do the job because $\gamma < \sqrt{2}$) ensures the convergence in probability of the potential

$\int_0^t e^{\gamma(c+B_s)} V(\varphi_s^{(N)}) ds$ towards $\int_0^t e^{\gamma(c+B_s)} V(\varphi_s) ds$. It is then easy to check that, for $\alpha \in \mathbb{C}$ with $\text{Im}(\alpha) \neq 0$ and $\text{Re}(\alpha) > 0$, the resolvent

$$\mathbf{R}_\alpha^{(N)} f = \int_0^\infty e^{-\alpha t} e^{-t\mathbf{H}^{(N)}} f dt$$

converges in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ towards $\int_0^\infty e^{-\alpha t} e^{-t\mathbf{H}^*} f dt$.

Now we check convergence in the strong resolvent sense towards \mathbf{H} . Let us fix $f \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. Then the resolvent $\mathbf{R}_\alpha^{(N)} f$ satisfies

$$(2.38) \quad \forall v \in \mathcal{D}(\mathcal{Q}^{(N)}), \quad \alpha \langle \mathbf{R}_\alpha^{(N)} f, v \rangle_2 + \mathcal{Q}^{(N)}(\mathbf{R}_\alpha^{(N)} f, v) = \langle f, v \rangle_2.$$

Taking $v = \mathbf{R}_\alpha^{(N)} f$ and then considering the real part of this relation, we deduce

$$\begin{aligned} \text{Re}(\alpha) \|\mathbf{R}_\alpha^{(N)} f\|_2^2 + \mathcal{Q}^{(N)}(\mathbf{R}_\alpha^{(N)} f) &= \text{Re}(\langle f, \mathbf{R}_\alpha^{(N)} f \rangle_2) \\ &\leq \|\mathbf{R}_\alpha^{(N)} f\|_2 \|f\|_2 \\ &\leq \frac{\text{Re}(\alpha)}{2} \|\mathbf{R}_\alpha^{(N)} f\|_2^2 + \frac{1}{2\text{Re}(\alpha)} \|f\|_2^2, \end{aligned}$$

which leads to the following a priori estimate

$$(2.39) \quad \frac{\text{Re}(\alpha)}{2} \|\mathbf{R}_\alpha^{(N)} f\|_2^2 + \mathcal{Q}^{(N)}(\mathbf{R}_\alpha^{(N)} f) \leq \frac{1}{2\text{Re}(\alpha)} \|f\|_2^2.$$

We deduce that the sequence $(\mathbf{R}_\alpha^{(N)} f, \mathbf{P}^{1/2} \mathbf{R}_\alpha^{(N)} f, (e^{\gamma c} V^{(N)})^{1/2} \mathbf{R}_\alpha^{(N)} f)_n$ is bounded in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})^3$ so that we can extract a subsequence weakly converging towards the triple $(u, p, w) \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})^3$. In fact, as explained above, we already know that convergence in the first component occurs in the strong sense. Furthermore, $\mathbf{P}^{1/2}$ being a closed operator, we have $p = \mathbf{P}^{1/2} u$. Finally, since $V^{(N)}$ converges to V almost surely, we have $w = e^{\gamma c/2} V^{1/2} u$ and, furthermore, Fatou's lemma entails that

$$\|e^{\gamma c/2} V^{1/2} u\|_2^2 \leq \liminf_N \|e^{\gamma c/2} (V^{(N)})^{1/2} \mathbf{R}_\alpha^{(N)} f\|_2^2.$$

The \liminf above is finite because of (2.39). Hence $u \in \mathcal{D}(\mathcal{Q})$. Taking $v \in \mathcal{C}$ in (2.38) and passing to the limit as $N \rightarrow \infty$ (notice that $(e^{\gamma c} V^{(N)})^{1/2} v$ converges towards $(e^{\gamma c} V)^{1/2} v$ in the strong sense in $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$) we get

$$(2.40) \quad \forall v \in \mathcal{C}, \quad \alpha \langle u, v \rangle_2 + \mathcal{Q}(u, v) = \langle f, v \rangle_2.$$

with $\mathbf{R}_\alpha f = u$ and \mathbf{R}_α the resolvent of \mathbf{H} . This completes the argument. \square

2.7. Basis in $L^2(\Omega_{\mathbb{T}})$. We will now study the completeness and analytic continuation of the polynomials $\mathcal{Q}_{\alpha, \nu, \tilde{\nu}}$ defined in (1.54) and (1.53). Recall the basis (2.30) of $L^2(\Omega_{\mathbb{T}})$ given by the normalized polynomials (1.43):

$$\pi_{\mathbf{k}, \mathbf{l}} = \|\mathbf{A}_{-\mathbf{k}} \tilde{\mathbf{A}}_{-1} \mathbf{1}\|_{L^2(\Omega_{\mathbb{T}})}^{-1} \mathbf{A}_{-\mathbf{k}} \tilde{\mathbf{A}}_{-1} \mathbf{1}$$

where $\mathbf{A}_{-\mathbf{k}} = \prod_n \mathbf{A}_{-n}^{k_n}$. Denote $|\mathbf{k}| = \sum n k_n$. Then

Proposition 2.12. *We have*

$$(2.41) \quad \mathcal{Q}_{\alpha, \nu, \tilde{\nu}} = \sum_{|\mathbf{k}|=|\nu|; |\mathbf{l}|=|\tilde{\nu}|} S_{\mathbf{k}, \nu}^{(\alpha)} S_{\mathbf{l}, \tilde{\nu}}^{(\alpha)} \pi_{\mathbf{k}, \mathbf{l}}$$

where the matrix $S_{\mathbf{k}, \nu}^{(\alpha)}$ is a polynomial in α . Furthermore

$$(2.42) \quad \langle \mathcal{Q}_{2Q-\bar{\alpha}, \nu, \tilde{\nu}} | \mathcal{Q}_{\alpha, \nu', \tilde{\nu}'} \rangle_{L^2(\Omega_{\mathbb{T}})} = \delta_{|\nu|, |\nu'|} \delta_{|\tilde{\nu}|, |\tilde{\nu}'|} F_\alpha(\nu, \nu') F_\alpha(\tilde{\nu}, \tilde{\nu}')$$

where the matrix $F_\alpha(\nu, \nu')$ is a polynomial in α as well. The functions $\mathcal{Q}_{\alpha, \nu, \tilde{\nu}}$ are linearly independent for $\alpha \neq \alpha_{r,s}$ where

$$\alpha_{r,s} = Q - r \frac{\gamma}{2} - s \frac{2}{\gamma}.$$

where $r, s \in \mathbb{N}^*$ with $rs \leq \max(|\nu|, |\tilde{\nu}|)$.

Proof. Recall that $\mathbf{A}_0 = \frac{i}{2}(\partial_c + Q)$. Hence $\mathbf{A}_0\psi_\alpha = \frac{i\alpha}{2}\psi_\alpha$. This implies

$$\mathbf{L}_{-\nu}^0\psi_\alpha = (\mathbf{L}_{-\nu}^{0,\alpha}1)\psi_\alpha$$

where $\mathbf{L}_n^{0,\alpha}$, $n \in \mathbb{Z}$ act in $L^2(\Omega_{\mathbb{T}})$ and are given by the expression (1.45) where we replace \mathbf{A}_0 by $\frac{i}{2}(\alpha + Q)$ i.e.

$$\mathbf{L}_n^{0,\alpha} := \begin{cases} i(\alpha - Q - nQ)\mathbf{A}_n + \sum_{m \neq n, 0} \mathbf{A}_{n-m}\mathbf{A}_m & n \neq 0 \\ \frac{\alpha}{2}(Q - \frac{\alpha}{2}) + 2\sum_{m>0} \mathbf{A}_{-m}\mathbf{A}_m & n = 0 \end{cases}$$

Hence

$$(2.43) \quad (\mathbf{L}_n^{0,\alpha})^* = \mathbf{L}_{-n}^{0,2Q-\bar{\alpha}}$$

and $\mathbf{L}_n^{0,\alpha}$ satisfies (1.48); in particular for $\alpha = Q + iP$ we have $(\mathbf{L}_n^{0,Q+iP})^* = \mathbf{L}_{-n}^{0,Q+iP}$. One can show (by recursion on the number of operators and using (1.48)) that for all $t_1, \dots, t_k \in \mathbb{N}^*$ such that $t_1 + \dots + t_k > 0$ that

$$(2.44) \quad \mathbf{L}_{t_1}^{0,\alpha} \dots \mathbf{L}_{t_k}^{0,\alpha} 1 = 0, \quad \tilde{\mathbf{L}}_{t_1}^{0,\alpha} \dots \tilde{\mathbf{L}}_{t_k}^{0,\alpha} 1 = 0$$

Let $\mathbf{L}_{\nu'}^{0,\alpha} := \mathbf{L}_{\nu_1}^{0,\alpha} \dots \mathbf{L}_{\nu_k}^{0,\alpha}$. Then $(\mathbf{L}_{\nu'}^{0,2Q-\bar{\alpha}})^* = \mathbf{L}_{-\nu'}^{0,\alpha}$ and since \mathbf{L} and $\tilde{\mathbf{L}}$ commute we have

$$(2.45) \quad \langle \mathcal{Q}_{2Q-\bar{\alpha},\nu,\tilde{\nu}} | \mathcal{Q}_{\alpha,\nu',\tilde{\nu}'} \rangle_{L^2(\Omega_{\mathbb{T}})} = \langle 1 | \tilde{\mathbf{L}}_{\tilde{\nu}'}^{0,\alpha} \tilde{\mathbf{L}}_{-\tilde{\nu}'}^{0,\alpha} \mathbf{L}_{\nu'}^{0,\alpha} \mathbf{L}_{-\nu'}^{0,\alpha} 1 \rangle_{L^2(\Omega_{\mathbb{T}})}.$$

If $|\nu| > |\nu'|$ or $|\tilde{\nu}| > |\tilde{\nu}'|$ then we get that (2.45) is equal to 0 by using (2.44). The case $|\nu| < |\nu'|$ or $|\tilde{\nu}| < |\tilde{\nu}'|$ can be dealt similarly and yields 0 also. Hence in what follows we suppose that $|\nu| = |\nu'|$ and $|\tilde{\nu}| = |\tilde{\nu}'|$.

Since $\mathbf{L}_n^{0,\alpha} 1 = 0$ for $n > 0$ we get by commuting the $(\mathbf{L}_{\nu_j}^{0,\alpha})_{1 \leq j \leq k}$ to the right thanks to (1.48)

$$(2.46) \quad \mathbf{L}_{\nu'}^{0,\alpha} \mathbf{L}_{-\nu'}^{0,\alpha} 1 = \sum_{k \geq 0} a_k (\mathbf{L}_0^{0,\alpha})^k 1$$

where the coefficients a_k are determined by the algebra (1.48) and are independent of α (note that for (2.46) to hold it is crucial that $|\nu| = |\nu'|$). Since $(\mathbf{L}_0^{0,\alpha})^k 1 = \Delta_\alpha^k$ and $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$ we conclude

$$\mathbf{L}_{\nu'}^{0,\alpha} \mathbf{L}_{-\nu'}^{0,\alpha} 1 = M_\alpha^{(N)}(\nu, \nu')$$

where $N = |\nu|$ and the matrix $M_\alpha^{(N)}$ is a polynomial in Δ_α and thus in α . Repeating the argument for $\tilde{\mathbf{L}}_{\tilde{\nu}'}^{0,\alpha} \tilde{\mathbf{L}}_{-\tilde{\nu}'}^{0,\alpha} 1$ yields (2.42) with

$$F_\alpha(\nu, \nu') = M_\alpha^{(|\nu|)}(\nu, \nu').$$

The determinant of the matrix $M_\alpha^{(N)}$ is given by the Kac determinant formula (see Feigin-Fuchs [FF84])

$$(2.47) \quad \det M_\alpha^{(N)} = \kappa_N \prod_{r,s=1; r+s \leq N} (\Delta_\alpha - \Delta_{\alpha_{r,s}})^{p(N-rs)}$$

where κ_N does not depend on α or c_L , $p(M)$ is the number of Young Diagrams of length M and

$$\alpha_{r,s} = Q - r\frac{\gamma}{2} - s\frac{2}{\gamma}.$$

The factorisation (2.41) follows in the same way from the fact that the the \mathbf{A} and the $\tilde{\mathbf{A}}$ algebras commute. The matrix $S_{\mathbf{k},\nu}^{(\alpha)}$ is given by

$$S_{\mathbf{k},\nu}^{(\alpha)} = \|\mathbf{A}_{-\mathbf{k}} 1\|_{L^2(\Omega_{\mathbb{T}})}^{-1} \langle 1 | \mathbf{A}_{\mathbf{k}} \mathbf{L}_{\nu}^{0,\alpha} 1 \rangle_{L^2(\Omega_{\mathbb{T}})}.$$

Since

$$[\mathbf{A}_n, \mathbf{L}_m^{0,\alpha}] = n\mathbf{A}_{n+m}^\alpha - \frac{i}{2}n(n+1)Q\delta_{n,-m}$$

where $\mathbf{A}_0^\alpha = \frac{i\alpha}{2}$ and $\mathbf{A}_n^\alpha = \mathbf{A}_n$ for $n \neq 0$ and $\mathbf{L}_0^{0,\alpha} 1 = \Delta_\alpha$ we conclude

$$S_{\mathbf{k},\nu}^{(\alpha)} = s_{\mathbf{k},\nu}(\alpha)$$

where $s_{\mathbf{k},\nu}(\alpha)$ is a polynomial in α . Note also that completeness of $\pi_{\mathbf{k},1}$ implies

$$F_\alpha(\nu, \nu') = \sum_{\mathbf{k}} s_{\mathbf{k},\nu}(2Q - \bar{\alpha}) \overline{s_{\mathbf{k},\nu'}(\alpha)}.$$

□

The proposition shows that $F_\alpha(\nu, \nu')$ is a non singular matrix for $\alpha \neq \alpha_{r,s}$ and so in particular $F_{Q+iP}(\nu, \nu')$ is a non singular matrix.

2.8. Stress Energy Field. In this section we construct a probabilistic representation for the Virasoro descendants (1.53). It is well known that this can be done in terms of a local field, the *stress-energy tensor*, formally given for $z \in \mathbb{D}$ by

$$(2.48) \quad T(z) := Q\partial_z^2 X(z) - (\partial_z X(z))^2 + \mathbb{E}[(\partial_z X(z))^2].$$

The stress tensor does not make sense as a random field but can be given sense at the level of correlation functions as the limit of a regularised field. Then $T_\epsilon(z)$ is defined by (2.48) with X replaced by the mollification $X_\epsilon = X * \theta_\epsilon$ with $\theta_\epsilon = \frac{1}{\epsilon^2} \theta(\frac{\cdot}{\epsilon})$ where θ is smooth with compact support of average 1, i.e. $\int_{\mathbb{C}} \theta(x) dx = 1$. We denote also by $\bar{T}(z)$ the complex conjugate of $T(z)$.

The action of the Virasoro generators will be expressed in terms of the states

$$(2.49) \quad U_0 \left(\prod_{i=1}^k T(u_i) \prod_{i=1}^j \bar{T}(v_i) F \right) = \lim_{\epsilon \rightarrow 0} U_0 \left(\prod_{i=1}^k T_\epsilon(u_i) \prod_{i=1}^j \bar{T}_\epsilon(v_i) F \right)$$

Let $\delta < 1$. We introduce the set \mathcal{E}_δ defined by

$$(2.50) \quad \mathcal{E}_\delta := \left\{ f \in C_0^\infty(\delta\mathbb{D}) \mid f(e^{-t+i\theta}) = \sum_{n \in \mathbb{Z}} f_n(t) e^{in\theta}, \text{ with } f_n \in C_0^\infty(0, \infty) \text{ and } f_n = 0 \text{ for } |n| \text{ large enough} \right\}.$$

We will study the limit (2.49) for $F \in \mathcal{F}_{\delta\mathbb{D}}$ where (recall that $\langle u, v \rangle_{\mathbb{D}} := \int_{\mathbb{D}} u(x)v(x)dx$)

$$(2.51) \quad \mathcal{F}_{\delta\mathbb{D}} := \left\{ \prod_{i=1}^l \langle g_i, c + X \rangle_{\mathbb{D}} e^{\langle f, c+X \rangle_{\mathbb{D}}}; l \geq 0, f, g_i \in \mathcal{E}_\delta \right\}$$

Note that for $f \in \mathcal{E}_\delta$ then $f_{-n} = \bar{f}_n$ since f is real. Hence

$$\langle X, f \rangle_{\mathbb{D}} = 2\pi \sum_{n \in \mathbb{Z}} \int_0^\infty e^{-2t} X_n(t) f_{-n}(t) dt$$

where X_n are the Fourier components of X and the sum is finite. We note that $U_0 F \in e^{\beta|c|} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for $\beta > |f_0 - Q|$ and is in the domain of the Virasoro algebras since it depends on a finite number of φ_n . Let

$$(2.52) \quad \mathcal{B}_\delta = \left\{ (\mathbf{u}, \mathbf{v}) \in \mathbb{C}^{m+n} \mid \delta < |u_j|, |v_j| < 1, \forall j \neq j', u_j \neq u_{j'}, v_j \neq v_{j'}, u_j \neq v_{j'} \right\}$$

We have the simple:

Proposition 2.13. *Let $F \in \mathcal{F}_{\delta\mathbb{D}}$ be of the form $\prod_i \langle c + X, g_i \rangle_{\mathbb{D}} e^{\langle f, c+X \rangle_{\mathbb{D}}}$ with $f, g_i \in \mathcal{E}_\delta$. Then the limit (2.49) exists in $e^{\beta|c|} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for $\beta > |f_0 - Q|$ where $f_0 = \langle f, 1 \rangle_{\mathbb{D}}$ and defines a function holomorphic in \mathbf{u} and anti holomorphic in \mathbf{v} in the region \mathcal{B}_δ taking values in $e^{\beta|c|} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$.*

Proof. To keep the notation simple we consider only the case $j = 0$ in (2.49) and the case where $F \in \mathcal{F}_{\delta\mathbb{D}}$ is of the form $F = e^{\langle f, c+X \rangle_{\mathbb{D}}}$ with $f \in \mathcal{E}_\delta$. By replacing f by $f + \sum_i \lambda_i g_i$ with $g_i \in \mathcal{E}_\delta$ and differentiating at $\lambda_i = 0$ we can deduce the result in the general case. For such F we have

$$(2.53) \quad (U_0 F)(c, \varphi) = e^{(f_0 - Q)c} e^{\langle P\varphi, f \rangle_{\mathbb{D}}} e^{\frac{1}{2} \langle f, G_{\mathbb{D}} f \rangle_{\mathbb{D}}}$$

By Gaussian integration by parts the right hand side is a sum of terms

$$(2.54) \quad \text{const} \times \prod_i ((\partial^{a_i} P\varphi_\epsilon(u_i))^{b_i} \prod_{j < k} (\partial_{u_j}^{b_{jk}} \partial_{u_k}^{c_{jk}} G_{\mathbb{D}, \epsilon}(u_j, u_k))^{d_{jk}} (\prod_l \partial_{u_l}^{d_l} (G_{\mathbb{D}, \epsilon} f)(u_l))^{e_l} U_0 F$$

where $a_i, b_{jk}, c_{jk} \in \{1, 2\}$, $b_i, d_{jk}, e_l \in \{0, 1, 2\}$, $d_l \in \{1, 2\}$ and we denoted $G_{\mathbb{D}, \epsilon} = \theta_\epsilon * G_{\mathbb{D}}$. Since only $G_{\mathbb{D}, \epsilon}(z, z')$ with $z \neq z'$ enters here, one can deduce that the limit of the $G_{\mathbb{D}, \epsilon}$ terms exists. From (2.2) we get

$$(2.55) \quad \partial_z G_{\mathbb{D}}(z, u) = -\frac{1}{2} \left(\frac{1}{z-u} - \frac{1}{z-\frac{1}{u}} \right)$$

Hence the limit of the second product in (2.54) is holomorphic since $b_{jk}, c_{jk} > 0$. For the third product, we get convergence to terms of the form

$$\int_{\mathbb{D}} \partial_u G_{\mathbb{D}}(u, z) f(z) dz = -\frac{1}{2} \sum_{n=0}^{\infty} (f_n u^{-n-1} + f_{-n-1} u^n)$$

or its ∂_u derivative where $f_n = \langle f, u^n \rangle_{\mathbb{D}}$ and $f_{-n} = \langle f, \bar{u}^n \rangle_{\mathbb{D}}$ and the sum is finite and holomorphic. Finally, the first product converges to a holomorphic function $g(\mathbf{u})$ taking values in $L^2(\Omega_{\mathbb{T}})$ and $g(\mathbf{u})U_0F \in e^{\beta|c|}L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. \square

We use the following notation for contour integrals of $f : \mathbb{D} \rightarrow \mathbb{C}$ for $\delta > 0$

$$\oint_{|z|=\delta} r f(z) dz := i\delta \int_0^{2\pi} f(\delta e^{i\theta}) e^{i\theta} d\theta, \quad \oint_{|z|=\delta} f(z) d\bar{z} := i\delta \int_0^{2\pi} f(\delta e^{i\theta}) e^{-i\theta} d\theta$$

Then we have

Lemma 2.14. *Let $F \in \mathcal{F}_{\delta\mathbb{D}}$ and $\delta' > \delta$. Then for all $n > 0$*

$$(2.56) \quad \frac{1}{2\pi i} \oint_{|z|=\delta'} z^{1-n} U_0(T(z)F) dz = \mathbf{L}_{-n}^0 U_0F, \quad \frac{1}{2\pi i} \oint_{|z|=\delta'} \bar{z}^{1-n} U_0(\bar{T}(z)F) d\bar{z} = \tilde{\mathbf{L}}_{-n}^0 U_0F.$$

Proof. Here again (for simplicity) we consider the case where $F \in \mathcal{F}_{\delta\mathbb{D}}$ is of the form $F = e^{\langle f, c+X \rangle_{\mathbb{D}}}$ with $f \in \mathcal{E}_{\delta}$. Let us write the integration by parts terms explicitly. First

$$U_0(\partial_z X(z)F) = \partial_z P\varphi(z)U_0F + \int_{\mathbb{D}} \partial_z G_{\mathbb{D}}(z, u) f(u) du U_0F$$

From (2.2) we get

$$(2.57) \quad \partial_z G_{\mathbb{D}}(z, u) = -\frac{1}{2} \left(\frac{1}{z-u} - \frac{1}{z-\frac{1}{\bar{u}}} \right) = -\frac{1}{2} \sum_{n=0}^{\infty} (u^n z^{-n-1} + \bar{u}^{n+1} z^n)$$

which converges since $|u| < |z| < 1$. Therefore

$$\int_{\mathbb{D}} \partial_z G_{\mathbb{D}}(z, u) f(u) du = -\frac{1}{2} \sum_{n=0}^{\infty} (f_n z^{-n-1} + f_{-n-1} z^n)$$

where $f_n = \langle f, u^n \rangle_{\mathbb{D}}$ and $f_{-n} = \langle f, \bar{u}^n \rangle_{\mathbb{D}}$ for $n \geq 0$. Recalling (2.6), we have obtained

$$(2.58) \quad U_0(\partial_z X(z)F) = \sum_{n \geq 0} (n\varphi_n z^{n-1} - \frac{1}{2}(f_n z^{-n-1} + f_{-n-1} z^n)) U_0F$$

$$(2.59) \quad = \sum_{n \in \mathbb{Z}} z^{n-1} (n\varphi_n 1_{n>0} - \frac{1}{2}f_{-n}) U_0F$$

By (2.60)

$$(2.60) \quad (U_0F)(c, \varphi) = e^{\langle f_0 - Q \rangle_c} e^{\sum_{n>0} (\varphi_n f_n + \varphi_{-n} f_{-n})} e^{\frac{1}{2} \langle f, G_{\mathbb{D}} f \rangle_{\mathbb{D}}}$$

so that

$$(2.61) \quad \begin{aligned} U_0(\partial_z X(z)F) &= \left(-\frac{1}{2} f_0 z^{-1} + \sum_{n \neq 0} z^{n-1} (n\varphi_n 1_{n>0} - \frac{1}{2} \partial_{-n}) \right) U_0F \\ &= i \sum_{n \in \mathbb{Z}} z^{n-1} \mathbf{A}_{-n} U_0F \end{aligned}$$

where we recalled that $\mathbf{A}_0 = \frac{i}{2}(\partial_c + Q)$. Hence

$$(2.62) \quad U_0(Q \partial_z^2 X(z)F) = -iQ \sum_{n \in \mathbb{Z}} (n+1) z^{-n-2} \mathbf{A}_n U_0F.$$

Next consider the quadratic terms in T :

$$\begin{aligned}
U_0(((\partial_z X(z))^2 - \mathbb{E}[(\partial_z X(z))^2])F) &= \left(\partial_z P\varphi(z) + \int_{\mathbb{D}} \partial_z G_{\mathbb{D}}(z, u) f(u) du\right)^2 U_0 F \\
&= \left(\sum_{n \in \mathbb{Z}} z^{n-1} (n\varphi_n 1_{n>0} - \frac{1}{2} f_{-n})\right)^2 U_0 F \\
&= \sum_{n, m} z^{n+m-2} (-\mathbf{A}_{-n} \mathbf{A}_{-m} + \frac{m}{2} \delta_{m, -n} 1_{m>0}) U_0 F \\
&= - \sum_{n, m} z^{n+m-2} : \mathbf{A}_{-n} \mathbf{A}_{-m} : U_0 F
\end{aligned}$$

where we used (2.58) in the second step, (2.61) in the third step and $\mathbf{A}_m \mathbf{A}_{-m} = \mathbf{A}_{-m} \mathbf{A}_m + \frac{m}{2}$ for $m > 0$ in the last step. The sum converges in $e^{\beta|c|} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. Combining this with (2.62) the claim (2.56) follows upon doing the contour integral. The claim for \bar{T} is proved in the same way. \square

Let us introduce some notation for the general correlations (2.49). Denote $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{D}^k$. We define nested contour integrals for $f : \mathbb{D}^k \times \mathbb{D}^j \rightarrow \mathbb{C}$ by

$$(2.63) \quad \oint_{|\mathbf{u}|=\delta} \oint_{|\mathbf{v}|=\tilde{\delta}} f(\mathbf{u}, \mathbf{v}) d\bar{\mathbf{v}} d\mathbf{u} := \oint_{|u_k|=\delta_k} \cdots \oint_{|u_1|=\delta_1} \oint_{|v_j|=\tilde{\delta}_j} \cdots \oint_{|v_1|=\tilde{\delta}_1} f(\mathbf{u}, \mathbf{v}) d\bar{v}_1 \dots d\bar{v}_j du_1 \dots du_k.$$

where $\delta := (\delta_1, \dots, \delta_k)$ with $0 < \delta_1 < \dots < \delta_k < 1$ and similarly for $\tilde{\delta}$. Furthermore we always suppose $\delta_i \neq \tilde{\delta}_j$ for all i, j . Next, for $\epsilon \in (\mathbb{R}^+)^k$ and $\epsilon' \in (\mathbb{R}^+)^j$ we set $T_{\epsilon}(\mathbf{u}) := \prod_{i=1}^k T_{\epsilon_i}(u_i)$ (and similarly for the anti-holomorphic part) and given two Young diagrams $\nu = (\nu_i)_{1 \leq i \leq k}$, $\tilde{\nu} = (\tilde{\nu}_i)_{1 \leq i \leq j}$, we denote $\mathbf{u}^{1-\nu} = \prod u_i^{1-\nu_i}$ and $\bar{\mathbf{v}}^{1-\tilde{\nu}} = \prod \bar{v}_i^{1-\tilde{\nu}_i}$. Recall that the limit (2.49) exists; in fact, the proof of Proposition 2.13 extends to show existence of the limit (2.49) where $\prod_{i=1}^k T_{\epsilon}(u_i)$ is replaced by $T_{\epsilon}(\mathbf{u})$ and one takes the successive limits $\lim := \lim_{\epsilon_1 \rightarrow 0} \cdots \lim_{\epsilon_k \rightarrow 0}$. With these notations we have:

Proposition 2.15. *Let $F \in \mathcal{F}_{\delta\mathbb{D}}$ be of the form $F = e^{\langle f, c+X \rangle_{\mathbb{D}}}$ with $f \in \mathcal{E}_{\delta}$ and with $\delta < \delta_1 \wedge \tilde{\delta}_1$. Then*

$$(2.64) \quad (2\pi i)^{-k-j} \oint_{|\mathbf{u}|=\delta} \oint_{|\mathbf{v}|=\tilde{\delta}} \mathbf{u}^{1-\nu} \bar{\mathbf{v}}^{1-\tilde{\nu}} U_0(T(\mathbf{u})\bar{T}(\mathbf{v})F) d\bar{\mathbf{v}} d\mathbf{u} = \mathbf{L}_{-\nu}^0 \mathbf{L}_{-\tilde{\nu}}^0 U_0 F.$$

Proof. For simplicity consider the case with only T insertions. We proceed by induction in k . By Lemma 2.14 the claim holds for $k = 1$. Suppose it holds for $k - 1$. Introduce the following regularization for T . Let $\rho \in C_0^{\infty}(\mathbb{R})$ be a smooth bump at origin with compact support such that $\rho(s) = \rho(-s)$ and $\rho_{\epsilon}(s) = \epsilon^{-1} \rho(\frac{s}{\epsilon})$. Let $z = e^{-t+i\theta} \in \mathbb{D}$ and define $f_{\epsilon, z} \in \mathcal{E}$ (recall (2.50)) by $f_{\epsilon, z, n}(s) = \frac{\epsilon^{-2s}}{2\pi} \rho_{\epsilon}(t-s) e^{-in\theta}$ for $|n| < \epsilon^{-1}$ and 0 otherwise. Define

$$(2.65) \quad X_{\epsilon}(z) := \langle X, f_{\epsilon, z} \rangle_{\mathbb{D}} = \sum_{|n| < \epsilon^{-1}} (\rho_{\epsilon} * X_n)(t) e^{in\theta} = \sum_{|n| < \epsilon^{-1}} \left(\int_0^{\infty} \rho(x) X_n(t + \epsilon x) dx \right) e^{in\theta}$$

and let $T_{\epsilon}(z)$ then be defined by 2.48 with X replaced by X_{ϵ} . We have the following covariance for $u = e^{-s} e^{i\theta'}$ with $e^{-s} < e^{-t}$

$$\begin{aligned}
\mathbb{E}[X_{\epsilon}(z) X(u)] &= \sum_{0 < |n| < \epsilon^{-1}} \frac{1}{2|n|} \left(\int_{\mathbb{R}} \rho(x) e^{\epsilon x|n|} dx \right) e^{-(s-t)|n|} e^{in(\theta-\theta')} - \frac{1}{2|n|} \left(\int_{\mathbb{R}} \rho(x) e^{-\epsilon x|n|} dx \right) e^{-(s+t)|n|} e^{in(\theta-\theta')} \\
&\quad + \int_{\mathbb{R}} \rho(x) (t + \epsilon x) dx \\
&= \sum_{0 < n < \epsilon^{-1}} \frac{1}{2n} \left(\int_{\mathbb{R}} \rho(x) e^{\epsilon x n} dx \right) \left(\frac{\bar{u}^n}{\bar{z}^n} + \frac{u^n}{z^n} \right) - \frac{1}{2n} \left(\int_{\mathbb{R}} \rho(x) e^{-\epsilon x n} dx \right) ((z\bar{u})^n + (\bar{z}u)^n) \\
(2.66) \quad &\quad - \ln|z|
\end{aligned}$$

Obviously the statement about the existence of the limit in Proposition 2.13 holds also with this regularization and by taking successive limits $\lim_{\epsilon \rightarrow 0} = \lim_{\epsilon_1 \rightarrow 0} \cdots \lim_{\epsilon_k \rightarrow 0}$ hence we have

$$\begin{aligned} \oint_{|\mathbf{u}|=\delta} \mathbf{u}^{1-\nu} U_0(T(\mathbf{u})F) d\mathbf{u} &= \lim_{\epsilon^k \rightarrow 0} \lim_{\epsilon_k \rightarrow 0} \oint_{|\mathbf{u}|=\delta} \mathbf{u}^{1-\nu} U_0(T_{\epsilon^k}(u_k) T_{\epsilon^k}(\mathbf{u}^k) F) d\mathbf{u} \\ &= \lim_{\epsilon^k \rightarrow 0} \oint_{|\mathbf{u}|=\delta} \mathbf{u}^{1-\nu} U_0(T(u_k) T_{\epsilon^k}(\mathbf{u}^k) F) d\mathbf{u} \end{aligned}$$

where $\epsilon^k = (\epsilon_1, \dots, \epsilon_{k-1})$. Now $T_{\epsilon^k}(\mathbf{u}^k)F \in \mathcal{F}_{\delta_k}$. Hence by Lemma 2.14

$$\frac{1}{2\pi i} \oint_{|u_k|=\delta_k} u_k^{1-\nu_k} U_0(T(u_k) T_{\epsilon^k}(\mathbf{u}^k) F) du_k = \mathbf{L}_{-\nu_k} U_0(T_{\epsilon^k}(\mathbf{u}^k) F)$$

and therefore

$$(2\pi i)^{-k} \oint_{|\mathbf{u}|=\delta} \mathbf{u}^{1-\nu} U_0(T(u_k) T_{\epsilon^k}(\mathbf{u}^k) F) d\mathbf{u} = \mathbf{L}_{-\nu_k} \left((2\pi i)^{-k+1} \oint_{|\mathbf{u}^k|=\delta^k} (\mathbf{u}^k)^{1-\nu^k} U_0(T_{\epsilon^k}(\mathbf{u}^k) F) d\mathbf{u}^k \right)$$

where $\mathbf{u}^k = (u_1, \dots, u_{k-1})$, $\delta^k = (\delta_1, \dots, \delta_{k-1})$ and $\nu^k = (\nu_1, \dots, \nu_{k-1})$. By the induction hypothesis the term

$$(2.67) \quad (2\pi i)^{-k+1} \oint_{|\mathbf{u}^k|=\delta^k} (\mathbf{u}^k)^{1-\nu^k} U_0(T_{\epsilon^k}(\mathbf{u}^k) F) d\mathbf{u}^k$$

converges to $\mathbf{L}_{-\nu^k} U_0(F)$ as ϵ^k goes to 0. By integration by parts, one can expand (2.67) into a sum of contour integrals of terms of the form (2.54) (where in this context the Green function is regularized in a different way, i.e. but cutting out high frequency modes: see expression (2.66)). Using expression (2.66)¹⁹ (and a similar expression for $\mathbb{E}[X_\epsilon(u_i) X_\epsilon(u_j)]$) one can then show that (2.67) is equal to $G_{\epsilon^k}((\varphi_j)_{|j| \leq N}) U_0 F$ where N is fixed (independent of ϵ^k) and G_{ϵ^k} is a polynomial of bounded degree (independent of ϵ^k) with coefficients depending on ϵ^k (this can be seen by first integrating with respect to u_1 then u_2 etc...). From the previous considerations, one can also see that the coefficients of G_{ϵ^k} converge to those of $\frac{1}{U_0 F} \mathbf{L}_{-\nu_{k-1}} \cdots \mathbf{L}_{-\nu_1} U_0 F$ when ϵ^k converges to 0. This strong form of convergence of $G_{\epsilon^k}((\varphi_j)_{|j| \leq N}) U_0 F$ towards $\mathbf{L}_{-\nu_{k-1}} \cdots \mathbf{L}_{-\nu_1} U_0 F$ implies then that $\mathbf{L}_{-\nu_k} (G_{\epsilon^k}((\varphi_j)_{|j| \leq N}) U_0 F)$ converges to $\mathbf{L}_{-\nu_k} \cdots \mathbf{L}_{-\nu_1} U_0 F$. \square

In the sequel we will apply Proposition 2.15 to the function

$$F = S_{e^{-t}} U_0^{-1} \psi_\alpha = e^\alpha \int_0^{2\pi} (c + X(e^{-t+i\theta})) \frac{d\theta}{2\pi} - \alpha Q t$$

where $\psi_\alpha(c, \varphi) = e^{(\alpha-Q)c}$ (in this case integration against a function $f \in \mathcal{E}_\delta$ is replaced by an average on a circle but the previous considerations apply also). Then $U_0 F = e^{-t\mathbf{H}^0} \psi_\alpha = e^{-2t\Delta_\alpha} \psi_\alpha$. Thus we arrive to the representation for the Virasoro descendants

$$(2.68) \quad \psi_{\alpha, \nu, \tilde{\nu}} = e^{2t\Delta_\alpha} (2\pi i)^{-k-j} \oint_{|\mathbf{u}|=\delta} \oint_{|\mathbf{v}|=\tilde{\delta}} \mathbf{u}^{1-\nu} \tilde{\mathbf{v}}^{1-\tilde{\nu}} U_0(T(\mathbf{u}) \tilde{T}(\mathbf{v}) S_{e^{-t}} U_0^{-1} \psi_\alpha) d\tilde{\mathbf{v}} d\mathbf{u}$$

where now $e^{-t} < \delta_1 \wedge \tilde{\delta}_1$.

3. SCATTERING OF THE LIOUVILLE HAMILTONIAN

In this section, we suppose that $\gamma \in (0, \sqrt{2})$ and we develop the scattering theory for the operator \mathbf{H} on $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ with underlying measure $dc \otimes \mathbb{P}_{\mathbb{T}}$ (where \mathbf{H} denotes the Friedrichs extension constructed in 2.4). This operator has continuous spectrum and can not be diagonalized with a complete set of $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ -eigenfunctions. We will rather use a stationary approach for this operator, in a way similar to what has been done in geometric scattering theory for manifolds with cylindrical ends in [Gu89, Me93]. The goal is to obtain a spectral resolution for \mathbf{H} in terms of generalized eigenfunctions, which will be shown to be analytic in the spectral parameter. In other words, we search to write the spectral measure of \mathbf{H} using these generalized eigenfunctions, which are similar to plane waves $(e^{i\lambda\omega \cdot x})_{\lambda \in \mathbb{R}, \omega \in S^{n-1}}$ in Euclidean scattering for the Laplacian Δ_x on \mathbb{R}^n . In our case, the generalized eigenfunctions will be functions in weighted spaces of the form $e^{-\beta c} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ with particular asymptotics at $c = -\infty$. Let us explain briefly the simplest one, corresponding to the functions $\Psi_\alpha := \Psi_{\alpha, \mathbf{0}, \mathbf{0}} = U(V_\alpha(0))$ defined in the Introduction and represented

¹⁹Here we use the fact that our regularization is a high frequency cutoff.

probabilistically by the expression (1.23) when $\alpha < Q$ is real. For $\alpha \in (Q - \gamma/2, Q)$, they will be the only eigenfunctions of \mathbf{H} in $e^{-\beta c} L^2(\mathbb{R} \times \Omega_T)$ for $\beta > Q - \alpha$ satisfying

$$(\mathbf{H} - 2\Delta_\alpha)\Psi_\alpha = 0, \quad \Psi_\alpha = e^{(\alpha-Q)c} + u, \quad \text{with } u \in L^2(\mathbb{R} \times \Omega_T).$$

One way to construct them will be to take the limit (see Proposition 3.7)

$$\Psi_\alpha = \lim_{t \rightarrow \infty} e^{-t\mathbf{H}}(e^{2t\Delta_\alpha} e^{(\alpha-Q)c})$$

where we observe that $e^{-t\mathbf{H}^0} e^{(\alpha-Q)c} = e^{-2t\Delta_\alpha} e^{(\alpha-Q)c}$ so that, formally speaking, Ψ_α is the limit of the intertwining $e^{-t\mathbf{H}} e^{t\mathbf{H}^0}(e^{(\alpha-Q)c})$ as $t \rightarrow +\infty$. An alternative expression is to write them as

$$(3.1) \quad \Psi_\alpha = e^{(\alpha-Q)c} \chi(c) - (\mathbf{H} - 2\Delta_\alpha)^{-1} (\mathbf{H} - 2\Delta_\alpha) (e^{(\alpha-Q)c} \chi(c))$$

where $\chi \in C^\infty(\mathbb{R})$ equal to 1 near $-\infty$ and 0 near $+\infty$ (see (3.22) and Lemma 3.5); here we notice that $e^{(\alpha-Q)c} \chi(c)$ is not $L^2(\mathbb{R} \times \Omega_T)$ but one can check that $(\mathbf{H} - 2\Delta_\alpha)(e^{(\alpha-Q)c} \chi(c)) \in L^2$ so that we can apply the resolvent $\mathbf{R}(\alpha) := (\mathbf{H} - 2\Delta_\alpha)^{-1}$ to it, and $(\mathbf{H} - 2\Delta_\alpha)^{-1} (\mathbf{H} - 2\Delta_\alpha)(e^{(\alpha-Q)c} \chi(c)) \in L^2$ is not equal to $(e^{(\alpha-Q)c} \chi(c))$. Our goal will be to extend analytically these Ψ_α (and their descendants $\Psi_{\alpha, \mathbf{k}, \mathbf{l}}$) to $\text{Re}(\alpha) \leq Q$ and in particular to the line $\alpha \in Q + i\mathbb{R}$ corresponding to the spectrum of \mathbf{H} . To perform this, we see that we need to extend analytically the resolvent operator $\mathbf{R}(\alpha)$ to $\text{Re}(\alpha) \leq Q$, which will be the main part of this section. In fact, we shall show that $\mathbf{R}(\alpha)$ extends analytically on an open set of a Riemann surface covering the complex plane, containing the real half-plane $\text{Re}(\alpha) \leq Q$. We note that the functions $\Psi_{\alpha, \mathbf{k}, \mathbf{l}}$ will be expressed as the elements in the range of some Poisson operator denoted $\mathcal{P}(\alpha)$, mapping (some subspaces of) $L^2(\Omega_T)$ to weighted spaces $e^{-\beta c} L^2(\mathbb{R} \times \Omega_T)$ for some $\beta > 0$ depending on α . The results proved here hold in some cases for geometric scattering in finite dimension [Gu89, Me93], but we are not aware of some results of this type in quantum field theory where the base space is infinite dimensional. The main difficulty will be to deal with the fact that the potential V is not bounded and the fact that the eigenfunctions of \mathbf{P} (Hermite polynomials) have $L^p(\Omega_T)$ norms which blow up very fast in terms of their eigenvalues.

In this section, we shall start by describing the resolvent of \mathbf{H} in the *probabilistic region* $\{\text{Re}((\alpha - Q)^2) > \beta^2\}$ acting on weighted spaces $e^{\beta c} L^2$ for $\beta \in \mathbb{R}$ and deduce the construction of the $\Psi_{\alpha, \mathbf{k}, \mathbf{l}}$ in this region. Next, we will show that the resolvent $\mathbf{R}(\alpha)$ admits an analytic extension in a neighborhood of $\{\text{Re}(\alpha) \leq Q\}$ (for α on some Riemann surface Σ). We shall use these results to prove the analytic continuation of the $\Psi_{\alpha, \mathbf{k}, \mathbf{l}}$ to $\text{Re}(\alpha) \leq Q$ and we shall finally construct the scattering operator $\mathbf{S}(\alpha)$ in Section 3.3 and write the spectral decomposition of \mathbf{H} in terms of the $\Psi_{\alpha, \mathbf{k}, \mathbf{l}}$ in Theorem 3.22 (written in terms of Poisson operator in this section).

In what follows, we will mostly consider the L^2 (or L^p) spaces on Ω_T or on $\mathbb{R} \times \Omega_T$ respectively equipped with the measure \mathbb{P}_T or $dc \otimes \mathbb{P}_T$, which we will denote by $L^2(\Omega_T)$ or $L^2(\mathbb{R} \times \Omega_T)$ for short. When the space is omitted, i.e. we simply write L^2 , this means that we consider $L^2(\mathbb{R} \times \Omega_T)$: this will relieve notations in some latter part of the paper. Recall that we denote by $\langle \cdot | \cdot \rangle_2$ the standard scalar product associated to $L^2(\mathbb{R} \times \Omega_T, dc \otimes \mathbb{P}_T)$ and $\|\cdot\|_2$ the associated norm; in general our scalar products will always be complex linear in the left component and anti-linear in the right component. Given two normed vector space E and F , the space of continuous linear mappings from E into F will be denoted by $\mathcal{L}(E, F)$ and when $E = F$ we will simply write $\mathcal{L}(E)$. The corresponding operator norms will be denoted by $\|\cdot\|_{\mathcal{L}(E, F)}$ or $\|\cdot\|_{\mathcal{L}(E)}$.

The operator \mathbf{H} is made up of several pieces. The first piece is the operator \mathbf{P} defined in (1.32), which is a self-adjoint non-negative unbounded operator on $L^2(\Omega_T)$. It has discrete spectrum $(\lambda_{\mathbf{k}\mathbf{l}})_{\mathbf{k}, \mathbf{l} \in \mathcal{N}}$, but to simply the indexing we shall order them in increasing order (without counting multiplicity) and denote them by

$$\sigma(\mathbf{P}) = \{\lambda_j \mid j \in \mathbb{N}, \lambda_j < \lambda_{j+1}\}.$$

We denote by

$$(3.2) \quad E_k := \{F \in L^2(\Omega_T) \mid 1_{[0, \lambda_k]}(\mathbf{P})F = F\} = \bigoplus_{j \leq k} \ker(\mathbf{P} - \lambda_j)$$

the sum of eigenspaces with eigenvalues less or equal to λ_k and $\Pi_k : L^2(\Omega_{\mathbb{T}}) \rightarrow E_k$ the orthogonal projection. We also will use the important fact

$$(3.3) \quad E_k \subset L^p(\Omega_{\mathbb{T}}), \quad \forall p < \infty.$$

We shall consider a measurable potential $V : \Omega_{\mathbb{T}} \rightarrow \mathbb{R}$ that satisfies for all $\varepsilon > 0$

$$(3.4) \quad V > 0, \quad V \in L^{\frac{2}{\gamma^2} - \varepsilon}(\Omega_{\mathbb{T}}) \subset L^{1+\varepsilon}(\Omega_{\mathbb{T}}), \quad \forall q \in [1, \infty), \quad V^{-1} \in L^q(\Omega_{\mathbb{T}}).$$

In particular, we shall use the important fact that the operator of multiplication by V satisfies for all $\varepsilon > 0$

$$(3.5) \quad \forall \varepsilon > 0, \quad V : E_k \mapsto L^{\frac{2}{\gamma^2} - \varepsilon}(\Omega_{\mathbb{T}}) \quad \text{is bounded.}$$

We then consider the operator that we introduced by (1.19) (here we include the μ inside V to simplify the notation)

$$(3.6) \quad \mathbf{H} = \mathbf{H}^0 + e^{\gamma c} V = -\frac{1}{2} \partial_c^2 + \frac{1}{2} Q^2 + \mathbf{P} + e^{\gamma c} V.$$

The quadratic form associated to \mathbf{H} is the form \mathcal{Q} defined in Section 2.4, with domain $\mathcal{D}(\mathcal{Q})$. We will consider the self-adjoint extension associated of \mathbf{H} on $\mathcal{D}(\mathcal{H}^0) \cap \mathcal{D}(e^{\gamma c} V)$, which is also the extension obtained using \mathcal{Q} . An important fact that we shall use is that, if $\mathcal{D}'(\mathcal{Q})$ is the dual of $\mathcal{D}'(\mathcal{Q})$, then

$$(3.7) \quad e^{\frac{\gamma}{2} c} V^{\frac{1}{2}} : L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow \mathcal{D}'(\mathcal{Q}).$$

First, we show a useful result for the spectral decomposition.

Lemma 3.1. *The operator \mathbf{H} does not have non-zero eigenvectors $u \in \mathcal{D}(\mathbf{H})$. If $\gamma \in (0, 1)$, the spectrum of \mathbf{H} is given by $\sigma(\mathbf{H}) = [\frac{Q^2}{2}, \infty)$ and consists of essential spectrum.*

Proof. In the case $\gamma \in (0, 1)$, the space \mathcal{C} is included in $\mathcal{D}(\mathbf{H})$. It is then easy to check that $\sigma(\mathbf{H}) = [\frac{Q^2}{2}, \infty)$ consists only of essential spectrum by using Weyl sequences $(e^{ipc} \chi(2^{-n} c) / \omega_n)_n \in \mathbb{N}$ where $\chi \in C_c^\infty(\mathbb{R})$ have support in $[-\frac{3}{2}, -1]$ and equal to 1 on some interval, with $\omega_n = \|\chi(2^{-n} \cdot)\|_{L^2(\mathbb{R})}$.

Let $u \in \mathcal{D}(\mathbf{H})$ such that $\mathbf{H}u = \lambda u$ with $\lambda \in [\frac{Q^2}{2}, \infty)$. Then $u \in \mathcal{D}(\mathcal{Q})$ (hence $\partial_c u \in L^2$), and it satisfies $\mathcal{Q}(u, v) = \langle \lambda u | v \rangle_2$ for all $v \in \mathcal{D}(\mathcal{Q})$. Now we claim

Lemma 3.2. *Assume we are given $f \in L^2$ such that $\partial_c f \in L^2$. Consider $u \in \mathcal{D}(\mathcal{Q})$ such that*

$$(3.8) \quad \mathcal{Q}(u, v) = \langle f | v \rangle_2, \quad \forall v \in \mathcal{D}(\mathcal{Q}).$$

Then $\partial_c u \in \mathcal{D}(\mathcal{Q})$ and

$$(3.9) \quad \mathcal{Q}(\partial_c u, v) = \langle \partial_c f | v \rangle_2 - \langle \gamma e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} u | e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} v \rangle_2, \quad \forall v \in \mathcal{D}(\mathcal{Q}).$$

We postpone the proof of this lemma and conclude first. Consider next (3.8) with $f = \lambda u$ and choose $v = \partial_c u \in \mathcal{D}(\mathcal{Q})$ to obtain $\mathcal{Q}(u, \partial_c u) = \langle \lambda u | \partial_c u \rangle_2 = 0$. Also, choosing $v = u$ in (3.9) we obtain $\mathcal{Q}(\partial_c u, u) = \langle \lambda \partial_c u | u \rangle_2 - \gamma \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} u\|_2^2$. These relations imply $\|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} u\|_2^2 = 0$, hence $u = 0$ as $V > 0$ almost surely. \square

Proof of Lemma 3.2. For $h > 0$, introduce the translation operator $T_h : L^2 \rightarrow L^2$ by $T_h v := v(c + h, \cdot)$ and the discrete derivative operator $D_h : L^2 \rightarrow L^2$ by $D_h v := (T_h v - v)/h$. Note that T_h maps $\mathcal{D}(\mathcal{Q})$ into itself, that $\|D_h v\|_2 \leq \|\partial_c v\|_2$ and we have the discrete IPP $\langle D_h u | v \rangle_2 = -\langle u | D_{-h} v \rangle_2$ for all $u, v \in L^2$. Now we can replace v by $D_{-h} v$ in (3.8) and use discrete IPP to obtain

$$(3.10) \quad \mathcal{Q}(D_h u, v) = \langle D_h f - (T_h(e^{\gamma c} V) - e^{\gamma c} V) D_h u - D_h(e^{\gamma c} V) u | v \rangle_2, \quad \forall v \in \mathcal{D}(\mathcal{Q}).$$

Next we choose $v = D_h u$ and, using repeatedly the inequality $|\langle f | g \rangle_2| \leq \frac{\varepsilon}{2} \|f\|_2^2 + \frac{1}{2\varepsilon} \|g\|_2^2$ for arbitrary $\varepsilon > 0$, we obtain the a priori estimate (for some constant $C > 0$ depending only on γ)

$$(3.11) \quad \begin{aligned} \mathcal{Q}(D_h u, D_h u) &\leq \frac{1}{2} (\|D_h f\|_2^2 + \|D_h u\|_2^2) + \left| \frac{1 - e^{\gamma h}}{2h} \right| \left(\frac{1}{\varepsilon} \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} T_h u\|_2^2 + \varepsilon \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} D_h u\|_2^2 \right) \\ &\leq C (\|\partial_c f\|_2^2 + \mathcal{Q}(u, u)) + C \varepsilon \mathcal{Q}(D_h u, D_h u) \end{aligned}$$

for all $h > 0$ small, and therefore the last term can be absorbed in the left hand side if $\varepsilon > 0$ is small enough. Then, writing (3.10) for h and h' , subtracting and then choosing $v = D_h u - D_{h'} u$, we find

$$\mathcal{Q}(D_h u - D_{h'} u, D_h u - D_{h'} u) \leq C(\|D_h f - D_{h'} f\|_2^2 + |h - h'| \mathcal{Q}(u, u) + |h - h'| \mathcal{Q}(D_{(h-h')} u, D_{(h-h')} u)).$$

Using (3.11) with h replaced by $h - h'$ to bound the last term, we obtain that the sequence $(D_h u)_h$ is Cauchy for the \mathcal{Q} -norm. Hence the limit $\partial_c u$ belongs to $\mathcal{D}(\mathcal{Q})$. \square

When $\gamma \in [1, \sqrt{2})$ the spectrum is also $[Q^2/2, \infty)$ and is made of essential spectrum, and this will follow from our analysis of the resolvent: in fact we will show for $\gamma \in (0, \sqrt{2})$ that the spectrum of \mathbf{H} is absolutely continuous.

3.1. Resolvent of \mathbf{H} . To describe the spectral measure of \mathbf{H} and construct its generalized eigenfunctions, the main step is to understand the resolvent of \mathbf{H} as a function of the spectral parameter, in particular when the spectral parameter approach the spectrum. Due to the fact that the spectrum of \mathbf{H} starts at $Q^2/2$, it is convenient to use the spectral parameter $2\Delta_\alpha$ where $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$ and $\alpha \in \mathbb{C}$. That way we have with $\alpha = Q + ip$

$$\mathbf{H} - 2\Delta_\alpha = -\frac{1}{2}\partial_c^2 + \mathbf{P} + e^{\gamma c} V - \frac{1}{2}p^2$$

where $p \in \mathbb{R}$ plays the role of a frequency: in particular $2\Delta_\alpha \in [Q^2/2, \infty)$ if and only if $p \in \mathbb{R}$. The half-plane $\{\alpha \in \mathbb{C} \mid \operatorname{Re}(\alpha) < Q\}$ is mapped by Δ_α to the resolvent set $\mathbb{C} \setminus [Q^2/2, \infty)$ of \mathbf{H} and will be called the *physical sheet*. By the spectral theorem

$$\mathbf{R}(\alpha) = (\mathbf{H} - 2\Delta_\alpha)^{-1} : L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow \mathcal{D}(\mathbf{H})$$

is bounded if $\operatorname{Re}(\alpha) < Q$. Our goal is to extend this resolvent up to the line $\operatorname{Re}(\alpha) = Q$ analytically, and we will actually do it in an even larger region. The price to pay is that $\mathbf{R}(\alpha)$ will not be bounded on L^2 but rather on certain weighted L^2 spaces, where the weights are $e^{\beta c}$ in the region $c \leq -1$, with $\beta \in \mathbb{R}$ tuned with respect to α .

Resolvent and propagator on weighted spaces in the probabilistic region. Our first task is to understand the resolvent on weighted spaces in a subregion of $\operatorname{Re}(\alpha) < Q$, that we call the *probabilistic region* due to the fact that the resolvent can be written in terms of the semigroup $e^{-t\mathbf{H}}$.

Let $\rho : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth non-decreasing function satisfying

$$\rho(c) = c + a \text{ for } c \leq -1, \quad \rho(c) = 0 \text{ for } c \geq 0, \quad 0 \leq \rho' \leq 1$$

for some $a \in \mathbb{R}$. We have for $\beta \geq 0$ the inclusion of weighted spaces

$$e^{\beta\rho(c)} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \subset L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \subset e^{-\beta\rho(c)} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}).$$

The weighted spaces $e^{\beta\rho(c)} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ are obviously Hilbert spaces with product $\langle u, v \rangle_{e^{\beta\rho} L^2} := \langle e^{-\beta\rho} u, e^{-\beta\rho} v \rangle_2$.

Lemma 3.3. *Let $\beta \in \mathbb{R}$. If $\operatorname{Re}((\alpha - Q)^2) > \beta^2$ and $\operatorname{Re}(\alpha) < Q$, the resolvent $\mathbf{R}(\alpha) = (\mathbf{H} - 2\Delta_\alpha)^{-1}$ extends to a bounded operator*

$$\begin{aligned} \mathbf{R}(\alpha) &: e^{-\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathbf{H}), \\ \mathbf{R}(\alpha) &: e^{-\beta\rho} \mathcal{D}'(Q) \rightarrow e^{-\beta\rho} \mathcal{D}(Q) \end{aligned}$$

which is analytic in α in this region. The operator $\mathbf{H} : e^{-\beta\rho} L^2 \rightarrow e^{-\beta\rho} L^2$ is closed with domain $e^{-\beta\rho} \mathcal{D}(\mathbf{H})$, it is a bijective mapping $e^{-\beta\rho} \mathcal{D}(\mathbf{H}) \rightarrow e^{-\beta\rho} L^2$ with inverse $\mathbf{R}(\alpha)$. Moreover, for $\alpha \in (-\infty, Q)$ and $0 \leq \beta < Q - \alpha$, the resolvent is bounded with norm $\|\mathbf{R}(\alpha)\|_{\mathcal{L}(e^{-\beta\rho} L^2)} \leq 2((\alpha - Q)^2 - \beta^2)^{-1}$ and is equal to the integral

$$(3.12) \quad \mathbf{R}(\alpha) = \int_0^\infty e^{-t\mathbf{H} + t2\Delta_\alpha} dt$$

where $e^{-t\mathbf{H}}$ is the semigroup on $e^{-\beta\rho} L^2$ obtained by Hille-Yosida theorem with norm

$$(3.13) \quad \forall t \geq 0, \quad \|e^{-t\mathbf{H}}\|_{\mathcal{L}(e^{-\beta\rho} L^2)} \leq e^{-t\frac{Q^2 - \beta^2}{2}}.$$

The integral (3.12) converges in $\mathcal{L}(e^{-\beta\rho}L^2)$ operator norm and $e^{-t\mathbf{H}} : e^{-\beta\rho}L^2 \rightarrow e^{-\beta\rho}L^2$ extends the semigroup defined in (1.68) and (2.21). Finally, $e^{-t\mathbf{H}} : e^{\beta\rho}\mathcal{D}'(\mathcal{Q}) \rightarrow \mathcal{D}'(\mathcal{Q})$ and $e^{-t\mathbf{H}} : e^{\beta\rho}\mathcal{D}'(\mathcal{Q}) \rightarrow \mathcal{D}'(\mathcal{Q})$ are bounded and for each $\varepsilon > 0$ there is some constant $C_\varepsilon > 0$ such that for all $t > 0$

$$(3.14) \quad \|e^{-t\mathbf{H}}\|_{\mathcal{L}(e^{\beta\rho}\mathcal{D}(\mathcal{Q}))} + \|e^{-t\mathbf{H}}\|_{\mathcal{L}(e^{\beta\rho}\mathcal{D}'(\mathcal{Q}))} \leq C_\varepsilon e^{-t(\frac{Q^2-\beta^2}{2}-\varepsilon)}.$$

Proof. Consider the operator for $\beta \in \mathbb{R}$, acting on the space \mathcal{C} (defined in (2.23)),

$$\mathbf{H}_\beta := e^{\beta\rho(c)}\mathbf{H}e^{-\beta\rho(c)} = \mathbf{H} - \frac{\beta^2}{2}(\rho'(c))^2 + \frac{\beta}{2}\rho''(c) + \beta\rho'(c)\partial_c.$$

Let $u \in \mathcal{C}$, then we have (using integration by parts)

$$(3.15) \quad \begin{aligned} \operatorname{Re}\langle \mathbf{H}_\beta u | u \rangle_2 &= \mathcal{Q}(u) - \frac{\beta^2}{2}\|\rho'u\|_2^2 + \frac{\beta}{2}\langle \rho''u | u \rangle_2 + \beta\operatorname{Re}\langle \rho'\partial_c u | u \rangle_2 \\ &= \mathcal{Q}(u) - \frac{\beta^2}{2}\|\rho'u\|_2^2 \\ &\geq \mathcal{Q}(u) - \frac{\beta^2}{2}\|u\|_2^2 \geq \frac{Q^2-\beta^2}{2}\|u\|_2^2 + \frac{1}{2}\|\partial_c u\|_2^2 + \|\mathbf{P}^{1/2}u\|_2^2 + \|e^{\frac{\gamma c}{2}}V^{\frac{1}{2}}u\|_2^2, \end{aligned}$$

where \mathcal{Q} was defined in Section 2.4. Consider the sesquilinear form $\mathcal{Q}_{\alpha,\beta}(u, v) := \langle (\mathbf{H}_\beta - 2\Delta_\alpha)u | v \rangle_2$ defined on \mathcal{C} . We easily see that if $-\operatorname{Re}(2\Delta_\alpha) > \frac{\beta^2-Q^2}{2}$, then

$$\{u \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \mid \mathcal{Q}_{\alpha,\beta}(u, u) < \infty\} = \mathcal{D}(\mathcal{Q}).$$

Let $\mathcal{D}'(\mathcal{Q})$ be the dual of $\mathcal{D}(\mathcal{Q})$ (note that $L^2 \subset \mathcal{D}'(\mathcal{Q})$). By Lax-Milgram, if $-\operatorname{Re}(2\Delta_\alpha) > \frac{\beta^2-Q^2}{2}$, then for each $f' \in \mathcal{D}'(\mathcal{Q})$, there is a unique $u \in \mathcal{D}(\mathcal{Q})$ such that

$$(3.16) \quad \forall v \in \mathcal{D}(\mathcal{Q}), \quad \mathcal{Q}_{\alpha,\beta}(u, v) = f'(v), \quad \mathcal{Q}(u)^{1/2} \leq C'\|f'\|_{\mathcal{D}'(\mathcal{Q})}$$

for all $v \in \mathcal{D}(\mathcal{Q})$, where $C' > 0$ depends only on $\operatorname{Re}(2\Delta_\alpha)$ and β^2 . This holds in particular for the linear form $f' : v \mapsto \langle f | v \rangle_2$ with norm $\|f'\|_{\mathcal{D}'(\mathcal{Q})} \leq C\|f\|_2$ for some $C > 0$ depending on $\operatorname{Re}(2\Delta_\alpha)$ and β^2 . We define $\tilde{\mathbf{R}}(\alpha)(e^{-\beta\rho}f) := e^{-\beta\rho}u$, this gives a bounded linear operator

$$(3.17) \quad \tilde{\mathbf{R}}(\alpha) : e^{-\beta\rho}\mathcal{D}'(\mathcal{Q}) \rightarrow e^{-\beta\rho}\mathcal{D}(\mathcal{Q}) \subset e^{-\beta\rho}L^2$$

inverting the bounded operator $e^{\beta\rho}(\mathbf{H} - 2\Delta_\alpha)e^{-\beta\rho} : \mathcal{D}(\mathcal{Q}) \rightarrow \mathcal{D}'(\mathcal{Q})$. Moreover, by (3.15), its weighted L^2 -norm is bounded by

$$(3.18) \quad \|\tilde{\mathbf{R}}(\alpha)\|_{\mathcal{L}(e^{-\beta\rho}L^2)} \leq 2(\operatorname{Re}((\alpha - Q)^2 - \beta^2))^{-1} = (\frac{Q^2-\beta^2}{2} - 2\operatorname{Re}(\Delta_\alpha))^{-1}$$

Using $\mathcal{D}(\mathcal{Q}) \subset e^{-\beta\rho}\mathcal{D}(\mathcal{Q})$ and the uniqueness property above, this means that for $f \in L^2$, we have $\mathbf{R}(\alpha)f = \tilde{\mathbf{R}}(\alpha)f$ and thus $\tilde{\mathbf{R}}(\alpha)$ is a continuous extension of $\mathbf{R}(\alpha)$ to the Hilbert space $e^{-\beta\rho}L^2$. The analyticity in α comes from Lax-Milgram, but can also alternatively be obtained by Cauchy formula (for $\varepsilon > 0$ small)

$$\tilde{\mathbf{R}}(\alpha)f = \frac{1}{2\pi i} \int_{|z-\alpha|=\varepsilon} \frac{\tilde{\mathbf{R}}(z)f}{z-\alpha} dz$$

which holds for all $f \in \mathcal{C}$ (since $\tilde{\mathbf{R}}(\alpha)f = \mathbf{R}(\alpha)f$ for such f), and can then be extended to $e^{-\beta\rho}L^2$ by density of \mathcal{C} in $e^{-\beta\rho}L^2$. The domain $\mathcal{D}(e^{\beta\rho}\mathbf{H}e^{-\beta\rho}) = \{u \in \mathcal{D}(\mathcal{Q}) \mid e^{\beta\rho}\mathbf{H}e^{-\beta\rho}u \in L^2\}$ of the operator $e^{\beta\rho}\mathbf{H}e^{-\beta\rho}$ is actually equal to $\mathcal{D}(\mathbf{H}) = \{u \in \mathcal{D}(\mathcal{Q}) \mid \mathbf{H}u \in L^2\}$ since

$$(3.19) \quad e^{-\beta\rho}\mathbf{H}(e^{\beta\rho}u) = \mathbf{H}u - \frac{\beta^2}{2}(\rho'(c))^2u - \frac{\beta}{2}\rho''(c)u - \beta\rho'(c)\partial_c u$$

with $\rho' \in C_c^\infty(\mathbb{R})$ (thus $\rho'(c)\partial_c u \in L^2$ for $u \in \mathcal{D}(\mathcal{Q})$). The operator $\mathbf{H} : e^{-\beta\rho}\mathcal{D}(\mathbf{H}) \rightarrow e^{-\beta\rho}L^2$ is thus closed. By Hille-Yosida theorem, there is an associated bounded semigroup $e^{-t\mathbf{H}}$ on $e^{-\beta\rho}L^2$, and by density of $L^2 \subset e^{-\beta\rho}L^2$ when $\beta \geq 0$, it is an extension of the $e^{-t\mathbf{H}}$ semigroup on L^2 . Let us check that the resolvent can be written as an integral of the propagator. For $f \in \mathcal{C} \subset L^2$, we have

$$\tilde{\mathbf{R}}(\alpha)f = \mathbf{R}(\alpha)f = \int_0^\infty e^{-t\mathbf{H}+t2\Delta_\alpha} f dt.$$

By Hille-Yosida theorem and (3.17), we have $\|e^{-t\mathbf{H}}\|_{\mathcal{L}(e^{-\beta\rho}L^2)} \leq e^{-t\frac{Q^2-\beta^2}{2}}$, so that the integral above converges if $Q-\alpha > \beta$ (for $\alpha \in (-\infty, Q)$) as a bounded operator on $e^{-\beta\rho}L^2$, showing the desired claim by density of \mathcal{C} in $e^{-\beta\rho}L^2$.

We conclude with a $e^{\beta\rho}\mathcal{D}'(\mathcal{Q})$ bound for $\tilde{\mathbf{R}}(\alpha)$ and $e^{-t\mathbf{H}}$. First, we note using (3.19) that for $u \in \mathcal{D}(\mathcal{Q})$

$$|\operatorname{Im}(\mathcal{Q}_{\alpha,\beta}(u, u))| \geq (|\operatorname{Im}(2\Delta_\alpha)| - \beta^2) \|u\|_2^2 - \frac{1}{4} \|\partial_c u\|_2^2,$$

which implies

$$|\mathcal{Q}_{\alpha,\beta}(u, u)| \geq \frac{1}{\sqrt{2}} \left(c_{\Delta_\alpha} \|u\|^2 + \frac{1}{4} \|\partial_c u\|_2^2 + \|\mathbf{P}^{1/2} u\|_2^2 + \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} u\|_2^2 \right).$$

$$c_z := \min \left(-2\operatorname{Re}(z) + 2|\operatorname{Im}(z)| + \frac{Q^2-3\beta^2}{2}, -2\operatorname{Re}(z) + \frac{Q^2-\beta^2}{2} \right).$$

This in turn gives that, provided $\operatorname{Re}(-2\Delta_\alpha) + |\operatorname{Im}(2\Delta_\alpha)| + \frac{Q^2-3\beta^2}{2} > 0$, $\tilde{\mathbf{R}}(\alpha) : e^{-\beta\rho}L^2 \rightarrow e^{-\beta\rho}L^2$ is also well-defined, analytic in α and bounded, and moreover satisfies for $|\Delta_\alpha| \gg 1$ and $\operatorname{Re}(\Delta_\alpha) \leq \frac{1}{2}|\operatorname{Im}(\Delta_\alpha)|$

$$\|\tilde{\mathbf{R}}(\alpha)\|_{\mathcal{L}(e^{-\beta\rho}L^2)} \leq C|\Delta_\alpha|^{-1}$$

where $C > 0$ is a uniform constant. First, for each $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that for all $f \in e^{-\beta\rho}\mathcal{D}(\mathcal{Q})$, $u := \mathbf{R}(\alpha)f$

$$\mathcal{Q}(u, u) \leq C_\varepsilon \operatorname{Re}(\mathcal{Q}_{\alpha,\beta}(u, u)) \leq C_\varepsilon |\langle u | f \rangle_2| \leq \frac{C_\varepsilon}{c_{\Delta_\alpha}} \|f\|_2^2$$

if $c_{\Delta_\alpha} > 0$, thus for $c_z > 0$

$$(3.20) \quad \begin{aligned} \|(\mathbf{H} - z)^{-1}\|_{\mathcal{L}(e^{-\beta\rho}\mathcal{D}(\mathcal{Q}))} &\leq C_\varepsilon^{1/2} c_z^{-1/2} \\ \|(\mathbf{H} - z)^{-1}\|_{\mathcal{L}(e^{-\beta\rho}L^2)} &\leq C_\varepsilon c_z^{-1}. \end{aligned}$$

Let us consider a contour $\Gamma = \Gamma_0 \cup \Gamma_+ \cup \Gamma_- \subset \mathbb{C}$ with $a := \frac{Q^2-\beta^2}{2} - \varepsilon$, $\Gamma_\pm := a \pm iN + e^{\pm 3i\pi/8}\mathbb{R}^+ \subset \mathbb{C}$ and $\Gamma_0 = a + i[-N, N]$ for some $N > 0$ large enough so that $c_z > 0$ on Γ , and where Γ oriented clockwise around $[a, +\infty)$. Using the holomorphic functional calculus, we have

$$e^{-t\mathbf{H}} = \frac{1}{2\pi i} \int_{\Gamma_+ \cup \Gamma_-} e^{-tz} (\mathbf{H} - z)^{-1} dz$$

and the integral converges both in $\mathcal{L}(e^{-\beta\rho}L^2)$ and $\mathcal{L}(e^{-\beta\rho}\mathcal{D}(\mathcal{Q}))$ using (3.20), with bound

$$\|e^{-t\mathbf{H}}\|_{e^{-\beta\rho}\mathcal{D}(\mathcal{Q})} \leq C e^{-ta}$$

for some some C depending only on $\varepsilon > 0$. Using duality, this gives (3.14). \square

In what follows, we will always write $\mathbf{R}(\alpha)$ for the resolvent, for both the operator acting on L^2 or acting on $e^{-\beta\rho}L^2$.

Poisson operator in the probabilistic region. For $\ell \in \mathbb{N}$, we shall define the Poisson operator $\mathcal{P}_\ell(\alpha)$ in the resolvent set. This operator is a way to construct the generalized eigenfunctions of \mathbf{H} : it takes an element $F \in E_\ell \subset L^2(\Omega_{\mathbb{T}})$ and produces a function $u = \mathcal{P}_\ell(\alpha)F$ solving $(\mathbf{H} - 2\Delta_\alpha)u = 0$ with a prescribed leading asymptotic in terms of F as $c \rightarrow -\infty$.

We will denote by $\sqrt{\cdot}$ the square root defined so that $\operatorname{Im}(\sqrt{z}) > 0$ if $z \in \mathbb{C} \setminus \mathbb{R}^+$, i.e. $\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}$ for $\theta \in [0, 2\pi)$ and $r > 0$. We note the following elementary property, which will be useful in the following.

Lemma 3.4. *For $z \in \mathbb{C} \setminus \mathbb{R}^+$, the following map is non-decreasing*

$$x \in \mathbb{R}^+ \mapsto \operatorname{Im}\sqrt{z-x}.$$

Proof. If $\operatorname{Arg}(z) \in (\pi/2, \pi)$ this is because for $x_2 > x_1$, $\pi > \operatorname{Arg}(z-x_2) > \operatorname{Arg}(z-x_1) \geq \pi/2$ and $|z-x_2| > |z-x_1|$ so, using that \sin is increasing on $(0, \pi/2)$ we get $\sin(\operatorname{Arg}(z-x_2)/2) > \sin(\operatorname{Arg}(z-x_1)/2)$ so $\operatorname{Im}\sqrt{z-x_2} > \operatorname{Im}\sqrt{z-x_1}$. If $\operatorname{Arg}(z) \in (\pi, 3\pi/2)$, we have $3\pi/2 > \operatorname{Arg}(z-x_1) > \operatorname{Arg}(z-x_2) \geq \pi$ and $|z-x_2| > |z-x_1|$ so using that \sin is decreasing on $[\pi/2, \pi]$ we get $\sin(\operatorname{Arg}(z-x_2)/2) > \sin(\operatorname{Arg}(z-x_1)/2)$ and thus $\operatorname{Im}\sqrt{z-x_2} > \operatorname{Im}\sqrt{z-x_1}$. Now if $\operatorname{Arg}(z) \in (0, \pi/2)$, let $X_j + iY_j := \sqrt{z-x_j}$ with $X_j, Y_j > 0$ (since $\sqrt{z-x_j}$ has argument in $(0, \pi/2)$) and write $z = x + iy$. Assume $Y_1 > Y_2$, then $y = \operatorname{Im}(z-x_j) = Y_j X_j$ for both

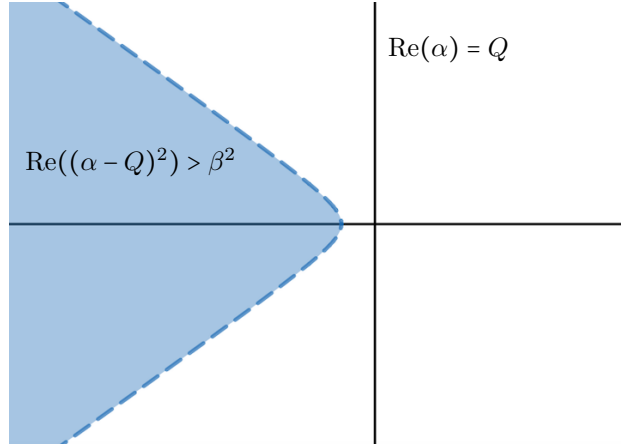


FIGURE 1. The blue region corresponds to the set of parameters $\alpha \in \mathbb{C}$ such that $\operatorname{Re}((\alpha - Q)^2) > \beta^2$, i.e. region of validity of Lemma 3.3 (here $\beta = 1$ on the plot).

$j = 1, 2$, so it would imply $X_1 < X_2$ and $x - x_1 = X_1^2 - Y_1^2 < X_2^2 - Y_2^2 = x - x_2$ which leads to a contradiction as $x_2 > x_1$. Finally if $\operatorname{Arg}(z) \in (3\pi/2, 2\pi)$, let $X_j + iY_j := \sqrt{z - x_j}$ with $Y_j > 0$ and $X_j < 0$ (since $\sqrt{z - x_j}$ has argument in $(\pi/2, \pi)$) and write $z = x + iy$. Assume $Y_1 > Y_2$, then $y = \operatorname{Im}(z - x_j) = Y_j X_j$ for both $j = 1, 2$ so $|X_1| < |X_2|$ and $x - x_1 = X_1^2 - Y_1^2 < X_2^2 - Y_2^2 = x - x_2$ which leads to a contradiction as $x_2 > x_1$. \square

Let $\chi \in C^\infty(\mathbb{R})$ be equal to 1 in $(-\infty, a - 1)$ and equal to 0 in $(a, +\infty)$ for some $a \in (0, 1/2)$, then for $\alpha = Q + ip$ with $\operatorname{Im}(p) > 0$ we choose

$$(3.21) \quad \beta_\ell > \max_{j=0, \dots, \ell} \operatorname{Im} \sqrt{p^2 - 2\lambda_j} - \gamma/2 = \operatorname{Im} \sqrt{p^2 - 2\lambda_\ell} - \gamma/2, \quad \text{and } \beta_\ell \geq 0.$$

Then for $\operatorname{Re}((\alpha - Q)^2) > \beta_\ell^2$ we define

$$(3.22) \quad \mathcal{P}_\ell(\alpha) : \begin{cases} E_\ell = \bigoplus_{j=0}^{\ell} \ker(\mathbf{P} - \lambda_j) & \rightarrow e^{-(\beta_\ell + \gamma/2)\rho} \mathcal{D}(\mathcal{Q}) \\ F = \sum_{0 \leq j \leq \ell} F_j & \mapsto \chi F_-(\alpha) - \mathbf{R}(\alpha)(\mathbf{H} - 2\Delta_\alpha)(\chi F_-(\alpha)), \end{cases}$$

with $F_-(\alpha) := \sum_{j=0}^{\ell} F_j e^{ic\sqrt{p^2 - 2\lambda_j}}$.

We will show in the following Lemma that this definition makes sense by using Lemma 3.3. Before going to the proof of it, recall that $\operatorname{Im} \sqrt{p^2 - 2\lambda_j} > 0$ for $\operatorname{Re}(\alpha) < Q$ by Lemma 3.4, and note that the condition $\operatorname{Re}((\alpha - Q)^2) > \beta_\ell^2$ implies that $\operatorname{Im} \sqrt{p^2 - 2\lambda_j} \geq \operatorname{Im}(p) > \beta_\ell$ for all $j = 0, \dots, \ell$. We then emphasize that the main reason for $\mathcal{P}_\ell(\alpha)F$ to be defined and non-trivial is that $\chi F_-(\alpha) \in e^{-(\beta_\ell + \gamma/2)\rho} \mathcal{D}(\mathcal{Q}) \setminus e^{-\beta_\ell \rho} \mathcal{D}(\mathcal{Q})$ and $(\mathbf{H} - 2\Delta_\alpha)(\chi F_-(\alpha)) \in e^{-\beta_\ell \rho} \mathcal{D}'(\mathcal{Q})$ so that $\mathbf{R}(\alpha)(\mathbf{H} - 2\Delta_\alpha)(\chi F_-(\alpha))$ is well-defined but not equal to $\chi F_-(\alpha)$.

Lemma 3.5. *For each $\ell \in \mathbb{N}$, let $\beta_\ell \geq 0$, then the operator $\mathcal{P}_\ell(\alpha)$ is well-defined, bounded and holomorphic in the region*

$$(3.23) \quad \left\{ \alpha = Q + ip \in \mathbb{C} \mid \operatorname{Re}(\alpha - Q) < 0, \operatorname{Re}((\alpha - Q)^2) > \beta_\ell^2, \beta_\ell > \operatorname{Im} \sqrt{p^2 - 2\lambda_\ell} - \gamma/2 \right\}$$

and it satisfies in $e^{-(\beta_\ell + \gamma/2)\rho} \mathcal{D}'(\mathcal{Q})$

$$(3.24) \quad (\mathbf{H} - 2\Delta_\alpha) \mathcal{P}_\ell(\alpha) = 0,$$

and in the region $c \leq -1$, one has the asymptotic behaviour, with $F_j := (\Pi_j - \Pi_{j-1})F$,

$$(3.25) \quad \mathcal{P}_\ell(\alpha)F = \sum_{j=0}^{\ell} F_j e^{ic\sqrt{p^2 - 2\lambda_j}} + F_+(\alpha), \quad F_+(\alpha) \in e^{-\beta_\ell \rho} \mathcal{D}(\mathcal{Q}).$$

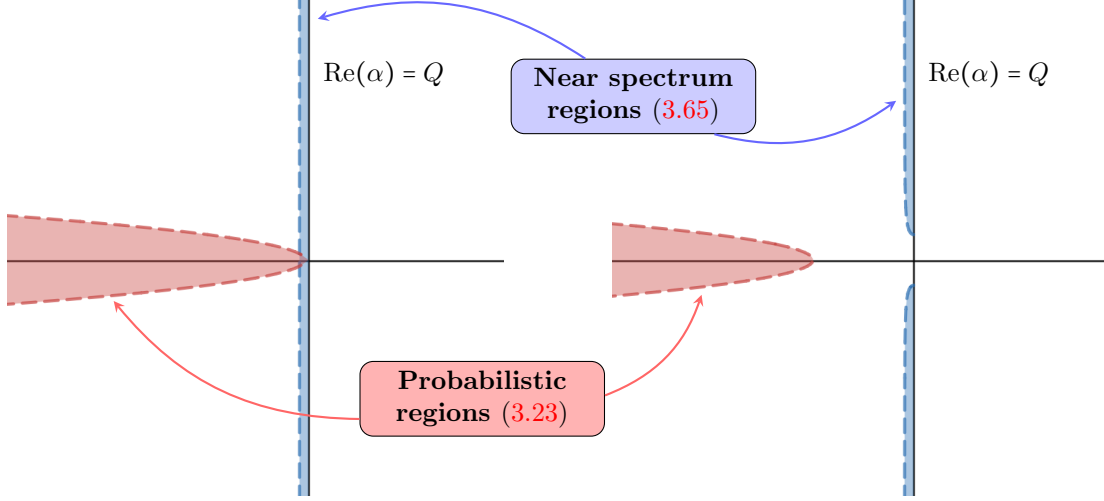


FIGURE 2. Left picture: regions of definition of the Poisson operator $\mathcal{P}_0(\alpha)$ (for the plot, we take $\gamma = 1/2$ and β_0 is optimized to obtain the largest possible region). Right picture: regions of definition of the Poisson operator $\mathcal{P}_\ell(\alpha)$ with $\ell > 0$ (for the plot we take $\lambda_\ell = 4, \gamma = 1/2$ and β_ℓ optimized).

Proof. First we observe that $(\mathbf{H}_0 - 2\Delta_\alpha)F_-(\alpha) = 0$ thus

$$(\mathbf{H} - 2\Delta_\alpha)\chi F_-(\alpha) = -\frac{1}{2}\chi''(c)F_-(\alpha) + e^{\gamma c}V\chi F_-(\alpha) - \chi'(c)\partial_c F_-(\alpha).$$

We note that χ', χ'' have compact support in \mathbb{R} , and also for each $u \in \mathcal{D}(\mathcal{Q})$

$$|\langle \chi e^{\gamma c + \beta_\ell \rho} V F_-(\alpha), u \rangle| \leq \sup_{j \leq \ell} \|e^{\rho \operatorname{Im} \sqrt{p^2 - 2\lambda_\ell}} V^{1/2} F_j \mathbf{1}_{\mathbb{R}^-}(c)\|_2 \|\mathbf{1}_{\mathbb{R}^-}(c) e^{\gamma c/2} V^{1/2} u\|_2 \leq C\mathcal{Q}(u)^{1/2}$$

thus, since $F_j \in L^p(\Omega_{\mathbb{T}})$ and $V^{1/2} \in L^{2+\varepsilon}(\Omega_{\mathbb{T}})$ for all $p < \infty$ and $\varepsilon > 0$, we obtain that

$$\chi F_-(\alpha) \in e^{-(\beta_\ell + \gamma/2)\rho} \mathcal{D}(\mathcal{Q}), \quad (\mathbf{H} - 2\Delta_\alpha)\chi F_-(\alpha) \in e^{-\beta_\ell \rho} \mathcal{D}'(\mathcal{Q}).$$

This shows, using Lemma 3.3, that $\mathbf{R}(\alpha)(\mathbf{H} - 2\Delta_\alpha)(\chi F_-(\alpha))$ is well-defined as an element of $e^{-\beta_\ell \rho} \mathcal{D}(\mathcal{Q})$, with holomorphic dependence in α , provided $\operatorname{Re}((\alpha - Q)^2) > \beta_\ell^2$. By construction, it clearly also solves (3.24) in $e^{-(\beta_\ell + \gamma/2)\rho} \mathcal{D}'(\mathcal{Q})$. \square

Note that the error term $F_+(\alpha)$ in (3.25) is smaller than the bigger term in $F_-(\alpha)$ but is not necessarily neglectible with respect to all terms of $F_-(\alpha)$.

We also notice that where they are defined, we have for $j \geq 0, \ell \geq 0$

$$(3.26) \quad \mathcal{P}_{\ell+j}(\alpha)|_{E_\ell} = \mathcal{P}_\ell(\alpha)|_{E_\ell}.$$

In (3.22), the definition of the operator $\mathcal{P}_\ell(\alpha)$ seemingly depends on the cutoff function χ . In fact, we can show that this is not the case. We state a lemma below in this direction

Lemma 3.6. *For $\ell \in \mathbb{N}$, β_ℓ satisfying (3.21) and for $\operatorname{Re}((\alpha - Q)^2) > \beta_\ell^2$ the definition of the operator $\mathcal{P}_\ell(\alpha)|_{E_\ell}$ does not depend on χ .*

Proof. Pick two functions $\chi, \hat{\chi}$ satisfying our assumptions and denote by $\mathcal{P}_\ell^\chi(\alpha), \mathcal{P}_\ell^{\hat{\chi}}(\alpha)$ the corresponding Poisson operators. Set $d(\chi) := \chi - \hat{\chi}$. Then observe that for $F \in E_\ell$

$$\mathcal{P}_\ell^\chi(\alpha)F - \mathcal{P}_\ell^{\hat{\chi}}(\alpha)F = d(\chi)F_-(\alpha) - \mathbf{R}(\alpha)(\mathbf{H} - 2\Delta_\alpha)(d(\chi)F_-(\alpha)).$$

Now we note that $d(\chi)F_-(\alpha) \in \mathcal{D}(\mathcal{Q})$ since $d(\chi)(c) = 0$ for $c \notin (-1, a)$ for some $a > 0$ and $V^{1/2}F \in L^2(\Omega_{\mathbb{T}})$. Then $\mathbf{R}(\alpha)(\mathbf{H} - 2\Delta_\alpha)(d(\chi)F_-(\alpha)) = d(\chi)F_-(\alpha)$ since $(\mathbf{H} - 2\Delta_\alpha) : \mathcal{D}(\mathcal{Q}) \rightarrow \mathcal{D}'(\mathcal{Q})$ is an isomorphism, hence $\mathcal{P}_\ell^\chi(\alpha)F - \mathcal{P}_\ell^{\hat{\chi}}(\alpha)F = 0$. \square

We can also rewrite the construction of the Poisson operator using the propagator.

Proposition 3.7. *We claim:*

1) Let $\ell \in \mathbb{N}$ and let $F \in L^2(\Omega_{\mathbb{T}}) \cap \ker(\mathbf{P} - \lambda_j)$ for $j \leq \ell$. Then we have the identity

$$\mathcal{P}_\ell(\alpha)F = \lim_{t \rightarrow +\infty} e^{t2\Delta_\alpha} e^{-t\mathbf{H}} \left(e^{ic\sqrt{p^2 - 2\lambda_j}} \chi(c) F \right)$$

in $e^{-(\beta+\gamma/2)\rho}\mathcal{D}'(\mathcal{Q})$ provided $\operatorname{Re}((\alpha - Q)^2) > \beta^2$ with $\beta > \operatorname{Im}(\sqrt{p^2 - 2\lambda_j}) - \gamma/2$ and $\alpha = Q + ip$. Furthermore if $\operatorname{Im}(\sqrt{p^2 - 2\lambda_j}) > \gamma$ then we can take $\chi = 1$ in the above statement.

2) Let $F \in L^2(\Omega_{\mathbb{T}}) \cap \ker(\mathbf{P})$. If $\alpha \in \mathbb{R}$ with $\alpha < Q$, then $dc \otimes \mathbf{P}$ almost everywhere

$$\mathcal{P}_0(\alpha)F = \lim_{t \rightarrow +\infty} e^{t2\Delta_\alpha} e^{-t\mathbf{H}} \left(e^{(\alpha-Q)c} F \right).$$

Proof. We first prove 1). Recall that $\mathbf{H} = \mathbf{H}_0 + e^{\gamma c} V$. We have

$$(\mathbf{H} - 2\Delta_\alpha)(\chi(c)e^{i\nu_j c} F) = \chi(c)e^{(i\nu_j + \gamma)c} V F - \tilde{\chi}(c)F e^{i\nu_j c}$$

where $\nu_j := \sqrt{p^2 - 2\lambda_j}$ and $\tilde{\chi}(c) := \frac{1}{2}\chi''(c) + i\nu_j\chi'(c) \in C_c^\infty(\mathbb{R})$, and this belongs to $e^{-\beta\rho}\mathcal{D}'(\mathcal{Q})$. Using Lemma 3.3,

$$\mathcal{P}_\ell(\alpha)F = \chi(c)e^{i\nu_j c} F - \mathbf{R}(\alpha)(e^{(i\nu_j + \gamma)c} V \chi F - \tilde{\chi}e^{i\nu_j c} F)$$

provided $\operatorname{Re}((\alpha - Q)^2) > \beta^2$ for any $\beta > \operatorname{Im}(\nu_j) - \gamma/2$. Noticing that the bound (3.13) remains valid with $V = 0$, we can make sense of $u(t) := e^{-t\mathbf{H}_0}(\chi(c)e^{i\nu_j c} F)$ as an element in $e^{-\beta_0\rho}L^2(\mathbb{R}; E_j)$ for any $\beta_0 > \operatorname{Im}(\nu_j)$. Then

$$(\partial_t + 2\Delta_\alpha)u(t) = e^{-t\mathbf{H}_0}(-\mathbf{H}_0 + 2\Delta_\alpha)(\chi(c)e^{i\nu_j c} F) = e^{-t\mathbf{H}_0}(e^{i\nu_j c} F \tilde{\chi}(c))$$

thus we get by integrating in t

$$(3.27) \quad \begin{aligned} e^{-t(\mathbf{H}_0 - 2\Delta_\alpha)}(\chi(c)e^{i\nu_j c} F) &= \chi(c)e^{i\nu_j c} F + \int_0^t e^{-s(\mathbf{H}_0 - 2\Delta_\alpha)}(e^{i\nu_j c} F \tilde{\chi}(c)) ds \\ &= \chi(c)e^{i\nu_j c} F + (1 - e^{-t(\mathbf{H}_0 - 2\Delta_\alpha)})\mathbf{R}_0(\alpha)(e^{i\nu_j c} F \tilde{\chi}(c)) \end{aligned}$$

where $\mathbf{R}_0(\alpha) := (\mathbf{H}_0 - 2\Delta_\alpha)^{-1}$ is defined also on $e^{-\beta\rho}L^2$ by taking the proof of Lemma 3.3 in the case of the trivial potential $V = 0$. We also note that $e^{-t\mathbf{H}_0}(e^{i\nu_j c} F \tilde{\chi}(c))$ and $e^{-t\mathbf{H}_0}\mathbf{R}_0(\alpha)(e^{i\nu_j c} F \tilde{\chi}(c))$ are in $L^2(\mathbb{R}; E_j)$ since $\mathbf{H}_0 F = (Q^2/2 + \lambda_j)F$ (and $\tilde{\chi} \in C_c^\infty(\mathbb{R})$), i.e. all terms above are functions of c with values in E_j .

We next claim that

$$e^{-t\mathbf{H}}(\chi(c)e^{i\nu_j c} F) = e^{-t\mathbf{H}_0}(\chi(c)e^{i\nu_j c} F) - \int_0^t e^{-(t-s)\mathbf{H}} e^{\gamma c} V e^{-s\mathbf{H}_0}(\chi(c)e^{i\nu_j c} F) ds.$$

Indeed, first, all terms are well-defined due to the fact that $e^{-s\mathbf{H}_0}(\chi(c)e^{i\nu_j c} F) \in e^{-\beta_0\rho}L^2(\mathbb{R}; E_j)$ and (3.5) so that $e^{\gamma c} V e^{-s\mathbf{H}_0}(\chi(c)e^{i\nu_j c} F) \in e^{-\beta_0\rho}\mathcal{D}'(\mathcal{Q})$. Then the identity above is obtained since both terms solve $(\partial_t + \mathbf{H})u(t) = 0$ in $e^{-\beta_0\rho}\mathcal{D}'(\mathcal{Q})$ with $u(0) = \chi(c)e^{i\nu_j c} F$ in $e^{-\beta_0\rho}\mathcal{D}'(\mathcal{Q})$. By applying twice (3.27), we thus obtain

$$\begin{aligned} e^{-t(\mathbf{H} - 2\Delta_\alpha)}(\chi(c)e^{i\nu_j c} F) &= \chi(c)e^{i\nu_j c} F - e^{t2\Delta_\alpha} \int_0^t e^{-(t-s)\mathbf{H}} e^{\gamma c} V e^{-s\mathbf{H}_0}(\chi(c)e^{i\nu_j c} F) ds \\ &\quad + (1 - e^{-t(\mathbf{H}_0 - 2\Delta_\alpha)})\mathbf{R}_0(\alpha)(e^{i\nu_j c} F \tilde{\chi}(c)) \\ &= \chi(c)e^{i\nu_j c} F - \int_0^t e^{-(t-s)(\mathbf{H} - 2\Delta_\alpha)}(e^{\gamma c} V \chi(c)e^{i\nu_j c} F) ds \\ &\quad + (1 - e^{-t(\mathbf{H}_0 - 2\Delta_\alpha)})\mathbf{R}_0(\alpha)(e^{i\nu_j c} F \tilde{\chi}(c)) \\ &\quad - \int_0^t e^{-(t-s)(\mathbf{H} - 2\Delta_\alpha)} e^{\gamma c} V (1 - e^{-s(\mathbf{H}_0 - 2\Delta_\alpha)})\mathbf{R}_0(\alpha)(e^{i\nu_j c} F \tilde{\chi}(c)) ds. \end{aligned}$$

Using (3.13) and (3.14) (applied with both $V > 0$ and $V = 0$) and $\operatorname{Re}((\alpha - Q)^2) > \beta^2$, we have as bounded operators on respectively $e^{-\beta\rho}\mathcal{D}'(\mathcal{Q})$ and $e^{-\beta\rho}L^2$

$$\begin{aligned} \lim_{t \rightarrow +\infty} \int_0^t e^{-(t-s)(\mathbf{H} - 2\Delta_\alpha)} ds &= \lim_{t \rightarrow +\infty} (1 - e^{-t(\mathbf{H} - 2\Delta_\alpha)})\mathbf{R}(\alpha) = \mathbf{R}(\alpha) \\ \lim_{t \rightarrow +\infty} (1 - e^{-t(\mathbf{H}_0 - 2\Delta_\alpha)})\mathbf{R}_0(\alpha) &= \mathbf{R}_0(\alpha). \end{aligned}$$

This gives in particular in $e^{-\beta\rho}\mathcal{D}'(\mathcal{Q})$

$$\lim_{t \rightarrow +\infty} \int_0^t e^{-(t-s)(\mathbf{H}-2\Delta_\alpha)} (e^{\gamma c} V \chi(c) e^{i\nu_j c} F) ds = \mathbf{R}(\alpha) (e^{\gamma c} V \chi(c) e^{i\nu_j c} F).$$

Similarly one has in $e^{-\beta\rho}\mathcal{D}'(\mathcal{Q})$

$$\lim_{t \rightarrow +\infty} \int_0^t e^{-(t-s)(\mathbf{H}-2\Delta_\alpha)} e^{\gamma c} V \mathbf{R}_0(\alpha) (e^{i\nu_j c} F \tilde{\chi}(c)) ds = \mathbf{R}(\alpha) e^{\gamma c} V \mathbf{R}_0(\alpha) (e^{i\nu_j c} F \tilde{\chi}(c)).$$

Finally we claim that

$$\lim_{t \rightarrow +\infty} \int_0^t e^{-(t-s)(\mathbf{H}-2\Delta_\alpha)} e^{\gamma c} V e^{-s(\mathbf{H}_0-2\Delta_\alpha)} \mathbf{R}_0(\alpha) (e^{i\nu_j c} F \tilde{\chi}(c)) ds = 0.$$

Indeed, we can apply Lebesgue theorem if one can show

$$\|e^{\gamma c} V e^{-s(\mathbf{H}_0-2\Delta_\alpha)} \mathbf{R}_0(\alpha) (e^{i\nu_j c} F \tilde{\chi}(c))\|_{e^{\beta\rho}\mathcal{D}'(\mathcal{Q})} \leq e^{-\delta s}$$

for some $\delta > 0$, since $\|e^{-(t-s)(\mathbf{H}-2\Delta_\alpha)}\|_{\mathcal{L}(e^{-\beta\rho}\mathcal{D}'(\mathcal{Q}))} \rightarrow 0$ by (3.14). But this estimate follows again from (3.13) with $V = 0$ and the fact that $\mathbf{R}_0(\alpha) (e^{i\nu_j c} F \tilde{\chi}(c)) \in e^{-\beta\rho} L^2(\mathbb{R}; E_j)$, which in turn implies that

$$e^{\gamma c} V e^{-s(\mathbf{H}_0-2\Delta_\alpha)} \mathbf{R}_0(\alpha) (e^{i\nu_j c} F \tilde{\chi}(c)) \in e^{\beta\rho}\mathcal{D}'(\mathcal{Q}).$$

We have thus proved that

$$\begin{aligned} \lim_{t \rightarrow +\infty} e^{-t(\mathbf{H}-2\Delta_\alpha)} (\chi(c) e^{i\nu_j c} F) &= \chi(c) e^{i\nu_j c} F - \mathbf{R}(\alpha) e^{\gamma c} V (e^{i\nu_j c} F \chi(c) + \mathbf{R}_0(\alpha) (e^{i\nu_j c} F \tilde{\chi}(c))) \\ &\quad + \mathbf{R}_0(\alpha) (e^{i\nu_j c} F \tilde{\chi}(c)). \end{aligned}$$

We conclude by observing that

$$\mathbf{R}(\alpha) (e^{\gamma c} V \chi e^{i\nu_j c} F - \tilde{\chi} e^{i\nu_j c} F) = -\mathbf{R}_0(\alpha) (e^{i\nu_j c} F \tilde{\chi}) + \mathbf{R}(\alpha) e^{\gamma c} V (\chi e^{i\nu_j c} F + \mathbf{R}_0(\alpha) (e^{i\nu_j c} F \tilde{\chi}(c))),$$

which can be established by applying $(\mathbf{H} - 2\Delta_\alpha)$ to this equation and using the injectivity of $(\mathbf{H} - 2\Delta_\alpha)$ on $e^{-\beta\rho}\mathcal{D}(\mathcal{Q})$ under our condition on α, β .

Notice that for $\text{Im}(\sqrt{p^2 - 2\lambda_j}) > \gamma$, we have $e^{(i\nu_j + \gamma)c} \mathbf{1}_{(0, +\infty)}(c) \in L^2(\mathbb{R})$ so that the above argument applies with $\chi = 1$.

Now we prove 2). As $F \in L^2(\Omega_{\mathbb{T}}) \cap \ker(\mathbf{P} - \lambda_0)$, we may assume $F = 1$ without loss of generality. With our assumptions, we can write $\alpha = Q + ip$ with $p = i(Q - \alpha)$ and choose $(Q - \alpha) > \beta > (Q - \alpha) - \gamma$. By applying 1), we get that

$$\mathcal{P}_\ell(\alpha) 1 = \lim_{t \rightarrow +\infty} e^{t2\Delta_\alpha} e^{-t\mathbf{H}} \left(e^{(\alpha-Q)c} \chi(c) \right)$$

in $e^{-\beta\rho} L^2$, hence $dc \otimes \mathbb{P}$ almost everywhere (up to extracting subsequence). We have to show that we can replace χ by 1. For this we will use the probabilistic representation (2.33): we have

$$\begin{aligned} e^{t2\Delta_\alpha} e^{-t\mathbf{H}} \left(e^{(\alpha-Q)c} (1 - \chi(c)) \right) &= e^{-Qc} \mathbb{E}_\varphi \left[S_{e^{-t}} (e^{\alpha c} (1 - \chi(c))) e^{-\mu M_\gamma(\mathbb{D}_t^i)} \right] \\ &= e^{t2\Delta_\alpha} e^{(\alpha-Q)c} \mathbb{E}_\varphi \left[e^{\alpha(X_t(0) - Qt)} (1 - \chi(c + X_t(0) - Qt)) e^{-\mu M_\gamma(\mathbb{D}_t^i)} \right]. \end{aligned}$$

By Girsanov's transform this expression can be rewritten as

$$e^{t2\Delta_\alpha} e^{-t\mathbf{H}} \left(e^{(\alpha-Q)c} (1 - \chi(c)) \right) = e^{(\alpha-Q)c} \mathbb{E}_\varphi \left[(1 - \chi(c + X_t(0) - (Q - \alpha)t)) \exp \left(-\mu \int_{\mathbb{D}_t^i} |z|^{-\gamma\alpha} M_\gamma(dz) \right) \right].$$

Recalling that $\chi = 1$ on $(-\infty, a - 1)$ and that $t \mapsto X_t(0)$ evolves as a Brownian motion independent of φ , then estimating the exponential term by 1 we obtain

$$\left| e^{t2\Delta_\alpha} e^{-t\mathbf{H}} \left(e^{(\alpha-Q)c} (1 - \chi(c)) \right) \right| \leq e^{(\alpha-Q)c} \mathbb{P}(c + X_t(0) - (Q - \alpha)t \geq a - 1).$$

The result easily follows from this estimate. \square

Meromorphic extension of the resolvent near the L^2 spectrum. We denote by Z the Riemann surface on which the functions $p \mapsto \sqrt{p^2 - 2\lambda_j}$ are single valued for all j . This is a ramified covering of \mathbb{C} with ramification points $\{\pm\sqrt{2\lambda_j} \mid j \in \mathbb{N}\}$, and in which we embed the region $\text{Im}(p) > 0$ that we call the *physical sheet*. We will call $\pi : Z \rightarrow \mathbb{C}$ the projection of the covering. The construction of Z can be done iteratively on j , as explained in Chapter 6.7 of Melrose's book [Me93]. The map $p \mapsto \alpha := Q + ip$ from $\text{Im}(p) > 0$ to $\text{Re}(\alpha) < Q$ (now called the physical sheet for the variable α) extends analytically as a map $Z \rightarrow \Sigma$ where Σ is an isomorphic Riemann surface to Z (it just amounts to a linear change of complex coordinates from p to α). We shall also denote by $\pi : \Sigma \rightarrow \mathbb{C}$ the projection. Finally we choose a specific function χ of the form indicated previously but we further impose that $\chi \in C^\infty(\mathbb{R})$ is equal to 1 in $(-\infty, -1 + \delta)$ and equal to 0 in $(0, +\infty)$ (for some small $\delta > 0$).

The goal of this section is to show the following:

Proposition 3.8. *Assume that $\gamma \in (0, \sqrt{2})$ and let $\beta < \gamma/2$. Then the following holds true:*

1) *The resolvent $\mathbf{R}(\alpha) := (\mathbf{H} - 2\Delta_\alpha)^{-1}$ is bounded as a map $L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow \mathcal{D}(\mathbf{H})$ for $\text{Re}(\alpha) < Q$ and for $k \geq 0$ large enough, the operator $(\mathbf{H} - 2\Delta_{\pi(\alpha)})^{-1}$ admits a meromorphic continuation to the region*

$$\{\alpha = Q + ip \in \Sigma \mid |\pi(p)|^2 \leq \lambda_k^{\frac{1}{4}}, \forall j \leq k, \text{Im}\sqrt{p^2 - 2\lambda_j} > -\min(\beta, \gamma/2 - \beta)\}$$

as a map

$$\mathbf{R}(\alpha) : e^{\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathbf{H}), \quad \mathbf{R}(\alpha) : e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$$

and the residue at each pole is a finite rank operator.

2) *If $f \in e^{\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \cap e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$, then for α as above and not a pole, one has in $c \leq 0$*

$$(3.28) \quad \mathbf{R}(\alpha)f = \sum_{j=0}^k a_j(\alpha, f) e^{-ic\sqrt{p^2 - 2\lambda_j}} + G(\alpha, f),$$

with $a_j(\alpha, f) \in \ker(\mathbf{P} - \lambda_j)$, and $G(\alpha, f), \partial_c G(\alpha, f) \in e^{\beta\rho} L^2(\mathbb{R}_-; E_k) + L^2(\mathbb{R}_-; E_k^\perp)$, all depending meromorphically in α in the region they are defined.

3) *There is no pole for $\mathbf{R}(\alpha)$ in $\{\alpha \in \Sigma \mid \text{Re}(\alpha) \leq Q\} \setminus \cup_{j=0}^\infty \{Q \pm i\sqrt{2\lambda_j}\}$ and $Q \pm i\sqrt{2\lambda_j}$ can be at most a pole of order 1.*

First, we recall the notation for the orthogonal projectors

$$\Pi_k = 1_{[0, \lambda_k]}(\mathbf{P}) : L^2(\Omega_{\mathbb{T}}, \mathbb{P}) \rightarrow L^2(\Omega_{\mathbb{T}}, \mathbb{P})$$

and we denote by E_k their range (which are Hilbert spaces) and E_k^\perp the range of $1 - \Pi_k$. Since $V \in L^p(\Omega_{\mathbb{T}})$ for some $p > 1$ and $\text{Ran} E_k \subset L^q(\Omega_{\mathbb{T}})$ for all $q < \infty$, the following operators are bounded

$$(3.29) \quad V^{\frac{1}{2}} \Pi_k : L^2(\Omega_{\mathbb{T}}) \rightarrow L^2(\Omega_{\mathbb{T}}), \quad \Pi_k V^{1/2} : L^2(\Omega_{\mathbb{T}}) \rightarrow L^2(\Omega_{\mathbb{T}}).$$

To prove this Proposition, we will construct parametrices for the operator $\mathbf{H} - 2\Delta_\alpha = \mathbf{H} - \frac{Q^2 + p^2}{2}$ in several steps and will split the argument. More concretely, we will search for some bounded model operator $\tilde{\mathbf{R}}(\alpha) : e^{\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$, holomorphic in α in the desired region of Σ , such that

$$(\mathbf{H} - 2\Delta_\alpha) \tilde{\mathbf{R}}(\alpha) = 1 - \mathbf{K}(\alpha)$$

where $\mathbf{K}(\alpha) \in \mathcal{L}(e^{\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}))$ is an analytic family of compact operators with $1 - \mathbf{K}(\alpha_0)$ invertible at some α_0 belonging to the physical sheet. Then the Fredholm analytic theorem will imply that $(1 - \mathbf{K}(\alpha))^{-1}$ exists as a meromorphic family and $\mathbf{R}(\alpha) := \tilde{\mathbf{R}}(\alpha)(1 - \mathbf{K}(\alpha))^{-1}$ gives us the desired meromorphic extension of $\mathbf{R}(\alpha)$. Our strategy will be based on that method with slight modifications. The continuous spectrum of \mathbf{H} near frequency $(Q^2 + p^2)/2 \in \mathbb{R}^+$ will come only from finitely many eigenmodes of \mathbf{P} , namely those λ_j for which $2\lambda_j \leq p^2$. This suggests, in order to construct the approximation $\tilde{\mathbf{R}}(\alpha)$ to split the modes of \mathbf{P} depending on the value of $\text{Im}(\alpha - Q)$. The parametrix will be constructed in three steps as follows:

- First, we deal with the large eigenmodes for the operator \mathbf{P} in the region $c \in (-\infty, 0]$ of $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. We will show that this part does not contribute to the continuous spectrum at frequency $(Q^2 + p^2)/2$, and we shall obtain a parametrix for that part by energy estimates.

- Then we consider the region $c \geq -1$ where we shall show that the model operator in that region (essentially \mathbf{H} on $L^2([-1, \infty) \times \Omega_{\mathbb{T}})$ with Dirichlet condition at $c = 0$) has compact resolvent, providing a compact operator for the parametrix of that region.
- Finally, we will deal with the $c \leq 0$ region corresponding to eigenmodes of \mathbf{P} of order $\mathcal{O}(|p|^2)$, where there is *scattering* at $c = -\infty$ for frequency $(Q^2 + p^2)/2$, producing continuous spectrum. The parametrix for this part is basically the exact inverse of $\mathbf{H}_0 - 2\Delta_\alpha$ restricted to finitely many modes of \mathbf{P} .

For $s \geq 0$ and $I \subset \mathbb{R}$ an open interval, we will denote by $H^s(I; L^2(\Omega_{\mathbb{T}}))$ the Sobolev space of order s in the c -variable

$$H^s(I; L^2(\Omega_{\mathbb{T}})) := \{u \in L^2(I \times \Omega_{\mathbb{T}}) \mid \forall j \leq \ell, \xi \mapsto \langle \xi \rangle^s \mathcal{F}(u)(\xi) \in L^2(I \times \Omega_{\mathbb{T}})\}$$

where \mathcal{F} denotes Fourier transform in c , and similarly $H_0^s(I; L^2(\Omega_{\mathbb{T}}))$ will be the completion of $C_c^\infty(I; L^2(\Omega_{\mathbb{T}}))$ with respect to the norm $\|\langle \xi \rangle^s \mathcal{F}(u)\|_{L^2(\mathbb{R}; L^2(\Omega_{\mathbb{T}}))}$.

1) Large \mathbf{P} eigenmodes in the region $c \leq 0$. Let $\chi \in C^\infty(\mathbb{R}, [0, 1])$ which satisfies $\chi(c) = 1$ for $c \leq -1 + \delta$ for some $\delta \in (0, \frac{1}{2})$ and $\chi(c) = 0$ in $[-1/2, \infty)$ and $\tilde{\chi} \in C^\infty(\mathbb{R}, [0, 1])$ such that $\tilde{\chi} = 1$ on the support of χ and $\text{supp}(\tilde{\chi}) \subset \mathbb{R}^-$, and we now view these functions as multiplication operators by $\chi(c)$ on the spaces $e^{\beta\rho} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}})$. We will first show the following

Lemma 3.9. *1) There is a constant $C > 0$ depending only on $|\tilde{\chi}'|_\infty, |\tilde{\chi}''|_\infty$ such that for each $k \in \mathbb{N}$, there is a bounded operator*

$$\mathbf{R}_k^\perp(\alpha) : L^2(\mathbb{R}; E_k^\perp) \rightarrow L^2(\mathbb{R}^-; E_k^\perp)$$

holomorphic in $\alpha = Q + ip \in \mathbb{C}$ in the region $\{\text{Re}(\alpha) < Q\} \cup \{|\alpha - Q|^2 \leq \lambda_k\}$, with

$$\tilde{\chi} \mathbf{R}_k^\perp(\alpha) \chi : L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow L^2(\mathbb{R}; E_k^\perp) \cap \mathcal{D}(\mathbf{H}), \quad \tilde{\chi} \mathbf{R}_k^\perp(\alpha) \chi : \mathcal{D}'(\mathcal{Q}) \rightarrow \mathcal{D}'(\mathcal{Q}) \cap L^2(\mathbb{R}; E_k^\perp)$$

bounded, so that

$$(3.30) \quad (\mathbf{H} - \frac{Q^2 + p^2}{2}) \tilde{\chi} \mathbf{R}_k^\perp(\alpha) (1 - \Pi_k) \chi = (1 - \Pi_k) \chi + \mathbf{L}_k^\perp(\alpha) + \mathbf{K}_k^\perp(\alpha) \quad \text{and} \quad \tilde{\chi} \mathbf{R}_k^\perp(\alpha) \Pi_k \chi = 0$$

with $\mathbf{L}_k^\perp(\alpha) : \mathcal{D}'(\mathcal{Q}) \rightarrow L^2(\mathbb{R}^-; E_k^\perp)$ and $\mathbf{K}_k^\perp(\alpha) : \mathcal{D}'(\mathcal{Q}) \rightarrow e^{\beta\rho} L^2(\mathbb{R}; E_k)$ bounded and holomorphic in α as above for each $0 < \beta < \gamma/2$. In the region where $|p^2| \leq \lambda_k$, one has the bound

$$(3.31) \quad \|\mathbf{L}_k^\perp(\alpha)\|_{\mathcal{L}(L^2)} \leq C \lambda_k^{-1/2}$$

and $\mathbf{K}_k^\perp(\alpha)$ is compact as a map $L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta\rho} L^2(\mathbb{R}; E_k)$.

2) Let $\beta \in \mathbb{R}$, then in the region $\text{Re}((\alpha - Q)^2) > \beta^2 - 2\lambda_k + 1$, the operator $\mathbf{R}_k^\perp(\alpha) : e^{-\beta\rho} \mathcal{D}'(\mathcal{Q}) \rightarrow e^{-\beta\rho} (L^2(\mathbb{R}^-; E_k^\perp) \cap \mathcal{D}(\mathcal{Q}))$ is a bounded holomorphic family, $\mathbf{K}_k^\perp(\alpha) : e^{-\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} L^2(\mathbb{R}^-; E_k)$ is a compact holomorphic family, $\mathbf{L}_k^\perp(\alpha) : e^{-\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} L^2(\mathbb{R}^-; E_k^\perp)$ is bounded analytic with norm

$$\|\mathbf{L}_k^\perp(\alpha)\|_{\mathcal{L}(e^{-\beta\rho} L^2)} \leq \frac{C(1 + |\beta|)}{\sqrt{\text{Re}((\alpha - Q)^2) + 2\lambda_k - \beta^2}}$$

for some $C > 0$ depending only on $|\tilde{\chi}'|_\infty$ and $|\tilde{\chi}''|_\infty$.

Proof. For $u \in C_c^\infty(\mathbb{R}^-; E_k^\perp) \cap \mathcal{C}$ and for each $\beta > 0$, we have for each $\varepsilon \geq 0$

$$(3.32) \quad \mathcal{Q}(u) \geq \frac{1}{2} \|\partial_c u\|_{L^2}^2 + (\frac{Q^2}{2} + (1 - \varepsilon)\lambda_k) \|u\|_{L^2}^2 + \varepsilon \|\mathbf{P}^{1/2} u\|_{L^2}^2 + \|e^{\gamma c/2} V^{\frac{1}{2}} u\|^2$$

and therefore the quadratic form $\mathcal{Q}_k^\perp(u) := \langle \mathbf{H}u | u \rangle$ is positive and bounded below by $\frac{\lambda_k}{2} \|u\|_{L^2}^2$ on (if ε is chosen small enough)

$$\mathcal{C}_k^- = \text{Span}\{\psi(c)F \mid \psi \in C_c^\infty(\mathbb{R}^-) \text{ and } F \in \mathcal{S} \cap E_k^\perp\}.$$

This form is associated to the symmetric operator $(1 - \Pi_k)\mathbf{H}(1 - \Pi_k)$ on the dense subspace \mathcal{C}_k^- of $L^2(\mathbb{R}^-; E_k^\perp)$ and it extends to the domain $\mathcal{D}(\mathcal{Q}_k^\perp)$ obtained by completion of \mathcal{C}_k^- with respect to \mathcal{Q}_k^\perp in the Hilbert space $L^2(\mathbb{R}^-; E_k^\perp)$; notice that $\mathcal{D}(\mathcal{Q}_k^\perp) = \mathcal{D}(\mathcal{Q}) \cap L^2(\mathbb{R}^-; E_k^\perp)$ by using that $V^{1/2}\Pi_k : L^2(\Omega_{\mathbb{T}}) \rightarrow L^2(\Omega_{\mathbb{T}})$ continuously. We also have $H_0^1(\mathbb{R}^-; E_k^\perp) \cap \mathcal{D}(\mathcal{Q}) \subset \mathcal{D}(\mathcal{Q}_k^\perp)$ for the same reason.

There is a self-adjoint extension denoted \mathbf{H}_k^\perp of the operator $(1 - \Pi_k)\mathbf{H}(1 - \Pi_k)$ on the domain

$$\mathcal{D}(\mathbf{H}_k^\perp) = \{u \in \mathcal{D}(\mathcal{Q}_k^\perp) \mid (1 - \Pi_k)\mathbf{H}u \in L^2(\mathbb{R}^-; E_k^\perp)\}$$

and the spectrum is contained in $[\frac{Q^2}{2} + \lambda_k, \infty)$ due to (3.32). It will be said to have Dirichlet condition at $c = 0$, by analogy with the Laplacian on finite dimensional manifolds. We can see that $\mathcal{D}(\mathbf{H}_k^\perp)$ embeds into $\mathcal{D}(\mathbf{H})$ using the natural embedding $E_k^\perp \rightarrow L^2(\Omega_{\mathbb{T}})$: indeed, for $u \in \mathcal{D}(\mathbf{H}_k^\perp) \subset \mathcal{D}(\mathcal{Q}_k^\perp) \subset \mathcal{D}(\mathcal{Q})$, we have

$$\mathbf{H}u = \mathbf{H}_k^\perp u + e^{\gamma c} \Pi_k(Vu).$$

Now, since $e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} u \in L^2(\mathbb{R}_- \times \Omega_{\mathbb{T}})$, we have by Hölder inequality with $p \in (1, 2)$

$$\int_{\mathbb{R}_-} e^{\gamma c} \|Vu\|_{L^p(\Omega_{\mathbb{T}})}^2 dc \leq \|V\|_{L^{\frac{p}{2-p}}(\Omega_{\mathbb{T}})} \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} u\|_{L^2(\mathbb{R}_- \times \Omega_{\mathbb{T}})}^2 < \infty$$

and $\Pi_k : L^p(\Omega_{\mathbb{T}}) \rightarrow L^2(\Omega_{\mathbb{T}})$ is bounded for all $p > 1$, thus

$$\int_{\mathbb{R}_-} e^{\gamma c} \|\Pi_k(Vu)\|_{L^2(\Omega_{\mathbb{T}})}^2 dc < \infty.$$

Thus the resolvent $\mathbf{R}_k^\perp(\alpha) := (\mathbf{H}_k^\perp - \frac{Q^2 + p^2}{2})^{-1}$ (with $\alpha = Q + ip$)

$$\mathbf{R}_k^\perp(\alpha) : L^2(\mathbb{R}^-; E_k^\perp) \rightarrow \mathcal{D}(\mathbf{H}_k^\perp) \quad \mathbf{R}_k^\perp(\alpha) : \mathcal{D}'(\mathcal{Q}_k^\perp) \rightarrow \mathcal{D}(\mathcal{Q}_k^\perp)$$

is well-defined and bounded if $p \in \mathbb{C}$ is such that $p^2 \notin [2\lambda_k, \infty)$, with L^2 norm

$$(3.33) \quad \|\mathbf{R}_k^\perp(\alpha)\|_{\mathcal{L}(L^2)} \leq 2/\lambda_k \text{ for } |p|^2 \leq \lambda_k.$$

It is also holomorphic in α in $\{\operatorname{Re}(\alpha) < Q\} \cup \{|\alpha - Q|^2 < \lambda_k\}$. We also notice that $\mathcal{D}'(\mathcal{Q}) \subset \mathcal{D}'(\mathcal{Q}_k^\perp)$ since $\mathcal{D}(\mathcal{Q}_k^\perp) \subset \mathcal{D}(\mathcal{Q})$, so this gives by embedding

$$\mathbf{R}_k^\perp(\alpha)(1 - \Pi_k) : L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow \mathcal{D}(\mathbf{H}) \quad \mathbf{R}_k^\perp(\alpha)(1 - \Pi_k) : \mathcal{D}'(\mathcal{Q}) \rightarrow \mathcal{D}(\mathcal{Q})$$

with the same properties. Using (3.32) with $u = \mathbf{R}_k^\perp(\alpha)f$ we also obtain that

$$(3.34) \quad \|\partial_c \mathbf{R}_k^\perp(\alpha)\|_{\mathcal{L}(L^2)} \leq \sqrt{\frac{4}{\lambda_k}}, \text{ for } |p|^2 \leq \lambda_k/2.$$

Thus, using the natural embedding $E_k^\perp \rightarrow L^2(\Omega_{\mathbb{T}})$ and that $\Pi_k \mathbf{H}_0 = \mathbf{H}_0 \Pi_k$, we get for $\alpha = Q + ip$

$$\begin{aligned} (\mathbf{H} - \frac{Q^2 + p^2}{2}) \widetilde{\chi} \mathbf{R}_k^\perp(\alpha)(1 - \Pi_k) \chi &= (1 - \Pi_k) \chi - \frac{1}{2} [\partial_c^2, \widetilde{\chi}] \mathbf{R}_k^\perp(\alpha)(1 - \Pi_k) \chi + e^{\gamma c} \widetilde{\chi} \Pi_k V \mathbf{R}_k^\perp(\alpha)(1 - \Pi_k) \chi \\ &=: (1 - \Pi_k) \chi + \mathbf{L}_k^\perp(\alpha) + \mathbf{K}_k^\perp(\alpha). \end{aligned}$$

Since $[\partial_c^2, \widetilde{\chi}]$ is a first order operator with compact support in c commuting with Π_k , we notice that $\mathbf{L}_k^\perp(\alpha) : \mathcal{D}'(\mathcal{Q}) \rightarrow L^2(\mathbb{R}^-; E_k^\perp)$ and we can use (3.34), (3.33) to deduce that there is $C > 0$ depending only on $|\widetilde{\chi}'|_\infty, |\widetilde{\chi}''|_\infty$ such that

$$\|\mathbf{L}_k^\perp(\alpha)\|_{\mathcal{L}(L^2)} \leq C \lambda_k^{-1/2}$$

as long as $|p|^2 \leq \lambda_k$. Let us now deal with $\mathbf{K}_k^\perp(\alpha)$. First, notice that $\mathbf{K}_k^\perp(\alpha)$ maps $\mathcal{D}'(\mathcal{Q})$ to $e^{\gamma \rho} L^2(\mathbb{R}^-; E_k)$, so we would like to prove some regularization property in c to deduce that $\mathbf{K}_k^\perp(\alpha)$ is compact on L^2 (or some weighted L^2 space). Since $V^{1/2} \in L^p$ for some $p > 2$ and $\Pi_k : L^q(\Omega_{\mathbb{T}}) \rightarrow L^2(\Omega_{\mathbb{T}})$ for all $q > 1$, we have $V_k := \Pi_k V^{(1+\varepsilon)/2} : L^2(\Omega_{\mathbb{T}}) \rightarrow E_k \subset L^2(\Omega_{\mathbb{T}})$ for some $\varepsilon > 0$ small depending on p . Let $\Delta_{\mathbb{R}} = -\partial_c^2$. We claim that there is $\varepsilon > 0$ small such that for each $A > 1$

$$(3.35) \quad e^{\frac{1-\varepsilon}{2} \gamma \rho} (A + \Delta_{\mathbb{R}})^{\varepsilon/2} e^{-\gamma \rho} \mathbf{K}_k^\perp(\alpha) : L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow L^2(\mathbb{R}; E_k)$$

is bounded. It suffices to show that $e^{\gamma \frac{1-\varepsilon}{2} \rho} (A^2 + \Delta_{\mathbb{R}})^{\varepsilon/2} \widetilde{\chi} V_k V^{(1-\varepsilon)/2} \mathbf{R}_k^\perp(\alpha)(1 - \chi)$ is bounded on L^2 . We can use Stein complex interpolation with the complex family of operator

$$T_z := e^{\frac{1-z}{2} \gamma \rho} (A^2 + \Delta_{\mathbb{R}})^{\frac{z}{2}} \widetilde{\chi} V_k V^{\frac{1-z}{2}} \mathbf{R}_k^\perp(\alpha)(1 - \chi)$$

for $\operatorname{Re}(z) \in [0, 1]$. The bound for $\operatorname{Re}(z) = 0$ is equivalent to a bound for $z = 0$, that is $\widetilde{\chi} e^{\gamma \rho/2} V_k V^{1/2} \mathbf{R}_k^\perp(\alpha)(1 - \chi) \in \mathcal{L}(L^2)$, and this bound follows from the fact that $\widetilde{\chi} \mathbf{R}_k^\perp(\alpha)(1 - \chi) : L^2 \rightarrow \mathcal{D}(\mathcal{Q})$. The bound for $\operatorname{Re}(z) = 1$ is equivalent to the bound for $z = 1$, that is $V_k (A^2 + \Delta_{\mathbb{R}})^{1/2} \widetilde{\chi} \mathbf{R}_k^\perp(\alpha)(1 - \chi) \in \mathcal{L}(L^2)$. But $\partial_c \widetilde{\chi} \mathbf{R}_k^\perp(\alpha)(1 - \chi) \in \mathcal{L}(L^2)$ and $\|(A^2 + \Delta_{\mathbb{R}})^{1/2} u\|_{L^2(\mathbb{R})}^2 = \|\partial_c u\|_{L^2}^2 + A^2 \|u\|_{L^2}^2$, so we obtain the desired result. For $\psi_1, \psi_2 \in \mathcal{C}$, the bound

$|(T_z \psi_1 | \psi_2)_2| \leq e^{C|z|}$ so we can apply Phragmen-Lindelof theorem to deduce (3.35). Next, for $A > 1$ large enough depending on γ and $\hat{\chi} \in C^\infty(\mathbb{R})$ with support in \mathbb{R}^- and equal to 1 on $\text{supp}(\tilde{\chi})$, the operator

$$\hat{\chi} e^{\gamma \rho} (A^2 + \Delta_{\mathbb{R}})^{-\frac{\varepsilon}{2}} e^{-\frac{1-\varepsilon}{2} \gamma \rho} : L^2(\mathbb{R}; E_k) \rightarrow e^{\beta \rho} L^2(\mathbb{R}; E_k)$$

is compact if $\beta < \gamma/2$: indeed, we can compute directly that the Schwartz kernel κ of $(A^2 + \Delta_{\mathbb{R}})^{-\frac{\varepsilon}{2}}$ is

$$\kappa(c, c') = C_{A,\varepsilon} |c - c'|^{\frac{\varepsilon-1}{2}} K_{\frac{\varepsilon-1}{2}}(A|c-c'|)$$

for some constant $C_{A,\varepsilon} \neq 0$, where $K_\nu(x)$ is the modified Bessel function of second kind, satisfying $K_\nu(x) = \mathcal{O}(e^{-x}/\sqrt{x})$ as $x \rightarrow +\infty$ and $K_\nu(x) = \mathcal{O}(x^\nu)$ as $x \rightarrow 0$ if $\nu < 0$. Thus, taking $A > 2\gamma$, the Schwartz kernel of $\hat{\chi} e^{\gamma \rho} (A^2 + \Delta_{\mathbb{R}})^{-\frac{\varepsilon}{2}} e^{-\frac{1-\varepsilon}{2} \gamma \rho}$ is bounded above by (for some uniform $C_{\gamma,\varepsilon} > 0$)

$$C_{\gamma,\varepsilon} |c - c'|^{\varepsilon-1} e^{\gamma \rho(c)/2} e^{\varepsilon \rho(c')/4} \mathbf{1}_{\mathbb{R}^-}(c).$$

It is then easy to see this is compact as announced since it maps boundedly $L^2(\mathbb{R}; E_k)$ to $e^{\frac{\gamma}{2} \rho} H_0^\varepsilon(\mathbb{R}^-; E_k)$, and this last space injects compactly to $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ by using that E_k has finite dimension. We conclude that $\mathbf{K}_k^\perp(\alpha)$ is compact as a map $L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta \rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. Moreover $\mathbf{K}_k^\perp(\alpha)$, $\mathbf{L}_k^\perp(\alpha)$ are holomorphic in $\alpha \in \mathbb{C}$ in the region $\{\text{Re}(\alpha) < Q\} \cup \{|\alpha - Q|^2 < \lambda_k\}$ since $\mathbf{R}_k^\perp(\alpha)$ is. This concludes the proof of 1).

Let us next consider the region $\{\text{Re}(\alpha) \leq Q\}$, and we proceed as in Lemma 3.3. Let $\mathbf{H}_{k,\beta}^\perp := e^{\beta \rho} \mathbf{H}_k^\perp e^{-\beta \rho}$ for $\beta \in \mathbb{R}$ which is also given by

$$\mathbf{H}_{k,\beta}^\perp = \mathbf{H}_k^\perp + (1 - \Pi_k) \left(-\frac{\beta^2}{2} (\rho'(c))^2 + \frac{\beta}{2} \rho''(c) + \beta \rho'(c) \partial_c \right)$$

and the associated sesquilinear form on $\mathcal{D}(\mathcal{Q}_k^\perp)$

$$\mathcal{Q}_{k,\beta}^\perp(u) := \mathcal{Q}_k^\perp(u) - \frac{\beta^2}{2} \|\rho' u\|_2^2 + \frac{\beta}{2} \langle \rho'' u, u \rangle_2 + \beta \langle \partial_c u, \rho' u \rangle_2.$$

Note that on $\mathcal{D}(\mathcal{Q}_k^\perp)$, we have

$$\text{Re}(\mathcal{Q}_{k,\beta}^\perp(u)) \geq \mathcal{Q}_k^\perp(u) - \frac{\beta^2}{2} \|u\|_2^2 \geq \frac{1}{2} \|\partial_c u\|_2^2 + \left(\frac{Q^2 - \beta^2}{2} + \lambda_k \right) \|u\|_2^2 + \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} u\|_2^2.$$

This implies by Lax-Milgram, just as in the proof of Lemma 3.3, that if $\text{Re}((\alpha - Q)^2) > \beta^2 - 2\lambda_k$, then for each $f \in \mathcal{D}'(\mathcal{Q}_k^\perp)$, there is a unique $u \in \mathcal{D}(\mathcal{Q}_k^\perp)$ such that

$$(3.36) \quad e^{\beta \rho} (\mathbf{H}_k^\perp - 2\Delta_\alpha) e^{-\beta \rho} u = f, \text{ and if } f \in L^2$$

$$(3.37) \quad \|u\|_2 \leq \frac{2\|f\|_2}{\text{Re}((\alpha - Q)^2) + 2\lambda_k - \beta^2}, \quad \|\partial_c u\|_2 \leq \frac{2\|f\|_2}{\sqrt{\text{Re}((\alpha - Q)^2) + 2\lambda_k - \beta^2}}.$$

In particular, this shows that, for $\text{Re}((\alpha - Q)^2) > \beta^2 - 2\lambda_k$, $\mathbf{R}_k^\perp(\alpha)$ extends as a map

$$\mathbf{R}_k^\perp(\alpha) : e^{-\beta \rho} \mathcal{D}'(\mathcal{Q}_k^\perp) \rightarrow e^{-\beta \rho} \mathcal{D}(\mathcal{Q}_k^\perp)$$

with $\|\mathbf{R}_k^\perp(\alpha)\|_{\mathcal{L}(e^{\beta \rho} L^2)} \leq 2(\text{Re}((\alpha - Q)^2) + 2\lambda_k - \beta^2)^{-1}$. If we further impose that $\text{Re}((\alpha - Q)^2) > \beta^2 - 2\lambda_k + 1$ then, since $e^{\beta \rho} [\partial_c, e^{-\beta \rho}] = -\beta \rho'$ and using (3.37),

$$\|\mathbf{L}_k^\perp(\alpha)\|_{\mathcal{L}(e^{-\beta \rho} L^2)} \leq \frac{2|\chi'|_\infty(1 + |\beta|) + |\chi''|_\infty}{\sqrt{\text{Re}((\alpha - Q)^2) + 2\lambda_k - \beta^2}}.$$

Finally, the same argument as above for $\mathbf{K}_k^\perp(\alpha)$ shows that for $\text{Re}((\alpha - Q)^2) + 2\lambda_k - \beta^2 > 1$, the operator $\mathbf{K}_k^\perp(\alpha)$ is compact from $e^{-\beta \rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ to $e^{-\beta \rho} L^2(\mathbb{R}^-; E_k)$. \square

Remark 3.10. We notice that the operators $\mathbf{R}_k^\perp(\alpha)$, $\mathbf{K}_k^\perp(\alpha)$, $\mathbf{L}_k^\perp(\alpha)$ lift as holomorphic family of operators to the region $\{\alpha \in \Sigma \mid \text{Re}(\pi(\alpha)) < Q, |\pi(\alpha) - Q|^2 < \lambda_k\}$ by simply composing with the projection $\pi : \Sigma \rightarrow \mathbb{C}$.

2) The region $c \geq -1$. Next, consider the operator $\mathbf{H} - \frac{Q^2 + p^2}{2}$ on $L^2([-1, \infty); L^2(\Omega_{\mathbb{T}}))$ with Dirichlet condition at $c = -1$ (i.e. the extension associated to the quadratic form on functions supported in $c \geq -1$), and $\hat{\chi} \in C^\infty(\mathbb{R}; [0, 1])$ such that $(1 - \hat{\chi}) = 1$ on $\text{supp}(1 - \chi)$ and $1 - \hat{\chi}$ supported in $(-1, \infty)$ (otherwise stated, $\hat{\chi} = 0$ on $(-1 + \delta, +\infty)$ and $\hat{\chi} = 1$ on $(-\infty, -1)$).

We will construct a quasi-compact approximate inverse to \mathbf{H} in $[-1, \infty)$ by using energy estimates and the properties of V , in particular the fact the region where $V > 0$ is small are somehow small. We show the following:

Lemma 3.11. *There is a uniform constant $C > 0$ and a bounded operator, independent of α ,*

$$\mathbf{R}_+ : L^2([-1, \infty) \times \Omega_{\mathbb{T}}) \rightarrow H_0^1([-1, \infty); L^2(\Omega_{\mathbb{T}})),$$

satisfying

$$(1 - \hat{\chi})\mathbf{R}_+(1 - \chi) : L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow \mathcal{D}(\mathbf{H}), \quad (1 - \hat{\chi})\mathbf{R}_+(1 - \chi) : \mathcal{D}'(\mathcal{Q}) \rightarrow \mathcal{D}'(\mathcal{Q})$$

and for $\alpha = Q + ip \in \mathbb{C}$ and $k \geq 1$

$$\left(\mathbf{H} - \frac{Q^2 + p^2}{2}\right)(1 - \hat{\chi})\mathbf{R}_+(1 - \chi) = (1 - \chi) + \mathbf{K}_{+,k}(\alpha) + \mathbf{L}_{+,k}(\alpha)$$

where $\mathbf{K}_{+,k}(\alpha) : L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow L^2([-1, \infty) \times \Omega_{\mathbb{T}})$ compact and holomorphic in $\alpha \in \mathbb{C}$, and the operator $\mathbf{L}_{+,k}(\alpha) : L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow L^2([-1, \infty) \times \Omega_{\mathbb{T}})$ is bounded and holomorphic in $\alpha \in \mathbb{C}$, such that

$$(3.38) \quad \|\mathbf{L}_{+,k}(\alpha)\|_{\mathcal{L}(L^2)} \leq C(1 + |p|^2 \lambda_k^{-1/2}), \quad \|\mathbf{L}_{+,k}(\alpha)^2\|_{\mathcal{L}(L^2)} \leq C(1 + |p|^2) \lambda_k^{-1/2} + C(1 + |p|^4) \lambda_k^{-1}$$

for some uniform constant C depending only on $\hat{\chi}$. Moreover $\mathbf{L}_{+,k}(\alpha)$ and $\mathbf{K}_{+,k}(\alpha)$ are bounded as maps $\mathcal{D}'(\mathcal{Q}) \rightarrow L^2([-1, \infty) \times \Omega_{\mathbb{T}})$.

Proof. We consider the quadratic form

$$\mathcal{Q}_+(u) = \frac{1}{2} \|\partial_c u\|_2^2 + \|u\|_2^2 + \|\mathbf{P}^{\frac{1}{2}} u\|_2^2 + \|e^{\frac{\gamma}{2}c} V^{\frac{1}{2}} u\|_2^2$$

with domain the Hilbert space $\mathcal{D}(\mathcal{Q}_+)$ obtained by completing $\mathcal{C}_+ := \{u \in \mathcal{C} \mid u = 0 \text{ in } c < -1\}$ with the norm $\sqrt{\mathcal{Q}_+(u)}$ and let $\mathcal{D}'(\mathcal{Q}_+)$ be the dual Hilbert space. We have the natural inclusion $\mathcal{D}(\mathcal{Q}_+) \subset \mathcal{D}(\mathcal{Q})$ and $\mathcal{D}'(\mathcal{Q}) \subset \mathcal{D}'(\mathcal{Q}_+)$. Note that for $u \in \mathcal{C}_+$, $\langle (\mathbf{H} - \frac{Q^2}{2} + 1)u, u \rangle_2 = \mathcal{Q}_+(u)$. We obtain the self-adjoint extension of $\mathbf{H} - \frac{Q^2}{2} + 1$ on $L^2([-1, \infty) \times \Omega_{\mathbb{T}})$ with domain $\mathcal{D}(\mathbf{H}_+) = \{u \in \mathcal{D}(\mathcal{Q}_+) \mid \mathbf{H}u \in L^2\}$ (corresponding to Dirichlet condition at $c = -1$) and the bound above implies that there is a bounded inverse

$$(3.39) \quad \mathbf{R}_+ : L^2([-1, \infty) \times \Omega_{\mathbb{T}}) \rightarrow L^2([-1, \infty) \times \Omega_{\mathbb{T}}) \text{ satisfying } (\mathbf{H} - Q^2 + 1)\mathbf{R}_+ = \text{Id}$$

$$\text{and } \frac{1}{2} \|\partial_c \mathbf{R}_+ f\|_2^2 + \|\mathbf{R}_+ f\|_2^2 + \|\mathbf{P}^{1/2} \mathbf{R}_+ f\|_2^2 + \|e^{\gamma c/2} V^{1/2} \mathbf{R}_+ f\|_2^2 \leq \|f\|_2^2,$$

that is $\mathbf{R}_+ : L^2([-1, \infty) \times \Omega_{\mathbb{T}}) \rightarrow \mathcal{D}(\mathbf{H}_+) \subset \mathcal{D}(\mathcal{Q}_+)$. Moreover $\mathbf{R}_+ : \mathcal{D}'(\mathcal{Q}_+) \rightarrow \mathcal{D}'(\mathcal{Q}_+)$ is bounded. Note that there is a natural inclusion $\mathcal{D}(\mathbf{H}_+) \subset \mathcal{D}(\mathbf{H})$ by using the inclusion $[-1, \infty) \subset \mathbb{R}$. We would like to prove that \mathbf{R}_+ is compact or quasi-compact. We have

$$(3.40) \quad \left(\mathbf{H} - \frac{Q^2 + p^2}{2}\right)(1 - \hat{\chi})\mathbf{R}_+(1 - \chi) = (1 - \chi) + \frac{1}{2} [\partial_c^2, \hat{\chi}] \mathbf{R}_+(1 - \chi) - \left(\frac{p^2}{2} - 1\right)(1 - \hat{\chi})\mathbf{R}_+(1 - \chi).$$

$$=: (1 - \chi) + \mathbf{K}_+^1 + \mathbf{K}_+^2(\alpha).$$

Notice that $\mathbf{K}_+^1, \mathbf{K}_+^2(\alpha)$ are bounded as maps $\mathcal{D}'(\mathcal{Q}) \rightarrow L^2$ by using that $[\partial_c^2, \hat{\chi}] : \mathcal{D}(\mathcal{Q}) \rightarrow L^2$ is bounded. We can use the orthonormal basis $(\psi_{\mathbf{k}l})_{\mathbf{k}, l \in \mathcal{N}}$ of eigenfunctions of \mathbf{P} defined as the normalization of (1.33). Then we use the fact that $\psi_{\mathbf{k}l} \in L^q(\Omega_{\mathbb{T}})$ for each $q < \infty$ to obtain, with $\Pi_j u = \sum_{\mathbf{k}, l \in \mathcal{N}, \lambda_{\mathbf{k}l} \leq \lambda_j} \langle u, \psi_{\mathbf{k}l} \rangle_2 \psi_{\mathbf{k}l}$ and $r \in (1, \infty)$,

$$(3.41) \quad \|\Pi_j u\|_{L^r(\Omega_{\mathbb{T}})} \leq \|\Pi_j u\|_{L^2(\Omega_{\mathbb{T}})} \sum_{\substack{\mathbf{k}, l \in \mathcal{N}, \\ \lambda_{\mathbf{k}l} \leq \lambda_j}} \|\psi_{\mathbf{k}l}\|_{L^r(\Omega_{\mathbb{T}})} \leq C_j \|\Pi_j u\|_{L^2(\Omega_{\mathbb{T}})}$$

where $C_j > 0$ is a constant depending only on j, r but not on u . Now, by Cauchy-Schwarz and Hölder inequality, we write for $u = \mathbf{R}_+ f$

$$(3.42) \quad \int_{-1}^{\infty} e^{\frac{\gamma c}{2}} \|\Pi_k u\|_{L^2(\Omega_{\mathbb{T}})}^2 dc \leq \int_{-1}^{\infty} e^{\frac{\gamma c}{2}} \mathbb{E} \left(\frac{V^{\frac{1}{2}} |\Pi_k u| |u|}{V^{\frac{1}{2}}} \right) dc$$

$$\leq \left(\int_{-1}^{\infty} \mathbb{E}(V^{-1} |\Pi_k u|^2) dc \right)^{\frac{1}{2}} \left(\int_{-1}^{\infty} e^{\gamma c} \mathbb{E}(V |u|^2) dc \right)^{\frac{1}{2}}$$

$$\leq \mathbb{E}(V^{-q})^{\frac{1}{2q}} \|f\|_2 \left(\int_{-1}^{\infty} \|\Pi_k u\|_{L^{2p}(\Omega_{\mathbb{T}})}^2 dc \right)^{\frac{1}{2}} \leq C_k \mathbb{E}(V^{-q})^{\frac{1}{2q}} \|f\|_2^2$$

where we used the bound (3.39), the bound (3.41) with $r = 2p$ if $1/p + 1/q = 1$, and the fact that $V^{-1} \in L^q(\Omega_T)$ for some $q > 2$. Now we claim that bootstrapping this argument gives the estimate

$$(3.43) \quad \|e^{\gamma c/2} \Pi_k u\|_2^2 \leq C_k^2 \mathbb{E}(V^{-q})^{\frac{1}{q}} \|f\|_2^2.$$

Indeed, define recursively the sequence $a_{n+1} = \frac{a_n+1}{2}$ ($n \geq 0$) with $a_0 = 0$, or equivalently $a_n = 1 - 2^{-n}$. Then the same argument as above produces the relation, using again Cauchy-Schwartz and Hölder inequality,

$$\begin{aligned} \int_{-1}^{\infty} e^{a_{n+1}\gamma c} \|\Pi_k u\|_{L^2(\Omega_T)}^2 dc &\leq \int_{-1}^{\infty} e^{\frac{a_n}{2}\gamma c} e^{\frac{\gamma}{2}c} \mathbb{E}\left(\frac{V^{\frac{1}{2}}|\Pi_k u||u|}{V^{\frac{1}{2}}}\right) dc \\ &\leq \left(\int_{-1}^{\infty} e^{a_n\gamma c} \mathbb{E}(V^{-1}|\Pi_k u|^2) dc\right)^{\frac{1}{2}} \left(\int_{-1}^{\infty} e^{\gamma c} \mathbb{E}(V|u|^2) dc\right)^{\frac{1}{2}} \\ &\leq \mathbb{E}(V^{-q})^{\frac{1}{2q}} \|f\|_2 \left(\int_{-1}^{\infty} e^{a_n\gamma c} \|\Pi_k u\|_{L^{2p}(\Omega_T)}^2 dc\right)^{\frac{1}{2}} \\ &\leq C_k \mathbb{E}(V^{-q})^{\frac{1}{2q}} \|f\|_2 \left(\int_{-1}^{\infty} e^{a_n\gamma c} \|\Pi_k u\|_{L^2(\Omega_T)}^2 dc\right)^{\frac{1}{2}}. \end{aligned}$$

This recursive relation can be solved to produce the bound

$$\int_{-1}^{\infty} e^{a_n\gamma c} \|\Pi_k u\|_{L^2(\Omega_T)}^2 dc \leq \left(C_k \mathbb{E}(V^{-q})^{\frac{1}{2q}}\right)^{2-2^{-n+1}} \|f\|_2^2.$$

Our claim (3.43) then follows from monotone convergence theorem.

We deduce that

$$(3.44) \quad \Pi_k \mathbf{R}_+ : L^2([-1, \infty) \times \Omega_T) \rightarrow e^{-\frac{\gamma c}{2}} L^2([-1, \infty); E_k)$$

is bounded. By using again (3.41) and the fact that $V \in L^q(\Omega_T)$ for some $q > 1$, this also implies using Hölder that

$$\int_{-1}^{\infty} e^{\gamma c} \|V^{\frac{1}{2}} \Pi_k u\|_{L^2(\Omega_T)}^2 dc \leq C_k^2 \|V\|_{L^q(\Omega_T)} \int_{-1}^{\infty} e^{\gamma c} \|\Pi_k u\|_{L^2(\Omega_T)}^2 dc.$$

Using (3.39) and (3.44), this implies that

$$(3.45) \quad e^{\gamma c/2} V^{1/2} (1 - \Pi_k) \mathbf{R}_+ : L^2 \rightarrow L^2,$$

is bounded. The argument above shows that if $u \in \mathcal{D}(\mathcal{Q}_+)$, then $e^{\gamma c/2} V^{1/2} (1 - \Pi_k) u \in L^2$. But since Π_k commutes with ∂_c, \mathbf{P} , we deduce that

$$u \in \mathcal{D}(\mathcal{Q}_+) \implies \Pi_k u \in \mathcal{D}(\mathcal{Q}_+).$$

Let us now consider the operator $\mathbf{R}_{+,k}^\perp := (1 - \Pi_k) \mathbf{R}_+$. Let $u = \mathbf{R}_+ f$ with $f \in L^2$, then we claim that

$$\begin{aligned} &\lambda_k \|(1 - \Pi_k) u\|_2^2 + \frac{1}{2} \|\partial_c (1 - \Pi_k) u\|_2^2 + \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} (1 - \Pi_k) u\|_2^2 \\ &\leq |\langle (1 - \Pi_k) f, (1 - \Pi_k) u \rangle| + |\langle e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} (1 - \Pi_k) u, e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} \Pi_k u \rangle| \\ &\leq \|(1 - \Pi_k) f\|_2 \|(1 - \Pi_k) u\|_2 + \frac{1}{2} (\|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} (1 - \Pi_k) u\|_2^2 + \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} \Pi_k u\|_2^2). \end{aligned}$$

To prove this, we first take $\varphi \in C_c^\infty((-1, \infty); E_j)$ (for $j \geq k$), compute $\langle (\mathbf{H} - \frac{Q^2}{2} + 1)(1 - \Pi_k) \varphi, (1 - \Pi_k) u \rangle = \mathcal{Q}_+(\langle (1 - \Pi_k) \varphi, (1 - \Pi_k) u \rangle)$ (the pairing is $\mathcal{D}'(\mathcal{Q}_+) \times \mathcal{D}(\mathcal{Q}_+)$), then use that

$$\langle (\mathbf{H} - \frac{Q^2}{2} + 1)(1 - \Pi_k) \varphi, (1 - \Pi_k) u \rangle = \langle (1 - \Pi_k)(\mathbf{H} - \frac{Q^2}{2} + 1) \varphi, (1 - \Pi_k) u \rangle - \langle e^{\gamma c/2} V^{\frac{1}{2}} \Pi_k \varphi, e^{\gamma c/2} V^{\frac{1}{2}} (1 - \Pi_k) u \rangle_2$$

and finally let φ converge to u in $\mathcal{D}(\mathbf{H}_+)$, which gives the desired bound by using also (3.45). This gives the bound

$$(3.46) \quad \lambda_k \|(1 - \Pi_k) u\|_2^2 + \frac{1}{2} \|\partial_c (1 - \Pi_k) u\|_2^2 + \frac{1}{2} \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} (1 - \Pi_k) u\|_2^2 \leq \|f\|_2 \|(1 - \Pi_k) u\|_2 + \frac{1}{2} \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} \Pi_k u\|_2^2.$$

Now we do the same computation with $(1 - \Pi_k)$ replaced with Π_k and get

$$(3.47) \quad \|\Pi_k u\|_2^2 + \frac{1}{2} \|\partial_c \Pi_k u\|_2^2 + \frac{1}{2} \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} \Pi_k u\|_2^2 \leq \|f\|_2 \|\Pi_k u\|_2 + \frac{1}{2} \|e^{\frac{\gamma c}{2}} V^{\frac{1}{2}} (1 - \Pi_k) u\|_2^2.$$

Combining (3.46) and (3.47) and using $\|u\|_2 \leq \|f\|_2$, we obtain (the bound is not optimal)

$$(3.48) \quad \|\mathbf{R}_{+,k}^1 f\|_2 = \|(1 - \Pi_k)u\|_2 \leq \frac{2}{\sqrt{\lambda_k}} \|f\|_2.$$

Since $[\partial_c^2, \hat{\chi}] = \hat{\chi}'' + 2\hat{\chi}'\partial_c$ and $\hat{\chi}' = 0$ on $\text{supp}(1 - \chi)$, we have $(\mathbf{K}_+^1)^2 = 0$ and $\|\mathbf{K}_+^1\|_{\mathcal{L}(L^2)} \leq C$ (using (3.39)) for some uniform C depending only on $\hat{\chi}$. By combining with (3.48), we deduce that

$$\begin{aligned} \|(\mathbf{K}_+^1 + (1 - \Pi_k)\mathbf{K}_+^2(\alpha))\|_{\mathcal{L}(L^2)} &\leq C(1 + |p|^2\lambda_k^{-1/2}), \\ \|(\mathbf{K}_+^1 + (1 - \Pi_k)\mathbf{K}_+^2(\alpha))^2\|_{\mathcal{L}(L^2)} &\leq C((1 + |p|^2)\lambda_k^{-1/2} + (1 + |p|^2)^2\lambda_k^{-1}) \end{aligned}$$

for some uniform C depending only on $\hat{\chi}$. Next we consider the operator $\Pi_k\mathbf{K}_+^2(\alpha)$. Recall that, by (3.39),

$$(3.49) \quad \partial_c \Pi_k \mathbf{R}_+ : L^2([-1, \infty) \times \Omega_{\mathbb{T}}) \rightarrow L^2([-1, \infty); E_k)$$

is bounded. Now we claim that the injection

$$(3.50) \quad F_k := \{u \in e^{-\frac{\gamma c}{2}} L^2([-1, \infty); E_k) \mid \partial_c u \in L^2([-1, \infty); E_k)\} \rightarrow e^{-\frac{\gamma c}{4}} L^2([-1, \infty) \times \Omega_{\mathbb{T}})$$

is compact if we put the norm $\|u\|_{F_k} := \|e^{\frac{\gamma c}{2}} u\|_2 + \|\partial_c u\|_2$ on F_k . Indeed, consider the operator $\eta_T \text{Id} : F_k \rightarrow e^{-\frac{\gamma c}{2}} L^2([-1, \infty); E_k)$ where $\eta_T(c) = \eta(c/T)$ if $\eta \in C_0^\infty((-2, 2))$ is equal to 1 on $(-1, 1)$ and $0 \leq \eta \leq 1$. Since E_k has finite dimension, this is a compact operator by the compact embedding $H^1([-1, T]; E_k) \rightarrow e^{-\frac{\gamma c}{4}} L^2([-1, \infty); E_k)$, and as $T \rightarrow \infty$ we have

$$\|e^{\frac{\gamma c}{4}}(\eta_T u - u)\|_2^2 \leq e^{-\frac{\gamma T}{2}} \int_{-1}^{\infty} (1 - \eta_T)^2 e^{\gamma c} \|u\|_{L^2(\Omega_{\mathbb{T}})}^2 dc \leq e^{-\frac{\gamma T}{2}} \|u\|_{F_k}^2$$

thus the injection (3.50) is a limit of compact operators for the operator norm topology, therefore is compact. By (3.44) and (3.49), the operator $\Pi_k \mathbf{R}_+ : L^2([-1, \infty) \times \Omega_{\mathbb{T}}) \rightarrow e^{-\frac{\gamma c}{4}} L^2([-1, \infty) \times \Omega_{\mathbb{T}})$ is compact. We get that the operators defined by (3.40) are such that

$$\Pi_k \mathbf{K}_+^2(\alpha) : L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow L^2(\mathbb{R}_- \times E_k)$$

is compact. This complete the proof by setting $\mathbf{K}_{+,k}(\alpha) := \Pi_k \mathbf{K}_+^2(\alpha)$ and $\mathbf{L}_{+,k}(\alpha) := \mathbf{K}_+^1 + (1 - \Pi_k)\mathbf{K}_+^2(\alpha)$. The holomorphy in $\alpha \in \mathbb{C}$ is clear since $\mathbf{K}_+^2(\alpha)$ is polynomial in α . \square

Remark 3.12. *As above, the operators \mathbf{R}_+ , $\mathbf{K}_{+,k}(\alpha)$ and $\mathbf{L}_{+,k}(\alpha)$, lift as holomorphic family of operators to Σ .*

3) Small P eigenmodes in the region $c \leq 0$, where there is scattering. We will view the potential $e^{\gamma c} V$ as a perturbation of the free Hamiltonian $\mathbf{H}_0 := -\frac{1}{2}\partial_c^2 + \frac{Q^2}{2} + \mathbf{P}$ on $L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}})$ with Dirichlet condition at $c = 0$. We show (recall that $\pi : \Sigma \rightarrow \mathbb{C}$ is the covering map)

Lemma 3.13. *1) Fix k and $0 < \beta < \gamma/2$. The operators*

$$(3.51) \quad \mathbf{R}_k(\alpha) := (\mathbf{H}_0 - \frac{Q^2 + p^2}{2})^{-1} \Pi_k : e^{\beta\rho} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} L^2(\mathbb{R}^-; E_k)$$

defined for $\text{Im}(p) > 0$ can be holomorphically continued to the region

$$(3.52) \quad \{\alpha = Q + ip \in \Sigma \mid \forall j \leq k, \text{Im}\sqrt{p^2 - 2\lambda_j} > -\beta\}.$$

This continuation, still denoted $\mathbf{R}_k(\alpha) : e^{\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} L^2(\mathbb{R}^-; E_k)$, satisfies

$$\begin{aligned} \tilde{\chi} \mathbf{R}_k(\alpha) \chi : e^{\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) &\rightarrow e^{-\beta\rho} (L^2(\mathbb{R}^-; E_k) \cap \mathcal{D}(\mathcal{Q})), \\ (\mathbf{H} - \frac{Q^2 + p^2}{2}) \tilde{\chi} \mathbf{R}_k(\alpha) \chi &= \Pi_k \chi + \mathbf{K}_{k,1}(\alpha) + \mathbf{K}_{k,2}(\alpha) \end{aligned}$$

where $\mathbf{K}_{k,1}(\alpha)$, $\mathbf{K}_{k,2}(\alpha)$ are such that for $\text{Im}\sqrt{p^2 - 2\lambda_j} > -\min(\beta, \gamma/2 - \beta)$

$$\mathbf{K}_{k,1}(\alpha) : e^{\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$$

$$\mathbf{K}_{k,2}(\alpha) : e^{\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta\rho} \mathcal{D}'(\mathcal{Q})$$

are holomorphic families of compact operators in (3.52), and we have $\mathbf{K}_{k,i}(\alpha)(1 - \Pi_k) = 0$ for $i = 1, 2$, $(1 - \Pi_k)\mathbf{K}_{k,1}(\alpha) = 0$ and $\Pi_k \mathbf{K}_{k,2}(\alpha) = 0$.

2) If $f \in e^{\beta\rho} L^2$, then there is $C_k > 0$ depending on k , some $a_j(\alpha, f)$ and $G(\alpha, f) \in H^1(\mathbb{R}; E_k)$ depending

linearly on f and holomorphic in $\{\alpha = Q + ip \in \Sigma \mid \forall j \leq k, \operatorname{Im}\sqrt{p^2 - 2\lambda_j} > -\min(\beta, \gamma/2 - \beta)\}$ such that in the region $c \leq 0$

$$(3.53) \quad (\mathbf{R}_k(\alpha)f) = \sum_{\lambda_j \leq \lambda_k} a_j(\alpha, f)e^{-ic\sqrt{p^2 - 2\lambda_j}} + G(\alpha, f),$$

$$\|G(\alpha, f)(c)\|_{L^2(\Omega_{\mathbb{T}})} + \|\partial_c G(\alpha, f)(c)\|_{L^2(\Omega_{\mathbb{T}})} \leq C_k e^{\beta\rho} \|e^{-\beta\rho} \Pi_k f\|_2.$$

3) For each $\beta \in \mathbb{R}$, the operator $\mathbf{R}_k(\alpha)$ extends as a bounded analytic family

$$\mathbf{R}_k(\alpha) : e^{-\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} (L^2(\mathbb{R}^-; E_k) \cap \mathcal{D}(\mathcal{Q}))$$

in the region $\operatorname{Im}(p) > |\beta|$ and $\mathbf{K}_{k,1}(\alpha) : e^{-\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$, $\mathbf{K}_{k,2}(\alpha) : e^{-\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}'(\mathcal{Q})$ are compact analytic families in that same region.

Proof. We first consider \mathbf{H}_0 on $(-\infty, 0]$ with Dirichlet condition at $c = 0$. Using the diagonalisation of \mathbf{P} on E_k , we can compute the resolvent $(\mathbf{H}_0 - \frac{Q^2 + p^2}{2})^{-1}$ on E_k by standard ODE methods (Sturm-Liouville theory): for $\operatorname{Im}(p) > 0$, this is the diagonal operator given for $j \leq k$ and $f \in L^2(\mathbb{R}^-)$ and $\phi_j \in \ker(\mathbf{P} - \lambda_j)$

$$\begin{aligned} (\mathbf{H}_0 - \frac{Q^2 + p^2}{2})^{-1} f(c) \phi_j &= \frac{2}{\sqrt{p^2 - 2\lambda_j}} \phi_j \left(\int_{-\infty}^c \sin(c\sqrt{p^2 - 2\lambda_j}) e^{-ic'\sqrt{p^2 - 2\lambda_j}} f(c') dc' \right. \\ &\quad \left. + \int_c^0 e^{-ic\sqrt{p^2 - 2\lambda_j}} \sin(c'\sqrt{p^2 - 2\lambda_j}) f(c') dc' \right) \end{aligned}$$

where our convention is that \sqrt{z} is defined with the cut on \mathbb{R}^+ , so that $\sqrt{p^2} = p$ if $\operatorname{Im}(p) > 0$. For $j = 0$, that is $\phi_0 = 1$, for each $\beta > 0$ the resolvent restricted to E_0 admits an analytic continuation from $\operatorname{Im}(p) > 0$ to $\operatorname{Im}(p) > -\beta$, as a map

$$(\mathbf{H}_0 - \frac{Q^2 + p^2}{2})^{-1} \Pi_0 : e^{-\beta|c|} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta|c|} L^2(\mathbb{R}^-; E_0).$$

This is easy to see by using Schur's lemma and the analyticity in p for the Schwartz kernel

$$\kappa_0(c, c') := \mathbf{1}_{\{c \geq c'\}} e^{-\beta(|c|+|c'|)} \sin(cp) e^{-ic'p} + \mathbf{1}_{\{c' \geq c\}} e^{-\beta(|c'|+|c|)} \sin(c'p) e^{-icp}$$

of the operator $\Pi_0 e^{-\beta|c|} (\mathbf{H}_0 - \frac{Q^2 + p^2}{2})^{-1} e^{-\beta|c|} \Pi_0$ that we view as an operator on $L^2(\mathbb{R}^-)$. Moreover, one directly also obtains that it maps $e^{-\beta|c|} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta|c|} H^2(\mathbb{R}^-; E_0) \cap H_0^1(\mathbb{R}^-; E_0)$. Similarly, the operators

$$(\mathbf{H}_0 - \frac{Q^2 + p^2}{2})^{-1} \phi_j \langle \phi_j, \cdot \rangle : e^{-\beta|c|} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta|c|} L^2(\mathbb{R}^-; \mathbb{C} \phi_j)$$

are analytic in p , which implies that

$$\mathbf{R}_k(\alpha) := (\mathbf{H}_0 - \frac{Q^2 + p^2}{2})^{-1} \Pi_k : e^{-\beta|c|} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta|c|} H^2(\mathbb{R}^-; E_k) \cap H_0^1(\mathbb{R}^-; E_k)$$

admits an analytic extension in p to the region $\{p \in Z \mid \forall j \geq 0, \operatorname{Im}(\sqrt{p^2 - 2\lambda_j}) > -\beta\}$ of the ramified Riemann surface Z . By using the fact that $V^{1/2} \Pi_k \in \mathcal{L}(L^2(\Omega_{\mathbb{T}}))$ and $\mathbf{P}^{1/2} \Pi_k \in \mathcal{L}(L^2(\Omega_{\mathbb{T}}))$, we deduce that

$$\tilde{\chi} \mathbf{R}_k(\alpha) \chi : e^{-\beta|c|} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta|c|} (L^2(\mathbb{R}^-; E_k) \cap \mathcal{D}(\mathcal{Q}))$$

is bounded. We have (using $\Pi_k \mathbf{R}_k(\alpha) = \mathbf{R}_k(\alpha)$)

$$\begin{aligned} (\mathbf{H} - \frac{Q^2 + p^2}{2}) \tilde{\chi} \mathbf{R}_k(\alpha) \chi &= \Pi_k \chi - \frac{1}{2} [\partial_c^2, \tilde{\chi}] \Pi_k \mathbf{R}_k(\alpha) \chi + e^{\gamma c} V \Pi_k \tilde{\chi} \mathbf{R}_k(\alpha) \chi \\ &= \Pi_k \chi + \frac{1}{2} [\partial_c^2, \tilde{\chi}] \Pi_k \mathbf{R}_k(\alpha) \chi + e^{\gamma c} \Pi_k V \Pi_k \tilde{\chi} \mathbf{R}_k(\alpha) \chi + e^{\gamma c} \tilde{\chi} (1 - \Pi_k) V \Pi_k \tilde{\chi} \mathbf{R}_k(\alpha) \chi \\ &= \Pi_k \chi + \mathbf{K}_{k,1}(\alpha) + \mathbf{K}_{k,2}(\alpha) \end{aligned}$$

where $\mathbf{K}_{k,2}(\alpha) := e^{\gamma c} \tilde{\chi} (1 - \Pi_k) V \Pi_k \tilde{\chi} \mathbf{R}_k(\alpha) \chi$ satisfies $\Pi_k \mathbf{K}_{k,2}(\alpha) = 0$. The operator $[\partial_c^2, \tilde{\chi}] \Pi_k \mathbf{R}_k(\alpha) \chi$ is compact on $e^{-\beta|c|} L^2(\mathbb{R}; L^2(\Omega_{\mathbb{T}}))$ since $e^{\beta|c|} [\partial_c^2, \tilde{\chi}]$ is a compactly supported first order operator in c , $E_k = \operatorname{Im}(\Pi_k)$ is finite dimensional in $L^2(\Omega_{\mathbb{T}})$ and $\mathbf{R}_k(\alpha) : e^{-\beta|c|} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta|c|} H^2(\mathbb{R}^-; E_k)$ (this amounts to the compact injection $H^2([-1, 0]; E_k) \rightarrow H^1(\mathbb{R}^-; E_k)$). Moreover, the operator $e^{(\beta - \gamma/2)|c|} \Pi_k \tilde{\chi} \mathbf{R}_k(\alpha) \chi e^{-\beta|c|}$ is

also compact from $L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ to $L^2(\mathbb{R} \times E_k)$ by using the same type of argument as for proving the compact injection (3.50): indeed, one has the pointwise bound on its Schwartz kernel restricted to $\ker(\mathbf{P} - \lambda_j)$

$$\begin{aligned} |\kappa_j(c, c')| &\leq C e^{(\beta-\gamma/2)|c|-\beta|c'|} (e^{\operatorname{Im}(\sqrt{p^2-2\lambda_j})(|c|-|c'|)} + e^{-\operatorname{Im}(\sqrt{p^2-2\lambda_j})(|c|+|c'|)}) \mathbf{1}_{|c'| \geq |c|} \\ &\quad + C e^{(\beta-\gamma/2)|c'|-\beta|c|} (e^{\operatorname{Im}(\sqrt{p^2-2\lambda_j})(|c'|+|c|)} + e^{-\operatorname{Im}(\sqrt{p^2-2\lambda_j})(|c|+|c'|)}) \mathbf{1}_{|c| \geq |c'|}. \end{aligned}$$

We see that for $\operatorname{Im}(\sqrt{p^2-2\lambda_j}) \geq 0$, if $0 < \beta < \gamma/2$, this is bounded by $C \max(e^{(\beta-\gamma/2)|c|-\beta|c'|}, e^{(\beta-\gamma/2)|c'|-\beta|c|})$, and is thus the integral kernel of a compact operator on $L^2(\mathbb{R}^-)$ since it is Hilbert-Schmidt (the kernel being in $L^2(\mathbb{R} \times \mathbb{R})$). If now $\operatorname{Im}(\sqrt{p^2-2\lambda_j}) < 0$, the same argument shows that a sufficient condition to be compact is that

$$\operatorname{Im}\sqrt{p^2-2\lambda_j} > -\beta \text{ and } \operatorname{Im}\sqrt{p^2-2\lambda_j} > \beta - \gamma/2.$$

But the multiplication operator $V^{1/2} : L^2(\mathbb{R}; E_k) \rightarrow L^2(\mathbb{R}; \Omega_{\mathbb{T}})$ is bounded since $\|Vu\|_{L^2(\Omega_{\mathbb{T}})} \leq C_k \|u\|_{L^2(\Omega_{\mathbb{T}})}$ by using that the eigenmodes of \mathbf{P} are in $L^q(\Omega_{\mathbb{T}})$ for all $q < \infty$ and $V \in L^r(\Omega_{\mathbb{T}})$ for some $r > 1$. Since $e^{\gamma c/2} V^{1/2} : L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow \mathcal{D}'(\mathcal{Q})$ is bounded, we deduce that if $0 < \beta < \gamma/2$

$$e^{\gamma c} V \Pi_k \tilde{\chi} \mathbf{R}_k(\alpha) \chi : e^{\beta \rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta \rho} \mathcal{D}'(\mathcal{Q})$$

is a compact operator. We also note that, since $\Pi_k V^{1/2} : L^2(\Omega_{\mathbb{T}}) \rightarrow E_k$ is bounded,

$$e^{\gamma c} \Pi_k V \Pi_k \tilde{\chi} \mathbf{R}_k(\alpha) \chi : e^{\beta \rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta \rho} L^2(\mathbb{R}^-; E_k)$$

is compact, thus $\mathbf{K}_{k,1}(\alpha) : e^{\beta \rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta \rho} L^2(\mathbb{R}^-; E_k)$ is also compact. This proves 1).

Now, if $f \in e^{\beta \rho} L^2(\mathbb{R}; E_k)$ we have for $c \leq 0$ and writing $f = \sum_{j \leq k} f_j$ with $f_j \in \ker(\mathbf{P} - \lambda_j)$

$$\begin{aligned} (\mathbf{R}_k(\alpha) \chi f)(c) &= 2 \sum_{j \leq k} \frac{e^{-ic\sqrt{p^2-2\lambda_j}}}{\sqrt{p^2-2\lambda_j}} \int_{-\infty}^0 \sin(c'\sqrt{p^2-2\lambda_j}) \chi(c') f_j(c') dc' \\ (3.54) \quad &\quad + 2 \sum_{j \leq k} \frac{e^{-ic\sqrt{p^2-2\lambda_j}}}{\sqrt{p^2-2\lambda_j}} \int_{-\infty}^c \sin((c-c')\sqrt{p^2-2\lambda_j}) f_j(c') dc' \end{aligned}$$

and the term in the second line, denoted $G(c)$, satisfies for $c < -1$

$$\begin{aligned} \|G(c)\|_{L^2(\Omega_{\mathbb{T}})} &\leq 2 \sum_{j \leq k} \int_{-\infty}^c e^{|\operatorname{Im}\sqrt{p^2-2\lambda_j}|(c-c')+\beta c'} \|e^{-\beta \rho(c')} f_j(c')\|_{L^2(\Omega_{\mathbb{T}})} dc' \\ &\leq C_k \|e^{-\beta \rho} f\|_2 e^{-\beta|c|}. \end{aligned}$$

and the same bounds hold for $\|\partial_c G(c)\|_{L^2(\Omega_{\mathbb{T}})}$; this completes the proof of 2).

We remark that if $\operatorname{Im}(p) > 0$, then the operator

$$(\mathbf{H}_0 - \frac{Q^2+p^2}{2})^{-1} \Pi_k : e^{\beta|c|} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta|c|} L^2(\mathbb{R}^-; E_k)$$

is bounded and analytic in p , provided $0 < \beta < \min_{j \leq k} \operatorname{Im}(\sqrt{p^2-2\lambda_j}) = \operatorname{Im}(p)$. Indeed, this follows again from Schur's lemma applied to the Schwartz kernel

$$\mathbf{1}_{\{c \geq c'\}} e^{-\beta(|c|-|c'|)} \sin(c\sqrt{p^2-2\lambda_j}) e^{-ic'\sqrt{p^2-2\lambda_j}} + \mathbf{1}_{\{c' \geq c\}} e^{-\beta(|c'|+|c|)} \sin(c'\sqrt{p^2-2\lambda_j}) e^{-ic\sqrt{p^2-2\lambda_j}}.$$

Using again that $V^{\frac{1}{2}} \Pi_k : E_k \rightarrow L^2(\Omega_{\mathbb{T}})$ is bounded, this implies that

$$\mathbf{R}_k(\alpha) : e^{\beta|c|} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow e^{\beta|c|} (L^2(\mathbb{R}^-; E_k) \cap \mathcal{D}(\mathcal{Q}))$$

is an analytic bounded family in p in the region $0 < \beta < \operatorname{Im}(p)$. The same argument works with $0 < -\beta < \operatorname{Im}(p)$ in case $\beta < 0$. The operator $\mathbf{K}_{k,1}(\alpha) : e^{-\beta \rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta \rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ is compact by the same argument as above since it is Hilbert-Schmidt and $\mathbf{K}_{k,2}(\alpha) : e^{-\beta \rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta \rho} \mathcal{D}'(\mathcal{Q})$ is compact. \square

4) Proof of Proposition 3.8. For $\beta \in \mathbb{R}$, define the Hilbert space for fixed k

$$\mathcal{H}_{k,\beta} := e^{\beta \rho} L^2(\mathbb{R}; E_k) \oplus L^2(\mathbb{R}; E_k^\perp)$$

with scalar product

$$\langle f, f' \rangle_{\mathcal{H}_{k,\beta}} := \int_{\mathbb{R}} e^{-2\beta\rho(c)} \langle \Pi_k f, \Pi_k f' \rangle_{L^2(\Omega_{\mathbb{T}})} dc + \langle (1 - \Pi_k) f, (1 - \Pi_k) f' \rangle_{L^2(\mathbb{R} \times \Omega_{\mathbb{T}})}.$$

We now fix β and β' as in the statement of Proposition 3.8. We now use the operators of Lemma 3.9, Lemma 3.11 and Lemma 3.13: let $\chi; \tilde{\chi}, \hat{\chi}$ be the cutoff functions of these Lemmas and let $\check{\chi} \in C^\infty(\mathbb{R})$ equal to 1 on $\text{supp}(\tilde{\chi})$ and supported in \mathbb{R}^- . We define

$$\tilde{\mathbf{R}}(\alpha) := \tilde{\chi} \mathbf{R}_k^\perp(\pi(\alpha)) \chi + (1 - \hat{\chi}) \mathbf{R}_+(1 - \chi) + \tilde{\chi} \mathbf{R}_k(\alpha) \chi - \check{\chi} \mathbf{R}_k^\perp(\alpha) \mathbf{K}_{k,2}(\alpha)$$

which in $\{\alpha = Q + ip \in \Sigma \mid \forall j \leq k, \text{Im} \sqrt{p^2 - 2\lambda_j} > -\beta\}$ is bounded and holomorphic (in α) as a map $\tilde{\mathbf{R}}(\alpha) : \mathcal{H}_{k,\beta} \rightarrow \mathcal{H}_{k,-\beta} \cap e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$. It moreover satisfies the identity

$$\begin{aligned} (\mathbf{H} - 2\Delta_{\pi(\alpha)}) \tilde{\mathbf{R}}(\alpha) &= 1 + \mathbf{L}_k^\perp(\pi(\alpha)) + \mathbf{K}_k^\perp(\pi(\alpha)) + \mathbf{K}_{+,k}(\pi(\alpha)) + \mathbf{L}_{+,k}(\pi(\alpha)) + \mathbf{K}_{k,1}(\alpha) \\ &\quad - (\check{\mathbf{L}}_k^\perp(\pi(\alpha)) + \check{\mathbf{K}}_k^\perp(\pi(\alpha))) \mathbf{K}_{k,2}(\alpha) \end{aligned}$$

where $\check{\mathbf{L}}_k^\perp(\pi(\alpha))$ and $\check{\mathbf{K}}_k^\perp(\pi(\alpha))$ are the operators of Lemma 3.9 with $\tilde{\chi}$ (resp. χ) is replaced by $\check{\chi}$ (resp. $\tilde{\chi}$). Let us define

$$(3.55) \quad \tilde{\mathbf{K}}_k(\alpha) := \mathbf{K}_{k,1}(\alpha) + \mathbf{K}_{+,k}(\pi(\alpha)) + \mathbf{K}_k^\perp(\pi(\alpha)) - (\check{\mathbf{L}}_k^\perp(\pi(\alpha)) + \check{\mathbf{K}}_k^\perp(\pi(\alpha))) \mathbf{K}_{k,2}(\alpha).$$

By Lemma 3.9, Lemma 3.11 and Lemma 3.13, $\tilde{\mathbf{K}}_k(\alpha) : \mathcal{H}_{k,\beta} \rightarrow \mathcal{H}_{k,\beta}$ is compact and holomorphic in α (recall that $\mathbf{K}_{k,j}(\alpha)(1 - \Pi_k) = 0$). Let us also check that

$$\tilde{\mathbf{K}}_k(\alpha) : e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow \mathcal{H}_{k,\beta} \quad \text{and} \quad \tilde{\mathbf{R}}(\alpha) : e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$$

are bounded. First, since $e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2 \subset \mathcal{D}'(\mathcal{Q})$, Lemma 3.9 and Lemma 3.11 show that $\mathbf{K}_{+,k}(\pi(\alpha)) + \mathbf{K}_k^\perp(\pi(\alpha))$ is bounded as map $e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2 \rightarrow \mathcal{H}_{k,\beta}$ and that $\tilde{\chi} \mathbf{R}_k^\perp(\pi(\alpha)) \chi + (1 - \hat{\chi}) \mathbf{R}_+(1 - \chi)$ is bounded as map $e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2 \rightarrow \mathcal{D}(\mathcal{Q})$; second, $\Pi_k V^{\frac{1}{2}} : L^2(\Omega_{\mathbb{T}}) \rightarrow L^2(\Omega_{\mathbb{T}})$ is bounded thus $\mathbf{K}_{k,j}(\alpha) : e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow \mathcal{H}_{k,\beta}$ is bounded as well as $\tilde{\chi} \mathbf{R}_k(\alpha) \chi : e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathbf{H})$ (using Lemma 3.13) and $\check{\chi} \mathbf{R}_k^\perp(\alpha) \mathbf{K}_{k,2}(\alpha) : e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2 \rightarrow \mathcal{D}(\mathbf{H})$, finally proving the desired claim.

Now if $|\pi(\alpha) - Q| \leq \lambda_k^{1/4}$ and if k is large enough, the operator $\tilde{\mathbf{L}}_k(\alpha) := \mathbf{L}_k^\perp(\pi(\alpha)) + \mathbf{L}_{+,k}(\alpha)$ is bounded as map

$$(3.56) \quad \tilde{\mathbf{L}}_k(\alpha) : \mathcal{H}_{k,\beta} \rightarrow L^2(\mathbb{R}; E_k^\perp) \quad \tilde{\mathbf{L}}_k(\alpha) : \mathcal{D}'(\mathcal{Q}) \rightarrow \mathcal{H}_{k,\beta}$$

(and thus as a map $e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2 \rightarrow \mathcal{H}_{k,\beta}$) with holomorphic dependance in α , and with bound (recall (3.31) and (3.38))

$$\|\tilde{\mathbf{L}}_k(\alpha)^2\|_{\mathcal{H}_{k,\beta} \rightarrow L^2(\mathbb{R}; E_k^\perp)} < 1/2.$$

In particular, $(1 + \tilde{\mathbf{L}}_k(\alpha))(1 - \tilde{\mathbf{L}}_k(\alpha)) = 1 - \tilde{\mathbf{L}}_k(\alpha)^2$ is invertible on $\mathcal{H}_{k,\beta}$ with holomorphic inverse given by the Neumann series $\sum_{j=0}^{\infty} \tilde{\mathbf{L}}_k(\alpha)^{2j}$; we write $(1 + \mathbf{T}_k(\alpha)) := (1 - \tilde{\mathbf{L}}_k(\alpha))(1 - \tilde{\mathbf{L}}_k(\alpha)^2)^{-1}$, with $\mathbf{T}_k(\alpha)$ mapping boundedly $\mathcal{H}_{k,\beta} \rightarrow \mathcal{H}_{k,\beta}$. Moreover we have

$$(\mathbf{H} - 2\Delta_{\pi(\alpha)}) \tilde{\mathbf{R}}(\alpha) (1 + \mathbf{T}_k(\alpha)) = 1 + \tilde{\mathbf{K}}_k(\alpha) (1 + \mathbf{T}_k(\alpha))$$

and the remainder $\hat{\mathbf{K}}(\alpha) := \tilde{\mathbf{K}}_k(\alpha) (1 + \mathbf{T}_k(\alpha))$ is now compact on $\mathcal{H}_{k,\beta}$, and $1 + \hat{\mathbf{K}}(\alpha)$ is thus Fredholm of index 0.

Let $p_0 = iq$ for some $q \gg \beta$, the operator $\mathbf{H} - \frac{Q^2}{2}$ being self-adjoint on its domain $\mathcal{D}(\mathbf{H})$ and non-negative, $\mathbf{H} - \frac{Q^2 + p_0^2}{2}$ is invertible with inverse denoted $\mathbf{R}(\alpha_0)$ if $\alpha_0 = Q + ip_0 = Q - q$. Now, let $(\psi_j)_{j \leq J} \subset \mathcal{H}_{k,\beta}$ be an orthonormal basis of $\ker(1 + \hat{\mathbf{K}}(\alpha_0)^*)$, and $(\varphi_j)_{j \leq J} \subset \mathcal{H}_{k,\beta}$ an orthonormal basis of $\ker(1 + \hat{\mathbf{K}}(\alpha_0))$. For each j , there is $w_j \in \mathcal{D}(\mathbf{H})$ such that $(\mathbf{H} - \frac{Q^2 + p_0^2}{2}) w_j = \psi_j$. If $\theta \in C^\infty(\mathbb{R})$ equal 1 in $c \in (-\infty, -1)$ and is supported in $c \in \mathbb{R}^-$, we have in $\mathcal{D}'(\mathcal{Q})$

$$(3.57) \quad (\mathbf{H}_0 - \frac{Q^2 + p_0^2}{2}) \theta w_j = \theta \psi_j - \theta e^{\gamma c} V w_j - \frac{1}{2} [\partial_c^2, \theta] w_j$$

and this implies by projecting this relation on E_k with Π_k that, setting $\psi_{j,k} = \Pi_k \psi_j$ and $\psi_{j,k}^\perp = (1 - \Pi_k) \psi_j$, similarly $w_{j,k} = \Pi_k w_j$ and $w_{j,k}^\perp = (1 - \Pi_k) w_j$

$$(3.58) \quad (\mathbf{H}_0 - \frac{Q^2 + p_0^2}{2}) \theta w_{j,k} = \theta \psi_{j,k} - \theta e^{\gamma c} \Pi_k (V w_j) - \frac{1}{2} [\partial_c^2, \theta] w_{j,k}.$$

Due to the fact that $V \in L^p(\Omega_{\mathbb{T}})$ for some $p > 1$ and $e^{\gamma c/2} V^{1/2} w_j \in L^2$ since $w_j \in \mathcal{D}(\mathcal{Q})$, then $e^{\gamma c} \Pi_k (V w_j) \in e^{\gamma \rho/2} L^2$ (recall $\Pi_k : L^{1+\varepsilon}(\Omega_{\mathbb{T}}) \rightarrow E_k \subset L^2(\Omega_{\mathbb{T}})$ is bounded). Since $[\partial_c^2, \theta]$ is a first order differential operator with compact support, we get $[\partial_c^2, \theta] w_{j,k} \in L^2$. This shows in particular that $\theta w_{j,k} \in H^2(\mathbb{R}^-; E_k) \cap H_0^1(\mathbb{R}^-; E_k)$ and since E_k is finite dimensional, it is direct to check that

$$\mathbf{R}_k(Q + ip_0) (\mathbf{H}_0 - \frac{Q^2 + p_0^2}{2}) \theta w_{j,k} = \theta w_{j,k}$$

with $\mathbf{R}_k(Q + ip_0)$ the operator of Lemma 3.13. We obtain in the region $c \in \mathbb{R}^-$

$$\theta w_{j,k} = \mathbf{R}_k(Q + ip_0) (\theta \psi_{j,k} - \theta e^{\gamma c} \Pi_k (V w_j) - \frac{1}{2} [\partial_c^2, \theta] w_{j,k}).$$

Using the properties of $\mathbf{R}_k(\alpha)$ in Lemma 3.13 and in particular (3.53), we see that for each $0 < \beta < \gamma/2$

$$\theta w_{j,k} \in e^{\beta \rho} L^2(\mathbb{R}^-; E_k)$$

and thus we deduce that

$$w_j \in e^{\beta \rho} L^2(\mathbb{R}; E_k) \oplus L^2(\mathbb{R}; E_k^\perp) = \mathcal{H}_{k,\beta}.$$

If we consider the finite rank operator \mathbf{W} defined by $\mathbf{W}f := \sum_{j=1}^J w_j \langle f, \varphi_j \rangle_{\mathcal{H}_{k,\beta}}$ for $f \in \mathcal{H}_{k,\beta}$, we have as operators

$$(\mathbf{H} - \frac{Q^2 + p_0^2}{2}) \mathbf{W} = \mathbf{Y} \quad \text{with } \mathbf{Y}f := \sum_{j=1}^J \psi_j \langle f, \varphi_j \rangle_{\mathcal{H}_{k,\beta}}.$$

But now it is direct to check that $1 + \hat{\mathbf{K}}(\alpha_0) + \mathbf{Y}$ is invertible on $\mathcal{H}_{k,\beta}$ and we obtain that

$$(\mathbf{H} - 2\Delta_{\pi(\alpha)}) (\tilde{\mathbf{R}}(\alpha) (1 + \mathbf{T}_k(\alpha)) + \mathbf{W}) = 1 + \hat{\mathbf{K}}(\alpha) + \mathbf{Y}.$$

The remainder $\mathbf{K}(\alpha) := \hat{\mathbf{K}}(\alpha) + \mathbf{Y}$ is compact on $\mathcal{H}_{k,\beta}$, analytic in α in the desired region and $1 + \mathbf{K}(\alpha)$ is invertible for $\alpha = \alpha_0$, therefore we can apply the Fredholm analytic theorem to conclude that the family of operator $(1 + \mathbf{K}(\alpha))^{-1}$ exists as a meromorphic family of bounded operators on $\mathcal{H}_{k,\beta}$ for α in $\{\alpha = Q + ip \in \Sigma \mid |\pi(\alpha) - Q|^2 \leq \lambda_k^{1/4}, \forall j \leq k, \text{Im}\sqrt{p^2 - \lambda_j} > -\beta\}$ except on a discrete set of poles with finite rank polar part. We can thus set

$$(3.59) \quad \mathbf{R}(\alpha) := (\tilde{\mathbf{R}}(\alpha) (1 + \mathbf{T}_k(\alpha)) + \mathbf{W}) (1 + \mathbf{K}(\alpha))^{-1}$$

which satisfies the desired properties. To get boundedness $e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$, we write

$$\mathbf{R}(\alpha) = \tilde{\mathbf{R}}(\alpha) - \mathbf{R}(\alpha) (\tilde{\mathbf{K}}_k(\alpha) + \tilde{\mathbf{L}}_k(\alpha))$$

and we have seen that $\tilde{\mathbf{K}}_k(\alpha) + \tilde{\mathbf{L}}_k(\alpha) : e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow \mathcal{H}_{k,\beta}$ is bounded and $\tilde{\mathbf{R}}(\alpha) : e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$ is bounded. We can use the mapping properties of $\tilde{\mathbf{R}}(\alpha)$ and \mathbf{W} , together with (3.53) to deduce (3.28).

We finally need to prove that there is no pole in the half plane $\text{Re}(\alpha) \leq Q$ except possibly at the points $\alpha = Q \pm i\sqrt{2\lambda_j}$. First, by the spectral theorem, one has for each $f \in e^{\beta\rho} L^2 \subset L^2$ with $\beta > 0$ and each α satisfying $\text{Re}(\alpha) < Q$

$$\|\mathbf{R}(\alpha)f\|_{e^{\beta\rho} L^2} \leq C \|\mathbf{R}(\alpha)f\|_2 \leq \frac{C \|f\|_2}{|\text{Im}(\alpha)| \cdot |\text{Re}(\alpha) - Q|}$$

which implies that a pole $\alpha_0 = Q + ip_0$ with $p_0 \notin \{\pm\sqrt{2\lambda_j} \mid j \geq 0\}$ on $\text{Re}(\alpha) = Q$ must be at most of order 1, while at $p_0 = \pm\sqrt{2\lambda_j}$ it can be at most of order 2 on Σ . Since $\tilde{\mathbf{R}}(\alpha) (1 + \mathbf{T}_k(\alpha)) + \mathbf{W}$ is analytic, a pole of $\tilde{\mathbf{R}}(\alpha)$ can only come from a pole of $(1 + \mathbf{K}(\alpha))^{-1}$, with polar part being a finite rank operator. We now assume that $p_0 \notin \{\pm\sqrt{2\lambda_j} \mid j \geq 0\}$. Let us denote by \mathbf{Z} the finite rank residue $\mathbf{Z} = \text{Res}_{\alpha_0} \mathbf{R}(\alpha)$. Then

$(\mathbf{H} - \frac{Q^2 + p_0^2}{2})\mathbf{Z} = 0$, which means that each element in $\text{Ran}(\mathbf{Z})$ is a $w \in e^{-\beta\rho}\mathcal{D}(\mathbf{H})$ such that $(\mathbf{H} - \frac{Q^2 + p_0^2}{2})w = 0$. There are finite rank operators $\mathbf{Z}_0, \dots, \mathbf{Z}_N$ on $\mathcal{H}_{k,\beta}$ so that for $\psi \in C^\infty((-\infty, -2)_c; [0, 1])$

$$\psi\mathbf{Z} = \sum_{n=0}^N \psi \partial_\alpha^n \tilde{\mathbf{R}}(\alpha_0)\mathbf{Z}_n = \sum_{n=1}^N \psi \partial_\alpha^n \mathbf{R}_k(\alpha_0)\chi\mathbf{Z}_n + \psi \tilde{\mathbf{R}}(\alpha_0)\mathbf{Z}_0 + \psi\mathbf{Z}_{L^2}$$

where \mathbf{Z}_{L^2} is a finite rank operator mapping to $\mathcal{H}_{k,\beta} \subset L^2$. For $f \in \mathcal{H}_{k,\beta}$, the expression of $\partial_\alpha^j \mathbf{R}_k(\alpha_0)f$ is explicit from (3.54), and one directly checks by differentiating (3.54) in α that it is of the form (for $c < -2$)

$$(\partial_\alpha^n \tilde{\mathbf{R}}(\alpha_0)f)(c) = \sum_{\lambda_j \leq \lambda_k} \sum_{m \leq n} \tilde{a}_{j,m}(\alpha, f) c^m e^{-ic\sqrt{p_0^2 - 2\lambda_j}} + \tilde{G}(\alpha_0, f)$$

for some $\tilde{a}_{j,m}(\alpha, f) \in \ker(\mathbf{P} - \lambda_j)$ and $\tilde{\chi}\tilde{G}(\alpha_0, f) \in \mathcal{H}_{k,\beta}$ satisfying $\mathbf{H}\tilde{\chi}\tilde{G}(\alpha_0, f) \in \mathcal{H}_{k,\beta}$. This implies that, in $c < -2$, $w \in \text{Ran}(\mathbf{Z})$ is necessarily of the form

$$w = \sum_{j \leq k} \sum_{m \leq N} b_{j,m} c^m e^{-ic\sqrt{p_0^2 - 2\lambda_j}} + \hat{G}$$

for some $b_{j,m} \in \ker(\mathbf{P} - \lambda_j)$ and $\hat{G} \in \mathcal{H}_{k,\beta}$ with $\mathbf{H}\hat{G} \in \mathcal{H}_{k,\beta}$. Using that $\tilde{\chi}(c)e^{\gamma c}\Pi_k(Vw) \in e^{\beta c}L^2$, we see from the equation $(\mathbf{H}_0 - \frac{Q^2 + p_0^2}{2})\Pi_k(w) = -e^{\gamma c}\Pi_k(Vw)$ that

$$(\mathbf{H}_0 - \frac{Q^2 + p_0^2}{2}) \left(\sum_{\lambda_j \leq \lambda_k} \sum_{m \leq N} b_{j,m} c^m e^{-ic\sqrt{p_0^2 - 2\lambda_j}} \right) \Big|_{c \leq 0} \in e^{\beta c}L^2$$

and by using the explicit expression of \mathbf{H}_0 , it is clear that necessarily $b_{j,m} = 0$ for all $m \neq 0$. Then we may apply Lemma 3.15 with $u_1 = u_2 = w$ to deduce that $b_{j,0} = 0$, and therefore $w \in \mathcal{D}(\mathbf{H})$, which implies $w = 0$ by Lemma 3.1.

It remains to show that $\alpha = Q \pm i\sqrt{2\lambda_j}$ is a pole of order at most 1. To simplify, we write the argument for $\alpha = Q$, the proof is the same for all j . The method is basically the same as in the proof of [Me93, Proposition 6.28]: the resolvent has Laurent expansion $\mathbf{R}(\alpha) = (\alpha - Q)^{-2}\mathbf{Q} + (\alpha - Q)^{-1}\mathbf{R}'(\alpha)$ for some holomorphic operator $\mathbf{R}'(\alpha)$ near $\alpha = Q$ and \mathbf{Q} has finite rank, then we also have $\|\mathbf{R}(\alpha)\phi\|_{L^2} \leq |Q - \alpha|^{-2}$ for $\alpha < Q$ and all $\phi \in \mathcal{C}$, thus we can deduce that

$$\mathbf{Q}\phi = \lim_{\alpha \rightarrow Q^-} (\alpha - Q)^2 \mathbf{R}(\alpha)\phi.$$

The limit holds in $e^{-\delta\rho}L^2$ for all $\delta > 0$ small, but the right hand side has actually a bounded L^2 -norm, so $\mathbf{Q}\phi \in L^2$ and thus $\text{Ran}(\mathbf{Q}) \subset L^2$. Since we also have $\mathbf{H}\mathbf{Q} = 0$ from Laurent expanding $(\mathbf{H} - 2\Delta_\alpha)\mathbf{R}(\alpha) = \text{Id}$ at $\alpha = Q$, we conclude that $\mathbf{Q} = 0$ by using Lemma 3.1. \square

The resolvent in the physical sheet on weighted spaces. We shall conclude this section on the resolvent of \mathbf{H} by analyzing its boundedness on weighted spaces $e^{-\beta\rho}L^2$ in the half-plane $\{\text{Re}(\alpha) < Q\}$. We recall that Lemma 3.3 was precisely proving such boundedness but the region of validity in α of this Lemma was not covering the whole physical-sheet, and in particular not the region close to the line $\text{Re}(\alpha) = Q$. Just as in Lemma 3.5, the main application of such boundedness on weighted spaces is to define the Poisson operator $\mathcal{P}_\ell(\alpha)$, and we aim to define it in a large connected region of $\{\text{Re}(\alpha) \leq Q\}$ relating the probabilistic region and the line $\alpha \in Q + i\mathbb{R}$ corresponding to the L^2 -spectrum of \mathbf{H} .

Proposition 3.14. *Let $\beta \in \mathbb{R}$ and $\text{Re}(\alpha) < Q$, then the resolvent $\mathbf{R}(\alpha)$ of \mathbf{H} extends as an analytic family of bounded operators*

$$\mathbf{R}(\alpha) : e^{-\beta\rho}L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho}\mathcal{D}(\mathcal{Q}) \quad \mathbf{R}(\alpha) : e^{(\frac{\gamma}{2} - \beta)\rho}V^{\frac{1}{2}}L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho}\mathcal{D}(\mathcal{Q})$$

in the region $\text{Re}(\alpha) < Q - |\beta|$.

Proof. We proceed as in the proof of Proposition 3.8: we let for $\text{Re}(\alpha) < Q$

$$\tilde{\mathbf{R}}(\alpha) := \tilde{\chi}\mathbf{R}_k^+(\alpha)\chi + (1 - \hat{\chi})\mathbf{R}_+(1 - \chi) + \tilde{\chi}\mathbf{R}_k(\alpha)\chi - \tilde{\chi}\mathbf{R}_k^+(\alpha)\mathbf{K}_{k,2}(\alpha)$$

and we get

$$(\mathbf{H} - \frac{Q^2 + p^2}{2})\tilde{\mathbf{R}}(\alpha) = \text{Id} + \tilde{\mathbf{K}}_k(\lambda) + \tilde{\mathbf{L}}_k(\alpha)$$

where we used the operators of the proof of Proposition 3.8 (see (3.55) and (3.56)). Now we take $\operatorname{Re}(\alpha) < Q$ and $|Q - \alpha| < A$ for some fixed constant $A > 0$ that can be chosen arbitrarily large, and we let $k > 0$ large enough so that

$$(3.60) \quad A^2 + 1 < \min\left(\frac{\lambda_k^{1/2}}{16(1+C_2)}, \lambda_k^{1/4}\right), \quad \lambda_k > (16^2 C_1^2 (1+C_2^2) + 1)(|\beta| + 1)^2,$$

where the constant C_1, C_2 above are the constants respectively given in Lemma 3.9 and Lemma 3.11. The conditions in (3.60) ensures both the condition $\operatorname{Re}((\alpha - Q)^2) > \beta^2 - 2\lambda_k + 1$ of Lemma 3.9 is satisfied and the operator $\widehat{\chi} \mathbf{R}_k^\perp(\alpha) \chi : e^{-\beta\rho} \mathcal{D}'(\mathcal{Q}) \rightarrow e^{-\beta\rho} (L^2(\mathbb{R}^-; E_k^\perp) \cap \mathcal{D}(\mathcal{Q}))$ is a bounded holomorphic family, and the norm estimate appearing in 2) of Lemma 3.9 gives

$$(3.61) \quad \|\mathbf{L}_k^\perp(\alpha)\|_{\mathcal{L}(e^{-\beta\rho} L^2)} \leq \frac{C_1(1+|\beta|)}{\sqrt{\operatorname{Re}((\alpha - Q)^2) + 2\lambda_k - \beta^2}} \leq \frac{1}{16(1+C_2)}.$$

The condition $|\beta| < Q - \operatorname{Re}(\alpha)$ (equivalent to $\operatorname{Im}(p) > |\beta|$) makes sure that we can apply 3) of Lemma 3.13: in particular the operator $\mathbf{R}_k(\alpha) : e^{-\beta\rho} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} (L^2(\mathbb{R}^-; E_k) \cap \mathcal{D}(\mathcal{Q}))$ is a bounded holomorphic family. Also, Lemma 3.11 ensures that $(1 - \widehat{\chi}) \mathbf{R}_+(1 - \chi) : e^{-\beta\rho} \mathcal{D}'(\mathcal{Q}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$ is a bounded holomorphic family (note that both cutoff functions $(1 - \widehat{\chi})$ and $(1 - \chi)$ kill the $c \rightarrow -\infty$ behaviour and this is why Lemma 3.11 extends to weighted spaces $e^{-\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$). Also the first condition in (3.60) ensures the norm estimate (as given by (3.38))

$$(3.62) \quad \|\mathbf{L}_{+,k}(\alpha)\|_{\mathcal{L}(L^2)} \leq 2C_2, \quad \|\mathbf{L}_{+,k}(\alpha)^2\|_{\mathcal{L}(L^2)} \leq \frac{1}{8}.$$

As a consequence

$$\widetilde{\mathbf{R}}(\alpha) : e^{-\beta\rho} L^2 \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$$

is bounded and holomorphic in $U := \{\alpha \in \mathbb{C} \mid |Q - \alpha| < A, \operatorname{Re}(\alpha) < Q - |\beta|\}$. Furthermore (3.61) and (3.62) provide the estimate

$$\|(\mathbf{L}_{+,k}(\alpha) + \mathbf{L}_k^\perp(\alpha))^2\|_{\mathcal{L}(e^{-\beta\rho} L^2)} < 1/2.$$

Moreover, 2) of Lemma 3.9, Lemma 3.11 and 3) of Lemma 3.13 also give that $\widetilde{\mathbf{K}}_k(\alpha)$ is compact on the Hilbert space $e^{-\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$. Exactly the same argument as in the proof of Proposition 3.8 gives that

$$\left(\mathbf{H} - \frac{Q^2 + p^2}{2}\right) \widetilde{\mathbf{R}}(\alpha) (1 + \mathbf{T}_k(\alpha)) = 1 + \widetilde{\mathbf{K}}_k(\alpha) (1 + \mathbf{T}_k(\alpha))$$

for some \mathbf{T}_k bounded holomorphic on $e^{-\beta\rho} L^2$ in U . Since by Lemma 3.3 we know that $(\mathbf{H} - 2\Delta_\alpha)$ is invertible on $e^{-\beta\rho} L^2$ for some $\alpha_0 \in U$, one can always add a finite rank operator $\mathbf{W} : e^{-\beta\rho} L^2 \rightarrow e^{-\beta\rho} \mathcal{D}(\mathbf{H})$, so that

$$(\mathbf{H} - 2\Delta_\alpha)(\mathbf{R}(\alpha)(1 + \mathbf{T}_k(\alpha)) + \mathbf{W}) = 1 + \mathbf{K}(\alpha)$$

for some compact remainder $\mathbf{K}(\alpha)$ on $e^{-\beta\rho} L^2$, analytic in $\alpha \in U$ in the desired region and $1 + \mathbf{K}(\alpha)$ being invertible for $\alpha = \alpha_0 \in U$. This implies by analytic Fredholm theorem that

$$\mathbf{R}(\alpha) = (\widetilde{\mathbf{R}}(\alpha)(1 + \mathbf{T}_k(\alpha)) + \mathbf{W})(1 + \mathbf{K}(\alpha))^{-1} : e^{-\beta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$$

is meromorphic for $\alpha \in U$. Now, using the density of the embeddings $e^{|\beta|\rho} L^2 \subset L^2 \subset e^{-|\beta|\rho} L^2$ and using that $\mathbf{R}(\alpha)$ is holomorphic in U as a bounded operator on L^2 , it is direct to check that $\mathbf{R}(\alpha) : e^{-\beta\rho} L^2 \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$ is analytic in U . Since A (and thus U) can be chosen arbitrarily large as long as the constraint $\operatorname{Re}(\alpha) < Q - |\beta|$ is satisfied, we obtain our desired result. To prove that $\mathbf{R}(\alpha)$ maps $e^{(\frac{\gamma}{2} - \beta)\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ to $e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$, we proceed as in the proof of Proposition 3.8 and write

$$\mathbf{R}(\alpha) = \widetilde{\mathbf{R}}(\alpha) - \mathbf{R}(\alpha)(\widetilde{\mathbf{K}}_k(\alpha) + \widetilde{\mathbf{L}}_k(\alpha)).$$

Using one more time that $\Pi_k e^{\frac{\gamma}{2}\rho} V^{\frac{1}{2}} \in \mathcal{L}(L^2(\mathbb{R} \times \Omega_{\mathbb{T}}))$, we get that

$$\mathbf{R}_k(\alpha) : e^{(\frac{\gamma}{2} - \beta)\rho} V^{\frac{1}{2}} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} (L^2(\mathbb{R}^-; E_k) \cap \mathcal{D}(\mathcal{Q}))$$

$$\mathbf{K}_{k,2}(\alpha) : e^{(\frac{\gamma}{2} - \beta)\rho} V^{\frac{1}{2}} L^2(\mathbb{R}^- \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$$

are bounded, and thus $\tilde{\mathbf{R}}(\alpha) : e^{(\frac{\gamma}{2}-\beta)\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$ is bounded by combining with the already described mapping properties of $\mathbf{R}_k^+(\alpha)$ and $\mathbf{R}_+(\alpha)$. The same arguments (just as in the proof of Proposition 3.8) also prove that that operators $\tilde{\mathbf{K}}_k(\alpha), \tilde{\mathbf{L}}_k(\alpha)$ are bounded as operators $e^{(\frac{\gamma}{2}-\beta)\rho} V^{\frac{1}{2}} L^2 \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$, thus we obtain that $\mathbf{R}(\alpha) : e^{(\frac{\gamma}{2}-\beta)\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) \rightarrow e^{-\beta\rho} \mathcal{D}(\mathcal{Q})$ is bounded. \square

3.2. The Poisson operator. We have seen in Lemma 3.5 that it is possible to construct a family of Poisson operators $\mathcal{P}_\ell(\alpha)$ in what we called the *probabilistic region*, which contains a half line $(-\infty, Q - c_\ell)$ for some $c_\ell \geq 0$ depending on ℓ . The construction was using the resolvent acting on weighted L^2 -spaces. In this section, we will use Proposition 3.8 and Proposition 3.14 to prove that the Poisson operators extend holomorphically in α in a connected region of $\text{Re}(\alpha) \leq Q$ containing the probabilistic region and the line $Q + i\mathbb{R}$.

Regime close to the continuous spectrum of H . We first start with a technical lemma that allows to define the Poisson operator on the continuous spectrum $Q + i\mathbb{R}$:

Lemma 3.15. *Let $p \in \mathbb{R}$ and for $m = 1, 2$, let $u_m \in e^{-\delta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ with $\delta > 0$ such that:*

- 1) *for each $\theta \in C^\infty(\mathbb{R}; [0, 1])$ supported in $(a, +\infty)$ for some $a \in \mathbb{R}$ then $\theta u_m \in \mathcal{D}(\mathcal{Q})$*
- 2) *u_m satisfies*

$$(\mathbf{H} - \frac{Q^2 + p^2}{2}) u_m = r_m \in e^{\delta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}).$$

Set $k = \max\{k \geq 0 \mid 2\lambda_k \leq p^2\}$. Then u_m has asymptotic behaviour

$$(3.63) \quad u_m = \sum_{j, 2\lambda_j \leq p^2} \left(a_m^j e^{-ic\sqrt{p^2 - 2\lambda_j}} + b_m^j e^{ic\sqrt{p^2 - 2\lambda_j}} \right) + G_m$$

with $a_m^j, b_m^j \in \ker(\mathbf{P} - \lambda_j)$, and both $G_m, \partial_c G_m \in e^{\delta\rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) + L^2(\Omega_{\mathbb{T}}; E_k^\perp)$. Then we have

$$\langle u_1 \mid r_2 \rangle - \langle r_1 \mid u_2 \rangle = i \sum_{j, 2\lambda_j \leq p^2} \sqrt{p^2 - 2\lambda_j} \left(\langle a_1^j \mid a_2^j \rangle_{L^2(\Omega_{\mathbb{T}})} - \langle b_1^j \mid b_2^j \rangle_{L^2(\Omega_{\mathbb{T}})} \right).$$

Proof. Let $\theta \in C^\infty(\mathbb{R})$ be non-negative satisfying $\theta_T = 1$ on $[-T, \infty)$ and $\text{supp}(\theta_T) \subset [-T - \varepsilon, \infty)$ where $T > 0$ is a large parameter and $\varepsilon > 0$ small, and let $\tilde{\theta}_T(\cdot) = \theta_T(\cdot + 1)$. In particular we have $\theta_T \theta_T = \theta_T$. First, $\tilde{\theta}_T u_m \in H^1(\mathbb{R}; L^2(\Omega_{\mathbb{T}}))$ satisfies

$$(\mathbf{H} - \frac{Q^2 + p^2}{2})(\tilde{\theta}_T u_m) = \tilde{\theta}_T r_m - \frac{1}{2} [\partial_c^2, \tilde{\theta}_T] u_m \in L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$$

thus $\tilde{\theta}_T u_m \in \mathcal{D}(\mathbf{H})$ (we used that $[\partial_c^2, \tilde{\theta}_T]$ is a first order differential operator with compactly supported coefficients). This implies, using $[\mathbf{H}, \tilde{\theta}_T] \theta_T = 0 = \theta_T [\mathbf{H}, \tilde{\theta}_T]$, that

$$(3.64) \quad \begin{aligned} \langle u_1, r_2 \rangle - \langle r_1, u_2 \rangle &= \lim_{T \rightarrow \infty} \langle \theta_T u_1, \mathbf{H}(\tilde{\theta}_T u_2) \rangle - \langle \mathbf{H}(\tilde{\theta}_T u_1), \theta_T u_2 \rangle \\ &= - \lim_{T \rightarrow \infty} \frac{1}{2} \langle [\partial_c^2, \theta_T] u_1, u_2 \rangle. \end{aligned}$$

We write $u_m = u_m^0 + G_m$ by using (3.63). Then we claim that, as $T \rightarrow \infty$,

$$|\langle [\partial_c^2, \theta_T] u_1^0, G_2 \rangle| + |\langle [\partial_c^2, \theta_T] G_1, u_2 \rangle| \rightarrow 0.$$

Indeed, we have $(|u_1^0| + |\partial_c u_1^0|) G_2 \in L^1(\mathbb{R} \times \Omega_{\mathbb{T}})$ and $(|G_1| + |\partial_c G_1|) u_2 \in L^1(\mathbb{R} \times \Omega_{\mathbb{T}})$, and the support of $[\partial_c^2, \theta_T]$ is contained in $[-T - 1, -T]$. We are left in (3.64) to study the limit of $\langle [\partial_c^2, \theta_T] u_1^0, u_2^0 \rangle$. But now we have $[\partial_c^2, \theta_T] u_1^0 = \theta_T'' u_1^0 + 2\theta_T' \partial_c u_1^0$ and for fixed $T > 0$ it is direct to check, using integration by parts and the fact that $(\mathbf{H}_0 - \frac{Q^2 + p^2}{2}) u_m^0 = 0$ that

$$\begin{aligned} \langle [\partial_c^2, \theta_T] u_1^0, u_2^0 \rangle &= \int_{-T-1}^{-T} \partial_c (\theta_T \langle \partial_c u_1^0, u_2^0 \rangle_{L^2(\Omega_{\mathbb{T}})} - \theta_T \langle u_1^0, \partial_c u_2^0 \rangle_{L^2(\Omega_{\mathbb{T}})}) dc \\ &= \langle \partial_c u_1^0(-T), u_2^0(-T) \rangle_{L^2(\Omega_{\mathbb{T}})} - \langle u_1^0(-T), \partial_c u_2^0(-T) \rangle_{L^2(\Omega_{\mathbb{T}})}. \end{aligned}$$

A direct computation gives that this is equal to

$$2i \sum_{j, \lambda_j \leq p^2} \sqrt{p^2 - 2\lambda_j} \left(\langle b_1^j, b_2^j \rangle_{L^2(\Omega_{\mathbb{T}})} - \langle a_1^j, a_2^j \rangle_{L^2(\Omega_{\mathbb{T}})} \right).$$

This completes the proof. \square

Now we extend the construction of the Poisson operator (3.22) in a neighborhood of the line spectrum $\alpha \in Q + i\mathbb{R}$.

Proposition 3.16. *Let $0 < \beta < \gamma/2$ and $\ell \in \mathbb{N}$. Then there is an analytic family of operators $\mathcal{P}_\ell(\alpha)$*

$$\mathcal{P}_\ell(\alpha) : E_\ell \rightarrow e^{-\beta\rho}\mathcal{D}(\mathcal{Q})$$

in the region

$$(3.65) \quad \{\alpha \in \mathbb{C} \mid \operatorname{Re}(\alpha) < Q, \operatorname{Im}\sqrt{p^2 - 2\lambda_\ell} < \beta\} \cup \{\alpha = Q + ip \mid p \in \mathbb{R} \setminus [-\sqrt{2\lambda_\ell}, \sqrt{2\lambda_\ell}]\},$$

satisfying $(\mathbf{H} - \frac{Q^2+p^2}{2})\mathcal{P}_\ell(\alpha)F = 0$ and

$$(3.66) \quad \mathcal{P}_\ell(\alpha)F = \sum_{j \leq \ell} \left(F_j^- e^{ic\sqrt{p^2-2\lambda_j}} + F_j^+(\alpha) e^{-ic\sqrt{p^2-2\lambda_j}} \right) + G_\ell(\alpha, F)$$

with $F_j^- = \Pi_{\ker(\mathbf{P}-\lambda_j)}F$, $F_j^+(\alpha) \in \ker(\mathbf{P}-\lambda_j)$, and $G_\ell(\alpha, F), \partial_c G_\ell(\alpha, F) \in e^{\frac{\beta}{2}\rho(c)}L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) + L^2(\mathbb{R}; E_\ell^\perp)$. Moreover, for each $\theta \in C_c^\infty(\mathbb{R})$, one has $\theta\mathcal{P}_\ell(\alpha)F \in \mathcal{D}(\mathbf{H})$. Such a solution $u \in e^{-\frac{\beta}{2}\rho}L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ to the equation $(\mathbf{H} - \frac{Q^2+p^2}{2})u = 0$ with the asymptotic expansion (3.66) is unique. The operator $\mathcal{P}_\ell(\alpha)$ admits a meromorphic extension to the region

$$(3.67) \quad \left\{ \alpha = Q + ip \in \Sigma \mid \forall j \in \mathbb{N}_0, \operatorname{Im}\sqrt{p^2 - 2\lambda_j} \in (\beta/2 - \gamma, \beta/2) \right\}$$

and $\mathcal{P}_\ell(\alpha)F$ satisfies (3.66) in that region. Finally, $F_j^+(\alpha)$ depends meromorphically on α in the region above.

Proof. We start by setting $u_-(\alpha) := \sum_{j=0}^\ell F_j^- e^{ic\sqrt{p^2-2\lambda_j}}$, and let $\chi \in C^\infty(\mathbb{R})$ equal to 1 in $(-\infty, -1)$ and with $\operatorname{supp}(\chi) \subset \mathbb{R}^-$. We get

$$(\mathbf{H} - \frac{Q^2+p^2}{2})(\chi u_-(\alpha)) = -\frac{1}{2}\chi''(c)u_-(\alpha) - \chi'(c)\partial_c u_-(\alpha) + e^{\gamma c}V\chi u_-(\alpha).$$

The first two terms are in $e^{N\rho}L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for all N , the term $e^{\gamma c}V\chi u_-(\alpha)$ belongs to $e^{\frac{\gamma}{2}\rho}V^{\frac{1}{2}}L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ if $\operatorname{Im}(\sqrt{p^2 - 2\lambda_j}) \leq \gamma/2$ by using $V^{1/2}F_j^- \in L^2(\Omega_{\mathbb{T}})$. We can thus define, using Proposition 3.8 (with $k \gg \ell$ large enough), if $\operatorname{Im}(\sqrt{p^2 - 2\lambda_j}) \in (-\min(\beta, \frac{\gamma}{2} - \beta), \gamma/2)$ for all $j \leq k$ and $|\pi(p)|^2 \leq \lambda_k^{1/4}$

$$u_+(\alpha) := \mathbf{R}(\alpha)(\mathbf{H} - \frac{Q^2+p^2}{2})(\chi u_-(\alpha)) \in e^{-\beta\rho}\mathcal{D}(\mathcal{Q})$$

so that $u(\alpha) := \chi u_-(\alpha) - u_+(\alpha)$ solves $(\mathbf{H} - \frac{Q^2+p^2}{2})u(\alpha) = 0$ in $e^{-\frac{\gamma}{2}\rho}\mathcal{D}(\mathcal{Q})$. We use Proposition 3.8 with $k \gg \ell$ large enough, and we see that $u_+(\alpha)$ is of the form, in $c \leq 0$,

$$u_+(\alpha) = \sum_{j \leq k} \tilde{a}_j(\alpha, F) e^{-ic\sqrt{p^2-2\lambda_j}} + G(\alpha, F) = \sum_{j \leq \ell} \tilde{a}_j(\alpha, F) e^{-ic\sqrt{p^2-2\lambda_j}} + G_\ell(\alpha, F)$$

with $\tilde{a}_j(\alpha, F) \in \ker(\mathbf{P}-\lambda_j)$ and $G(\alpha, F), \partial_c G(\alpha, F) \in e^{\beta\rho}L^2(\mathbb{R}_- \times \Omega_{\mathbb{T}}) + L^2(\mathbb{R}_-; E_k^\perp)$ and $G_\ell(\alpha, F) \in e^{\beta\rho}L^2(\mathbb{R}_- \times \Omega_{\mathbb{T}}) + L^2(\mathbb{R}_-; E_\ell^\perp)$ if $\operatorname{Re}(\alpha) < Q$. Here we have used the fact that $\operatorname{Im}\sqrt{p^2 - 2\lambda_j} > 0$ (since either $\operatorname{Re}(\alpha) < Q$ or $p^2 < 2\lambda_j$ for all $j > \ell$ if $\alpha = Q + ip$ with $p \in \mathbb{R} \setminus [-2\sqrt{2\lambda_\ell}, \sqrt{2\lambda_\ell}]$) to place all terms corresponding to all $\ell < j \leq k$, which belong to $L^2(\mathbb{R}; E_\ell^\perp)$, in the remainder term $G_\ell(\alpha, F)$. This shows that

$$(3.68) \quad \mathcal{P}_\ell(\alpha)F := u(\alpha) = \chi(c) \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2-2\lambda_j}} - \mathbf{R}(\alpha)(\mathbf{H} - 2\Delta_\alpha) \left(\chi(c) \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2-2\lambda_j}} \right)$$

satisfies all the required properties. The analyticity in α except possibly at the points $Q \pm i\sqrt{2\lambda_j}$ for $j \in \mathbb{N}_0$ follows from Proposition 3.8, in particular 3) of that Proposition. At the points $Q \pm i\sqrt{2\lambda_j}$, the analyticity is a consequence of the Lemma 3.17, in particular (3.70) and the fact that $Q \pm i\sqrt{2\lambda_j}$ is at most a pole of order 1 of $\mathbf{R}(\alpha)$ and $a_j(\alpha, \varphi)$. We notice that the expression of $\mathcal{P}_\ell(\alpha)$ is the same as in (3.22), thus when the regions of α considered in Lemma 3.5 and here have an intersection, then this corresponds to the same operator, by analytic continuation.

The uniqueness of the solution with such an asymptotic is direct if $\operatorname{Re}(\alpha) < Q$: the difference of two such solutions would be in $\mathcal{D}(\mathcal{Q})$ and the operator \mathbf{H} has no L^2 eigenvalues (Lemma 3.1), hence the difference is

identically 0. For the case $\alpha = Q + ip$ with $p \in \mathbb{R}$, denote by $\hat{u}(\alpha)$ the difference of two such solutions. Then $\hat{u}(\alpha)$ can be written under the form

$$\hat{u}(\alpha) = \sum_{j \leq \ell} \hat{F}_j^+(\alpha) e^{-ic\sqrt{p^2-2\lambda_j}} + \hat{G}_\ell(\alpha, F)$$

where $\hat{F}_j^+(\alpha) \in \ker(\mathbf{P} - \lambda_j)$ and $\hat{G}_\ell(\alpha, F) \in e^{\beta\rho(c)}L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) + L^2(\mathbb{R}; E_\ell^\perp)$. We can split the sum above as $\sum_{j, 2\lambda_j \leq p^2} \dots + \sum_{j, p^2 < j \leq 2\lambda_\ell} \dots$. The sum $\sum_{j, p^2 < j \leq 2\lambda_\ell} \dots$ belongs to some $e^{\delta\rho}L^2$ as well as its ∂_c derivative. We can use Lemma 3.15 to see that $\sum_{j, 2\lambda_j \leq p^2} \|\hat{F}_j^+(\alpha)\|_{L^2(\Omega_{\mathbb{T}})}^2 = 0$, hence again $\hat{u}(\alpha) \in L^2$ and we can conclude as previously.

The meromorphic extension of $\mathcal{P}_\ell(\alpha)$ is a direct consequence of the meromorphic extension of $\mathbf{R}(\alpha)$ in Proposition 3.8. \square

We notice that for $\alpha = Q + ip$ with $p \in \mathbb{R}$, the function $\overline{\mathcal{P}(\bar{\alpha})F}$ is another solution of $(\mathbf{H} - \frac{Q^2+p^2}{2})u = 0$ satisfying

$$\overline{\mathcal{P}_\ell(\bar{\alpha})F} = \sum_{j \leq \ell} \left(\overline{F_j^-} e^{-ic\sqrt{p^2-2\lambda_j}} + \overline{F_j^+(\bar{\alpha})} e^{ic\sqrt{p^2-2\lambda_j}} \right) + \overline{G_\ell(\bar{\alpha}, F)}.$$

This implies that for each $F = \sum_{j \leq \ell} F_j^- \in E_\ell$, there is a unique solution $u = \widehat{\mathcal{P}}_\ell(\alpha)F$ to $(\mathbf{H} - \frac{Q^2+p^2}{2})u = 0$ of the form

$$(3.69) \quad \widehat{\mathcal{P}}_\ell(\alpha)F = \sum_{j \leq \ell} \left(F_j^- e^{-ic\sqrt{p^2-2\lambda_j}} + \widehat{F}_j^+(\alpha) e^{ic\sqrt{p^2-2\lambda_j}} \right) + \widehat{G}_\ell(\alpha, F)$$

with $\widehat{G}_\ell(\alpha, F) \in e^{\frac{\beta}{2}\rho(c)}L^2(\mathbb{R} \times \Omega_{\mathbb{T}}) + L^2(\mathbb{R}; E_\ell^\perp)$ and $\widehat{F}_j^+ \in \mathbb{C}\rho_j$, and $\widehat{\mathcal{P}}_\ell(\alpha)$ extends meromorphically on an open set of Σ just like $\mathcal{P}_\ell(\alpha)$.

Lemma 3.17. *Let $\ell \in \mathbb{N}$, $0 < \beta < \gamma$ and α in (3.65), the Poisson operator $\mathcal{P}_\ell(\alpha)$ can be obtained from the resolvent as follows: for $F = \sum_{j \leq \ell} F_j^- \in E_\ell$ and $\varphi \in e^{\beta\rho}L^2$*

$$(3.70) \quad \langle \mathcal{P}_\ell(\alpha)F, \varphi \rangle_2 = i \sum_{j \leq \ell} \sqrt{p^2 - 2\lambda_j} \left\langle F_j^-, a_j(\bar{\alpha}, \varphi) \right\rangle_{L^2(\Omega_{\mathbb{T}})}$$

where $a_j(\alpha, \varphi)$ are the functionals obtained from (3.28), holomorphic in α and linear in φ .

Proof. Let $\alpha = Q + ip$ with $p \in \mathbb{R} \setminus ([-\sqrt{2\lambda_\ell}, \sqrt{2\lambda_\ell}] \cup_{j \geq \ell} \{\pm\sqrt{2\lambda_j}\})$ and let us take $F = \sum_{j \leq \ell} F_j^-$ with $F_j^- \in \text{Ker}(\mathbf{P} - \lambda_j)$ for $j \leq \ell$. Then from the construction of $\mathcal{P}_\ell(\alpha)F$ (with a function $\chi = \chi(c)$) in the proof of Proposition 3.16

$$\begin{aligned} \langle \mathcal{P}_\ell(\alpha)F, \varphi \rangle &= \left\langle \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2-2\lambda_j}} \chi, \varphi \right\rangle - \left\langle \mathbf{R}(\alpha)(\mathbf{H} - 2\Delta_\alpha) \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2-2\lambda_j}} \chi, \varphi \right\rangle \\ &= \left\langle \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2-2\lambda_j}} \chi, \varphi \right\rangle - \left\langle (\mathbf{H} - 2\Delta_\alpha) \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2-2\lambda_j}} \chi, \mathbf{R}(\bar{\alpha})\varphi \right\rangle. \end{aligned}$$

Here we used $\mathbf{R}(\alpha)^* = \mathbf{R}(\bar{\alpha}) = \mathbf{R}(2Q - \alpha)$. Let θ_T be as in the proof of Lemma 3.15. We have

$$\begin{aligned} &\lim_{T \rightarrow \infty} \left\langle \theta_T(c)(\mathbf{H} - 2\Delta_\alpha) \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2-2\lambda_j}} \chi, \mathbf{R}(\bar{\alpha})\varphi \right\rangle \\ &= \left\langle \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2-2\lambda_j}} \chi, \varphi \right\rangle - \frac{1}{2} \lim_{T \rightarrow \infty} \left\langle \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2-2\lambda_j}} \chi, \theta_T'' \mathbf{R}(\bar{\alpha})\varphi \right\rangle \\ &\quad - \lim_{T \rightarrow \infty} \left\langle \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2-2\lambda_j}} \chi, \theta_T' \partial_c \mathbf{R}(\bar{\alpha})\varphi \right\rangle. \end{aligned}$$

Using now the asymptotic form (3.28), the last two limits above can be rewritten as

$$\begin{aligned} &\lim_{T \rightarrow \infty} \left\langle \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2-2\lambda_j}} \chi, \theta_T'' \mathbf{R}(\bar{\alpha})\varphi \right\rangle = \lim_{T \rightarrow \infty} \sum_{j \leq \ell} \left\langle F_j^- e^{ic\sqrt{p^2-2\lambda_j}}, \theta_T'' a_j(\bar{\alpha}, \varphi) e^{ic\sqrt{p^2-2\lambda_j}} \right\rangle \\ &\lim_{T \rightarrow \infty} \left\langle \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2-2\lambda_j}} \chi, \theta_T' \partial_c \mathbf{R}(\bar{\alpha})\varphi \right\rangle = \lim_{T \rightarrow \infty} \sum_{j \leq \ell} \left\langle F_j^- e^{ic\sqrt{p^2-2\lambda_j}}, \theta_T' a_j(\bar{\alpha}, \varphi) \partial_c (e^{ic\sqrt{p^2-2\lambda_j}}) \right\rangle \end{aligned}$$

and this easily yields

$$\langle \mathcal{P}_\ell(\alpha)F, \varphi \rangle = i \sum_{j \leq \ell} \sqrt{p^2 - \lambda_j} \left\langle F_j^-, a_j(\bar{\alpha}, \varphi) \right\rangle_{L^2(\Omega_T)}.$$

where a_j are the functional obtained from (3.28). The result then extends holomorphically to the region (3.65) and meromorphically to (3.67) \square

The Poisson operator far from $\text{Re}(\alpha) = Q$. We have seen in Lemma 3.5 that the Poisson operator can be defined far from the spectrum. The problem is that the region of analyticity of $\mathcal{P}_\ell(\alpha)$ in Lemma 3.5 does not intersect (for ℓ large at least) the region of analyticity of $\mathcal{P}_\ell(\alpha)$ from Proposition 3.16. The proposition below extends the construction of the Poisson operator to a region overlapping both regions in Lemma 3.5 and Proposition 3.16 (see figure 3).

Proposition 3.18. *For ℓ fixed, the Poisson operator $\mathcal{P}_\ell(\alpha)$ of Lemma 3.5 extends analytically to the region*

$$(3.71) \quad \left\{ \alpha = Q + ip \mid \text{Re}(\alpha) < Q, \text{Im}(p) > \text{Im}(\sqrt{p^2 - 2\lambda_\ell}) - \gamma/2 \right\}$$

Proof. As before, for $F = \sum_{j=0}^\ell F_j^- \in E_\ell$ with $F_j^- \in \text{Ker}(\mathbf{P} - \lambda_j)$, we set $u_-(\alpha) := \sum_{j \leq \ell} F_j^- e^{ic\sqrt{p^2 - 2\lambda_j}}$, and let $\chi \in C^\infty(\mathbb{R})$ equal to 1 in $(-\infty, -1)$ and with $\text{supp}(\chi) \subset \mathbb{R}^-$. We get

$$\left(\mathbf{H} - \frac{Q^2 + p^2}{2} \right) (\chi u_-(\alpha)) = -\frac{1}{2} \chi''(c) u_-(\alpha) - \chi'(c) \partial_c u_-(\alpha) + e^{\gamma c} V \chi u_-(\alpha) \in e^{(-\text{Im}\sqrt{p^2 - 2\lambda_\ell} + \gamma)\rho} V^{\frac{1}{2}} L^2(\mathbb{R} \times \Omega_T).$$

Using Proposition 3.14, we can thus define, with the same formula as in Lemma 3.22 and Proposition 3.16, the Poisson operator

$$\begin{aligned} u_+(\alpha) &:= \mathbf{R}(\alpha) \left(\mathbf{H} - \frac{Q^2 + p^2}{2} \right) (\chi u_-(\alpha)) \in e^{(-\text{Im}\sqrt{p^2 - 2\lambda_\ell} + \gamma/2)\rho} \mathcal{D}(\mathcal{Q}), \\ \mathcal{P}_\ell(\alpha)F &:= u_-(\alpha)\chi - u_+(\alpha) \end{aligned}$$

in the region

$$\text{Im}(p) = \text{Re}(Q - \alpha) > \max_{j \leq \ell} \text{Im}\sqrt{p^2 - 2\lambda_j} - \gamma/2 = \text{Im}\sqrt{p^2 - 2\lambda_\ell} - \gamma/2.$$

\square

Remark 3.19. *Notice that this region of holomorphy is non-empty and connected, as for ℓ and $|p| = R \gg \lambda_\ell$*

$$\text{Im}(p) - \text{Im}\sqrt{p^2 - 2\lambda_\ell} + \gamma/2 = \gamma/2 + \mathcal{O}\left(\frac{\lambda_\ell}{R^2}\right) > 0.$$

3.3. The Scattering operator.

Definition 3.20. *Let $\ell \in \mathbb{N}$ and $\alpha = Q + ip$ with $p \in \mathbb{R} \setminus (-\sqrt{2\lambda_\ell}, \sqrt{2\lambda_\ell})$. The scattering operator $\mathbf{S}_\ell(\alpha) : E_\ell \rightarrow E_\ell$ for the ℓ -th layer (also called ℓ -scattering operator) is the operator defined as follows: let $F = \sum_{j \leq \ell} F_j \in E_\ell$ (with $F_j \in \text{Ker}(\mathbf{P} - \lambda_j)$) and let $F_j^- := (p^2 - 2\lambda_j)^{-1/4} F_j$, then we set*

$$\mathbf{S}_\ell(\alpha)F := \begin{cases} \sum_{j \leq \ell} F_j^+(\alpha) (p^2 - 2\lambda_j)^{1/4}, & \text{if } p > \sqrt{2\lambda_\ell}, \\ \sum_{j \leq \ell} \widehat{F}_j^+(\alpha) (p^2 - 2\lambda_j)^{1/4}, & \text{if } p < -\sqrt{2\lambda_\ell}. \end{cases}$$

where $F_j^+(\alpha), \widehat{F}_j^+(\alpha)$ are the functions in (3.66) and (3.69). We will call more generally

$$\mathbf{S}(\alpha) := \begin{cases} \bigcup_{\ell, 2\lambda_\ell < p^2} \text{ker}(\mathbf{P} - \lambda_j) & \rightarrow \bigcup_{\ell, 2\lambda_\ell < p^2} \text{ker}(\mathbf{P} - \lambda_j) \\ F \in E_\ell & \mapsto \mathbf{S}_\ell(\alpha)F \end{cases}$$

the scattering operator, where we use $\mathbf{S}_\ell(\alpha)|_{E_{\ell'}} = \mathbf{S}_{\ell'}(\alpha)$ for $\ell' < \ell$.

Let us define the map $\omega_\ell : \Sigma \rightarrow \Sigma$ by the following property: if $r_j(\alpha) = \sqrt{p^2 - 2\lambda_j}$ are the analytic functions on Σ used to define this ramified covering (with $\alpha = Q + ip$ in the physical sheet), then $\omega_\ell(\alpha)$ is the point in Σ so that

$$r_j(\omega_\ell(\alpha)) = \begin{cases} -r_j(\alpha) & \text{if } j \leq \ell, \\ r_j(\alpha) & \text{if } j > \ell \end{cases}.$$

As a consequence of Proposition 3.16 and Lemma 3.15, we obtain the

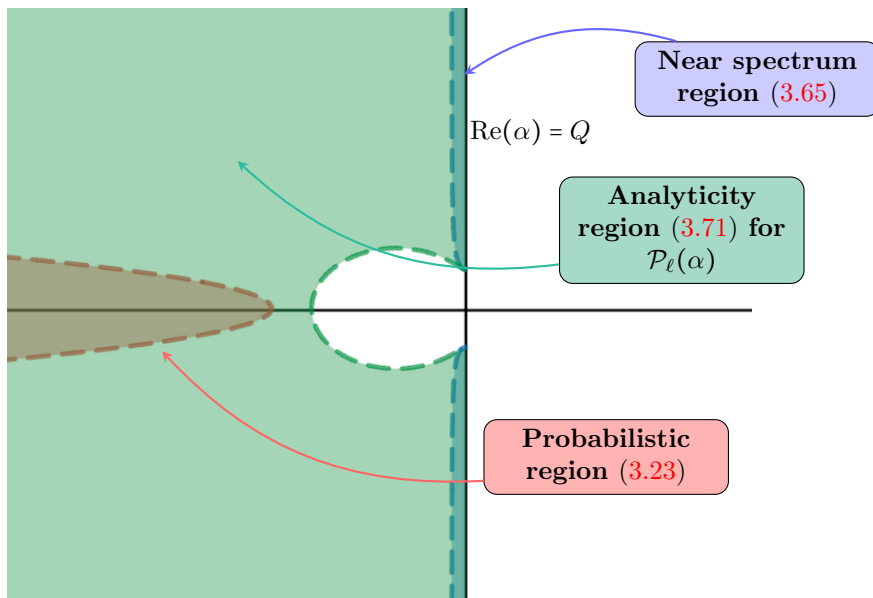


FIGURE 3. The green colored region correspond to the region (3.71) of validity of Lemma 3.18 for the analyticity of the Poisson operator $\mathcal{P}_\ell(\alpha)$ with $\ell > 0$ (for the plot, we take $\lambda_\ell = 4$, $\gamma = 1/2$). It overlaps the probabilistic (3.23) and near spectrum (3.65) regions.

Corollary 3.21. *For each $\ell \in \mathbb{N}$, the ℓ -th scattering operator $\mathbf{S}_\ell(\alpha)$ is unitary on E_ℓ if $\alpha = Q + ip$ ($p \in \mathbb{R}$) is such that $\lambda_\ell < p^2 < \min\{\lambda_j \mid \lambda_\ell < \lambda_j\}$. It also satisfies the functional equation*

$$(3.72) \quad \mathbf{S}_\ell(\alpha)\mathbf{S}_\ell(\omega_\ell(\alpha)) = \text{Id}.$$

Moreover it extends meromorphically in (3.67) if $\beta < \gamma$. It also satisfies the following functional equation for each $F = \sum_{j=0}^{\ell} F_j \in E_\ell$

$$(3.73) \quad \mathcal{P}_\ell(\alpha) \sum_{j=0}^{\ell} (p^2 - 2\lambda_j)^{-1/4} F_j = \mathcal{P}_\ell(\omega_\ell(\alpha)) \mathbf{S}_\ell(\alpha) \sum_{j=0}^{\ell} F_j.$$

Proof. The unitarity of $\mathbf{S}_\ell(\alpha)$ on the line $\text{Re}(\alpha) = Q$ follows directly from Lemma 3.15 applied with $u_\ell = \mathcal{P}_\ell(\alpha)F$. The functional equation (3.72) reads $\mathbf{S}_\ell(\alpha)\mathbf{S}_\ell(2Q - \alpha) = \text{Id}$ on the line $\text{Re}(\alpha) = Q$ and that comes directly from the uniqueness statement in Proposition 3.16 on the line. The extension of $\mathbf{S}(\alpha)$ with respect to α comes directly from the meromorphy of the $a_j(\alpha, F)$ in Proposition 3.8. The functional identity extends meromorphically under the formula (3.72). The functional equation (3.72) also comes from uniqueness of the Poisson operator. \square

Theorem 3.22. *For each $j \in \mathbb{N}$, let $(h_{jk})_{k=1, \dots, k(j)}$ be an orthonormal basis of $\ker_{L^2(\Omega_{\mathbb{T}})}(\mathbf{P} - \lambda_j)$. The spectral resolution holds for all $\varphi, \varphi' \in e^{\beta\rho} L^2(\Omega_{\mathbb{T}} \times \mathbb{R})$ with $\beta > 0$*

$$\langle \varphi | \varphi' \rangle_2 = \frac{1}{2\pi} \sum_{j=0}^{\infty} \sum_{k=1}^{k(j)} \int_0^{\infty} \langle \varphi | \mathcal{P}_j(Q + i\sqrt{p^2 + 2\lambda_j}) h_{jk} \rangle \langle \mathcal{P}_j(Q + i\sqrt{p^2 + 2\lambda_j}) h_{jk} | \varphi' \rangle dp.$$

As a consequence the spectrum of \mathbf{H} is absolutely continuous.

Proof. We recall the Stone formula: for $\varphi, \varphi' \in e^{\beta\rho}L^2$ for $\beta > 0$

$$\begin{aligned} \langle \varphi | \varphi' \rangle_2 &= \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \langle [(\mathbf{H} - \frac{Q^2}{2} - t - i\varepsilon)^{-1} - (\mathbf{H} - \frac{Q^2}{2} - t + i\varepsilon)^{-1}] \varphi | \varphi' \rangle dt \\ &= \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \langle [(\mathbf{H} - \frac{Q^2+p^2}{2} - i\varepsilon)^{-1} - (\mathbf{H} - \frac{Q^2+p^2}{2} + i\varepsilon)^{-1}] \varphi | \varphi' \rangle p dp \\ &= \frac{1}{2\pi i} \int_0^\infty \langle [(\mathbf{R}(Q+ip) - \mathbf{R}(Q-ip))] \varphi | \varphi' \rangle p dp. \end{aligned}$$

Here $\alpha = Q + ip$ (with $p > 0$) has to be viewed as an element in Σ obtained by limit $Q + ip - \varepsilon$ as $\varepsilon \rightarrow 0^+$ and, if ℓ is the largest integer such that $2\lambda_\ell \leq p^2$, we write $\bar{\alpha}$ for the point $\omega_\ell(\alpha)$ on Σ . For $\alpha = Q + ip$ with $p \in \mathbb{R}^+$, we have for $\varphi \in e^{\beta\rho}L^2$ with $\beta > 0$

$$\begin{aligned} \frac{1}{2i} \langle (\mathbf{R}(\alpha) - \mathbf{R}(\bar{\alpha})) \varphi | \varphi \rangle &= \text{Im} \langle \mathbf{R}(\alpha) \varphi | \varphi \rangle \\ &= \text{Im} \langle \mathbf{R}(\alpha) (\mathbf{H} - 2\Delta_\alpha) \mathbf{R}(\bar{\alpha}) \varphi | \varphi \rangle \\ &= \text{Im} \langle (\mathbf{H} - 2\Delta_\alpha) \mathbf{R}(\bar{\alpha}) \varphi | \mathbf{R}(\bar{\alpha}) \varphi \rangle \end{aligned}$$

Here we have used that $(\mathbf{H} - 2\Delta_\alpha) \mathbf{R}(\bar{\alpha}) = \text{Id}$ on $e^{\beta\rho}L^2$ provided $p \in \mathbb{R}$ and that $\langle \mathbf{R}(\bar{\alpha}) \varphi | \varphi' \rangle = \langle \varphi, \mathbf{R}(\alpha) \varphi' \rangle$ for $\varphi | \varphi' \in e^{\beta\rho}L^2$, this last fact coming from the identity $\mathbf{R}(\bar{\alpha}) = \mathbf{R}(\alpha)^*$ for $\text{Re}(\alpha) < Q$ and passing to the limit $\text{Re}(\alpha) \rightarrow Q$. Let $\theta_T(c)$ as in the proof of Lemma 3.15. We have

$$\begin{aligned} \langle (\mathbf{H} - 2\Delta_\alpha) \mathbf{R}(\bar{\alpha}) \varphi, \mathbf{R}(\bar{\alpha}) \varphi \rangle &= \lim_{T \rightarrow +\infty} \langle \theta_T (\mathbf{H} - 2\Delta_\alpha) \mathbf{R}(\bar{\alpha}) \varphi, \mathbf{R}(\bar{\alpha}) \varphi \rangle \\ &= \lim_{T \rightarrow +\infty} \langle \theta_T \mathbf{R}(\bar{\alpha}) \varphi | \varphi \rangle + \lim_{T \rightarrow +\infty} \frac{1}{2} \langle [\partial_c^2, \theta_T] \mathbf{R}(\bar{\alpha}) \varphi | \mathbf{R}(\bar{\alpha}) \varphi \rangle. \end{aligned}$$

Using (3.28) and arguing as in the proof of Lemma 3.15, as $T \rightarrow \infty$ we get

$$\begin{aligned} \langle (\mathbf{H} - 2\Delta_\alpha) \mathbf{R}(\bar{\alpha}) \varphi, \mathbf{R}(\bar{\alpha}) \varphi \rangle &= \langle \mathbf{R}(\bar{\alpha}) \varphi, \varphi \rangle + \frac{1}{2} \lim_{T \rightarrow \infty} \sum_{j \leq \ell} \|a_j(\bar{\alpha}, \varphi)\|_{L^2(\Omega_T)}^2 \partial_c (e^{ic\sqrt{p^2-2\lambda_j}})_{|c=-T} e^{iT\sqrt{p^2-2\lambda_j}} \\ &\quad - \frac{1}{2} \lim_{T \rightarrow \infty} \sum_{j \leq \ell} \|a_j(\bar{\alpha}, \varphi)\|_{L^2(\Omega_T)}^2 e^{-iT\sqrt{p^2-2\lambda_j}} \partial_c (e^{-ic\sqrt{p^2-2\lambda_j}})_{|c=-T} \\ &= \langle \mathbf{R}(\bar{\alpha}) \varphi, \varphi \rangle - i \sum_{j \leq \ell} \sqrt{p^2-2\lambda_j} \|a_j(\bar{\alpha}, \varphi)\|_{L^2(\Omega_T)}^2. \end{aligned}$$

We conclude that

$$(3.74) \quad -\text{Im} \langle \mathbf{R}(\bar{\alpha}) \varphi, \varphi \rangle = \frac{1}{2} \sum_{j \leq \ell} \sqrt{p^2-2\lambda_j} \|a_j(\bar{\alpha}, \varphi)\|_{L^2(\Omega_T)}^2.$$

By Lemma 3.17, we have

$$(3.75) \quad \mathcal{P}_\ell(\alpha)^* \varphi = -i \sum_{j \leq \ell} a_j(\bar{\alpha}, \varphi) \sqrt{p^2-2\lambda_j}.$$

By polarisation and by denoting Π_j the orthogonal projectors on $\text{Ker}(\mathbf{P} - \lambda_j)$, we deduce from (3.74) and (3.75) that

$$-\text{Im} \langle \mathbf{R}(\bar{\alpha}) \varphi, \varphi \rangle = \frac{1}{2} \sum_{j \leq \ell} \frac{1}{\sqrt{p^2-2\lambda_j}} \langle \Pi_j \mathcal{P}_\ell(\alpha)^* \varphi, \Pi_j \mathcal{P}_\ell(\alpha)^* \varphi \rangle_{L^2(\Omega_T)}.$$

Rewriting $\mathcal{P}_\ell(\alpha)^* \varphi = \sum_{j=0}^\ell \sum_{k=1}^{k(j)} \langle \varphi, \mathcal{P}_\ell(\alpha) h_{jk} \rangle_2 h_{jk}$, we obtain

$$\langle \varphi, \varphi' \rangle_2 = \frac{1}{2\pi} \int_0^\infty \sum_{\ell=0}^\infty \mathbb{1}_{[\sqrt{2\lambda_\ell}, \sqrt{2\lambda_{\ell+1}})}(p) \sum_{j=0}^\ell \sum_{k=1}^{k(j)} \langle \varphi, \mathcal{P}_\ell(Q+ip) h_{jk} \rangle_2 \overline{\langle \varphi', \mathcal{P}_\ell(Q+ip) h_{jk} \rangle_2} \frac{p}{\sqrt{p^2-2\lambda_j}} dp.$$

This finally can be rewritten, using (3.26), as

$$\begin{aligned}
\langle \varphi, \varphi' \rangle_2 &= \frac{1}{2\pi} \sum_{j=0}^{\infty} \sum_{\ell \geq j} \int_0^{\infty} \mathbf{1}_{[\sqrt{2\lambda_\ell}, \sqrt{2\lambda_{\ell+1}})}(p) \sum_{k=1}^{k(j)} \langle \varphi, \mathcal{P}_\ell(Q+ip)h_{jk} \rangle \overline{\langle \varphi', \mathcal{P}_\ell(Q+ip)h_{jk} \rangle} \frac{p}{\sqrt{p^2 - 2\lambda_j}} dp \\
&= \frac{1}{2\pi} \sum_{j=0}^{\infty} \int_{\sqrt{2\lambda_j}}^{\infty} \sum_{k=1}^{k(j)} \langle \varphi, \mathcal{P}_j(Q+ip)h_{jk} \rangle \overline{\langle \varphi', \mathcal{P}_j(Q+ip)h_{jk} \rangle} \frac{p}{\sqrt{p^2 - 2\lambda_j}} dp \\
&= \frac{1}{2\pi} \sum_{j=0}^{\infty} \sum_{k=1}^{k(j)} \int_0^{\infty} \langle \varphi, \mathcal{P}_j(Q+i\sqrt{r^2+2\lambda_j})h_{jk} \rangle \overline{\langle \varphi', \mathcal{P}_j(Q+i\sqrt{r^2+2\lambda_j})h_{jk} \rangle} dr
\end{aligned}$$

where we performed the change of variables in the last line $r = \sqrt{p^2 - 2\lambda_j}$. \square

4. PROOF OF THEOREM 1.1

We write the spectrum $\sigma(\mathbf{P})$ of \mathbf{P} in increasing order, without repeating multiplicity, and denote the eigenvalues by $0 = \lambda_0 < \dots < \lambda_j < \dots$, and we recall from Section 1.3 that $(\psi_{\mathbf{k}\mathbf{l}})_{\mathbf{k}, \mathbf{l} \in \mathcal{N}}$ is an orthonormal basis of $L^2(\Omega_{\mathbb{T}})$ with $\mathbf{P}\psi_{\mathbf{k}\mathbf{l}} = \lambda_{\mathbf{k}\mathbf{l}}\psi_{\mathbf{k}\mathbf{l}}$. First, in Theorem 3.22, we prove that for any function $u_1, u_2 \in e^{\delta c} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ with $\delta > 0$,

$$\begin{aligned}
\langle u_1 | u_2 \rangle_2 &= \frac{1}{2\pi} \int_0^{\infty} \sum_{j=0}^{\infty} \sum_{\substack{\mathbf{k}, \mathbf{l} \in \mathcal{N}, \\ \lambda_{\mathbf{k}\mathbf{l}} = \lambda_j}} \langle u_1 | \mathcal{P}_j(Q+i\sqrt{P^2+2\lambda_j})\psi_{\mathbf{k}\mathbf{l}} \rangle_2 \langle \mathcal{P}_j(Q+i\sqrt{P^2+2\lambda_j})\psi_{\mathbf{k}\mathbf{l}} | u_2 \rangle_2 dP \\
(4.1) \quad &= \frac{1}{2\pi} \int_0^{\infty} \sum_{j=0}^{\infty} \sum_{\substack{\mathbf{k}, \mathbf{l} \in \mathcal{N}, \\ \lambda_{\mathbf{k}\mathbf{l}} = \lambda_j}} \langle u_1 | \Psi_{Q+iP, \mathbf{k}, \mathbf{l}} \rangle_2 \langle \Psi_{Q+iP, \mathbf{k}, \mathbf{l}} | u_2 \rangle_2 dP
\end{aligned}$$

where, to simplify the notations, we have defined for each \mathbf{k}, \mathbf{l} ,

$$\Psi_{Q+iP, \mathbf{k}, \mathbf{l}} := \mathcal{P}_j(Q+i\sqrt{P^2+2\lambda_{\mathbf{k}\mathbf{l}}})\psi_{\mathbf{k}\mathbf{l}}$$

for $j \in \mathbb{N}$ satisfying $\lambda_j = \lambda_{\mathbf{k}\mathbf{l}}$. Now, we will perform a change of basis to recover the bootstrap formalism. Let us fix an eigenvalue λ_j for the operator \mathbf{P} . As in Section 1.5, we set the following definition for the Liouville descendant fields for P (possibly complex): for each $\nu, \tilde{\nu} \in \mathcal{T}$, there is $j \in \mathbb{N}_0$ such that $|\nu| + |\tilde{\nu}| = \lambda_j \in \sigma(\mathcal{P})$, we then define

$$(4.2) \quad \Psi_{Q+iP, \nu, \tilde{\nu}} := \frac{1}{\sqrt{2\pi}} \sum_{\mathbf{k}, \mathbf{l}, \lambda_{\mathbf{k}\mathbf{l}} = \lambda_j} (M_{Q+iP}^{\lambda_j})_{\mathbf{k}\mathbf{l}, \nu \tilde{\nu}}^{-1} \Psi_{Q+iP, \mathbf{k}, \mathbf{l}}$$

where the matrix $M_{Q+iP}^{\lambda_j}$ are defined by the change of basis of (1.56). Just as in (1.58), it is direct to see by using this change of basis in (4.1) that

$$(4.3) \quad \langle u_1 | u_2 \rangle_2 = \lim_{\substack{N \rightarrow \infty \\ L \rightarrow \infty}} \sum_{\substack{\nu, \tilde{\nu}, \nu', \tilde{\nu}' \in \mathcal{T}, \\ |\nu| + |\tilde{\nu}| \leq N, \\ |\nu'| = |\nu|, |\tilde{\nu}'| = |\tilde{\nu}|}} \int_0^L \langle u_1 | \Psi_{Q+iP, \nu', \tilde{\nu}'} \rangle_2 \langle \Psi_{Q+iP, \nu, \tilde{\nu}} | u_2 \rangle_2 F_{Q+iP}^{-1}(\nu, \nu') F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}') dP$$

where the coefficients $F_{Q+iP}^{-1}(\nu, \nu')$ are matrix coefficients defined right below equation (1.57). To justify the formula, we can write (4.1) as a limit of $\sum_{j=0}^N \int_0^L$ and notice that for each $(\nu, \tilde{\nu}, \nu', \tilde{\nu}') \in \mathcal{T}$ with $|\nu| + |\tilde{\nu}| = |\nu'| + |\tilde{\nu}'| = \lambda_j$ the matrix valued function $P \mapsto F_{Q+iP}^{-1}(\nu, \nu')$ is continuous in P .

Let us define, for $N \in \mathbb{N}$, the sets

$$\mathcal{T}_N = \{\nu \in \mathcal{T} \mid |\nu| = N\}.$$

Next, we notice that the matrices $(F_{Q+iP}(\nu, \nu'))_{\nu, \nu' \in \mathcal{T}_N}$ appear as Gram matrices in (1.55) and can thus be taken to be positive definite (F_{Q+iP} are defined up to sign in (1.55)): indeed, for each $\tilde{\nu}$ of length $|\tilde{\nu}| = N'$, one has $F_{Q+iP}(\nu, \nu) F_{Q+iP}(\tilde{\nu}, \tilde{\nu}) > 0$ by (1.55), thus we can choose all the F_{Q+iP} to satisfy $F_{Q+iP}(\nu, \nu) > 0$ for all $|\nu| = N$ and all N , but for each $\tilde{\nu}$ the matrix $(F_{Q+iP}(\nu, \nu') F_{Q+iP}(\tilde{\nu}, \tilde{\nu}'))_{\nu, \nu' \in \mathcal{T}_N}$ is non-negative as the Gram matrix of the vectors $(\mathcal{Q}_{Q+iP, \nu, \tilde{\nu}})_{\nu \in \mathcal{T}_N}$ and thus positive definite since $(F_{Q+iP}(\nu, \nu'))_{\nu, \nu' \in \mathcal{T}_N}$ is also invertible.

In particular, one can define the matrices $(F_{Q+iP}^{-1/2}(\nu, \nu'))_{\nu, \nu' \in \mathcal{T}_N}$ to be the square root of the positive definite matrix $(F_{Q+iP}^{-1}(\nu, \nu'))_{\nu, \nu' \in \mathcal{T}_N}$. We then define the new elements for $\nu \in \mathcal{T}_N$ and $\tilde{\nu} \in \mathcal{T}_{N'}$

$$H_{Q+iP, \nu, \tilde{\nu}} := \sum_{|\nu_1|=N, |\nu_2|=N'} F_{Q+iP}^{-1/2}(\nu, \nu_1) \Psi_{Q+iP, \nu_1, \nu_2} F_{Q+iP}^{-1/2}(\tilde{\nu}, \nu_2).$$

One can first notice that for all $u_1, u_2 \in e^{\delta c} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ with $\delta > 0, L > 0, N \in \mathbb{N}$, we have

$$(4.4) \quad \int_0^L \sum_{|\nu| \leq N, |\tilde{\nu}| \leq N} \langle u_1 | H_{Q+iP, \nu, \tilde{\nu}} \rangle_2 \langle H_{Q+iP, \nu, \tilde{\nu}} | u_2 \rangle_2 dP \\ = \sum_{j, j'=1}^N \sum_{\substack{\nu_1, \nu_2, \nu_3, \nu_4 \\ |\nu_1|=|\nu_3|=j, |\nu_2|=|\nu_4|=j'}} \int_0^L \langle u_1 | \Psi_{Q+iP, \nu_1, \nu_2} \rangle_2 \langle \Psi_{Q+iP, \nu_3, \nu_4} | u_2 \rangle_2 F_{Q+iP}^{-1}(\nu_1, \nu_3) F_{Q+iP}^{-1}(\nu_2, \nu_4) dP$$

and similarly

$$(4.5) \quad \int_0^L \sum_{|\nu|+|\tilde{\nu}| \leq N} \langle u_1 | H_{Q+iP, \nu, \tilde{\nu}} \rangle_2 \langle H_{Q+iP, \nu, \tilde{\nu}} | u_2 \rangle_2 dP \\ = \sum_{j+j' \leq N} \sum_{\substack{\nu_1, \nu_2, \nu_3, \nu_4 \\ |\nu_1|=|\nu_3|=j, |\nu_2|=|\nu_4|=j'}} \int_0^L \langle u_1 | \Psi_{Q+iP, \nu_1, \nu_2} \rangle_2 \langle \Psi_{Q+iP, \nu_3, \nu_4} | u_2 \rangle_2 F_{Q+iP}^{-1}(\nu_1, \nu_3) F_{Q+iP}^{-1}(\nu_2, \nu_4) dP \\ = \int_0^L \sum_{\mathbf{k}, \mathbf{l}, \lambda_{\mathbf{k}\mathbf{l}} \leq N} \langle u_1 | \Psi_{Q+iP, \mathbf{k}, \mathbf{l}} \rangle_2 \langle \Psi_{Q+iP, \mathbf{k}, \mathbf{l}} | u_2 \rangle_2 dP.$$

Now, we claim that for all $u_1, u_2 \in e^{\delta c} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for some $\delta > 0$, we have

$$(4.6) \quad \langle u_1 | u_2 \rangle_2 = \lim_{L \rightarrow \infty} \lim_{N \rightarrow \infty} \int_0^L \sum_{|\nu| \leq N, |\tilde{\nu}| \leq N} \langle u_1 | H_{Q+iP, \nu, \tilde{\nu}} \rangle_2 \langle H_{Q+iP, \nu, \tilde{\nu}} | u_2 \rangle_2 dP.$$

In the case $u_1 = u_2 = u$, the limit (4.6) is just a consequence of the inequality

$$\int_0^L \sum_{|\nu|+|\tilde{\nu}| \leq N} |\langle u | H_{Q+iP, \nu, \tilde{\nu}} \rangle_2|^2 dP \leq \int_0^L \sum_{\substack{|\nu| \leq N, \\ |\tilde{\nu}| \leq N}} |\langle F | H_{Q+iP, \nu, \tilde{\nu}} \rangle_2|^2 dP \leq \int_0^L \sum_{|\nu|+|\tilde{\nu}| \leq 2N} |\langle u | H_{Q+iP, \nu, \tilde{\nu}} \rangle_2|^2 dP$$

along with identity (4.5) and (4.1). In the general case, we can take the difference of (4.4) and (4.5), use the Cauchy-Schwarz inequality and the results for the $u_1 = u_2$ case to deduce that the difference tends to 0 as $(N, L) \rightarrow \infty$. Then (4.3) allows to conclude that (4.6) holds.

For $|z_1|, |z_2| < 1$ and $|z_3|, |z_4| > 1$, the 4-point correlation function is given by

$$(4.7) \quad \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) V_{\alpha_4}(z_4) \rangle_{\gamma, \mu} = |z_3|^{-4\Delta_{\alpha_3}} |z_4|^{-4\Delta_{\alpha_4}} \left\langle U_{\alpha_1, \alpha_2}(z_1, z_2) \middle| U_{\alpha_4, \alpha_3} \left(\frac{1}{z_4}, \frac{1}{z_3} \right) \right\rangle_2$$

where recall that for $|z_i|, |z_j| < 1$

$$U_{\alpha_i, \alpha_j}(z_i, z_j; c, \varphi) = \lim_{\epsilon \rightarrow 0} \mathbb{E}_{\varphi} \left[V_{\alpha_i, \epsilon}(z_i) V_{\alpha_j, \epsilon}(z_j) \exp \left(-\mu e^{\gamma c} \int_{\mathbb{D}} M_{\gamma}(dx) \right) \right].$$

By using the Girsanov theorem, one gets the following explicit expression

$$U_{\alpha_i, \alpha_j}(z_i, z_j; c, \varphi) = (1 - |z_i|^2)^{\frac{\alpha_i^2}{2}} (1 - |z_j|^2)^{\frac{\alpha_j^2}{2}} e^{(\alpha_i + \alpha_j - Q)c} e^{\alpha_i P \varphi(z_i) + \alpha_j P \varphi(z_j) + \alpha_i \alpha_j G_{\mathbb{D}}(z_i, z_j)} \\ \times \mathbb{E}_{\varphi} \left[\exp \left(-\mu e^{\gamma c} \int_{\mathbb{D}} e^{\gamma \alpha_i G_{\mathbb{D}}(x, z_i) + \gamma \alpha_j G_{\mathbb{D}}(x, z_j)} M_{\gamma}(dx) \right) \right].$$

The formula (4.7) can be extended to the case $|z_4| = \infty$ by taking the limit $|z_4| \rightarrow \infty$ of both sides multiplied by $|z_4|^{4\Delta_{\alpha_4}}$, and we define

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) V_{\alpha_4}(\infty) \rangle_{\gamma, \mu} := \lim_{|z_4| \rightarrow \infty} |z_4|^{4\Delta_{\alpha_4}} \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) V_{\alpha_4}(\infty) \rangle_{\gamma, \mu} \\ = |z_3|^{-4\Delta_{\alpha_3}} \left\langle U_{\alpha_1, \alpha_2}(z_1, z_2) \middle| U_{\alpha_4, \alpha_3} \left(0, \frac{1}{z_3} \right) \right\rangle_2.$$

We want to compute the right hand side for $z_1 = 0$, $z_2 = z$ with $|z| < 1$, $z_3 = z'$ with $|z'| > 1$ and let $z' \rightarrow 1$, and we recall that for $|z| < 1$ the vector $U_{\alpha_1, \alpha_2}(0, z) \in e^{\delta c_-} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for some $\delta > 0$ if $\alpha_1 + \alpha_2 > Q$, and the same holds for $U_{\alpha_4, \alpha_3}(0, \frac{1}{z'})$ if $\alpha_3 + \alpha_4 > Q$. Using the identity (4.6) and (4.4), we get

(4.8)

$$\left\langle U_{\alpha_1, \alpha_2}(z_1, z_2) \middle| U_{\alpha_4, \alpha_3}(0, \frac{1}{z'}) \right\rangle_2 = \lim_{(N, L) \rightarrow \infty} \sum_{\substack{\nu, \tilde{\nu}, \nu', \tilde{\nu}' \in \mathcal{T}, \\ |\nu'| = |\nu| \leq N, \\ |\tilde{\nu}'| = |\tilde{\nu}| \leq N}} \int_0^L \langle U_{\alpha_1, \alpha_2}(0, z) | \Psi_{Q+iP, \nu, \tilde{\nu}} \rangle_2 \langle \Psi_{Q+iP, \nu', \tilde{\nu}'} | U_{\alpha_4, \alpha_3}(0, \frac{1}{z'}) \rangle_2 F_{Q+iP}^{-1}(\nu, \nu') F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}') dP.$$

Now, we can give *exact* analytic expressions for the scalar products $\langle U_{\alpha_1, \alpha_2}(0, z) | \Psi_{Q+iP, \nu, \tilde{\nu}} \rangle_2$. This is the content of Proposition 1.10, proved in Section 5. For $\nu = (\nu_i)_{i \in [1, k]}$ a Young diagram and some real numbers $\Delta, \Delta', \Delta''$, let

$$v(\Delta, \Delta', \Delta'', \nu) := \prod_{j=1}^k (\nu_j \Delta' - \Delta + \Delta'' + \sum_{u < j} \nu_u).$$

With this notation, we can state that for $P > 0$, the following expression holds

$$\begin{aligned} & \langle U_{\alpha_1, \alpha_2}(0, z) | \Psi_{Q+iP, \nu, \tilde{\nu}} \rangle \\ &= \frac{1}{2} C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, Q - iP) z^{|\nu|} \bar{z}^{|\tilde{\nu}|} |z|^{2(\Delta_{Q+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \nu) v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \tilde{\nu}) \end{aligned}$$

where $\Delta_{\alpha} = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$ are conformal weights and $C^{\text{DOZZ}}(\alpha_1, \alpha_2, Q - iP)$ is the constant defined in (A.3). Let us now take $\alpha_3 = \alpha_2$ and $\alpha_1 = \alpha_4$, with $\alpha_1 + \alpha_2 > Q$ as before and choose $z' = \frac{1}{t}$ with $t \in (0, 1)$. In view of the above considerations then (4.8) is, up to the multiplicative factor $|tz|^{-2\Delta_{\alpha_1} - 2\Delta_{\alpha_2}}$, the limit as $N, L \rightarrow \infty$ of

$$\begin{aligned} & \int_0^L |C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, Q - iP)|^2 |tz|^{2\Delta_{Q+iP}} \sum_{\substack{\nu, \tilde{\nu} \in \mathcal{T}, \\ |\nu| \leq N, |\tilde{\nu}| \leq N}} \sum_{\substack{\nu', \tilde{\nu}' \in \mathcal{T}, \\ |\nu'| = |\nu|, |\tilde{\nu}'| = |\tilde{\nu}|}} z^{|\nu|} \bar{z}^{|\tilde{\nu}|} t^{|\nu| + |\tilde{\nu}|} F_{Q+iP}^{-1}(\tilde{\nu}, \tilde{\nu}') F_{Q+iP}^{-1}(\nu, \nu') \\ & \times v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \nu) v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \tilde{\nu}) v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \nu') v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \tilde{\nu}') dP \\ &= \int_0^L |C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, Q - iP)|^2 |tz|^{2\Delta_{Q+iP}} \left| \sum_{n=0}^N \beta_n(\Delta_{Q+iP}, \Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{\alpha_2}, \Delta_{\alpha_1}) (zt)^n \right|^2 dP \end{aligned}$$

where the coefficients β_n are given by

$$\beta_n(\Delta_{Q+iP}, \Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{\alpha_3}, \Delta_{\alpha_4}) = \sum_{\nu, \nu' \in \mathcal{T}_n} v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+iP}, \nu) F_{Q+iP}^{-1}(\nu, \nu') v(\Delta_{\alpha_4}, \Delta_{\alpha_3}, \Delta_{Q+iP}, \nu').$$

If $z \in (0, 1)$ the above expression is increasing in the variables N, L and zt since $\beta_n \geq 0$, but since it is also bounded by its limit, it implies that the series

$$\sum_{n=0}^{\infty} z^n \beta_n(\Delta_{Q+iP}, \Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{\alpha_2}, \Delta_{\alpha_1})$$

is convergent for $|z| < 1$ for almost all $P \geq 0$ and

$$\begin{aligned} & \langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_2}(\frac{1}{t}) V_{\alpha_1}(\infty) \rangle_{\gamma, \mu} = \\ & t^{4\Delta_{\alpha_2}} |tz|^{-2\Delta_{\alpha_1} - 2\Delta_{\alpha_2}} \int_0^{\infty} |C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, Q - iP)|^2 |tz|^{2\Delta_{Q+iP}} \left| \sum_{n=1}^{\infty} \beta_n(\Delta_{Q+iP}, \Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{\alpha_2}, \Delta_{\alpha_1}) (zt)^n \right|^2 dP. \end{aligned}$$

The formula then extends to $t = 1$ by continuity.

Next, in the general case $\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1/t)V_{\alpha_4}(\infty) \rangle_{\gamma,\mu}$ with $|z| < 1$ and $t < 1$. We have by Cauchy-Schwartz

$$\begin{aligned} \beta_n(\Delta_{Q+iP}, \Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{\alpha_3}, \Delta_{\alpha_4}) &\leq \sqrt{\beta_n(\Delta_{Q+iP}, \Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{\alpha_2}, \Delta_{\alpha_1})} \sqrt{\beta_n(\Delta_{Q+iP}, \Delta_{\alpha_4}, \Delta_{\alpha_3}, \Delta_{\alpha_3}, \Delta_{\alpha_4})} \\ &\leq \frac{1}{2}(\beta_n(\Delta_{Q+iP}, \Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{\alpha_2}, \Delta_{\alpha_1}) + \beta_n(\Delta_{Q+iP}, \Delta_{\alpha_4}, \Delta_{\alpha_3}, \Delta_{\alpha_3}, \Delta_{\alpha_4})). \end{aligned}$$

We can then take (4.8) and get similarly to the previous case that

$$\begin{aligned} \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_2}(\frac{1}{t})V_{\alpha_1}(\infty) \rangle_{\gamma,\mu} &= t^{4\Delta_{\alpha_3}} t^{-2\Delta_{\alpha_3}-2\Delta_{\alpha_4}} |z|^{-2\Delta_{\alpha_1}-2\Delta_{\alpha_2}} \\ &\times \lim_{(N,L) \rightarrow \infty} \int_0^L C_{\gamma,\mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, Q-iP) \overline{C_{\gamma,\mu}^{\text{DOZZ}}(\alpha_4, \alpha_3, Q-iP)} |tz|^{2\Delta_{Q+iP}} \left| \sum_{n=1}^N \beta_n(zt)^n \right|^2 dP. \end{aligned}$$

where $\beta_n := \beta_n(\Delta_{Q+iP}, \Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{\alpha_3}, \Delta_{\alpha_4})$. We can use Cauchy-Schwarz and the estimate above for the case $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$ to see that the sum/integral are convergent and the limit exists. This achieves the proof by finally taking $t \rightarrow 1^+$ (note that $\overline{C_{\gamma,\mu}^{\text{DOZZ}}(\alpha_4, \alpha_3, Q-iP)} = C_{\gamma,\mu}^{\text{DOZZ}}(\alpha_3, \alpha_4, Q+iP)$ by (A.3)): we obtain the bootstrap formula (1.71).

5. PROBABILISTIC REPRESENTATION OF THE POISSON OPERATOR

In Section 3 we constructed the generalized eigenstates (by means of the Poisson operator) of the Liouville Hamiltonian \mathbf{H} on the spectrum line $Q+i\mathbb{R}$ and showed that these generalized eigenstates can be analytically continued in the parameter $\alpha = Q+ip$ to the physical²⁰ region $\text{Re}(\alpha) < Q$. It turns out that for α real in the physical region, which we will call probabilistic region, we will be able to give a probabilistic representation for the generalized eigenstates. The probabilistic representation of the Poisson operator will be instrumental to reveal the combinatorial relations between the generalized eigenstates, which are encoded in the Ward identities. Then we will analytically continue these relations back to the spectrum line.

5.1. Highest weight states. Recall the definition (1.51) of the highest weight state $\psi_\alpha(c, \varphi) = e^{(\alpha-Q)c}$ for $\alpha \in \mathbb{C}$ of the $\mu = 0$ theory. From Proposition 3.7 item 2 (applied with $F = 1$), we know that for $\alpha \in \mathbb{R}$ with $\alpha < Q$, the state

$$(5.1) \quad \Psi_\alpha = \mathcal{P}_0(\alpha)1$$

is given by the large time limit

$$(5.2) \quad \Psi_\alpha = \lim_{t \rightarrow \infty} e^{2t\Delta_\alpha} e^{-t\mathbf{H}} \psi_\alpha, \quad dc \otimes \mathbb{P} \text{ a.e.}$$

where Δ_α denotes the conformal weight (1.8). In physics (or representation theory) terminology the state Ψ_α is the highest weight state corresponding to the *primary field* V_α . Combining (5.2) with the Feynman-Kac formula (2.33) leads to the probabilistic representation for $\alpha < Q$

$$(5.3) \quad \Psi_\alpha(c, \varphi) := e^{(\alpha-Q)c} \mathbb{E}_\varphi \left[\exp \left(-\mu e^{\gamma c} \int_{\mathbb{D}} |x|^{-\gamma\alpha} M_\gamma(dx) \right) \right].$$

We recall here that the integrability of $|x|^{-\gamma\alpha}$ with respect to $M_\gamma(dx)$ are detailed in [DKRV16].

Remark 5.1. *In forthcoming work, we will show that, for $\alpha \in (\frac{2}{\gamma}, Q)$, we have as $c \rightarrow -\infty$*

$$\Psi_\alpha(c, \varphi) = e^{(\alpha-Q)c} + e^{(Q-\alpha)c} R(\alpha) + e^{(Q-\alpha)c} o(1)$$

with $o(1) \rightarrow 0$ in $L^2(\Omega_{\mathbb{T}})$ as $c \rightarrow -\infty$ with R the reflection coefficient defined in [KRV20], which thus appears as the scattering coefficient of constant functions: for $\ell = 0$ is given by $\mathbf{S}_0(\alpha) = R(\alpha)\mathbf{Id}$. More generally, we will show that the scattering matrix is diagonal.

²⁰To avoid confusion, we recall that, in scattering theory, "physical region" refers to the region where the resolvent operator is L^2 -bounded.

5.2. Descendant states. Recall that for $\mu = 0$ we have the Virasoro algebra descendant states given by

$$\psi_{\alpha,\nu,\tilde{\nu}}(c, \varphi) = \mathcal{Q}_{\alpha,\nu,\tilde{\nu}}(\varphi) e^{(\alpha-Q)c}$$

where $\mathcal{Q}_{\alpha,\nu,\tilde{\nu}}$ are eigenstates of the operator \mathbf{P} :

$$\mathbf{P}\mathcal{Q}_{\alpha,\nu,\tilde{\nu}} = (|\nu| + |\tilde{\nu}|)\mathcal{Q}_{\alpha,\nu,\tilde{\nu}}$$

so that

$$\mathbf{H}^0\psi_{\alpha,\nu,\tilde{\nu}} = (2\Delta_\alpha + |\nu| + |\tilde{\nu}|)\psi_{\alpha,\nu,\tilde{\nu}}.$$

From Proposition 3.7 we infer the following

Proposition 5.2. *Let $\alpha < -\frac{|\nu|+|\tilde{\nu}|-2}{\gamma} \wedge Q - \gamma$. Then the limit*

$$(5.4) \quad \lim_{t \rightarrow +\infty} e^{t(2\Delta_\alpha + |\nu| + |\tilde{\nu}|)} e^{-t\mathbf{H}}\psi_{\alpha,\nu,\tilde{\nu}} := \Psi_{\alpha,\nu,\tilde{\nu}}$$

exists in $e^{-\beta\rho}L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for $\beta > Q - \alpha - \gamma$.

Proof. Write $|\nu| + |\tilde{\nu}| = \lambda_j$ for some j . We apply Proposition 3.7 item 1 but we have to make a small notational warning: indeed recall that in Section 3, eigenvalues are parametrized by $2\Delta_\alpha$ whereas here eigenvalues correspond to $2\Delta_\alpha + |\nu| + |\tilde{\nu}| = 2\Delta_\alpha + \lambda_j$. So let us call α' the α in the statement of Proposition 3.7, and write it as $\alpha' = Q + ip$ with $p^2 = 2\lambda_j - (Q - \alpha)^2$ ($p \in i\mathbb{R}$) in such a way that

$$2\Delta_{\alpha'} = 2\Delta_\alpha + \lambda_j, \quad \text{otherwise stated } \sqrt{p^2 - 2\lambda_j} = i(Q - \alpha).$$

Let $\ell \geq 1$, $j \leq \ell$ and $F = \mathcal{Q}_{\alpha,\nu,\tilde{\nu}}$ (and $\chi = 1$). Then the limit (5.4) exists in $e^{-\beta\rho}L^2$ if $\beta > \text{Im}\sqrt{p^2 - 2\lambda_j} - \gamma = Q - \alpha - \gamma > 0$ and $-p^2 > \beta^2$. In conclusion we get $\alpha < Q - \gamma$ (first condition) and $-p^2 > (Q - \alpha - \gamma)^2$. Substituting $p^2 = 2\lambda_j - (Q - \alpha)^2$ in the latter, we arrive at the relation $(Q - \alpha)^2 - 2\lambda_j > (Q - \alpha - \gamma)^2$, which can be solved to find our second condition. \square

5.3. Ward Identities. In this section we state the main identity relating a LCFT correlation function with a V_α insertion to a scalar product with the descendant states $\Psi_{\alpha,\nu,\tilde{\nu}}$ given by (5.4) (recall our convention for contour integrals in (2.63)). We recall the representation (2.68) of the free field states

$$\psi_{\alpha,\nu,\tilde{\nu}} = e^{2\Delta_\alpha s} (2\pi i)^{-k-j} \oint_{|\mathbf{u}|=\delta} \oint_{|\mathbf{v}|=\tilde{\delta}} \mathbf{u}^{1-\nu} \bar{\mathbf{v}}^{1-\tilde{\nu}} U_0(T(\mathbf{u})\bar{T}(\mathbf{v})S_{e^{-s}}U_0^{-1}\psi_\alpha) d\bar{\mathbf{v}}d\mathbf{u}$$

where $s > 0$ arbitrary, $e^{-s} < \delta_1 \wedge \delta'_1$ (and $S_{e^{-s}}U_0^{-1}\psi_\alpha = e^{\alpha(c+B_s-Qs)}$) and the integrand is defined as the limit

$$U_0(T(\mathbf{u})\bar{T}(\mathbf{v})S_{e^{-s}}U_0^{-1}\psi_\alpha) := \lim_{\epsilon \rightarrow 0} \lim_{\epsilon' \rightarrow 0} U_0(T_\epsilon(\mathbf{u})\bar{T}_{\epsilon'}(\mathbf{v})S_{e^{-s}}U_0^{-1}\psi_\alpha).$$

The meaning of the limit is as follows. The state on the RHS is given by

$$e^{2\Delta_\alpha s} U_0(T_\epsilon(\mathbf{u})\bar{T}_{\epsilon'}(\mathbf{v})S_{e^{-s}}U_0^{-1}\psi_\alpha) = q_{\alpha,\epsilon,\epsilon'}(\varphi, \mathbf{u}, \mathbf{v}) e^{(\alpha-Q)c}$$

where $q_{\alpha,\epsilon,\epsilon'}(\cdot, \mathbf{u}, \mathbf{v}) \in L^2(\Omega_{\mathbb{T}})$ do not depend on s . Furthermore

$$(5.5) \quad \lim_{\epsilon \rightarrow 0} \lim_{\epsilon' \rightarrow 0} q_{\alpha,\epsilon,\epsilon'}(\varphi, \mathbf{u}, \mathbf{v}) = q_\alpha(\varphi, \mathbf{u}, \mathbf{v})$$

where the convergence is in $L^2(\Omega_{\mathbb{T}})$ (and pointwise a.s.) uniformly in \mathbf{u}, \mathbf{v} on compact sets in the region

$$(5.6) \quad \mathcal{O}_s = \{e^{-s} < |u_j|, |v_j| < 1, u_j \neq u_{j'}, v_j \neq v_{j'}, j \neq j', u_j \neq v_{j'}\}$$

for all $s > 0$. Finally the limit $q_\alpha(\cdot, \mathbf{u}, \mathbf{v})$ is analytic in \mathbf{u} and anti-analytic in \mathbf{v} on \mathcal{O}_s for all $s > 0$. We have then

Lemma 5.3. *We have*

$$e^{-t\mathbf{H}}U_0(T(\mathbf{u})\bar{T}(\mathbf{v})S_{e^{-s}}U_0^{-1}\psi_\alpha) = \lim_{\epsilon \rightarrow 0} \lim_{\epsilon' \rightarrow 0} e^{-t\mathbf{H}}U_0(T_\epsilon(\mathbf{u})\bar{T}_{\epsilon'}(\mathbf{v})S_{e^{-s}}U_0^{-1}\psi_\alpha)$$

where the limit is in $e^{-\beta\rho}L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for all $\beta > Q - \alpha$, uniformly in $(\mathbf{u}, \mathbf{v}) \in \mathcal{O}_s$ and the LHS is analytic in \mathbf{u} and anti-analytic in \mathbf{v} on \mathcal{O}_s .

Proof. Convergence of $e^{-t\mathbf{H}}(q_{\alpha,\epsilon,\epsilon'}\psi_\alpha)$ in $e^{-\beta\rho}L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ follows from the $L^2(\Omega_{\mathbb{T}})$ convergence (5.5) and (3.13) applied with $\beta > Q - \alpha$. \square

The following lemma gives a probabilistic expression for $e^{-t\mathbf{H}}\psi_{\alpha,\nu,\tilde{\nu}}$:

Lemma 5.4. *Let $\delta_k \wedge \tilde{\delta}_k < e^{-t}$. Then*

$$e^{-t\mathbf{H}}\psi_{\alpha,\nu,\tilde{\nu}} = \frac{e^{-(2\Delta_\alpha + |\nu| + |\tilde{\nu}|)t}}{(2\pi i)^{k+j}} \oint_{|\mathbf{u}|=\delta} \oint_{|\mathbf{v}|=\tilde{\delta}} \mathbf{u}^{1-\nu} \bar{\mathbf{v}}^{1-\tilde{\nu}} e^{-Qc} \mathbb{E}_\varphi \left(T(\mathbf{u}) \bar{T}(\mathbf{v}) V_\alpha(0) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D} \setminus \mathbb{D}_t)} \right) d\bar{\mathbf{v}} d\mathbf{u}$$

where

$$\mathbb{E}_\varphi \left(T(\mathbf{u}) \bar{T}(\mathbf{v}) V_\alpha(0) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D} \setminus \mathbb{D}_t)} \right) := \lim_{\epsilon \rightarrow 0} \lim_{\epsilon' \rightarrow 0} \mathbb{E}_\varphi \left(T_\epsilon(\mathbf{u}) \bar{T}_{\epsilon'}(\mathbf{v}) V_\alpha(0) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D} \setminus \mathbb{D}_t)} \right)$$

and the limit exists in $e^{-\beta\rho}L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ for all $\beta > Q - \alpha$ and is analytic in \mathbf{u} and anti-analytic in \mathbf{v} .

Proof. For the sake of readability we write the proof in the case when $\tilde{\nu} = 0$. Thus, consider

$$(5.7) \quad \psi_{\alpha,\nu,0} = \frac{e^{2\Delta_\alpha s}}{(2\pi i)^k} \oint_{|\mathbf{u}|=\delta} \mathbf{u}^{1-\nu} U_0 \left(T(\mathbf{u}) S_{e^{-s}} U_0^{-1} \psi_\alpha \right) d\mathbf{u}.$$

By Lemma 5.3

$$\begin{aligned} e^{-t\mathbf{H}} U_0 \left(T(\mathbf{u}) S_{e^{-s}} U_0^{-1} \psi_\alpha \right) &= \lim_{\epsilon \rightarrow 0} e^{-t\mathbf{H}} U_0 \left(T_\epsilon(\mathbf{u}) S_{e^{-s}} U_0^{-1} \psi_\alpha \right) \\ &= \lim_{\epsilon \rightarrow 0} e^{-t\mathbf{H}} U \left(T_\epsilon(\mathbf{u}) (S_{e^{-s}} U_0^{-1} \psi_\alpha) e^{\mu e^{\gamma c} M_\gamma(\mathbb{D})} \right) \\ &= \lim_{\epsilon \rightarrow 0} U \left(S_{e^{-t}} (T_\epsilon(\mathbf{u}) (S_{e^{-s}} U_0^{-1} \psi_\alpha) e^{\mu e^{\gamma c} M_\gamma(\mathbb{D})}) \right) \\ &= \lim_{\epsilon \rightarrow 0} e^{-2t} U \left(T_{e^{-t}\epsilon}(e^{-t}\mathbf{u}) (S_{e^{-s-t}} U_0^{-1} \psi_\alpha) S_{e^{-t}} (e^{\mu e^{\gamma c} M_\gamma(\mathbb{D})}) \right) \\ &= e^{-2t} e^{-Qc} \lim_{\epsilon \rightarrow 0} \mathbb{E}_\varphi \left(T_{e^{-t}\epsilon}(e^{-t}\mathbf{u}) (S_{e^{-s-t}} U_0^{-1} \psi_\alpha) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D} \setminus \mathbb{D}_t)} \right) \\ &:= e^{-2t} e^{-Qc} \mathbb{E}_\varphi \left(T(e^{-t}\mathbf{u}) (S_{e^{-s-t}} U_0^{-1} \psi_\alpha) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D} \setminus \mathbb{D}_t)} \right) \end{aligned}$$

where we used $S_{e^{-t}}(T_\epsilon(\mathbf{u})) = e^{-2t} T_{e^{-t}\epsilon}(e^{-t}\mathbf{u})$. By Lemma 5.3 the last expression is analytic in \mathbf{u} and since $e^{2\Delta_\alpha s} e^{-t\mathbf{H}} U_0 \left(T(\mathbf{u}) S_{e^{-s}} U_0^{-1} \psi_\alpha \right)$ is independent on s we can take the limit $s \rightarrow \infty$. For this we note that

$$S_{e^{-s-t}} U_0^{-1} \psi_\alpha = e^{\alpha(c-(t+s)Q)} e^{\alpha(1, X(e^{-t-s}\cdot))_{\mathbb{T}}} = e^{-2\Delta_\alpha(s+t)} e^{\alpha c} e^{\alpha(1, X(e^{-t-s}\cdot))_{\mathbb{T}} - \frac{1}{2}\alpha^2 \mathbf{E}(1, X(e^{-t-s}\cdot))_{\mathbb{T}}^2}$$

so that

$$e^{2\Delta_\alpha s} \mathbb{E}_\varphi \left(T(e^{-t}\mathbf{u}) (S_{e^{-s-t}} U_0^{-1} \psi_\alpha) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D} \setminus \mathbb{D}_t)} \right) = e^{-2\Delta_\alpha t} \mathbb{E}_\varphi \left(T(e^{-t}\mathbf{u}) V_\alpha(0) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D} \setminus \mathbb{D}_t)} \right)$$

and the last expression is analytic in \mathbf{u} . Hence by a change of variables in the \mathbf{u} -integral

$$\begin{aligned} e^{-t\mathbf{H}}\psi_{\alpha,\nu,0} &= \frac{e^{2\Delta_\alpha s}}{(2\pi i)^k} \oint_{|\mathbf{u}|=\delta} \mathbf{u}^{1-\nu} e^{-t\mathbf{H}} U_0 \left(T(\mathbf{u}) S_{e^{-s}} U_0^{-1} \psi_\alpha \right) d\mathbf{u} \\ &= \frac{e^{-(2\Delta_\alpha + |\nu|)t}}{(2\pi i)^k} \oint_{|\mathbf{u}|=e^{-t}\delta} \mathbf{u}^{1-\nu} e^{-Qc} \mathbb{E}_\varphi \left(T(\mathbf{u}) V_\alpha(0) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D} \setminus \mathbb{D}_t)} \right) d\mathbf{u} \\ &= \frac{e^{-(2\Delta_\alpha + |\nu|)t}}{(2\pi i)^k} \oint_{|\mathbf{u}|=\delta} \mathbf{u}^{1-\nu} e^{-Qc} \mathbb{E}_\varphi \left(T(\mathbf{u}) V_\alpha(0) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D} \setminus \mathbb{D}_t)} \right) d\mathbf{u} \end{aligned}$$

where in the last step we used analyticity to move the contours to $|\mathbf{u}| = \delta$. \square

In what follows, for fixed $n \geq 1$, we will denote

$$(5.8) \quad \mathcal{Z} := \{ \mathbf{z} = (z_1, \dots, z_n) \mid \forall i \neq j, z_i \neq z_j \text{ and } \forall i, |z_i| < 1 \}.$$

Denoting $\theta(\mathbf{z}) = (\theta(z_1), \dots, \theta(z_n)) \in \mathbb{C}^n$ we have $\theta\mathcal{Z} = \{ (z_1, \dots, z_n) \mid \forall i \neq j, z_i \neq z_j \text{ and } \forall i, |z_i| > 1 \}$.

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ such that $\alpha_i < Q$ for all i we define the function $U_\alpha(\mathbf{z}) : \mathbb{R} \times \Omega_{\mathbb{T}} \rightarrow \mathbb{R}$ by

$$(5.9) \quad U_\alpha(\mathbf{z}, c, \varphi) := \lim_{\epsilon \rightarrow 0} e^{-Qc} \mathbb{E}_\varphi \left[\left(\prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \right) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D})} \right], \quad \text{for } \mathbf{z} \in \mathcal{Z}$$

where $V_{\alpha_i, \epsilon}$ stands for the regularized vertex operator (1.7). Let us set

$$(5.10) \quad s := \sum_{i=1}^n \alpha_i.$$

Remark 5.5. *It follows directly from the construction of correlation functions that for $\mathbf{z} \in \theta\mathcal{Z}$*

$$\langle V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_{\gamma, \nu} = \left(\prod_{i=1}^n |z_i|^{-4\Delta_{\alpha_i}} \right) \langle \Psi_\alpha | U_\alpha(\theta(\mathbf{z})) \rangle_2$$

and these expressions are finite if $\alpha + s > 2Q$ and $\alpha, \alpha_i < Q$.

Lemma 5.6. *Let $\mathbf{z} \in \theta\mathcal{Z}$. Then almost everywhere in c, φ and for all $R > 0$*

$$U_\alpha(\theta(\mathbf{z}))(c, \varphi) \leq e^{(s-Q)(c\wedge 0) - R(c\vee 0)} A(\varphi)$$

where $A \in L^2(\Omega_{\mathbb{T}})$.

Proof. Let $r = \max_i |\theta(z_i)|$ and $\iota(\varphi) = \inf_{x \in \mathbb{D}_r} P\varphi(x)$ and $\sigma(\varphi) = \sup_{x \in \mathbb{D}_r} P\varphi(x)$ with \mathbb{D}_r the disk centered at 0 with radius r . Then

$$U_\alpha(\theta(\mathbf{z}))(c, \varphi) \leq C e^{-Qc} e^{(c+\sigma(\varphi))s} \mathbb{E} e^{-\mu e^{\gamma(c+\iota(\varphi))} Z}$$

where the expectation is over the Dirichlet GFF $X_{\mathbb{D}}$ and

$$Z = \int_{\mathbb{D}_r} (1 - |z|^2)^{\frac{\gamma^2}{2}} e^{\sum_i \gamma \alpha_i G_{\mathbb{D}}(z, \theta(z_i))} M_{\gamma, \mathbb{D}}(dz)$$

where $M_{\gamma, \mathbb{D}}$ is the GMC of $X_{\mathbb{D}}$. For $c < 0$ we use the trivial bound

$$U_\alpha(\theta(\mathbf{z}))(c, \varphi) \leq C e^{(s-Q)c} e^{\sigma(\varphi)s}$$

and for $c > 0$ we note that Z has all negative moments so that for $a > 0$

$$\mathbb{E} e^{-aZ} = \mathbb{E}(aZ)^{-n} (aZ)^n e^{-aZ} \leq n! \mathbb{E}(aZ)^{-n} \leq C_n a^{-n}$$

implying

$$U_\alpha(\theta(\mathbf{z}))(c, \varphi) \leq C_n e^{c(s-Q-\gamma n)} e^{s\sigma(\varphi) - n\gamma\iota(\varphi)}$$

Since $e^{s\sigma(\varphi) - n\iota(\varphi)}$ is in $L^2(\mathbb{P})$ for all s, n the claim follows. \square

Define now the modified Liouville expectation (with now \mathbb{D}_t the disk centered at 0 with radius e^{-t})

$$(5.11) \quad \langle F \rangle_t = \int_{\mathbb{R}} e^{-2Qc} \mathbb{E} \left[F(c, X) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{C} \setminus \mathbb{D}_t)} \right] dc.$$

Also, in the contour integrals below, for vectors $\delta, \tilde{\delta}$ registering the radii of the respective contours, we will put a subscript t when these variables are multiplied by e^{-t} , namely $\delta_t := e^{-t}\delta$ and similarly for $\tilde{\delta}_t$. Then we get

Corollary 5.7. *Let $\mathbf{z} \in \theta\mathcal{Z}$. For $\alpha \in \mathbb{R}$ as in Proposition 5.2 and $\alpha + \sum_i \alpha_i > 2Q$, we have*

$$(5.12) \quad \langle \Psi_{\alpha, \nu, \nu'} | U_\alpha(\theta(\mathbf{z})) \rangle_2$$

$$(5.13) \quad = \left(\prod_{i=1}^n |z_i|^{4\Delta_{\alpha_i}} \right) \times \frac{1}{(2\pi i)^{k+j}} \lim_{t \rightarrow \infty} \oint_{|\mathbf{u}|=\delta_t} \oint_{|\mathbf{v}|=\delta'_t} \mathbf{u}^{1-\nu} \bar{\mathbf{v}}^{1-\nu'} \langle T(\mathbf{u}) \bar{T}(\mathbf{v}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t d\bar{\mathbf{v}} d\mathbf{u}.$$

Proof. Combining Proposition 5.2 and Lemma 5.6 the existence of the limit

$$\langle \Psi_{\alpha, \nu, \nu'} | U_\alpha(\theta(\mathbf{z})) \rangle_2 = \lim_{t \rightarrow \infty} e^{(2\Delta_\alpha + |\nu| + |\nu'|)t} \langle e^{-t\mathbf{H}} \psi_{\alpha, \nu, \nu'} | U_\alpha(\theta(\mathbf{z})) \rangle_2$$

follows. By Lemma 5.4 the RHS is given by the RHS of (5.12). \square

Here is the main result of this section:

Proposition 5.8. *Let $\mathbf{z} \in \theta\mathcal{Z}$. For $\alpha \in \mathbb{R}$ as in Proposition 5.2 and $\alpha + \sum_i \alpha_i > 2Q$ and for all i $\alpha_i < Q$, we have in the distributional sense*

$$\langle \Psi_{\alpha, \nu, \bar{\nu}} | U_{\alpha}(\theta(\mathbf{z})) \rangle_2 = \left(\prod_{i=1}^n |z_i|^{4\Delta_{\alpha_i}} \right) \times \mathbf{D}_{\nu} \bar{\mathbf{D}}_{\bar{\nu}} (V_{\alpha}(0) \prod_{i=1}^n V_{\alpha_i}(z_i))_{\gamma, \mu}$$

where the differential operators \mathbf{D}_{ν} , $\bar{\mathbf{D}}_{\bar{\nu}}$ are defined by

$$(5.14) \quad \mathbf{D}_{\nu} = \mathbf{D}_{\nu_k} \dots \mathbf{D}_{\nu_1} \quad \text{and} \quad \bar{\mathbf{D}}_{\bar{\nu}} = \bar{\mathbf{D}}_{\bar{\nu}_j} \dots \bar{\mathbf{D}}_{\bar{\nu}_1}$$

where for $n \in \mathbb{N}$

$$(5.15) \quad \mathbf{D}_n = \sum_{i=1}^n \left(-\frac{1}{z_i^{n-1}} \partial_{z_i} + \frac{(n-1)}{z_i^n} \Delta_{\alpha_i} \right)$$

$$(5.16) \quad \bar{\mathbf{D}}_n = \sum_{i=1}^n \left(-\frac{1}{\bar{z}_i^{n-1}} \partial_{\bar{z}_i} + \frac{(n-1)}{\bar{z}_i^n} \Delta_{\alpha_i} \right)$$

Proof. Section 6 will be devoted to the proof of this proposition. \square

5.4. Proof of Proposition 1.10. We are now in position to deduce the structure of 3 point correlation functions involving descendant fields. For this we first need the following lemma concerning analyticity of $U_{\alpha}(\mathbf{z})$ in the parameter $\alpha \in \mathbb{C}^n$, proved in Appendix B.1

Lemma 5.9. *For fixed $\mathbf{z} \in \mathcal{Z}$ the mapping $\alpha \mapsto U_{\alpha}(\mathbf{z}) \in e^{\beta' \rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ extends analytically over a complex neighborhood $\mathcal{A}_{\mathbb{T}}^n$ in \mathbb{C}^n of the set $\{\alpha \in \mathbb{R}^n \mid \forall i, \alpha_i < Q\}$, for arbitrary $\beta' < \text{Re}(s) - Q$ (recall (5.10)). This analytic extension is continuous over $(\alpha, \mathbf{z}) \in \mathcal{A}_{\mathbb{T}}^n \times \mathcal{Z}$.*

The first conclusion we want to draw is the fact that the pairing of Ψ_{α} with $U_{\alpha}(\theta(\mathbf{z}))$ (in the case $n = 2$) is related to the DOZZ formula when α is on the spectrum line $Q + i\mathbb{R}$.

Lemma 5.10. *Here we fix $n = 2$ and we consider $\mathbf{z} \in \theta\mathcal{Z}$. The mapping $(\alpha, \alpha) \mapsto \langle \Psi_{\alpha} | U_{\alpha}(\theta(\mathbf{z})) \rangle_2$ is analytic over the set*

$$\{(\alpha, \alpha) \mid Q - \frac{\gamma}{2} < \text{Re}(\alpha) \leq Q, \text{Re}(\alpha + \alpha_1 + \alpha_2) > 2Q, \alpha \in \mathcal{A}_{\mathbb{T}}^2\}$$

and, over this set, we have the relation

$$\langle \Psi_{\alpha} | U_{\alpha}(\theta(\mathbf{z})) \rangle_2 = |z_1|^{2\Delta_{\alpha_2} - 2\Delta_{\alpha} + 2\Delta_{\alpha_1}} |z_1 - z_2|^{2\Delta_{\alpha} - 2\Delta_{\alpha_1} - 2\Delta_{\alpha_2}} |z_2|^{2\Delta_{\alpha_1} - 2\Delta_{\alpha} + 2\Delta_{\alpha_2}} \frac{1}{2} C_{\gamma, \mu}^{\text{DOZZ}}(\alpha, \alpha_1, \alpha_2).$$

In particular this relation holds for $\alpha = Q + iP$ with $P \in (0, +\infty)$ and $\alpha_1, \alpha_2 \in (-\infty, Q)$ with $\alpha_1 + \alpha_2 > Q$.

Proof. By Proposition 3.16, the mapping $\alpha \mapsto \Psi_{\alpha} \in e^{-\frac{\beta}{2} \rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ is analytic provided that $0 < \beta < \gamma$ in the region $\{\alpha \in \mathbb{C} \mid Q - \text{Re}(\alpha) < \beta/2\}$. Combining with Lemma 5.9 (with $n = 2$) produces directly the region of analyticity we claim. Furthermore when all the parameters are real, by Remark 5.5 we have

$$\langle V_{\alpha}(0) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \rangle_{\gamma, \nu} = \left(\prod_{i=1}^2 |z_i|^{-4\Delta_{\alpha_i}} \right) \langle \Psi_{\alpha} | U_{\alpha}(\theta(\mathbf{z})) \rangle_2.$$

Also, for real parameters, the RHS coincide with the DOZZ formula [KRV20], namely

$$\langle V_{\alpha}(0) V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \rangle_{\gamma, \nu} = |z_1|^{2\Delta_{\alpha_2} - 2\Delta_{\alpha} - 2\Delta_{\alpha_1}} |z_1 - z_2|^{2\Delta_{\alpha} - 2\Delta_{\alpha_1} - 2\Delta_{\alpha_2}} |z_2|^{2\Delta_{\alpha_1} - 2\Delta_{\alpha} - 2\Delta_{\alpha_2}} \frac{1}{2} C_{\gamma, \mu}^{\text{DOZZ}}(\alpha, \alpha_1, \alpha_2).$$

This proves the claim. \square

Now we would like to use Ward identities, i.e. Proposition 5.8, to express the correlations of descendant fields with two insertions $\langle \Psi_{\alpha, \nu, \bar{\nu}} | U_{\alpha}(\theta(\mathbf{z})) \rangle_2$ (here with $n = 2$) in terms of differential operators applied to to correlation of primaries $\langle \Psi_{\alpha} | U_{\alpha}(\theta(\mathbf{z})) \rangle_2$ when the parameter α is close to the spectrum line $\alpha \in Q + i\mathbb{R}$. This is not straightforward because Proposition 5.8 is not only restricted to real values of the parameter but also because the constraint on α , which forces it to be negatively large, implies to have n large in order for the global Seiberg bound $\alpha + \sum_i \alpha_i > 2Q$ to be satisfied. Transferring Ward's relations close to the line spectrum is thus our next task.

For this, recall Proposition 3.18 (and its remark just below) that establishes analyticity of the mappings $\alpha \mapsto \Psi_{\alpha, \nu, \bar{\nu}} \in e^{-\beta \rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ and $\alpha \mapsto \Psi_{\alpha} \in e^{-\beta \rho} L^2(\mathbb{R} \times \Omega_{\mathbb{T}})$ are analytic over a connected domain $\mathcal{A}_{\nu, \bar{\nu}} \subset \mathbb{C}$ such that

- $\mathcal{A}_{\nu, \bar{\nu}}$ contains a complex neighborhood of the spectrum line $\{\alpha = Q + iP \mid P \in (0, +\infty)\}$ over which $\beta < 0$ can be chosen arbitrarily close to 0, see Propositions 3.16,
- $\mathcal{A}_{\nu, \bar{\nu}}$ contains a complex neighborhood of the half-line $\{\alpha \in \mathbb{R} \mid \alpha < (-\frac{|\nu|+|\bar{\nu}|-2}{\gamma}) \wedge (Q - \gamma)\}$ over which $\beta < Q - \text{Re}(\alpha) - \gamma$ (as explained in Proposition 5.2).

Therefore, for arbitrarily fixed n and $\mathbf{z} \in \theta\mathcal{Z}$, the pairings $(\alpha, \boldsymbol{\alpha}) \mapsto \langle \Psi_{\alpha, \nu, \bar{\nu}} | U_{\boldsymbol{\alpha}}(\theta(\mathbf{z})) \rangle_2$ and $(\alpha, \boldsymbol{\alpha}) \mapsto \langle \Psi_{\alpha} | U_{\boldsymbol{\alpha}}(\theta(\mathbf{z})) \rangle_2$ are holomorphic in the region $\mathcal{A}_{\nu, \bar{\nu}} \odot \mathcal{A}_U^n := \{(\alpha, \boldsymbol{\alpha}) \in \mathcal{A}_{\nu, \bar{\nu}} \times \mathcal{A}_U^n \mid \beta < \text{Re}(s) - Q\}$. Now we want to make sure that the subsets

$$\mathcal{S} := \{(\alpha, \boldsymbol{\alpha}) \mid \forall i \alpha_i \in \mathbb{R} \text{ and } \alpha_i < Q, s - Q > 0, \alpha = Q + iP \text{ with } P \in (0, +\infty)\}$$

and

$$\mathcal{R}_{\nu, \bar{\nu}} := \{(\alpha, \boldsymbol{\alpha}) \mid \forall i \alpha_i \in \mathbb{R} \text{ and } \alpha_i < Q, \alpha + s - 2Q > 0, \alpha \in \mathbb{R}, \alpha < (-\frac{|\nu|+|\bar{\nu}|-2}{\gamma}) \wedge (Q - \gamma)\}$$

are both non-empty and in the same connected component of $\mathcal{A}_{\nu, \bar{\nu}} \odot \mathcal{A}_U^n$. For this, we can choose a continuous path $\sigma : [0, 1] \rightarrow \mathbb{C}$ joining both sets and a complex neighborhood $\mathcal{A}_{\sigma} \subset \mathcal{A}_{\nu, \bar{\nu}}$ of this path such that $\sup_{\alpha \in \mathcal{A}_{\sigma}} \beta < +\infty$ (β being a function of α). Now we choose n large enough in such a way that $(n-1)Q > \sup_{\alpha \in U} \beta$ and $(n-2)Q > (-\frac{|\nu|+|\bar{\nu}|-2}{\gamma}) \wedge (\gamma - Q)$. These two conditions makes sure that \mathcal{S} and $\mathcal{R}_{\nu, \bar{\nu}}$ are both non-empty and in the same connected component of $\mathcal{A}_{\nu, \bar{\nu}} \odot \mathcal{A}_U^n$.

Now we exploit the Ward identities, valid on $\mathcal{R}_{\nu, \bar{\nu}}$. Let us consider a smooth compactly supported function φ on $\theta\mathcal{Z}$. The mapping

$$(\alpha, \boldsymbol{\alpha}) \in \mathcal{A}_{\nu, \bar{\nu}} \odot \mathcal{A}_U^n \mapsto \int \langle \Psi_{\alpha, \nu, \bar{\nu}} | U_{\boldsymbol{\alpha}}(\theta(\mathbf{z})) \rangle_2 \bar{\varphi}(\mathbf{z}) \, d\mathbf{z}$$

is thus analytic. Furthermore on $\mathcal{R}_{\nu, \bar{\nu}}$ and by Proposition 5.8, it coincides with the mapping

$$\begin{aligned} (\alpha, \boldsymbol{\alpha}) \in \mathcal{A}_{\nu, \bar{\nu}} \odot \mathcal{A}_U^n &\mapsto \int \langle V_{\alpha}(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_{\gamma, \mu} \overline{\bar{\mathbf{D}}_{\bar{\nu}}^* \mathbf{D}_{\nu}^*} \left(\varphi(\mathbf{z}) \prod_{i=1}^n |z_i|^{4\Delta_{\alpha_i}} \right) d\mathbf{z} \\ &= \int \left(\prod_{i=1}^n |z_i|^{-4\Delta_{\alpha_i}} \right) \langle \Psi_{\alpha} | U_{\boldsymbol{\alpha}}(\theta(\mathbf{z})) \rangle_2 \overline{\bar{\mathbf{D}}_{\bar{\nu}}^* \mathbf{D}_{\nu}^*} \left(\varphi(\mathbf{z}) \prod_{i=1}^n |z_i|^{4\Delta_{\alpha_i}} \right) d\mathbf{z} \end{aligned}$$

where we have introduced the (adjoint) operator \mathbf{D}_{ν}^* by

$$\int_{\mathbb{C}^n} \mathbf{D}_{\nu} f(\mathbf{z}) \bar{\varphi}(z) \, d\mathbf{z} = \int_{\mathbb{C}^n} f(\mathbf{z}) \overline{\bar{\mathbf{D}}_{\bar{\nu}}^* \varphi(z)} \, d\mathbf{z}$$

for all functions f in the domain of \mathbf{D}_{ν} and all smooth compactly supported functions φ in \mathbb{C}^n (and similarly for $\bar{\mathbf{D}}_{\bar{\nu}}$). Therefore both mappings are analytic and coincide on \mathcal{R} , thus on the connected component of $\mathcal{A}_{\nu, \bar{\nu}} \odot \mathcal{A}_U$ containing \mathcal{R} , and finally on \mathcal{S} . Notice that, on \mathcal{S} , we can take all the α_i 's equal to 0 but the first two of them provided they satisfy $\alpha_1 + \alpha_2 > Q$. This fact being valid for all test function φ , we deduce that the relation

$$\begin{aligned} &\langle \Psi_{Q+iP, \nu, \bar{\nu}} | U(V_{\alpha_1}(\theta(z_1))V_{\alpha_2}(\theta(z_2))) \rangle_2 \\ &= \frac{1}{2} C_{\gamma, \mu}^{\text{DOZZ}}(Q+iP, \alpha_1, \alpha_2) |z_1|^{4\Delta_{\alpha_1}} |z_2|^{4\Delta_{\alpha_2}} \\ &\quad \times \mathbf{D}_{\nu} \bar{\mathbf{D}}_{\bar{\nu}} \left(|z_1|^{2\Delta_{\alpha_2} - 2\Delta_{Q+iP} - 2\Delta_{\alpha_1}} |z_1 - z_2|^{2\Delta_{Q+iP} - 2\Delta_{\alpha_1} - 2\Delta_{\alpha_2}} |z_2|^{2\Delta_{\alpha_1} - 2\Delta_{Q+iP} - 2\Delta_{\alpha_2}} \right) \end{aligned}$$

holds for almost every z_1, z_2 and $\alpha_1, \alpha_2 < Q$ such that $\alpha_1 + \alpha_2 > Q$, and thus for every $z_1, z_2 \in \theta\mathcal{Z}$ as both sides are continuous in these variables. From this relation and after some algebra, sending $z_2 \rightarrow \infty$, we end up with the claimed relation. \square

6. PROOF OF PROPOSITION 5.8

6.1. Preliminary remarks. Before proceeding to computations, we stress that the reader should keep in mind that the SET field $T(u)$ is not a proper random field. In particular the expectation in (5.12) is a notation for the object constructed in the limit as $\epsilon \rightarrow 0$ and $t \rightarrow \infty$. In LCFT the construction of the correlation functions of the SET is subtle. This was done in [KRV19] only for one or two SET insertions. However the situation is here much simpler and we will not have to rely on [KRV19]. The reason is that we need to deal with correlation functions with a regularized LCFT expectation $\langle \cdot \rangle_t$ where we have replaced

$M_\gamma(\mathbb{C})$ by $M_\gamma(\mathbb{C} \setminus \mathbb{D}_t)$ in (5.11), and all the SET insertions that we will consider are located in \mathbb{D}_t which as we will see makes them much more regular than in the full LCFT.

The regularized SET field $T_\epsilon(u)$ is a proper random field and its correlation functions in the presence of the vertex operators are defined as limits of the corresponding ones with regularized vertex operators

$$(6.1) \quad \langle T_\epsilon(\mathbf{u}) \bar{T}_{\epsilon'}(\mathbf{v}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t = \lim_{\epsilon'' \rightarrow 0} \langle T_\epsilon(\mathbf{u}) \bar{T}_{\epsilon'}(\mathbf{v}) V_{\alpha, \epsilon''}(0) \prod_{i=1}^n V_{\alpha_i, \epsilon''}(z_i) \rangle_t.$$

The existence of this limit follows from the representation of the expectation on the RHS as a GFF expectation of an explicit function of a GMC integral [DKRV16]. And in particular the limit is independent of the regularization procedure used for the vertex operators. For simplicity we will use in this section the following:

$$(6.2) \quad V_{\alpha, \epsilon}(x) = \epsilon^{\frac{\alpha^2}{2}} e^{\alpha \phi_\epsilon(x)} = |x|_+^{-4\Delta_\alpha} e^{\alpha c} e^{\alpha X_\epsilon(x) - \frac{\alpha^2}{2} \mathbb{E} X_\epsilon(x)^2} (1 + \mathcal{O}(\epsilon))$$

where X_ϵ is the same regularization as in T_ϵ . The $\mathcal{O}(\epsilon)$ will drop out from all terms in the $\epsilon \rightarrow 0$ limit and will not be displayed below.

The proof of Proposition 5.8 consists of using Gaussian integration by parts to the the $T_\epsilon(\mathbf{u})$ and $\bar{T}_{\epsilon'}(\mathbf{v})$ factors in (6.1) to which we now turn.

6.2. Gaussian integration by parts. (6.1) is analysed using Gaussian integration by parts. For a centered Gaussian vector (X, Y_1, \dots, Y_N) and a smooth function f on \mathbb{R}^N , the Gaussian integration by parts formula is

$$\mathbb{E}[X f(Y_1, \dots, Y_N)] = \sum_{k=1}^N \mathbb{E}[X Y_k] \mathbb{E}[\partial_k f(Y_1, \dots, Y_N)].$$

Applied to the LCFT this leads to the following formula. Let $\phi = c + X - 2Q \log |z|_+$ be the Liouville field and F a smooth function on \mathbb{R}^N . Define for $u, v \in \mathbb{C}$

$$(6.3) \quad C(u, v) = -\frac{1}{2} \frac{1}{u-v}, \quad C_{\epsilon, \epsilon'}(u, v) = \int \rho_\epsilon(u-u') \rho_{\epsilon'}(v-v') C(u', v') du' dv'$$

with $(\rho_\epsilon)_\epsilon$ a mollifying family of the type $\rho_\epsilon(\cdot) = \epsilon^{-2} \rho(|\cdot|/\epsilon)$. Then for $z, x_1, \dots, x_N \in \mathbb{C}$

$$(6.4) \quad \langle \partial_z \phi_\epsilon(z) F(\phi_{\epsilon'}(x_1), \dots, \phi_{\epsilon'}(x_N)) \rangle_t = \sum_{k=1}^N C_{\epsilon, \epsilon'}(z, x_k) \langle \partial_k F(\phi_{\epsilon'}(x_1), \dots, \phi_{\epsilon'}(x_N)) \rangle_t \\ - \mu \gamma \int_{\mathbb{C}_t} C_{\epsilon, 0}(z, x) \langle V_\gamma(x) F(\phi_{\epsilon'}(x_1), \dots, \phi_{\epsilon'}(x_N)) \rangle_t dx$$

where F in the applications below is such that all the terms here are well defined. Note that $\partial_u G(u, v) = C(u, v) + \frac{1}{2} u^{-1} 1_{|u|>1}$. The virtue of the Liouville field is that the annoying metric dependent terms $u^{-1} 1_{|u|>1}$ drop out from the formulae. This fact is nontrivial and it was proven in [KRV19, Subsection 3.2], for the case $t = \infty$ with F corresponding to product of vertex operators. The proof goes the same way to produce (6.4) with a finite t .

The first application of this formula is a direct proof of the existence of the $\epsilon, \epsilon' \rightarrow 0$ limit of (6.1) which will be useful also later in the proof. We have

Proposition 6.1. *The functions (6.1) converge uniformly on compact subsets of $(\mathbf{u}, \mathbf{v}, \mathbf{z}) \in \mathbb{D}_t \times \mathbb{D}_t \times \theta \mathcal{Z}$*

$$(6.5) \quad \lim_{\epsilon, \epsilon' \rightarrow 0} \langle T_\epsilon(\mathbf{u}) \bar{T}_{\epsilon'}(\mathbf{v}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t := \langle T(\mathbf{u}) \bar{T}(\mathbf{v}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t$$

where the limit is analytic in $\mathbf{u} \in \mathcal{O}_t$ and anti-analytic in $\mathbf{v} \in \mathcal{O}_t$.

Proof. We consider for simplicity only the case $\tilde{\nu} = 0$. The LHS is defined as the limit $\epsilon'' \rightarrow 0$ in (6.1) but we will for clarity work directly with $\epsilon'' = 0$ as it will be clear below that this limit trivially exists. Indeed, the functions $C_{\epsilon, \epsilon'}(u, v)$ are smooth for $\epsilon, \epsilon' > 0$ and they converge together with their derivatives uniformly on $|u| < e^{-s}$, $|v| \geq e^{-t}$ for all $s > t$ to the derivatives of $C(u, v)$.

We will now apply (6.4) to all the ϕ_ϵ factors in the SET tensors in (6.5) one after the other. To make this systematic let us introduce the notation

$$(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3) := (\partial_z \phi_\epsilon, \partial_z^2 \phi_\epsilon, (\partial_z \phi_\epsilon)^2 - \mathbb{E}(\partial_z \phi_\epsilon)^2).$$

Applying the integration by parts formula to $\partial_z \phi_\epsilon(u_k)$ (or to $\partial_z^2 \phi_\epsilon(u_k)$ if $i_k = 2$ below) we obtain

$$\begin{aligned} \langle \prod_{j=1}^k \mathcal{O}_{i_j}(u_j) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t &= \sum_{l=1}^{k-1} \langle \{ \mathcal{O}_{i_k}(u_k) \mathcal{O}_{i_l}(u_l) \} \prod_{j \neq k, l} \mathcal{O}_{i_j}(u_j) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ &\quad + \sum_{l=0}^n \alpha_l \langle \{ \mathcal{O}_{i_k}(u_k) \phi(z_l) \} \prod_{j \neq k} \mathcal{O}_{i_j}(u_j) V_\alpha(0) \prod_{i \neq l} V_{\alpha_i}(z_i) \rangle_t \\ &\quad - \mu \gamma \int_{\mathbb{C}_t} \langle \{ \mathcal{O}_{i_k}(u_k) \phi(x) \} \prod_{j \neq k} \mathcal{O}_{i_j}(u_j) V_\alpha(0) V_\gamma(x) \prod_{i \neq l} V_{\alpha_i}(z_i) \rangle_t dx \end{aligned}$$

where $\alpha_0 = \alpha$ and $z_0 = 0$ and the contractions are defined by

$$\begin{aligned} \{ \mathcal{O}_1(u) \mathcal{O}_1(v) \} &= \partial_v C_{\epsilon, \epsilon}(u, v) & \{ \mathcal{O}_2(u) \mathcal{O}_2(v) \} &= \partial_u \partial_v^2 C_{\epsilon, \epsilon}(u, v) \\ \{ \mathcal{O}_1(u) \mathcal{O}_3(v) \} &= 2 \mathcal{O}_1(v) \partial_v C_{\epsilon, \epsilon}(u, v) & \{ \mathcal{O}_1(u) \mathcal{O}_2(v) \} &= \partial_v^2 C_{\epsilon, \epsilon}(u, v) \\ \{ \mathcal{O}_3(u) \mathcal{O}_1(v) \} &= \mathcal{O}_1(u) \partial_v C_{\epsilon, \epsilon}(u, v) & \{ \mathcal{O}_2(u) \mathcal{O}_1(v) \} &= \partial_u \partial_v C_{\epsilon, \epsilon}(u, v) \\ \{ \mathcal{O}_3(u) \mathcal{O}_3(v) \} &= 2 \mathcal{O}_1(u) \mathcal{O}_1(v) \partial_v C_{\epsilon, \epsilon}(u, v) & \{ \mathcal{O}_2(u) \mathcal{O}_3(v) \} &= 2 \mathcal{O}_1(v) \partial_u \partial_v C_{\epsilon, \epsilon}(u, v) \\ \{ \mathcal{O}_3(u) \mathcal{O}_2(v) \} &= \mathcal{O}_1(u) \partial_v^2 C_{\epsilon, \epsilon}(u, v) \end{aligned}$$

etc. Similarly

$$\{ \mathcal{O}_1(u) \phi(x) \} = C_{\epsilon, 0}(u, x), \quad \{ \mathcal{O}_2(u) \phi(x) \} = \partial_u C_{\epsilon, 0}(u, x), \quad \{ \mathcal{O}_3(u) \phi(x) \} = \mathcal{O}_1(u) C_{\epsilon, 0}(u, x).$$

Since $T_\epsilon = Q \mathcal{O}_2 - \mathcal{O}_3$ we can iterate the integration by parts formula to obtain an expression for $\langle T_\epsilon(\mathbf{u}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t$ as a sum of terms of the form

$$(6.6) \quad C \prod_{\alpha, \beta} \partial^{i_{\alpha\beta}} C_{\epsilon, \epsilon}(u_\alpha, u_\beta) \prod_{\alpha, l} \partial^{j_{\alpha, l}} C_{\epsilon, 0}(u_\alpha, z_l) \int_{\mathbb{C}_t^m} \prod_{\alpha, k} \partial^{l_{\alpha, k}} C_{\epsilon, 0}(u_\alpha, x_k) \langle V_\alpha(0) \prod_{j=1}^m V_\gamma(x_j) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx$$

where the products run through some subsets of the index values. We have for all compact $K \subset \mathbb{D}_t$

$$\sup_{u \in K, x \in \mathbb{C}_t} \sup_{\epsilon} |\partial_u^l \partial_u^{l'} C_{\epsilon, 0}(u, x)| < \infty.$$

Hence the expression (6.6) is bounded together with all its derivatives in \mathbf{u} uniformly in ϵ by

$$(6.7) \quad C \int_{\mathbb{C}_t^m} \langle V_\alpha(0) \prod_{j=1}^m V_\gamma(x_j) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx.$$

which by the lemma below is finite. The limit of (6.6) and all its derivatives exist by dominated convergence. Clearly the ∂_u derivatives vanish so the limit is analytic in \mathbf{u} . \square

We need the following KPZ identity for the a priori bound (6.7) (recall (5.10))

Lemma 6.2. *The functions $\mathbf{x} \in \mathbb{C}_t^n \rightarrow \langle V_\alpha(0) \prod_{j=1}^m V_\gamma(x_j) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t$ are integrable and*

$$(6.8) \quad \int_{\mathbb{C}_t^m} \langle V_\alpha(0) \prod_{j=1}^m V_\gamma(x_j) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx = C \int_{\mathbb{C}_t^m} \langle V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx$$

where $C = (\mu\gamma)^{-m} \prod_{l=0}^{m-1} (\alpha + s + \gamma l - 2Q)$.

Proof. See [KRV19, Lemma 3.3]. Briefly,

$$\langle V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t = \mu^{\frac{2Q-s-\alpha}{\gamma}} \langle V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \Big|_{\mu=1}$$

and the LHS of (6.8) equals $(-\partial_\mu)^m \langle V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t$. \square

The representation (6.6) will not be useful for a direct proof of the Ward identity due to the V_γ insertions. We will rather use the integration by parts inductively, first to $T_\epsilon(u_k)$ which corresponds to the largest contour in the contour integrals in expression (5.12), and by showing then that at each step the V_γ insertions give rise to the derivatives in the Proposition 5.8. We give now the inductive step to prove this claim, stated for simplicity for the case $\tilde{\nu} = 0$. For this, we introduce the (adjoint) operator \mathbf{D}_n^* defined (by duality) by

$$\int_{\mathbb{C}^n} \mathbf{D}_n f(\mathbf{z}) \bar{\varphi}(z) d\mathbf{z} = \int_{\mathbb{C}^n} f(\mathbf{z}) \overline{\mathbf{D}_n^* \varphi}(z) d\mathbf{z}$$

for all functions f in the domain of \mathbf{D}_n and all smooth compactly supported functions φ in \mathbb{C}^n . Then

Proposition 6.3. *Let $\varphi \in C_0^\infty(\theta\mathcal{Z})$ be a smooth compactly supported function in $\theta\mathcal{Z} \subset \mathbb{C}^n$ and define*

$$\mathcal{T}_t(\nu, \varphi) := \int \left(\frac{1}{(2\pi i)^k} \oint_{|\mathbf{u}|=\delta_t} \mathbf{u}^{1-\nu} \langle T(\mathbf{u}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t d\mathbf{u} \right) \bar{\varphi}(\mathbf{z}) d\mathbf{z}.$$

Then for $\nu^{(k)} = (\nu_1, \dots, \nu_{k-1})$

$$\mathcal{T}_t(\nu, \varphi) = \mathcal{T}_t(\nu^{(k)}, \mathbf{D}_{\nu_k}^* \varphi) + \mathcal{B}_t(\nu, \varphi)$$

where

$$(6.9) \quad |\mathcal{B}_t(\nu, \varphi)| \leq C e^{(\alpha+|\nu|-2)t} \int \langle V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t |\varphi(\mathbf{z})| d\mathbf{z}.$$

Proof of Proposition 5.8. Iterating Proposition 6.3 we get

$$\mathcal{T}_t(\nu, \varphi) = \int \langle V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \overline{\mathbf{D}_\nu^* \varphi}(\mathbf{z}) d\mathbf{z} + \mathcal{B}_t(\varphi)$$

where $\mathcal{B}_t(\varphi)$ satisfies

$$\mathcal{B}_t(\varphi) \leq C \sum_{\ell=1}^k e^{(\alpha+|\nu|-2)t} \int \langle V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t |\mathbf{D}_{\nu_{\ell+1}}^* \cdots \mathbf{D}_{\nu_k}^* \varphi(\mathbf{z})| d\mathbf{z}$$

where by convention $\mathbf{D}_{\nu_{\ell+1}}^* \cdots \mathbf{D}_{\nu_k}^* \varphi = \varphi$ if $\ell = k$. The functions $\mathbf{z} \rightarrow \langle V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t$ are continuous on $\theta\mathcal{Z}$ and converge uniformly as $t \rightarrow \infty$ on compact subsets of $\theta\mathcal{Z}$ to the function $\langle V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle$. Hence $\mathcal{T}_t(\nu, \cdot)$ converges in the Frechet topology of $\mathcal{D}'(\theta\mathcal{Z})$ to the required limit since $\mathcal{B}_t(\varphi)$ goes to 0 as t goes to infinity (recall that $\alpha + |\nu| - 2 < 0$). \square

6.3. Proof of Proposition 6.3. We start the proof of Proposition 6.3 by applying Gaussian integration by parts formula twice to the $T(u_k)$. This produces plenty of terms which we group in four contributions:

$$(6.10) \quad \langle T(\mathbf{u}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t = R(\mathbf{u}, \mathbf{z}) + M(\mathbf{u}, \mathbf{z}) + N(\mathbf{u}, \mathbf{z}) + D(\mathbf{u}, \mathbf{z}).$$

In $R(\mathbf{u})$ we group all the contractions with $C(u_k, u_l)$ between u_k and u_l , $l < k$ and u_k and 0. These terms do not contribute to the contour integral of the u_k variable since they give rise to integrals of the form

$$\oint_{|u_k|=e^{-t}\delta_k} u_k^{1-\nu_k} (u_k - v)^{-a} (u_k - w)^{-b} du_k$$

where $v, w \in \{0, u_1, \dots, u_{k-1}\}$ and $a + b \geq 2$. Since $|v|, |w| < e^{-t}\delta_k$ and $\nu_k > 0$, this integral vanishes. We conclude

$$(6.11) \quad \oint_{|u_k|=e^{-t}\delta_k} u_k^{1-\nu_k} R(\mathbf{u}, \mathbf{z}) du_k = 0.$$

For the benefit of the reader we display all the terms in $R(\mathbf{u}, \mathbf{z})$ in the Appendix B.

Let us now introduce the notations $\mathbf{u}^{(k)} := (u_1, \dots, u_{k-1})$ and $\mathbf{u}^{(k, \ell)} := (u_1, \dots, u_{\ell-1}, u_{\ell+1}, \dots, u_{k-1})$. The second contribution in (6.10) collects the contractions hitting only one V_{α_p} :

$$(6.12) \quad M(\mathbf{u}, \mathbf{z}) = \sum_{p=1}^n \left(\frac{Q\alpha_p}{2} - \frac{\alpha_p^2}{4} \right) \frac{1}{(u_k - z_p)^2} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t.$$

We can then do the u_k integral explicitly to obtain

$$(6.13) \quad \int \left(\frac{1}{(2\pi i)^k} \oint_{|\mathbf{u}|=\delta_t} \mathbf{u}^{1-\nu} \mathbf{M}(\mathbf{u}, \mathbf{z}) d\mathbf{u} \right) \bar{\varphi}(\mathbf{z}) d\mathbf{z}$$

$$(6.14) \quad = \int \left(\sum_{p=1}^n \frac{\nu_k - 1}{z_p^{\nu_k}} \Delta_{\alpha_p} \right) \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \bar{\varphi}(\mathbf{z}) d\mathbf{z}$$

Note that this term contains the constant part of the differential operator \mathbf{D}_{ν_k} .

The third contribution is given by terms where all contractions hit V_γ :

$$(6.15) \quad \begin{aligned} N(\mathbf{u}, \mathbf{z}) &= \left(\frac{\mu\gamma^2}{4} - \frac{\mu\gamma Q}{2} \right) \int_{\mathbb{C}_t} \frac{1}{(u_k - x)^2} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \\ &= -\mu \int_{\mathbb{C}_t} \frac{1}{(u_k - x)^2} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx. \end{aligned}$$

Finally D gathers all the other terms

$$(6.16) \quad D(\mathbf{u}, \mathbf{z}) = \sum_{i=1}^9 T_i(\mathbf{u}, \mathbf{z})$$

with

$$(6.17) \quad \begin{aligned} T_1(\mathbf{u}, \mathbf{z}) &= - \sum_{\ell=1}^{k-1} \sum_{p=1}^n \frac{Q\alpha_p}{(u_k - u_\ell)^3 (u_k - z_p)} \langle T(\mathbf{u}^{(\ell,k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ T_2(\mathbf{u}, \mathbf{z}) &= \sum_{\ell=1}^{k-1} \sum_{p=1}^n \frac{\alpha_p}{(u_k - u_\ell)^2 (u_k - z_p)} \langle \partial_z X(u_\ell) T(\mathbf{u}^{(\ell,k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ T_3(\mathbf{u}, \mathbf{z}) &= - \sum_{p=1}^n \frac{\alpha\alpha_p}{2} \frac{1}{(u_k - z_p) u_k} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ T_4(\mathbf{u}, \mathbf{z}) &= \frac{\mu\gamma}{2} \sum_{p=1}^n \alpha_p \int_{\mathbb{C}_t} \frac{1}{(u_k - z_p)(u_k - x)} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \\ T_5(\mathbf{u}, \mathbf{z}) &= - \sum_{p \neq p'=1}^n \frac{\alpha_p \alpha_{p'}}{4} \frac{1}{(u_k - z_p)(u_k - z_{p'})} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ T_6(\mathbf{u}, \mathbf{z}) &= \mu Q \gamma \sum_{\ell=1}^{k-1} \int_{\mathbb{C}_t} \frac{1}{(u_k - u_\ell)^3 (u_k - x)} \langle T(\mathbf{u}^{(\ell,k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \\ T_7(\mathbf{u}, \mathbf{z}) &= -\mu\gamma \sum_{\ell=1}^{k-1} \int_{\mathbb{C}_t} \frac{1}{(u_k - u_\ell)^2 (u_k - x)} \langle \partial_z X(u_\ell) T(\mathbf{u}^{(\ell,k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \\ T_8(\mathbf{u}, \mathbf{z}) &= \frac{\mu\gamma\alpha}{2} \int_{\mathbb{C}_t} \frac{1}{u_k (u_k - x)} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \\ T_9(\mathbf{u}, \mathbf{z}) &= -\frac{\mu^2\gamma^2}{4} \int_{\mathbb{C}_t} \int_{\mathbb{C}_t} \frac{1}{(u_k - x)(u_k - x')} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) V_\gamma(x') \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx. \end{aligned}$$

We need to show that N and D will give rise (after contour integration) to the ∂_{z_i} -derivatives in the expression $\mathbf{D}_{\nu_k} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t$. To show this we need to analyse N further.

Regularizing the vertex insertions (beside the V_γ insertion, we also regularize the V_{α_i} 's for later need) in $N(\mathbf{u}, \mathbf{z})$ given by (6.15), and performing an integration by parts (Green formula) in the x integral we get

$$\begin{aligned} N(\mathbf{u}, \mathbf{z}) &= -\mu \lim_{\epsilon \rightarrow 0} \int_{\mathbb{C}_t} \partial_x \frac{1}{u_k - x} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_{\gamma, \epsilon}(x) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_t dx \\ &= B_t(\mathbf{u}, \mathbf{z}) + \mu\gamma \lim_{\epsilon \rightarrow 0} \int_{\mathbb{C}_t} \frac{1}{u_k - x} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \partial_x \phi_\epsilon(x) V_{\gamma, \epsilon}(x) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_t dx \\ &=: B_t(\mathbf{u}, \mathbf{z}) + \tilde{N}(\mathbf{u}, \mathbf{z}) \end{aligned}$$

where the boundary term appearing in Green formula has $\epsilon \rightarrow 0$ limit given by

$$(6.18) \quad B_t(\mathbf{u}, \mathbf{z}) := i\mu \oint_{|x|=e^{-t}} \frac{1}{u_k - x} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t d\bar{x}$$

and we used (1.7) to write

$$\partial_x V_{\gamma, \epsilon}(x) = \alpha \partial_x \phi_\epsilon(x) V_{\gamma, \epsilon}(x).$$

In $\tilde{N}(\mathbf{u}, \mathbf{z})$ we integrate by parts the $\partial_x \phi_\epsilon(x)$ and end up with

$$(6.19) \quad \begin{aligned} \tilde{N}(\mathbf{u}, \mathbf{z}) &= -\mu Q \gamma \sum_{\ell=1}^{k-1} \int_{\mathbb{C}_t} \frac{1}{(u_k - x)(x - u_\ell)^3} \langle T(\mathbf{u}^{(\ell, k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \\ &\quad + \mu \gamma \sum_{\ell=1}^{k-1} \int_{\mathbb{C}_t} \frac{1}{(u_k - x)(x - u_\ell)^2} \langle \partial_z X(u_\ell) T(\mathbf{u}^{(\ell, k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \\ &\quad - \frac{\mu \gamma \alpha}{2} \int_{\mathbb{C}_t} \frac{1}{(u_k - x)x} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \\ &\quad + \frac{\mu^2 \gamma^2}{2} \int_{\mathbb{C}_t} \int_{\mathbb{C}_t} \frac{1}{(u_k - x)(x - x')} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) V_\gamma(x') \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \\ &\quad + \mu \gamma \lim_{\epsilon \rightarrow 0} \sum_{p=1}^n \int_{\mathbb{C}_t} \frac{\alpha_p}{u_k - x} C_{\epsilon, 0}(x, z_p) \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_t dx \\ &=: T'_6(\mathbf{u}, \mathbf{z}) + T'_7(\mathbf{u}, \mathbf{z}) + T'_8(\mathbf{u}, \mathbf{z}) + T'_9(\mathbf{u}) + T'_4(\mathbf{u}, \mathbf{z}) \end{aligned}$$

where again we took the $\epsilon \rightarrow 0$ limit in the terms where it was obvious. In particular this identity proves that the limit on the RHS, denoted by $T'_4(\mathbf{u}, \mathbf{z})$, exists. The numbering of these terms and the ones below will be used when comparing with (6.17).

Derivatives of correlation functions. We want to compare the expression (6.10) to derivatives of the function $\langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t$. We need to treat separately the cases $\nu_k \geq 2$ and $\nu_k = 1$.

Case $\nu_k \geq 2$. We have

Lemma 6.4. *Let*

$$I_\epsilon(\mathbf{u}, \mathbf{z}) := \sum_{p=1}^n \frac{1}{u_k - z_p} \partial_{z_p} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_t.$$

Then $\lim_{\epsilon \rightarrow 0} I_\epsilon(\mathbf{u}, \mathbf{z}) := I(\mathbf{u}, \mathbf{z})$ exists and defines a continuous function in $\mathbf{z} \in \theta \mathcal{Z}$ satisfying

$$(6.20) \quad \int \left(\frac{1}{(2\pi i)^k} \oint_{|\mathbf{u}|=\delta_t} \mathbf{u}^{1-\nu} I(\mathbf{u}, \mathbf{z}) d\mathbf{u} \right) \bar{\varphi}(\mathbf{z}) d\mathbf{z} = \mathcal{T}_t(\nu^{(k)}, \hat{\mathbf{D}}_{\nu_k} \varphi)$$

for all $\varphi \in C_0^\infty(\theta \mathcal{Z})$ with $\hat{\mathbf{D}}_n = \mathbf{D}_n^* - (n-1) \sum_i \Delta_{\alpha_i} z_i^{-n}$.

Proof. We have

$$I_\epsilon(\mathbf{u}, \mathbf{z}) = \sum_{p=1}^n \frac{\alpha_p}{u_k - z_p} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \partial_{z_p} \phi_\epsilon(z_p) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_t = K_\epsilon(\mathbf{u}, \mathbf{z}) + L_\epsilon(\mathbf{u}, \mathbf{z})$$

where we integrate by parts the $\partial_{z_p} \phi_\epsilon(z_p)$ and $K_\epsilon(\mathbf{u}, \mathbf{z})$ collects the terms with an obvious $\epsilon \rightarrow 0$ limit $K(\mathbf{u}, \mathbf{z})$:

$$\begin{aligned} K(\mathbf{u}, \mathbf{z}) &= - \sum_{p=1}^n \sum_{\ell=1}^{k-1} \frac{Q\alpha_p}{(u_k - z_p)(z_p - u_\ell)^3} \langle T(\mathbf{u}^{(\ell, k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ &\quad + \sum_{p=1}^n \sum_{\ell=1}^{k-1} \frac{\alpha_p}{(u_k - z_p)(z_p - u_\ell)^2} \langle \partial_z X(u_\ell) T(\mathbf{u}^{(\ell, k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ &\quad - \sum_{p=1}^n \frac{\alpha_p \alpha}{2} \frac{1}{(u_k - z_p) z_p} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ &\quad - \sum_{p \neq p'=1}^n \frac{\alpha_p \alpha_{p'}}{2} \frac{1}{(u_k - z_p)(z_p - z_{p'})} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ &=: D_1(\mathbf{u}, \mathbf{z}) + D_2(\mathbf{u}, \mathbf{z}) + D_3(\mathbf{u}, \mathbf{z}) + D_5(\mathbf{u}, \mathbf{z}), \end{aligned}$$

whereas

$$L_\epsilon(\mathbf{u}, \mathbf{z}) = -\mu\gamma \sum_{p=1}^n \alpha_p \int_{\mathbb{C}_t} \frac{1}{u_k - z_p} C_{\epsilon,0}(z_p, x) \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_t dx.$$

Since $C_{0,0}(z_p, x) = -\frac{1}{2} \frac{1}{z_p - x}$ and since it is not clear that $\frac{1}{z_p - x} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t$ is integrable the $\epsilon \rightarrow 0$ limit of L_ϵ is problematic²¹. However, we can compare it with the term T'_4 in (6.19). Writing

$$\frac{1}{u_k - z_p} = \frac{1}{u_k - x} + \frac{z_p - x}{(u_k - z_p)(u_k - x)},$$

we conclude that L_ϵ converges:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} L_\epsilon(\mathbf{u}, \mathbf{z}) &= -\mu\gamma \lim_{\epsilon \rightarrow 0} \sum_{p=1}^n \alpha_p \left(\int_{\mathbb{C}_t} \frac{1}{u_k - x} C_{\epsilon,0}(z_p, x) \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_t dx \right. \\ &\quad \left. + \int_{\mathbb{C}_t} \frac{z_p - x}{(u_k - z_p)(u_k - x)} C_{\epsilon,0}(z_p, x) \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \right) \\ &=: T'_4(\mathbf{u}, \mathbf{z}) + T_4(\mathbf{u}, \mathbf{z}). \end{aligned}$$

Indeed, setting $z = z_p - x$ the function

$$(z_p - x) C_{\epsilon,0}(z_p, x) = -\frac{1}{2} \int \rho_\epsilon(y) \frac{z}{z+y} dy = \frac{i}{2\epsilon^2} \int_{\mathbb{R}^+} \rho\left(\frac{r}{\epsilon}\right) \oint_{|u|=1} \frac{z}{z+ru} \frac{du}{u} r dr = -\pi \int_{\mathbb{R}^+} \rho(r) 1_{r < |z|/\epsilon} r dr$$

is uniformly bounded and converges almost everywhere to $-\frac{1}{2}$.

The same argument can be repeated to the smeared functions to show that (because convergence is uniform over compact subsets of $\theta\mathcal{Z}$)

$$\lim_{\epsilon \rightarrow 0} \sum_{p=1}^n \int \left(\frac{\alpha_p}{u_k - z_p} \partial_{z_p} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_t \right) \bar{\varphi}(\mathbf{z}) d\mathbf{z} := \ell(\nu, \varphi)$$

exists. Then integrating ∂_{z_p} by parts and using that $\lim_{\epsilon \rightarrow 0} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_t$ exists we conclude

$$j(\nu, \varphi) = - \int \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_t \sum_{p=1}^n \partial_{z_p} \left(\frac{\alpha_p}{u_k - z_p} \varphi(\mathbf{z}) \right) d\mathbf{z}$$

which proves (6.20). \square

We have obtained the relation

$$(6.21) \quad N(\mathbf{u}, \mathbf{z}) - I(\mathbf{u}, \mathbf{z}) = B_t(\mathbf{u}, \mathbf{z}) + \sum_{i=6}^9 T_i(\mathbf{u}, \mathbf{z}) - \sum_{i=1}^3 D_i(\mathbf{u}, \mathbf{z}) - D_5(\mathbf{u}, \mathbf{z}) - T_4(\mathbf{u}, \mathbf{z}).$$

²¹Actually, this fact was shown in [KRV20] without the SET insertions and could be proven here as well but we will follow another route because the recursion to prove this extension is painful.

Let us consider the expression

$$\begin{aligned} K(\mathbf{u}, \mathbf{z}) &:= \langle T(\mathbf{u})V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t - I(\mathbf{u}, \mathbf{z}) - M(\mathbf{u}, \mathbf{z}) \\ &= R(\mathbf{u}, \mathbf{z}) + N(\mathbf{u}, \mathbf{z}) + D(\mathbf{u}, \mathbf{z}) - I(\mathbf{u}, \mathbf{z}). \end{aligned}$$

By (6.13) and (6.20) we have

$$(6.22) \quad \int \left(\oint_{|\mathbf{u}|=\delta_t} \mathbf{u}^{1-\nu} K(\mathbf{u}, \mathbf{z}) d\mathbf{u} \right) \varphi(\mathbf{z}) d\mathbf{z} = \mathcal{T}(\nu, \varphi) - \mathcal{T}(\nu^{(k)}, \mathbf{D}_{\nu_k}^* \varphi).$$

On the other hand combining (6.10), (6.16) and (6.21) we obtain

$$(6.23) \quad K = R + \sum_{i=1}^3 (T_i - D_i) + T_5 - D_5 + \sum_{i=6}^9 (T_i + T'_i) + B_t.$$

Now some simple algebra, see Appendix B gives:

$$(6.24) \quad T'_9 = -T_9, \quad T_5 = D_5$$

$$(6.25) \quad \oint_{|u_k|=e^{-t}\delta_k} u_k^{1-\nu_k} (T'_i(\mathbf{u}, \mathbf{z}) + T'_i(\mathbf{u}, \mathbf{z})) du_k = 0 \quad i = 6, 7, 8,$$

$$(6.26) \quad \oint_{|u_k|=e^{-t}\epsilon_k} u_k^{1-\nu_k} (T_i(\mathbf{u}, \mathbf{z}) - D_i(\mathbf{u}, \mathbf{z})) du_k = 0 \quad i = 1, 2, 3.$$

Hence using these relations and (6.11) we conclude

$$\int \left(\oint_{|\mathbf{u}|=\delta_t} \mathbf{u}^{1-\nu} K(\mathbf{u}, \mathbf{z}) d\mathbf{u} \right) \bar{\varphi}(\mathbf{z}) d\mathbf{z} = \int \left(\oint_{|\mathbf{u}|=\delta_t} \mathbf{u}^{1-\nu} B_t(\mathbf{u}, \mathbf{z}) d\mathbf{u} \right) \bar{\varphi}(\mathbf{z}) d\mathbf{z} =: B_t(\nu, \varphi).$$

Thus to prove Proposition 5.8 for $\nu_k \geq 2$ we need to prove the bound (6.9) for $B_t(\nu, \varphi)$. Recalling (6.18) we get by residue theorem

$$\oint_{|\mathbf{u}|=\delta_t} \mathbf{u}^{1-\nu} B_t(\mathbf{u}, \mathbf{z}) d\mathbf{u} = -2\pi \oint_{|\mathbf{u}^{(k)}|=\delta_t^{(k)}} (\mathbf{u}^{(k)})^{1-\nu^{(k)}} \oint_{|x|=e^{-t}} x^{1-\nu_k} \langle T(\mathbf{u}^{(k)})V_\alpha(0)V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t d\bar{x} d\mathbf{u}.$$

By (6.6) (at $\epsilon = 0$ and an extra $V_{\alpha_{n+1}}(z_{n+1}) = V_\gamma(x)$) the expectation on the RHS is a sum of terms of the form

$$\int_{\mathbb{C}_t^m} I(\mathbf{u}^{(k)}, \mathbf{x}, \mathbf{z}, x) \langle V_\alpha(0)V_\gamma(x) \prod_{\ell=1}^m V_\gamma(x_\ell) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t d\mathbf{x}$$

where

$$I(\mathbf{u}^{(k)}, \mathbf{x}, \mathbf{z}, x) = C \prod_{\alpha, \beta} \frac{1}{(u_\alpha - u_\beta)^{k_{\alpha\beta}}} \prod_{\alpha, i} \frac{1}{(u_\alpha - z_i)^{l_{\alpha i}}} \prod_{\alpha, \ell} \frac{1}{(u_\alpha - x_\ell)^{m_{\alpha\ell}}} \prod_{\alpha} \frac{1}{(u_\alpha - x)^{n_\alpha}}$$

where $\sum k_{\alpha\beta} + \sum l_{\alpha k} + \sum m_{\alpha\ell} + \sum n_\alpha = 2(k-1)$. Performing the u -integrals in the order u_{k-1}, u_{k-2}, \dots by the residue theorem we get

$$\oint_{|\mathbf{u}^{(k)}|=\delta_t^{(k)}} (\mathbf{u}^{(k)})^{1-\nu^{(k)}} I(\mathbf{u}^{(k)}, \mathbf{x}, \mathbf{z}, x) d\mathbf{u} = \sum C(a, \mathbf{b}, \mathbf{c}) x^{-a} \prod_{\ell} x_\ell^{-b_\ell} \prod_i z_i^{-c_i}$$

with $a + \sum b_\ell + \sum c_i = |\nu^{(k-1)}|$. Since $|x_\ell| \geq e^{-t}$ and $|x| = e^{-t}$ we conclude

$$\left| \oint_{|\mathbf{u}|=\delta_t} \mathbf{u}^{1-\nu} B_t(\mathbf{u}, \mathbf{z}) d\mathbf{u} \right| \leq C e^{t(|\nu|-2)} \max_{m \leq 2(k-1)} \sup_{|x|=e^{-t}} \int_{\mathbb{C}_t^m} \langle V_\alpha(0)V_\gamma(x) \prod_{\ell=1}^m V_\gamma(x_\ell) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t d\mathbf{x}.$$

By Lemma 6.2

$$\int_{\mathbb{C}_t^m} \langle V_\alpha(0)V_\gamma(x) \prod_{\ell=1}^m V_\gamma(x_\ell) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t d\mathbf{x} = C \langle V_\alpha(0)V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \leq C|x|^{-\gamma\alpha} = C e^{t\gamma\alpha}.$$

where we used the formula (1.10) and this estimate is uniform over the compact subsets of $\mathbf{z} \in \theta\mathcal{Z}$. Hence

$$\left| \oint_{|\mathbf{u}|=\delta_t} \mathbf{u}^{1-\nu} B_t(\mathbf{u}, \mathbf{z}) d\mathbf{u} \right| \leq C e^{t(|\nu|+\alpha-2)}$$

as claimed.

Case $\nu_k = 1$. Here we need to regularize also the Liouville expectation: let $\langle - \rangle_{t,\epsilon}$ be as in (5.11) except we replace $e^{\gamma c} M_\gamma(\mathbb{C}_t)$ by the regularized version $\int_{\mathbb{C}_t} V_{\gamma,\epsilon}(x) dx$. We use following variant of Lemma 6.4.

Lemma 6.5. *Let*

$$I'_\epsilon(\mathbf{u}, \mathbf{z}) := \frac{1}{u_k} \sum_{p=1}^n \partial_{z_p} \langle T(\mathbf{u}^k) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i,\epsilon}(z_i) \rangle_{t,\epsilon}.$$

Then $\lim_{\epsilon \rightarrow 0} I'_\epsilon(\mathbf{u}, \mathbf{z}) := I'(\mathbf{u}, \mathbf{z})$ exists and defines a continuous function in $\mathbf{z} \in \theta\mathcal{Z}$ satisfying

$$(6.27) \quad \int \left(\frac{1}{(2\pi i)^{|\nu|}} \oint_{|\mathbf{u}|=\delta_t} \mathbf{u}^{1-\nu} I'(\mathbf{u}, \mathbf{z}) d\mathbf{u} \right) \bar{\varphi}(\mathbf{z}) d\mathbf{z} = \mathcal{T}_t(\nu^{(k)}, \mathbf{D}_{\nu_k}^* \varphi)$$

for all $\varphi \in C_0^\infty(\theta\mathcal{Z})$.

Proof. The proof is similar to Lemma 6.4 but cancellations occur for other reasons and we explain how. First, the integration by parts gives

$$I'_\epsilon(\mathbf{u}, \mathbf{z}) = K'_\epsilon(\mathbf{u}, \mathbf{z}) + L'_\epsilon(\mathbf{u}, \mathbf{z})$$

where $\lim_{\epsilon \rightarrow 0} K'_\epsilon = K'$ exists and is given by

$$\begin{aligned} K'(\mathbf{u}, \mathbf{z}) &= - \sum_{p=1}^n \sum_{\ell=1}^{k-1} \frac{Q\alpha_p}{u_k(z_p - u_\ell)^3} \langle T(\mathbf{u}^{(\ell,k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ &\quad + \sum_{p=1}^n \sum_{\ell=1}^{k-1} \frac{\alpha_p}{u_k(z_p - u_\ell)^2} \langle \partial_z X(u_\ell) T(\mathbf{u}^{(\ell,k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ &\quad - \sum_{p=1}^n \frac{\alpha_p \alpha}{2} \frac{1}{u_k z_p} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ &\quad - \sum_{p \neq p'=1}^n \frac{\alpha_p \alpha_{p'}}{2} \frac{1}{u_k(z_p - z_{p'})} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ &=: C_1(\mathbf{u}, \mathbf{z}) + C_2(\mathbf{u}, \mathbf{z}) + C_3(\mathbf{u}, \mathbf{z}) + C_5(\mathbf{u}, \mathbf{z}) \end{aligned}$$

whereas

$$L'_\epsilon(\mathbf{u}, \mathbf{z}) = - \frac{\mu\gamma}{u_k} \sum_{p=1}^n \alpha_p \int_{\mathbb{C}_t} C_{\epsilon,\epsilon}(z_p, x) \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_{\gamma,\epsilon}(x) \prod_{i=1}^n V_{\alpha_i,\epsilon}(z_i) \rangle_{t,\epsilon} dx$$

is the term that needs analysis. Let us define

$$B'_t(\mathbf{u}, \mathbf{z}) := - \frac{i\mu}{u_k} \oint_{|x|=e^{-t}} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t d\bar{x}.$$

Then

$$\begin{aligned}
B'_t(\mathbf{u}, \mathbf{z}) &= -\frac{i\mu}{u_k} \lim_{\epsilon \rightarrow 0} \oint_{|x|=e^{-t}} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_{\gamma, \epsilon}(x) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_{t, \epsilon} d\bar{x} \\
&= \frac{\mu}{u_k} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{C}_t} \partial_x \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_{\gamma, \epsilon}(x) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_{t, \epsilon} dx \\
&= -\frac{\mu Q \gamma}{u_k} \sum_{\ell=1}^{k-1} \int_{\mathbb{C}_t} \frac{1}{(x-u_\ell)^3} \langle T(\mathbf{u}^{(\ell, k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \\
&\quad + \frac{\mu \gamma}{u_k} \sum_{\ell=1}^{k-1} \int_{\mathbb{C}_t} \frac{1}{(x-u_\ell)^2} \langle \partial_z X(u_\ell) T(\mathbf{u}^{(\ell, k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \\
&\quad - \mu \frac{\gamma \alpha}{2u_k} \int_{\mathbb{C}_t} \frac{1}{x} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx \\
&\quad - \frac{\mu \gamma}{u_k} \lim_{\epsilon \rightarrow 0} \sum_{p=1}^n \alpha_p \int_{\mathbb{C}_t} C_{\epsilon, \epsilon}(z_p, x) \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_{\gamma, \epsilon}(x) \prod_{i=1}^n V_{\alpha_i, \epsilon}(z_i) \rangle_{t, \epsilon} dx \\
&=: P_6(\mathbf{u}, \mathbf{z}) + P_7(\mathbf{u}, \mathbf{z}) + P_8(\mathbf{u}, \mathbf{z}) + P_4(\mathbf{u}, \mathbf{z})
\end{aligned}$$

This proves the existence of $\lim_{\epsilon \rightarrow 0} L'_\epsilon = L' = P_4$ and furthermore

$$(6.28) \quad I' = \lim_{\epsilon \rightarrow 0} I'_\epsilon = C_1 + C_2 + C_3 + C_5 + B'_t - P_6 - P_7 - P_8.$$

The claim (6.27) follows as in Lemma 6.4. □

Let us consider the expression

$$\begin{aligned}
K'(\mathbf{u}, \mathbf{z}) &:= \langle T(\mathbf{u}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t + I'(\mathbf{u}, \mathbf{z}) - M(\mathbf{u}, \mathbf{z}) \\
&= R(\mathbf{u}, \mathbf{z}) + N(\mathbf{u}, \mathbf{z}) + D(\mathbf{u}, \mathbf{z}) + I'(\mathbf{u}, \mathbf{z}).
\end{aligned}$$

By (6.13) and (6.27) we have

$$(6.29) \quad \int \left(\oint_{|\mathbf{u}|=\delta_t} \mathbf{u}^{1-\nu} K(\mathbf{u}, \mathbf{z}) d\mathbf{u} \right) \bar{\varphi}(\mathbf{z}) d\mathbf{z} = \mathcal{T}(\nu, \varphi) - \mathcal{T}(\nu^{(k)}, \mathbf{D}_{\nu_k}^* \varphi).$$

On the other hand combining (6.10), (6.16) and (6.21) we obtain

$$(6.30) \quad K = R + N + \sum_{i=1}^9 T_i + \sum_{i=1}^3 C_i + C_5 - \sum_{i=6}^8 P_i + B'_t$$

As before it is easy to check that the following relations hold (see Appendix B)

$$(6.31) \quad \oint_{|u_k|=e^{-t}\epsilon_k} T_i(\mathbf{u}, \mathbf{z}) du_k = 0 \quad \text{for } i = 4, 5, 9$$

$$(6.32) \quad \oint_{|u_k|=e^{-t}\epsilon_k} (T_i(\mathbf{u}, \mathbf{z}) + C_i(\mathbf{u}, \mathbf{z})) du_k = 0 \quad \text{for } i = 1, 2, 3$$

$$(6.33) \quad \oint_{|u_k|=e^{-t}\epsilon_k} (T_i(\mathbf{u}, \mathbf{z}) - P_i(\mathbf{u}, \mathbf{z})) du_k = 0 \quad \text{for } i = 6, 7, 8$$

$$(6.34) \quad \oint_{|u_k|=e^{-t}\epsilon_k} C_5(\mathbf{u}, \mathbf{z}) du_k = 0$$

$$(6.35) \quad \oint_{|u_k|=e^{-t}\epsilon_k} N(\mathbf{u}, \mathbf{z}) du_k = 0$$

We can now conclude as in the case $\nu_k > 1$. □

APPENDIX

APPENDIX A. THE DOZZ FORMULA

We set $\ell(z) = \frac{\Gamma(z)}{\Gamma(1-z)}$ where Γ denotes the standard Gamma function. We introduce Zamolodchikov's special holomorphic function $\Upsilon_{\frac{\gamma}{2}}(z)$ by the following expression for $0 < \operatorname{Re}(z) < Q$

$$(A.1) \quad \ln \Upsilon_{\frac{\gamma}{2}}(z) = \int_0^\infty \left(\left(\frac{Q}{2} - z \right)^2 e^{-t} - \frac{(\sinh((\frac{Q}{2} - z)\frac{t}{2}))^2}{\sinh(\frac{t\gamma}{4}) \sinh(\frac{t}{\gamma})} \right) \frac{dt}{t}.$$

The function $\Upsilon_{\frac{\gamma}{2}}$ is then defined on all \mathbb{C} by analytic continuation of the expression (A.1) as expression (A.1) satisfies the following remarkable functional relations:

$$(A.2) \quad \Upsilon_{\frac{\gamma}{2}}(z + \frac{\gamma}{2}) = \ell(\frac{\gamma}{2}z) \left(\frac{\gamma}{2}\right)^{1-\gamma z} \Upsilon_{\frac{\gamma}{2}}(z), \quad \Upsilon_{\frac{\gamma}{2}}(z + \frac{2}{\gamma}) = \ell(\frac{2}{\gamma}z) \left(\frac{\gamma}{2}\right)^{\frac{4}{\gamma}z-1} \Upsilon_{\frac{\gamma}{2}}(z).$$

The function $\Upsilon_{\frac{\gamma}{2}}$ has no poles in \mathbb{C} and the zeros of $\Upsilon_{\frac{\gamma}{2}}$ are simple (if $\gamma^2 \notin \mathbb{Q}$) and given by the discrete set $(-\frac{\gamma}{2}\mathbb{N} - \frac{2}{\gamma}\mathbb{N}) \cup (Q + \frac{\gamma}{2}\mathbb{N} + \frac{2}{\gamma}\mathbb{N})$. With these notations, the DOZZ formula is defined for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ by the following formula where we set $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$

$$(A.3) \quad C_{\gamma, \mu}^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3) = \left(\pi \mu \ell\left(\frac{\gamma^2}{4}\right) \left(\frac{\gamma}{2}\right)^{2-\frac{\gamma^2}{2}} \right)^{\frac{2Q-\bar{\alpha}}{\gamma}} \frac{\Upsilon'_{\frac{\gamma}{2}}(0) \Upsilon_{\frac{\gamma}{2}}(\alpha_1) \Upsilon_{\frac{\gamma}{2}}(\alpha_2) \Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2} - Q) \Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2} - \alpha_1) \Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2} - \alpha_2) \Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}}{2} - \alpha_3)}$$

The DOZZ formula is meromorphic with poles corresponding to the zeroes of the denominator of expression (A.3). Note that it is symmetric in $\alpha_1, \alpha_2, \alpha_3$ and real valued when α_j are real.

APPENDIX B. INTEGRATION BY PARTS CALCULATIONS

R-terms. Here we list explicitly the terms in the integration by parts formula in the proof of Proposition 6.3 giving zero contribution to the contour integral:

$$\begin{aligned} R(\mathbf{u}, \mathbf{z}) := & 3Q^2 \sum_{\ell=1}^{k-1} \frac{1}{(u_k - u_\ell)^4} \langle T(\mathbf{u}^{(\ell, k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ & - 2Q \sum_{\ell=1}^{k-1} \frac{1}{(u_k - u_\ell)^3} \langle T(\mathbf{u}^{(\ell, k)}) \partial_z X(u_\ell) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ & + \frac{Q\alpha}{2} \frac{1}{u_k^2} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ & - Q^2 \sum_{\ell, \ell' \neq 1}^{k-1} \frac{1}{(u_k - u_\ell)^3 (u_k - u_{\ell'})^3} \langle T(\mathbf{u}^{\ell, \ell', k}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ & + 2Q \sum_{\ell, \ell' = 1}^{k-1} \frac{1}{(u_k - u_\ell)^3 (u_k - u_{\ell'})^2} \langle \partial_z X(u_{\ell'}) T(\mathbf{u}^{\ell, \ell', k}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ & - Q\alpha \sum_{\ell=1}^{k-1} \frac{1}{(u_k - u_\ell)^3 u_k} \langle T(\mathbf{u}^{(\ell, k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ & - \sum_{\ell, \ell' = 1}^{k-1} \frac{1}{(u_k - u_\ell)^2 (u_k - u_{\ell'})^2} \langle \partial_z X(u_\ell) \partial_z X(u_{\ell'}) T(\mathbf{u}^{\ell, \ell', k}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ & + \alpha \sum_{\ell=1}^{k-1} \frac{1}{(u_k - u_\ell)^2 u_k} \langle \partial_z X(u_\ell) T(\mathbf{u}^{(\ell, k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t \\ & - \frac{\alpha^2}{4} \frac{1}{u_k^2} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t. \end{aligned}$$

Proof of (6.24)-(6.26). The claim $T'_9 = -T_9$ results from the relation

$$\frac{1}{(u_k - x)(u_k - x')} = \frac{1}{x - x'} \left(\frac{1}{u_k - x} - \frac{1}{u_k - x'} \right)$$

and the fact that the mapping $(x, x') \mapsto \langle T(\mathbf{u}^{(k)})V_\alpha(0)V_\gamma(x)V_\gamma(x') \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t$ is symmetric.

Also, $T_5 = D_5$ comes from the relation

$$\frac{1}{(u_k - z_p)(u_k - z_{p'})} = \frac{1}{z_p - z_{p'}} \left(\frac{1}{u_k - z_p} - \frac{1}{u_k - z_{p'}} \right)$$

and a re-indexation of the double sum.

Using

$$\frac{1}{(u_k - u_\ell)^2(u_k - x)} = \frac{1}{(u_k - u_\ell)^2(u_\ell - x)} - \frac{1}{(u_k - u_\ell)(x - u_\ell)^2} + \frac{1}{(x - u_\ell)^2(u_k - x)}$$

we find that

$$T_7 + T'_7 = -\mu\gamma \sum_{\ell=1}^{k-1} \int_{\mathbb{C}_t} \left(\frac{1}{(u_k - u_\ell)^2(u_\ell - x)} - \frac{1}{(u_k - u_\ell)(x - u_\ell)^2} \right) \langle \partial_z X(u_\ell) T(\mathbf{u}^{(\ell,k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx$$

satisfies

$$\oint_{|u_k|=e^{-t}\delta_k} u_k^{1-\nu_k} (T_7(\mathbf{u}, \mathbf{z}) + T'_7(\mathbf{u}, \mathbf{z})) du_k = 0$$

by moving the contour to ∞ ($\nu_k > 1$ is used here).

Using that

$$\frac{1}{u_k(u_k - x)} - \frac{1}{x(u_k - x)} = -\frac{1}{u_k x}$$

and $\oint_{|u_k|=e^{-t}\delta_k} u_k^{1-\nu_k} \frac{1}{u_k} du_k = 0$ for $\nu_k > 1$ we deduce that

$$\oint_{|u_k|=e^{-t}\delta_k} u_k^{1-\nu_k} (T_8(\mathbf{u}, \mathbf{z}) + T'_8(\mathbf{u}, \mathbf{z})) du_k = 0.$$

The relation

$$\frac{1}{(u_k - u_\ell)^3(u_k - x)} = \frac{1}{(u_k - u_\ell)^3(u_\ell - x)} - \frac{1}{(u_k - u_\ell)^2(x - u_\ell)^2} + \frac{1}{(u_k - u_\ell)(u_\ell - x)^3} - \frac{1}{(u_k - x)(u_\ell - x)^3}$$

entails in the same way (6.25) for $i = 6$.

Finally the relations (6.26) follow by computing residues at the pole $u_k = z_p$.

Proof of (6.31)-(6.35).

The relations (6.31) and (6.35) holds because all the corresponding $T_i(\mathbf{u})$ are holomorphic in $u_k \in \mathbb{D}_t$. For (6.34) we observe that

$$\frac{1}{2\pi i} \oint_{|u_k|=e^{-t}\epsilon_k} C_5(\mathbf{u}) du_k = - \sum_{p \neq p'=1}^n \frac{\alpha_p \alpha_{p'}}{2} \frac{1}{(z_p - z_{p'})} \langle T(\mathbf{u}^k) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t$$

and that this expression is null for antisymmetry reasons. (6.32) and (6.33) follow from the residue at z_p and x respectively

$$\oint_{|u_k|=e^{-t}\epsilon_k} T_1(\mathbf{u}, \mathbf{z}) du_k = 2\pi i \sum_{\ell=1}^{k-1} \sum_{p=1}^n \frac{Q\alpha_p}{(z_p - u_\ell)^3} \langle T(\mathbf{u}^{(\ell,k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t = - \oint_{|u_k|=e^{-t}\epsilon_k} C_1(\mathbf{u}, \mathbf{z}) du_k$$

$$\oint_{|u_k|=e^{-t}\epsilon_k} T_2(\mathbf{u}, \mathbf{z}) du_k = -2\pi i \sum_{\ell=1}^{k-1} \sum_{p=1}^n \frac{\alpha_p}{(z_p - u_\ell)^2} \langle \partial_z X(u_\ell) T(\mathbf{u}^{(\ell,k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t = - \oint_{|u_k|=e^{-t}\epsilon_k} C_2(\mathbf{u}, \mathbf{z}) du_k$$

$$\oint_{|u_k|=e^{-t}\epsilon_k} T_3(\mathbf{u}, \mathbf{z}) du_k = 2\pi i \sum_{p=1}^n \frac{\alpha_p}{2} \frac{1}{z_p} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t = - \oint_{|u_k|=e^{-t}\epsilon_k} C_3(\mathbf{u}, \mathbf{z}) du_k,$$

thus proving (6.32). Finally, we compute

$$\begin{aligned} \frac{1}{2\pi i} \oint_{|u_k|=e^{-t}\epsilon_k} T_6(\mathbf{u}, \mathbf{z}) du_k &= -\mu Q \gamma \sum_{\ell=1}^{k-1} \int_{\mathbb{C}_t} \frac{1}{(x-u_\ell)^3} \langle T(\mathbf{u}^{(\ell,k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx = P_6(\mathbf{u}, \mathbf{z}) \\ \frac{1}{2\pi i} \oint_{|u_k|=e^{-t}\epsilon_k} T_7(\mathbf{u}, \mathbf{z}) du_k &= \mu \gamma \sum_{\ell=1}^{k-1} \int_{\mathbb{C}_t} \frac{1}{(x-u_\ell)^2} \langle \partial_z X(u_\ell) T(\mathbf{u}^{(\ell,k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx = P_7(\mathbf{u}, \mathbf{z}) \\ \frac{1}{2\pi i} \oint_{|u_k|=e^{-t}\epsilon_k} T_8(\mathbf{u}, \mathbf{z}) du_k &= -\frac{\mu \gamma \alpha}{2} \int_{\mathbb{C}_t} \frac{1}{x} \langle T(\mathbf{u}^{(k)}) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx = P_8(\mathbf{u}, \mathbf{z}) \\ \frac{1}{2\pi i} \oint_{|u_k|=e^{-t}\epsilon_k} C_4(\mathbf{u}, \mathbf{z}) du_k &= \mu \gamma \sum_{p=1}^n \frac{\alpha_p}{2} \int_{\mathbb{C}_t} \frac{1}{(z_p-x)} \langle T(\mathbf{u}^k) V_\alpha(0) V_\gamma(x) \prod_{i=1}^n V_{\alpha_i}(z_i) \rangle_t dx = P_4(\mathbf{u}, \mathbf{z}). \end{aligned}$$

B.1. Analyticity of the vertex operators. Recall the definition of $U_\alpha(\mathbf{z})$ in (5.9) for $\mathbf{z} \in \mathcal{Z}$ and real α_i 's such that $\alpha_i < Q$. It is plain to see that $U_\alpha(\mathbf{z})$ agrees with the following slightly different regularization

$$U_\alpha^{\text{holes}}(\mathbf{z}) = \lim_{k \rightarrow \infty} U_{\alpha,k}^{\text{holes}}(\mathbf{z})$$

where

$$U_{\alpha,k}^{\text{holes}}(\mathbf{z}) = e^{\sum_i \alpha_i - Qc} e^{\sum_i \alpha_i P\varphi(z_i)} \mathbb{E}_\varphi \left[\left(\prod_{i=1}^n \epsilon_k^{\frac{\alpha_i^2}{2}} e^{\alpha_i X_{\mathbb{D}, \epsilon_k}(z_i)} \right) e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D}_k)} \right]$$

where $\epsilon_k = 2^{-k}$, $X_{\mathbb{D}, \epsilon_k}$ is the circle average of the Dirichlet GFF and \mathbb{D}_k is the unit disk with small holes removed around each insertion, namely $\mathbb{D}_k := \mathbb{D} \setminus \bigcup_{i=1}^n B(z_i, \epsilon_k)$. Recall that we get the following explicit expression by using the Girsanov theorem:

$$(B.1) \quad U_\alpha^{\text{holes}}(\mathbf{z}) = e^{(\sum_{j=1}^n \alpha_j - Q)c} e^{\sum_i \alpha_i P\varphi(z_i) + \sum_{i < j} \alpha_i \alpha_j G_{\mathbb{D}}(z_i, z_j)} \mathbb{E}_\varphi \left[e^{-\mu e^{\gamma c} \int_{\mathbb{D}} e^{\gamma \sum_{i=1}^n \alpha_i G_{\mathbb{D}}(x, z_i)} M_\gamma(dx)} \right].$$

We fix k_0 such that the open balls $B(z_i, 2^{-k_0})$ are disjoint and included in \mathbb{D} . Set $\mathcal{O}^n := \{\alpha \in \mathbb{R}^n; \alpha_i < Q, \forall i\}$. Then we have the following analyticity result:

Proposition B.1. *The (random) function $\alpha \rightarrow U_\alpha^{\text{holes}}(\mathbf{z})$ admits an analytic extension in a complex neighborhood of \mathcal{O}^n such that for all real $\alpha \in \mathcal{O}^n$ there exists some $\epsilon > 0$ (depending on α) and (non random) $C, \tilde{C} > 0$ satisfying*

$$\begin{aligned} & \sup_{\beta \in [-\epsilon, \epsilon]^n} |U_{\alpha+i\beta}^{\text{holes}}(\mathbf{z}) - U_{\alpha+i\beta, k_0}^{\text{holes}}(\mathbf{z})| \\ & \leq C(1 + e^{\gamma c}) e^{(\sum_{j=1}^n \alpha_j - Q)c} e^{\tilde{C} \sup_i \sup_{u \in B(z_i, 2^{-k_0})} P\varphi(u)} \mathbb{E}_\varphi \left[e^{-\mu e^{\gamma c} \int_{\mathbb{D}_{k_0}} e^{\gamma \sum_{i=1}^n \alpha_i G_{\mathbb{D}}(x, z_i)} M_\gamma(dx)} \right]. \end{aligned}$$

Proof. To simplify we will suppose that $n = 1$ and $z_1 = 0$ and set $G_{\mathbb{D}, k}(0, u) = \mathbb{E}[X_{\mathbb{D}, \epsilon_k}(0) X_{\mathbb{D}}(u)]$. This is no restriction as the same analysis can be performed around each insertion in case $n > 1$. In this context, we get by using the Markov property of the Dirichlet GFF that:

$$\begin{aligned} & |U_{\alpha+i\beta, k+1}^{\text{holes}} - U_{\alpha+i\beta, k}^{\text{holes}}(\mathbf{z})| \\ & = e^{(\alpha-Q)c} |\mathbb{E}_\varphi \left[\epsilon_{k+1}^{(\alpha+i\beta)^2/2} e^{(\alpha+i\beta) X_{\mathbb{D}, \epsilon_{k+1}}(0)} (e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D}_{k+1})} - e^{-\mu e^{\gamma c} M_\gamma(\mathbb{D}_k)}) \right]| \\ & \leq e^{(\alpha-Q)c} 2^{(k+1)\beta^2/2} |\mathbb{E}_\varphi \left[\left(e^{-\mu e^{\gamma c} \int_{\mathbb{D}_{k+1}} e^{\gamma \alpha_1 G_{\mathbb{D}, k+1}(x, 0)} M_\gamma(dx)} - e^{-\mu e^{\gamma c} \int_{\mathbb{D}_k} e^{\gamma \alpha_1 G_{\mathbb{D}, k+1}(x, 0)} M_\gamma(dx)} \right) \right]| \\ & = e^{(\alpha-Q)c} 2^{(k+1)\beta^2/2} |\mathbb{E}_\varphi \left[(e^{-\mu e^{\gamma c} (Y_k + \delta Y_k)} - e^{-\mu e^{\gamma c} Y_k}) \right]|, \end{aligned}$$

where we have set

$$Y_k = \int_{\mathbb{D}_k} e^{\gamma P\varphi(x)} e^{\gamma \alpha_1 G_{\mathbb{D}, \epsilon_{k+1}}(0, x)} M_\gamma(dx) \quad \text{and} \quad \delta Y_k = \int_{\mathbb{D}_k \setminus \mathbb{D}_{k+1}} e^{\gamma P\varphi(x)} e^{\gamma \alpha_1 G_{\mathbb{D}, k+1}(0, u)} M_\gamma(dx).$$

Now, we consider the cases $\delta Y_k > 1$ and $\delta Y_k \leq 1$. By FKG inequality for the Dirichlet GFF, we have

$$\begin{aligned} & \mathbb{E}_\varphi \left[\mathbb{1}_{\delta Y_k > 1} |e^{-\mu e^{\gamma c} Y_k} - e^{-\mu e^{\gamma c} (Y_k + \delta Y_k)}| \right] \\ & \leq 2 \mathbb{E}_\varphi \left[\mathbb{1}_{\delta Y_k > 1} e^{-\mu e^{\gamma c} Y_k} \right] \\ & \leq 2 \mathbb{E}_\varphi \left[\mathbb{1}_{\delta Y_k > 1} \right] \mathbb{E}_\varphi \left[e^{-\mu e^{\gamma c} Y_k} \right] \\ & \leq 2 \mathbb{E}_\varphi \left[(\delta Y_k)^\eta \right] \mathbb{E}_\varphi \left[e^{-\mu e^{\gamma c} Y_k} \right]. \end{aligned}$$

Next, we choose $\beta > 0$ and $\eta > 0$ such that $2^{(k+1)\beta^2/2} \mathbb{E}_\varphi \left[(\delta Y_k)^\eta \right] \leq C e^{\gamma \eta \sup_{|u| \leq \epsilon_k} P\varphi(u)} 2^{-\theta k}$ with $\theta > 0$.

In the case $\delta Y_k \leq 1$, we get (using the inequality $x \mathbb{1}_{\{x \leq 1\}} \leq x^\eta$ for $x > 0$ and then FKG for the Dirichlet GFF)

$$\begin{aligned} & 2^{(k+1)\beta^2/2} \mathbb{E}_\varphi \left[\mathbb{1}_{\delta Y_k \leq 1} |e^{-\mu e^{\gamma c} Y_k} - e^{-\mu e^{\gamma c} (Y_k + \delta Y_k)}| \right] \\ & \leq 2^{(k+1)\beta^2/2} \mu e^{\gamma c} \mathbb{E}_\varphi \left[\mathbb{1}_{\delta Y_k \leq 1} \delta Y_k e^{-\mu e^{\gamma c} Y_k} \right] \\ & \leq 2^{(k+1)\beta^2/2} \mu e^{\gamma c} \mathbb{E}_\varphi \left[(\delta Y_k)^\eta e^{-\mu e^{\gamma c} Y_k} \right] \\ & \leq 2^{(k+1)\beta^2/2} \mu e^{\gamma c} \mathbb{E}_\varphi \left[(\delta Y_k)^\eta \right] \mathbb{E}_\varphi \left[e^{-\mu e^{\gamma c} Y_k} \right] \\ & \leq C 2^{-k\theta} e^{\gamma c} e^{\gamma \eta \sup_{|u| \leq \epsilon_k} P\varphi(u)} \mathbb{E}_\varphi \left[e^{-\mu e^{\gamma c} Y_k} \right]. \end{aligned}$$

Gathering the above considerations, we get

$$|U_{\alpha+i\beta, k+1}^{\text{holes}} - U_{\alpha+i\beta, k}^{\text{holes}}(\mathbf{z})| \leq C e^{(\alpha-Q)c} 2^{-k\theta} (1 + e^{\gamma c}) e^{\gamma \eta \sup_{|u| \leq \epsilon_n} P\varphi(u)} \mathbb{E}_\varphi \left[e^{-\mu e^{\gamma c} Y_n} \right]$$

This shows that the (random) analytic function $U_{\alpha+i\beta, k}^{\text{holes}}(\mathbf{z})$ converges as $k \rightarrow \infty$ with probability 1 and for all c towards an analytic function that satisfies

$$|U_{\alpha+i\beta, k+1}^{\text{holes}} - U_{\alpha+i\beta, k_0}^{\text{holes}}(\mathbf{z})| \leq C e^{(\alpha-Q)c} (1 + e^{\gamma c}) e^{\gamma \eta \sup_{|u| \leq \frac{1}{2}} P\varphi(u)} \mathbb{E}_\varphi \left[e^{-\mu e^{\gamma c} Y_{k_0}} \right].$$

□

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