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Analysis of base-stock perishable inventory systems with general lifetime and lead-time

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Abstract

A base-stock inventory system for perishables with Markovian demand and general lead-time and lifetime distributions is investigated. Using a queueing network model, we derive explicit expressions of the stationary distribution of the inventory state together with the total expected cost in a base-stock system with lost-sales. Next, we show some monotonicity properties of the cost function and propose a procedure to derive the optimal base-stock. Extensions to the backorders case and to a dual-sourcing system with multiple warehouses are also provided. Our results generalize existing ones from the literature where the lifetime and the lead-time follow either deterministic or exponential distributions. Finally, we investigate the effect of the lifetime and lead-time variability on the system cost and the optimal base-stock level. In particular, we show the substantial errors made when assuming deterministic or exponential distributions for the lifetime. This further shows the need to have results beyond exponential or deterministic assumptions.

Keywords: Base-stock; perishable items; queueing network; general lifetime, general lead-time.

1 Introduction

Perishable products, or products with a finite lifetime, have a major role in our economy. Fruits, vegetables, meat, and frozen food constitute the majority of supermarket sales. Progressive Grocers magazine (2015) reported that, in 2014, perishables accounted for 59.1% of total sales in US supermarkets. Perishable products also include pharmaceuticals like drugs or vaccines, which represent a very large industry. The blood bank is another example of perishables. The lifetime of platelets is typically five days, while the whole blood has a legal lifetime of 21 days, after which it has to be discarded because of contamination (Chao et al., 2017). Fontaine et al. (2009) reported that, in 2006, 10.9% of blood platelets outdated without being transfused. Therefore, companies have a strong need to implement an efficient inventory management of perishables to save costs and avoid wastes. To answer these challenges, the academic literature has extensively developed mathematical models for decision-makers. We mention for instance Karaesmen et al. (2011); Bakker et al. (2012); Janssen et al. (2016); Minner and Transchel (2017).

In this paper, we focus on a base-stock policy for perishables. The particularity of base-stock policies is that a replenishment order to restore the base-stock level S is made whenever the inventory position is below S . Therefore, the reorder point is $S - 1$ and the policy is commonly referred to as an $(S - 1, S)$ policy in the inventory literature. The base-stock model is often used to control inventory systems with a zero or negligible ordering cost and/or when the demand per time unit is low. This policy is extensively implemented in practice for perishables such as pharmaceutical products or stored blood (Olsson and Tydesjö (2010)). In blood banks, the orders are placed on a daily basis and the demand for some rare blood types (such as B-negative) requires a continuous review base-stock inventory policy. Haijema et al. (2007) reported that the daily demand for blood platelets can be modelled as a Poisson process and the ordering cost can be ignored since many blood platelets types share a common fixed transportation cost. Note also that expensive pharmaceutical products are characterized with low demands and different expiration dates, which are often replenished using continuous review base-stock systems.

From the theoretical point of view, the attractiveness of the base-stock policy is partially due to its simplicity as the expected cost and the state distribution under this policy can be expressed in closed-form. Yet, existing results in the literature are restricted to either deterministic or exponential lifetime distributions. A deterministic lifetime corresponds to products with a specific expiry date like meat or fish, while an exponential one may be more adapted to deteriorating products like fruits or vegetables. The deterministic assumption has a practical reality, but the exponential one has been chosen for tractability reasons. Recall that the product deterioration is due to a growing quantity

of bacteria and fungi over time. Beyond a certain quantity, these products become unhealthy and should not be consumed. Determining the appropriate distribution for the evolution of a population is an ongoing research question. However, clearly, most studies on this question have shown that such distribution is non-exponential (Brockwell et al., 1982; Kapodistria et al., 2016; Logachov et al., 2019). Therefore, to overcome the limitation of existing results in the literature and have a better fit with real lifetime distributions, we assumed that lifetimes are generally distributed.

There is a rich body of literature on perishable inventory systems. The proposed models deal with periodic and continuous review systems. In both cases, various assumptions are considered in the literature to develop tractable (heuristic) models. Among these assumptions, one can mention a positive or zero replenishment lead-time, an exponential or deterministic lifetime, a lost-sales or backordered excess demands. Interested readers can refer to the latest literature review by Bakker et al. (2012) and Janssen et al. (2016) for more details. Our work falls into the literature of continuous review perishable inventory systems with a positive random lead-time and a general lifetime distribution.

Investigating a positive replenishment lead-time together with a general lifetime distribution is rather challenging. The complexity can be understood in Chao et al. (2015), where an inventory with a zero-replenishment lead-time and a deterministic lifetime with a fixed number of periods is considered. When the number of periods exceeds 2, the characterization of the exact operating costs becomes very complicated. The complexity stands in the dimension of the state space. For a complete description of the state space, one need to keep track of the age of each items on-hand. With N items on-hand, each with a lifetime of k periods, $N \times k$ states can be encountered. This prohibits the direct determination of a tractable solution.

In this work, we overcome this difficulty by employing a queuing network approach. It allows us, in the case of a Markovian demand, to characterize the stationary probabilities of the inventory level and subsequently to determine the expression of the operating costs. Specifically, we consider a perishable base-stock inventory system under a Poisson demand where both the lifetime and lead-time are generally distributed. The lost-sales case, the backorders case and a dual-sourcing case with multiple warehouses are investigated. The classical approach consists in formulating and solving a set of partial differential equations. With generally distributed lifetime and lead-time, this approach is unlikely to be successful due to the model complexity. We overcome this difficulty by modeling the inventory as a queueing network. This approach is first detailed in the lost-sales case and next extended to the two aforementioned cases. For each case, we derive the stationary probabilities and the total cost function in closed-form. In addition, we show some relevant monotonicity properties of the total inventory cost in the lost-sales case. A procedure to compute the optimal base-stock

level in each case is also provided. Finally, we develop an extensive numerical study to investigate the effect of the lifetime variability on the optimal inventory level and the optimal cost. Our analysis highlights the substantial cost error that can be made when assuming deterministic or exponential lifetime distributions to compute the optimal base-stock level. These results further show the need to go beyond exponential or deterministic assumptions.

Structure of the article. Section 2 presents a literature review. Sections 3, 4 and 5 are devoted to the lost-sales case, the backorders case, and the dual-sourcing case with multiple warehouses, respectively. Section 6 evaluates the impact of the lifetime variability and determines the errors made when assuming exponential or deterministic distributions. Section 7 concludes the paper and highlights avenues for future research. In the appendix, we provide a table of notations and the details of our numerical investigations.

2 Literature review

The literature dealing with the base-stock policy in perishable inventory systems spans two main approaches. The first is concerned with the analysis of continuous inventory review while the second one deals with periodic inventory review systems. Our approach falls in the first category. In this literature review, for both approaches, the contributions dealing with the base-stock policy in the lost-sales and backorder cases are presented. Under the continuous review, for most contributions, the lifetime is assumed to be either exponential or deterministic. We structure the presentation of the references based on the lifetime assumption in the two first paragraphs of this section. The third paragraph presents the references that deal with the continuous review under non-exponential and non-deterministic lifetime. Under the periodic review, more general assumptions are encountered for the lifetime. In the last paragraph, we present the references treating inventory models for lost-sales and backorders.

Under the continuous review, we first present references in which a *deterministic* lifetime is assumed. Schmidt and Nahmias (1985) analyzed a lost-sales base-stock inventory system with deterministic lifetime and lead-time. They derived explicit expressions for the steady-state probabilities and the total expected cost. Olsson and Tydesjö (2010) extended Schmidt and Nahmias (1985)'s results to the backorders case. Olsson and Turova (2016) studied a similar model with the additional feature of a double stochastic Poisson demand process. They showed the uniqueness of the stationary distribution. The approach in these studies is to solve partial differential equations and their boundary conditions so as to obtain closed-form expressions of the cost components. In our paper,

we instead employ a queueing network approach which significantly simplifies the analysis. Berk and Gürlér (2008) studied a lost-sales (r, Q) inventory system with a service-level constraint and a positive lead-time. They derived the optimal system parameters using an embedded Markov process approach of order's lifetime and showed that the performance of the (r, Q) policy deteriorates when the service-level increases. Under partial backorders, Olsson (2014) developed a heuristic method to find near-optimal parameters of an (r, Q) model with a Poisson demand process. More recently, Olsson and Turova (2016) extended the work by Olsson (2014) to the case where the demand follows a general doubly stochastic Poisson process.

We now provide references involving an *exponential* lifetime distribution. Kalpakam and Sapna (1994) pioneered the first study under the exponential lifetime distribution assumption. They considered a lost-sales (s, S) model with a Poisson demand process, exponential lifetimes and lead-times. By restricting the number of outstanding replenishment orders to at most one at any time, they derived the steady-state probabilities and obtained the explicit expression of the cost function. Kalpakam and Sapna (1995) extended this work to the base-stock system with a general lead-time distribution. Liu and Yang (1999) analyzed the same model as the one of Kalpakam and Sapna (1994) by considering the backorders case and no restriction on the number of outstanding replenishment orders. Kalpakam and Shanthi (2000) considered a base-stock policy where orders are triggered only at demand instances. With partial backordering, Liu and Cheung (1997) investigated an $(S - 1, S)$ inventory policy with a Poisson demand and exponential lead-times. Finally, Kouki et al. (2018) derived the expression of the total cost of a continuous review base-stock system under fixed and exponential lifetime distributions. The inventory system operates under a dual-sourcing with constant lead-times.

It is worthy pointing out that Gürlér and Özkaya (2008) is, to the best of our knowledge, the only work that focuses on general lifetime distributions (e.g. Gamma, Uniform, Triangular) with zero and constant positive lead-time. With zero-lead time, they examined the impact of a random shelf life on the performance of the (s, S) model under a general renewal arrival process. With constant positive lead-time, they proposed a heuristic allowing them to express the total cost. However, the work of Gürlér and Özkaya (2008) does not deal with the case of a generally distributed lead-time. Hence, our paper bridges the gap in the above-cited literature by developing the expression of the total cost function of a base-stock inventory system under a continuous review when both the lead-time and the lifetime are generally distributed. The analytical results generalize the findings of the literature for both the lost-sales and backorders cases.

Under periodic review setting, perishable inventory systems have been extensively studied. We review here the literature that is most closely related to our work. Minner and Transchel (2010),

studied a dynamic order policy under service level constraints. Their policy considerably outperforms periodic review base-stock and constant order policies. Haijema and Minner (2016), compared several periodic review constant order and base-stock policies with positive lead-time and deterministic lifetime. Through extensive numerical experiments, they determined situations in terms of lead-time, lifetime and cost parameters under which one of the proposed policies should be used in practice. Haijema and Minner (2019) extended their work by considering the stock-age vector to derive the optimal age-based order quantity and the expected total cost under various policies. They showed that the integration of the aging categories yields a higher cost saving. Broekmeulen and Van Donselaar (2009) and Tekin et al. (2001) also proposed inventory systems that take into account the stock-age to manage perishable inventory, they found that the age integration lead to substantial cost savings. Ketzenberg et al. (2018) studied a periodic review inventory policy with random lifetime and a one-period lead-time. They focused on the impact the optimal expiration date taken from lifetime distribution on best order quantities that minimize the total expected cost per time unit. In contrast to these existing works, we consider a base-stock system with random lifetime and a random lead-time but under a continuous review.

3 The lost-sales case

In this section, we investigate the lost-sales case. Section 3.1 gives the model description and explains the queueing network approach for analyzing this model. Next, Section 3.2 derives the stationary distribution of the number of items in the inventory together with the expected cost. An algorithm to compute the optimal base-stock level is also provided. Finally, in Section 3.3, we show how our results allow us to retrieve those from the literature in the exponential and deterministic cases.

Table of notations

In Table 1, we introduce the notations used throughout the paper. The index i is omitted whenever a unique warehouse is considered (for instance, in Sections 3 and 4).

3.1 Model description and queueing network approach

We first provide the model description of the inventory model. Our aim is to obtain the inventory cost expression with general lifetime and lead-time distributions, which extend the existing literature. The idea is to obtain a simple tool for decision-makers in perishable inventory control, and to gain insights into the conditions under which existing models with exponential or deterministic

Table 1: Table of notations

λ_i	Demand rate at warehouse i ($i = 1, 2, \dots, J$)
$f_i(\cdot)$	Lifetime probability density function at warehouse i ($i = 1, 2, \dots, J$)
$F_i(\cdot)$	Cumulative lifetime distribution function at warehouse i ($i = 1, 2, \dots, J$); $F_i(x) = \int_0^x f_i(t) dt$
m_i	Expected lifetime at warehouse i ($i = 1, 2, \dots, J$); $m_i = \int_0^\infty t f_i(t) dt$
CV_i	Ratio between the standard deviation and the expected value of the lifetime at warehouse i ($i = 1, 2, \dots, J$)
γ_i	Lifetime rate at warehouse i ($i = 1, 2, \dots, J$); $\gamma_i = 1/m_i$
$\delta_{i,n}$	Departure rate due to items expiring at warehouse i ($i = 1, 2, \dots, J$) when there are n items on-hand
$g(\cdot)$	Lead-time probability density function
$G(\cdot)$	Cumulative lead-time distribution function; $G(x) = \int_0^x g(t) dt$
L	Expected lead-time; $L = \int_0^\infty t g(t) dt$
L_e	Expected lead-time of an emergency order
β	Replenishment rate; $\beta = 1/L$
β_n	Departure rate due to the end of the lead-time given that there are n orders
h	Holding cost per item and per time unit
b	Cost per unsold item due to lost demand or backorder unit cost
w	Outdated cost per perished item
c_r	Purchasing cost per regular order
c_e	Purchasing cost per emergency order
S_i	Base-stock level at warehouse i ($i = 1, 2, \dots, J$)
\tilde{S}_i	Threshold level for emergency orders at warehouse i ($i = 1, 2, \dots, J$)
Z	Total expected cost
T	Normalising constant
σ_e	Rate of emergency orders
σ_r	Rate of regular orders
$\pi(n_1, n_2)$	Probability to have n_1 items on-hand and n_2 orders
$p_{n_i}(n_i)$	Stationary probability to find n_i items on-hand at warehouse i ($i = 1, 2, \dots, J$)
$P(n_1, n_2, \dots, n_J)$	Stationary probability that warehouses $1, 2, \dots, J$ have n_1, n_2, \dots, n_J , on-hand items, respectively
LS	Lost-sales
MLS	Lost-sales with multiple warehouses
BO	Backorders
DS	Dual-sourcing
MDS	Dual-sourcing with multiple warehouses

distributions fail to capture the system's real cost. The lost-sales case of this section serves as the basis of our analysis. The backorders and the multiple warehouses cases in the following sections can be seen as extensions of this model.

We consider an inventory model with a unique item where the product lifetime and lead-time are generally distributed with probability-density functions $f(\cdot)$, and $g(\cdot)$, respectively. We also make the assumptions that the demand for this item follows a Poisson process with parameter λ and that the stored items are served under a first in, first out manner. The system manager controls the inventory according to a base-stock policy. As such, each time a demand is satisfied from the stock on-hand or a product has perished, an order is immediately placed.

The particularity of the lost-sales case is that excess demands are lost. For this inventory model, the system manager is concerned by the volume of stored items, the rate of lost customers, and the rate of outdated items. To capture these features, we assume that items on-hand incur a holding cost per time unit, h . Moreover, a penalty, b , is counted each time a demand is lost due to item unavailability. Finally, a cost of w is counted per outdated item. This cost is also called the expiration cost.

Before presenting the methodological approach, we conclude our model description with two remarks related to assumptions chosen for the tractability of the analysis. First, as mentioned above, we consider a demand which follows a Poisson process. This assumption is restrictive as one could consider any other arrival processes. The Poisson assumption is a common one for the arrival process in inventory models (Silver et al., 1998). Statistical analyses have shown that the homogeneous-Poisson assumption well models customers' arrivals in service and inventory systems (Lengu et al., 2014; Kim and Whitt, 2014). One of the specific advantages of the Poisson process is that it exactly captures a similar effect as the short-term non-stationarity encountered in some arrival processes. Finally, we may also use the characteristic that the Poisson distribution - and a number of other distributions - approximates the normal distribution if it is accumulated over a certain time period (Fransoo et al., 1995). Therefore, we believe that the Poisson assumption allows us to capture or to well approximate a wide variety of arrival process.

Second, we assume that the service discipline is a first-in-first-out one. This assumption is also restrictive as many other discipline could be implemented like the last-in-first-out, or the random selection (Haijema et al., 2007; Minner and Transchel, 2010; Haijema, 2014; Haijema and Minner, 2019). The first-in-first-out assumption can be justified by practical reasons for its simplicity to implement. This discipline is also optimal when the lifetime has the increasing failure rate property (Legros, 2020). Given that the oldest item has the highest chance to perish at the next period, it makes sense to remove this item in priority in order to avoid the outdated cost. For the analysis,

the value of the first-in-first-out discipline is the simplicity of the offered wait distribution of a given item when the lifetime is infinite; it follows an Erlang distribution with parameter λ . This allowed Movaghar (1998) and Brandt and Brandt (2002) to explicitly determine the rate out of each state in the corresponding queueing model when abandonment is involved. Their results are employed in this paper. Extending their result for instance to the last-come-first-served case would be very involving as the offered wait would then follow a complex distribution requiring modified Bessel functions (Kleinrock, 1975). We therefore decided not to pursue this analysis.

In the lost-sales case, whenever an item is removed from the on-hand inventory, either due to an item expiration or to a satisfied demand, an order is automatically generated. Each order results in a new item in the on-hand inventory after a lead-time. Therefore, the total number of items in the on-hand inventory plus the number of outstanding orders is constant and equal to S . The value of S is called the base-stock level.

We employ a queueing network approach to evaluate the inventory cost. With this approach, the base-stock level, S , is seen as a finite population which makes the network a closed network. The population S is divided in between two parts; the on-hand inventory, and, the outstanding orders. These two parts are viewed as connected queues, called Queue 1, and, Queue 2, respectively. Queue 1 can be understood as a single server queue of impatient customers waiting for service. The arrival process for this queue is generated by the outstanding orders from Queue 2. The service is made by the Poisson demand and customers' patience is represented by the items' lifetime. Using Kendall's notation, Queue 1 is referred to as the $./M/1+G$ queue. The outstanding orders are executed independently and in parallel. As such, Queue 2 can then be viewed as a queue with an infinite number of parallel servers. The service time in this queue is determined by the outstanding orders' lead-time and the arrival process is generated by items' removal from Queue 1. This queue is referred in the queueing literature to as the $./G/\infty$ queue. This queueing network is depicted in Figure 1.

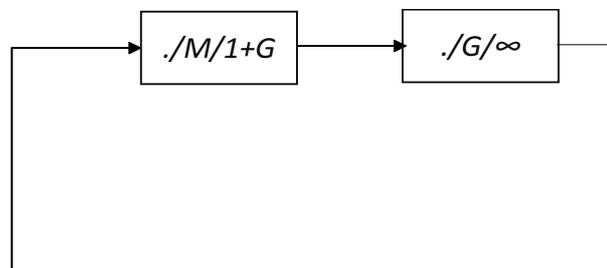


Figure 1: Modeling the base-stock inventory systems with lost-sales as a closed network

3.2 Stationary distribution, expected cost, and computation of the optimal base-stock level

In this section, we first derive the steady-states probabilities of the system to express the total cost function. Then, we provide some properties of the cost function, which are needed to develop the algorithm we use to calculate the optimal base-stock level. Our model is not Markovian due to the generally distributed lifetime and lead-time. However, since we have Markovian service times in Queue 1 and an infinite number of servers in Queue 2, a state of the system can be described by the number of items in each queue (Barrer, 1957; Movaghar, 1998) and the system performance can be derived from a Markov chain analysis.

Let n_1 and n_2 be the number of items on-hand (i.e., number in Queue 1) and the number of outstanding orders (i.e., number in Queue 2) respectively, and denote by $\Omega^{LS} = \{(n_1, n_2) \in \mathbb{N}^2 | n_1 + n_2 = S\}$, the set of admissible states. We are interested in calculating the steady-states probability of the system, denoted by $\pi^{LS}(n_1, n_2)$. To this end, we introduce the notations δ_{n_1} and β_{n_2} to represent the departure rate due to items expiring given that there are $n_1 > 0$ items on-hand and the departure rate due to the end of the lead-time in the second queue given that there are $n_2 > 0$ items in this queue, respectively. To simplify the notations, we use the convention that $\delta_0 = \beta_0 = 0$. For $0 < n_1, n_2 < S$, we may write

$$\left\{ \begin{array}{c} (\lambda + \delta_{n_1}) \mathbb{1}_{n_1 > 0} \pi^{LS}(n_1, n_2) \\ + \\ \beta_{n_2} \mathbb{1}_{n_2 > 0} \pi^{LS}(n_1, n_2) \end{array} \right\} = \left\{ \begin{array}{c} (\lambda + \delta_{n_1+1}) \pi^{LS}(n_1 + 1, n_2 - 1) \\ + \\ \beta_{n_2+1} \pi^{LS}(n_1 - 1, n_2 + 1) \end{array} \right\}, \quad (1)$$

where $\mathbb{1}_A$ is the indicator function of a given subset A .

Since $n_1 + n_2 = S$, we have a finite one-dimensional system. Therefore, it is simpler to redefine the system state by n where $n = n_1$ and $n_2 = S - n$. The stationary probabilities are thus defined as $p_n^{LS} = \pi^{LS}(n, S - n)$, for $0 \leq n \leq S$. Therefore, Equation (1) can be simplified into

$$\begin{aligned} (\lambda + \delta_n + \beta_{S-n}) p_n^{LS} &= \beta_{S-n+1} p_{n-1}^{LS} + (\lambda + \delta_{n+1}) p_{n+1}^{LS}, \text{ for } 0 < n < S, \\ \beta_S p_0^{LS} &= (\lambda + \delta_1) p_1^{LS}, \text{ and,} \\ (\lambda + \delta_S) p_S^{LS} &= \beta_1 p_{S-1}^{LS}. \end{aligned} \quad (2)$$

Equation (2) enables us to show by induction that

$$p_n^{LS} = p_0^{LS} \prod_{k=1}^n \frac{\beta_{S-k+1}}{\lambda + \delta_k}, \text{ for } 0 \leq n \leq S. \quad (3)$$

Finally, since the sum of the overall probabilities is equal to one, we get

$$p_0^{LS} = \left[\sum_{n=0}^S \prod_{k=1}^n \frac{\beta_{S-k+1}}{\lambda + \delta_k} \right]^{-1}. \quad (4)$$

There remains to provide the expressions of δ_n and β_n . The expression of δ_n is known from Brandt and Brandt (2002) and Movaghar (1998) for an $\cdot/M/1 + G$ queue;

$$\delta_i = \delta_i(\lambda) = \frac{i\Phi_{i-1}(\lambda)}{\Phi_i(\lambda)} - \lambda, \text{ where } \Phi_i(\lambda) = \int_0^\infty \left(\int_0^x (1 - F(t)) dt \right)^i e^{-\lambda x} dx, \quad (5)$$

for $\lambda > 0$. Queue 2 can also be viewed as a particular $\cdot/M/1 + G$ queue where the abandonment is generated by lead-times and the service process is zero. One way to obtain the β_i 's is then to let λ tend to zero in Equation (5). Interestingly, in Lemma 1, by letting λ tend to 0, we prove that $\beta_i = i\beta$, where β is the expected lead-time rate. This means that the departure rate from Queue 2 only depends on the first moment of the lead-time. This shows that the heuristic in Gürlér and Özkaya (2008) is valid for a non-deterministic distribution as the distribution of the lead-time is proven to have no impact beyond its mean. Note that the result of Lemma 1 holds only if the first two moments of the lead-time are finite.

Lemma 1. *If $L < \infty$ and $\int_0^x t^2 g(t) dt < \infty$, the departure rate from Queue 2 when n items are present is given by $\beta_n = n\beta$.*

Proof. After substituting G to F in δ_n , one can obtain β_n as the limit of δ_n by letting λ tend to zero if this limit exists. We hence have to find the limit of $\frac{\Phi_{n-1}(\lambda)}{\Phi_n(\lambda)}$ as λ tends to 0. Observe that $\Phi_n(\lambda)$ is the Laplace transform of the function $H(x) = \left(\int_0^x (1 - G(t)) dt \right)^n$ at the point λ . Since $|e^{-\lambda x} H'(x)| \leq H'(x)$, one may apply the dominated convergence theorem and write

$$\lim_{\lambda \rightarrow 0} \int_0^\infty H'(x) e^{-\lambda x} dx = \int_0^\infty H'(x) dx = H(\infty) - H(0) = H(\infty).$$

Moreover, the property of derivation for Laplace transforms leads to

$$\int_0^\infty H'(x) e^{-\lambda x} dx = \lambda \int_0^\infty H(x) e^{-\lambda x} dx - H(0) = \lambda \int_0^\infty H(x) e^{-\lambda x} dx = \lambda \Phi_n(\lambda).$$

This proves that

$$\lim_{\lambda \rightarrow 0} \lambda \Phi_n(\lambda) = H(\infty).$$

There remains to obtain the limit of $H(x)$ as x tends to infinity. By applying an integration by

parts to $1 - G(t)$, we get

$$\int_0^x (1 - G(t)) dt = x(1 - G(x)) + \int_0^x tg(t) dt.$$

The Chebyshev's inequality indicates that $1 - G(x) \leq \frac{1}{x^2}E(X^2)$, where $E(X^2)$ is the second order moment of the lead-time distribution. So, $0 \leq x(1 - G(x)) \leq \frac{1}{x}E(X^2)$. This proves that $\lim_{x \rightarrow \infty} x(1 - G(x)) = 0$. Finally, $\int_0^x tg(t)dt = L$. This proves that $H(\infty) = L^n$. By writing $\frac{\Phi_{n-1}(\lambda)}{\Phi_n(\lambda)} = \frac{\lambda\Phi_{n-1}(\lambda)}{\lambda\Phi_n(\lambda)}$, we then may write

$$\beta_n = \lim_{\lambda \rightarrow 0} \frac{\lambda n \Phi_{n-1}(\lambda)}{\lambda \Phi_n(\lambda)} - \lambda = \frac{nL^{n-1}}{L^n} = n\beta.$$

□

Lemma 1, enables us to express the steady-states probabilities as

$$p_n^{LS} = \frac{L^{S-n}\Phi_n(\lambda)}{n!(S-n)!\Phi_0(\lambda)T_{LS}(S)}, \text{ for } 0 \leq n \leq S, \quad (6)$$

where $T_{LS}(S) = \sum_{n=0}^S \frac{L^{S-n}}{(S-n)!} \frac{\Phi_n(\lambda)}{n!\Phi_0(\lambda)} = \frac{1}{S!\Phi_0(\lambda)} \int_0^\infty \left(L + \int_0^\xi (1 - F(x))dx \right)^S e^{-\lambda\xi} d\xi$, for $S > 0$.

We are now in position to derive the total cost function, $Z^{LS}(S)$. We obtain

$$Z_{LS}(S) = h \sum_{n=1}^S np_n^{LS} + w \sum_{n=1}^S \delta_n p_n^{LS} + b\lambda p_0^{LS} \quad (7)$$

$$= h \left(S - \frac{LT_{LS}(S-1)}{T_{LS}(S)} \right) + w \frac{T_{LS}(S-1)}{T_{LS}(S)} + \lambda \frac{(w+b)L^S}{S!T_{LS}(S)} - w\lambda, \quad (8)$$

for $S \geq 0$, with the convention $T_{LS}(-1) = 0$, where the three components of the cost function correspond to the holding cost, the outdated cost and the lost-sales cost, respectively. From extensive numerical tests, we observed that $Z_{LS}(S)$ is quasi-convex in S . However, the complexity of the components of $Z_{LS}(S)$ does not enable us to analytically show this result. Instead, in Proposition 1, we show the first order monotonicity properties of the components of $Z_{LS}(S)$. This result may be used as a first step to show the second order monotonicity properties of $Z_{LS}(S)$.

Proposition 1. *The following holds:*

1. $\frac{T_{LS}(S-1)}{T_{LS}(S)}$ is increasing in S and tends to infinity as S tends to infinity.
2. $S - \frac{LT_{LS}(S-1)}{T_{LS}(S)}$ is increasing in S and tends to infinity as S tends to infinity.

3. $\frac{L^S}{S!T_{LS}(S)}$ is decreasing in S .

Proof. 1) From Equation (5), we have

$$\frac{T_{LS}(S-1)}{T_{LS}(S)} = S \cdot \frac{\int_0^\infty \left(L + \int_0^\xi (1 - F(x)) dx \right)^{S-1} e^{-\lambda\xi} d\xi}{\int_0^\infty \left(L + \int_0^\xi (1 - F(x)) dx \right)^S e^{-\lambda\xi} d\xi}.$$

Let us define the function $I(\xi) = L + \int_0^\xi (1 - F(x)) dx$. Since $F(x) \leq 1$, $I(\xi)$ is increasing in ξ .

Moreover, $\lim_{\xi \rightarrow \infty} I(\xi) = L + \int_0^\infty (1 - F(x)) dx = L + m$. Thus, $I(\xi) \leq L + m$.

Let us prove that $\frac{T_{LS}(S-1)}{T_{LS}(S)}$ is increasing in S . To prove this result, we show that $\frac{\frac{T_{LS}(S-1)}{T_{LS}(S)}}{\frac{T_{LS}(S)}{T_{LS}(S+1)}} < \frac{S}{S+1} < 1$. Using Fubini's Theorem, we have

$$\frac{\frac{T_{LS}(S-1)}{T_{LS}(S)}}{\frac{T_{LS}(S)}{T_{LS}(S+1)}} - \frac{S}{S+1} = -\frac{1}{2} \frac{S}{S+1} \frac{\int_0^\infty \int_0^\infty e^{-\lambda(x+y)} I(x)^{S-1} I(y)^{S-1} (I(x) - I(y))^2 dx dy}{\left(\int_0^\infty I(\xi)^S e^{-\lambda\xi} d\xi \right)^2} \leq 0,$$

which proves the result.

We now show that $\frac{T_{LS}(S-1)}{T_{LS}(S)}$ tends to infinity as S tends to infinity. We may write

$$\frac{1}{L+m} = \frac{\int_0^\infty (L+m)^{S-1} e^{-\lambda\xi} d\xi}{\int_0^\infty (L+m)^S e^{-\lambda\xi} d\xi}.$$

Therefore,

$$\begin{aligned} & \frac{\int_0^\infty (I(\xi))^{S-1} e^{-\lambda\xi} d\xi}{\int_0^\infty (I(\xi))^S e^{-\lambda\xi} d\xi} - \frac{\int_0^\infty (L+m)^{S-1} e^{-\lambda\xi} d\xi}{\int_0^\infty (L+m)^S e^{-\lambda\xi} d\xi} \\ &= \frac{\int_0^\infty \int_0^\infty e^{-\lambda(x+y)} (L+m)^{S-1} I(x)^{S-1} (L+m - I(y)) dx dy}{\int_0^\infty (I(\xi))^S e^{-\lambda\xi} d\xi \cdot \int_0^\infty (L+m)^S e^{-\lambda\xi} d\xi} \geq 0. \end{aligned}$$

This leads to $\frac{T_{LS}(S-1)}{T_{LS}(S)} \geq \frac{S}{m+L}$ and proves the result.

2) To prove the second statement of Proposition 1, we need to prove that $\frac{T_{LS}(S-1)}{ST_{LS}(S)}$ is decreasing in S . For $S \geq 1$, using the same approach as for 1), from Fubini's Theorem, we may write

$$\begin{aligned} \frac{T_{LS}(S-1)}{ST_{LS}(S)} - \frac{T_{LS}(S)}{(S+1)T_{LS}(S+1)} &= \frac{\int_0^\infty I(\xi)^{S-1} e^{-\lambda\xi} d\xi}{\int_0^\infty I(\xi)^S e^{-\lambda\xi} d\xi} - \frac{\int_0^\infty I(\xi)^S e^{-\lambda\xi} d\xi}{\int_0^\infty I(\xi)^{S+1} e^{-\lambda\xi} d\xi} \\ &= -\frac{1}{2} \frac{\int_0^\infty \int_0^\infty e^{-\lambda(x+y)} I(x)^{S-1} I(y)^{S-1} (I(x) - I(y))^2 dx dy}{\int_0^\infty I(\xi)^S e^{-\lambda\xi} d\xi \cdot \int_0^\infty I(\xi)^{S+1} e^{-\lambda\xi} d\xi} \leq 0. \end{aligned}$$

Note that

$$S - \frac{LT_{LS}(S-1)}{T_{LS}(S)} = S \left(1 - \frac{LT_{LS}(S-1)}{ST_{LS}(S)} \right).$$

Since the term $\frac{LT_{LS}(S-1)}{ST_{LS}(S)}$ is decreasing in S and tends to 0 as S tends to infinity, $S \left(1 - \frac{LT_{LS}(S-1)}{ST_{LS}(S)} \right)$ tends to infinity as S tends to infinity.

3) We may write

$$\frac{\frac{L^{S-1}}{(S-1)!T_{LS}(S-1)}}{\frac{L^S}{S!T_{LS}(S)}} = \frac{1}{L} \frac{ST_{LS}(S)}{T_{LS}(S-1)}.$$

From 2), the function $\frac{ST_{LS}(S)}{T_{LS}(S-1)}$ is increasing in S . So, $\frac{1}{L} \frac{ST_{LS}(S)}{T_{LS}(S-1)} \geq \frac{1}{L} \frac{1 \cdot T_{LS}(1)}{T_{LS}(0)}$. We have $T_{LS}(0) = \frac{1}{\lambda}$ and $T_{LS}(1) = \int_0^\infty I(\xi) e^{-\lambda\xi} d\xi \geq \int_0^\infty L e^{-\lambda\xi} d\xi = \frac{L}{\lambda}$. Therefore

$$\frac{\frac{L^{S-1}}{(S-1)!T_{LS}(S-1)}}{\frac{L^S}{S!T_{LS}(S)}} = \frac{1}{L} \frac{ST_{LS}(S)}{T_{LS}(S-1)} \geq 1,$$

which proves 3). □

Computation of the optimal base-stock level. A classical algorithm to obtain the optimal base-stock level would stop at the first local minimum of $Z_{LS}(S)$ obtained by increasing S . However, we are not able to prove that $Z_{LS}(S)$ has a unique local minimum. Alternatively, we propose to use an algorithm in the spirit of the one of Olsson and Tydesjö (2010). The idea is to limit the search of the optimal threshold level to a finite number of iterations by computing an upper bound for the base-stock level. The algorithm of Olsson and Tydesjö (2010) is valid under the condition that all cost components are positive and one part of the cost function is increasing and tends to infinity as S tends to infinity. From Proposition 1, we deduce that the term $Z_{LS}^1(S) = h \left(S - \frac{LT_{LS}(S-1)}{T_{LS}(S)} \right) + w \frac{T_{LS}(S-1)}{T_{LS}(S)}$ has these properties if either $w > 0$ or $h > 0$. In the case $w = h = 0$, $S = 0$ is optimal. The algorithm is as follows:

- Step 1: Set $S := 0$, $S^* := 0$, and $Z_{min} := \infty$, and go to Step 2.
- Step 2: Compute $Z_{LS}(S)$. If $Z_{LS}(S) < Z_{min}$ then $Z_{min} = Z_{LS}(S)$ and $S^* = S$. Set $S := S+1$, and go to Step 3.
- Step 3: Compute $Z_{LS}^1(S)$. If $Z_{LS}^1(S) \leq Z_{min}$ then go to Step 2, otherwise stop the procedure.

In Section 4, we will show that the same algorithm can be employed for the backorders case.

3.3 Special cases: Deterministic and exponential lifetime

In this section, we retrieve [the stationary probabilities found in Schmidt and Nahmias \(1985\)](#) for the deterministic case and those of [Liu and Cheung \(1997\)](#) for the exponential case using the results of Section 3.2. [This also shows that the expected cost in these references can be obtained from our results.](#) The idea is to show that our analysis generalizes existing ones.

Deterministic case. As mentioned in the introduction, the deterministic case is the most studied one. In what follows, we use the cumulative distribution function of the deterministic distribution to derive the stationary probabilities in this case from our results. Let us introduce the cdf of the deterministic distribution defined as

$$F(x) = \begin{cases} 0 & \text{if } x < m, \\ 1 & \text{if } x \geq m, \end{cases}$$

where we recall that m is the expected duration of the lifetime. After some algebra, we obtain $\Phi_n(\lambda) = \frac{\Gamma[n+1] - \Gamma[n+1, m\lambda]}{\lambda^{n+1}} + \frac{m^n e^{-m\lambda}}{\lambda}$, where $\Gamma[n+1, m\lambda] = \int_{m\lambda}^{\infty} t^n e^{-t} dt$ is the incomplete gamma function. Since

$$\Gamma[n+1] - \Gamma[n+1, m\lambda] = \int_0^{m\lambda} t^n e^{-t} dt,$$

we get

$$\Phi_n(\lambda) = \frac{1}{\lambda^{n+1}} \int_0^{m\lambda} t^n e^{-t} dt + \frac{m^n e^{-m\lambda}}{\lambda}.$$

In the case $n = 0$, we get $\Phi_0(\lambda) = \frac{1}{\lambda}$. For $n \geq 1$, replacing the variable t by z/λ followed by an integration by parts leads to

$$\Phi_n(\lambda) = \frac{n}{\lambda} \int_0^m z^{n-1} e^{-\lambda z} dz.$$

Replacing now the variable z by $x - L$, we obtain

$$\Phi_n(\lambda) = \frac{ne^{\lambda L}}{\lambda} \int_L^{m+L} (x - L)^{n-1} e^{-\lambda x} dx.$$

Substituting this expression into Equation (5) leads to

$$p_n^{LS} = \frac{e^{\lambda L}}{\lambda \Phi_0(\lambda)} \frac{L^{S-n}}{(n-1)!(S-n)!T_{LS}(S)} \int_L^{m+L} (x - L)^{n-1} e^{-\lambda x} dx, \text{ for } 1 \leq n \leq S.$$

The last expression is the one derived in Schmidt and Nahmias (1985) with the appropriate normalizing constant.

Exponential case. We now develop the same approach for finding the steady state probabilities of the inventory level under the case of exponential lifetime distribution. For an exponential distribution with rate $\gamma = 1/m$, we may write

$$F(x) = 1 - e^{-\gamma x}, \text{ for } x > 0.$$

Using the expression of $\phi_n(\lambda)$ in Equation (5), we obtain

$$\Phi_n(\lambda) = n! \prod_{i=0}^n \frac{1}{\lambda + i\gamma},$$

and

$$p_n^{LS} = \frac{L^{S-n}}{(S-n)!T_{LS}(S)} \prod_{i=1}^n \frac{1}{\lambda + i\gamma}, \text{ for } 0 \leq n \leq S.$$

This corresponds to the steady-state probability derived by Liu and Cheung (1997) as given in Equation (2) of their paper.

4 The backorders case

In this section, we extend our analysis to the backorders case. This case can be seen as an extension to the lost-sales case. Instead of having lost-sales due to an empty Queue 1, we assume here that an order is generated so as to provide the item to the customer later. Therefore, the sale is not lost but postponed after a lead-time. Whenever an item is available in Queue 1, the system behaves as in the lost-sales case. We employ the same approach as in Section 3; our inventory model can be depicted as a semi-open network with a minimal population equal to S (see Figure 2). As in Section 3, the state space is defined by the couple (n_1, n_2) , where n_i is the number in Queue i , for $i = 1, 2$. The set space, Ω^{BO} , is then $\Omega^{BO} = \{(n_1, n_2) \in \mathbb{N}^2 | n_1 + n_2 = S \text{ or } n_1 = 0, n_2 \geq S\}$. The

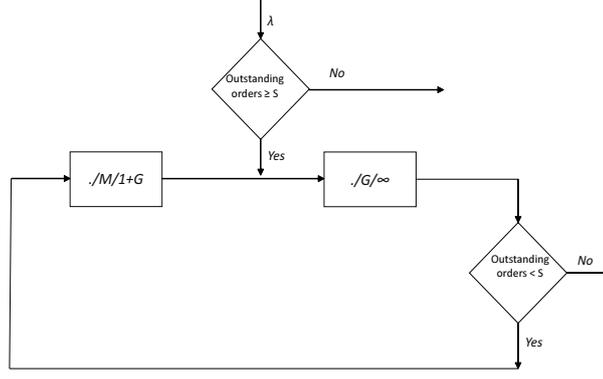


Figure 2: Modeling a base-stock inventory systems with backorders as a queueing network

global balance equation of our queueing network is

$$\begin{aligned}
(\lambda + \delta_S)\pi^{BO}(S, 0) &= \beta_1\pi^{BO}(S - 1, 1), \text{ for } n_1 = S, n_2 = 0, \\
(\lambda + n_2\beta + \delta_{n_1})\pi^{BO}(n_1, n_2) &= (n_2 + 1)\beta\pi^{BO}(n_1 - 1, n_2 + 1) \\
&\quad + (\lambda + \delta_{n_1+1})\pi^{BO}(n_1 + 1, n_2 - 1), \text{ for } 0 < n_1, n_2 < S, \\
(\lambda + S\beta)\pi^{BO}(0, S) &= (S + 1)\beta\pi^{BO}(0, S + 1) + (\lambda + \delta_1)\pi^{BO}(1, S - 1), \text{ for } n_1 = 0, n_2 = S, \\
(\lambda + n_2\beta)\pi^{BO}(0, n_2) &= (n_2 + 1)\beta\pi^{BO}(0, n_2 + 1) + \lambda\pi^{BO}(0, n_2 - 1), \text{ for } n_1 = 0, n_2 > S.
\end{aligned} \tag{9}$$

As in the lost-sales case, Equation (9) is a one-dimensional system. We can show by induction that

$$\pi^{BO}(n_1, n_2) = \begin{cases} \frac{1}{T_{BO}(S)} \left(\prod_{i=1}^{n_1} \frac{\lambda}{\lambda + \delta_i} \right) \frac{1}{n_2!} \left(\frac{\lambda}{\beta} \right)^{n_2}, & \text{if } n_1 > 0, \\ \frac{1}{T_{BO}(S)} \frac{1}{n_2!} \left(\frac{\lambda}{\beta} \right)^{n_2}, & \text{if } n_1 = 0, n_2 \geq S, \end{cases} \tag{10}$$

where $T_{BO}(S)$ is the normalizing constant;

$$T_{BO}(S) = \sum_{n_1+n_2=S} \left(\prod_{i=1}^{n_1} \frac{\lambda}{\lambda + \delta_i} \frac{1}{n_2!} \left(\frac{\lambda}{\beta} \right)^{n_2} \right) + \sum_{n_2=S+1}^{\infty} \frac{1}{n_2!} \left(\frac{\lambda}{\beta} \right)^{n_2} = \lambda^S T_{LS}(S) + Q(S),$$

where $T_{LS}(S)$ is the normalizing constant given in Section 3 for the lost-sales case and $Q(S) = \sum_{n_2=S+1}^{\infty} \frac{1}{n_2!} \left(\frac{\lambda}{\beta}\right)^{n_2}$. Using Equation (5), $\pi^{BO}(n_1, n_2)$ can be further simplified into

$$\pi^{BO}(n_1, n_2) = \begin{cases} \frac{1}{T_{BO}(S)} \frac{\lambda^S L^{S-n_1} \Phi_{n_1}(\lambda)}{n_1!(S-n_1)! \Phi_0(\lambda)}, & \text{if } n_1 > 0 \text{ and, } n_1 + n_2 = S, \\ \frac{1}{T_{BO}(S)} \frac{1}{n_2!} \left(\frac{\lambda}{\beta}\right)^{n_2}, & \text{if } n_1 = 0, n_2 \geq S. \end{cases} \quad (11)$$

As for the lost-sales case, Equation (11) enables us to retrieve the formula of Olsson (2014) found under a deterministic lifetime assumption. Our stationary probabilities are an extension of those derived in Liu and Yang (1999) under exponential lead-time and lifetime assumptions. Specifically, Liu and Yang (1999) considered a general (s, S) inventory model and used a matrix-analytic approach to numerically derive the stationary probabilities and the total cost function. Equation (11) in our paper provides a closed-form expression for the stationary probabilities in the case of a base-stock policy (i.e., for $s = S - 1$).

We consider the same cost components as in the lost-sales case. Instead of counting a penalty b per lost sale, we count the same penalty for backorders in order to penalize the postpone of the service. The total cost function can be written as

$$\begin{aligned} Z_{BO}(S) &= \sum_{n=1}^S hn\pi^{BO}(n, S-n) + \sum_{n=1}^S w\delta_n\pi^{BO}(n, S-n) + \lambda b \left(1 - \sum_{n=1}^S \pi^{BO}(n, S-n)\right) \\ &= \lambda b + \frac{\lambda^S}{T_{BO}(S)\Phi_0(\lambda)} \sum_{n=1}^S \left(\frac{L^{S-n}(hn\Phi_n(\lambda) + wn\Phi_{n-1}(\lambda) - \lambda(b+w)\Phi_n(\lambda))}{(S-n)!n!}\right). \end{aligned} \quad (12)$$

As the expression of the stationary probabilities given in Equation (11) and that of the lost-sales case are related, we can express the total cost of the backorders case, $Z_{BO}(S)$, as a function of the total cost in the lost-sales case, $Z_{LS}(S)$ as follows:

$$Z_{BO}(S) = \lambda b \left[1 - \frac{\lambda^S T_{LS}(S)}{T_{BO}(S)}\right] + \frac{\lambda^S T_{LS}(S)}{T_{BO}(S)} Z_{LS}(S). \quad (13)$$

Computation of the optimal base-stock level. To prove that the algorithm of Olsson and Tydesjö (2010) can be employed for the backorders case, we need to prove that one part of the cost function is increasing in S and tends to infinity as S tends to infinity. We consider $Z_{BO}^2(S) = Z_{LS}^1(S) \cdot \frac{\lambda^S T_{LS}(S)}{T_{BO}(S)}$. As proven in Proposition 1, $Z_{LS}^1(S)$ is increasing in S and tends to infinity as S tends to infinity. In Proposition 2, we prove that $\frac{\lambda^S T_{LS}(S)}{T_{BO}(S)}$ is also increasing in S . Therefore, by replacing the function $Z_{LS}^1(S)$ by $Z_{BO}^2(S)$ in the algorithm of Section 3, we can compute the

optimal base-stock level after a finite number of iterations in the same way. The algorithm is as follows:

- Step 1: Set $S := 0$, $S^* := 0$, and $Z_{min} := \infty$, and go to Step 2.
- Step 2: Compute $Z_{BO}(S)$. If $Z_{BO}(S) < Z_{min}$ then $Z_{min} = Z_{BO}(S)$ and $S^* = S$. Set $S := S + 1$, and go to Step 3.
- Step 3: Compute $Z_{BO}^2(S)$. If $Z_{BO}^2(S) \leq Z_{min}$ then go to step 2, otherwise stop the procedure.

Proposition 2. *The function $\frac{\lambda^S T_{LS}(S)}{T_{BO}(S)}$ is increasing in S .*

Proof. We may write

$$\frac{\lambda^S T_{LS}(S)}{T_{BO}(S)} = \frac{1}{1 + \frac{Q(S)}{\lambda^S T_{LS}(S)}}.$$

Therefore, we need to prove that $\frac{Q(S)}{\lambda^S T_{LS}(S)}$ is decreasing in S . We rewrite $\frac{Q(S)}{\lambda^S T_{LS}(S)}$ as $\frac{S!Q(S)}{\lambda^S L^S} \cdot \frac{L^S}{S!T_{LS}(S)}$. From the third statement of Proposition 1, the function $\frac{L^S}{S!T_{LS}(S)}$ is decreasing in S . There remains to prove that $\frac{S!Q(S)}{\lambda^S L^S}$ is also decreasing in S . We have

$$\begin{aligned} \frac{S!Q(S)}{\lambda^S L^S} - \frac{(S+1)!Q(S+1)}{\lambda^{S+1} L^{S+1}} &= \sum_{n=S+1}^{\infty} \frac{S!(\lambda L)^{n-S}}{n!} - \sum_{n=S+2}^{\infty} \frac{(S+1)!(\lambda L)^{n-(S+1)}}{n!} \\ &= \sum_{n=S+1}^{\infty} \frac{S!(\lambda L)^{k+1}}{(S+1+k)!} \left(1 - \frac{S+1}{S+2+k}\right) \geq 0, \end{aligned}$$

and proves the result. □

5 Model extensions

In this section, we propose different extensions of the lost-sales case previously developed in Section 3. In particular, we consider three extensions: (i) with multiple warehouses with lost-sales, each with a product having a different lifetime (Section 5.1), (ii) with dual-sourcing in a single-warehouse (Section 5.2), and (iii) with dual-sourcing with multiple warehouses (Section 5.3).

5.1 Multi-warehouse with lost-sales

We first extend the analysis of Section 3 with lost-sales for a system with J local warehouses, as shown in Figure 3. Each warehouse behaves independently from the others and apply a base-stock policy with base-stock level S_i , for $1 \leq i \leq J$. At warehouse i , the demand is Poisson with parameter

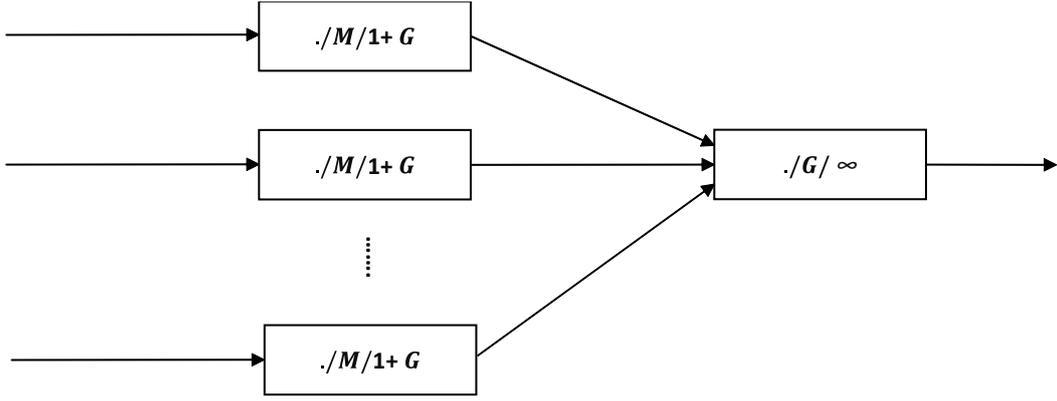


Figure 3: The multi-class base-stock system with different lifetime distributions

λ_i and the lifetime of each product follows a general distribution with cdf F_i , for $1 \leq i \leq J$. As in Section 3, a demand of class i is lost if no item is available at its associated local warehouse. Once an item is consumed or expired, an order is generated. The delivery lead-time is generally distributed with a cdf G and mean L common for all warehouses. After a lead-time, an order from warehouse i generates a new item at the same warehouse. As such, each warehouse together with its related outstanding orders can be viewed as a closed network as in Section 3.

Let n_i be the number of class i on-hand items at warehouse i , for $1 \leq i \leq J$. Given that the number of items on-hand plus the number of orders for warehouse i is equal to S_i , the vector (n_1, n_2, \dots, n_J) completely defines the state of the system;

$\Omega^{MLS} = \{(n_1, n_2, \dots, n_J) \mid 0 \leq n_i \leq S_i, \text{ for } i = 1, \dots, J\}$. We denote by $P^{MLS}(n_1, n_2, \dots, n_J)$, the stationary probability to be in state (n_1, n_2, \dots, n_J) . Using the independence of each warehouse and the results of Section 3, we may write

$$P^{MLS}(n_1, n_2, \dots, n_J) = \prod_{i=1}^J p_{n_i}^{LS}, \quad (14)$$

for $(n_1, n_2, \dots, n_J) \in \Omega^{MLS}$. The probability to have n items in total in the different warehouses, $P(n)$, is given by $P(n) = \sum_{\sum_{j=1}^J n_j = n} P^{MLS}(n_1, n_2, \dots, n_J)$, and the total expected cost is

$Z_{MLS}(S_1, S_2, \dots, S_J) = \sum_{i=1}^J Z_{LS}(S_i)$. Due to the independence of the different warehouses, the optimization of the optimal base-stock levels S_1, S_2, \dots, S_J can be done individually for each warehouse using the algorithm presented in Section 3.

5.2 Dual-sourcing for a single warehouse

We further extend the model of Section 3 to a case with *dual-sourcing*. The idea of dual-sourcing is that an emergency but costlier order can be sent to refill the on-hand inventory when the number of items in the inventory becomes too low (Bijvank and Vis, 2012). The idea is to reduce the risk of lost-sales. We consider the dual-sourcing policy of Moinzadeh and Schmidt (1991) and Song and Zipkin (2009). In comparison with these references, we add the feature of item deterioration where the lifetime is generally distributed. The dual sourcing policy is analyzed here for deterministic lead-times in order to be consistent with the queueing network approach developed in this paper. However, as shown in Section 3 of Song and Zipkin (2009), the results in this section are valid for any distribution of the lead-time.

The regular lead-time duration, L , is decomposed into $L = L - L_e + L_e$, where L_e is the duration of an emergency lead-time, with $L_e < L$. As such the lead-time of a regular order is decomposed into a two-step process with duration $L - L_e$ and L_e whereas the lead-time of an emergency order follows a one-step process with duration L_e . In Figure 4, we depict the queueing network associated to the dual-sourcing inventory model. The closed queueing network associated with this model is

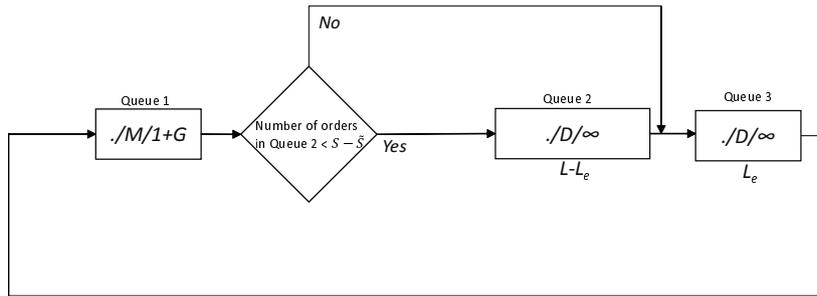


Figure 4: Queueing network under dual-sourcing for a single warehouse

composed of three connected queues: the one of on-hand items called Queue 1, the one of regular orders in the first part of the lead-time process with duration $L - L_e$, called Queue 2, and the one of regular orders in the second part of the lead-time process plus the emergency orders both with duration L_e , called Queue 3. Queue 1 is an $\cdot/M/1 + G$ queue as in the lost-sales and backorders cases while Queue 2 and Queue 3 are $\cdot/D/\infty$ queues. We denote by n_i , the number of items in Queue i , for $i = 1, 2, 3$. We have $n_1 + n_2 + n_3 = S$. The dual-sourcing policy is controlled by the threshold level \tilde{S} , with $0 \leq \tilde{S} \leq S$, such that regular orders are made if the number of items in Queue 2 is strictly below $S - \tilde{S}$ (i.e., $n_2 < S - \tilde{S}$), otherwise emergency orders are made when $n_2 = S - \tilde{S}$.

We use the notations $\beta_{n_2}^1$, and $\beta_{n_3}^2$ to represent the departure rates from Queue 2, and Queue 3, respectively, when there are n_i items in Queue i , for $i = 1, 2, 3$. The departure rate from Queue 1 is identical to the one in the lost-sales case. With similar arguments as in Lemma 1, we can show that

$\beta_{n_2}^1 = \frac{n_2}{L-L_e}$, and $\beta_{n_3}^2 = \frac{n_3}{L}$. We denote by $\pi^{DS}(n_1, n_2, n_3)$ the stationary probability to be in state (n_1, n_2, n_3) , for $0 \leq n_1, n_3 \leq S$, $0 \leq n_2 + n_3 \leq S$, and $0 \leq n_2 \leq S - \tilde{S}$. Given that $n_1 + n_2 + n_3 = S$, we define $q^{DS}(n_2, n_3) = \pi^{DS}(S - n_2 - n_3, n_2, n_3)$, for $0 \leq n_2 + n_3 \leq S$, and $0 \leq n_2 \leq S - \tilde{S}$. The system balance equations can then be written as follows:

$$((\lambda + \delta_{S-n_2-n_3})\mathbb{1}_{n_2+n_3 < S} + \beta_{n_2}^1 + \beta_{n_3}^2) q^{DS}(n_2, n_3) = (\lambda + \delta_{S-(n_2-1)-n_3}) q^{DS}(n_2-1, n_3) \mathbb{1}_{n_2 > 0} \quad (15)$$

$$+ \beta_{n_2+1}^1 q^{DS}(n_2+1, n_3-1) \mathbb{1}_{n_3 > 0} + \beta_{n_3+1}^2 q^{DS}(n_2, n_3+1) \mathbb{1}_{n_2+n_3 < S}, \text{ for } 0 \leq n_2 < S - \tilde{S},$$

and,

$$\begin{aligned} & \left((\lambda + \delta_{\tilde{S}-n_3}) \mathbb{1}_{n_3 < \tilde{S}} + \beta_{S-\tilde{S}}^1 + \beta_{n_3}^2 \right) q^{DS}(S - \tilde{S}, n_3) = (\lambda + \delta_{\tilde{S}-n_3+1}) q^{DS}(S - \tilde{S} - 1, n_3) \mathbb{1}_{S > \tilde{S}} \\ & + (\lambda + \delta_{\tilde{S}-n_3+1}) q^{DS}(S - \tilde{S}, n_3 - 1) \mathbb{1}_{n_3 > 0} + \beta_{n_3+1}^2 q^{DS}(S - \tilde{S}, n_3 + 1) \mathbb{1}_{n_3 < \tilde{S}}, \text{ for } n_2 = S - \tilde{S}, \end{aligned}$$

for $0 \leq n_2 + n_3 \leq S$, and $0 \leq n_2 \leq S - \tilde{S}$. From Equation (5), we get $\prod_{i=1}^{S-n_2-n_3} \frac{1}{\lambda + \delta_i} = \frac{\Phi_{S-n_2-n_3}(\lambda)}{(S-n_2-n_3)! \Phi_0(\lambda)}$. Therefore, the solution of the balance equations is given by

$$q^{DS}(n_2, n_3) = T_{DS}^{-1}(S, \tilde{S}) \frac{\Phi_{S-n_2-n_3}(\lambda)}{(S-n_2-n_3)! \Phi_0(\lambda)} \frac{(L-L_e)^{n_2}}{n_2!} \frac{L_e^{n_3}}{n_3!}, \quad (16)$$

where $T_{DS}(S, \tilde{S}) = \sum_{0 \leq n_2+n_3 \leq S, 0 \leq n_2 \leq S-\tilde{S}} \frac{\Phi_{S-n_2-n_3}(\lambda)}{(S-n_2-n_3)! \Phi_0(\lambda)} \frac{(L-L_e)^{n_2}}{n_2!} \frac{L_e^{n_3}}{n_3!}$, for $0 \leq n_2 + n_3 \leq S$, and $0 \leq n_2 \leq S - \tilde{S}$. We rewrite $T_{DS}(S, \tilde{S})$ as

$$T_{DS}(S, \tilde{S}) = \sum_{n_2=0}^{S-\tilde{S}} \frac{(L-L_e)^{n_2}}{n_2!} \left[\sum_{n_3=0}^{S-n_2} \frac{\Phi_{n_3}(\lambda)}{n_3! \Phi_0(\lambda)} \frac{L_e^{S-n_2-n_3}}{(S-n_2-n_3)!} \right] = \sum_{n_2=0}^{S-\tilde{S}} \frac{(L-L_e)^{n_2}}{n_2!} T_{LS}(S-n_2),$$

where $T_{LS}(S-n_2)$ is the normalizing constant for the lost-sales case with expected lead-time L_e , and base-stock level $S-n_2$.

We are interested in the probability to have exactly n_1 items on-hands in Queue 1, $p_{n_1}^{DS}$. For $0 \leq n_1 \leq \tilde{S}$, we obtain

$$p_{n_1}^{DS} = T_{DS}^{-1}(S, \tilde{S}) \frac{\Phi_{n_1}(\lambda)}{n_1! \Phi_0(\lambda)} \sum_{n_2=0}^{\min(S-\tilde{S}, S-n_1)} \frac{(L-L_e)^{n_2}}{n_2!} \frac{L_e^{S-n_1-n_2}}{(S-n_1-n_2)!}. \quad (17)$$

If $\tilde{S} \leq n_1 \leq S$, then the expression of $p_{n_1}^{DS}$ can be further simplified into

$$p_{n_1}^{DS} = T_{DS}^{-1}(S, \tilde{S}) \frac{\Phi_{n_1}(\lambda)}{n_1! \Phi_0(\lambda)} \frac{L^{S-n_1}}{(S-n_1)!}.$$

Equation (17) allows us to retrieve the steady-state probability found by Kouki et al. (2018) for deterministic or exponential lifetimes. This equation also gives the steady-state probability of Moinzadeh and Schmidt (1991) and Song and Zipkin (2009) for cases without product deterioration.

We now provide the total expected cost. The total cost is the sum of the holding cost, the outdating cost, the lost-sales cost and the purchasing cost. The cost parameters for holding, outdating and lost-sales are identical to those of Section 3. Additionally, we count a purchasing cost of c_r per regular order and c_e per emergency order ($c_e > c_r$). The expected holding cost is proportional to the expected number of items in Queue 1; $\sum_{n_1=1}^S n_1 p_{n_1}^{DS}$. From a similar approach as in Section 3, we obtain

$$\sum_{n_1=1}^S n_1 p_{n_1}^{DS} = T_{DS}^{-1}(S, \tilde{S}) \sum_{n_2=0}^{S-\tilde{S}} \frac{(L-L_e)^{n_2}}{n_2!} ((S-n_2)T_{LS}(S-n_2) - L_e T_{LS}(S-1-n_2)).$$

The outdating rate is given by $\sum_{n_1=1}^S \delta_{n_1} p_{n_1}^{DS}$. It can be computed as the difference between the rate of orders (emergency+regular) and the rate of satisfied demand. The rate of lost-sales is $\lambda p_0^{DS} = \lambda T_{DS}^{-1}(S, \tilde{S}) \sum_{n_2=0}^{S-\tilde{S}} \frac{(L-L_e)^{n_2}}{n_2!} \frac{L_e^{S-n_2}}{(S-n_2)!}$. Emergency orders are triggered when Queue 1 is non-empty and the number of items in Queue 2 is equal to $S - \tilde{S}$. The rate of emergency orders, σ_e , is then

$$\begin{aligned} \sigma_e &= \sum_{n_3=0}^{\tilde{S}-1} q^{DS}(S - \tilde{S}, n_3)(\lambda + \delta_{\tilde{S}-n_3}) = T_{DS}^{-1}(S, \tilde{S}) \frac{(L-L_e)^{S-\tilde{S}}}{(S-\tilde{S})!} \sum_{n_3=0}^{\tilde{S}-1} \frac{\Phi_{\tilde{S}-n_3-1}(\lambda)}{(\tilde{S}-n_3-1)! \Phi_0(\lambda)} \frac{L_e^{n_3}}{n_3!} \\ &= \frac{(L-L_e)^{S-\tilde{S}} T_{LS}(\tilde{S}-1)}{(S-\tilde{S})! T_{DS}(S, \tilde{S})}. \end{aligned}$$

The rate of regular orders, σ_r , is

$$\begin{aligned} \sigma_r &= T_{DS}^{-1}(S, \tilde{S}) \sum_{n_2=0}^{S-\tilde{S}-1} \frac{(L-L_e)^{n_2}}{n_2!} \sum_{n_3=0}^{S-1-n_2} \frac{\Phi_{S-n_2-n_3-1}(\lambda)}{(S-n_2-n_3-1)! \Phi_0(\lambda)} \frac{L_e^{n_3}}{n_3!} \\ &= \sum_{n_2=0}^{S-\tilde{S}-1} \frac{(L-L_e)^{n_2}}{n_2!} \frac{T_{LS}(S-1-n_2)}{T_{DS}(S, \tilde{S})}. \end{aligned}$$

Therefore, the total expected cost can be written as

$$\begin{aligned}
Z_{DS}(S, \tilde{S}) &= \sum_{n_2=0}^{S-\tilde{S}} \frac{(L-L_e)^{n_2}}{n_2!} \left[h \cdot \frac{(S-n_2)T_{LS}(S-n_2) - L_e T_{LS}(S-1-n_2)}{T_{DS}(S, \tilde{S})} \right. \\
&\quad \left. + w \cdot \left(\frac{T_{LS}(S-1-n_2)}{T_{DS}(S, \tilde{S})} - \lambda \left(1 - \frac{L_e^{S-n_2}}{T_{DS}(S, \tilde{S})(S-n_2)!} \right) \right) + b\lambda \cdot \frac{L_e^{S-n_2}}{T_{DS}(S, \tilde{S})(S-n_2)!} \right. \\
&\quad \left. + c_r \frac{T_{LS}(S-1-n_2)}{T_{DS}(S, \tilde{S})} \right] + (c_e - c_r) \frac{(L-L_e)^{S-\tilde{S}} T_{LS}(\tilde{S}-1)}{(S-\tilde{S})! T_{DS}(S, \tilde{S})}.
\end{aligned} \tag{18}$$

We can then relate the cost in the dual-sourcing case to the one in the lost-sales case with lead-time L_e :

$$\begin{aligned}
Z_{DS}(S, \tilde{S}) &= \sum_{n_2=0}^{S-\tilde{S}} \frac{(L-L_e)^{n_2}}{n_2!} \left[Z_{LS}(S-n_2) \frac{T_{LS}(S-n_2)}{T_{DS}(S, \tilde{S})} + c_r \frac{T_{LS}(S-1-n_2)}{T_{DS}(S, \tilde{S})} \right] \\
&\quad + (c_e - c_r) \frac{(L-L_e)^{S-\tilde{S}} T_{LS}(\tilde{S}-1)}{(S-\tilde{S})! T_{DS}(S, \tilde{S})}.
\end{aligned} \tag{19}$$

Computation of the optimal base-stock parameters. As in the previous sections, we want to provide a numerical method to compute the base-stock parameters S and \tilde{S} . Specifically, we want to find S and \tilde{S} such that $(S, \tilde{S}) = \arg \min_{S \geq 0, 0 \leq \tilde{S} \leq S} Z_{DS}(S, \tilde{S})$. For $\tilde{S} = S$ and $c_e = c_r = 0$, we obtain $Z_{DS}(S, S) = Z_{LS}(S)$, with a lead-time of L_e and for $\tilde{S} = 0$ and $c_e = c_r = 0$, we obtain $Z_{DS}(S, 0) = Z_{LS}(S)$, with a lead-time of L . Therefore, the algorithm presented in Section 3 for the lost-sales case is valid to obtain a maximal value for S when either $\tilde{S} = 0$ or $\tilde{S} = S$. However, this only provides an upper bound for the search of the optimal base-stock in two special cases. Supported by our numerical experiments, we conjecture that the maximal value of the base-stock level between these two cases of upper bounds is also the upper bound for S , for any value of \tilde{S} such that $0 \leq \tilde{S} \leq S$. This conjecture is however difficult to prove analytically due to the complex expression of the cost function. Based on this conjecture, the algorithm is as follows:

- Step 1: Set $\tilde{S} := 0$, and apply the algorithm of Section 3 to determine a first upper bound for the base-stock level, S_1 , and go to Step 2.
- Step 2: Set $\tilde{S} := S$ and apply the algorithm of Section 3 to determine a second upper bound for the base-stock level, S_2 , and go to Step 3.
- Step 3: Determine the optimal values for S and \tilde{S} such that

$$(S, \tilde{S}) = \arg \min_{0 \leq S \leq \max(S_1, S_2), 0 \leq \tilde{S} \leq S} Z_{DS}(S, \tilde{S}).$$

This procedure allows us to determine the optimal values for S and \tilde{S} after a finite number of

iterations.

5.3 Multi-warehouse with dual-sourcing

We now extend our analysis to the case with J independent warehouses. The same dual-sourcing policy applies at warehouse i , with parameters \tilde{S}_i and S_i , with $0 \leq \tilde{S}_i \leq S_i$, for $i = 1, 2, \dots, J$. The lead-times at each warehouse are identical with expected values of L and L_e for regular and emergency orders, respectively, with $L_e < L$. We denote by $P^{MDS}(n_1, n_2, n_3, \dots, n_J)$ the stationary probability to have n_i on-hand items at warehouse i . Since the warehouses are independent, we deduce from Equation (17) that

$$P^{MDS}(n_1, n_2, n_3, \dots, n_J) = \prod_{i=1}^J p_{n_i}^{DS}, \quad (20)$$

and the total expected cost is $Z_{MDS}(S_1, \tilde{S}_1, S_2, \tilde{S}_2, \dots, S_J, \tilde{S}_J) = \sum_{i=1}^J Z_{DS}(S_i, \tilde{S}_i)$. Due to the independence of the different warehouses, the optimization of the optimal base-stock levels $S_1, \tilde{S}_1, S_2, \tilde{S}_2, \dots, S_J, \tilde{S}_J$ can be done individually for each warehouse using the algorithm presented in the single-warehouse case.

6 Numerical Analysis

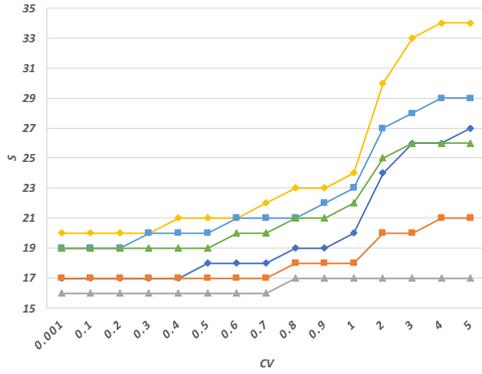
In this section, we primarily conduct a numerical investigation to explore how the lifetime variability affects the optimal base-stock level and the optimal cost for the different base-stock models presented in this paper. In particular, we are interested in the errors made when making decisions based on deterministic or exponential assumptions. We also show the impact of the cost parameters and the base-stock policy.

For this purpose, we assume that the lifetime follows a Gamma distribution. This distribution enables us to have full flexibility for setting the lifetime variability. Moreover, the Gamma distribution is known to well model failure processes (Teunter et al., 2010). We present the optimal cost and the optimal base-stock levels as functions of CV , defined as the ratio between the standard-deviation and the expected value of the lifetime. For consistency, we choose the same expected value for the expected lifetime for all numerical experiments; $m = 3$. We choose to vary CV from 0.001 to 5 in order to reflect a large range of variability. The effects of the holding cost, the demand rate, and the expected lead-time are known and expected. As such, we decided not to present them and set $h = 1, \lambda = 4$, and $L = 3$. The optimal base-stock level under the lost-sales case is obtained

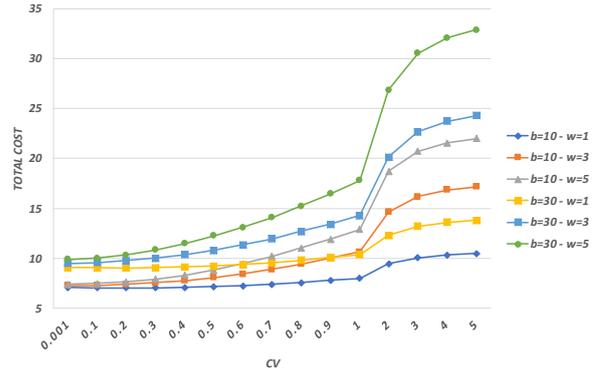
using the optimization procedure given in Section 3 while similar search algorithms are employed for the backorders and dual-sourcing cases.

6.1 Sensitivity analysis

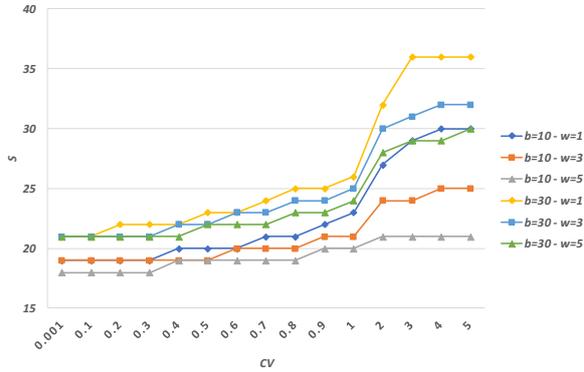
We conduct a sensitivity analysis to show the effect of the lifetime variability, the cost parameters, and the base-stock policy on the optimal base-stock level and the related optimal cost. The results are given in Figure 5 with $b = 10, 30$ and $w = 1, 3, 5$. The main observations can be summarized as



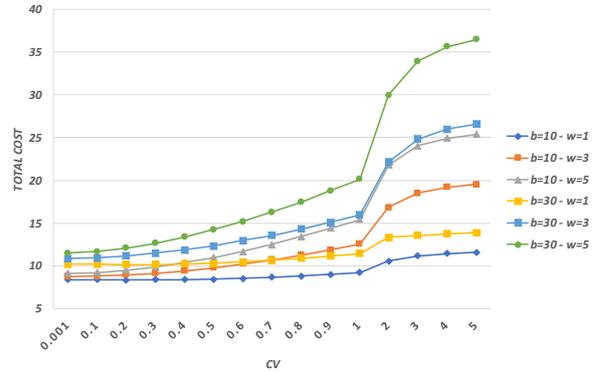
(a) Optimal base-stock level in the lost-sales case



(b) Optimal cost in the lost-sales case



(c) Optimal base-stock level in the backorders case



(d) Optimal cost in the backorders case

Figure 5: Numerical results ($h = 1, \lambda = 4, L = 3$ and $m = 3$)

follows:

- *Effect of CV.* We observe that the optimal base-stock level and the optimal cost increase with the lifetime variability. With a high variability, the risk of having a short life for one item is high. For the overall items, this risk can be reduced by having a large number of items on-hand. Thus, the risk of a short lifetime can be compensated by the chance of a long one. This result follows from the law of large numbers. The speed convergence of a sum of i.i.d. random variables to the sum of their expected value decreases with the variability of

each random variable. Moreover, we observe that the sensitivity of the base-stock level and the optimal cost to CV is the highest when CV is slightly above 1. For low values of CV , the optimal decisions are mostly driven by the expected value of the lifetime. For high values of CV , we observe a saturation regarding the effect of CV as the number of items in the system seems to be sufficiently high to absorb the negative effect of the variability.

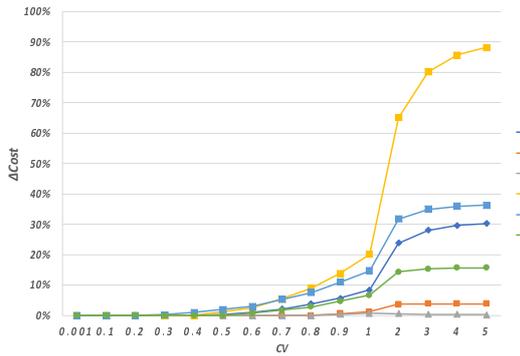
- *Effect of the cost parameters.* As expected, the optimal base-stock level and the optimal cost increase with the lost-sales/backorders cost. The larger is the base-stock level, the higher would be the quantity of expired items. Therefore, the base-stock level reduces with the expiration/outdating cost although the system cost increases with it.
- *Effect of the base-stock policy.* While comparing the lost-sales case with the backorders cases, we observe a higher sensitivity to CV and to the cost parameters in the latter case. In the backorders case when the number of items on-hand reaches zero, all orders are used to serve customers who couldn't be directly satisfied. This slows down the speed of refilling of the on-hand queue as compared to the lost-sales case where customers are rejected when there is no available item. Consequently, the base-stock level and the optimal cost are higher in the backorders case.

Our numerical results show the significant effect of the lifetime variability for decision-makers. This may suggest the need of a better control of the lifetime by investing in the cold chain or in items' composition. In the following section, we further investigate the errors made by employing exponential or deterministic models when deciding for the optimal base-stock level.

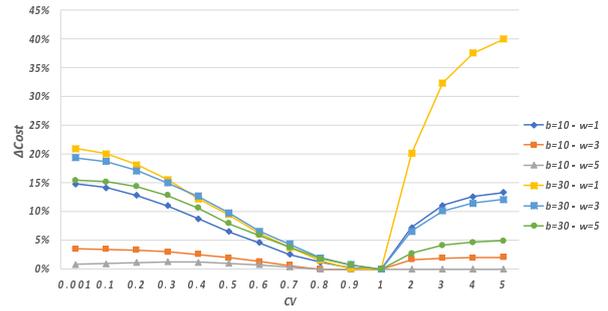
6.2 Exponential or deterministic models; how wrong can they be?

In this section, we numerically analyse the errors made by employing exponential or deterministic models in both the lost-sales and the backorders cases and in an inventory system with dual-sourcing.

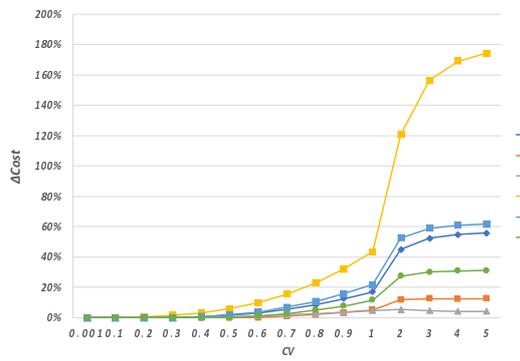
Lost-sales and backorders cases. We first consider the lost-sales and backorders cases. Let ΔS and ΔCost denote the percentage of errors for the base-stock level and the total cost; $\Delta S = \frac{S_{\text{Determ/Expo}} - S_{\text{Gamma}}}{S_{\text{Gamma}}}$, and $\Delta \text{Cost} = \frac{\text{Cost}(S_{\text{Determ/Expo}}) - \text{Cost}(S_{\text{Gamma}})}{\text{Cost}(S_{\text{Gamma}})}$, where $S_{\text{Determ/Expo}}$, and $\text{Cost}(S_{\text{Determ/Expo}})$ denote the optimal base-stock level and the total cost obtained with deterministic/exponential assumptions and S_{Gamma} and $\text{Cost}(S_{\text{Gamma}})$ are the ones obtained with a Gamma lifetime distribution. The cost and base-stock level errors are given in Figures 6 and 7. The errors for the optimal base-stock level and the total cost are substantial, exceeding 40% and 100%, respectively. These errors are the highest for high values of CV . As the deterministic distribution is



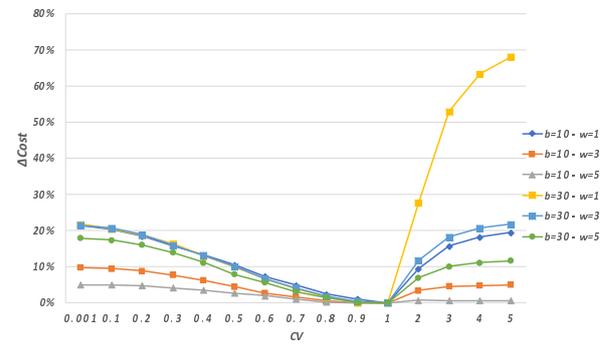
(a) Lost-sales with deterministic lifetime



(b) Lost-sales with exponential lifetime

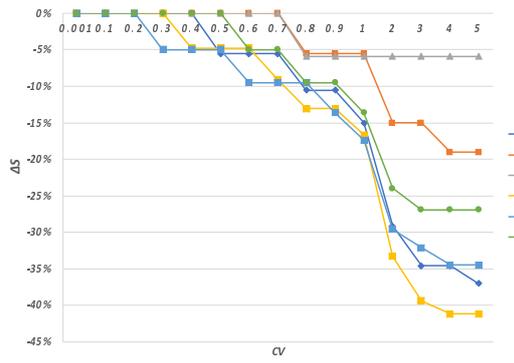


(c) Backorders with deterministic lifetime

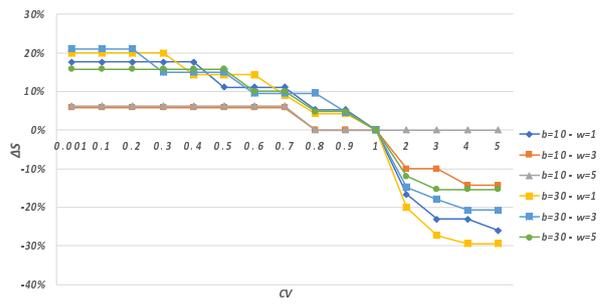


(d) Backorders with exponential lifetime

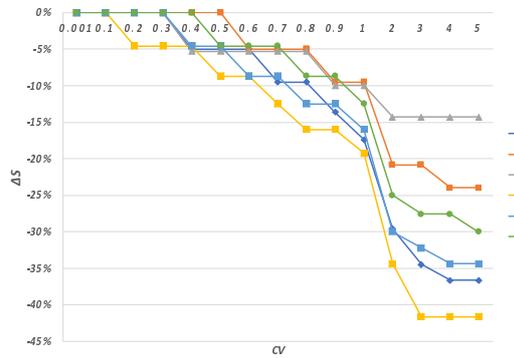
Figure 6: Cost errors ($h = 1, \lambda = 4, L = 3$ and $m = 3$)



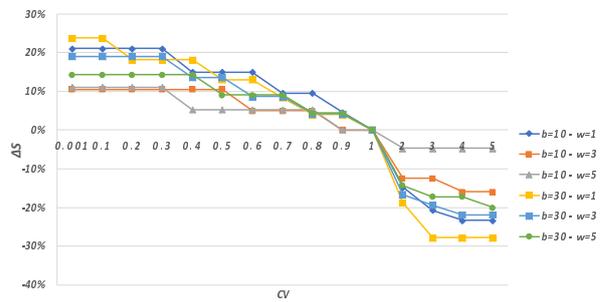
(a) Lost-sales with deterministic lifetime



(b) Lost-sales with exponential lifetime



(c) Backorders with deterministic lifetime



(d) Backorders with exponential lifetime

Figure 7: Base-stock level errors ($h = 1, \lambda = 4, L = 3$ and $m = 3$)

an extreme case of variability, the error is significantly higher when employing this distribution for decision making as compared to the exponential assumption. For instance, with $b = 30$ and $w = 1$, the cost error of a deterministic assumption under the backorders case is 175% and it is 88% in the lost-sales case. This error is reduced to 68% and 40% in the case of an exponential assumption under the backorders case and the lost-sales case, respectively.

As the optimal base-stock increases (respectively, decreases) with the lost-sales/backorders cost (respectively, with the expiry cost), the errors also increase (respectively, decrease) with the lost-sales/backorders cost (respectively, with the expiry cost). This result might be surprising; it means that when items are complicated to reprocess, then existing results of the literature may be applicable. When some items can be recycled as they are still valuable, then the effective expiry cost is reduced which in turn reduce the validity of the exponential assumption.

The dual-sourcing case. In Figure 8, we evaluate the cost errors for an inventory system with dual-sourcing in the lost-sales case. The results are consistent with the previous ones; the errors

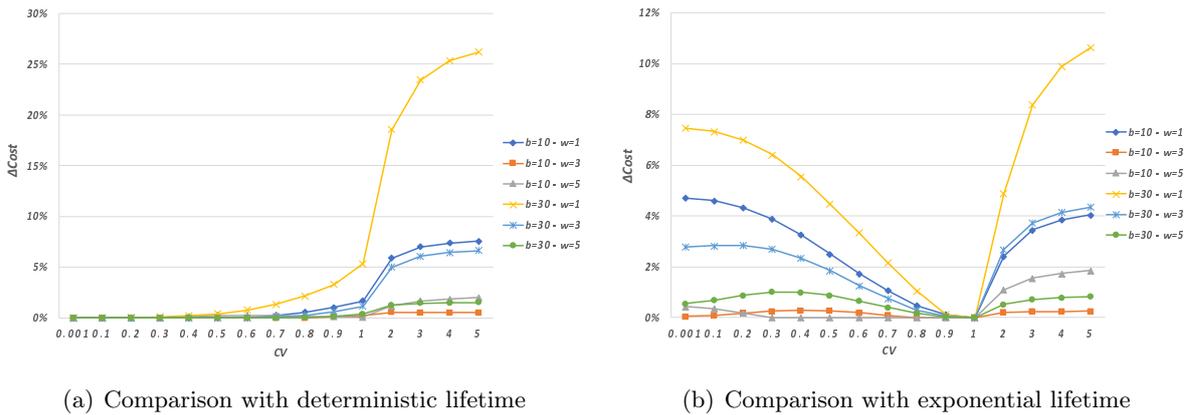


Figure 8: Cost errors in the dual-sourcing case ($h = 1, \lambda = 4, L = 3, c_r = 0.5, c_e = 0.75$ and $m = 3$)

increase with the lifetime variability. However, the effect of CV is reduced with dual-sourcing as compared to the single item case. Dual-sourcing enables to mitigate the impact of the lifetime variability, which consequently reduce the errors when deterministic or exponential assumptions are made. For example, with $CV = 5, b = 30$ and $w = 1$, the cost errors in Figure 8 under the deterministic and the exponential assumptions are equal to 26% and 11% respectively, whereas these were equal to 88% and 40% with single-sourcing.

7 Conclusion

The perishable base-stock inventory system is nowadays used in many contexts including pharmaceutical industry, blood banks, and supermarkets. However, existing results in the literature are restricted to exponential or deterministic assumptions for the lifetime and lead-time. These assumptions may not cover the wide variety of deterioration processes. The aim of this work was to extend the existing results to cases with generally distributed lifetime and lead-time. Based on a queueing network framework, we derived explicit expressions for the stationary distribution of the system state and for the total expected cost in the lost-sales case, the backorders case and in a more complex case with several warehouses and a dual-sourcing policy. In addition, we showed some monotonicity properties of the cost components in the lost-sales case. Next, employing a Gamma distribution for the lifetime, we conducted a numerical investigation to analyze the impact of the lifetime variability and the cost parameters on the optimal base-stock level and on the expected cost. Our results assessed the intuition that the optimal base-stock level increases with the lifetime variability and the lost-sales or backorders cost. Surprisingly, we showed that the base-stock level decreases and the total cost increases with the cost per expired item. Finally, our study revealed that the errors resulting from deterministic or exponential assumptions can be substantial.

In practice, this means that ignoring the real value of the lifetime variability can lead to significant costs. This is particularly the case in the context of blood bank inventory management. The lifetime of blood platelets received by a given hospital can vary considerably (Nahmias, 1977). Therefore, assuming that all replenishment orders have the same deterministic lifetime is unrealistic and may lead to wrong decisions regarding the base-stock level. In the context of perishable products sold in supermarkets or by retailers, the orders also have different shelf-lives (Gürler and Özkaya, 2008). In practice, a periodic review base-stock policy is commonly implemented to replenish inventories in supermarkets (Haijema and Minner, 2019). However, the literature assumes that orders have the same deterministic lifetime. We believe that adjusting our results to inventory policies under different shelf-lives for items that form the same replenishment order could significantly reduce costs in supermarkets.

An interesting avenue for future research would be to extend this work to other inventory control policies used in the literature such as the reorder point order-up-to-level (s, S) policy. Recall that we used a "first-in-first-out" discipline for delivering an item to a customer. This policy makes sense when the lifetime of each item has the increasing failure rate property like in the deterministic case. However, in general, this policy may not be optimal. It would be interesting to extend the paper results to cases with service disciplines different from first-in-first-out, such as low perish time first.

Our results could also be used as a first step to analyze spare parts inventory control systems with preventive maintenance.

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Appendix

This appendix provides the tables for errors in percentages in support of the numerical analysis provided in Section 6.

Table 2: Error in (%) of the total cost when the lifetime is assumed to be deterministic or exponentially distributed

b	CV	Errors from deterministic lifetime						Errors from exponential lifetime					
		Lost-sales			Backorders			Lost-sales			Backorders		
		$w = 1$	$w = 3$	$w = 5$	$w = 1$	$w = 5$	$w = 5$	$w = 1$	$w = 3$	$w = 5$	$w = 1$	$w = 5$	$w = 5$
10	0.001	0	0	0	0	0	0	15	4	1	21	10	5
10	0.1	0	0	0	0	0	0	14	3	1	20	10	5
10	0.2	0	0	0	0	0	0	13	3	1	18	9	5
10	0.3	0	0	0	0	0	0	11	3	1	16	8	4
10	0.4	0	0	0	1	0	0	9	3	1	13	6	4
10	0.5	0	0	0	2	0	1	6	2	1	11	4	3
10	0.6	1	0	0	3	0	1	5	1	1	7	3	2
10	0.7	2	0	0	6	1	2	3	1	0	5	2	1
10	0.8	4	0	0	9	2	3	1	0	0	2	1	0
10	0.9	6	1	0	13	4	4	0	0	0	1	0	0
10	1	8	1	1	17	5	5	0	0	0	0	0	0
10	2	24	4	1	45	12	5	7	2	0	9	3	1
10	3	28	4	0	52	13	5	11	2	0	16	5	1
10	4	30	4	0	55	13	4	13	2	0	18	5	1
10	5	30	4	0	56	13	4	13	2	0	19	5	1
30	0.001	0	0	0	0	0	0	21	19	15	22	21	18
30	0.1	0	0	0	0	0	0	20	19	15	21	21	17
30	0.2	0	0	0	1	0	0	18	17	14	19	19	16
30	0.3	0	0	0	2	0	0	16	15	13	16	16	14
30	0.4	0	1	0	3	1	0	12	13	11	13	13	11
30	0.5	1	2	0	6	2	0	10	10	8	10	10	8
30	0.6	3	3	1	10	4	1	6	7	6	7	7	6
30	0.7	5	5	2	16	7	3	4	4	4	4	4	3
30	0.8	9	8	3	23	11	5	1	2	2	2	2	1
30	0.9	14	11	5	32	16	8	0	1	1	0	0	0
30	1	20	15	7	43	22	12	0	0	0	0	0	0
30	2	65	32	14	121	53	28	20	7	3	28	12	7
30	3	80	35	15	157	59	30	32	10	4	53	18	10
30	4	86	36	16	169	61	31	38	11	5	63	21	11
30	5	88	36	16	175	62	31	40	12	5	68	22	12

Deterministic parameters: $\lambda = 4, L = 3, h = 1$.

Table 3: Error in (%) of the optimal base-stock level S when the lifetime is assumed to be deterministic or exponentially distributed

b	CV	Errors from deterministic lifetime						Errors from exponential lifetime					
		Lost-sales			Backorders			Lost-sales			Backorders		
		$w = 1$	$w = 3$	$w = 5$	$w = 1$	$w = 5$	$w = 5$	$w = 1$	$w = 3$	$w = 5$	$w = 1$	$w = 5$	$w = 5$
10	0.001	0	0	0	0	0	0	18	6	6	21	11	11
10	0.1	0	0	0	0	0	0	18	6	6	21	11	11
10	0.2	0	0	0	0	0	0	18	6	6	21	11	11
10	0.3	0	0	0	0	0	0	18	6	6	21	11	11
10	0.4	0	0	0	-5	0	-5	18	6	6	15	11	5
10	0.5	-6	0	0	-5	0	-5	11	6	6	15	11	5
10	0.6	-6	0	0	-5	-5	-5	11	6	6	15	5	5
10	0.7	-6	0	0	-10	-5	-5	11	6	6	10	5	5
10	0.8	-11	-6	-6	-10	-5	-5	5	0	0	10	5	5
10	0.9	-11	-6	-6	-14	-10	-10	5	0	0	5	0	0
10	1	-15	-6	-6	-17	-10	-10	0	0	0	0	0	0
10	2	-29	-15	-6	-30	-21	-14	-17	-10	0	-15	-13	-5
10	3	-35	-15	-6	-34	-21	-14	-23	-10	0	-21	-13	-5
10	4	-35	-19	-6	-37	-24	-14	-23	-14	0	-23	-16	-5
10	5	-37	-19	-6	-37	-24	-14	-26	-14	0	-23	-16	-5
30	0.001	0	0	0	0	0	0	20	21	16	24	19	14
30	0.1	0	0	0	0	0	0	20	21	16	24	19	14
30	0.2	0	0	0	-5	0	0	20	21	16	18	19	14
30	0.3	0	-5	0	-5	0	0	20	15	16	18	19	14
30	0.4	-5	-5	0	-5	-5	0	14	15	16	18	14	14
30	0.5	-5	-5	0	-9	-5	-5	14	15	16	13	14	9
30	0.6	-5	-10	-5	-9	-9	-5	14	10	10	13	9	9
30	0.7	-9	-10	-5	-13	-9	-5	9	10	10	8	9	9
30	0.8	-13	-10	-10	-16	-13	-9	4	10	5	4	4	4
30	0.9	-13	-14	-10	-16	-13	-9	4	5	5	4	4	4
30	1	-17	-17	-14	-19	-16	-13	0	0	0	0	0	0
30	2	-33	-30	-24	-34	-30	-25	-20	-15	-12	-19	-17	-14
30	3	-39	-32	-27	-42	-32	-28	-27	-18	-15	-28	-19	-17
30	4	-41	-34	-27	-42	-34	-28	-29	-21	-15	-28	-22	-17
30	5	-41	-34	-27	-42	-34	-30	-29	-21	-15	-28	-22	-20

Deterministic parameters: $\lambda = 4, L = 3, h = 1$.

Table 4: Error in (%) of the total cost when the lifetime is assumed to be deterministic or exponentially distributed under dual-sourcing and lost-sales case

CV	Deterministic lifetime distribution						Exponential lifetime distribution					
	b=10			b=30			b=10			b=30		
	w=1	w=3	w=5	w=1	w=3	w=5	w=1	w=3	w=5	w=1	w=3	w=5
0.001	0	0	0	0	0	0	5	0	0	7	3	1
0.1	0	0	0	0	0	0	5	0	0	7	3	1
0.2	0	0	0	0	0	0	4	0	0	7	3	1
0.3	0	0	0	0	0	0	4	0	0	6	3	1
0.4	0	0	0	0	0	0	3	0	0	6	2	1
0.5	0	0	0	0	0	0	3	0	0	4	2	1
0.6	0	0	0	1	0	0	2	0	0	3	1	1
0.7	0	0	0	1	0	0	1	0	0	2	1	0
0.8	1	0	0	2	0	0	0	0	0	1	0	0
0.9	1	0	0	3	1	0	0	0	0	0	0	0
1	2	0	0	5	1	0	0	0	0	0	0	0
2	6	1	1	19	5	1	2	0	1	5	3	1
3	7	1	2	23	6	1	3	0	2	8	4	1
4	7	1	2	25	6	2	4	0	2	10	4	1
5	8	1	2	26	7	2	4	0	2	11	4	1

Deterministic parameters: $h = 1, \lambda = 4, L = 3, c_r = 0.5, c_e = 0.75$